Layered models

In a stratified system, as density increases monotonically downward, it can be used as the vertical coordinate

$$\rho(x,y,z,t) \longrightarrow z(x,y,\rho,t)$$

$$a = a(x,y,\rho(x,y,z,t),t)$$

$$\frac{\partial}{\partial x} \longrightarrow \left.\frac{\partial a}{\partial x}\right|_z = \left.\frac{\partial a}{\partial x}\right|_\rho + \frac{\partial a}{\partial \rho}\left.\frac{\partial \rho}{\partial x}\right|_z$$

$$\frac{\partial}{\partial y} \longrightarrow \left.\frac{\partial a}{\partial y}\right|_z = \left.\frac{\partial a}{\partial y}\right|_\rho + \frac{\partial a}{\partial \rho}\left.\frac{\partial \rho}{\partial y}\right|_z$$

$$\frac{\partial}{\partial z} \longrightarrow \frac{\partial a}{\partial z} = \frac{\partial a}{\partial \rho}\frac{\partial \rho}{\partial z}$$

$$\frac{\partial}{\partial t} \longrightarrow \left.\frac{\partial a}{\partial t}\right|_z = \left.\frac{\partial a}{\partial t}\right|_\rho + \frac{\partial a}{\partial \rho}\left.\frac{\partial \rho}{\partial t}\right|_z$$

If $a = z$

$$0 = z_x + z_\rho \rho_x$$

$$1 = z_\rho \rho_z$$

$$0 = z_t + z_\rho \rho_t$$

$$\left.\frac{\partial a}{\partial x}\right|_z = \left.\frac{\partial a}{\partial x}\right|_\rho - \frac{z_x}{z_\rho}\frac{\partial a}{\partial \rho}$$

$$\frac{\partial a}{\partial z} = \frac{1}{z_\rho}\frac{\partial a}{\partial \rho}$$

$$\frac{\partial a}{\partial t}|_z = \frac{\partial a}{\partial t}|_\rho - \frac{z_t}{z_\rho}\frac{\partial a}{\partial \rho}$$

Hydrostatic balance: $\dfrac{\partial p}{\partial z} = -\rho g$

$$\boxed{\frac{\partial a}{\partial z} = \frac{1}{z_\rho}\frac{\partial a}{\partial \rho}}$$

$$\boxed{\frac{\partial a}{\partial x}\Big|_z = \frac{\partial a}{\partial x}\Big|_\rho - \frac{z_x}{z_\rho}\frac{\partial a}{\partial \rho}}$$

$$\frac{\partial p}{\partial \rho} = -\rho g\frac{\partial z}{\partial \rho}$$

$$\boxed{\frac{\partial a}{\partial t}|_z = \frac{\partial a}{\partial t}|_\rho - \frac{z_t}{z_\rho}\frac{\partial a}{\partial \rho}}$$

$$\frac{\partial p}{\partial x}\Big|_z = \frac{\partial p}{\partial x}\Big|_\rho - \frac{z_x}{z_\rho}\frac{\partial p}{\partial \rho} = \frac{\partial p}{\partial x}\Big|_\rho + \rho g\frac{\partial z}{\partial x} = \frac{\partial P}{\partial x}\Big|_\rho$$

$$\boxed{P = p + \rho g z}$$

$$\frac{\partial P}{\partial \rho} = \frac{\partial p}{\partial \rho} + gz + \rho g\frac{\partial z}{\partial \rho} = gz$$

Let $a = \rho$:

$$\frac{\partial \rho}{\partial x}|_z = -\frac{z_x}{z_\rho} \qquad \frac{\partial \rho}{\partial y}|_z = -\frac{z_y}{z_\rho} \qquad \frac{\partial \rho}{\partial z}|_z = \frac{1}{z_\rho} \qquad \frac{\partial \rho}{\partial t}|_z = -\frac{z_t}{z_\rho}$$

For incompressible fluid: $\dfrac{d\rho}{dt} = 0$

z-coordinate

ρ-coordinate

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = 0 \Rightarrow -z_t - uz_x - vz_y + w = 0 \Rightarrow \frac{dz}{dt} = z_t + uz_x + vz_y$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

There is no across-isopycnal advection, and the model is ideal for studying along-isopycnal processes

Without friction, the momentum equations in density coordinate become:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -\frac{1}{\rho_0}\frac{\partial P}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -\frac{1}{\rho_0}\frac{\partial P}{\partial y}$$

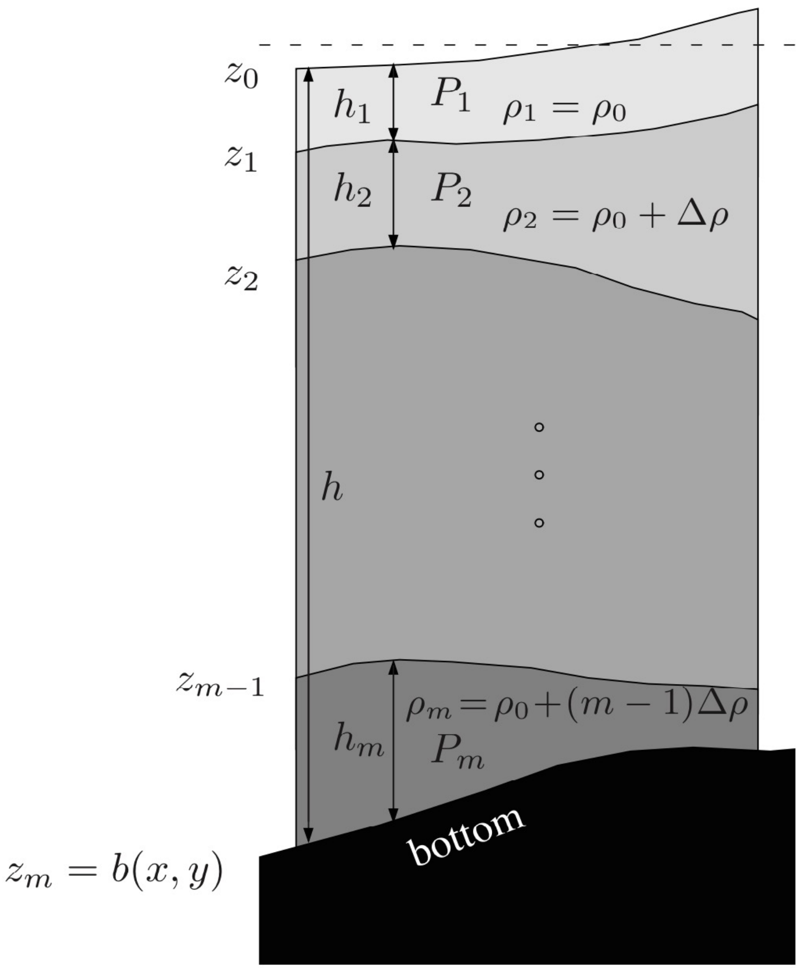
$$\frac{\partial P}{\partial \rho} = gz$$

$$\frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} + \frac{\partial hv}{\partial y} = 0$$

$$h = -\Delta\rho\frac{\partial z}{\partial \rho}$$

the thickness of a fluid layer between ρ and $\rho + \Delta\rho$

Layered models



(a)

$$z_m = b$$

upward:

$$z_{k-1} = z_k + h_k, \quad k = m \text{ to } 1.$$

downward:

$$\boxed{P = p + \rho g z}$$

$$P_1 = p_{\text{atm}} + \rho_0 g z_0$$

$$\boxed{\frac{\partial P}{\partial \rho} = gz}$$

$$P_{k+1} = P_k + \Delta\rho g z_k, \quad k = 1 \text{ to } m-1.$$

$$\boxed{z_m = b} \qquad \boxed{z_{k-1} = z_k + h_k} \qquad \boxed{P_1 = p_{\text{atm}} + \rho_0 g z_0} \qquad \boxed{P_{k+1} = P_k + \Delta\rho g z_k}$$

One layer:

$$z_0 = h_1 + b \qquad P_1 = \rho_0 g(h_1 + b)$$
$$z_1 = b$$

$$\boxed{g' = \frac{\Delta\rho}{\rho_0} g}$$

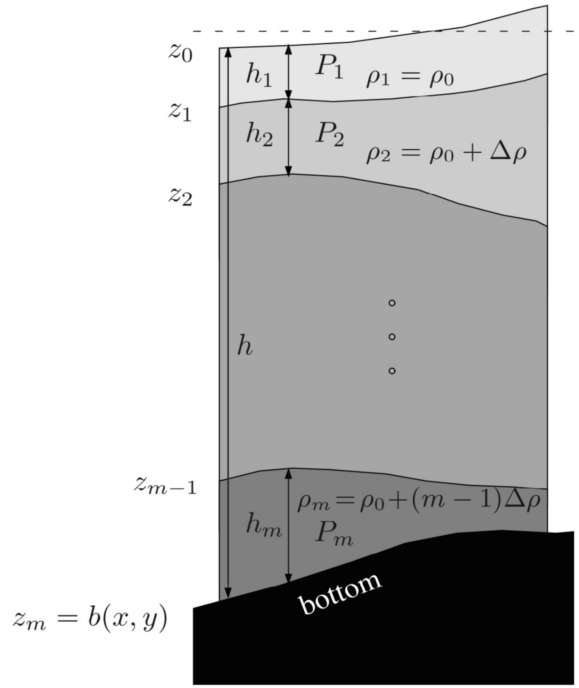
reduced gravity

Two layers:

$$z_0 = h_1 + h_2 + b \qquad P_1 = \rho_0 g(h_1 + h_2 + b)$$
$$z_1 = h_2 + b \qquad P_2 = \rho_0 g h_1 + \rho_0 (g + g')(h_2 + b)$$
$$z_2 = b$$

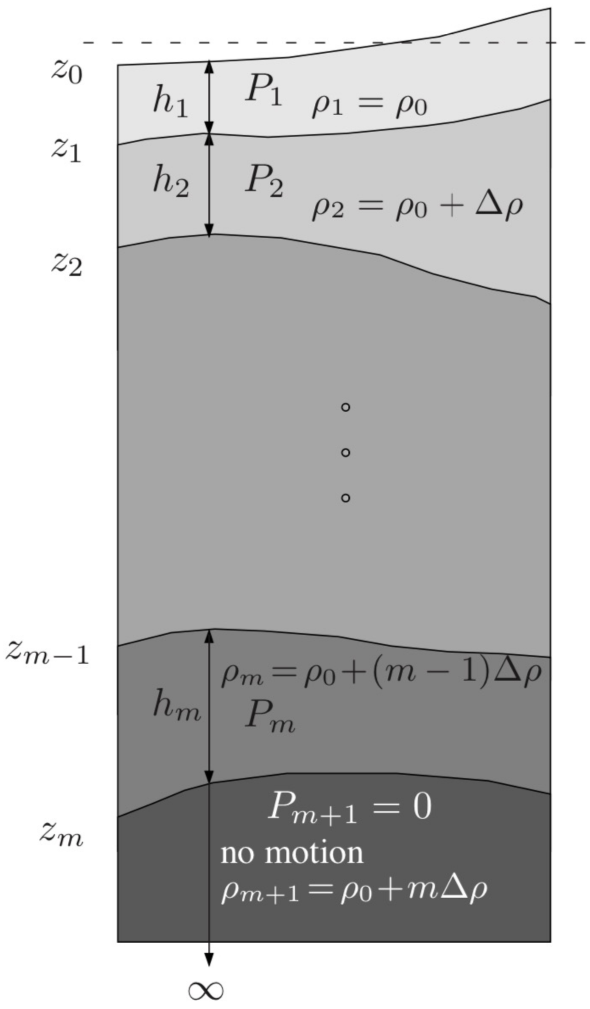
Three layers:

$$z_0 = h_1 + h_2 + h_3 + b \qquad P_1 = \rho_0 g(h_1 + h_2 + h_3 + b)$$
$$z_1 = h_2 + h_3 + b \qquad P_2 = \rho_0 g h_1 + \rho_0 (g + g')(h_2 + h_3 + b)$$
$$z_2 = h_3 + b \qquad P_3 = \rho_0 g h_1 + \rho_0 (g + g') h_2 + \rho_0 (g + 2g')(h_3 + b)$$
$$z_3 = b$$



(a)

Reduced gravity model



The lowest layer may be imagined to be infinitely deep and at rest

$$P_{m+1} = 0$$

Rigid-lid approximation: $z_0 = 0$

$$\boxed{P_{k+1} = P_k + \Delta\rho g z_k}$$

One layer:

$$z_1 = -h_1 \qquad P_1 = \rho_0 g' h_1$$

Two layers:

$$z_1 = -h_1 \qquad P_1 = \rho_0 g'(2h_1 + h_2)$$
$$z_2 = -h_1 - h_2 \qquad P_2 = \rho_0 g'(h_1 + h_2)$$

Three layers:

$$z_1 = -h_1 \qquad P_1 = \rho_0 g'(3h_1 + 2h_2 + h_3)$$
$$z_2 = -h_1 - h_2 \qquad P_2 = \rho_0 g'(2h_1 + 2h_2 + h_3)$$
$$z_3 = -h_1 - h_2 - h_3 \qquad P_3 = \rho_0 g'(h_1 + h_2 + h_3)$$

shallow-water reduced gravity model – one layer

One layer:

$$z_1 = -h_1 \qquad\qquad P_1 = \rho_0 g' h_1$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g'\frac{\partial h}{\partial x} \quad (1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g'\frac{\partial h}{\partial y} \quad (2)$$

$$\frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} + \frac{\partial hv}{\partial y} = 0 \quad \Longrightarrow \quad \frac{dh}{dt} + h(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = 0$$

$\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(1)$:

$$\frac{d(f+\zeta)}{dt} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(f+\zeta) = 0 \ (1)$$

$$\frac{d}{dt}\left(\frac{f+\zeta}{h}\right) = 0$$

PV conservation

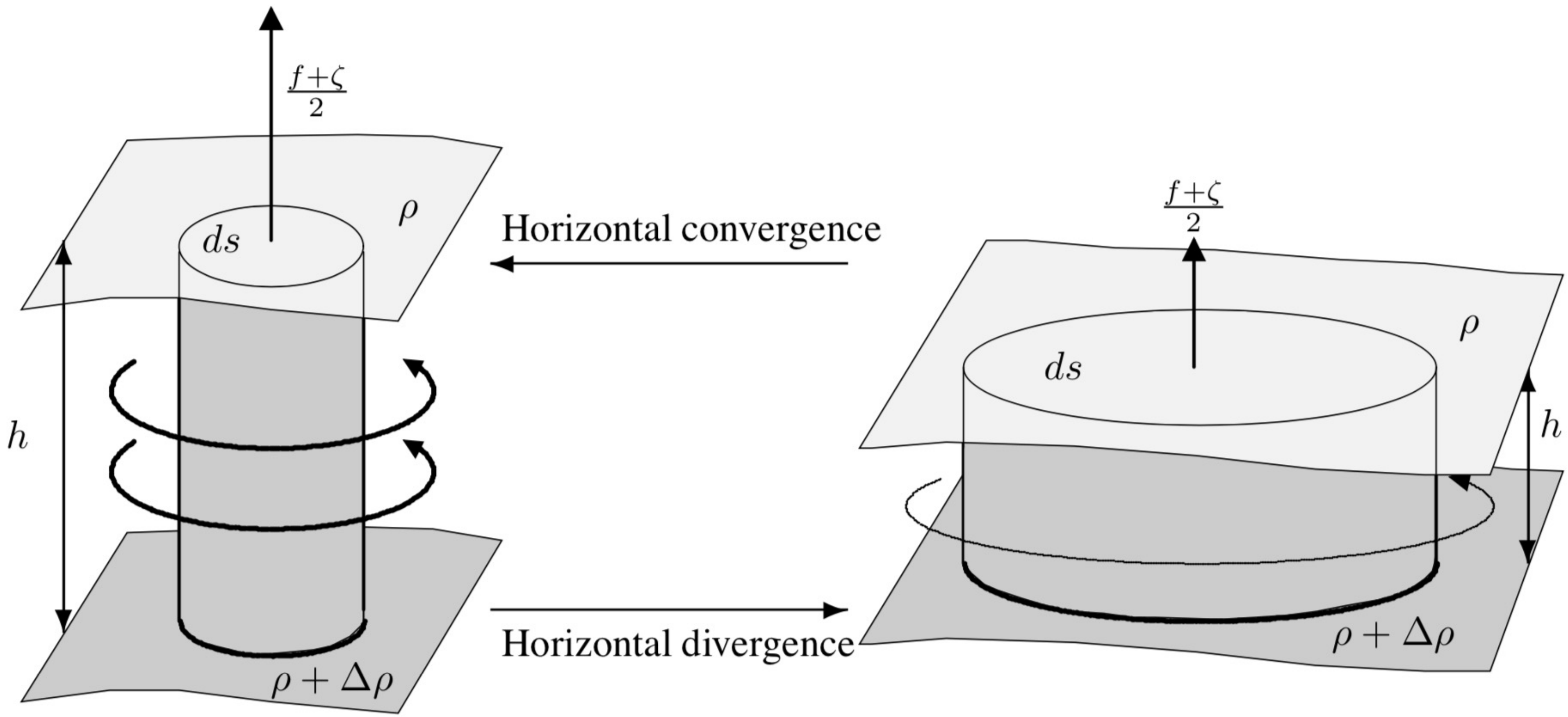
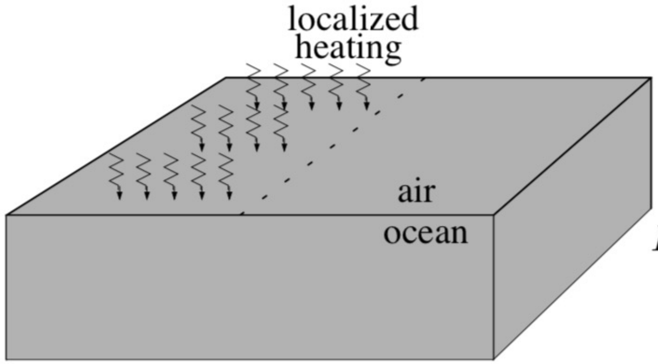
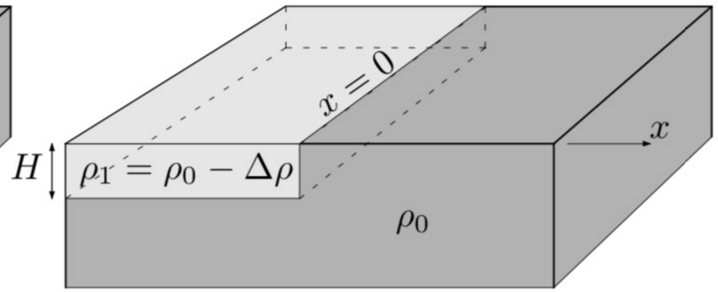


Figure 12-4 Conservation of volume and circulation in a fluid undergoing divergence (squeezing) or convergence (stretching). The products of $h\ ds$ and $(f + \zeta)\ ds$ are conserved during the transformation, implying conservation of $(f + \zeta)/h$, too.

Baroclinic geostrophic adjustment



(a) Initial state



(b) Immediately after heating event

$$\frac{\partial}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} - fv = -g'\,\frac{\partial h}{\partial x}$$

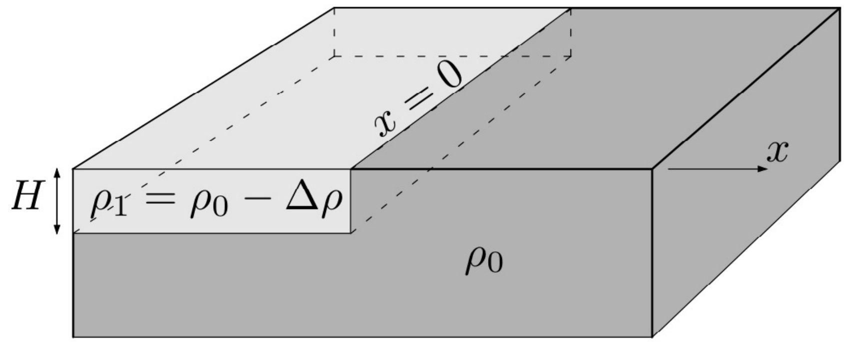
$$\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} + fu = 0$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0.$$

Initial state (unbalanced):

$$h = \begin{cases} H, & x < 0 \\ 0, & x > 0 \end{cases}$$

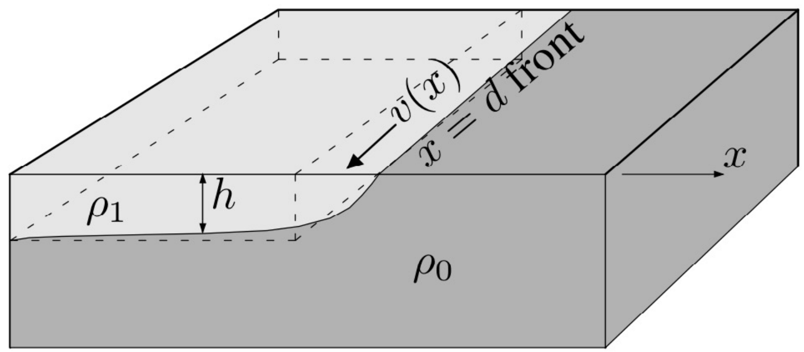
$$u = v = 0$$



Boundary conditions:

$$x \to -\infty, \quad h \to \mathrm{H}, \quad \mathrm{u}, \mathrm{v} \to 0$$

$$x \to d, \quad h \to 0$$



(c) After adjustment

Final state (steady): $\frac{\partial}{\partial t} = 0$

$$\boxed{\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0} \Rightarrow \frac{\partial hu}{\partial x} = 0$$

at $x = d$, $h = 0$, $hu = 0$: $hu = 0$ everywhere $\Rightarrow$ $u = 0$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - fv = -g'\frac{\partial h}{\partial x}$$

$$\cancel{\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + fu = 0}$$

$$-fv = -g'\frac{dh}{dx}$$

PV conservation:

$$\frac{f}{H} = \frac{f + \frac{\partial v}{\partial x}}{h} \Rightarrow \frac{f}{H} = \frac{f + \frac{g'}{f}\frac{d^2h}{\partial x^2}}{h}$$

$$fh = fH + \frac{g'H}{f}\frac{d^2h}{\partial x^2}$$

baroclinic deformation radius

$$R = \frac{\sqrt{g'H}}{f}$$

$$R^2\frac{d^2h}{\partial x^2} - h + H = 0$$

$$\mathrm{h} = \mathrm{f(x)} + H \qquad \mathrm{f(x)}:\ R^2\frac{d^2h}{\partial x^2} - h = 0$$

$$x \to -\infty, h \to \mathrm{H}$$

$$\mathrm{f(x)} = Ae^{\lambda x} \Rightarrow R^2\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm\frac{1}{R} \Rightarrow \lambda = \frac{1}{R} \qquad \mathrm{f(x)} = Ae^{x/R}$$

$$f(x) = Ae^{x/R}$$

$x \to d, \qquad h \to 0: \qquad f(x) \to -\mathrm{H}$ $\qquad\qquad$ $\mathrm{h} = \mathrm{f(x)} + H$

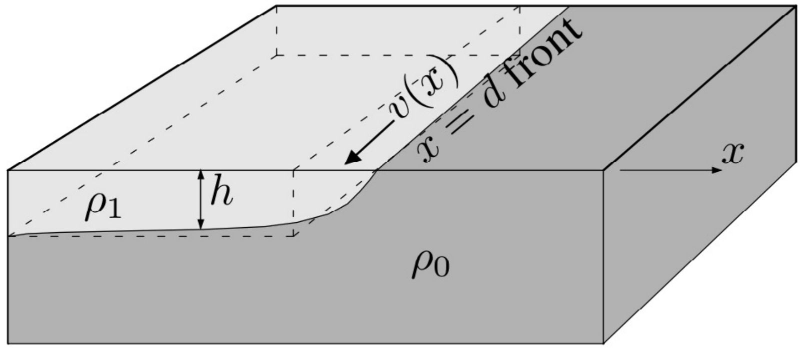
$$f(x) = Be^{(x-d)/R} \qquad B = -H$$

$$h = H(1 - e^{\frac{x-d}{R}})$$

$$-fv = -g'\frac{dh}{dx}$$

$$v = -\sqrt{g'H}\, e^{\frac{x-d}{R}}$$

The depleted volume of light water in x < 0 should be equal to the volume of light water in x > 0:



(c) After adjustment

$$\int_{-\infty}^{0}(H-h)dx = \int_{0}^{d} h dx$$

$$H\int_{-\infty}^{0} e^{\frac{x-d}{R}}dx = Hd - H\int_{0}^{d} e^{\frac{x-d}{R}}dx$$

$$Re^{\frac{x-d}{R}}\Big|_{-\infty}^{0} = d - Re^{\frac{x-d}{R}}\Big|_{0}^{d}$$

$d = R$ $\qquad$ adjustment spatial scale is R

Energetics

Initial state: $KE_i = 0$ $PE_i = \frac{1}{2}\rho_0 \int_{-\infty}^{0} g' H^2 dx = \frac{1}{2}\rho_0 g' H^2 x|_{-\infty}^{0}$

Final state: $KE_f = \frac{1}{2}\rho_0 \int_{-\infty}^{R} h v^2 dx = \frac{1}{2}\rho_0 g' H^2 \int_{-\infty}^{R} (1 - e^{\frac{x-R}{R}}) e^{2\frac{x-R}{R}} dx$

$v = -\sqrt{g'H}\, e^{\frac{x-R}{R}}$

$$= \frac{1}{2}\rho_0 g' H^2 (\frac{R}{2} e^{2\frac{x-R}{R}}|_{-\infty}^{R} - \frac{R}{3} e^{3\frac{x-R}{R}}|_{-\infty}^{R}) = \frac{1}{12}\rho_0 g' H^2 R \qquad \Delta KE$$

$h = H(1 - e^{\frac{x-R}{R}})$

$$PE_f = \frac{1}{2}\rho_0 \int_{-\infty}^{R} g' h^2 dx = \frac{1}{2}\rho_0 g' H^2 \int_{-\infty}^{R} (1 - e^{\frac{x-R}{R}})^2 dx$$

$$= \frac{1}{2}\rho_0 g' H^2 (x|_{-\infty}^{R} - 2Re^{\frac{x-R}{R}}|_{-\infty}^{R} + \frac{R}{2} e^{2\frac{x-R}{R}}|_{-\infty}^{R})$$

$$= \frac{1}{2}\rho_0 g' H^2 (x|_{-\infty}^{R} - \frac{3}{2} R)$$

$$\Delta PE = PE_i - PE_f = \frac{1}{4}\rho_0 g' H^2 R \qquad \frac{\Delta KE}{\Delta PE} = 1/3$$

Burger number:

$$Bu = \left(\frac{R_0}{Fr}\right)^2$$

a measure of relative importance of rotation and stratification

$$= \frac{U^2}{f^2L^2} \Big/ \frac{U^2}{N^2H^2}$$

$$= \frac{N^2H^2}{f^2L^2} = \frac{g'H}{f^2L^2} = \frac{R^2}{L^2}$$

$$\boxed{N^2 = -\frac{g}{\rho_0}\frac{d\rho}{dz} \simeq \frac{g}{\rho_0}\frac{\Delta\rho}{H} = \frac{g'}{H}}$$

$L < R,\quad Bu > 1,\quad Fr < R_0,$ motion is more affected by stratification

$L > R,\quad Bu < 1,\quad R_0 < Fr,$ motion is more affected by rotation

$$Fr^2 = \frac{U^2}{N^2H^2} = \frac{U^2}{g'H} \qquad Fr = \frac{U}{\sqrt{g'H}}$$

$\sqrt{g'H}$: internal gravity wave speed

Some advantages of using z-coordinates are:

- Simple to implement and use!
- The full equation of state can be used.
- Diabatic processes, including the mixed-layer, can be represented easily.
- Non-hydrostatic and non-Boussinesq terms can be included.

Some dis-advantages are:

- Representation of along isopycnal processes are awkward.
- Representing the bottom boundary layer is awkward.

Isopycnal models have several advantages over the height and terrain-following coordinates:

- Ideal for modeling lateral transfer processes. Adiabatic motions modeled without any spurious diabatic terms.
- Smooth representation of topography. The bottom topography is represented as piecewise-linear and is included in the model through a vanishing of the layer thickness.
- Conserves volume of density classes

Some dis-advantages are:

- Full or non-linear equation of state is difficult.
- Non-hydrostatic effects/dynamics are not possible.
- Density is not a natural coordinate for representing mixing processes such as the surface BBL (shallow and deep mixed layers).
- Vertical and horizontal resolution are tightly connected in regions where isopycnals outcrop. This can lead to inadequate horizontal resolution in regions such as the ACC.