Numerical Solutions to PDEs - FALL 2024

Problem Set #1

Due 12/10 in class

Describe the setup and each step in your solutions with words and clearly label your final answers. Use Matlab for plotting and programming and include your code as an appendix to your problem set.

1. If you are not already comfortable with Matlab, spend 1-2 hours going through the Mathworks tutorial introduction step by step: http://www.mathworks.com/support/learn-with-matlab-tutorials.html (You don't have to show any work for this part.)

2. Consider the central finite difference operator δ defined by

$$\delta u_n = \frac{u_{n+1} - u_{n-1}}{2\Delta}$$

   (a) In calculus we have

   $$\frac{duv}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

   Show that the analogous finite difference expression does not hold, i.e.: $\delta(u_n v_n \neq u_n \delta v_n + v_n \delta u_n$.

   (b) Show that this expression holds instead:

   $$\delta(u_n v_n) = \bar{u}\delta v_n + \bar{v}\delta u_n$$

   where the overbar indicates an average over the nearest neighbors, $\bar{u} = \frac{1}{2}(u_{n+1} + u_{n-1})$.

3. Consider the first-order forward difference, second-order central difference, and fourth-order central difference approximations (as given in lecture) to the first derivative of $f(x) = sin(5x)$. Plot the exact derivative and the three approximations on the same plot for $0 \leq x \leq 3$ and $N = 16$. (Don't worry about points near the boundaries, just leave out those values.) Evaluate the derivative at $x = 1.5$ and plot the absolute value of the differences from the exact solution as a function of Δ on a log-log plot. Use $N = 8, 16, 32, ..., 2048$ where N is the number of grid cells. Discuss your plot.

   Note: On the course website you will find a Matlab script to assist you with this problem. There are key places in the script that have been erased - they are marked by three question marks (???) and you need to replace all of those with the correct information for the script to work.

4. Use a Taylor table to construct the most accurate formula for the first derivative at x_i using known function values of f at x_{i-1}, x_i, x_{i+1} and x_{i+2}, assuming the points are uniformly spaced. Include the leading error term and state the "order" of the method.

5. Recall that the second derivative of $f = exp(ikx)$ is $-k^2 f$. Application of a finite difference operator for the second derivative to f would lead to $-k'^2 f$, where k'^2 is the 'modified wavenumber' for the second derivative. The modified wave number method for assessing the accuracy of second derivative finite-difference formulas is then to compare the corresponding k'^2 with k^2 (in a plot with $k'^2\Delta^2$ vs. $k^2\Delta^2$, $0 \le k\Delta \le \pi$).

   Use modified wavenumber analysis to compare the accuracy of the second-order central difference formula
   $$f_j'' = \frac{f_{j+1} - 2f_j + f_{j-1}}{\Delta^2}$$

   and the fourth-order Pade formula

   $$\frac{1}{12} f_{j-1}'' + \frac{10}{12} f_j'' + \frac{1}{12} f_{j+1}'' = \frac{f_{j+1} - 2f_j + f_{j-1}}{\Delta^2}$$

   Hint: to derive the modified wavenumber for Pade type schemes, replace f'' with $-k'^2 exp(ikx_j)$, etc.