The momentum equations

$$\left(\frac{d\boldsymbol{v}_I}{dt}\right)_I = \left(\frac{d\boldsymbol{v}_R}{dt}\right)_R + 2\boldsymbol{\Omega}\times\boldsymbol{v}_R + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}) = \text{force terms}$$

x direction:

$$\underbrace{\frac{\partial u}{\partial t}}_{\text{Local acceleration}} + \underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}}_{\text{Nonlinear advection term}} \underbrace{- fv + f_* w}_{\text{Coriolis term}} = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial x}}_{\text{Pressure gradient term}} + \underbrace{\nu\nabla^2 u}_{\text{Viscosity term}} + \cdots$$

y direction:

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v + \cdots$$

z direction:

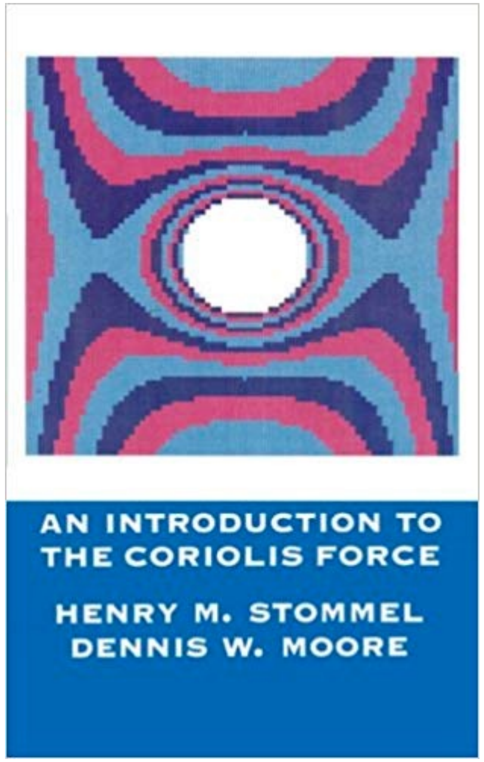
$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} - f_* u = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w + \cdots$$

References about Coriolis force

- ***An introduction to the Coriolis Force***, H.M. Stommel and D.W. Moore, 1989

- Persson (2000). ***What is the Coriolis force***? Weather, 55, 165–170.

- Persson (1998). ***How to understand the Coriolis force?*** Bulletin of the American Meteorological Society, 79(7), 1373–1385.



Centrifugal Force

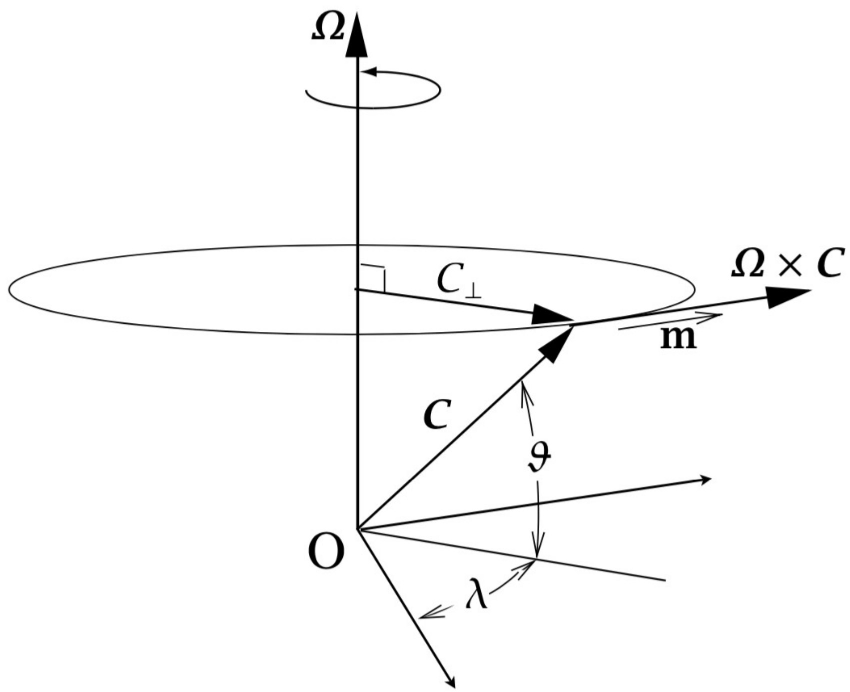
$$-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})$$

Substitute $\boldsymbol{C}$ with $\boldsymbol{r}$:

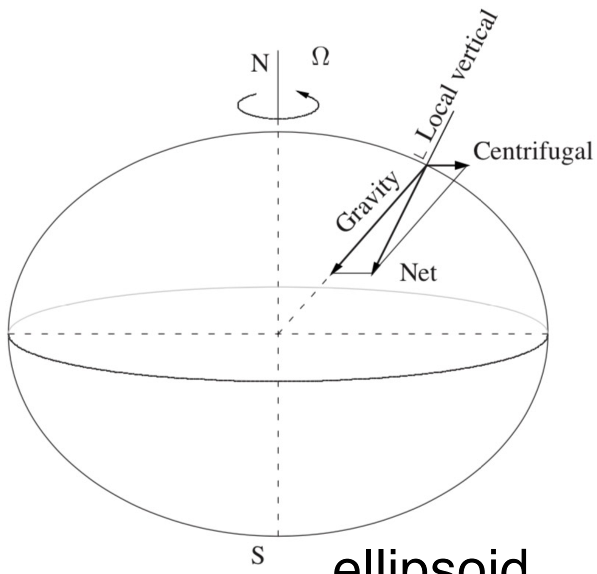
$$= -\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}_\perp)$$

$$= -(\boldsymbol{\Omega}\cdot\boldsymbol{r}_\perp)\boldsymbol{\Omega} + (\boldsymbol{\Omega}\cdot\boldsymbol{\Omega})\boldsymbol{r}_\perp$$

$$= \Omega^2\boldsymbol{r}_\perp$$



Gravity Force



ellipsoid

The Gravity Term in the Momentum Equation The gravitational attraction of two masses M_1 and m is:

$$\mathbf{F}_g = \frac{G\, M_1\, m}{R^2}$$

where R is the distance between the masses, and G is the gravitational constant. The vector force $\mathbf{F}_g$ is along the line connecting the two mases.

The force per unit mass due to gravity is:

$$\frac{\mathbf{F}_g}{m} = \mathbf{g}_f = \frac{G\, M_E}{R^2} \tag{7.15}$$

where M_E is the mass of Earth. Adding the centrifugal acceleration to (7.15) gives gravity $\mathbf{g}$ (Figure 7.5):

effective gravity $$\mathbf{g} = \mathbf{g}_f - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{R}) \tag{7.16}$$

The momentum equations

x direction:

$$\underbrace{\frac{\partial u}{\partial t}}_{\text{Local acceleration}} + \underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}}_{\text{Nonlinear advection term}} \underbrace{- fv + f_* w}_{\text{Coriolis term}} = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial x}}_{\text{Pressure gradient term}} + \underbrace{\nu\nabla^2 u}_{\text{Viscosity term}}$$

y direction:

$$\underline{\frac{\partial v}{\partial t}} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \underline{+ fu} = \underline{-\frac{1}{\rho}\frac{\partial p}{\partial y}} + \underline{\nu\nabla^2 v}$$

z direction:

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \underline{f_* u} = \underline{-\frac{1}{\rho}\frac{\partial p}{\partial z}} + \underline{\nu\nabla^2 w} - \underbrace{g}_{\text{gravity term}}$$

Continuity equation

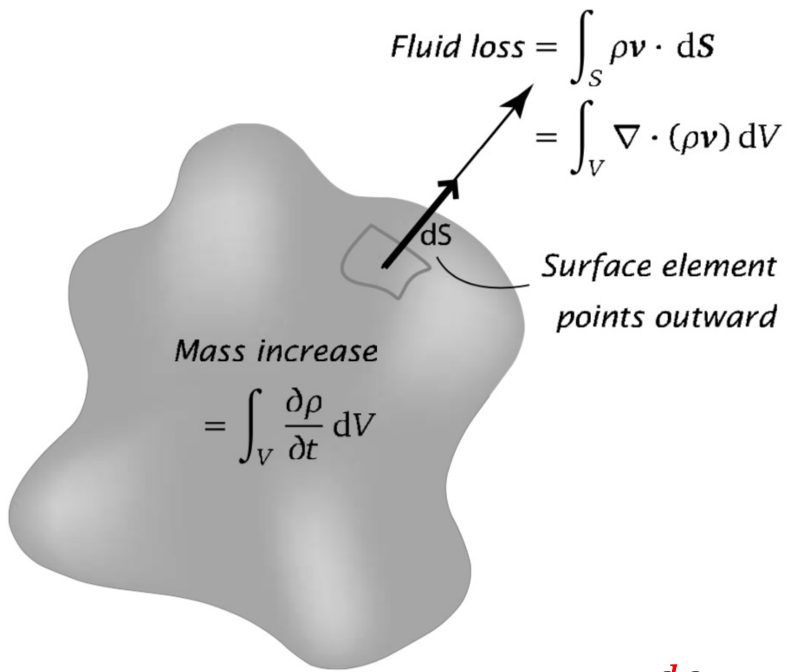
$$\text{fluid loss} = \int_S \rho \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = \int_V \nabla \cdot (\rho \boldsymbol{v})\, \mathrm{d}V$$

Divergence (Gaussian) theorem

Vallis (Eq. 1.19)

$$= -\frac{dM}{dt} = -\frac{d}{dt}\int_v \rho\, dV = -\int_v \frac{\partial \rho}{\partial t} dV$$

$$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v})\right] \mathrm{d}V = 0$$



For any arbitrary volume V,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \quad \text{or} \quad \frac{d\rho}{dt} + \rho \nabla \cdot \boldsymbol{v} = \mathbf{0}$$

For incompressible fluids ($\frac{d\rho}{dt} = 0$)

$$\nabla \cdot \boldsymbol{v} = 0$$

Boussinesq approximation

$$\begin{aligned}\rho = \ & \rho_0 + \Delta\rho(x,y,z,t) \\ = \ & \boldsymbol{\rho_0} + \boldsymbol{\bar{\rho}(z)} + \boldsymbol{\rho'(x,y,z,t)} \\ = \ & \tilde{\rho}\,(z) + \rho'(x,y,z,t)\end{aligned}$$

$$\boxed{\bar{\rho},\quad \rho' \ll \rho_0}$$

$\boldsymbol{\rho_0}$: mean (reference) density (~1025 kg m^{-3})

$\boldsymbol{\bar{\rho}(z)}$: density variation due to stratification

$\boldsymbol{\rho'}$: density variation due to perturbation

For pressure:

dynamic pressure

$$p = \tilde{p}(z) + p'(x,y,z,t)$$

hydrostatic pressure

$$\tilde{p}(z) = P_0 - \rho_0 g z$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = -\frac{1}{\rho}\left(\frac{d\tilde{p}}{dz} + \frac{\partial p'}{\partial z}\right) = \frac{\rho_0}{\rho}g - \frac{1}{\rho}\frac{\partial p'}{\partial z}$$

Then the z-momentum equation becomes:

$b = -\Delta\rho g$: buoyancy force

$$\frac{\mathrm{dw}}{\mathrm{dt}} + f_* u = -\frac{1}{\rho}\frac{\partial p'}{\partial z} - \frac{\Delta\rho}{\rho} g + \nu\nabla^2 \mathrm{w}$$

Buoyancy gain or loss?

ρ_0 ρ_0

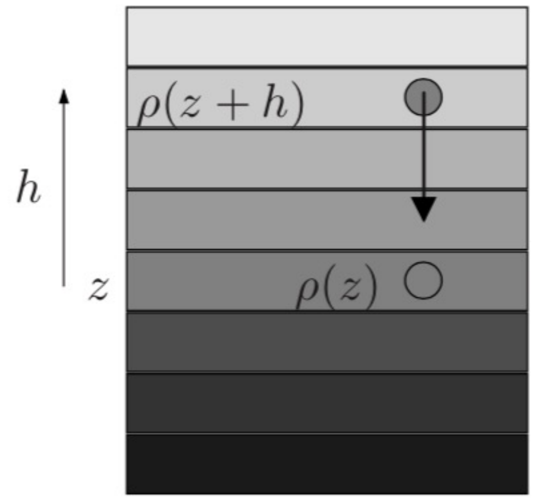
Boussinesq approximation: ρ can be replaced by ρ_0 (or $\Delta\rho$ can be neglected) everywhere except in the gravity term

Horizontal momentum equations

$$\frac{du}{dt} + f_* w - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu \nabla^2 u$$

$$\frac{dv}{dt} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \nu \nabla^2 v.$$

$$\frac{\mathrm{dw}}{\mathrm{dt}} + f_* u = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{\rho}{\rho_o} g + \nu\nabla^2 \mathrm{w}$$



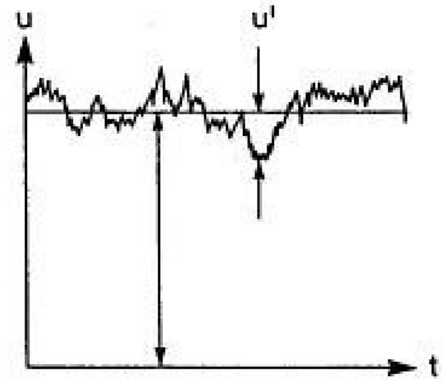
Reynolds-averaged (time-average) equations

The velocity can be decomposed into (time) mean velocity and perturbation (turbulent) velocity

$$u=\overline{u}+u', \quad v=\overline{v}+v', \quad w=\overline{w}+w', \quad p=\overline{p}+p'$$

$$\overline{u'}=0, \ \overline{v'}=0, \ \overline{w'}=0, \ \overline{p'}=0$$

$$\overline{uv}=\overline{(\bar{u}+u')(\bar{v}+v')}=\bar{u}\bar{v}+\overline{u'v'}$$



The advection term in the x momentum equation can be written as:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=\frac{\partial\left(u^2\right)}{\partial x}+\frac{\partial(uv)}{\partial y}+\frac{\partial(uw)}{\partial z}-u\underbrace{\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)}_{=0}$$

$$=\frac{\partial\left(u^2\right)}{\partial x}+\frac{\partial(uv)}{\partial y}+\frac{\partial(uw)}{\partial z}$$

$$\overline{\rho_0\left\{\frac{\partial(\bar{u}+u')}{\partial t}+\frac{\partial(\bar{u}+u')^2}{\partial x}+\frac{\partial(\bar{u}+u')(\bar{v}+v')}{\partial y}+\frac{\partial(\bar{u}+u')(\bar{w}+w')}{\partial z}\right\}} \quad +\rho_0(-f\bar{v}+f_*\bar{u})$$

$$= \overline{-\frac{\partial(\bar{p}+p')}{\partial x}+\mu\left(\frac{\partial^2(\bar{u}+u')}{\partial x^2}+\frac{\partial^2(\bar{u}+u')}{\partial y^2}+\frac{\partial^2(\bar{u}+u')}{\partial z^2}\right)}$$

$$F^x = \frac{\partial\tau^{xx}}{\partial x}+\frac{\partial\tau^{yx}}{\partial y}+\frac{\partial\tau^{zx}}{\partial z}$$

$$= \frac{1}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{1}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)+\frac{1}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)$$

$$= \mu\,\nabla^2 u$$

$$\boxed{\Delta = \nabla^2}$$

$$\rho_0\left(\frac{\partial\bar{u}}{\partial t}+\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial\bar{u}}{\partial y}+\bar{w}\frac{\partial\bar{u}}{\partial z}\right)+\rho_0(-f\bar{v}+f_*\bar{u}) = -\frac{\partial\bar{p}}{\partial x}+\underline{\mu\Delta\bar{u}-\rho_0\left(\frac{\partial\overline{u'u'}}{\partial x}+\frac{\partial\overline{u'v'}}{\partial y}+\frac{\partial\overline{u'w'}}{\partial z}\right)}$$

$$= \rho_0\,(\underbrace{\upsilon+\upsilon_E}_{A})\Delta\bar{u}$$

Eddy viscosity

$$\overline{u'u'} = -\upsilon_E\frac{\partial\bar{u}}{\partial x} \qquad \overline{u'v'} = -\upsilon_E\frac{\partial\bar{u}}{\partial y} \qquad \overline{u'w'} = -\upsilon_E\frac{\partial\bar{u}}{\partial z}$$

Reynolds stress: stress induced by turbulence acting on mean flow

Reynolds-averaged equations

$$x:\ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv + f_* w = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(A_H\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_H\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_V\frac{\partial u}{\partial z}\right)$$

$$y:\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(A_H\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_H\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_V\frac{\partial v}{\partial z}\right)$$

$$z:\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + f_* u = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left(A_H\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_H\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_V\frac{\partial w}{\partial z}\right) - \frac{\rho}{\rho_0} g$$