

# Excellent Notes

Sarah (79297)

### Math Exploration 1

A line that passes through the origin intersects both the line  $x = 1$  and the line  $y = 1 + \frac{\sqrt{3}}{3}x$ . The three lines create an equilateral triangle. What is the perimeter of the triangle? (2015 AMC 10A Problem, Question #17)

4 not long  
nearest tenth  
line length  
pass the origin

**Practice 1**

First, we visualize the graph:  
 To find the perimeter, we can just find one side (and multiply by 3) to get our answer (equilateral triangle)  
 Find AB: when  $x=1$ , plugging it into  $y=1 + \frac{\sqrt{3}}{3}x$ , we see that  $y = 1 + \frac{\sqrt{3}}{3}$  is  $AD$ ,  
 Find BD:  $ZB = 60^\circ$  because the equal.  $A = 90^\circ$  so  $\triangle BCD$  is a  $45-45-90^\circ$  triangle  
 Since side  $CD = 1$  (due to intersection of origin), we use that to get side  $BD = \sqrt{2}$   
 Finally, we add  $AD$  and  $BD$  and multiply by 3:  $3 + 2\sqrt{2}$

Sarah

**Geometry 12. Solving Systems by Elimination**

**Problem 1:** Use an ordered pair to solve the system of equations.

$$\begin{aligned} A. \quad & x + y = 7 \\ & x - y = 1 \\ B. \quad & x + y = 7 \\ & x - y = 3 \\ C. \quad & x + y = 7 \\ & x - y = 5 \\ D. \quad & x + y = 7 \\ & x - y = 7 \end{aligned}$$

**Problem 2:** The area of the region bounded by the graph of  $x^2 + y^2 = 25$  and the line  $y = 3x$ . (2018 AMC 10A, Question #17)

**Solution:**

The graph shows a circle centered at the origin with radius 5, and a line passing through the origin with slope 3. The line intersects the circle at two points. The region between the line and the circle is shaded.

$$\begin{aligned} x^2 + y^2 = 25 \\ y = 3x \end{aligned}$$

The intersection points are found by solving the system:

$$\begin{aligned} x^2 + (3x)^2 &= 25 \\ 10x^2 &= 25 \\ x^2 &= 2.5 \\ x &= \pm\sqrt{2.5} \end{aligned}$$

Substituting  $x = \pm\sqrt{2.5}$  into  $y = 3x$  gives  $y = \pm 3\sqrt{2.5}$ .

The region is a sector of the circle with central angle  $2\arctan(3)$ .

The area of the sector is:

$$\text{Area} = \frac{1}{2} r^2 \theta = \frac{1}{2} \cdot 25 \cdot 2\arctan(3) = 25\arctan(3)$$

# Braydon

Sherry Du (B9947)  
 $\Rightarrow AB \perp l_1 \left( 1 + \frac{1}{2} \right) = \frac{\sqrt{5}}{2} = |t| \Rightarrow t = \pm \frac{\sqrt{5}}{2}$   
 $P_{\text{triangle } ABC} = 5\sqrt{3} + 2 + 2\sqrt{3}$

**Practice 1**

Line  $l_1$  has equation  $3x - 2y = 1$  and goes through point  $A(-1, -2)$ . Line  $l_2$  has equation  $y = 1$  and meets line  $l_1$  at point  $B$ . Line  $l_3$  has positive slope, goes through point  $A$ , and meets  $l_2$  at point  $C$ . The area of  $\triangle ABC$  is 3. What is the slope of  $l_3$ ?

A  $\frac{2}{3}$       B  $\frac{3}{2}$       C. 1      D.  $-\frac{4}{3}$       E.  $\frac{3}{2}$

$\therefore l_1: y = \frac{3}{2}x + \frac{1}{2}$   
 $\therefore B = \left( -\frac{1}{2}, -\frac{1}{2} \right)$   
 $\therefore C = \left( \frac{1}{2}, 1 \right)$

$\Delta ABC: \text{height } = 3$ , so base  $= 2$ , so  $2|x_B - x_C| = 2 \Rightarrow x_C - x_B = 1$   
 $\Rightarrow C \in \{3, 1\}$   
 $\Rightarrow C(3, 1)$

Slope  $\{l_3: \frac{(x_2 - x_1)}{(y_2 - y_1)} = \frac{3}{4}\}$

# Sherry

**Math Exploration 2**

The area of the region bounded by the graph of  $x^2 + y^2 = 3|x| - 1$  and  $3x + y = m$ , where  $m$  and  $n$  are integers. What is  $m + n$ ? (2021 AMC Spring 10A, Question 10)

**A. 18      B. 27      C. 36      D. 45**

**Practice 2**

1. How many ordered pairs of real numbers  $(x, y)$  satisfy the following system of equations?  $x + 3y = 3$ ,  $|x| - y = 1$ . (2018 AMC 10A, Question #12)

**B. 2      C. 3      D. 4**

**Quintus**

# Michelle

Emma

# Lesson 3

# Polynomial Functions

## Concept 1

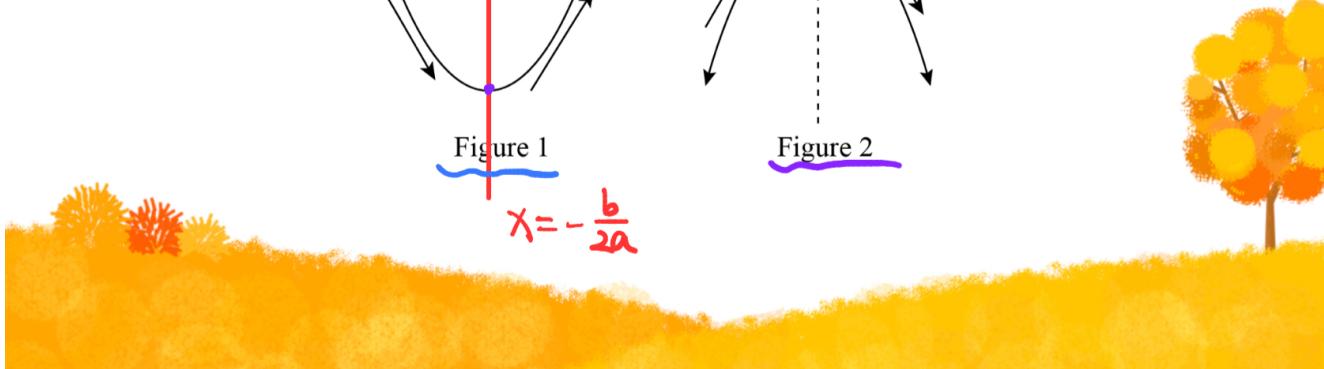
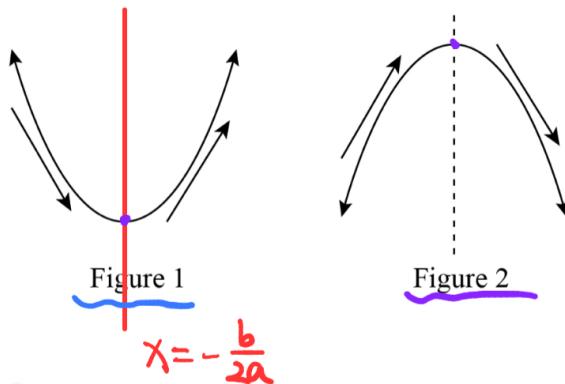
# Quadratic Functions



## Concept 1: Quadratic Functions

Consider the parabola  $y = ax^2 + bx + c$  or  $y = a(x - h)^2 + k$  with  $a \neq 0$ .

- If  $a > 0$ , the parabola opens upward; if  $a < 0$ , the parabola opens downward.
- The axis of symmetry is  $x = -\frac{b}{2a}$  or  $x = h$ .
- The vertex is at  $(-\frac{b}{2a}, \frac{4ac - b^2}{4a})$  or  $(h, k)$ .



## Math Exploration 1.1

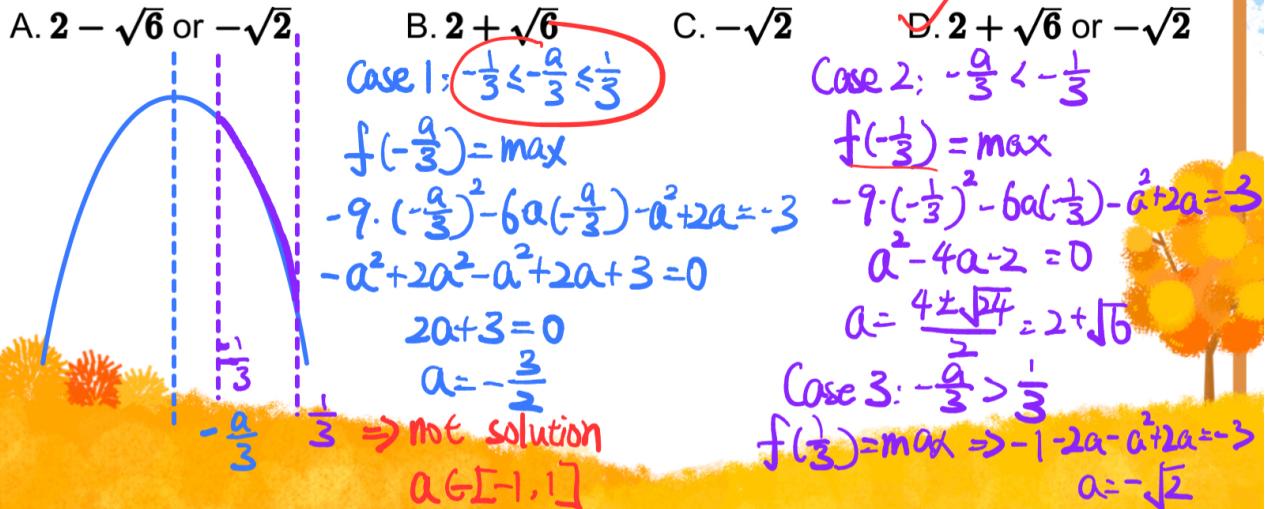
The given function is:

$$f(x) = -9x^2 - 6ax - a^2 + 2a, \quad \text{for } -\frac{1}{3} \leq x \leq \frac{1}{3}$$

It is known that  $f(x)$  has a maximum value of  $-3$ .

Find the value of the real number  $a$ . Axis of Symmetry:  $x = -\frac{-6a}{2(-9)} = -\frac{a}{3}$

A.  $2 - \sqrt{6}$  or  $-\sqrt{2}$



B.  $2 + \sqrt{6}$

Case 1:  $-\frac{1}{3} \leq -\frac{a}{3} \leq \frac{1}{3}$

$f(-\frac{a}{3}) = \max$

$$-9 \cdot (-\frac{a}{3})^2 - 6a(-\frac{a}{3}) - a^2 + 2a = -3$$

$$-a^2 + 2a^2 - a^2 + 2a + 3 = 0$$

$$2a + 3 = 0$$

$$a = -\frac{3}{2}$$

$\Rightarrow$  no solution  
 $a \in [-1, 1]$

C.  $-\sqrt{2}$

D.  $2 + \sqrt{6}$  or  $-\sqrt{2}$

Case 2:  $-\frac{a}{3} < -\frac{1}{3}$

$f(-\frac{1}{3}) = \max$

$$-9 \cdot (-\frac{1}{3})^2 - 6a(-\frac{1}{3}) - a^2 + 2a = -3$$

$$a^2 - 4a - 2 = 0$$

$$a = \frac{4 \pm \sqrt{14}}{2} = 2 \pm \sqrt{6}$$

Case 3:  $-\frac{a}{3} > \frac{1}{3}$

$$f(\frac{1}{3}) = \max \Rightarrow -1 - 2a - a^2 + 2a = -3$$

$$a = -\sqrt{2}$$

## Practice 1.1

Given the quadratic function:

$$y = x^2 - 2mx \Rightarrow x = -\frac{-2m}{2} = m$$

where  $m$  is a constant, the minimum value of  $y$  on the interval  $-1 \leq x \leq 2$  is given as  $-2$ . Find the value of  $m$ .

A.  $\frac{3}{2}$

B.  $\sqrt{2}$

C.  $\frac{3}{2}$  or  $\sqrt{2}$

D.  $-\frac{3}{2}$  or  $\sqrt{2}$

Case 1:  $-1 \leq m \leq 2$

$$f(m) = -2$$

$$m^2 - 2m^2 = -2$$

$$m = \pm \sqrt{2}$$

$$\Rightarrow m = \sqrt{2} \in [-1, 2]$$

Case 2:  $m < -1$

$$f(-1) = -2$$

$$1 + 2m = -2$$

$$m = -\frac{3}{2}$$

$$\Rightarrow -\frac{3}{2} \in (-\infty, -1)$$

Case 3:  $m > 2$

$$f(2) = -2$$

$$4 - 4m = -2$$

$$m = \frac{3}{2} < 2$$

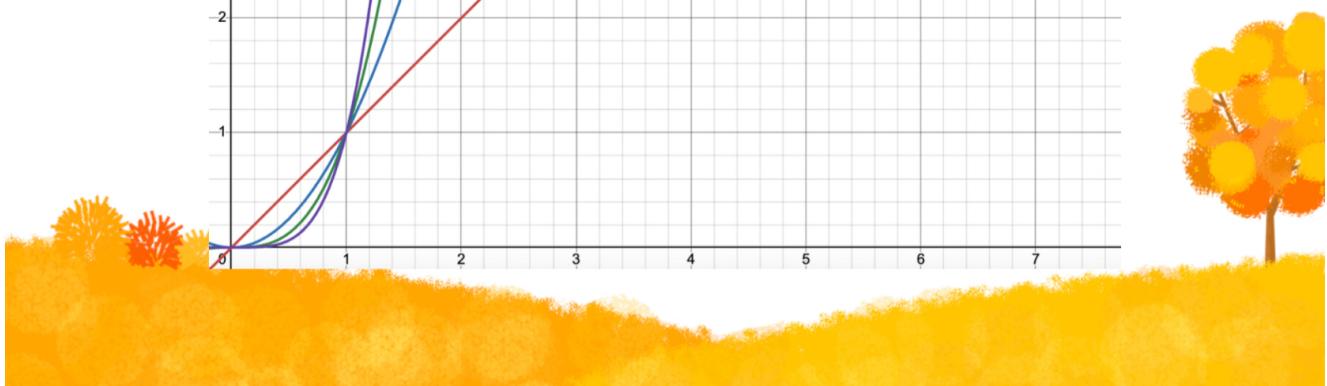
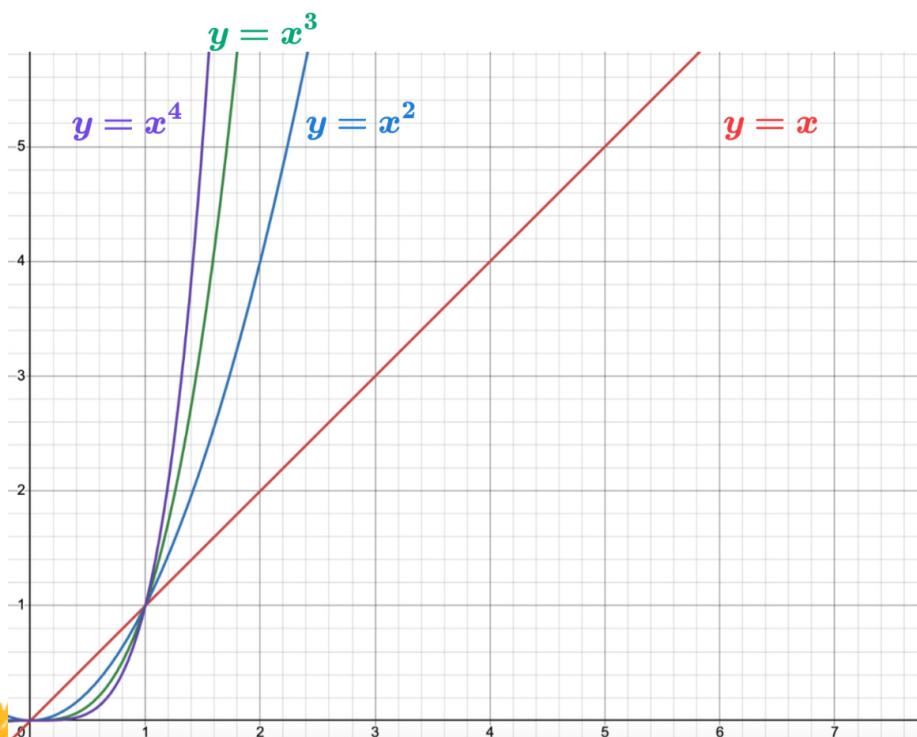
$\Rightarrow$  no solution

## Concept 2

# Graph of Polynomial Functions



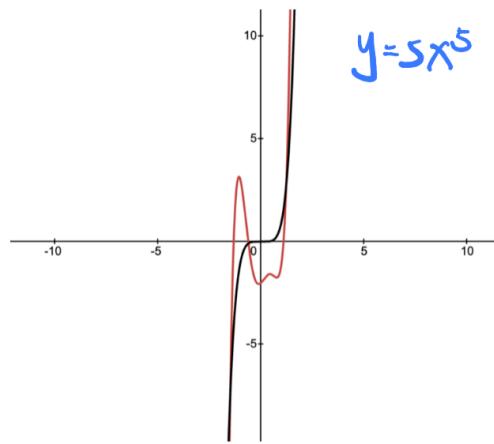
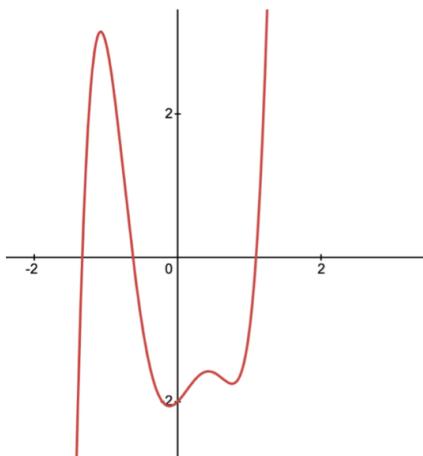
How to graph higher degree polynomials?



## How to graph higher degree polynomials?

How to graph this function?

$$y = \boxed{5x^5} - 8x^3 + 3x^2 + x - 2$$



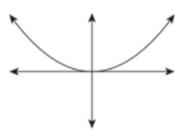
## Concept 2: Graph of Polynomial Functions

The Leading-Term Test

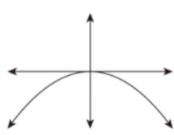
If  $a_n x^n$  is the leading term of a polynomial. Then the behavior of the graph as

$x \rightarrow \infty$  or  $x \rightarrow -\infty$  can be known by one the four following behaviors:

1. If  $a_n > 0$  and  $n$  even:



2. If  $a_n < 0$  and  $n$  even:

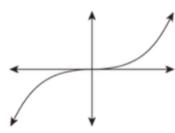


$$y = -2x^5 + \dots$$

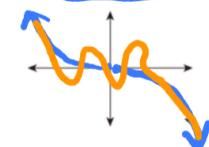
$x \rightarrow +\infty, y \rightarrow -\infty$

$x \rightarrow -\infty, y \rightarrow \infty$

3. If  $a_n > 0$  and  $n$  odd:

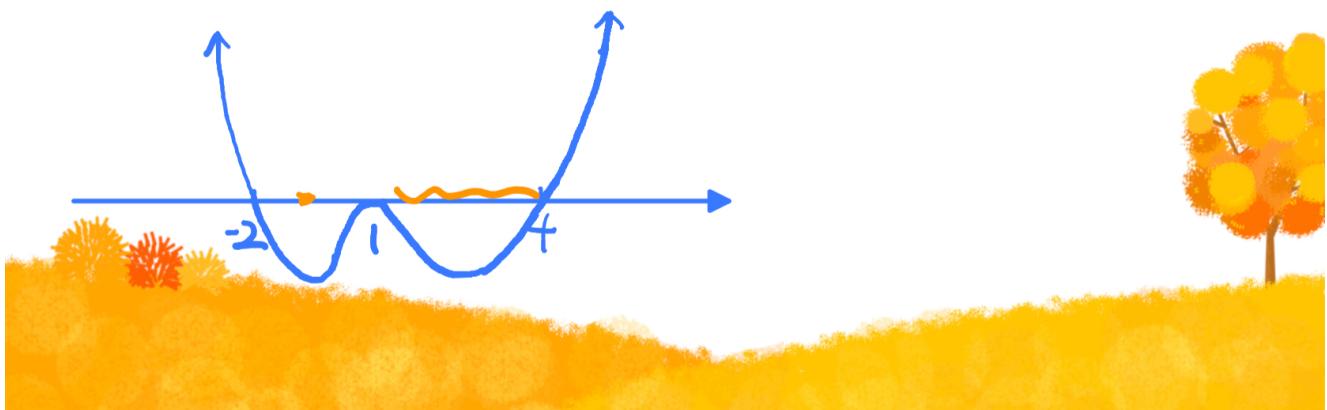


4. If  $a_n < 0$  and  $n$  odd:



For even powers, such as 2, 4, 6, and 8, the graph will still touch and bounce off of the  $x$ -axis, but for each increasing even power the graph will appear flatter as it approaches and leaves the  $x$ -axis.

For odd powers, such as 1, 5, 7, and 9, the graph will still cross through the  $x$ -axis, but for each increasing odd power, the graph will appear flatter as it approaches and leaves the  $x$ -axis.  $y = (x-4)(x-1)^2(x+2)^3$

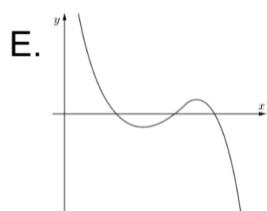
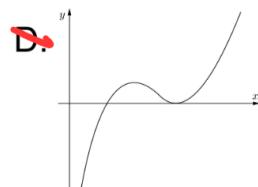
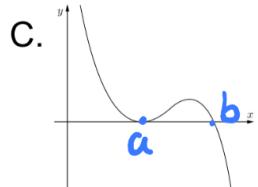
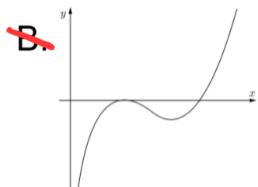
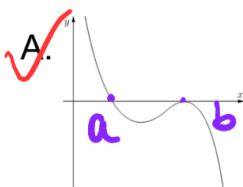


MC(1)

### Math Exploration 2.1

Among the graphs shown below there is the graph of the function

$$f(x) = (a-x)(b-x)^2 \text{ with } a < b. \text{ Which is it?}$$



① End Behavior:  
 $(-x)(-x)^2 = -x^3$

②  $x$ -intercepts:  
 $a, b$   
 pass      bounce



## Math Exploration 2.2

When the roots of the polynomial

$$P(x) = (x-1)^1(x-2)^2(x-3)^3 \cdots (x-10)^{10}$$

are removed from the real number line, what remains is the union of 11 disjoint open intervals. On how many of these intervals is  $P(x)$  positive?

(2023 AMC 10B Problems, Question #12)

A. 3

B. 7

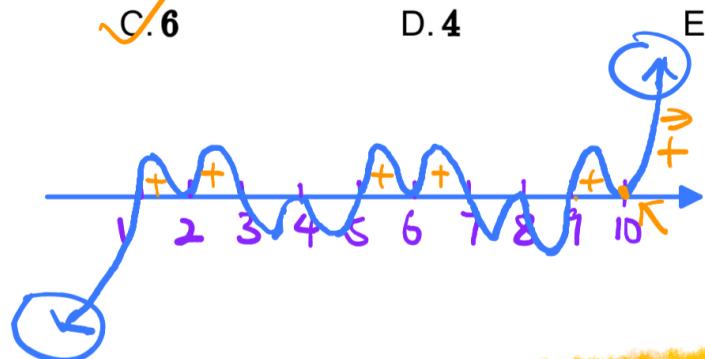
C. 6

D. 4

E. 5

① End Behavior:  
 $x \cdot x^2 \cdots x^{10} = x^{55}$

② Roots:



## Practice 2.1

Define  $P(x) = (x-1^2)(x-2^2) \cdots (x-100^2)$ . How many integers  $n$  are there such that  $P(n) \leq 0$ ? (2020 AMC 10A, Question #17)

A. 4900

B. 4950

C. 5000

D. 5050

E. 5100

Intervals:

$$[99^2, 100^2], [97^2, 98^2], \dots, [1^2, 2^2]$$

$$N = (100^2 - 99^2 + 1) + (98^2 - 97^2 + 1) + \dots$$

$$= 50 + (100^2 - 99^2) + (98^2 - 97^2) + \dots$$

$$= 50 + (100 - 99)(100 + 99) + \dots$$

$$= 50 + 100 + 99 + 98 + 97 + \dots + 2 + 1$$

$$= 50 + \frac{(1+100) \times 100}{2} = 5100$$

① End Behavior:  $x^{100}$

② Zeros:  $1^2, 2^2, \dots, 100^2$



### Concept 3

# Remainder Theorem and Factor Theorem



### Review: Polynomial Division

$$\begin{array}{r} 4x + 3 \\ \hline x^2 - 1 \) 4x^3 + 3x^2 + x - 1 \\ 4x^3 \quad \quad \quad - 4x \\ \hline 3x^2 + 5x - 1 \\ 3x^2 \quad \quad \quad - 3 \\ \hline 5x + 2 \end{array}$$

$(4x^3 + 3x^2 + x - 1) \div (x^2 - 1)$   
 $= (4x + 3) R (5x + 2)$



### Concept 3: Remainder Theorem and Factor Theorem

The polynomials with respect to  $x$  can be simply represented by  $f(x)$  or  $g(x)$ .

The quotient of the division of  $f(x)$  divided by  $g(x)$  is  $q(x)$ , and the remainder is

$$r(x). \frac{f(x) \div g(x)}{=} q(x) R r(x), \text{ i.e., } f(x) = g(x)q(x) + r(x).$$

$$\Delta \quad 21 \div 5 = 4 R 1 \quad 21 = 5 \times 4 + 1$$

Especially, when the divisor  $g(x)$  is a linear term  $(x - a)$ , the remainder  $r(x)$  must be a constant. In this case, the remainder can also be written as  $r$ , therefore  $f(x) = (x - a) \cdot q(x) + r$

Degree of remainder must be smaller than degree of divisor.



### Concept 3: Remainder Theorem and Factor Theorem

Polynomial **Remainder Theorem** states that the remainder of the division of a polynomial  $f(x)$  by a linear polynomial  $x - a$  is equal to  $f(a)$ .

$$\begin{aligned} f(x) &= (x-a)q(x) + r \\ \Rightarrow f(a) &= \underline{0} \cdot q(x) + r = r \end{aligned}$$

In particular,  $x - a$  is a factor of  $f(x)$  if and only if  $f(a) = 0$ , and this is the **Factor Theorem**.



**Math Exploration 3.1**

Let  $P(x)$  be the unique polynomial of minimal degree with the following properties:

- $P(x)$  has leading coefficient 1,
- 1 is a root of  $P(x) - 1$ ,
- 2 is a root of  $P(x - 2)$ ,
- 3 is a root of  $P(3x)$ , and
- 4 is a root of  $4P(x)$ .

The roots of  $P(x)$  are integers, with one exception. The root that is not an integer can be written as  $\frac{m}{n}$ , where  $m$  and  $n$  are relatively prime positive integers. What is  $m + n$ ?

A. 41

B. 43

C. 45

D. 47

E. 49

Roots:

$$\begin{cases} P(1)-1=0 \\ P(2-2)=0 \\ P(3\cdot 3)=0 \\ 4P(4)=0 \end{cases} \Rightarrow \begin{cases} P(1)=1 \\ P(0)=0 \\ P(9)=0 \\ P(4)=0 \end{cases}$$

$\Rightarrow P(x)$  has roots:  
0, 4, 9

Suppose  $r = \text{non-integer root}:$ 

$$P(x) = (x-0)(x-4)(x-9)(x-r)$$

$\Rightarrow$  plug in  $x=1$ :

$$1 \cdot (-3)(-8)(1-r) = 1$$

$$1-r = \frac{1}{24}$$

$$r = \frac{23}{24} \quad \text{Ans} = 23+24=47$$



PhotoPost

**Practice 3.1**

$$f(x) = bx + c$$

$$f(3x) = 3bx + c$$

$$f(3x) - f(x) = 2bx$$

Let  $f(x)$  be a polynomial of degree 5. If  $f(1) = 0$ ,  $f(3) = 1$ ,  $f(9) = 2$ ,

$f(27) = 3$ ,  $f(81) = 4$  and  $f(243) = 5$ , find the coefficient of  $x$  in  $f(x)$ .

Pattern:  $f(3x) - f(x) = 1$  for all  $x = 1, 3, 9, 27, 81$   $\Rightarrow$  coefficient is  $b$ .

Make  $P(x) = f(3x) - f(x) - 1 = 0$  for all  $x = 1, 3, 9, 27, 81$

$$\Rightarrow P(1) = P(3) = P(9) = P(27) = P(81) = 0, \quad P(0) = f(0) - f(0) - 1 = -1$$

$$P(x) = a(x-1)(x-3)(x-9)(x-27)(x-81)$$

$\Rightarrow$  plug in  $x=0$ :

$$-1 = a(-1)(-3)(-9)(-27)(-81)$$

$$a = \frac{1}{3 \cdot 9 \cdot 27 \cdot 81}$$

Coefficient of  $x$  in  $P(x)$ :

$$\frac{1 \cdot 3 \cdot 9 \cdot 27 + 1 \cdot 3 \cdot 9 \cdot 81 + 1 \cdot 3 \cdot 27 \cdot 81 + 1 \cdot 9 \cdot 27 \cdot 81}{3 \cdot 9 \cdot 27 \cdot 81}$$

$$= 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} = \frac{242}{81}$$

$$\Rightarrow 3b - b = \frac{242}{81} \Rightarrow b = \frac{121}{81}$$



### Practice 3.2

Let  $P$  be a cubic polynomial with  $P(0) = k$ ,  $P(1) = 2k$ , and  $P(-1) = 3k$ .

What is  $P(2) + P(-2)$ ?

A. 0

B.  $k$

C.  $6k$

D.  $7k$

E.  $14k$  ✓

$$\text{Let } P(x) = ax^3 + bx^2 + cx + d$$

$$\Rightarrow \begin{cases} P(0) = d = k \\ P(1) = a + b + c + d = 2k \\ P(-1) = -a + b - c + d = 3k \end{cases}$$

*add* ( )

$$\begin{cases} P(0) = d = k \\ P(1) = a + b + c + d = 2k \\ P(-1) = -a + b - c + d = 3k \end{cases}$$

$$\Rightarrow \begin{cases} d = k \\ b = \frac{3}{2}k \end{cases}$$

$$\begin{aligned} 2b + 2d &= 5k \\ 2b &= 3k \end{aligned}$$

$$P(2) + P(-2)$$

$$= \cancel{8a} + \cancel{4b} + \cancel{2c} + \cancel{d} + \cancel{(-8)a} + \cancel{4b} + \cancel{(-2)c} + \cancel{d}$$

$$= 8b + 2d$$

$$= 8 \cdot \frac{3}{2}k + 2k$$

$$= 14k$$

### Concept 4

# Functional Equations



**Definition** of Functional Equations:

If an equation includes a function, then we call it as a **functional equation**.

For example,  $f(2x + 3) = 2f(x) + 1$ ,  $f(x + 2y) = f(x) + 2y$  are functional equations.

### Key Points:

1. You can treat function  $f(x)$  as a variable
2. Use recursion to substitute function.



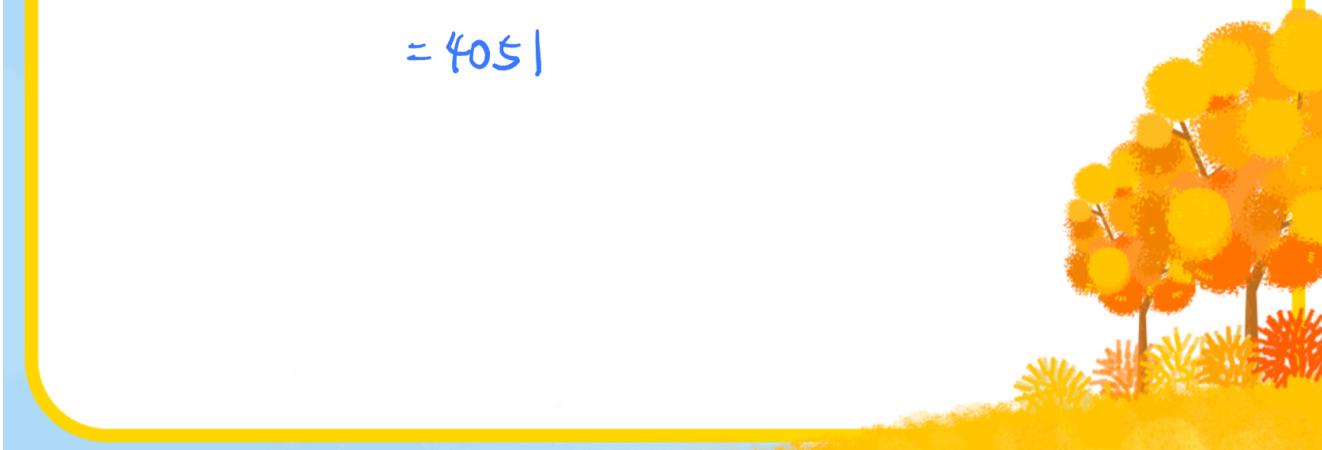
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### Warm UP

Let  $f(x)$  be a function such that  $f(x + y) = 2x + f(y)$  for any two real numbers  $x$  and  $y$ , and  $f(0) = 5$ . The value of  $f(2023)$  is 405.

Let  $x=2023, y=0$

$$\begin{aligned}f(2023+0) &= 2 \cdot 2023 + f(0) \\&= 4046 + 5 \\&= 4051\end{aligned}$$



## Math Exploration 5.1

$$a = \frac{ab}{b} \quad f(32) = f(2^5)$$

Let  $f$  be a function defined on the set of positive rational numbers with the property that  $f(a \cdot b) = f(a) + f(b)$  for all positive rational numbers  $a$  and  $b$ .

Furthermore, suppose that  $f$  also has the property that  $f(p) = p$  for every prime number  $p$ . For which of the following numbers  $x$  is  $f(x) < 0$ ? (2021 AMC Spring 10A, Question #18)

A.  $\frac{17}{32} \Rightarrow$

B.  $\frac{11}{16} = 1 - 8 = 3$

C.  $\frac{7}{9} = 7 - 6 = 1$

D.  $\frac{7}{6} = 7 - 5 = 2$

E.  $\frac{25}{11} \checkmark$

$$f(a \cdot b) = f\left(\frac{a \cdot b}{b}\right) + f(b)$$

$$f\left(\frac{17}{32}\right) = f(17) - f(32)$$

$$f\left(\frac{ab}{b}\right) = f(ab) - f(b)$$

$$= 17 - (f(2) + f(16))$$

$$f\left(\frac{c}{b}\right) = f(c) - f(b)$$

$$= 17 - (5 \times 2)$$

$$\Leftrightarrow c = ab$$

$$= 7$$



## Practice 5.1

$$f(n) = f(n-1) - f(n-2) + n-1 = f(n-2) - f(n-3) + n-1$$

A function  $f$  is defined recursively by  $f(1) = f(2) = 1$  and

$$f(n) = f(n-1) - f(n-2) + n \text{ for all integers } n \geq 3. \text{ What is } f(2018)?$$

(2018 AMC 10B Problem, Question#20)

A. 2016

B. 2017

C. 2018

D. 2019

E. 2020

$$f(n) = [f(n-2) - f(n-3) + n-1] - f(n-2) + n$$

$$f(n) = -f(n-3) + 2n-1 \Rightarrow f(n-3)$$

$$f(n) = -[-f(n-6) + 2(n-3)-1] + 2n-1$$

$$f(n) = f(n-6) + 6$$

$$f(2018) = f(2018 - 2016) + 2016 = f(2) + 2016 = 2017$$

multiple of 6

