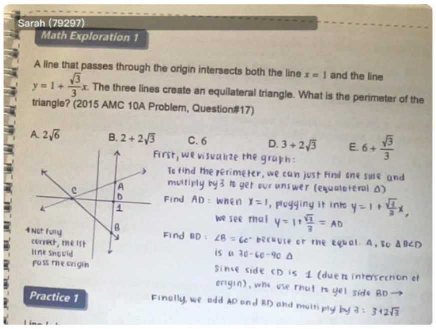
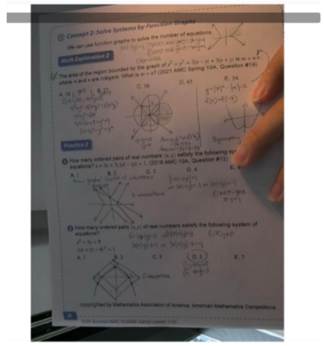
Excellent Notes



Sarah



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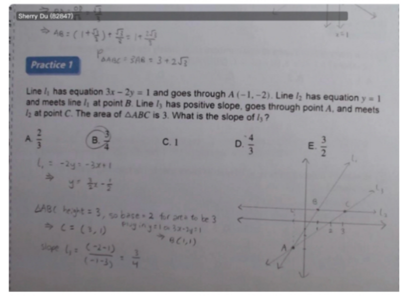
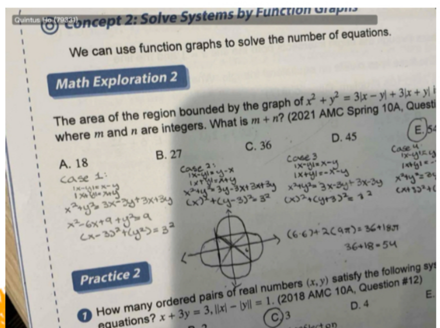


Sherry



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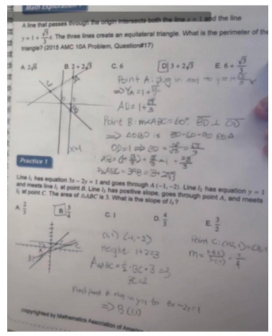
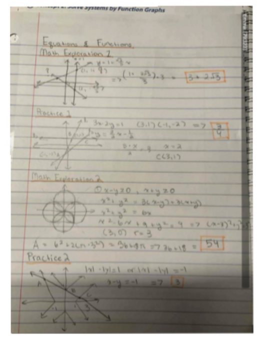


Michelle



Emma

Lesson 3
Polynomial Functions

Concept 1

Quadratic Functions

Concept 1: Quadratic Functions

Consider the parabola $y = \boxed{ax^2} + bx + c$ or $y = a(x-h)^2 + k$ with $a \neq 0$.

- If $a > 0$, the parabola opens upward; if $a < 0$, the parabola opens downward.
- The axis of symmetry is $x = -\dfrac{b}{2a}$ or $x = h$.
- The vertex is at $\left(-\dfrac{b}{2a}, \dfrac{4ac-b^2}{4a}\right)$ or (h, k).

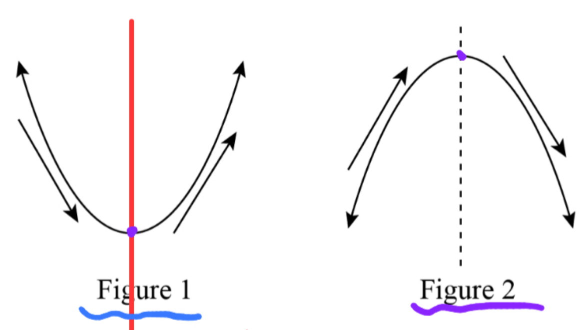


Figure 1 Figure 2

$x = -\frac{b}{2a}$

MC(1)

Math Exploration 1.1

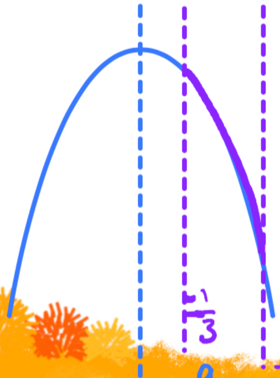
The given function is:

$$f(x) = -9x^2 - 6ax - a^2 + 2a, \quad \text{for} \quad -\frac{1}{3} \leqslant x \leqslant \frac{1}{3}$$

It is known that $f(x)$ has a maximum value of -3.

Find the value of the real number a. Axis of Symmetry: $x=-\frac{-6a}{2\cdot(-9)}=-\frac{a}{3}$

A. $2-\sqrt{6}$ or $-\sqrt{2}$ B. $2+\sqrt{6}$ C. $-\sqrt{2}$ D. $2+\sqrt{6}$ or $-\sqrt{2}$



Case 1: $-\frac{1}{3} \le -\frac{a}{3} \le \frac{1}{3}$

$f(-\frac{a}{3}) = max$

$-9\cdot(-\frac{a}{3})^2-6a(-\frac{a}{3})-a^2+2a=-3$

$-a^2+2a^2-a^2+2a+3=0$

$2a+3=0$

$a=-\frac{3}{2}$

$\Rightarrow$ not solution

$a\in[-1,1]$

Case 2: $-\frac{a}{3} < -\frac{1}{3}$

$f(-\frac{1}{3}) = max$

$-9\cdot(-\frac{1}{3})^2-6a(-\frac{1}{3})-a^2+2a=-3$

$a^2-4a-2=0$

$a=\frac{4\pm\sqrt{24}}{2}=2+\sqrt{6}$

Case 3: $-\frac{a}{3} > \frac{1}{3}$

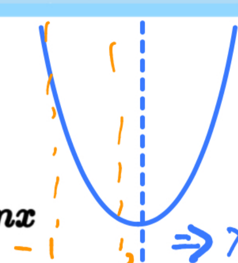
$f(\frac{1}{3})=max \Rightarrow -1-2a-a^2+2a=-3$

$a=-\sqrt{2}$

MC(1)

Practice 1.1

Given the quadratic function:



$$y = x^2 - 2mx$$

$\Rightarrow x=-\frac{-2m}{2}=m$

where m is a constant, the minimum value of y on the interval $-1 \leqslant x \leqslant 2$ is given as -2. Find the value of m.

A. $\frac{3}{2}$ B. $\sqrt{2}$ C. $\frac{3}{2}$ or $\sqrt{2}$ D. $-\frac{3}{2}$ or $\sqrt{2}$

Case 1: $-1\le m\le 2$

$f(m)=-2$

$m^2-2m^2=-2$

$m=\pm\sqrt{2}$

$\Rightarrow m=\sqrt{2}\in[-1,2]$

Case 2: $m<-1$

$f(-1)=-2$

$1+2m=-2$

$m=-\frac{3}{2}$

$\Rightarrow -\frac{3}{2}\in(-\infty,-1)$

Case 3: $m>2$

$f(2)=-2$

$4-4m=-2$

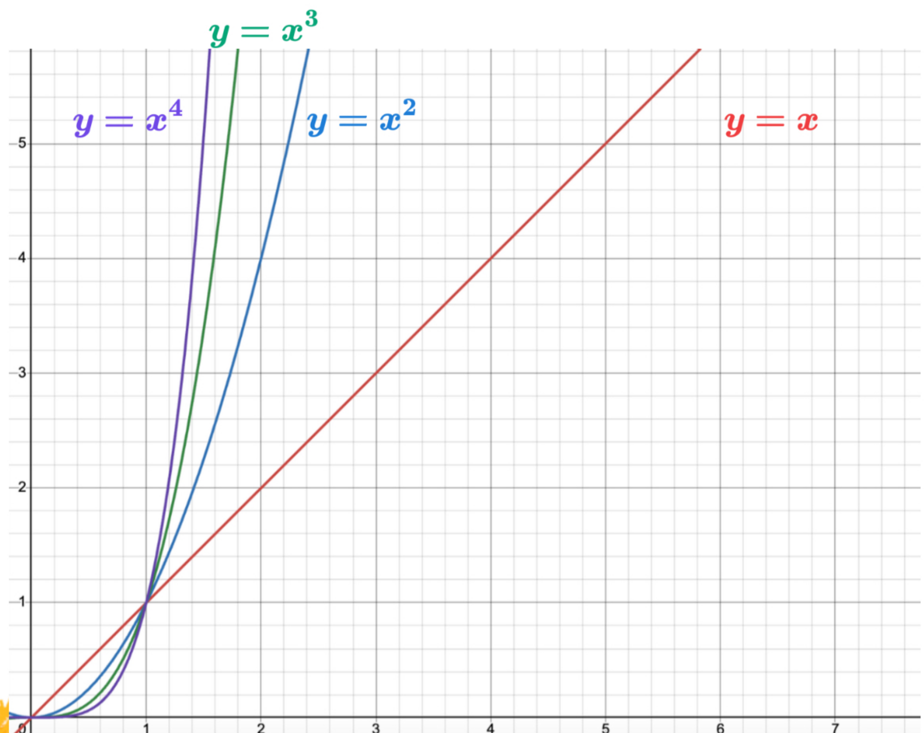
$m=\frac{3}{2}<2$

$\Rightarrow$ no solution

Concept 2

Graph of Polynomial Functions

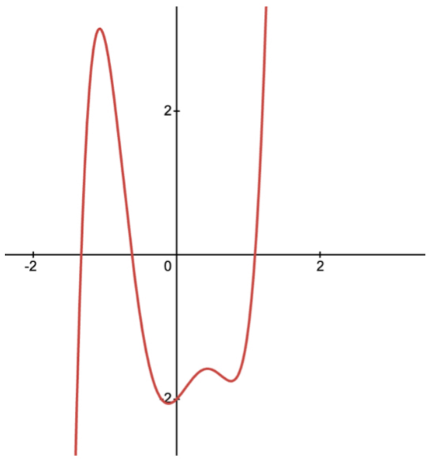
How to graph higher degree polynomials?

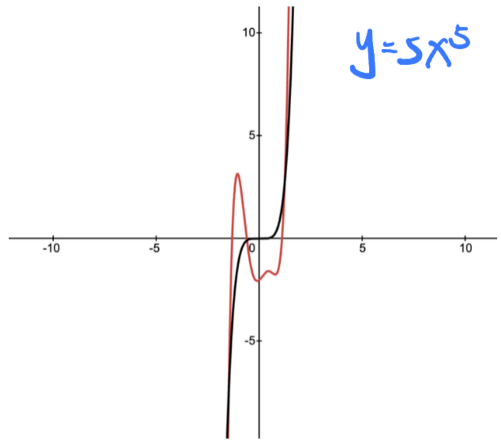


How to graph higher degree polynomials?

How to graph this function?

$$y = \boxed{5x^5} - 8x^3 + 3x^2 + x - 2$$





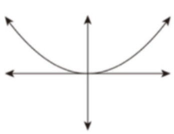
$y = 5x^5$

Concept 2: Graph of Polynomial Functions

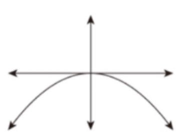
The Leading-Term Test

If $a_n x^n$ is the leading term of a polynomial. Then the behavior of the graph as $x \to \infty$ or $x \to -\infty$ can be known by one the four following behaviors:

1. If $a_n > 0$ and n even:



2. If $a_n < 0$ and n even:

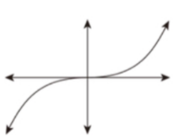


$y = -2x^5 + \cdots$

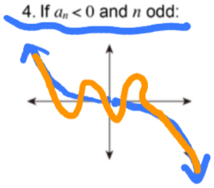
$x \to +\infty$, $y \to -\infty$

$x \to -\infty$, $y \to \infty$

3. If $a_n > 0$ and n odd:

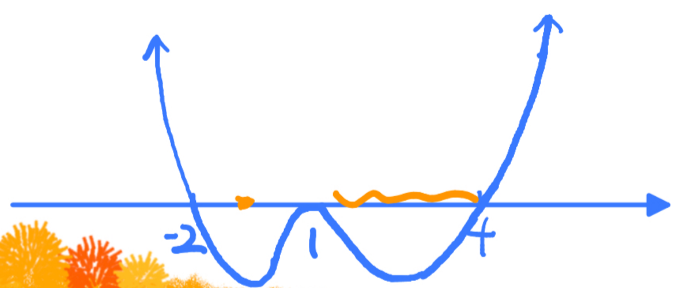


4. If $a_n < 0$ and n odd:



For even powers, such as 2, 4, 6, and 8, the graph will still touch and bounce off of the x-axis, but for each increasing even power the graph will appear flatter as it approaches and leaves the x-axis.

For odd powers, such as 1, 5, 7, and 9, the graph will still cross through the x-axis, but for each increasing odd power, the graph will appear flatter as it approaches and leaves the x-axis. $y=(x-4)(x-1)^2(x+2)^3$

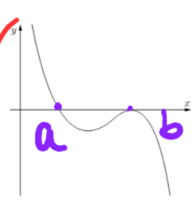


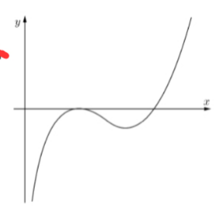
MC(1)

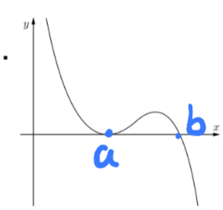
Math Exploration 2.1

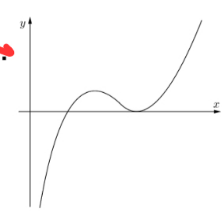
Among the graphs shown below there is the graph of the function

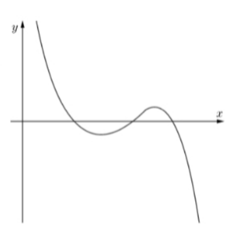
$f(x) = (a - x)(b - x)^2$ with $a < b$. Which is it?

A. 

~~B.~~ 

C. 

~~D.~~ 

E. 

① End Behavior:
$(-x)(-x)^2 = -x^3$

② x-intercepts:
a, b
pass bounce

MC(1)

Math Exploration 2.2

When the roots of the polynomial

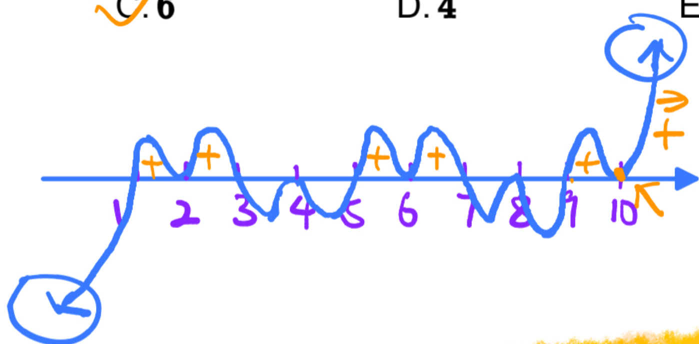
$$P(x) = (x-1)^1(x-2)^2(x-3)^3\cdots(x-10)^{\boxed{10}}$$

are removed from the real number line, what remains is the union of **11** disjoint open intervals. On how many of these intervals is $P(x)$ positive? (2023 AMC 10B Problems, Question #12)

A. **3** B. **7** C. **6** D. **4** E. **5**

① End Behavior: $x^1 \cdot x^2 \cdots x^{10} = x^{55}$

② Roots:



MC(1)

Practice 2.1

Define $P(x) = (x-1^2)(x-2^2)\cdots(x-100^2)$. How many integers n are there such that $P(n) \le 0$? (2020 AMC 10A, Question #17)

A. **4900** B. **4950** C. **5000** D. **5050** E. **5100**

① End Behavior: x^{100}

② Zeros: $1^2, 2^2, \dots, 100^2$

Intervals:

$[99^2, 100^2], [97^2, 98^2], \dots, [1^2, 2^2]$

$N = (100^2 - 99^2 + 1) + (98^2 - 97^2 + 1) + \cdots$

$= 50 + (100^2 - 99^2) + (98^2 - 97^2) + \cdots$

$= 50 + (100-99)(100+99) + \cdots$

$= 50 + 100 + 99 + 98 + 97 + \cdots + 2 + 1$

$= 50 + \frac{(1+100)\times 100}{2} = 5100$

Concept 3

Remainder Theorem and Factor Theorem

Review: Polynomial Division

$$
\begin{array}{r|l}
 & 4x + 3 \\
\hline
x^2 - 1 & 4x^3 + 3x^2 + x - 1 \\
 & 4x^3 \qquad\quad - 4x \\
\hline
 & \qquad 3x^2 + 5x - 1 \\
 & \qquad 3x^2 \qquad - 3 \\
\hline
 & \qquad\qquad 5x + 2
\end{array}
$$

$$(4x^3+3x^2+x-1) \div (x^2-1)$$
$$= (4x+3) \text{ R } (5x+2)$$

Concept 3: Remainder Theorem and Factor Theorem

The polynomials with respect to x can be simply represented by $f(x)$ or $g(x)$.

The quotient of the division of $f(x)$ divided by $g(x)$ is $q(x)$, and the remainder is $r(x)$. $f(x) \div g(x) = q(x)\ R\ r(x)$, i.e., $f(x) = g(x)q(x) + r(x)$.

$21 \div 5 = 4\ R\ 1$ $\quad$ $21 = 5 \times 4 + 1$

Especially, when the divisor $g(x)$ is a linear term $(x-a)$, the remainder $r(x)$ must be a constant. In this case, the remainder can also be written as r, therefore $f(x) = (x-a) \cdot q(x) + r$

Degree of remainder must be smaller than degree of divisor.

Concept 3: Remainder Theorem and Factor Theorem

Polynomial **Remainder Theorem** states that the remainder of the division of a polynomial $f(x)$ by a linear polynomial $x-a$ is equal to $f(a)$.

$f(x) = (x-a)q(x) + r$

$\Rightarrow f(a) = 0 \cdot q(x) + r = r$

In particular, $x-a$ is a factor of $f(x)$ if and only if $f(a) = 0$, and this is the **Factor Theorem**.

MC(1)

Math Exploration 3.1

\# of roots

Let $P(x)$ be the unique polynomial of minimal degree with the following properties:

- $P(x)$ has leading coefficient 1,
- 1 is a root of $P(x)-1$,
- 2 is a root of $P(x-2)$,
- 3 is a root of $P(3x)$, and
- 4 is a root of $4P(x)$.

The roots of $P(x)$ are integers, with one exception. The root that is not an integer can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

A. 41 B. 43 C. 45 D. 47 ✓ E. 49

Roots:

$$\begin{cases} P(1)-1=0 \\ P(2-2)=0 \\ P(3\cdot 3)=0 \\ 4P(4)=0 \end{cases} \Rightarrow \begin{cases} P(1)=1 \\ P(0)=0 \\ P(9)=0 \\ P(4)=0 \end{cases}$$

$\Rightarrow P(x)$ has roots: $0, 4, 9$

Suppose r = non-integer root:

$P(x)=(x-0)(x-4)(x-9)(x-r)$

$\Rightarrow$ plug in $x=1$:

$1\cdot(-3)(-8)(1-r)=1$

$1-r=\frac{1}{24}$

$r=\frac{23}{24}$ Ans $=23+24=47$

PhotoPost

Practice 3.1

$f(x)=bx+c$ $f(3x)=3bx+c$ $f(3x)-f(x)=2bx$

Let $f(x)$ be a polynomial of degree 5. If $f(1)=0$, $f(3)=1$, $f(9)=2$, $f(27)=3$, $f(81)=4$ and $f(243)=5$, find the coefficient of x in $f(x)$.

$\Rightarrow$ coefficient is b.

Pattern: $f(3x)-f(x)=1$ for all $x=1,3,9,27,81$

Make $P(x)=f(3x)-f(x)-1=0$ for all $x=1,3,9,27,81$

$\Rightarrow P(1)=P(3)=P(9)=P(27)=P(81)=0$, $P(0)=f(0)-f(0)-1=-1$

$P(x)=a(x-1)(x-3)(x-9)(x-27)(x-81)$

$\Rightarrow$ plug in $x=0$:

$-1=a(-1)(-3)(-9)(-27)(-81)$

$a=\frac{1}{3\cdot 9\cdot 27\cdot 81}$

Coefficient of x in $P(x)$:

$\frac{1\cdot 3\cdot 9\cdot 27+1\cdot 3\cdot 9\cdot 81+1\cdot 3\cdot 27\cdot 81+1\cdot 9\cdot 27\cdot 81+3\cdot 9\cdot 27\cdot 81}{3\cdot 9\cdot 27\cdot 81}$

$=1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}=\frac{242}{81}$

$\Rightarrow 3b-b=\frac{242}{81} \Rightarrow b=\frac{121}{81}$

MC(1)

Practice 3.2

Let P be a cubic polynomial with $P(0) = k$, $P(1) = 2k$, and $P(-1) = 3k$.

What is $P(2) + P(-2)$?

A. 0 B. k C. $6k$ D. $7k$ E. $14k$

Let $P(x) = ax^3 + bx^2 + cx + d$

$$\Rightarrow \begin{cases} P(0) = d = k \\ P(1) = a+b+c+d = 2k \\ P(-1) = -a+b-c+d = 3k \end{cases}$$

add

$$\Rightarrow \begin{cases} d = k \\ b = \frac{3}{2}k \end{cases}$$

$2b + 2d = 5k$

$2b = 3k$

$P(2) + P(-2)$

$= \cancel{8a} + 4b + \cancel{2c} + d + \cancel{(-8)a} + 4b + \cancel{(-2)c} + d$

$= 8b + 2d$

$= 8 \cdot \frac{3}{2}k + 2k$

$= 14k$

Concept 4

Functional Equations

Definition of Functional Equations:
If an equation includes a function, then we call it as a **functional equation**.

For example, $f(2x+3) = 2f(x)+1$, $f(x+2y) = f(x)+2y$ are functional equations.

Key Points:

1. You can treat function $f(x)$ as a variable
2. Use recursion to substitute function.

Blank filling

Warm UP

Let $f(x)$ be a function such that $f(x+y) = 2x + f(y)$ for any two real numbers x and y, and $f(0) = 5$. The value of $f(2023)$ is 4051 .

Let $x=2023$, $y=0$

$$f(2023+0) = 2\cdot 2023 + f(0)$$
$$= 4046+5$$
$$= 4051$$

MC(1)

Math Exploration 5.1

$a = \frac{ab}{b}$ $f(32) = f(2^5)$

Let f be a function defined on the set of positive rational numbers with the property that $f(a \cdot b) = f(a) + f(b)$ for all positive rational numbers a and b. Furthermore, suppose that f also has the property that $f(p) = p$ for every prime number p. For which of the following numbers x is $f(x) < 0$? (2021 AMC Spring 10A, Question #18)

A. $\frac{17}{32}$ $=7$ B. $\frac{11}{16}$ $=11-8=3$ C. $\frac{7}{9}$ $=7-6=1$ D. $\frac{7}{6}$ $=7-5=2$ E. $\frac{25}{11}$ $10-11=-1$

$f(a \cdot b) = f(\frac{a \cdot b}{b}) + f(b)$

$f(\frac{ab}{b}) = f(ab) - f(b)$

$f(\frac{c}{b}) = f(c) - f(b)$ $\Leftarrow c = ab$

$f(\frac{17}{32}) = f(17) - f(32)$

$= 17 - (f(2) + f(16))$

$= 17 - (5 \times 2)$

$= 7$

MC(1)

Practice 5.1

$f(n-1) = f(n-1-1) - f(n-1-2) + n-1 = f(n-2) - f(n-3) + n-1$

A function f is defined recursively by $f(1) = f(2) = 1$ and $f(n) = f(n-1) - f(n-2) + n$ for all integers $n \geqslant 3$. What is $f(2018)$? (2018 AMC 10B Problem, Question#20)

A. **2016** B. **2017** C. **2018** D. **2019** E. **2020**

$f(n) = [\cancel{f(n-2)} - f(n-3) + n-1] - \cancel{f(n-2)} + n$

$f(n) = -f(n-3) + 2n-1 \Rightarrow f(n-3)$

$f(n) = -[-f(n-6) + 2(n-3) - 1] + 2n-1$

$f(n) = f(n-6) + 6$

$f(2018) = f(2018 - \underbrace{2016}_{\text{multiple of 6}}) + \underbrace{2016}_{} = f(2) + 2016 = 2017$