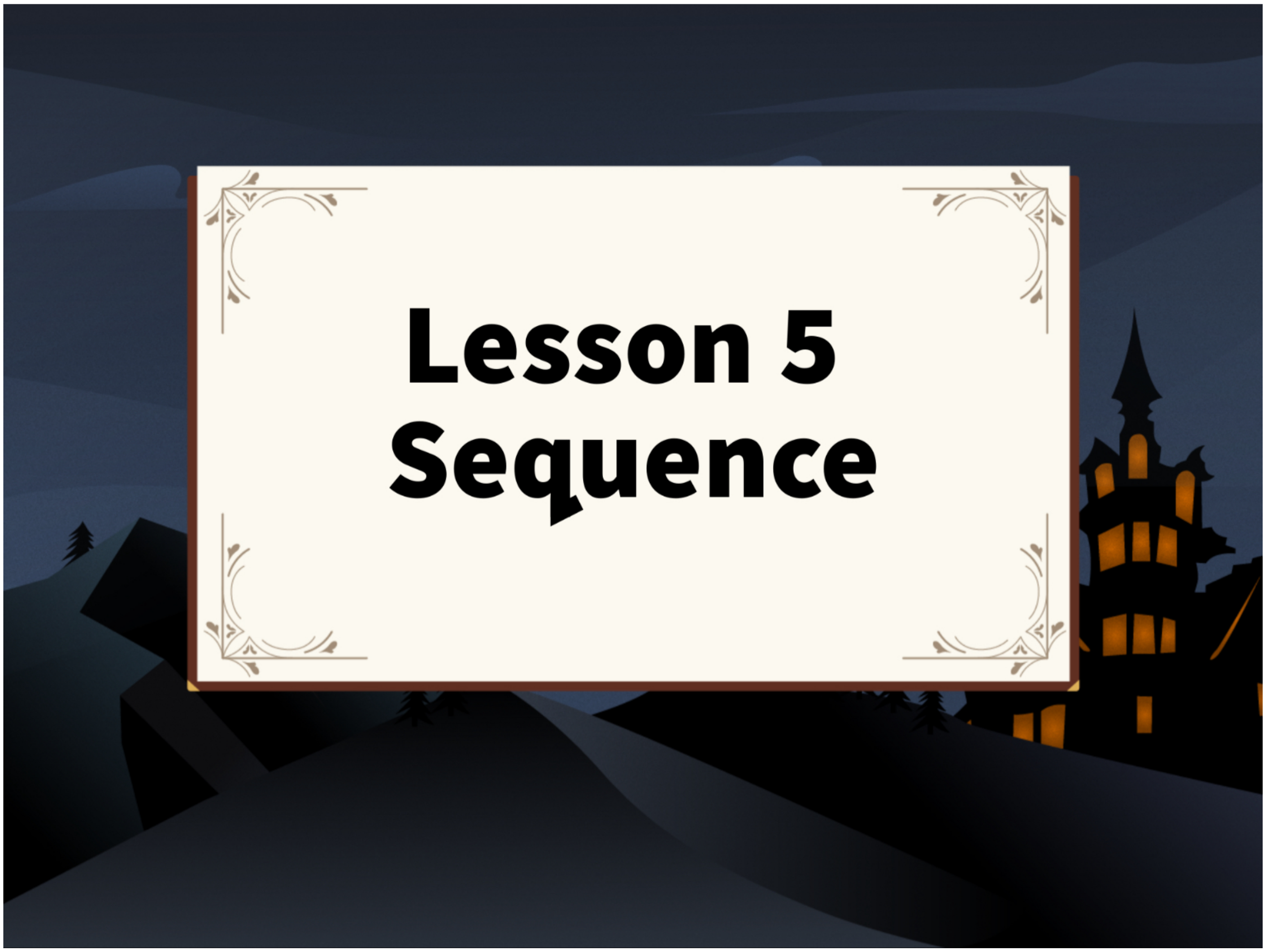


Announcement

1. Mock Exam: Available under Lesson 4 Learning Materials

2. Mock Exam Explanation Session: 7/16 Wed 5-7PM PDT

Concept 1

Comprehensive Application of Arithmetic and Geometric Sequence

Properties of Arithmetic Sequence

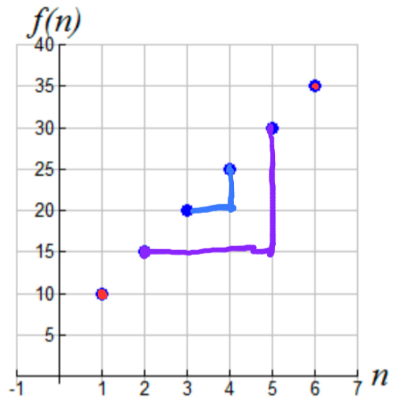
1. Definition: $a_{n+1} - a_n = d$, $a_n = a_1 + (n-1)d$

2. Sum: $S_n = \frac{\boxed{n}\boxed{(a_1 + a_n)}}{2}$

\# Average

3. Midpoint Formula:

If $m + n = p + q$, then $a_m + a_n = a_p + a_q$



a_1

$a_2 = a_1 + d$

$a_3 = a_1 + 2d$

$a_m = a_n + (m-n)d$

Properties of Geometric Sequence

1. Definition: $\frac{a_{n+1}}{a_n} = r$, $a_n = a_1 r^{n-1}$

2. Sum: $S_n = a_1 \frac{1-r^n}{1-r}$ if $r \neq 1$

3. Sum of infinite geometric series: $S = \frac{a_1}{1-r}$ if $-1 < r < 1$

a_1
$a_2 = a_1 \cdot r$
$a_3 = a_1 \cdot r \cdot r = a_1 \cdot r^2$

$r^n \ (n \to \infty) \to 0$

$S_n = a_1 + \cancel{a_1 r} + \cancel{\cdots} + \cancel{a_1 r^{n-1}}$

$r \cdot S_n = \cancel{a_1 r} + \cancel{a_1 r^2} + \cdots + \cancel{a_1 r^{n-1}} + a_1 \cdot r^n$

$\Rightarrow S_n - r \cdot S_n = a_1 - a_1 \cdot r^n$

$S_n = \frac{a_1(1-r^n)}{1-r}$

MC(1)

Math Exploration 1.1

A four-term sequence is formed by adding each term of a four-term arithmetic sequence of positive integers to the corresponding term of a four-term geometric sequence of positive integers. The first three terms of the resulting four-term sequence are **57**, **60**, and **91**. What is the fourth term of this sequence? (2022 AMC 10A Problems, Question #20)

A. **190** B. **194** C. **198** D. **202** E. **206**

Arithmetic: $a, a+d, a+2d, a+3d$

Geometric: b, br, br^2, br^3

$\Rightarrow \begin{cases} 60-57 = d + br - b \\ 91-60 = d + br^2 - br \end{cases}$

$\Rightarrow \begin{cases} 3 = d + b(r-1) \\ 31 = d + br(r-1) \end{cases}$

$\Rightarrow 28 = br(r-1) - b(r-1)$

$28 = b(r-1)(r-1)$

$28 = b(r-1)^2$

pos. integer

$\Rightarrow (r-1)^2 = 1$ or 4

Check:

① $(r-1)^2 = 1 \Rightarrow r = 2$, $b = 28$

29 4 -21

28 56 112 X

② $(r-1)^2 = 4 \Rightarrow r = 3$, $b = 7$

50 39 28 17

7 21 63 189

MC(1)

Math Exploration 1.2

Ans $= 8+9+10+17+18+19+26+27+28+35 = 197$

For each integer $n \geq 2$, let S_n be the sum of all products jk, where j and k are integers and $1 \leq j < k \leq n$. What is the sum of the 10 least values of n such that S_n is divisible by 3? (2021 Fall AMC 10B Problems, Question #22)

A. **196** B. **197** ✓ C. **198** D. **199** E. **200**

$n=2$: $S_2 = 1\times2$

$n=3$: $S_3 = 1\times2 + 1\times3 + 2\times3$

$n=4$: $S_4 = 1\times2 + 1\times3 + 1\times4 + 2\times3 + 2\times4 + 3\times4$

$\Rightarrow S_{n+1} - S_n = 1\times(n+1) + 2\times(n+1) + \cdots + n\cdot(n+1)$

$= (n+1)(1+2+\cdots+n) = (n+1)\dfrac{(n+1)}{2}\cdot n = \boxed{\dfrac{n(n+1)^2}{2}}$

If $n \equiv 0$ or $2 \pmod 3$

$\Rightarrow \dfrac{n(n+1)^2}{2}$ is divisible by 3

If $n \equiv 1 \pmod 3$

$\Rightarrow \dfrac{n(n+1)^2}{2} \equiv 2 \pmod 3$

Divided by 3:

$S_2 = 2$

$S_3 = 2 + \boxed{\dfrac{2(2+1)^2}{2}}$ divisible by 3

S_n	2	3	4	5	6	7	8
Remainder	2	2	2	$2+2 \equiv 1$	1	1	0

MC(1)

Practice 1.1

Given an arithmetic sequence $\{a_n\}$ with a nonzero common difference, and that the terms a_2, a_3, a_9 form a geometric sequence, what is the value of $\dfrac{a_4 + a_5 + a_6}{a_2 + a_3 + a_4}$?

Midpoint: $a_4 + a_6 = 2a_5$

$a_5 - d$ $a_5 + d$

A. $\dfrac{8}{3}$ ✓ B. $\dfrac{7}{3}$ C. 3 D. $\dfrac{5}{3}$

$a_3 = a_2 + d$

$a_9 = a_2 + 7d$

$\Rightarrow \dfrac{a_3}{a_2} = \dfrac{a_9}{a_3} \Rightarrow a_3^2 = a_2 \cdot a_9$

$(a_2+d)^2 = a_2(a_2+7d)$

$\cancel{a_2^2} + 2a_2d + d^2 = \cancel{a_2^2} + 7a_2d$

$d = 5a_2 \Rightarrow a_2 = \dfrac{d}{5}$

$\dfrac{a_4+a_5+a_6}{a_2+a_3+a_4} = \dfrac{3a_5}{3a_3}$

$= \dfrac{a_2+3d}{a_2+d}$

$= \dfrac{\frac{16}{5}d}{\frac{6}{5}d} = \dfrac{16}{6} = \dfrac{8}{3}$

MC(1)

Practice 1.2

From the sequence $\{a_n\}$ satisfying the recurrence relation $a_1 = a_2 = 1$, and $a_{n+2} = a_{n+1} + a_n (n \geqslant 1)$, extract the terms that are divisible by $\mathbf{3}$ to form a new sequence $\{b_n\}$. What is the value of b_{100}?

A. a_{100} B. a_{200} C. a_{300} D. a_{400}

$a_1 = 1$

$a_2 = 1$

$a_3 = 2$

$a_4 = 3$ ✗

$a_5 = 5$

$a_6 = 8$

$a_7 = 13$

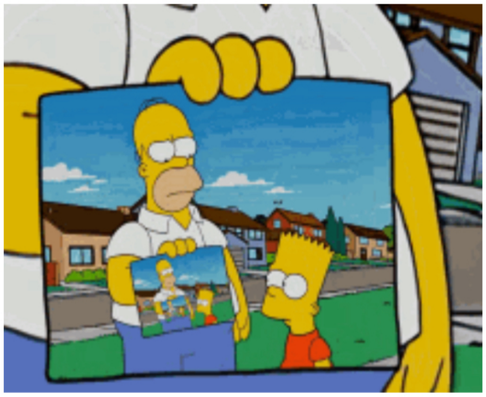
$a_8 = 21$ ✗

$\Rightarrow b_{100} = a_{4\times 100} = a_{400}$

Concept 2

Recursive Sequence

A **recursive sequence** is a sequence in which terms are defined using one or more previous terms which are given.
Addition Form: $a_{n+1} = a_n + f(n)$
Multiplication Form: $a_{n+1} = a_n \cdot f(n)$.



Ways to Solve Recursive sequence:

1. Find patterns through enumeration

$f(n) = f(f(n))$

5, ... , 5, ... , 5, ...

Default

0 1 1 2 3 5

0 + 1 = 1
1 + 1 = 2
1 + 2 = 3
2 + 3 = 5

2. Express the sequence as $\boldsymbol{f(n) = f(f(n))}$ and solve for $\boldsymbol{f(n)}$

$$F(n) = \overbrace{\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n}^{G(n)} - \overbrace{\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n}^{E(n)}$$

*Almost every problem can be solved by finding patterns, but it could be time consuming.

MC(1)

Math Exploration 2.1

Let $a_1, a_2 \ldots,$ be a sequence with the following properties.

(i) $a_1 = 1$, and

(ii) $a_{2n} = n \cdot a_n$ for any positive integer n.

What is the value of $a_{2^{100}}$?

A. 1 B. 2^{99} C. 2^{100} D. 2^{4950} E. 2^{9999}

$$a_{2^{100}} = a_{2\cdot 2^{99}} = 2^{99} \cdot a_{2^{99}} = 2^{99} \cdot 2^{98} \cdot a_{2^{98}}$$
$$= 2^{99} \cdot 2^{98} \cdot \ldots \cdot 2^{1} \cdot 1 \cdot a_1$$
$$= 2^{99+98+\cdots+1}$$
$$= 2^{\frac{(99+1)}{2} \cdot 99}$$
$$= 2^{4950}$$

MC(1)

Math Exploration 2.2

Define a sequence recursively by $F_0 = 0,\ F_1 = 1,$ and $F_n =$ the remainder when $F_{n-1} + F_{n-2}$ is divided by $3,$ for all $n \geq 2.$ Thus the sequence starts $0, 1, 1, 2, 0, 2, \ldots$ What is $F_{2017} + F_{2018} + F_{2019} + F_{2020} + F_{2021} + F_{2022} + F_{2023} + F_{2024}$? (2017 AMC 10A Problems, Question #13)

A. 6 B. 7 C. 8 D. 9 E. 10

Pattern: $\underbrace{0, 1, 1, 2, 0, 2, 2, 1,}_{\text{8 numbers in a cycle.}} 0, 1, 1, 2, 0$

Sum $= 0+1+1+2+0+2+2+1$
$= 9$

MC(1)

Math Exploration 2.3

The Fibonacci numbers are defined by $F_1 = 1, F_2 = 1,$ and

$F_n = F_{n-1} + F_{n-2}$ for $n \geq 3.$ What is $\frac{F_2}{F_1} + \frac{F_4}{F_2} + \frac{F_6}{F_3} + \cdots + \frac{F_{20}}{F_{10}}$?

(2024 AMC 10B Problems, Question #23)

A. **318** B. **319** C. **320** D. **321** E. **322**

$\frac{F_{2n}}{F_n} = \frac{F_{2n-2}}{F_{n-1}} + \frac{F_{2n-4}}{F_{n-2}}$

$F_n: 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

$\frac{F_2}{F_1} = 1 \quad \frac{F_4}{F_2} = 3 \quad \frac{F_6}{F_3} = \frac{8}{2} = 4 \quad \frac{F_8}{F_4} = \frac{21}{3} = 7$

$\text{Sum} = 1+3+4+7+11+18+29+47+76+123$
$= 319$

Combination

Practice 2.1

Let S_n denote the sum of the first n terms of the sequence $\{a_n\}$. Given that $a_1 = 1$, and that $S_{n+1} = 4a_n + 2$.

(1) Let $b_n = a_{n+1} - 2a_n$. Prove that $\{b_n\}$ is a geometric sequence.

$S_n = 4a_{n-1} + 2$ and $S_{n+1} - S_n = a_{n+1}$

(2) Find the general formula of the sequence $\{b_n\}$. Let $T_n = b_1 + b_2 + \cdots + b_n$. Find T_n.

$a_2 = S_2 - a_1 = 4a_1 + 2 - a_1 = 5$

(1) $S_{n+1} - S_n = a_{n+1}$

$(4a_n \cancel{+2}) - (4a_{n-1} \cancel{+2}) = a_{n+1}$

$4a_n - 4a_{n-1} = a_{n+1}$

$2a_n - 4a_{n-1} = a_{n+1} - 2a_n$

$2(a_n - 2a_{n-1}) = a_{n+1} - 2a_n$

$2b_{n-1} = b_n$

(2) $b_1 = a_2 - 2a_1 = 5 - 2 = 3$

$\{b_n\}$ is a geometric sequence, $b_n = 3 \cdot 2^{n-1}$

$\Rightarrow T_n = \frac{3(1-2^n)}{1-2} = 3(2^n - 1)$

MC(1)

Practice 2.2

Sequence $\{a_n\}$ satisfies $a_1 = 3$, and $a_{n+1} = \dfrac{5a_n - 13}{3a_n - 7}$. $a_{2016} =$ ______ .

A. 4 B. 3 C. 2 D. 1 E. 0

Pattern: $a_1 = 3$

$a_2 = \dfrac{5\cdot 3 - 13}{3\cdot 3 - 7} = 1$

$a_3 = \dfrac{5\cdot 1 - 13}{3\cdot 1 - 7} = 2$

$a_4 = \dfrac{5\cdot 2 - 13}{3\cdot 2 - 7} = 3$

$\Rightarrow \dot{3}, \dot{1}, \dot{2}, 3, 1, 2, \ldots$

$2016 \div 3 = R0 \Rightarrow a_{2016} = a_3 = 2$

MC(1)

Practice 2.3

Consider the following operation. Given a positive integer n, if n is a multiple of 3, then you replace n by $\dfrac{n}{3}$. If n is not a multiple of 3, then you replace n by $n + 10$. Then continue this process. For example, beginning with $n = 4$, this procedure gives $4 \to 14 \to 24 \to 8 \to 18 \to 6 \to 2 \to 12 \to \cdots$. Suppose you start with $n = 100$. What value results if you perform this operation exactly 100 times? (2024 AMC 10A Problems, Question #10)

A. 10 B. 20 C. 30 D. 40 E. 50

$100 \xrightarrow[a_1]{} 110 \xrightarrow[a_2]{} 120 \xrightarrow[a_3]{} 40 \xrightarrow[a_4]{} 50 \xrightarrow[a_5]{} 60 \boxed{\xrightarrow[a_6 \,\triangle]{} 20} \xrightarrow[a_7]{} 30 \xrightarrow[a_8]{} 10 \boxed{\xrightarrow[a_9 \,\triangle]{} 20}$

$100 \div 3 = 33R1$

$\Rightarrow a_{100} = a_7 = 30$

MC(1)

Math Exploration 3.1

$\frac{3}{3}$

A sequence of numbers is defined recursively by $a_1 = 1$, $a_2 = \frac{3}{7}$, and

$a_n = \dfrac{a_{n-2} \cdot a_{n-1}}{2a_{n-2} - a_{n-1}}$ for all $n > 3$. Then a_{2019} can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p + q$? (2019 AMC 10A Problems, Question #15)

A. $\mathbf{2020}$ B. $\mathbf{4039}$ C. $\mathbf{6057}$ D. $\mathbf{6061}$ E. $\mathbf{8078}$ ✓

$$a_3 = \frac{1 \cdot \frac{3}{7}}{2 - \frac{3}{7}} = \frac{\frac{3}{7}}{\frac{11}{7}} = \frac{3}{11}$$

$$a_{2019} = \frac{3}{4 \cdot 2019 - 1} = \frac{3}{8075}$$

Pattern: $\frac{3}{3}, \frac{3}{7}, \frac{3}{11}, \ldots \frac{3}{4m-1}$

Ans $= 8078$

MC(1)

Math Exploration 3.2 Skip

Define a sequence recursively by $x_0 = 5$ and $x_{n+1} = \dfrac{x_n^2 + 5x_n + 4}{x_n + 6}$ for all nonnegative integers n. Let m be the least positive integer such that $x_m \le 4 + \frac{1}{2^{20}}$. In which of the following intervals does m lie? (2019 AMC 10B Problems, Question #24)

$$x_{n+1} = \frac{(x_n^2 + 6x_n)(-x_n + 4)}{x_n + 6} = x_n + \frac{4 - x_n}{x_n + 6}$$

A. $[9, 26]$ B. $[27, 80]$ C. $[81, 242]$ ✓ D. $[243, 728]$ E. $[729, \infty)$

$$\Rightarrow x_{n+1} - x_n = \frac{4 - x_n}{x_n + 6}$$

Let $x_n = 4 + f_n$

$$\Rightarrow x_{n+1} - x_n = \frac{-f_n}{10 + f_n}$$

$$x_n - x_{n+1} = \frac{f_n}{10 + f_n} \approx \frac{f_n}{10}$$

Find $x_m \le 4 + \frac{1}{2^{20}}$ and $x_0 = 4 + \frac{1}{2^0}$

$\Rightarrow$ Step to $\frac{1}{20^{n+1}} = x_k$ from $\frac{1}{20^n} = x_j$

$$x_j - x_k = \frac{1}{20^n} - \frac{1}{20^{n+1}} = \frac{f_j}{10} + \frac{f_{j+1}}{10} + \cdots + \frac{f_k}{10}$$

$$\Rightarrow \frac{1}{2^n} = f_j > f_k = \frac{1}{2^{n+1}}$$

$\Rightarrow$ from f_0 to f_{20} : $100 < m < 200$

MC(1)

Practice 4.1

Same as Lesson 3 Practice 5.1

A function f is defined recursively by $f(1) = f(2) = 1$ and $f(n) = f(n-1) - f(n-2) + n$ for all integers $n \geq 3$. What is $f(2018)$?

(2018 AMC 10B Problems, Question #20)

A. 2016 B. 2017 C. 2018 D. 2019 E. 2020