Math 6008 Numerical PDEs–Lecture 7

FDM for elliptic equations and hyperbolic equations

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1 The properties of the finite difference method

The stability can also be analyzed using the spectral radius of the matrix A. In this section, we will introduce another way: the maximal principle (极值原理).

Here, we will take the 5-point scheme as the example.

Theorem 1. *Let D_h be the set of all interior grid points and ∂D_h is the set of boundary point. Let u_{ij} be a grid function defined on $D_h \cup \partial D_h$.*

- *If $\Delta_h u_{i,j} \ge 0$ for all $(x_i, y_j) \in D_h$, then $\max_{D_h} u \le \max_{\partial D_h} u$. Further, if $\max_{\Omega_h} u = \max_{\Gamma_h}$, then u is a constant.*
- *If $\Delta_h u_{i,j} \le 0$ for all $(x_i, y_j) \in D_h$, then $\min_{D_h} u \ge \min_{\partial D_h} u$. Further, if $\min_{\Omega_h} u = \min_{\Gamma_h}$, then u is a constant.*

证明. We only need to show the first claim as the second claim can be obtained using the first by considering $-u$.

To show the first claim, we prove by contradiction. Suppose that

$$\max_{\Omega_h} u \ge \max_{\Gamma_h},$$

and the maximum M is achieved at some (x_{i_0}, y_{j_0}). Since $\Delta_h u_{i,j} \ge 0$, then

$$u_{i_0,j_0} \le \frac{1}{1/h^2 + 1/k^2}[\frac{u_{i_0+1,j_0} + u_{i_0-1,j_0}}{2h^2} + \frac{u_{i_0,j_0+1} + u_{i_0,j_0-1}}{2k^2}]$$
$$\le \frac{1}{1/h^2 + 1/k^2}[\frac{M+M}{2h^2} + \frac{M+M}{2k^2}] = M.$$

Since the left is the maximum, we know that the four values must all equal to M. This argument can be repeated until the neighborhood becomes a point on the boundary. This means u is a constant. The first claim then follows. $\square$

Using the maximal principle, we can conclude the ℓ^∞-stability

Theorem 2. *Let g be the boundary value of u and*

$$-\Delta_h u_{ij} = f_{ij}, \quad (i,j) \in D_h.$$

Then, there exists a which is independent of h, k and u such that

$$\max_{D_h} |u_{ij}| \leq \max_{\partial D_h} |g_{ij}| + \frac{a^2}{2} \max_{D_h} |f_{ij}|.$$

In other words, the FDM is ℓ^∞ stable:

$$\|u\|_\infty \leq \|g\|_\infty + C\|f\|_\infty.$$

证明. For the ℓ^∞ stability, consider an auxiliary function ϕ such that $\Delta_h \phi = 1$. Then,

$$\Delta_h(u + \phi\|f\|_\infty) = -f + \|f\|_\infty \geq 0.$$

The discrete maximum principle implies that

$$u + \phi\|f\|_\infty \leq \max_{\Gamma_h}(g + \phi\|f\|_\infty) \Rightarrow u \leq \|g\|_\infty + 2\|\phi\|_\infty\|f\|_\infty.$$

Then, one applies the same argument for $-u$. The claims follows.

To finish the proof, we must show that ϕ exists. The example given in the book is

$$\phi(x,y) = \frac{1}{2}x^2.$$

Another frequently used example is

$$\phi = \frac{1}{4}((x - \frac{1}{2})^2 + (y - \frac{1}{2})^2).$$

$\square$

Lastly, we conclude easily that

Theorem 3. *For any given data g and f, the FDM has a unique solution. Moreover, if the solution to the PDE is smooth enough, then the error decays like*

$$\|u(x_i, y_j) - u_{ij}\|_\infty \leq C(h^2 + k^2).$$

证明. For the uniqueness, suppose there are two solutions u_1 and u_2. Then, $\Delta_h^2(u_1 - u_2) = 0$ and the boundary values of $u_1 - u_2$ are zero. The discrete maximum principle implies that $u_1 - u_2 \leq 0$ for all interior points. Then, switching the roles of u_1 and u_2, we have $u_2 - u_1 \leq 0$. Hence, $u_1 = u_2$.

For the convergence, define the error $e_{ij} = u_{ij} - u(x_i, y_j)$. It is easy to see that

$$-\Delta_h e_{ij} = -\tau_{ij}.$$

The boundary value of e_{ij} is zero. Then, by the stability estimate above,

$$\|e_{ij}\|_\infty \leq \frac{a^2}{2}\|\tau_{ij}\|_\infty \leq C(h^2 + k^2).$$

$\square$

2 Other topics

- For other boundary conditions, like the mixed boundary condition or curved boundaries, there are several ways like interpolation etc. Read section 5.3 of the reference book.

  For example, consider that $\Omega = [0, 1] \times [0, 1]$. We approximate

  $$\frac{\partial u}{\partial n} + \gamma u = \beta$$

  at the boundary $y = 0$ (the bottom).

  Here, the normal direction is $-\hat{y} = -\vec{j}$. Hence, the condition is actually,

  $$-\partial_y u + \gamma u = \beta.$$

We apply the ghost point method:

$$\frac{u_{i,-1} - u_{i,1}}{2k} + \gamma_{i,0} u_{i,0} = \beta_{i,0}.$$

To elminate $u_{i,-1}$, we also impose the equation at this point so that we can eliminate $u_{i,-1}$ to get a second order approximation for the boundary condition.

- For the variable coefficient $-\partial_x(a(x,y)\partial_x u) - \partial_y(b(x,y)\partial_y u) = f$, the treatment is similar to the 1D case. We omit.

- Regarding the eigenvalue problem or the biharmonic equations, the treatment should be similar.

Advection equation

We consider the simplest hyperbolic equation, the advection equation $u_t + au_x = 0$ with constant coefficient.

The analytical solution is known to be

$$u(x,t) = u_0(x - at).$$

3 First order schemes

3.1 Lax-Friedrichs

In the previous lectures, we have the following method where the centered difference in space and forward Euler in time are applied:

$$\frac{u_j^{n+1} - u_j^n}{\tau} = -\frac{a}{2h}(u_{j+1}^n - u_{j-1}^n).$$

By our analysis before, this scheme is unstable for a fixed τ/h ratio, but if $\tau = O(h^2)$, we can have convergence.

The requirement for $\tau = O(h^2)$ is difinitely not good because this equation is not very stiff. To overcome this difficulty, Lax and Friedrichs introduced the following where u_j^n is replaced with $\frac{1}{2}(u_{j-1}^n + u_{j+1}^n)$

$$u_j^{n+1} = \frac{1}{2}(u_{j-1}^n + u_{j+1}^n) - \frac{a\tau}{2h}(u_{j+1}^n - u_{j-1}^n).$$

This scheme is called the Lax-Friedrichs scheme. It turns out that this has a better stability compared with the method just mentioned.

Taylor expansion shows that the local truncation error is

$$T_j^n = O(\tau + h^2 + h^2/\tau).$$

If one fixes the parameter $\lambda := \tau/h$, one sees that this method is first order method.

Let us perform the von Neumann analysis. The growth factor is given by

$$G(\xi;\tau,h) = \cos(\xi h) - ia\lambda\sin(\xi h) \Rightarrow |G|^2 = 1 + (a^2\lambda^2 - 1)\sin^2(\xi h).$$

Hence, if $a\lambda \le 1$, the method is (ℓ^2) stable. Hence, we conclude that

Proposition 1. *If $a\lambda \le 1$, the Lax-Friedrichs scheme is stable, and if λ is fixed, the method is convergent with first order accuracy.*

Remark 1. *One can also compute the eigenvalues to check the stability condition using the theory of stability region.*

3.2 Upwind schemes

Another way is to use one-sided finite difference for u_x. There are then two options

$$\begin{aligned}\frac{u_j^{n+1} - u_j^n}{\tau} &= -a\frac{1}{h}(u_j^n - u_{j-1}^n)\\ \frac{u_j^{n+1} - u_j^n}{\tau} &= -a\frac{1}{h}(u_{j+1}^n - u_j^n).\end{aligned}$$

Performing von-Neumann analysis, we obtain the growth factor

$$G(\xi;\tau,h) = 1 - a\lambda(1 - e^{-ih\xi}) = (1 - a\lambda(1 - \cos(\xi h))) - ia\lambda\sin(h\xi)$$
$$G(\xi;\tau,h) = 1 - a\lambda(e^{ih\xi} - 1) = (1 + a\lambda(1 - \cos(\xi h))) - ia\lambda\sin(h\xi).$$

For the first, one has

$$|G(\xi;\tau,h)|^2 = 1 - 4a\lambda(1 - a\lambda)\sin^2(\frac{\xi h}{2}),$$

while for the second,

$$|G(\xi;\tau,h)|^2 = 1 + 4a\lambda(1 + a\lambda)\sin^2(\frac{\xi h}{2}).$$

Clearly, if $a > 0$, the first scheme satisfies that

$$|G|^2 \le 1, \quad \text{if } a\lambda \le 1.$$

The second scheme is unstable for $a > 0$.

Correspondingly, if $a < 0$, the first scheme is unstable while the second is stable if $|a|\lambda \le 1$.

When $a > 0$, the information of the PDE moves to right. The information flowing to j is from $j-1$. Hence, we use u_{j-1}^n and u_j^n. When $a < 0$, we should use u_j^n and u_{j+1}^n. The one-sided differences therefore follows the direction of the moving or direction of 'wind'. Since the point where were are (i.e. x_j) is the pointing where the 'wind' is flowing to, we call these schemes 'upwind schemes'.

3.3 Discussion on the two first order schemes

Let us now perform some discussion on the two schemes to gain our understanding.

- According to the discussion, the Lax-Friedrichs scheme can be applied without conisdering the sign of a, while the upwind needs to consider the sign of a. The upwind scheme may be written as a form without considering the sign of a:

$$\frac{u_j^{n+1}-u_j^n}{\tau}+\frac{1}{2}(a+|a|)\frac{u_j^n-u_{j-1}^n}{h}+\frac{1}{2}(a-|a|)\frac{u_{j+1}^n-u_j^n}{h}=0.$$

- Diffusion coefficient.

  The Lax-Friedrichs can be rewritten as

  $$\frac{u_j^{n+1}-u_j^n}{\tau}+aD_0u_j=\frac{h^2}{2\tau}\Delta_h u_j^n.$$

  The upwind scheme for $a>0$ can be rewritten as

  $$\frac{u_j^{n+1}-u_j^n}{\tau}+aD_0u_j=\frac{ah}{2}\Delta_h u_j^n.$$

  For the upwind scheme, we see clearly that if $a<0$, the coefficient for the diffusion is negative and such model is ill-posed and it must be unstable. Hence, this upwind scheme must be used for $a>0$.

  Often, if $a\lambda<1$, the error of the Lax-Friedrichs can be larger as the coefficient for the modified diffusion is bigger.

  Further, assume that we have periodic boundary condition. The eigenvalues for

  $$\frac{u_j^{n+1}-u_j^n}{\tau}+aD_0u_j=\epsilon\Delta_h u_j^n.$$

  are

  $$\lambda_p=-\frac{ia}{h}\sin(2\pi ph)-\frac{2\epsilon}{h^2}(1-\cos(2\pi ph)).$$

  Hence, $\tau\lambda_p$ falls on the ellipse $(\frac{x}{2\tau\epsilon/h^2}+1)^2+y^2/(a^2\tau^2/h^2)=1$. To ensure this ellipse to be in the stability region of the forward Euler, we must have $|a\tau/h|\le 1$ and $\frac{2\tau\epsilon}{h^2}\le 1$.

4 Courant-Friedrichs-Lewy(CFL) condition

Condition 1. *A numerical method can be convergent only if its numerical domain of dependence contains the true domain of dependence of the PDE, at least in the limit as* $\tau,h\to 0$.

We look at the upwind scheme.

For the upwind scheme for $a > 0$, u_j^{n+1} depends on u_j^n and u_{j-1}^n. For the PDE, $u(x_j, t^{n+1})$ depends on $u(x_j - a\tau, t^n)$. Hence, we need $x_j - a\tau$ to be between x_{j-1} and x_j, or $a\tau \leq h$.

For the heat equation with forward Euler, u_j^{n+1} depends on $u_{j-1}^n, u_j^n, u_{j+1}^n$. For the PDE, the dependence domain is the whole axis. However, if $\tau = O(h^2)$, as $h \to 0$, x_{j-1}, x_j, x_{j+1} will be the whole axis as $h \to 0$ since $2h/\tau \to \infty$.

Note that the CFL condition is not sufficient. Even if the condition is satisifed, the method can be unstable. Using the centered forward Euler scheme as the exaple, we can see that the CFL condition is satisfied if $a\lambda \leq 1$. However, it can be unstable.

5 Second order schemes

The schemes above have first order accuracy. We now look at some second order schemes.

5.1 Leapfrog and Crank-Nicolson

If we use the midpoint method for time and centered difference for the space, we have the leapfrog scheme:

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\tau} = -\frac{a}{2h}(u_{j+1}^n - u_{j-1}^n).$$

The accuracy is $O(k^2 + h^2)$. This is the **leapfrog** scheme. The reason to use this three-level scheme is that the eigenvalues of the finite difference are $\lambda_p = -i\frac{a}{h}\sin(2\pi ph)$ by assuming the boundary condition. Recall the stability region of the midpoint method is $(-i, i)$. Hence, we need $|a\lambda| < 1$ for the method to be stable (at this point, we are not claiming it is sufficient, but it is indeed sufficient).

By converting this into a two-level system, one may find the growth matrix in the Fourier domain as

$$G(\xi;\tau,h)=\begin{bmatrix}-2a\lambda i\sin(\xi h) & 1\\ 1 & 0\end{bmatrix}.$$

The characteristic polynomial is

$$\lambda^2+2a\lambda i\sin(\xi h)-1=0.$$

Since $\lambda_1\lambda_2=-1$, for $\rho(G)\le 1$, we must require both magnitudes to be 1. This implies that $-2a\lambda i\sin(\xi h)=-2i\sin\theta$ for some θ. Hence, one must equire $a\lambda\le 1$. If, however, $a\lambda=1$, when $\sin(\xi h)=1$, there are two eigenvalues to be equal. $\|G^n\|$ can have unbounded norm. Hence, $a\lambda<1$. By Theorem 3.7 in the book, this is a sufficient condition for stability.

The Crank-Nicolson is to apply trapezoidal in time. It is two-time level but it is implicit. The implicit schemes are not common in hyperbolic equations since hyperbolic equations are not stiff. Besides, the discretization does not yield symmetric matrices, so that inverting is not desired in numerics.

5.2 Using the interpolation or the Taylor expansion to construct the schemes

Charateristics and interpolation

In Figure 5.2, the solution at P is determined by the solution at Q in the original advection equation. The length of QC is $a\lambda h$.

By linear interpolation,

$$u(P)=u(Q)\approx(1-a\lambda)u(C)+a\lambda u(B)\Rightarrow u_j^{n+1}=u_j^n-a\lambda(u_j^n-u_{j-1}^n).$$

If one uses B and D to do linear interpolation, then

$$u_j^{n+1}=\frac{1-a\lambda}{2}u_{j+1}^n+\frac{1}{2}(1+a\lambda)u_{j-1}^n.$$

Ths is just Lax-Friedrichs.

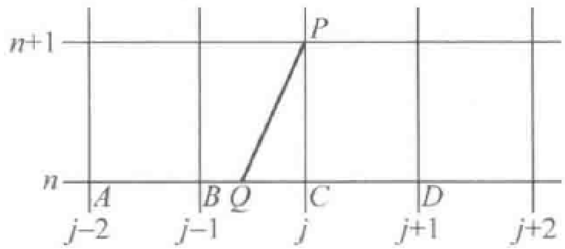

图 1: Illustration of derivation using characteristics

To achieve better accuracy, one may use second order interpolation. If one uses the BCD to approximate, then one has by Lagrange interpolation

$$u_j^{n+1} = \frac{(x_Q - x_C)(x_Q - x_D)}{(x_B - x_C)(x_B - x_D)}u(B) + \frac{(x_Q - x_B)(x_Q - x_D)}{(x_C - x_B)(x_C - x_D)}u(C) + \frac{(x_Q - x_B)(x_Q - x_C)}{(x_D - x_B)(x_D - x_C)}u(D).$$

Then,

$$\frac{(-a\lambda)(-1 - a\lambda)}{(-1)(-2)}u_{j-1}^n + \frac{(1 - a\lambda)(-1 - a\lambda)}{1 * (-1)}u_j^n + \frac{(1 - a\lambda)(-a\lambda)}{2 * 1}u_{j+1}^n.$$

Hence, one has

$$u_j^{n+1} = u_j^n - \frac{1}{2}a\lambda(u_{j+1}^n - u_{j-1}^n) + \frac{1}{2}a^2\lambda^2(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

This is called the **Lax-Wendroff scheme**. Clearly, there is a difference whether a is positive or negative.

The local truncation error is $O(\tau^2 + h^2)$. Using von-Neumann or MOL eigenvalue approach, the stability condition is $|a\tau/h| \leq 1$.

If one considers the flow of information, one may use A, B, C to do the Lagrange interpolation. The resulted formula is

$$u_j^{n+1} = u_j^n - a\lambda(u_j^n - u_{j-1}^n) - \frac{a\lambda}{2}(1 - a\lambda)(u_{j+1}^n - 2u_{j-1}^n + u_{j-2}^n)$$

This is the **Beam-Warming method**.

Direct Taylor expansion shows that the local truncation error is $O(\tau^2 + h^2)$. The stability condition is better:

$$\frac{a\tau}{h} \le 2.$$

You can use the von-Neumann, or CFL condition to derive this.

Taylor expansion

Consider

$$u(x_j, t^{n+1}) = u(x_j, t^n) + u_t(x_j, t^n)\tau + \frac{1}{2}u_{tt}(x_j, t^n)\tau^2 + O(\tau^3).$$

By the equation, $u_t = -au_x$ and $u_{tt} = -au_{xt} = a^2 u_{xx}$. Hence,

$$u(x_j, t^{n+1}) = u(x_j, t^n) - a\tau u_x + \frac{1}{2}a^2\tau^2 u_{xx} + O(\tau^3).$$

We then obtain the Lax-Wendroff method:

$$u_j^{n+1} = u_j^n - a\tau D_0 u_j^n + \frac{1}{2}a^2\tau^2 D^2 u_j^n.$$

In the Beam-Warming, the upwind idea is used. For $a > 0$, we use $u_{j-2}^n, u_{j-1}^n, u_j^n$ to approximate u_x and u_{xx}.Then, we have

$$u_j^{n+1} = u_j^n - \frac{a\tau}{2h}(3u_j^n - 4u_{j-1}^n + u_{j-2}^n) + \frac{a\tau^2}{2h^2}(u_j^n - 2u_{j-1}^n + u_{j-2}^n).$$