Reynolds-averaged equations

$$x:\ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv + f_* w = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(A_H\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_H\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_V\frac{\partial u}{\partial z}\right)$$

$$y:\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(A_H\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_H\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_V\frac{\partial v}{\partial z}\right)$$

$$z:\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + f_* u = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left(A_H\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_H\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_V\frac{\partial w}{\partial z}\right)$$

$$-\frac{\rho}{\rho_0} g$$

Scale analysis

Characteristic scale: a typical scale for a variable

Table 4.1 TYPICAL SCALES OF ATMOSPHERIC AND OCEANIC FLOWS

Variable	Scale	Unit	Atmospheric value	Oceanic value	
x, y	L	m	100 km = 10^5 m	10 km = 10^4 m	10^4~10^6 km
z	H	m	1 km = 10^3 m	100 m = 10^2 m	10^1~10^3 km
t	T	s	$\geq \frac{1}{2}$ day $\simeq 4 \times 10^4$ s	≥ 1 day $\simeq 9 \times 10^4$ s	
u, v	U	m/s	10 m/s	0.1 m/s	
w	W	m/s			
p	P	$\mathrm{kg\,m^{-1}\,s^{-2}}$	variable		
ρ	$\Delta\rho$	$\mathrm{kg/m^3}$			

The continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial x} = \frac{\Delta u}{\Delta x} \sim \frac{U}{L}$$

scale $\quad \frac{U}{L} \quad \frac{U}{L} \quad \frac{W}{H}$

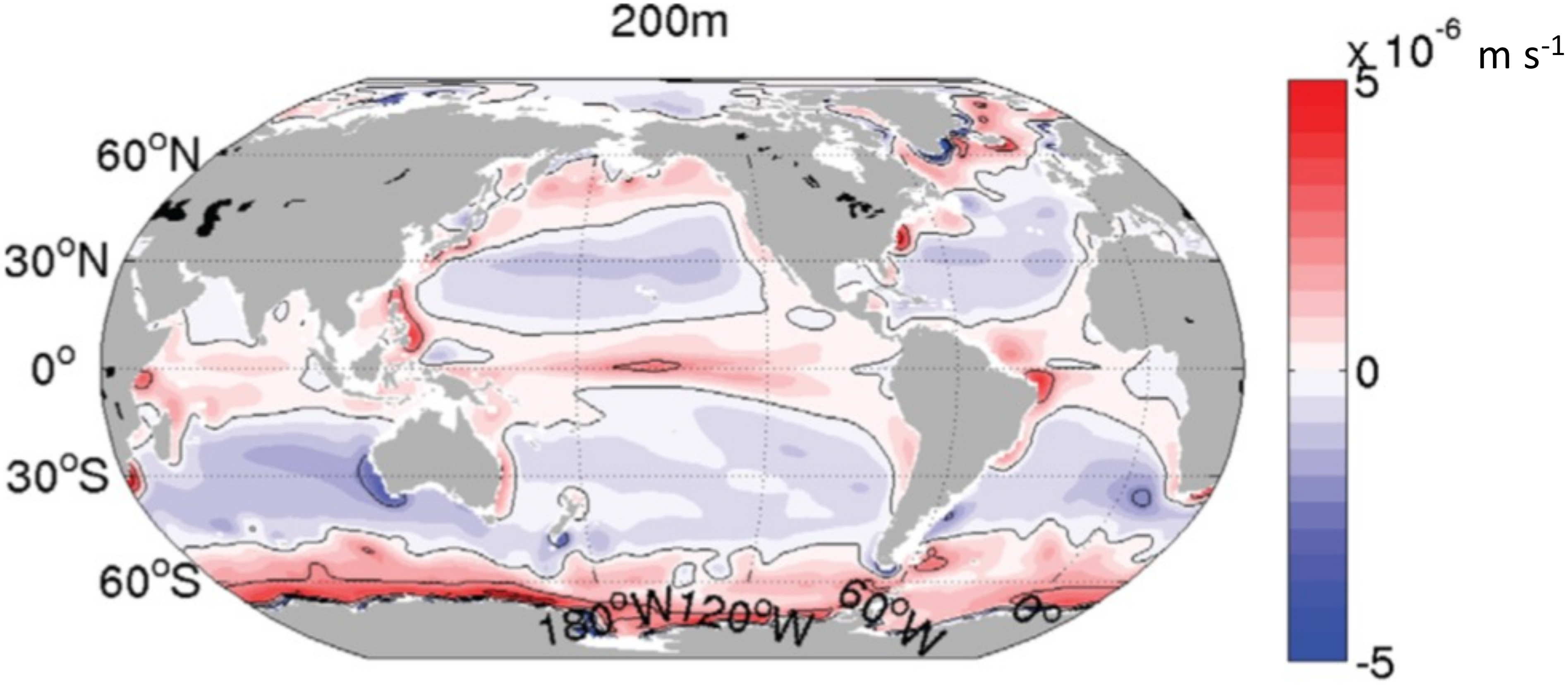
$$\frac{U}{L} \sim \frac{W}{H}$$

$$W \sim U\frac{H}{L} \sim U\delta$$

$\delta = \frac{H}{L}$: aspect ratio ($10^{-4} \sim 10^{-2}$)

$$W \ll U$$

Vertical velocity in the global ocean



Liang et al. (2017, JGR)

The momentum equation

x direction: $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv + \cancel{f_* w} =$$

$$\frac{U}{T} \qquad \frac{U^2}{L} \qquad \frac{U^2}{L} \qquad W\frac{U}{H} \qquad fU \qquad \cancel{fW}$$

$W \ll U$

$$-\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(A_H\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_H\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_V\frac{\partial u}{\partial z}\right)$$

$$\frac{P}{\rho_0 L} \qquad A_H\frac{U}{L^2} \qquad A_H\frac{U}{L^2} \qquad A_V\frac{U}{H^2}$$

Non-dimensional numbers

Rossby number (R_o) = scale of nonlinear advection term / scale of Coriolis term

$$= \frac{U^2}{L} / \mathrm{f}\mathrm{U} = U/fL$$

measure of the role of Earth's rotation in motions

$R_o \ll 1$: rotation is important for the motion

Ekman number (E_k) = scale of vertical viscosity term / scale of Coriolis term

$$= A_V \frac{U}{H^2} / fU = \frac{A_V}{fH^2}$$

measure of the importance of frictional force

E_k <<1: friction can be neglected

Reynolds number (Re) = scale of the nonlinear term / scale of viscous term

$$= \frac{U^2}{L} / \nu \frac{U}{L^2} = \frac{UL}{\nu}$$

small Re: viscous flow

large Re: inviscous, turbulent flow

The momentum equation

From the x-direction equation:

$$\frac{P}{\rho_0 L} \sim \mathrm{fU} \sim \frac{U^2}{L} \sim A_H \frac{U}{L^2} \sim A_V \frac{U}{H^2}$$

As $H \ll L, \quad W \ll U$

z direction:

$$\cancel{\frac{\partial \mathrm{w}}{\partial \mathrm{t}}} + \cancel{\mathrm{u}\frac{\partial \mathrm{w}}{\partial \mathrm{x}}} + \cancel{\mathrm{v}\frac{\partial \mathrm{w}}{\partial \mathrm{y}}} + \cancel{\mathrm{w}\frac{\partial \mathrm{w}}{\partial \mathrm{z}}} + \cancel{f_* u} =$$

$$\cancel{\frac{W}{T}} \qquad \cancel{\frac{UW}{L}} \qquad \cancel{\frac{UW}{L}} \qquad \cancel{W\frac{W}{H}} \qquad \cancel{fU}$$

$$-\frac{1}{\rho_0}\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left(\cancel{A_H \frac{\partial \mathrm{w}}{\partial \mathrm{x}}}\right) + \frac{\partial}{\partial y}\left(\cancel{A_H \frac{\partial \mathrm{w}}{\partial \mathrm{y}}}\right) + \frac{\partial}{\partial \mathrm{z}}\left(\cancel{A_V \frac{\partial \mathrm{w}}{\partial \mathrm{z}}}\right) - \frac{\rho}{\rho_0} g$$

$$\frac{P}{\rho_0 H} \qquad \cancel{A_H \frac{W}{L^2}} \qquad \cancel{A_H \frac{W}{L^2}} \qquad \cancel{A_V \frac{W}{H^2}} \qquad \frac{\Delta\rho}{\rho_0}\mathrm{g}$$

$$\frac{P}{\rho_0 H} \gg \mathrm{fU}$$

$$\frac{P}{\rho_0 H} \gg \frac{UW}{L}$$

$$\frac{P}{\rho_0 H} \gg A_H \frac{W}{L^2}$$

$$\frac{P}{\rho_0 H} \gg A_V \frac{W}{H^2}$$

$$\frac{\partial p}{\partial z} + \rho g = 0$$

Hydrostatic equation (balance)

Hydrostatic balance: $$\frac{\partial p}{\partial z} + \rho g = 0$$

$$p = \tilde{p}(z) + p'(x, y, z, t) \qquad \tilde{p}(z) = P_0 - \rho_0 g z$$

$$\frac{\partial \tilde{p}}{\partial z} + \rho_0 g = 0 \qquad \frac{\partial p'}{\partial z} + \Delta\rho g = 0$$

$$\frac{P}{H} \quad g\Delta\rho$$

$$\frac{P}{\rho_0 L} \sim \mathrm{fU} \qquad \frac{gH\Delta\rho}{P} = \frac{gH\Delta\rho}{\rho_0 f L U} = \frac{U}{fL} \cdot \frac{gH\Delta\rho}{\rho_0 U^2}$$

$$= R_o \frac{N^2}{(U/H)^2}$$ Richardson number

The momentum equations

$$\frac{\partial p'}{\partial x}$$

x direction: $$\frac{\partial \mathrm{u}}{\partial \mathrm{t}} + \mathrm{u}\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \mathrm{v}\frac{\partial \mathrm{u}}{\partial \mathrm{y}} + \mathrm{w}\frac{\partial \mathrm{u}}{\partial \mathrm{z}} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(A_H\frac{\partial \mathrm{u}}{\partial \mathrm{x}}\right) + \frac{\partial}{\partial y}\left(A_H\frac{\partial \mathrm{u}}{\partial \mathrm{y}}\right) + \frac{\partial}{\partial \mathrm{z}}\left(A_V\frac{\partial \mathrm{u}}{\partial \mathrm{z}}\right)$$

y direction: $$\frac{\partial \mathrm{v}}{\partial \mathrm{t}} + \mathrm{u}\frac{\partial \mathrm{v}}{\partial \mathrm{x}} + \mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{y}} + \mathrm{w}\frac{\partial \mathrm{v}}{\partial \mathrm{z}} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(A_H\frac{\partial \mathrm{v}}{\partial \mathrm{x}}\right) + \frac{\partial}{\partial y}\left(A_H\frac{\partial \mathrm{v}}{\partial \mathrm{y}}\right) + \frac{\partial}{\partial \mathrm{z}}\left(A_V\frac{\partial \mathrm{v}}{\partial \mathrm{z}}\right)$$

$$\frac{\partial p'}{\partial y}$$

z direction: $$\frac{\partial p}{\partial z} + \rho g = 0$$ Hydrostatic equation

$$\frac{\partial p'}{\partial z} + \Delta\rho g = 0$$

The thermodynamic equation

1st Law of Thermodyamics (energy conservation, and for per unit mass):

$$\frac{dI}{dt} = \mathrm{Q} - \mathrm{W}$$

I: internal energy

Q: rate of heat gain

W: rate of work done by pressure force onto surrounding fluids

C_v: specific heat capacity

$$I = C_v T$$

Fourier's Law of heat conduction:

$$Q = \frac{k_T}{\rho} \nabla^2 T$$

Rate of work done by pressure force:

specific volume

$$W = p \frac{d\alpha}{dt} \qquad \boxed{\alpha = 1/\rho}$$

$$C_v \frac{dT}{dt} = \frac{k_T}{\rho} \nabla^2 T - p \frac{d\alpha}{dt}$$

$$\boxed{\alpha = 1/\rho}$$

$$= \frac{k_T}{\rho} \nabla^2 T + \frac{p}{\rho^2} \frac{d\rho}{dt}$$

From the continuity equation:

$$\frac{d\rho}{dt} = -\rho \nabla \cdot \boldsymbol{v}$$

$$C_v \frac{dT}{dt} = \frac{k_T}{\rho} \nabla^2 T - \frac{p}{\rho} \nabla \cdot \boldsymbol{v}$$

For incompressible fluids:

K_T: thermal diffusivity coefficient

$$\frac{dT}{dt} = \frac{k_T}{\rho C_v} \nabla^2 T$$

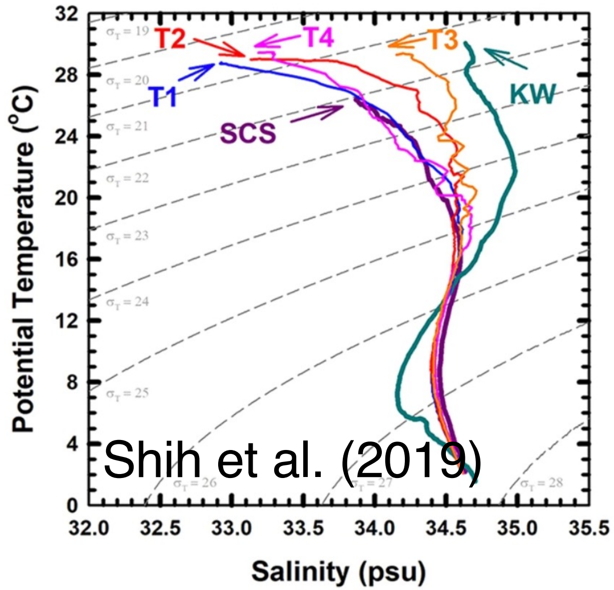
$$\frac{dS}{dt} = K_S \nabla^2 S$$

Equation of State

Specific volume:

$$\alpha = 1/\rho$$

$$\alpha = \alpha(\mathrm{T}, \mathrm{S}, \mathrm{P})$$



For small variations of α around a reference value:

$$d\alpha = \left(\frac{\partial \alpha}{\partial T}\right)_{S,p} dT + \left(\frac{\partial \alpha}{\partial S}\right)_{T,p} dS + \left(\frac{\partial \alpha}{\partial p}\right)_{T,S} dp = \alpha(\beta_T\, dT - \beta_S\, dS - \beta_p\, dp)$$

β_T:thermal expansion coefficient

β_s: salt contraction coefficient

β_p: compressibility coefficient

$$\alpha = \alpha_0 \left[1 + \beta_T (T - T_0) - \beta_S (S - S_0) - \beta_p (p - p_0)\right]$$

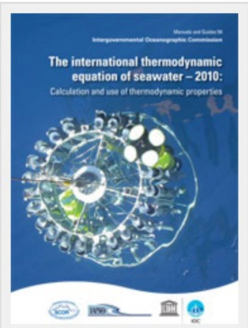
$$\rho = \rho_0 \left[1 - \beta_T (T - T_0) + \beta_S (S - S_0) + \beta_p (p - p_0)\right]$$

nonlinear nonlinear

$$\alpha = \alpha_0 \left[1 + \beta_T (1 + \gamma^* p)(T - T_0) + \frac{\beta_T^*}{2} (T - T_0)^2 - \beta_S (S - S_0) - \beta_p (p - p_0)\right]$$

TEOS-10

Thermodynamic Equation Of Seawater - 2010



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HOME

This site is the official source of information about the Thermodynamic Equation Of Seawater - 2010 (TEOS-10), and the way in which it should be used.

TEOS-10 is based on a Gibbs function formulation from which all thermodynamic properties of seawater (density, enthalpy, entropy sound speed, etc.) can be derived in a thermodynamically consistent manner. TEOS-10 was adopted by the Intergovernmental Oceanographic Commission at its 25th Assembly in June 2009 to replace EOS-80 as the official description of seawater and ice properties in marine science.

A significant change compared with past practice is that TEOS-10 uses Absolute Salinity S_A (mass fraction of salt in seawater) as opposed to Practical Salinity S_P (which is essentially a measure of the conductivity of seawater) to describe the salt content of seawater. Ocean salinities now have units of g/kg.

Absolute Salinity (g/kg) is an SI unit of concentration. The thermodynamic properties of seawater, such as density and enthalpy, are now correctly expressed as functions of Absolute Salinity rather than being functions of the conductivity of seawater. Spatial variations of the composition of seawater mean that Absolute Salinity is not simply proportional to Practical Salinity; TEOS-10 contains procedures to correct for these effects.

Summary of the governing equations

Momentum equation:

$$\frac{d\boldsymbol{v}}{dt} + f\boldsymbol{k}\times\boldsymbol{v} = -\frac{1}{\rho_0}\nabla p + \frac{\rho}{\rho_0}\boldsymbol{g} + \nu_E\nabla^2\boldsymbol{v}$$

Continuity equation:

$$\frac{d\rho}{dt} + \rho\nabla\cdot\boldsymbol{v} = 0$$

Tracer equations:

$$\frac{dT}{dt} = K\nabla^2 T$$

$$\frac{dS}{dt} = K\nabla^2 S$$

Equation of state:

$$\rho = \rho(\mathrm{T}, \mathrm{S}, \mathrm{P})$$

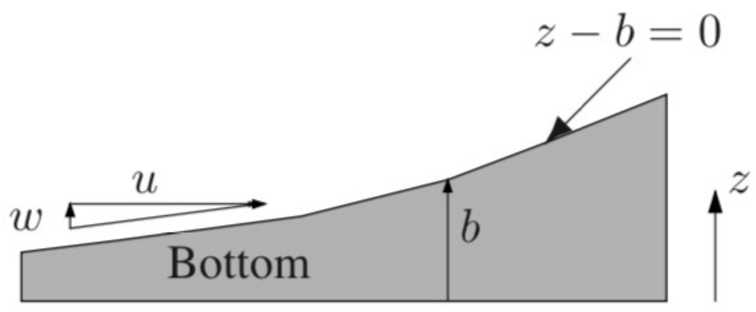
Boundary conditions

Kinematic conditions — velocity

Principle: flow cannot penetrate a solid wall

Bottom boudary: $z - b(x, y) = 0$

$$\nabla G = \left(-\frac{\partial b}{\partial x}, -\frac{\partial b}{\partial y}, 1\right)$$



Considering impermeability:

$$\boldsymbol{u} \cdot \nabla G = 0$$

$$\boxed{w = u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y}}$$

For flat bottom:

$$w = 0$$

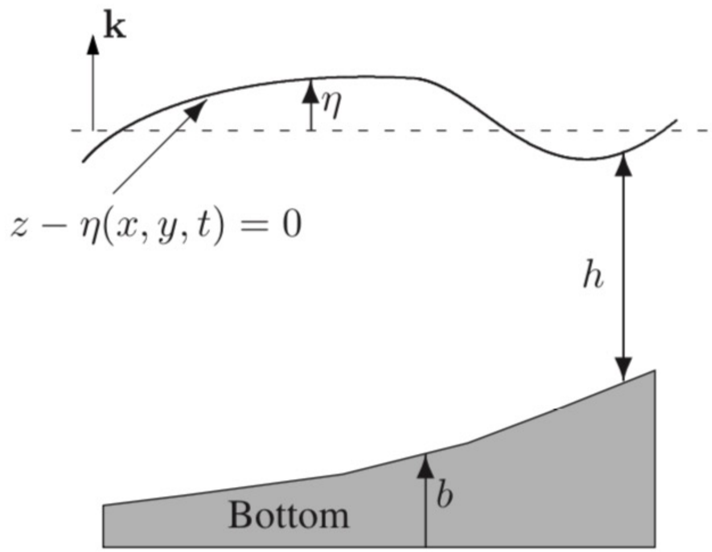
Surface boudary: the boundary is moving with the fluid (free surface)

$$z - \eta(x, y, t) = 0$$

Without precipication and evaporation, it can be considered as a material surface:

$$\frac{d}{dt}(z - \eta) = 0$$

$$\boxed{w = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}}$$



Rigid-lid approximation:

$$\eta = C$$

$$w = 0$$

Tangential velocity for fixed boundaries

No-slip boundary condition:

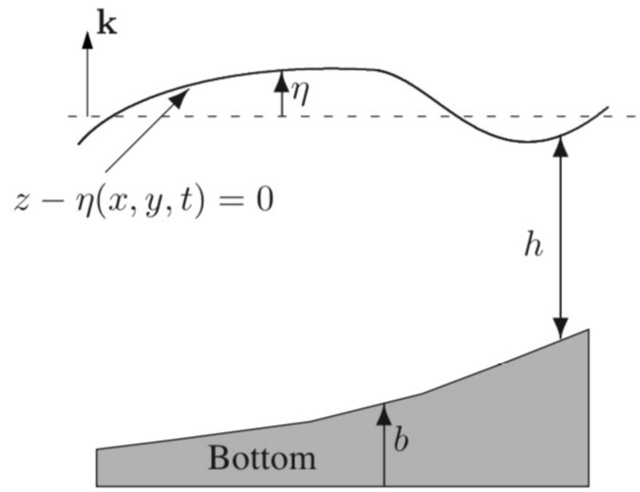
$$\boldsymbol{u} \cdot \boldsymbol{s} = \boldsymbol{0}$$

Free-slip boundary condition (for very thin boundary layer):

$$\boldsymbol{u} \cdot \boldsymbol{s} \neq \boldsymbol{0}$$

$\boldsymbol{s}$: unit vector in the tangential direction of the boundary

Dynamic conditions — force



Pressure:

$$p_{\rm atm} = p_{\rm sea} \quad \text{at air-sea interface.}$$

$$p_{\rm sea}(z=0) = p_{\text{atm at sea level}} + \rho_0 g \eta$$

Stress:

Stress must be cotinuous along moving boudaries (sea surface)

$$\rho_0 \nu_E \left(\frac{\partial u}{\partial z}\right)\bigg|_{\text{at surface}} = \tau^x, \quad \rho_0 \nu_E \left(\frac{\partial v}{\partial z}\right)\bigg|_{\text{at surface}} = \tau^y$$

The stress must be equal to the wind stress that is parameterized by:

$$U_{10} = \sqrt{u_{10}^2 + v_{10}^2} \quad \tau^x = C_d\, \rho_{\rm air}\, U_{10} u_{10}, \quad \tau^y = C_d\, \rho_{\rm air}\, U_{10} v_{10}$$