Functional Analysis and PDE

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ABSTRACT. Preliminary Lecture notes on

Contents

CHAPTER 1

Banach spaces, Hilbert spaces

1.1. The Banach fixed-point theorem and the iteration method

- $\mathbb{K}$ complete field, e.g. $\mathbb{R}, \mathbb{C}, \ldots$
- $\cancel{V}$ Banach space over $\mathbb{K}$.
- $A : M \to M$ an operator.
- M sub-space of X.

THEOREM 1.1. *We assume that:*

(a) M *is a closed nonempty set of* X *over* $\mathbb{K}$*,*
(b) $A : M \to M$ *is* k*-contractive, i.e.*

$$\|Au - Av\| \leq k\|u - v\| \quad \forall u, v \in M,$$

k *fixed,* $0 \leq k < 1$.

Then,

(1) The equation

$$Au = u \quad u \in M \tag{1.1}$$

has a unique solution. The solution of (1.1) *is called fixed point of the operator* A*,*

(2) The sequence

$$u_{n+1} = Au_n \quad \forall n \in \mathbb{N}$$

where $u_0 \in M$ *converges to the fixed point of* (1.1).

@We consider the the variational problem

$$\min_{v \in V} J(v) = ? \tag{1.2}$$

where $J(v) = \frac{1}{2}a(v, v) - L(v)$, and the corresponding the variational equation

$$a(u, v) = L(v) \quad \text{for fixed } u \in V, \forall v \in V. \tag{1.3}$$

$$\boxed{\text{Boundary value problem} \implies \text{variational equation} \Leftrightarrow \text{variational problem.}}$$

1.2. Quadratic variational problems

@

We consider the minimum problem

$$\min_{v\in V}\left(\frac{1}{2}a(v,v)-L(v)\right)=? \tag{1.4}$$

where

(i)
$$a : V\times V \longrightarrow \mathbb{R}$$
is a bounded, symmetric, strongly positive bilinear form, i.e. there exist positive constants μ, M such that

$$|a(w,v)|\leq M\|w\|\|v\| \quad a(v,v)\geq \nu\|v\|^2 \quad v,w\in V. \tag{1.5}$$

$$u\mapsto (u,v), \quad v\mapsto a(u,v)$$

are linear functionals.

(ii) The functional $L: V\to\mathbb{R}$ is linear and continuous.

Then,

(1) The variational problem (1.5) has a unique solution.
(2) The problem (1.5) is equivalent to the following so-called variational equation:

$$a(u,v)=L(v) \quad \text{for fixed } u\in V, \text{ and all } v\in V. \tag{1.6}$$

.

Sketch of the proof. Let $u,v\in V$. We set

$$\phi(t):=\frac{1}{2}a(u+tv,u+tv)-L(u+tv) \quad t\in\mathbb{R}$$

Then,

$$\phi(t)=2^{-1}t^2a(v,v)+t(a(u,v)-L(v))+2^{-1}a(u,u)-L(u). \tag{1.7}$$

(a) u is a solution to (1.4) then $\phi(t)\geq\phi(0) \quad \forall t$. That is ϕ has a minimum at $t=0$, i.e.

$$\phi'(0)=0.$$

This equation is identical to

$$a(u,v)=L(v) \quad \forall v\in V.$$

(b) If u is solution of 1.6, then 1.7 becomes

$$\phi(t) = 2^{-1}t^2 a(v,v) - 2^{-1}a(u,u).$$

Then,

$$\frac{1}{2}a(w,w) - L(w) \geq -2^{-1}a(u,u)(= \frac{1}{2}a(u,u) - L(u)) \quad \forall w \in V.$$

So, u solution of 1.4. □

1.3. The Dirichlet principle-Dirichlet problem I

@

$$\boxed{\text{Quadratic variational problems} \Longrightarrow \text{(the generalized) Dirichlet problem}}$$

- We want to study the following variational problem:

$$\min_u J(u), \quad u = g \text{ on } \partial\Omega, \tag{1.8}$$

where $J(u) = 2^{-1}\int_\Omega \sum_{j=1}^N (\frac{\partial u}{\partial x_j})^2 dx - \int_\Omega f u dx$. This problem is called also Dirichlet problem.

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$$\min_u J(u), \quad u - g \in H_0^1(\Omega), \tag{1.9}$$

called also the generalized Dirichlet problem.

$$\boxed{\textit{The Dirichlet problem} \text{ says that problem 1.8 (or 1.9) has a solution } u.}$$

@

- Boundary-value problem for the Poisson equation:

$$\begin{cases} -\Delta u = f & \text{on } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.10}$$

where

$$\Delta = \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2}$$

1.10 is called the classical Euler-Lagrange equation.

- $$\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx \quad \forall v \in \mathcal{C}_0^\infty(\Omega).$$

  where
  $$\nabla(\cdot) = (\frac{\partial}{\partial x_1}(\cdot), \dots, \frac{\partial}{\partial x_N}(\cdot)).$$

PROPOSITION 1.2. Let Ω be a nonempty bounded open set in $\mathbb{R}^N$.Let $g : \partial\Omega \to \mathbb{R}$, $f : \overline{\Omega} \to \mathbb{R}$ continuous functions. Suppose that $u \in \mathcal{C}^2(\overline{\Omega})$. Then,

$$\boxed{\text{Variational problem 1.8} \Longrightarrow \text{the boundary-value problem 1.10}}.$$

PROOF. Let u be a solution of 1.8. Let $v \in \mathcal{C}_0^\infty(\Omega)$. Let $t \in \mathbb{R}$. The function
$$w : u + tv$$
is admissible for the variational problem 1.8, i.e. $w = g$ on $\partial\Omega$, and $w \in \mathcal{C}^2(\Omega)$.

We set
$$\phi(t) = J(u + tv).$$

Explicitly,
$$\phi(t) = 2^{-1} \int_\Omega \sum_j (\partial_j u + t\partial_j v)^2 dx - \int_\Omega f \cdot (u + tv) dx.$$
The quadratic function $\phi : \mathbb{R} \to \mathbb{R}$ has a minimum at $t = 0$. Hence,
$$\phi'(0) = 0$$
That is
$$\int_\Omega \nabla u \cdot \nabla v dx - \int_\Omega f v dx = 0 \quad \forall v \in \mathcal{C}_0^\infty(\Omega)$$
□

@

EXERCISE 1.3. Consider the minimum problem
$$\min_u J(u), \quad u \in \mathcal{C}^1[-1, 1], u(-1) = 0, u(1) = 1.$$
where
$$J(u) = \int_{-1}^1 (xu'(x))^2 dx.$$
Use the sequence
$$u_n(x) = \frac{1}{2} + \frac{1}{2}\frac{\arctan(nx)}{\arctan(n)}, \quad n = 1, 2, \dots$$
to show that this variational problem has no solution.

1.4. Linear Lax-Milgram theory

We consider the the variational problem

$$\min_{v\in V} J(v) = ? \tag{1.11}$$

where $J(v) = \frac{1}{2}a(v,v) - L(v)$, and the corresponding the variational equation

$$a(u,v) = L(v) \quad \text{for fixed } \ u \in V, \forall v \in V. \tag{1.12}$$

We assume the following:

(i) V is a real separable Hilbert space, i.e, there exists a sequence $(V_n)_{n\in\mathbb{N}}$ of finite dimensional linear subspaces V_n of V such that

$$\lim_n \mathrm{dist}_V(u, V_n) = 0 \quad \forall u \in V.$$

(ii)

$$a : V \times V \longrightarrow \mathbb{R}$$

is a bounded, symmetric, and strongly positive, i.e. there exist positive constants μ, M such that

$$|a(w,v)| \le M\|w\|\|v\| \quad a(v,v) \ge \nu\|v\|^2 \quad v, w \in V. \tag{1.13}$$

(iii) The functional $L : V \to \mathbb{R}$ is linear and continuous.

Let us consider the following problem

$$\min_{v\in V_n} \frac{1}{2}a(v,v) - L(v), \tag{1.14}$$

and

$$a(u_n, v) = L(v) \quad \text{for fixed } u_n \in V_n \text{ and for any } v \in V_n. \tag{1.15}$$

Let $\{e_{n,1}, \dots, e_{n,N_n}\}$ be a basis of V_n ($N_n = \dim V_n$). (1.15) is equivalent to the following system of linear equations:

$$\sum_{i=1}^{N_n} \alpha_{n,i} a(e_{n,i}, e_{n,j}) = L(e_{n,j}) \quad j = 1, \dots, N_n. \tag{1.16}$$

where $u_n = \sum_{i=1}^{N_n} \alpha_{n,i} e_{n,i}$ (we take $v = e_{n,i}$ in (1.15)).

From (1.13), the following matrix

$$(a(e_{n,i}, e_{n,j}))_{1\le i,j\le N_n}$$

is invertible.

We have

(a) (1.11) has a unique solution u. This is also the unique solution of the variational equation (1.12). We have a prior estimate

$$\|u\| \leq \nu^{-1}\|L\|. \tag{1.17}$$

(b) u_n is a solution for (1.16), and

$$\|u_n - u\| \leq \frac{M}{\nu}\operatorname{dist}(u, X_n),$$

(approximation of u by u_n.)

1.4.1. Applications to Boundary-Value problems, the method of finite elements, and Elasiticity.

@ Let us consider the following boundary-value problem:

$$\begin{cases} -u''(x) = f(x) & \text{if } \alpha < x \leq \beta \\ u(\alpha) = u(\beta) = 0 & u \in \mathcal{C}^2[\alpha, \beta]. \end{cases} \tag{1.18}$$

where $-\infty < \alpha < \beta < +\infty$. In other words, $u \in \mathcal{C}_0^2[\alpha, \beta]$.

The corresponding variational problem is

$$\begin{cases} \min_u J(u), \\ \text{under the conditions } \; u(\alpha) = u(\beta) = 0 \text{ and } u \in \mathcal{C}^2[\alpha, \beta]. \end{cases} \tag{1.19}$$

where $J(u) = \int_\alpha^\beta (\frac{1}{2}(u')^2 - uf)dx$.

- $u(x) =$ deflection of a string at the point x under the vertical outer force density $\mathbf{f}(x) = f(x)e$.
- $J(u)=$ total potential energy of the string.
- $\int_\alpha^\beta 2^{-1}u'^2 dx$ elastic energy of the string.
- $L(u) := \int_\alpha^\beta fudx=$ work of the outer force density $\mathbf{f}$ with respect to the vertical displacement $\mathbf{u}(x) = u(x)\mathbf{e}$ of the string , which is also $-$(potential energy stored by the force density f).

Let us consider the (generalized) variational problem

$$(1.20)\qquad \begin{cases} \min_u J(u), \\ \text{under the conditions } \ u \in H_0^1(\alpha, \beta). \end{cases}$$

Recall (Definition 1.26)

$$(H_0^1(\alpha,\beta), \|\cdot\|_{H_0^1(\alpha,\beta)})$$

is a real Hilbert space, with the associated scalar product

$$(u,v)_{H_0^1(\alpha,\beta)} = \int_\alpha^\beta (uv + u'v')dx, \ \|u\|_{H_0^1(\alpha,\beta)} = \left(\int_\alpha^\beta (u^2 + (u')^2)dx\right)^{\frac{1}{2}}.$$

We set

$$(u,v)_E = \int_\alpha^\beta u'v'dx, \quad \|u\|_E = (u,v)_E{}^{\frac{1}{2}},$$

called the the energetic inner product and the energetic norm[1]. Obviously,

$$2^{-1}(u,u)_{H_0^1(\alpha,\beta)} = \text{elastic potential energy with respect to the displacement } u$$

The variational equation reads as follows

$$(1.21)\qquad \frac{dJ(u+tv)}{dt}_{|t=0} = \int_\alpha^\beta (u'v' - fv)dx = 0 \quad \forall v \in H_0^1(\alpha,\beta)$$

This means the following: The time derivative of the energy $J(u+tv)$ at $t=0$ equals zero.

[1] This is, indeed, a norm which moreover is EQUIVALENT to $\|\cdot\|_{H_0^1(\alpha,\beta)}$. This is a consequence, of 1.24.

1.4.2. The Poincaré-Friedrichs Inequality. Let $f \in \mathcal{C}^1[\alpha, \beta]$. We have, for any $\alpha \le x \le \beta$,

$$\begin{aligned} f(x)^2 =& f(x)^2 - f(\alpha)^2 + f(\alpha)^2 \\ =& 2\int_\alpha^x f'(y)f(y)dy + f(\alpha)^2 \\ \le& 2\left(\int_\alpha^x (f'(y))^2 dy\right)^{\frac{1}{2}} \left(\int_\alpha^x f(y)^2 dy\right)^{\frac{1}{2}} + f(\alpha)^2 \\ \le& 2\left(\int_\alpha^\beta (f'(y))^2 dy\right)^{\frac{1}{2}} \left(\int_\alpha^\beta f(y)^2 dy\right)^{\frac{1}{2}} + f(\alpha)^2 \end{aligned}$$

That is

$$f(x)^2 \le 2\left(\int_\alpha^\beta (f'(y))^2 dy\right)^{\frac{1}{2}} \left(\int_\alpha^\beta f(y)^2 dy\right)^{\frac{1}{2}} + f(\alpha)^2.$$

Then

$$\int_\alpha^\beta f(x)^2 \le 2(\beta-\alpha)\left(\int_\alpha^\beta (f'(y))^2 dy\right)^{\frac{1}{2}} \left(\int_\alpha^\beta f(y)^2 dy\right)^{\frac{1}{2}} + (\beta-\alpha)f(\alpha)^2. \tag{1.22}$$

It follows that

(1.23)

$$\left(\int_\alpha^\beta f(x)^2 dx\right)^{\frac{1}{2}} \le (\beta-\alpha)\left(\int_\alpha^\beta (f'(y))^2 dy\right)^{\frac{1}{2}} + \sqrt{(\beta-\alpha)^2\int_\alpha^\beta (f'(y))^2 dy + (\beta-\alpha)f(\alpha)^2}.$$

Put $X = \left(\int_\alpha^\beta f(x)^2 dx\right)^{\frac{1}{2}}$, then 1.22 becomes $X^2 \le 2aX + b$..

We have proved the following lemma

Lemma 1.4 (The Poincaré-Friedrichs inequality for regular functions in dimension 1). *Let $f \in \mathcal{C}_0^1[\alpha, \beta]$. We have*

$$(\beta-\alpha)^{-1}\int_\alpha^\beta f(x)^2 dx \le \int_\alpha^\beta (f'(x))^2 dx$$

Now, let $u \in H_0^1(\alpha, \beta)$, (see Definition 1.26). By definition, $\exists (u_n) \subset \mathcal{C}_0^\infty(\alpha, \beta)$ such that

$$u_n \to u \text{ in } H_0^1(\alpha, \beta),$$

as $n \to \infty$. We have

$$(\beta-\alpha)^{-1}\int_\alpha^\beta u_n(x)^2 dx \le \int_\alpha^\beta (u_n'(x))^2 dx \quad \forall n \in \mathbb{N}.$$

Letting $n \to \infty$, we get

$$(\beta-\alpha)^{-1}\int_\alpha^\beta u^2dx \le \int_\alpha^\beta (u')^2dx \quad \forall u \in H_0^1(\alpha,\beta) \tag{1.24}$$

@ More generally, we have the Poincaré-Friedrichs inequality:

PROPOSITION 1.5. Let Ω be a nonempty bounded open set in $\mathbb{R}^N$. There there exists a constant $C>0$ such that

$$C\int_\Omega u^2dx \le \int_\Omega \left(\sum_{j=1}^N \frac{\partial u}{\partial x_j}\right)^2 dx \quad \forall u \in H_0^1(\Omega). \tag{1.25}$$

Application: With the notations above, we have

$$J(u) = 2^{-1}(u,u)_E - L(u) \quad \forall u \in H_0^1(\alpha,\beta).$$

We see that $a(u,v) := (u,v)_E$ is a bounded, symmetric, strongly positive bilinear form. By Theorem 1.13, the problem 1.20 has a unique solution $u \in H_0^1(\alpha,\beta)$, and hence

$$-\Delta u = f \quad \text{on } H_0^1(\alpha,\beta).$$

@

EXERCISE 1.6. Let $G(x,y)$ be the function defined on $[\alpha,\beta]\times[\alpha,\beta]$ as follows

$$G(x,y)=\begin{cases}\frac{(\beta-y)(x-\alpha)}{\beta-\alpha} & \text{if } \alpha\leq x\leq y\leq\beta\\ \frac{(\beta-x)(y-\alpha)}{\beta-\alpha} & \text{if } \alpha\leq y< x\leq\beta,\end{cases}$$

Show that the function u given by

$$u(x)=\int_\alpha^\beta G(x,y)f(y)dy.$$

is a solution of 1.18.

Hint: Show that

$$G(x,y)=G(y,x).$$

Let $\phi\in\mathcal{C}_0^\infty(\alpha,\beta)$. Compute

$$\int_\alpha^\beta G(x,y)\phi''(y)dy=?$$

Use Fubini's theorem to determine

$$\int_\alpha^\beta u(x)\phi''(x)dx=?$$

1.5. Generalized derivatives=Weak differentiation

@Ω open subset of $\mathbb{R}^n$.

$$L^2(\Omega)=\left\{u:\Omega\to\mathbb{R}\cup\{\infty\}\mid u\text{ measurable }\int_\Omega|u|^2dx<\infty\right\}$$

$$\mathcal{C}_0^\infty(\Omega),$$

denotes the space of smooth functions on Ω with compact support in Ω. Also denoted by $\mathcal{C}_c^\infty(\Omega)$.

DEFINITION 1.7. Let $u,w\in L^2(\Omega)$. Assume that

$$\int_\Omega u\,\partial_j v dx=-\int_\Omega w\,v dx\quad\forall v\in\mathcal{C}_0^\infty(\Omega).$$

Then w is called the generalized derivative (or the weak partial derivative) of the function u on the set Ω of type ∂_j. We write

$$\frac{\partial u}{\partial x_j}:=w.$$

EXAMPLE 1.8. Consider the function $u(x) = |x|$ for $x \in (-1, 1)$.

$$u(x) = \begin{cases} u(x) = f(x) & \text{if } \ -1 < x \leq 0 \\ u(x) = g(x) & \text{if } \ 0 \leq x < 1, \end{cases}$$

where f and g are differentiable functions such that $f \in L^2(-1, 0)$, $g \in L^2(0, 1)$ and $f(0) = g(0)$. We have, for any $v \in \mathcal{C}_0^\infty((-1, 1))$,

$$\begin{aligned}
\int_{-1}^{1} uv' dx &= \int_{-1}^{0} uv' dx + \int_{0}^{1} uv' dx \\
&= -\int_{-1}^{0} u'v dx - \int_{0}^{1} u'v dx + [uv]_{-1}^{0} + [uv]_{0}^{1} \\
&= -\int_{-1}^{0} f'v dx - \int_{0}^{1} g'v dx + f(0)v(0) - g(0)v(0) \\
&= -\int_{-1}^{1} wv dx
\end{aligned}$$

where w is in $L^2(-1, 1)$.

EXAMPLE 1.9. Recall the definition of G in Exercise 1.6. We have the following equality of currents

$$-\frac{\partial^2}{\partial x^2} G(x, y) = \delta_y$$

equivalently

$$-\Delta G(x, y) = \delta_y$$

where δ_y is the Dirac measure at y,i.e. δ_y is a "generalized function"="distribution"="continuous functional" such that

$$\delta_y(\phi) = \phi(y) \quad \forall \phi \in \mathcal{C}_0^\infty(\alpha, \beta)$$

@

DEFINITION 1.10.

(i) The Sobolev space:

$$H^1(\Omega) := \left\{ u \in L^2(\Omega) \mid \forall i = 1, \ldots, N, \ \frac{\partial u}{\partial x_i} \in L^2(\Omega) \right\}$$

also denoted by $W_2^1(\Omega)$. This is a Hilbert space for the scalar product

(a)

$$(u, v) = \int_\Omega (uv + \nabla u \cdot \nabla v) \, dx.$$

(b) The associated norm,

$$\|u\|_{H^1(\Omega)} := \left(\int_\Omega (|u|^2 + |\nabla u|^2)dx\right)^{1/2}$$

(ii) The space $H_0^1(\Omega)$.

$$H_0^1(\Omega) := \overline{\mathcal{C}_0^\infty(\Omega)}(\subset H^1(\Omega)) \tag{1.26}$$

the closure wrt the metric of $H^1(\Omega)$.

EXAMPLE 1.11. $\Omega = (a,b) \subset \mathbb{R}$. If $u \in H_0^1(a,b)$, then there exits a unique continuous function $v : [a,b] \to \mathbb{R}$ such that $u(x) = v(x)$ for almost $x \in (a,b)$ and

$$v(a) = v(b) = 0.$$

1.6. Nonlinear Lax-Milgram theory

@

We want to solve the nonlinear operator equation

$$Au = z, \quad u \in V, \tag{1.27}$$

where

(1) $A : V \to A$ is an operator on the real Hilbert space V, such that

$$(Au - Av, u - v) \geq \nu\|u - v\|^2 \quad \forall u, v \in V$$

(2)

$$\|Au - Av\| \leq L\|u - v\| \quad \forall u, v \in V.$$

THEOREM 1.12. *For each given $z \in V$, problem (1.27) has a unique solution u.*

PROOF. Let $t > 0$, and we consider the operator

$$Bu := u - t(Au - z), \quad u \in V.$$

We have

$$Bu = u \Leftrightarrow Au = z.$$

For any $u, v \in V$,

$$\begin{aligned}\|Bu - Bv\|^2 =& \|u - v\|^2 - 2t(Au - Av, u - v) + t^2\|Au - Av\|^2 \\ \leq& (1 - 2\nu t + L^2t^2)\|u - v\|^2 \\ =& \left(1 - L^2t(\frac{2\nu}{L^2} - t)\right)\|u - v\|^2\end{aligned}$$

So, if $0 < t < \frac{2\nu}{L^2}$, then

$$k := (1 - 2\nu t + L^2t^2) < 1$$

In this case,

$$\|Bu - Bv\| \leq \sqrt{k}\|u - v\| < \|u - v\| \quad \forall u, v \in V.$$

This means that B is $\sqrt{k}$-contractive. We conclude by using Theorem 1.1.

□

For each given $u_0 \in V$ and $t \in (0, 2\nu/L^2)$, the *iteration method*

$$u_{n+1} = u_n - t(Au_n - z), \quad n = 1, 2, \ldots$$

converges to the unique solution u of 1.27. We have the error estimate

$$\|u - u_n\| \leq k^{n/2}(1 - \sqrt{k})^{-1}\|u_1 - u_0\| \quad \forall n \in \mathbb{N}.$$

1.6.1. Application to the Nonlinear Lax-Milgram Theorem. @

Let

(i) $b : V \to \mathbb{R}$ linear continuous functional on the real Hilbert space V.

(ii) $a : V \times V \to \mathbb{R}$ a function such that, for each $w \in V$,

$$v \mapsto a(w, v)$$

is a linear continuous functional on V.

(iii) $\exists \nu, L > 0$ such that

$$a(u, u - v) - a(v, u - v) \geq \nu\|u - v\|^2,$$

and

$$|a(u, w) - a(v, w)| \leq L\|u - v\|\|w\|,$$

for all $u, v, w \in V$.

THEOREM 1.13. *The equation*

$$a(u, v) = L(v) \quad \forall v \in V \tag{1.28}$$

has a unique solution u.

PROOF. By Riesz theorem, for each $w \in X$, there is an element called Aw such that

$$a(w, u) = (Aw, u) \quad \forall u \in V.$$

Then (iii) becomes

$$(Au, u - v) - (Av, u - v) \geq \nu\|u - v\|^2,$$

and

$$|(Au - Av, w)| \leq L\|u - v\|\|w\|.$$

Again by Riesz theorem. there is a $z \in V$ such that

$$L(u) = (z, u) \quad \forall u \in V$$

Consequently, (1.28) is equivalent to the operator equation

$$Au = z, \quad u \in V.$$

We conclude by Theorem 1.12. □

1.7. The Dirichlet principle-Dirichlet problem 2

- Let us consider the generalized Dirichlet problem

$$\min_u 2^{-1}\int_\Omega \sum_{j=1}^{N}(\frac{\partial u}{\partial x_j})^2 dx - \int_\Omega fudx, \quad u-g\in H_0^1(\Omega)^2, \tag{1.29}$$

- and the generalized boundary-value problem:

$$\int_\Omega \nabla u\cdot\nabla v dx = \int_\Omega fvdx \quad \forall v\in H_0^1(\Omega), u-g\in H_1^0(\Omega). \tag{1.30}$$

THEOREM 1.14 (Dirichlet principle). *Let Ω be a nonempty bounded open subset of $\mathbb{R}^N$. Let $f\in L^2(\Omega), g\in H_0^1(\Omega)$. Then the following hold true:*

(i) 1.29 has unique solution $u\in H^1(\Omega)$.
(ii) This is also the unique solution $u\in H^1(\Omega)$ of 1.30.

PROOF. Let $V = H^1(\Omega)$, and

$$a(u,v)=\int_\Omega \nabla u\cdot\nabla v dx, \quad L(v)=\int_\Omega fvdx.$$

□

[2]This condition is a generalization of the following : $u=g$ on $\partial\Omega$

CHAPTER 2

Introduction to mathematical modeling

2.1. Some classical models; Quelques modèles classiques

Ω is a domain in $\mathbb{R}^N$.

2.1.1. The convection-diffusion equation.

$$
\begin{cases}
c\frac{\partial\theta}{\partial t} + cV \cdot \nabla\theta - k\Delta\theta = f & \text{in } \Omega \times \mathbb{R}_*^+ \\
\theta = 0 & \text{on } \partial\Omega \times \mathbb{R}_*^+ \\
\theta(t = 0, x) = \theta_0(x) & \text{in } \Omega.
\end{cases} \tag{2.1}
$$

$V(t, x)$ velocity (a vector valued function in $\mathbb{R}^N$).

2.1.1.1. *The convection-diffusion equation The case $N = 1,\ f \equiv 0$.*

$$\begin{cases} \frac{\partial\theta}{\partial t} + V\frac{\partial\theta}{\partial x} - \nu\frac{\partial^2\theta}{\partial x^2} = 0 & \text{in } \mathbb{R}\times\mathbb{R}_*^+ \\ \theta(t=0,x) = \theta_0(x) & \text{in } \mathbb{R}. \end{cases} \tag{2.2}$$

with $\nu = k/c$. 2.2 has the following solution

$$\theta(t,x) = \frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty}\theta_0(y)\exp\left(-\frac{(x-Vt-y)^2}{4\nu t}\right)dy. \tag{2.3}$$

2.1.2. The advection equation. $k = 0$ in 2.1.

$$\begin{cases} c\frac{\partial\theta}{\partial t} + cV\cdot\nabla\theta = f & \text{in } \Omega\times\mathbb{R}_*^+ \\ \theta = 0 & \text{on } \partial\Omega\times\mathbb{R}_*^+ \text{ if } V(x)\cdot n(x) < 0 \\ \theta(t=0,x) = \theta_0(x) & \text{in } \Omega. \end{cases} \tag{2.4}$$

2.1.2.1. *The advection equation, The case of $N = 1, f \equiv 0$.*

$$\begin{cases} \frac{\partial\theta}{\partial t} + V\frac{\partial\theta}{\partial x} = 0 & \text{in } \mathbb{R}\times\mathbb{R}_*^+ \\ \theta(t=0,x) = \theta_0(x) & \text{in } \mathbb{R}. \end{cases} \tag{2.5}$$

$$\theta(t,x) = \theta_0(x - Vt)$$

is a solution of (2.5).

EXERCISE 2.1. Let φ be a smooth function with support in $[-1,1]$. Show that

$$\lim_{\mu \to 0^+} \frac{1}{\sqrt{4\pi\mu}} \int_{-\infty}^{\infty} \varphi(y) \exp(-\frac{y^2}{\sqrt{\mu}}) dy = \varphi(0).$$

PROPOSITION 2.2 (The maximum principle). Let $\theta(t,x)$ be solution of (2.2) or (2.5). We have

$$\min_{x \in \mathbb{R}} \theta_0(x) \le \theta(t,x) \le \max_{x \in \mathbb{R}} \theta_0(x) \quad \forall (x,t) \in \mathbb{R} \times \mathbb{R}^+. \tag{2.6}$$

2.1.3. The heat flow equation.

$$
(2.7) \qquad \begin{cases} \frac{\partial u}{\partial t} - \Delta u = f & \text{in } \ \Omega \times \mathbb{R}_*^+ \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}_*^+ \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases}
$$

We have

$$
(2.8) \qquad \frac{1}{2}\frac{\partial}{\partial t}\left(\int_\Omega u^2(t,x)dx\right) = \int_\Omega u(t,x)f(x)dx + \int_\Omega u\Delta u dx
$$

$$
\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}\left(\int_\Omega u^2(t,x)dx\right) &= \int_\Omega \frac{1}{2}\frac{\partial}{\partial t}u^2(t,x)dx \\
&= \int_\Omega u(t,x)\frac{\partial}{\partial t}u(t,x)dx \\
&= \int_\Omega u(t,x)(f(x) + \Delta u)dx \\
&= \int_\Omega u(t,x)f(x)dx + \int_\Omega u\Delta u dx
\end{aligned}
$$

When $\Omega = (0,1)$ and $f = 0$, this equation becomes

$$
(2.9) \qquad \begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 & \text{in } \ \mathbb{R}_*^+ \times \Omega \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}_*^+ \\ u(t=0,x) = u_0(x) & \text{in } \Omega. \end{cases}
$$

In this case (2.8) becomes

$$
\frac{1}{2}\frac{\partial}{\partial t}\left(\int_\Omega u^2(t,x)dx\right) = \int_\Omega u\frac{\partial^2 u}{\partial x^2}dx = [u\frac{\partial u}{\partial x}]_{\partial\Omega} - \int_\Omega (\frac{\partial u}{\partial u})^2 dx = -\int_\Omega (\frac{\partial u}{\partial x}(t,x))^2 dx.
$$

EXERCISE 2.3. Let $\Omega = (0,1)$ Let $v \in \mathcal{C}^1[0,1]$ such that $v(0) = 0$. Show that

$$
\int_\Omega v^2(x)dx \leq \int_\Omega |\frac{dv}{dx}|^2 dx
$$

Proof. Let $x \in \Omega$.

$$\begin{aligned} v(x)^2 =& v(x)^2 - v(0)^2 \\ =& 2\int_0^x v'(y)v(y)dy \\ & \leq 2(\int_0^x (v'(y))^2 dy)^{\frac{1}{2}}(\int_0^x v(y)^2 dy)^{\frac{1}{2}} \quad \text{Cauchy-Schwartz} \\ & \leq 2(\int_0^1 (v'(y))^2 dy)^{\frac{1}{2}}(\int_0^1 v(y)^2 dy)^{\frac{1}{2}}. \end{aligned}$$

□

2.1.4. The wave equation.

2.1.5. The Laplacian.

2.1.6. Schrödinger's equation.

2.1.7. The Lamé equation.

2.1.8. The Stokes system.

2.1.9. The plate equations. Let u

2.2. Numerical calculation by finite differences

@

Let $(E, \|\cdot\|_E), (F, \|\cdot\|_F)$ be two Banach spaces, e.g. $E = \mathbb{R}^k, F = \mathbb{R}^l$. Let Ω be a nonempty open subset of E, and $f : \Omega \to F$ a $\mathcal{C}^p$-function. We have the Taylor's expansion of f at x,

$$f(x+h) = f(x)+f'(x)\cdot h+\frac{f''(x)}{2!}\cdot h^{(2)}+\ldots+\frac{f^{(p-1)}(x)}{(p-1)!}h^{(p-1)}+\int_0^1 \frac{(1-\xi)^{p-1}}{(p-1)!}f^{(p)}(x+\xi t)\cdot h^{(p)}$$

where $h^{(j)}$ stands for $(h, \ldots, h)$. From this, we get

$$\left\| f(x+h) - \left(f(x) + f'(x)\cdot h + \frac{f''(x)}{2!}\cdot h^{(2)} + \ldots + \frac{f^{(p)}(x)}{p!}h^{(p)} \right) \right\|_F \leq \epsilon \|h\|_E^p,$$

for $\varepsilon > 0$ and h sufficiently small. Let $\varepsilon > 0$. The continuity of $f^{(p)}$ implies of the continuity of $t \mapsto f(x+\xi t)$ for any $t \in \mathbb{R}$ such that $x + \xi t \in \Omega$. Let $\varepsilon > 0$, there exists a $\eta > 0$ such that

$$|t| < \eta \implies \|f(x+\xi t) - f(x)\|_F \leq \varepsilon$$

CHAPTER 3

Sobolev spaces

@

Ω open subset of $\mathbb{R}^n$, $s \in \mathbb{N}$.

$$H^s(\Omega) = \{u \in L^2(\Omega) \mid D^\alpha u \in L^2(\Omega)\ \forall |\alpha| \leq s\} \tag{3.1}$$

is called the Sobolev space, where D^α is the derivative of u in the sense of distributions.

$$H^s(\Omega) \subset L^2(\Omega), \quad H^0(\Omega) = L^2(\Omega).$$

$$\|u\|_s = \left(\sum_{|\alpha| \leq s} \|D^\alpha u\}_0^2 \right)^{1/2} \tag{3.2}$$

where $\|\cdot\|$ denotes the norm on $L^2(\Omega)$.

$$(u, v)_s = \sum_{|\alpha| \leq s} (D^\alpha u, D^\alpha v)_0$$

where $(\cdot, \cdot)$ is the scalar product on L^2.

PROPOSITION 3.1. Let $s \in \mathbb{N}$, $H^s(\Omega)$ is a Hilbert space.

CHAPTER 4

La formule de Taylor

@

Soit f une fonction de classe $\mathcal{C}^k$. Soit $j \leq k$

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \cdots + \frac{1}{j!}f^{(j)}(x)h^j + R_j(h).$$

$$f(x+h) = \sum$$