Improved Linear Quadratic Regulator for Spacecraft Docking using Krotov Conditions

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I. Introduction

Autonomous Spacecraft and Rendezvous is one of the most complex and challenging component of modern spacecraft and satellite missions. It requires two vehicles to come together to an common orbit, rendezvous and dock together for payload supply or repair. It has application on debris-management in close proximity analysis[]. Various control strategies are introduced in the literature with varying system complexity. Vision based navigation system is discussed in []. Coordinate transformation on the image plane is used to get the relative pose which can then be reduce using suitable controllers. Feedback law control like MPC are also discussed in [] which gives advantage over open loop Δv sequencing discussed in [].

With increasing mission complexity, dynamic controllers are needed so as to adjust to mission design. Conventional control methods that are generally used in literature is LQR or modification of it. But dynamic change of cost function is necessary to adjust to specific mission requirements. Calculation of variation and Hamilton-Jacobi-Bellman equation-based approaches are widely employed for solving this [], there are some assumptions associated with these approaches in their solution procedure that provide hindrance to solutions in some cases that can be a part of mission cost constraints. These loopholes are not addressed yet. We try to overcome this problem by using Krotov functions for developing global optimum which will provide optimal solutions for all cost scenarios.

The paper is divided into 5 sections. In section 1 Problem statement is described. In section 2, mathematical model for system dynamics is describes using Clohessy–Wiltshire–Hill (CWH) relative motion model[] using Cartesian coordinates in Hill frame. In Section 3 we develop the optimal control using Krotov functions on our system dynamics. In section 4 we analyse our control on various mission cost scenarios and Section 5 concluding remarks are given.

II. System Description

A general docking scenario consist of a target vehicle A and a chaser vehicle B. A is in an keplerian circular orbit with radius r_0 while B performs maneuvers at position vector δr from A in an non-keplerian orbit. From the figure,

$$\vec{r} = \vec{r}_0 + \vec{\delta} r$$

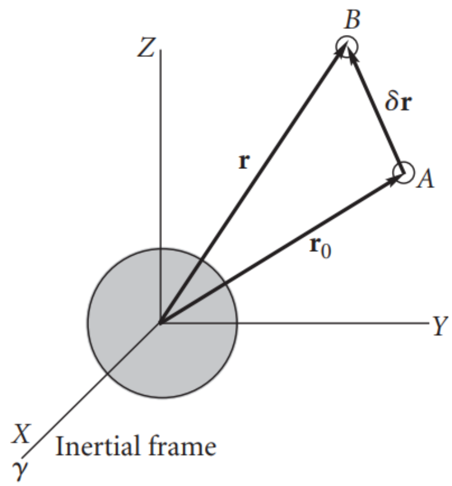

Fig. 1 Orbital framework of Docking: A is target and B is chaser.

with

$$\frac{\delta r}{r_0} << 1$$

We define a moving frame(CWH frame) xyz with origin on A with,

$$\hat{i} = \frac{\vec{r}_0}{r_0}$$

y-axis is the local horizon and $\hat{k} = \hat{i} \times \hat{j}$

Let angular velocity of A(and so of the CWH frame) be Ω

III. Mathematical Model

The equations of motion for the chaser spacecraft are nonlinear and can be expressed in vector form as:

$$\ddot{\vec{r}} = -\mu \frac{\vec{r}}{r^3} + \frac{\vec{F}}{m} \tag{1}$$

$\vec{F}$ is force applied on the spacecraft.

Putting $\vec{r} = \vec{r}_0 + \vec{\delta} r$ in (1),

$$\ddot{\vec{\delta r}} = -\ddot{\vec{r}}_0 - \mu \frac{\vec{r}_0 + \delta \vec{r}}{r^3} + \frac{\vec{F}}{m} \tag{2}$$

Now,

$$r^{-3} = r_0^{-3} \left(1 + \frac{2\vec{r}_0 \cdot \vec{\delta} r}{r_0^2} \right)^{-3/2}$$

by neglecting $\left(\frac{\delta}{r_0}\right)^2$,

Using Binomial theorem this simplifies as:

$$\left(1+\frac{2\vec{r}_0\cdot\vec{\delta r}}{r_0^2}\right)^{-3/2} = 1+\left(\frac{-3}{2}\right)\left(1+\frac{2\vec{r}_0\cdot\vec{\delta r}}{r_0^2}\right)$$

$$\therefore \frac{1}{r^3} = \frac{1}{r_0^3} - \frac{3}{r_0^5}\vec{r}_0\cdot\vec{\delta r}$$

Putting this in (2):

$$\ddot{\vec{\delta r}} = \ddot{\vec{r}}_0 - \mu\left(\frac{1}{r_0^3} - \frac{3}{r_0^5}\vec{r}_0\cdot\vec{\delta r}\right)(\vec{r}_0+\delta\vec{r}) + \frac{\vec{F}}{m}$$

$$= -\ddot{\vec{r}}_0 - \mu\left[\frac{\vec{r}_0}{r_0^3} + \frac{\vec{\delta r}}{r_0^3} - \frac{3(\vec{r}_0\cdot\vec{\delta r})\vec{r}_0}{r_0^5} + \text{Higher order terms in } \delta r \text{ (which we neglect)}\right] + \frac{\vec{F}}{m}$$

$$\therefore \ddot{\vec{\delta r}} = -\ddot{\vec{r}}_0 - \mu\left[\frac{\vec{r}_0}{r_0^3} + \frac{\vec{\delta r}}{r_0^3} - \frac{3(\vec{r}_0\cdot\vec{\delta r})\vec{r}_0}{r_0^5}\right] + \frac{\vec{F}}{m}$$

Now dynamical equation of the target vehicle is

$$\ddot{\vec{r}}_0 = -\mu\frac{\vec{r}_0}{r_0^3}$$

Substituting above,

$$\ddot{\vec{\delta r}} = -\frac{\mu}{r_0^3}\left[\delta\vec{r} - \frac{3(\vec{r}_0\cdot\vec{\delta r})\vec{r}_0)}{r_0^2}\right] + \frac{\vec{F}}{m} \tag{3}$$

Now,

$$\vec{\delta r} = \delta x\hat{i} + \delta y\hat{j} + \delta z\hat{k}$$

Using vector kinematics, we know that,

$$\ddot{\vec{r}} = \ddot{\vec{r}}_0 + \vec{\Omega}\times\delta r + \dot{\vec{\Omega}}\times(\vec{\Omega}\times\vec{\delta r}) + 2\vec{\Omega}\times\delta\vec{v}_{rel} + \vec{\delta}a_{rel}$$

Putting $\vec{r} = \vec{r}_0 + \delta r$ above,

$$\ddot{\vec{\delta r}} = \dot{\vec{\Omega}}\times\vec{\delta r} + \vec{\Omega}\times(\vec{\Omega}\times\vec{\delta r}) + 2\vec{\Omega}\times\vec{\delta}v_{rel} + \vec{\delta}a_{rel}$$

Assuming target vehicle to be in **circular orbit**, $\dot{\vec{\Omega}} = 0$,

$$\ddot{\vec{\delta r}} = \vec{\Omega}\times(\vec{\Omega}\times\vec{\delta r}) + 2\vec{\Omega}\times\vec{\delta}v_{rel} + \vec{\delta}a_{rel}$$

Expanding the RHS,

$$\ddot{\vec{\delta r}} = \vec{\Omega}(\vec{\Omega}\cdot\vec{\delta r}) - \Omega^2\vec{\delta r} + 2\vec{\Omega}\times\vec{\delta}v_{rel} + \vec{\delta}a_{rel}$$

Putting $\vec{\Omega} = n\hat{k}$, $\vec{\delta}v_{rel} = \delta\dot{x}\hat{\imath} + \delta\dot{y}\hat{\jmath} + \delta\dot{z}\hat{k}$ and $\vec{\delta}a_{rel} = \delta\ddot{x}\hat{\imath} + \delta\ddot{y}\hat{\jmath} + \delta\ddot{z}\hat{k}$ and recollecting terms this simplifies as,

$$\ddot{\vec{\delta r}} = (-n^2\delta x - 2n\delta\dot{y} + \delta\ddot{x})\hat{\imath} + (-n^2\delta y - 2n\delta\dot{x} + \delta\ddot{y})\hat{\jmath} + \delta\ddot{z}\hat{k} \tag{4}$$

Now from (3),by substituting values of $\vec{\delta r}$, $\vec{r}_0 \cdot \delta r = r_0\delta x$ and using $n^2 = \frac{\mu}{r_0^3}$ we get,

$$\ddot{\vec{\delta r}} = -n^2\left[\delta x\hat{\imath} + \delta y\hat{\jmath} + \delta z\hat{k} - \frac{3}{r_0^2}(r_0\delta x)r_0\hat{\imath}\right] + \frac{F_x}{m}\hat{\imath} + \frac{F_y}{m}\hat{\jmath} + \frac{F_z}{m}\hat{k} = (2n^2\delta x + \frac{F_x}{m})\hat{\imath} + (-n^2\delta y + \frac{F_y}{m})\hat{\jmath} + (-n^2\delta z + \frac{F_z}{m})\hat{k} \tag{5}$$

Equating the value of $\ddot{\vec{\delta r}}$ from equation (4) and (5) and collecting terms,

$$(\delta\ddot{x} - 3n^2\delta x - 2n\delta\dot{y})\hat{\imath} + (\delta\ddot{y} + 2n\delta\dot{x})\hat{\jmath} + (\delta\ddot{z} + n^2\delta z)\hat{k} = \frac{F_x}{m}\hat{\imath} + \frac{F_y}{m}\hat{\jmath} + \frac{F_z}{m}\hat{k}$$

This can be simplified to give linear CWH equation:

$$\delta\ddot{x} - 3n^2\delta x - 2n\delta\dot{y} = \frac{F_x}{m} = u_x \tag{6}$$

$$\delta\ddot{y} + 2n\delta\dot{x} = \frac{F_y}{m} = u_y \tag{7}$$

$$\delta\ddot{z} + n^2\delta z = \frac{F_z}{m} = u_z \tag{8}$$

This model can be formulated as:

$$\dot{X} = AX + BU$$

where X $\in \mathcal{R}^6$ is the state vector and U $\in \mathcal{R}^3$ is the control vector.

$$X = \begin{bmatrix} \delta x \\ \delta y \\ \delta z \\ \delta\dot{x} \\ \delta\dot{y} \\ \delta\dot{z} \end{bmatrix} \quad , \quad U = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$$

and,

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3n^2 & 0 & 0 & 0 & 2n & 0 \\ 0 & 0 & 0 & -2n & 0 & 0 \\ 0 & 0 & -n^2 & 0 & 0 & 0 \end{bmatrix} , \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A. Controllability:

Now as we have derived our system model, we must check if the system is controllable. This is necessary as system must have ability to move the internal states from any initial state to any final state in a finite time interval using any control input. Control input is subjected to noise from sensor and controllable system guarantees that any final states can be reached. Now,

$$rank \begin{bmatrix} A & A^2 B \ldots & A^{(n-1)} B \end{bmatrix} = 6 = rank(A)$$

∴ The derived system satisfies the necessary and sufficient rank criteria for controllability.

IV. Optimal Control Design Using Krotov Conditions

LQR control is selected for the controlling the system dynamics. LQR, as an optimal in terms of energy-like regulator, provides robust stability with a minimized energy-like performance index. It is also very computationally efficient as gain matrix can be calculated offline. The cost function used in infinite horizon LQR is,

$$min\ J = \int_0^\infty X^T QX + U^T RU\, dt \tag{9}$$

where Q≥0 and R>0 ∀ t are gain matrices appropriately chosen related to the system dynamics.

Although the calculation of variation and HJB equation-based(derived from Dynamic Programming) approaches are widely employed for solving this, there are some assumptions associated with these approaches in their solution procedure. Specifically, the calculus of variation approach uses costates and their relationship with states to compute the optimal control law. Similarly, the HJB equation-based approach requires the existence of the continuously differentiable optimal cost function, and it's gradient with respect to the state is the costate corresponding to the optimal trajectory.So, the information about the optimal cost function must be known a priori.

This poses some challenges as different cost functions are required for different mission scenarios. We require a universal method that will work for all types of cost functions.

We now develop optimal control by using Krotov functions[] which will give sufficient conditions and the control law derived will provide global optimum without encountering issues stated above which will be valid for all types of cost functions.This will provide optimal solutions for all type of cost functions depending on mission requirements during Docking.

From the equivalency principle, J from equation (9) can be restated as:

$$min\ J_{eq} = L(x(0), t)) + \int_0^\infty S(X, U, t))\, dt \tag{10}$$

where L is a Krotov function which when chosen appropriately reduces the non-convex function and

$$S = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial X}[AX + BU] + X^T QX + U^T RU \tag{11}$$

Now,

Theorem:- Krotov sufficient conditions: If $(x^*(t), u^*(t))$ is and admissible process such that

$$S(x^*(t), u^*(t)) = \min_{(X,U)\in\mathcal{R}^6\times\mathcal{R}^3} S(X, U, t), \forall t \in [t, \infty]$$

then then $(x^*(t), u^*(t))$ is an optimal process.

Proof: Refer to Section 2.3 of [2]

So, the problem reduces to

$$\min_{(X,U)\in\mathcal{R}^6\times\mathcal{R}^3} S(X, U, t), \forall t \in [t, \infty] \tag{12}$$

This gives us a sufficient condition as minimum S will minimise the cost function. As S is non-convex, we choose our Krotov function to be $L = X^T PX$,

Hence, putting L in (11) we get,

$$S = X^T \dot{P}X + X^T (P + P^T)[AX + BU] + X^T QX + U^T RU$$

$$= X^T \dot{P}X + X^T PAX + X^T P^T AX + X^T PBX + X^T P^T BX + X^T QX + U^T RU$$

Adding and subtracting $X^T (\frac{1}{2}PBR^{-1}B^T P + \frac{1}{4}PBR^{-1}B^T P^T + \frac{1}{4}P^T BR^{-1}B^T P)X$ and collecting terms:

$$S = X^T (\dot{P} + A^T P + Q - \frac{1}{2}PBR^{-1}B^T P - \frac{1}{4}PBR^{-1}B^T P^T - \frac{1}{4}P^T BR^{-1}B^T P)X + \text{scalar terms}$$

Now S will be convex if,

$$\dot{P} + A^T P + Q - \frac{1}{2} PBR^{-1} B^T P - \frac{1}{4} PBR^{-1} B^T P^T - \frac{1}{4} P^T BR^{-1} B^T P \geq 0$$

We set $\dot{P} = 0$, as a static P is achieved for infinite horizon case and it has been also demonstrated that calculating P iteratively is computationally complex and provides little benfit over static case. For optimality P must satisfy,

$$A^T P + Q - \frac{1}{2} PBR^{-1} B^T P - \frac{1}{4} PBR^{-1} B^T P^T - \frac{1}{4} P^T BR^{-1} B^T P = 0 \tag{13}$$

Therefore the optimal Control solution is:

$$U^* = -\frac{1}{2} R^{-1} B^T (P^T + P) X^* \tag{14}$$

This provides a feedback controller for the system dynamics where required P can be calculated offline or online based on mission requirements.

Docking requires a robust control as parameter error can lead to unwanted deviations from the calculated optimal solution. System is prone to parameter uncertainties and checking stability is necessary for convergence to equilibrium states.

For checking stability of the calculated control solution, we choose Lyapunov function $V = X^T (P^T + P) X$, assuming $P + P^T > 0$, then:

$$\dot{V} = X^T (P^T + P) \dot{X} + \dot{X}^T (P^T + P) X$$

Now, using $\dot{X} = AX + BU$ where U is substituted from (14)

$$\dot{V} = 2X^T (-Q - \frac{1}{2} PBR^{-1} B^T P - \frac{1}{4} PBR^{-1} B^T P^T - \frac{1}{4} P^T BR^{-1} B^T P) X \leq 0$$

This inequality is holds truth as, assuming Q≥0, the other terms within the bracket are negative since R>0,B>0 and $P + P^T > 0$. Hence, for, $P + P^T > 0$ the closed loop system is Lyapunov stable for $Q \geq 0$ and asymptotically stable for Q>0

NOW, Gain matrix Q and R are selected as diagonal matrix with Q>0 for asymptotic stability.

$$Q = \begin{pmatrix} \frac{\alpha_1^2}{(X_1)^2_{max}} & 0 & 0 & 0 & 0 & \\ 0 & \frac{\alpha_2^2}{(X_2)^2_{max}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\alpha_3^2}{(X_3)^2_{max}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\alpha_4^2}{(X_4)^2_{max}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\alpha_5^2}{(X_5)^2_{max}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\alpha_2^6}{(X_6)^2_{max}} \end{pmatrix}$$

and

$$R = \begin{pmatrix} \frac{\beta_1^2}{(U_1)^2_{max}} & 0 & 0 \\ 0 & \frac{\beta_2^2}{(U_2)^2_{max}} & 0 \\ 0 & 0 & \frac{\beta_3^2}{(U_3)^2_{max}} \end{pmatrix}$$

where $\sum_{i=1}^{6} \beta_i^2 = 1$ and $\sum_{i=1}^{6} \alpha_i^2 = 1$ which are used to add an additional relative weighting on the various components of the state/control and $(X_i)_{max}$and $(U_i)_{max}$ represent the largest desired response/control input for that component of the state/actuator signal.

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