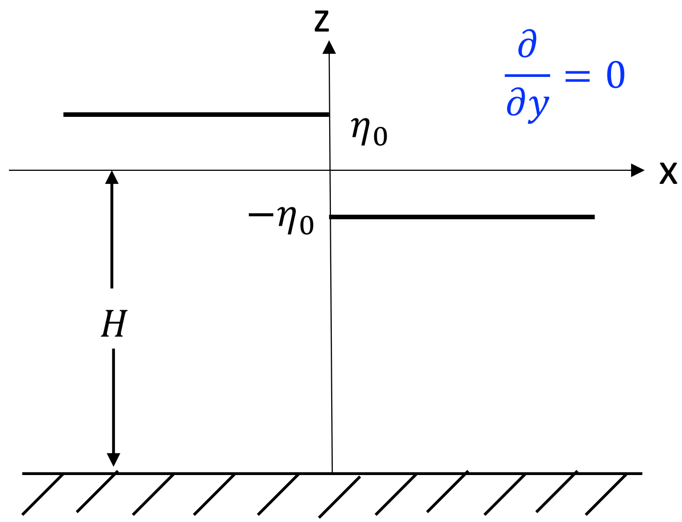
Barotropic geostrophic adjustment

Initial state (unbalanced):

$$\eta = \begin{cases} \eta_0, & x < 0 \\ -\eta_0, & x > 0 \end{cases}$$

η_0<<H

$$u = v = 0$$



$$\frac{\partial}{\partial y} = 0$$

For a steady final state (based on shallow water model):

$$\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}} - fv = -g\frac{\partial \eta}{\partial x}$$

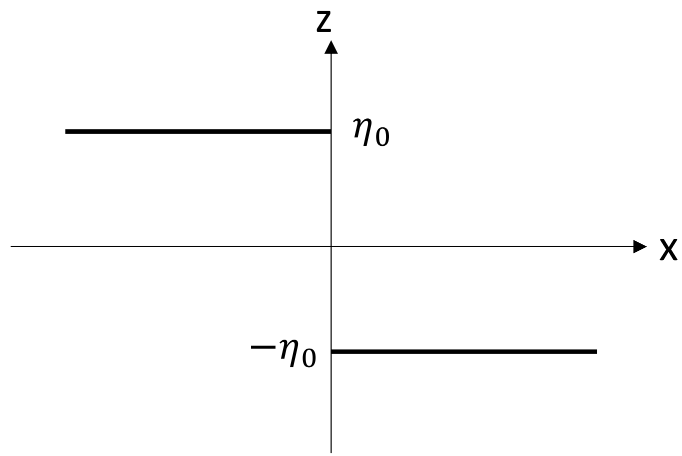
$$\cancel{\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial x}} + \cancel{v\frac{\partial v}{\partial y}} + fu = -g\frac{\partial \eta}{\partial y}$$

geostrophic flow

$$\cancel{\frac{\partial \eta}{\partial t}} + \frac{\partial hu}{\partial x} + \cancel{\frac{\partial hv}{\partial y}} = 0$$

sufficient condition: u=0 everywhere

For PV conservation:



x < 0:

$$\frac{f}{H+\eta_0} = \frac{f+\frac{\partial v}{\partial x}}{H+\eta} = \frac{f+\frac{g}{f}\frac{\partial^2\eta}{\partial x^2}}{H+\eta}$$

η_0<<H

$$\cancel{fH} + f\eta = \cancel{fH} + f\eta_0 + \frac{gH}{f}\frac{\partial^2\eta}{\partial x^2} + \cancel{\frac{g\eta_0}{f}\frac{\partial^2\eta}{\partial x^2}}$$

$$R^2\frac{\partial^2\eta}{\partial x^2} - \eta = -\eta_0$$

$$R = \frac{\sqrt{gH}}{f}$$

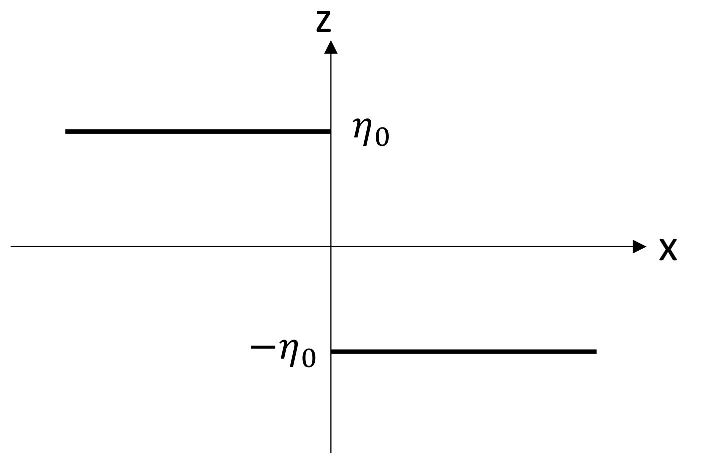
$$\eta = \mathrm{f(x)} + \eta_0$$

$$e^{-\frac{1}{R}x} \to \infty \; as \; x \to -\infty$$

$$\mathrm{f(x)} = \mathrm{C}e^{\lambda x} \Longrightarrow R^2\lambda^2 - 1 = 0 \Longrightarrow \lambda = \pm\frac{1}{R} \Longrightarrow \lambda = \frac{1}{R}$$

$$\eta = Ce^{\frac{x}{R}} + \eta_0 \xrightarrow{x=0,\,\eta=0,\,\mathrm{C}=-\eta_0} \eta = -\eta_0 e^{\frac{x}{R}} + \eta_0$$

For PV conservation:



x > 0:

$$\frac{f}{H-\eta_0} = \frac{f+\frac{\partial v}{\partial x}}{H+\eta} = \frac{f+\frac{g}{f}\frac{\partial^2\eta}{\partial x^2}}{H+\eta}$$

$$\cancel{fH} + f\eta = \cancel{fH} - f\eta_0 + \frac{gH}{f}\frac{\partial^2\eta}{\partial x^2} - \cancel{\frac{g\eta_0}{f}\frac{\partial^2\eta}{\partial x^2}}$$

$$R^2\frac{\partial^2\eta}{\partial x^2} - \eta = \eta_0$$

$$\eta = \mathrm{f(x)} - \eta_0$$

$$e^{\frac{1}{R}x} \rightarrow \infty \ as\ x \rightarrow \infty$$

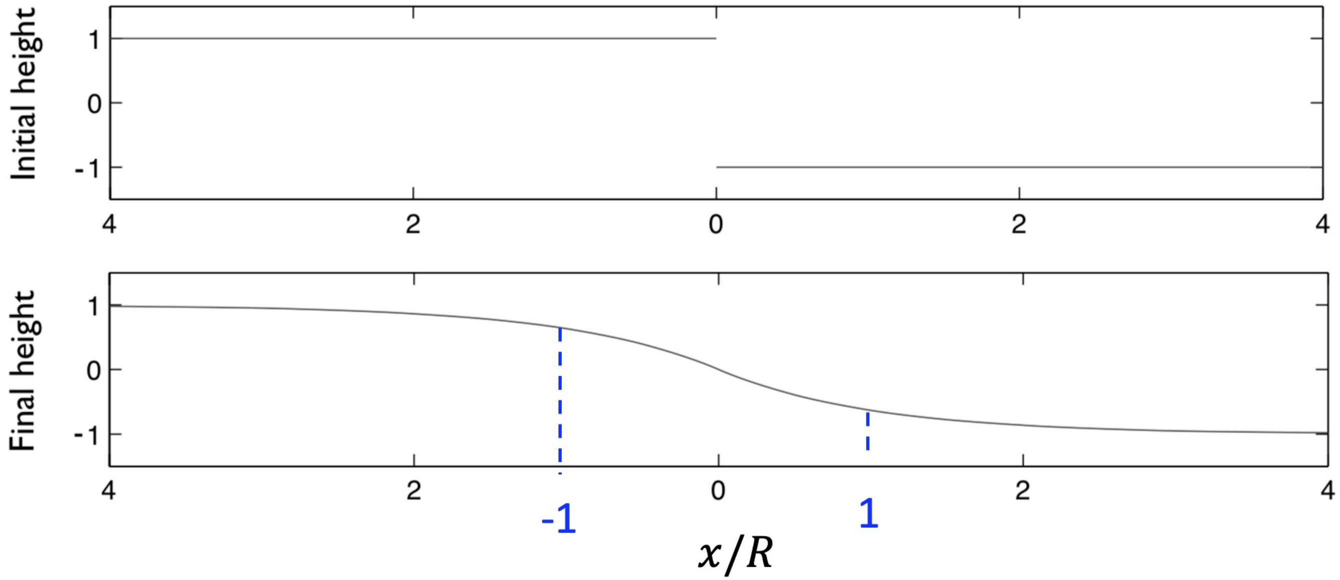
$$\mathrm{f(x)} = \mathrm{C}e^{\lambda x} \Rightarrow R^2\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm\frac{1}{R} \Rightarrow \lambda = -\frac{1}{R}$$

$$\eta = Ce^{-\frac{x}{R}} - \eta_0 \xrightarrow{x=0,\eta=0,\mathrm{C}=\eta_0} \eta = \eta_0 e^{-\frac{x}{R}} - \eta_0$$

$x \to -R, \quad \eta \to \eta_0$

$$\eta = \begin{cases} -\eta_0 e^{\frac{x}{R}} + \eta_0, & x < 0 \\ \eta_0 e^{-\frac{x}{R}} - \eta_0, & x > 0 \end{cases}$$

$x \to R, \quad \eta \to -\eta_0$

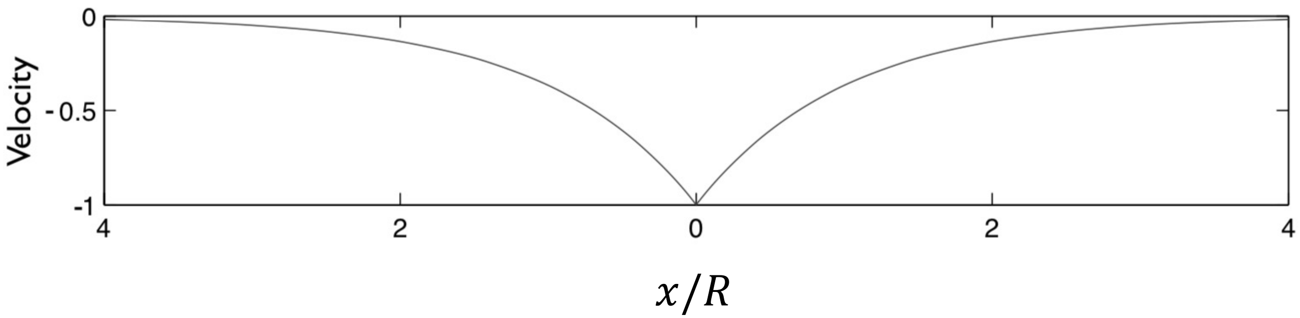


The adjustment spatial scale is the Rossby deformation radius

$x \to -R, \quad v \to 0$

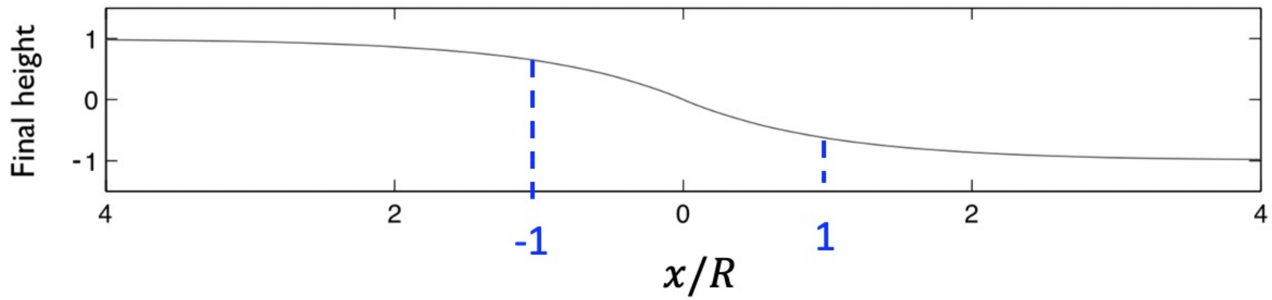
$$v = \frac{g}{f}\frac{\partial \eta}{\partial x} = \begin{cases} -\sqrt{\frac{g}{H}}\,\eta_0 e^{\frac{x}{R}}, & x < 0 \\ -\sqrt{\frac{g}{H}}\,\eta_0 e^{-\frac{x}{R}}, & x > 0 \end{cases}$$

$x \to R, \quad v \to 0$



Energetics of geostrophic adjustment

For unit length in the y-direction:



Potential energy:

PE change is limited to $(-R\!: R,\ -\eta_0\!: \eta_0)$

$$PE_I = \int_0^{\eta_0}\int_{-R}^{0}\rho_0 gz\,dx\,dz + \int_0^{-\eta_0}\int_0^{R}\rho_0 gz\,dx\,dz = \frac{1}{2}\rho_0 g\mathrm{R}{\eta_0}^2 + \frac{1}{2}\rho_0 g\mathrm{R}{\eta_0}^2 = \rho_0 g\mathrm{R}{\eta_0}^2$$

$$PE_F = \int_0^{\eta}\int_{-R}^{0}\rho_0 gz\,dx\,dz + \int_0^{\eta}\int_0^{R}\rho_0 gz\,dx\,dz$$

$$\boxed{\eta = \begin{cases} -\eta_0 e^{\frac{x}{R}} + \eta_0, & x < 0 \\ \eta_0 e^{-\frac{x}{R}} - \eta_0, & x > 0 \end{cases}}$$

$$= \frac{1}{2}\rho_0 g{\eta_0}^2\int_{-R}^{0}(1 - e^{\frac{x}{R}})^2 dx + \frac{1}{2}\rho_0 g{\eta_0}^2\int_0^{R}(1 - e^{-\frac{x}{R}})^2 dx$$

$$\Delta PE = -\frac{3}{2}\rho_0 gR{\eta_0}^2$$

$$= \frac{1}{2}\rho_0 g{\eta_0}^2\left(x|_{-R}^{0} - 2Re^{\frac{x}{R}}|_{-R}^{0} + \frac{R}{2}e^{\frac{2x}{R}}|_{-R}^{0} + x|_0^R + 2Re^{-\frac{x}{R}}|_0^R - \frac{R}{2}e^{-\frac{2x}{R}}|_0^R\right) = -\frac{1}{2}\rho_0 gR{\eta_0}^2$$

Kinetic energy:

$$KE_I = 0$$

$$v = \begin{cases} -\sqrt{\frac{g}{H}}\eta_0 e^{\frac{x}{R}}, & x < 0 \\ -\sqrt{\frac{g}{H}}\eta_0 e^{-\frac{x}{R}}, & x > 0 \end{cases}$$

$$KE_F = \int_{-H}^{0}\int_{-R}^{R}\frac{1}{2}\rho_0 v^2 dxdz$$

$$= \int_{-H}^{0}\int_{-R}^{0}\frac{1}{2}\rho_0{\eta_0}^2\frac{g}{H}e^{\frac{2x}{R}}dxdz + \int_{-H}^{0}\int_{0}^{R}\frac{1}{2}\rho_0{\eta_0}^2\frac{g}{H}e^{-\frac{2x}{R}}dxdz$$

$$= \frac{1}{2}\rho_0{\eta_0}^2 g(\int_{-R}^{0}e^{\frac{2x}{R}}dx + \int_0^R e^{-\frac{2x}{R}}dx)$$

$$= \frac{1}{2}\rho_0{\eta_0}^2 g(\frac{R}{2}e^{\frac{2x}{R}}|_{-R}^{0} - \frac{R}{2}e^{-\frac{2x}{R}}|_0^R)$$

$$= \frac{1}{2}\rho_0 gR{\eta_0}^2$$

$$\Delta PE = -\frac{3}{2}\rho_0 gR{\eta_0}^2$$

$$\frac{\Delta KE}{|\Delta PE|} = 1/3$$

Only 1/3 of released PE is converted into KE, and the rest of lost PE is radiated away by waves

inertial-gravity waves