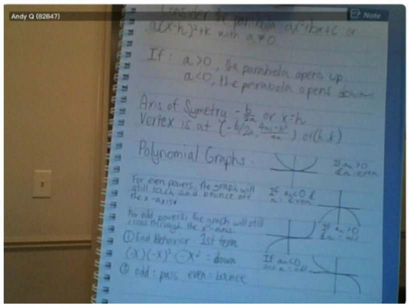
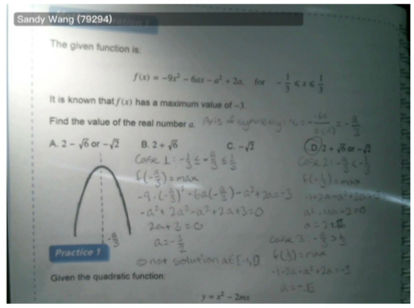
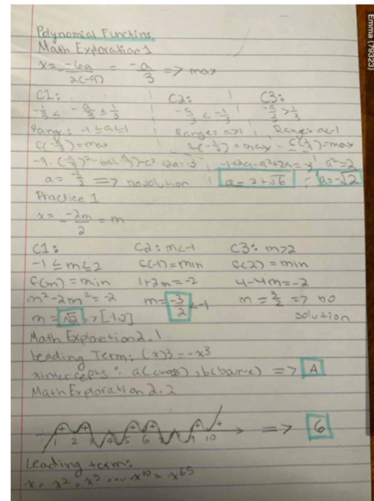
Excellent Notes



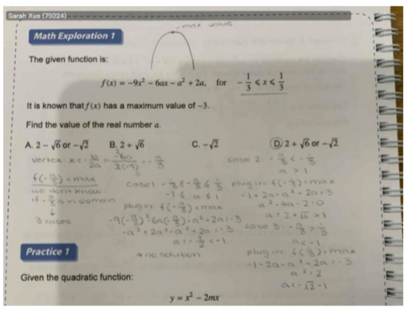
Andy



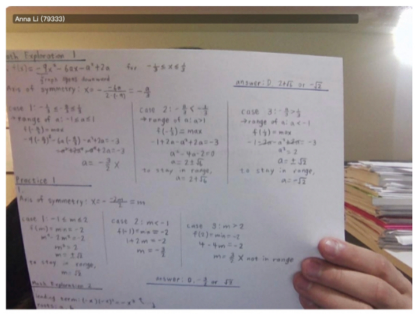
Sandy



Emma



Sarah



Anna

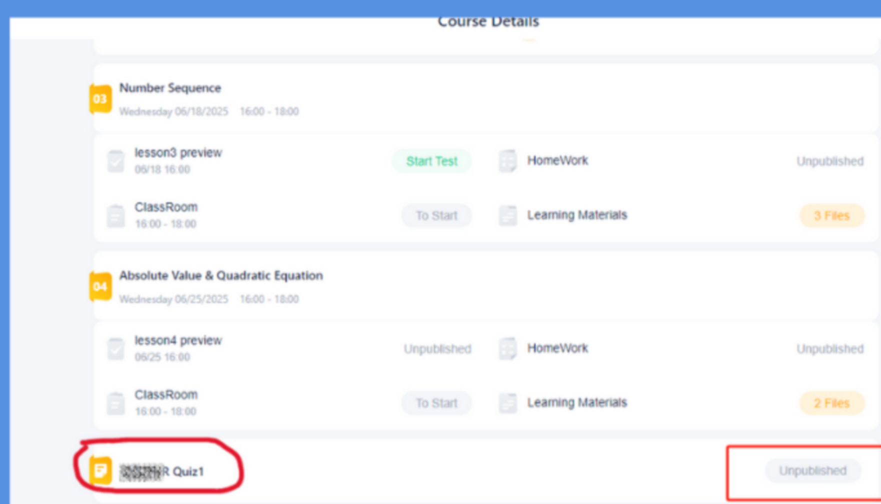
---

0 4 : 2 8 ×

AMC10 Mock Exam 1

Exam link and instruction in Learning Materials (Lesson 4)

75 mins, 25 questions



04:19

ANNOUNCEMENT:

We ~~DO~~ have classes next week
Do not

Wishing you an early Happy July 4th!

L4 Floor and Ceiling Functions

Concept 1

Applications of Diophantine Equations

Review: Common Ways to Solve Diophantine Equations

In a diophantine equation $\underline{A + B = C}$, if A and C are both divisible by an integer k, then B must be divisible by k.

$x + 3y = 21$

In a diophantine equation $\underline{A} \cdot \underline{B} = C$, A and B must be complementary factors of C

$xy = 24$

1	24
2	12
3	8
4	6
6	4
8	3

Review: Factorization Formulas

1. Simon's favorite Factoring Tricks: $xy+ax+by+ab=(x+b)(y+a)$

2. Multiplication Formula with 3 variables:
$abc-ab-ac-bc+a+b+c-1=(a-1)(b-1)(c-1)$

3. Sum of Two Cubes Pattern: $a^3+b^3=(a+b)(a^2-ab+b^2)$,

4. Difference of Two Cubes Pattern: $a^3-b^3=(a-b)(a^2+ab+b^2)$

5. Cube of a Binomial Pattern: $a^3+3a^2b+3ab^2+b^3=(a+b)^3$
$a^3-3a^2b+3ab^2-b^3=(a-b)^3$

PhotoPost

Math Exploration 1.1

Find integers p, q, r such that $p+q+r=26$, $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}+\frac{360}{pqr}=1$.

Multiply by $pqr \Rightarrow pr+pq+qr+360=pqr$

$$360=pqr-(pr+pq+qr)$$

$$360+26-1=pqr-(pr+pq+qr)+(p+q+r)-1$$

$$385=\underbrace{(p-1)(q-1)(r-1)}_{\text{Integers}}$$

$385=5\times7\times11$

$$\Rightarrow \begin{cases} p-1=5 \\ q-1=7 \\ r-1=11 \end{cases} \Rightarrow (6, 8, 12)$$

PhotoPost

Practice 1.1

How many positive integer solutions are there of $\frac{1}{x}+\frac{1}{y}=\frac{1}{2002}$?

Multiply by $2002xy \Rightarrow 2002x+2002y=xy$

$$xy-2002x-2002y=0$$

$$x(y-2002)-2002y=0$$

$$x(y-2002)-2002y-2002\cdot(-2002)=2002^2$$

$$x(y-2002)-2002(y-2002)=2002^2$$

$$\underbrace{(x-2002)}_{\geq -2001}\underbrace{(y-2002)}_{\geq -2001}=2002^2$$

$(-2001)(-2001)=2001^2<2002^2$

number of factors: $(2+1)^4=\boxed{81}$

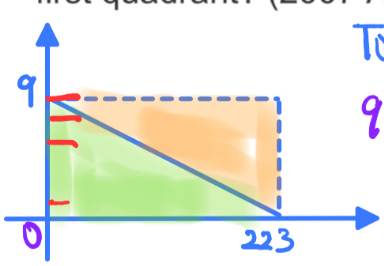
$\Rightarrow (x-2002),\ (y-2002)$ must be positive.

$$(2002)^2=(2\times7\times11\times13)^2$$
$$=2^2\times7^2\times11^2\times13^2$$

PhotoPost

Math Exploration 2.1

The graph of the equation $9x+223y=2007$ is drawn on graph paper with each square representing one unit in each direction. How many of the 1 by 1 graph paper squares have interiors lying entirely below the graph and entirely in the first quadrant? (2007 AIME II Problem, Question#5)

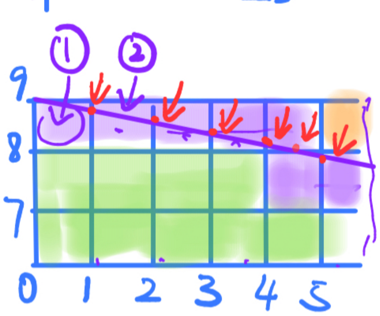


Total squares: $9\times223=2007$ squares

$9x+223y=2007$ has no positive integer solutions

$\Rightarrow$ squares on the line:

$$(9-1)+(223-1)+1=231$$



squares below: $\frac{2007-231}{2}=888$

PhotoPost

Practice 2.1

0 0 : 1 3 ×

The product N of three positive integers (a, b, c) is $\mathbf{6}$ times their sum, and one of the integers is the sum of the other two. Find the sum of all possible values of N. (2003 AIME II Problem, Question#1)

$$\begin{cases} abc = 6(a+b+c) \\ a = b+c \end{cases}$$

$$\Rightarrow (b+c)bc = 6(b+c+b+c)$$

$$\cancel{(b+c)}bc = 12\cancel{(b+c)} \Leftarrow b+c > 0$$

$$\Rightarrow bc = 12$$

① $\begin{cases} a=13 \\ b=1 \\ c=12 \end{cases}$ ② $\begin{cases} a=8 \\ b=2 \\ c=6 \end{cases}$ ③ $\begin{cases} a=7 \\ b=3 \\ c=4 \end{cases}$

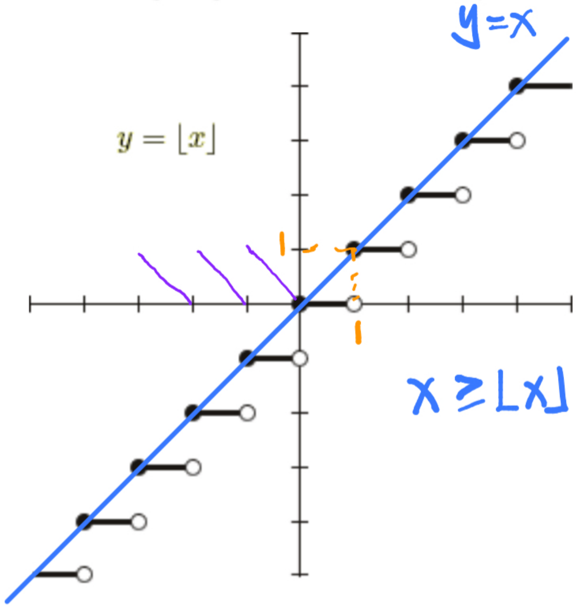
$\Rightarrow N = 156, 96, 84$ sum $= 336$

Concept 2

Definition and Properties of Floor and Ceiling Functions

Definition

The **floor function** is a function that takes the input as a real number x, and gives the output as the greatest integer less than or equal to x, denoted by floor(x) or $\lfloor x \rfloor$. Similarly, the **ceiling function** maps x to the least integer greater than or equal to x, denoted by ceil(x) or $\lceil x \rceil$



$x \geq \lfloor x \rfloor$

$\lfloor 2.3 \rfloor = 2$

$\lfloor 5 \rfloor = 5$

$\lfloor -1.2 \rfloor = -2$

Integer and Fraction Part

We hereby define the integer part, regardless of the sign of x, as the greatest integer not exceeding x, which is the same as the floor.

The **fractional part** of x is the sawtooth function, denoted by $\{x\}$ for real x and defined by the formula $\{x\} = x - \lfloor x \rfloor$.

$\lfloor 2.3 \rfloor = 2$

$2.3 = \underbrace{2}_{\lfloor x \rfloor} + \underbrace{0.3}_{\{x\}}$

Useful Properties

(1) $x = [x] + \{x\}$, where $0 \leq \{x\} < 1$;

(2) $[x] \leq x < [x] + 1$, and $x - 1 < [x] \leq x$;

(3) the domain of $f(x) = [x]$ is $\mathbb{R}$, and the range is $\mathbb{Z}$;

(4) the domain of $f(x) = \{x\}$ is $\mathbb{R}$, and the range is $[0, 1)$;

Useful Properties

(5) $[x + n] = n + [x]$, and $\{x + n\} = \{x\}$, where $x \in \mathbb{R}$ and $n \in \mathbb{Z}$;

(6) $[x + y] \geqslant [x] + [y]$, $\{x\} + \{y\} \geqslant \{x + y\}$,

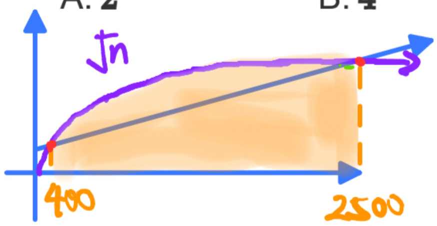
MC(1)

Math Exploration 3.1

How many positive integers n satisfy $\frac{n+1000}{70} = \lfloor\sqrt{n}\rfloor$? (Recall that $\lfloor x\rfloor$ is the greatest integer not exceeding x.) (2020 AMC 10B Problems, Question #24)

$\le \sqrt{n}$

A. 2 B. 4 C. 6 D. 30 E. 32



Intersections:

$\frac{n+1000}{70} = \sqrt{n}$

$n^2+2000n+1000000=4900n$

$n^2-2900n+1000000=0$

$(n-400)(n-2500)=0$

$n=400$ or 2500

⇒ 400 and 2500 satisfy the equation ($\sqrt{400}=20$ and $\sqrt{2500}=50$)

Also, $\frac{n+1000}{70}$ is an integer

⇒ Try 470: $21=\lfloor\sqrt{470}\rfloor$ ✓

Try 540: $22=\lfloor\sqrt{540}\rfloor$ ✗ ($\sqrt{540}=23\ldots$)

Try 2430: $49=\lfloor\sqrt{2430}\rfloor$ ✓

Try 2360, 2290, They work ✓ ✓

Try 2220, $46=\lfloor\sqrt{2220}\rfloor$ ✗

⇒ 6 solutions

MC(1)

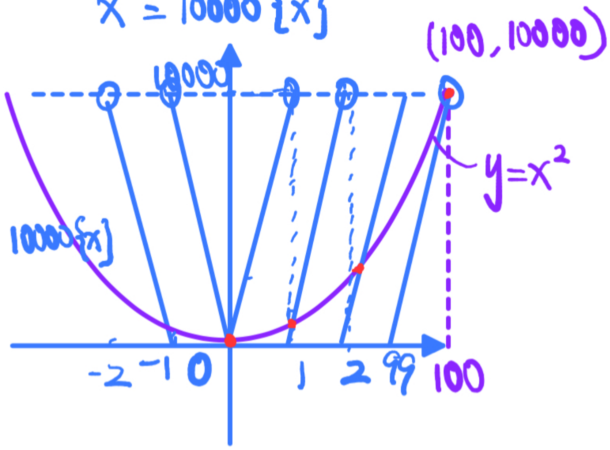
Math Exploration 3.2

Let $\lfloor x\rfloor$ denote the greatest integer less than or equal to x. How many real numbers x satisfy the equation $x^2 + 10,000\lfloor x\rfloor = 10,000x$? (2018 AMC 10B Problems, Question #25)

A. 197 B. 198 C. 199 D. 200 E. 201

⇒ $x^2 = 10000x - 10000\lfloor x\rfloor$

$x^2 = 10000\{x\}$



From $x=0\sim100$

⇒ 99 intersections.

$99\times2+1=199$

MC(1)

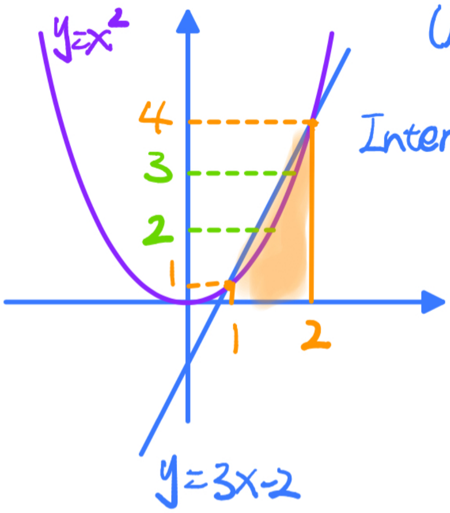
Practice 3.1

How many distinct values of x satisfy $\lfloor x \rfloor^2 - 3x + 2 = 0$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x? (2023 AMC 10B Problems, Question #22)

A. an infinite number   B. $\mathbf{4}$ C. $\mathbf{2}$ D. $\mathbf{3}$ E. $\mathbf{0}$

$\lfloor x \rfloor^2 = 3x-2 \Rightarrow \lfloor x \rfloor \leq x$

$(\lfloor x \rfloor)^2 \leq x^2$ if $x \geq 0$



Intersections: $x^2 = 3x-2$

$(x-1)(x-2) = 0$

$x = 1$ or 2

$\Rightarrow 4$ solutions

Useful Method # I

Does $\lfloor x \rfloor < \lfloor x + k \rfloor$?

① $0 \leq k < 1$, $\lfloor x+k \rfloor = \lfloor x \rfloor$ or $\lfloor x \rfloor + 1$

② $k \geq 1$, $\lfloor x+k \rfloor > \lfloor x \rfloor$

MC(1)

Math Exploration 4.1

For how many positive integers $n \leqslant 1000$ is $\lfloor \frac{998}{n} \rfloor + \lfloor \frac{999}{n} \rfloor + \lfloor \frac{1000}{n} \rfloor$ not divisible by 3? (Recall that $\lfloor x \rfloor$ is the greatest integer less than or equal to x)

$+\frac{1}{n}$ $+\frac{1}{n}$

$\cdots \frac{5}{7}\ \frac{6}{7}\ \boxed{\frac{7}{7}}$

A. 22 B. 23 C. 24 D. 25 E. 26

① $n=1$, $\lfloor \frac{998}{n} \rfloor + \lfloor \frac{999}{n} \rfloor + \lfloor \frac{1000}{n} \rfloor$

$\Rightarrow$ Expression $= 998+999+1000$, divisible by 3. ✗

② $n>1$ $(0<\frac{1}{n}<1)$: next floor function is the same or larger by 1.

Let $\lfloor \frac{998}{n} \rfloor = A$:

① $A+A+A$ ✗

② $A+(A+1)+(A+2)$ ✗

③ $A+(A+1)+(A+1)=3A+2$ ✓

④ $A+A+(A+1)=3A+1$ ✓

How: $\lfloor \boxed{\frac{999}{n}} \rfloor = \lfloor \frac{998}{n} \rfloor + 1$

if n is a factor of 999

Similarly, n is a factor of 1000

$999=3^3 \times 37$, $1000=2^3 \times 5^3$

$\Rightarrow 8+16=24$ factors

$24-1-1=22$

Useful Method #2

Find number of factors in a factorial.

Example:

What is the largest integer value of k so that $\frac{20!}{3^k}$ is an integer?

$20! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times \cdots \times 20$

3 3 3
3

$\Rightarrow \lfloor \frac{20}{3} \rfloor + \lfloor \frac{20}{3^2} \rfloor + \lfloor \frac{20}{3^3} \rfloor + \cdots$

$= 6+2+0 \cdots$

$= 8$

PhotoPost

Practice 4.1

Let $\boldsymbol{P}$ be the product of the first $\mathbf{100}$ positive odd integers. Find the largest integer $\boldsymbol{k}$ such that $\boldsymbol{P}$ is divisible by $\mathbf{3}^{\boldsymbol{k}}$. (2006 AIME II Problem, Question#3)

$$P = 1\times3\times5\times\dots\times199$$
$$= \frac{200!}{2\times4\times6\times\dots\times200}$$
$$= \frac{200!}{2^{100}(1\times2\times\dots\times100)}$$
$$= \frac{200!}{2^{100}\cdot100!}$$

200! has factors of 3:

$$\left\lfloor\frac{200}{3}\right\rfloor+\left\lfloor\frac{200}{9}\right\rfloor+\left\lfloor\frac{200}{27}\right\rfloor+\left\lfloor\frac{200}{81}\right\rfloor=97$$

100! has factors of 3:

$$\left\lfloor\frac{100}{3}\right\rfloor+\left\lfloor\frac{100}{9}\right\rfloor+\left\lfloor\frac{100}{27}\right\rfloor+\left\lfloor\frac{100}{81}\right\rfloor=48$$

$$\text{Total} = 97-48$$
$$= 49$$

Blank filling

Math Exploration 5.1

Let $\lfloor x\rfloor$ represent the greatest integer not exceeding x. For example, $\lfloor 2021\rfloor = 2021$ and $\lfloor 2.3\rfloor = 2$. If $\lfloor\frac{a}{2}\rfloor+\lfloor\frac{b}{3}\rfloor+\lfloor\frac{c}{4}\rfloor=6$, and a, b, and c are all integers greater than or equal to 4, then this equation has 120 solutions.

$\lfloor\frac{a}{2}\rfloor\ge2$, $\lfloor\frac{b}{3}\rfloor\ge1$, $\lfloor\frac{c}{4}\rfloor\ge1$

Possible cases:

$\lfloor\frac{a}{2}\rfloor = 4, 3, 3, 2, 2, 2$

$\lfloor\frac{b}{3}\rfloor = 1, 2, 1, 2, 3, 1$

$\lfloor\frac{c}{4}\rfloor = 1, 1, 2, 2, 1, 3$

① ① ①

For every value of $\lfloor\frac{a}{2}\rfloor$, a has 2 options.

① If $\lfloor\frac{b}{3}\rfloor\ge2$:

$3\times(2\times3\times4)=72$

② If $\lfloor\frac{b}{3}\rfloor=1$, $b=4$ or 5

$3\times(2\times2\times4)=48$

$\text{Total} = 72+48=120$

Blank filling

Math Exploration 5.2

There exists positive integers n and p such that $\{\frac{n}{2}\}+\{\frac{n}{4}\}+\{\frac{n}{6}\}+\{\frac{n}{p}\}=3$. When p is at its minimum, the number of n that $n \leqslant 2023$ is 168 . ($\lfloor x \rfloor$ denotes the largest integer that does not exceed x, $\{x\}=x-\lfloor x \rfloor$)

$\{\frac{n}{2}\}=0, \frac{1}{2}$

$\{\frac{n}{3}\}=0, \frac{1}{3}, \frac{2}{3}$

$\{\frac{n}{4}\}=0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}$

$\Rightarrow \{\frac{n}{p}\} \geq \frac{1}{2}+\frac{1}{3}+\frac{1}{4}$

$\geq \frac{11}{12}$

p is minimum $\Rightarrow p=12$

$\Rightarrow n=12k+11$ (k is integer ≥ 0)

$n \equiv 11 \pmod{12}$

$n \leq 2023$

$12k+11 \leq 2023$

$k \leq 167$

k is integer from $0 \sim 167$

$\Rightarrow$ 168 solutions

Blank filling

Practice 5.1

In $\lfloor\frac{1^2}{2017}\rfloor, \lfloor\frac{2^2}{2017}\rfloor, \lfloor\frac{3^2}{2017}\rfloor, \cdots, \lfloor\frac{2017^2}{2017}\rfloor$, there are 1513 distinct numbers. (Here, $\lfloor x \rfloor$ represents the greatest integer not exceeding x).

Find k so that:

$\frac{(k+1)^2}{2017}-\frac{k^2}{2017} \geq 1$

$2k+1 \geq 2017$

$k \geq 1008$

$\Rightarrow n=1009 \sim 2017$ $(\lfloor\frac{n^2}{2017}\rfloor)$

Each value is distinct.

$2017-1008=1009$ different values.

For $n=1,2,\ldots,1008$

Each value increases by at most 1

$\Rightarrow$ don't skip any integer $0 \sim \lfloor\frac{1008^2}{2017}\rfloor$

$\lfloor\frac{1008^2}{2017}\rfloor=\lfloor\frac{1008}{2017}\cdot 1008\rfloor$

$<\lfloor\frac{1008}{2016}\cdot 1008\rfloor$

$<\lfloor 504 \rfloor$

$\Rightarrow \lfloor\frac{1008^2}{2017}\rfloor=503$

$503+1=504$ distinct values

$\Rightarrow 1009+504=1513$