1

General principles

When the density of a fluid is determined by two components, which diffuse at different rates, the fluid at rest can be unstable even if its density increases downward. This simple, although seemingly counterintuitive, idea is the cornerstone of the theory of double-diffusive convection. As with any other instability, double-diffusion requires a finite amount of energy to sustain the growth of perturbations. If the basic state is motionless, instability can be driven by the potential energy of one of the density components. The ensuing convection depends very strongly, in terms of its pattern and dynamics, on whether the destabilizing component is of higher or lower diffusivity. The configuration in which the required energy is supplied by the slower diffuser is called salt fingering; the instability driven by the faster diffuser is known as diffusive convection. Of course, both density components could be concurrently destabilizing. In this case, the total density stratification is unstable and the result is top-heavy convection, a very different and much more violent process, which is beyond the scope of this book.

Because the interest in double-diffusion was originally motivated by oceanic applications, we follow conventional practice and introduce the key concepts in the oceanographic context. For instance, the faster diffuser will be conveniently referred to as temperature (T) and the slower diffuser as salinity (S) – two major components of seawater density. However, it is our intention to present the basic theory of double-diffusion in its most general form. Aside from parameter values and notation, the analysis is generally applicable to a variety of other physical systems, including double-diffusion in geology, astrophysics and metallurgy. Specifics of each field are discussed in Chapter 12.

1.1 Salt fingers

In much of the upper kilometer of tropical and subtropical oceans, warm and salty waters are located above cold and fresh; the mean vertical stratification in
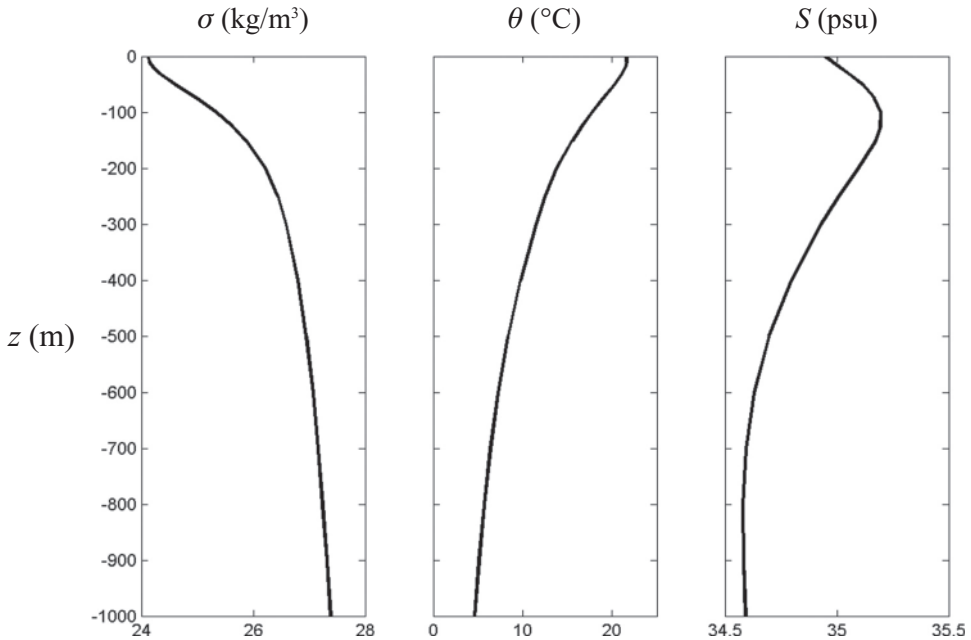
Figure 1.1 The vertical profiles of potential density (left), potential temperature (center) and salinity (right) in the upper kilometer of the ocean, horizontally averaged over the latitude band from 50° S to 50° N. Potential density (σ) and potential temperature (θ) are used to take into account effects of compressibility of seawater. Data are taken from the Levitus world ocean database.

the latitude band from 50° S to 50° N is shown in Figure 1.1. Since the density of seawater decreases with temperature but increases with salinity, the available potential energy of the system is stored in the salinity component. Vertical mixing of salinity tends to lower the center of gravity, thereby releasing potential energy, whilst mixing of temperature does the opposite. If the salinity stratification is losing energy at a higher rate than the temperature gains it, there will be a continuous supply of kinetic energy that could maintain and enhance vertical mixing. In the ocean, the amount of energy contained in the salinity stratification is enormous, and the release of even a small fraction of it can substantially affect the large-scale circulation. But does it really happen? So far, we have only argued that the instability of a two-component bottom-heavy fluid at rest does not contradict the principle of energy conservation – an intriguing, suggestive, but not exactly conclusive statement.

The first paper that hints at the possibility of releasing the potential energy of salt in the gravitationally stable environment appeared in 1956: *An Oceanographical Curiosity: The Perpetual Salt Fountain* by Stommel, Arons and Blanchard. The
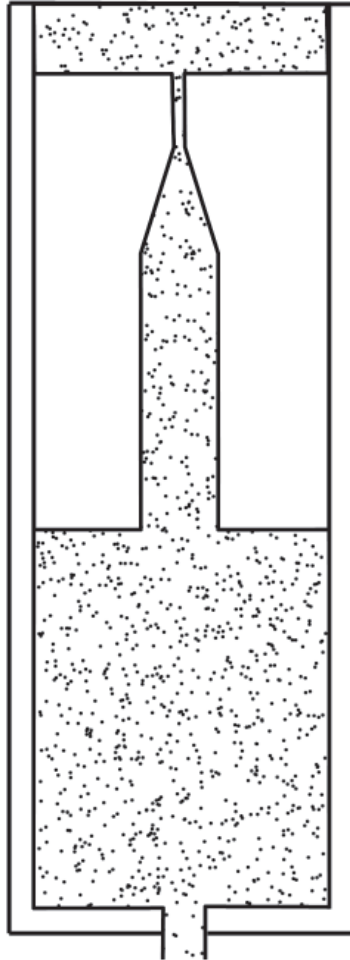
Figure 1.2 Schematic diagram of the salt fountain. From Stommel *et al.* (1956).

authors inserted a narrow heat-conducting pipe into a tank filled with doubly stratified water: warm and salty above cold and fresh. They discovered that if the water in the pipe is pushed upward, the circulation will be maintained for as long as there is a vertical salinity gradient. The reason was attributed to heat conduction through the wall of the pipe. As shown in Figure 1.2, the rising water in the pipe comes into thermal equilibrium with the surrounding fluid while remaining fresher, therefore lighter, and continues to move upward, reinforcing the initial circulation pattern. Stommel *et al.*'s study did not explain the connection between the laboratory experiment and the ocean – the heat-conducting pipe was considered to be essential in driving the circulation, and the defensive term "curiosity" slipped into the title. At this stage, it still seemed like there was a long way to go to take
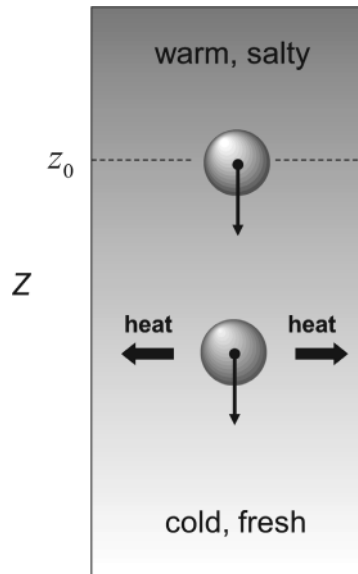
Figure 1.3 Illustration of the physical mechanism of salt fingering.

the idea of double-diffusive convection from the "it is not impossible" to the "it is likely to occur" level. However, the critical step was made only four years later (Stern, 1960). In a surprisingly simple argument, Melvin Stern showed that not only can instability naturally arise in the bottom-heavy stratification, but it should also be very common in the ocean.

The key, Stern argued, is in the two orders of magnitude difference between the molecular diffusivities of density components: $k_T \approx 1.4 \cdot 10^{-7}$ m^2 s^{-1} for temperature and $k_s \approx 1.1 \cdot 10^{-9}$ m^2 s^{-1} for salt. The significance of unequal T–S diffusivities can be illustrated as follows. Consider continuously stratified fluid at rest as shown in Figure 1.3; temperature and salinity do not vary horizontally and their vertical gradients are positive. Imagine perturbing this basic state by displacing a small parcel of fluid downward from its equilibrium position (z_0). It rapidly adjusts its temperature (fast diffuser) to that of the surrounding fluid but largely retains its original salinity (slow diffuser). The parcel is saltier than the ambient water at the same level and, since density increases with salinity, heavier. It continues to sink, moving further away from its equilibrium location, which implies that the basic stratification is unstable. The proposed mechanism is not unlike that of the perpetual salt fountain and, in retrospect, it becomes clear that in 1956 Stommel and his collaborators were very close to discovering double-diffusion. They just failed to realize that Stommel's pipe is not necessary to release the potential energy of a doubly stratified system – the low diffusivity of salt can be almost as effective in preserving the salinity of water parcels.
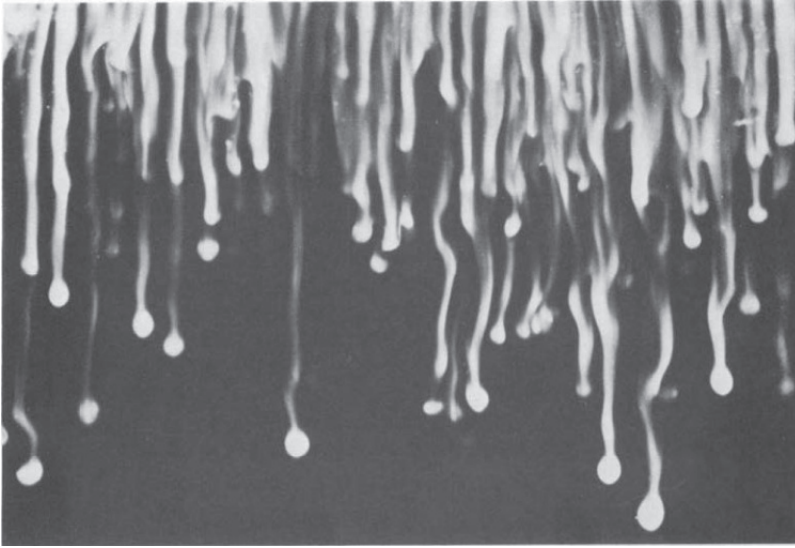
Figure 1.4 Laboratory experiment on fingering convection. An array of salt fingers is created by setting up a stable temperature gradient and pouring salt solution on top. From Huppert and Turner (1981).

Since salt fingering is a fundamentally diffusive process, the spatial scale of convection cells is limited by the range of effective molecular conduction of heat. In the ocean, salt fingers operate on scales of a few centimeters; in the laboratory, even smaller. Often, but not always, salt fingers come in the form of vertically elongated narrow filaments, very much as their colorful name suggests. Figure 1.4 presents an example of a laboratory experiment in which warm salty solution was poured on top of a stable temperature gradient. The emerging pattern consists of long parallel fingers with small round jelly-fish eddies forming at their extremities, a pattern dramatically different from that of ordinary thermal convection.

1.2 The early years: from Jevons to Stommel

While there is no doubt that full credit for the discovery of salt fingers belongs to Melvin Stern, it is interesting to consider some of the earlier missed opportunities. The complete account of the pre-Stern history of double-diffusion, rich with fascinating and lively details of scientific missteps, is given by Schmitt (1995a,b). What is particularly striking in this story is how close science has come to discovering double-diffusion on so many occasions but failed to make one final step.

The first recorded salt-finger experiment was performed by the English–Australian Stanley Jevons. To say that Jevons was a talented fluid dynamicist does not even start to describe his interests – he was a scientist in the broadest

sense of the word. He was one of the first photographers and an active researcher in chemistry and meteorology. By the age of twenty-two, Jevons had become a professor of logic, moral and mental philosophy, and political economy. But the coolest thing that he ever did was to accidentally create double-diffusion in the laboratory. After placing warm sugar solute on top of cold and fresh water, he observed an "infiltration of minute, thread-like streams" (Jevons, 1857). In retrospect, it is clear that this phenomenon is double-diffusion: sugar is a relatively slow diffuser and placing it in the upper warmer layer triggers fingering instability. Unfortunately, Jevons misinterpreted his observations. He viewed his experiment as a form of top-heavy convection, missing his opportunity to discover double-diffusion more than a century before Stern. However, we should not feel too sorry for Jevons since he got himself into a rather distinguished group of scientists who almost solved the salt-finger puzzle.

In 1880, Lord Rayleigh, perhaps the most prolific and reputable scientist of his period, reproduced the experiments of Jevons, also observed fingering convection, and also failed to explain it physically. In a peculiar twist of fate, Rayleigh's attempt to understand the origin of the sugar fingers in Jevons' experiments led to the first rigorous stability analysis of a stratified non-diffusive fluid (Rayleigh, 1883). His treatment of the top-heavy configuration describes what is now known as the Rayleigh–Taylor instability. For the bottom-heavy case, Rayleigh predicted the maximum frequency of free oscillations, arriving at the classical expression for the buoyancy frequency $N^2 = -\frac{g}{\rho}\frac{\partial \rho}{\partial z}$. However, the failure to recognize the destabilizing role of thermal diffusion prevented Rayleigh from adding salt fingers to his string of scientific victories.

The next opportunity to discover double-diffusion presented itself in 1906, when Vagn Walfrid Ekman, another giant of fluid dynamics, performed laboratory experiments on the "dead water" phenomenon. To visualize the interface displacements, Ekman used a two-layer system of milk over seawater and observed "a shower of small vortex-rings" – undoubtedly, milk fingers. Ekman came a bit closer than his predecessors to pinning down the elusive phenomenon. He pointed out that when a milk parcel comes into close contact with salty water, it gains, by molecular diffusion, extra salinity. Salt makes the parcel denser than the surrounding fluid and it rapidly sinks. While Ekman was right on target in explaining the physics of milk fingers, he did not realize that the analogous dynamics occur naturally in the ocean, where heat and salt play the same role as salt and milk in his experiments. Apparently Ekman considered the analysis of milk fingering by itself to be overly esoteric and frivolous and did not pursue the subject any further. Another half-a-century had gone by before two crucial steps were made: Stommel's salt fountain idea in 1956 and, finally, Melvin Stern's salt-finger paper in 1960. It is interesting that neither Stommel nor Stern was familiar with the earlier experiments. Stern's
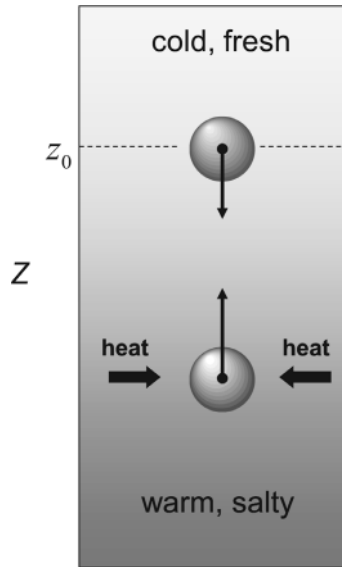
Figure 1.5 Illustration of the physical mechanism of oscillatory diffusive instability.

paper cites only one study (Stommel *et al.*, 1956) and Stommel's paper contains no references at all. The field of double-diffusion was created from scratch.

One can only speculate about possible reasons for a century-long delay between the first experimental realization of fingering and the first physical model. Would Rayleigh have discovered salt fingers if a personal encounter had permitted him to discuss his experiments with Jevons, as suggested by Schmitt (1995a,b)? Was it familiarity with more advanced mathematical methods that helped Stern to make the critical connection, or did the salt fountain idea provide the valuable hint? Could it be that the level of conceptual understanding of hydrodynamic instabilities, developed by the middle of the twentieth century, was a prerequisite for his discovery? We will never know for sure, but I have always suspected that the reason is much simpler: Melvin was just a bit sharper than the rest of the group.

1.3 Diffusive convection

In a footnote to his seminal salt-finger paper – perhaps the most important footnote in the history of fluid dynamics – Stern (1960) suggested the possibility of the oscillatory diffusive instability of cold and fresh water located above warm and salty. The dynamics of diffusive convection can be explained by reversing the arguments used for the fingering case (Section 1.1). The schematic in Figure 1.5 represents a thought experiment in which the diffusively favorable stratification

Figure 1.6 Laboratory experiment on diffusive convection. The two-layer diffusive system is created by pouring haline solution on top of a denser sucrose solution. From Turner (1985).

is perturbed by displacing a small parcel downward. As in the salt-finger case (cf. Fig. 1.3), it rapidly adjusts its temperature but retains salinity. However, since the background salinity now increases with depth, the parcel becomes lighter than the ambient fluid at the same level, and the buoyancy force drives it upward. The parcel is not only lighter than its surroundings but, because of heat gain, it is also lighter than it was originally. Thus, on its way back to the point of origin, it experiences a buoyancy force that is greater than on its way down. As a result, the parcel gains some energy and overshoots its original equilibrium position. Above the equilibrium level, the parcel again quickly adjusts its temperature but not its salinity; it is now saltier and therefore heavier than the surrounding fluid. Eventually, gravity forces it downward, back to the original location. The parcel overshoots again, and the process repeats over and over. The energy gain by the particle at each cycle leads to a gradual increase in the amplitude of oscillations, resulting in the so-called "overstable" mode of instability.

Since oscillatory modes could be easily damped by viscous drag, diffusive instability in the uniformly stratified fluid is restricted to a rather narrow range of parameters (quantified in Chapter 2). More common in nature and in laboratory realizations is a stepped configuration consisting of well-mixed layers separated by a thin diffusive interface. Figure 1.6 shows an experiment set up by pouring a layer of salty water on top of a layer of denser sugar solute. In this case, salt is the fast

diffuser and sugar the slow one. The more rapid molecular diffusion of salt across the interface produces the downward density flux. The region immediately below (above) the interface becomes denser (lighter), which maintains the top-heavy convection in both mixed layers. Despite the apparent differences, convection in layered systems and overstable oscillations in a continuously stratified fluid are both described by the generic term diffusive convection.

1.4 Scale analysis

The foregoing examples leave us with a sense that double-diffusive convection operates in a most unusual way. All its forms, fingering and diffusive, are driven by the net release of potential energy and therefore fluid necessarily lowers its center of mass: double-diffusive mixing makes the relatively light fluid in upper layers even lighter; the heavy fluid at depth becomes heavier. The primary instability is driven by molecular diffusion, a stabilizing agent in most fluid dynamical problems. The eddy diffusivities of density constituents are different – yet another unexpected consequence of two-component dynamics. Overall, it seems that our physical intuition, built on experience with simple one-component fluids, fails miserably when it comes to double-diffusion. Therefore to fully understand double-diffusive convection, one has to rely, perhaps more strongly than in other branches of fluid dynamics, on formal mathematical results.

The analytical explorations throughout this book are based on the well-known (e.g., Pedlosky, 1979) Boussinesq equations of motion:

$$
\begin{cases}
\dfrac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\dfrac{\nabla p}{\rho_0} + g \dfrac{\rho - \rho_0}{\rho_0} + \nu \nabla^2 \vec{v}, \\[2ex]
\dfrac{\partial T}{\partial t} + \vec{v} \cdot \nabla T = k_T \nabla^2 T, \\[2ex]
\dfrac{\partial S}{\partial t} + \vec{v} \cdot \nabla S = k_S \nabla^2 S, \\[2ex]
\nabla \cdot \vec{v} = 0,
\end{cases}
\tag{1.1}
$$

where $\vec{v}$ is the (non-divergent) velocity field, p is the dynamic pressure, ν is the kinematic viscosity, ρ is the density and ρ_0 is a reference value. T and S represent two scalar quantities affecting the density of the fluid (e.g., temperature and salinity in the oceanographic context). Molecular diffusivities k_T and k_S are assumed to be uniform but unequal ($k_T > k_S$). Rotational effects are neglected and the fluid is regarded as incompressible. We also assume the linear equation of state:

$$
\frac{\rho - \rho_0}{\rho_0} = \beta(S - S_0) - \alpha(T - T_0),
\tag{1.2}
$$

where (α, β) are the constant expansion/contraction coefficients and (T_0, S_0) are the reference temperature and salinity.

We now proceed to establish, tentatively at first, the scales relevant to double-diffusive convection. Typical temporal and spatial scales are denoted as $\langle t \rangle$ and $\langle L \rangle$; the scales of velocity, temperature, salinity, density and pressure perturbations are $\langle v' \rangle$, $\langle T' \rangle$, $\langle S' \rangle$, $\langle \rho' \rangle$ and $\langle p' \rangle$ respectively. We are particularly interested in the dependence of these quantities on the background vertical temperature and salinity gradients $(\bar{T}_z, \bar{S}_z)$ and therefore the temperature scale is expressed as follows:

$$\langle T' \rangle \sim \langle L \rangle \, |\bar{T}_z|. \tag{1.3}$$

Because molecular dissipation plays a central role in double-diffusive convection, the magnitude of the diffusive term in the temperature equation should be comparable to the local rate of change in temperature. Since our focus will be on fully developed instabilities, it is also reasonable to assume that the nonlinear terms are equally important:

$$\frac{1}{\langle t \rangle} \sim \frac{\langle v' \rangle}{\langle L \rangle} \sim \frac{k_T}{\langle L \rangle^2}. \tag{1.4}$$

For the equation of state (1.2), we expect comparable effects of temperature and salinity on density distribution:

$$\frac{\langle \rho' \rangle}{\rho_0} \sim \alpha \langle T' \rangle \sim \beta \langle S' \rangle. \tag{1.5}$$

Finally, in the momentum equation, we anticipate that the buoyancy force is of the same order as viscous dissipation and the pressure gradient term, which is equivalent to setting

$$g \frac{\langle \rho' \rangle}{\rho_0} \sim \nu \frac{\langle v' \rangle}{\langle L \rangle^2} \sim \frac{\langle p \rangle}{\rho_0 \langle L \rangle}. \tag{1.6}$$

Drawing together Eqs. (1.3)–(1.6), we arrive at the following magnitudes of key variables:

$$\langle t \rangle \sim \frac{d^2}{k_T}, \ \langle L \rangle \sim d, \ \langle v' \rangle \sim \frac{k_T}{d}, \ \langle p' \rangle \sim \frac{\rho_0 \nu k_T}{d^2}, \ \langle T' \rangle \sim d|\bar{T}_z|, \ \langle S' \rangle \sim \frac{\alpha}{\beta} \langle T' \rangle, \tag{1.7}$$

where

$$d = \left(\frac{k_T \nu}{g \alpha \, |\bar{T}_z|} \right)^{\frac{1}{4}}. \tag{1.8}$$

The combination (1.8) can be interpreted as the nominal length scale expected for primary double-diffusive instabilities. A typical value of $\bar{T}_z$

in the mid-latitude thermocline is about 0.01 °C m^{-1} and therefore (1.7) and (1.8) yield $\langle t \rangle \sim 10^3$ s, $\langle L \rangle \sim 0.01$ m, $\langle v' \rangle \sim 10^{-5}$ m s^{-1}, and $\langle T' \rangle \sim 10^{-4}$ °C. Such scales place oceanic double-diffusive convection among the so-called microstructure processes. It should be emphasized, however, that this estimate pertains only to primary instabilities. These scales do not apply to secondary double-diffusive phenomena – collective instability waves, intrusions and thermohaline staircases – which fall into the "finescale" ($\sim$10–100 m) category and will be discussed in Chapters 6–9.

1.5 Non-dimensionalization and governing parameters

In order to systematically explore any hydrodynamic phenomenon, it is essential to determine the key non-dimensional variables that control its pattern and strength. We fully appreciate, for instance, the significance of the Rayleigh number in thermal convection and the Reynolds number in viscous shear flows. For double-diffusion, the optimal choice of governing parameters is not as clear-cut as one would hope. The literature contains two approaches to the analysis of double-diffusive convection, each motivated by a distinct view of the dynamics at play.

The older approach builds on experience with the Rayleigh–Bénard thermal convection problem, in which a fluid confined between two rigid planes is heated from below. If the analogy between double-diffusion and thermal convection is justified, then it is sensible to classify the double-diffusive regimes in terms of Rayleigh numbers based on temperature and salinity variation:

$$R = \frac{g\alpha \Delta T H^3}{k_T \nu}, \quad R_s = \frac{g\beta \Delta S H^3}{k_T \nu}, \quad (1.9)$$

where H is the thickness of a diffusive layer and ΔT, ΔS are the temperature and salinity variations across it. However, for many double-diffusive problems, such a description is misleading. The principal feature of double-diffusion is that, unlike thermal convection, it defines its own internal length scale d in (1.8) – the scale that is independent of H. If H greatly exceeds d, the vertical boundaries become irrelevant for the processes in the interior, and therefore inclusion of H among the principal governing parameters is unwarranted.

In subsequent developments we tend to exploit an alternative approach – the more modern "unbounded gradient layer" model. Instead of focusing on the integral measures of a double-diffusive layer ($\Delta T, \Delta S, H$), we ascribe greater physical significance to the local large-scale gradients of temperature and salinity. The temperature and salinity fields are separated into the linear background stratification ($\bar{T}_z = const$, $\bar{S}_z = const$) and perturbations (T', S'):

$$\begin{cases} T = \bar{T}(z) + T', \\ S = \bar{S}(z) + S'. \end{cases} \quad (1.10)$$

In most models considered here, the background state is at rest and therefore $\vec{v}' = \vec{v}$. On many occasions, analytical development can be simplified by non-dimensionalizing the governing equations using the scales established in (1.7). The dimensional variables are replaced by their non-dimensional counterparts as follows:

$$
\begin{cases}
(x, y, z) \rightarrow d \cdot (x, y, z), \\
\vec{v} \rightarrow \dfrac{k_T}{d} \cdot \vec{v}, \, t \rightarrow \dfrac{d^2}{k_T} \cdot t, \, p' \rightarrow \dfrac{\rho_0 \nu k_T}{d^2} p', \\
\alpha T' \rightarrow \alpha \left| \bar{T}_z \right| d \cdot T', \, \beta S' \rightarrow \alpha \left| \bar{T}_z \right| d \cdot S',
\end{cases}
\tag{1.11}
$$

where the nominal double-diffusive scale d is given in (1.8). The expansion/contraction coefficients (α, β) are incorporated in (T', S'), and $\alpha \left| \bar{T}_z \right| d$ is used as the scale for both temperature and salinity. The background gradients $\left(\bar{T}_z, \bar{S}_z \right)$ are consistently treated as dimensional variables. We shall refer to the transformation (1.11) as the standard system of non-dimensionalization. Considerations of clarity require dimensional or non-dimensional treatment for different topics and the type of variables will be explicitly specified.

For the finger case $(\bar{T}_z > 0, \bar{S}_z > 0)$, expressing the Boussinesq equations (1.1) in terms of (T', S') and non-dimensionalizing the result yields:

$$
\begin{cases}
\dfrac{1}{Pr} \left(\dfrac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} \right) = -\nabla p + (T' - S')\vec{k} + \nabla^2 \vec{v}, \\
\dfrac{\partial T'}{\partial t} + \vec{v} \cdot \nabla T' + w = \nabla^2 T', \\
\dfrac{\partial S'}{\partial t} + \vec{v} \cdot \nabla S' + \dfrac{w}{R_\rho} = \tau \nabla^2 S', \\
\nabla \cdot \vec{v} = 0,
\end{cases}
\tag{1.12}
$$

where $Pr = \frac{\nu}{k_T}$ is the Prandtl number, $\tau = \frac{k_S}{k_T}$ is the diffusivity ratio, and $R_\rho = \frac{\alpha \bar{T}_z}{\beta \bar{S}_z}$ is the density ratio.

For the diffusive case $(\bar{T}_z < 0, \bar{S}_z < 0)$ the equivalent set of non-dimensional equations is

$$
\begin{cases}
\dfrac{1}{Pr} \left(\dfrac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} \right) = -\nabla p + (T' - S')\vec{k} + \nabla^2 \vec{v}, \\
\dfrac{\partial T'}{\partial t} + \vec{v} \cdot \nabla T' - w = \nabla^2 T', \\
\dfrac{\partial S'}{\partial t} + \vec{v} \cdot \nabla S' - \dfrac{w}{R_\rho} = \tau \nabla^2 S', \\
\nabla \cdot \vec{v} = 0.
\end{cases}
\tag{1.13}
$$

The only difference between (1.12) and (1.13) appears in the signs of w terms – terms representing the influence of the background T–S gradients. However, in studies of diffusive convection it is more common to use, instead of R_ρ, the diffusive density ratio $R_\rho^* = \frac{1}{R_\rho} = \frac{\beta \bar{S}_z}{\alpha \bar{T}_z}$. Adopting different conventions for diffusive and fingering cases is not as unreasonable as it first seems. This way, the density ratio always exceeds unity for the bottom-heavy stratification and increases (decreases) with the gradient of the stabilizing (destabilizing) component.

The non-dimensional systems (1.12) and (1.13) suggest that the dynamics of unbounded double-diffusive systems, fingering or diffusive, are controlled by three key parameters: Pr, τ, and R_ρ. The Prandtl number and the diffusivity ratio reflect the properties of a particular fluid; for seawater in the mid-latitude upper ocean, $Pr \sim 7$ and $\tau \sim 10^{-2}$. The density ratio R_ρ, on the other hand, is determined by local environmental conditions and can be highly inhomogeneous. Therefore, a central problem in double-diffusive convection involves prediction of its control by the background density ratio. Physical interpretation of the density ratio is straightforward – it measures the degree of compensation between temperature and salinity gradients in terms of their effects on density stratification. Thus, for instance, $R_\rho = 1$ corresponds to full T–S compensation, resulting in the uniform background density ($\bar{\rho}_z = 0$). Large values of R_ρ imply that the density distribution is controlled by its thermal component.

1.6 Turner angle

The density ratio is closely related to another common measure of the stratification pattern – the Turner angle (Ruddick, 1983). While the Turner angle (Tu) generally serves the same purpose as R_ρ, it can offer a more convenient and transparent interpretation of the environment in terms of susceptibility to double-diffusion. Tu can be defined as a polar angle in the $(\alpha \bar{T}_z, \beta \bar{S}_z)$ plane, measured relative to the $\alpha \bar{T}_z = -\beta \bar{S}_z > 0$ ray, as indicated in Figure 1.7. This definition can be cast in a more explicit form:

$$Tu = 135° - \arg(\beta \bar{S}_z + i \alpha \bar{T}_z), \tag{1.14}$$

which, in turn, can be used to express the T–S gradients in terms of the Turner angle:

$$\begin{cases} \alpha \bar{T}_z = A \sin(Tu + 45°), \\ \beta \bar{S}_z = -A \cos(Tu + 45°), \end{cases} \tag{1.15}$$

where $A = \sqrt{(\alpha \bar{T}_z)^2 + (\beta \bar{S}_z)^2}$. Since the types of instability are determined by the signs of background T–S gradients, rather than by their absolute values, various instabilities occupy distinct sectors in the parameter space of Figure 1.7. Thus,

Figure 1.7 Schematic diagram of the Turner angle in the $(\alpha\bar{T}_z, \beta\bar{S}_z)$ parameter space. After Ruddick (1983).

for $Tu = 0$, temperature and salinity gradients are both stabilizing and contribute equally to the density gradient. As Tu is increased, the absolute value of the salinity gradient decreases, vanishing at $Tu = 45°$. For $45° < Tu < 90°$, the salinity gradient is destabilizing but its magnitude is less than that of the stabilizing temperature gradient. In this region, stratification is susceptible to salt fingering. As we further increase Tu, the system enters into the gravitationally unstable regime ($90° < Tu < 270°$). The region predisposed to diffusive convection is located at $-90° < Tu < -45°$ and the stratification is doubly stable for $-45° < Tu < 45°$.

As apparent from (1.15), there is a direct connection between the Turner angle and the density ratio:

$$R_\rho = -\tan(Tu + 45°). \tag{1.16}$$

However, Ruddick (1983) notes several advantages of focusing on Tu rather than on R_ρ in the analysis of water-mass composition. For instance, complications associated with the singularity of the density ratio in the nearly homogeneous salinity stratification ($\bar{S}_z \rightarrow 0$) do not arise if Tu is used instead. Another undesirable
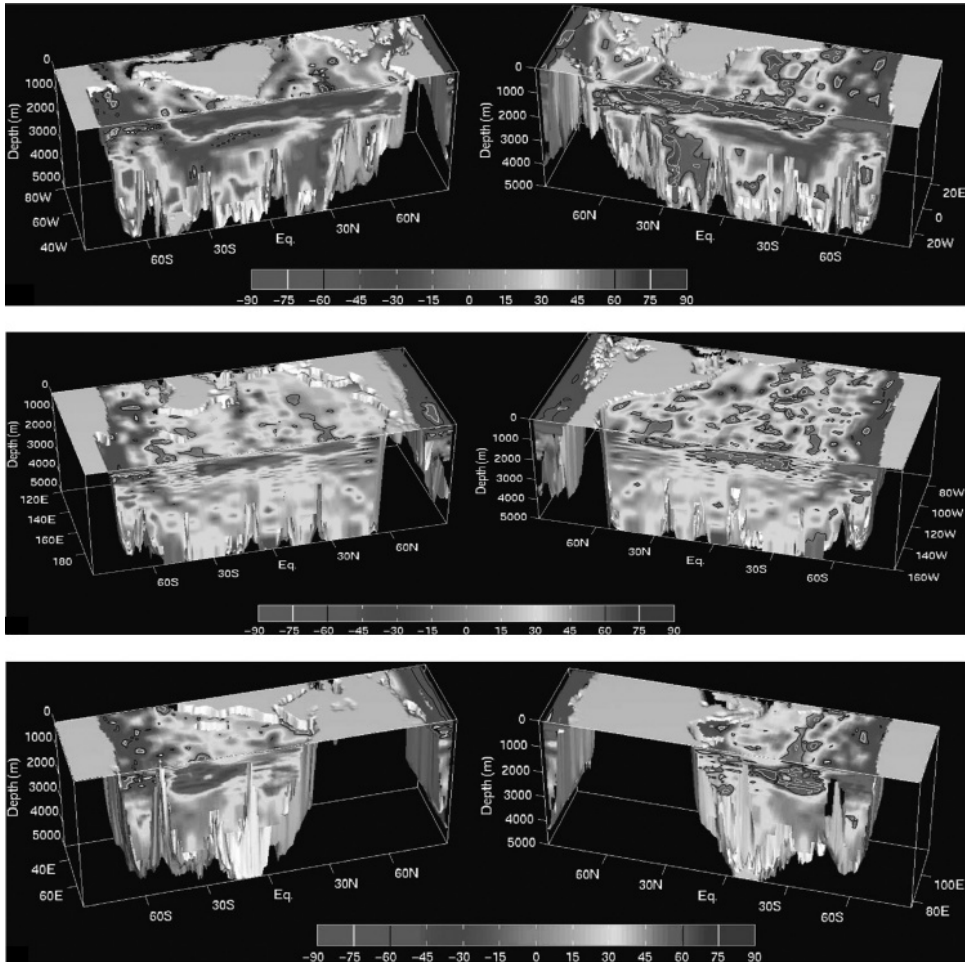
Figure 1.8 Distribution of the Turner angle in the Atlantic (top), Pacific (center) and Indian (bottom) Oceans. From You (2002). See color plates section.

feature of the density ratio is its ambiguity. Concurrently reversing the signs of *T–S* gradients does not affect the density ratio, although the background stratification and the character of resulting instabilities change dramatically. The Turner angle, on the other hand, uniquely defines the signs of $\bar{T}_z$ and $\bar{S}_z$. Finally, *Tu* replaces the infinite scale of R_ρ by a finite range of 2π.

Figure 1.8 shows the distribution of the Turner angle in the Atlantic, Pacific and Indian Oceans. In all basins, conditions favoring double-diffusion are quite common. In the world ocean, about 30% of the volume is finger favorable and 15% is diffusively stratified. Double-diffusion is particularly widespread in the main thermocline, which has significant large-scale implications (discussed in greater

detail in Chapter 11). For instance, double-diffusion contributes to the vertical transport of heat and carbon dioxide in the thermocline, affecting air–sea fluxes and thereby the global climate. The prevalence of double-diffusion in the upper ocean also leads to enhanced mixing of nutrients, which directly controls biological productivity of the ocean.

It should be emphasized, however, that the classification of vertical stratification as finger favorable for $\bar{T}_z > 0$, $\bar{S}_z > 0$ and diffusive for $\bar{T}_z < 0$, $\bar{S}_z < 0$ offers only a crude and preliminary assessment of double-diffusive characteristics of the environment. These conditions are necessary but not always sufficient. A more precise description of double-diffusive instabilities is based on linear stability analysis, which is discussed next.