第 1 次作业

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摘 要：摘要.
关键词：关键词 1，关键词 2.

Title

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Keywords: keyword1, keyword2.

目 录

1 第 1 题

在数学分析课程（陈纪修 et al., 2019, p. 315）中，已从数学上严格证明了一维的**积分号下求导定理**

定理 15. 1. 3（积分号下求导定理） 设 $f(x,y)$，$f_y(x,y)$ 都在闭矩形 $[a,b]\times[c,d]$ 上连续，则 $I(y)=\int_a^b f(x,y)\,\mathrm{d}x$ 在 $[c,d]$ 上可导，并且在 $[c,d]$ 上成立

$$\frac{\mathrm{d}I(y)}{\mathrm{d}y}=\int_a^b f_y(x,y)\,\mathrm{d}x.$$

和 **Leibniz 定理**

定理 15.1.4 设 $f(x,y)$, $f_y(x,y)$ 都是闭矩形 $[a,b]\times[c,d]$ 上的连续函数，又设 $a(y)$, $b(y)$ 是在 $[c,d]$ 上的可导函数，满足 $a\leqslant a(y)\leqslant b$, $a\leqslant b(y)\leqslant b$，则函数

$$F(y)=\int_{a(y)}^{b(y)}f(x,y)\,\mathrm{d}x$$

在 $[c,d]$ 上可导，并且在 $[c,d]$ 上成立

$$F'(y)=\int_{a(y)}^{b(y)}f_y(x,y)\,\mathrm{d}x+f(b(y),y)b'(y)-f(a(y),y)a'(y).$$

Leibniz 定理在二维、三维情形下的推广，便是著名的**雷诺输运定理**(Reynolds transport theorem, RTT). Kundu et al. (2016, pp. 99-103) 给出了推导，其结果是

$$\frac{d}{dt}\int_{V^*(t)}F(\mathbf{x},t)dV=\int_{V^*(t)}\frac{\partial F(\mathbf{x},t)}{\partial t}dV+\int_{A^*(t)}F(\mathbf{x},t)\mathbf{b}\cdot\mathbf{n}dA. \tag{3.35}$$

式中各字母的含义示于下图 (Kundu et al., 2016, Fig 3.20).

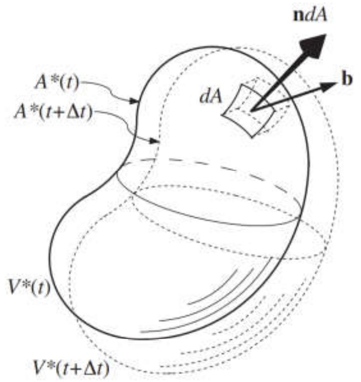


FIGURE 3.20 Geometrical depiction of a control volume $V^*(t)$ having a surface $A^*(t)$ that moves at a nonuniform velocity **b** during a small time increment Δt. When Δt is small enough, the volume increment $\Delta V = V^*(t+\Delta t) - V^*(t)$ will lie very near $A^*(t)$, so the volume-increment element adjacent to dA will be $(\mathbf{b}\Delta t)\cdot\mathbf{n}dA$ where **n** is the outward normal on $A^*(t)$.

吴望一（1982）从定义出发，推导了线、面、体元的随体导数 (1982a, pp. 135-138) 和体、面、线积分的随体导数 (1982b, pp. 479-484). 这些都是流体力学中十分基本且重要的结果.

从 RTT (3.35) 出发，还可以得到一些重要推论. 例如，在 (3.35) 中取 $F\equiv 1, |V^*|=\delta V\to 0^+$, 就得到

$$\lim_{\delta V\to 0^+}\frac{1}{\delta V}\frac{\mathrm{d}}{\mathrm{d}t}\delta V=\nabla\cdot\boldsymbol{b}. \tag{1}$$

利用（1），在（3.35）中取 $|V^*|=\delta V\to 0^+$, 就得到

$$\frac{\mathrm{d}F}{\mathrm{d}t}=\frac{\partial F}{\partial t}+(\boldsymbol{b}\cdot\nabla)F. \tag{2}$$

在式（1）中取 $\boldsymbol{b}=\boldsymbol{u}$, 便得到我们熟悉的速度散度的物理意义（相对体积膨胀率）. 在（2）中取 $\boldsymbol{b}=\boldsymbol{u}$, 便得到随体导数公式，或称“the *Eulerian representation of the Lagrangian derivative as applied to a field*.” (Vallis, 2017, p. 5)

关于 RTT (3.35)，受 Vallis (2017, pp. 3-7) 和吴望一 (1982a, pp. 135-138) 的启发，还可以给出如下的推导. 这种推导方法在数学上不严格，然而物理意义较清晰，便于记忆. 注意，我不认为这种方法属于我的原创.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V^*(t)}F\,\mathrm{d}V=\int_{V^*(t)}\frac{\mathrm{d}}{\mathrm{d}t}(F\,\delta V)=\int_{V^*(t)}\left(\frac{\mathrm{d}F}{\mathrm{d}t}+F\frac{1}{\delta V}\frac{\mathrm{d}\delta V}{\mathrm{d}t}\right)\delta V$$

$$= \int_{V^*(t)} \left(\frac{\mathrm{d}F}{\mathrm{d}t} + F\nabla \cdot \boldsymbol{b} \right) \mathrm{d}V = \int_{V^*(t)} \left(\frac{\partial F}{\partial t} + \nabla \cdot (\boldsymbol{b}F) \right) \mathrm{d}V$$

$$= \int_{V^*(t)} \frac{\partial F}{\partial t} \mathrm{d}V + \int_{A^*(t)} F\boldsymbol{b} \cdot \boldsymbol{n} \, \mathrm{d}S.$$

在推导中利用了（1）（2）.

1.1 问题描述

1. For a fluid volume, show that $\frac{\partial}{\partial t} \int_v \rho \, dV = \int_v \frac{\partial \rho}{\partial t} dV$.

1.2 解决方案

在 RTT（3.35）中取 $\boldsymbol{b} \equiv \boldsymbol{0}$ 立得

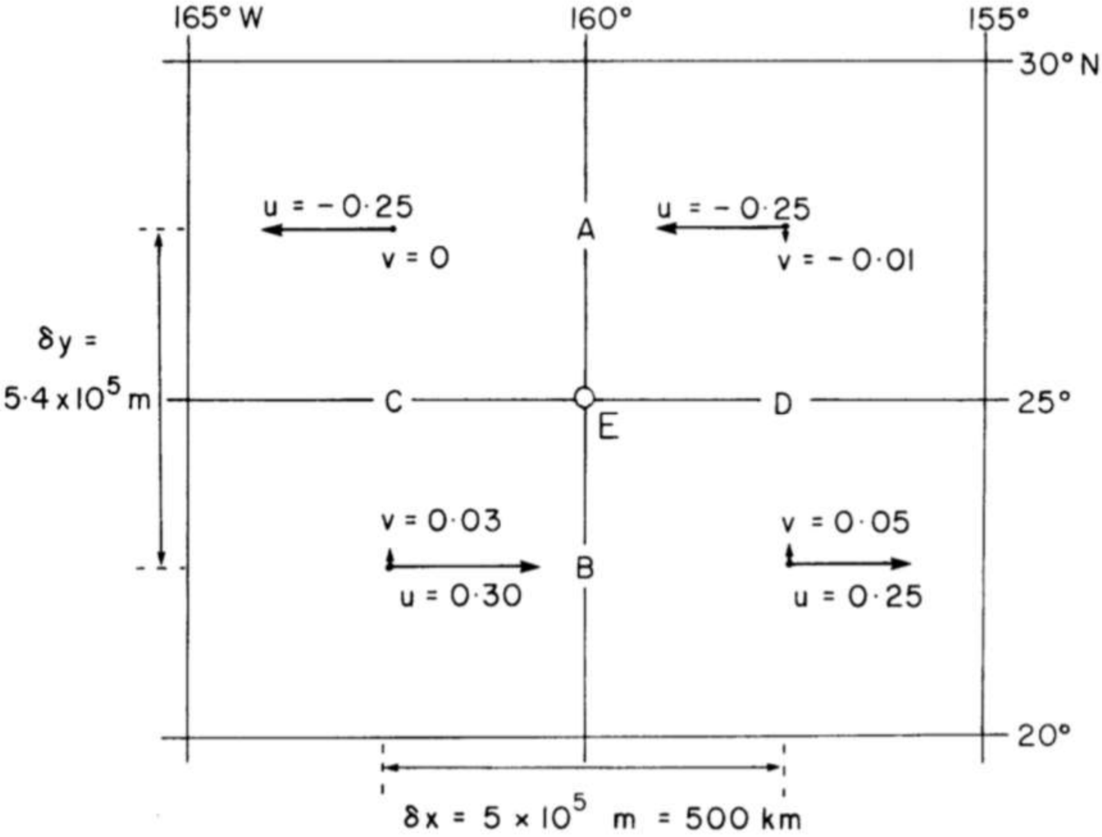
$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V_0} F \, \mathrm{d}V = \int_{V_0} \frac{\partial F}{\partial t} \mathrm{d}V. \tag{3}$$

此时，积分区域 V_0 是空间位置固定的**控制体** (Control Volume, CV). 事实上，上式就是定理 15.1.3 在三维情形下的推广.

2 第 2 题

2.1 问题描述

2. Use the following configuration for a domain in the ocean, derive the vertical velocity *w* at 50 m (assuming incompressible fluids and *w=0* at surface) based on the continuity equation. Hint: you can obtain the horizontal velocity at points A, B, C and D first, and then use these values to compute the horizonal divergence at point E.


165° W
160°
155°
30° N
u = −0·25
v = 0
A
u = −0·25
v = −0·01
δy =
5·4 x 10^5 m
C
E
D
25°
v = 0·03
u = 0·30
B
v = 0·05
u = 0·25
20°
δx = 5 × 10^5 m = 500 km

2.2 解决方案

3 第 3 题

3.1 问题描述

3. There are two sites in the ocean, A and B. The distributions of temperature (C°) and salinity (‰) with pressure (P, dbar) at these sites are shown in the following table.

	Site-A		Site-B	
P	S	T	S	T
0	35.10	28.50	33.50	2.50
20	34.99	28.45	33.50	3.74
40	34.88	28.35	34.25	4.02
60	34.78	24.55	34.55	4.10
80	34.68	22.75	34.65	4.15
100	34.60	20.55	34.74	4.20
200	34.45	15.50	34.90	4.30
250	34.35	13.00	35.10	4.35
500	34.25	6.58	35.23	4.25
1000	34.53	4.20	35.40	3.75

1) Calculate density with the linear Equation of State (EOS) as shown below, and plot the density profiles separately for A and B.

$$\rho = \rho_0 \left[1 - \beta_T (T - T_0) + \beta_S (S - S_0) + \beta_p (p - p_0)\right]$$

where $\rho_0 = 1027$ kg m^{-3}, β_T=0.15 kg m^{-3} C$^{\circ -1}$, β_S=0.78 kg m^{-3} ‰$^{-1}$, β_P=4.5 kg m^{-3} dbar^{-1}, and p_0=0 dbar。

2) Use the Thermodynamic Equation of Seawater – 2010 (TEOS2010) (matlab or python packages are available at Github, named as "Gibbs Sea Water (GSW)") to compute the density again, and plot the profiles on the figure drawn in 1) to make a comparison between the density distributions obtained from linear and nonlinear EOS.

3.2 解决方案

References

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附录 1