Math 6008 Numerical PDEs–Lecture 3

Stability and the Lax equivalence theorem

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1 Stability

A natural question is: from consistency to convergence, what do we need? The answer is **stability** (稳定性). Now try to investigate this issue.

In the numerical simulation, there could be some errors introduced. One type is the initial error. Another one is the one step error introduced $\tau T(x_j, t_n)$. These errors will be propagated along the numerical computation and then eventually accumulate in the final global error. Hence, one very important requirement is that these errors will not be amplified very much to conceal the true soluion. This is the notion of stability .

Consider again the following scheme for the advection equation

$$u_j^{n+1} = u_j^n + a\lambda(u_j^n - u_{j+1}^n) = u_j^n - a\lambda(u_{j+1}^n - u_j^n).$$

The error e_j^n satisfies

$$e_j^{n+1} = e_j^n - a\lambda(e_{j+1}^n - e_j^n) - \tau T(x_j, t_n).$$

Suppose that there is error at $t_n = 0$: $-\tau T(x_j, 0) = \epsilon(-1)^j$ where ϵ is very small and $\tau T(x_j, t_n) = 0$ for $n \ge 1$.

Hence, equivalently, we have

$$e_j^1 = \epsilon(-1)^j$$

and

$$e_j^{n+1} = e_j^n - a\lambda(e_{j+1}^n - e_j^n), \quad n \ge 1.$$

For this relation, we can find that $e_j^n = (-1)^j v_n$ (this is can be obtained by the discrete Fourier analysis) and thus

$$v_{n+1} = v_n(1 + 2a\lambda).$$

Hence,

$$v_n = (1+2a\lambda)^{n-1}\epsilon \Rightarrow e_j^n = (-1)^j(1+2a\lambda)^{n-1}\epsilon.$$

Clearly, the local error truncation error will be amplified very quickly if we fix $\lambda = \tau/h$ with $\tau, h \to 0$. This reflects the fact that the scheme is unstable.

Exercise Implement this method and run numerical simulation. Also run the numerical simulation for the following method

$$\frac{u_j^{n+1} - u_j^n}{\tau} + a\frac{u_j^n - u_{j-1}^n}{h} = 0$$

for $\lambda = 0.9$ and 2.

1.1 The concept of stability

The stability considers the numerical method itself without comparing to the exact solution. The notions for evolutionary equations and elliptic equations can be different.

Consider the time evolutionary equations. We need a quantity to gauge the bigness of the data at time level t_n. We need some "norm" (范数) to achieve this. The frequently used norms include

$$\|v^n\|_\infty = \sup_j |v_j^n|$$

and

$$\|v^n\|_h := \sqrt{\sum_j h(v_j^n)^2}.$$

This is the ℓ^2 norm for discrete data. (The difference is that we have extra factor h to be consistent with the continuous L^2 integral.)

Definition 1. *Suppose we have a numerical scheme*

$$u^n = Y^n(u^{n-1}, \cdots, u^0, h, \tau)$$

with initial data u^0. Let the numerical error for u^0 be ϵ^0 (or we have initial data $v^0 = u^0 + \epsilon^0$) and the numerical error at t_n be $\epsilon^n = v^n - u^n$ where v^n is *obtained using the scheme with initial data* v^0. *We say the scheme is stable (稳定的) if there is* $K > 0$ *such that for all* $\tau \le \tau_0$ *and* $n\tau \le T$ *one has*

$$\|\epsilon^n\|_h \le K\|\epsilon^0\|_h.$$

Consider the scheme

$$u_j^{n+1} = u_j^n - a\lambda(u_{j+1}^n - u_j^n).$$

Clearly, if u^0 has error ϵ^0, then the error (difference between two solutions to the scheme) is evolved by

$$e_j^{n+1} = e_j^n - a\lambda(e_{j+1}^n - e_j^n).$$

The stability means that e^n should be controlled. As we have seen, the local truncation error at t_0 can be viewed as the initial error for t_1. Hence, the stability can also be used to gauge how the local truncation error can be amplified. This is why it is useful.

For linear equations, the scheme is often given by

$$u_j^{n+1} = L_h u_j^n,$$

where L_h is a linear operator. Then, the stability can be equivalently be given by

$$\sup_{n\tau \le T} \|L_h^n\| \le K$$

or

$$\|u^n\|_h \le K\|u^0\|_h.$$

For nonlinear equations, we have to use the definition to define the stability.

For elliptic equations, the stability is similar.

Definition 2. *Consider the scheme* $L(U) = F$ *for some elliptic equation. If* F *has some error* δF *so that* $\tilde{F} = F + \delta F$ *and the corresponding numerical solution* $\tilde{U}$ *has error*

$$e = \tilde{U} - U.$$

The method is said to be stable if

$$\|e\| \leq C\|\delta F\|$$

for some constant C independent of the step size $h \leq h_0$.

2 The Lax equivalence theorem for convergence

Roughly speaking, this theorem says: for linear equations, a method is convergent if and only if it is consistent and stable.

Suppose a method for a well-posed **linear evolutionary equation** with initial condition can be written as

$$u^{n+1} = B(\tau)u^n + b^n(\tau),$$

where $u^n = (u_j^n)$ is a vector and $B(\tau)$ is a linear operator.

The method is called Lax-Richtmyer stable if for any fixed $T > 0$, there exists a constant C_T such that

$$\|B(\tau)^n\| \leq C_T,$$

whenever $n\tau \leq T$.

Theorem 1. *Any* **consistent** *method of the above form is convergent if and only if it is Lax-Richtmyer stable.*

The 'if' part is straightforward. Suppose $u(x,t)$ is the exact solution, and $\bar{u}^n := (u(x_j,t_n))_{j=0,\pm 1,\cdots}$ consists of the values of the exact solutions, then we have

$$\bar{u}^{n+1} = B(\tau)\bar{u}^n + b^n(\tau) + \tau T_n,$$

where T_n is the local truncation error and goes to zero as $\tau \to 0$ since the method is consistent. Then, $E^n = u^n - \bar{u}^n$ and we have

$$E^{n+1} = B(\tau)E^n - kT_n.$$

This relation implies that

$$\|E^N\| \le C_T\|E^0\| + TC_T \max_n \|T_n\|.$$

For the 'only if' part, it involves some uniform boundedness principle. We ignore it. Those who are interested can read the paper by Richtmyer.

As we can guess: the scheme we just considered for advection equation is not stable while the scheme for the diffusion equation considered is stable for $2a\mu \le 1$.

- By Lax equivalence theorem, we only need to consider the stability to prove convergence, which might be easier.

- For nonlinear equations, there is no such clean equivalence. However, the concept of stability is still important. Often, one needs to establish a Grönwall-type equality for the errors, which is again the stability plus consistency.

- For linear elliptic equation, we also have analogues for the Lax equivalence theorem. We will come back to this issue when we talk about the elliptic equations.

3 Some typical methods to study stability

There are some typical methods for the stability analysis.

- For linear equations with constant coefficients, one may use the Fourier analysis, which is also called the von Neumann analysis for evolutionary equations.

- For schemes of the MOL type, we may make use of the stability region to understand the stability intuitively.

- For general nonlinear or equations of variable coefficients, the energy methods can be more useful.

- The maximal principles are also useful for ℓ^∞ stability.
- Other methods like direct esimation of the matrix etc. We'll not go into details.

The energy methods and maximal principles are used frequently for rigorous analysis, but they are quite involved. In our course, we will give some brief introduction to the stability region analysis and the Fourier analysis.

3.1 linear stability and weak stability region of ODE solvers

(Chap. 7 in Leveque.)

Consider solving the ODE

$$u' = f(t, u) \quad u(0) = u_0.$$

For most ODE solvers, as the step size h tends to zero, one may get convergence. In practice, h is usually finite and one may be curious how big h can be chosen. Studying the linear stability could give us some insight. The concepte of (weak) linear stability region can be useful.

We consider do linearization on the equation

$$u' = f(t, u)$$

around (t_0, u_0). Then,

$$u' \approx f(t_0, y_0) + J(t_0, u_0)(u - u_0)$$

where

$$J = \frac{\partial f(t, u)}{\partial u}$$

is the Jacobian matrix. The equation is reduced to

$$u' = Ju + C.$$

We remark that the stability of this system is the same as the homogeneous system

$$u' = Ju.$$

Hence, investigating the effects of one method applied on such linear equations can be useful. Consider one eigenmode and the equation is given by

$$u' = \lambda u, \quad \lambda \in \mathbb{C}.$$

Apply the method on the test equation $u' = \lambda u$ and define $z = \tau\lambda$. Usually, the method yields

$$u^{n+1} = R(z)u^n$$

for one step method. Clearly, as long as $|R(z)| \leq 1$, the solution can stay bounded. Hence, we define the following.

Definition 3. *For one-step method, we define the weak linear stability region to be*

$$\{z : |R(z)| \leq 1\}.$$

Remark 1. *The stability region often refers to the region $\{z : |R(z)| < 1\}$. However, in some literature, the word "stability region" actually means the weak linear stability region we defined above.*

Consider the linear multistep methods (LMM)

$$\sum_{j=0}^{r} \alpha_j u^{n+j} = \tau \sum_{j=0}^{r} \beta_j f(u^{n+j}, t_{n+j}).$$

Applying the method to the test equation $u' = \lambda u$, we can find that the whether the numerical solution grows or not depends on whether or not the roots of the polynomial

$$\pi(\zeta; z) = \rho(\zeta) - z\sigma(\zeta) = \sum_{j=0}^{r} (\alpha_j - z\beta_j)\zeta^j$$

satisfy the **root-condition**:

> Suppose ζ_j are the roots of the characteristic equation. The root conditions means that $|\zeta_j| \leq 1$ and $|\zeta_j| < 1$ if it is repeated.

Definition 4. *For LMM, the weak linear stability region is*

$$\mathcal{D} = \{z : \pi(\cdot; z) \text{ satisfies the root condition.}\}$$

We should choose τ so that $\tau\lambda$ falls into the stability region for any eigenvalue, at least in the limit $\tau \to 0$. (If there is $Re(\lambda) > 0$ that means the system itself has growing modes and we can hope the convergence only in the limit $\tau \to 0$.)

Example 1. *The Euler's method*

$$y_{n+1} = y_n + \tau f(t_n, y_n).$$

We have $y_{n+1} = y_n + \tau\lambda y_n = (1+z)y_n$. For the test problem, $y_n = (1+z)y_{n-1}$. Hence, the stability region is $\{z : |1+z| \le 1\}$.

Example 2. *The implicit Euler's method*

$$y_{n+1} = y_n + \tau f(t_{n+1}, y_{n+1}).$$

We have $y_{n+1} = y_n + z y_{n+1}$. Hence, $y_{n+1} = y_n/(1-z)$. Hence, the stability region is $\{z : |1-z| \ge 1\}$. It's an unbounded region.

Example 3. *Consider the* **midpoint method (leapfrog method)** $u^{n+1} = u^{n-1} + 2\tau f(t_n, u^n)$.

$$u^{n+1} = u^{n-1} + 2zu^n \Rightarrow \zeta^2 - 2z\zeta - 1 = 0.$$

For $|\zeta| \le 1$, both roots must have magnitude 1 since their product is -1. $\zeta_1 = e^{i\theta}$ and $\zeta_2 = -e^{-i\theta}$. $z = \frac{1}{2}(\zeta - \frac{1}{\zeta}) = i\sin\theta$ if $\zeta = e^{i\theta}$. $\theta \ne \pm\pi/2$ since $\zeta_1 \ne \zeta_2$. Hence, the stability region is the open interval from $-i$ to i.

3.2 stability analysis for MOL type schemes

The first method of stability is to use the ODE theory if the scheme can be derived from MOL. (This is in fact for the ℓ^2 stability analysis.)

As in the ODE theory, we require $\lambda\tau$ to be in the stability region of the ODE method for any eigenvalue λ of the spatial discretization, at least in the $\tau \to 0$ limit.

Example 4. *Consider the scheme*

$$\frac{u_j^{n+1} - u_j^n}{\tau} = \frac{1}{h^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

for $u_t = u_{xx}$ with Dirichlet boundary conditions, the matrix A is tridiagonal and the eigenvalues of the matrix are given by

$$\lambda_p = \frac{2}{h^2}(\cos(p\pi h) - 1), \quad p = 1, 2, \ldots, m.$$

Note that $\cos(\xi) - 1$ is decreasing on $[0, \pi]$, so the eigenvalues are roughly in the interval $(\frac{-4}{h^2}, \lambda_1) \approx (\frac{-4}{h^2}, -\pi^2)$. Hence, we require $-\frac{4\tau}{h^2}$ to be in the stability region of the ODE method. The stability region of the forward Euler is $|1 + z| \leq 1$. Hence, the condition for the scheme to be stable is

$$\frac{4\tau}{h^2} \leq 2.$$

Hence, the time step is roughly the square of the spatial step. This is a severe restriction. Explicit schemes for parabolic equations usually have such restrictions.

Example 5. *For the scheme*

$$\frac{u_j^{n+1} - u_j^n}{\tau} = -\frac{a}{h}(u_{j+1}^n - u_j^n)$$

with **periodic boundary conditions** *on $x \in [0, 1]$ and $h = 1/(m+1)$. The matrix A has eigenvalues*

$$\lambda_p = -\frac{a}{h}(e^{i2\pi ph} - 1), \quad p = 1, 2, \ldots, m+1.$$

We therefore need $\tau\lambda_p$ to be in the stability region $|1 + z| \leq 1$. Clearly, $\mathrm{Re}(\tau\lambda_p) \geq 0$ and the imaginary part is $\mathrm{Im}(\tau\lambda_p) = -a\frac{\tau}{h}\sin(2\pi ph)$. Hence, for $h, \tau \to 0$ but with τ/h fixed, the method is **unstable** *as there is an imaginary part that does not go to zero. If the iteration $n_{max} = T/\tau$ is big enough, the solution blows up.*

However, if $\tau, h \to 0$ but the ratio $\lambda = \tau/h$ also goes to zero like $\tau = O(h^2)$. Then, $\tau\lambda_p$ will fall into the stability region eventually. We may expect the convergence for this case.