



## Excellent Notes

**adria (79313)** and  $C_1$  each have radius 1, and the distance between their centers is 3. Internally tangent to both  $C_1$  and  $C_2$ , Circle  $C_3$  is externally tangent to  $C_1$ . If the radius of  $C_3$  is 100th of the value of  $C_1$ , what is the radius of  $C_3$ ? (2023 AMC 10A Problems, Question #22)

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Adria's work:

Let radius of  $C_3$  be  $r$ .  
 $\Rightarrow r = \frac{1}{100}$  of radius of  $C_1$ .  
 $\Rightarrow r = \frac{1}{100} \times 1 = \frac{1}{100}$ .  
 $\Rightarrow r = 0.01$ .  
 $\Rightarrow r = \frac{1}{100}$ .

common tangent theorem  
 regions

external:  $(R+r)^2 - (r-R)^2 = 2rR$   
 internal:  $(R-r)^2 - (R+r)^2 = -2r^2$

$AB = 0, L = \sqrt{(R+r)^2 - (R-r)^2} = \sqrt{4R^2} = 2R$   
 $PQ = \sqrt{(3-4)^2 + (3-4)^2} = \sqrt{2}R$   
 $RS = \sqrt{(4-3)^2 + (4-3)^2} = \sqrt{2}R$

$\frac{PQ}{RS} = \frac{\sqrt{2}R}{\sqrt{2}R} = 1$

Adria's answer:  $\frac{1}{10}$

Adria

**Frank (93224)** Circle  $C_1$  is the largest circle internally tangent to both  $C_2$  and  $C_3$ . Circle  $C_3$  is internally tangent to both  $C_1$  and  $C_2$  and externally tangent to  $C_1$ . What is the radius of  $C_3$ ? (2023 AMC 10A Problems, Question #22)

Frank's work:

Let radius of  $C_3$  be  $r$ , radius of  $C_2$  be  $R$ .  
 $\Rightarrow AD = BD = 1-r$   
 $\Rightarrow DC = R+r$   
 $\Rightarrow AC = 4 = r + r + 2R = \frac{3}{2}r$   
 $\Rightarrow 8R = 4AC$   
 $\Rightarrow AC^2 = DC^2 + AD^2$   
 $\Rightarrow (\frac{3}{2}r)^2 = (\frac{3}{2}r)^2 + (1-r)^2$   
 $\Rightarrow \frac{9}{4}r^2 = \frac{9}{4}r^2 + 1 - 2r + r^2$   
 $\Rightarrow \frac{9}{4}r^2 - \frac{9}{4}r^2 = 1 - 2r + r^2$   
 $\Rightarrow \frac{9}{4}r^2 = \frac{9}{4}r^2$   
 $\Rightarrow 1 = 1 - 2r + r^2$   
 $\Rightarrow 2r = 1$   
 $\Rightarrow r = \frac{1}{2}$

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Frank

**Sherry**

Given four points  $P$ ,  $Q$ ,  $R$ ,  $S$  lie on the circle  $x^2 + y^2 = 25$  and have integer coordinates. The distances  $PQ$  and  $RS$  are irrational numbers. What is the greatest possible value of the ratio  $\frac{PQ}{RS}$ ? (2017 AMC 10A Problems, Question #17)

A. 3      B. 5      C.  $3\sqrt{3}$       D. 7      E.  $5\sqrt{2}$

Sherry's work:

Make  $PQ$  longest,  $RS$  shortest.

$PQ = \sqrt{(4-(-3))^2 + (-3-4)^2} = 7\sqrt{2}$

$RS = \sqrt{(4-3)^2 + (4-3)^2} = \sqrt{2}$

$\frac{PQ}{RS} = \frac{7\sqrt{2}}{\sqrt{2}} = 7$

**Cindy Long (79331)**

Two straight pipes (circular cylinders), with radii 1 and  $\frac{1}{4}$ , lie parallel and in contact on a flat floor. The figure below shows a head-on view. What is the sum of the possible radii of a third parallel pipe lying on the same floor and in contact with both? (2024 AMC 10B Problems, Question #21)

Common tangents:  
 $CPB = 2\sqrt{1+\frac{1}{4}} = \frac{5}{2}$   
 $AC = 2\sqrt{1+r} = 2\sqrt{r}$   
 $BC = 2\sqrt{r} = \sqrt{r}$

$AC = AB + BC$   
 $2\sqrt{r} = 1 + 2\sqrt{r} \Rightarrow \text{Ans: } \frac{1}{9} + 1 = \frac{10}{9}$

Cindy's work:

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Cindy

**L8**

# Analytical Geometry and Geometric Transformations



## Second Mock Exam

Link can be found in Lesson 8 Learning Material!

or enter the link below:

<https://www.thethinkacademy.com/quiz/evaluation/instruction/7006>



## Concept 1

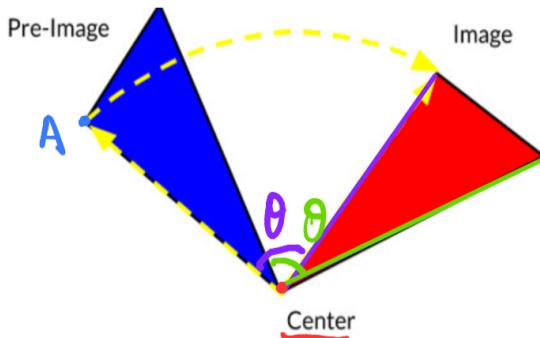
# Rotation





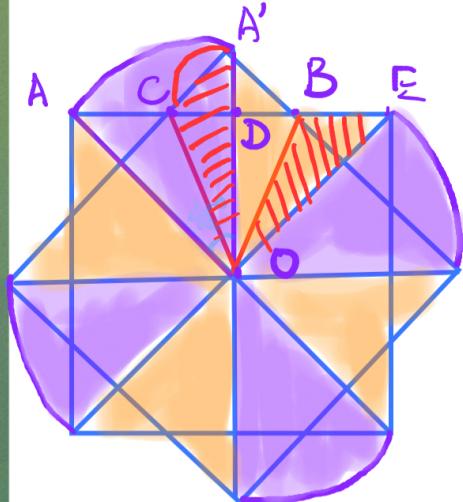
## How does rotation work?

1. The distance from the center to any point on the shape stays the same.
2. Every point makes a sector around the center



## Math Exploration 1.1

0 0 : 0 0 X



$$\begin{aligned} \text{Total} &= 4 \text{ sectors} + 4 \text{ darts} \\ &= 4 \cdot \left(\frac{\pi}{16}\right) + 4 \left(\frac{1}{2} \cdot \frac{\sqrt{2}}{4}\right) \\ &= 2 - \sqrt{2} + \frac{\pi}{4} \end{aligned}$$

Sector: radius =  $\frac{\sqrt{2}}{2}$ , angle =  $45^\circ$

$$A_{\text{sector}} = \frac{45^\circ}{360^\circ} \cdot \pi \cdot \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{\pi}{16}$$

Dart:  $A A' B O C = \frac{1}{2} \cdot \underline{B C} \cdot A' O$

Find BC:

$$AC = A'C, \Delta A'CD \text{ is } 45-45-90 \text{ RCD}$$

$$\Rightarrow A'C = \sqrt{2} \cdot CD, BC = 2CD \quad (CD = \frac{C}{2})$$

$$AC + BC + BE = 1$$

$$\frac{\sqrt{2}}{2} BC + BC + \frac{\sqrt{2}}{2} BC = 1$$

$$(\sqrt{2} + 1)BC = 1$$

$$BC = \frac{1}{\sqrt{2}+1} = \sqrt{2}-1$$

$$\begin{aligned} \rightarrow A &= \frac{1}{2} BC \cdot A' O \\ &= \frac{1}{2} (\sqrt{2}-1) \frac{\sqrt{2}}{2} \\ &= \frac{1}{2} - \frac{\sqrt{2}}{4} \end{aligned}$$



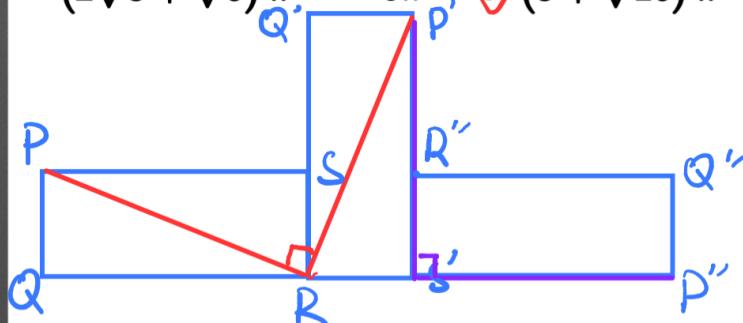


## Practice 1.1

Rectangle  $PQRS$  lies in a plane with  $PQ = RS = 2$  and  $QR = SP = 6$ . The rectangle is

rotated  $90^\circ$  clockwise about  $R$ , then rotated  $90^\circ$  clockwise about the point  $S$  moved to after the first rotation. What is the length of the path traveled by point  $P$ ? (2008 AMC 10A Problems, Question #19)

- A.  $(2\sqrt{3} + \sqrt{5})\pi$    B.  $6\pi$    C.  $\checkmark (3 + \sqrt{10})\pi$    D.  $(\sqrt{3} + 2\sqrt{5})\pi$    E.  $2\sqrt{10}\pi$



$$\text{diagonal} = \sqrt{2^2 + 6^2} \\ = 2\sqrt{10}$$

First:  $\frac{90^\circ}{360^\circ} \cdot 2\pi \cdot 2\sqrt{10} = \pi\sqrt{10}$

Second:  $\frac{90^\circ}{360^\circ} \cdot 2\pi \cdot 6 = 3\pi$

$$\begin{aligned} \text{Total} &= \pi\sqrt{10} + 3\pi \\ &= (3 + \sqrt{10})\pi \end{aligned}$$



## Concept 2

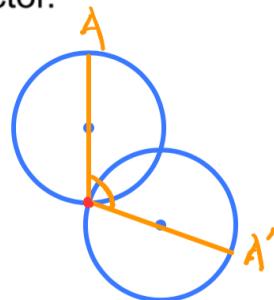
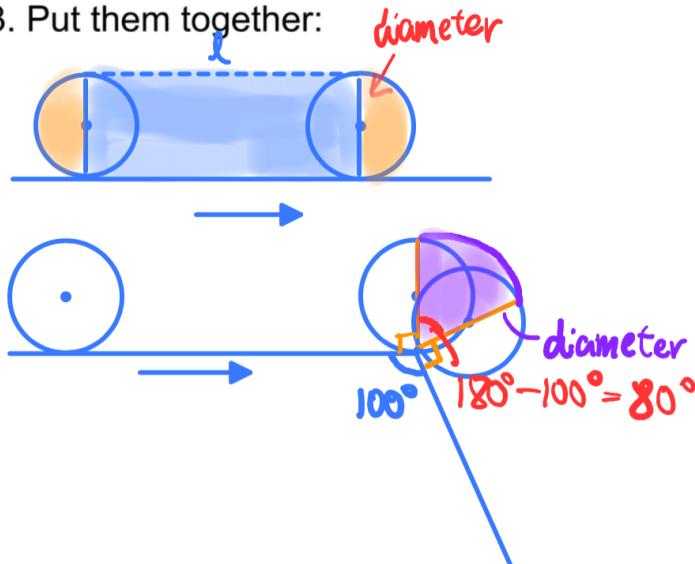
# Rolling





## How do shapes roll?

1. Rolling on a tangent line: the region swept out is a rectangle.
2. Rolling around a point: the region swept out is a sector.
3. Put them together:



MC(1)



## Math Exploration 2

A disk of radius 1 rolls all the way around the inside of a square of side length  $s > 4$  and sweeps out a region of area  $\mathbf{A}$ . A second disk of radius 1 rolls all the way around the outside of the same square and sweeps out a region of area  $\mathbf{2A}$ . The value of  $s$  can be written as  $a + \frac{b\pi}{c}$ , where  $a$ ,  $b$ , and  $c$  are positive integers and  $b$  and  $c$  are relatively prime. What is  $a + b + c$ ? (2021 AMC Fall 10A, Question #19)

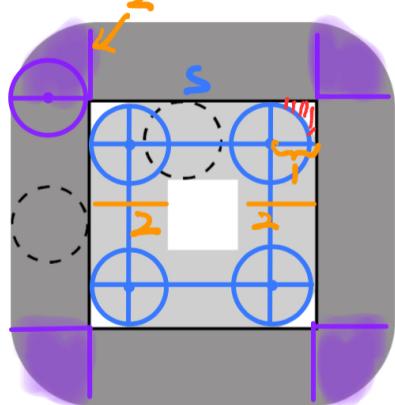
A. 10

B. 11

C. 12

D. 13

E. 14



$$\text{Inside: Unshaded} = (s-4)^2 + 4 \cdot (1^2 - \frac{1}{4}\pi \cdot 1^2)$$

$$= s^2 - 8s + 16 + 4 - \pi$$

$$= s^2 - 8s + 20 - \pi$$

$$A = s^2 - (s^2 - 8s + 20 - \pi) = 8s - 20 + \pi$$

$$\text{Outside: } 2A = 4 \cdot (2s) + \pi \cdot 2^2 = 8s + 4\pi$$

$$\Rightarrow 2(8s - 20 + \pi) = 8s + 4\pi$$

$$8s = 40 + 2\pi$$

$$s = s + \frac{1}{4}\pi$$

$$s + 1 + 4 = 10$$





## Practice 2.1

An equilateral triangle has side length **6**. What does the area of the region contain all points that are outside the triangle but not more than **3** units from a point of the triangle? (2008 AMC 10A Problems, Question #17)

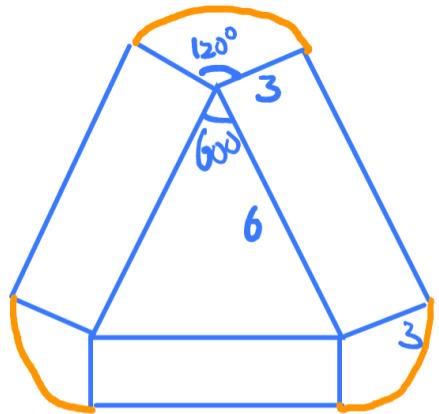
A.  $36 + 24\sqrt{3}$

B.  $\checkmark 54 + 9\pi$

C.  $54 + 18\sqrt{3} + 6\pi$

D.  $(2\sqrt{3} + 3)^2\pi$

E.  $9(\sqrt{3} + 1)^2\pi$



$$\text{Rectangle} = 3 \cdot 6 = 18$$

$$\text{Sectors} = \frac{120^\circ}{360^\circ} \cdot \pi \cdot 3^2 = 3\pi$$

$$\begin{aligned}\text{Total} &= 3 \cdot (18) + 3 \cdot (3\pi) \\ &= 54 + 9\pi\end{aligned}$$



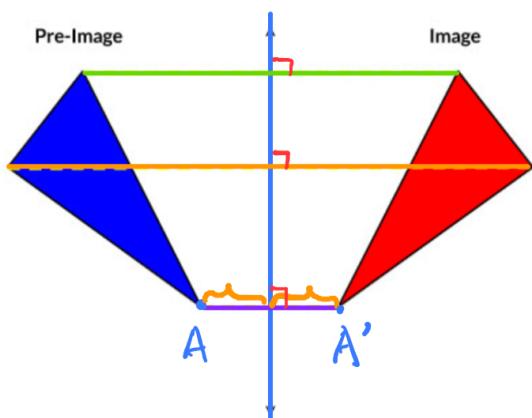
## Concept 3

# Symmetry





## How does symmetry work?



1. Perpendicular

2. Midpoint

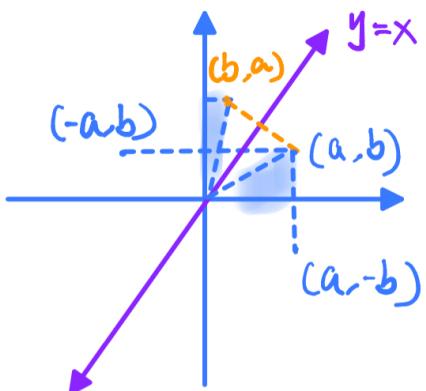


## Common Symmetry in Coordinate Plane

Reflection of functions:

1. Reflection over x-axis:  $y$  is replaced by  $-y$
2. Reflection over y-axis:  $x$  is replaced by  $-x$
3. Reflection over  $y = x$ , replace  $x$  by  $y$  and replace  $y$  by  $x$
4. Relection over  $y = -x$ , replace  $x$  by  $-y$  and replace  $x$  by  $-y$ .

Use symmetry of point to memorize!



 **Math Exploration 3.1**

The  $y$ -intercepts,  $P$  and  $Q$ , of two perpendicular lines intersecting at the point  $A(6, 8)$  have a sum of zero. What is the area of  $\triangle APQ$ ?

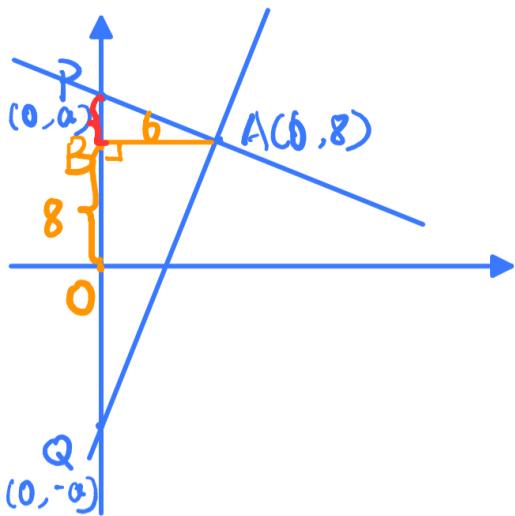
A. 45

B. 48

C. 54

D. 60

E. 72



$$AB = 6, BO = 8 \Rightarrow PB = a - 8 \\ QB = a + 8$$

$$AB^2 = PB \cdot QB$$

$$6^2 = (a - 8)(a + 8)$$

$$a = 10$$

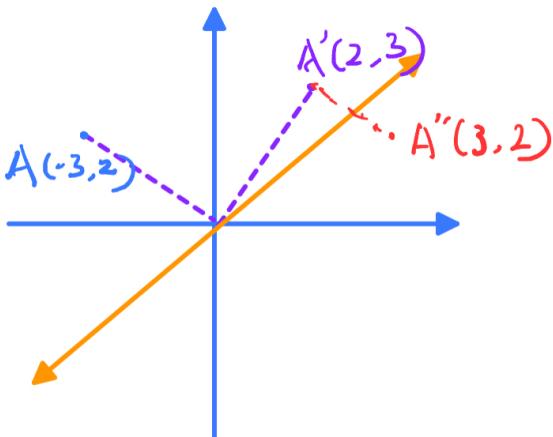
$$\Rightarrow PQ = 2a = 20$$

$$A_{\triangle PAQ} = \frac{1}{2} \cdot 20 \cdot 6 = 60$$


 **Practice 3.1**

The point  $(-3, 2)$  is rotated  $90^\circ$  clockwise around the origin to point  $B$ .

Point  $B$  is then reflected over the line  $x = y$  to point  $C$ . What are the coordinates of  $C$ ? (2004 AMC 12B Problems, Question #9)

A.  $(-3, -2)$ B.  $(-2, -3)$ C.  $(2, -3)$ D.  $(2, 3)$ E.  $(3, 2)$ 

### Practice 3.2

slope of line  $m$  is  $\frac{-1}{3} = \frac{2}{3} \Rightarrow y = \frac{2}{3}x$

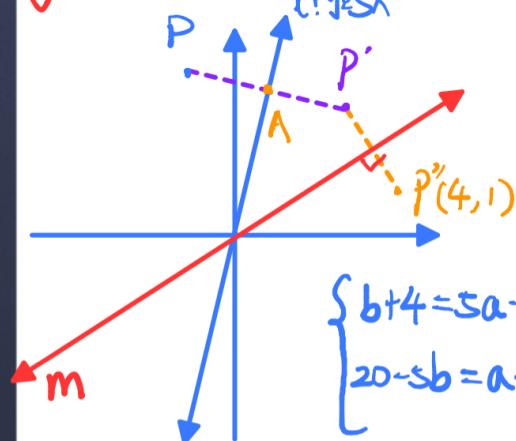
Distinct lines  $\ell$  and  $m$  lie in the  $xy$ -plane. They intersect at the origin. Point  $P(-1, 4)$  is reflected about line  $\ell$  to point  $P'$ , and then  $P'$  is reflected about line  $m$  to point  $P''$ .

The equation of line  $\ell$  is  $5x - y = 0$ , and the coordinates of  $P''$  are  $(4, 1)$ . What is the equation of line  $m$  ?(2021 AMC Fall 10B Problem, Question #17)

- A.  $5x + 2y = 0$       B.  $3x + 2y = 0$       C.  $x - 3y = 0$

D.  $2x - 3y = 0$

E.  $5x - 3y = 0$



$$\begin{cases} b+4 = 5a - 5 \\ 20 - 5b = a + 1 \end{cases}$$

Find  $P'$ : suppose  $P'(a, b)$

① Midpoint of  $PP'$  is  $(\frac{a-1}{2}, \frac{b+4}{2})$

$$\text{plug in to } y = 5x \Rightarrow \frac{b+4}{2} = 5 \cdot \frac{a-1}{2}$$

②  $PP' \perp l$ , slope of  $PP' = \frac{4-b}{-1-a}$

$$\Rightarrow \frac{4-b}{-1-a} \cdot (5) = -1$$

$$\Rightarrow \begin{cases} a = \frac{32}{13} \\ b = \frac{43}{13} \end{cases} \quad \begin{array}{l} \text{Find slope of } P'P'' \\ \frac{43}{13} - 1 = \frac{30}{20} = -\frac{3}{2} \end{array}$$

THINK ACADEMY



### Concept 4

# Analytical Geometry



MC(1)

## Math Exploration 4.1

A triangle with vertices  $A(0, 2)$ ,  $B(-3, 2)$ , and  $C(-3, 0)$  is reflected about the  $x$ -axis, then the image  $\Delta A'B'C'$  is rotated counterclockwise about the origin by  $90^\circ$  to produce  $\Delta A''B''C''$ . Which of the following transformations will return  $\Delta A''B''C''$  to  $\Delta ABC$ ? (2016 AMC 10A Problems, Question #16)

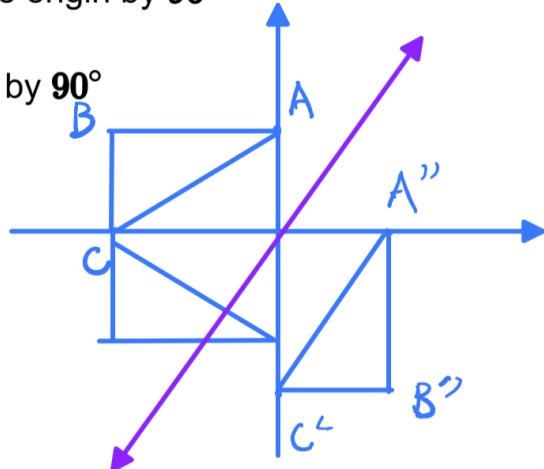
A. counterclockwise rotation about the origin by  $90^\circ$

B. clockwise rotation about the origin by  $90^\circ$

C. reflection about the  $x$ -axis

D. reflection about the line  $y = x$

E. reflection about the  $y$ -axis



MC(1)

## Math Exploration 4.2

The line segment formed by  $A(1, 2)$  and  $B(3, 3)$  is rotated to the line segment formed by  $A'(3, 1)$  and  $B'(4, 3)$  about the point  $P(r, s)$ . What is  $|r - s|$ ? (2023 AMC 10A Problems, Question #19)

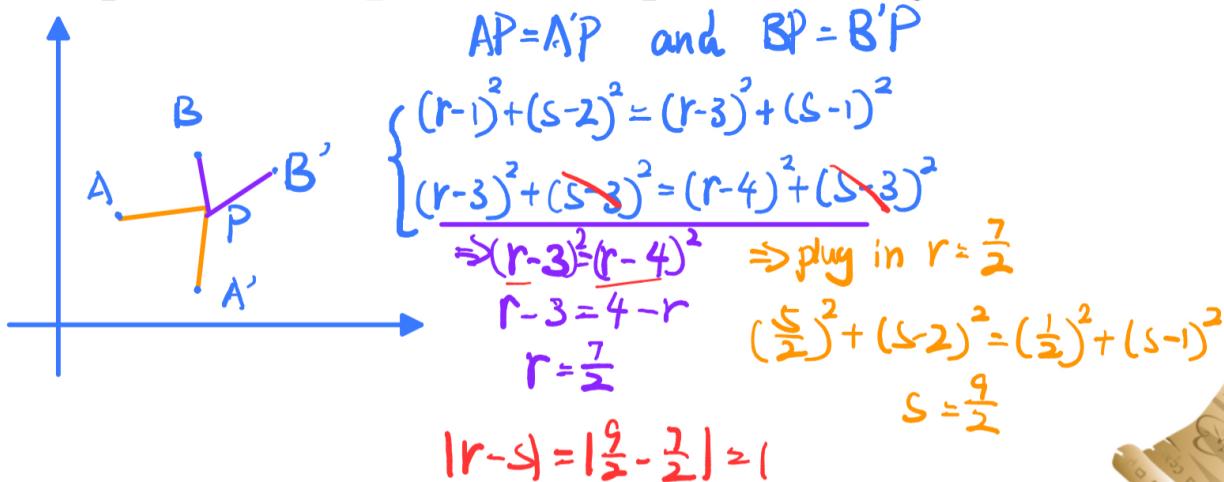
A.  $\frac{1}{4}$

B.  $\frac{1}{2}$

C.  $\frac{3}{4}$

D.  $\frac{2}{3}$

E. 1



## Practice 4.1 Lesson 6 Practice 2.1

A triangle with vertices  $(6, 5)$ ,  $(8, -3)$ , and  $(9, 1)$  is reflected about the line  $x = 8$  to create a second triangle. What is the area of the union of the two triangles? (2013 AMC 10A Problems, Question #16)

- A. 9      B.  $\frac{28}{3}$       C. 10      D.  $\frac{31}{3}$       E.  $\frac{32}{3}$



## Practice 4.2

Triangle  $OAB$  has  $O = (0, 0)$ ,  $B = (5, 0)$ , and  $A$  in the first quadrant. In addition,  $\angle ABO = 90^\circ$  and  $\angle AOB = 30^\circ$ . Suppose that  $OA$  is rotated  $90^\circ$  counterclockwise about  $O$ . What are the coordinates of the image of  $A$ ?

(2008 AMC 10B Problems, Question #14)

- A.  $\left(-\frac{10}{3}\sqrt{3}, 5\right)$     B.  $\checkmark \left(-\frac{5}{3}\sqrt{3}, 5\right)$     C.  $(\sqrt{3}, 5)$     D.  $\left(\frac{5}{3}\sqrt{3}, 5\right)$     E.  $\left(\frac{10}{3}\sqrt{3}, 5\right)$

