I. Governing equations of GFD

Objectives:

1. Know the governing equations (momentum, continuity, tracer, density) and how their terms are derived

2. Boussinesq approximation and Reynolds average

3. Scale analysis (dimensionless numbers)

Momentum equation

Newton's second Law:

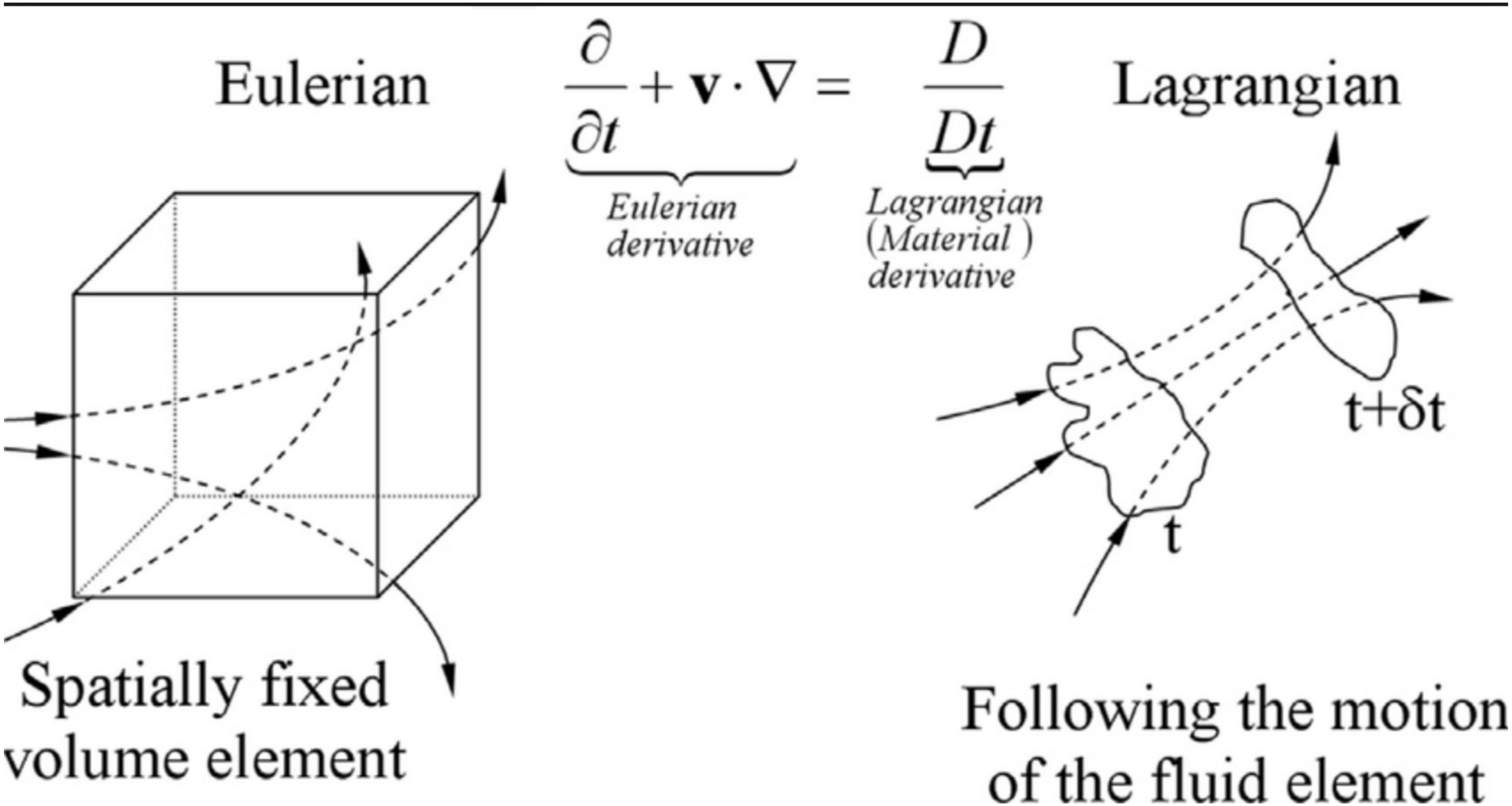
$$m\mathbf{a} = \mathbf{F}$$

For unit volume:

$$\rho \, d\mathbf{u}/dt = \mathbf{F}$$

$$d\mathbf{u}/dt = \mathbf{F}/\rho$$

Eulerian and Lagrangian methods



$\boldsymbol{u} = (u, v, w)$

$$\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} \longrightarrow \text{non-linear advection term}$$

local acceleration term

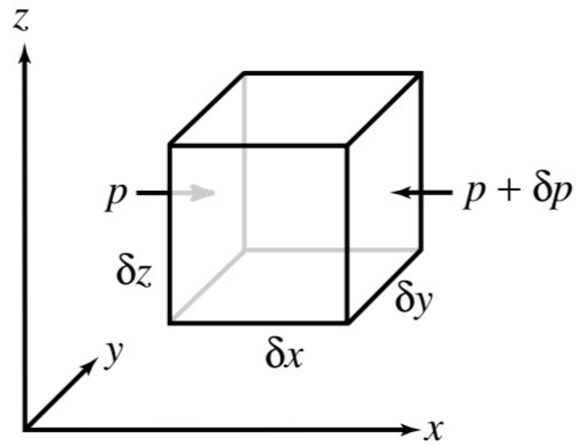
$$\text{x direction: } \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$\text{y direction: } \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$\text{z direction: } \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

Pressure gradient force

Derivation of Pressure Term Consider the forces acting on the sides of a small cube of fluid (Figure 7.4). The net force δF_x in the x direction is



$$\delta F_x = p\,\delta y\,\delta z - (p+\delta p)\,\delta y\,\delta z$$
$$\delta F_x = -\delta p\,\delta y\,\delta z$$

But

$$\delta p = \frac{\partial p}{\partial x}\,\delta x$$

and therefore

$$\delta F_x = -\frac{\partial p}{\partial x}\,\delta x\,\delta y\,\delta z$$
$$\delta F_x = -\frac{\partial p}{\partial x}\,\delta V$$

$$f(x) = f(x_0) + \left.\frac{df(x)}{dx}\right|_{x=x_0}(x-x_0) + \left.\frac{d^2 f(x)}{2!\,dx^2}\right|_{x=x_0}(x-x_0)^2 + \ldots$$

Dividing by the mass of the fluid δm in the box, the acceleration of the fluid in the x direction is:

$$a_x = \frac{\delta F_x}{\delta m} = -\frac{\partial p}{\partial x}\,\frac{\delta V}{\delta m}$$

$$\boxed{a_x = -\frac{1}{\rho}\frac{\partial p}{\partial x}} \qquad (7.13)$$

The momentum equations

x direction: $$\frac{\partial \mathrm{u}}{\partial \mathrm{t}} + \mathrm{u}\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \mathrm{v}\frac{\partial \mathrm{u}}{\partial \mathrm{y}} + \mathrm{w}\frac{\partial \mathrm{u}}{\partial \mathrm{z}} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \cdots$$

y direction: $$\frac{\partial \mathrm{v}}{\partial \mathrm{t}} + \mathrm{u}\frac{\partial \mathrm{v}}{\partial \mathrm{x}} + \mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{y}} + \mathrm{w}\frac{\partial \mathrm{v}}{\partial \mathrm{z}} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \cdots$$

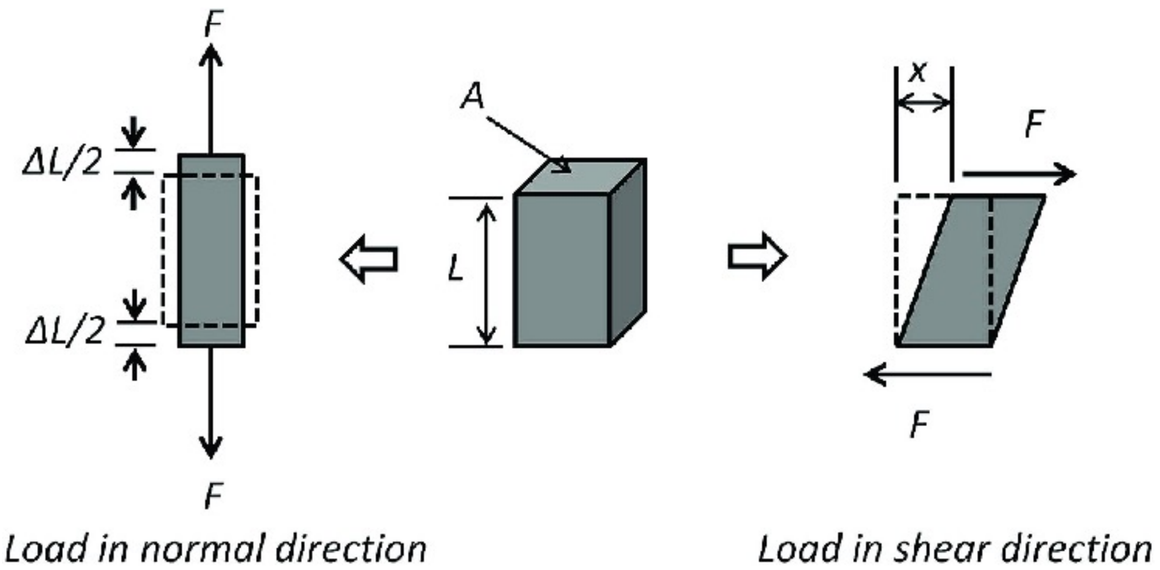
z direction: $$\frac{\partial \mathrm{w}}{\partial \mathrm{t}} + \mathrm{u}\frac{\partial \mathrm{w}}{\partial \mathrm{x}} + \mathrm{v}\frac{\partial \mathrm{w}}{\partial \mathrm{y}} + \mathrm{w}\frac{\partial \mathrm{w}}{\partial \mathrm{z}} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \cdots$$

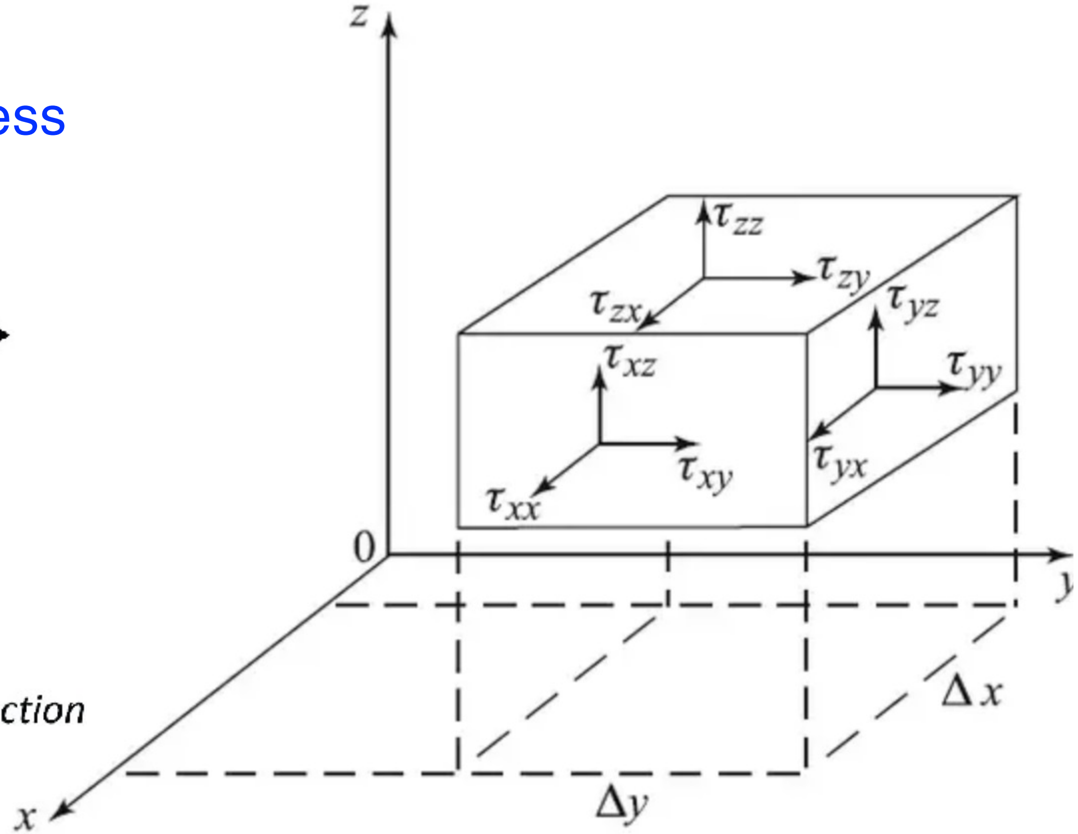
Frictional force term

Stress – second-order tensor ($N\ m^{-2}$)

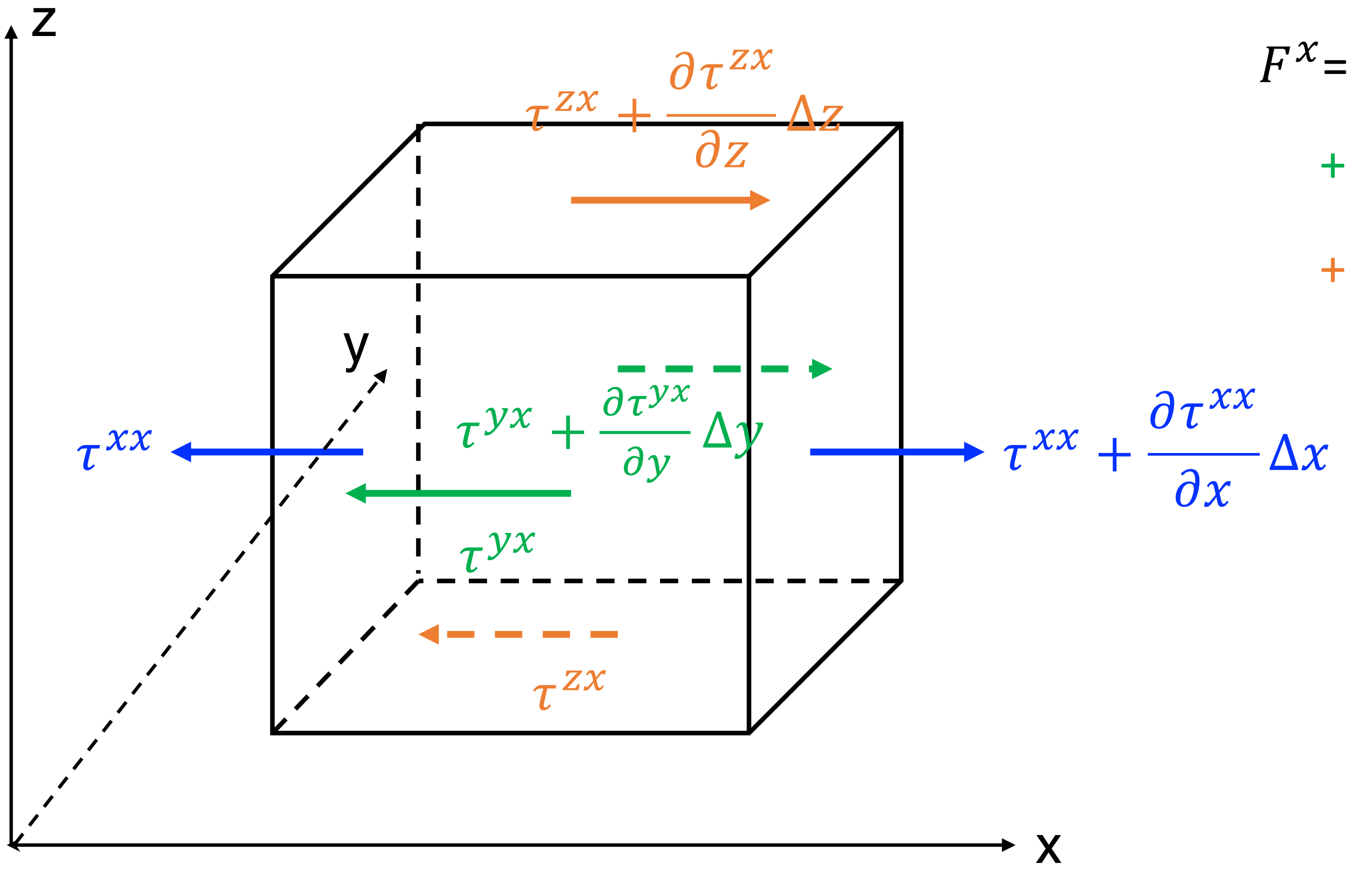
normal stress

shear stress





9 components



$$
\begin{aligned}
F^{x} = & \left(\tau^{xx} + \frac{\partial \tau^{xx}}{\partial x}\Delta x\right)\Delta y \Delta z - \tau^{xx}\Delta y \Delta z \\
& + \left(\tau^{yx} + \frac{\partial \tau^{yx}}{\partial y}\Delta y\right)\Delta x \Delta z - \tau^{yx}\Delta x \Delta z \\
& + \left(\tau^{zx} + \frac{\partial \tau^{zx}}{\partial z}\Delta z\right)\Delta x \Delta y - \tau^{zx}\Delta x \Delta y
\end{aligned}
$$

For per unit volume

$$
F^{x} = \frac{\partial \tau^{xx}}{\partial x} + \frac{\partial \tau^{yx}}{\partial y} + \frac{\partial \tau^{zx}}{\partial z}
$$

For Newtonian fluids, viscous stress is: $\tau^{zx} = \mu \frac{\partial u}{\partial z}$

μ: dynamic viscosity coefficient

Assumption:
μ is constant

$$F^x = \frac{\partial \tau^{xx}}{\partial x} + \frac{\partial \tau^{yx}}{\partial y} + \frac{\partial \tau^{zx}}{\partial z}$$
$$= \frac{1}{\partial x}\left(\mu \frac{\partial u}{\partial x}\right) + \frac{1}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) + \frac{1}{\partial z}\left(\mu \frac{\partial u}{\partial z}\right)$$
$$= \mu \nabla^2 u$$

Laplacian operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$F^y = \frac{\partial \tau^{xy}}{\partial x} + \frac{\partial \tau^{yy}}{\partial y} + \frac{\partial \tau^{zy}}{\partial z}$$
$$= \frac{1}{\partial x}\left(\mu \frac{\partial v}{\partial x}\right) + \frac{1}{\partial y}\left(\mu \frac{\partial v}{\partial y}\right) + \frac{1}{\partial z}\left(\mu \frac{\partial v}{\partial z}\right)$$
$$= \mu \nabla^2 v$$

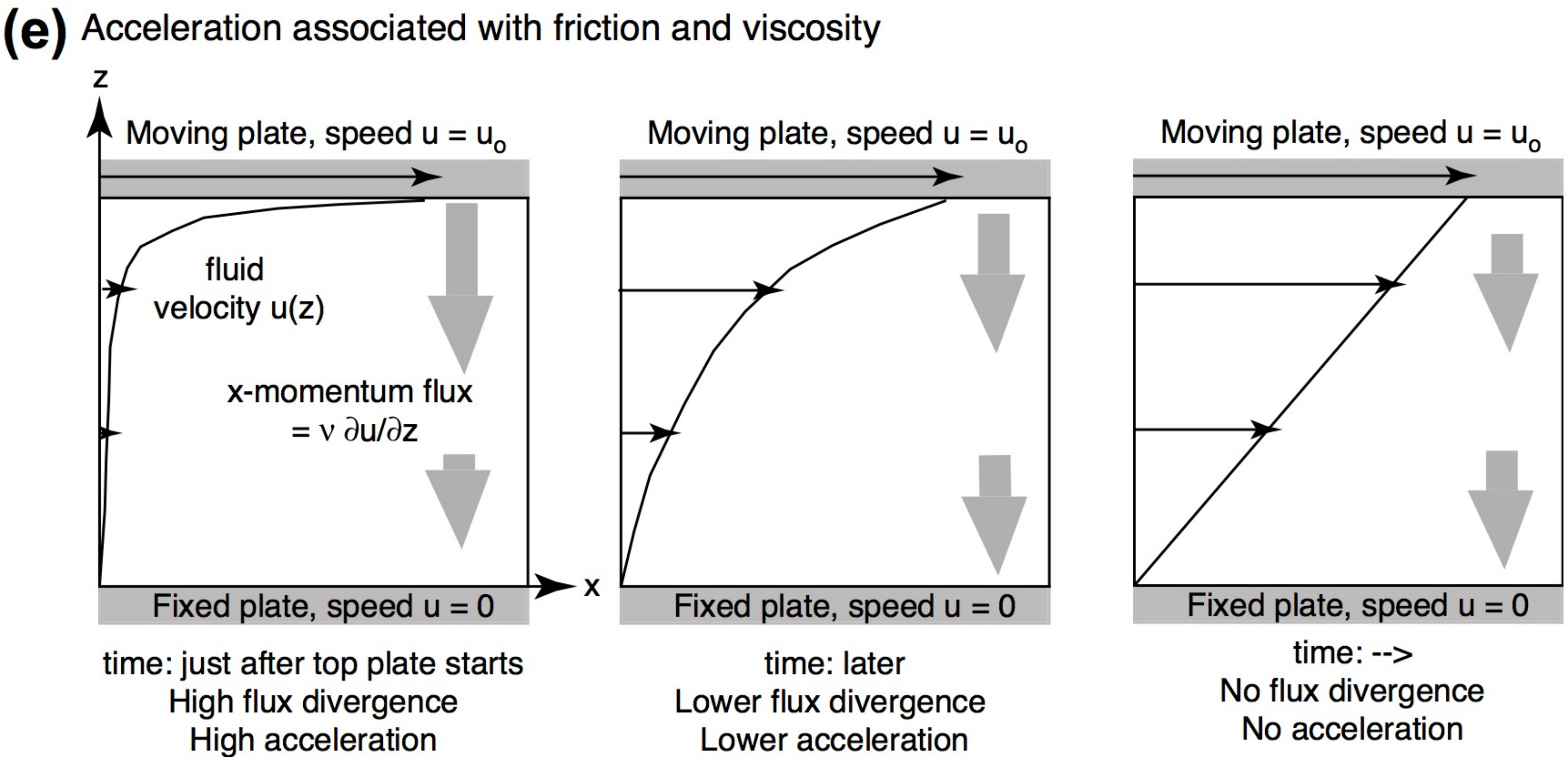
$$\frac{F^x}{\rho} = \frac{\mu}{\rho} \nabla^2 u$$
$$= \nu \nabla^2 u$$

ν : kinematic viscosity coefficient

$$F^z = \frac{\partial \tau^{xz}}{\partial x} + \frac{\partial \tau^{yz}}{\partial y} + \frac{\partial \tau^{zz}}{\partial z}$$
$$= \frac{1}{\partial x}\left(\mu \frac{\partial w}{\partial x}\right) + \frac{1}{\partial y}\left(\mu \frac{\partial w}{\partial y}\right) + \frac{1}{\partial z}\left(\mu \frac{\partial w}{\partial z}\right)$$
$$= \mu \nabla^2 w$$

	μ ($\mathrm{kg\,m^{-1}\,s^{-1}}$)	ν ($\mathrm{m^2\,s^{-1}}$)
Air	1.8×10^{-5}	1.5×10^{-5}
Water	1.1×10^{-3}	1.1×10^{-6}
Mercury	1.6×10^{-3}	1.2×10^{-7}

$$\tau^{zx} = \mu \frac{\partial u}{\partial z}$$



Acceleration is finally determined by the divergence of the viscous stress.

$$\frac{\partial}{\partial z}\left(\nu \frac{\partial u}{\partial z}\right)$$

The momentum equations

x direction: $$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+$$

$\nu\nabla^2 u$ (for constant ν)

$$\underline{\frac{\partial}{\partial x}\left(\nu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\nu\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(\nu\frac{\partial u}{\partial z}\right)}+\cdots$$

y direction: $$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+$$

$\nu\nabla^2 v$

$$\frac{\partial}{\partial x}\left(\nu\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(\nu\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(\nu\frac{\partial v}{\partial z}\right)+\cdots$$

z direction: $$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+$$

$\nu\nabla^2 w$

$$\frac{\partial}{\partial x}\left(\nu\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(\nu\frac{\partial w}{\partial y}\right)+\frac{\partial}{\partial z}\left(\nu\frac{\partial w}{\partial z}\right)+\cdots$$

Expressions for the force terms are given in the inertial frame.

So is the acceleration term.

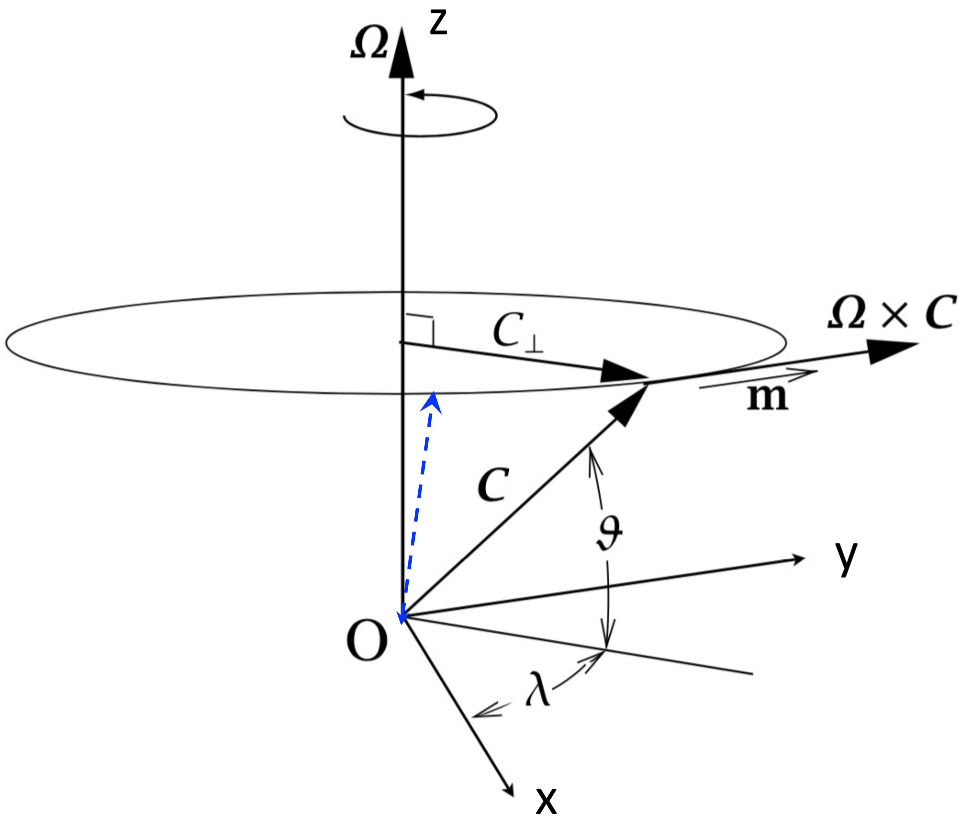


Fig. 2.1 A vector C rotating at an angular velocity Ω. It appears to be a constant vector in the rotating frame, whereas in the inertial frame it evolves according to $(\mathrm{d}C/\mathrm{d}t)_I = \Omega \times C$.

Coriolis Force

The change in C *in* δt with respect to the Intertial frame

$$\delta \boldsymbol{C} = |\boldsymbol{C}| \cos \vartheta \, \delta\lambda \, \boldsymbol{m},$$

$$\delta\lambda = |\boldsymbol{\Omega}| \delta t$$

Let $\hat{\vartheta} = (\pi/2 - \vartheta)$

$$\delta \boldsymbol{C} = |\boldsymbol{C}||\boldsymbol{\Omega}| \sin \hat{\vartheta} \, \boldsymbol{m} \, \delta t = \boldsymbol{\Omega} \times \boldsymbol{C} \, \delta t.$$

$$\left(\frac{\mathrm{d}\boldsymbol{C}}{\mathrm{d}t}\right)_I = \boldsymbol{\Omega} \times \boldsymbol{C}$$

Non-constant vector in the rotating frame

For a vector ***B*** that changes in the inertial frame:

$$(\delta \boldsymbol{B})_I = (\delta \boldsymbol{B})_R + (\delta \boldsymbol{B})_{rot}$$

$$\boxed{\begin{aligned} \delta \boldsymbol{C} &= \boldsymbol{\Omega} \times \boldsymbol{C}\, \delta t \\ \left(\frac{\mathrm{d}\boldsymbol{C}}{\mathrm{d}t}\right)_I &= \boldsymbol{\Omega} \times \boldsymbol{C} \end{aligned}}$$

With $(\delta \boldsymbol{B})_{rot} = \boldsymbol{\Omega} \times \boldsymbol{B}\, \delta t$

$$\left(\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega} \times \boldsymbol{B}$$

$\boldsymbol{r}$ is a vector from the Earth center pointing to the Earth surface

$$\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega} \times \boldsymbol{r}$$

$$\left(\frac{\mathrm{d}\boldsymbol{v}_I}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\boldsymbol{v}_R}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega} \times \boldsymbol{v}_R + \cancel{\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t} \times} \boldsymbol{r} + \boldsymbol{\Omega} \times \underbrace{\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_I}_{\boldsymbol{v}_I}$$

$$\boldsymbol{v}_I = \boldsymbol{v}_R + \boldsymbol{\Omega} \times \boldsymbol{r}$$

Centrifugal acceleration

$$\left(\frac{\mathrm{d}\boldsymbol{v}_R}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\boldsymbol{v}_R}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega} \times \boldsymbol{v}_R$$

$$\left(\frac{d\boldsymbol{v}_I}{dt}\right)_I = \left(\frac{d\boldsymbol{v}_R}{dt}\right)_R \underline{+2\boldsymbol{\Omega}\times\boldsymbol{v}_R} + \underline{\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})}$$

Coriolis acceleration

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v}_I - \boldsymbol{\Omega} \times \boldsymbol{r})\right)_I = \left(\frac{\mathrm{d}\boldsymbol{v}_R}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega} \times \boldsymbol{v}_R$$

The Coriolis acceleration

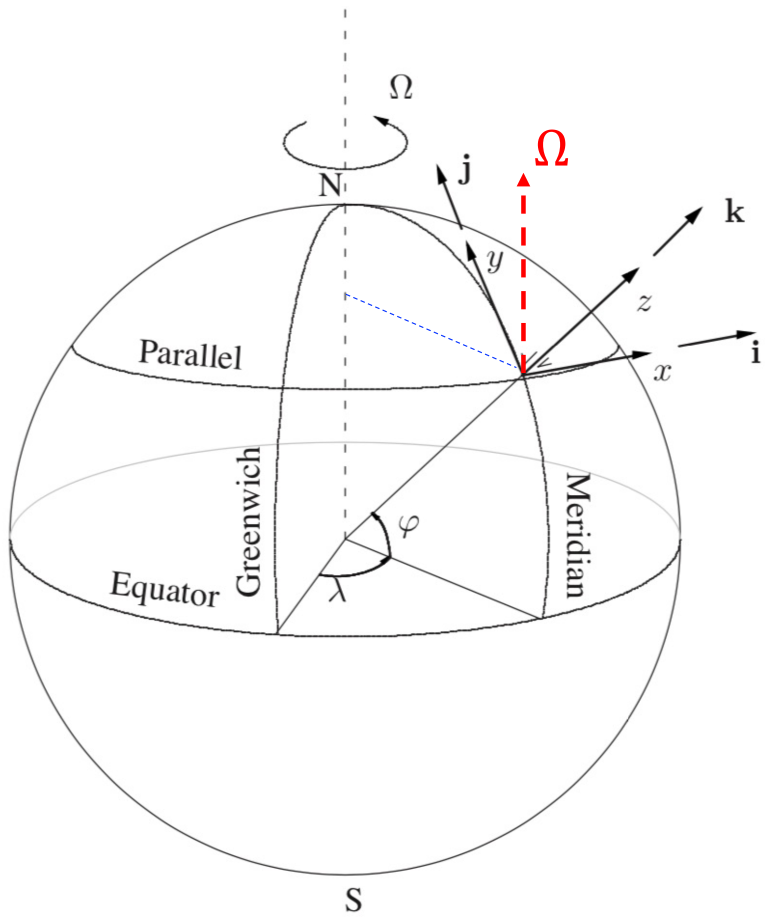
$$2\boldsymbol{\Omega}\times\boldsymbol{v}_R$$

$$\boldsymbol{\Omega} = \Omega \cos\varphi\,\mathbf{j} + \Omega \sin\varphi\,\mathbf{k}$$

x: $2\Omega \cos\varphi\, w - 2\Omega \sin\varphi\, v$

y: $2\Omega \sin\varphi\, u$

z: $-2\Omega \cos\varphi\, u.$



$$f = 2\Omega \sin\varphi$$
$$f_* = 2\Omega \cos\varphi.$$

f: Coriolis parameter

The momentum equations

$$\left(\frac{d\boldsymbol{v}_I}{dt}\right)_I = \left(\frac{d\boldsymbol{v}_R}{dt}\right)_R + 2\boldsymbol{\Omega}\times\boldsymbol{v}_R + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}) = \text{ force terms}$$

Nonlinear advection term Coriolis term Pressure gradient term

x direction: $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv + f_* w = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u + \cdots$$

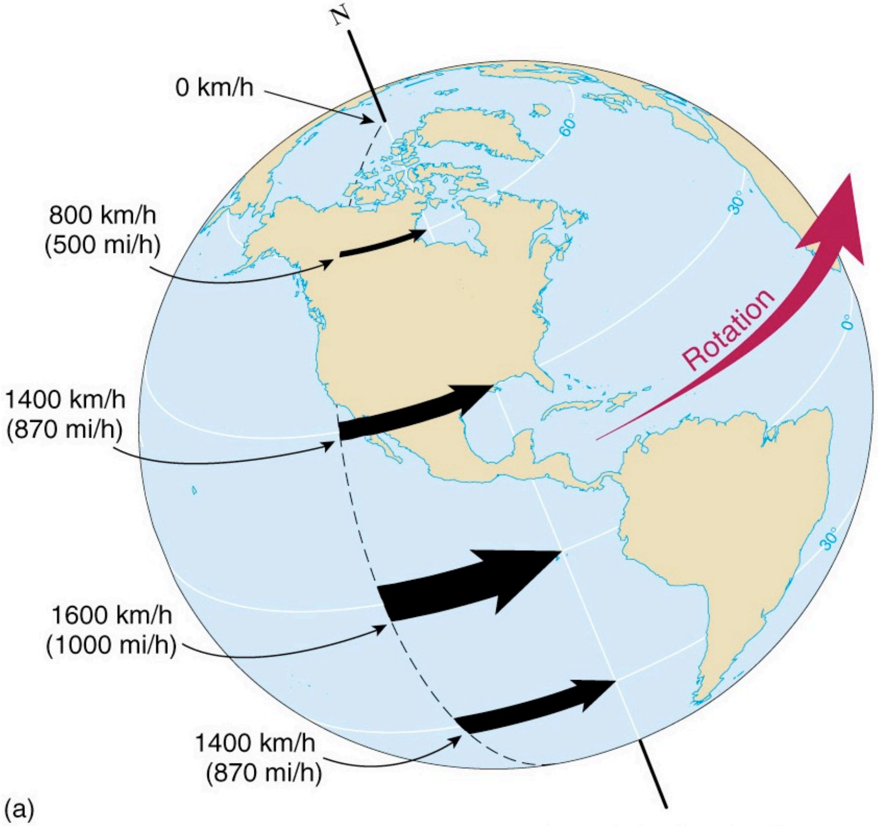
Viscosity term

Local acceleration

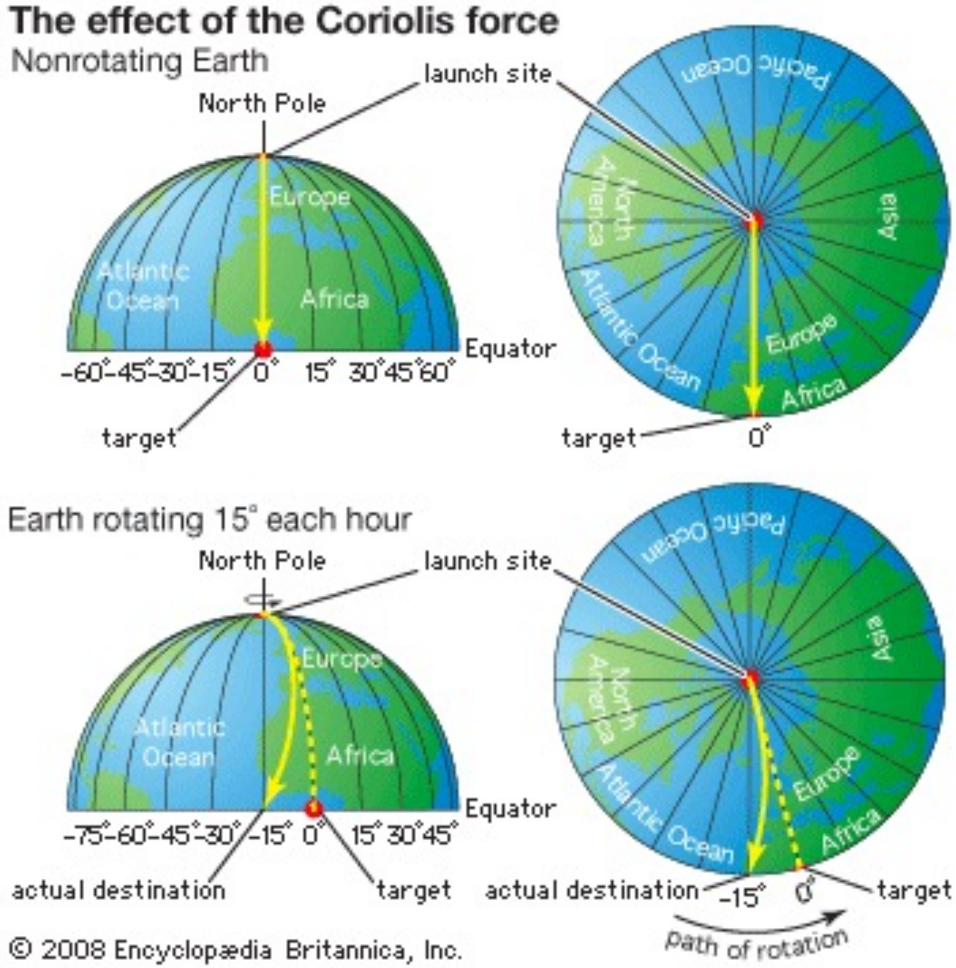
y direction: $$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v + \cdots$$

z direction: $$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} - f_* u = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w + \cdots$$

The Coriolis Force

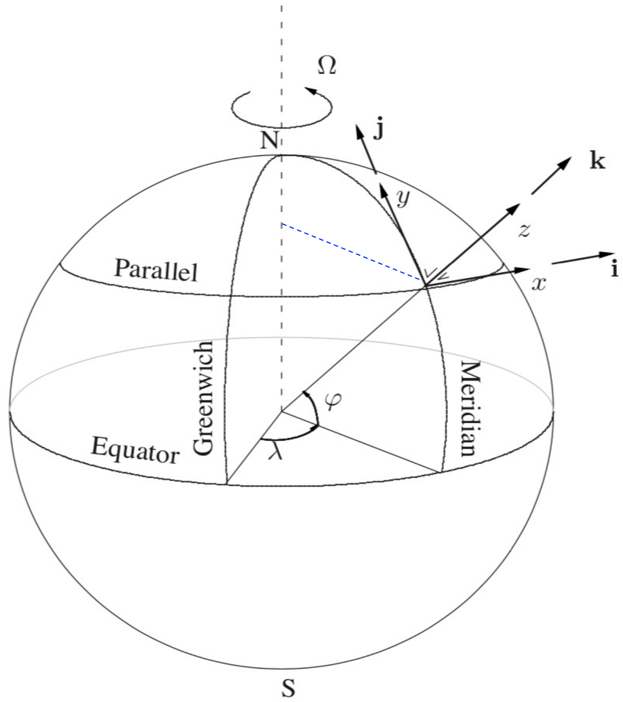


(a)



f-plane and β-plane

$$f = 2\Omega\sin\varphi = \underbrace{2\Omega\big(sin\varphi_0}_{f_0} + cos\varphi_0(\varphi - \varphi_0)\big)$$



If $\varphi - \varphi_0$ is small:

f-plane: $f = f_0$ is a constant.

If $\varphi - \varphi_0$ cannot be neglected:

β-plane: $f = f_0 + 2\Omega cos\varphi_0 \underbrace{(\varphi - \varphi_0)}_{y/a} = f_0 + \underbrace{\frac{2\Omega\cos\varphi_0}{a}}_{\beta} y$