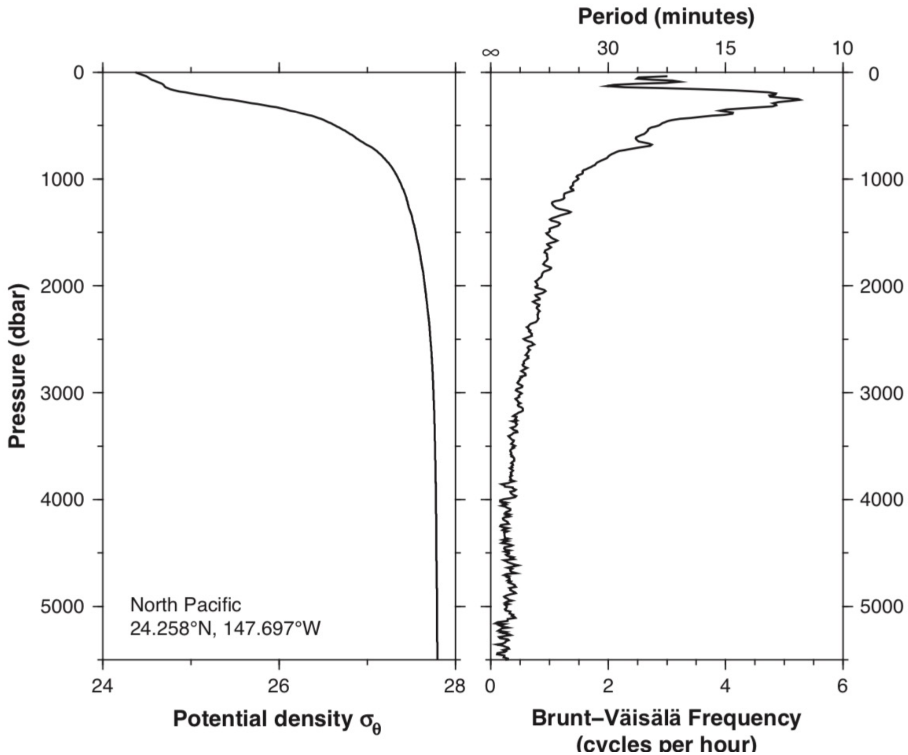
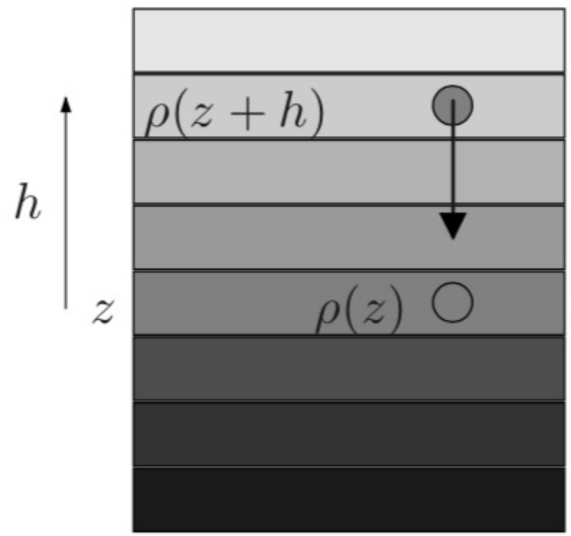
$$N^2 = -\frac{g}{\rho(z)}\frac{d\rho}{dz}$$

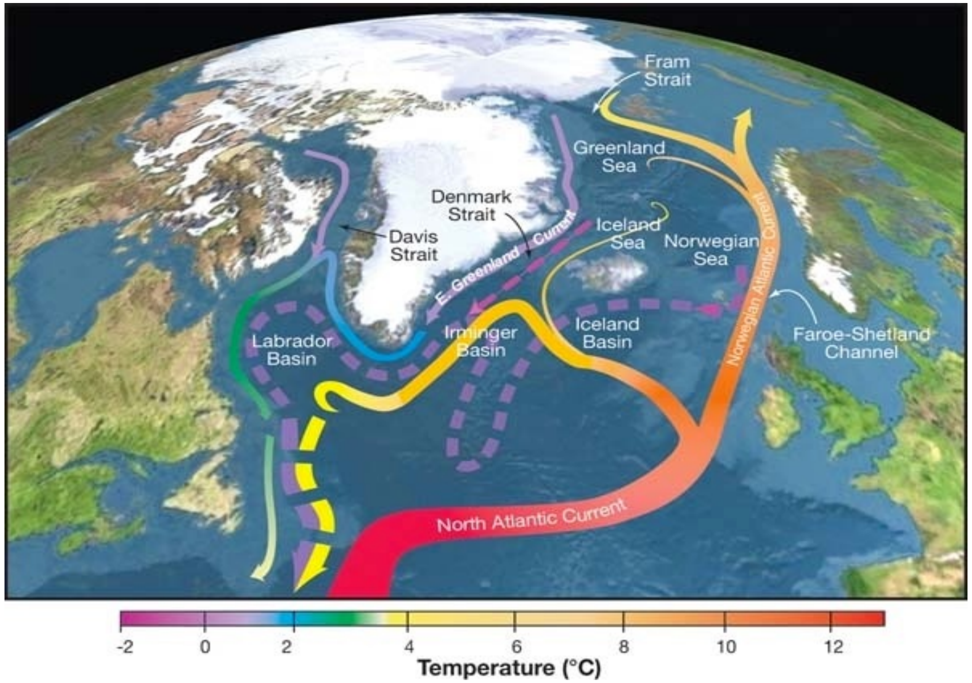
$N^2 > 0, \quad \frac{d\rho}{dz} < 0,$ stable water column

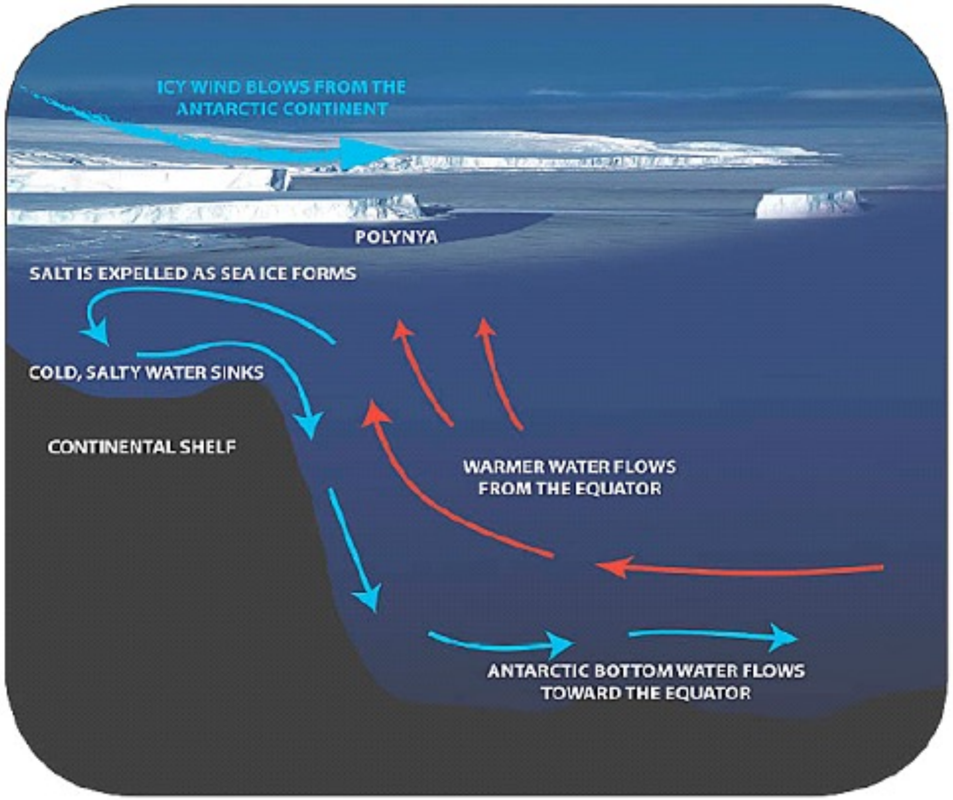
$N^2 < 0, \quad \frac{d\rho}{dz} > 0,$ unstable water column (convection)





Deep convection in the Nordic Sea and the Southern Ocean





Internal waves

Assumptions: stratified, large-scale, inviscid, incompressible, small perturbation

The density in a stratified system with wave perturbation is:

$$\rho = \rho_0 + \bar{\rho}(z) + \rho'(x, y, z, t) \qquad \rho' \ll \bar{\rho}(z) << \rho_0$$

The governing equations are:

$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{\rho}{\rho_0}g$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial \rho}{\partial t} + w\frac{d\bar{\rho}}{dz} = 0$$

incompressible fluid:

$$\frac{d\rho}{dt} = 0$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = 0$$

$$\frac{\partial \rho}{\partial t} + u\cancel{\frac{\partial \rho'}{\partial x}} + v\cancel{\frac{\partial \rho'}{\partial y}} + w\frac{\partial \bar{\rho}}{\partial z} = 0$$

From the horizontal momentum equations:

$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \qquad (1)$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \qquad (2)$$

$\frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(2)$:

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - f\zeta = -\frac{1}{\rho_0}{\nabla_h}^2 p$$

$$-\frac{\partial^2 w}{\partial z \partial t} - f\zeta = -\frac{1}{\rho_0}{\nabla_h}^2 p$$

$$-\frac{\partial^2 w}{\partial z \partial t^2} - f\frac{\partial \zeta}{\partial t} = -\frac{1}{\rho_0}\frac{\partial}{\partial t}{\nabla_h}^2 p \qquad (3)$$

$\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(1)$:

$$\frac{\partial \zeta}{\partial t} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

$$\frac{\partial \zeta}{\partial t} = f\frac{\partial w}{\partial z} \qquad (4)$$

Combine (3) and (4), and eliminate ζ:

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial w}{\partial z}\right) + f^2\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial}{\partial t}{\nabla_h}^2 p \qquad (5)$$

From the vertical momentum equation and the density equation:

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{\rho}{\rho_0}g$$

$$\frac{\partial \rho}{\partial t} + w\frac{d\bar{\rho}}{dz} = 0$$

$$\frac{\partial^2 w}{\partial t^2} = -\frac{1}{\rho_0}\frac{\partial^2 p}{\partial z \partial t} - N^2 w \qquad (6)$$

Combine (5) and (6), and eliminate p:

$$\frac{\partial^2}{\partial t^2}\nabla^2 w + f^2\frac{\partial^2 w}{\partial z^2} + N^2{\nabla_h}^2 w = 0$$

$$\frac{\partial^2}{\partial t^2}\nabla^2 w + f^2\frac{\partial^2 w}{\partial z^2} + N^2{\nabla_h}^2 w = 0$$

Assume constant N^2, and apply a wavelike solution $w = w_0 e^{i(kx+ly+mz-\omega t)}$:

$$(-i\omega)^2[(ik)^2 + (il)^2 + (im)^2] + f^2(im)^2 + N^2[(ik)^2 + (il)^2] = 0$$

$$\omega^2(k^2 + l^2 + m^2) - f^2m^2 - N^2(k^2 + l^2) = 0$$

$$\omega^2 = \frac{N^2(k^2 + l^2) + f^2m^2}{k^2 + l^2 + m^2}$$

$$= \frac{N^2{\boldsymbol{K_h}}^2 + f^2m^2}{\boldsymbol{K}^2}$$

$$= N^2cos^2\theta + f^2sin^2\theta$$

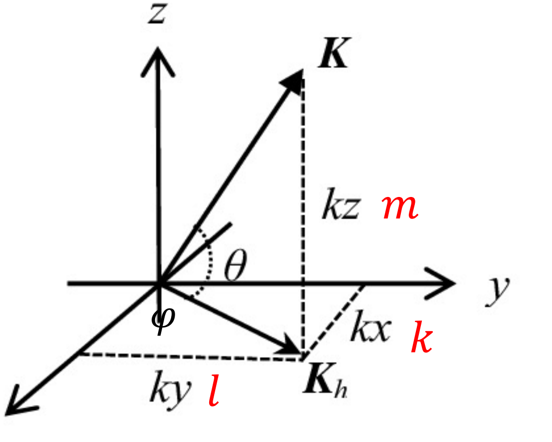


Fig. 1. Wave number vector and its components.

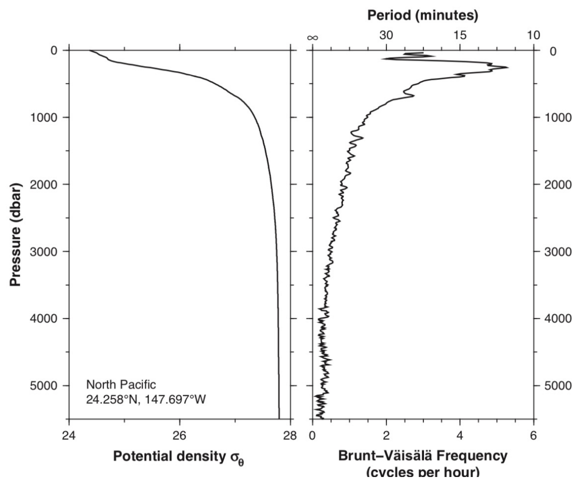
What if $\theta = 0$?

$f^2 \le N^2$

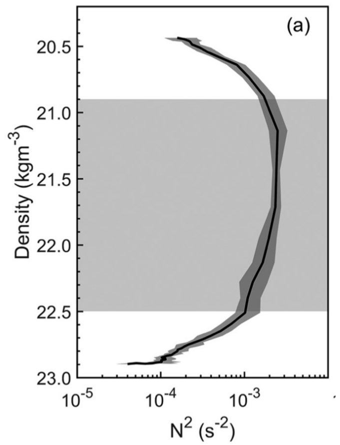
$$\omega^2 - N^2 = (f^2 - N^2)sin^2\theta \le 0$$

$$\omega^2 - f^2 = (N^2 - f^2)cos^2\theta \ge 0$$

$f^2 \le \omega^2 \le N^2$



Yang et al. (2020)



$(u, v, w) = (u_0, v_0, w_0)e^{i(kx+ly+mz-\omega t)}$, and from the continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$(iku_0 + ilv_0 + imw_0)e^{i(kx+ly+mz-\omega t)} = 0 \qquad \boldsymbol{K} \cdot \boldsymbol{u} = 0$$

The flow direction is perpendicular to the wave propagation

Consider $w = w_0 e^{i(kx+mz-\omega t)}$:

$$\omega^2 = \frac{N^2(k^2 + l^2) + f^2 m^2}{k^2 + l^2 + m^2}$$

$$\omega^2 = N^2 \frac{k^2}{k^2 + m^2} + f^2 \frac{m^2}{k^2 + m^2}$$

$$c_{gx} = \frac{\partial \omega}{\partial k} = (N^2 - f^2)\frac{m^2 k}{\omega K^4} \qquad c_{gz} = \frac{\partial \omega}{\partial m} = -(N^2 - f^2)\frac{mk^2}{\omega K^4}$$

$$c_z \cdot c_{gz} = -\frac{\omega}{m} \cdot (N^2 - f^2)\frac{mk^2}{\omega K^4} \quad \leq 0$$

$$\boldsymbol{K} \cdot \boldsymbol{c_g} = kc_{gx} + mc_{gz} = (N^2 - f^2)\frac{m^2 k^2}{\omega K^4} - (N^2 - f^2)\frac{m^2 k^2}{\omega K^4} \quad = 0$$

The energy propagation direction is perpendicular to the wave propagation direction

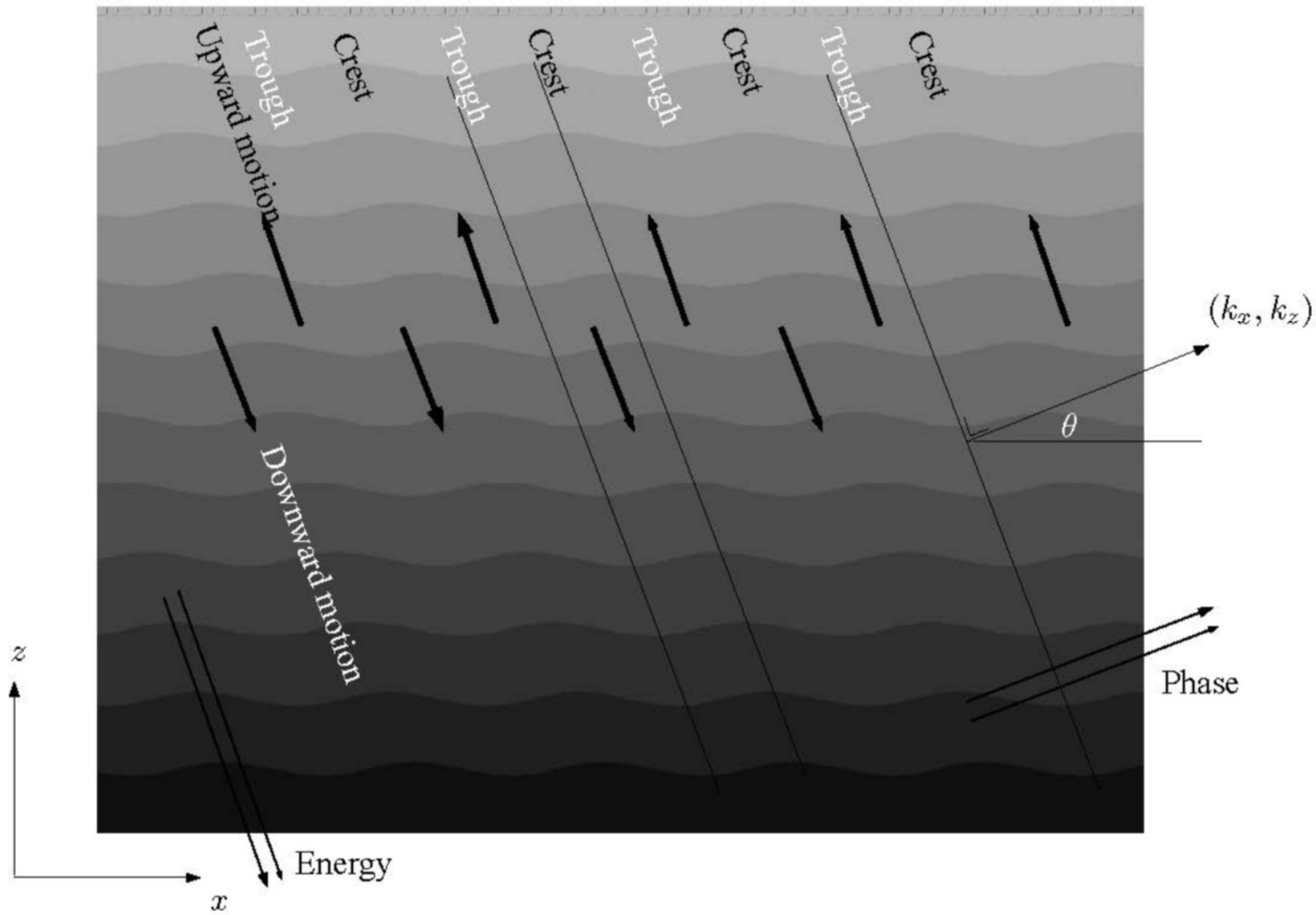


Figure 13-3 Vertical structure of an internal wave.

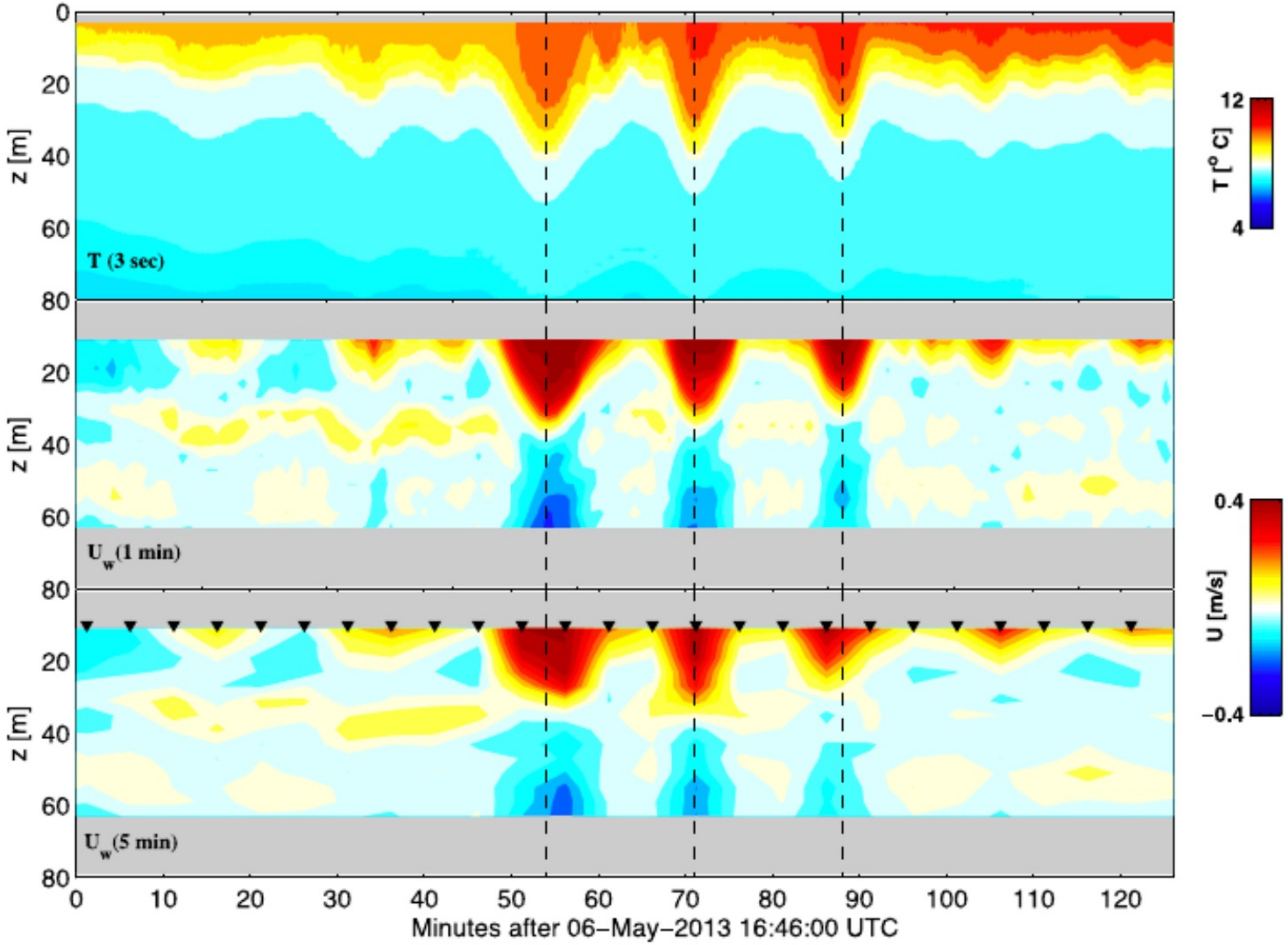
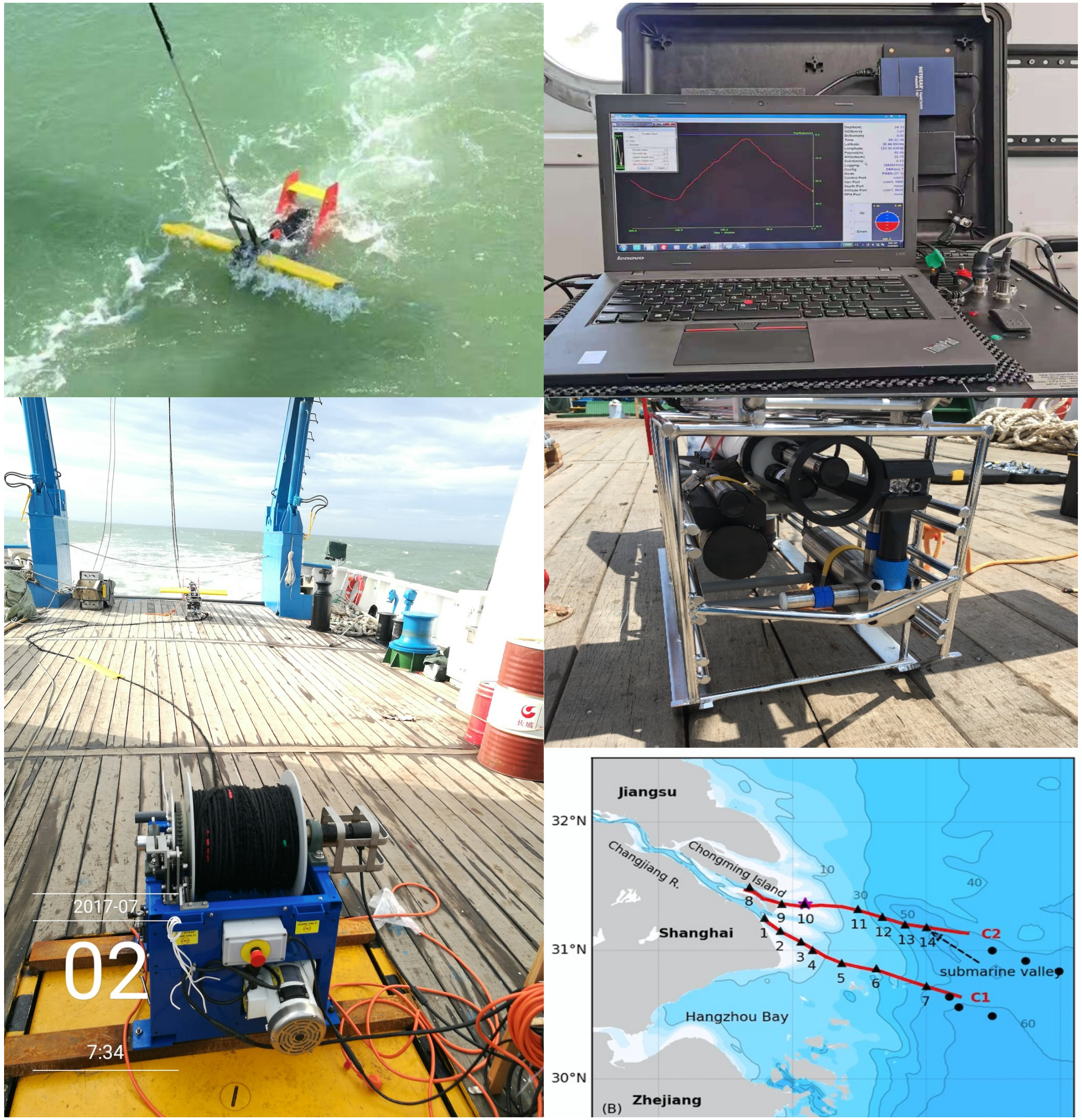


Figure 6. Temperature, 1 min velocity and subsampled 5 min velocity for a wave detected on 6 May 2013. Vertical dashed lines denote detected wave troughs. Black triangles denote the 5 min subsampling time.

Zhang et al. (2014, JGR)

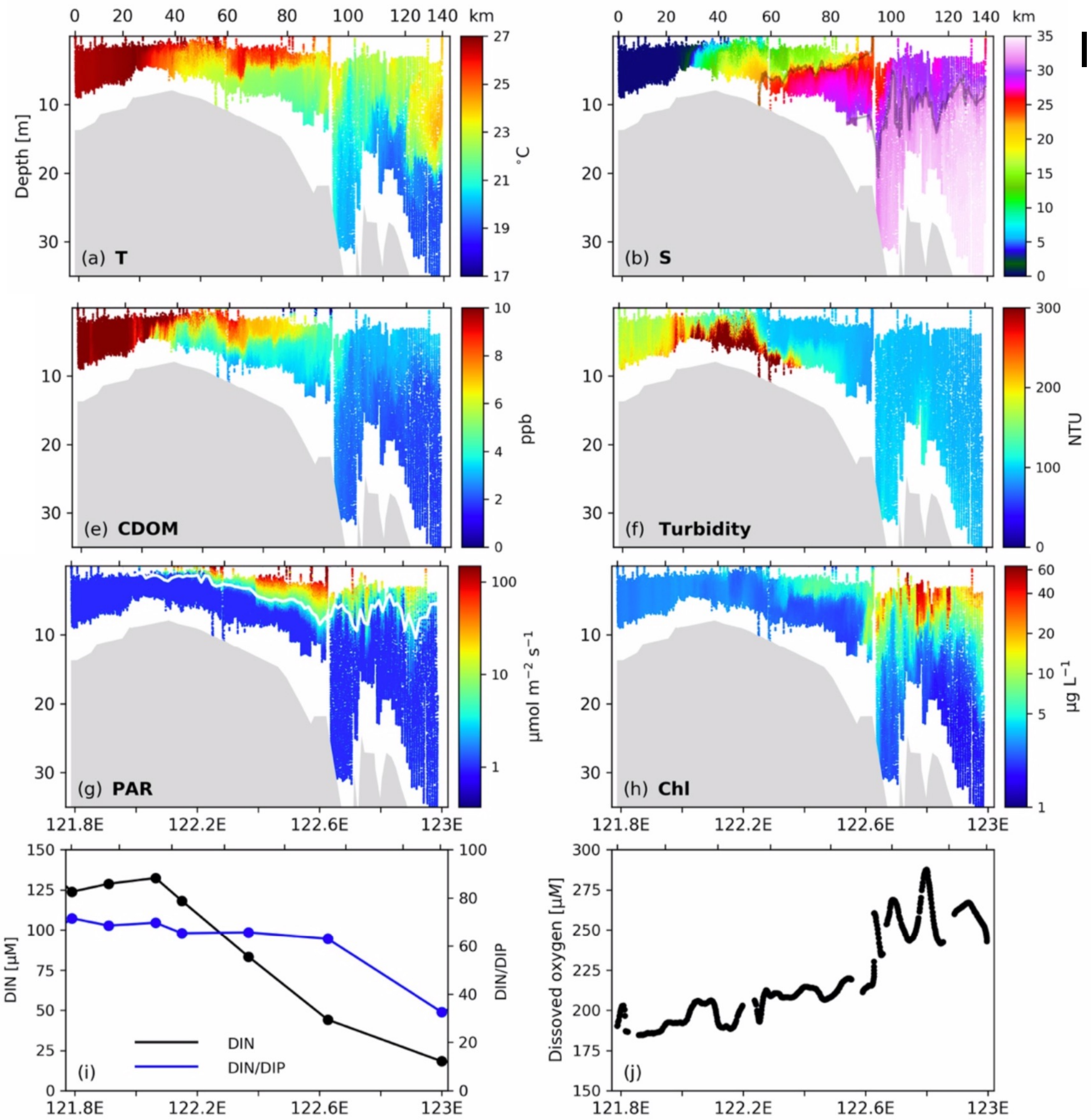
近海拖曳式走航观测系统



- ❑ **Acrobat (Sea Sciences)**
- ❑ **AML-MVP CTD**
- ❑ **RBR Concerto logger**
  - CDOM
  - Turbitidy
  - PAR
  - Fluorescence

Horizontal resolution:

- 200 -1000 m



Internal waves generated by plume front

Nash and Moum (2005, Nature)

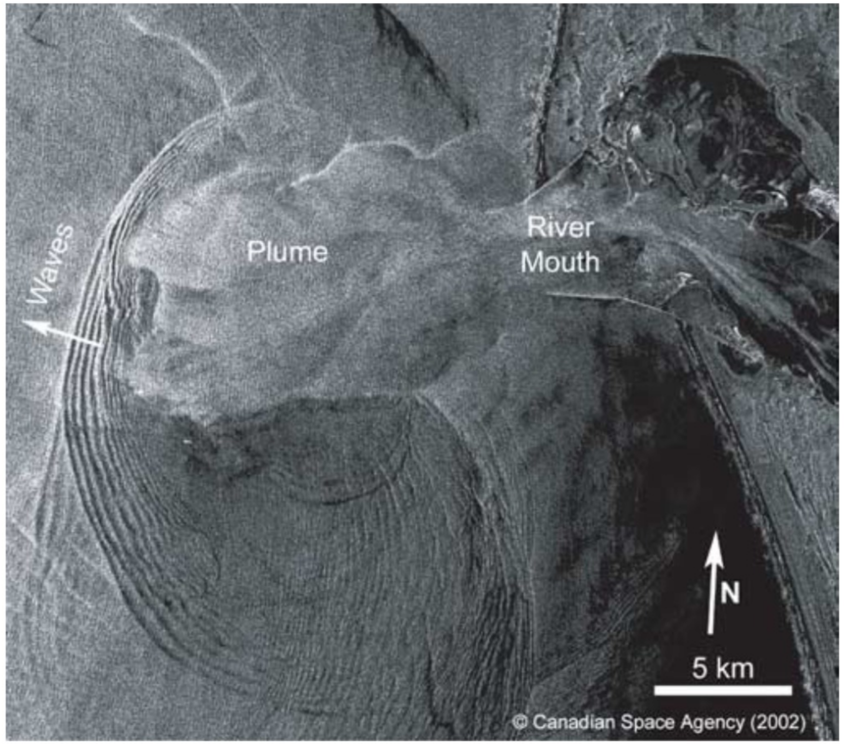


Figure 1 | Synthetic aperture radar (SAR) image of the Columbia River plume on 9 August 2002. Image indicates regions of enhanced surface roughness associated with plume-front and internal wave velocity convergences. Similar features appear in images during all summertime months (April–October; see http://oceanweb.ocean.washington.edu/rise/data.htm for more Columbia River plume images) and from other regions[1,2]. SAR image courtesy of P. Orton, T. Sanders and D. Jay; image was processed at the Alaska Satellite Facility, and is copyright Canadian Space Agency.