Ekman Surface Layer

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（按汇报顺序）

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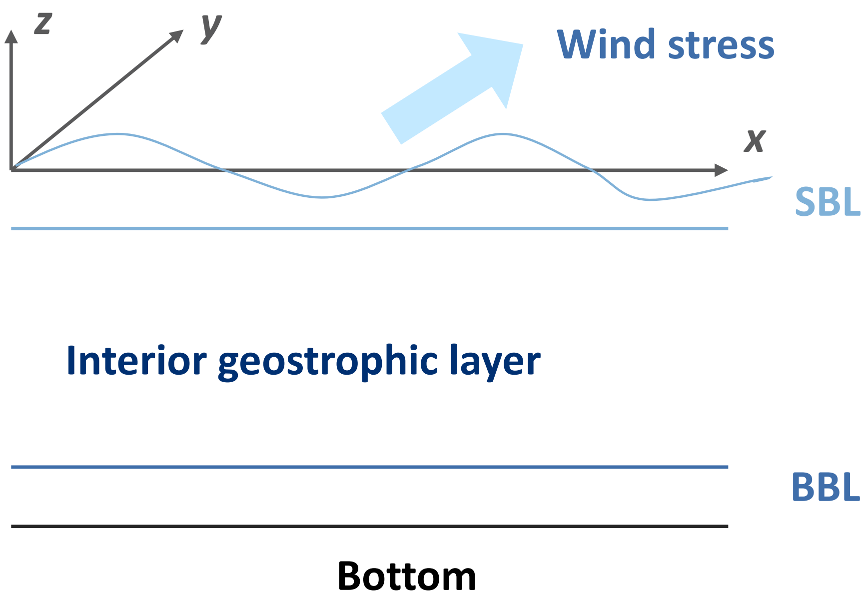
Outline

1. Assumptions and equation
2. Deduction and solution
3. Ekman spiral and Ekman depth
4. Ekman transport and pumping

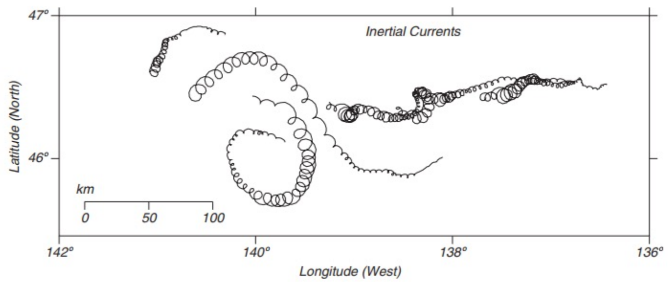
The Ekman surface layer

An Ekman layer occurs wherever there is **a horizontal frictional stress**.(Cushman-Roisin and Beckers., 2011)



- A strong wind blew across the sea

$$\frac{du}{dt} = fv \qquad \frac{dv}{dt} = -fu \quad \longrightarrow \quad u = V\sin(ft+\phi), \quad v = V\cos(ft+\phi), \quad V^2 = u^2 + v^2$$



- How about **steady winds blowing on the sea surface continuously** ?

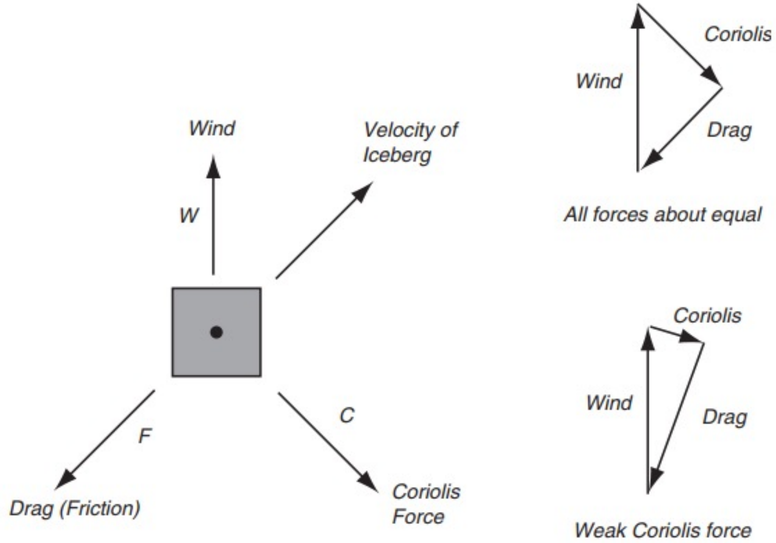
Background

Nansen observed that icebergs moved at angles of between 20-40° to the right of the surface winds in the Arctic (1893).

The balance between three forces exists:

- Wind Stress, $\boldsymbol{W}$;
- Friction, $\boldsymbol{F}$;
- Coriolis Force, $\boldsymbol{C}$.

$$\boldsymbol{W} + \boldsymbol{F} + \boldsymbol{C} = 0$$



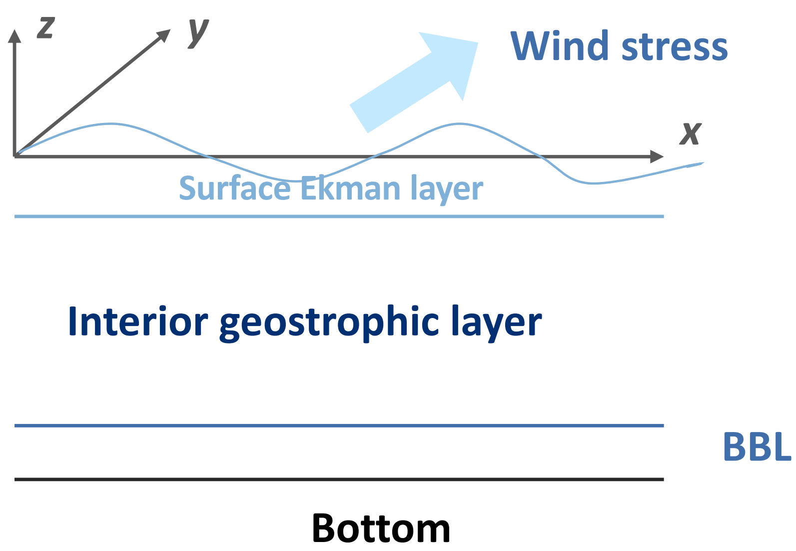
The balance of forces in a wind **on a rotating earth**

1. **Drag** must be opposite the direction of the ice's velocity;
2. **Coriolis force** must be perpendicular to the velocity;
3. The forces must balance for **steady flow**.

Assumptions/Conditions

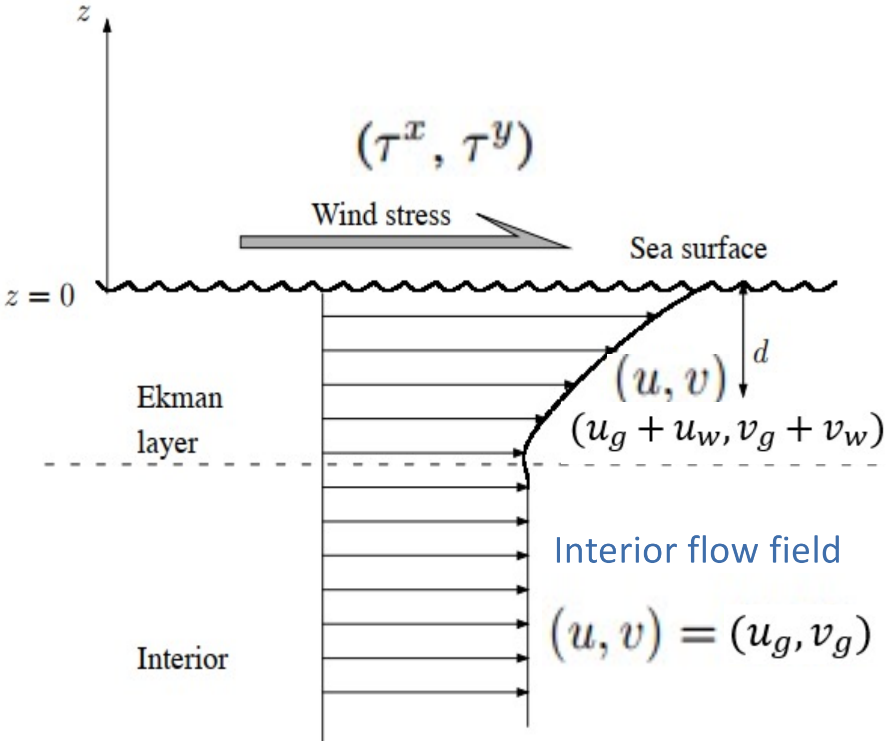
Ekman layer theory - oversimplified based on the assumptions:

- No boundaries (open ocean)
- Infinitely deep
- A_Z constant
- Steady-state (wind and currents)
- No pressure gradient, no mixing, no waves
- Homogeneous water
- f-plane ($f = f_0$)
- Motions are horizontal



Below the surface layer, the current is geostrophic.

Equation



The surface Ekman layer generated by a wind stress on the ocean

Ekman assumed **steady, homogeneous and horizontal flow** with friction on a rotating Earth.

The Horizontal and time derivatives are zero,

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$$

$$\boxed{\begin{aligned} -f(\cancel{v_g} + v_w) &= \cancel{-\frac{1}{\rho_0}\frac{\partial p}{\partial x}} + \frac{\partial}{\partial z}\left(K_z \frac{\partial(\cancel{u_g} + u_w)}{\partial z}\right) \\ f(\cancel{u_g} + u_w) &= \cancel{-\frac{1}{\rho_0}\frac{\partial p}{\partial y}} + \frac{\partial}{\partial z}\left(K_z \frac{\partial(\cancel{v_g} + v_w)}{\partial z}\right) \end{aligned}}$$

A constant vertical eddy viscosity of the form:

$$T_{xz} = \rho A_z \frac{\partial u}{\partial z} \qquad T_{yz} = \rho A_z \frac{\partial v}{\partial z}$$

$\boldsymbol{T}_{xz}$, $\boldsymbol{T}_{yz}$ - the vertical viscosity term
ρ - the density of sea water

Equation

$$T_{xz} = \rho A_z \frac{\partial u}{\partial z} \qquad T_{yz} = \rho A_z \frac{\partial v}{\partial z} \longrightarrow \begin{cases} \dfrac{\partial T_{xz}}{\partial z} = \dfrac{\partial}{\partial z}\left(\rho A_z \dfrac{\partial u}{\partial z}\right) \approx \rho A_z \dfrac{\partial^2 u}{\partial z^2} \\ \dfrac{\partial T_{yz}}{\partial z} = \dfrac{\partial}{\partial z}\left(\rho A_z \dfrac{\partial v}{\partial z}\right) \approx \rho A_z \dfrac{\partial^2 v}{\partial z^2} \end{cases}$$

Equation

$$\boxed{\begin{cases} fv + A_z \dfrac{\partial^2 u}{\partial z^2} = 0 \\ -fu + A_z \dfrac{\partial^2 v}{\partial z^2} = 0 \end{cases}} \qquad \boxed{\begin{cases} \rho f v + \rho A_z \dfrac{\partial^2 u}{\partial z^2} = 0 \\ -\rho f u + \rho A_z \dfrac{\partial^2 v}{\partial z^2} = 0 \end{cases}}$$

Solution

$$\begin{cases} u = V_0 \exp(az) \cos\left(\dfrac{\pi}{4} + az\right) \\ v = V_0 \exp(az) \sin\left(\dfrac{\pi}{4} + az\right) \end{cases}$$

In $T = T_{yz}$,the constant $\boldsymbol{V}_0$

$$V_0 = \frac{T}{\sqrt{\rho_w^2 f A_z}}, a = \sqrt{\frac{f}{2A_z}}$$

The transformation of equation's form

- We know: $\begin{cases} fv + A_z \dfrac{\partial^2 u}{\partial z^2} = 0 \quad (1) \\ -fu + A_z \dfrac{\partial^2 v}{\partial z^2} = 0 \quad (2) \end{cases}$ and $\begin{cases} z = 0, \quad \tau = \tau_y = \rho A_z \dfrac{\partial v}{\partial z} \\ z = -\infty, \quad u = \mathrm{v} = 0 \end{cases}$

y wind

- How does u v merge into one parameter w? 👉 $w = u + vi, \quad \tau = 0 + i\tau_y$

👉 $(1) + (2) \times i$

Convert to: $A_z \dfrac{\partial^2 w}{\partial z^2} - ifw = 0$ 👉 $\dfrac{\partial^2 w}{\partial z^2} - j^2 w = 0, \quad j = (1+i)a$

The general solution is: $w = c_1 e^{jz} + c_2 e^{-jz}$

The solution of velocity’ s expression

- Using the boundary condition, we can solve c_1 、 c_2 （easy! you can try!）

Result: $\begin{cases} C_2 = 0 \\ C_1 = \dfrac{i\tau_y}{\rho A_z j} \end{cases}$ 👉 $w = \dfrac{i\tau_y}{\rho A_z j} e^{jz}$ 👉 replace j with i: $j = (1+i)a$

- Decompose w into u v (you can practice it after class)

$$\cos\left(az + \frac{\pi}{4}\right) \times \frac{2}{\sqrt{2}}$$

$$\begin{cases} u = \dfrac{\tau_y}{2\rho A_z a} e^{az}(\cos az - \sin az) \\ v = \dfrac{\tau_y}{2\rho A_z a} e^{az}(\cos az + \sin az) \end{cases}$$

$$\sin\left(az + \frac{\pi}{4}\right) \times \frac{2}{\sqrt{2}}$$

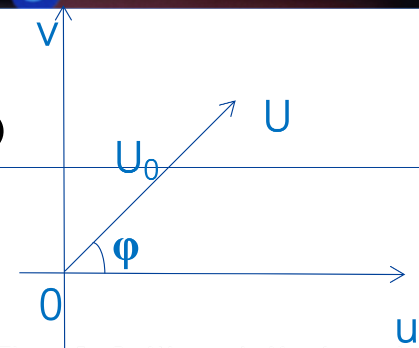
- Replace $a = \sqrt{\dfrac{f}{2A_z}}$, Coefficient: $\dfrac{\tau_y}{\sqrt{A_z f \rho^2}} = V_0$

Velocity solution: $\begin{cases} u = V_0 e^{az} \cos\left(az + \frac{\pi}{4}\right) \\ v = V_0 e^{az} \sin\left(az + \frac{\pi}{4}\right) \end{cases}$

Ekman spiral

θ: wind direction

$$(\theta \pm \frac{\pi}{4})$$



$$u = V_0 e^{az}\cos(az + \frac{\pi}{4})$$

$$v = V_0 e^{az}\sin(az + \frac{\pi}{4})$$

$$\xrightarrow[e^{i\theta} = \cos\theta + i\sin\theta]{U = u + iv}$$

$$U = \underbrace{\frac{\tau_y}{\sqrt{fA_z\rho^2}}}_{U_0} e^{\frac{\pi}{D_E}z} e^{\underbrace{i(\frac{\pi}{D_E}z + \frac{\pi}{4})}_{\Phi}}$$

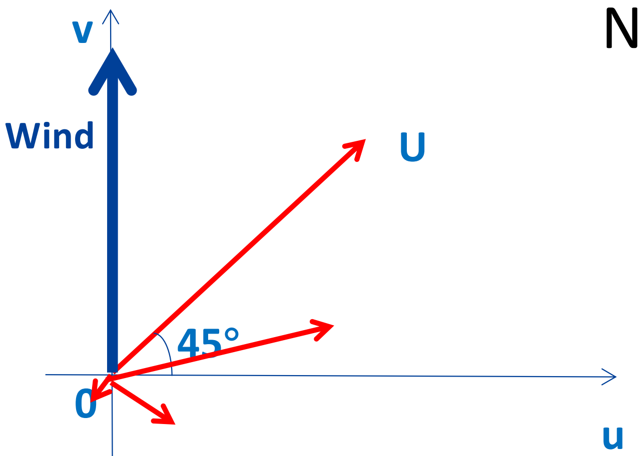
here,

$$\tau_y = \rho_{air} C_D u_{10}^2$$

$$V_0 = \frac{\tau_y}{\sqrt{\rho^2 f A_z}}$$

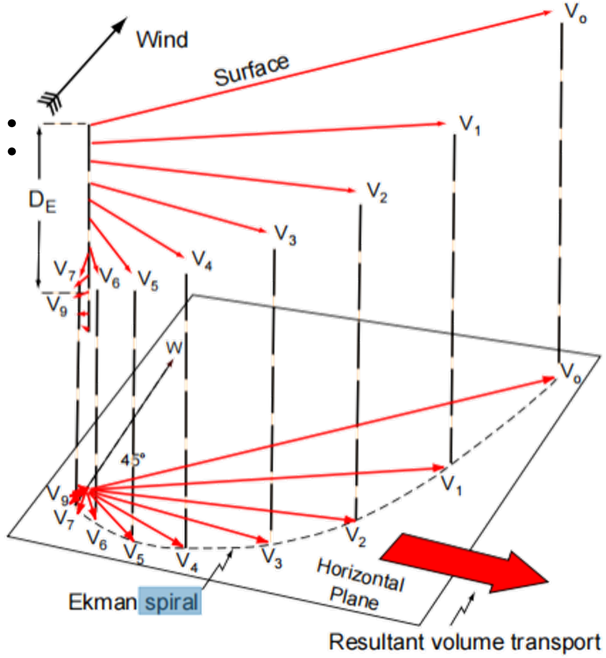
$$a = \sqrt{\frac{f}{2A_z}} = \sqrt{\omega \sin\varphi / A_z} = \frac{\pi}{D_E}$$

$$D_E = \pi\sqrt{\frac{2A_z}{f}}$$



NH：

"Ekman flow" - Water velocities decreasing and rotating with increasing depth:



Lynne D. Talley, George L. Pickard, Willaim J. Emery, James H. Swift. 2011. *Descriptive Physical Oceanography: An Introduction (Sixth Edition).* Published by Elsevier Ltd. 197-199.

Ekman depth D_E

$$D_E = \pi\sqrt{\frac{2A_z}{f}} \qquad U = \frac{\tau_y}{\sqrt{fA_z\rho^2}} e^{\frac{\pi}{D_E}z} e^{i(\frac{\pi}{D_E}z+\frac{\pi}{4})}$$

$z = 0$:

$$U_0 = \frac{\tau_y}{\sqrt{\rho^2 fA_z}} \qquad \varphi = \frac{\pi}{4}$$

$z = -D_E$

$$U_{-D_E} = \frac{\tau_y}{\sqrt{\rho^2 fA_z}} e^{-\pi} \qquad \varphi = -\frac{3\pi}{4}$$

$= 0.043U_0$ (e-fold)

the opposite direction from the surface velocity

Determined value:

$\rho = 1027kg/m^3$ $\rho_{air} = 1.25kg/m^3$

$C_D = 2.6\times10^{-3}$ $A_z = 10^{-2} - 10^{-5}$

$\Omega = 7.29\times10^{-5} rad/s$

$$U_0 = \frac{\rho_{air}C_D u_{10}^2}{\sqrt{2\rho^2\Omega\sin|\varphi|A_z}} \approx 2.5\% - 1.1\% u_{10}, |\varphi| \geq 10^{\circ}$$

$$D_E = \pi\sqrt{\frac{2A_z}{2\Omega\sin|\varphi|}} \approx 45 \sim 300m$$

Vertical Ekman number Ez

$$E_z = \frac{A_z \dfrac{\partial^2 u}{\partial z^2}}{fu} = \frac{A_z \dfrac{U}{D_E^2}}{fU} = \frac{A_z}{fD_E^2}$$ **(vertical viscosity term/Coriolis term)**

$$D_E = \pi\sqrt{\frac{2A_z}{f}} \qquad D_E = \sqrt{\frac{A_z}{fE_z}}$$

$$E_z = \frac{1}{2\pi^2} \approx 0.05$$

$z = -D_E$:vertical viscosity term<<Coriolis term

Ekman transport

$$\begin{cases} fv + A_z \dfrac{\partial^2 u}{\partial z^2} = 0 \\ -fu + A_z \dfrac{\partial^2 v}{\partial z^2} = 0 \end{cases}$$

Ekman transport: vertical integral of ekman velocity u, v times density ρ from bottom to surface

$$M_{\mathrm{E}x} = \int_{-\infty}^{0} \rho u \mathrm{d}z,\ M_{\mathrm{E}y} = \int_{-\infty}^{0} \rho v \mathrm{d}z$$

$$\begin{cases} -fM_{\mathrm{E}y} = \displaystyle\int_{-\infty}^{0} f\rho u dz = \rho A_z \int_{-\infty}^{0} \frac{\partial^2 u}{\partial z^2} dz = \rho A_z \frac{\partial u}{\partial z}\bigg|_0 - \rho A_z \frac{\partial u}{\partial z}\bigg|_{-\infty} = \tau_x \\ fM_{\mathrm{E}x} = \displaystyle\int_{-\infty}^{0} f\rho v dz = \rho A_z \int_{-\infty}^{0} \frac{\partial^2 v}{\partial z^2} dz = \rho A_z \frac{\partial v}{\partial z}\bigg|_0 - \rho A_z \frac{\partial v}{\partial z}\bigg|_{-\infty} = \tau_y \end{cases} \longrightarrow \begin{cases} M_{\mathrm{E}y} = -\dfrac{\tau_x}{\rho f} \\ M_{Ex} = \dfrac{\tau_y}{\rho f} \end{cases}$$

South wind: $\tau_x = 0, M_{\mathrm{E}x} = \frac{\tau_y}{\rho f}, M_{\mathrm{E}y} = 0$

Ekman transport is perpendicular and to th **right** (left) of the wind in the **Northern** (Southern) Hemisphere

Ekman pumping

For incompressible fluids: $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$

Integrate vertically from $-D_E$ to 0: $\rho\int_{-D_E}^{0}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\mathrm{dz}=0$

$$\longrightarrow \quad \frac{\partial M_x}{\partial x}+\frac{\partial M_y}{\partial y}=-\rho\big(w(0)-w(-D_E)\big)$$

Surface: $w(0)=\frac{\partial \eta}{\partial t}=0$ $\longrightarrow$ $\nabla\cdot M=\rho w(-D_E)$

The relationship between Ekman pumping and wind stress curl:

$$w(-D_E)=\left(\frac{\partial}{\partial x}\frac{\tau_y}{f}-\frac{\partial}{\partial y}\frac{\tau_x}{\rho f}\right)=curl\left(\frac{\tau}{\rho f}\right)$$

Ekman pumping

$$w(-D_E) = \left(\frac{\partial}{\partial x}\frac{\tau_y}{\rho f} - \frac{\partial}{\partial y}\frac{\tau_x}{\rho f}\right) = curl\left(\frac{\tau}{\rho f}\right)$$

➢ Divergence: upwelling

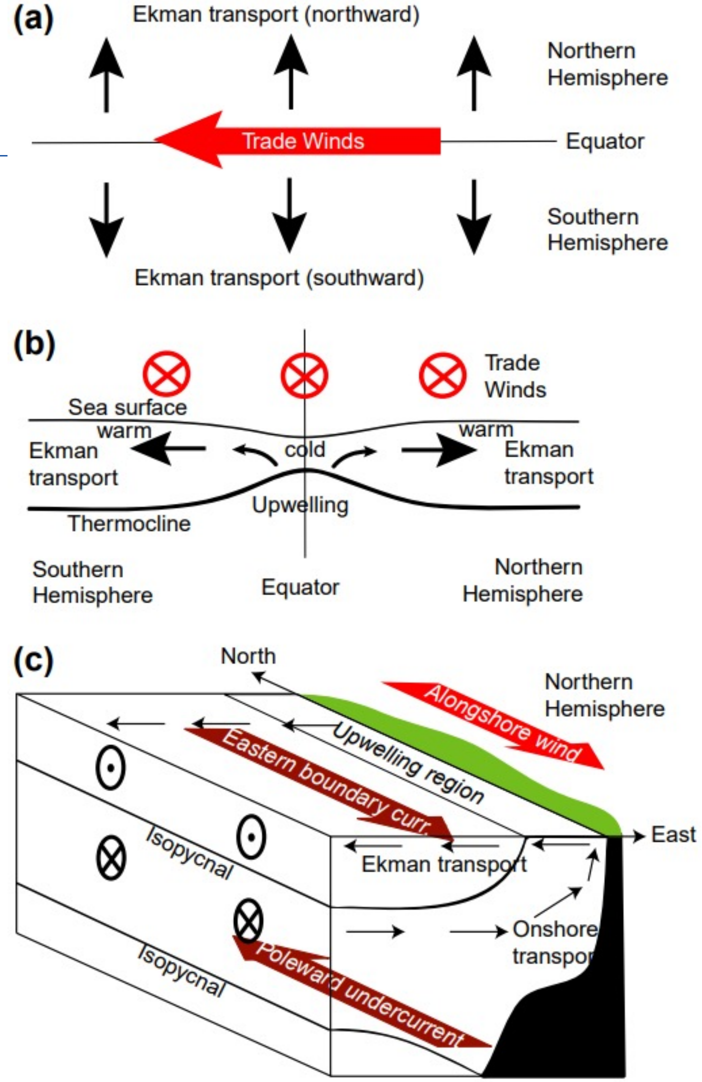
$$\frac{\partial}{\partial x}\frac{\tau_y}{\rho f} - \frac{\partial}{\partial y}\frac{\tau_x}{\rho f} > 0 \qquad w(-D_E) > 0$$

The positive wind stress curl causes the divergence of the Ekman layer and the Ekman upwelling

➢ Convergence: downwelling

$$\frac{\partial}{\partial x}\frac{\tau_y}{\rho f} - \frac{\partial}{\partial y}\frac{\tau_x}{\rho f} < 0 \qquad w(-D_E) < 0$$

The negetive wind stress curl causes the convergence of the Ekman layer and the Ekman downwelling



Summary

Northern Hemisphere:

- The angle of surface current is 45 degrees to the right of the wind.
- The Ekman transport is exactly perpendicular and to the right of the wind.
- Upwelling into the Ekman layer results from positive wind stress curl, and downwelling results from negative wind stress curl.

Thank you！

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