Assignment #3 for NMPDE

Exercise 1 Determine the variational formulation of following boundary problem:

$$\begin{cases} -u'' + g\,u = f,\ x \in (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

and state the definition of the weak solution to the problem where $f, g \in C^\infty(0,1)$.

Exercise 2 Determine the variational formulation of following boundary problem:

$$\begin{cases} -u'' + u = f,\ x \in (0,1) \\ u(0) = 0,\ u'(1) + u(1) = 0 \end{cases}$$

and state the definition of the weak solution to the problem.

Exercise 3 Which variational problem is associated to the boundary-value problem an ordinary differential equation

$$u''(x) = \mathrm{e}^x,\ x \in (0,1)$$

where $u(0) = u(1) = 0$.

Exercise 4 Consider the elliptic, but not uniformly elliptic, bilinear form

$$a(u,v) = \int_0^1 x^2 u'v'\,\mathrm{d}x$$

on $[0,1]$. Show that the problem

$$\frac{1}{2}a(u,u) - \int_0^1 u\,\mathrm{d}x \to \min$$

does not have a solution in $H_0^1(0,1)$. Determine the associated differential equation.

Exercise 5 Consider a bounded, connected and regular domain $\Omega \subseteq \mathbb{R}^n$ and the

$$\begin{cases} -\Delta u = \lambda u,\ \forall x \in \Omega. \\ u(x) = 0,\ \forall x \in \partial\Omega. \end{cases}$$

deduce its variational formulation where λ is a constant.

Exercise 6 Justify that $u(x) = 1/|x|^\alpha \in H^1$ for $\alpha < (n-2)/2$ and $n > 3$ where $x \in \mathbb{R}^n$ and $|x| := \sqrt{x_1^2 + \cdots + x_n^2}$ is its norm.

Exercise 7 For the following equation with Robin boundary condition

$$\begin{cases} -u'' = f, \ \forall x \in (0,1). \\ u'(0) + \gamma_0 u(0) = \alpha_0 \\ u'(1) + \gamma_1 u(1) = \alpha_1 \end{cases}$$

show its weak formulation is given as follows: determine $u \in H^1$ satisfying

$$a(u,v) = (f,v) + (\alpha_1 - \gamma_1 u(1))v(1) - (\alpha_0 - \gamma_0 u(0))v(0), \forall v \in H^1.$$

also show that the function $w \in H^1$ that minimizes

$$J[w] = a(w,w) - 2(f,w) - 2\alpha_1 w(1) + \gamma_1 w(1)^2 + 2\alpha_0 w(0) - \gamma_0 w(0)^2$$

is u.

Exercise 8 Suppose Ω is contained in an $n-$dimensional cube with side length s, prove that

$$||v||_{L^2(\Omega)} \leq s \cdot ||\nabla v||_{L^2(\Omega)}, \forall v \in H_0^1(\Omega).$$

Hint: establish the inequality for $v \in C_0^\infty$ and use the density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$. Notice that $v(x_1, x_2, \cdots, x_n) = \displaystyle\int_0^{x_1} \partial_1 v(t, x_2, \cdots, x_n)\, \mathrm{d}t$ and use Cauchy-Schwarz inequality.