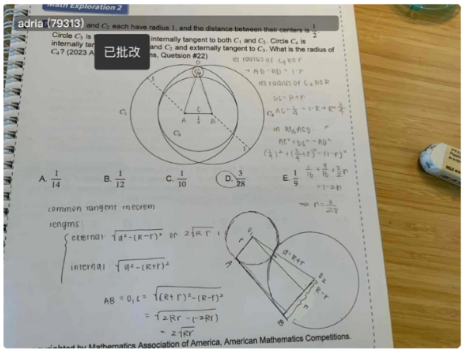
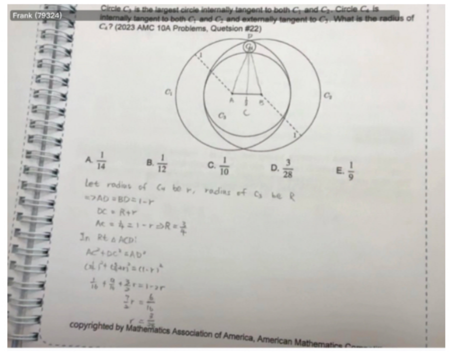
## Excellent Notes



Adria



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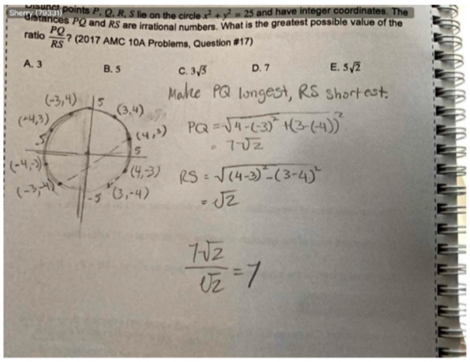
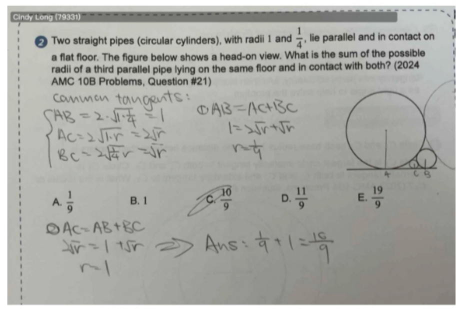


Sherry



Cindy

# L8
# Analytical Geometry and Geometric Transformations

## Second Mock Exam

Link can be found in Lesson 8 Learning Material!

or enter the link below:

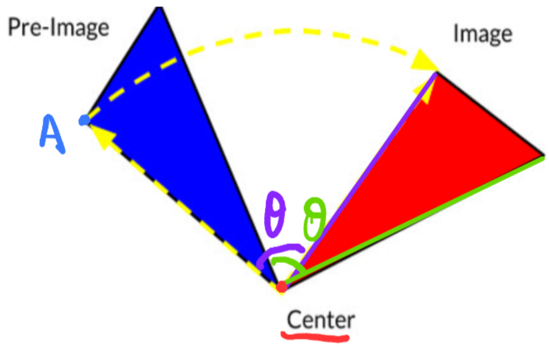
https://www.thethinkacademy.com/quiz/evaluation/instruction/7006

## Concept 1

# Rotation

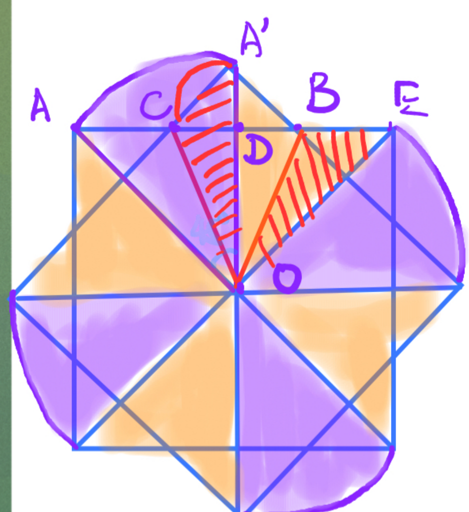
## How does rotation work?

1. The distance from the center to any point on the shape stays the same.
2. Every point makes a sector around the center




## Math Exploration 1.1

0 0 : 0 0




diagonal

Sector: radius = $\frac{\sqrt{2}}{2}$, angle = 45°

$A_{sector} = \frac{45°}{360°} \cdot \pi \cdot \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{\pi}{16}$

Dart: $A_{A'BOC} = \frac{1}{2} \cdot BC \cdot A'O$

Find BC:

$AC = A'C$, $\triangle A'CD$ is 45-45-90 Rt$\triangle$

$\Rightarrow A'C = \sqrt{2} \cdot CD$, $BC = 2CD$ $\left(CD = \frac{BC}{2}\right)$

$AC + BC + BE = 1$

$\frac{\sqrt{2}}{2}BC + BC + \frac{\sqrt{2}}{2}BC = 1$

$(\sqrt{2}+1)BC = 1$

$BC = \frac{1}{\sqrt{2}+1} = \sqrt{2} - 1$

$\Rightarrow A = \frac{1}{2} BC \cdot A'O$

$= \frac{1}{2}(\sqrt{2}-1)\frac{\sqrt{2}}{2}$

$= \frac{1}{2} - \frac{\sqrt{2}}{4}$

Total = 4 sectors + 4 darts

$= 4 \cdot \left(\frac{\pi}{16}\right) + 4\left(\frac{1}{2} - \frac{\sqrt{2}}{4}\right)$

$= 2 - \sqrt{2} + \frac{\pi}{4}$

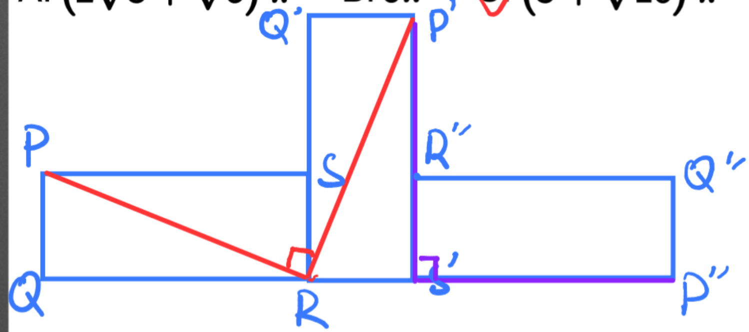
$\boxed{C}$

MC(1)

## Practice 1.1

Rectangle $PQRS$ lies in a plane with $PQ = RS = 2$ and $QR = SP = 6$. The rectangle is
rotated $90^\circ$ clockwise about $R$, then rotated $90^\circ$ clockwise about the point $S$ moved to after the first rotation. What is the length of the path traveled by point $P$? (2008 AMC 10A Problems, Question #19)

A. $(2\sqrt{3}+\sqrt{5})\pi$ B. $6\pi$ C. $(3+\sqrt{10})\pi$ D. $(\sqrt{3}+2\sqrt{5})\pi$ E. $2\sqrt{10}\pi$




diagonal $= \sqrt{2^2+6^2}$

$= 2\sqrt{10}$

First: $\frac{90^\circ}{360^\circ}\cdot 2\pi\cdot 2\sqrt{10} = \pi\sqrt{10}$

Second: $\frac{90^\circ}{360^\circ}\cdot 2\pi\cdot 6 = 3\pi$

Total $= \pi\sqrt{10} + 3\pi$

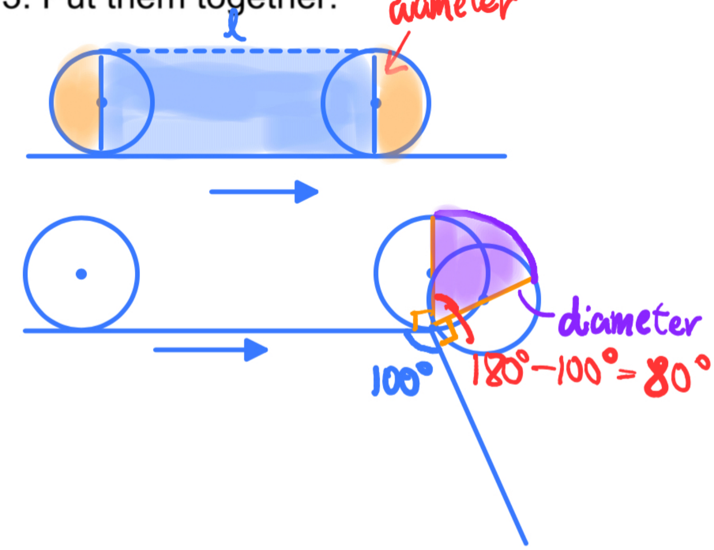
$= (3+\sqrt{10})\pi$

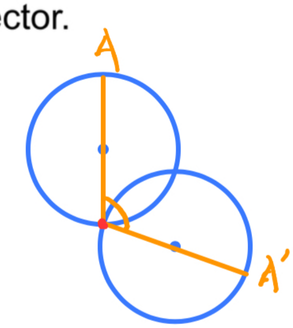
## Concept 2

# Rolling

## How do shapes roll?

1. Rolling on a tangent line: the region swept out is a rectangle.
2. Rolling around a point: the region swept out is a sector.
3. Put them together:



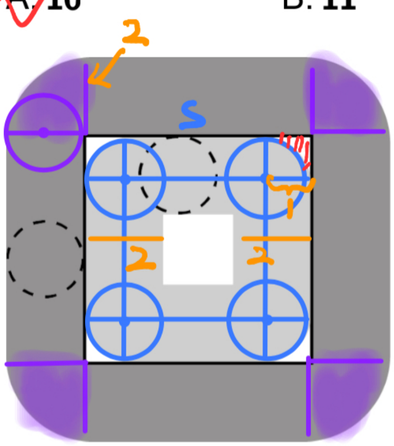




MC(1)

## Math Exploration 2

A disk of radius 1 rolls all the way around the inside of a square of side length $s > 4$ and sweeps out a region of area $A$. A second disk of radius 1 rolls all the way around the outside of the same square and sweeps out a region of area $2A$. The value of $s$ can be written as $a + \frac{b\pi}{c}$, where $a, b$, and $c$ are positive integers and $b$ and $c$ are relatively prime. What is $a + b + c$? (2021 AMC Fall 10A, Question #19)

A. **10** ✓　　B. **11**　　C. **12**　　D. **13**　　E. **14**




Inside: unshaded $= (s-4)^2 + 4\cdot(1^2 - \frac{1}{4}\cdot\pi\cdot1^2)$

$= s^2 - 8s + 16 + 4 - \pi$

$= s^2 - 8s + 20 - \pi$

$A = s^2 - (s^2 - 8s + 20 - \pi) = 8s - 20 + \pi$

Outside: $2A = 4\cdot(2\cdot s) + \pi\cdot 2^2 = 8s + 4\pi$

$\Rightarrow 2(8s - 20 + \pi) = 8s + 4\pi$

$8s = 40 + 2\pi$

$s = 5 + \frac{1}{4}\pi$

$5 + 1 + 4 = 10$

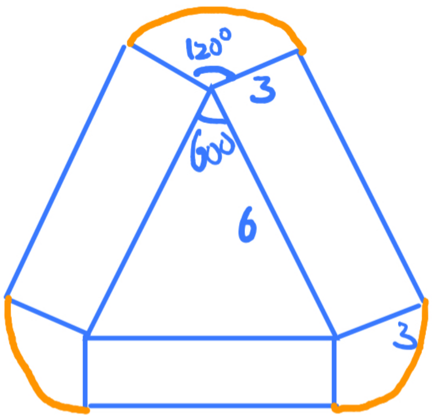
MC(1)

## Practice 2.1

An equilateral triangle has side length **6**. What does the area of the region contain all points that are outside the triangle but not more than **3** units from a point of the triangle? (2008 AMC 10A Problems, Question #17)

A. $36+24\sqrt{3}$ B. $54+9\pi$ C. $54+18\sqrt{3}+6\pi$

D. $(2\sqrt{3}+3)^2\pi$ E. $9(\sqrt{3}+1)^2\pi$




Rectangle $= 3\cdot 6 = 18$

Sectors $= \frac{120^\circ}{360^\circ}\cdot\pi\cdot 3^2 = 3\pi$

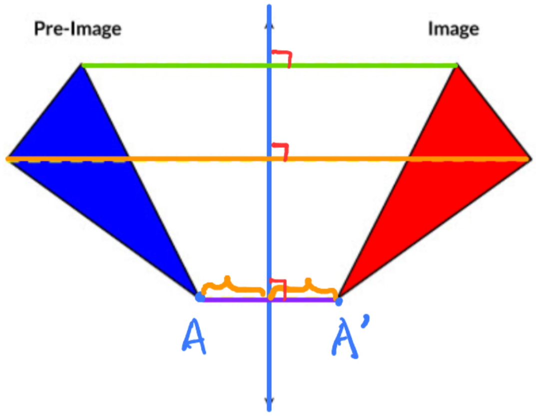
Total $= 3\cdot(18) + 3\cdot(3\pi)$

$= 54 + 9\pi$

## Concept 3

# Symmetry

## How does symmetry work?



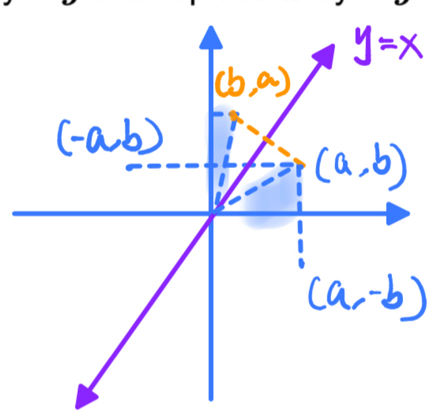

1. Perpendicular
2. Midpoint

## Common Symmetry in Coordinate Plane

Reflection of functions:

1. Reflection over x-axis: $y$ is replaced by $-y$
2. Reflection over y-axis: $x$ is replaced by $-x$
3. Reflection over $y = x$, replace $x$ by $y$ and replace $y$ by $x$
4. Relection over $y = -x$, replace $x$ by $-y$ and replace $x$ by $-y$.

Use symmetry of point to memorize!

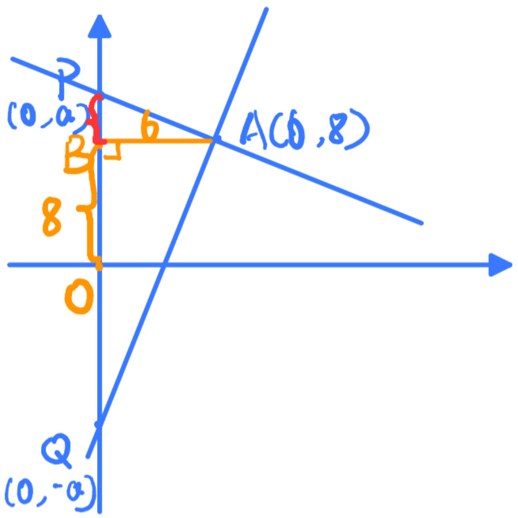


MC(1)

## Math Exploration 3.1

The $y$—intercepts, $P$ and $Q$, of two perpendicular lines intersecting at the point $A(6,8)$ have a sum of zero. What is the area of $\triangle APQ$?

A. $45$ B. $48$ C. $54$ D. $60$ E. $72$




$AB=6$, $BO=8 \Rightarrow PB=a-8$
$QB=a+8$

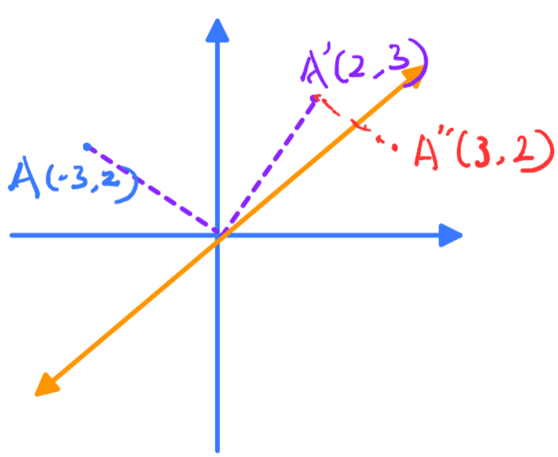
$AB^2=PB\cdot QB$
$6^2=(a-8)(a+8)$
$a=10$
$\Rightarrow PQ=2a=20$
$A_{\triangle PAQ}=\frac{1}{2}\cdot 20\cdot 6=60$

MC(1)

## Practice 3.1

The point $(-3,2)$ is rotated $90^\circ$ clockwise around the origin to point $B$. Point $B$ is then reflected over the line $x=y$ to point $C$. What are the coordinates of $C$? (2004 AMC 12B Problems, Question #9)

A. $(-3,-2)$ B. $(-2,-3)$ C. $(2,-3)$ D. $(2,3)$ E. $(3,2)$



MC(1)

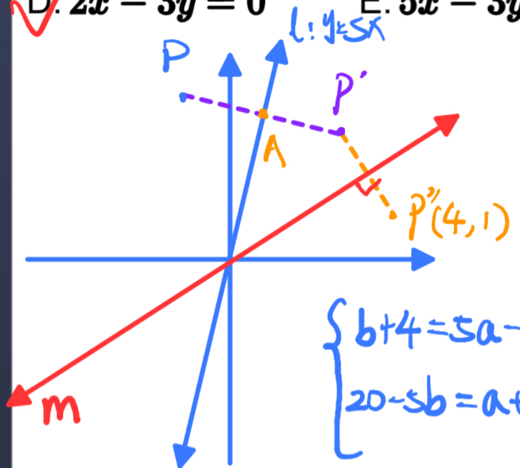
## Practice 3.2

slope of line m is $\frac{-1}{-\frac{3}{2}} = \frac{2}{3} \Rightarrow y = \frac{2}{3}x$

Distinct lines $\ell$ and $m$ lie in the $xy$-plane. They intersect at the origin. Point $P(-1,4)$ is reflected about line $\ell$ to point $P'$, and then $P'$ is reflected about line $m$ to point $P''$. The equation of line $\ell$ is $5x-y=0$, and the coordinates of $P''$ are $(4,1)$. What is the equation of line $m$ ?(2021 AMC Fall 10B Problem, Question #17)

A. $5x+2y=0$ B. $3x+2y=0$ C. $x-3y=0$

D. $2x-3y=0$ ✓ E. $5x-3y=0$




Find P': suppose P'(a,b)

① Midpoint of PP' is $\left(\frac{a-1}{2}, \frac{b+4}{2}\right)$

plug in to $y=5x \Rightarrow \frac{b+4}{2} = 5\cdot\frac{a-1}{2}$.

② $PP' \perp \ell$, slope of $PP' = \frac{4-b}{-1-a}$

$\Rightarrow \frac{4-b}{-1-a}\cdot(5) = -1$

$\begin{cases} b+4=5a-5 \\ 20-5b=a+1 \end{cases} \Rightarrow \begin{cases} a=\frac{32}{13} \\ b=\frac{43}{13} \end{cases}$

Find slope of P'P''

$\frac{\frac{43}{13}-1}{\frac{32}{13}-4} = \frac{30}{-20} = -\frac{3}{2}$



## Concept 4

# Analytical Geometry

MC(1)

## Math Exploration 4.1

A triangle with vertices $A(0,2)$, $B(-3,2)$, and $C(-3,0)$ is reflected about the $x$-axis, then the image $\triangle A'B'C'$ is rotated counterclockwise about the origin by $90°$ to produce $\triangle A''B''C''$. Which of the following transformations will return $\triangle A''B''C''$ to $\triangle ABC$? (2016 AMC 10A Problems, Question #16)

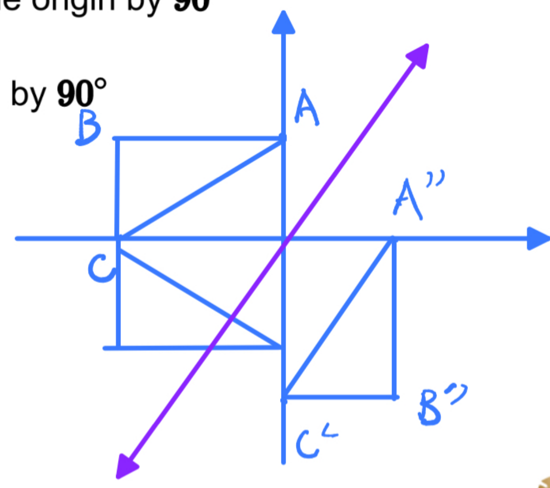
A. counterclockwise retation about the origin by $90°$

B. clockwise rotation about the origin by $90°$

C. reflection about the $x$-axis

D. reflection about the line $y = x$ ✓

E. reflection about the $y$-axis

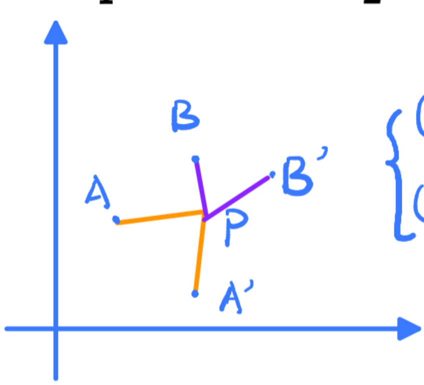



MC(1)

## Math Exploration 4.2

The line segment formed by $A(1,2)$ and $B(3,3)$ is rotated to the line segment formed by $A'(3,1)$ and $B'(4,3)$ about the point $P(r,s)$. What is $|r-s|$? (2023 AMC 10A Problems, Question #19)

A. $\frac{1}{4}$ B. $\frac{1}{2}$ C. $\frac{3}{4}$ D. $\frac{2}{3}$ E. $1$ ✓




$AP = A'P$ and $BP = B'P$

$$\begin{cases}(r-1)^2+(s-2)^2=(r-3)^2+(s-1)^2 \\ (r-3)^2+(\cancel{s-3})^2=(r-4)^2+(\cancel{s-3})^2\end{cases}$$

$\Rightarrow (r-3)^2=(r-4)^2$ $\Rightarrow$ plug in $r=\frac{7}{2}$

$r-3=4-r$

$r=\frac{7}{2}$

$(\frac{5}{2})^2+(s-2)^2=(\frac{1}{2})^2+(s-1)^2$

$s=\frac{9}{2}$

$|r-s|=|\frac{9}{2}-\frac{7}{2}|=1$

MC(1)

## Practice 4.1 Lesson 6 Practice 2.1

A triangle with vertices $(6,5)$, $(8,-3)$, and $(9,1)$ is reflected about the line $x=8$ to create a second triangle. What is the area of the union of the two triangles? (2013 AMC 10A Problems, Question #16)

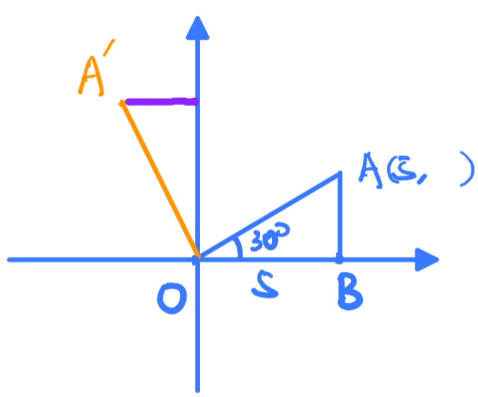
A. $9$ B. $\frac{28}{3}$ C. $10$ D. $\frac{31}{3}$ E. $\frac{32}{3}$

MC(1)

## Practice 4.2

Triangle $OAB$ has $O=(0,0)$, $B=(5,0)$, and $A$ in the first quadrant. In addition, $\angle ABO = 90^\circ$ and $\angle AOB = 30^\circ$. Suppose that $OA$ is rotated $90^\circ$ counterclockwise about $O$. What are the coordinates of the image of $A$? (2008 AMC 10B Problems, Question #14)

A. $\left(-\frac{10}{3}\sqrt{3},5\right)$ B. $\left(-\frac{5}{3}\sqrt{3},5\right)$ C. $(\sqrt{3},5)$ D. $\left(\frac{5}{3}\sqrt{3},5\right)$ E. $\left(\frac{10}{3}\sqrt{3},5\right)$




$AB = \frac{OB}{\sqrt{3}} = \frac{5}{\sqrt{3}} = \frac{5}{3}\sqrt{3}$

$\Rightarrow A(5, \frac{5}{3}\sqrt{3})$

$\Rightarrow A'(-\frac{5}{3}\sqrt{3}, 5)$