Numerical methods for PDEs: HW4

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1 Theory

(10 pts)

1. (路金甫，第三版) 第三章习题 3, 6, 7.
2. Derive the modified equation for Lax-Wendroff method. Explain why Lax-Wendroff will give fake oscillation near discontinuity.
3. Consider the following scheme for $u_t + au_x = 0$:
$$\frac{u_j^{n+1} - u_j^n}{\tau} = -a\frac{u_{j+1}^n - u_{j-1}^n}{2h}$$
Compute the modified equation. Using the modified equation to justify that choosing $\tau \sim h$ makes the scheme unstable.

2 Numerics

(8 pts)

1. (路金甫，第三版) 第三章习题 5
2. Consider the equation $u_t + au_x = 0$.
   (a) Derive the finite difference for u_x using $u_{j-2}^n, u_{j-1}^n, u_j^n$ and u_{j+1}^n.
   (b) Using the finite difference scheme in (a) for space and RK3 for time, compute the solution at $t = 2$ numerically, with initial condition
   $$u(x,0) = e^{-20(x-1)^2}.$$
   Explain why the RK3 in time is appropriate (Hint: use the theory of stability region).

3. Consider the Burgers' equation

$$\partial_t u + u u_x = 0$$

with initial data

$$u_0(x) = \frac{1}{1+x^2}.$$

Let $f(u) = u^2/2$. Consider the following two upwind schemes with conservative form

$$u_j^{n+1} = \begin{cases} u_j^n - \lambda(\frac{1}{2}(u_j^n)^2 - \frac{1}{2}(u_{j-1}^n)^2), & f'(u_j^n) \geq 0 \\ u_j^n - \lambda(\frac{1}{2}(u_{j+1}^n)^2 - \frac{1}{2}(u_j^n)^2), & f'(u_j^n) < 0 \end{cases}$$

and nonconservative form

$$u_j^{n+1} = \begin{cases} u_j^n - \lambda u_j^n(u_j^n - u_{j-1}^n), & u_j^n \geq 0 \\ u_j^n - \lambda u_j^n(u_{j+1}^n - u_j^n), & u_j^n < 0 \end{cases}$$

Do the simulation up to $t = 2$. Discuss the solutions obtained by these two schemes before and after the discontiuity forms.

We remark that a more reasonable generalization of the conservative upwind scheme here is the Gudonov scheme.