2025 AMC10 必刷 20 题-毒舌版解析

前言： AMC10 是整个 AMC 系列考试中，考试题目难度和题目范围都最克制的一场。AMC10 相比于国际学科课程，知识在深度和广度上都有着较大的提升。可选择的题目和题目类型都相当的多。从传统数学竞赛的角度来看，一般会被分为代数，几何，数论和组合。我们在近几年考过的典型题目中，挑选了最有代表性的 20 个题，拆分到了四个模块，对于 2025 年需要准备 AMC10 的学生来说，这些近几年的典型题型是最合适的来准备 2025 年 AMC10 的材料。

板块一： Algebra

AMC10 里的代数内容涵盖了三角函数，复数，指数对数函数前的所有高中以前知识。涉及到重要的代数变化技巧，变形公式以及一系列高级数学问题所必备的代数基础，这样的基础不仅对于解决单独出现的代数类问题，典型的方程问题有帮助，还会悄悄出现在勾股定理，高阶数论问题中，是同学们必须掌握的基本内容。

题目 1：2023 AMC 12B Q14

Problem

For how many ordered pairs (a, b) of integers does the polynomial $x^3 + ax^2 + bx + 6$ have 3 distinct integer roots?

$\textbf{(A)}\ 5 \qquad \textbf{(B)}\ 6 \qquad \textbf{(C)}\ 8 \qquad \textbf{(D)}\ 7 \qquad \textbf{(E)}\ 4$

答案：A

Solution

Denote three roots as $r_1 < r_2 < r_3$. Following from Vieta's formula, $r_1 r_2 r_3 = -6$.

Case 1: All roots are negative.

We have the following solution: $(-3, -2, -1)$.

Case 2: One root is negative and two roots are positive.

We have the following solutions: $(-3, 1, 2)$, $(-2, 1, 3)$, $(-1, 2, 3)$, $(-1, 1, 6)$.

Putting all cases together, the total number of solutions is $\boxed{\textbf{(A) } 5}$.

Algebra C 位公式韦达定理 Vieta’ s Formula，如果 Algebra 部分只能背一个定理，那么这个定理就是一定是韦达定理，每年都会闭眼考。这题三个根之积为 6，分类讨论。

题目 2：2023 AMC12A Q12

Problem

What is the value of

$$2^3 - 1^3 + 4^3 - 3^3 + 6^3 - 5^3 + \cdots + 18^3 - 17^3?$$

$\textbf{(A) } 2023 \qquad \textbf{(B) } 2679 \qquad \textbf{(C) } 2941 \qquad \textbf{(D) } 3159 \qquad \textbf{(E) } 3235$

答案：D

Solution 1

To solve this problem, we will be using difference of cube, sum of squares and sum of arithmetic sequence formulas.

$$2^3 - 1^3 + 4^3 - 3^3 + 6^3 - 5^3 + \dots + 18^3 - 17^3$$

$= (2-1)(2^2 + 1\cdot 2 + 1^2) + (4-3)(4^2 + 4\cdot 3 + 3^2) + (6-5)(6^2 + 6\cdot 5 + 5^2) + \dots + (18-17)(18^2 + 18\cdot 17 + 17^2)$

$= (2^2 + 1\cdot 2 + 1^2) + (4^2 + 4\cdot 3 + 3^2) + (6^2 + 6\cdot 5 + 5^2) + \dots + (18^2 + 18\cdot 17 + 17^2)$

$= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 \dots + 17^2 + 18^2 + 1\cdot 2 + 3\cdot 4 + 5\cdot 6 + \dots + 17\cdot 18$

$= \frac{18(18+1)(36+1)}{6} + 1\cdot 2 + 3\cdot 4 + 5\cdot 6 + \dots + 17\cdot 18$

we could rewrite the second part as $\sum_{n=1}^{9}(2n-1)(2n)$

$(2n-1)(2n) = 4n^2 - 2n$

$\sum_{n=1}^{9} 4n^2 = 4(\frac{9(9+1)(18+1)}{6})$

$\sum_{n=1}^{9} -2n = -2(\frac{9(9+1)}{2})$

Hence,

$1\cdot 2 + 3\cdot 4 + 5\cdot 6 + \dots + 17\cdot 18 = 4(\frac{9(9+1)(18+1)}{6}) - 2(\frac{9(9+1)}{2})$

Adding everything up:

$2^3 - 1^3 + 4^3 - 3^3 + 6^3 - 5^3 + \dots + 18^3 - 17^3$

$= \frac{18(18+1)(36+1)}{6} + 4(\frac{9(9+1)(18+1)}{6}) - 2(\frac{9(9+1)}{2})$

$= 3(19)(37) + 6(10)(19) - 9(10)$

$= 2109 + 1140 - 90$

$= \boxed{\textbf{(D) } 3159}$

立方差公式加上平方求和，算数列 Series 再也不是套公式这么简单了。

题目 3：2024 AMC12A Q9 / AMC10A Q15

Problem

Let M be the greatest integer such that both $M + 1213$ and $M + 3773$ are perfect squares. What is the units digit of M?

$\textbf{(A) } 1 \qquad \textbf{(B) } 2 \qquad \textbf{(C) } 3 \qquad \textbf{(D) } 6 \qquad \textbf{(E) } 8$

答案：E

Solution 1

Let $M+1213=P^2$ and $M+3773=Q^2$ for some positive integers P and Q. We subtract the first equation from the second, then apply the difference of squares:

$$(Q+P)(Q-P)=2560.$$

Note that $Q+P$ and $Q-P$ have the same parity, and $Q+P>Q-P$.

We wish to maximize both P and Q, so we maximize $Q+P$ and minimize $Q-P$. It follows that

$$Q+P=1280,$$
$$Q-P=2,$$

from which $(P,Q)=(639,641)$.

Finally, we get $M=P^2-1213=Q^2-3773\equiv 1-3\equiv 8 \pmod{10}$, so the units digit of M is $\boxed{\textbf{(E) } 8}$.

Perfect Square (完全平方式)，一个每个同学都懂但是又不太会用的知识点。比如它和它的 Number Theory 搭子一同出现的时候。这题用到了奇偶性（Parity)。

题目 4：2023 AMC12B Q9/ AMC10B Q13

Problem

What is the area of the region in the coordinate plane defined by

$||x|-1|+||y|-1|\le 1$?

$\textbf{(A)}\ 2 \qquad \textbf{(B)}\ 8 \qquad \textbf{(C)}\ 4 \qquad \textbf{(D)}\ 15 \qquad \textbf{(E)}\ 12$

答案： B

Solution 1

First consider, $|x-1|+|y-1|\le 1$. We can see that it is a square with a side length of $\sqrt{2}$ (diagonal 2). The area of the square is $\sqrt{2}^2=2$.

Next, we insert an absolute value sign into the equation and get $|x-1|+||y|-1|\le 1$. This will double the square reflecting over x-axis.

So now we have 2 squares.

Finally, we add one more absolute value and obtain $||x|-1|+||y|-1|\le 1$. This will double the squares as we reflect the 2 squares we already have over the y-axis.

Concluding, we have 4 congruent squares. Thus, the total area is $4\cdot 2=\boxed{\text{(B) } 8}$

~Technodoggo ~Minor formatting change: e_is_2.71828, mathkiddus ~Grammar and clarity: NSAoPS j

Solution 2 (Graphing)

We first consider the lattice points that satisfy $||x|-1|=0$ and $||y|-1|=1$. The lattice points satisfying these equations are $(1,0)$, $(1,2)$, $(1,-2)$, $(-1,0)$, $(-1,2)$, and $(-1,-2)$. By symmetry, we also have points $(0,1)$, $(2,1)$, $(-2,1)$, $(0,-1)$, $(2,-1)$, and $(-2,-1)$ when $||x|-1|=1$ and $||y|-1|=0$. Graphing and connecting these points, we form 5 squares. However, we can see that any point within the square in the middle does not satisfy the given inequality (take $(0,0)$, for instance). As noted in the above solution, each square has a diagonal 2 for an area of $\frac{2^2}{2}=2$, so the total area is $4\cdot 2=\boxed{\text{(B) } 8}$.

Absolute Function 搭配 Symmetry 使用，可以省略多种分情况讨论，做题速度 up up up!

题目 5：2024 AMC12A Q10

Problem

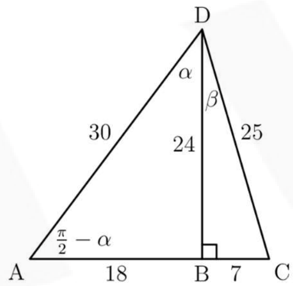
Let α be the radian measure of the smallest angle in a $3-4-5$ right triangle. Let β be the radian measure of the smallest angle in a $7-24-25$ right triangle. In terms of α, what is β?

$\textbf{(A)}\ \frac{\alpha}{3} \qquad \textbf{(B)}\ \alpha - \frac{\pi}{8} \qquad \textbf{(C)}\ \frac{\pi}{2} - 2\alpha \qquad \textbf{(D)}\ \frac{\alpha}{2} \qquad \textbf{(E)}\ \pi - 4\alpha$

答案：C

Solution 2: Scaling and combining triangles

We can scale the 3-4-5 triangle up by a factor of 6 to make its side lengths 18, 24, and 30, then glue its side of length 24 to the corresponding side in the 7-24-25 triangle:



Angles $\angle DAB$ and $\angle BDA$ are complementary in $\triangle ABD$, so $\angle DAB = \frac{\pi}{2} - \alpha$. We also have $AC = 18 + 7 = 25 = CD$, so $\triangle ACD$ is isosceles. That means that its base angles $\angle CDA$ and $\angle CAD$ are congruent, so $\alpha + \beta = \frac{\pi}{2} - \alpha$, and hence $\beta = \boxed{\textbf{(C)}\ \frac{\pi}{2} - 2\alpha}$.

三角函数 Trigonometry 倍角公式，和差化积还记得吗？AMC 考试可是不能带 Formula Sheet 的，如果实在不行，画个图也不是不能解决。

题目 6：2023 AMC10B Q22

Problem

How many distinct values of x satisfy $\lfloor x \rfloor^2 - 3x + 2 = 0$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x?

$\textbf{(A)}\ \text{an infinite number} \qquad \textbf{(B)}\ 4 \qquad \textbf{(C)}\ 2 \qquad \textbf{(D)}\ 3 \qquad \textbf{(E)}\ 0$

答案：B

Solution 1

To further grasp at this equation, we rearrange the equation into

$$\lfloor x \rfloor^2 = 3x - 2.$$

Thus, $3x-2$ is a perfect square and nonnegative. It is now much more apparent that $x \geq 2/3$, and that $x = 2/3$ is a solution.

Additionally, by observing the RHS, $x < 4$, as

$$\lfloor 4 \rfloor^2 > 3 \cdot 4,$$

since squares grow quicker than linear functions.

Now that we have narrowed down our search, we can simply test for intervals $[2/3, 1], [1, 2], [2, 3], [3, 4)$. This intuition to use intervals stems from the fact that $x = 1, 2$ are observable integral solutions.

Notice how there is only one solution per interval, as $3x-2$ increases while $\lfloor x \rfloor^2$ stays the same.

Finally, we see that $x = 3$ does not work, however, through setting $\lfloor x \rfloor^2 = 9$, $x = 11/3$ is a solution and within our domain of $[3, 4)$.

This provides us with solutions $\left(\frac{2}{3}, 1, 2, \frac{11}{3}\right)$, thus the final answer is $\boxed{\text{(B) } 4}$.

高斯函数之取整函数是近几年 AMC 考察大热门，抓紧安排上。利用 $\lfloor x \rfloor \leq x < \lfloor x+1 \rfloor$，得到两个关于$\lfloor x \rfloor$的一元二次不等式，求出$\lfloor x \rfloor$范围，反向带回有 4 个解。千万不要闭眼蒙（C）2，一看出来的解肯定少。

题目 7：2021 AMC10B Q15

Problem

The real number x satisfies the equation $x + \frac{1}{x} = \sqrt{5}$. What is the value of $x^{11} - 7x^7 + x^3$?

$\textbf{(A)}\ -1 \qquad \textbf{(B)}\ 0 \qquad \textbf{(C)}\ 1 \qquad \textbf{(D)}\ 2 \qquad \textbf{(E)}\ \sqrt{5}$

答案：B

Solution 1

We square $x + \frac{1}{x} = \sqrt{5}$ to get $x^2 + 2 + \frac{1}{x^2} = 5$. We subtract 2 on both sides for $x^2 + \frac{1}{x^2} = 3$ and square again, and see that $x^4 + 2 + \frac{1}{x^4} = 9$ so $x^4 + \frac{1}{x^4} = 7$. We can factor out x^7 from our original expression of $x^{11} - 7x^7 + x^3$ to get that it is equal to $x^7(x^4 - 7 + \frac{1}{x^4})$. Therefore because $x^4 + \frac{1}{x^4}$ is 7, it is equal to $x^7(0) = \boxed{\textbf{(B)}\ 0}$.

Solution 2

Multiplying both sides by x and using the quadratic formula, we get $\frac{\sqrt{5} \pm 1}{2}$. We can assume that it is $\frac{\sqrt{5}+1}{2}$, and notice that this is also a solution the equation $x^2 - x - 1 = 0$, i.e. we have $x^2 = x + 1$. Repeatedly using this on the given (you can also just note Fibonacci numbers),

$$\begin{aligned}
(x^{11}) - 7x^7 + x^3 &= (x^{10} + x^9) - 7x^7 + x^3 \\
&= (2x^9 + x^8) - 7x^7 + x^3 \\
&= (3x^8 + 2x^7) - 7x^7 + x^3 \\
&= (3x^8 - 5x^7) + x^3 \\
&= (-2x^7 + 3x^6) + x^3 \\
&= (x^6 - 2x^5) + x^3 \\
&= (-x^5 + x^4 + x^3) \\
&= -x^3(x^2 - x - 1) = \boxed{\textbf{(B)}\ 0}
\end{aligned}$$

提取 x^7,剩下的交给完全平方式（完全平方式：是谁又在 Call 我？ ）

题目 8：2021 AMC 12A Q18/ AMC 10A Q18

Problem

Let f be a function defined on the set of positive rational numbers with the property that $f(a \cdot b) = f(a) + f(b)$ for all positive rational numbers a and b. Suppose that f also has the property that $f(p) = p$ for every prime number p. For which of the following numbers x is $f(x) < 0$?

$\textbf{(A)}\ \frac{17}{32} \qquad \textbf{(B)}\ \frac{11}{16} \qquad \textbf{(C)}\ \frac{7}{9} \qquad \textbf{(D)}\ \frac{7}{6} \qquad \textbf{(E)}\ \frac{25}{11}$

答案：E

Solution 1 (Intuitive)

From the answer choices, note that

$$\begin{aligned} f(25) &= f\left(\frac{25}{11} \cdot 11\right) \\ &= f\left(\frac{25}{11}\right) + f(11) \\ &= f\left(\frac{25}{11}\right) + 11. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} f(25) &= f(5 \cdot 5) \\ &= f(5) + f(5) \\ &= 5 + 5 \\ &= 10. \end{aligned}$$

Equating the expressions for $f(25)$ produces

$$f\left(\frac{25}{11}\right) + 11 = 10,$$

from which $f\left(\frac{25}{11}\right) = -1$. Therefore, the answer is $\boxed{\textbf{(E)}\ \frac{25}{11}}$.

AMC 特爱考的抽象函数，找几个数字写一写，马上会发现这个函数不抽象了。比如 f(1) = 0, f(25)=2f(5)=10, f(1) = f(11) +f(1/11)，答案就自取吧。

题目 9：2021 AMC12A Q16/AMC10A Q16

Problem

In the following list of numbers, the integer n appears n times in the list for $1 \leq n \leq 200$.

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots, 200, 200, \ldots, 200$$

What is the median of the numbers in this list?

$\textbf{(A)}\ 100.5 \qquad \textbf{(B)}\ 134 \qquad \textbf{(C)}\ 142 \qquad \textbf{(D)}\ 150.5 \qquad \textbf{(E)}\ 167$

答案：C

Solution 1

There are $1+2+..+199+200=\frac{(200)(201)}{2}=20100$ numbers in total. Let the median be k. We want to find the median k such that

$$\frac{k(k+1)}{2}=20100/2,$$

or

$$k(k+1)=20100.$$

Note that $\sqrt{20100}\approx 142$. Plugging this value in as k gives

$$\frac{1}{2}(142)(143)=10153.$$

$10153-142<10050$, so 142 is the 152nd and 153rd numbers, and hence, our desired answer. $\boxed{\text{(C) } 142}$.

统计值常考的有 median,mean 和 mode。这题带一点点等差数列求和，一点点掐指一算，最重要的是 AMC 不能带计算器（敲黑板！！！）。

板块二：Number Theory

数论是很完整的竞赛知识体系，且这些知识互相关联穿插。一方面，数论内容如果没有进行专门的，深入的学习，很多知识在标准的校内课程里学不到；另一方面，数论内容几乎不会单独考察某一方面，通常是综合进行数论的考察。所以，对于数论部分的处理应该是综合的。

同学们需要注意的是，我们应当把数论的知识体系作为一个整体，理解它们之间的相互联系。

题目 10：2024 AMC12A Q12/ AMC10A Q19

Problem

The first three terms of a geometric sequence are the integers $a,\ 720,$ and $b,$ where $a<720<b.$ What is the sum of the digits of the least possible value of $b?$

$\textbf{(A) } 9 \qquad \textbf{(B) } 12 \qquad \textbf{(C) } 16 \qquad \textbf{(D) } 18 \qquad \textbf{(E) } 21$

答案：E

Solution 1

For a geometric sequence, we have $ab=720^2=2^8 3^4 5^2$, and we can test values for b. We find that $b=768$ and $a=675$ works, and we can test multiples of 5 in between the two values. Finding that none of the multiples of 5 divide 720^2 besides 720 itself, we know that the answer is $7+6+8=\boxed{\textbf{(E) } 21}$.

(Note: To find the value of b without bashing, we can observe that $2^8=256$, and that multiplying it by 3 gives us 768, which is really close to 720. ~ YTH)

Note: The reason why $ab=720^2$ is because $b/720=720/a$. Rearranging this gives $ab=720^2$

~eevee9406

Note: Another reason that $ab=720^2$ is because the $\sqrt{ab}=720$ (as the middle term in a geometric series is always the geometric mean [the geometric mean is the square root of the product of the first and last terms of the series]) and squaring on both sides results in $ab=720^2$.

~ThatPrimePunnyGuy

Note: Because it is a geometric sequence, it is clear that $a*k=720$ and $720*k=b$. Then, $a*k=720=b/k$. Multiply the first and third term together and it is equal to the middle term multiplied by itself.

~RON-WEASLEY

Solution 2

We have $720=2^4\cdot 3^2\cdot 5$. We want to find factors x and y where $y>x$ such that $\frac{y}{x}$ is minimized, as $720\cdot\frac{y}{x}$ will then be the least possible value of b. After experimenting, we see this is achieved when $y=16$ and $x=15$, which means our value of b is $720\cdot\frac{16}{15}=768$, so our sum is $7+6+8=\boxed{\textbf{(E) } 21}$.

等比数列 Geometric Series 不是重点，重点是整除性 Divisibility。这题坑在于（1）等比数列的公比不是整数。（2）需要稍微试一下数字（符合 AMC 一贯特性）。

题目 11：2024 AMC12B Q14/AMC10B Q18

Problem

How many different remainders can result when the 100th power of an integer is divided by 125?

$\textbf{(A) } 1 \qquad \textbf{(B) } 2 \qquad \textbf{(C) } 5 \qquad \textbf{(D) } 25 \qquad \textbf{(E) } 125$

答案：B

Solution 1

First note that the Euler's totient function of 125 is 100. We can set up two cases, which depend on whether a number is relatively prime to 125.

If n is relatively prime to 125, then $n^{100} \equiv 1 \pmod{125}$ because of Euler's Totient Theorem.

If n is not relatively prime to 125, it must be have a factor of 5. Express n as $5m$, where m is some integer. Then $n^{100} \equiv (5m)^{100} \equiv 5^{100} \cdot m^{100} \equiv 125 \cdot 5^{97} \cdot m^{100} \equiv 0 \pmod{125}$.

Therefore, n^{100} can only be congruent to 0 or $1 \pmod{125}$. Our answer is $\boxed{2}$.

欧拉定理 Euler Totient Theorem 听说过吗？ 没有的话没关系，二项式定理 Binomial Theorem 来救场。

题目 12：2023 AMC12B Q15/AMC10B Q18

Problem

Suppose a, b, and c are positive integers such that

$$\frac{a}{14}+\frac{b}{15}=\frac{c}{210}.$$

Which of the following statements are necessarily true?

I. If $\gcd(a,14)=1$ or $\gcd(b,15)=1$ or both, then $\gcd(c,210)=1$.

II. If $\gcd(c,210)=1$, then $\gcd(a,14)=1$ or $\gcd(b,15)=1$ or both.

III. $\gcd(c,210)=1$ if and only if $\gcd(a,14)=\gcd(b,15)=1$.

$\textbf{(A) }\text{I, II, and III}\qquad\textbf{(B) }\text{I only}\qquad\textbf{(C) }\text{I and II only}\qquad\textbf{(D) }\text{III only}\qquad\textbf{(E) }\text{II and III only}$

答案： E

Solution 2

The equation given in the problem can be written as

$$15a + 14b = c. \qquad (1)$$

First, we prove that Statement I is not correct.

A counter example is $a = 1$ and $b = 3$. Thus, $\gcd(c, 210) = 3 \neq 1$.

Second, we prove that Statement III is correct.

First, we prove the ``if *part.*

Suppose $\gcd(a, 14) = 1$ and $\gcd(b, 15) = 1$. However, $\gcd(c, 210) \neq 1$.

Thus, c must be divisible by at least one factor of 210. W.L.O.G, we assume c is divisible by 2.

Modulo 2 on Equation (1), we get that $2|a$. This is a contradiction with the condition that $\gcd(a, 14) = 1$. Therefore, the ``if *part in Statement III is correct.*

Second, we prove the ``only if *part.*

Suppose $\gcd(c, 210) \neq 1$. Because $210 = 14 \cdot 15$, there must be one factor of 14 or 15 that divides c. W.L.O.G, we assume there is a factor $q > 1$ of 14 that divides c. Because $\gcd(14, 15) = 1$, we have $\gcd(q, 15) = 1$. Modulo q on Equation (1), we have $q|a$.

Because $q|14$, we have $\gcd(a, 14) \geq q > 1$.

Analogously, we can prove that $\gcd(b, 15) > 1$.

Third, we prove that Statement II is correct.

This is simply a special case of the ``only if *part of Statement III. So we omit the proof.*

All analyses above imply $\boxed{\textbf{(E) II and III only}}$.

常用技巧包括举反例(这题的 I）和举不出反例(这题的 II &III)。举不出反例的话从还是要深入理解 GCD 这个知识点。偷懒可以，每回偷懒可不行。

题目 13：2021 AMC10B Q13

Problem

Let n be a positive integer and d be a digit such that the value of the numeral $\underline{32d}$ in base n equals 263, and the value of the numeral $\underline{324}$ in base n equals the value of the numeral $\underline{11d1}$ in base six. What is $n + d?$

$\textbf{(A)}\ 10 \qquad \textbf{(B)}\ 11 \qquad \textbf{(C)}\ 13 \qquad \textbf{(D)}\ 15 \qquad \textbf{(E)}\ 16$

答案：B

Solution 1

We can start by setting up an equation to convert $\underline{32d}$ base n to base 10. To convert this to base 10, it would be $3n^2 + 2n + d$. Because it is equal to 263, we can set this equation to 263. Finally, subtract d from both sides to get $3n^2 + 2n = 263 - d$.

We can also set up equations to convert $\underline{324}$ base n and $\underline{11d1}$ base 6 to base 10. The equation to covert $\underline{324}$ base n to base 10 is $3n^2 + 2n + 4$. The equation to convert $\underline{11d1}$ base 6 to base 10 is $6^3 + 6^2 + 6d + 1$.

Simplify $6^3 + 6^2 + 6d + 1$ so it becomes $6d + 253$. Setting the above equations equal to each other, we have

$$3n^2 + 2n + 4 = 6d + 253.$$

Subtracting 4 from both sides gets $3n^2 + 2n = 6d + 249$.

We can then use equations

$$3n^2 + 2n = 263 - d$$

$$3n^2 + 2n = 6d + 249$$

to solve for d. Set $263 - d$ equal to $6d + 249$ and solve to find that $d = 2$.

Plug $d = 2$ back into the equation $3n^2 + 2n = 263 - d$. Subtract 261 from both sides to get your final equation of $3n^2 + 2n - 261 = 0$. We solve using the quadratic formula to find that the solutions are 9 and $-29/3$. Because the base must be positive, $n = 9$.

Adding 2 to 9 gets $\boxed{\textbf{(B)}\ 11}$

进制数的表示是 Number Theory 中的必会题。这题只要知道怎么表示 一个 base n 的数

怎么转化成 base 10,就包对的。

题目 14：2021 AMC12B Q7/AMC10B Q12

Let $N = 34 \cdot 34 \cdot 63 \cdot 270$. What is the ratio of the sum of the odd divisors of N to the sum of the even divisors of N?

$\textbf{(A)}\ 1:16 \qquad \textbf{(B)}\ 1:15 \qquad \textbf{(C)}\ 1:14 \qquad \textbf{(D)}\ 1:8 \qquad \textbf{(E)}\ 1:3$

答案：C

Solution 1

Prime factorize N to get $N = 2^3 \cdot 3^5 \cdot 5 \cdot 7 \cdot 17^2$. For each odd divisor n of N, there exist even divisors $2n, 4n, 8n$ of N, therefore the ratio is $1:(2+4+8) = \boxed{\textbf{(C)}\ 1:14}$

Solution 2

Prime factorizing N, we see $N = 2^3 \cdot 3^5 \cdot 5 \cdot 7 \cdot 17^2$. The sum of N's odd divisors are the sum of the factors of N without 2, and the sum of the even divisors is the sum of the odds subtracted by the total sum of divisors.

BY SUM OF FACTORS FORMULA (search if you don' know): The formula is actually for all factors, but we can just take out the 2^3, so we have:

The sum of odd divisors is given by

$$a = (1+3+3^2+3^3+3^4+3^5)(1+5)(1+7)(1+17+17^2)$$

and the total sum of divisors is

$$(1+2+4+8)(1+3+3^2+3^3+3^4+3^5)(1+5)(1+7)(1+17+17^2) = 15a.$$

Thus, our ratio is

$$\frac{a}{15a-a} = \frac{a}{14a} = \boxed{\textbf{(C)}\ 1:14}.$$

一个数的因子个数（Number of Divisors）大家都会，奇偶因子的数目之比大家会吗？那

奇偶因子的和之比呢？关键的公式还是要理解怎么来的。

板块三：Geometry

AMC10 里的几何可以理解为是平面几何的过关考试。因为在这个年龄阶段之后，大部分的

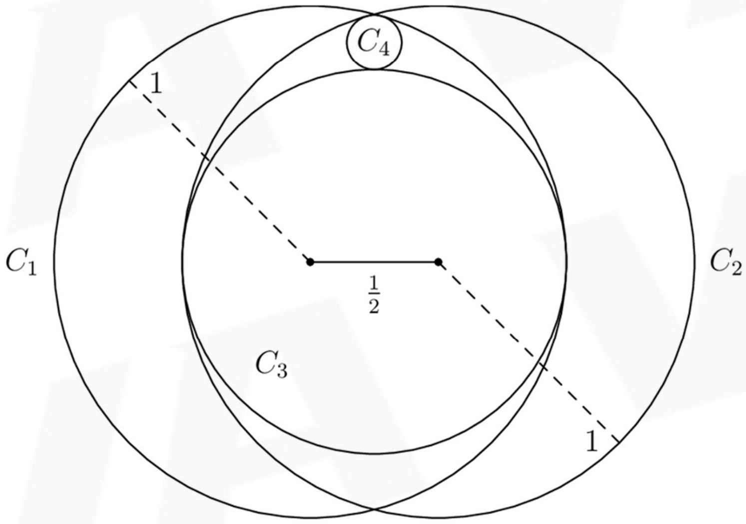
平面几何都会被转化为解析几何或和三角函数关联起来的解三角形等内容。所以 AMC10 的

考察重点，还在几何本身，无论是三角形的勾股定理，相似，圆和不规则图形的面积计算，甚至解析几何中，都在频繁考察几何性质和对图形的理解。另外，AMC10 对于几何的考察也相对较广，还会涉及到立体几何等大家不太熟悉的内容。

题目 15：2023 AMC12A Q18/ AMC10A Q22

Problem

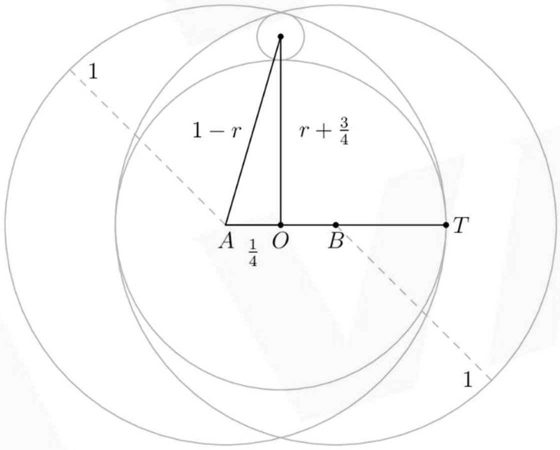
Circle C_1 and C_2 each have radius 1, and the distance between their centers is $\frac{1}{2}$. Circle C_3 is the largest circle internally tangent to both C_1 and C_2. Circle C_4 is internally tangent to both C_1 and C_2 and externall tangent to C_3. What is the radius of C_4?



$\textbf{(A)}\ \frac{1}{14} \qquad \textbf{(B)}\ \frac{1}{12} \qquad \textbf{(C)}\ \frac{1}{10} \qquad \textbf{(D)}\ \frac{3}{28} \qquad \textbf{(E)}\ \frac{1}{9}$

答案：D

Solution



Let O be the center of the midpoint of the line segment connecting both the centers, say A and B.

Let the point of tangency with the inscribed circle and the right larger circles be T.

Then $OT = BO + BT = BO + AT - \frac{1}{2} = \frac{1}{4} + 1 - \frac{1}{2} = \frac{3}{4}$.

Since C_4 is internally tangent to C_1, center of C_4, C_1 and their tangent point must be on the same line.

Now, if we connect centers of C_4, C_3 and C_1/C_2, we get a right angled triangle.

Let the radius of C_4 equal r. With the pythagorean theorem on our triangle, we have

$$\left(r+\frac{3}{4}\right)^2 + \left(\frac{1}{4}\right)^2 = (1-r)^2$$

Solving this equation gives us

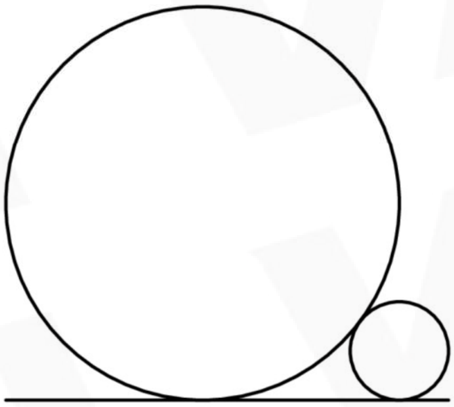
$$r = \boxed{\textbf{(D)}\ \frac{3}{28}}$$

图越复杂，题越简单（一般来说）。考察圆的位置关系，Pythagorean Theorem 勾股定理一步解决。

题目 16：2024AMC10B Q21

Problem

Two straight pipes (circular cylinders), with radii 1 and $\frac{1}{4}$, lie parallel and in contact on a flat floor. The figure below shows a head-on view. What is the sum of the possible radii of a third parallel pipe lying on the same floor and in contact with both?



$\textbf{(A)}\ \frac{1}{9} \qquad \textbf{(B)}\ 1 \qquad \textbf{(C)}\ \frac{10}{9} \qquad \textbf{(D)}\ \frac{11}{9} \qquad \textbf{(E)}\ \frac{19}{9}$

答案：C

Solution 1

Notice that the sum of radii of two circles tangent to each other will equal to the distance from center to center. Set the center of the big circle be at $(0,1)$. Since both circles are tangent to a line (in this case, $y=0$), the y-coordinates of the centers are just its radius.

Hence, the center of the smaller circle is at $\left(x_2, \frac{1}{4}\right)$. From the the sum of radii and distance formula, we have:

$$1+\frac{1}{4}=\sqrt{x_2^2+\left(\frac{3}{4}\right)^2} \Rightarrow x_2=1.$$

So, the coordinates of the center of the smaller circle are $(1, \frac{1}{4})$. Now, let (x_3, r_3) be the coordinates of the new circle. Then, from the fact that sum of radii of this circle and the circle with radius 1 is equal to the distance from the two centers, you have:

$$\sqrt{(x_3-0)^2+(r_3-1)^2}=1+r_3.$$

Similarly, from the fact that the sum of radii of this circle and the circle with radius $\frac{1}{4}$, you have:

$$\sqrt{(x_3-1)^2+\left(r_3-\frac{1}{4}\right)^2}=\frac{1}{4}+r_3.$$

Squaring the first equation, you have:

$$x_3^2+r_3^2-2r_3+1=1+2r_3+r_3^2 \Rightarrow 4r_3=x_3^2 \Rightarrow x_3=2\sqrt{r_3}.$$

Squaring the second equation, you have:

$$x_3^2-2x_3+1+r_3^2-\frac{r_3}{2}+\frac{1}{16}=\frac{1}{16}+\frac{r_3}{2}+r_3^2 \Rightarrow x_3^2-2x_3+1=r_3$$

Plugging in from the first equation we have

$$r_3-1=x_3^2-2x_3=4r_3-4\sqrt{r_3} \Rightarrow 3r_3-4\sqrt{r_3}+1=0 \Rightarrow (3\sqrt{r_3}-1)(\sqrt{r_3}-1)=0 \Rightarrow r_3=1, \frac{1}{9}.$$

Summing these two yields $\boxed{\frac{10}{9}}$.

这题为啥配 21 题，因为有两个符合条件的圆。考察圆和圆的外切时，那么必要操作就是圆心切点连连线，勾股定理安排上，一般来说，都能解决。遇到不一般的问题，那祝大家蒙的全对！

板块四：Probability

好消息：知识点比较少，

坏消息：题目不按常理出牌。

好消息：题不多，

坏消息：题有点难。

题目 17：2023 AMC12A Q7/ AMC10A Q9

Problem

A digital display shows the current date as an 8-digit integer consisting of a 4-digit year, followed by a 2-digit month, followed by a 2-digit date within the month. For example, Arbor Day this year is displayed as 20230428. For how many dates in 2023 will each digit appear an even number of times in the 8-digital display for that date?

$\textbf{(A) } 5 \qquad \textbf{(B) } 6 \qquad \textbf{(C) } 7 \qquad \textbf{(D) } 8 \qquad \textbf{(E) } 9$

答案： E

Solution 1 (Casework)

Do careful casework by each month. In the month and the date, we need a 0, a 3, and two digits repeated (which has to be 1 and 2 after consideration). After the casework, we get $\boxed{\textbf{(E) } 9}$. For curious readers, the numbers (in chronological order) are:

```
20230113
20230131
20230223
20230311
20230322
20231013
20231031
20231103
20231130
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分类讨论 Casework，没有感情，也没有技巧。看起来有 12 个月份，讨论起来很快，就是不要漏算了。

题目 18：2024 AMC12A Q16

Problem

A set of 12 tokens — 3 red, 2 white, 1 blue, and 6 black — is to be distributed at random to 3 game players, 4 tokens per player. The probability that some player gets all the red tokens, another gets all the white tokens, and the remaining player gets the blue token can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

$\textbf{(A) } 387 \qquad \textbf{(B) } 388 \qquad \textbf{(C) } 389 \qquad \textbf{(D) } 390 \qquad \textbf{(E) } 391$

答案： C

Solution 1A (Trivial/Easy solve)

We have $\binom{12}{4,4,4}$ ways to handle the red/white/blue balls distribution on the denominator. Now we simply $\binom{6}{1}\binom{5}{2}$ $3!$ for the numerator in order to handle the black balls and distinguishable persons. The solution is therefore $\frac{6\cdot 6\cdot 10}{70\cdot 45\cdot 11}=\frac{4}{385}$ or $4+385=\boxed{\textbf{(C) } 389}$.

Remarks - Notice we let balls and persons be distinguishable to increase ease of calculations

~polya_mouse

Solution 1B (12fact bash)

We have $12!$ total possible arrangements of 12 distinct tokens. If we imagine the first 4 tokens of our arrangement go to the first player, the next 4 go to the second, and the final 4 go to the third, then we can view this problem as counting the number of valid arrangements.

Firstly, the tokens are not all distinct, so we multiply by $3!$, $2!$, $1!$, and $6!$ to account for the fact that the red, white, blue, and black tokens, respectively can switch around from where they are.

Letting R denote red, W denote white, B denote blue, and L denote black, then our arrangement must be something like $RRRLWWLLBLLL$. The three players are arbitrary, so we multiply by $3!$; then, the player who gets the reds has $\binom{4}{1}=4$ possible arrangements, the player who gets the whites has $\binom{4}{2}=6$ possibilities, and the player who gets the blacks has $\binom{4}{3}=4$ possibilities. Our total on top is thus $3!\cdot 2!\cdot 1!\cdot 6!\cdot 3!\cdot 4\cdot 6\cdot 4$, and the denominator is $12!$. Firstly, we have the $6!$ in the numerator cancel out part of the denominator; we thus have the following:

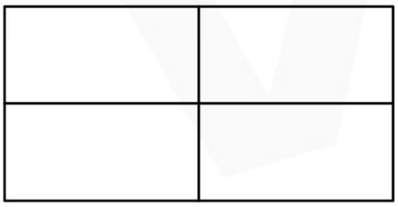
$$\frac{3\cdot 2\cdot 2\cdot 3\cdot 2\cdot 4\cdot 6\cdot 4}{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7}=\frac{2^8 3^3}{2^6 3^3 5\cdot 7\cdot 11}=\frac{4}{385}.$$

Our answer is $4+385=\boxed{\textbf{(C) } 389}$.

相同颜色的球不做区分，但是 Player 区分，Grouping 怎么用是一门学问。

题目 19：2021 Fall AMC 10A Q18

A farmer's rectangular field is partitioned into 2 by 2 grid of 4 rectangular sections as shown in the figure. In each section the farmer will plant one crop: corn, wheat, soybeans, or potatoes. The farmer does not want to grow corn and wheat in any two sections that share a border, and the farmer does not want to grow soybeans and potatoes in any two sections that share a border. Given these restrictions, in how many ways can the farmer choose crops to plant in each of the four sections of the field?



$\textbf{(A)}\ 12 \qquad \textbf{(B)}\ 64 \qquad \textbf{(C)}\ 84 \qquad \textbf{(D)}\ 90 \qquad \textbf{(E)}\ 144$

答案： C

Solution 1 (Casework)

There are 4 possibilities for the top-left section. It follows that the top-right and bottom-left sections each have 3 possibilities, so they have $3^2 = 9$ combinations. We have two cases:

1. **The top-right and bottom-left sections have the same crop.**

   Note that 3 of the 9 combinations of the top-right and bottom-left sections satisfy this case, from which the bottom-right section has 3 possibilities. Therefore, there are $4 \cdot 3 \cdot 3 = 36$ ways in this case.

2. **The top-right and bottom-left sections have different crops.**

   Note that 6 of the 9 combinations of the top-right and bottom-left sections satisfy this case, from which the bottom-right section has 2 possibilities. Therefore, there are $4 \cdot 6 \cdot 2 = 48$ ways in this case.

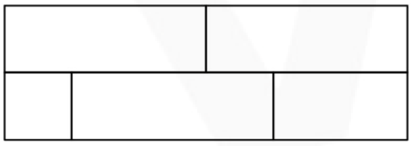
Together, the answer is $36 + 48 = \boxed{\textbf{(C)}\ 84}$.

看到这种方格子，不是农民伯伯要种庄稼了，就是粉刷匠要刷格子了-_-||。这题方法很多，主要是要考虑对角的庄稼是不是要种一样的，还要考虑死对头不能当邻居。可以分类讨论（Casework），也可以试试树形图（Tree diagram）。图就不画了，答案放出。

题目 20：2022 AMC12A Q7/AMC10A Q9

Problem

A rectangle is partitioned into 5 regions as shown. Each region is to be painted a solid color - red, orange, yellow, blue, or green - so that regions that touch are painted different colors, and colors can be used more than once. How many different colorings are possible?



$\textbf{(A)}\ 120 \qquad \textbf{(B)}\ 270 \qquad \textbf{(C)}\ 360 \qquad \textbf{(D)}\ 540 \qquad \textbf{(E)}\ 720$

答案：D

Solution 1

The top left rectangle can be 5 possible colors. Then the bottom left region can only be 4 possible colors, and the bottom middle can only be 3 colors since it is next to the top left and bottom left. Similarly, we have 3 choices for the top right and 3 choices for the bottom right, which gives us a total of $5\cdot4\cdot3\cdot3\cdot3=\boxed{\textbf{(D)}\ 540}$.

~Txu

Solution 2 (Casework)

Case 1: All the rectangles are different colors. It would be $5!=120$ choices.

Case 2: Two rectangles that are the same color. Grouping these two rectangles as one gives us $5\cdot4\cdot3\cdot2=120$. But, you need to multiply this number by three because the same-colored rectangles can be chosen at the top left and bottom right, the top right and bottom left, or the bottom right and bottom left, which gives us a grand total of 360.

Case 3: We have two sets of rectangles chosen from these choices (top right & bottom left, top left & bottom right) that have the same color. However, the choice of the bottom left and bottom right does not work for this case, as the second pair would be chosen from two touching rectangles. Again, grouping the same-colored rectangles gives us $5\cdot4\cdot3=60$.

Therefore, we have $120+360+60=\boxed{\textbf{(D)}\ 540}$.

论在不规则方格上之熟练的刷涂料。先从最麻烦的格子开始搞起（下面中间那个格子），剩下的问题就是斜对角的两个格子要不要刷一样颜色的。排列组合中最基本的乘法原理和加法原理要灵活应用。