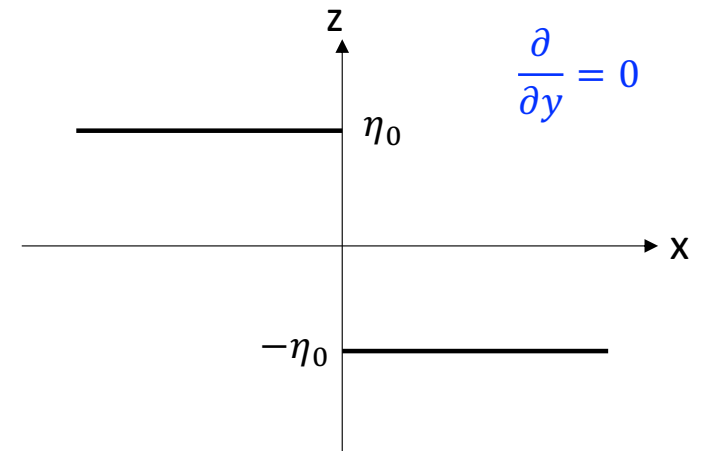


Barotropic geostrophic adjustment

Initial state (unbalanced):

$$\eta = \begin{cases} \eta_0, & x < 0 \\ -\eta_0, & x > 0 \end{cases} \quad \eta_0 \ll H$$

$$u = v = 0$$



For a steady final state (based on shallow water model):

$$\cancel{\frac{\partial u}{\partial t}} + u \cancel{\frac{\partial u}{\partial x}} + v \cancel{\frac{\partial u}{\partial y}} - fv = -g \frac{\partial \eta}{\partial x}$$

$$\cancel{\frac{\partial v}{\partial t}} + u \cancel{\frac{\partial v}{\partial x}} + v \cancel{\frac{\partial v}{\partial y}} + fu = -g \frac{\partial \eta}{\partial y}$$

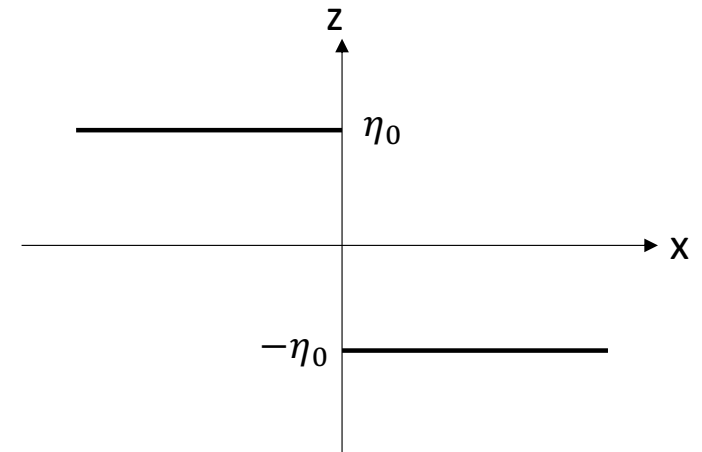
geostrophic flow

$$\cancel{\frac{\partial \eta}{\partial t}} + \frac{\partial hu}{\partial x} + \cancel{\frac{\partial hv}{\partial y}} = 0 \quad \text{necessary condition: } u=0 \text{ everywhere}$$

For PV conservation:

$x < 0$:

$$\frac{f}{H + \eta_0} = \frac{f + \frac{\partial v}{\partial x}}{H + \eta} = \frac{f + \frac{g}{f} \frac{\partial^2 \eta}{\partial x^2}}{H + \eta} \quad \eta_0 \ll H$$



~~$$fH + f\eta = fH + f\eta_0 + \frac{gH}{f} \frac{\partial^2 \eta}{\partial x^2} + \frac{g\eta_0}{f} \frac{\partial^2 \eta}{\partial x^2}$$~~

$$R^2 \frac{\partial^2 \eta}{\partial x^2} - \eta = -\eta_0 \quad R = \frac{\sqrt{gH}}{f}$$

$$\eta = f(x) + \eta_0 \quad e^{-\frac{1}{R}x} \rightarrow \infty \text{ as } x \rightarrow -\infty$$

$$f(x) = Ce^{\lambda x} \longrightarrow R^2 \lambda^2 - 1 = 0 \longrightarrow \lambda = \pm \frac{1}{R} \longrightarrow \lambda = \frac{1}{R}$$

$$\eta = Ce^{\frac{x}{R}} + \eta_0 \quad x = 0, \eta = 0, C = -\eta_0 \longrightarrow \eta = -\eta_0 e^{\frac{x}{R}} + \eta_0$$

For PV conservation:

$x > 0$:

$$\frac{f}{H - \eta_0} = \frac{f + \frac{\partial v}{\partial x}}{H + \eta} = \frac{f + \frac{g}{f} \frac{\partial^2 \eta}{\partial x^2}}{H + \eta}$$

~~$$fH + f\eta = fH - f\eta_0 + \frac{gH}{f} \frac{\partial^2 \eta}{\partial x^2} - \frac{g\eta_0}{f} \frac{\partial^2 \eta}{\partial x^2}$$~~

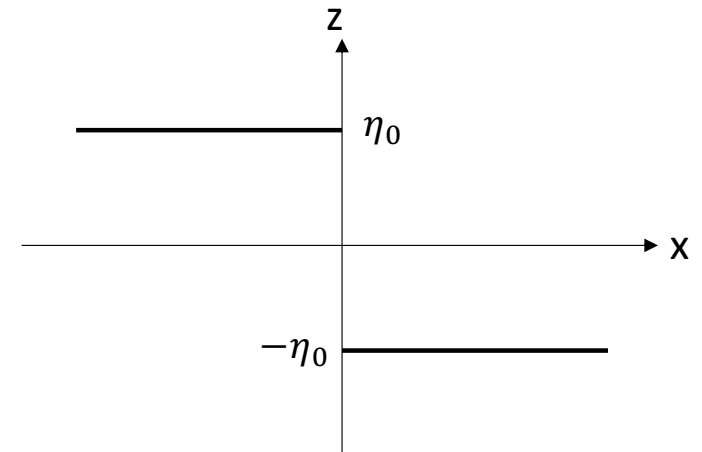
$$R^2 \frac{\partial^2 \eta}{\partial x^2} - \eta = \eta_0$$

$$\eta = f(x) - \eta_0$$

$$e^{\frac{1}{R}x} \rightarrow \infty \text{ as } x \rightarrow \infty$$

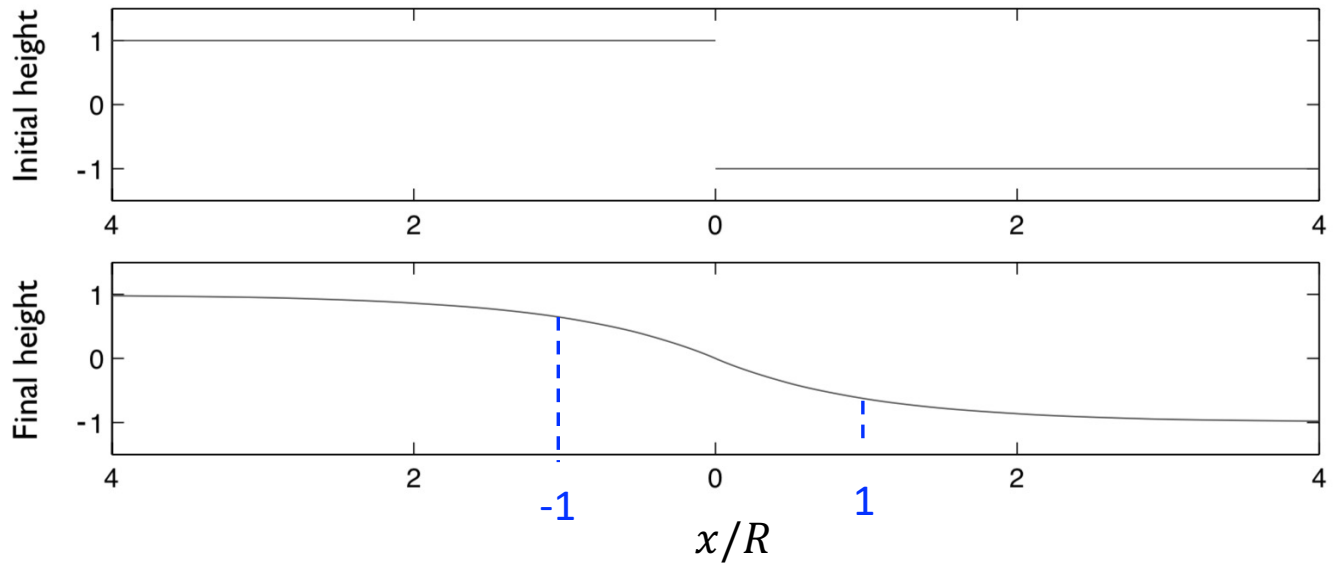
$$f(x) = Ce^{\lambda x} \longrightarrow R^2 \lambda^2 - 1 = 0 \longrightarrow \lambda = \pm \frac{1}{R} \longrightarrow \lambda = -\frac{1}{R}$$

$$\eta = Ce^{-\frac{x}{R}} - \eta_0 \xrightarrow{x=0, \eta=0, C=\eta_0} \eta = \eta_0 e^{-\frac{x}{R}} - \eta_0$$



$$\eta = \begin{cases} -\eta_0 e^{\frac{x}{R}} + \eta_0, & x < 0 \\ \eta_0 e^{-\frac{x}{R}} - \eta_0, & x > 0 \end{cases}$$

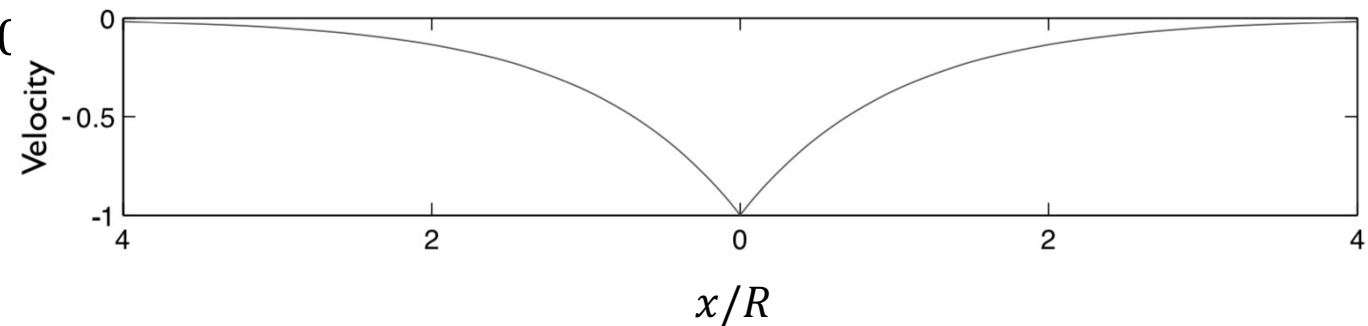
$x \rightarrow -R, \quad \eta \rightarrow \eta_0$
 $x \rightarrow R, \quad \eta \rightarrow -\eta_0$



the adjustment spatial scale is the **Rossby deformation radius**

$$v = \frac{g}{f} \frac{\partial \eta}{\partial x} = \begin{cases} -\sqrt{\frac{g}{H}} \eta_0 e^{\frac{x}{R}}, & x < 0 \\ \sqrt{\frac{g}{H}} \eta_0 e^{-\frac{x}{R}}, & x > 0 \end{cases}$$

$x \rightarrow -R, \quad v \rightarrow 0$
 $x \rightarrow R, \quad v \rightarrow 0$

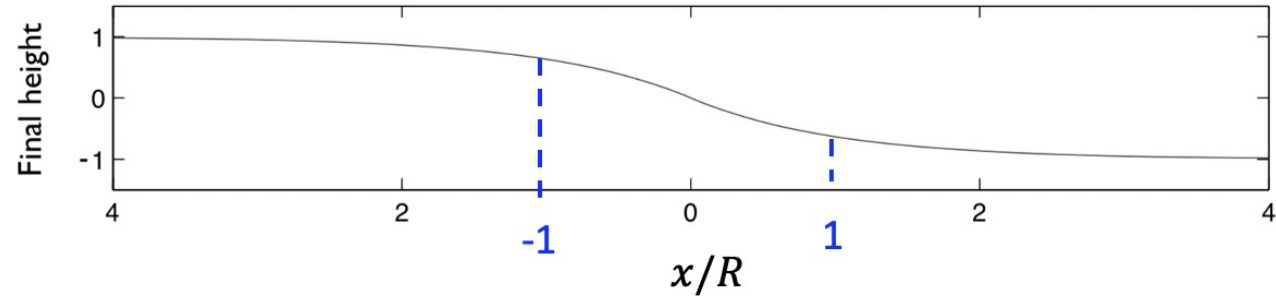


Energetics of geostrophic adjustment

For unit length in the y-direction:

Potential energy:

PE change is limited to $(-R: R, -\eta_0: -\eta_0)$



$$PE_I = \int_0^{\eta_0} \int_{-R}^0 \rho_0 g z dx dz + \int_0^{-\eta_0} \int_0^R \rho_0 g z dx dz = \frac{1}{2} \rho_0 g R \eta_0^2 + \frac{1}{2} \rho_0 g R \eta_0^2 = \rho_0 g R \eta_0^2$$

$$PE_F = \int_0^{\eta} \int_{-R}^0 \rho_0 g z dx dz + \int_0^{\eta} \int_0^R \rho_0 g z dx dz$$

$$\eta = \begin{cases} -\eta_0 e^{\frac{x}{R}} + \eta_0, & x < 0 \\ \eta_0 e^{-\frac{x}{R}} - \eta_0, & x > 0 \end{cases}$$

$$= \frac{1}{2} \rho_0 g \eta_0^2 \int_{-R}^0 (1 - e^{\frac{x}{R}})^2 dx + \frac{1}{2} \rho_0 g \eta_0^2 \int_0^R (1 - e^{-\frac{x}{R}})^2 dx$$

$$\Delta PE = -\frac{3}{2} \rho_0 g R \eta_0^2$$

$$= \frac{1}{2} \rho_0 g \eta_0^2 \left(x|_{-R}^0 - 2R e^{\frac{x}{R}}|_{-R}^0 + \frac{R}{2} e^{\frac{2x}{R}}|_{-R}^0 + x|_0^R + 2R e^{-\frac{x}{R}}|_0^R - \frac{R}{2} e^{-\frac{2x}{R}}|_0^R \right) = -\frac{1}{2} \rho_0 g R \eta_0^2$$

Kinetic energy:

$$KE_I = 0$$

$$KE_F = \int_{-H}^0 \int_{-R}^R \frac{1}{2} \rho_0 v^2 dx dz$$

$$= \int_{-H}^0 \int_{-R}^0 \frac{1}{2} \rho_0 \eta_0^2 \frac{g}{H} e^{\frac{2x}{R}} dx dz + \int_{-H}^0 \int_0^R \frac{1}{2} \rho_0 \eta_0^2 \frac{g}{H} e^{-\frac{2x}{R}} dx dz$$

$$= \frac{1}{2} \rho_0 \eta_0^2 g \left(\int_{-R}^0 e^{\frac{2x}{R}} dx + \int_{-R}^0 e^{-\frac{2x}{R}} dx \right)$$

$$= \frac{1}{2} \rho_0 \eta_0^2 g \left(\frac{R}{2} e^{\frac{2x}{R}} \Big|_{-R}^0 - \frac{R}{2} e^{-\frac{2x}{R}} \Big|_0^R \right)$$

$$= \frac{1}{2} \rho_0 g R \eta_0^2$$

$$v = \begin{cases} -\sqrt{\frac{g}{H}} \eta_0 e^{\frac{x}{R}}, & x < 0 \\ -\sqrt{\frac{g}{H}} \eta_0 e^{-\frac{x}{R}}, & x > 0 \end{cases}$$

$$\Delta PE = -\frac{3}{2} \rho_0 g R \eta_0^2$$

$$\frac{\Delta KE}{|\Delta PE|} = 1/3$$

Only 1/3 of released PE is converted into KE, and the rest of lost PE is radiated away by waves