Numerical methods for PDEs: HW6

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1 Theory

(15 pts)

1. (路金甫，第三版) 第六章习题 1, 4, 6(2),(3) (第6题只需写出两个变分问题，不需要证明三个问题之间关系), 10.

2. (路金甫，第三版) 第七章习题 4, 9.

3. Consider the following equation

$$\frac{d^4u}{dx^4} = f,$$
$$u(0) = u'(0) = u(1) = u'(1) = 0$$

(a). Find a weak formulation for this problem and specify the space where the weak solution is from.

(b). In the conditions $u(0) = \alpha_0, u'(0) = \alpha_1, u''(0) = \alpha_2, u'''(0) = \alpha_3$, which are essential boundary conditions? Which of them are natural boundary conditions?

(c). To formulate an FEM, we need a finite element space V_h such that if $v \in V_h$, then $v|_K \in P_3(K)$ (here P_3 means polynomials of third degree) where K is an interval. Find the basis functions for V_h and explain why $P_2(K)$ is not enough.

2 Numerics

(10 pts)

1. (路金甫，第三版) 第七章习题 1(2). 同时说明，边界条件$u'(1) = 0$如何在你构造的方程组种体现出来的.

2. Consider

$$-\Delta u + u = f \quad (x, y) \in \Omega = [0, 1] \times [0, 1]$$
$$\frac{\partial u}{\partial n} = g \quad \partial\Omega$$

   Use uniform square elements (正方形单元) and on each of them, you may use the bilinear functions. For $f = \sin(\pi x) + \sin(\pi y)$ and $g = 0$, program and solve the problem. Use your code to determine the order p for approximating the function values.

3. Determine numerically which one of the following two Fourier spectral methods for computing derivatives is better:

   - Given $v = (v_1, \ldots, v_N)$, compute $\hat{v}$.
   - Define $\hat{w}_k = ik\hat{v}_k$, for $k = -N/2+1, \ldots, N/2$ (or $k = 0, \ldots, N/2, -N/2+1, \ldots, -1$ in Matlab.).
   - Compute the inverse DFT (inverse FFT) and get w_k, the real part of which is the approximation of the derivative.

   and

   - Given $v = (v_1, \ldots, v_N)$, compute $\hat{v}$.
   - Define $\hat{w}_k = ik\hat{v}_k$, for $k = 0, 1, \ldots, N$.
   - Compute the inverse DFT (inverse FFT) and get w_k, the real part of which is the approximation of the derivative.