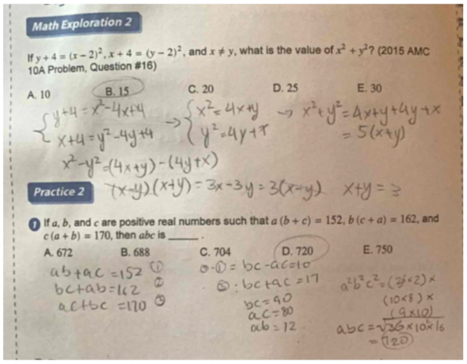
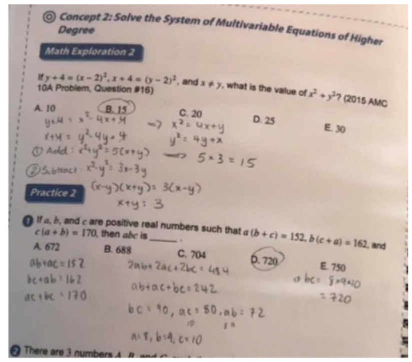
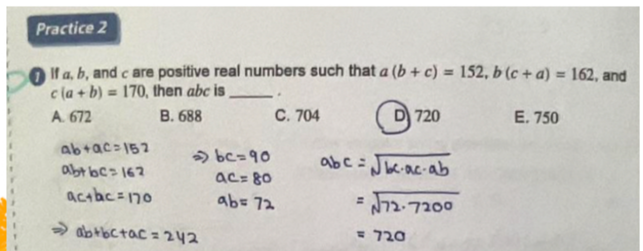
Excellent Notes



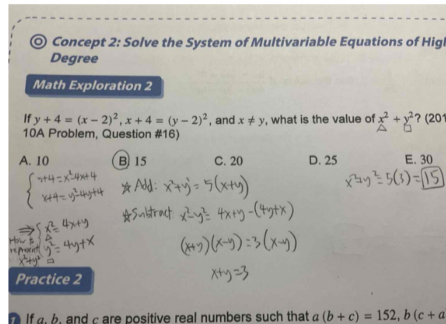
Sherry



Rachel



Alison



Lilian

Lesson 2
Equations and Functions

Concept 1

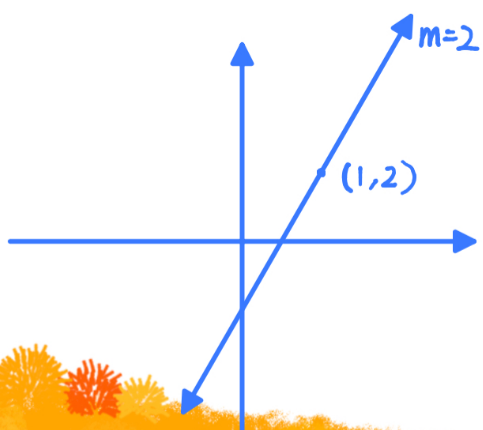
Geometric Applications of Linear Functions

Linear Function Reviews

Slope-intercept form: $y = mx + b$ ($m \neq 0, m, b$ are constant values).

Slope formula: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

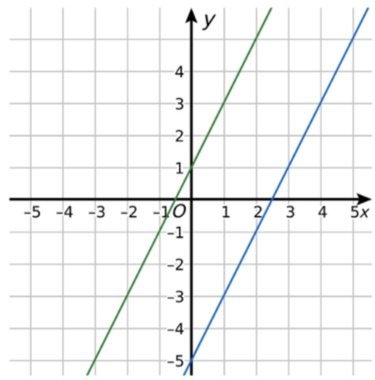
Point-slope form $y - y_1 = m(x - x_1)$.



$$y - 2 = \overset{m}{2}(x - 1)$$

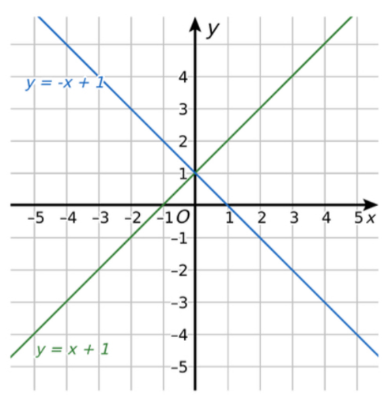
Parallel Lines

If two lines are parallel in a coordinate plane, their slopes are the same.



Perpendicular Lines

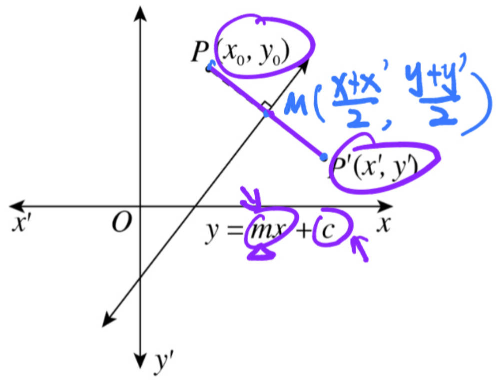
If two nonvertical lines are perpendicular in a coordinate plane, the product of their slopes is -1.



Reflecting a Point over a Line

Key Properties:

1. Midpoint
2. Perpendicular

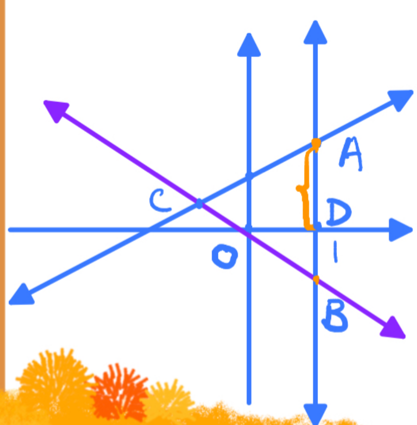


MC(1)

Math Exploration 1.1

A line that passes through the origin intersects both the line $x = 1$ and the line $y = 1 + \frac{\sqrt{3}}{3}x$. The three lines create an $\boxed{\text{equilateral triangle}}$. What is the perimeter of the triangle? (2015 AMC 10A Problem, Question#17)

A. $2\sqrt{6}$ B. $2 + 2\sqrt{3}$ C. 6 D. $3 + 2\sqrt{3}$ ✓ E. $6 + \frac{\sqrt{3}}{3}$



Point A: plug $x=1$, $y=1+\frac{\sqrt{3}}{3}$

$\Rightarrow AD = 1+\frac{\sqrt{3}}{3}$

In $\triangle BOD$, $m\angle DBO = 60°$, 30-60-90 Rt$\triangle$

$OD = 1$

$\Rightarrow BD = \frac{OD}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\Rightarrow AB = \left(1+\frac{\sqrt{3}}{3}\right)+\frac{\sqrt{3}}{3} = 1+\frac{2\sqrt{3}}{3}$

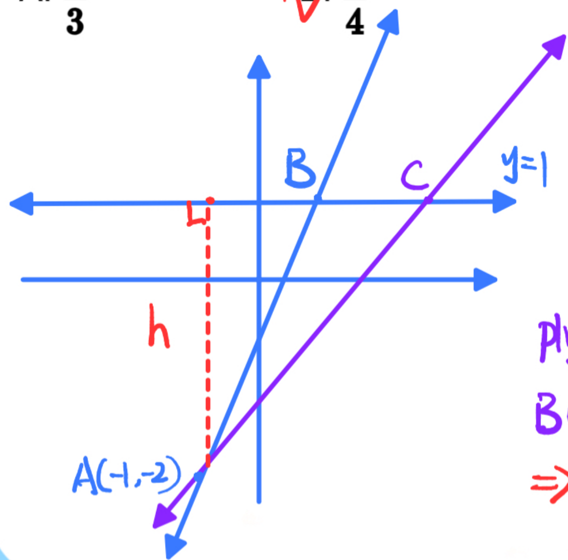
$P_{\triangle ABC} = 3AB = 3+2\sqrt{3}$

MC(1)

Practice 1.1

Line l_1 has equation $3x - 2y = 1$ and goes through $A(-1, -2)$. Line l_2 has equation $y = 1$ and meets line l_1 at point B. Line l_3 has positive slope, goes through point A, and meets l_2 at point C. The area of $\triangle ABC$ is 3. What is the slope of l_3?

A. $\frac{2}{3}$ B. $\frac{3}{4}$ C. 1 D. $\frac{4}{3}$ E. $\frac{3}{2}$



Height $= 1-(-2) = 3$

$A_{\triangle ABC} = \frac{1}{2} \cdot BC \cdot h = 3$

$BC = 2$

Find Point B:

Plug in $y=1$ to $3x-2y=1 \Rightarrow x=1$

$B(1,1)$

$1-2$

$\Rightarrow C$ is $(1+2, 1) = (3,1)$

$m = \frac{1-(-2)}{3-(-1)} = \frac{3}{4}$

Concept 2

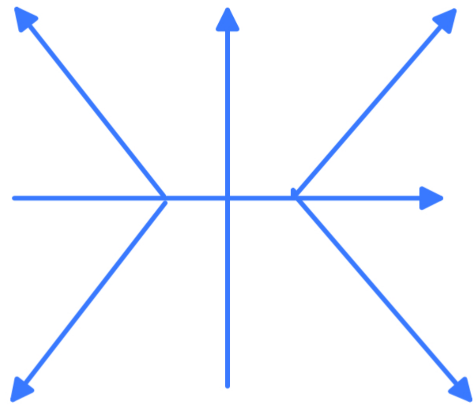
Solve Systems by Function Graphs

Basic Rules to Graph Functions with Absolute Values

Casework! $|x| - |y| = 1$

① $x \geq 0$ and $y \geq 0$: $x - y = 1$

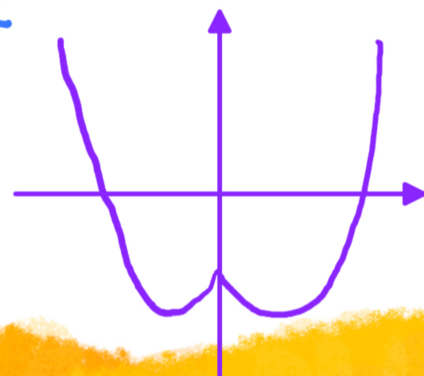
② $x \geq 0$ and $y < 0$: $x + y = 1$



Symmetry $y = x^2 - |x| - 3$

$y = |x|^2 - |x| - 3$

$f(x) = f(-x)$



MC(1)

Math Exploration 2.1

The area of the region bounded by the graph of

$x^2 + y^2 = 3|x - y| + 3|x + y|$ is $m + n\pi$, where m and n are integers. What is $m + n$? (2021 AMC Spring 10A, Question #19)

A. **18** $y \leq x$ B. **27** $y \geq -x$ C. **36** D. **45** E. **54**

① $x - y \geq 0$, $x + y \geq 0$

$x^2 + y^2 = 3(x - \cancel{y}) + 3(x + \cancel{y})$

$x^2 + y^2 = 6x$

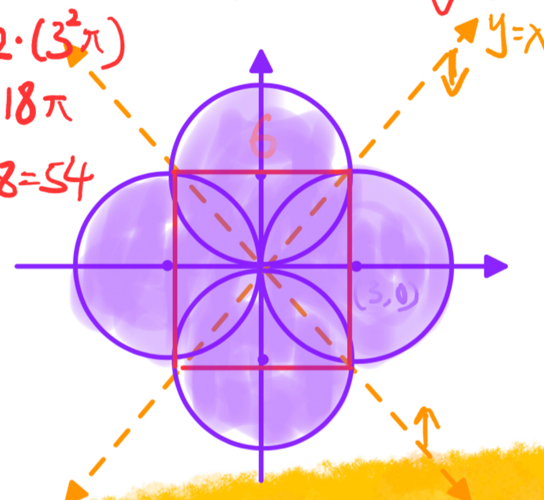
$x^2 - 6x + 9 + y^2 = 9$

$(x-3)^2 + y^2 = 3^2$

$\text{Area} = 6^2 + 2 \cdot (3^2\pi)$

$= 36 + 18\pi$

$\text{Ans} = 36 + 18 = 54$



MC(1)

Practice 2.1

How many ordered pairs of real numbers (x, y) satisfy the following system of equations? $x + 3y = 3, ||x| - |y|| = 1$. (From 2018 AMC 10A, Question #12)

A. 1 B. 2 C. 3 D. 4 E. 8

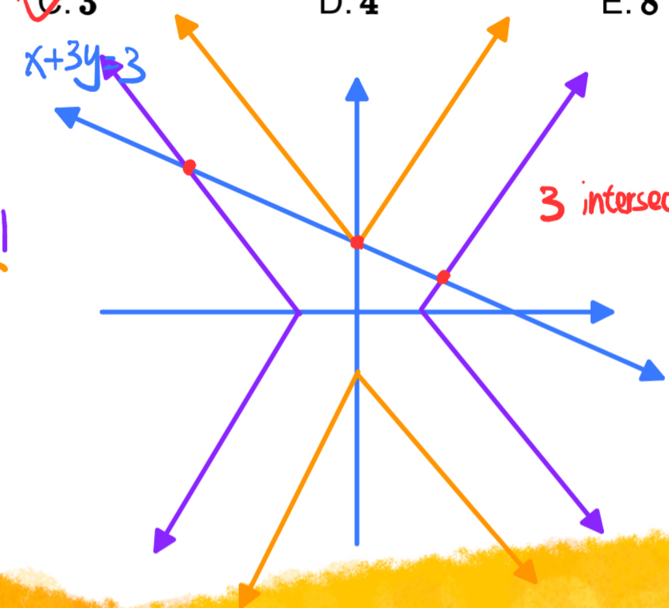
Draw graphs!

$||x|-|y||=1$

$\Rightarrow |x|-|y|=1$ $\boxed{\text{or}}$ $|x|-|y|=-1$

① $x\geq 0, y\geq 0$ $x-y=1$

① $x\geq 0, y\geq 0$ $x-y=-1$



3 intersections

MC(1)

Practice 2.2

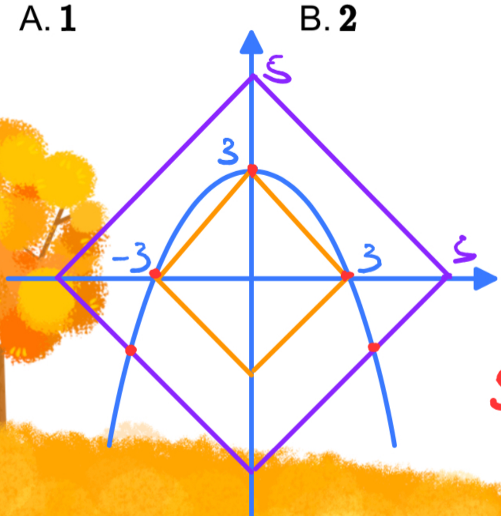
0 0 : 0 2

How many ordered pairs (x, y) of real numbers satisfy the following system of equations?

$x^2 + 3y = 9 \Rightarrow y = -\frac{1}{3}x^2+3$

$(|x| + |y| - 4)^2 = 1 \Rightarrow |x|+|y|-4=1$ or -1 $\Rightarrow$ $|x|+|y|=5$ or $|x|+|y|=3$

A. 1 B. 2 C. 3 D. 5 E. 7



① $x\geq 0, y\geq 0$: $x+y=5$

② $x\geq 0, y\geq 0$: $x+y=3$

5 intersections

Concept 3

Vieta's Formulas of Quadratic Equations

A **quadratic function** is a function that the greatest degree of any term is 2.

The **standard form of a quadratic function** is $y = ax^2 + bx + c$ or $f(x) = ax^2 + bx + c$, where a, b and c are real numbers and $a \neq 0$.

For any quadratic equations of the form $ax^2 + bx + c = 0$,

the **sum of roots** is $-\dfrac{b}{a}$,

the **product of roots** is $\dfrac{c}{a}$.

Blank filling

Math Exploration 3.1

Given that α and β are two distinct real roots of the quadratic equation $x^2+(2m+3)x+m^2=0$, and $\frac{1}{\alpha}+\frac{1}{\beta}=-1$, the value of m is 3.

$\Rightarrow \frac{\alpha+\beta}{\alpha\beta}=-1$

$\alpha+\beta=-\alpha\beta$

$-\frac{2m+3}{1}=-\frac{m^2}{1}$

$m^2-2m-3=0$

$(m-3)(m+1)=0$

$m=3$ or -1

Check discriminant:

① $m=3$, $x^2+9x+9=0$

$\Delta=9^2-4\cdot 9\cdot 1>0$

② $m=-1$, $x^2+x+1=0$

$\Delta=1^2-4\cdot 1\cdot 1=-3<0$

$\boxed{m=3}$

MC(1)

Math Exploration 3.2

$(-2)(x_2-2)$

The zeroes of the function $f(x)=x^2-ax+2a$ are integers. What is the sum of the possible values of a? (2015 AMC 10A Problems, Question #23)

$9+8+(-1)+0=16$

A. **7** B. **8** C. **16** ✓ D. **17** E. **18**

Let x_1, x_2 be zeros:

$\begin{cases} x_1+x_2=a \\ x_1x_2=2a \end{cases}$

$\Rightarrow x_1x_2=2(x_1+x_2)$

$x_1x_2-2x_1-2x_2=0$

$x_1(x_2-2)\boxed{-2}x_2-(-2)\cdot 2=4$

$x_1(x_2-2)-2(x_2-2)=4$

$(x_1-2)(x_2-2)=4$

Because x_1 and x_2 are integers:

(x_1-2) and (x_2-2) are factors of 4

① $x_1-2=4$, $x_2-2=1$

$\Rightarrow x_1=6$, $x_2=3 \Rightarrow a=9$

② $x_1-2=x_2-2=2$

$\Rightarrow x_1=4$, $x_2=4 \Rightarrow a=8$

③ $x_1-2=-4$, $x_2-2=-1$

$\Rightarrow x_1=-2$, $x_2=1 \Rightarrow a=-1$

④ $x_1-2=x_2-2=-2$

$x_1=x_2=0 \Rightarrow a=0$

MC(1)

Practice 3.1

Let a and b be the roots of the equation $x^2 - mx + \boxed{2} = 0$. Suppose that $a + (\frac{1}{b})$ and $b + (\frac{1}{a})$ are the roots of the equation $x^2 - px + q = 0$. What is q? (2006 AMC 10B Problem, Question#14)

A. $\frac{5}{2}$ B. $\frac{7}{2}$ C. 4 D. $\frac{9}{2}$ E. 8

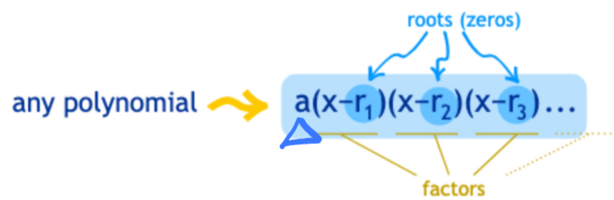
$$q = (a+\tfrac{1}{b})(b+\tfrac{1}{a})$$
$$= ab + 1 + 1 + \frac{1}{ab}$$
$$= 2 + 1 + 1 + \frac{1}{2}$$
$$= \frac{9}{2}$$

$ab = 2$

Concept 4

Vieta's Formulas on Higher Degree Equations

Fundamental Theorem of Algebra



$$2x^3+3x^2+4x+5=2(x-r_1)(x-r_2)(x-r_3)$$

Cubic Equation

Vieta's Formulas can also be applied to higher degree equations.

For a cubic equation $ax^3+bx^2+cx+d=0(a\neq 0)$, suppose that the three roots are x_1, x_2, and x_3, respectively. Then we have:

$$\begin{cases} x_1+x_2+x_3=-\frac{b}{a} \\ x_1x_2+x_1x_3+x_2x_3=\frac{c}{a} \\ x_1x_2x_3=-\frac{d}{a}. \end{cases}$$

$$ax^3+\boxed{b}x^2+\boxed{c}x+\boxed{d}=a(x-r_1)(x-r_2)(x-r_3)$$
$$=ax^3\boxed{-a(r_1+r_2+r_3)}x^2+\boxed{a(r_1r_2+r_1r_3+r_2r_3)}x\boxed{-ar_1r_2r_3}$$

Higher Degree Equation

Similar conclusions also apply for quartic (fourth-degree) or even higher degree equations:

For the equation $a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 = 0$, we have

$$\begin{cases} x_1 + x_2 + \cdots + x_{n-1} + x_n = -\frac{a_{n-1}}{a_n} \\ (x_1x_2 + x_1x_3 + \cdots + x_1x_n) + (x_2x_3 + x_2x_4 + \cdots + x_2x_n) + \cdots + x_{n-1}x_n = \frac{a_{n-2}}{a_n} \\ \vdots \\ x_1x_2\ldots x_n = (-1)^n \frac{a_0}{a_n}. \end{cases}$$

degree

MC(1)

Math Exploration 4.1

The polynomial $x^3 \boxed{-a}x^2 + bx - 2010$ has three positive integer roots. What is the smallest possible value of a? (2010 AMC 10A Problems, Question #21)

A. 78 B. 88 C. 98 D. 108 E. 118

$-(-a) = \text{Sum} = r_1 + r_2 + r_3$

$a = r_1 + r_2 + r_3$

$\text{Product} = -(-2010)$

$r_1 \cdot r_2 \cdot r_3 = 2010 = 2 \times 3 \times 5 \times 67$

$a = 67 + 5 + 6 = 78$

close to each other

MC(1)

Practice 4.1

All the roots of the polynomial $z^6 - 10z^5 + Az^4 + Bz^3 + Cz^2 + Dz + 16$ are positive integers, possibly repeated. What is the value of B? (2021 AMC Spring 10A, Question #14)

A. -88 B. -80 C. -64 D. -41 E. -40

Degree $=6$:

$$\begin{cases} r_1+r_2+r_3+r_4+r_5+r_6=10 \\ r_1\cdot r_2\cdot r_3\cdot r_4\cdot r_5\cdot r_6=16=2^4 \\ \text{All integers} \end{cases}$$

$\Rightarrow$ 4 roots are 2

2 roots are 1.

$\Rightarrow$ polynomial $=(z-2)^4(z-1)^2$

$= \cdots + (-88)z^3 + \cdots$

MC(1)

Practice 4.2

The roots of the polynomial $\boxed{10}x^3 - 39x^2 + 29x\boxed{-6}$ are the height, length, and width of a rectangular box (right rectangular prism). A new rectangular box is formed by lengthening each edge of the original box by 2 units. What is the volume of the new box? (2022 AMC 10A Problems, Question #16)

a b c

A. $\frac{24}{5}$ B. $\frac{42}{5}$ C. $\frac{81}{5}$ D. 30 E. 48

$V=(a+2)(b+2)(c+2)$

$=abc+2(ab+ac+bc)+4(a+b+c)+8$

$=-\left(\frac{-6}{10}\right)+2\cdot\frac{29}{10}+4\cdot\frac{39}{10}+8$

$=30$