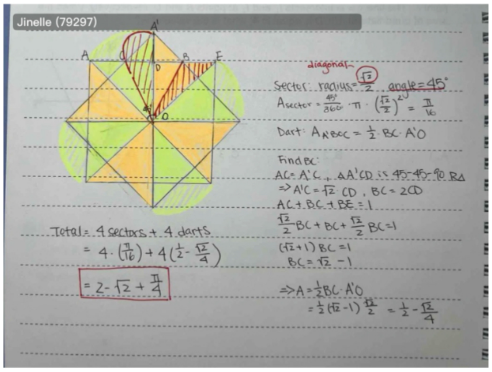
Excellent Notes



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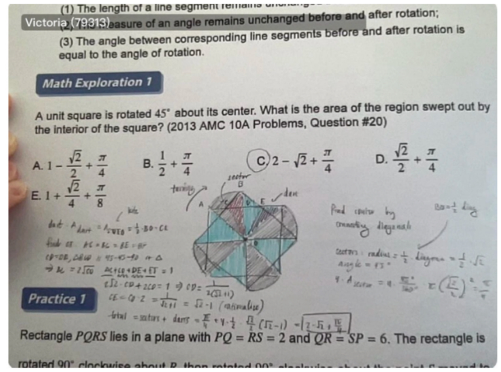

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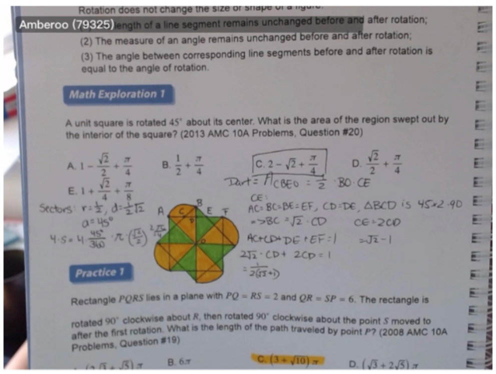

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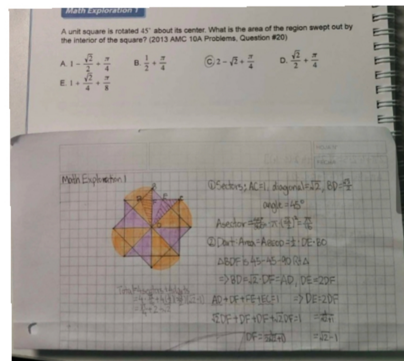
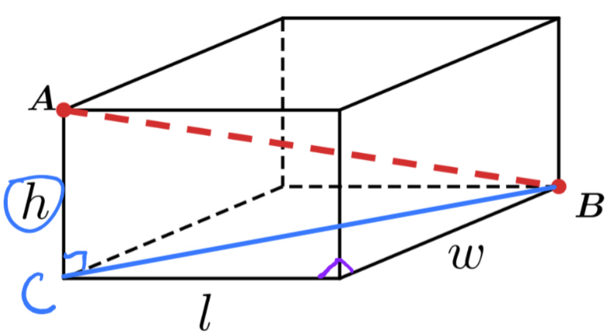

Hanxi

Lesson 9
Polyhedron

Concept 1

Space Pythagorean Theorem

The 3D Pythagorean Theorem is a natural extension of the 2D Pythagorean Theorem into three-dimensional space.
One of the common applications is finding the length of the diagonal of the rectangle:
the diagonal $AB = \sqrt{l^2 + w^2 + h^2}$



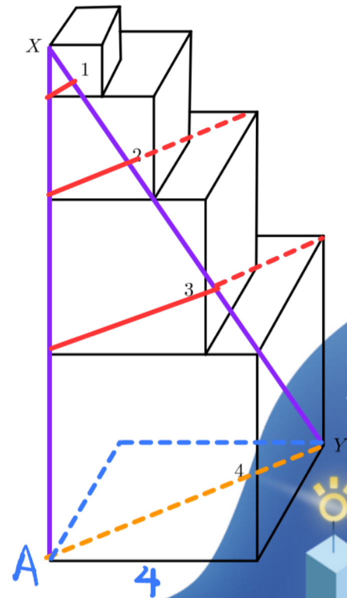
$BC^2 = l^2 + w^2$

$AB^2 = AC^2 + BC^2$
$= \underline{h^2 + l^2 + w^2}$

MC(1)

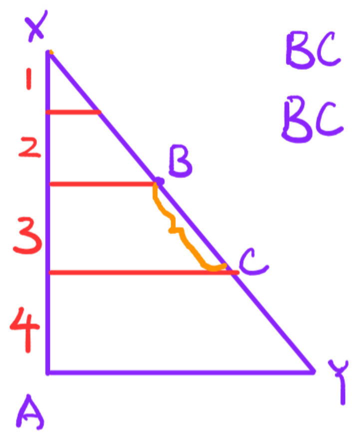
Math Exploration 1.1

Four cubes with edge lengths **1**, **2**, **3**, and **4** are stacked as shown. What is the length of the portion of $\overline{XY}$ contained in the cube with edge length **3**? (2014 AMC 10A Problems, Question #19)



A. $\frac{3\sqrt{33}}{5}$ B. $2\sqrt{3}$ C. $\frac{2\sqrt{33}}{3}$ D. 4 E. $3\sqrt{2}$

$AX = 1+2+3+4 = 10 \Rightarrow XY = \sqrt{10^2+4^2+4^2} = 2\sqrt{33}$



$BC : XY = 3 : 10$

$BC = \frac{6\sqrt{33}}{10}$

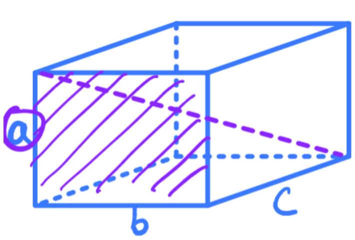
$= \frac{3\sqrt{33}}{5}$

MC(1)

Practice 1.1

A rectangular box $\mathcal{P}$ has distinct edge lengths a, b, and c. The sum of the lengths of all **12** edges of $\mathcal{P}$ is **13**, the areas of all **6** faces of $\mathcal{P}$ is $\frac{11}{2}$, and the volume of $\mathcal{P}$ is $\frac{1}{2}$. What is the length of the longest interior diagonal connecting two vertices of $\mathcal{P}$? (2023 AMC 10B Problems, Question #17)

A. 2 B. $\frac{3}{8}$ C. $\frac{9}{8}$ D. $\frac{9}{4}$ E. $\frac{3}{2}$



Find: $\sqrt{a^2+b^2+c^2}$

given:

$$\begin{cases} 4a+4b+4c=13 \\ 2ab+2ac+2bc=\frac{11}{2} \\ abc=\frac{1}{2} \end{cases} \Rightarrow \begin{cases} a+b+c=\frac{13}{4} \\ ab+ac+bc=\frac{11}{4} \\ abc=\frac{1}{2} \end{cases}$$

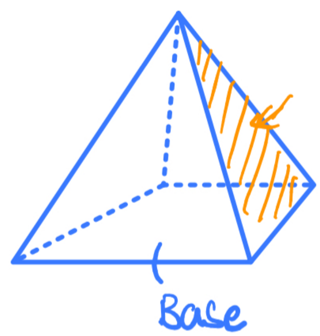
$\Rightarrow \underline{(a+b+c)^2} = \underline{a^2+b^2+c^2} + \underline{2ab+2ac+2bc}$

$a^2+b^2+c^2 = \left(\frac{13}{4}\right)^2 - \frac{11}{2} = \frac{81}{16}$

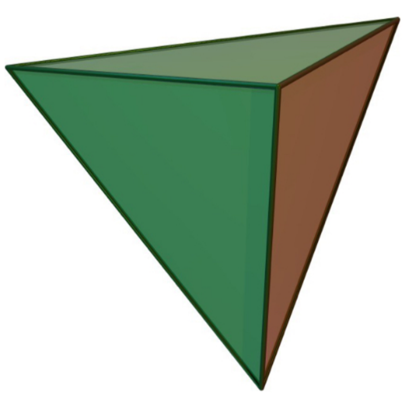
$\Rightarrow \sqrt{\frac{81}{16}} = \frac{9}{4}$

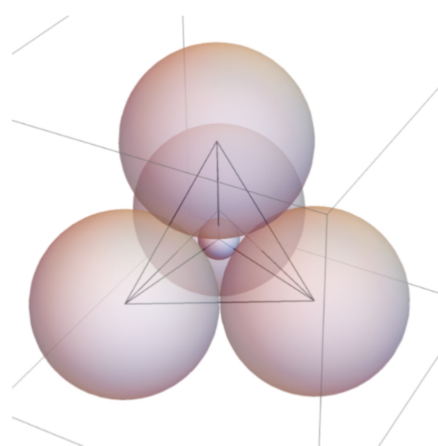
Concept 2

Tetrahedron

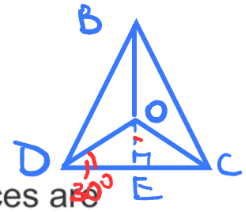


In geometry, a **tetrahedron**, also known as a triangular pyramid, is a polyhedron composed of four triangular faces, six straight edges, and four vertices. The tetrahedron is the simplest of all the ordinary convex polyhedra.

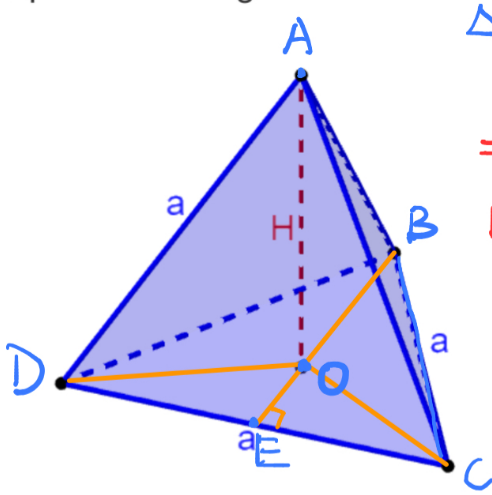




Tetrahedrons can be classified into three types.
1. regular tetrahedrons: a tetrahedron in which all four faces are equilateral triangles.



$$\triangle AOB \cong \triangle AOC \Leftarrow \begin{cases} AB = AC \\ AO = AO \\ m\angle AOB = m\angle AOC = 90^\circ \end{cases}$$



$$\Rightarrow BO = CO = DO$$

$$BO : EO = 2 : 1 \quad ☆$$

$$BE = \frac{\sqrt{3}}{2}a \Rightarrow BO = \frac{2}{3}BE = \frac{\sqrt{3}}{3}a$$

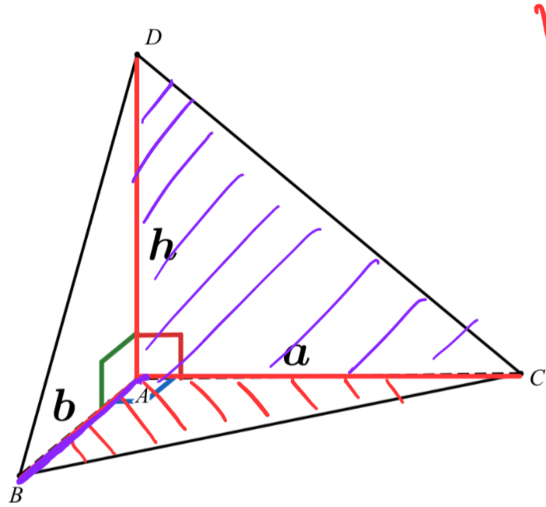
$\triangle AOB$ is a Rt $\triangle$

$$\Rightarrow AO = \sqrt{AB^2 - BO^2} = \sqrt{a^2 - \frac{a^2}{3}} = \sqrt{\frac{2}{3}}a$$

$$V = \frac{a^3\sqrt{2}}{12}$$

$$V = \frac{1}{3}B\cdot h = \frac{1}{3}\left(\frac{\sqrt{3}}{4}a^2\right)\cdot\sqrt{\frac{2}{3}}a = \frac{\sqrt{2}}{12}a^3$$

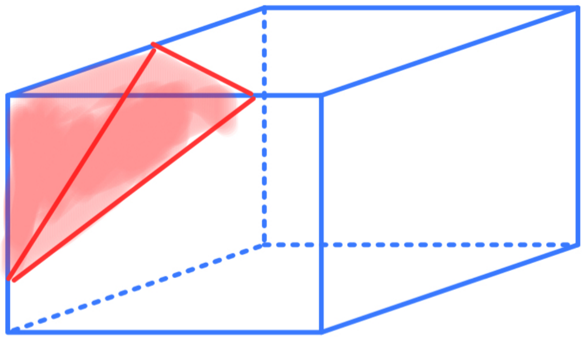
2. 3 perpendicular tetrahedrons, like a cornor: three edges are mutually perpendicular



$$V = \frac{1}{3} \times \frac{1}{2}ab \times h = \frac{1}{6}abh$$

$$V = \frac{1}{3}Bh = \frac{1}{3}\left(\frac{1}{2}ab\right)\cdot h = \frac{1}{6}abh$$

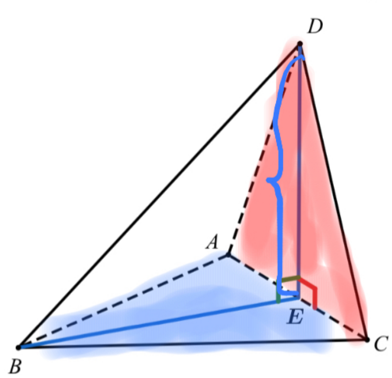
2 (Extension). A corner of Rectangular Prism



3. the other irregular tetrahedrons.
In order to determine the volume, we have to find the height and the base of the tetrahedron first, where the height is perpendicular to the base.
Theorem for determining when a line is perpendicular to a plane:
If a line is perpendicular to two intersecting lines that lie in a plane, then the line is perpendicular to the plane.

Example: in the tetrahedron $\boldsymbol{ABCD}$, $\overline{\boldsymbol{DE}} \perp \overline{\boldsymbol{AC}}$, and $\overline{\boldsymbol{DE}} \perp \overline{\boldsymbol{BE}}$.
Then, $\overline{\boldsymbol{DE}}$ must be perpendicular to the base $\boldsymbol{ABC}$.
Then, $\boldsymbol{DE}$ is the height of the tetrahedron $\boldsymbol{ABCD}$.

$$\boldsymbol{V_{ABCD}} = \frac{\mathbf{1}}{\mathbf{3}} \times \boldsymbol{DE} \times \boldsymbol{A_{ABC}}.$$

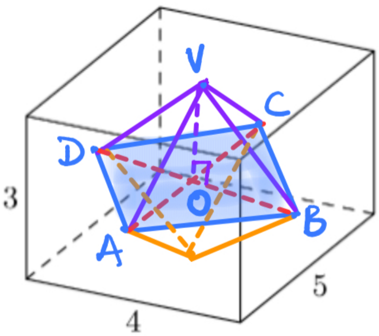


$$V = \frac{1}{3} \times \text{base} \times \text{height}$$

MC(1)

Math Exploration 2.1

The centers of the faces of the right rectangular prism shown below are joined to create an octahedron. What is the volume of this octahedron? (2015 AMC 10B Problems, Question #17)



A. $\frac{75}{12}$ B. $\mathbf{10}$ C. $\mathbf{12}$ D. $\mathbf{10\sqrt{2}}$ E. $\mathbf{15}$

$A_{ABCD} = \frac{1}{2} AC \cdot BD = \frac{1}{2} \cdot 4 \cdot 5 = 10$

$VO = \frac{1}{2} \cdot 3 = \frac{3}{2}$

$\Rightarrow V_{V\text{-}ABCD} = \frac{1}{3} \cdot B \cdot h = \frac{1}{3} \cdot 10 \cdot \frac{3}{2} = 5$

Ans $= 2 \cdot V_{V\text{-}ABCD} = 2 \cdot 5 = \boxed{10}$

MC(1)

Practice 2.1

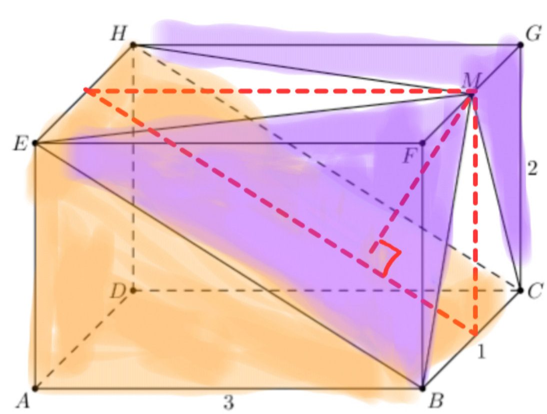
In the rectangular parallelepiped shown, $AB = 3$, $BC = 1$, and $CG = 2$. Point M is the midpoint of $\overline{FG}$. What is the volume of the rectangular pyramid with base $BCHE$ and apex M? (2018 AMC 10B Problems, Question #10)

A. $\mathbf{1}$ B. $\frac{4}{3}$ C. $\frac{3}{2}$ D. $\frac{5}{3}$ E. $\mathbf{2}$

$V_{corner} = \frac{1}{6} EF \cdot BF \cdot FM$

$= \frac{1}{6} \cdot 3 \cdot 2 \cdot \frac{1}{2}$

$= \frac{1}{2}$



Ans $= \frac{1}{2} \cdot (3 \cdot 1 \cdot 2) - 2V_{corner}$

$= 3 - 1$

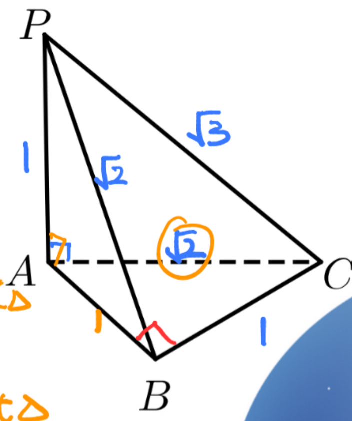
$= 2$

MC(1)

Math Exploration 3.1

Tetrahedron $PABC$ has $\overline{PA} \perp \overline{AB}$, and $PA = BC = 1$, $PB = AC = \sqrt{2}$, $PC = \sqrt{3}$. What is the volume of the tetrahedron?

A. $\frac{1}{2}$ B. $\frac{1}{3}$ C. $\frac{1}{4}$ ✓D. $\frac{1}{6}$

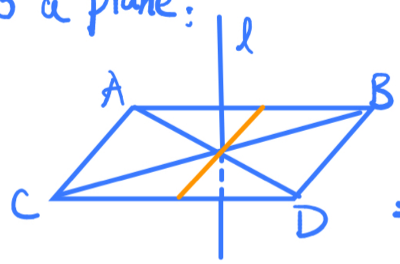


In $\triangle PAC$:

$\begin{cases} PA = 1 \\ AC = \sqrt{2} \\ PC = \sqrt{3} \end{cases} \Rightarrow \triangle PAC$ is a Rt$\triangle$, $\overline{PA} \perp \overline{AC}$

$\Rightarrow \overline{PA} \perp$ Plane ABC.

Prove a line perpendicular to a plane:



$\begin{cases} l \perp \overline{AD} \\ l \perp \overline{BC} \end{cases}$

$\Rightarrow l \perp ABCD$

$\triangle PAB$ is a Rt$\triangle$

$\Rightarrow AB = 1$

$\Rightarrow \triangle ABC$ is Rt$\triangle$

$A_{\triangle ABC} = \frac{1}{2} \cdot AB \cdot BC = \frac{1}{2}$

$V = \frac{1}{3} \cdot B \cdot h$

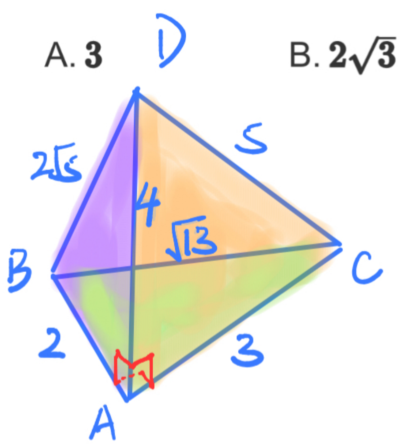
$= \frac{1}{3} \cdot \frac{1}{2} \cdot 1 = \frac{1}{6}$

MC(1)

Math Exploration 3.2

What is the volume of tetrahedron $ABCD$ with edge lengths $AB = 2$, $AC = 3$, $AD = 4$, $BC = \sqrt{13}$, $BD = 2\sqrt{5}$, and $CD = 5$? (2021 Spring AMC 10A Problems, Question #13)

A. 3 B. $2\sqrt{3}$ ✓C. 4 D. $3\sqrt{3}$ E. 6



Show $\triangle ABD$, $\triangle ADC$, and $\triangle ABC$ are right triangle by length

$\Rightarrow \overline{AB}$, $\overline{AD}$, and $\overline{AC}$ are perpendicular

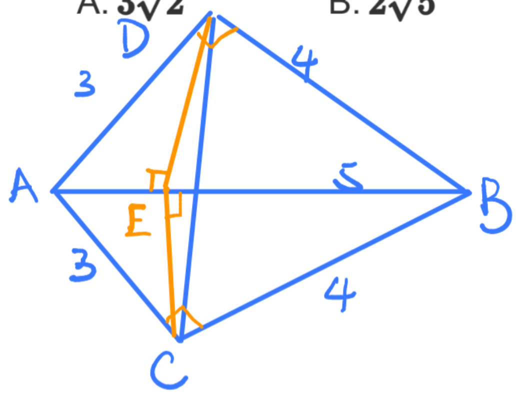
$V = \frac{1}{6} \cdot 2 \cdot 3 \cdot 4 = 4$

MC(1)

Practice 3.1

Tetrahedron $ABCD$ has $AB=5$, $AC=3$, $BC=4$, $BD=4$, $AD=3$, and $CD=\frac{12}{5}\sqrt{2}$. What is the volume of the tetrahedron? (2015 AMC 10A Problems, Question #21)

A. $3\sqrt{2}$ B. $2\sqrt{5}$ C. $\frac{24}{5}$ D. $3\sqrt{3}$ E. $\frac{24}{5}\sqrt{2}$



$\triangle ACB$ and $\triangle ADB$ are Rt$\triangle$.

Draw $\overline{CE}\perp\overline{AB}$ and $\overline{DE}\perp AB$

$\Rightarrow CE = DE = \frac{3\times4}{5} = \frac{12}{5}$

$\Rightarrow \triangle CDE$ is a Rt$\triangle$ ($DE^2+EC^2=DC^2$)

$\overline{DE}\perp\overline{AB}$ and $\overline{DE}\perp\overline{CE}$

$\Rightarrow \triangle ABC$ is base

DE is height.

$V=\frac{1}{3}B\cdot h$

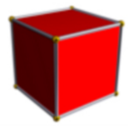
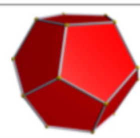
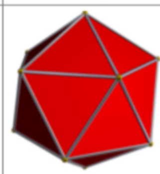
$=\frac{1}{3}\left(\frac{1}{2}\cdot3\cdot4\right)\cdot\frac{12}{5}$

$=\frac{24}{5}$

Concept 3

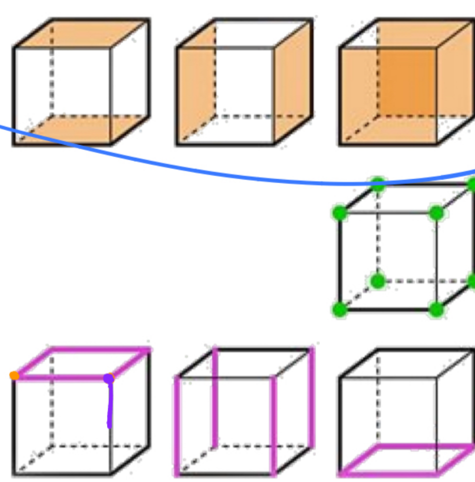
Euler's Polyhedral Formula

Euler's Polyhedral Formula: Let P be any convex polyhedron, and let V, E and F denote the number of vertices, edges, and faces, respectively. Then $V - E + F = 2$.

Polyhedra	Graph	Vertices (V)	Edges (E)	Faces (F)	Euler Characteristic
Tetrahedron		4	6	4	2
Hexahedron or Cube		8	12	6	2
Octahedron		6	12	8	2
Dodecahedron		20	30	12	2
Icosahedron		12	30	20	2

Additional Facts

- An edge is shared by 2 faces.
- An edge connects 2 vertices.



6 faces

8 vertices

12 edges

MC(1)

Math Exploration 4.1

A rhombic dodecahedron is a solid with **12** congruent rhombus faces. At every vertex, **3** or **4** edges meet, depending on the vertex. How many vertices have exactly **3** edges meet? (2023 AMC 10A Problems, Question #18)

A. **5** B. **6** C. **7** D. **8** E. **9**

Including duplicates, there are $12 \times 4 = 48$ edges.

$\Rightarrow 48 \div 2 = 24$ edges in solid (each edge connects 2 faces).

$V - E + F = 2 \Rightarrow V - 24 + 12 = 2 \Rightarrow V = 14$

Let x be the number of vertices adjacent to 3 edges

$\Rightarrow (14 - x)$ vertices are adjacent to 4 edges.

Therefore: $3x + 4(14 - x) = 24 \cdot 2$

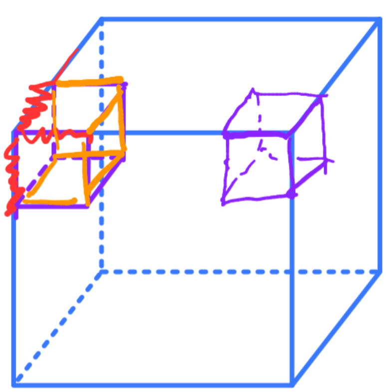
$x = 8$

MC(1)

Practice 4.1

A solid cube of side length **1** is removed from each corner of a solid cube of side length **3**. How many edges does the remaining solid have? (2013 AMC 10A Problems, Question #14)

A. **36** B. **60** C. **72** D. **84** E. **108**



Each corner adds $12 - 3 = 9$ edges.

Originally, there are 12 edges.

$Ans = 12 + 8 \times 9 = 84$