HOMEWORK 1
SOME DIFFERENTIAL CALCULUS

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1.

exercise 1.1. *Let I be an open interval of $\mathbb{R}$, let $a \in I$. Let F be a normed vector space. Let*

$$f : I \to F$$

be a continuous function at a.

Show that f is derivable at a iff the expression

$$\Delta(h,k) = \frac{f(a+h) - f(a-k)}{h+k}$$

admits a limit as (h,k) tends to $(0,0)$ with $h, k > 0$. Compute this limit.

exercise 1.2. *Let E be a normed space $\neq 0$. Show that*

$$\|\cdot\| : E \to \mathbb{R}$$

is not derivable at the origin.

exercise 1.3. *Let F be a normed space, $f : [a,b] \to F$ a continuous function, which is right differentiable on (a,b).*

If $f'_d : (a,b) \to F$ is continuous at a point $x_0 \in (a,b)$, show that f is derivable at x_0. (hint : use the mean value theorem); where f'_d is the right derivative of f.

exercise 1.4. *Let E, F be normed spaces. We suppose that $\dim E > 1$. Let Ω be a convex open subset of E, $a \in \Omega$ and $f : \Omega \setminus \{a\} \to F$ a differentiable function with*

$$\|Df(x)\| \leq k \quad \forall x \in \Omega \setminus \{a\}$$

(Df is the differential of f). Show that

$$\|f(x) - f(y)\| \leq k\|x - y\| \quad \forall x, y \in \Omega \setminus \{a\}.$$

exercise 1.5. *Let φ be a smooth function with support in $[-1,1]$. Show that*

$$\lim_{\mu \to 0^+} \frac{1}{\sqrt{4\pi\mu}} \int_{-\infty}^{\infty} \varphi(y) \exp(-\frac{y^2}{\sqrt{\mu}}) dy = \varphi(0).$$

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exercise 1.6. *Let θ_0 be a continuous and uniformly bounded on $\mathbb{R}$. Show that*

$$\theta(t,x) = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} \theta_0(y) \exp(-\frac{(x - Vt - y)^2}{4\nu t})dy. \tag{1}$$

exists, and is a solution to the following equation

$$\begin{cases} \frac{\partial \theta}{\partial t} + V\frac{\partial \theta}{\partial x} - \nu\frac{\partial^2 \theta}{\partial x^2} = 0 & \textit{in } \mathbb{R} \times \mathbb{R}_*^+ \\ \theta(t=0,x) = \theta_0(x) & \textit{in } \mathbb{R}. \end{cases} \tag{2}$$

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