T1.(路金甫，第三版) 第五章习题 4.

4. 在 $D=\{(x,y)\mid 0\leqslant x,y\leqslant 1\}$ 上给出边值问题

$$\begin{cases} -\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right)=16, & 0<x,y<1, \\ u\big|_{x=1}=0, \dfrac{\partial u}{\partial y}\Big|_{y=1}=-u, \\ \dfrac{\partial u}{\partial x}\Big|_{x=0}=\dfrac{\partial u}{\partial y}\Big|_{y=0}=0. \end{cases}$$

取 $h=\frac{1}{4}$，试用五点差分格式求此问题的数值解.

解：

(1)节点取为:

$x_i=(i-1)h, y_j=(j-1)h$,这里 $i=1,2,\dots,m-1, j=1,2,\dots,m;m=\frac{1}{h}+1$。网格点的个数为(m-1)*m。

(2)使用五点差分格式:

$$-\Delta_h u_{i,j}=-\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}-\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{h^2}=f_{i,j}=16$$
$$i=2,\dots,m-2, j=2,\dots,m-1$$

(3)边界处理

其中$u_x(x=0)=0$，所以由鬼点法得$u_{i,0}=u_{i,2}$。又因为 $u(x=1)=0$, 所以$u_{i,m}=0$；令 C 为

$$C=\begin{pmatrix} -2 & 2 & & & & & & & \\ 1 & -2 & 1 & & & & & & \\ & 1 & -2 & 1 & & & & & \\ & & \dots & \dots & \dots & & & & \\ & & & \dots & \dots & \dots & & & \\ & & & & \dots & \dots & \dots & & \\ & & & & & 1 & -2 & 1 & \\ & & & & & & 1 & -2 & 1 \\ & & & & & & & 1 & -2 \end{pmatrix}$$

且$u_y(y=0)=0$，所以由鬼点法得$u_{0,j}=u_{2,j}$。又因为$u_y(y=1)=-u$，同样由鬼点法得$u_{m+1,j}=u_{m-1,j}-2hu_{m,j}$，所以 A 为

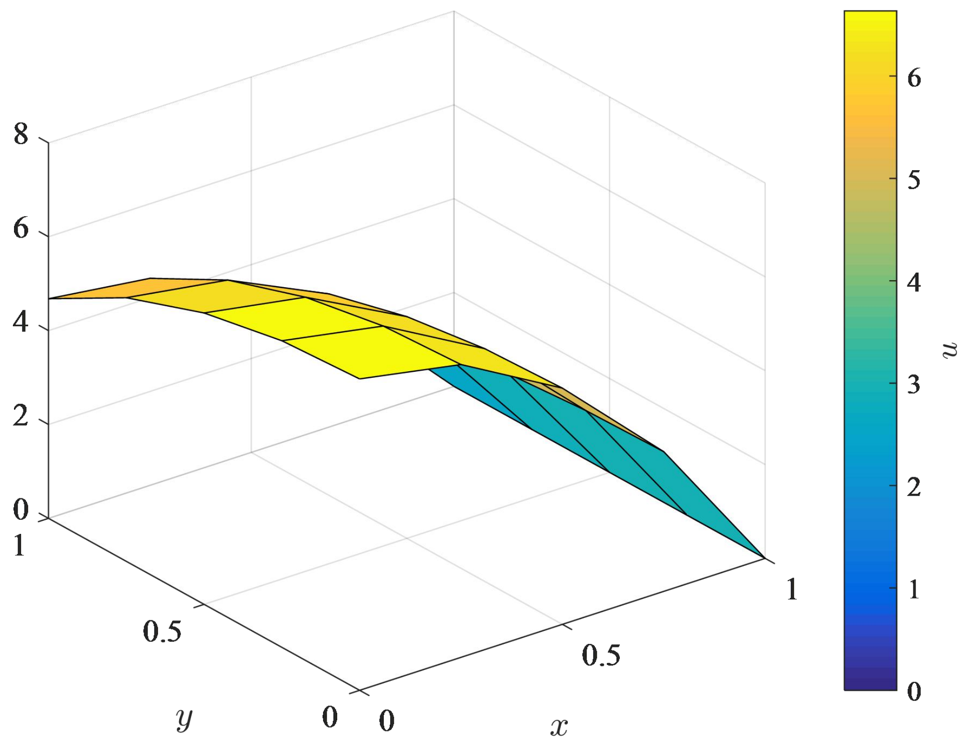
$$A=\begin{pmatrix} C-2I_{m-1} & 2I_{m-1} & & & & & \\ I_{m-1} & C-2I_{m-1} & I_{m-1} & & & & \\ & I_{m-1} & C-2I_{m-1} & I_{m-1} & & & \\ & & \dots & \dots & \dots & & \\ & & & \dots & \dots & \dots & \\ & & & & \dots & \dots & \dots \\ & & & & I_{m-1} & C-2I_{m-1} & I_{m-1} \\ & & & & & I_{m-1} & C-2I_{m-1} & I_{m-1} \\ & & & & & & 2I_{m-1} & C-2(h+1)I_{m-1} \end{pmatrix}$$

（4）格式

令$u_j=(u_{1,j},u_{2,j},\dots,u_{m-1,j})^T;u=(u_1,u_2,\dots,u_m)^T$,则差分格式可以表示为:

$$Au=[16]_{m(m-1)*1}$$

结果图



$h=1/4$ 的解

T2

2. Consider the mixed type boundary value problem:

$$u''+u=f, 0\le x\le \pi$$
$$u'(0)-u(0)=0,\quad u'(\pi)+u(\pi)=0$$

Construct a second order accurate FDM. For $f=-e^x$, plot the error versus the spatial step h in loglog scale.

采用中心差分，则差分格式为

$$\frac{u_{i+1}-2u_i+u_{i-1}}{h^2}+u_i=f_i, i=2,\dots,m-1,$$

其中 $m=\frac{\pi}{h}$。

因为 $u'(0)-u(0)=0$, 所以由鬼点法有$u_0=u_2-2hu_1$。又因为 $u'(\pi)+u(\pi)=0$, 所以$u_{m+1}=u_{m-1}-2hu_m$, 则

$$\frac{u_2-2u_1+u_2-2hu_1}{h^2}+u_1=f_1$$
$$\frac{u_{m+1}-2u_m+u_{m-1}-2hu_m}{h^2}+u_m=f_m$$

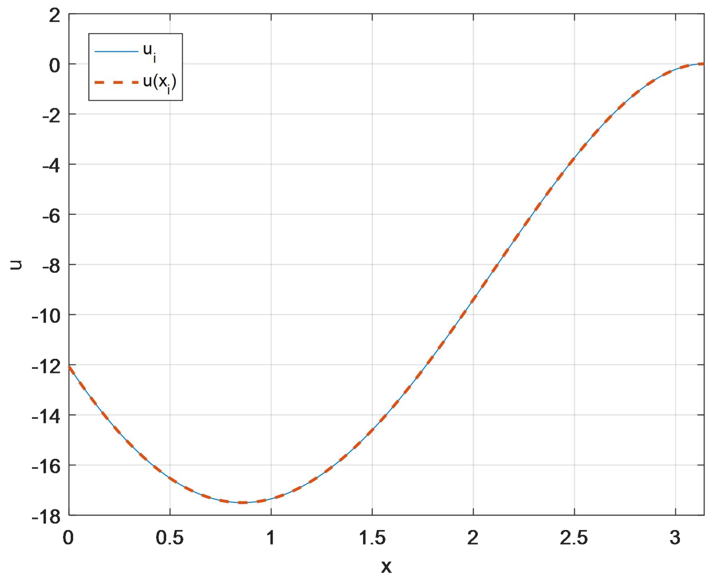
令 $u=(u_1,\dots,u_m)^T$，$f=(f_1,\dots,f_m,)^T$则令 A 为

$$
\begin{pmatrix}
1-2/h^2-2/h & 2/h^2 & & & & & & \\
1/h^2 & 1-2/h^2 & 1/h^2 & & & & & \\
 & 1/h^2 & 1-2/h^2 & 1/h^2 & & & & \\
 & \cdots & & \cdots & \cdots & & & \\
 & & & \cdots & \cdots & \cdots & & \\
 & & & & \cdots & \cdots & \cdots & \\
 & & & & 1/h^2 & 1-2/h^2 & 1/h^2 & \\
 & & & & & 1/h^2 & 1-2/h^2 & 1/h^2 \\
 & & & & & & 2/h^2 & 1-2/h^2-2/h
\end{pmatrix}
$$

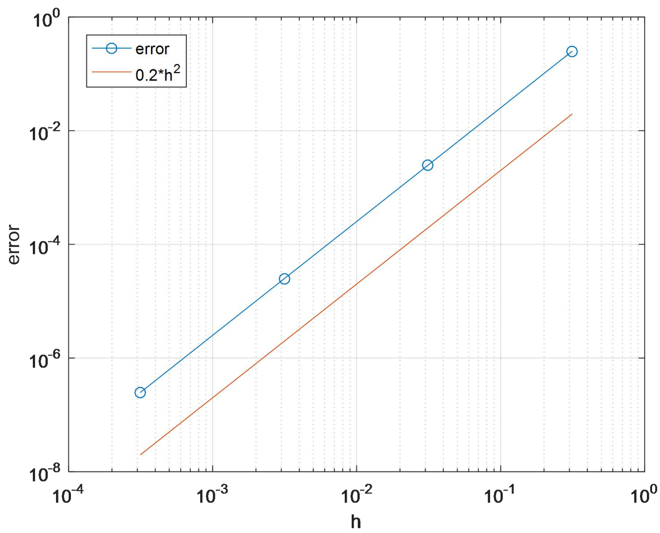
格式为 Au=f

理论解为 $u(x) = -\frac{e^{\pi}}{2}(\sin x + \cos x) - \frac{e^{x}}{2}$, 作图讨论误差。

结果：



$h = \pi/10^4$的解



误差-步长关系

二阶精度

T3

3. Solve the nonlinear equation $\theta'' = -\sin(\theta)$, $\theta(0) = \alpha, \theta(1) = \beta$. For the nonlinear system of equations you obtained, use Newton's iteration to solve. Plot the error versus h.

$$\theta^{n} = -\sin(\theta), \theta(0) = \alpha, \theta(1) = \beta$$

使用中心差分，得到差分格式为

$$\frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{h^2} + \sin\theta_i = 0,\ i = 1,2,\ldots,m$$

其中$\theta_0 = \alpha, \theta_{m+1} = \beta$. 令$\theta = (\theta_1,\ldots,\theta_m)^T$, 差分格式写为

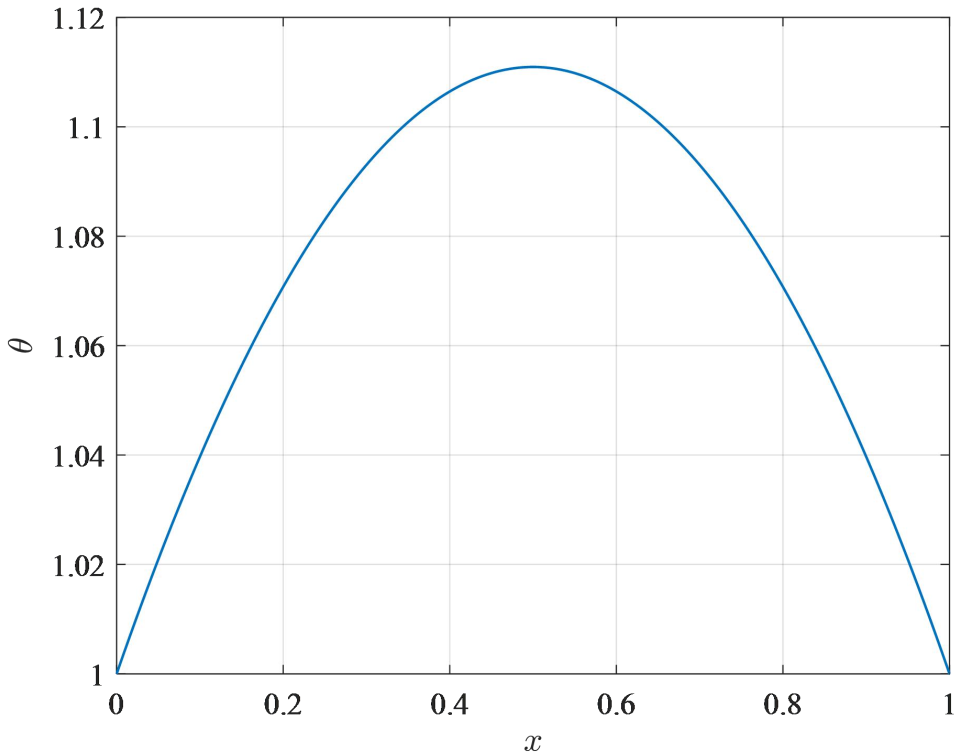
$$F(\theta) = \begin{pmatrix} \theta_2 - 2\theta_1 + \alpha + h^2 sin\theta_1 \\ \theta_3 - 2\theta_2 + \theta_1 + h^2 sin\theta_2 \\ \ldots \\ \theta_m - 2\theta_{m-1} + \theta_{m-2} + h^2 sin\theta_{m-1} \\ \beta - 2\theta_m + \theta_{m-1} + h^2 sin\theta_m \end{pmatrix}$$

使用牛顿迭代法对$\min\limits_{\theta} F(\theta)$进行求解，由

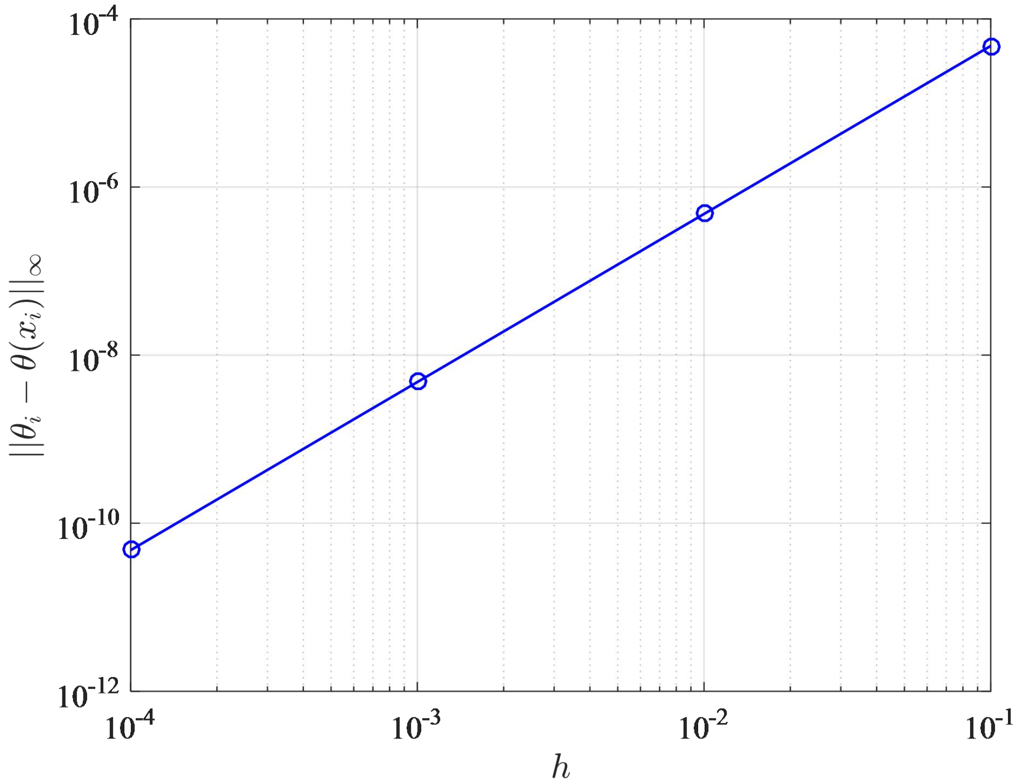
$$F'(\theta) = \begin{pmatrix} -2 + h^2\cos\theta_1 & 1 & & & & \\ 1 & -2 + h^2\cos\theta_2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & & \ddots \\ & & & 1 & -2 + h^2\cos\theta_m & 1 \\ & & & & 1 & -2 + h^2\cos\theta_m \end{pmatrix}$$

牛顿迭代法迭代步为$\theta^{(k+1)} = \theta^{(k)} - F'\left(\theta^{(k)}\right)^{-1} F\left(\theta^{(k)}\right), k = 0,1,\ldots$

取$\alpha = \beta = 1$，设定迭代误差限为$\varepsilon = 10^{-10}$，进行计算。



h = 10^{-7}时的计算结果



误差随 h 变化的曲线图

二阶精度

T4

4. Consider the 2D elliptic equation

$$-(a(x,y)u_x)_x - (a(x,y)u_y)_y = f(x,y), \Omega = [-1,1]\times[-1,1]$$
$$u = 0,\ on\ \partial\Omega$$

$a = 1 + 3\exp(-3(x+y)^2 - (x-y)^2)$ and $f = 1$. Apply a five-point scheme for this equation. Determine the order of accuracy of your scheme, using the calculation with a small h as the 'exact' solution.

Hint: To solve the linear system $AU = F$, one option is to construct A directly and do $A\backslash F$. Here, the coefficient is not a constant, constructing this matrix might be a little tricky (remember to keep the matrix sparse in Matlab) (One bad way is to set the point value of U to be 1 at a single point, then output the action of the scheme on this U, which will be the corresponding column of your matrix.) Another better option is to write a function that returns AU when the input is U and then apply an iterative method (such as conjugate gradient) to find the solution. By doing this, you do not have to construct A.

因为边界上的 u 已知，所以将区域$\Omega = [-1,1]\times[-1,1]$划分成 $m \times m$ 个点，其中 $m = 2/h - 1$。则五点差分格式为:

$$-\frac{a_{i+\frac{1}{2},j}\left(u_{i+1,j}-u_{i,j}\right)-a_{i-\frac{1}{2},j}\left(u_{i,j}-u_{i-1,j}\right)}{h^2}$$
$$-\frac{a_{i,j+\frac{1}{2}}\left(u_{i,j+1}-u_{i,j}\right)-a_{i,j-\frac{1}{2}}\left(u_{i,j}-u_{i,j-1}\right)}{h^2} = f_{i,j}, i = 1,\ldots,m, j-1,\ldots,m$$

令$u_j = \left(u_{1,j},u_{2,j},\ldots,u_{m,j}\right)^T, u = \left(u_1^T,u_2^T,\ldots,u_m^T\right)^T, u_j = \left(f_{1,j},f_{2,j},\ldots,f_{m,j}\right)^T, f = \left(f_1^T,f_2^T,\ldots,f_m^T\right)^T$

类似题目 1 的表示。那么方程组可表示成

$$Au = f$$

与题目 1 不同之处在于边界条件的处理，由于四个边均为第一类边界条件，因此只要将最外侧的节点外的点设置为 0，即迭代格式中的中出现的边界值$u_{1,j},u_{m,j},u_{i,1},u_{i,m}$都设为 0。

用共轭梯度法进行求解:

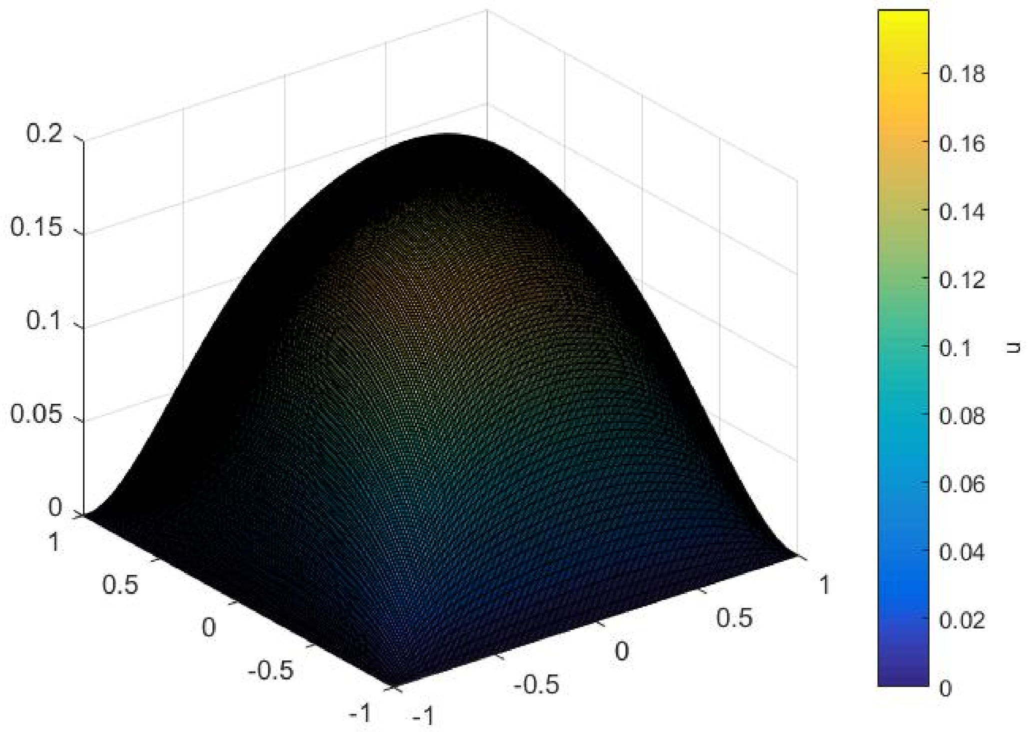
(1) 任取 $u^0 \in R^n$，计算$r^0 = F - Au^0$，取$p^0 = r^0$；

(2) 对 $k = 0,1,\ldots,$ 计算 A 并计算

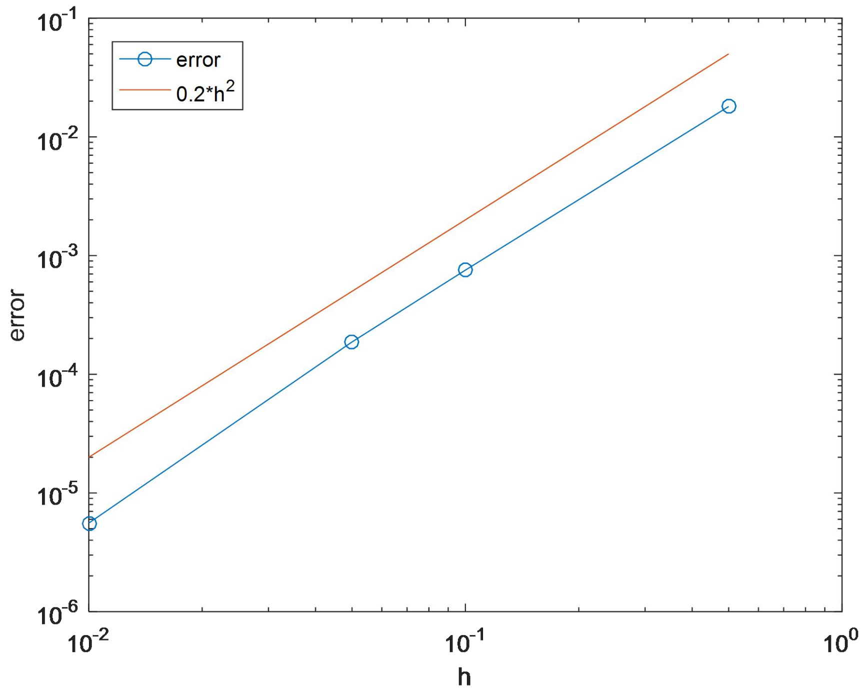
$$\alpha_k = \frac{\left(r^{(k)},r^{(k)}\right)}{\left(p^{(k)},Ap^{(k)}\right)}$$

$$u^{(k+1)} = u^{(k)} + \alpha_k p^{(k)}$$
$$r^{(k+1)} = r^{(k)} - \alpha_k A p^{(k)}$$
$$\beta_k = \frac{(r^{(k+1)}, r^{(k+1)})}{(r^{(k)}, r^{(k)})}$$
$$p^{(k+1)} = p^{(k+1)} + \beta_k p^{(k)}$$

(3)若$||r^k|| \le \epsilon$，计算停止，$u^* = u^{(k)}$.



h = 0.01 的解



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