Numerical methods for PDEs: HW3

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1 Theory

(10 pts)

1. (路金甫，第三版) 第五章习题 3, 5.

2. Show that the discretization
$$-\frac{\kappa_{j+1/2}(u_{j+1}-u_j)-\kappa_{j-1/2}(u_j-u_{j-1})}{h^2}=f(x_j)$$
is of a second order accuracy for the 1D equation $-(\kappa u')'=f(x)$. Show that the resulted matrix is symmetric (assuming Dirichlent boundary condition).

3. Consider the equation in 2D:
$$u-\Delta u=f,\quad u|_{\partial\Omega}=0.$$
Suppose Ω is a square (正方形). Show the maximal principle of the five point scheme:
$$u_{ij}-\Delta_h u_{ij}=f_{ij}.$$
Use this to show the convergence of the method.

4. (Bonus. Not required to submit. Credits obtained can be added to other homework scores provided that the score is no bigger than the total score) Consider the 1D elliptic problem
$$-u''(x)=f(x),\quad u'(0)=u'(1)=0.$$
Assume that $\int_0^1 f(x)\,dx=0$.

- Show that the eigenfunctions of $-u''(x) = \lambda u$, $u'(0) = u'(1) = 0$ are given by $u_n(x) = \cos(n\pi x)$.
- The solution can be written as

$$u = \sum_n c_n u_n(x), \quad f(x) = \sum_n d_n u_n(x).$$

  Given the expression of d_n, explain how to find c_n. This is the cosine transform.
- Conisder discrete case. We set $x_j = jh - \frac{h}{2}$, $j = 1, \cdots, m$ for $h = \frac{1}{m}$. For the FDM

$$-\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} = f_j, \quad \frac{u_1 - u_{-1}}{h} = 0, \quad \frac{u_{m+1} - u_m}{h} = 0,$$

  one can show that $\{u_j\}$ can be written as linear combination of $\cos(n\pi x_j)$. Explain how you can mimick the continuous case to solve this FDM. Explain why it can be fast.

2 Numerics

(10 pts)

1. (路金甫，第三版) 第五章习题4.
2. Consider the mixed type boundary value problem:

$$u'' + u = f, 0 \leq x \leq \pi$$
$$u'(0) - u(0) = 0, \quad u'(\pi) + u(\pi) = 0$$

   Construct a second order accurate FDM. For $f = -e^x$, plot the error versus the spatial step h in loglog scale.
3. Solve the nonlinear equation $\theta'' = -\sin(\theta)$, $\theta(0) = \alpha, \theta(1) = \beta$. For the nonlinear system of equations you obtained, use Newton's iteration to solve. Plot the error versus h.

4. Consider the 2D elliptic equation

$$-(a(x,y)u_x)_x - (a(x,y)u_y)_y = f(x,y), \Omega = [-1,1] \times [-1,1]$$
$$u = 0, \ \ on \ \partial\Omega$$

$a = 1 + 3\exp(-3(x+y)^2 - (x-y)^2)$ and $f = 1$. Apply a five-point scheme for this equation. Determine the order of accuracy of your scheme, using the calculation with a small h as the 'exact' solution.

Hint: To solve the linear system $AU = F$*, one option is to construct* A *directly and do* $A\backslash F$*. Here, the coefficient is not a constant, constructing this matrix might be a little tricky (remember to keep the matrix sparse in Matlab) (One bad way is to set the point value of* U *to be* 1 *at a single point, then output the action of the scheme on this* U*, which will be the corresponding column of your matrix.) Another better option is to write a function that returns* AU *when the input is* U *and then apply an iterative method (such as conjugate gradient) to find the solution. By doing this, you do not have to construct* A*.*