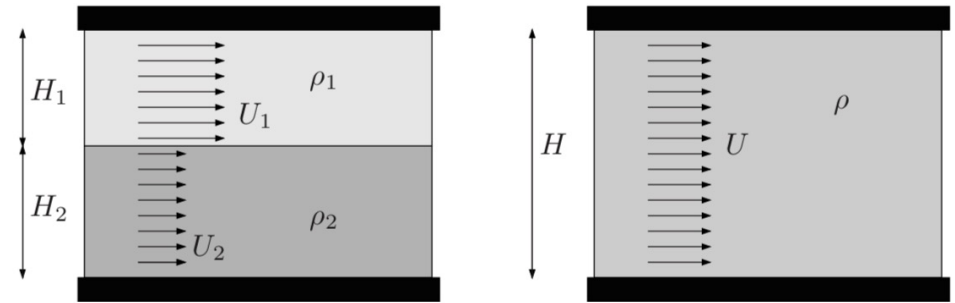


Mixing in stratified fluids

Assume unit length in the x and y directions:

$$PE \text{ gain} = \int_0^H \rho_{\text{final}} g z \, dz - \int_0^H \rho_{\text{initial}} g z \, dz$$

$$= \int_0^{H/2} \rho_2 g z \, dz + \int_{H/2}^H \rho_1 g z \, dz$$



$$= \frac{1}{2} \rho g H^2 - \left[\frac{1}{2} \rho_2 g \frac{H^2}{4} + \frac{1}{2} \rho_1 g \frac{3H^2}{4} \right]$$

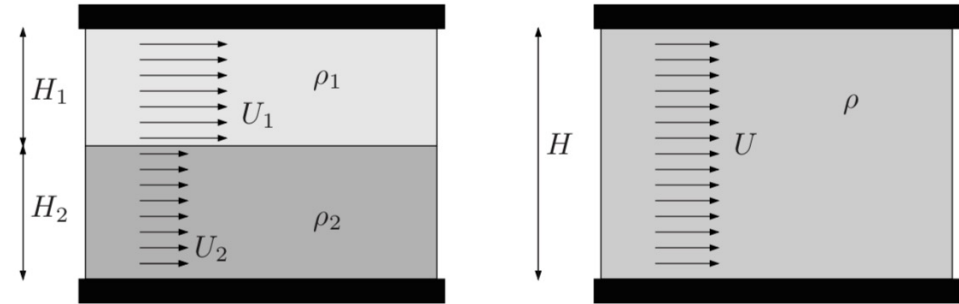
From mass conservation:

$$\rho_1 \frac{H}{2} + \rho_2 \frac{H}{2} = \rho H \quad \longrightarrow \quad \rho = \frac{1}{2} (\rho_1 + \rho_2)$$

$$PE \text{ gain} = \frac{1}{8} (\rho_2 - \rho_1) g H^2$$

For the KE change, ignore the density variation ($\rho = \rho_0$):

$$\begin{aligned}
 KE \text{ loss} &= \int_0^H \frac{1}{2} \rho_0 u_{\text{initial}}^2 dz - \int_0^H \frac{1}{2} \rho_0 u_{\text{final}}^2 dz \\
 &= \left(\int_0^{H/2} \rho_0 U_2^2 dz + \int_{H/2}^H \rho_0 U_1^2 dz \right) \\
 &= \frac{1}{2} \rho_0 U_2^2 \frac{H}{2} + \frac{1}{2} \rho_0 U_1^2 \frac{H}{2} - \frac{1}{2} \rho_0 U^2 H
 \end{aligned}$$



Complete vertical mixing is possible if KE loss is larger than PE gain:

From momentum conservation:

$$\begin{aligned}
 \rho_0 \frac{H}{2} U_1 + \rho_0 \frac{H}{2} U_2 &= \rho_0 U H \\
 U &= \frac{U_1 + U_2}{2}
 \end{aligned}$$

$$KE \text{ loss} = \frac{1}{8} \rho_0 (U_1 - U_2)^2 H$$

$$PE \text{ gain} = \frac{1}{8} (\rho_2 - \rho_1) g H^2$$

$$\frac{(\rho_2 - \rho_1) g H}{\rho_0 (U_1 - U_2)^2} < 1$$

$$\frac{\frac{g}{\rho_0} \frac{\Delta \rho}{H}}{\left(\frac{U_1 - U_2}{H}\right)^2} < 1$$

Richardson number $\frac{N^2}{\left(\frac{\partial u}{\partial z}\right)^2} < 1$

Kelvin-Helmholtz instability

Assumptions: stratified, inviscid, irrotational, two-dimensional (x-z) flow

The governing equations are:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = - \frac{1}{\rho_0} \frac{\partial p}{\partial x}$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} = - \frac{1}{\rho_0} \frac{\partial p}{\partial z} - \frac{\rho g}{\rho_0}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + w \frac{\partial \rho}{\partial z} = 0$$

At steady state:

$$u = \bar{u}(z), w = 0 \quad \rho = \bar{\rho}(z) \quad d\bar{p}/dz = -g\bar{\rho}(z)$$

After **small perturbation**:

$$u = \bar{u} + u', \quad w = w', \quad p = \bar{p} + p' \quad \rho = \bar{\rho} + \rho'$$

The equations become:

$$\frac{\partial(\bar{u} + u')}{\partial t} + (\bar{u} + u') \frac{\partial(\bar{u} + u')}{\partial x} + w' \frac{\partial(\bar{u} + u')}{\partial z} = -\frac{1}{\rho_0} \frac{\partial(\bar{p} + p')}{\partial x}$$

$$\frac{\partial u'}{\partial t} + \bar{u} \frac{\partial u'}{\partial x} + w' \frac{d\bar{u}}{dz} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial x}$$

$$\frac{\partial w'}{\partial t} + \bar{u} \frac{\partial w'}{\partial x} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial z} - \frac{\rho' g}{\rho_0}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0$$

$$\frac{\partial \rho'}{\partial t} + \bar{u} \frac{\partial \rho'}{\partial x} + w' \frac{d\bar{\rho}}{dz} = 0$$

perturbation streamfunction:

$$u' = -\frac{\partial \psi}{\partial z} \quad w' = \frac{\partial \psi}{\partial x}$$

$$\frac{\partial u'}{\partial t} + \bar{u} \frac{\partial u'}{\partial x} + w' \frac{d\bar{u}}{dz} = - \frac{1}{\rho_0} \frac{\partial p'}{\partial x} \quad (1)$$

$$\frac{\partial w'}{\partial t} + \bar{u} \frac{\partial w'}{\partial x} = - \frac{1}{\rho_0} \frac{\partial p'}{\partial z} - \frac{\rho' g}{\rho_0} \quad (2)$$

$\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial z}(1):$

$$\frac{\partial}{\partial t} \nabla^2 \psi + \bar{u} \frac{\partial}{\partial x} \nabla^2 \psi - \frac{d^2 \bar{u}}{dz^2} \frac{\partial \psi}{\partial x} = - \frac{g}{\rho_0} \frac{\partial \rho'}{\partial x}$$

$$\frac{\partial \rho'}{\partial t} + \bar{u}(z) \frac{\partial \rho'}{\partial x} + \frac{d\bar{\rho}}{dz} \frac{\partial \psi}{\partial x} = 0$$

$$\boxed{\frac{\partial \rho'}{\partial t} + \bar{u} \frac{\partial \rho'}{\partial x} + w' \frac{d\bar{\rho}}{dz} = 0}$$

Apply a wave solution: $\psi = \Psi(z) e^{i(kx - \omega t)}$ $\rho' = R(z) e^{i(kx - \omega t)}$

$$(-i\omega)[(ik)^2 \Psi + \frac{d^2 \Psi}{dz^2}] + \bar{u} ik[(ik)^2 \Psi + \frac{d^2 \Psi}{dz^2}] - \frac{d^2 \bar{u}}{dz^2} ik \Psi = -\frac{g}{\rho_0} ik R$$

Divide the equation by ik :

$$-c(\frac{d^2 \Psi}{dz^2} - k^2 \Psi) + \bar{u}(\frac{d^2 \Psi}{dz^2} - k^2 \Psi) - \frac{d^2 \bar{u}}{dz^2} \Psi = -\frac{g}{\rho_0} R$$

For the density equation:

$$(-i\omega)R + \bar{u} ikR + \frac{d\bar{\rho}}{dz} ik \Psi = 0$$

$$(\bar{u} - c)R + \frac{d\bar{\rho}}{dz} \Psi = 0$$

$$(\bar{u} - c) \left(\frac{d^2 \Psi}{dz^2} - k^2 \Psi \right) - \frac{d^2 \bar{u}}{dz^2} \Psi = - \frac{g}{\rho_0} R$$

$$(\bar{u} - c) R + \frac{d\bar{\rho}}{dz} \Psi = 0$$

$$c = c_r + ic_i$$

Eliminate R :

$$\begin{aligned} \psi &= \Psi(z) e^{ik(x - c_r t - ic_i t)} \\ &= \Psi(z) e^{kc_i t} e^{ik(x - c_r t)} \end{aligned}$$

$$(\bar{u} - c) \left(\frac{d^2 \Psi}{dz^2} - k^2 \Psi \right) + \left(\frac{N^2}{\bar{u} - c} - \frac{d^2 \bar{u}}{dz^2} \right) \Psi = 0$$

Taylor–Goldstein equation

Let $\phi = \frac{\Psi}{\sqrt{\bar{u} - c}} :$

$$\frac{d}{dz} \left[(\bar{u} - c) \frac{d\phi}{dz} \right] + \left[k^2 (\bar{u} - c) + \frac{1}{2} \frac{d^2 \bar{u}}{dz^2} + \frac{(d\bar{u}/dz)^2 - 4N^2}{4(\bar{u} - c)} \right] \phi = 0$$

Multiply the equation by ϕ^* and vertically integrate it across the domain:

$$\begin{aligned} \int (\bar{u} - c) \left(\left| \frac{d\phi}{dz} \right|^2 + k^2 |\phi|^2 \right) dz + \frac{1}{2} \int \frac{d^2 \bar{u}}{dz^2} |\phi|^2 dz \\ + \frac{1}{4} \int \frac{(d\bar{u}/dz)^2 - 4N^2}{(\bar{u} - c)} |\phi|^2 dz = 0. \end{aligned}$$

The imaginary part is:

$$c_i \int \left(\left| \frac{d\phi}{dz} \right|^2 + k^2 |\phi|^2 \right) dz = \frac{c_i}{4} \int \frac{(d\bar{u}/dz)^2 - 4N^2}{|\bar{u} - c|^2} |\phi|^2 dz \quad > 0$$

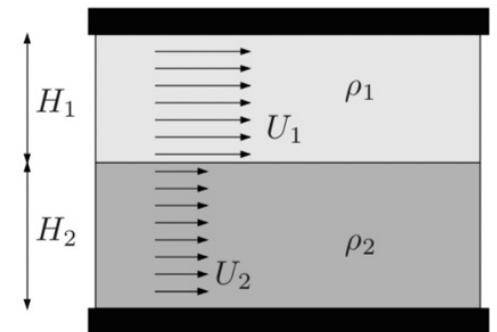
The necessary condition for instability to occur ($c_i \neq 0$) is:

Richardson number $Ri = \frac{N^2}{(d\bar{u}/dz)^2} < 1/4$

For a two-layer system:

$$\left| \frac{d\bar{u}}{dz} \right| = \frac{|U_1 - U_2|}{H} \quad N^2 = - \frac{g}{\rho_0} \frac{d\rho}{dz} = \frac{g}{\rho_0} \frac{\rho_2 - \rho_1}{H}$$

$$\frac{(\rho_2 - \rho_1)gH}{\rho_0(U_1 - U_2)^2} < \frac{1}{4}$$



Ri is essentially the ratio of the potential energy barrier that mixing must overcome to the kinetic energy that the sheared flow must supply

Kelvin-Helmholtz Instability

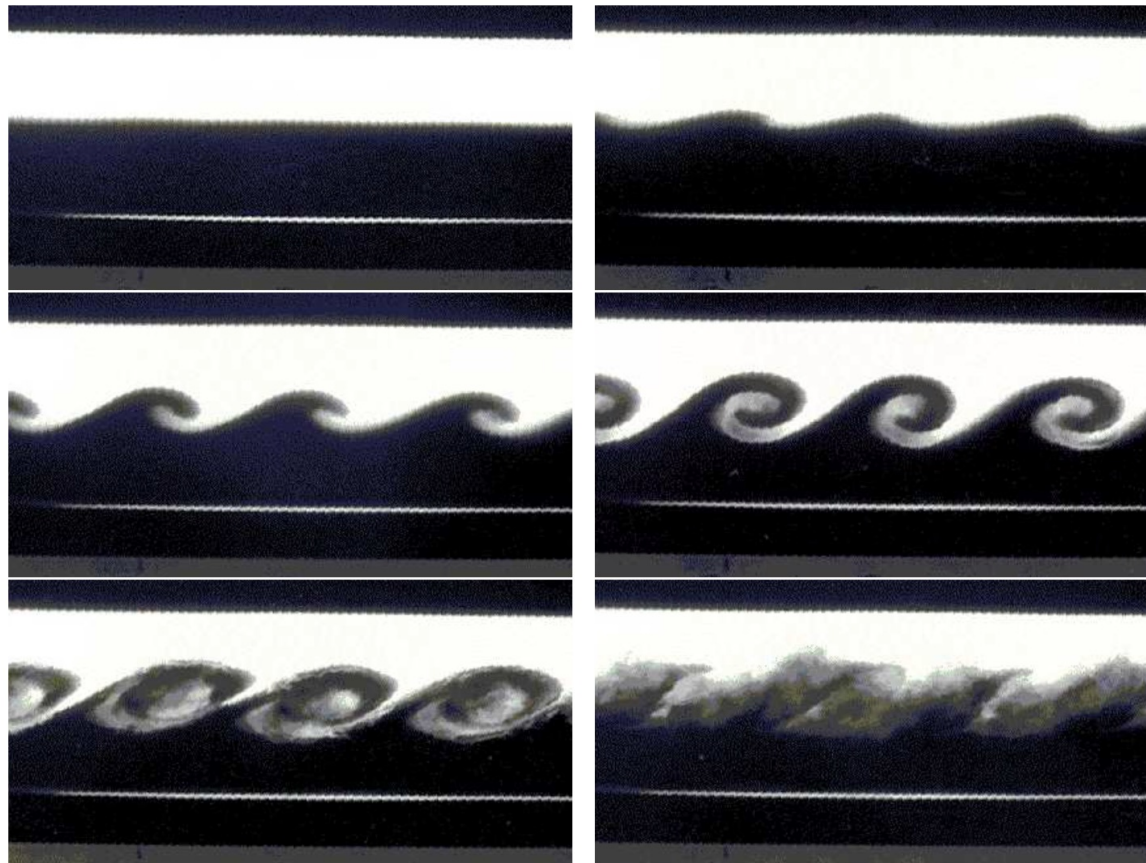
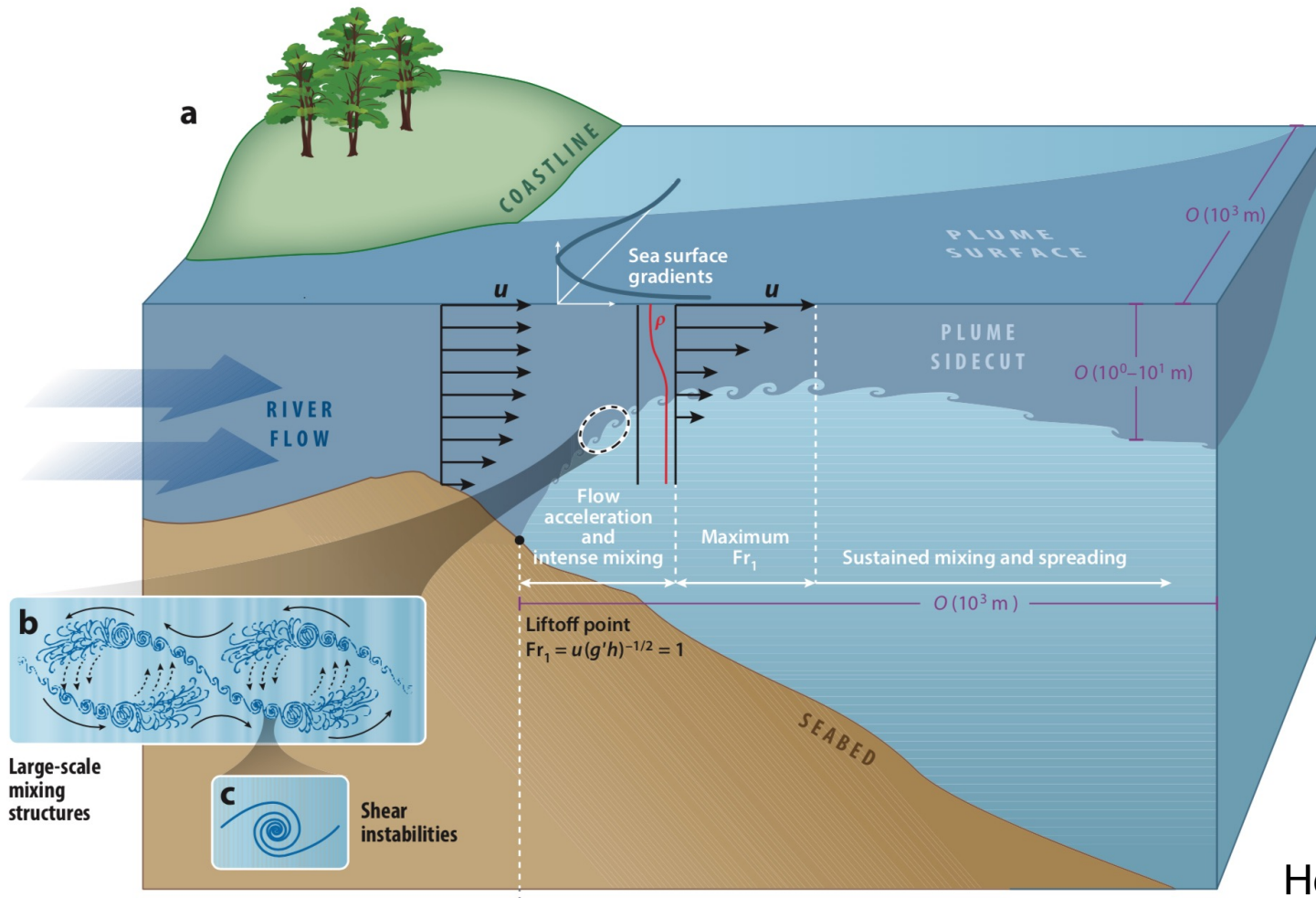


Figure 14-4 Kelvin–Helmholtz instability generated in a laboratory with fluids of two different densities and colors. (Adapted from *GFD-online*, Satoshi Sakai, Isawo Iizawa, Eiji Aramaki)



Kelvin-Helmholtz instability in estuaries



Horner-Devine et al. (2014)