Computer modeling of spacecraft constellations

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October 28, 2022

1 Preliminaries

Equations of motion:

$$\ddot{x} - 2\dot{y} = U_x + a_x$$
$$\ddot{y} + 2\dot{x} = U_y + a_y$$
$$\ddot{z} = U_z + a_z$$

where the potential U

$$U = \frac{1}{2}(x^2 + y^2) + \frac{\mu}{r_2} + \frac{1-\mu}{r_1}$$

and a is the acceleration due to the solar sail force, given by

$$a_i = \beta \frac{(1-\mu)}{r_1^2} \cos^2 \alpha n_i$$

Here n_i is components of the normal solar sail vector $\hat{n}$ along the coordinate system.

These equations can be used to find the artificial liberation points. As the sail lightness increases, the liberation points move closer to the Sun.

1.1 Periodic Solutions near Artificial Equilibrium Points

Define effective potential as:

$$\Omega = \frac{(1-\mu)(1-\beta)}{r_1} + \frac{1}{2}(x^2 + y^2) + \frac{\mu}{r_2}$$

Then the linearized equation of motion can be written using Taylor's expansion near the equilibrium point:

$$\nabla\Omega(\delta X) = \nabla^2\Omega \cdot \delta X$$

Hence,

$$\begin{aligned} \ddot{\xi} - 2\dot{\eta} &= \nabla^2\Omega \cdot \delta X_1 \\ \ddot{\eta} + 2\dot{\xi} &= \nabla^2\Omega \cdot \delta X_2 \\ \ddot{\zeta} &= \nabla^2\Omega \cdot \delta X_3 \end{aligned}$$

where

$$(\xi, \eta, \zeta)^{\intercal} = \delta X$$

A third order solution is given by Collins which can be used to generate approximate periodic orbits.

The method of differential corrections can be applied to get exact periodic orbits. For that we write the equations above in state space form. Define perturbation from reference trajectory as $\delta\bar{x}(t) = (\delta x, \delta y, \delta z, \delta\dot{x}, \delta\dot{y}, \delta\dot{z})^{\intercal}$. Then the variational equation can be written in state space form

$$\delta\dot{\bar{x}}(t) = A(t)\delta\bar{x}(t)$$

where,

$$A(t) = \begin{bmatrix} 0 & I_3 \\ B(t) & C \end{bmatrix}$$

The general form solution is

$$\delta\bar{x}(t) = \Phi(t, t_0)\delta\bar{x}(t_0)$$

where the STM satisfies

$$\dot{\Phi}(t.t_0) = \Phi(t, t_0)A(t)$$

with the initial condition

$$\Phi(t_0, t_0) = I_6$$

For differential correction, we chose a desired state $\bar{x}(t_f)$ at $t = t_f$ and vary the initial state $\bar{x}(t_0)$ to achieve that state. The effect on the final state of a modification in the initial state can be estimated by examining the first variations in the final state, that is,

$$\delta\bar{x}(t_f) = \frac{\partial\bar{x}(t_f)}{\partial\bar{x}(t_0)}\bar{x}(t_0) + \dot{\bar{x}}(t_f)\delta(t_f - t_0)$$

or,

$$\delta\bar{x}(t_f) = \Phi(t_f, t_0)\bar{x}(t_0) + \dot{\bar{x}}(t_f)\delta(t_f - t_0)$$

Finally, the correction in the final state can be written as,

$$\delta\bar{x}(t_f) = \bar{x}(t_f)_{des} - \bar{x}(t_f)$$

Hence using this method, with starting trajectory as the approximate third order solution, we can get the exact trajectory.

2 An improved formation keeping algorithm for Fresnel constellation

2.1 Family of periodic orbits

Family of periodic orbits can be generated by varying the β-parameter. The variation period and amplitude of the trajectory with β is shown in Figure 1. By generating the reference trajectory for the lens formation, we can maintain a constellation.

2.2 Proposed Formation keeping Strategy

Define the sail parameter vector θ as,

$$\theta = \begin{bmatrix} \alpha & \gamma & \beta \end{bmatrix}^{\intercal}$$

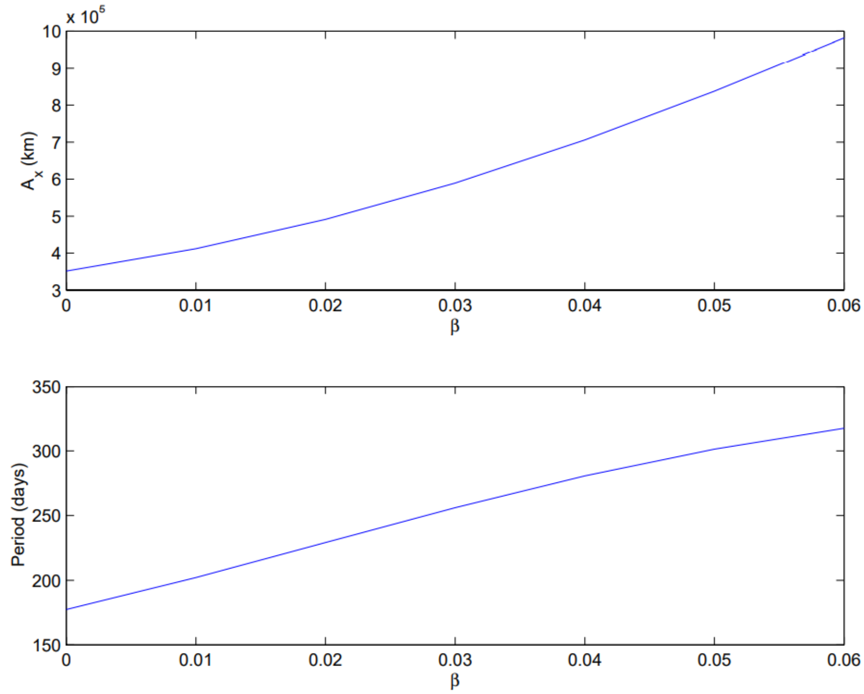

Figure 1: Variation of Period and Amplitude with β

Then the variation from nominal trajectory can be represented by $\delta\theta$ given by

$$\delta\theta = \begin{bmatrix} \delta\alpha & \delta\gamma & \delta\beta \end{bmatrix}^{\mathsf{T}}$$

To facilitate expression of the linearized system in state space form, the variations in the orientation variables are incorporated into an augmented variational state vector, defined as follows :

$$\delta\Gamma = \begin{bmatrix} \delta x & \delta y & \delta z & \delta\alpha & \delta\gamma & \delta\beta \end{bmatrix}^{\mathsf{T}}$$

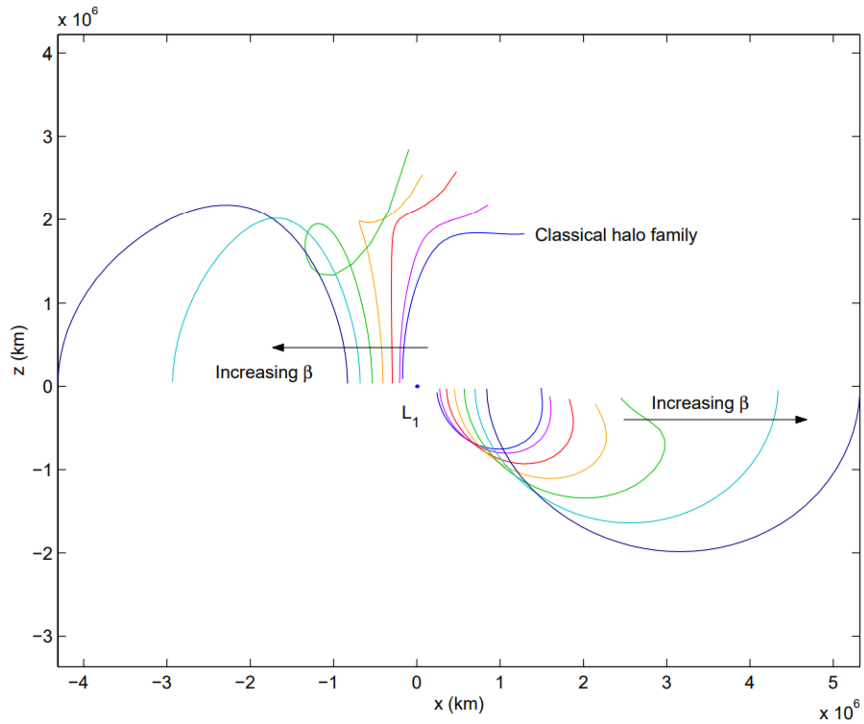

Figure 2: Family of periodic orbits w.r.t β

Thus, the state space form of the modified variational equations can be written as

$$\delta\dot{\Gamma} = A(t)\delta\Gamma$$

where

$$A(t) = \begin{bmatrix} 0 & I_3 & 0 \\ B(t) & C & D(t) \\ 0 & 0 & 0 \end{bmatrix}$$

is a 9×9 matrix. The time varying D(t) is a 3×3 matrix given by

$$D(t) = \begin{bmatrix} a_{x\alpha} & a_{x\gamma} & a_{x\beta} \\ a_{y\alpha} & a_{y\gamma} & a_{y\beta} \\ a_{z\alpha} & a_{z\gamma} & a_{z\beta} \end{bmatrix}$$

where a is the sail acceleration and $a_{jk} = \frac{\partial a_j}{\partial a_k}$.

The solution of above equation can be expressed as

$$\delta\Gamma(t_p) = \Phi(t_p, t_q)\delta\Gamma(t_q)$$

where $\Phi(t_p, t_q)$ is the state transition matrix. Following the method by Collins, the state transition matrix can be partitioned as

$$\Phi(t_p, t_q) = \begin{bmatrix} K_{pq} & L_{pq} & E_{pq} \\ M_{pq} & N_{pq} & F_{pq} \\ 0 & 0 & I_3 \end{bmatrix}$$

where

$$K_{pq} = \frac{\partial r_{t_p}}{\partial r_{t_q}}$$
$$L_{pq} = \frac{\partial r_{t_p}}{\partial v_{t_q}}$$
$$M_{pq} = \frac{\partial v_{t_p}}{\partial r_{t_q}}$$
$$N_{pq} = \frac{\partial v_{t_p}}{\partial v_{t_q}}$$
$$E_{pq} = \frac{\partial r_{t_p}}{\partial \theta_{t_q}}$$
$$F_{pq} = \frac{\partial v_{t_p}}{\partial \theta_{t_q}}$$

We will now state and solve the optimal control problem.

2.3 A Genetic Algorithm approach to Optimal sail orientation

Let m_i denote the deviation from reference trajectory at time t_i, i.e,

$$m_i = \begin{bmatrix} \delta x(t_i) & \delta y(t_i) & \delta z(t_i) \end{bmatrix}^\intercal$$

Let us define the cost function which we will be going to minimize:

$$\mathcal{J} = \delta\theta^\intercal Q \delta\theta + \sum_{i=1}^{n} m_i^\intercal S_i m_i$$

where Q and S_i are positive semi definite matrices. We have to minimize the cost function ensuring minimum $||\delta\theta||$ Note here that n determines how much the cost function takes into account the future predicted error (based on the current error). Hence the problem can be converted into a multi-objective optimization. Define the vector x of decision variables

$$x = \begin{bmatrix} n & \delta\alpha & \delta\gamma & \delta\beta \end{bmatrix}^\intercal$$

Next we define our objective function

$$f(x) = \begin{bmatrix} n & ||\delta\theta|| \end{bmatrix}^\intercal$$

subjected to the constraint

$$\frac{\partial \mathcal{J}}{\partial \delta\theta} = 0$$

or,

$$\delta\theta = [-Q + \sum_{i=1}^{n} E_{i0}^{\intercal} S_i E_{i0}]^{-1} \times [(\sum_{i=1}^{n} E_{i0}^{\intercal} S_i L_{i0}) e_0 + (\sum_{i=1}^{n} E_{i0}^{\intercal} S_i K_{i0}) p_0]$$

where p_0, e_0 are the initial position and velocity errors respectively.
Additional inequality constraints on the decision variable can be imposed depending on the mission requirements.

2.4 NSGA-II with TOPSIS

The NSGA-II is one of the most efficient and popular multi- objective evolutionary algorithm, which is first proposed by Deb. In addition to standard GA operators including selection, crossover and mutation, a fast non-dominated sorting approach, an elitist strategy and an efficient crowding distance estimation procedure have been introduced into the NSGA-II. It should be noted that maintaining a diverse population is an important consideration in multi-objective evolutionary algorithm. Therefore, the NSGA-II adopts crowding distance measure, instead of fitness sharing parameter used by NSGA, to obtain a uniform spread of solutions along the Pareto front. The crowding distance provides an estimate of the density of solutions surrounding a particular solution in the population. Thus, to promote diversity, a solution having higher value of crowding distance is preferred over the solution with a lower crowding distance in removal process.
TOPSIS method is of great use for solving multiple criteria decision making problems by ranking the possible alternatives through measuring Euclidean distances. Its fundamental concept is that the selected alternative should simultaneously have the shortest distance from the Positive ideal solution (PIS) and the longest distance from the Negative ideal solution (NIS).
We apply the above discussed algorithms for our problem. The algorithm is

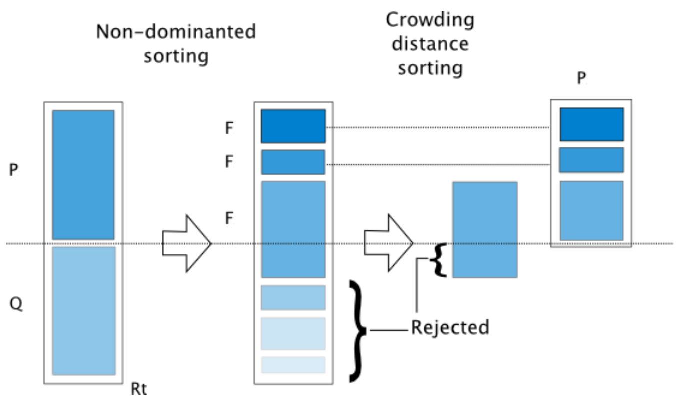

Figure 3: NSGA-II

applied on normalized objective function. A sample Pareto front obtained is shown in Figure 4.

3 Future work to be done

- Derive the parametric curve equation for the Fresnel lens formation
- Decide weights for TOPSIS algorithm on the Pareto front.
- Simulate the full system for proof of concept.
- Test robustness of the solution by introducing uncertainties in the formation.

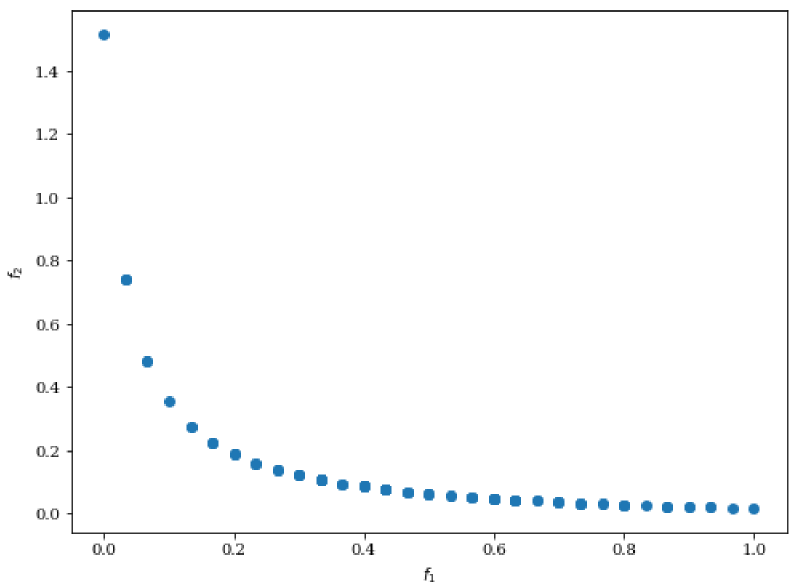

Figure 4: Pareto front for the optimal sail orientation and n