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Collective instability of salt fingers

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We first consider a steady laminar model of salt fingers and show that the latter become unstable with respect to internal gravity waves when the finger Reynolds number exceeds a critical value. The criterion is then used in speculations about the statistically steady state in a fully developed similarity model where horizontally averaged temperature and salinity gradients are constant at all depths. Dimensional reasoning is used to obtain the asymptotic dependence of the turbulent flux on the molecular salt diffusivity. From this and other relationships order-of-magnitude estimates are obtained and compared with laboratory experiments and ocean observations.

1. Introduction

A salt solution which is in diffusive and hydrostatic equilibrium, with salinity $S^*(z)$, temperature $T^*(z)$, and specific volume $1/\rho^*(z)$ increasing linearly in the upward (z) direction, is known to be unstable in the gravity field because the molecular diffusivity κ_S is two orders of magnitude smaller than the thermal diffusivity κ_T (Stern 1960). By considering the evolution of one-dimensional perturbations varying in the horizontal as $\sin(x/L)$ and independent of z, one shows that the most unstable disturbance has a dimension of order

$$L = [g\alpha(\partial T^*/\partial z)/\kappa_T \nu]^{-\frac{1}{4}},$$

where α is the coefficient of thermal expansion and ν the kinematic viscosity. It is easy to show that the linear solution will also satisfy the non-linear equations of Boussinesq, and this solution will be referred to as the laminar 'similarity solution'. This laminar similarity solution does not modify the horizontally averaged temperature and salinity, and the convective salt flux ρF_S (g/cm^2/sec) continues to increase exponentially with time. The foregoing ignores the finite vertical dimensions which are imposed by the boundary reservoirs. Important qualitative modifications will also be produced by another effect. As the amplitude of the salt fingers increases in time there will be additional instabilities leading to a break-up of the fingers and to their complete mixing. One may therefore examine the time dependent similarity solution for the critical value of F_S which leads to this transition.

In order to avoid the inherent time-dependence of the similarity model it seems desirable first to investigate the stability of a closely related 'equilibrium' model (figure 1 (a)). This model was first proposed by W. V. R. Malkus (private communication); it differs from the 'similarity' model in so far as the salinity field $\bar{S}(z)$ is

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uniform in z and the salt diffusivity κ_S is neglected. Under certain conditions the mean $\overline{T}(z)$, $\overline{S}(z)$ fields of the equilibrium model, supplemented by boundary layers at the reservoirs, may adequately represent the convective modification of a $T^*, S^*(z)$ field originally in hydrostatic equilibrium. The stability analysis of § 3 is undertaken for the equilibrium model, but it seems reasonable that with minor re-interpretation of the parameters the results should be applicable to the corresponding instability in the similarity model. There are many different modes of instability for the equilibrium state shown in figures 1 (*a*), (*b*). The mechanism explored in § 3 is rather novel in that it results in a systematic transfer of energy to a scale of motion which includes many salt fingers. It will be shown that this can occur at small Reynolds number (based on the undisturbed velocity of the salt fingers and L) when the Prandtl number is large.

By making use of this instability mechanism in § 5, we sketch a sequence of events in the 'similarity' model; starting from the state of diffusive equilibrium and ending with the 'fully developed' state of the thermohaline convective régime. The speculations lead to an order-of-magnitude estimate of the salt flux across a discontinuity layer (Turner 1967), and to an estimate of the dimensions of recently observed micro-structure in the ocean thermocline [see, for example, Cooper & Stommel (1968), Turner (1967), Cooper (1967)]. Additional indirect evidence for the suspected salt finger régime is given by the halocline formula of Stern (1968).

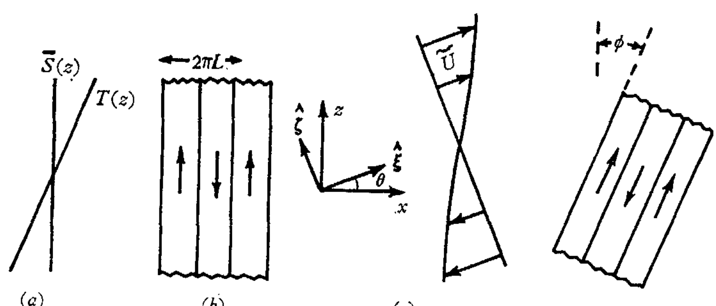

FIGURE 1. The 'equilibrium' model. (*a*) shows the vertical (z) variation of mean temperature ($\overline{T}$) and salinity ($\overline{S}$) in the undisturbed state. (*b*) shows a group of salt fingers in the undisturbed state. (*c*) is a schematic diagram of the infinitesimal-amplitude perturbation $\tilde{U}(\zeta, t)$, where the wavelength of $\tilde{U}$ in the $\hat{\zeta}$-direction is much larger than the width (πL) of the fingers in figure (*b*). (*d*) shows the infinitesimal rotation (ϕ) of a group of salt fingers by the shear of $\tilde{U}$. All quantities are assumed to be independent of the direction normal to the (x, z)-plane.

2. Equilibrium model of salt fingers

Consider an unbounded liquid in which the horizontally averaged temperature field $\overline{T}(z)$ increases linearly upward as shown in figure 1 (*a*). In this model the horizontally averaged salinity field $\overline{S}(z)$ is independent of z, but there are horizontal variations of salinity $S_0(x)$, temperature $T_0(x)$, and vertical velocity $w_0(x)$. These describe the salt fingers illustrated in figure 1 (*b*). A further simplification used in §§ 2–4 is the assumption that $\kappa_S = 0$. We shall now show that the

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equations of motion are exactly satisfied by fields with the following horizontal variation:

$$\left.\begin{aligned} S_0(x) &= \Delta S \sin(x/L), \\ T_0(x) &= \Delta T \sin(x/L), \\ w_0(x) &= -\Delta w \sin(x/L). \end{aligned}\right\} \tag{2.1}$$

It is easily seen that mass and salt conservation are immediately satisfied by the assertion of a one-dimensional motion and by the assertion $\kappa_S = 0$. Since there are no pressure gradients beyond that which is necessary to balance $g\alpha\overline{T}(z)$, the vertical momentum equation will be satisfied if $0 = \nu\,\partial^2 w_0/\partial x^2 + g(\alpha T_0 - \beta S_0)$, where βS_0 is the fractional increase in density due to a salinity increment S_0. In order to satisfy the heat equation we require that $w_0\,\partial\overline{T}/\partial z = \kappa_T\,\partial^2 T_0/\partial x^2$. By substituting (2.1) into these two preceding equations and solving for the amplitudes one obtains

$$\left.\begin{aligned} \Delta w &= \frac{g\beta\Delta S L^2}{\nu(1+\lambda)}, \\ \alpha\Delta T &= \beta\Delta S\left(\frac{\lambda}{1+\lambda}\right), \\ \lambda &\equiv \frac{g\alpha\,\partial\overline{T}/\partial z\, L^4}{\kappa_T \nu}. \end{aligned}\right\} \tag{2.2}$$

The horizontally averaged flux of heat F_H and salt F_S(cm/sec) obtained from (2.1) and (2.2) are

$$\left.\begin{aligned} F_S &\equiv -\beta\overline{S_0 w_0} = \frac{g(\beta\Delta S)^2 L^2}{2\nu(1+\lambda)}, \\ F_H &\equiv -\alpha\overline{T_0 w_0} = \left(\frac{\lambda}{1+\lambda}\right) F_S. \end{aligned}\right\} \tag{2.3}$$

We shall choose $\lambda = 1$ in all numerical work to follow (for the reason mentioned in the second sentence of the introduction and also because it gives a heat- to salt-flux ratio which agrees with Turner's (1967) measurements), and we also prefer to regard F_S (rather than ΔS) as the primary independent physical variable.

3. Collective instability of salt fingers

The equilibrium field of § 2 may be unstable to different kinds of small perturbations, each of which is associated with different spatial scales and different overall effects on the salt fingers. The term 'collective instability' refers to the transfer of energy to relatively large-scale perturbations by means of the cumulative stress exerted by a group of salt fingers moving together. The velocity field $\tilde{U}(\zeta, t)$ in figure 1 (*c*) illustrates such a large-scale perturbation. It consists of a shear wave whose fronts are parallel to the unit vector $\hat{\boldsymbol{\xi}}$, which is in turn rotated θ radians from the x-axis. The $\hat{\boldsymbol{\zeta}}$ unit vector is perpendicular to $\hat{\boldsymbol{\xi}}$ and there is to be no motion perpendicular to the plane of the figure. Let (tilde) $[(\tilde{\ })]$ denote the average of a quantity over ξ. In the absence of salt fingers the average velocity ($\tilde{U}$) would oscillate with the Väisälä frequency $[g\alpha\,\partial\overline{T}/\partial z \sin^2\theta]^{\frac{1}{2}}$ of internal gravity waves, and we now intend to investigate the coupling with the salt

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fingers present. Let $\mathbf{V} = \hat{\boldsymbol{\xi}}\tilde{U} + \mathbf{V}'$ denote the total velocity at a point (with $\tilde{V}' = 0$), let $T = \overline{T}(z) + T'$ be the total temperature (with $\tilde{T}'(\zeta, t) \neq 0$), and let S be the total salinity. Note that part of $T'(\xi, \zeta, t)$ is associated with the undisturbed salt fingers and part with the $\tilde{U}$ motion. When the Boussinesq equation for $\hat{\boldsymbol{\xi}}$ momentum is averaged along the ξ-axis, there results

$$\frac{\partial \tilde{U}}{\partial t} - \nu \frac{\partial^2 \tilde{U}}{\partial \zeta^2} - g \sin\theta(\alpha\tilde{T}' - \beta\tilde{S}) = -\frac{\partial \sigma}{\partial \zeta}, \tag{3.1}$$

where $\sigma = \boldsymbol{\xi}.\widetilde{\mathbf{V}'\mathbf{V}'}.\hat{\boldsymbol{\zeta}}$ is the *Reynolds* stress due to the salt fingers. When the heat equation is averaged in the $\boldsymbol{\xi}$ direction one gets

$$\frac{\partial \tilde{T}'}{\partial t} + \hat{\boldsymbol{\xi}}\tilde{U}.\nabla\overline{T} + \widetilde{\mathbf{V}'.\nabla T'} = \kappa_T \nabla^2 \tilde{T}',$$

or

$$\left(\frac{\partial}{\partial t} - \kappa_T \frac{\partial^2}{\partial \zeta^2}\right)\alpha\tilde{T}' + \alpha\,\partial\overline{T}/\partial z \sin\theta\,\tilde{U} = -\frac{\partial \mathscr{F}_H}{\partial \zeta}, \tag{3.2}$$

where

$$\mathscr{F}_H \equiv \alpha\widetilde{T'\mathbf{V}'}.\hat{\boldsymbol{\zeta}},$$

and, when the non-diffusive salt equation is averaged, the corresponding relation is

$$\frac{\partial \beta\tilde{S}}{\partial t} = -\frac{\partial \mathscr{F}_S}{\partial \zeta}, \tag{3.3}$$

$$\mathscr{F}_S \equiv \beta\widetilde{S\mathbf{V}'}.\hat{\boldsymbol{\zeta}}.$$

Using (3.3) and (3.2) to eliminate $\tilde{T}'$ and $\tilde{S}$ in (3.1) there results

$$\left(\frac{\partial}{\partial t} - \kappa_T \frac{\partial^2}{\partial \zeta^2}\right)\left(\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial \zeta^2}\right)\frac{\partial \tilde{U}}{\partial t} + g\alpha\frac{\partial \overline{T}}{\partial z}\sin^2\theta\frac{\partial \tilde{U}}{\partial t}$$
$$= -g\sin\theta\frac{\partial}{\partial \zeta}\left[\frac{\partial \mathscr{F}_H}{\partial t} - \left(\frac{\partial}{\partial t} - \kappa_T\frac{\partial^2}{\partial \zeta^2}\right)\mathscr{F}_S\right] - \left(\frac{\partial}{\partial t} - \kappa_T\frac{\partial^2}{\partial \zeta^2}\right)\frac{\partial^2 \sigma}{\partial t\,\partial \zeta}. \tag{3.4}$$

4. The advective approximation

The right-hand side of (3.4) contains two different types of stress, the Reynolds stress σ and the buoyancy 'stresses' $\mathscr{F}_H$, $\mathscr{F}_S$. Before we can discuss the effect of these on the internal gravity waves it is necessary to discuss the modification of the salt fingers by the large-scale motion. Since salinity is conserved in our basic model the vertical isohalines of the undisturbed fingers will move with, and be rotated by, the local motion of the large-scale wave as shown in figure 1(*c*). To some extent this will occur for the ($T_0(x)$) isotherms (if κ_T is small compared with ν, but large compared with κ_S). We will assume that this is the case and this restricts the analysis of this section to large Prandtl number. Consequently the viscous (and gravity) force in the undisturbed finger is large compared to the inertial force. Furthermore, a preliminary order-of-magnitude calculation leads one to expect (and verify at the end of the calculation) that the term associated with σ in (3.4) is small compared with the terms containing ($\mathscr{F}_H, \mathscr{F}_S$). When this simplification

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is made (3.4) becomes

$$\left(\frac{\partial}{\partial t}-\nu\frac{\partial^2}{\partial\zeta^2}\right)\frac{\partial^2\tilde{U}}{\partial t^2}+g\alpha\frac{\partial\overline{T}}{\partial z}\sin^2\theta\frac{\partial\tilde{U}}{\partial t} = -g\sin\theta\frac{\partial^2}{\partial\zeta\,\partial t}(\mathscr{F}_H-\mathscr{F}_S). \tag{4.1}$$

A more important assumption appears now when the fluxes $(\mathscr{F}_H, \mathscr{F}_S)$ are related to $\tilde{U}$. First we note that $\partial\tilde{U}/\partial\zeta$ produces a rotation of the isohalines and therefore a variation of the $(\mathscr{F}_S, \mathscr{F}_H)$ flux vectors. This *kinematical* effect gives rise to a net convergence of heat and salt in a lamina parallel to the wave fronts, thereby modulating the amplitude of the $\tilde{U}$ oscillation. However, the rotation of the isolines may also alter the magnitude of the flux vectors because the perturbation of the buoyancy force relative to the density gradient in the finger modifies the local thickness and amplitude inside the finger. However this *dynamical* effect is strongly dependent on the structure of the salt finger in the undisturbed state, whereas the kinematical effect is not. For example, we will subsequently show that the kinematical effect does not depend on the ratio of the width of the salt finger to the wavelength of the internal wave. Furthermore, if one generalizes our two-dimensional model to the three-dimensional case the kinematical effect will again be unchanged, whereas the dynamical effect will undoubtedly depend on the planform structure of the undisturbed state. Because of this physical and parametrical difference it seems reasonable to investigate the influence of the kinematical effect separately, by neglecting the dynamical effect in the simplest model. Subsequently, one may search for a salt finger régime where this approximation is formally valid and then treat the dynamical effect as a perturbation in certain structural parameters. With this preliminary rationalization we may restate the advective approximation as follows: $\tilde{U}(\zeta, t)$ is regarded as a slightly modified internal gravity wave which translates and rotates the salt fingers in groups. The local salt and heat flux is assumed to be parallel to isohalines of each group and the *magnitude of the flux is the same as in the undisturbed state.*

The foregoing assumption allows us to write the salt flux as

$$\mathscr{F}_S = \beta\overline{S\mathbf{V}'}\,.\,\hat{\boldsymbol{\zeta}} = -F_S\cos(\theta+\phi),$$

where the undisturbed F_S is given by (2.3). The quantity $\phi(\zeta, t)$ is the infinitesimal angle between a local group of salt fingers and the vertical direction, as shown in figure 1 (*d*). According to simple kinematical considerations of the rotation of an isohaline by a shear we obtain the relation

$$\partial\phi/\partial t = \cos^2(\theta+\phi)\,\partial\tilde{U}/\partial\zeta \approx \cos^2\theta\,\partial\tilde{U}/\partial\zeta$$

plus higher-order terms in $\partial\tilde{U}/\partial\zeta$. Therefore

$$\partial\mathscr{F}_S/\partial t = F_S\sin\theta\,\partial\phi/\partial t = F_S\cos^2\theta\sin\theta\,\partial\tilde{U}/\partial\zeta.$$

Moreover, $\partial\mathscr{F}_H/\partial t = [\lambda/(1+\lambda)]\,\partial\mathscr{F}_S/\partial t$, according to (2.3), and the introduction of these flux relations into (4.1) gives

$$\left(\frac{\partial}{\partial t}-\nu\frac{\partial^2}{\partial\zeta^2}\right)\frac{\partial^2\tilde{U}}{\partial t^2}+g\alpha\frac{\partial\overline{T}}{\partial z}\sin^2\theta\frac{\partial\tilde{U}}{\partial t} = \frac{gF_S\sin^2\theta\cos^2\theta}{1+\lambda}\frac{\partial^2\tilde{U}}{\partial\zeta^2}, \tag{4.2}$$

which is a complete linear equation for $\tilde{U}(\zeta, t)$.

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Since (4.2) above has constant coefficients, we look for solutions of the form $\tilde{U} \propto \exp(\Omega t + im\zeta)$, where $2\pi/m$ is a real wavelength, and the complex growth rate Ω satisfies the cubic

$$\Omega\left(\Omega^2 + \nu m^2\Omega + g\alpha\frac{\partial \overline{T}}{\partial z}\sin^2\theta\right) + \frac{m^2 g F_S \sin^2\theta\cos^2\theta}{1+\lambda} = 0. \tag{4.3}$$

This cubic has no real positive root, but it has at least one negative (stable) root which we designate by $\Omega_1 < 0$. Since the sum of the roots is $-\nu m^2$, it follows that the remaining two roots are purely imaginary when $\Omega_1 = -\nu m^2$. By substituting this latter relation for Ω in (4.3) we obtain the condition for marginal stability: $\nu\alpha\, \partial\overline{T}/\partial z \sin^2\theta = [F_S/(1+\lambda)]\sin^2\theta\cos^2\theta$. The minimum critical condition for instability, corresponding to small θ, is given by

$$\frac{F_S}{\nu\alpha(\partial\overline{T}/\partial z)} \geqslant 1+\lambda \simeq 2. \tag{4.4}$$

With the aid of (2.1), (2.2) and (2.3) one may rewrite (4.4) in terms of a finger Reynolds number defined by $R = \Delta w L \nu^{-1}$. This alternative criterion for instability is

$$R \geqslant (2\lambda\kappa_T/\nu)^{\frac{1}{2}}.$$

From the foregoing idealized and simplified model we are led to the following generalization which has a certain degree of independent plausibility. One does not expect to find quasi-laminar salt fingers in water ($\nu/\kappa_T = 7$) with a concomitant Reynolds number which is much greater than unity. The Reynolds number concept is incorporated in the following discussion of an entirely different model.

5. The fully developed similarity model

Consider a 'very deep' layer of water which is arranged so that the temperature ($T^*(z)$) and salinity fields ($S^*(z)$) increase linearly upwards. This is initially in a state of hydrostatic and diffusive equilibrium with $\kappa_S \neq 0$. A hypothetical sequence of events will now be sketched which transforms this unstable state to a new régime that is steady and homogeneous in a statistical sense. Dimensional reasoning will then be used to obtain interesting relations for the turbulent régime, including an estimate of the depth of water that is necessary to justify the assumption of homogeneity.

The onset of the first salt finger instability (Stern 1960) in the starting state of diffusive equilibrium may produce a laminar field of vertical motion which increases exponentially with time until the critical Reynolds number for the collective instability is reached (§4). The latter sets in as a growing quasi-horizontal oscillation which draws on some form of the available potential energy. The kinetic energy of the fingers and the larger-scale convective mode (identified as a wave in §4) cease to grow when the shear of the latter is sufficient to disrupt the fingers.

In our model of the statistically steady state, quasi-laminar salt fingers will be retained in relatively thin (h, centimetres) and broad 'discontinuity' layers.

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Separating these are relatively deep (H, centimetres) 'mixing' layers containing convective eddies with relatively large motion and disorder. As the convective eddies sweep away the base of the discontinuity layer, groups of salt fingers are injected into them in such a way as to add to the circulation of the eddy. A similar account of the maintenance of each region is implicit in Turner's (1967) paper.

Let $\Delta T''$ denote the average increase in temperature from the bottom to the top of a discontinuity layer and let $\Delta S''$ denote the corresponding salinity increase. Then the temperature gradient across this layer is of order $\Delta T''/h$ and the salt flux is of the order of the statistical mean (F_S^*). By substituting the foregoing for the corresponding quantities appearing in (4.4) we obtain

$$\frac{F_S^* h}{\nu\alpha\Delta T''} \approx 1. \tag{5.1}$$

The symbol ($\approx$) denotes equality in order of magnitude; a sharper definition (implied by subsequent arguments) is that the symbol ($\approx$) denotes the asymptotic dependence (of a relation) on κ_S when $\kappa_S \to 0$. In these asymptotic relations the Prandtl number and the number

$$N \equiv \frac{\beta\,\partial S^*/\partial z}{\alpha\,\partial T^*/\partial z} < 1$$

are fixed at order unity values (in the central ocean N is about $\frac{1}{2}$). Notice that when the temperature (salinity) gradient is averaged over many steps in our similarity model the result is *still* $\partial T^*/\partial z(\partial S^*/\partial z)$. This novel feature of the deep thermohaline régime requires that the vertical separation of the reservoirs must be large compared with a vertical scale depth (cf. (5.10) below). Let us first estimate the mean turbulent salt flux (F_S^*) which accompanies a given $\partial S^*/\partial z$. Since the external parameters which govern the dynamics of the similarity model are $g\beta\,\partial S^*/\partial z$, ν, κ_T, κ_S, $g\alpha\,\partial T^*/\partial z$, it follows on formal dimensional grounds that the ratio of F_S^* to $(\nu^2/\kappa_S)\,\partial S^*\beta/\partial z$ can depend only on ν/κ_T, ν/κ_S and N. Some simple physical considerations will now show that the aforementioned ratio is independent of the Schmidt number (ν/κ_S) when the latter is very large. These and other considerations have been used by Stern (1968) to relate the gradient of the oceanic halocline to the evaporation boundary condition. The main assumptions in the halocline formula (5.4) should be more transparent in the relatively simple model of this paper.

The so-called 'power-integral' relating the r.m.s. (fluctuating) salinity gradient $(|\nabla S|)$ to the 'average flux of salt down the mean-gradient' will be obtained first. If $S+S^*(z)$ denotes total salinity then $d(S+S^*)/dt = \kappa_S\nabla^2(S+S^*)$. Multiply by S and average the result over a large volume of fluid and over a long period of time. Assuming no net vertical flux of mass and assuming vertical symmetry (i.e. the flux of salt and salt variance is independent of z) gives the thermodynamic equality

$$\kappa_S|\nabla S|^2 = F_S^* \frac{\partial S^*}{\partial z}. \tag{5.2}$$

The dimensionless quantity β has been set equal to unity (its approximate value)

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for convenience here and in what follows. We now take the r.m.s. value of the vorticity equation in the following manner: $|\nabla \times \dot{\mathbf{V}} - \nu \nabla^2 \boldsymbol{\eta}| = |g\nabla_2 S - g\alpha \nabla_2 T|$, where $\boldsymbol{\eta} = \nabla \times \mathbf{V}$ is the vorticity, $\dot{\mathbf{V}}$ is the particle acceleration and ∇_2 is the horizontal component of the gradient. For small κ_S it is apparent that the right-hand side of the above equation is dominated by $|\nabla_2 S| \gg |\alpha \nabla_2 T|$, and apart from a geometrical factor we have $|g\nabla_2 S - g\alpha\nabla_2 T| \approx g|\nabla S|$. Next consider the left-hand side of the r.m.s. vorticity budget. Its order of magnitude (in κ_S) is either determined by $|\nabla \times \dot{\mathbf{V}}| \approx \eta^2$ or by $|\nu \nabla^2 \boldsymbol{\eta}| \approx \nu \eta L_D^{-2}$, where L_D is a dissipation scale length. Let us tentatively assume that both of the preceding terms are of the same order in κ_S, i.e. the dissipation scale length is associated with a unit Reynolds number. By equating the curl of the inertial acceleration to the curl of the buoyancy force we get $\eta^2 \approx g|\nabla S|$. On the other hand the mechanical energy equation tells us that the dissipation ($\nu\eta^2$) is equal to the work done by buoyancy forces. The latter is g times the difference of the downward salt and heat flux (density units), i.e. $gF_S^*(1-\gamma)$, where γ is about $\frac{1}{2}$ when $N = \frac{1}{2}$ (Turner 1967). Thus the mechanical energy equation may be written as $\nu\eta^2 \approx gF_S^*$, and this may be used to eliminate η^2 from the vorticity balance. Doing this we get

$$|\nabla S| \approx \frac{F_S^*}{\nu} \tag{5.3}$$

and when (5.3) is substituted in (5.2) we obtain the important formula

$$F_S^* \approx \frac{\nu^2}{\kappa_S} \frac{\partial S^*}{\partial z}. \tag{5.4}$$

Also note that $|\nabla S| \approx (\nu/\kappa_S)\,\partial S^*/\partial z$ is implied by the foregoing. In the central ocean the downward salt flux is determined by the evaporation rate and if this consideration is introduced into (5.4) one obtains the halocline formula (for $\partial S^*/\partial z$) derived previously by Stern (1968) in a more complex context. Next we consider the viscous term in the r.m.s. vorticity balance and solve for

$$L_D \approx (\nu/\eta)^{\frac{1}{2}} \approx [\nu\kappa_S/(g\,\partial S^*/\partial z)]^{\frac{1}{4}}.$$

This must now be reconciled with the width of the salt fingers.

Turner (1967) has measured F_S^* as a function of $\Delta S''$ for a single layer which was 'artificially' produced in the laboratory and he obtained a $\frac{4}{3}$-power law. Although our model is quite different the same relationship is also required here on dimensional grounds. However, the analysis given below leads to a different dependence on Schmidt number than Turner suggested (but did not measure). We first write (5.3) as $\epsilon\Delta S''/L'' \approx F_S^*/\nu$, where L'' is the width of a salt finger in the discontinuity layer and $\epsilon\Delta S'' < \Delta S''$ is the salinity difference between adjacent salt fingers. By substituting $\Delta T''/h$ for the temperature gradient in (2.2) we obtain the corresponding value of L'' as

$$L'' \approx \left[\frac{\kappa_T \nu}{g\alpha(\Delta T''/h)}\right]^{\frac{1}{4}}.$$

Since $\Delta T''/h$ depends on F_S^* according to (5.1) we may write the preceding in the

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form
$$L'' \approx \left(\frac{\kappa_T \nu^2}{g F_S^*}\right)^{\frac{1}{4}}. \tag{5.5}$$

Now if the salt fingers extended across the discontinuity layer as laminar entities then the relation between 'salt flux' and $\Delta S''$ could be obtained directly from a relation like (2.3), viz. 'salt flux' $\approx g(\Delta S'')^2 (L'')^2 \nu^{-1}$. We do not, however, think that the preceding supposition is correct. We think that F_S^* will be somewhat less than 'salt flux' (given above). The salinity contrast $\epsilon \Delta S''$ between adjacent fingers should be less than $\Delta S''$ for the same reason. Since ϵ is a function of the ratio of the two previously mentioned salt fluxes, it may be represented by the power law
$$\epsilon \approx \left[\frac{F_S^*}{g(\Delta S'')^2 (L'')^2 \nu^{-1}}\right]^p, \tag{5.6}$$

where $p \approx 1$ is an undetermined positive constant of order unity. The substitution of (5.5) into the relation $\epsilon(\Delta S''/L'') \approx F_S^*/\nu$ gives
$$F_S^* \approx \epsilon^{\frac{4}{3}} (\Delta S'')^{\frac{4}{3}} \left[\frac{g \nu^2}{\kappa_T}\right]^{\frac{1}{3}}. \tag{5.7}$$

Equations (5.7) and (5.5) are now employed to eliminate F_S^* and L'' from (5.6):
$$\epsilon \approx \left[\epsilon^2 \frac{\nu}{\kappa_T}\right]^p,$$
$$\epsilon \approx \left(\frac{\kappa_T}{\nu}\right)^{[2-1/p]^{-1}}. \tag{5.8}$$

Although the order-unity constant $p > \frac{1}{2}$ is not yet determined, we see from (5.8) and (5.7) that the salinity difference across the discontinuity layer is proportional to the $\frac{3}{4}$ power of the salt flux. This relation between F_S^* and $\Delta S''$ was found by Turner (1967) for layers produced in the laboratory, and applied by him to compute the salt flux across layers observed in the ocean. It is therefore of great importance to make some theoretical estimate of the constant of proportionality in (5.7) and to compare it with laboratory measurements. The supposition that p is large compared to unity reduces (5.8) to $\epsilon \approx (\kappa_T/\nu)^{\frac{1}{2}}$, whereas if $p = 1$ then $\epsilon \approx \kappa_T/\nu$. Since p must be greater than $\frac{1}{2}$ our 'hedge' estimate is $p = 1$ when the Prandtl number is in the vicinity of one. Using this (5.7) becomes
$$F_S^* \approx \left(\frac{g \kappa_T^3}{\nu^2}\right)^{\frac{1}{3}} (\Delta S'')^{\frac{4}{3}}. \tag{5.9}$$

The value of $(g\kappa_T^3/\nu^2)^{\frac{1}{3}}$ is about $\frac{1}{3}$ cm/sec. The corresponding number in Turner's measurements is about $\frac{1}{10}$ cm/sec. On intuitive grounds one expects that, for fixed $\Delta S''$, F_S^* decreases with increasing ν and decreases as $\kappa_T \to \kappa_S$. Therefore the error in using (5.9) should be no greater than a factor of $(\nu/\kappa_T)^{\frac{2}{3}}$.

The foregoing can also be used to estimate the thickness of the 'mixing layer' for the oceanic case. The difference in salinity between two successive mixed layers is $\Delta S''$, and by equating this to $H\,\partial S^*/\partial z$ one obtains $H \approx \Delta S''(\partial S^*/\partial z)^{-1}$.

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Simplification of the latter equation by the use of (5.9) and (5.4) gives

$$H \approx \left[\frac{\nu^8}{g\,\partial S^*/\partial z\,\kappa_T^3\kappa_S^3}\right]^{\frac{1}{4}}. \tag{5.10}$$

To obtain h we set $\alpha\Delta T'' \approx \Delta S''$ in (5.1) and use (5.9) to eliminate $\Delta S''$. The result may be written as

$$h \approx \left[\frac{\kappa_S\,\nu^4}{g(\partial S^*/\partial z)\,\kappa_T^3}\right]^{\frac{1}{4}}. \tag{5.11}$$

Finally we get the salt finger thickness from (5.5):

$$L'' \approx \left[\frac{\kappa_T\kappa_S}{g\,\partial S^*/\partial z}\right]^{\frac{1}{4}}. \tag{5.12}$$

Recalling our previous result for the dissipation scale ($L_D \approx [\nu\kappa_S/(g\,\partial S^*/\partial z)]^{\frac{1}{4}}$) we see that the smallest scale of motion is of order $\kappa_S^{\frac{1}{4}}$ whereas the largest vertical scale we have found is (5.10) of order $\kappa_S^{-\frac{3}{4}}$. The value of H is $10^{\frac{1}{4}} \times 10^3$ cm and h is $10^{\frac{1}{4}}$ cm for a value of $\partial S^*/\partial z = 10^{-8}\,\text{cm}^{-1}$, the latter number being the gradient of the halocline in the central ocean. The value of H compares favourably with Mediterranean outflow observations (Tait & Howe 1968) but recent measurements off Bermuda (Cooper & Stommel 1968) show smaller and more variable step structure. Variations in synoptic microstructure may be expected for a variety of theoretical reasons (Stern 1967).

6. Summary

The highly idealized model of §§ 2–4 has led to the notion that quasi-laminar salt fingers of unit Reynolds number exist in thin discontinuity layers separated by deeper mixing layers. The discussion of the fully developed thermohaline convection régime is an attempt to connect salt fingers observed in the laboratory to certain observed features of the main thermocline on the micro- and macro-scales. The order-of-magnitude calculations are not in conflict with the thesis that the sinking of salt (on the small scale) is the dominant mechanism for the vertical heat flux in the central ocean. Additional theoretical and experimental work appears to be possible and of potential value to the oceanographer trying to decipher the puzzle of microstructure in the thermocline.

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