Barotropic instability

Assumptions: shallow water model + flat bottom and surface

$$p = p_0(z) + \tilde{p}(x,y,z,t) \qquad p_0(z) = P_0 - \rho_0 g z$$

$$\frac{\partial \mathrm{u}}{\partial \mathrm{t}} + \mathrm{u}\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \mathrm{v}\frac{\partial \mathrm{u}}{\partial \mathrm{y}} - fv = -\frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial x}$$

$$\frac{\partial \mathrm{v}}{\partial \mathrm{t}} + \mathrm{u}\frac{\partial \mathrm{v}}{\partial \mathrm{x}} + \mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{y}} + fu = -\frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial y}$$

$$\boxed{\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0}$$

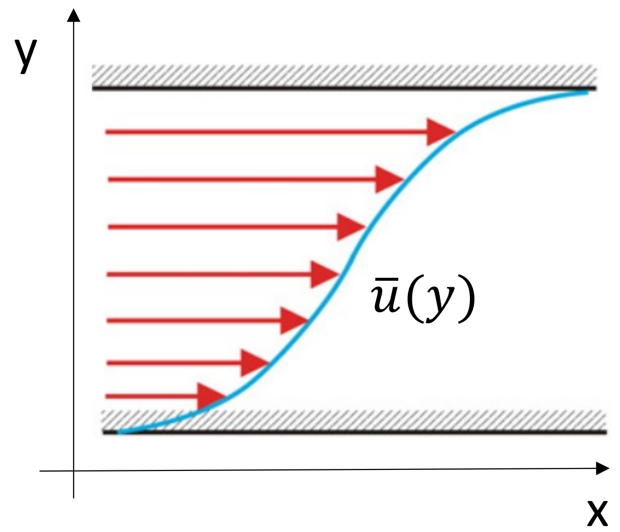
$$\frac{\partial \mathrm{u}}{\partial x} + \frac{\partial v}{\partial y} + \cancel{\frac{\partial w}{\partial z}} = 0$$

Take the z-derivative of the continuity equation: $\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial z}\right) = 0 \qquad w = az + b$

flat bottom and surface: $w = 0$

Assume the background (mean) flow as:

$$u = \bar{u}(y) \qquad v = 0.$$



The momentum equations for the mean flow:

$$\cancel{\frac{\partial u}{\partial t}} + u\cancel{\frac{\partial u}{\partial x}} + v\cancel{\frac{\partial u}{\partial y}} - \cancel{fv} = -\frac{1}{\rho_0}\cancel{\frac{\partial \tilde{p}}{\partial x}}$$

$$\cancel{\frac{\partial v}{\partial t}} + u\cancel{\frac{\partial v}{\partial x}} + v\cancel{\frac{\partial v}{\partial y}} + fu = -\frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial y}$$

$$(f_0 + \beta_0 y)\,\bar{u}(y) = -\frac{1}{\rho_0}\frac{d\bar{p}}{dy}$$

Assume small perturbations of u, v and p due to waves:

$$
\begin{aligned}
u &= \bar{u}(y) + u'(x,\, y,\, t) \\
v &= v'(x,\, y,\, t) \\
\tilde{p} &= \bar{p}(y) + p'(x,\, y,\, t),
\end{aligned}
$$

perturbation much smaller than mean

Subsitution into the momentum equations:

$$
\boldsymbol{x}:\ \frac{\partial(\bar{u}+u')}{\partial t} + (\bar{u}+u')\frac{\partial(\bar{u}+u')}{\partial x} + v'\frac{\partial(\bar{u}+u')}{\partial y} - f v' = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}
$$

$$
\cancel{\frac{\partial \bar{u}}{\partial t}} + \frac{\partial u'}{\partial t} + \cancel{\bar{u}\frac{\partial \bar{u}}{\partial x}} + \bar{u}\frac{\partial u'}{\partial x} + \cancel{u'\frac{\partial \bar{u}}{\partial x}} + \cancel{u'\frac{\partial u'}{\partial x}} + v'\frac{\partial \bar{u}}{\partial y} + \cancel{v'\frac{\partial u'}{\partial y}} - f v' = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}
$$

$$
\boldsymbol{y}:\ \frac{\partial v'}{\partial t} + (\bar{u} + \cancel{u'})\frac{\partial v'}{\partial x} + \cancel{v'\frac{\partial v'}{\partial y}} + f(\cancel{\bar{u}} + u') = -\frac{1}{\rho_0}\frac{\partial(\cancel{\bar{p}} + p')}{\partial y}
$$

$$\frac{\partial u'}{\partial t} + \bar{u}\,\frac{\partial u'}{\partial x} + v'\,\frac{d\bar{u}}{dy} - (f_0 + \beta_0 y)v' = -\frac{1}{\rho_0}\,\frac{\partial p'}{\partial x} \qquad (1)$$

$$\frac{\partial v'}{\partial t} + \bar{u}\,\frac{\partial v'}{\partial x} + (f_0 + \beta_0 y)u' = -\frac{1}{\rho_0}\,\frac{\partial p'}{\partial y} \qquad (2)$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0.$$

$$\boxed{u' = -\frac{\partial \psi}{\partial y}\,, \quad v' = +\frac{\partial \psi}{\partial x}}$$

$$\boxed{\zeta' = \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} = \nabla^2 \psi}$$

$$\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(1):$$

$$\left(\frac{\partial}{\partial t} + \bar{u}\,\frac{\partial}{\partial x}\right)\nabla^2\psi + \left(\beta_0 - \frac{d^2\bar{u}}{dy^2}\right)\frac{\partial \psi}{\partial x} = 0.$$

$$\left(\frac{\partial}{\partial t} + \bar{u}\,\frac{\partial}{\partial x}\right)\nabla^2\psi + \left(\beta_0 - \frac{d^2\bar{u}}{dy^2}\right)\frac{\partial\psi}{\partial x} = 0.$$

Asume a wave solution (note the coefficient is a function of y):

$$\psi(x,\ y,\ t) = \underbrace{\phi(y)}_{\text{eigenfunction}}e^{\mathrm{i}\,(kx-\omega t)}$$

$$(-i\omega + \bar{u}ik)\left(-k^2\phi + \frac{d^2\phi}{dy^2}\right)e^{i(kx-\omega t)} + \left(\beta_0 - \frac{d^2\bar{u}}{dy^2}\right)ike^{i(kx-\omega t)} = 0$$

$$(-c + \bar{u})\left(-k^2\phi + \frac{d^2\phi}{dy^2}\right) + \beta_0 - \frac{d^2\bar{u}}{dy^2} = 0$$

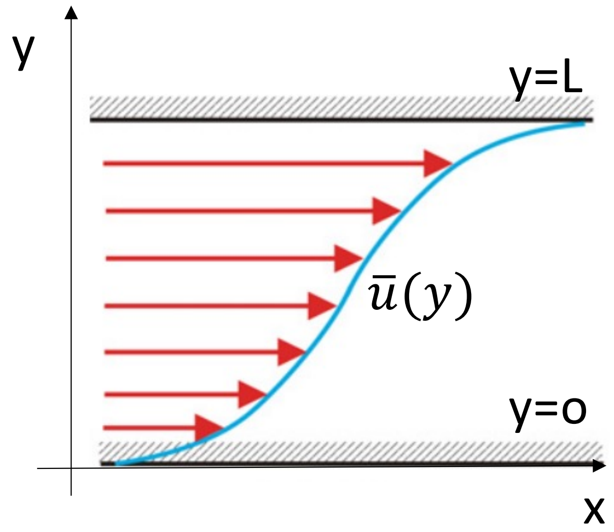
$$\frac{d^2\phi}{dy^2} - k^2\phi + \frac{\beta_0 - d^2\bar{u}/dy^2}{\bar{u}(y) - c}\,\phi = 0 \quad \text{Rayleigh equation}$$

Assume two lateral boundaries y=0 and y=L:

$$\boxed{\psi(x,y,t) = \phi(y)e^{i(kx-\omega t)}}$$

$$v = v' = \frac{\partial \psi}{\partial x} = 0$$

$$\phi(y=0) = \phi(y=L) = 0$$



The phase speed:

$$c = c_r + ic_i$$

$$\psi(x,y,t) = \phi(y)e^{ik(x-c_rt-ic_it)} = \phi(y)e^{kc_it}e^{ik(x-c_rt)}$$

growing mode - instability

Multiply $\frac{d^2\phi}{dy^2} - k^2\phi + \frac{\beta_0 - d^2\bar{u}/dy^2}{\bar{u}(y) - c}\phi = 0$ by ϕ^*, and integrate across the domain:

$$-\int_0^L \left(\left|\frac{d\phi}{dy}\right|^2 + k^2|\phi|^2\right) dy + \int_0^L \frac{\beta_0 - d^2\bar{u}/dy^2}{\bar{u} - c}|\phi|^2\, dy = 0$$

The imaginary part is:

$$c_i \int_0^L \left(\beta_0 - \frac{d^2\bar{u}}{dy^2}\right) \frac{|\phi|^2}{|\bar{u} - c|^2}\, dy = 0.$$

For growing mode to exist (instability), $c_i \neq 0$, and then

$$\int_0^L \left(\underline{\beta_0 - \frac{d^2\bar{u}}{dy^2}}\right) \frac{|\phi|^2}{|\bar{u} - c|^2}\, dy = 0$$

necessary condition – Rayleigh's criterion

$$\boxed{\frac{d}{dy}\left(f_0 + \beta_0 y - \frac{d\bar{u}}{dy}\right)}$$

must change sign (vanish) somewhere in the domain (total vorticity reaches a extremum)

The real part:

$$\int_0^L (\bar{u} - c_r)\left(\beta_0 - \frac{d^2\bar{u}}{dy^2}\right) \frac{|\phi|^2}{|\bar{u} - c|^2}\, dy = \int_0^L \left(\left|\frac{d\phi}{dy}\right|^2 + k^2|\phi|^2\right) dy \quad (1)$$

With the result from the imaginary part: $$\int_0^L \left(\beta_0 - \frac{d^2\bar{u}}{dy^2}\right) \frac{|\phi|^2}{|\bar{u} - c|^2}\, dy = 0 \quad (2)$$

(2)*($c_r - \underline{\bar{u}_0}$)+(1):

any arbitrary velocity

$$\int_0^L (\bar{u} - \bar{u}_0)\left(\beta_0 - \frac{d^2\bar{u}}{dy^2}\right) \frac{|\phi|^2}{|\bar{u} - c|^2}\, dy > 0$$

$\bar{u}_0$ is taken as the value where $\beta_0 - \frac{d^2\bar{u}}{dy^2} = 0$

$(\bar{u} - \bar{u}_0)\left(\beta_0 - \frac{d^2\bar{u}}{dy^2}\right)$ must be positive in some finite portion of the domain

necessary condition – FjØrtoft's criterion

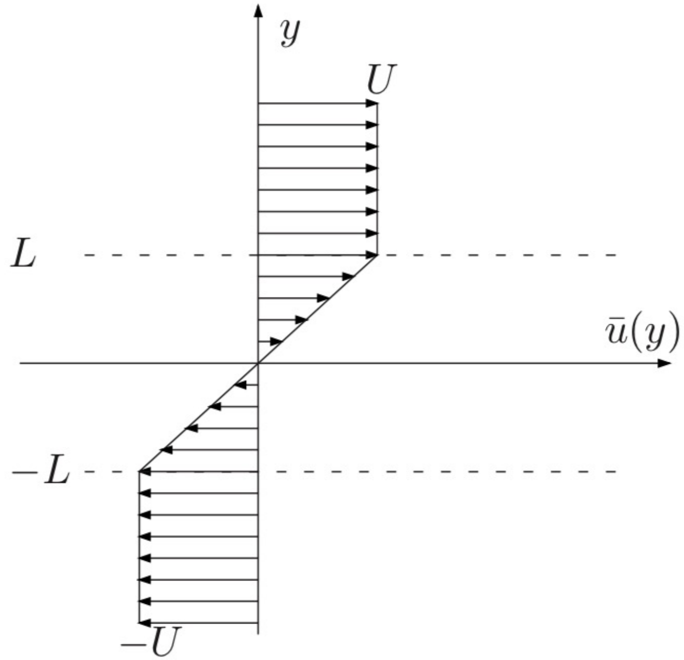
A simple example for barotropic instability

$\boxed{f\text{-plane } (\beta_0 = 0)}$

$$y < -L: \quad \bar{u} = -U, \quad \frac{d\bar{u}}{dy} = 0, \quad \frac{d^2\bar{u}}{dy^2} = 0$$

$$-L < y < +L: \quad \bar{u} = \frac{U}{L}y, \quad \frac{d\bar{u}}{dy} = \frac{U}{L}, \quad \frac{d^2\bar{u}}{dy^2} = 0$$

$$+L < y: \quad \bar{u} = +U, \quad \frac{d\bar{u}}{dy} = 0, \quad \frac{d^2\bar{u}}{dy^2} = 0$$



At y = -L: $\frac{d^2\bar{u}}{dy^2} > 0$ At y = L: $\frac{d^2\bar{u}}{dy^2} < 0$

$\frac{d^2\bar{u}}{dy^2}$ changes sign in the domain – Rayleigh's criterion

$\cancel{\beta_0} - \frac{d^2\bar{u}}{dy^2}$ changes sign

Take $\bar{u}_0 = 0$,

$(\bar{u} - \bar{u}_0)\left(\cancel{\beta_0} - \frac{d^2\bar{u}}{dy^2}\right)$ > 0 in some portion

y = -L: $-\bar{u}\frac{d^2\bar{u}}{dy^2} > 0$ y = L: $-\bar{u}\frac{d^2\bar{u}}{dy^2} > 0$ – Fjørtoft's criterion