Numerical methods for PDEs: HW1

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1 Theory

(5 pts)

1. Let $t_n = n\tau$. Derive the finite difference formulas if we use the data at t_{n-2}, t_{n-1} and t_n to approximate $u'(t_n)$ and $u''(t_n)$. Find the local truncation errors for these two formulas.

2. Solve the advection equation
$$u_t + u_x = e^t, \quad u(x,0) = e^{-x^2}.$$
If we change $u(x,0)$ on $[0,1]$, which part of $u(x,1)$ will be affected?

3. Solve the diffusion equation
$$u_t = u_{xx}, \quad u(x,0) = e^{-x^2}.$$
If we change $u(x,0)$ on $[0,1]$, which part of $u(x,1)$ will be affected?

4. Solve the diffusion equation on $x \in [0,1]$
$$u_t = u_{xx}, \quad u(x,0) = e^{-x}, \quad u(0,t) = u(1,t) = 0,$$
by expanding the solutions using the Fourier basis $\{\sin(anx)\}$ (where you need to determine a). Discuss the behavior for different modes.

2 Numerics

(5 pts)

Solve the advection equation

$$u_t + u_x = 0, \quad u(x,0) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0, \end{cases}$$

numerically using the two schemes mentioned in class, namely

$$\frac{u_j^{n+1} - u_j^n}{\tau} + a\frac{u_{j+1}^n - u_j^n}{h} = 0, \quad n = 0, 1, 2, \cdots, \quad , j = 0, \pm 1, \pm 2, \cdots,$$

and

$$\frac{u_j^{n+1} - u_j^n}{\tau} + a\frac{u_j^n - u_{j-1}^n}{h} = 0, \quad n = 0, 1, 2, \cdots, \quad , j = 0, \pm 1, \pm 2, \cdots.$$

Choose the parameters to be $\lambda = \tau/h = 0.9$ and 2, for both schemes.

To numerically compute the solution using these methods, you need to truncate the domain $\mathbb{R}$. In this example, let us truncate it to be $[-5, 5]$. Compute the solution at time $T = 4$.