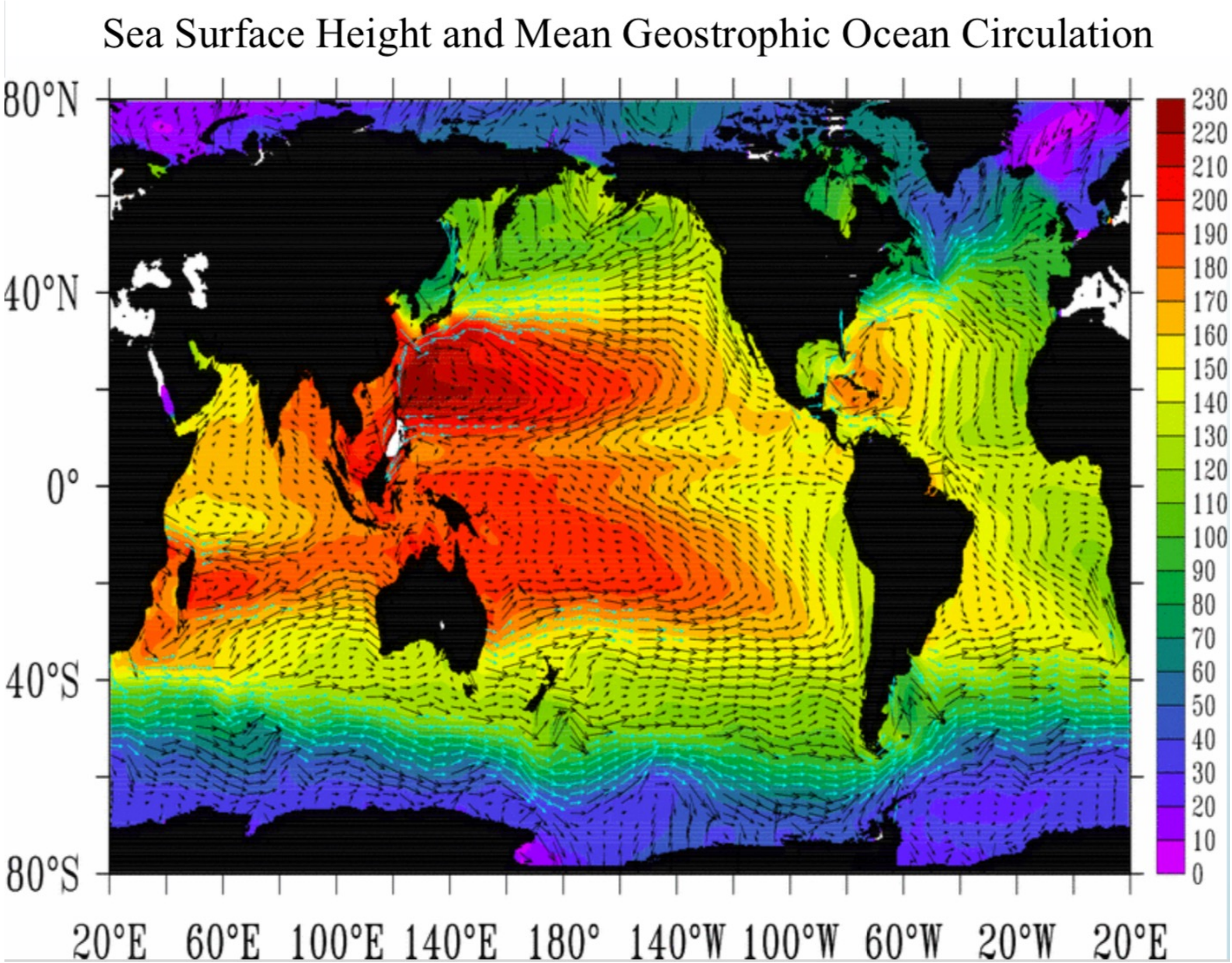
Homogeneous model for subtropical gyre



For the interior ocean (shallow-water model):

$$\frac{\partial u}{\partial \mathrm{t}} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$

$$\frac{\partial v}{\partial \mathrm{t}} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}$$

Take the curl of the equation:

$$\frac{\partial}{\partial \mathrm{t}}\left(\frac{\partial \mathrm{v}}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{\partial u}{\partial x}\frac{\partial \mathrm{v}}{\partial x} + u\frac{\partial^2 v}{\partial x^2} - \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + v\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} - v\frac{\partial^2 u}{\partial y^2} + f\frac{\partial u}{\partial x} + \frac{\partial f}{\partial y}v + f\frac{\partial v}{\partial y} = 0$$

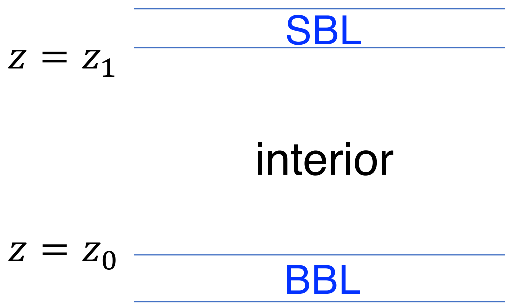
$$\frac{\partial \zeta}{\partial t} + \frac{\partial u}{\partial x}\zeta + u\frac{\partial \zeta}{\partial x} + \frac{\partial v}{\partial y}\zeta + v\frac{\partial \zeta}{\partial y} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta_0 v = 0$$

$$\frac{d\zeta}{dt} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(f + \cancel{\zeta}) + \beta_0 v = 0$$

large scale: $\zeta \ll f$

$$\frac{d\zeta}{dt} + \beta_0 v = f\frac{\partial w}{\partial z}$$

$$\frac{d\zeta}{dt} + \beta_0 v = f\frac{\partial w}{\partial z}$$



Take the vertical integral, and given that $\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0, \frac{\partial \zeta}{\partial z} = 0$

$$\left(\frac{d\zeta}{dt} + \beta_0 v\right) H = f\left(w|_{z_1} - w|_{z_0}\right)$$

$$= f\left\{\frac{1}{\rho_0}\left[\frac{\partial}{\partial x}\left(\frac{\tau^y}{f}\right) - \frac{\partial}{\partial y}\left(\frac{\tau^x}{f}\right)\right] - \frac{d}{2}\,\zeta\right\}$$

$$= f\left\{\frac{1}{\rho_0 f}\frac{\partial \tau^y}{\partial x} - \frac{1}{\rho_0 f}\frac{\partial \tau^x}{\partial y} + \frac{\tau^x}{\rho_0 f^2}\beta_0 - \frac{d}{2}\,\zeta\right\}$$

Define non-dimensional variables:

$$(u, v) = U(u', v') \qquad (x, y) = L(x', y') \qquad t = Tt' \qquad \tau = \tau_0 \tau'$$

$$H\left(\frac{U}{LT}\frac{d\zeta'}{dt'} + \beta_0 U v'\right) = \frac{1}{\rho_0}\frac{\tau_0}{L}\, curl\,\tau' + \frac{\tau_0}{\rho_0 f}\beta_0 \tau^{x'} - \frac{d}{2} f \frac{U}{L}\zeta'$$

$$H\left(\frac{U}{LT}\frac{d\zeta'}{dt}+\beta_0 U v'\right)=\frac{1}{\rho_0}\frac{\tau_0}{L}curl\tau'+\frac{\tau_0}{\rho_0 f}\beta_0\tau^{x'}-\frac{d}{2}f\frac{U}{L}\zeta'$$

$$E_k=\frac{\nu_E}{fH^2}$$

$$H\frac{U^2}{L^2}\left(\frac{d\zeta'}{dt}+\frac{\beta_0 L^2}{U}v'\right)=\frac{\tau_0}{\rho_0 L}\left(curl\tau'+\frac{\beta_0 L}{f}\tau^{x'}\right)-\frac{d}{2}f\frac{U}{L}\zeta'$$

$$d=\sqrt{\frac{\nu_E}{f}}=E_k{}^{1/2}\mathrm{H}$$

<<1

$$\frac{d\zeta'}{dt}+\frac{\beta_0 L^2}{U}v'=\frac{\tau_0 L}{\rho_0 H U^2}\left(curl\tau'+\cancel{\frac{\beta_0 L}{f}}\tau^{x'}\right)-\frac{1}{2}\frac{fL}{U}\frac{d}{H}\zeta'$$

β

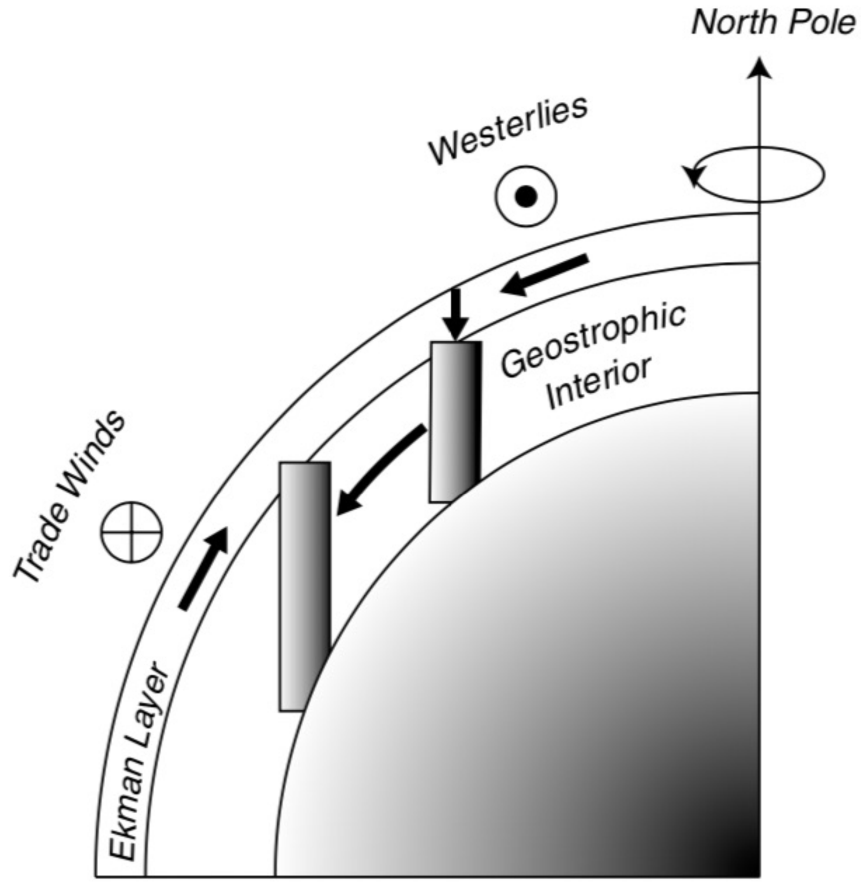
$$-\frac{1}{2}\frac{1}{R_o}E_k{}^{1/2}\zeta'$$

Remove the prime *symbol* ′

$$\cancel{\frac{d\zeta}{dt}}+\beta v=\frac{\tau_0 L}{\rho_0 H U^2}curl\tau-\cancel{r\zeta}$$

For steady, large-scale flow, and assuming the bottom friction is negligible:

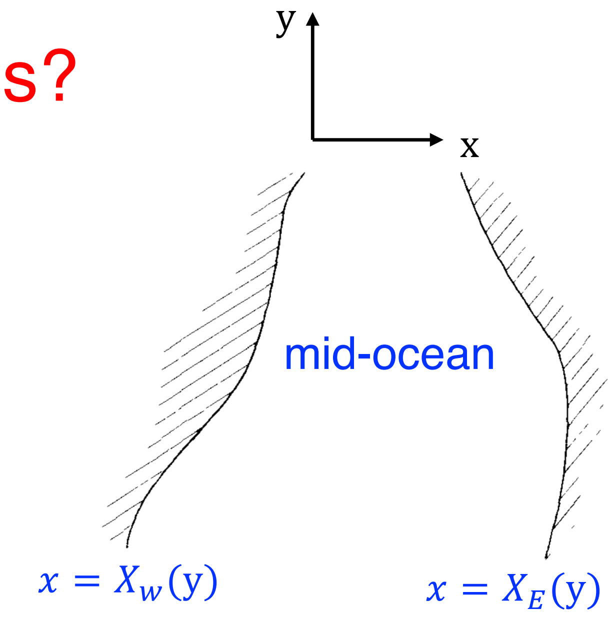
$v=curl\tau$ Sverdrup balance



$$\frac{d}{dt}\left(\frac{f\ +\ \zeta}{h}\right)\ =\ 0$$

Figure 12.7 Ekman pumping that produces a downward velocity at the base of the Ekman layer forces the fluid in the interior of the ocean to move southward. (From Niiler, 1987)

How about at the laternal boundaries?



Lateral boundaries:

West: $x = X_w(y)$ East: $x = X_E(y)$

$$x - X_w(y) = 0 \qquad x - X_E(y) = 0$$

No normal flows:

$$\boldsymbol{u} \cdot \nabla\big(x - X_w(y)\big) = 0 \Longrightarrow u - v\frac{\partial X_w}{\partial y} = 0 \; at \; x = X_w$$

$$\boldsymbol{u} \cdot \nabla\big(x - X_E(y)\big) = 0 \Longrightarrow u - v\frac{\partial X_E}{\partial y} = 0 \; at \; x = X_E$$

For interior geostrophic flow:

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} = -\frac{\partial curl\tau}{\partial y}$$

For a point in the mid-ocean x_0:

$$\int_{x_0}^{x} \frac{\partial u}{\partial x} dx' = -\int_{x_0}^{x} \frac{\partial curl\tau}{\partial y} dx'$$

$$u(x,y) = -\int_{x_0}^{x} \frac{\partial curl\tau}{\partial y} dx' + \mathrm{U}(x_0,\mathrm{y})$$ unknown, and needs to be determined

If the Sverdrup relation is valid at the eastern boundary: $u = v\frac{\partial X_E}{\partial y}$ at $x = X_E$

$$-\int_{x_0}^{X_E} \frac{\partial curl\tau}{\partial y} dx' + \mathrm{U}(x_0,\mathrm{y}) \quad = v\frac{\partial X_E}{\partial y} \quad = curl\tau(X_E,y)\frac{\partial X_E}{\partial y}$$

$$\mathrm{U}(x_0,\mathrm{y}) = \int_{x_0}^{X_E} \frac{\partial curl\tau}{\partial y} dx' + curl\tau(X_E,y)\frac{\partial X_E}{\partial y}$$

$$u(x,y) = -\int_{x_0}^{x} \frac{\partial curl\tau}{\partial y} dx' + \int_{x_0}^{X_E} \frac{\partial curl\tau}{\partial y} dx' + curl\tau(X_E,y)\frac{\partial X_E}{\partial y}$$

$$= \int_{x}^{X_E} \frac{\partial curl\tau}{\partial y} dx' + curl\tau(X_E,y)\frac{\partial X_E}{\partial y}$$

$$u(x,y) = \int_{x}^{X_E} \frac{\partial curl\tau}{\partial y} dx' + curl\tau(X_E, y)\frac{\partial X_E}{\partial y}$$

If the Sverdrup relation is also valid at the western boundary: $u = v\dfrac{\partial X_W}{\partial y}\ at\ x = X_W$

$$u(X_W,y) = \int_{X_W}^{X_E} \frac{\partial curl\tau}{\partial y} dx' + curl\tau(X_E, y)\frac{\partial X_E}{\partial y} \quad = v\frac{\partial X_W}{\partial y} \quad = curl\tau(X_W, y)\frac{\partial X_W}{\partial y}$$

$$\int_{X_W}^{X_E} \frac{\partial curl\tau}{\partial y} dx' + curl\tau(X_E, y)\frac{\partial X_E}{\partial y} - curl\tau(X_W, y)\frac{\partial X_W}{\partial y} = 0$$

$$\frac{\partial}{\partial y}\int_{X_W}^{X_E} curl\tau dx' = 0$$

condition for Sverdrup relation to be valid at both boundaries

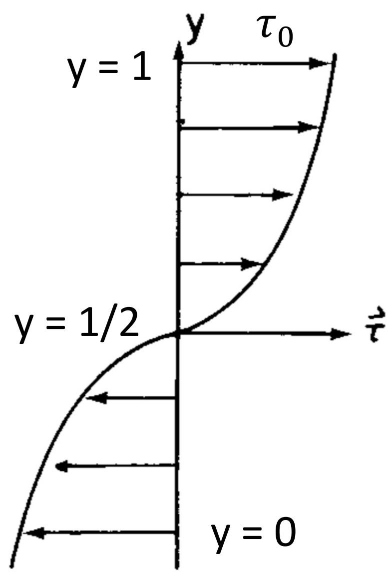
$$\frac{\partial}{\partial y}\int_{X_W}^{X_E} curl\tau dx' = 0$$

For the trade wind and westerly wind:

$$\boldsymbol{\tau} = (-\tau_0 cos\pi y, \qquad 0)$$

$$0 < y < 1$$

$$0 < \pi y < \pi$$



$$v = curl\tau = -\frac{\partial \tau^x}{\partial y} = -\pi\tau_0 sin\pi y < 0$$

$$X_E = C, X_W = C$$

$$\frac{\partial}{\partial y}\int_{X_W}^{X_E} curl\tau dx' = \frac{\partial}{\partial y}\int_{X_W}^{X_E} -\pi\tau_0 sin\pi y dx' = -\pi^2\tau_0 cos\pi y(X_E - X_W)$$

only =0 at y=1/2

Sverdrup relation cannot hold at both boundaries

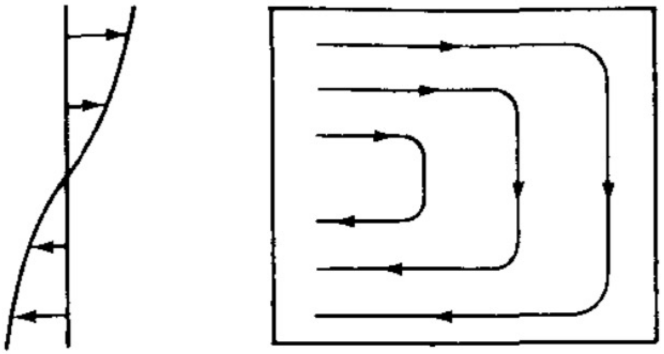
If Sverdrup relation holds at the eastern boundary:

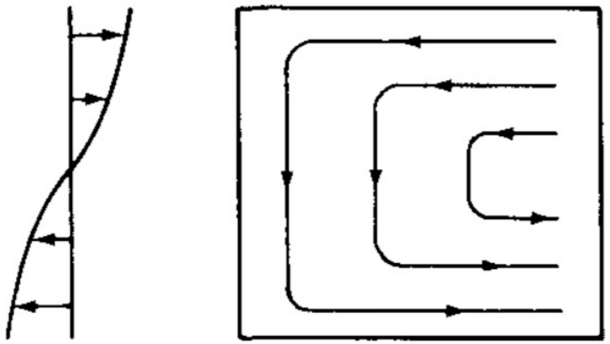
$$u(x,y) = \int_x^{X_E} \frac{\partial curl\tau}{\partial y} dx' + curl\tau(X_E, y)\cancel{\frac{\partial X_E}{\partial y}}$$

straight coastline $X_E = C$

$$\boxed{curl\tau = -\frac{\partial \tau^x}{\partial y} = -\pi\tau_0 sin\pi y}$$

$$= \frac{\partial}{\partial y}\int_x^{X_E} curl\tau dx' \quad = -\pi^2\tau_0 cos\pi y\overset{>0}{(X_E - x)} \quad \begin{cases} > 0, y > 1/2 \\ < 0, y < 1/2 \end{cases}$$





opposite to the wind direction (X)

If Sverdrup relation holds at the western boundary:

$$u(x,y) = \int_x^{X_W} \frac{\partial curl\tau}{\partial y} dx' + curl\tau(X_W, y)\cancel{\frac{\partial X_W}{\partial y}}$$

straight coastline $X_W = C$

$$= \frac{\partial}{\partial y}\int_x^{X_W} curl\tau dx' \quad = -\pi^2\tau_0 cos\pi y\overset{<0}{(X_W - x)} \quad \begin{cases} < 0, y > 1/2 \\ > 0, y < 1/2 \end{cases}$$

Stommel's model for western boundary intensification — **bottom friction**

$$\cancel{\frac{d\zeta}{dt}} + \beta v = \frac{\tau_0 L}{\rho_0 H U^2} curl\tau - r\zeta$$

For the boundary layers, retain bottom friction, and divide the equation by β:

$$v = curl\tau - \frac{r}{\beta}\zeta$$

For geostrophic flow:

$$\frac{\partial\psi}{\partial x} = curl\tau - \varepsilon_s \nabla^2 \psi$$

$$\psi = \underline{\psi_I(x,y)} + \underline{\psi_B(x,y)}$$ boundary layer correction

interior (mid-ocean) solution

$$\frac{\partial\psi_I}{\partial x}(x,y) = curl\tau$$

$$\varepsilon_s(\nabla^2\psi_I + \nabla^2\psi_B) + \cancel{\frac{\partial\psi_I}{\partial x}} + \frac{\partial\psi_B}{\partial x} = \cancel{curl\tau}$$

West: $\alpha = \frac{x-0}{\varepsilon} \sim O(1)$ $\qquad \frac{\partial}{\partial x} = \frac{1}{\varepsilon}\frac{\partial}{\partial\alpha}$ $\qquad \frac{\partial^2}{\partial x^2} = \frac{1}{\varepsilon^2}\frac{\partial^2}{\partial\alpha^2}$

ε: *boundary layer thickness* ($\ll 1$)

East: $\alpha = \frac{x-1}{\varepsilon} \sim O(1)$

$$\varepsilon_s\left(\cancel{\nabla^2\psi_I} + \frac{1}{\varepsilon^2}\frac{\partial^2\psi_B}{\partial\alpha^2} + \cancel{\frac{\partial^2\psi_B}{\partial y^2}}\right) + \frac{1}{\varepsilon}\frac{\partial\psi_B}{\partial\alpha} = 0$$

Only retain the large terms:

$$\frac{\partial^2\psi_B}{\partial\alpha^2} + \frac{\partial\psi_B}{\partial\alpha} = 0$$

$$\frac{\partial^2 \psi_B}{\partial \alpha^2} + \frac{\partial \psi_B}{\partial \alpha} = 0$$

$$\psi_B = A(y)e^{\lambda\alpha} \Longrightarrow \lambda^2 + \lambda = 0 \Longrightarrow \lambda = -1$$

$$\psi_B = A(y)e^{-\alpha}$$

If ψ_B applies to the western boudary: $\alpha = \frac{x-0}{\varepsilon} > 0$

ψ_B decays expotentially toward the mid-ocean ✓

If ψ_B applies to the eastern boudary: $\alpha = \frac{x-1}{\varepsilon} < 0$

ψ_B grows expotentially toward the mid-ocean ×

The correction applies to the western boundary, and Sverdrup relation holds for the eastern boudary

For the interior and the eastern boundary: $\frac{\partial \psi_I}{\partial x}(x,y) = curl\tau$

$$\int_x^1 \frac{\partial \psi_I}{\partial x} dx' = \int_x^1 curl\tau \, dx'$$

$$\boxed{curl\tau = -\frac{\partial \tau^x}{\partial y} = -\pi\tau_0 sin\pi y}$$

$$\psi_I(1,y) - \psi_I(x,y) = -\pi\tau_0 sin\pi y(1-x)$$

$$\psi_I(x,y) = \pi\tau_0 sin\pi y(1-x)$$

At the western boundary: $\psi_B = A(y)e^{-\alpha}$

$$\psi(0,y) = \psi_I(0,y) + \psi_B(0,y) = \pi\tau_0 \; sin\pi y + A(y) = 0$$

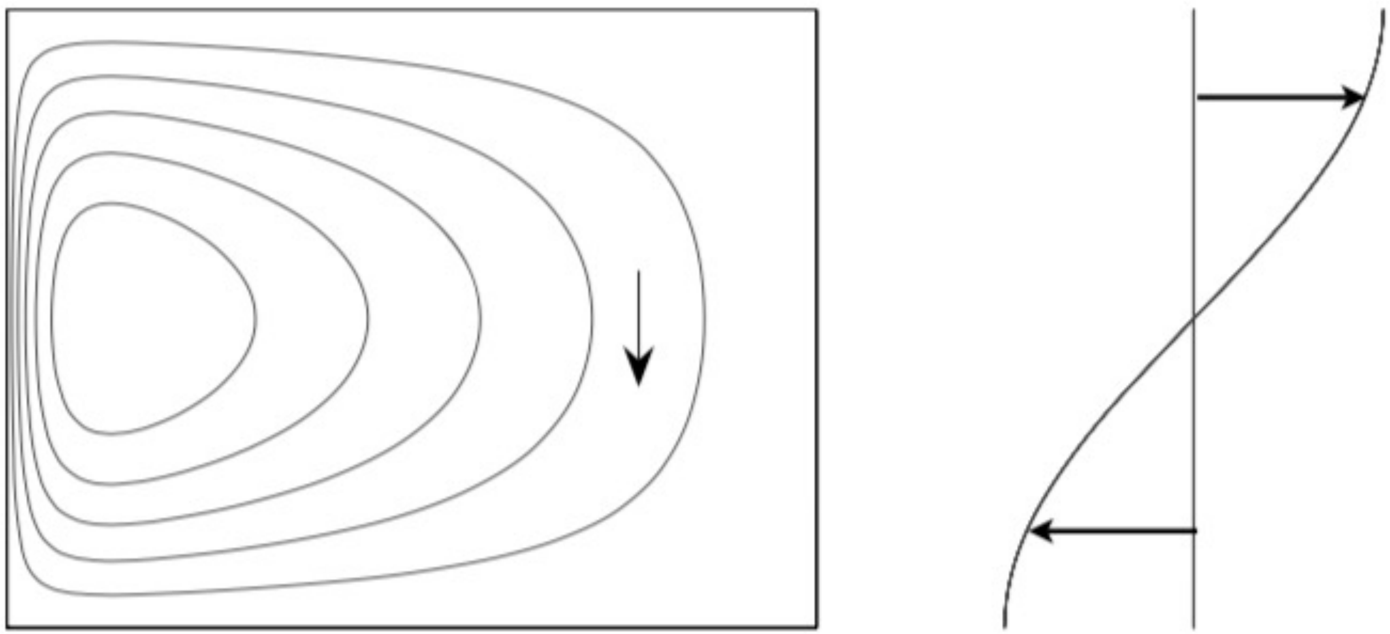
$$A(y) = -\pi\tau_0 sin\pi y$$

$$\boxed{\alpha = \frac{x-0}{\varepsilon}}$$

$$\psi(x,y) = \psi_I(x,y) + \psi_B\,(x,y) = \pi\tau_0 sin\pi y(1-x) - \pi\tau_0 \; sin\pi y e^{-x/\varepsilon}$$

$$= \left(1 - x - e^{-x/\varepsilon}\right) \pi\tau_0 \; sin\pi y$$

Streamfunction Wind stress



Vorticity dynamics

$$\boxed{\frac{d}{dt}\left(\frac{f\ +\ \zeta}{h}\right)\ =\ 0}$$

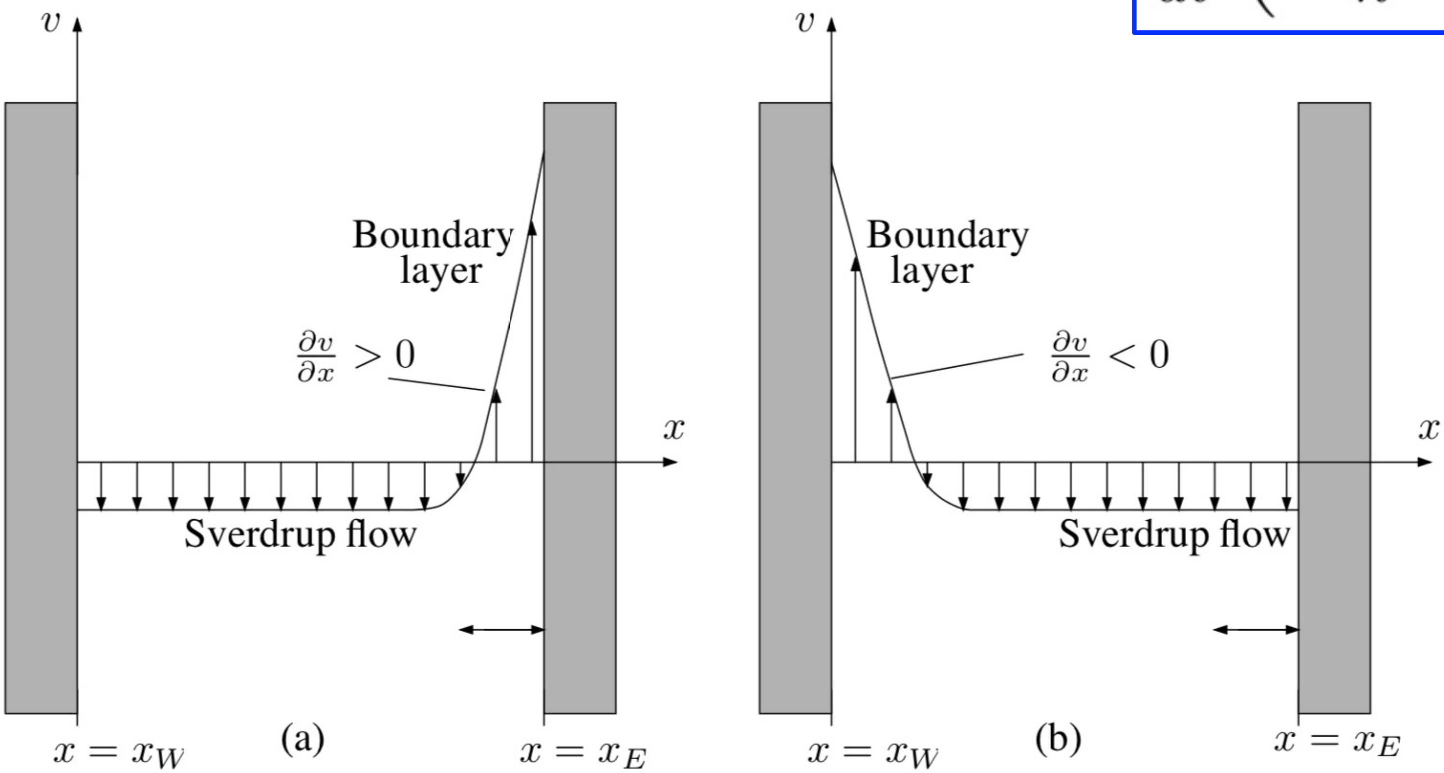


Figure 20-7 The two possible configurations for a northward boundary current to return the southward Sverdrup flow that exists across most of an ocean basin in the mid-latitudes of the Northern Hemisphere: (a) boundary current on the eastern side, (b) boundary current on the western side. The former is to be rejected on dynamic grounds, leaving the latter as the correct configuration.