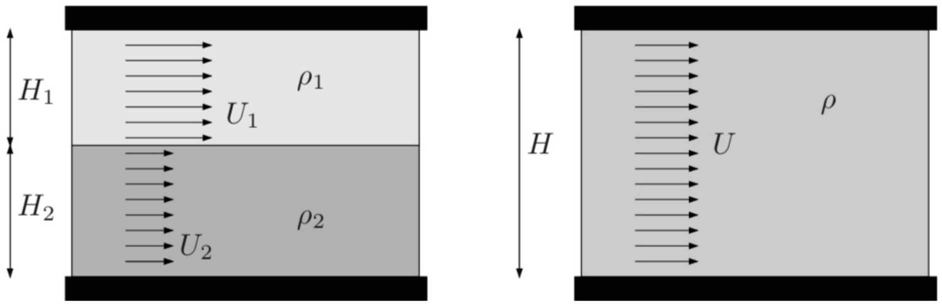
Mixing in stratified fluids

Assume unit length in the x and y directions:



$$PE \text{ gain} = \int_0^H \rho_{\text{final}}\, gz\, dz - \int_0^H \rho_{\text{initial}}\, gz\, dz$$

$$\left(\int_0^{H/2} \rho_2 gzdz + \int_{H/2}^{H} \rho_1 gzdz\right)$$

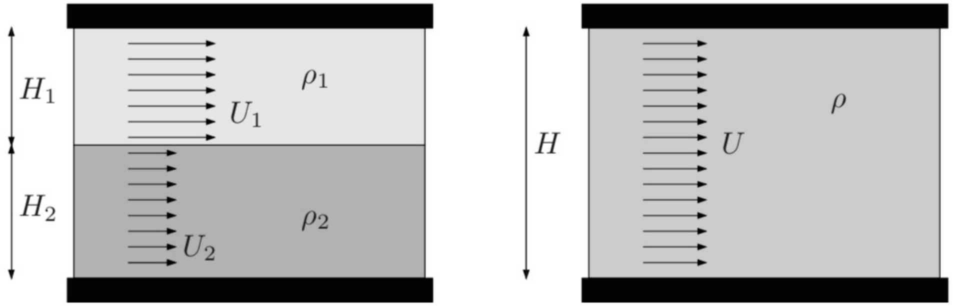
$$= \frac{1}{2}\rho g H^2 - \left[\frac{1}{2}\rho_2 g\frac{H^2}{4} + \frac{1}{2}\rho_1 g\frac{3H^2}{4}\right]$$

From mass conservation:

$$\rho_1\frac{H}{2} + \rho_2\frac{H}{2} = \rho H \quad \Rightarrow \quad \rho = \frac{1}{2}(\rho_1 + \rho_2)$$

$$PE \text{ gain} = \frac{1}{8}(\rho_2 - \rho_1) g H^2$$

For the KE change, ingore the density variation ($\rho = \rho_0$):



$$KE\,\text{loss} = \int_0^H \frac{1}{2}\rho_0 u_{\text{initial}}^2\,dz - \int_0^H \frac{1}{2}\rho_0 u_{\text{final}}^2\,dz$$

$$\left(\int_0^{H/2} \rho_0 {U_2}^2 dz + \int_{H/2}^{H} \rho_0 {U_1}^2 dz\right)$$

$$= \frac{1}{2}\rho_0 U_2^2 \frac{H}{2} + \frac{1}{2}\rho_0 U_1^2 \frac{H}{2} - \frac{1}{2}\rho_0 U^2 H$$

Complete vertical mixing is possible if KE loss is larger than PE gain:

From momentum conservation:

$$\rho_0 U_1 + \rho_0 U_2 = \rho_0 U$$

$$U = \frac{U_1 + U_2}{2}$$

$$KE\,\text{loss} = \frac{1}{8}\rho_0 (U_1 - U_2)^2 H.$$

$$PE\,\text{gain} = \frac{1}{8}(\rho_2 - \rho_1) g H^2$$

$$\frac{(\rho_2 - \rho_1) g H}{\rho_0 (U_1 - U_2)^2} < 1$$

$$\frac{\frac{g}{\rho_0}\frac{\Delta\rho}{H}}{\left(\frac{U_1 - U_2}{H}\right)^2} < 1$$

Richardon number $\frac{N^2}{\left(\frac{\partial u}{\partial z}\right)^2} < 1$

Kelvin-Helmholtz instability

Assumptions: stratified, inviscid, irrotational, two-dimensional (x-z) flow

The governing equations are:

$$
\begin{aligned}
\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + w\,\frac{\partial u}{\partial z} &= -\frac{1}{\rho_0}\,\frac{\partial p}{\partial x} \\
\frac{\partial w}{\partial t} + u\,\frac{\partial w}{\partial x} + w\,\frac{\partial w}{\partial z} &= -\frac{1}{\rho_0}\,\frac{\partial p}{\partial z} - \frac{\rho g}{\rho_0} \\
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0 \\
\frac{\partial \rho}{\partial t} + u\,\frac{\partial \rho}{\partial x} + w\,\frac{\partial \rho}{\partial z} &= 0
\end{aligned}
$$

At steady state:

$$u = \bar{u}(z),\, w = 0 \qquad \rho = \bar{\rho}(z) \qquad d\bar{p}/dz = -g\bar{\rho}(z)$$

After small perturbation:

$$u = \bar{u} + u', \qquad w = w', \qquad p = \bar{p} + p' \qquad \rho = \bar{\rho} + \rho'$$

The equations become:

$$\frac{\partial(\cancel{\bar{u}} + u')}{\partial t} + (\bar{u} + \cancel{u'})\frac{\partial(\cancel{\bar{u}} + u')}{\partial x} + w'\frac{\partial(\bar{u} + \cancel{u'})}{\partial z} = -\frac{1}{\rho_0}\frac{\partial(\cancel{\bar{p}} + p')}{\partial x}$$

$$\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} + w'\frac{d\bar{u}}{dz} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$

$$\frac{\partial w'}{\partial t} + \bar{u}\frac{\partial w'}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\rho' g}{\rho_0}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0$$

$$\frac{\partial \rho'}{\partial t} + \bar{u}\frac{\partial \rho'}{\partial x} + w'\frac{d\bar{\rho}}{dz} = 0$$

perturbation streamfunction:

$$u' = -\frac{\partial \psi}{\partial z} \qquad w' = \frac{\partial \psi}{\partial x}$$

$$\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} + w'\frac{d\bar{u}}{dz} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} \quad (1)$$

$$\frac{\partial w'}{\partial t} + \bar{u}\frac{\partial w'}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\rho' g}{\rho_0} \quad (2)$$

$\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial z}(1)$:

$$\frac{\partial}{\partial t}\nabla^2\psi + \bar{u}\frac{\partial}{\partial x}\nabla^2\psi - \frac{d^2\bar{u}}{dz^2}\frac{\partial\psi}{\partial x} = -\frac{g}{\rho_0}\frac{\partial\rho'}{\partial x}$$

$$\frac{\partial\rho'}{\partial t} + \bar{u}(z)\frac{\partial\rho'}{\partial x} + \frac{d\bar{\rho}}{dz}\frac{\partial\psi}{\partial x} = 0.$$

$$\boxed{\frac{\partial\rho'}{\partial t} + \bar{u}\frac{\partial\rho'}{\partial x} + w'\frac{d\bar{\rho}}{dz} = 0}$$

Apply a wave solution: $\psi = \Psi(z)\,\mathrm{e}^{i(kx-\omega t)}$ $\quad \rho' = R(z)\,\mathrm{e}^{i(kx-\omega t)}$

$$(-i\omega)[(ik)^2\Psi + \frac{d^2\Psi}{dz^2}] + \bar{u}ik[(ik)^2\Psi + \frac{d^2\Psi}{dz^2}] - \frac{d^2\bar{u}}{dz^2}ik\Psi = -\frac{g}{\rho_0}ikR$$

Divide the equation by ik:

$$-c(\frac{d^2\Psi}{dz^2} - k^2\Psi) + \bar{u}(\frac{d^2\Psi}{dz^2} - k^2\Psi) - \frac{d^2\bar{u}}{dz^2}\Psi = -\frac{g}{\rho_0}R$$

For the density equation:

$$(-i\omega)R + \bar{u}ik\mathrm{R} + \frac{d\bar{\rho}}{dz}ik\Psi = 0$$

$$(\bar{u} - c)\mathrm{R} + \frac{d\bar{\rho}}{dz}\Psi = 0$$

$$(\bar{u}-c)\left(\frac{d^2\Psi}{dz^2} - k^2\Psi\right) - \frac{d^2\bar{u}}{dz^2}\Psi = -\frac{g}{\rho_0}R$$

$$(\bar{u}-c)\,R + \frac{d\bar{\rho}}{dz}\Psi = 0$$

$c = c_r + ic_i$

$\psi = \Psi(z)e^{ik(x-c_rt-ic_it)}$
$= \Psi\,(z)e^{kc_it}e^{ik(x-c_rt)}$

Eliminate R:

$$(\bar{u}-c)\left(\frac{d^2\Psi}{dz^2} - k^2\Psi\right) + \left(\frac{N^2}{\bar{u}-c} - \frac{d^2\bar{u}}{dz^2}\right)\Psi = 0$$

Taylor–Goldstein equation

Let $\phi = \dfrac{\Psi}{\sqrt{\bar{u}-c}}$:

$$\frac{d}{dz}\left[(\bar{u}-c)\frac{d\phi}{dz}\right] + \left[k^2(\bar{u}-c) + \frac{1}{2}\frac{d^2\bar{u}}{dz^2} + \frac{(d\bar{u}/dz)^2 - 4N^2}{4(\bar{u}-c)}\right]\phi = 0$$

Mutiply the equation by ϕ^* and vertically integrate it across the domain:

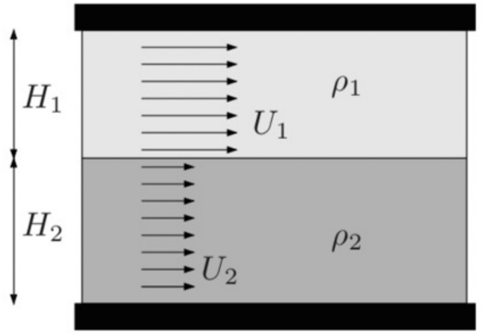
$$\int(\bar{u}-c)\left(\left|\frac{d\phi}{dz}\right|^2 + k^2|\phi|^2\right)dz + \frac{1}{2}\int\frac{d^2\bar{u}}{dz^2}|\phi|^2\,dz$$
$$+\frac{1}{4}\int\frac{(d\bar{u}/dz)^2 - 4N^2}{(\bar{u}-c)}|\phi|^2\,dz = 0.$$

The imaginary part is:

$$> 0$$

$$c_i \int \left(\left| \frac{d\phi}{dz} \right|^2 + k^2 |\phi|^2 \right) dz = \frac{c_i}{4} \int \frac{(d\bar{u}/dz)^2 - 4N^2}{|\bar{u} - c|^2} |\phi|^2 \, dz$$

The necessary condition for instability to occur ($c_i \neq 0$) is:



Richardson number $\quad Ri = \dfrac{N^2}{(d\bar{u}/dz)^2} < 1/4$

For a two-layer system:

$$\left| \frac{d\bar{u}}{dz} \right| = \frac{|U_1 - U_2|}{H} \qquad N^2 = -\frac{g}{\rho_0} \frac{d\rho}{dz} = \frac{g}{\rho_0} \frac{\rho_2 - \rho_1}{H}$$

$$\frac{(\rho_2 - \rho_1) g H}{\rho_0 (U_1 - U_2)^2} < \frac{1}{4}$$

Ri is essentially the ratio of the potential energy barrier that mixing must overcome to the kinetic energy that the sheared flow must supply

Kelvin-Helmholtz Instability

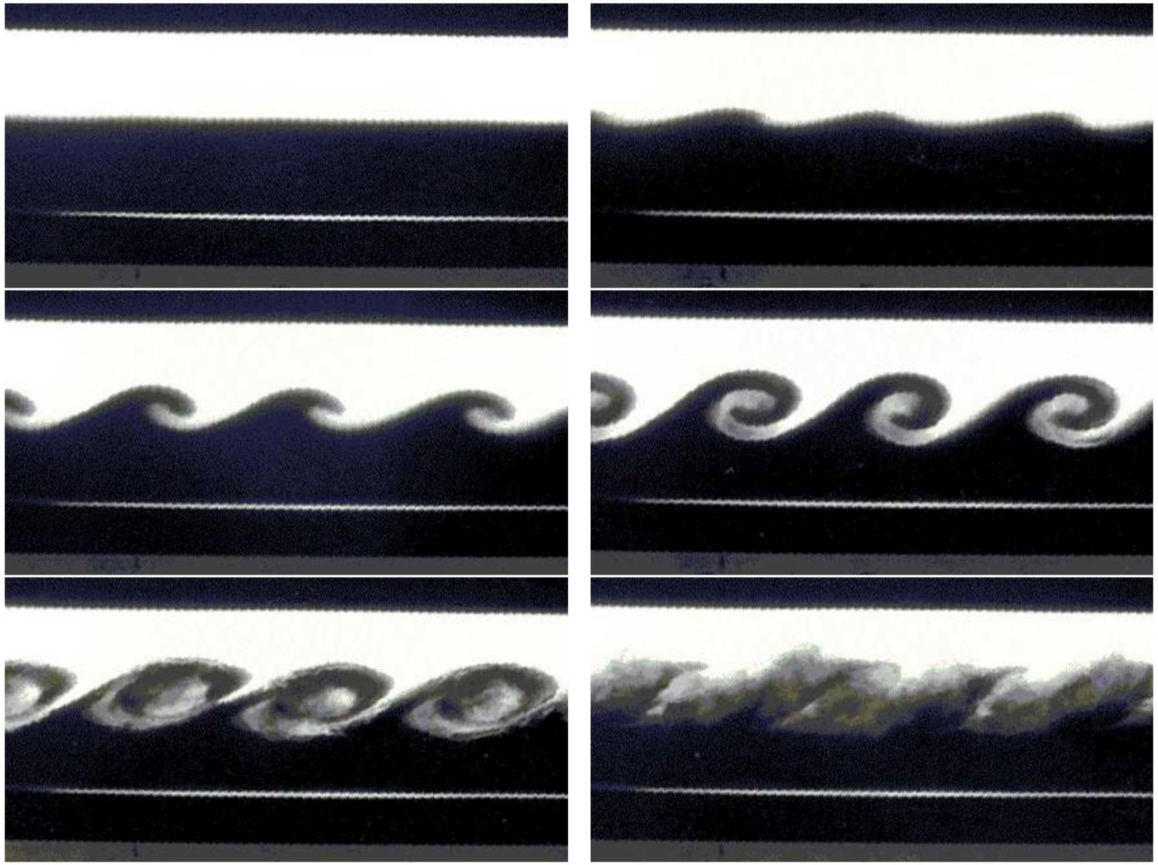
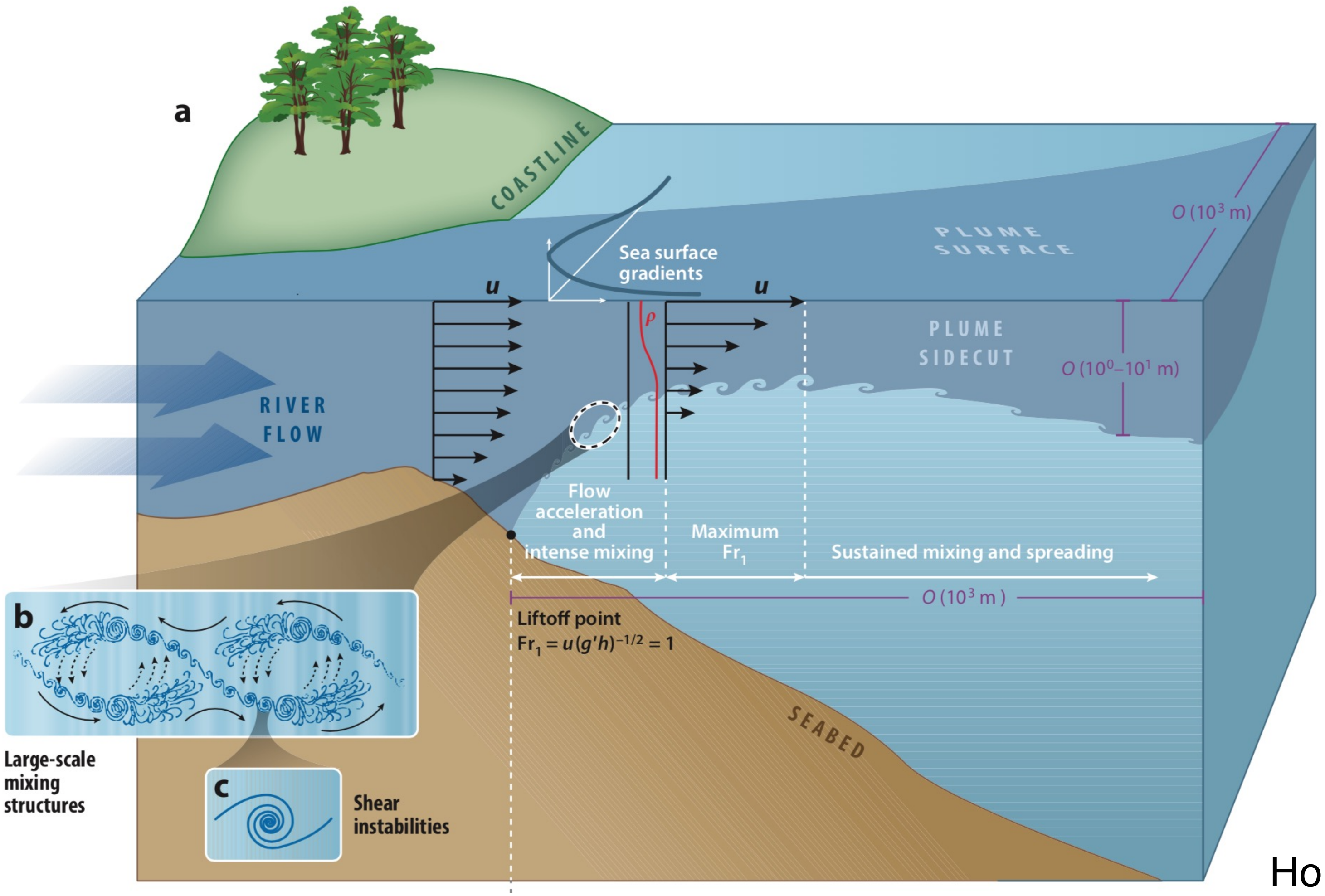


Figure 14-4 Kelvin–Helmholtz instability generated in a laboratory with fluids of two different densities and colors. (*Adapted from GFD-online, Satoshi Sakai, Isawo Iizawa, Eiji Aramaki*)

Kelvin-Helmholtz instability in estuaries



Horner-Devine et al. (2014)

Layered models

In a stratified system, as density increases monotonically downward, it can be used as the vertical coordinate

$$\rho(x,y,z,t) \longrightarrow z(x,y,\rho,t)$$

$$a = a(x,y,\rho(x,y,z,t),t)$$

$$\frac{\partial}{\partial x} \longrightarrow \left.\frac{\partial a}{\partial x}\right|_z = \left.\frac{\partial a}{\partial x}\right|_\rho + \frac{\partial a}{\partial \rho}\left.\frac{\partial \rho}{\partial x}\right|_z$$

$$\frac{\partial}{\partial y} \longrightarrow \left.\frac{\partial a}{\partial y}\right|_z = \left.\frac{\partial a}{\partial y}\right|_\rho + \frac{\partial a}{\partial \rho}\left.\frac{\partial \rho}{\partial y}\right|_z$$

$$\frac{\partial}{\partial z} \longrightarrow \frac{\partial a}{\partial z} = \frac{\partial a}{\partial \rho}\frac{\partial \rho}{\partial z}$$

$$\frac{\partial}{\partial t} \longrightarrow \left.\frac{\partial a}{\partial t}\right|_z = \left.\frac{\partial a}{\partial t}\right|_\rho + \frac{\partial a}{\partial \rho}\left.\frac{\partial \rho}{\partial t}\right|_z$$

If $a = z$

$$0 = z_x + z_\rho \rho_x$$
$$1 = z_\rho \rho_z$$
$$0 = z_t + z_\rho \rho_t$$

$$\left.\frac{\partial a}{\partial x}\right|_z = \left.\frac{\partial a}{\partial x}\right|_\rho - \frac{z_x}{z_\rho}\frac{\partial a}{\partial \rho}$$

$$\frac{\partial a}{\partial z} = \frac{1}{z_\rho}\frac{\partial a}{\partial \rho}$$

$$\frac{\partial a}{\partial t}\Big|_z = \frac{\partial a}{\partial t}\Big|_\rho - \frac{z_t}{z_\rho}\frac{\partial a}{\partial \rho}$$

Hydrostatic balance: $$\frac{\partial p}{\partial z} = -\rho g$$

$$\boxed{\frac{\partial a}{\partial z} = \frac{1}{z_\rho}\frac{\partial a}{\partial \rho}}$$

$$\boxed{\left.\frac{\partial a}{\partial x}\right|_z = \left.\frac{\partial a}{\partial x}\right|_\rho - \frac{z_x}{z_\rho}\frac{\partial a}{\partial \rho}}$$

$$\frac{\partial p}{\partial \rho} = -\rho g \frac{\partial z}{\partial \rho}$$

$$\boxed{\frac{\partial a}{\partial t}\Big|_z = \frac{\partial a}{\partial t}\Big|_\rho - \frac{z_t}{z_\rho}\frac{\partial a}{\partial \rho}}$$

$$\left.\frac{\partial p}{\partial x}\right|_z = \left.\frac{\partial p}{\partial x}\right|_\rho - \frac{z_x}{z_\rho}\frac{\partial p}{\partial \rho} = \left.\frac{\partial p}{\partial x}\right|_\rho + \rho g\frac{\partial z}{\partial x} = \left.\frac{\partial P}{\partial x}\right|_\rho$$

$$\boxed{P = p + \rho g z}$$

$$\frac{\partial P}{\partial \rho} = \frac{\partial p}{\partial \rho} + gz + \rho g\frac{\partial z}{\partial \rho} = gz$$

Let $a = \rho$:

$$\frac{\partial \rho}{\partial x}\Big|_z = -\frac{z_x}{z_\rho} \qquad \frac{\partial \rho}{\partial y}\Big|_z = -\frac{z_y}{z_\rho} \qquad \frac{\partial \rho}{\partial z}\Big|_z = \frac{1}{z_\rho} \qquad \frac{\partial \rho}{\partial t}\Big|_z = -\frac{z_t}{z_\rho}$$

For incompressible fluid: $$\frac{d\rho}{dt} = 0$$

z-coordinate

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = 0$$

ρ-coordinate

$$\Rightarrow \quad -z_t - uz_x - vz_y + w = 0 \quad \Rightarrow \quad \frac{dz}{dt} = z_t + uz_x + vz_y$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

There is no across-isopycnal advection, and the model is ideal for studying along-isopycnal processes

Without friction, the momentum equations in density coordinate become:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -\frac{1}{\rho_0}\frac{\partial P}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -\frac{1}{\rho_0}\frac{\partial P}{\partial y}$$

$$\frac{\partial P}{\partial \rho} = gz$$

$$\frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} + \frac{\partial hv}{\partial y} = 0$$

$$h = -\Delta\rho\frac{\partial z}{\partial \rho}$$

the thickness of a fluid layer between ρ and $\rho + \Delta\rho$