Math 6008 Numerical PDEs–Lecture 10

FDM for parabolic equations

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1 1D diffusion equation: two level schemes

1.1 Typical two-level schemes

We have introduced the scheme where the centered difference is used in space and forward Euler is used in time:

$$\frac{1}{\tau}(u_j^{n+1} - u_j^n) = \frac{a}{h^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n).$$

We have shown that the local truncation error is $O(\tau + h^2)$. We have already seen that it is convergent provided that $a\mu = a\frac{\tau}{h^2} \le \frac{1}{2}$ in ℓ^∞ norm. Moreover, using the theory of stability region or von Neumann analysis, it is also ℓ^2 stable if $a\frac{\tau}{h^2} \le 1/2$.

Clearly, one needs

$$\tau \sim h^2.$$

This is a restriction for the time step size. For hyperbolic equtions, we know that this restriction for forward Euler is due to that the eigenvalues of hyperbolic equations are imaginary. How about the diffusion equation? Can we choose other explicit scheme to relax this?

In fact, after we discretize in space using the centered difference, the highest frequency (short wavelength) will roughly be $1/h^2$. The eigenvalues are all **real** and nonpositive! Hence, the numerical methods containing negative real axis is fine. The reason here is that the eigenvalues indeed has a large range $0 \sim 1/h^2$. Often, we care about the modes with eigenvalues $O(1) \sim O(1/h)$ since we use step size h to resolve the spatial variations. We do not care those modes that have big eigenvalues, but they bring restrictions on the step size for stability. This is the typical issue in the **stiff problems**.

The restriction is exactly because the problem is stiff and we use the explicit method.

Hence, it is better that we adopt some implicit schemes. Consider a more general θ-method for time discretization,

$$u^{n+1} = u^n + \tau((1-\theta)f(t^n, u^n) + \theta f(t^{n+1}, u^{n+1})).$$

Applying this to the diffusion equation, we have the **weighted implicit scheme** (加权隐式格式):

$$\frac{u_j^{n+1} - u_j^n}{\tau} - a\left[\theta \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{h^2} + (1-\theta)\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}\right] = 0.$$

As we know, the backward Euler ($\theta = 1$) and the trapezoidal method $\theta = 1/2$ are A-stable ODE methods. Then, one can expect that the scheme with backward Euler

$$\frac{1}{\tau}(u_j^{n+1} - u_j^n) = a\frac{1}{h^2}(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}),$$

and the **Crank-Nicolson** method

$$\frac{1}{\tau}(u_j^{n+1} - u_j^n) = \frac{a}{2h^2}[(u_{j+1}^n - 2u_j^n + u_{j-1}^n) + (u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1})].$$

should be **unconditionally stable**.

In fact, with the Fourier analysis, the growth matrix is

$$G(\xi; \tau, h) = \frac{1 - 4(1-\theta)a\mu\sin^2(\xi h/2)}{1 + 4\theta a\mu \sin^2(\xi h/2)}.$$

One may find for $|G| \le 1$, the stability condition is

$$2a\mu(1-2\theta) \le 1.$$

Then, for $\theta \ge 1/2$, it is unconditionally stable.

Below, we briefly discuss the case $\theta = 1$ and $\theta = 1/2$:

- If we do Taylor expansion at $(x_{j+1/2}, t^{n+1/2})$, we can show that the local truncation error for Cranck-Nicolson is $O(\tau^2 + h^2)$. Hence it is of second order accuracy in both time and space.

- They are unconditionally stable.
- The matrices are tridiagonal and easy to be inverted to find u^{n+1}.
- The trapezoidal is not L-stable. The short wavelength modes will not decay that fast. Hence, if you desire those modes to decay fast, you probably want the backward Euler. Otherwise, the Crank-Nicolson is good.

2 1D heat equation: three level schemes

Previously, we discussed a family of two-level schemes. Here, we explore some three-level schemes to achieve good stability and accuracy.

2.1 Three-level implicit scheme

As known, the BDF-2 method in the class of implicit linear multistep method (隐式线性多步法):

$$y_{n+2}-\frac{4}{3}y_{n+1}+\frac{1}{3}y_n=\frac{2}{3}\tau f(t_{n+2},y_{n+2})\Rightarrow \frac{3}{2}\frac{y_{n+1}-y_n}{\tau}-\frac{1}{2}\frac{y_n-y_{n-1}}{\tau}=f(t_{n+1},y_{n+1})$$

is A-stable. This has second order accuracy.

Recall that the weak stability region is determined by

$$\left\{z:\pi(\zeta;z)=(\frac{3}{2}-z)\zeta^2-2\zeta+\frac{1}{2}=0\text{ satisfies the root condition.}\right\}$$

We claim that the stability region contains $\mathbb{C}^-$. In fact, the roots are given by

$$\zeta=\frac{2\pm\sqrt{1+2z}}{3-2z}=\frac{1}{2\mp\sqrt{1+2z}}.$$

If $\mathrm{Re}(z)\le 0$, then

$$\sqrt{1+2z}=:a+bi$$

satisfies

$$a^2-b^2\le 1.$$

Geometrically, one needs to bound the distance from 2 to the region between the hyperbola $a^2 - b^2 = 1$ below by 1.

$$|2 \mp (a+bi)| = \sqrt{|2 \mp a|^2 + b^2} \geq \sqrt{a^2 \mp 4a + 4 + a^2 - 1} \geq \sqrt{2(a \mp 1)^2 + 1} \geq 1.$$

If the root is repeated, then $1 + 2z = 0$ or $z = -\frac{1}{2}$, but in this case the magnitude of the roots is less than 1.

Now that the scheme is A-stable and the right hand side has only non-positive eigenvalues, we expect the method to be **unconditionally stable**.

In fact, introducing $v_j^{n+1} = u_j^n$ and reducing it to a two-level system, one may compute that the growth matrix (this is HW 2) that

$$G = \begin{bmatrix} \frac{4}{3+8a\lambda\sin^2(\xi h/2)} & \frac{-1}{3+8a\lambda\sin^2(\xi h/2)} \\ 1 & 0 \end{bmatrix}.$$

The characteristic equation is

$$\mu^2 - \frac{4}{3 + 8a\lambda\sin^2(\xi h/2)}\mu + \frac{1}{3 + 8a\lambda\sin^2(\xi h/2)} = 0.$$

Using the following lemma, one finds that $\rho(G) \leq 1$ and that when the root is repeated, the magnitude is less than 1. Hence, it is unconditionally stable.

Lemma 1. *For the quadratic equation with* **real** *coefficients*

$$\mu^2 - b\mu - c = 0.$$

The sufficient and necessary condition for the roots $|\mu_i| \leq 1$ *is*

$$|b| \leq 1 - c, \quad |c| \leq 1.$$

证明. By Vieta's Theorem,

$$\mu_1 + \mu_2 = b, \quad \mu_1\mu_2 = -c.$$

Both are real numbers, $\mu_1 = re^{i\theta}$, $\mu_2 = (-c/r)e^{-i\theta}$. Since $\mu_1 - \mu_2$ is real, $(r + c/r)\sin\theta = 0$. If $\sin\theta \neq 0$, then both roots are complex. It is easy to see that in this case $\mu_1 = \bar{\mu}_2$. If $\sin\theta = 0$, then both roots are real.

We claim that the sufficient and necessary condition is

$$|\mu_1\mu_2| \le 1, \quad (1-\mu_1)(1-\mu_2) \ge 0, \quad (1+\mu_1)(1+\mu_2) \ge 0. \tag{1}$$

no matter which case happens.

In fact, if both are complex numbers with $\mu_1 = \bar{\mu}_2$, then $|\mu_1\mu_2| \le 1$ implies the second condition $1 \pm (\mu_1 + \mu_2) + \mu_1\mu_2 \ge 0$. $|\mu_1\mu_2| \le 1$ is equivalent to $|\mu_i| \le 1$. If both are real numbers, the conditions are clearly sufficient and necessary.

Condition (1) clearly gives the desired result. □

2.2 Du Fort-Frankel scheme

The Richardson scheme

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\tau} - a\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} = 0,$$

is unstable. The reason is clear to see: the eigenvalues of the spatial discretization are negative and real. The stability region of the leapfrog is $(-i, i)$ so it cannot be stable.

Now, as one does in Lax-Friedrichs, one could consider

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\tau} - a\frac{u_{j+1}^n - (u_j^{n+1} + u_j^{n-1}) + u_{j-1}^n}{h^2} = 0.$$

This is called the Du Fort-Frankel scheme.

The local truncation error is, however,

$$O(\tau^2 + h^2 + \frac{\tau^2}{h^2})$$

Clearly, this requires $\tau/h \to 0$ to be consistent. It is unconditionally stable.

The benefit of this scheme is that it can be implemented by the jumping method (跳点格式) , which has good computational efficiency, less memory requirement, and might be good for parallel computing.

3 Initial boundary value problems

If the problem is on a bounded domain, then besides the initial conditions, one should also impose the boundary conditions.

For the diffusion equation, the boundary conditions are similar as for elliptic equations, which are simpler compared with the hyperbolic equations.

For example, one may impose the Dirichlet boundary conditions.

$$\begin{aligned}
&\partial_t u - a\partial_{xx} u = 0,\\
&u(x,0) = g(x),\\
&u(0,t) = \varphi(t), \quad u(1,t) = \psi(t).
\end{aligned}$$

The treatment is similar as before: one may use directly the boundary value for the numerical solution $u_0^n = \varphi(t_n)$ and $u_J^n = \psi(t_n)$.

Below, we consider again the boundary conditions of the third type (which includes the Neumann conditions as the special case).

$$\begin{aligned}
&\partial_t u - a\partial_{xx} u = 0,\\
&u(x,0) = g(x),\\
&u_x(0,t) = \alpha u(0,t) + \mu(t), \quad t \geq 0,\\
&u_x(1,t) = \beta u(1,t) + \nu(t), \quad t \geq 0.
\end{aligned}$$

As before, the treatment is similar.

- One may use the one-sided finite difference
$$\frac{u_1^n - u_0^n}{h} = \alpha u_0^n + \mu(t_n).$$
- One may also use the ghost point method:
$$\frac{u_1^n - u_{-1}^n}{2h} = \alpha u_0^n + \mu(t_n).$$
To eliminate u_{-1}^n, one may impose the equation at (x_0, t_n). For example, one may impose
$$\frac{u_0^{n+1} - u_0^n}{\tau} - a\frac{u_1^n - 2u_0^n + u_{-1}^n}{h^2} = 0.$$

This is only first order in time but it is fine as this is a correction in the second order term in the boundary approximation.

With this, one can eliminate u_{-1} and thus form an approximation using u_0, u_1 for the boundary condition.

$$u_0^{n+1} = [1 - 2a\mu(1 + \alpha h)]u_0^n + 2a\mu u_1^n - 2a\mu h\mu(t_n).$$

Then, one may solve the linear system.

4 Variable coefficient problems

Consider

$$u_t = (a(x)u_x)_x, \quad a(x) \geq a_0 > 0.$$

4.1 Schemes obtained from integration

We apply forward Euler in time and the typical elliptic difference in space and may have

$$\frac{1}{\tau}(u_j^{n+1} - u_j^n) = \frac{1}{h^2}\Big(a_{j+1/2}(u_{j+1}^n - u_j^n) - a_{j-1/2}(u_j^n - u_{j-1}^n)\Big).$$

Such kind of scheme can be derived using the integration method (finite volume).

Introduce

$$w := au_x.$$

Take

$$\mathscr{D} = [x_{j-1/2}, x_{j+1/2}] \times [t_n, t_{n+1}].$$

Integrate on $\mathscr{D}$:

$$\int_{x_{j-1/2}}^{x_{j+1/2}} [u(x, t_{n+1}) - u(x, t_n)]dx = \int_{t_n}^{t_{n+1}} [w(x_{j+1/2}, t) - w(x_{j-1/2}, t)]dt.$$

Consider w: one has $u_x = \frac{1}{a}w$. Then,

$$u(x_{j+1},t) - u(x_j,t) = \int_{x_j}^{x_{j+1}} \frac{w(x,t)}{a(x)}dx \approx w(x_{j+1/2},t)\int_{x_j}^{x_{j+1}} \frac{1}{a(x)}dx$$
$$\Rightarrow w_{j+1/2} = [\frac{1}{h}\int_{x_j}^{x_{j+1}} \frac{1}{a(x)}dx]^{-1}\frac{u_{j+1}-u_j}{h}.$$

The reason to do this is that w has better continuity compared with u. For example, if a is discontinuous as discussed below, u_x is often discontinuous but w is often continuous.

If we define

$$A_{j+1/2} := [\frac{1}{h}\int_{x_j}^{x_{j+1}} \frac{1}{a(x)}dx]^{-1},$$

then one may have the discretization using the two-level methods:

$$\frac{u_j^{n+1}-u_j^n}{\tau} = \frac{1}{h^2}[\theta\delta(A\delta u^{n+1})_j + (1-\theta)\delta(A\delta u^n)_j].$$

Remark 1. *If you call*

$$A_{j+1} = [\frac{1}{h}\int_{x_j}^{x_{j+1}} \frac{1}{a(x)}dx]^{-1},$$

then the method becomes

$$\frac{u_j^{n+1}-u_j^n}{\tau} = \theta D_+(AD_-u^{n+1})_j + (1-\theta)D_+(AD_-u^n)_j,$$

as in the book (where the book used Δ and ∇ to denote the forward and backward difference, which is not very suitable because we reserve these for Laplacian and gradient). Note that $(D_+u)_j = (D_-u)_{j+1} = \frac{1}{h}(\delta u)_{j+1/2}$.

Clearly, if one approximates

$$w_{j+1/2} = (au_x)_{j+1/2} \approx a_{j+1/2}\frac{u_{j+1}-u_j}{h},$$

then the method we used at the beginning is obtained.

4.2 Stability analsysis **(Free reading)

For the ℓ^2 stability, the von Neumann analysis is not suitable here since a is varying. However, you may use the frozen coefficients method to see the stability intuitively. Here, we provide the energy method for ℓ^2 stability for your convenience for the simplest method (forward Euler and the simple treatment $w_{j+1/2} = (au_x)_{j+1/2} \approx a_{j+1/2}\frac{u_{j+1}-u_j}{h}$).

Multiplying u_j^n and taking the sum (just like multiplying u and integrating in PDEs), assuming that $|u_j^n| \to 0$ as $|j| \to \infty$ and that $u^n \in \ell^2$, we have

$$
\begin{aligned}
LHS = \frac{1}{\tau}\sum_j (u_j^{n+1}u_j^n - (u_j^n)^2) &= \frac{1}{\tau}\sum_j \frac{1}{2}\Big((u_j^{n+1})^2 - (u_j^n)^2 - (u_j^{n+1} - u_j^n)^2\Big) \\
&= \frac{1}{2\tau h}\Big[\|u^{n+1}\|_2^2 - \|u^n\|_2^2 - \sum_j h(u_j^{n+1} - u_j^n)^2\Big]
\end{aligned}
$$

$$
RHS = \frac{1}{h^2}\Big[\sum_j (a_{j+1/2}u_{j+1}^n u_j^n - (u_j^n)^2) - \sum_j a_{j-1/2}((u_j^n)^2 - u_j^n u_{j-1}^n)\Big]
$$

$$
= \frac{1}{h^2}\sum_j -a_{j+1/2}[(u_{j+1}^n)^2 + (u_j^n)^2 - 2u_{j+1}^n u_j^n] = -\frac{1}{h^2}\sum_j a_{j+1/2}(u_{j+1}^n - u_j^n)^2.
$$

Hence,

$$
\|u^{n+1}\|_2^2 = \|u^n\|_2^2 + \sum_j h(u_j^{n+1} - u_j^n)^2 - \frac{2\tau}{h}\sum_j a_{j+1/2}(u_{j+1}^n - u_j^n)^2.
$$

The inequality $(a+b)^2 \le 2(a^2+b^2)$ implies that

$$
(u_j^{n+1} - u_j^n)^2 \le \frac{2\tau^2}{h^4}[a_{j+1/2}^2(u_{j+1}^n - u_j^n)^2 + a_{j-1/2}^2(u_j^n - u_{j-1}^n)^2].
$$

Hence,

$$
\|u^{n+1}\|_2^2 \le \|u^n\|_2^2 + \frac{2\tau^2}{h^3}2\sum_j a_{j+1/2}^2(u_{j+1}^n - u_j^n)^2 - \frac{2\tau}{h}\sum_j a_{j+1/2}(u_{j+1}^n - u_j^n)^2.
$$

If $\frac{4\tau^2 a_{j+1/2}}{h^3} - \frac{2\tau}{h} \le 0$, then $\|u_j^n\|_2$ will be controlled. Hence, the requirement is $\frac{2\tau\|a\|_\infty}{h^2} \le 1$.

4.3 Discontinuous coefficients

Here, we have used the idea of finite volume method (FVM). This idea can also be used for equations with discontinuous coefficients.

Consider the equation

$$\partial_t u + b(x)u = \partial_x(a(x)\partial_x u), \quad a(x) > 0.$$

Here, the coefficients a, b may be discontinuous! The coniditons near the discontinuity ξ are

$$u(\xi-, t) = u(\xi+, t), \quad a(\xi-)\partial_x u(\xi-, t) = a(\xi+)\partial_x u(\xi+, t).$$

Remark 2. *These two conditions are required so that the terms in the PDE make sense. For example, in the $\partial_x u$ term, if u is not continous, the distributional derivative has atoms (delta function), and it cannot balance other terms (the "$\partial_t u$" and "bu" do not have such deltas).*

The scheme can be derived again using integration method. The key is to use the continuity. For example,

$$\int_{x_{j-1/2}}^{x_{j+1/2}} bu\, dx \approx u(x_j, t) \int_{x_{j-1/2}}^{x_{j+1/2}} b(x)\, dx.$$

The reason is that u is continuous. If $\xi = x_j$, this is $u(x_j, t)\frac{1}{2}(b(x_j-) + b(x_j+))h$.

The term for $\partial_x(a\partial_x u)$ can be similarly derived as before. Introducing $w = a\partial_x u$, one has

$$\int_{x_{j-1/2}}^{x_{j+1/2}} \partial_x w\, dx = w(x_{j+1/2}, t) - w(x_{j-1/2}, t).$$

To obtain $w_{j+1/2}$, one may do using the continuity of w

$$u(x_{j+1}, t) - u(x_{j,t}) = \int_{x_j}^{x_{j+1}} \frac{w}{a(x)}\, dx \approx w(x_{j+1/2}) \int_{x_j}^{x_{j+1}} \frac{1}{a}\, dx.$$

The eventual scheme can be obtained as

$$\frac{u_j^{n+1} - u_j^n}{\tau} + u_j^n \int_{x_{j-1/2}}^{x_{j+1/2}} b(x)\, dx = \frac{1}{h^2}[\theta D_+(AD_- u^{n+1})_j + (1-\theta)D_+(AD_- u^n)_j].$$

Note that the term for bu is not treated using the θ-method. We used the explicit treatment simply. We will explain this later for mixed type equations.

4.4 Keller's Box Scheme **(free reading)

Consider again

$$\partial_t u = \partial_x w, \quad a(x)\partial_x u = w.$$

Recall that the first scheme we had is

$$\frac{u_j^n - u_j^{n-1}}{\tau} = \frac{w_{j+1/2}^{n-1} - w_{j-1/2}^{n-1}}{h},$$
$$w_{j+1/2} = a(x_{j+1/2})\frac{u_{j+1/2} - u_{j-1/2}}{h}$$

This scheme is first order in time and is not absolutely stable. To improve the accuracy, one needs to evaluate the derivative of w_x at time $t_{n-1/2}$.

The Keller;s Box scheme is to put the unknowns w and u all on (x_j, t^n). Note that here one is not going to approximate using w directly using the values of u. Instead, one goes to establish the equations for u and w, which are coupled together. In particular

$$\frac{1}{2}(w_j^n + w_{j+1}^n) = a_{j+1/2}\frac{u_{j+1}^n - u_j^n}{h},$$
$$\frac{(u_j^n + u_{j+1}^n) - (u_{j+1}^{n-1} + u_j^{n-1})}{2\tau} = \frac{(w_{j+1}^n + w_{j+1}^{n-1}) - (w_j^n + w_j^{n-1})}{2h}.$$

5 Parabolic equations in multi-dimensional case

We'll consider 2D heat equation for simplicity (3D will be similar):

$$u_t = a\Delta u = a(u_{xx} + u_{yy}).$$

5.1 Generalization of 1D schemes; MOL (method of lines) type schemes

Let Δ_h be the five-point difference for Laplacian:

$$\Delta_h = D_x^2 + D_y^2 = \frac{1}{h^2}(\delta_x^2 + \delta_y^2).$$

The generalization of 1D scheme where one uses forward Euler in time and centered difference in space can be written as

$$\frac{u_{j\ell}^{n+1} - u_{j\ell}^n}{\tau} = \Delta_h u_{j\ell}^n.$$

The local truncation error is $O(\tau + h^2 + h^2)$. Using von Neumann analysis (we should use $u_{j\ell}^n = e^{ix_j\xi_1 + iy_k\xi_2}$), we find

$$G(\xi) = G(\xi_1, \xi_2; h, \tau) = 1 - 4\frac{\tau}{h^2}(\sin^2(\xi_1 h/2) + \sin^2(\xi_2 h/2)).$$

Then, we need $4\tau/h^2 \le 1$ for this to be stable. This is more severe compared with the 1D case. In general d dimensional case, the condition would be

$$\frac{2d\tau}{h^2} \le 1.$$

This is not good for higher dimension.

To gain better stability schemes, one may consider using implicit schemes. For example, the *Crank-Nicolson* is an option

$$\frac{u_{j\ell}^{n+1} - u_{j\ell}^n}{\tau} = \frac{1}{2}[\Delta_h u_{j\ell}^n + \Delta_h u_{j\ell}^{n+1}].$$

The LTE is $O(\tau^2 + h^2)$ and it is unconditionally stable.

However, the issue arises for how to solve this linear system.

$$(I - \frac{\tau}{2}\Delta_h)u_{j\ell}^{n+1} = (I + \frac{\tau}{2}\Delta_h)u_{j\ell}^n.$$

Here, the matrix for $I - \frac{\tau}{2}\Delta_h$ is not tridiagonal and the direct method is not that fast. Hence, the iterative methods are desired. The condition number for this matrix is $O(\tau/h^2)$, which is not very big. Using the initial guess $(u^{n+1})^{(0)} = u^n$, the convergence can be obtained quickly.

Below, we will introduce two approaches to resolve this problem.

5.2 Locally one dimensional method (LOD)

From above we see that the Crank-Nicolson for 2D will produce a matrix that is not tridiagonal. The idea of LOD is to use a time splitting method. Form t^n to t^{n+1}, the equation

$$u_t = u_{xx} + u_{yy},$$

is split into two steps

$$\begin{aligned} u_t &= u_{xx} \\ u_t &= u_{yy} \end{aligned}$$

(The time splitting method is to split $u_t = Au + Bu$ into $u_t = Au$ and $u_t = Bu$. We will come to this later for more details.)

We then apply the Crank-Nicolson for both and obtain the LOD method:

$$\begin{aligned} \frac{u_{ij}^* - u_{ij}^n}{\tau} &= \frac{1}{2}(D_x^2 u_{ij}^n + D_x^2 u_{ij}^*), \\ \frac{u_{ij}^{n+1} - u_{ij}^*}{\tau} &= \frac{1}{2}(D_y^2 u_{ij}^* + D_y^2 u_{ij}^{n+1}) \end{aligned}$$

Then, for each step, we have tridiagonal matrices and we can invert them easily. Since we have two 1D Crank-Nicolson for each step, the method is unconditionally stable.

For this method, one must determine the boundary conditions for u^* carefully. Note that u^* is not a physical quantity and it is **not** $u^{n+1/2}$. Read P198 of the book by Leveque to understand how to impose boundary conditions for LOD.

5.3 Alternating Direction Implicit Methods (ADI)

In the LOD method, the intermediate quantity u^* is not physical. Then, the following ADI method gives physical intermediate step quantities:

$$\frac{u_{ij}^* - u_{ij}^n}{\tau/2} = D_x^2 u_{ij}^* + D_y^2 u_{ij}^n$$
$$\frac{u_{ij}^{n+1} - u_{ij}^*}{\tau/2} = D_x^2 u_{ij}^* + D_y^2 u_{ij}^{n+1}$$

In the first $\tau/2$ time, we make x implicit; in the second $\tau/2$ time, we make y implicit. For each $\tau/2$ step, the local truncation error is $O(\tau + h^2 + h^2)$ but the whole local truncation error is $O(\tau^2 + h^2 + h^2)$ because the errors in the two steps cancel.

To see this, we are going to eliminate u^* and find the formula from u^n to u^{n+1} directly. We have

$$(1 + \frac{1}{4}\frac{\tau^2}{h^4} D_x^2 D_y^2)\frac{u_{ij}^{n+1} - u_{ij}^n}{\tau} = \frac{1}{h^2}(D_x^2 + D_y^2)\frac{u_{ij}^{n+1} + u_{ij}^n}{2}$$

The method is unconditionally stable. Compared with the LOD, the unconditional stability is not so obvious. Here, we use Von Neumann analysis to see this. The factor for the first half step is given by

$$\frac{G_1(\xi_1, \xi_2) - 1}{\tau/2} = \frac{1}{h^2}(G(\xi_1, \xi_2)(2\cos(\xi_1 h) - 2) + 2\cos(\xi_2 h) - 2)$$

This gives

$$G_1(\xi_1, \xi_2) = \frac{1 - 2\sin^2(\xi_2 h/2)\frac{\tau}{h^2}}{1 + 2\sin^2(\xi_1 h/2)\frac{\tau}{h^2}}$$

Similarly, we have G_2. Clearly, a single g_i will not guarantee stability, but the product of them is

$$G = G_1 G_2 = \frac{(1 - 2\sin^2(\xi_2 h/2)\frac{\tau}{h^2})(1 - 2\sin^2(\xi_1 h/2)\frac{\tau}{h^2})}{(1 + 2\sin^2(\xi_1 h/2)\frac{k}{h^2})(1 + 2\sin^2(\xi_2 h/2)\frac{\tau}{h^2})}$$

We find

$$|G| \leq 1.$$

The advantage of this method is that we have a series decouple linear systems whose matrices are tridiagonal.

For example, in the first $\tau/2$ step, we need to figure out u^*. Then, we can solve the values for a fixed y_j. Then, it is 1D and the matrix is tridiagonal.

Code presentation: Consider the diffusion equation

$$u_t = \Delta u, \quad (x,y) \in \Omega = [0,1] \times [0,1].$$
$$u_0(x,y) = \sin(2\pi x)\sin(2\pi y), \quad u = 0, \partial\Omega.$$

Simulate the problem using ADI. We plot the error versus the spatial step h. Check how error changes with time step by yourself.