Quiz 5 (2022-05-23)

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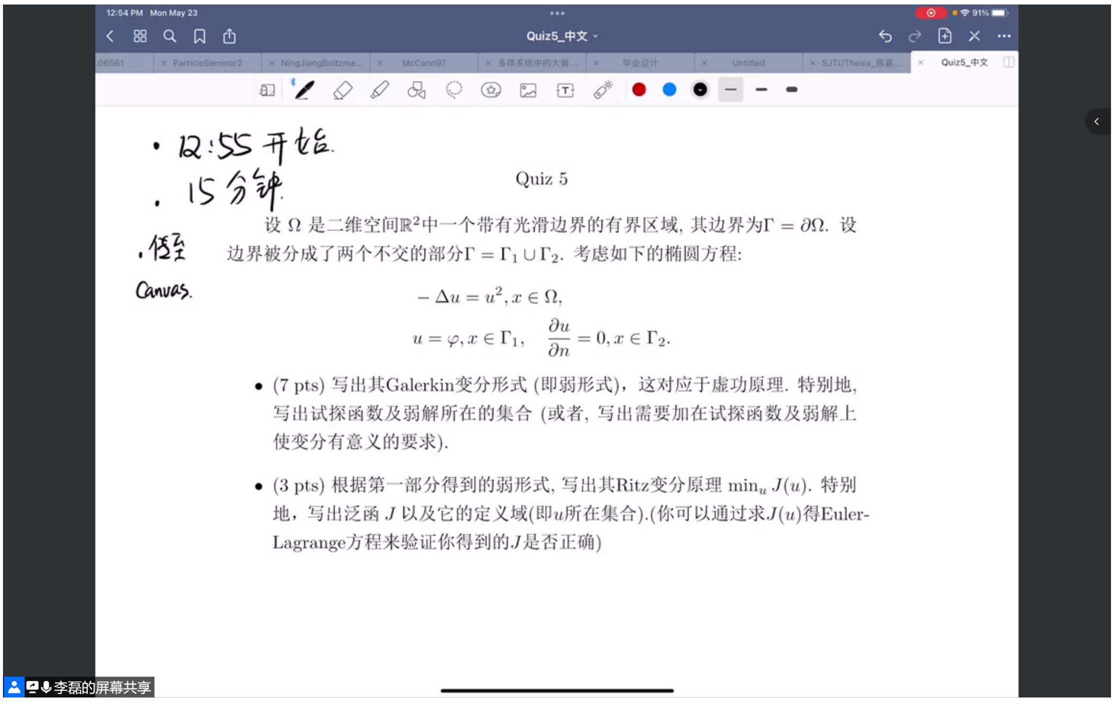

• 12:55 开始

• 15 分钟

• 传至 Canvas.

Quiz 5

设 Ω 是二维空间$\mathbb{R}^2$中一个带有光滑边界的有界区域，其边界为$\Gamma = \partial\Omega$. 设边界被分成了两个不交的部分$\Gamma = \Gamma_1 \cup \Gamma_2$. 考虑如下的椭圆方程:

$$-\Delta u = u^2, x \in \Omega,$$

$$u = \varphi, x \in \Gamma_1, \quad \frac{\partial u}{\partial n} = 0, x \in \Gamma_2.$$

- (7 pts) 写出其Galerkin变分形式 (即弱形式)，这对应于虚功原理. 特别地，写出试探函数及弱解所在的集合 (或者，写出需要加在试探函数及弱解上使变分有意义的要求).
- (3 pts) 根据第一部分得到的弱形式, 写出其Ritz变分原理 $\min_u J(u)$. 特别地，写出泛函 J 以及它的定义域(即u所在集合).(你可以通过求$J(u)$得Euler-Lagrange方程来验证你得到的J是否正确)

MATH6008 quiz 5 2022.05.23. $\boxed{13:10}$ (due)

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$$\begin{cases} -\Delta u = u^2, & x \in \Omega \subset \mathbb{R}^2, \\ u\big|_{\Gamma_1} = \varphi, & \frac{\partial u}{\partial n}\Big|_{\Gamma_2} = 0. \end{cases}$$

$$-\int_\Omega \underbrace{v \Delta u}_{= \nabla\cdot(v\Delta u) - \nabla v\cdot\nabla u} dx = \int_\Omega u^2 v \, dx$$

$$\Rightarrow -\int_{\Gamma_1} v \frac{\partial u}{\partial n} ds - \cancel{\int_{\Gamma_2} v \underbrace{\frac{\partial u}{\partial n}}_{=0} ds} + \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega u^2 v \, dx.$$

令 $v\big|_{\Gamma_1} = 0$. $\Rightarrow D(u,v) := \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega u^2 v \, dx =: F(v)$.

$$u \in K := \left\{ u : \int_\Omega \left(u^2 + |\nabla u|^2 \right) dx < +\infty, \ u\big|_{\Gamma_1} = \varphi \right\},$$

$$v \in H := \left\{ v : \int_\Omega \left(v^2 + |\nabla v|^2 \right) dx < +\infty, \ v\big|_{\Gamma_1} = 0 \right\}.$$

$\therefore$ Galerkin 变分形式（弱形式）：找一个 $u \in K$，使得

$$D(u,v) := \int_\Omega (\nabla u \cdot \nabla v) dx = \int_\Omega u^2 v \, dx =: F(v) \qquad (1)$$

对一切 $v \in H$ 成立.

$v \leftarrow \delta u$，则(1)成为 $\delta \int_\Omega \frac{1}{2} |\nabla u|^2 dx = \delta \int_\Omega \frac{u^3}{3} dx$

$$\Rightarrow J(u) := \int_\Omega \frac{1}{2} |\nabla u|^2 dx - \int_\Omega \frac{1}{3} u^3 dx = \frac{1}{2} D(u,v) - \frac{1}{3} F(v), \ u \in K.$$

$\therefore$ Ritz 变分问题：寻求 $\min\limits_{u \in K} J(u)$.

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~~验~~验证：

$$J(u) = \int_\Omega \frac{1}{2}|\nabla u|^2\,dx \overset{-}{\cancel{+}} \int_\Omega \frac{1}{3}u^3\,dx\,,\quad u \in \underline{\underline{K}}$$

$$\Rightarrow \delta J(u) = \int_\Omega \underbrace{\nabla u \cdot \delta\nabla u}\,dx \overset{-}{\cancel{+}} \int_\Omega u^2 \cancel{dx}\,\delta u\,dx.$$

$$= \nabla\cdot(\cancel{u\nabla\delta u}\ \cancel{u\,\delta u}\ \delta u\,\nabla u) - \delta u\,\Delta u$$

$$= \int_\Gamma \underbrace{\frac{\partial u}{\partial n}}_{\cancel{\Gamma_2=}0}\ \underbrace{\delta u}_{\cancel{\Gamma_2}:\ \delta u|_{\Gamma_2}=0}\,dS \overset{-}{\cancel{+}} \int_\Omega (+\Delta u + u^2)\,\delta u\,dx.,\ \forall \delta u \in H.$$

$$\Rightarrow \begin{cases} -\Delta u = u^2, & x \in \Omega, \\ u|_{\Gamma_1} = \varphi\ (\text{几何边界}), & \dfrac{\partial u}{\partial n}\Big|_{\Gamma_2} = 0\ (\text{自然边界}). \end{cases}$$

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