Shallow water waves

Linear wave dynamics

Assumption: $R_O \ll 1$

c: wave speed

$$R_{OT} = \frac{\frac{U}{T}}{fU} = \frac{1}{fT} \sim \frac{1}{fL/c} = \frac{c}{fL} \,(c \gg U) \sim 1$$

The horizontal momentum equations are:

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$

The continuity equation is:

$$\frac{\partial \eta}{\partial t} + \frac{\partial hu}{\partial x} + \frac{\partial hv}{\partial y} = 0$$

$$\frac{\partial \eta}{\partial t} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} = 0$$

For flat bottom, $\eta = h - H$:

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \eta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} = 0$$

$$\boxed{\Delta H\frac{c}{L}} \qquad \frac{\Delta H}{T} \qquad H\frac{U}{L} \qquad \cancel{\Delta H\frac{U}{L}} \qquad \cancel{U\frac{\Delta H}{L}}$$

$$\Delta H\frac{c}{L} \sim H\frac{U}{L}$$

Then the continuity equation is reduced to:

$$\Delta H \ll H$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

small-amplitude waves

Inertia-gravity waves (Poincaré waves)

Assumption: flat bottom

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

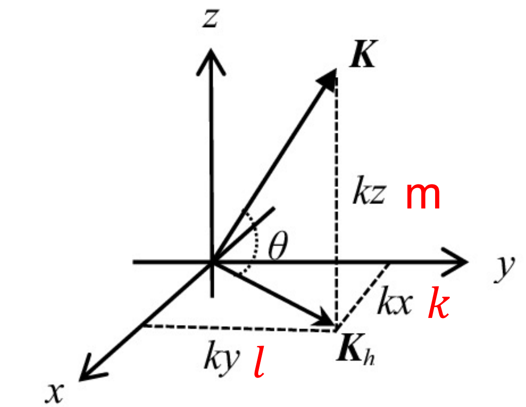


Fig. 1. Wave number vector and its components.

Give a wave solution:

$$u = Ue^{i(kx+ly-\omega t)}$$

$$v = Ve^{i(kx+ly-\omega t)}$$

$$\eta = Ae^{i(kx+ly-\omega t)}$$

$$-i\omega U - fV = -gikA$$

$$-i\omega V + fU = -gilA$$

$$-i\omega A + ikHU + ilHV = 0$$

$$\begin{pmatrix} -i\omega & -f & gik \\ f & -i\omega & gil \\ ikH & ilH & -i\omega \end{pmatrix} \begin{bmatrix} U \\ V \\ A \end{bmatrix} = 0$$

This homogeneous equation has non-trivial solutions only if the determinant of the coefficient matrix vanishes:

$$\omega[\omega^2 - f^2 - gH(k^2 + l^2)] = 0$$ dispersion relation

$$\boxed{R_d = \frac{\sqrt{gH}}{f}}$$

Rossby deformation radius (barotropic)

1. $\omega = 0, \ \dfrac{\partial}{\partial t} = 0,$ geostrophic flow

2. $\omega = \sqrt{f^2 + gH\underbrace{(k^2 + l^2)}_{K^2}}$

a. rotation is weak, $f^2 \ll gHK^2, \ \ \lambda \ll R_d$ (short-wave limit)

$$\omega = \sqrt{gH}K, \ \ c = \sqrt{gH},$$ gravity waves

b. rotation is important, $f^2 \gg gHK^2, \ \ \lambda \gg R_d$ (long-wave limit)

$K\ (k, l)\ is\ small$

pressure gradient term is negliable, equations reduced to inertial-motion

$\omega \sim f,$ inertial oscillations

Dispersion relation diagram

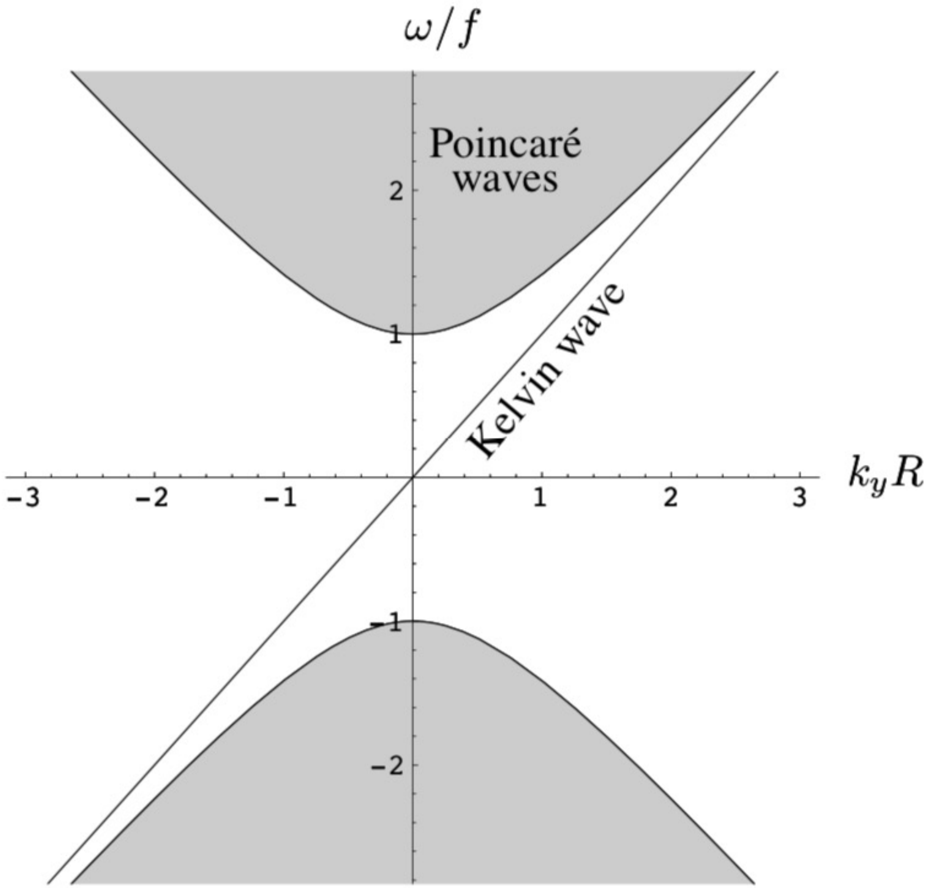


Figure 9-3 Recapitulation of the dispersion relation of Kelvin and Poincaré waves on the f-plane and on a flat bottom. While Poincaré waves (gray shades) can travel in all directions and occupy therefore a continuous spectrum in terms of k_y, the Kelvin wave (diagonal line) propagates only along a boundary.

Kelvin wave

Assumptions: flat bottom

one side boundary (y-axis)

velocity normal to the boundary is zero everywhere (u=0)

The momentum equations:

geostrophic flow

$$\cancel{\frac{\partial u}{\partial t}} - fv = -g\frac{\partial \eta}{\partial x} \quad (1)$$

$$\frac{\partial v}{\partial t} + \cancel{fu} = -g\frac{\partial \eta}{\partial y} \quad (2) \Rightarrow \frac{\partial^2 v}{\partial t^2} = -g\frac{\partial}{\partial y}\left(\frac{\partial \eta}{\partial t}\right) = \underbrace{gH}_{c^2}\frac{\partial^2 v}{\partial y^2}$$

The continuity equation:

$$v = F_1(x)e^{i(ly-\omega t)} + F_2(x)e^{i(ly+\omega t)}$$

$$\frac{\partial \eta}{\partial t} + H\left(\cancel{\frac{\partial u}{\partial x}} + \frac{\partial v}{\partial y}\right) = 0 \quad (3)$$

$$v = F_1(x)e^{i(ly-\omega t)} + F_2(x)e^{i(ly+\omega t)}$$

$$\frac{\partial v}{\partial t} = -g\frac{\partial \eta}{\partial y} \quad (2)$$

$$\eta = -\frac{1}{g}\int \frac{\partial v}{\partial t}dy = -\frac{1}{g}\int [-i\omega F_1(x)e^{i(ly-\omega t)} + i\omega F_2(x)e^{i(ly+\omega t)}]dy$$

$$= \frac{\omega}{g}\int [F_1(x)e^{i(ly-\omega t)} - F_2(x)e^{i(ly+\omega t)}]d(iy)$$

$$= \frac{\omega}{gl}\int [F_1(x)e^{i(ly-\omega t)} - F_2(x)e^{i(ly+\omega t)}]d(ily)$$

$$c = \frac{\omega}{l} = \sqrt{gH} \qquad = \frac{\omega}{gl}[F_1(x)e^{i(ly-\omega t)} - F_2(x)e^{i(ly+\omega t)}]$$

$$\sqrt{\frac{H}{g}}$$

$$v = F_1(x)e^{i(ly-\omega t)} + F_2(x)e^{i(ly+\omega t)}$$

$$\eta = \sqrt{\frac{H}{g}}\left[F_1(x)e^{i(ly-\omega t)} - F_2(x)e^{i(ly+\omega t)}\right]$$

$$fv = g\frac{\partial \eta}{\partial x} \qquad (1)$$

toward the open ocean $(x \to \infty), v \to \infty$

$$\boxed{R_d = \frac{\sqrt{gH}}{f}}$$

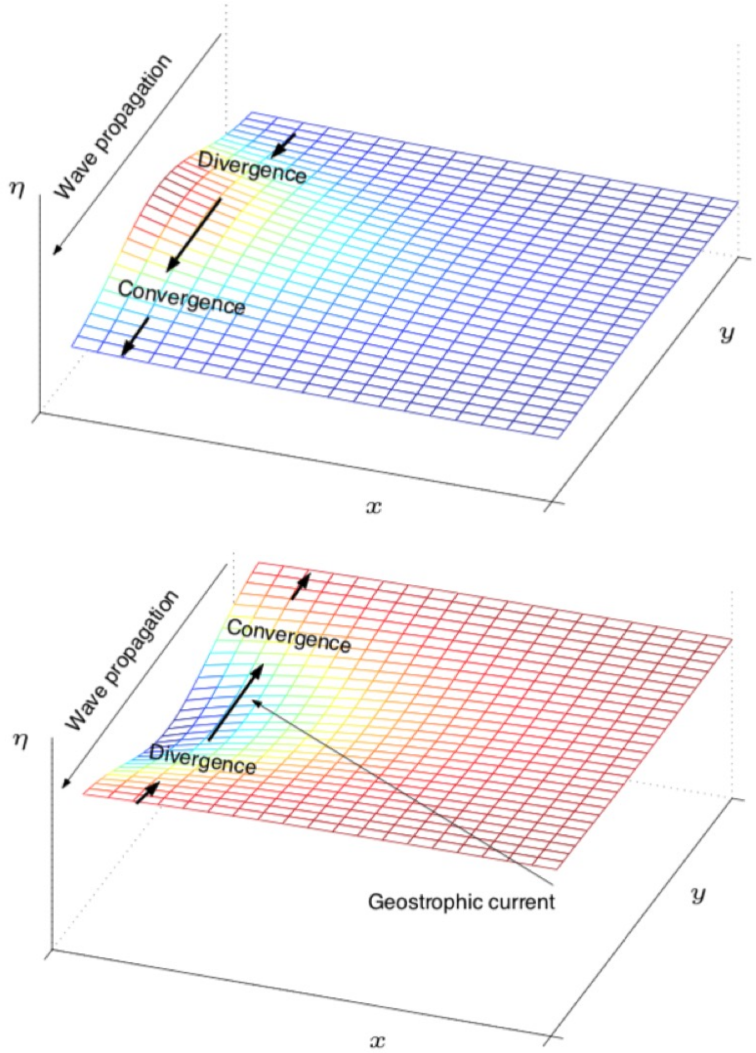
Rossby deformation radius (barotropic)

$$\frac{\partial F_1(x)}{\partial x} - \frac{f}{\sqrt{gH}}F_1(x) = 0 \Rightarrow \cancel{F_1(x) = C_1 e^{x/R_d}}$$

$$\frac{\partial F_2(x)}{\partial x} + \frac{f}{\sqrt{gH}}F_2(x) = 0 \Rightarrow F_2(x) = C_2 e^{-x/R_d}$$

$$v = Ae^{-x/R_d}e^{i(ly+\omega t)}$$

$$\eta = -A\sqrt{\frac{H}{g}}e^{-x/R_d}e^{i(ly+\omega t)}$$



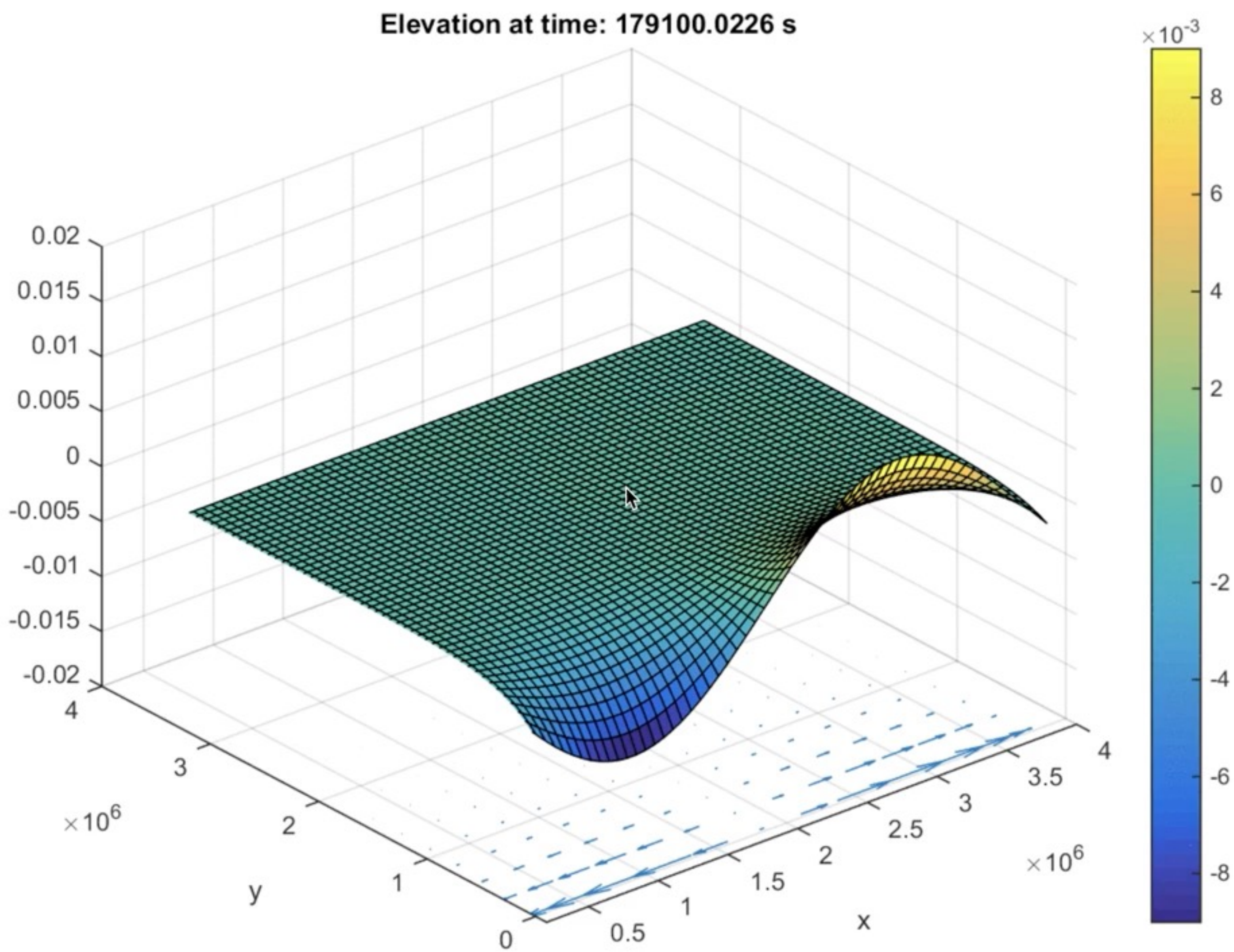
$$u = 0$$

$$v = Ae^{-x/R_d}e^{i(ly+\omega t)}$$

$$\eta = -A\sqrt{\frac{H}{g}}\,e^{-x/R_d}e^{i(ly+\omega t)}$$

- Kelvin waves propagate with the boundary on the right (left) in the Northern (Southern) Hemisphere (homework)
- The wave speed is gravity wave speed ($c = \sqrt{gH}$)
- Velocity perpendicular to the boundary is nail; along-boundary flow is geostrophic
- Surface elevation and along-boundary velocity decay from the boundary to the interior ocean, and the decay scale is R_d (trapped wave)

An upwelling wave ($\eta > 0$) has currents flowing in the direction of wave propagation ($v < 0$)


Elevation at time: 179100.0226 s
0.02
0.015
0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
y
×10^6
x
×10^6
×10^-3
8
6
4
2
0
-2
-4
-6
-8

Dispersion relation diagram

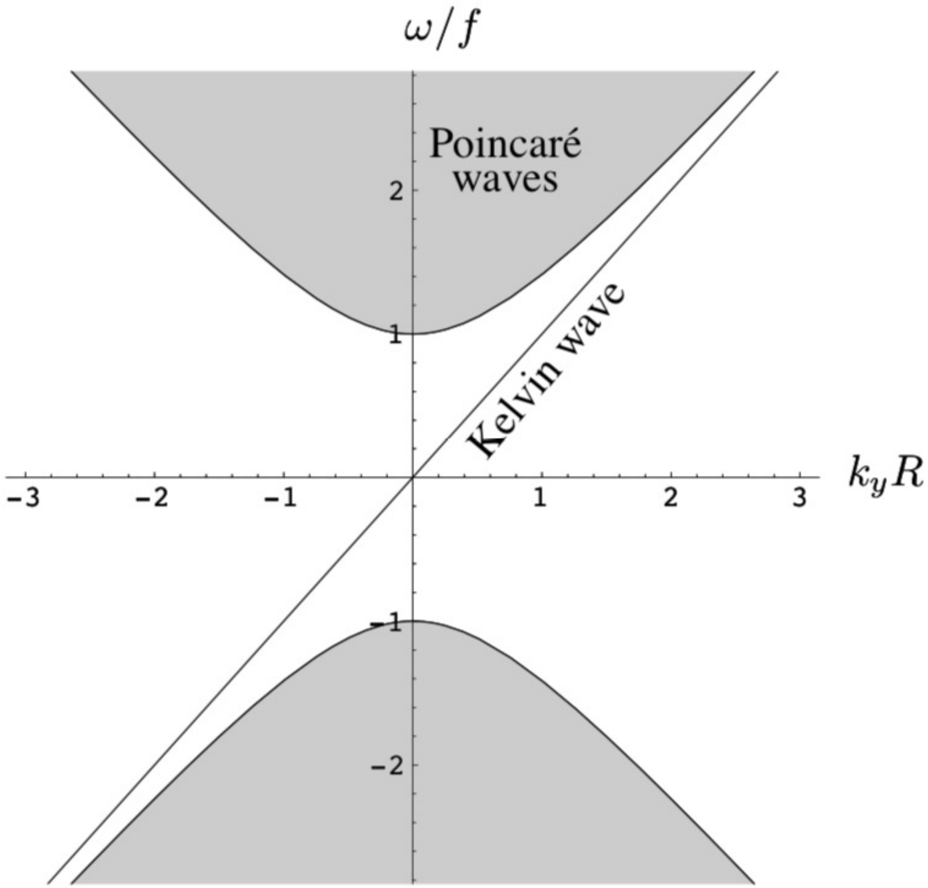


Figure 9-3 Recapitulation of the dispersion relation of Kelvin and Poincaré waves on the f-plane and on a flat bottom. While Poincaré waves (gray shades) can travel in all directions and occupy therefore a continuous spectrum in terms of k_y, the Kelvin wave (diagonal line) propagates only along a boundary.

Planetary Rossby Waves

β – plane approximation: $f = f_0 + \beta_0 y$ $\boxed{\beta_0 = 2(\Omega/a)\cos\varphi_0}$

Assumptions: $\dfrac{\beta_0 L}{f_0} \ll 1$

flat bottom

The governing equations:

$$
\begin{aligned}
\underline{\frac{\partial u}{\partial t}} - (\underline{f_0} + \underline{\beta_0 y})v &= \underline{-g\frac{\partial \eta}{\partial x}} \\
\underline{\frac{\partial v}{\partial t}} + (\underline{f_0} + \underline{\beta_0 y})u &= \underline{-g\frac{\partial \eta}{\partial y}} \\
\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= 0,
\end{aligned}
$$

large terms

small terms

To 1st order approximation – geostrophic balance:

$$\frac{\partial u}{\partial t} - (f_0 + \beta_0 y)v = -g\frac{\partial \eta}{\partial x}$$
$$\frac{\partial v}{\partial t} + (f_0 + \beta_0 y)u = -g\frac{\partial \eta}{\partial y}$$

$$-f_0 v = -g\frac{\partial \eta}{\partial x}$$
$$f_0 u = -g\frac{\partial \eta}{\partial y}$$

Subsitute the solutions into the small terms of the governing equations:

$$-\frac{g}{f_0}\frac{\partial^2 \eta}{\partial y \partial t} - f_0 v - \frac{\beta_0 g}{f_0} y \frac{\partial \eta}{\partial x} = -g\frac{\partial \eta}{\partial x}$$
$$+\frac{g}{f_0}\frac{\partial^2 \eta}{\partial x \partial t} + f_0 u - \frac{\beta_0 g}{f_0} y \frac{\partial \eta}{\partial y} = -g\frac{\partial \eta}{\partial y}$$

ageostrophic flow

The solutions for u and v are:

$$u = -\frac{g}{f_0}\frac{\partial \eta}{\partial y} \underline{- \frac{g}{f_0^2}\frac{\partial^2 \eta}{\partial x \partial t} + \frac{\beta_0 g}{f_0^2} y \frac{\partial \eta}{\partial y}}$$
$$v = +\frac{g}{f_0}\frac{\partial \eta}{\partial x} \underline{- \frac{g}{f_0^2}\frac{\partial^2 \eta}{\partial y \partial t} - \frac{\beta_0 g}{f_0^2} y \frac{\partial \eta}{\partial x}}$$

$$
\begin{aligned}
u &= -\frac{g}{f_0}\frac{\partial \eta}{\partial y} - \frac{g}{f_0^2}\frac{\partial^2 \eta}{\partial x \partial t} + \frac{\beta_0 g}{f_0^2}\, y\, \frac{\partial \eta}{\partial y} \\
v &= +\frac{g}{f_0}\frac{\partial \eta}{\partial x} - \frac{g}{f_0^2}\frac{\partial^2 \eta}{\partial y \partial t} - \frac{\beta_0 g}{f_0^2}\, y\, \frac{\partial \eta}{\partial x}
\end{aligned}
$$

Substitution into the continuity equation: $\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0.$

$$
\frac{\partial \eta}{\partial t} - R^2 \frac{\partial}{\partial t}\nabla^2 \eta - \beta_0 R^2 \frac{\partial \eta}{\partial x} = 0 \qquad \boxed{R = \sqrt{gH}/f_0}
$$

Apply a wave solution $\eta = Ae^{i(kx+ly-\omega t)}$:

The dispersion relation is:

$$
\omega = -\beta_0 R^2 \frac{k}{1 + R^2(k^2 + l^2)}
$$

$$\omega = -\beta_0 R^2 \frac{k}{1 + R^2(k^2 + l^2)}$$

If $\beta_0 = 0, \quad \omega = 0,$ gesotrophic flow

If $R^2K^2 \ll 1, \; L \gg R,$ long wave:

$$\omega \sim \frac{\beta_0 R^2}{L} \ll \beta_0 \mathrm{L} \ll f_0$$

$$\boxed{\frac{\beta_0 L}{f_0} \ll 1}$$

If $R^2K^2 \geq 1, \; L \leq R,$ short wave:

$$\omega \sim \beta_0 \mathrm{L} \ll f_0$$

Planetary Rossby waves are subinertial (low frequency) waves

Homework

7-8. In Utopia, a narrow 200-m deep channel empties in a broad bay of varying bottom topography (Figure 7-14). Trace the path to the sea and the velocity profile of the channel outflow. Take $f = 10^{-4}\ \mathrm{s}^{-1}$. Solve only for straight stretches of the flow and ignore corners.

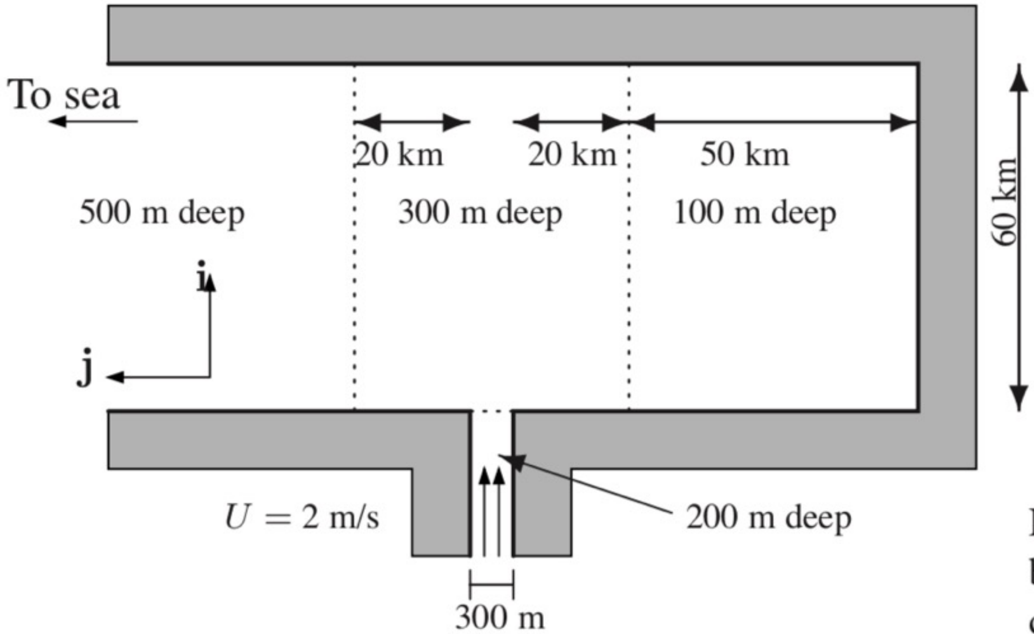


Figure 7-14 Geometry of the idealized bay and channel mentioned in Analytical Problem 7-8.