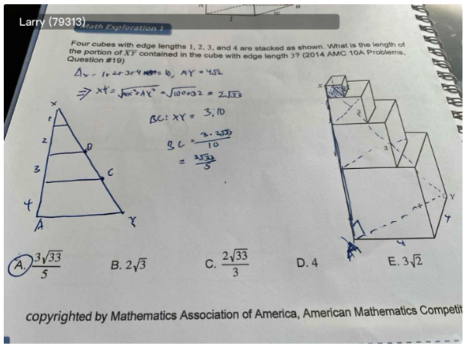
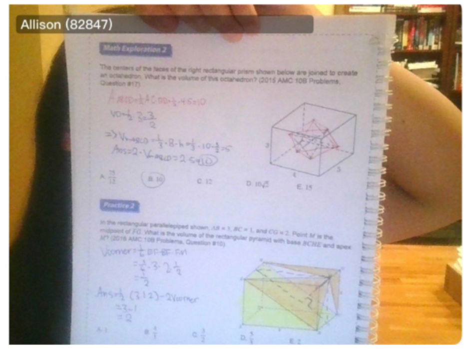
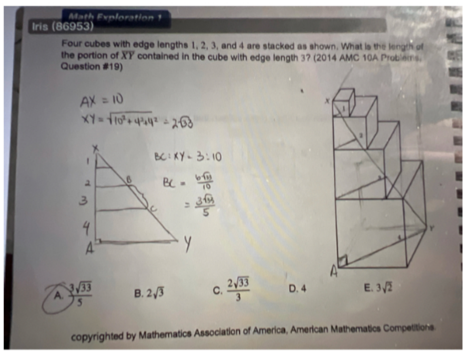
Excellent Notes



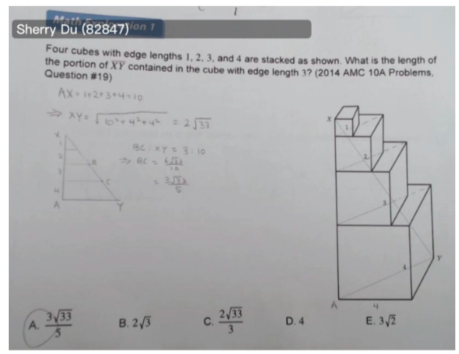
Larry



Allison



Iris



Sherry

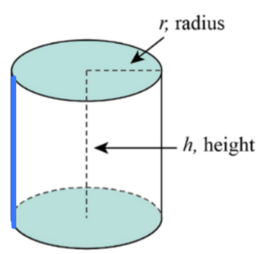
Lesson 10 Cylinder, Cone and Sphere

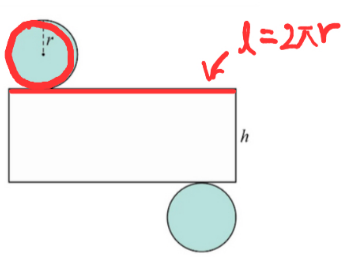
Concept 1

Cylinder and Cone

Cylinder

A cylinder is a solid figure with two congruent circular bases that lie in parallel lines.

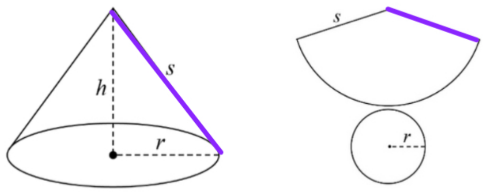




Base area $B = \pi r^2$
Surface area $S = 2B + 2\pi r h = 2\pi r^2 + 2\pi r h$
Volume $V = Bh = \pi r^2 h$

Cone

A **cone** is a solid figure with a circular base. As we can see, the lateral area of a cone is a sector.



The **slant height** s of a cone is the distance between any point on the edge of the base and the vertex.

Slant height $s = \sqrt{r^2 + h^2}$

Base area $B = \pi r^2$

Surface area $S = B + \pi r s = \pi r^2 + \boxed{\pi r s}$, where slant height $s = \sqrt{r^2 + h^2}$

Volume $V = \dfrac{1}{3}\pi r^2 h$

MC(1)

Math Exploration 1.1

Two right circular cones with vertices facing down as shown in the figure below contain the same amount of liquid. The radii of the tops of the liquid surfaces are $\mathbf{3}$ cm and $\mathbf{6}$ cm. Into each cone is dropped a spherical marble of radius $\mathbf{1}$ cm, which sinks to the bottom and is completely submerged without spilling any liquid. What is the ratio of the rise of the liquid level in the narrow cone to the rise of the liquid level in the wide cone? (2021 Spring AMC 10A Problems, Question #12)

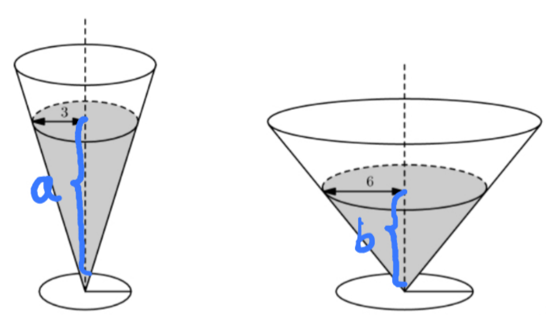
A. $\mathbf{1:1}$ B. $\mathbf{47:43}$ C. $\mathbf{2:1}$ D. $\mathbf{40:13}$ E. $\mathbf{4:1}$

Before: $\frac{1}{3}\cdot\pi\cdot 3^2\cdot a = \frac{1}{3}\cdot\pi\cdot 6^2\cdot b$

$a = 4b$

After: $a_{new} = 4b_{new}$

Ratio $= 4:1$

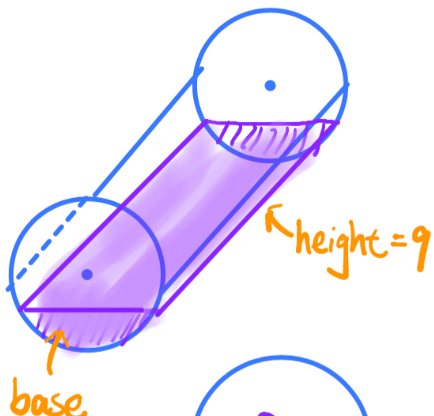


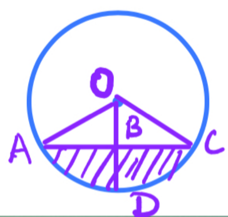
MC(1)

Math Exploration 1.2

A cylindrical tank with radius **4** feet and height **9** feet is lying on its side. The tank is filled with water to a depth of **2** feet. What is the volume of water, in cubic feet? (2008 AMC 10B Problems, Question #19)

A. $24\pi - 36\sqrt{2}$　B. $24\pi - 24\sqrt{3}$　C. $36\pi - 36\sqrt{3}$　D. $36\pi - 24\sqrt{2}$　E. $48\pi - 36\sqrt{3}$





$BD = 2$, $OD = OA = OC = 4$

$\Rightarrow OB = OD - BD = 2$

Also, $\overline{OB} \perp \overline{AC}$ and $AB = BC$

$\begin{cases} OB = \frac{1}{2} OC \\ m\angle OBC = 90^\circ \end{cases} \Rightarrow \triangle OBC$ is a 30-60-90 $\triangle$

$m\angle AOC = 2m\angle BOC = 120^\circ \Rightarrow A_{sector} = \frac{120^\circ}{360^\circ} \cdot \pi \cdot 4^2 = \frac{16}{3}\pi$

$BC = \sqrt{3} \cdot OB = 2\sqrt{3}$

$A_{\triangle AOC} = 2A_{\triangle OBC} = 2 \cdot 2\sqrt{3} = 4\sqrt{3}$

$\text{Base} = \frac{16}{3}\pi - 4\sqrt{3} \Rightarrow V = B \cdot h = 9\left(\frac{16}{3}\pi - 4\sqrt{3}\right) = 48\pi - 36\sqrt{3}$

MC(1)

Practice 1.1

An inverted cone with base radius **12cm** and height **18cm** is full of water. The water is poured into a tall cylinder whose horizontal base has radius of **24cm**. What is the height in centimeters of the water in the cylinder? (2021 Spring AMC 10B Problems, Question #10)

A. **1.5**　B. **3**　C. **4**　D. **4.5**　E. **6**

$V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi \cdot 12^2 \cdot 18$

$= 6 \cdot 144 \cdot \pi$

$\pi \cdot \cancel{24}^{2} \cdot h = 6 \cdot \cancel{144} \cdot \pi$ (4 written above 24)

$4h = 6$

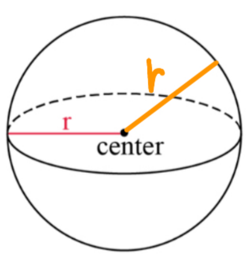
$h = 1.5$

Concept 2

Sphere

Just as a circle is the set of all points in a plane that are the same distance from a given point, a **sphere** is the set of all points in space that are equidistant from a given point. Most balls and globes are examples of spheres.

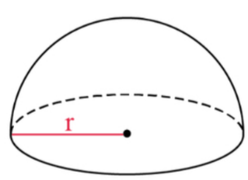
Sphere



$$V = \frac{4}{3}\pi r^3$$

$$SA = 4\pi r^2$$

Hemisphere



$$V = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = \frac{2}{3}\pi r^3$$

$$SA = \frac{1}{2}(4\pi r^2) + \pi r^2 = 3\pi r^2$$

Focus on the cross-section!

MC(1)

Math Exploration 2.1

Four congruent semicircles are drawn on the surface of a sphere with radius $\mathbf{2}$, as shown, creating a close curve that divides the surface into two congruent regions. The length of the curve is $\pi\sqrt{n}$. What is n? (2023 AMC 10B Problems, Question #20)

A. $\mathbf{32}$ ✓ B. $\mathbf{12}$ C. $\mathbf{48}$ D. $\mathbf{36}$ E. $\mathbf{27}$

ABCD is a square.

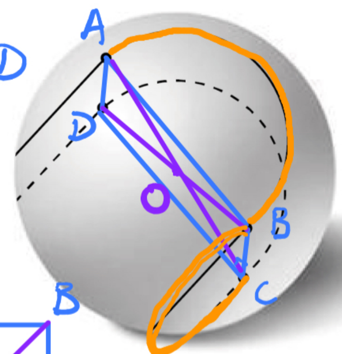
center of sphere is at center of ABCD

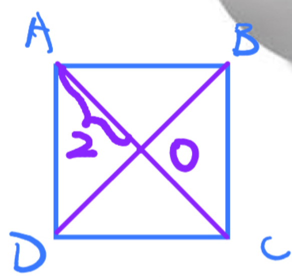
$\Rightarrow AO=2$, $AD=\sqrt{2}\cdot AO=2\sqrt{2}$

Ans: $4\cdot\underbrace{\left(\frac{1}{2}\cdot\pi\cdot 2\sqrt{2}\right)}$ ← semicircle

$=(4\sqrt{2})\pi$

$=\pi\sqrt{32}$



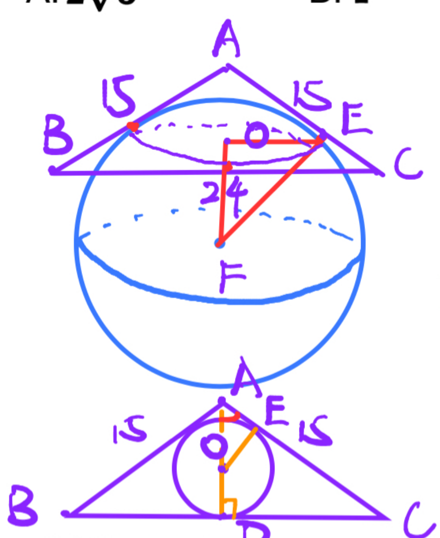


MC(1)

Practice 2.1

A sphere with center O has radius $\mathbf{6}$. A triangle with sides of length $\mathbf{15, 15,}$ and $\mathbf{24}$ is situated in space so that each of its sides is tangent to the sphere. What is the distance between O and the plane determined by the triangle? (2019 AMC 10A Problems, Question #21)

A. $\mathbf{2\sqrt{3}}$ B. $\mathbf{4}$ C. $\mathbf{3\sqrt{2}}$ D. $\mathbf{2\sqrt{5}}$ ✓ E. $\mathbf{5}$



$AB=AC=15$, $BD=CD=\frac{1}{2}BC=12$

$\Rightarrow \triangle ADC$ is a Rt$\triangle$, $AD=9$

$OD=OE=r$, $AO=9-r$

$\triangle AOE \sim \triangle ACD \Rightarrow \frac{OE}{AD}=\frac{4}{5} \Rightarrow r=4$

$EF=6$, $OE=4$, $\triangle OEF$ is a Rt$\triangle$

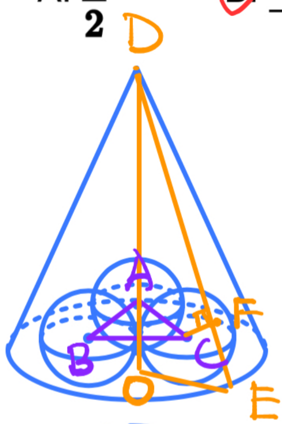
$\Rightarrow OF=\sqrt{EF^2-OE^2}=\sqrt{36-16}=2\sqrt{5}$

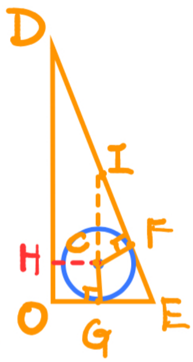
MC(1)

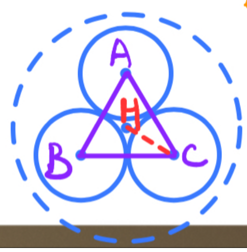
Math Exploration 3.1

Inside a right circular cone with base radius **5** and height **12** are three congruent spheres with radius r. Each sphere is tangent to the other two spheres and also tangent to the base and side of the cone. What is r? (2021 Fall AMC 10 A Problems, Question #22)

A. $\frac{3}{2}$ B. $\frac{90-40\sqrt{3}}{11}$ C. 2 D. $\frac{144-25\sqrt{3}}{44}$ E. $\frac{5}{2}$







$CG = CF = r$, $AB = BC = AC = 2r$

In $\triangle ABC$, $HC = \frac{2}{3} \cdot AC \cdot \frac{\sqrt{3}}{2} = \frac{2r\sqrt{3}}{3}$

$\Rightarrow CH = OG = \frac{2r\sqrt{3}}{3}$

In Rt$\triangle DOE$, $OD = 12$, $OE = 5$

$\triangle CIF \sim \triangle EDO \Rightarrow \frac{CI}{CF} = \frac{13}{5}$

$CI = \frac{13}{5}r \Rightarrow GI = CG + CI = r + \frac{13}{5}r = \frac{18}{5}r$

$GE = OE - OG = 5 - \frac{2r\sqrt{3}}{3} = \frac{15-2r\sqrt{3}}{3}$

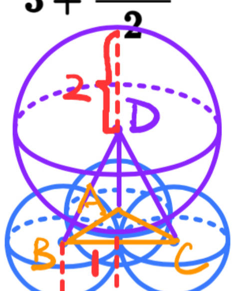
Lastly, $\frac{GI}{GE} = \frac{12}{5} \Rightarrow \frac{18}{5}r \Big/ \left(\frac{15-2r\sqrt{3}}{3}\right) = \frac{12}{5}$

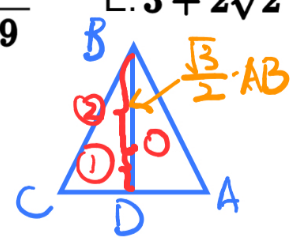
MC(1)

Math Exploration 3.2

Three mutually tangent spheres of radius **1** rest on a horizontal plane. A sphere of radius **2** rests on them. What is the distance from the plane to the top of the larger sphere? (2004 AMC 10A Problems, Question #25)

A. $3+\frac{\sqrt{30}}{2}$ B. $3+\frac{\sqrt{69}}{3}$ C. $3+\frac{\sqrt{123}}{4}$ D. $\frac{52}{9}$ E. $3+2\sqrt{2}$





$AB = AC = BC = 1+1 = 2$

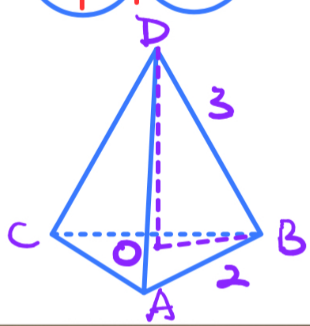
$AD = BD = CD = 1+2 = 3$

$OB = \frac{2}{3} \cdot AB \cdot \frac{\sqrt{3}}{2} = \frac{2\sqrt{3}}{3}$

$\triangle DOB$ is a Rt$\triangle$, $DO = \sqrt{BD^2 - OB^2}$

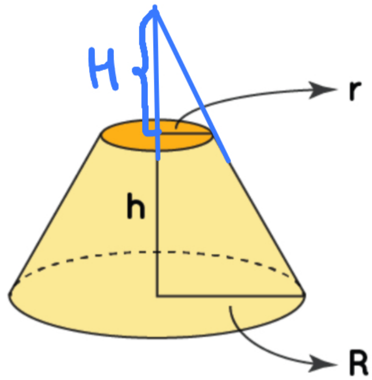
$= \sqrt{9 - \frac{4}{3}}$

$= \frac{\sqrt{23}}{\sqrt{3}} = \frac{\sqrt{69}}{3}$



Ans $= 2+1+\frac{\sqrt{69}}{3} = 3+\frac{\sqrt{69}}{3}$

Volume of a Partial (Truncated) Cone



Partial Cone

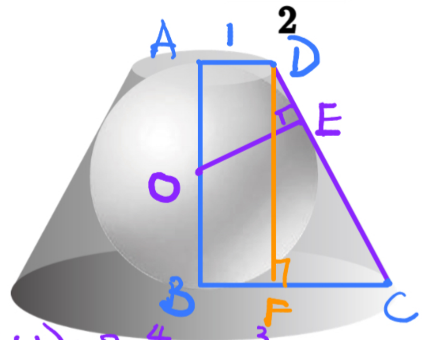
$$V = \frac{1}{3}\pi h(R^2 + Rr + r^2)$$

MC(1)

Practice 3.1

A sphere is inscribed in a truncated right circular cone as shown. The volume of the truncated cone is twice that of the sphere. What is the ratio of the radius of the bottom base of the truncated cone to the radius of the top base of the truncated cone? (2014 AMC 10B Problems, Question #23)

A. $\frac{3}{2}$ B. $\frac{1+\sqrt{5}}{2}$ C. $\sqrt{3}$ D. 2 E. $\frac{3+\sqrt{5}}{2}$



Make top radius be 1 and bottom be R

Let $AO=BO=EO=x$, $\overline{EO}\perp\overline{DC}$

$\Rightarrow DE=AD=1$, $BC=EC=R$

$\Rightarrow DC=R+1$, $DF=2x$, $CF=R-1$

$(2x)^2+(R-1)^2=(R+1)^2$

$4x^2=4R$

$x=\sqrt{R}$

Volume: $\frac{1}{3}\pi\cdot(2x)(R^2+R+1)=2\cdot\frac{4}{3}\pi\cdot x^3$

$2\sqrt{R}(R^2+R+1)=8R\cdot\sqrt{R}$

$R^2-3R+1=0$

$R=\frac{3+\sqrt{5}}{2}$

Concept 3: Irregular 3D-figures

Irregular 3D-figures

In the AMC 10, **irregular solid figure** are often tested by Cross-Section Problems.

Features:
A regular solid is cut by a plane (flat or slanted)
Find remaining volume, visible surface area, or shape of the cross-section

How to solve:
Draw a cross-sectional diagram
Use geometric reasoning or proportional methods to find height/area
Useful formulas: frustum volume, triangle area in a section, etc.

MC(1)

Math Exploration 4.1

A bowl is formed by attaching four regular hexagons of side $\mathbf{1}$ to a square of side $\mathbf{1}$. The edges of the adjacent hexagons coincide, as shown in the figure. What is the area of the octagon obtained by joining the top eight vertices of the four hexagons, situated on the rim of the bowl? (2022 AMC 10A Problems, Question #21)

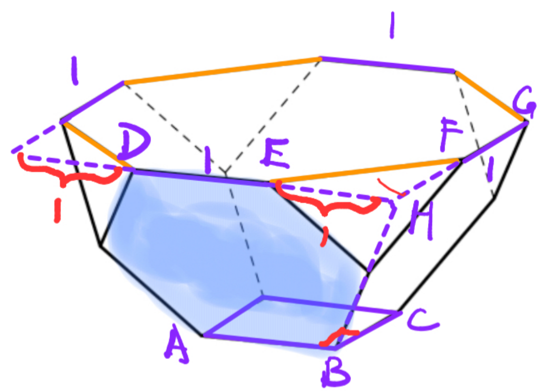
A. $\mathbf{6}$ B. $\mathbf{7}$ C. $\mathbf{5+2\sqrt{2}}$ D. $\mathbf{8}$ E. $\mathbf{9}$

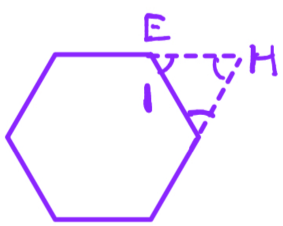
$\overline{AB} \perp \overline{BC}$, $\overline{DE} \parallel \overline{AB}$, $\overline{FG} \parallel \overline{BC}$

$\Rightarrow \overline{DE} \perp \overline{FG}$, $m\angle EHF = 90^\circ$

$EH = EF = 1$

Area $= 3^2 - 4 \cdot (\frac{1}{2} \cdot 1 \cdot 1) = 7$



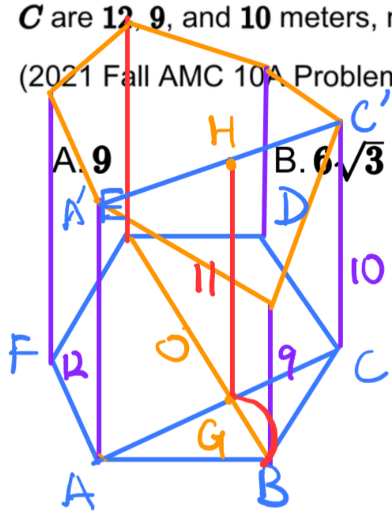


MC(1)

Practice 4.1

An architect is building a structure that will place vertical pillars at the vertices of regular hexagon $\boldsymbol{ABCDEF}$, which is lying horizontally on the ground. The six pillars will hold up a flat solar panel that will not be parallel to the ground. The heights of pillars at $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are $\mathbf{12}$, $\mathbf{9}$, and $\mathbf{10}$ meters, respectively. What is the height, in meters, of the pillar at $\boldsymbol{E}$? (2021 Fall AMC 10A Problems, Question #17)

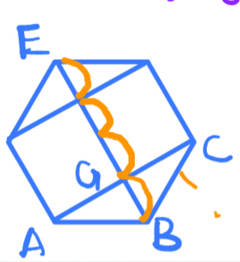
A. $\mathbf{9}$ B. $\mathbf{6\sqrt{3}}$ C. $\mathbf{8\sqrt{3}}$ D. $\mathbf{17}$ E. $\mathbf{12\sqrt{3}}$



$AG = GC$, GH is midline of trapezoid $ACC'A'$

$\Rightarrow GH = \frac{1}{2}(12+10) = 11$

Also, $EG : BG = 3:1$



$EE' = BB' + 4 \cdot (11-9)$

$= 9+8$

$= 17$