Stratification

董铠溢、高志斌、郑璐曦、危国锐

上海交通大學
SHANGHAI JIAO TONG UNIVERSITY

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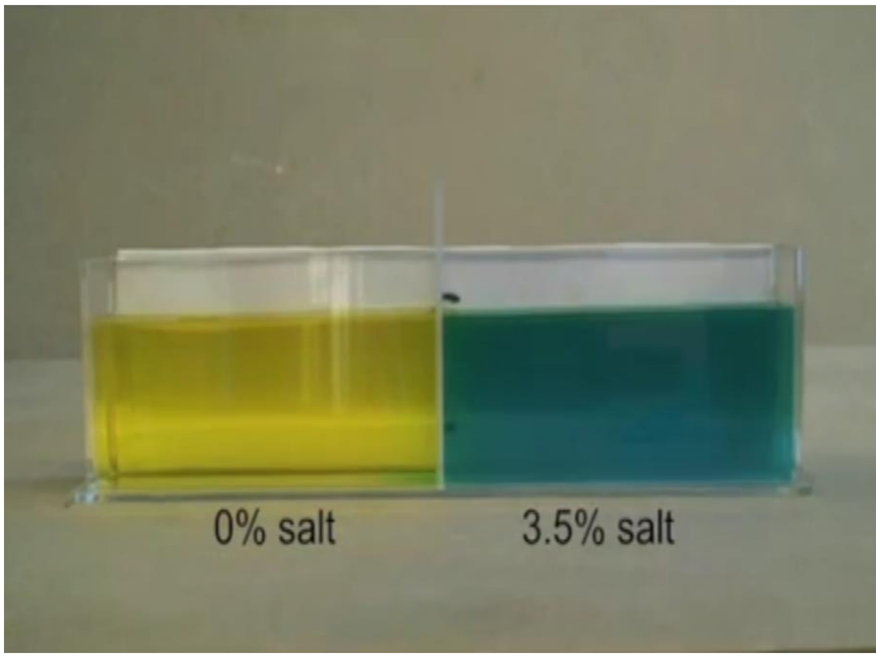
Introduction



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- Assumption： static equilibrium
- （absence of horizontal forces）

$$g[\rho(z)-\rho(z+h)]V$$

(Archimedes’ buoyancy principle)

$$\rho(z)\, V\frac{\mathrm{d}^2h}{\mathrm{d}t^2}=g\,[\rho(z+h)-\rho(z)]\,V.$$

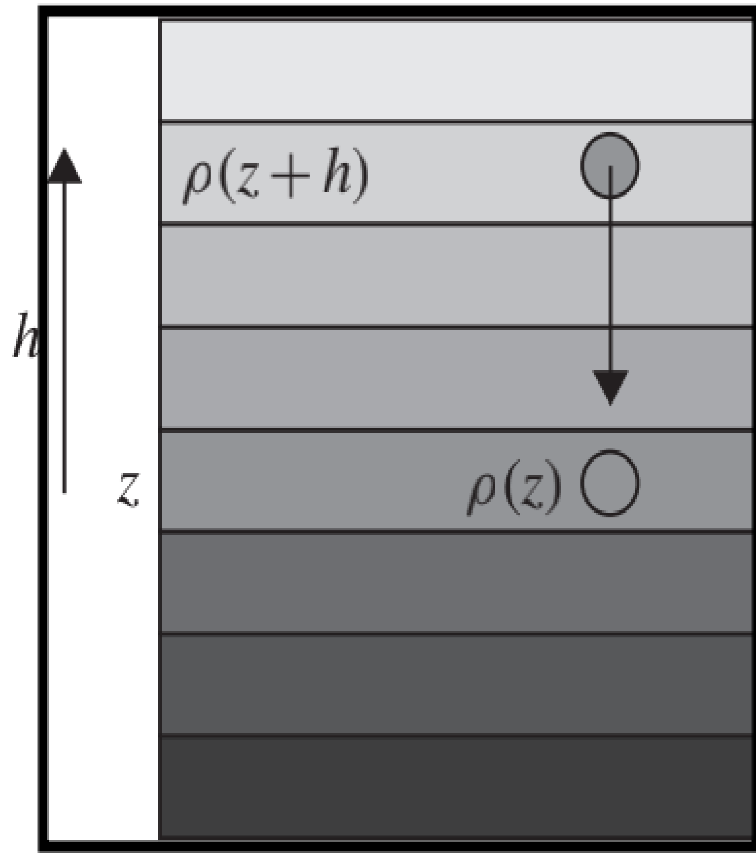
(Newton’s law)

$$\rho(z+h)-\rho(z)\simeq\frac{\mathrm{d}\rho}{\mathrm{d}z}h.$$

(Taylor expansion)

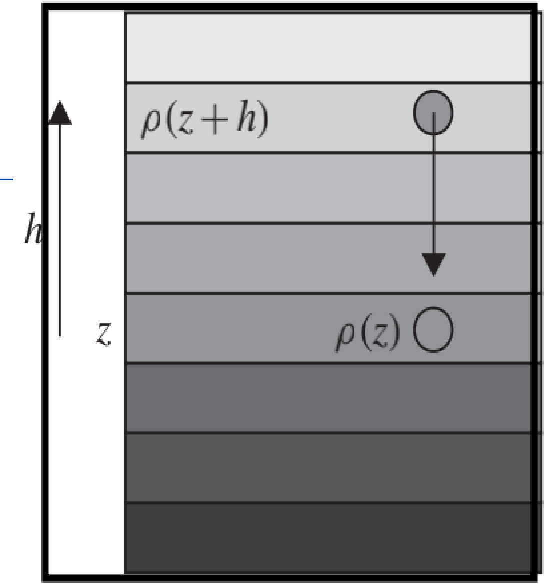
$$\rho(z)=\rho_0$$

(Boussinesq approximation)



$$\frac{\mathrm{d}^2h}{\mathrm{d}t^2}-\frac{g}{\rho_0}\frac{\mathrm{d}\rho}{\mathrm{d}z}h=0,$$

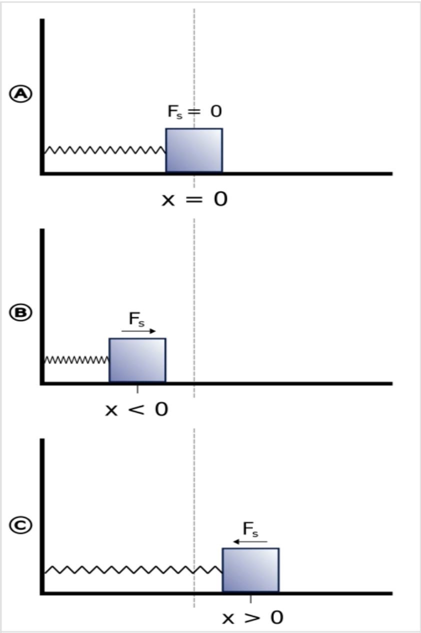
$$\frac{d^2h}{dt^2} - \frac{g}{\rho_0}\frac{d\rho}{dz}h = 0,$$



- Assumption: The coefficient $-\frac{g}{\rho_0}\frac{d\rho}{dz}$ is positive
- 常系数齐次微分方程:

$$h''(t) + N^2 * h(t) = 0$$

$$N^2 = -\frac{g}{\rho_0}\frac{d\rho}{dz},$$



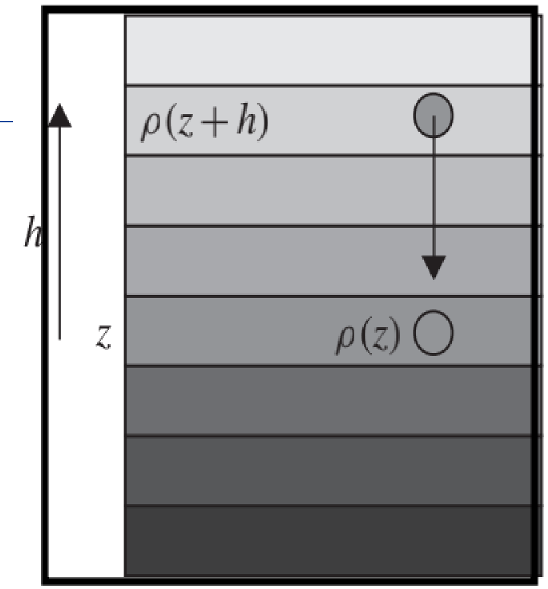
N:

1.stratification frequency

2. Brunt–Väisälä frequency

3.buoyancy frequency

$$\frac{\mathrm{d}^2 h}{\mathrm{d}t^2} - \frac{g}{\rho_0}\frac{\mathrm{d}\rho}{\mathrm{d}z}h = 0,$$



- Assumption: The coefficient $-\frac{g}{\rho_0}\frac{\mathrm{d}\rho}{\mathrm{d}z}$ is negative
- 常系数齐次微分方程:

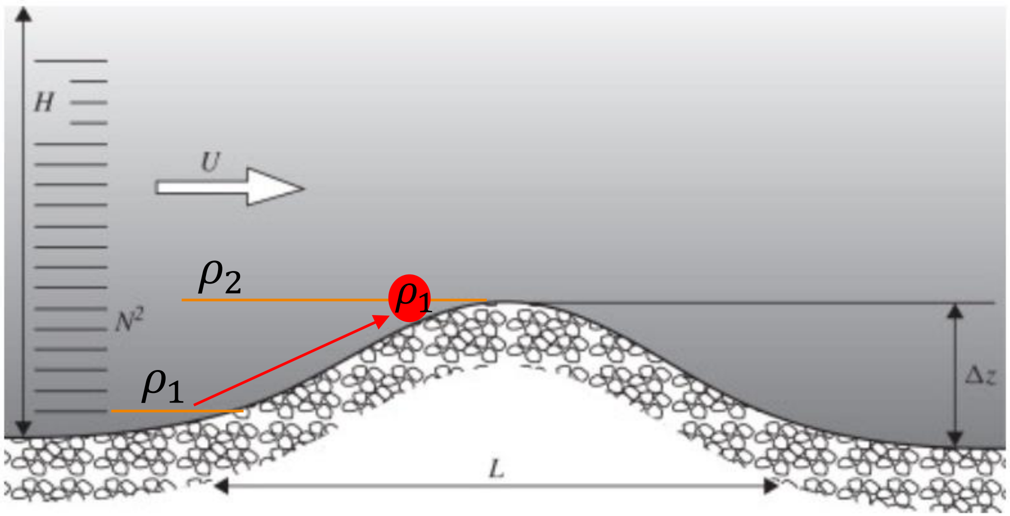
$$N^2 = +\frac{g}{\rho_0}\frac{d\rho}{dz}$$

$$h''(t) - N^2 * h(t) = 0$$

exhibits exponential growth

Instability!!!

The Froude Number



$$T = \frac{L}{U}$$

$$\Longrightarrow W = \frac{\Delta z}{T} = \frac{U \Delta z}{L}$$

$$\Delta\rho = \left| \frac{\mathrm{d}\bar{\rho}(z)}{\mathrm{d}z} \right| \Delta z$$

$\bar{\rho}(z)$: density variation due to stratification

$$= \frac{\rho_0 N^2}{g} \Delta z$$

$$N^2 = -\frac{g}{\rho_0} \frac{\mathrm{d}\rho}{\mathrm{d}z}$$

the hydrostatic balance $\Delta P = \Delta\rho g\mathrm{H} = \rho_0 N^2 \Delta z \mathrm{H}$

The Froude Number

x direction: $\cancel{\frac{\partial u}{\partial t}} + u\frac{\partial u}{\partial \lambda} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + \cancel{fv} + \cancel{\vartheta\nabla^2 u}$

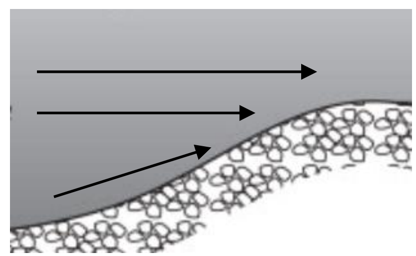
$$\frac{U^2}{L} \qquad\qquad \frac{1}{\rho_0}\frac{\Delta P}{L}$$

$$\because \Delta P = \rho_0 N^2 \Delta z \mathrm{H}$$

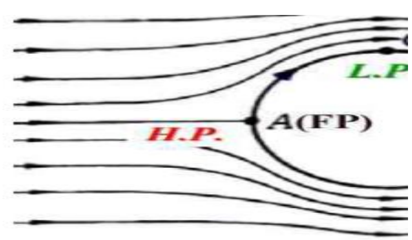
$$\Longrightarrow \quad U^2 = N^2 H \Delta z$$

$$\mathrm{div}\vec{\mathrm{u}} = \frac{\partial \mathrm{u}}{\partial x} + \frac{\partial \mathrm{v}}{\partial \mathrm{y}} + \frac{\partial \mathrm{w}}{\partial \mathrm{z}} = 0$$

$$\frac{U}{L} \qquad\qquad \frac{W}{H}$$



vertical convergence



horizontal divergence

The Froude Number

$$\left.\begin{aligned} W &= \frac{U\Delta z}{L} \\ U^2 &= N^2 H \Delta z \end{aligned}\right\} \Longrightarrow \frac{W/H}{U/L} = \frac{\Delta z}{H} = \frac{U^2}{N^2 H^2}$$

$$\boxed{Fr = \frac{U}{NH}}$$ Froude number

Fr<<1: W/H<<U/L

vertical convergence cannot meet horizontal divergence

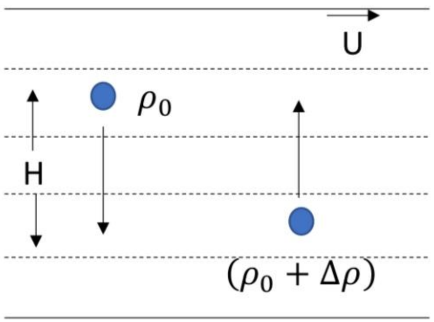
$$\mathrm{div}\vec{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$ deflect horizontally

Fr shows the Importance of Stratification：

Stratification act to restrict or minimize vertical displacements

the stronger the stratification, the smaller *Fr*

The Froude Number



For per unit volume,

Potential energy change:

$$\Delta PE = (\rho_0 + \Delta\rho) gH - \rho_0 gH = \Delta\rho gH$$

Kinetic energy:

$$KE = \frac{1}{2}\rho_0 U^2 + \frac{1}{2}(\rho_0 + \Delta\rho)\, U^2 \approx \rho_0 U^2$$

$$\sigma = \frac{KE}{\Delta PE} = \frac{\rho_0 U^2}{\Delta\rho gH} \sim \frac{U^2}{N^2 H^2}$$ Froude number

$\sigma > 1$, PE change consumes a small portion of the KE of the system, so it takes little cost to break stratification, stratification is unimportant

$\sigma \leq 1$, PE change consumes all KE of the system, or KE is not sufficient to supply ΔPE, stratification cannot be broken and is important

水平–无地转: $\boldsymbol{V}_{\mathrm{h}} \cdot \nabla_{\mathrm{h}} \boldsymbol{V}_{\mathrm{h}} \sim -\dfrac{1}{\rho} \nabla_{\mathrm{h}} p \Rightarrow \dfrac{U^2}{L} = \dfrac{\Delta P}{\rho_0 L}.$

竖直-层化 & 静力平衡:

$$\left.\begin{aligned} p = \rho g H &\Rightarrow \Delta P = g H \Delta\rho \\ N^2 = -\frac{g}{\rho}\frac{\partial\rho}{\partial z} &\Rightarrow \Delta\rho = \frac{N^2 \rho_0 \Delta z}{g} \end{aligned}\right\} \Rightarrow \Delta P = N^2 H \rho_0 \Delta z.$$

水平–地转流: $\boldsymbol{V}_{\mathrm{h}} \sim \dfrac{\boldsymbol{k}}{\rho f} \times \nabla_{\mathrm{h}} p \Rightarrow \Omega U = \dfrac{\Delta P}{\rho_0 L}.$

$$U^2 = N^2 H \Delta z. \cdots (1)$$

$$W = \frac{\Delta z}{T} = \frac{\Delta z}{L/U} \Rightarrow \frac{W/H}{U/L} = \frac{\Delta z}{H}. \cdots (3)$$

$$U = \frac{N^2 H \Delta z}{\Omega L}. \cdots (2)$$

Case 1-仅(竖直)层化、无地转:

$$(1)\ \&\ (3) \Rightarrow \frac{W/H}{U/L} = \frac{\Delta z}{H} = \left(\frac{U}{NH}\right)^2 =: \mathrm{Fr}^2.$$

Case 2-仅地转、无层化: [1, pp.357-358]

$$\frac{W/H}{U/L} = \mathrm{Ro} := \frac{U}{\Omega L}.$$

Case 3-(竖直)层化 + 地转:

$$(2)\ \&\ (3) \Rightarrow \frac{W/H}{U/L} = \frac{\Delta z}{H} = \frac{\Omega L U}{N^2 H^2} = \frac{\mathrm{Fr}^2}{\mathrm{Ro}}.$$

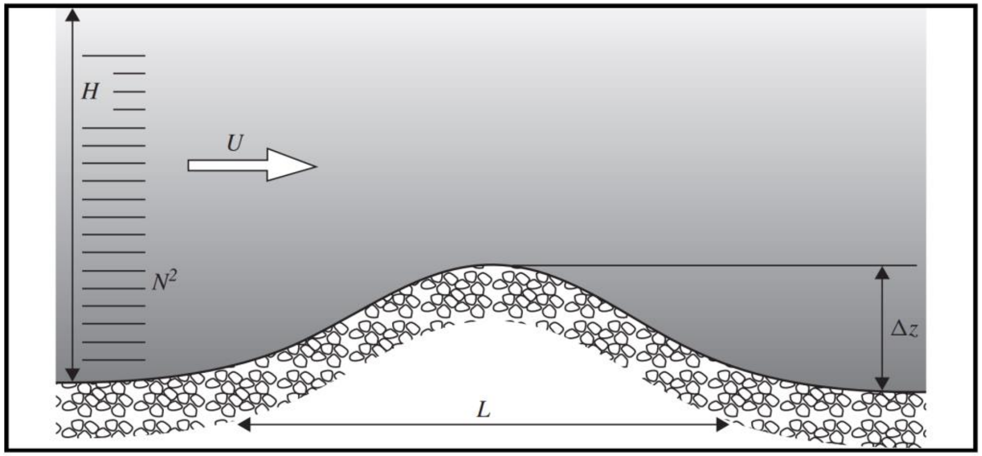
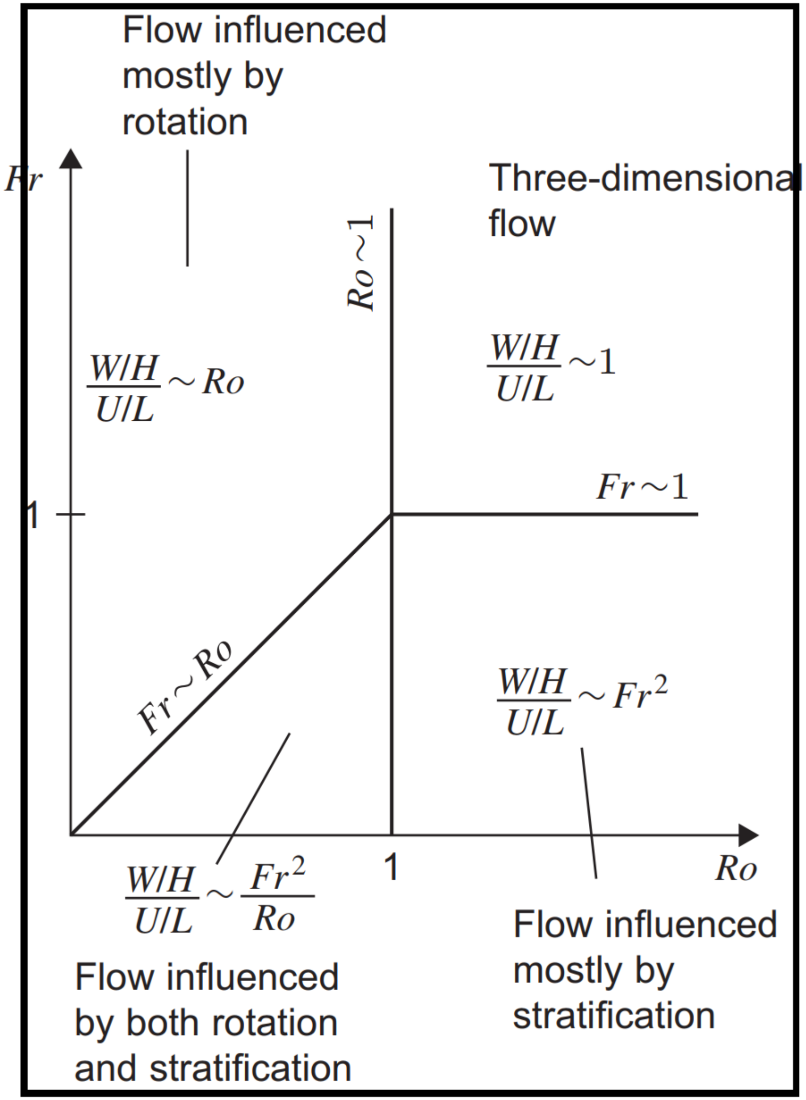


FIGURE 11.5 Situation in which a stratified flow encounters an obstacle, forcing some fluid parcels to move vertically against a buoyancy force.

上图来自: [1, p.356]

[1] Cushman-Roisin, B., & Beckers, J.-M. (2011). Chapter 11 - Stratification. In B. Cushman-Roisin & J.-M. Beckers (Eds.), *International Geophysics (Vol. 101, pp. 347-364). Academic Press.* https://doi.org/10.1016/B978-0-12-088759-0.00011-0



竖直辐合(散) 与 水平辐散(合) 的量级之比

Case 1-层化为主、地转可略 ($\mathrm{Fr} \lesssim 1, \mathrm{Ro} \gg 1$)

$$(1)\ \&\ (3) \Rightarrow \frac{W/H}{U/L} = \frac{\Delta z}{H} = \left(\frac{U}{NH}\right)^2 =: \mathrm{Fr}^2.$$

Case 2-地转为主、层化可略 ($\mathrm{Fr} \gg 1, \mathrm{Ro} \lesssim 1$)

$$\frac{W/H}{U/L} = \mathrm{Ro} := \frac{U}{\Omega L}.$$

Case 3-(竖直)层化 + 地转 ($\mathrm{Fr} \lesssim 1, \mathrm{Ro} \lesssim 1$)

$$(2)\ \&\ (3) \Rightarrow \frac{W/H}{U/L} = \frac{\Delta z}{H} = \frac{\Omega L U}{N^2 H^2} = \frac{\mathrm{Fr}^2}{\mathrm{Ro}}.$$

Burger number:

$$\mathrm{Bu} := \left(\frac{\mathrm{Ro}}{\mathrm{Fr}}\right)^2 = \left(\frac{NH}{\Omega L}\right)^2$$

FIGURE 11.6 Recapitulation of the various scalings of the ratio of vertical convergence (divergence), W/H, to horizontal divergence (convergence), U/L, as a function of the Rossby number, $Ro = U/(\Omega L)$, and Froude number, $Fr = U/(NH)$.

上图来自：[1, p.359]

当 $\mathrm{Fr}, \mathrm{Ro} \lesssim 1$，净浮力和科氏力比惯性力相当或更大，此时可用 Bu 衡量层化相对地转的重要性：

$\mathrm{Bu} \ll 1$：地转主导；
$\mathrm{Bu} \sim 1$：同等重要；
$\mathrm{Bu} \gg 1$：层化主导.

[1] Cushman-Roisin, B., & Beckers, J.-M. (2011). Chapter 11 - Stratification. In B. Cushman-Roisin & J.-M. Beckers (Eds.), *International Geophysics (Vol. 101, pp. 347-364). Academic Press.* https://doi.org/10.1016/B978-0-12-088759-0.00011-0

谢谢！