Shallow water model

Assumptions:

thin layer ($H \ll L$)

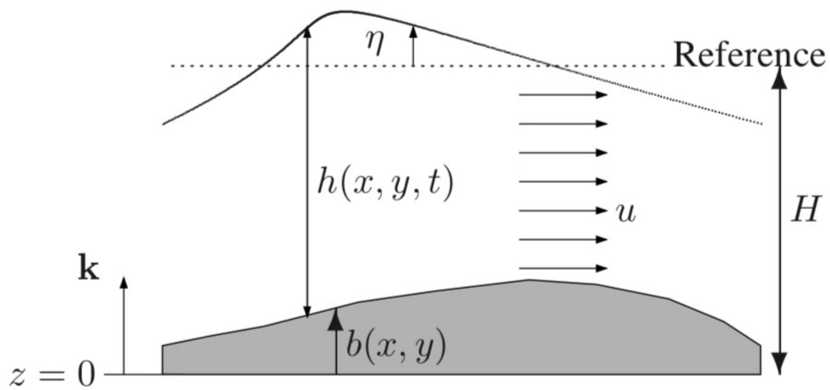
free surface

inviscid

motions initially independent of z

$$\frac{\partial}{\partial z} = 0 \text{ always}$$

homogeneous (constant density)



$$-g\frac{\partial \eta}{\partial x}$$

$$\text{x:}\quad \frac{\partial \mathrm{u}}{\partial \mathrm{t}} + \mathrm{u}\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \mathrm{v}\frac{\partial \mathrm{u}}{\partial \mathrm{y}} + \cancel{\mathrm{w}\frac{\partial \mathrm{u}}{\partial \mathrm{z}}} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$\text{y:}\quad \frac{\partial \mathrm{v}}{\partial \mathrm{t}} + \mathrm{u}\frac{\partial \mathrm{v}}{\partial \mathrm{x}} + \mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{y}} + \cancel{\mathrm{w}\frac{\partial \mathrm{v}}{\partial \mathrm{z}}} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

$$-g\frac{\partial \eta}{\partial x}$$

Continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

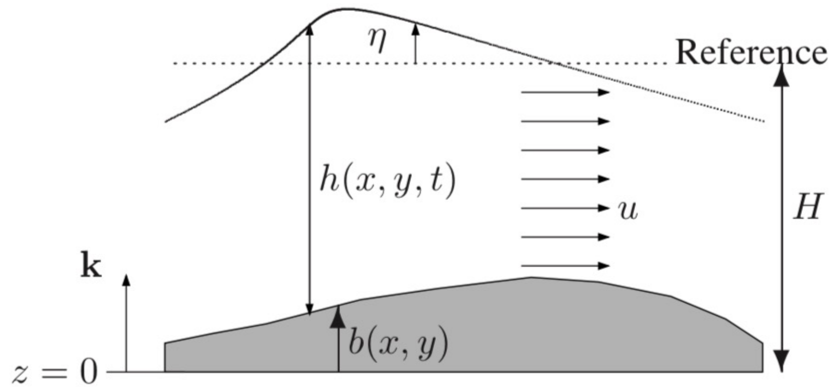
Integrate vertically from b to η:

$$\int_b^\eta \frac{\partial w}{\partial z}dz = -\int_b^\eta\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dz$$

$$w|_{z=\eta}-w|_{z=b}$$

$$\frac{\partial \eta}{\partial t}+u\frac{\partial h}{\partial x}+v\frac{\partial h}{\partial y}=-h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

$$\boxed{\frac{\partial \eta}{\partial t}+\frac{\partial hu}{\partial x}+\frac{\partial hv}{\partial y}=0}$$



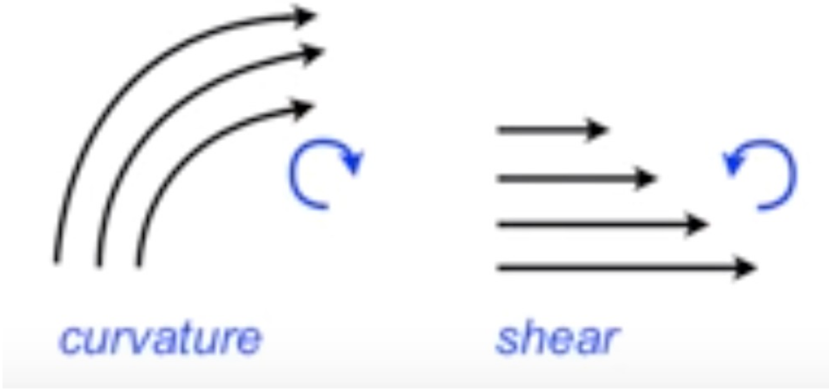
Boudary conditions

$$w|_{z=\eta}=\frac{\partial \eta}{\partial t}+u\frac{\partial \eta}{\partial x}+v\frac{\partial \eta}{\partial y}$$

$$w|_{z=-h}=u\frac{\partial b}{\partial x}+v\frac{\partial b}{\partial y}$$

$$\boxed{\eta=h+b-H}$$

Vorticity



Vorticity: curl of velocity (a measure of spin)

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$$

$$= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\boldsymbol{k}$$

For 2-D flow on the horizontal plane: $\boldsymbol{u} = u\mathbf{i} + v\mathbf{j}$

$$\boldsymbol{\omega} = \mathbf{k}\left(\underbrace{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}_{\zeta}\right)$$

Properties:

$$\nabla \cdot \boldsymbol{\omega} = 0$$

PV conservation for the shallow water model

The horizontal momentum equations (**homogeneous, invisicd,** $\frac{\partial}{\partial z} = \mathbf{0}$):

$$\frac{\partial \mathrm{u}}{\partial \mathrm{t}} + \mathrm{u}\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \mathrm{v}\frac{\partial \mathrm{u}}{\partial \mathrm{y}} - fv = -g\frac{\partial \eta}{\partial x} \quad (1)$$

$$\frac{\partial \mathrm{v}}{\partial \mathrm{t}} + \mathrm{u}\frac{\partial \mathrm{v}}{\partial \mathrm{x}} + \mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{y}} + fu = -g\frac{\partial \eta}{\partial y} \quad (2)$$

Taking the curl of the momentum equations by doing $\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(1)$:

$$\frac{\partial}{\partial \mathrm{t}}\left(\frac{\partial \mathrm{v}}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{\partial u}{\partial x}\frac{\partial \mathrm{v}}{\partial x} + u\frac{\partial^2 v}{\partial x^2} - \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + v\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} - v\frac{\partial^2 u}{\partial y^2} + f\frac{\partial u}{\partial x} + \frac{\partial f}{\partial y}v + f\frac{\partial v}{\partial y} = 0$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial u}{\partial x}\zeta + u\frac{\partial \zeta}{\partial x} + \frac{\partial v}{\partial y}\zeta + v\frac{\partial \zeta}{\partial y} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v = 0$$

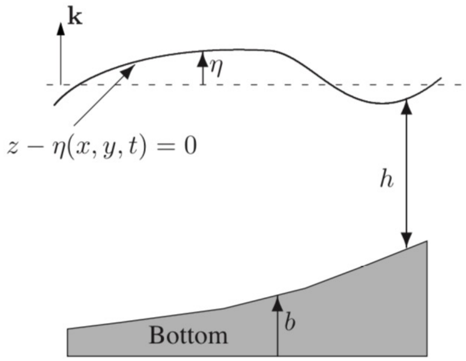
$$\frac{d\zeta}{dt} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(f + \zeta) + \beta v = 0$$

$$\boxed{\frac{df}{dt} = \frac{\partial f}{dt} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} = \beta v}$$

$$\frac{d(f + \zeta)}{dt} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(f + \zeta) = 0$$

$$\frac{d(f+\zeta)}{dt} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(f+\zeta) = 0 \; (1)$$

The continuity equation:



$$\frac{\partial \eta}{\partial t} + \frac{\partial hu}{\partial x} + \frac{\partial hv}{\partial y} = 0$$

$$\frac{dh}{dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \;\; (2)$$

$$\boxed{\eta = h + b - H}$$

Combining (1) and (2):

$$\frac{d(f+\zeta)}{dt} - \frac{f+\zeta}{h}\frac{dh}{dt} = 0$$

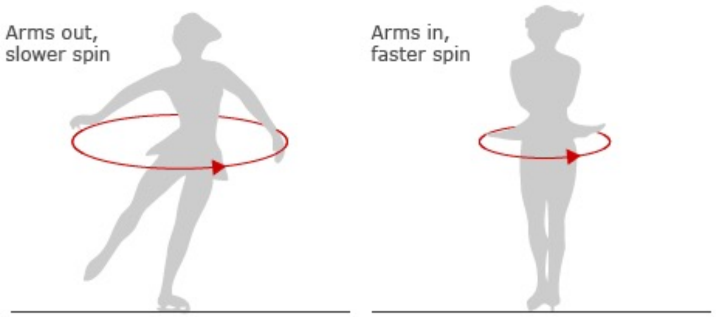
f: planetary vorticity

ζ: relative vorticity

$f+\zeta$: absolute vorticity

$$\boxed{\frac{d}{dt}\left(\frac{f+\zeta}{h}\right) = 0}$$

potential vorticity



$$\boxed{\frac{d}{dt}\left(\frac{f+\zeta}{h}\right) = 0}$$

Valid for barotropic, invisic fluids

If the density is only a function of pressure:

$$\rho = \rho(\mathrm{p})$$

Isolines of pressure and density are parallel

$$\nabla\rho \times \nabla p = 0$$

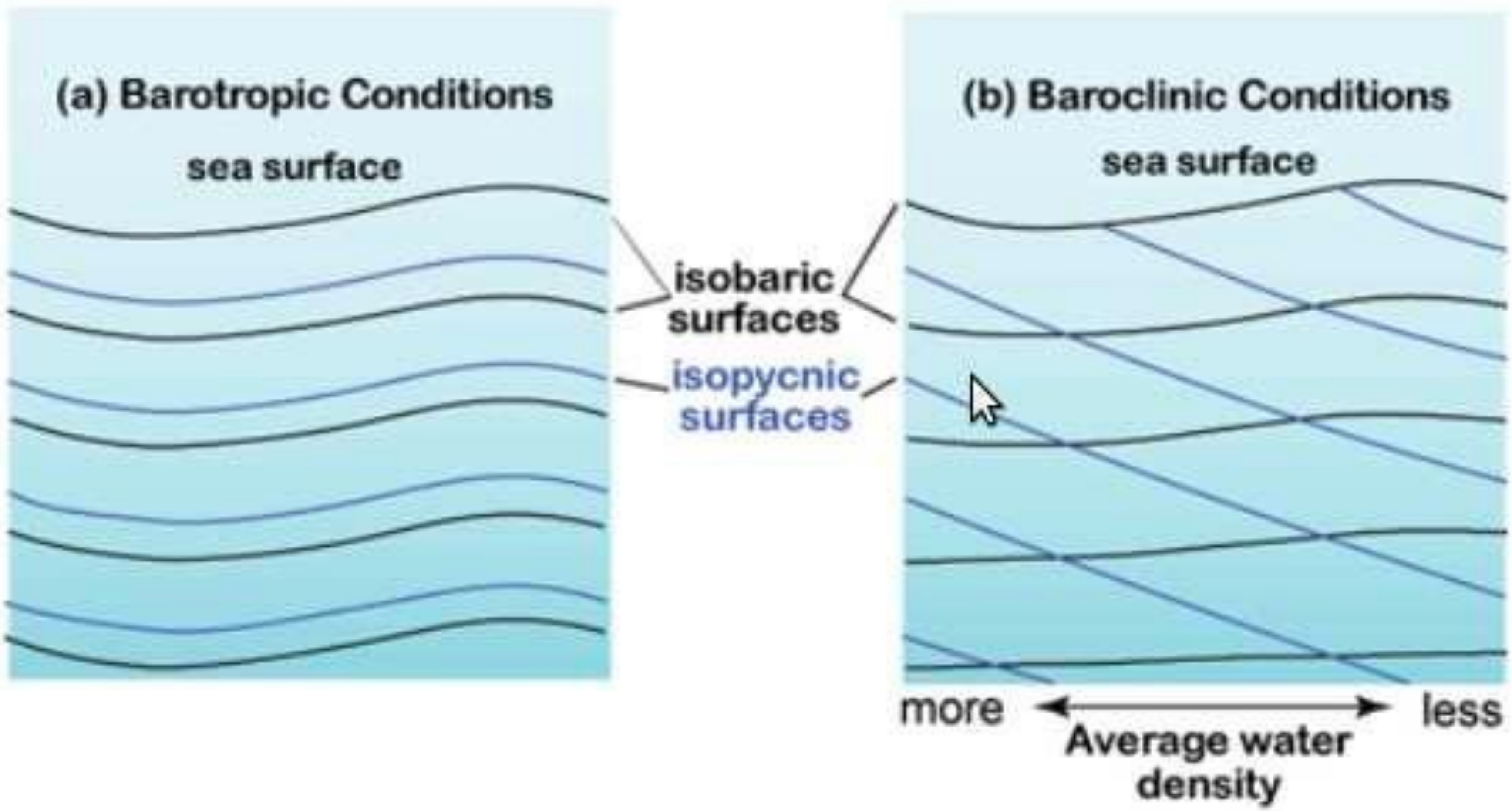
barotropic fluid

(for constant density?)

Otherwise:

$$\nabla\rho \times \nabla p \neq 0$$

baroclinic fluid



(a) Barotropic Conditions
sea surface
isobaric surfaces
isopycnic surfaces
(b) Baroclinic Conditions
sea surface
more
less
Average water density

Horizontal divergence/convergence in PV conservation

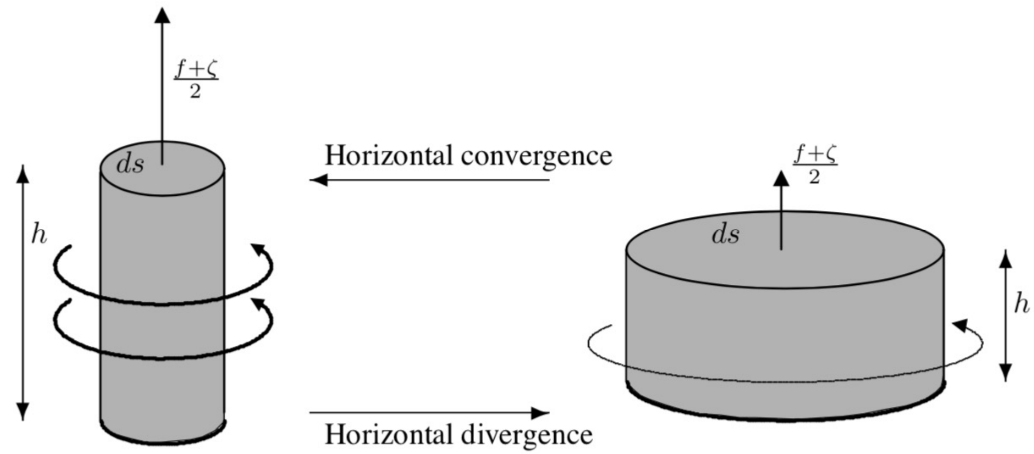
For volume conservation of a fluid column with a cross-section of ds and thickness of h :

$$\frac{d}{dt}(hds) = 0$$

$$\frac{dh}{dt}ds + h\frac{d}{dt}(ds) = 0$$

$$\frac{dh}{dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \quad (2)$$

$$\frac{d}{dt}(ds) = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)ds$$

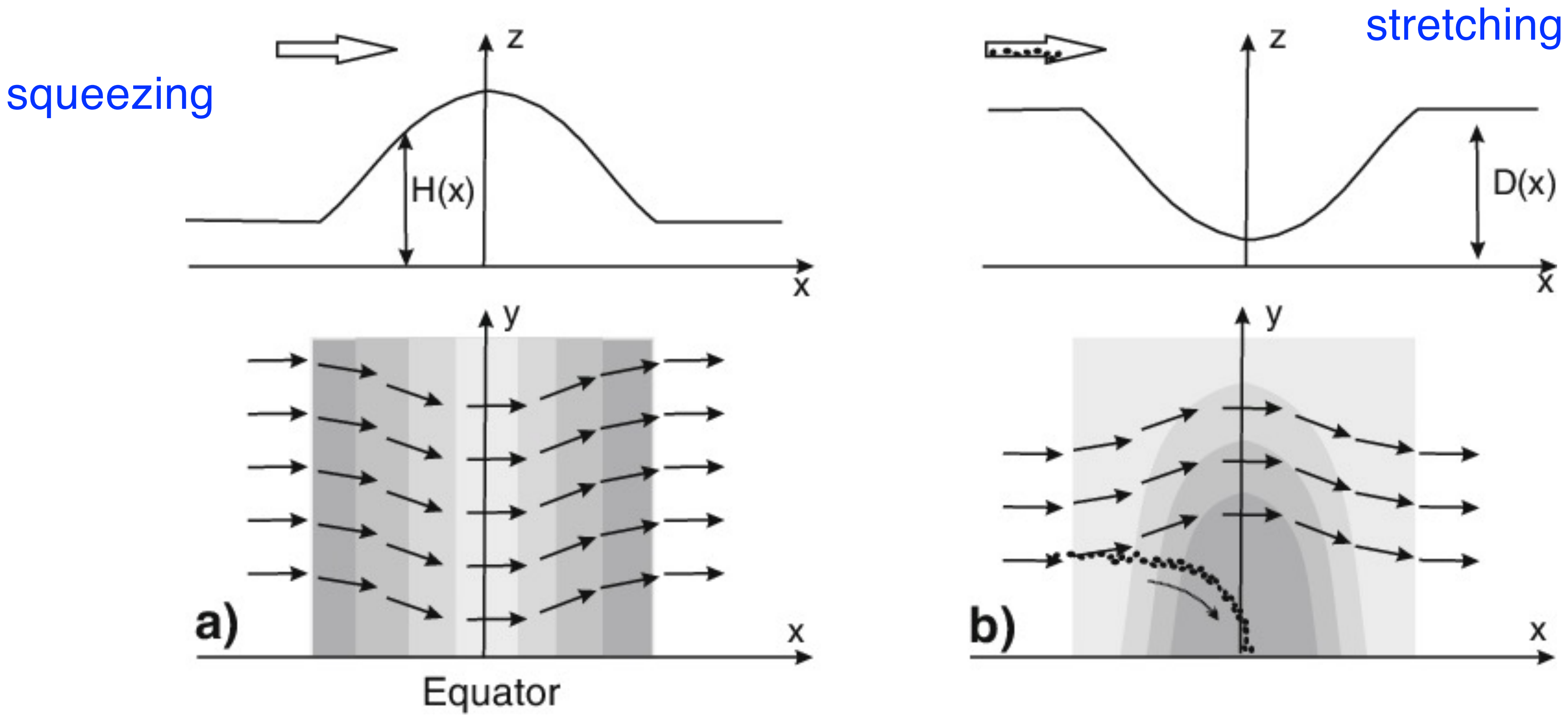


divergent flow, $\frac{d}{dt}(ds) > 0$, h decreases

convergent flow, $\frac{d}{dt}(ds) < 0$, h increases

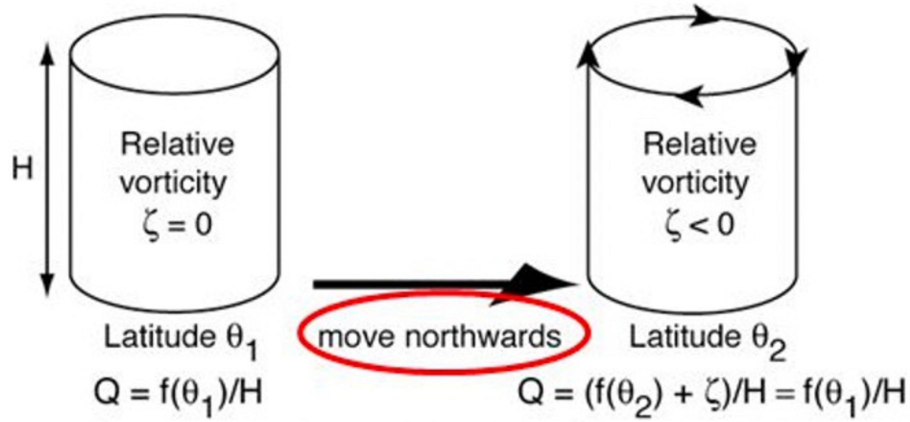
$$\frac{d}{dt}\left(\frac{f + \zeta}{h}\right) = 0$$

For large scale flow, $\zeta \ll f$



$$\frac{d}{dt}\left(\frac{f + \zeta}{h}\right) = 0$$

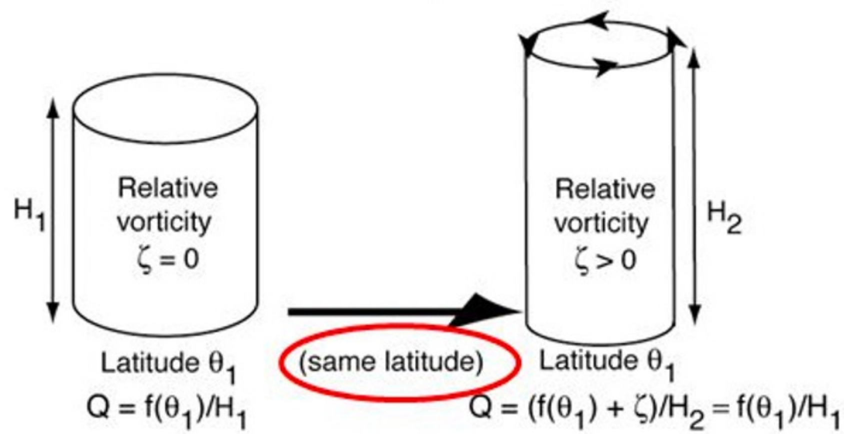
h is constant



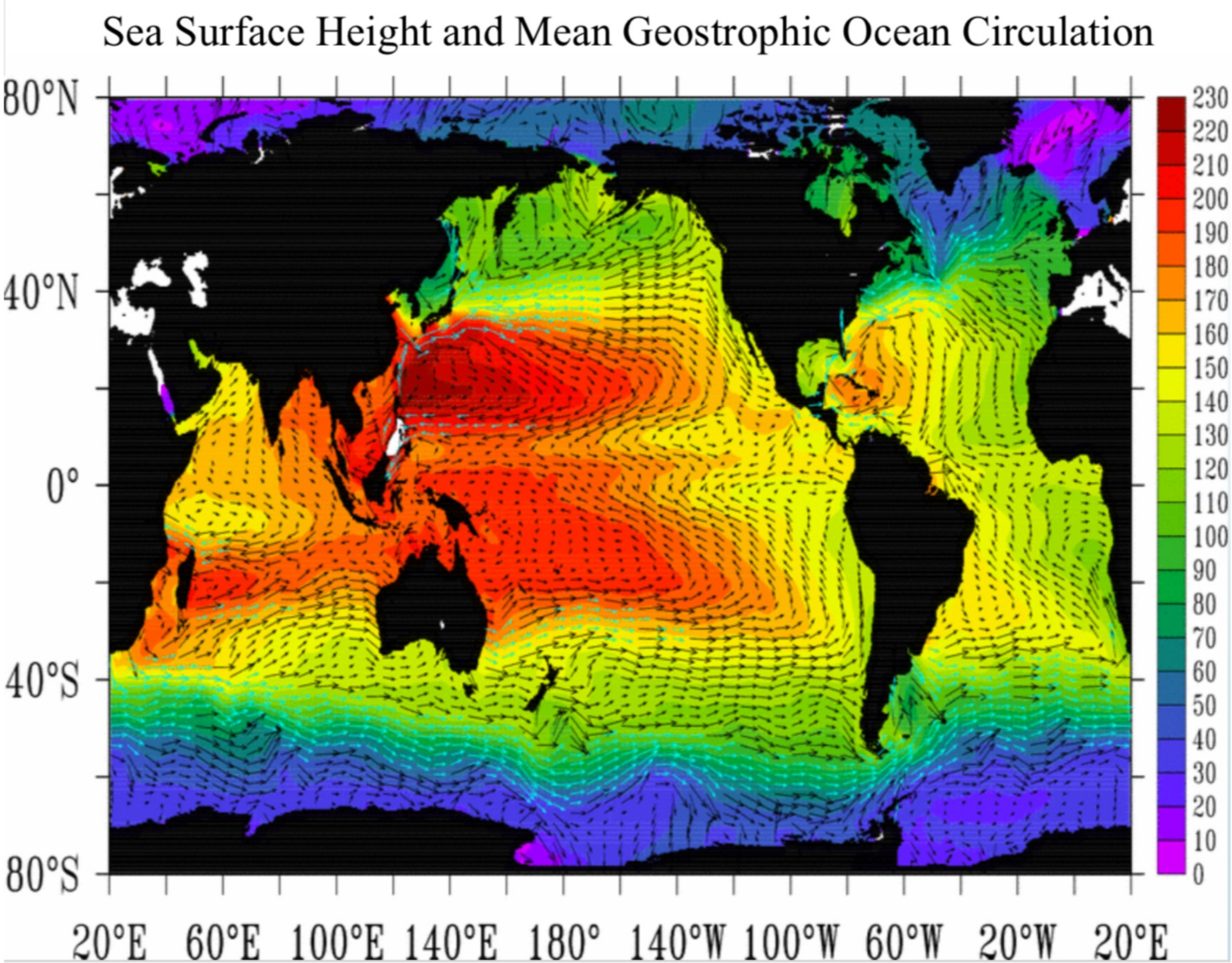
Conservation of potential vorticity Q in the absence of stretching (northern hemisphere): balance of planetary vorticity and relative vorticity

f is constant

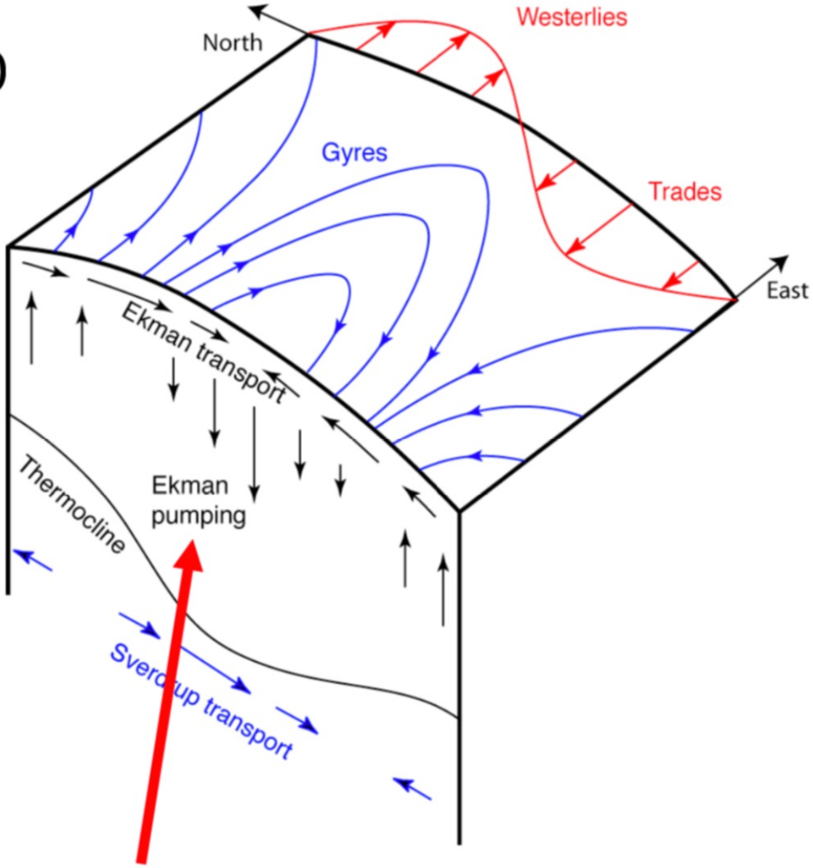
eddies generated by topography



Conservation of potential vorticity Q in the absence of planetary vorticity change (northern hemisphere): balance of relative vorticity and stretching


Sea Surface Height and Mean Geostrophic Ocean Circulation
80°N
40°N
0°
40°S
80°S
20°E
60°E
100°E
140°E
180°
140°W
100°W
60°W
20°W
20°E
230
220
210
200
190
180
170
160
150
140
130
120
110
100
90
80
70
60
50
40
30
20
10
0

Sverdrup



$$\frac{d}{dt}\left(\frac{f + \zeta}{h}\right) = 0$$

- Ekman pumping provides the squashing or stretching.
- The water columns must respond. They do this by changing latitude.
- (They do not spin up in place for the large-scale circulation.)

Squashing -> equatorward movement   Stretching -> poleward

TRUE in both Northern and Southern Hemisphere

DPO Fig. 7.13

Shallow water waves

Linear wave dynamics

Assumption: $R_O \ll 1$

c: wave speed

$$R_{OT} = \frac{\frac{U}{T}}{fU} = \frac{1}{fT} \sim \frac{1}{fL/c} = \frac{c}{fL}\,(c \gg U) \sim 1$$

The horizontal momentum equations are:

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$

The continuity equation is:

$$\frac{\partial \eta}{\partial t} + \frac{\partial hu}{\partial x} + \frac{\partial hv}{\partial y} = 0$$

$$\frac{\partial \eta}{\partial t} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} = 0$$

For flat bottom, $\eta = h - H$:

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \eta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} = 0$$

$$\boxed{\Delta H \frac{c}{L}} \qquad \frac{\Delta H}{T} \qquad H\frac{U}{L} \qquad \cancel{\Delta H \frac{U}{L}} \qquad \cancel{U\frac{\Delta H}{L}}$$

$$\Delta H \frac{c}{L} \sim H\frac{U}{L}$$

Then the continuity equation is reduced to:

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

$$\Delta H \ll H$$

small-amplitude waves

Inertia-gravity waves (Poincaré waves)

Assumption: flat bottom

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

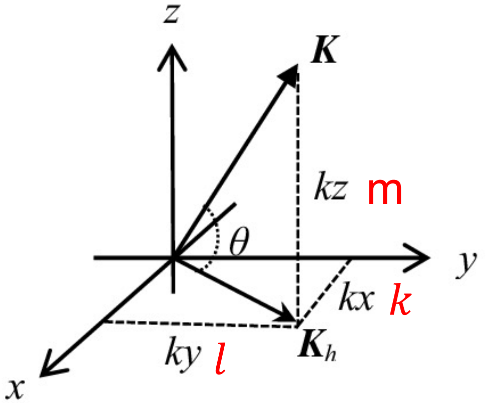


Fig. 1. Wave number vector and its components.

$$-i\omega U - fV = -gikA$$

$$-i\omega V + fU = -gilA$$

$$-i\omega A + ikHU + ilHV = 0$$

Give a wave solution:

$$u = Ue^{i(kx+ly-\omega t)}$$

$$v = Ve^{i(kx+ly-\omega t)}$$

$$\eta = Ae^{i(kx+ly-\omega t)}$$

$$\begin{pmatrix} -iw & -f & gik \\ f & -i\omega & gil \\ ikH & ilH & -i\omega \end{pmatrix} \begin{pmatrix} U \\ V \\ A \end{pmatrix} = 0$$

This homogeneous equation has non-trivial solutions only if the determinant of the coefficient matrix vanishes:

$$\omega[\omega^2 - f^2 - gH(k^2 + l^2)] = 0$$ dispersion relation

$$\boxed{R_d = \frac{\sqrt{gH}}{f}}$$

Rossby deformation radius (barotropic)

1. $\omega = 0, \ \frac{\partial}{\partial t} = 0,$ geostrophic flow

2. $\omega = \sqrt{f^2 + gH\underbrace{(k^2 + l^2)}_{K^2}}$

a. rotation is weak, $f^2 \ll gHK^2, \ \lambda \ll R_d$ (short-wave limit)

$\omega = \sqrt{gH}K, \ c = \sqrt{gH},$ gravity waves

b. rotation is important, $f^2 \gg gHK^2, \ \lambda \gg R_d$ (long-wave limit)

$\omega \sim f,$ inertial oscillations

Dispersion relation diagram

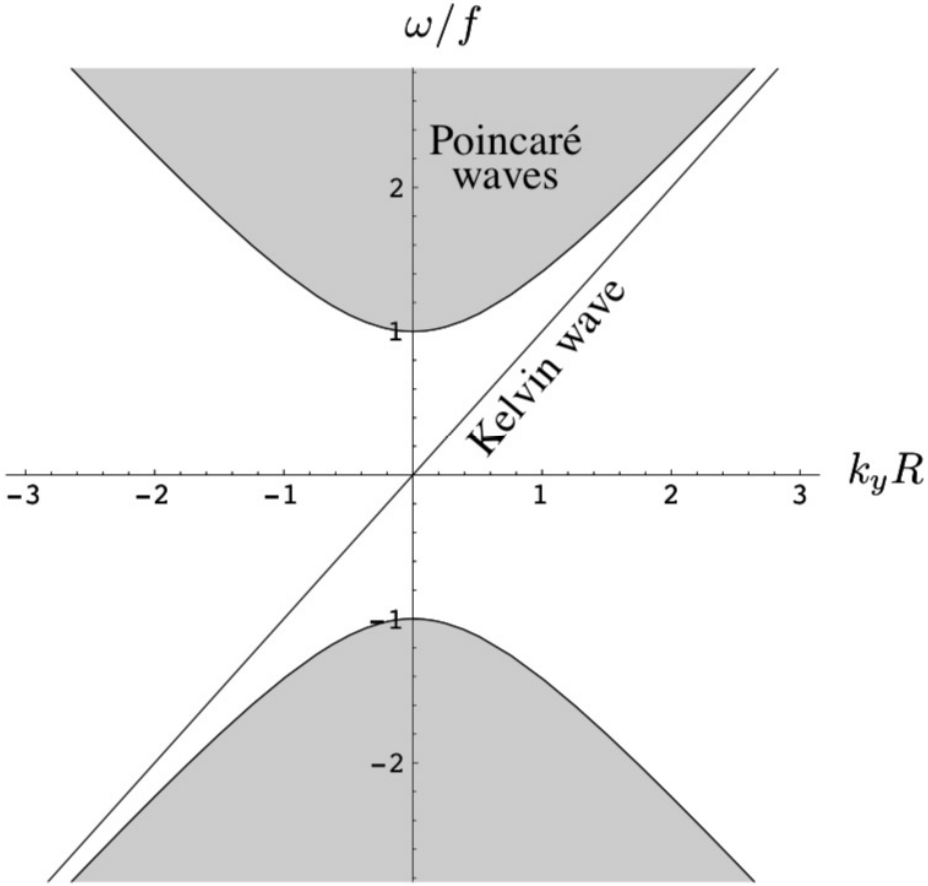


Figure 9-3 Recapitulation of the dispersion relation of Kelvin and Poincaré waves on the f-plane and on a flat bottom. While Poincaré waves (gray shades) can travel in all directions and occupy therefore a continuous spectrum in terms of k_y, the Kelvin wave (diagonal line) propagates only along a boundary.

Homework

7-8. In Utopia, a narrow 200-m deep channel empties in a broad bay of varying bottom topography (Figure 7-14). Trace the path to the sea and the velocity profile of the channel outflow. Take $f = 10^{-4}\ \mathrm{s}^{-1}$. Solve only for straight stretches of the flow and ignore corners.

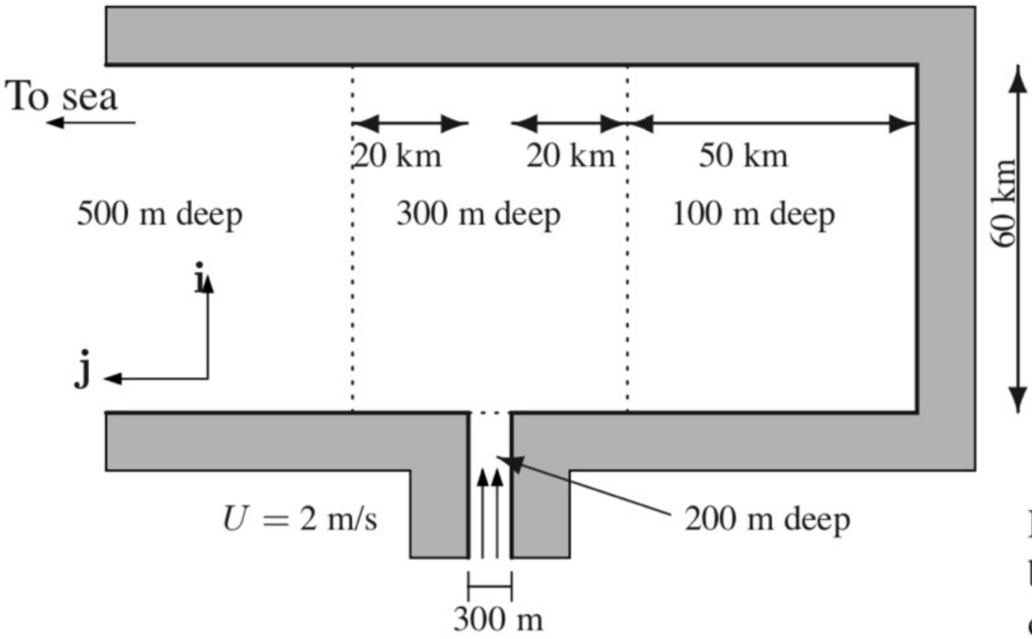


Figure 7-14 Geometry of the idealized bay and channel mentioned in Analytical Problem 7-8.