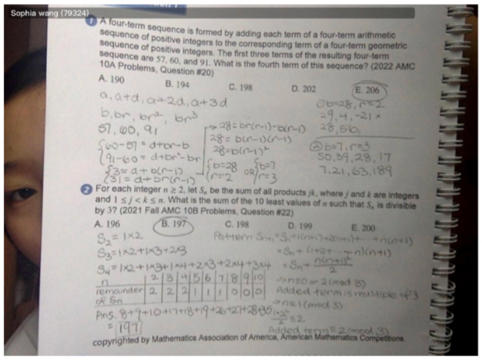
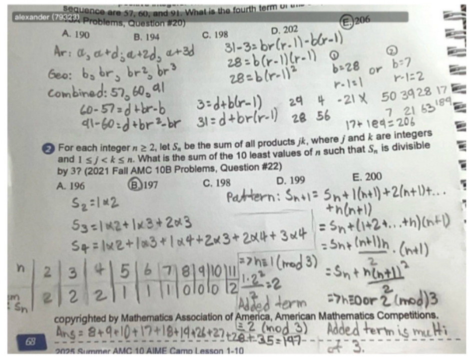
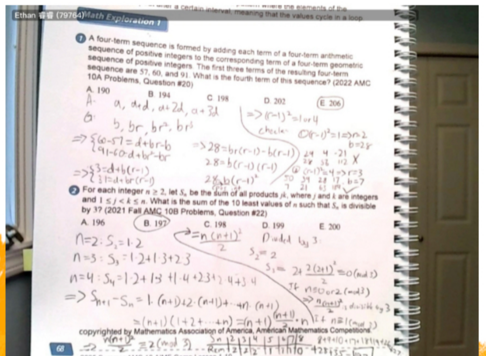
Excellent Notes



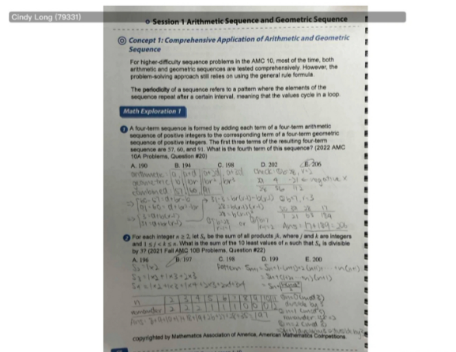
Sophia Wang



Alexander



Ethan



Cindy Long

Lesson 6 Coordinate Bash and Euclid's Theorem

Announcement

Parent's Meeting (Brief QA Session) at the end of the class

Answer questions about exam registration, mock exam and more.

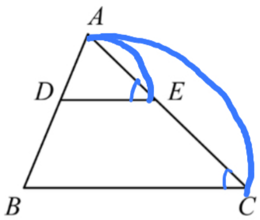
Concept 1

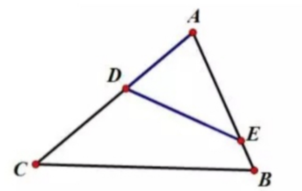
Comprehensive Applications of Congruence and Similarity

Review – "A" Model

If $\overline{DE} \parallel \overline{BC}$, we have $\triangle ADE \sim \triangle ABC$, and then

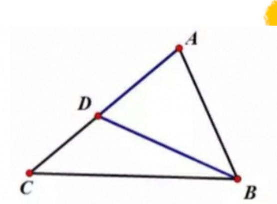
$$\frac{AD}{AB}=\frac{AE}{AC}=\frac{DE}{BC},\ \frac{AD}{DB}=\frac{AE}{EC},\ \frac{DB}{AB}=\frac{EC}{AC}.$$





Reverse A Model

$$m\angle AED = m\angle C \Rightarrow \triangle ADE \sim \triangle ABC$$



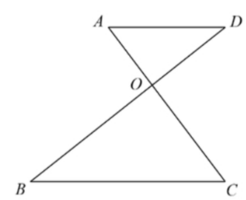
Reverse A Model (Common Side)

$$m\angle ABD = m\angle C \Rightarrow \triangle ADB \sim \triangle ABC$$

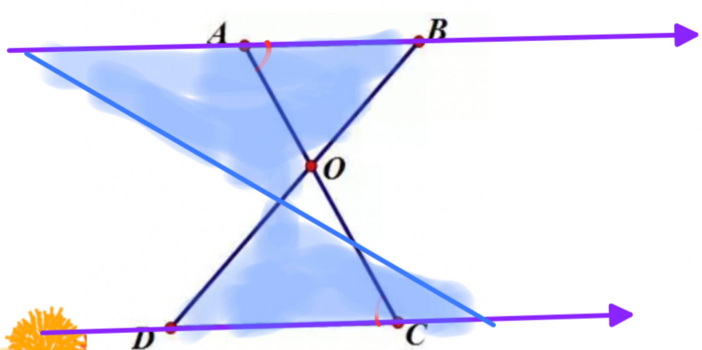
Review – "8" Model

If $\overline{AD} \parallel \overline{BC}$, we have $\triangle OAD \sim \triangle OCB$, and then

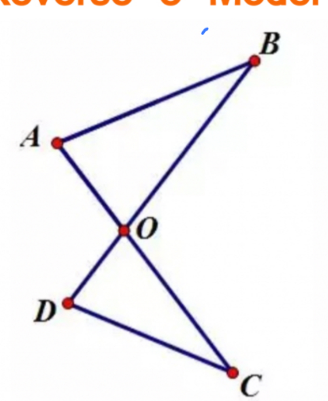
$$\frac{AD}{CB}=\frac{AO}{CO}=\frac{DO}{BO},\ \frac{AO}{AC}=\frac{DO}{BD},\ \frac{CO}{CA}=\frac{BO}{DB}.$$



"8" Model



Reverse "8" Model

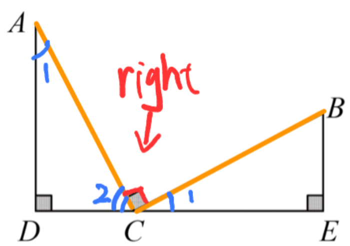


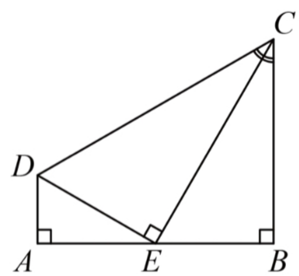
Review – K Model

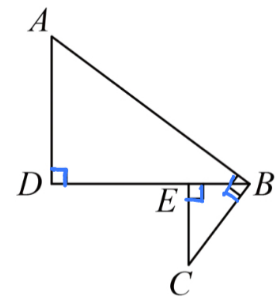
Summary of "K" congruence models in congruent models:

① Signs of the Existence: The presence of an isosceles right triangle.

② Conclusion: Two right triangles with their two legs as the hypotenuses are congruent.







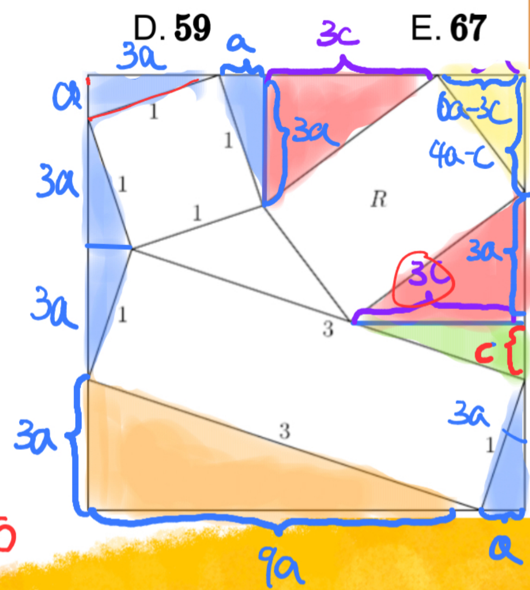
MC(1)

Math Exploration 1.1

$(3a)^2+a^2=1 \Rightarrow a^2=\frac{1}{10}$

A rectangle with side lengths **1** and **3,** a square with side length **1,** and a rectangle R are inscribed inside a larger square as shown. The sum of all possible values for the area of R can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$? (2021 Fall AMC 10B Problems, Question #25)

A. **14** B. **23** C. **46** D. **59** E. **67**



Use similar and congruent Δ:

$a+b+b+3a=3b+a \Rightarrow 3a=b$

Yellow Δ ~ Red Δ:

$\frac{3c}{4a-c} = \frac{3a}{6a-3c}$

$\Rightarrow 18ac-9c^2=12a^2-3ac$

$12a^2-21ac+9c^2=0$

$4a^2-7ac+3c^2=0$

$(4a-3c)(a-c)=0$

$\Rightarrow 4a=3c$ or $a=c$

① $a=c$, for R:

length $=3\sqrt{2}\cdot a$

width $=3\sqrt{2}\,a$

Area $=18a^2$

② $4a=3c$, for R:

Area $=\frac{50}{3}a^2$

Ans $=18\cdot\frac{1}{10}+\frac{50}{3}\cdot\frac{1}{10}$

$=\frac{52}{15} \Rightarrow 67$

MC(1)

Practice 1.1

Let $\mathcal{K}$ be the kite formed by joining two right triangles with legs 1 and $\sqrt{3}$ along a common hypotenuse. Eight copies of $\mathcal{K}$ are used to form the polygon shown below. What is the area of triangle $\triangle ABC$? (2024 AMC 10A Problems, Question #22)

A. $2+3\sqrt{3}$ B. $\frac{9}{2}\sqrt{3}$ C. $\frac{10+8\sqrt{3}}{3}$ D. 8 E. $5\sqrt{3}$

$\triangle ADE$ is 30-60-90 $\triangle$

$AE = AD \cdot \frac{\sqrt{3}}{2} = \frac{3}{2}$

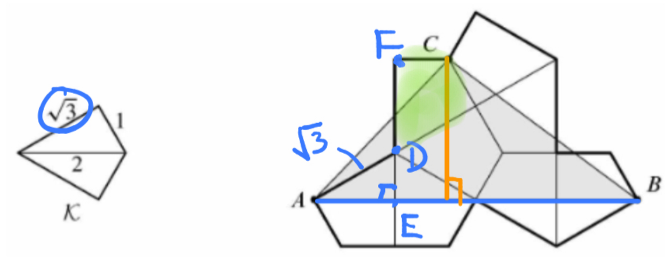
$AB = 4AE = 6$

Also, $DE = AD \cdot \frac{1}{2} = \frac{\sqrt{3}}{2}$

and $DF = \sqrt{3}$

$\Rightarrow h = EF = \sqrt{3} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$

$A_{\triangle ABC} = \frac{1}{2}bh = \frac{1}{2} \cdot 6 \cdot \frac{3\sqrt{3}}{2} = \frac{9\sqrt{3}}{2}$

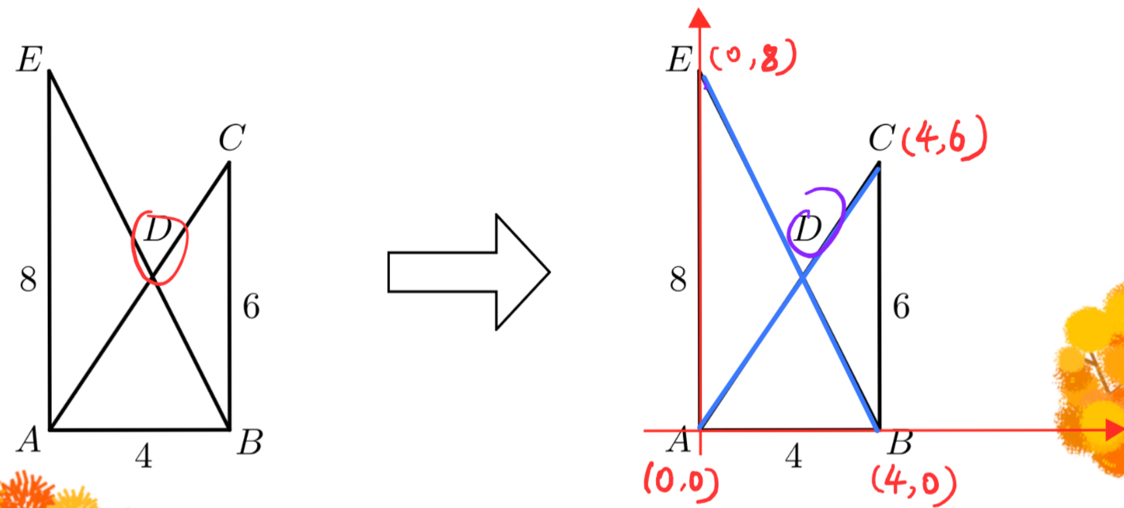


Concept 2: Coordinate Bash

Coordinate Bash

Concept 2: Coordinate Bash

Coordbash (Coordinate Bash) is a strategy in geometry where coordinate geometry is used to solve problems, especially when constructing auxiliary lines is difficult or when pure geometric methods are not straightforward.

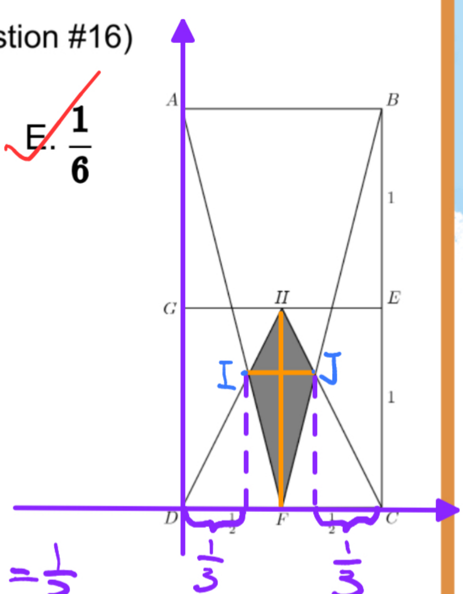


MC(1)

Math Exploration 2.1

In rectangle $ABCD$, $AB = 1$, $BC = 2$, and points E, F, and G are midpoints of $\overline{BC}$, $\overline{CD}$, and $\overline{AD}$, respectively. Point H is the midpoint of $\overline{GE}$. What is the area of the shaded region? (2014 AMC 10A Problems, Question #16)

A. $\frac{1}{12}$ B. $\frac{\sqrt{3}}{18}$ C. $\frac{\sqrt{2}}{12}$ D. $\frac{\sqrt{3}}{12}$ E. $\frac{1}{6}$ ✓



$FH = 1$, need to find IJ

Set a coordinate plane:

$D(0,0)$, $H(\frac{1}{2},1)$, $A(0,2)$, $F(\frac{1}{2},0)$

$\Rightarrow$ Equation of $\overline{DH}$: $y = 2x$

Equation of $\overline{AF}$: $y = -4x + 2$

Find intersection point I: $\Rightarrow IJ = 1 - 2\cdot\frac{1}{3} = \frac{1}{3}$

$2x = -4x + 2$

$x = \frac{1}{3}$

Area $= \frac{1}{2}\cdot 1\cdot\frac{1}{3} = \frac{1}{6}$

MC(1)

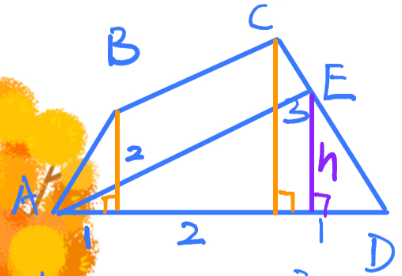
Math Exploration 2.2

$m = \frac{0-3}{4-3} = -3$

Let points $A = (0,0)$, $B = (1,2)$, $C = (3,3)$, and $D = (4,0)$. Quadrilateral $ABCD$ is cut into equal area pieces by a line passing through A. This line intersects $\overline{CD}$ at point $\left(\frac{p}{q}, \frac{r}{s}\right)$, where these fractions are in lowest terms. What is $p + q + r + s$?

(2013 AMC 10A Problems, Question #18)

A. **54** B. **58** C. **62** D. **70** E. **75**



$A_{ABCD} = 1 + 5 + \frac{3}{2}$
$= \frac{15}{2}$

$A_{\triangle AED} = \frac{1}{2} A_{ABCD}$

$\frac{1}{2} \cdot 4 \cdot h = \frac{1}{2} \cdot \frac{15}{2}$

$h = \frac{15}{8}$

Find equation of $\overline{CD}$.

$y = -3x + 12$

Find point E

plug in $y = \frac{15}{8}$

$\Rightarrow \frac{15}{8} = -3x + 12$

$x = \frac{27}{8}$

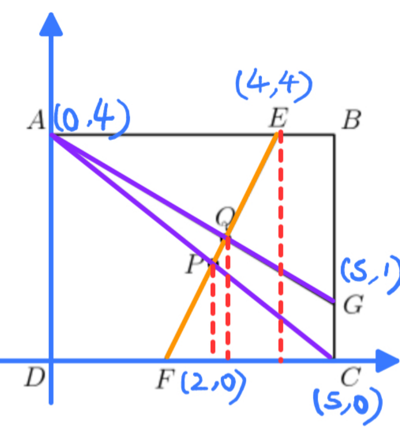
Ans $= 15 + 8 + 27 + 8$
$= 58$

MC(1)

Math Exploration 2.3

Rectangle $ABCD$ has $AB = 5$ and $BC = 4$. Point E lies on $\overline{AB}$ so that $EB = 1$, point G lies on $\overline{BC}$ so that $CG = 1$, and point F lies on $\overline{CD}$ so that $DF = 2$. Segments $\overline{AG}$ and $\overline{AC}$ intersect $\overline{EF}$ at Q and P, respectively. What is the value of $\frac{PQ}{EF}$? (2016 AMC 10B Problems, Question #19)

A. $\frac{\sqrt{13}}{16}$ B. $\frac{\sqrt{2}}{13}$ C. $\frac{9}{82}$ D. $\frac{10}{91}$ E. $\frac{1}{9}$



Equation
$$\begin{cases} EF: y = 2x - 4 \\ AC: y = -\frac{4}{5}x + 4 \\ AG: y = -\frac{3}{5}x + 4 \end{cases}$$

$$\boxed{\frac{PQ}{EF} = \frac{|x_P - x_Q|}{|x_E - x_F|} = \left(\frac{40}{13} - \frac{20}{7}\right) \div 2 = \frac{10}{91}}$$

Intersection;

① point Q: $2x - 4 = -\frac{3}{5}x + 4$

$\frac{13}{5}x = 8$

$x = \frac{40}{13}$

② Point P: $2x - 4 = -\frac{4}{5}x + 4$

$\frac{14}{5}x = 8$

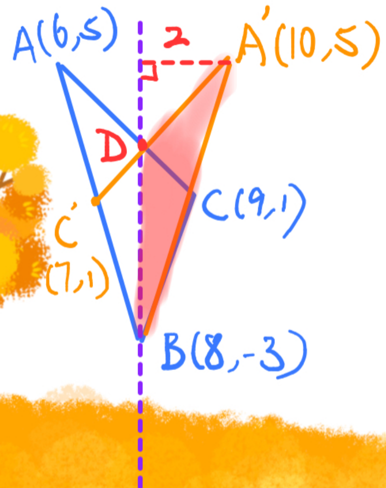
$x = \frac{20}{7}$

MC(1)

Practice 2.1

A triangle with vertices $(6,5)$, $(8,-3)$, and $(9,1)$ is reflected about the line $x=8$ to create a second triangle. What is the area of the union of the two triangles? (2013 AMC 10A Problems, Question #16)

A. 9 B. $\frac{28}{3}$ C. 10 D. $\frac{31}{3}$ E. $\frac{32}{3}$



Prove: A, C' and B are co-linear.

slope of AC' is $\frac{5-1}{6-7}=-4$

slope of C'B is $\frac{1-(-3)}{7-8}=-4$

Find equation AC: slope $=\frac{5-1}{6-9}=-\frac{4}{3}$

$y=-\frac{4}{3}x+13$

plug in $x=8 \Rightarrow D(8, \frac{7}{3}) \Rightarrow BD=\frac{7}{3}-(-3)$
$=\frac{16}{3}$

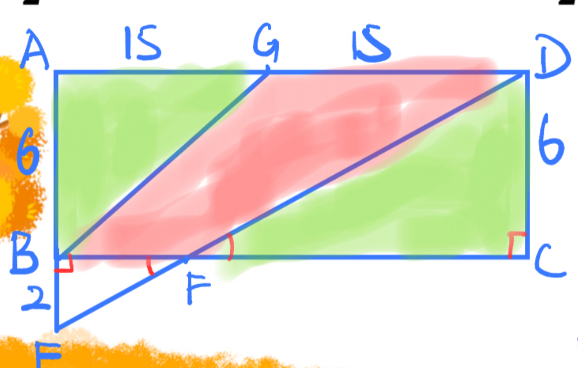
Area $=2\cdot\left(\frac{1}{2}\cdot\frac{16}{3}\cdot 2\right)=\frac{32}{3}$

MC(1)

Practice 2.2

In rectangle $ABCD$, $AB=6$, $AD=30$, and G is the midpoint of $\overline{AD}$. Segment AB is extended 2 units beyond B to point E, and F is the intersection of $\overline{ED}$ and $\overline{BC}$. What is the area of quadrilateral $BFDG$? (2012 AMC 10B Problems, Question #19)

A. $\frac{133}{2}$ B. 67 C. $\frac{135}{2}$ D. 68 E. $\frac{137}{2}$



$\triangle BFE \sim \triangle DFC$

$\Rightarrow \frac{BF}{FC}=\frac{BE}{DC}=\frac{2}{6}$

$\Rightarrow FC=3BF$

$FC=\frac{3}{4}BC=\frac{3}{4}\cdot 30=\frac{45}{2}$

Area $=6\times 30-\left(\frac{1}{2}\times 6\times 15\right)-\left(\frac{1}{2}\times 6\times\frac{45}{2}\right)$

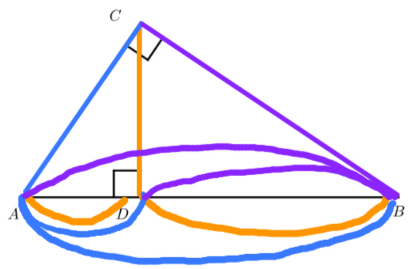
$=180-45-\frac{135}{2}$

$=\frac{135}{2}$

Concept 3: Euclid's Theorem

As shown in the figure below, if CD is the altitude drawn to the hypotenuse of right triangle ABC, then these conclusions can be deduced:

① $CD^2 = AD \cdot BD$;
② $AC^2 = AD \cdot AB$;
③ $BC^2 = BD \cdot AB$;
④ $AC \cdot BC = AB \cdot CD$.



$\triangle ADC \sim \triangle CDB$

$\Rightarrow \dfrac{CD}{DB} = \dfrac{AD}{CD} \Rightarrow CD^2 = AD \cdot BD$

Longer leg   shorter leg

Blank filling

Math Exploration 5.1

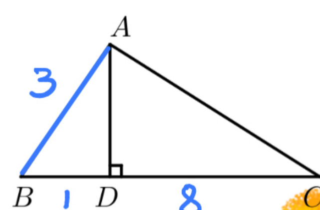
As shown in the figure, AD is the height on the hypotenuse of $\mathrm{Rt}\triangle ABC$.

Given $AB = 3$, $CD = 8$, try to use the Euclid's theorem to find the length of other line segments $BD =$ 1 , $AD =$ $2\sqrt{2}$, $AC =$ $6\sqrt{2}$.

$AB^2 = BD \cdot BC$
$3^2 = BD \cdot (BD+8)$
$BD^2 + 8BD - 9 = 0$
$(BD+9)(BD-1) = 0$
$BD = 1$

$AD^2 = BD \cdot DC$
$AD^2 = 1 \cdot 8$
$AD = 2\sqrt{2}$

$AC^2 = DC \cdot BC$
$AC^2 = 8 \cdot 9$
$AC = 6\sqrt{2}$

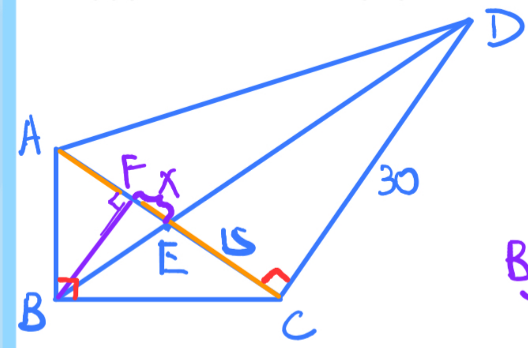


MC(1)

Math Exploration 5.2

Quadrilateral $ABCD$ satisfies $\angle ABC = \angle ACD = 90^\circ$, $AC = 20$, and $CD = 30$. Diagonals $\overline{AC}$ and $\overline{BD}$ intersect at point E, and $AE = 5$. What is the area of quadrilateral $ABCD$? (2020 AMC 10A Problem, Question #20)

A. 330 B. 340 C. 350 D. 360 E. 370



Draw $\overline{BF} \perp \overline{AC}$, given $AE=5$, $CE=20-5=15$

Let $EF = x$, $\frac{EF}{CE} = \frac{BF}{CD} \Rightarrow BF = 2x$

$\Rightarrow AF = 5-x$ and $CF = 15+x$

By Geometric Mean Theorem:

$BF^2 = AF \cdot CF$

$(2x)^2 = (5-x)(15+x)$

$x^2+2x-15=0$

$x=3$

$\Rightarrow BF = 2x = 6$

Area $= 300 + \frac{1}{2}\cdot 20 \cdot 6$

$= 360$

$A_{\triangle ACD} = \frac{1}{2}\cdot AC \cdot CD$

$= 300$

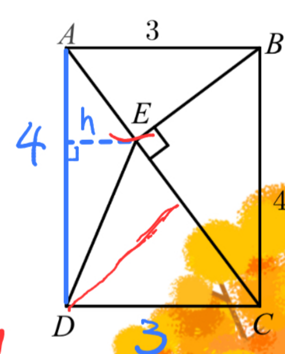
Find $\triangle ABC$.

MC(1)

Practice 5.1

Rectangle $ABCD$ has $AB = 3$ and $BC = 4$. Point E is the foot of the perpendicular from B to diagonal $\overline{AC}$. What is the area of $\triangle AED$? (2017 AMC 10B Problem, Question #15)

A. 1 B. $\frac{42}{25}$ C. $\frac{28}{15}$ D. 2 E. $\frac{54}{25}$



In $\triangle ABC$, $AC=5$ and $BE = \frac{3\times 4}{5} = \frac{12}{5}$

$\Rightarrow AB^2 = AE \cdot AC$

$3^2 = AE \cdot 5$

$AE = \frac{9}{5}$

Similar triangles: $\frac{h}{DC} = \frac{AE}{AC}$

$h = \frac{9}{5} \div 5 \cdot 3$

$h = \frac{27}{25}$

$A_{\triangle ADE} = \frac{1}{2} bh$

$= \frac{1}{2}\cdot 4 \cdot \frac{27}{25}$

$= \frac{54}{25}$