Math 6008 Numerical PDEs–Lecture 5

von Neumann condition and FDM for elliptic equations

Instructor: Lei Li, INS, Shanghai Jiao Tong University;
Email: leili2010@sjtu.edu.cn

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1 Cotinuation of the Fourier analysis for stability

1.1 von Neumann condition

The following is direct observation:

Proposition 1. *For the linear evolutionary equation with constant coefficients, a scheme is stable if and only if there exists τ_0, K such that for wall $\tau \le \tau_0$, $n\tau \le T$, $\xi \in [-\pi/h, \pi/h)$ such that*

$$\|G(\xi, \tau, h)^n\| \le K.$$

Using a fact from linear algebra, which says that

$$\rho(A) \le \|A\|$$

for any matrix A and operator norm $\|\cdot\|$ where $\rho(A) = \sup_i |\lambda_i|$, the largest magnitude of the eigenvalues, is called the spectral radius. [For general operators, the spectrum may include other points than eigenvalues]. Then,

$$(\rho(G))^n = \rho(G^n) \le \|G^n\| \le K.$$

Hence,

$$\rho(G(\xi, \tau, h)) \le K^{1/n} = K^{\tau/T} \le 1 + M\tau, \forall \tau \le \tau_0.$$

for some constant M that depends on K and τ_0. Hence,

Theorem 1. *A necessary condition for the stability is that*

$$|\lambda_j| \le 1 + M\tau, \forall j, \tau \le \tau_0.$$

This condition is called the **von Neumann condition**.

Theorem 2. *If there exists $M > 0$ such that*

$$\|G(\xi, \tau, h)\| \leq 1 + M\tau$$

for any $\xi \in [-\pi/h, \pi/h)$ and all $n\tau \leq T$, then the scheme is stable.

证明. The proof is straightforward:

$$\|G^n\| \leq \|G\|^n \leq (1 + M\tau)^n \leq \exp(Mn\tau) \leq \exp(MT) =: K.$$

Then,

$$\|\hat{u}^n\|_2 \leq \sup_{\xi} \|G^n(\xi)\| \|\hat{u}_0\|_2 \leq K \|\hat{u}_0\|_2.$$

Using the Parseval equality, the conclusion follows. □

Hence, if there are some conditions such that

$$\rho(G) = \|G\|,$$

the the von Neumann condition is both sufficient and necessary. One possible condition is that G is a normal matrix (正规矩阵, $GG^* = G^*G$ where $G^* = \bar{G}^T$). In fact, if G is a scalar, this holds.

Corollary 1. *If $G(\xi, \tau, h) = g(\xi, \tau, h)$ is a scalar (i.e., u is a scalar), then a sufficient and necessary condition for the ℓ^2-stability is that there exists $M \geq 0$ such that the amplification factor g satisfies*

$$|g(\xi, \tau, h)| \leq 1 + M\tau.$$

1.2 More examples

Example 1. *For the method,*

$$\frac{u_j^{n+1} - u_j^n}{\tau} = -\frac{a}{2h}(u_{j+1}^n - u_{j-1}^n),$$

we find

$$G(\xi) = 1 - a\lambda i \sin(\xi h).$$

Then,

$$|G| = \sqrt{1 + a^2\lambda^2 \sin^2(\xi h)}.$$

Clearly, if in the limit $\tau, h \to 0$, the ratio $\lambda = \tau/h$ is fixed, the method cannot be stable.

However, since

$$\sqrt{1 + a^2 \frac{\tau^2 \sin^2(\xi h)}{h^2}} \lesssim 1 + a^2 \frac{\tau^2}{2h^2}$$

Then, if τ/h^2 is fixed as $\tau, h \to 0$. The method is stable.

Example 2. *Consider again the scheme*

$$\frac{u_j^{n+1} - u_j^n}{\tau} = \frac{1}{h^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n).$$

We have

$$v_{n+1} = [1 + \frac{\tau}{h^2}(e^{ih\xi} - 2 + e^{-ih\xi})]v_n e^{ijh\xi}.$$

Hence, we find

$$G(\xi) = 1 + \frac{\tau}{h^2} 2(\cos(\xi h) - 1).$$

We find $2(\cos(\xi h) - 1) \in [-4, 0]$. Then, if $-4\tau/h^2 \geq -2$, $|G| \leq 1$, the method is stable. (Due to $1/h^2$, it hard to get relaxed conditions and get $|G| \leq 1 + M\tau$.) We obtain the same requirement.

Example 3. *Consider the implicit sheme corresponding to the backward Euler*

$$\frac{u_j^{n+1} - u_j^n}{\tau} = \frac{1}{h^2}(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}).$$

We find

$$G(\xi) = \frac{1}{1 - \frac{2\tau}{h^2}(\cos(\xi h) - 1)}.$$

Clearly, we always have

$$0 \leq G(\xi; h, \tau) \leq 1.$$

Hence, it is unconditionally stable.

Example 4. *Consider the Richardson scheme (applying the leapfrog in time and centered difference in space) for the diffusion equation* $u_t = au_{xx}$*,*

$$u_j^{n+1} = u_j^n + 2a\mu(u_{j+1}^n - 2u_j^n + u_{j-1}^n).$$

This is a three-level scheme.

Introducing $v_j^{n+1} = u_j^n$ *and defining* $U_j^n = [u_j^n, v_j^n]$*, we can get the growth matrix*

$$G = \begin{bmatrix} -8a\mu\sin^2(\xi h/2) & 1 \\ 1 & 0 \end{bmatrix}$$

For this matrix, one eigenvalue is

$$\mu_1 = -4a\mu\sin^2(\xi h/2) - (1 + 16a^2\mu^2\sin^4(\xi h/2))^{1/2}.$$

Then, one sees that

$$\rho \geq 1 + 4a\mu\sin^2(\xi h/2) = 1 + \tau\frac{4a\sin^2(\xi h/2)}{h^2}.$$

This violates the von Neumann condition and it cannot be stable. In fact, for $\tau \to 0, h \to 0$*, the condition is violated regardless of the relative speed of convergence to zero.*

2 FDM for 1D elliptic equations

2.1 Dirichlet boundary conditions

Consider a one-dimensional elliptic equation

$$-u''(x) = f, \quad u(0) = \alpha, u(1) = \beta.$$

This elliptic equation can be regarded as the steady state of the heat equation $u_t = u_{xx} + f$ which is parabolic.

Let $h = 1/(m+1)$, $x_j = jh$ and u_j is the numerical value at the node x_j that solves the finite difference equations. Approximating the derivatives by centered difference D^2, we have

$$-D^2 u_j = -\frac{1}{h^2}(u_{j+1} - 2u_j + u_{j-1}) = f(x_j), j = 1, 2, \ldots, m.$$

$$u_0 = \alpha, \quad u_{m+1} = \beta$$

The relations can be written in the matrix form:

$$AU = F, \tag{1}$$

with $F_1 = f(x_1) + \frac{1}{h^2}\alpha, F_m = f(x_m) + \frac{1}{h^2}\beta$. The matrix A can be constructed using the commands:

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e = ones(m,1);
A = spdiags( [-e 2*e -e], -1:1, m, m)/h^2;
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Exercise: Show that A is nonsingular.

Code presentation. Let's take $\alpha = 1, \beta = 0$ and $f = -\sin(x)$.

The numerical results are good. Well, how do we prove that the error goes to zero as we refine the mesh **rigorously**?

consistency and stability

Define the local truncation error (LTE)

$$T_j = -\frac{1}{h^2}(u(x_{j+1}) - 2u(x_j) + 2u(x_{j-1})) - f(x_j),$$

where $u(x)$ is the exact solution. Taylor expansion shows that $T_j = O(h^2)$. Let $\hat{U} = (u(x_1), \ldots, u(x_m))$ which consist of the true values. We have

$$A\hat{U} = F + T,$$

where $T = [T_1, \cdots, T_m]^T$.

We take the difference:

$$A(U - \hat{U}) = AE = -T.$$

Consistency (true solution almost satisfies the discrete equation): A numerical method (scheme) is consistent with the differential equation and boundary conditions if

$$\|T_j\| \to 0, \ as \ h \to 0.$$

Stability (the numerical solution can be well-solved):

Definition 1. *For the finite difference method $AU = F$, if for all $h < h_0$ ($h_0 > 0$) U is solvable and there exists a constant C independent of h, such that for each perturbation $F \to F + \delta F$,*

$$\|\delta U\| \leq C\|\delta F\|, \forall h < h_0$$

then the method is said to be stable.

Hence, we need $\|A^{-1}\| \leq C$.

Convergence: the method is convergent if $\|E\| \to 0, \ as \ h \to 0$.

Theorem 3.

$$consistency + stability \to convergence.$$

证明. Note that:

$$AE = \tau.$$

Then,

$$\|E\| \leq C\|\tau\|$$

by the stability condition where C is independent of h. Taking $h \to 0$, $\|\tau\| \to 0$ by the consistency condition. The claim then follows. □

Eigenvalues of some special matrices

Consider the circulant matrix

$$A = \begin{pmatrix} d_0 & d_1 & d_2 & \dots & d_m \\ d_m & d_0 & d_1 & \dots & d_{m-1} \\ \dots & & & & \\ d_2 & d_3 & d_4 & \dots & d_1 \\ d_1 & d_2 & d_3 & \dots & d_0 \end{pmatrix}.$$

Such kind of matrix is usually discretization of some differential operators on $[0, 2\pi]$ with periodic boundary conditions. Hence, we can introduce $h = \frac{2\pi}{m+1}$ and set $x_j = jh$. We guess the eigenvector is the Fourier mode (Fourier modes are eigenfunctions of constant coefficient differential operators)

$$v_j^{(k)} = \exp(ikx_j)$$

We set $d_{j+m+1} = d_j$ for index out of $[0, m+1]$. Then, for the p-th row, we have

$$\sum_{j=1}^{m+1} d_{m+1-p+j} \exp(ikx_j) = \sum_j d_j \exp(ikh(j+p-m-1))$$
$$= \sum_j d_j \exp(ikh(j+p)) = \exp(ikx_p) \sum_j d_j \exp(ik\frac{2\pi j}{m+1})$$

Hence, it is really an-eigenfunction with eigenvalue

$$\lambda_k = \sum_j d_j \exp(ik\frac{2\pi j}{m+1}).$$

We now consider a second type of size $m \times m$

$$A = \begin{pmatrix} d_0 & d_1 & & \dots & \\ d_1 & d_0 & d_1 & \dots & \\ & & & \dots & \\ & & \dots & d_0 & d_1 \\ & & \dots & d_1 & d_0 \end{pmatrix}.$$

Such kind of matrices are discretizations of differential operators with Dirichlet boundary conditions. Then, we imagine that the domain is $[0, \pi]$ and $h = \frac{\pi}{m+1}$. The is because we have m interior points. We guess the eigenfunctions are

$$v_j^{(k)} = \sin(k\frac{j\pi}{m+1})$$

Then, the eigenvalues are found to be

$$\lambda_k = d_0 + 2d_1 \cos(k\frac{\pi}{m+1})$$

2.2 ℓ^2 stability of the finite difference method

Consider that we use ℓ^2 norm. The usual ℓ^2 norm for a sequence $a = (a_0, a_1, \ldots, a_n)$ is defined by $\|a\|_2 = \sqrt{\sum_i |a_i|^2}$. The ℓ^p norm of a matrix M is given by $\|M\|_p = \sup_{x \neq 0} \|Mx\|_p / \|x\|_p$. In numerical PDEs, we usually use the following scaled ℓ^2 norm:

$$\|a\|_2 = \sqrt{h \sum_i |a_i|^2}.$$

No matter which ℓ^2 norm we use, the ℓ^2 norm of the matrix is the same.

If M is real and symmetric, the ℓ^2 norm equals the spectral radius (the largest absolute value of eigenvalue). If we show that $\|A^{-1}\|_2 \leq C$, the method is stable under the L^2 norm.

The eigenvectors of the matrix A are some Fourier modes on the grid. The eigenvalues are

$$\lambda_p = -\frac{2}{h^2}(\cos(p\pi h) - 1), p = 1, \ldots, m.$$

The smallest eigenvalue of A is $|\lambda_1| = \frac{2}{h^2}(1 - \cos(\pi h)) = \pi^2 + O(h^2)$. Hence, the spectral radius of A^{-1} is $\sim 1/\pi^2$. In other words, there exists C, such that

$$\|A^{-1}\|_2 \leq C.$$

By Theorem 3, the error goes to zero under the ℓ^2 norm.