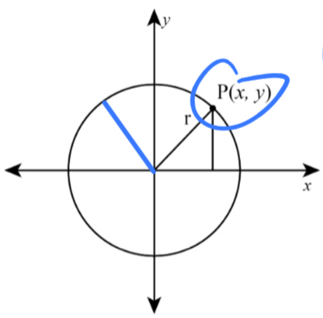
Lesson 7 Lines and Circles

Concept 1

The Equation of Circles

A **circle** is the set of all points in a plane that are equidistant from a given point in that plane. The given point is called the **center** of the circle and the distance bewteen the center and any point on the circle is the **radius**.

The standard form of the equation of a circle with center at $(0, 0)$ and radius r is:
$x^2 + y^2 = r^2$.
The standard form of the equation of a circle with center at (h, k) and radius r is:
$(x - h)^2 + (y - k)^2 = \boxed{r^2}$
The general form of the equation of a circle is given as
$x^2 + y^2 + Dx + Ey + F = 0$.



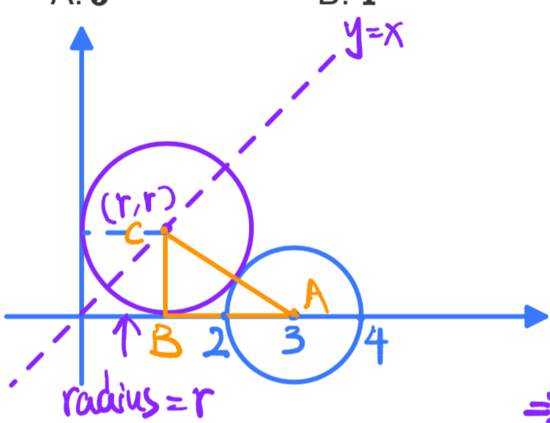
$(x-0)^2+(y-0)^2=r^2$

MC(1)

Math Exploration 1.1

A circle with center C is tangent to the positive x and y-axes and externally tangent to the circle centered at $(3, 0)$ with radius 1. What is the sum of all possible radii of the circle with center C?

A. 3 B. 4 C. 6 D. 8 E. 9



$$\begin{cases} AC = r+1 \\ BC = r \\ AB = 3-r \end{cases}$$

$\Rightarrow (r+1)^2 = r^2 + (3-r)^2$

$r^2+2r+1 = r^2+r^2-6r+9$

$r^2-8r+8=0$

$\Rightarrow r_1+r_2 = -\frac{(-8)}{1} = 8$

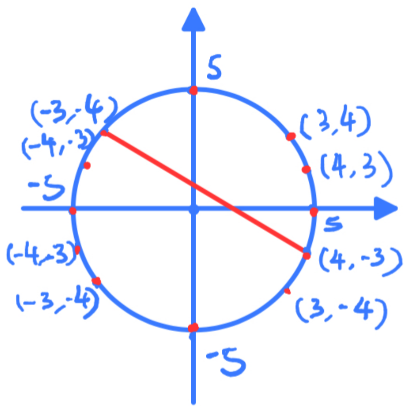
sum=8

MC(1)

Practice 1.1

Distinct points P, Q, R, S lie on the circle $x^2 + y^2 = \underline{25} = r^2$ and have integer coordinates. The distances PQ and RS are irrational numbers. What is the greatest possible value of the ratio $\frac{PQ}{RS}$? (2017 AMC 10A Problems, Question #17)

A. 3 B. 5 C. $3\sqrt{5}$ D. 7 ✓ E. $5\sqrt{2}$



Make PQ longest and RS shortest.

$\Rightarrow PQ = \sqrt{(4-(-3))^2 + (3-(-4))^2}$

$= \sqrt{7^2 + 7^2} = 7\sqrt{2}$

$\Rightarrow RS = \sqrt{(4-3)^2 + (3-4)^2}$

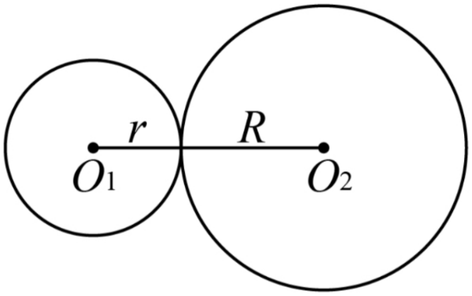
$= \sqrt{2}$

$\frac{PQ}{RS} = \frac{7\sqrt{2}}{\sqrt{2}} = 7$

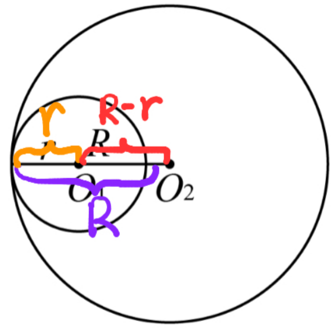
Concept 2

Advanced Tangent Problems

Tangent Circles



$O_1O_2 = r + R$



$O_1O_2 = R - r$

MC(1)

Math Exploration 2.1

Circle C_1 and C_2 each have radius 1, and the distance between their centers is $\frac{1}{2}$. Circle C_3 is the largest circle internally tangent to both C_1 and C_2. Circle C_4 is internally tangent to both C_1 and C_2 and externally tangent to C_3. What is the radius of C_4? (2023 AMC 10A Problems, Quetsion #22)

A. $\frac{1}{14}$ B. $\frac{1}{12}$ C. $\frac{1}{10}$ D. $\frac{3}{28}$ ✓ E. $\frac{1}{9}$

Let radius of C_4 be r

$AD = BD = 1 - r$

$AC = \frac{1}{2}AB = \frac{1}{4}$

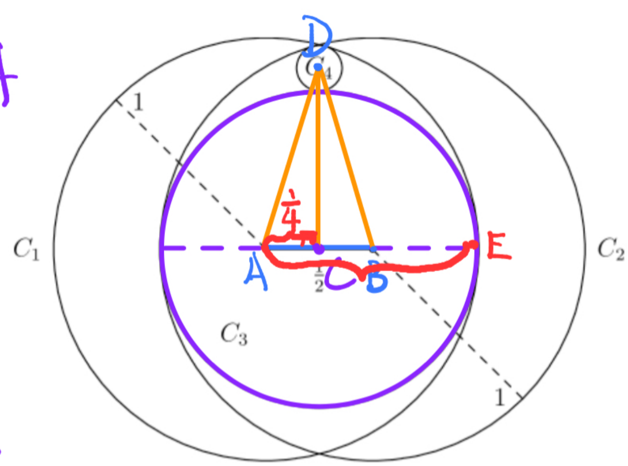
Also, $CE = AE - AC = 1 - \frac{1}{4} = \frac{3}{4}$ ← radius of C_3

$\Rightarrow DC = \frac{3}{4} + r$

In Rt$\triangle ACD$: $AD^2 = AC^2 + DC^2$

$(1-r)^2 = (\frac{1}{4})^2 + (r + \frac{3}{4})^2$

$1 - 2r = \frac{1}{16} + \frac{3}{2}r + \frac{9}{16} \Rightarrow r = \frac{3}{28}$

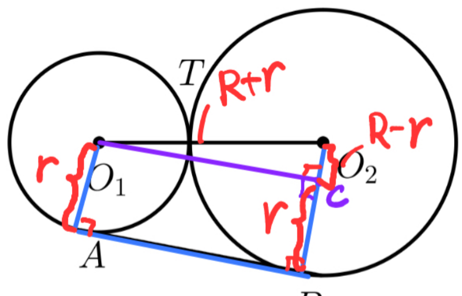


Common Tangent Theorem

Lengths of Common Tangent:

External Common Tangent: $\sqrt{d^2-(R-r)^2}$, or $2\sqrt{R\cdot r}$ if two circles are externally tangent.

Internal Common Tangent: $\sqrt{d^2-(R+r)^2}$.



$AB=O_1C=\sqrt{(R+r)^2-(R-r)^2}$

$=\sqrt{2Rr-(-2Rr)}$

$=2\sqrt{Rr}$

$ABCO_1$ is a rectangle.

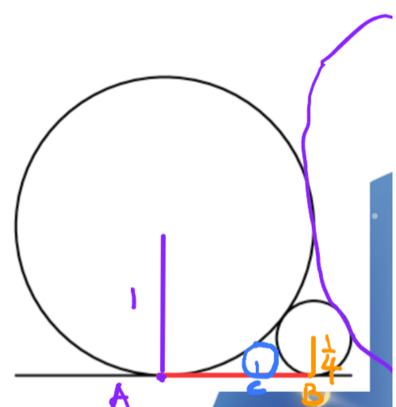
$\Rightarrow AB=O_1C$

MC(1)

Math Exploration 2.2

Two straight pipes (circular cylinders), with radii $\mathbf{1}$ and $\dfrac{\mathbf{1}}{\mathbf{4}}$, lie parallel and in contact on a flat floor. The figure below shows a head-on view. What is the sum of the possible radii of a third parallel pipe lying on the same floor and in contact with both? (2024 AMC 10B Problems, Question #21)

A. $\dfrac{\mathbf{1}}{\mathbf{9}}$ B. $\mathbf{1}$ C. $\dfrac{\mathbf{10}}{\mathbf{9}}$ D. $\dfrac{\mathbf{11}}{\mathbf{9}}$ E. $\dfrac{\mathbf{19}}{\mathbf{9}}$



$AB=2\sqrt{Rr}=2\sqrt{1\cdot\frac{1}{4}}=1$, Let radius of third be x

$\Rightarrow AC=2\sqrt{R\cdot x}=2\sqrt{x}$ or $\Rightarrow AC=AB+BC$

$BC=2\sqrt{r\cdot x}=\sqrt{x}$ longest

$AC+BC=AB$

$2\sqrt{x}+\sqrt{x}=1$

$x=\frac{1}{9}$

$2\sqrt{x}=1+\sqrt{x}$

$x=1$

$SUM=\frac{1}{9}+1=\frac{10}{9}$

MC(1)

Practice 2.1

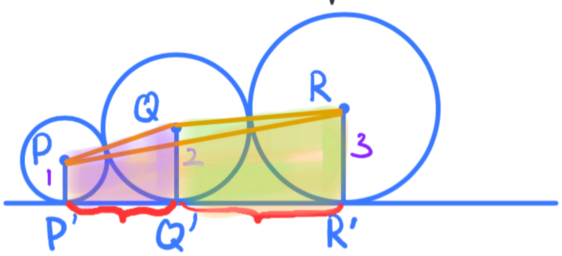
Circles with centers $\boldsymbol{P}, \boldsymbol{Q}$ and $\boldsymbol{R}$, having radii $\mathbf{1}, \mathbf{2}$ and $\mathbf{3}$, respectively, lie on the same side of line $\boldsymbol{l}$ and are tangent to $\boldsymbol{l}$ at $\boldsymbol{P'}, \boldsymbol{Q'}$ and $\boldsymbol{R'}$, respectively, with $\boldsymbol{Q'}$ between $\boldsymbol{P'}$ and $\boldsymbol{R'}$. The circle with center $\boldsymbol{Q}$ is externally tangent to each of the other two circles. What is the area of triangle $\boldsymbol{PQR}$? (2016 AMC 10A Problems, Question #21)

A. $\mathbf{0}$ B. $\sqrt{\frac{2}{3}}$ C. $\mathbf{1}$ D. $\sqrt{6}-\sqrt{2}$ ✓ E. $\sqrt{\frac{3}{2}}$

$A_{PQQ'P'} = \frac{1}{2}(1+2)\cdot 2\sqrt{2} = 3\sqrt{2}$

$A_{QRR'Q'} = \frac{1}{2}(2+3)\cdot 2\sqrt{6} = 5\sqrt{6}$

$A_{PRR'P'} = \frac{1}{2}(1+3)(2\sqrt{2}+2\sqrt{6}) = 4\sqrt{2}+4\sqrt{6}$



$\Rightarrow A_{\triangle PQR} = (3\sqrt{2}+5\sqrt{6}) - (4\sqrt{2}+4\sqrt{6})$
$= \sqrt{6}-\sqrt{2}$

$A_{\triangle PQR} = (A_{PQQ'P'} + A_{QRR'Q'}) - A_{PRR'P'}$

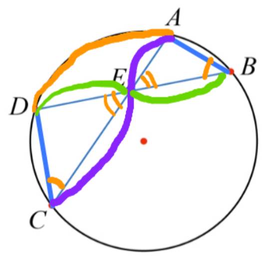
$P'Q' = 2\sqrt{1\cdot 2} = 2\sqrt{2} \Rightarrow P'R' = 2\sqrt{2}+2\sqrt{6}$

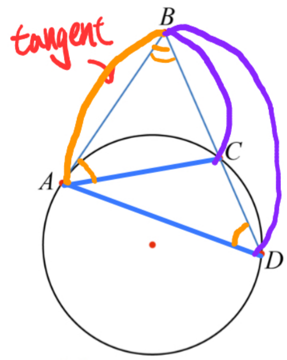
$Q'R' = 2\sqrt{2\cdot 3} = 2\sqrt{6}$

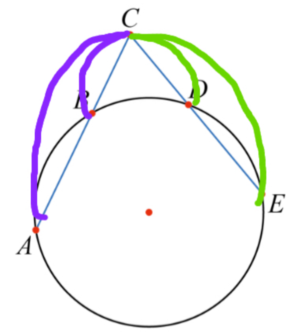
Concept 3

Power of A Point

Through a point draw two lines intersecting a circle. The products of segments on the same straight line from this point are equal.







$\frac{AE}{DE} = \frac{BE}{CE}$

$\Rightarrow AE \cdot CE = BE \cdot DE$

$\frac{BA}{BD} = \frac{BC}{BA}$

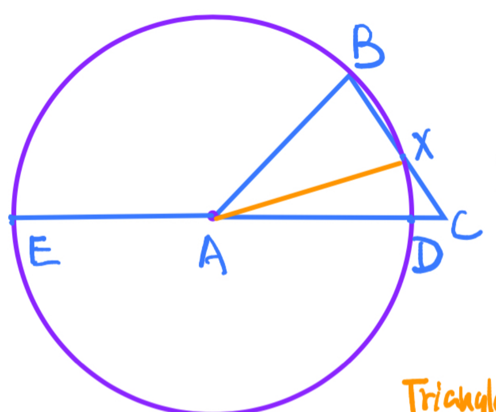
$BA^2 = BC \cdot BD$

MC(1)

Math Exploration 3.1

In $\triangle ABC$, $AB = 86$, and $AC = 97$. A circle with center A and radius AB intersects $\overline{BC}$ at points B and X. Moreover $\overline{BX}$ and $\overline{CX}$ have integer lengths. What is BC? (2013 AMC 10A Problems, Question #23)

A. 11 B. 28 C. 33 D. 61 E. 72



Let $BX = a$, $CX = b \Rightarrow BC = a+b$

$\Rightarrow CD = AC - \underbrace{AD}_{\text{radius}} = 97 - 86 = 11$

$CE = CD + DE = 11 + 2 \times 86 = 183$

Power of a Point:

$CX \cdot CB = CD \cdot CE$

$b(a+b) = 11 \cdot 183 = 3 \cdot 11 \cdot 61$

Triangle inequality: $CX > AC - AX \Rightarrow CX = 33$

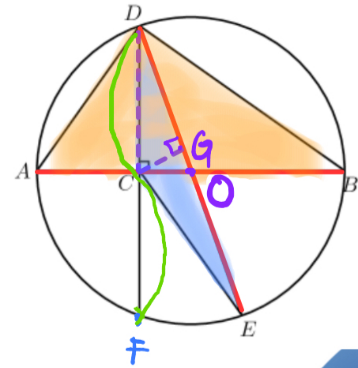
$CX > 11 \Rightarrow BC = 61$

MC(1)

Practice 3.1

Let AB be a diameter of a circle and let C be a point on AB with $2 \cdot AC = BC$. Let D and E be points on the circle such that $DC \perp AB$ and DE is a second diameter. What is the ratio of the area of ΔDCE to the area of ΔABD? (2005 AMC 10A Problems, Question #23)

A. $\frac{1}{6}$ B. $\frac{1}{4}$ C. $\frac{1}{3}$ D. $\frac{1}{2}$ E. $\frac{2}{3}$



Let $AC=1$, $BC=2$, $AB=DE=3$

Because $AB \perp CD$, $DC=CF$

Power of a point: $CD \cdot CF = CA \cdot CB$

$CD^2 = 1 \cdot 2$

height of $\triangle ABD \Rightarrow CD=\sqrt{2}$

In Rt$\triangle DCO$, $CO = AO - AC$

$= 1 - \frac{1}{2} = \frac{1}{2}$

$CG = \frac{CO \cdot CD}{DO} = \frac{\frac{1}{2}\cdot\sqrt{2}}{\frac{3}{2}} = \frac{\sqrt{2}}{3}$

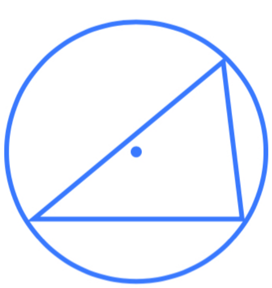
$\frac{A_{\triangle DCE}}{A_{\triangle ABD}} = \frac{CG}{CD}$

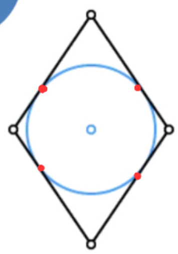
$= \frac{\frac{\sqrt{2}}{3}}{\sqrt{2}}$

$= \frac{1}{3}$

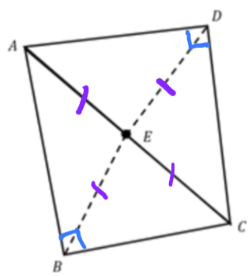
Concept 4

Cyclic Quadrilateral and Ptolemy's Theorem

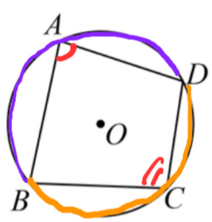




1. Four points are equidistant to another fixed point

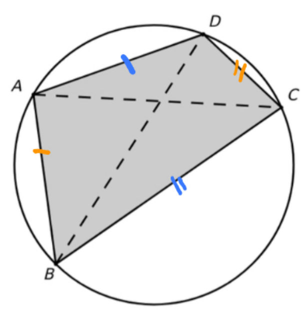


2. Two right triangels with common hypotenuse

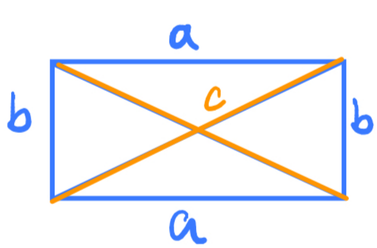


3. Opposite angles are supplementary

Ptolemy's Theorem: given a cyclic quadrilateral $ABCD$, $AD \cdot BC + AB \cdot CD = AC \cdot BD$.



Ptolemy's Theorem is also a "more powerful" Pythagorean Theorem



$a \cdot a + b \cdot b = c^2$

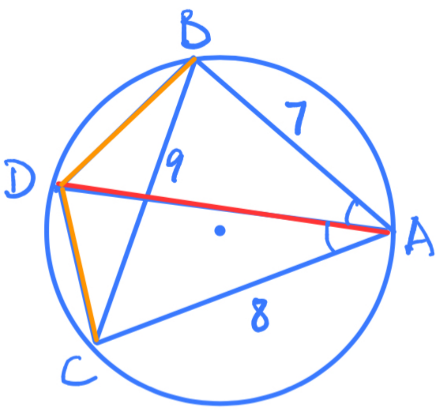
$a^2 + b^2 = c^2$

MC(1)

Math Exploration 4.1

In triangle ABC we have $AB = 7$, $AC = 8$, $BC = 9$. Point D is on the circumscribed circle of the triangle so that AD bisects angle BAC. What is the value of $\frac{AD}{CD}$? (2004 AMC 10B Problems, Question #24)

A. $\frac{9}{8}$ B. $\frac{5}{3}$ ✓ C. 2 D. $\frac{17}{7}$ E. $\frac{5}{2}$



$m\angle BAD = m\angle CAD$

$\Rightarrow BD = CD$

Ptolemy' Therom:

$BD \cdot AC + DC \cdot AB = AD \cdot BC$

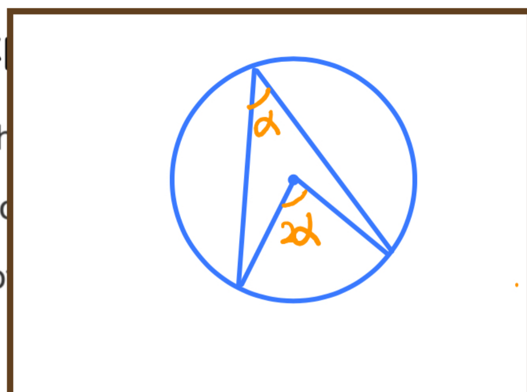
$8 \cdot CD + 7 \cdot CD = 9 \cdot AD$

$15\,CD = 9AD$

$\Rightarrow \frac{AD}{CD} = \frac{15}{9} = \frac{5}{3}$

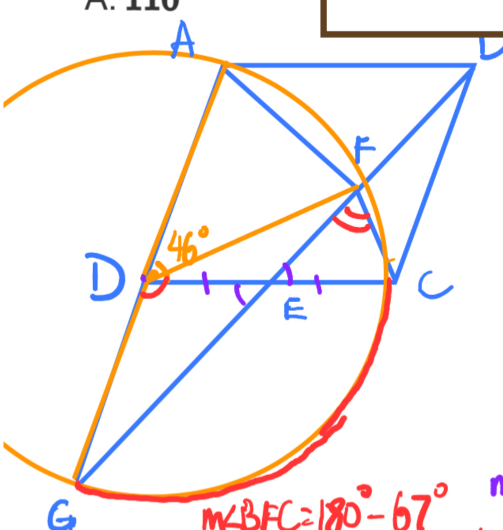
MC(1)

Math Ex



Let $ABCD$ be a rh … be the midpoint of $\overline{CD}$, and let F be the point … ular to $\overline{BE}$. What is the degree measure o … ms, Question #20)

A. 110 D. 113 ✓ E. 114



Extend $\overline{BE}$ and $\overline{AD}$ to meet at G.

$\Rightarrow \triangle BEC \cong \triangle DEG$

$BC = DG = AD \Rightarrow D$ is midpoint of AG

$\overline{AF} \perp \overline{BG} \Rightarrow m\angle AFG = 90°$, $\triangle AFG$ is Rt$\triangle$

$\Rightarrow AD = GD = FD = CD$

A, F, C, G are on a circle centered at Point D.

$m\angle CFG = \frac{1}{2} m\angle CDG$ (Extended by $\overline{CG}$)

$= \frac{1}{2}(180° - 46°) = 67°$

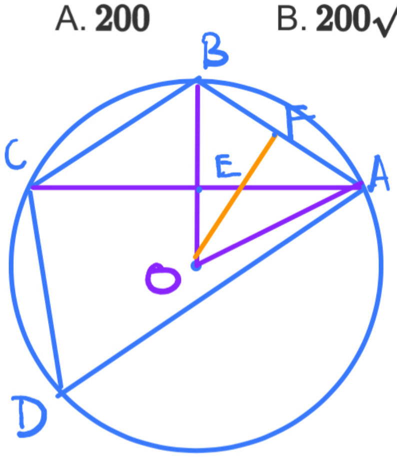
$m\angle BFC = 180° - 67° = 113°$ $\Leftarrow$

MC(1)

Practice 4.1

A quadrilateral is inscribed in a circle of radius $200\sqrt{2}$. Three of the sides of this quadrilateral have length 200. What is the length of the fourth side? (2016 AMC 10A Problems, Question #24)

A. 200 B. $200\sqrt{2}$ C. $200\sqrt{3}$ D. $300\sqrt{2}$ E. 500 ✓



$AB=CB \Rightarrow BO\perp AC \Rightarrow AE=CE$

$AF=\frac{1}{2}AB=100$, $AO=200\sqrt{2}$

$\Rightarrow OF=\sqrt{AO^2-AF^2}=\sqrt{80000-10000}=100\sqrt{7}$

$A_{\triangle AOB}=\frac{1}{2}AB\cdot OF=\frac{1}{2}\cdot 200\cdot 100\sqrt{7}=100^2\sqrt{7}$

$\Rightarrow A_{\triangle AOB}=AE\cdot OB$

$100^2\sqrt{7}=AE\cdot 200\sqrt{2}\cdot\frac{1}{2}$, $AE=\frac{100\sqrt{7}}{\sqrt{2}}$

$\Rightarrow AC=2AE=200\sqrt{\frac{7}{2}}$

$200AD+200^2=AC^2$ $\Rightarrow AD=500$

$200AD=200^2\cdot\frac{7}{2}-200^2$

MC(1)

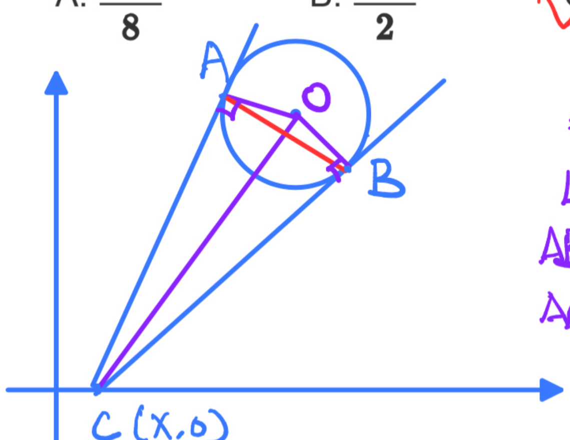
Practice 4.2

Also, $BC^2+OB^2=CO^2 \Rightarrow 170+r^2=17r^2$

$170+r^2=d^2$ $r^2=\frac{170}{16}=\frac{85}{8}$

Points $A=(6,13)$ and $B=(12,11)$ lie on circle ω in the plane. Suppose that the tangent lines to ω at A and B intersect at a point on the x-axis. What is the area of ω? (2019 AMC 10B Problems, Question #23)

A. $\frac{83\pi}{8}$ B. $\frac{21\pi}{2}$ C. $\frac{85\pi}{8}$ ✓ D. $\frac{43\pi}{4}$ E. $\frac{87\pi}{8}$



$AC=BC \Rightarrow (6-x)^2+13^2=(12-x)^2+11^2$

$x=5$

$m\angle OBC=m\angle OAC=90^\circ$

$\Rightarrow A,O,B,C$ are cyclic quadrilateral.

Let $OB=r$, $OC=d$.

$AB=\sqrt{(12-6)^2+(13-11)^2}=\sqrt{40}$

$AC=BC=\sqrt{(6-5)^2+13^2}=\sqrt{170}$

$\Rightarrow AO\cdot BC+BO\cdot AC=AB\cdot CO$

$2r\sqrt{170}=d\sqrt{40}$

$\Rightarrow d=\sqrt{17}\,r$