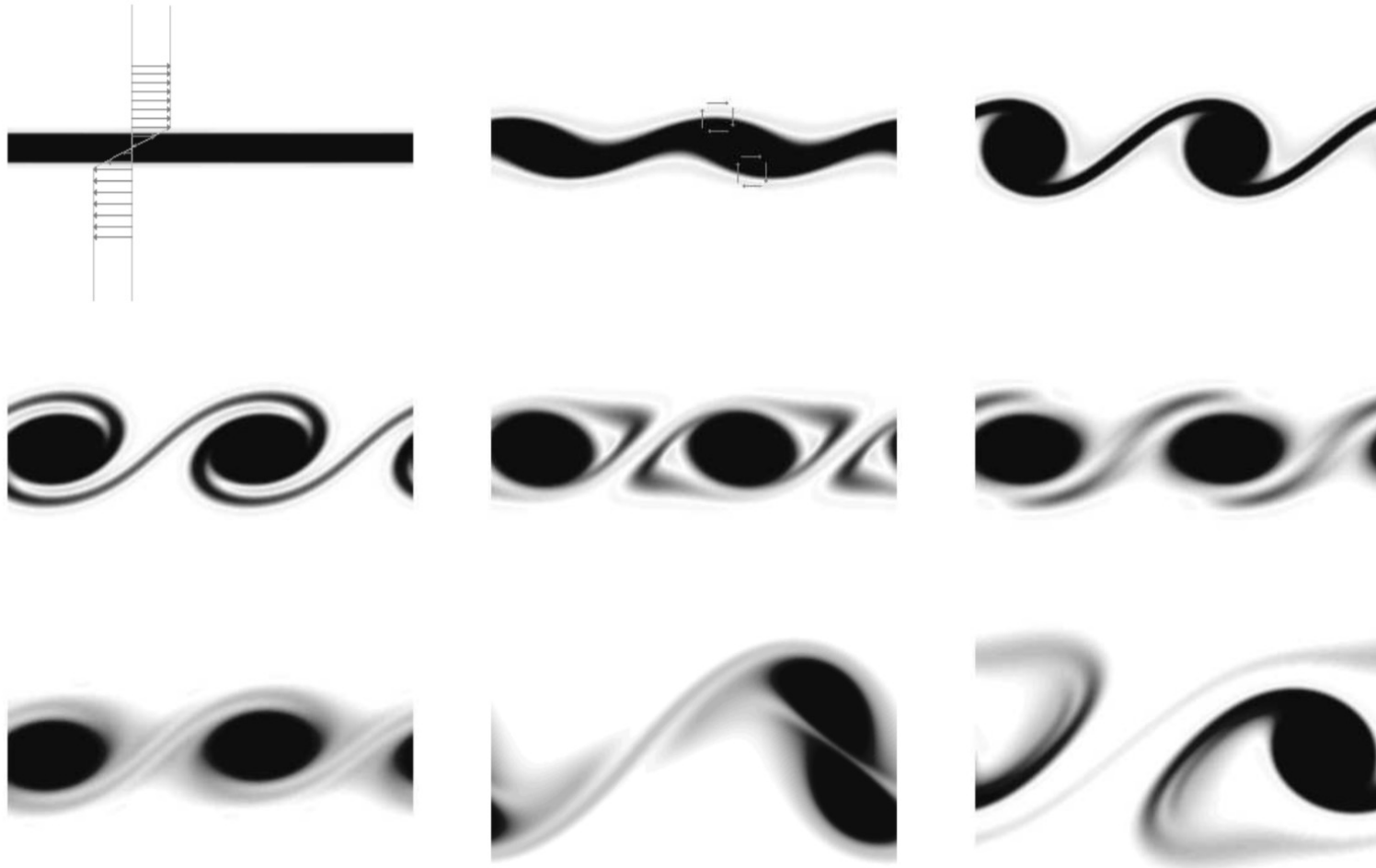


Barotropic instability – absolute vorticity has spatial variations



Barotropic instability

Assumptions: shallow water model + flat bottom and surface

$$p = p_0(z) + \tilde{p}(x, y, z, t) \quad p_0(z) = P_0 - \rho_0 g z$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - f v = -\frac{1}{\rho_0} \frac{\partial \tilde{p}}{\partial x}$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + f u = -\frac{1}{\rho_0} \frac{\partial \tilde{p}}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \cancel{\frac{\partial w}{\partial z}} = 0$$

Take the z-derivative of the continuity equation: $\frac{\partial}{\partial z} \left(\frac{\partial w}{\partial z} \right) = 0$ $w = az + b$

flat bottom and surface: $w = 0$

Assume the background (mean) flow as:

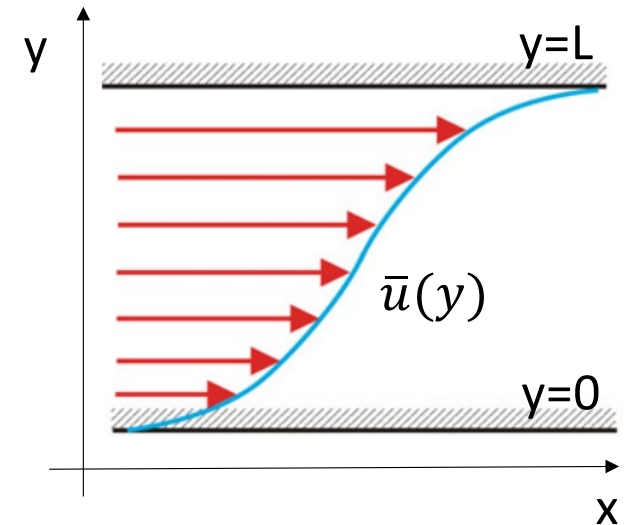
$$u = \bar{u}(y) \quad v = 0.$$

The momentum equations for the mean flow:

$$\cancel{\frac{\partial u}{\partial t}} + u \cancel{\frac{\partial u}{\partial x}} + v \cancel{\frac{\partial u}{\partial y}} - \cancel{f}v = -\cancel{\frac{1}{\rho_0}} \cancel{\frac{\partial \tilde{p}}{\partial x}}$$

$$\cancel{\frac{\partial v}{\partial t}} + u \cancel{\frac{\partial v}{\partial x}} + v \cancel{\frac{\partial v}{\partial y}} + fu = -\frac{1}{\rho_0} \frac{\partial \tilde{p}}{\partial y}$$

$$(f_0 + \beta_0 y) \bar{u}(y) = -\frac{1}{\rho_0} \frac{d\bar{p}}{dy}$$



Assume **small perturbations** of u , v and p due to waves:

$$u = \bar{u}(y) + u'(x, y, t)$$

$$v = v'(x, y, t)$$

$$\tilde{p} = \bar{p}(y) + p'(x, y, t)$$

perturbation much smaller than mean

Substitution into the momentum equations:

$$x: \frac{\partial(\bar{u} + u')}{\partial t} + (\bar{u} + u') \frac{\partial(\bar{u} + u')}{\partial x} + v' \frac{\partial(\bar{u} + u')}{\partial y} - f v' = -\frac{1}{\rho_0} \frac{\partial p'}{\partial x}$$

$$\cancel{\frac{\partial \bar{u}}{\partial t}} + \frac{\partial u'}{\partial t} + \cancel{\bar{u} \frac{\partial \bar{u}}{\partial x}} + \bar{u} \frac{\partial u'}{\partial x} + \cancel{u' \frac{\partial \bar{u}}{\partial x}} + \cancel{u' \frac{\partial u'}{\partial x}} + v' \frac{\partial \bar{u}}{\partial y} + \cancel{v' \frac{\partial u'}{\partial y}} - f v' = -\frac{1}{\rho_0} \frac{\partial p'}{\partial x}$$

$$y: \frac{\partial v'}{\partial t} + (\bar{u} + \cancel{u'}) \frac{\partial v'}{\partial x} + \cancel{v' \frac{\partial v'}{\partial y}} + \cancel{f(\bar{u} + u')} = -\frac{1}{\rho_0} \frac{\partial(\bar{p} + \cancel{p'})}{\partial y}$$

$$\frac{\partial u'}{\partial t} + \bar{u} \frac{\partial u'}{\partial x} + v' \frac{d\bar{u}}{dy} - (f_0 + \beta_0 y)v' = -\frac{1}{\rho_0} \frac{\partial p'}{\partial x} \quad (1)$$

$$\frac{\partial v'}{\partial t} + \bar{u} \frac{\partial v'}{\partial x} + (f_0 + \beta_0 y)u' = -\frac{1}{\rho_0} \frac{\partial p'}{\partial y} \quad (2)$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0.$$

$$u' = -\frac{\partial \psi}{\partial y}, \quad v' = +\frac{\partial \psi}{\partial x}$$

$$\frac{\partial}{\partial x} (2) - \frac{\partial}{\partial y} (1):$$

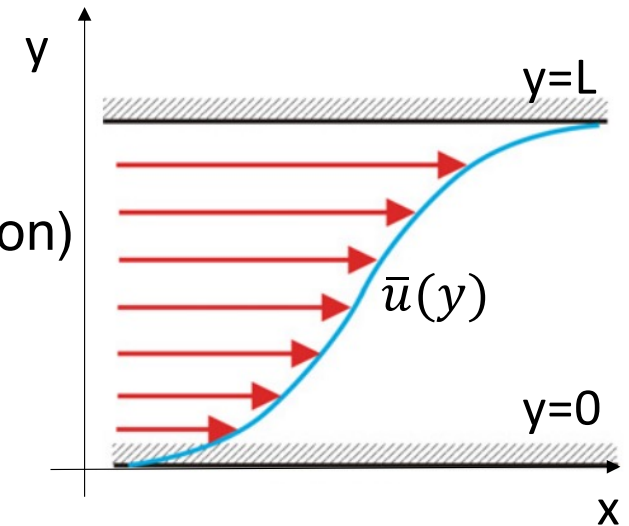
$$\zeta' = \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} = \nabla^2 \psi$$

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \nabla^2 \psi + \left(\beta_0 - \frac{d^2 \bar{u}}{dy^2} \right) \frac{\partial \psi}{\partial x} = 0.$$

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \nabla^2 \psi + \left(\beta_0 - \frac{d^2 \bar{u}}{dy^2} \right) \frac{\partial \psi}{\partial x} = 0.$$

Assume a wave solution (note there are boundaries in y-direction)

$$\psi(x, y, t) = \underbrace{\phi(y)}_{\text{eigenfunction}} e^{i(kx - \omega t)}$$



$$(-i\omega + \bar{u}ik) \left(-k^2 \phi + \frac{d^2 \phi}{dy^2} \right) e^{i(kx - \omega t)} + \left(\beta_0 - \frac{d^2 \bar{u}}{dy^2} \right) \phi i k e^{i(kx - \omega t)} = 0$$

$$(-c + \bar{u}) \left(-k^2 \phi + \frac{d^2 \phi}{dy^2} \right) + \left(\beta_0 - \frac{d^2 \bar{u}}{dy^2} \right) \phi = 0$$

$$\boxed{\frac{d^2 \phi}{dy^2} - k^2 \phi + \frac{\beta_0 - d^2 \bar{u}/dy^2}{\bar{u}(y) - c} \phi = 0}$$

Rayleigh equation

growing mode - instability

The phase speed: $c = c_r + ic_i$ $\psi(x, y, t) = \phi(y) e^{ik(x - c_r t - ic_i t)} = \phi(y) e^{k c_i t} e^{ik(x - c_r t)}$

Multiply $\frac{d^2\phi}{dy^2} - k^2\phi + \frac{\beta_0 - d^2\bar{u}/dy^2}{\bar{u}(y) - c} \phi = 0$ by ϕ^* , and integrate across the domain:

$$- \int_0^L \left(\left| \frac{d\phi}{dy} \right|^2 + k^2 |\phi|^2 \right) dy + \int_0^L \frac{\beta_0 - d^2\bar{u}/dy^2}{\bar{u} - c} |\phi|^2 dy = 0$$

The imaginary part is:

$$c_i \int_0^L \left(\beta_0 - \frac{d^2\bar{u}}{dy^2} \right) \frac{|\phi|^2}{|\bar{u} - c|^2} dy = 0.$$

For growing mode to exist (instability), $c_i \neq 0$, and then

$$\int_0^L \left(\beta_0 - \frac{d^2\bar{u}}{dy^2} \right) \frac{|\phi|^2}{|\bar{u} - c|^2} dy = 0$$

necessary condition – Rayleigh's criterion

$$\frac{d}{dy} \left(f_0 + \beta_0 y - \frac{d\bar{u}}{dy} \right)$$

must **change sign (vanish)** somewhere in the domain (**total vorticity reaches a extremum**)

The real part:

$$\int_0^L (\bar{u} - c_r) \left(\beta_0 - \frac{d^2 \bar{u}}{dy^2} \right) \frac{|\phi|^2}{|\bar{u} - c|^2} dy = \int_0^L \left(\left| \frac{d\phi}{dy} \right|^2 + k^2 |\phi|^2 \right) dy \quad (1)$$

With the result from the imaginary part:

$$\int_0^L \left(\beta_0 - \frac{d^2 \bar{u}}{dy^2} \right) \frac{|\phi|^2}{|\bar{u} - c|^2} dy = 0 \quad (2)$$

(2)*($c_r - \bar{u}_0$)+(1):

any arbitrary velocity $\int_0^L (\bar{u} - \bar{u}_0) \left(\beta_0 - \frac{d^2 \bar{u}}{dy^2} \right) \frac{|\phi|^2}{|\bar{u} - c|^2} dy > 0$ \bar{u}_0 is taken as the value where $\beta_0 - \frac{d^2 \bar{u}}{dy^2} = 0$

$(\bar{u} - \bar{u}_0) \left(\beta_0 - \frac{d^2 \bar{u}}{dy^2} \right)$ must be positive in some finite portion of the domain
necessary condition – Fjørtoft's criterion

A simple example for barotropic instability

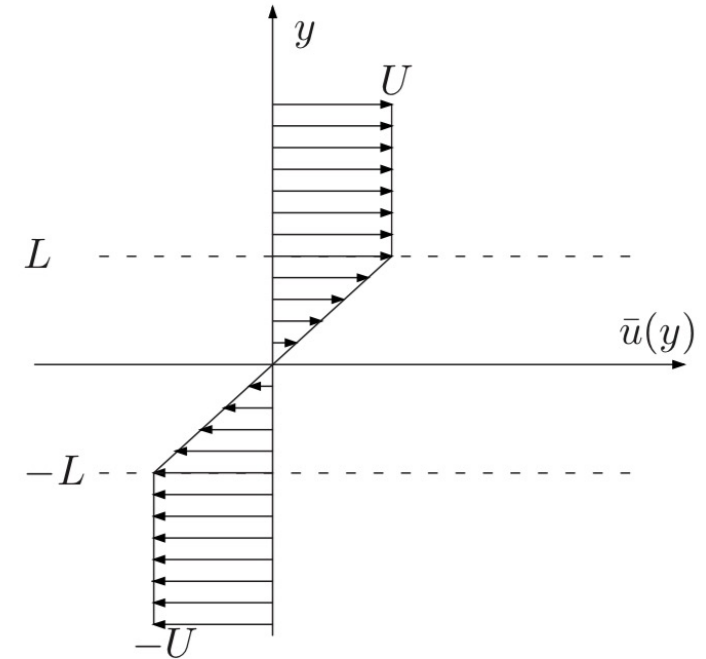
f -plane ($\beta_0 = 0$)

$$y < -L : \quad \bar{u} = -U, \quad \frac{d\bar{u}}{dy} = 0, \quad \frac{d^2\bar{u}}{dy^2} = 0$$

$$-L < y < +L : \quad \bar{u} = \frac{U}{L} y, \quad \frac{d\bar{u}}{dy} = \frac{U}{L}, \quad \frac{d^2\bar{u}}{dy^2} = 0$$

$$+L < y : \quad \bar{u} = +U, \quad \frac{d\bar{u}}{dy} = 0, \quad \frac{d^2\bar{u}}{dy^2} = 0$$

$$\text{At } y = -L: \quad \frac{d^2\bar{u}}{dy^2} > 0 \qquad \text{At } y = L: \quad \frac{d^2\bar{u}}{dy^2} < 0$$



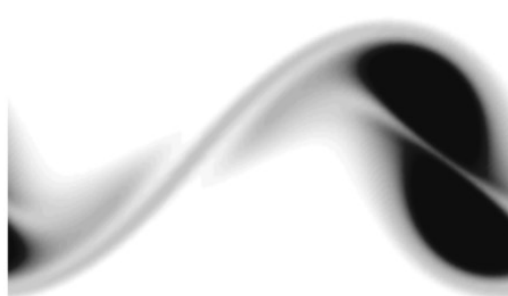
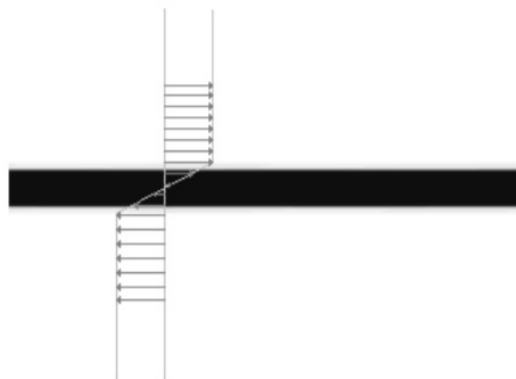
$\frac{d^2\bar{u}}{dy^2}$ changes sign in the domain – Rayleigh's criterion

Take $\bar{u}_0 = 0$,

$$y = -L: -\bar{u} \frac{d^2\bar{u}}{dy^2} > 0 \qquad y = L: -\bar{u} \frac{d^2\bar{u}}{dy^2} > 0 \quad - \text{Fjørtoft's criterion}$$

~~$\beta_0 - \frac{d^2\bar{u}}{dy^2}$~~ changes sign

$$(\bar{u} - \bar{u}_0) \left(\beta_0 - \frac{d^2\bar{u}}{dy^2} \right) > 0 \text{ in some portion}$$



Barotropic Instability with Coriolis at Equator

with Original Amplitude , time = 0.00 hr

