NMPDE 2022
PROBLEM 1

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exercise 1.1. *Let* $c, f \in \mathcal{C}^0([0,1])$*,* α, β *two constants. We consider the following problem: The existence of* $u \in \mathcal{C}^2([0,1])$ *satisfying*

$$\begin{cases} -u''(x) + c(x)u(x) = f(x) & 0 < x < 1 \\ u(0) = \alpha, u(1) = \beta. \end{cases} \tag{1}$$

Let $N \in \mathbb{N} \setminus \{0\}$*. We set*

$$x_i = ih, \quad i = 0, 1, \dots, N+1 \qquad \text{(uniform mesh with step } h \text{ in } [0,1].)$$

where $h = 1/(N+1)$.

$$x_i = \textit{are the nodes}$$

Let u *be an exact solution to* (5)*. We assume that* $u \in \mathcal{C}^4([0,1])$.

(i) Using Taylor expansion, determine $u(x_{i+1})$ *and* $u(x_{i-1})$ *in terms of* $u(x_i), u'(x_i), \dots$*. Deduce that*

$$-u(x_{i+1}) + 2u(x_i) - u(x_{i-1}) = ?$$

and

$$-u''(x_i) = \frac{-u(x_{i-1}) + 2u(x_i) - u(x_{i+1})}{h^2} + \frac{h^2}{12}u^{(4)}(x_i + \theta_i h), \text{ \textit{with} } |\theta_i| < 1, 1 \le i \le N.$$

We set

$$u_i = u(x_i), c_i = c(x_i), f_i = f(x_i) \quad i = 1, \dots, N.$$

(ii) Using equation (5)*, deduce that*

$$\begin{cases} -\frac{\alpha}{h^2} + \frac{2u_1 - u_2}{h^2} + c_1 u_1 = & f_1 - \frac{h^2}{12}u^{(4)}(x_1 + \theta_1 h) \\ \frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} + c_i u_i = & f_i - \frac{h^2}{12}u^{(4)}(x_i + \theta_i h) \; i = 2, \dots, N-1, \\ \frac{-u_{N-1} + 2u_N}{h^2} - \frac{\beta}{h^2} + c_N u_N = f_N - \frac{h^2}{12}u^{(4)}(x_N + \theta_N h). \end{cases} \tag{2}$$

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(iii) *In the sequel, we suppose that $\alpha = \beta = 0$.*

Verfiy that (2) *can be written as follows:*

$$A_h u_h = b_h - \frac{h^2}{12} r_h(u) \tag{3}$$

where $u_h =$

$$u_h = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix}$$

A_h *is a matrix of type* $N \times N$, b_h *and* $r_h(u)$ *column vectors.*

(a) *Determine* A_h, b_h *and* r_h.

(b) *We consider the norm* $\|\cdot\|_2$, [a] *, show that*

$$\|A_h u_h - b_h\|_2 = O(h^{3/2})$$

(Hint: recall that u is C^4 on $[0,1]$.). Now, consider the sup norme $\|\cdot\|_\infty$*:*

$$\|A_h u_h - b_h\|_\infty = O(h^\alpha),$$

determine α.

(iv) *We consider the following discrete problem:*

$$A_h v_h = b_h \tag{4}$$

where A_h *and* b_h *are given in the previous question.*

Now, we suppose that

$$c(x) \geq 0 \quad \forall x \in [0,1].$$

(a) *Verify the following*

$$v^t A_h v = \sum_{i=1}^{N} c_i v_i^2 + \frac{1}{h^2}\left(v_1^2 + v_N^2 + \sum_{i=2}^{N}(v_i - v_{i-1})^2\right), \quad \forall v \in \mathbb{R}^N.$$

Show that A_h *is positive.*

(b) *Deduce that the linear system has a unique solution.*

(v) *Let* $v \in \mathbb{R}^N$*, such that* $A_h v \geq 0$ *(i.e. the coordinates of* $A_h v$ *are* ≥ 0*).*

(a) *Verify that*

$$\begin{aligned} v_2 &\leq (2 + c_1 h^2) v_1 \\ v_i + v_{i+1} &\leq (2 + c_i h^2) v_i, \quad i = 2, \dots, N-1 \\ v_{N-1} &\leq (2 + c_N h^2) v_N. \end{aligned}$$

[a] Recall that $\|(x_1, \dots, x_N)\|_2 = (x_1^2 + \dots + x_N^2)^{\frac{1}{2}}$

(b) *Prove that* $\min_{1\le i\le N} v_i \ge 0$, *for* $i = 2, \dots, N-1$. *Hint: Consider* $p \in \{1, \dots, N\}$ *verifying* $v_p \le v_i$ *for every* i. *For example, you may use*

$$0 \le -v_{i-1} + (2 + c_i h^2)v_i - v_{i+1} \le$$

(c) *We may assume that* p *is the smallest integer* $\in \{1, \dots, N\}$ *such that* $v_p = \min_i v_i$. *By adapting the proof of the preceding question, show that* $v_p \ge 0$. *We have then established the following :*

$$\textit{si } v \in \mathbb{R}^N \textit{ avec} A_h v \ge 0 \implies v \ge 0.$$

(d) *Show that*

$$A_h^{-1} \ge 0^b.$$

Observe that $A_h(A_h^{-1}e_i) \ge 0$, *where* e_i *is an element of the standard basis of* $\mathbb{R}^N$.

(e) *Let* A_{0h} *be the matrix which corresponds to* A_h *but assuming* $c \equiv 0$. *Show that*

$$A_{0h}^{-1} - A_h^{-1} \ge 0$$

Note that $A_h - A_{0h} \ge 0$.

(f) *Deduce that*

$$\|A_h^{-1}\|_\infty \le \|A_{0h}^{-1}\|_\infty$$

(g) *Verify that* $\psi(x) = \frac{1}{2}x(1-x)$ *is an exact solution to*

$$\begin{cases} -u''(x) = 1 & 0 < x < 1 \\ u(0) = 0, u(1) = 0. \end{cases} \tag{5}$$

and

$$(A_{0h}^{-1}e)_i = \psi(x_i), \forall i$$

where $e = (1, \dots, 1)$.

Observe that

$$\|A_{0h}^{-1}e\|_\infty = \max_{1\le i\le N} |\psi(x_i)|,$$

and, deduce that

$$\|A_{0h}\|_\infty \le \frac{1}{8}.$$

(vi) *Show that*

$$u_h - v_h = -\frac{h^2}{12}A_h^{-1}r_h(u_h)$$

(recall that u_h *exact solution , and* v_h *is the discrete solution.)*

(vii) *Conclude that*

$$\|u_h - v_h\|_\infty \le \frac{h^2}{96} \max_{x\in[0,1]} |u^{(4)}(x)|.$$

This shows that the method of the exercise converges as $h \to 0$.

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[b] This means. the following , for $M = (m_{ij})_{ij}$ square matrix. We say that $M \ge 0$ if $m_{ij} \ge 0, \forall i, j$.