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AE691A Project Report

Optimality of Error Dynamics in Flight Guidance

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AE691A Course Project

I. Abstract

Optimal guidance for flying objects is a heavily researched area. Need for minimum cost trajectories help us cross the practical limitations. In the project we present guidance strategies using optimal error dynamics. Application to missile guidance and spacecraft docking problem is presented. This project demonstrates use of optimal error dynamics as an efficient and viable guidance strategy design.

II. Introduction

Missile guidance problem holds rich history with extensive research study. Development of missile guidance laws for successful interception of target is hold motivation due to its application in many fields: from heavy object deflection to protection against attacking vehicles/missiles. Traditionally proportional guidance navigation (PNG) law was used, where acceleration command was based on derivative of LOS angle, chosen so as to nullify it. Recent developed non-linear controllers use tracking error for developing guidance laws. The aim is to asymptotically converge error to zero in finite time. To regulate the tracking error to be zero, various control theories such as SMC, the Lyapunov function, and feedback linearization have been applied to system in previous works. In these approaches, the desired error dynamics is first selected, and then appropriate control input is calculated to ensure that the system equation shown in follows the selected error dynamics. All these method only focuses on asymptotic convergence. The attention on how fast the error converges, a major constraint of finite time missile guidance, seems to be missing. Finite time convergence is not guaranteed. Also, the optimality of the developed law with respect to meaningful cost function/performance index is not considered. This is addressed in this project. For higher dimension case, this theorem is extended and application to spacecraft docking problem is presented. Numerical simulations are also presented for both cases.

III. Optimal Error Dynamics

The general form of tracking problem that is often observed in missile guidance is :

$$\dot{\epsilon}(t) = g(t)u(t) \tag{1}$$

where $\epsilon(t)$ represents the tracking error, $g(t)$ denotes a known time varying function and $u(t)$ is the control input. Depending on geometry and mission requirement, $\epsilon(t)$ can be chosen. For example, in missile guidance tracking error can be ZEM, the LOS rate, the impact-angle error, the impact-time error, the heading error, etc. In spacecraft attitude control problem, this can be taken as attitude error. The function $g(t)$ can be chosen as any continuous function which is not identically zero(as tracking problem is controllable).

Popular control theories used in missile guidance like SMC, feedback linearisation etc. set $g(t)$ equal a constant k and derive control necessary for following resulting error dynamics. This ensures the tracking error converges to zero asymptotically with an exponential rate governed by the guidance gain k. This desired error dynamics has two major drawbacks:

- The finite-time convergence is not strictly guaranteed.
- It only focuses on how to drive the tracking error to 0 and never considers how to achieve zero tracking error optimally with respect to a meaningful performance index.

To get around this, we derive optimal error dynamics that minimises the control effort required.

Theorem 1. *For the system dynamics given by Eq. 1, the cost function*

$$J = \frac{1}{2}\int_{t}^{t_f} R(\tau)u^2(\tau)d\tau \tag{2}$$

is minimised if we choose error dynamics given by

$$\dot{\epsilon}(t) + \frac{\Gamma(t)}{t_{go}} = 0 \tag{3}$$

where

$$\Gamma(t) = \frac{t_{go}R^{-1}(\tau)g^2(\tau)}{\int_t^{t_f} R^{-1}(\tau)g^2(\tau)d\tau} \tag{4}$$

Proof. Optimal control problem is

$$\min_{u} J = \frac{1}{2}\int_{t}^{t_f} R(\tau)u^2(\tau)d\tau$$

subject to:

$$\dot{\epsilon}(t) = g(t)u(t) \qquad \epsilon(t_f) = 0$$

Integrating the dynamical equation from $t = t$ to $t = t_f$, we get,

$$\epsilon(t_f) - \epsilon(t) = \int_{t}^{t_f} g(\tau)u(\tau)d\tau$$

As $\epsilon(t_f) = 0$,

$$-\epsilon(t) = \int_{t}^{t_f} g(\tau)u(\tau)d\tau$$

Right hand side can be decomposed by introducing slack variable $R(t)$,

$$-\epsilon(t) = \int_t^{t_f} g(\tau)R^{-\frac{1}{2}}(\tau)R^{\frac{1}{2}}(\tau)u(\tau)d\tau$$

Applying Schwarz's inequality to the preceding equation yields,

$$[-\epsilon(t)]^2 \leq \int_t^{t_f} g^2(\tau)R^{-1}(\tau)d\tau \int_t^{t_f} R(\tau)u^2(\tau)d\tau$$

Rewriting,

$$\int_t^{t_f} R(\tau)u^2(\tau)d\tau \geq \frac{[-\epsilon(t)]^2}{\int_t^{t_f} g^2(\tau)R^{-1}(\tau)d\tau}$$

Equality (hence minimum value) is achieved when for some constant C,

$$u(t) = CR^{-1}(t)g(t)$$

Hence,

$$-\epsilon(t) = C\int_t^{t_f} g^2(\tau)R^{-1}(\tau)d\tau$$

Solving for C and substituting,

$$u(t) = \frac{R^{-1}(\tau)g^2(\tau)}{\int_t^{t_f} R^{-1}(\tau)g^2(\tau)d\tau}\epsilon(t) \tag{5}$$

which is the same as obtained from error dynamics proposed, completing the proof. □

A. Extension of argument to n-dimensional system

The theorem presented above in [1] is only for 1-dimensional systems. We extend the argument to n-dimension:

Theorem 2. *For the system dynamics given by vectored form Eq. 1 with $\epsilon \in \mathcal{R}^n$, the cost function*

$$J = \frac{1}{2}\int_t^{t_f} U^T(\tau)R(\tau)U(\tau)d\tau \tag{6}$$

*is minimised only if R is **symmetrical** matrix and we choose error dynamics given by*

$$\dot{\epsilon}(t) + \frac{\Gamma(t)}{t_{go}} = 0 \tag{7}$$

where

$$\Gamma(t) = \frac{t_{go}G(\tau)R^{-1}(\tau)G^T(\tau)}{\int_t^{t_f} G(\tau)R^{-1}(\tau)G^T(\tau)d\tau} \tag{8}$$

Proof. Proceeding same as above,

$$\begin{aligned} -\epsilon(t) &= \int_t^{t_f} G(\tau)U(\tau)d\tau \\ &= \int_t^{t_f} G(\tau)e^{-\frac{1}{2}\log R(\tau)}e^{\frac{1}{2}\log R(\tau)}U(\tau)d\tau \end{aligned}$$

where $\log R$ is defined as,

$$\log R = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{(R-I)^k}{k}$$

Applying Schwarz's inequality,

$$[-\epsilon(t)]^2 \leq \int_t^{t_f} G(\tau)e^{-\frac{1}{2}\log R(\tau)}e^{-\frac{1}{2}(\log R(\tau))^T}G^T(\tau)d\tau \int_t^{t_f} U^T(\tau)e^{\frac{1}{2}(\log R(\tau))^T}e^{\frac{1}{2}\log R(\tau)}U(\tau)d\tau$$

where we used the property:

$$(e^A)^T = e^{A^T}$$

Similarly using matrix logarithm property:

$$(\log(A))^T = \log(A^T)$$

the inequality simplifies as

$$[-\epsilon(t)]^2 \leq \int_t^{t_f} G(\tau)e^{-\frac{1}{2}\log R(\tau)}e^{-\frac{1}{2}\log R^T(\tau)}G^T(\tau)d\tau \int_t^{t_f} U^T(\tau)e^{\frac{1}{2}\log R^T(\tau)}e^{\frac{1}{2}\log R(\tau)}U(\tau)d\tau$$

Now using the **Baker–Campbell–Hausdorff formula** which states:

$$\begin{aligned} &e^X e^Y = e^Z \\ &Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \ldots \end{aligned}$$

where $[X,Y] = XY - YX$ is the commutator.

Hence finally notice that above inequality reduces to

$$[-\epsilon(t)]^2 \leq \int_t^{t_f} G(\tau)R^{-1}(\tau)G^T(\tau)d\tau \int_t^{t_f} U^T(\tau)R(\tau)U(\tau)d\tau$$

if $[R, R^T] = 0$ which is true if

$$R = R^T$$

Now proceeding similarly as theorem 1 completes the proof. □

B. General comments on derived result

The following properties can be observed about the derived results:

1) Proposed optimal error dynamics is that it guarantees finite-time convergence since it is obtained by directly solving the finite-time optimal tracking problem.
2) Instead of constant gain k, here the gain $\frac{\Gamma(t)}{t_{go}}$ is time varying with

$$\lim_{t_{go}\to 0} \frac{\Gamma(t)}{t_{go}} = \infty$$

IV. Application to Missile guidance

A. System Model

Fig 1 below shows the system model for the problem. M and T denote the missile and target, respectively. The notation of (X_I,Y_I) represents the inertial frame. For the purpose of introducing the linearized kinematics, a new frame called the reference frame (X_R,Y_R) is also defined. This frame is rotated from the inertial frame by σ_0, which is the reference angle. The variables of σ and γ stand for the LOS angle and flight-path angle, respectively; r denotes the relative distance between the target and the missile; y is the relative distance between the target and the missile perpendicular to the X_R direction; and a_M and a_T are the missile and target accelerations normal to the velocity vectors, respectively. The variables of a_{M_σ} and a_{T_σ} denote the missile and target accelerations normal to the LOS direction, respectively.

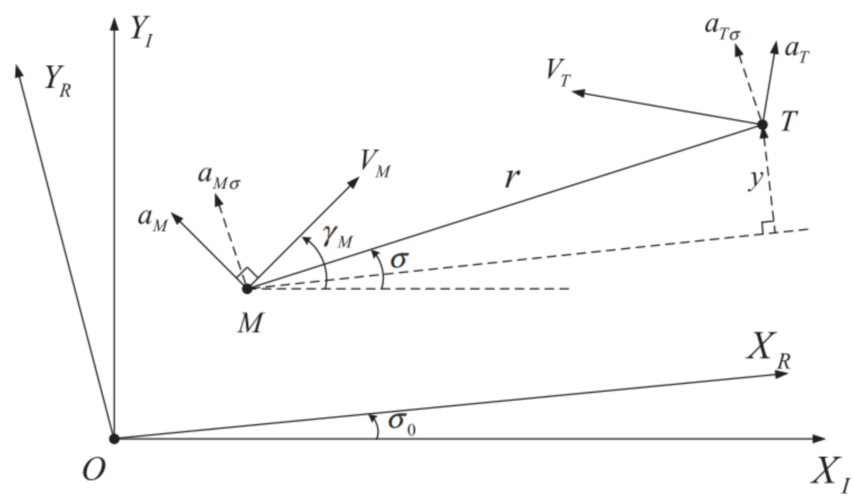

Fig. 1 The homing engagement geometry and parameter definitions.

Fig. 1

From Figure 1, the governing kinematics can be expressed as,

$$\dot{y} = v \tag{9}$$

$$\dot{v} = a_{T_\sigma} \cos(\sigma - \sigma_0) - a_{M_\sigma} \cos(\sigma - \sigma_0) \tag{10}$$

$$a_{M_\sigma} = a_M \cos(\gamma_M - \sigma) \tag{11}$$

$$\dot{\sigma} = \frac{y + v t_{go}}{V_c t_{go}^2} \tag{12}$$

where V_c is the closing velocity. These equations define the kinematic engagement of the missile system.

B. Zero effort miss control

ZEM is the distance that the missile will miss the target, if there is no corrective measure. It is given by:

$$z = y + v t_{go} + \frac{1}{2} a_{T_\sigma} t_{go}^2 \tag{13}$$

Differentiating w.r.t time we get kinematic equation,

$$\dot{z} = -t_{go} a_{M_\sigma} \tag{14}$$

Choosing $\epsilon = z$, we can compare Eq. (14) with Eq. (12) to get $g(t) = -t_{go}$ and $u = a_{M_\sigma}$.

Let us choose the error dynamics as

$$\dot{\epsilon} + \frac{N}{t_{go}} = 0 \tag{15}$$

Then this dynamics minimises

$$J = \frac{1}{2} \int_t^{t_f} \frac{u^2(\tau)}{t_{go}^{N-3}} d\tau \tag{16}$$

Notice that for N=3, this cost function becomes energy like minimisation. Hence using Eq. (15) in Eq. (14) and using Eq. (13) and Eq. (4) we get

$$a_{M_\sigma} = N V_c \dot{\sigma} + \frac{N}{2} a_{T_\sigma} \tag{17}$$

The obtained law is well known augmented PNG.

C. Guidance-to-Collision for Exoatmospheric Interception

In the case of exoatmospheric interception, there is no aerodynamic force, and the interception trajectory is only controlled by an instantaneous rotation of the missile's body. Guidance to collision is achieved by nullifying the heading

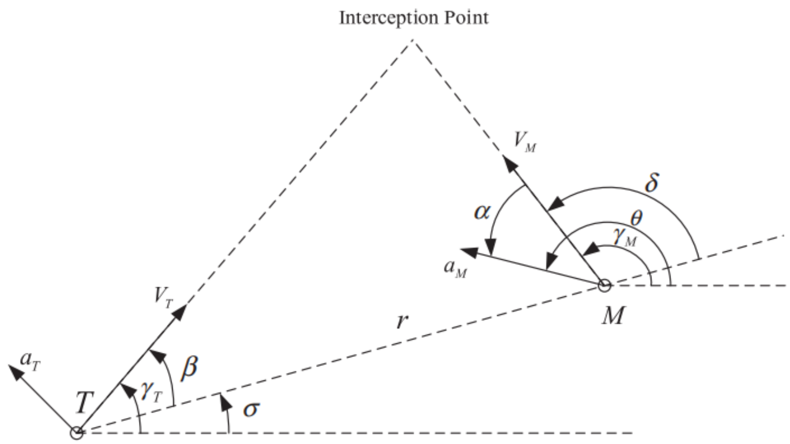

Fig. 2 Guidance to Collision

error to follow a straight line toward the expected collision point.

To maintain the collision triangle, it is necessary to equalise between the distances travelled by the interceptor and the target perpendicular to the LOS as:

$$\int_0^{t_{go}} V_M(t)\sin\delta dt = \int_0^{t_{go}} V_T(t)\sin\beta dt \tag{18}$$

Using Leibniz rule,

$$(V_M(t)t_{go} + \frac{a_M}{2}t_{go}^2)\sin\delta = V_T(t)\sin\beta t_{go} \tag{19}$$

Now,

$$r = V_T(t)\cos\beta t_{go} + (V_M(t)t_{go} + \frac{a_M}{2}t_{go}^2)\cos(\pi - \delta) \tag{20}$$

This equation is quadratic in t_{go}. Coupled equations Eq. (18) and Eq. (19) is solved to get required $\delta = \delta_r$. Hence the required flight path angle is:

$$\gamma_M^* = \delta_r + \sigma \tag{21}$$

Hence we choose tracking error as $\epsilon = \gamma_M^* - \gamma_M$. Then the dynamics, assuming slow variation rate of γ_M^*, becomes

$$\dot{\epsilon} = -\frac{a_M \sin\alpha}{V_M(t)} \tag{22}$$

Now if we choose error dynamics as

$$\dot{\epsilon} + \frac{K}{t_{go}} = 0 \tag{23}$$

Table 1

Parameters	Values
Missile–target initial relative range r(0), km	5
Initial LOS angle $\sigma(0)$, deg	0
Missile initial velocity $V_M(0)$, m/s	2500
Missile initial flight-path angle $\gamma_M(0)$, deg	160
Target velocity V_T, m/s	3000
Target initial flight-path angle $\gamma_M(0)$, deg	20

Hence from above two equations, the guidance command is:

$$a_M \sin\alpha = \frac{KV_M(t)\epsilon}{t_{go}} \tag{24}$$

and this minimises

$$J = \frac{1}{2}\int_t^{t_f} \frac{u^2(\tau)}{V_M^2(\tau)t_{go}^{K-1}} d\tau \tag{25}$$

which is energy optimal case for $K = 1$.

D. Numerical simulation

To validate the proposed algorithms, a numerical simulation is carried out in MATLAB R2020a with initial conditions given in Table 1. Proposed law shown in figures is Guidance-to-Collision law.

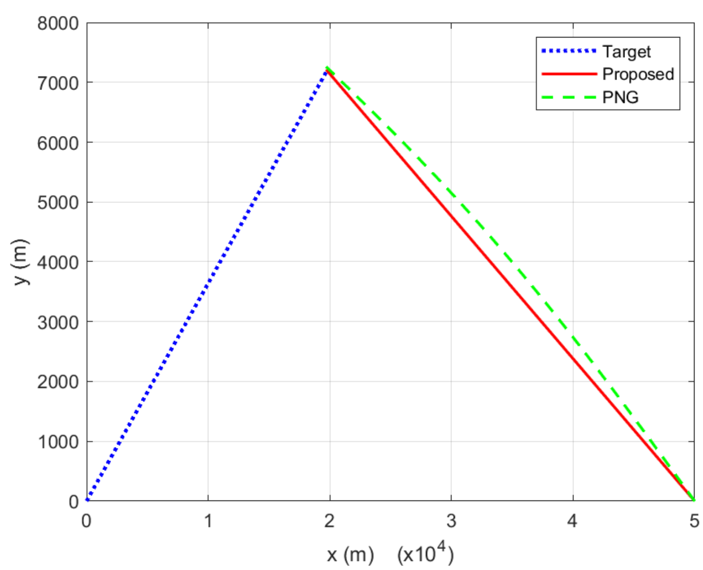

Fig. 3 Missile trajectory

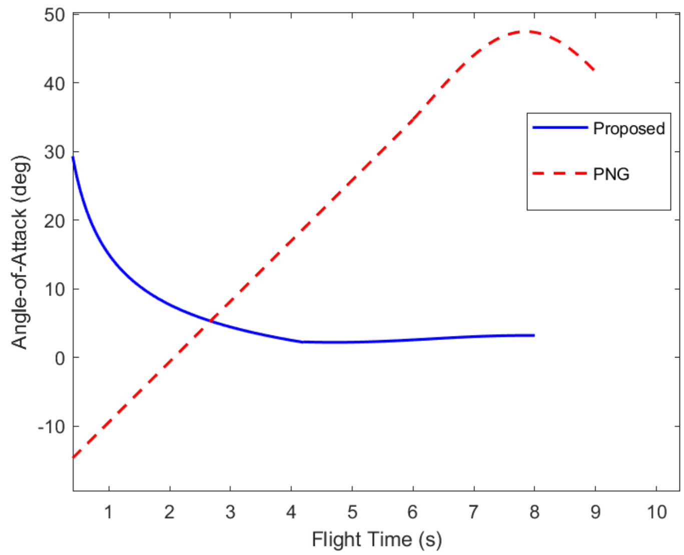

Fig. 4 Angle of attack

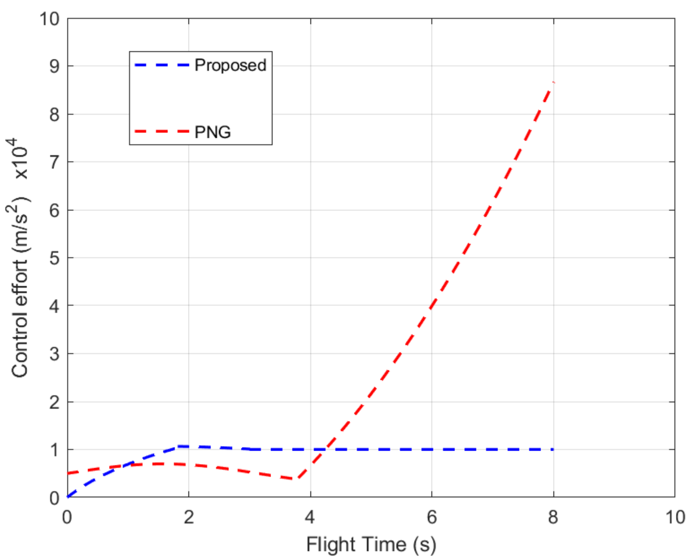

Fig. 5 Control effort required

E. Results

From this figure, it can be noted that the axial acceleration of PNG does not align with the velocity vector, thereby forcing the missile to fly along a curved path to intercept the target. Proposed law follows a straight line. Compared to the PNG, Fig. 5 reveals that the proposed guidance law requires less control effort.

V. Application to Spacecraft Docking Problem

Autonomous Spacecraft docking and rendezvous is one of the most complex and challenging component of modern spacecraft and satellite missions. It requires two vehicles to come together to an common orbit, rendezvous and dock together for payload supply or repair. It has application on debris-management in close proximity analysis. Various control strategies are introduced in the literature with varying system complexity.

A. System Description

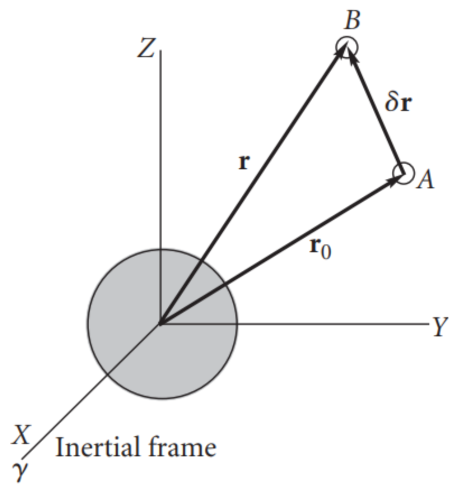

Fig. 6 Orbital framework of Docking: A is target and B is chaser.

A general docking scenario consist of a target vehicle A and a chaser vehicle B. A is in an keplerian circular orbit with radius r_0 while B performs maneuvers at position vector δr from A in an non-keplerian orbit. From the figure 6,

$$\vec{r} = \vec{r}_0 + \vec{\delta} r$$

with

$$\frac{\delta r}{r_0} << 1$$

We define a moving frame(CWH frame) xyz with origin on A with,

$$\hat{i} = \frac{\vec{r}_0}{r_0}$$

y-axis is the local horizon and $\hat{k} = \hat{i} \times \hat{j}$

Let angular velocity of A(and so of the CWH frame) be Ω

B. Mathematical Model

The equations of motion for the chaser spacecraft are nonlinear and can be expressed in vector form as:

$$\ddot{\vec{r}} = -\mu \frac{\vec{r}}{r^3} + \frac{\vec{F}}{m} \tag{26}$$

$\vec{F}$ is force applied on the spacecraft.

Putting $\vec{r} = \vec{r}_0 + \vec{\delta} r$ in (1),

$$\ddot{\vec{\delta r}} = -\ddot{\vec{r}}_0 - \mu \frac{\vec{r}_0 + \delta\vec{r}}{r^3} + \frac{\vec{F}}{m} \tag{27}$$

Now,

$$r^{-3} = r_0^{-3} \left(1 + \frac{2\vec{r}_0 \cdot \vec{\delta} r}{r_0^2}\right)^{-3/2}$$

by neglecting $\left(\frac{\delta}{r_0}\right)^2$,

Using Binomial theorem this simplifies as:

$$\left(1 + \frac{2\vec{r}_0 \cdot \vec{\delta} r}{r_0^2}\right)^{-3/2} = 1 + \left(\frac{-3}{2}\right)\left(1 + \frac{2\vec{r}_0 \cdot \vec{\delta} r}{r_0^2}\right)$$

$$\therefore \frac{1}{r^3} = \frac{1}{r_0^3} - \frac{3}{r_0^5}\vec{r}_0 \cdot \vec{\delta} r$$

Putting this in Eq. (27):

$$\ddot{\vec{\delta r}} = \ddot{\vec{r}}_0 - \mu \left(\frac{1}{r_0^3} - \frac{3}{r_0^5}\vec{r}_0 \cdot \vec{\delta} r\right)(\vec{r}_0 + \delta\vec{r}) + \frac{\vec{F}}{m}$$

$$= -\ddot{\vec{r}}_0 - \mu \left[\frac{\vec{r}_0}{r_0^3} + \frac{\vec{\delta} r}{r_0^3} - \frac{3(\vec{r}_0 \cdot \vec{\delta} r)\vec{r}_0}{r_0^5} + \text{Higher order terms in } \delta r \text{ (which we neglect)}\right] + \frac{\vec{F}}{m}$$

$$\therefore \ddot{\vec{\delta r}} = -\ddot{\vec{r}}_0 - \mu \left[\frac{\vec{r}_0}{r_0^3} + \frac{\vec{\delta} r}{r_0^3} - \frac{3(\vec{r}_0 \cdot \vec{\delta} r)\vec{r}_0}{r_0^5}\right] + \frac{\vec{F}}{m}$$

Now dynamical equation of the target vehicle is

$$\ddot{\vec{r}}_0 = -\mu \frac{\vec{r}_0}{r_0^3}$$

Substituting above,

$$\ddot{\vec{\delta r}} = -\frac{\mu}{r_0^3}\left[\delta\vec{r} - \frac{3(\vec{r}_0 \cdot \vec{\delta} r)\vec{r}_0)}{r_0^2}\right] + \frac{\vec{F}}{m} \tag{28}$$

Now,

$$\vec{\delta} r = \delta x \hat{i} + \delta y \hat{j} + \delta z \hat{k}$$

Using vector kinematics, we know that,

$$\ddot{\vec{r}} = \ddot{\vec{r}}_0 + \dot{\vec{\Omega}} \times \delta r + \dot{\vec{\Omega}} \times (\vec{\Omega} \times \vec{\delta r}) + 2\vec{\Omega} \times \delta\vec{v}_{rel} + \vec{\delta}a_{rel}$$

Putting $\vec{r} = \vec{r}_0 + \delta r$ above,

$$\ddot{\vec{\delta r}} = \dot{\vec{\Omega}} \times \vec{\delta r} + \vec{\Omega} \times (\vec{\Omega} \times \vec{\delta r}) + 2\vec{\Omega} \times \vec{\delta}v_{rel} + \vec{\delta}a_{rel}$$

Assuming target vehicle to be in **circular orbit**, $\dot{\vec{\Omega}} = 0$,

$$\ddot{\vec{\delta r}} = \vec{\Omega} \times (\vec{\Omega} \times \vec{\delta r}) + 2\vec{\Omega} \times \vec{\delta}v_{rel} + \vec{\delta}a_{rel}$$

Expanding the RHS,

$$\ddot{\vec{\delta r}} = \vec{\Omega}(\vec{\Omega} \cdot \vec{\delta r}) - \Omega^2\vec{\delta r} + 2\vec{\Omega} \times \vec{\delta}v_{rel} + \vec{\delta}a_{rel}$$

Putting $\vec{\Omega} = n\hat{k}$, $\vec{\delta}v_{rel} = \delta\dot{x}\hat{i} + \delta\dot{y}\hat{j} + \delta\dot{z}\hat{k}$ and $\vec{\delta}a_{rel} = \delta\ddot{x}\hat{i} + \delta\ddot{y}\hat{j} + \delta\ddot{z}\hat{k}$ and recollecting terms this simplifies as,

$$\ddot{\vec{\delta r}} = (-n^2\delta x - 2n\delta\dot{y} + \delta\ddot{x})\hat{i} + (-n^2\delta y - 2n\delta\dot{x} + \delta\ddot{y})\hat{j} + \delta\ddot{z}\hat{k} \tag{29}$$

Now from Eq. (28),by substituting values of $\vec{\delta r}$, $\vec{r}_0 \cdot \delta r = r_0\delta x$ and using $n^2 = \frac{\mu}{r_0^3}$ we get,

$$\ddot{\vec{\delta r}} = -n^2\left[\delta x\hat{i} + \delta y\hat{j} + \delta z\hat{k} - \frac{3}{r_0^2}(r_0\delta x)r_0\hat{i}\right] + \frac{F_x}{m}\hat{i} + \frac{F_y}{m}\hat{j} + \frac{F_z}{m}\hat{k} = (2n^2\delta x + \frac{F_x}{m})\hat{i} + (-n^2\delta y + \frac{F_y}{m})\hat{j} + (-n^2\delta z + \frac{F_z}{m})\hat{k} \tag{30}$$

Equating the value of $\ddot{\vec{\delta r}}$ from equation Eq. (29) and Eq. (30) and collecting terms,

$$(\delta\ddot{x} - 3n^2\delta x - 2n\delta\dot{y})\hat{i} + (\delta\ddot{y} + 2n\delta\dot{x})\hat{j} + (\delta\ddot{z} + n^2\delta z)\hat{k} = \frac{F_x}{m}\hat{i} + \frac{F_y}{m}\hat{j} + \frac{F_z}{m}\hat{k}$$

This can be simplified to give linear CWH equation:

$$\delta\ddot{x} - 3n^2\delta x - 2n\delta\dot{y} = \frac{F_x}{m} = u_x \tag{31}$$

$$\delta\ddot{y} + 2n\delta\dot{x} = \frac{F_y}{m} = u_y \tag{32}$$

$$\delta\ddot{z} + n^2\delta z = \frac{F_z}{m} = u_z \tag{33}$$

This model can be formulated as:

$$\dot{X} = AX + BU$$

where X $\in \mathcal{R}^6$ is the state vector and U $\in \mathcal{R}^3$ is the control vector.

$$X = \begin{bmatrix} \delta x \\ \delta y \\ \delta z \\ \delta \dot{x} \\ \delta \dot{y} \\ \delta \dot{z} \end{bmatrix}, \qquad U = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$$

and,

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3n^2 & 0 & 0 & 0 & 2n & 0 \\ 0 & 0 & 0 & -2n & 0 & 0 \\ 0 & 0 & -n^2 & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

C. Optimal Error Dynamics

We choose error $\epsilon = X$. This error represents the relative position and velocity between the bodies. Then the dynamics can be written as:

$$\dot{\epsilon} = A\epsilon + BU \tag{34}$$

We choose the required error dynamics as

$$\dot{\epsilon} + \frac{kI_6}{t_{go}} = 0 \tag{35}$$

where k is a constant and I_6 is 6×6 identity matrix, thus ensuring R is symmetric matrix. Now comparing above two equations, we get the optimal control effort as,

$$BU = -\left(\frac{kI_6}{t_{go}} + A\right)\epsilon \tag{36}$$

This minimises the cost function:

$$J = \frac{1}{2}\int_t^{t_f} \frac{U^T(\tau)U(\tau)}{t_{go}^{k-1}} d\tau \tag{37}$$

which is energy like cost function for $k = 1$.

D. Numerical Results

Table 2 Simulation Parameters

Parameter	Value	Unit
m_1	200	kg
m_2	200	kg
n	7.29×10^{-5}	rad/sec
$\delta x(0)$	-15	m
$\delta y(0)$	-25	m
$\delta z(0)$	5	m
$\delta\dot{x}(0)$	0.5	m/s
$\delta\dot{y}(0)$	0.5	m/s
$\delta\dot{z}(0)$	0	m/s
t_f	12	days

Numerical simulation is carried out on the spacecraft docking system in MATLAB R2020a. Simulation parameters are specified in Table 2.

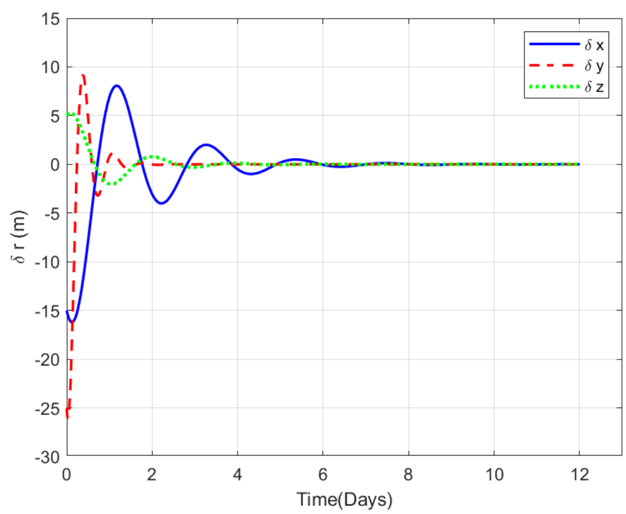

Fig. 7 Position Error

E. Results

Figures 7 and 8 shows the plot for error and control effort respectively. The error seems to converge in about 8 days for a given t_{go} of 12 days. As the motion is mostly in a plane, error in $x - y$ plane is higher. The max control effort required is $10\mu N$ which is within the range of a practical thruster.

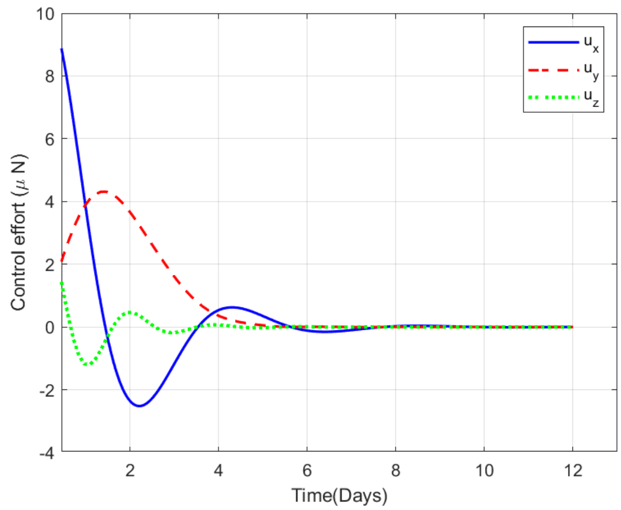

Fig. 8 Control effort required

VI. Conclusion

This project present a derivation and use of optimal error dynamics guaranteeing the finite time convergence and optimality of derived guidance law. Formal proof for both scalar and n-dimensional systems are given. Application to missile guidance and spacecraft docking problem is presented. Simulations are carried out in MATLAB to verify and test the derived results. The control required in both cases are shown to be less than conventional controllers like PNG.

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$$
\begin{aligned}
&\frac{\partial L}{\partial C_T} = 0 \\
&\Rightarrow \begin{pmatrix} 2C_{T_1} \\ 2C_{T_2} \\ 2C_{T_3} \\ 2C_{T_4} \end{pmatrix} - \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \mu \begin{pmatrix} \frac{Tkl}{J_y} \\ \frac{Tkl}{J_y} \\ -\frac{Tkl}{J_y} \\ -\frac{Tkl}{J_y} \end{pmatrix}
\end{aligned}
\tag{38}
$$