Math 6008 Numerical PDEs–Lecture 8

FDM for hyperbolic equations

Instructor: Lei Li, INS, Shanghai Jiao Tong University;
Email: leili2010@sjtu.edu.cn

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1 Second order schemes

The schemes above have first order accuracy. We now look at some second order schemes.

1.1 Leapfrog and Crank-Nicolson

If we use the midpoint method for time and centered difference for the space, we have the leapfrog scheme:

$$\frac{u_j^{n+1}-u_j^{n-1}}{2\tau}=-\frac{a}{2h}(u_{j+1}^n-u_{j-1}^n).$$

The accuracy is $O(k^2+h^2)$. This is the **leapfrog** scheme. The reason to use this three-level scheme is that the eigenvalues of the finite difference are $\lambda_p=-i\frac{a}{h}\sin(2\pi ph)$ by assuming the boundary condition. Recall the stability region of the midpoint method is $(-i,i)$. Hence, we need $|a\lambda|<1$ for the method to be stable (at this point, we are not claiming it is sufficient, but it is indeed sufficient).

By converting this into a two-level system, one may find the growth matrix in the Fourier domain as

$$G(\xi;\tau,h)=\begin{bmatrix}-2a\lambda i\sin(\xi h) & 1\\ 1 & 0\end{bmatrix}.$$

The characteristic polynomial is

$$\lambda^2+2a\lambda i\sin(\xi h)-1=0.$$

Since $\lambda_1\lambda_2=-1$, for $\rho(G)\le 1$, we must require both magnitudes to be 1. This implies that $-2a\lambda i\sin(\xi h)=-2i\sin\theta$ for some θ. Hence, one must equire $a\lambda \leq 1$. If, however, $a\lambda = 1$, when $\sin(\xi h) = 1$, there are two eigenvalues to be equal. $\|G^n\|$ can have unbounded norm. Hence, $a\lambda < 1$. By Theorem 3.7 in the book, this is a sufficient condition for stability.

The Crank-Nicolson is to apply trapezoidal in time. It is two-time level but it is implicit. The implicit schemes are not common in hyperbolic equations since hyperbolic equations are not stiff. Besides, the discretization does not yield symmetric matrices, so that inverting is not desired in numerics.

1.2 Using the interpolation or the Taylor expansion to construct the schemes

Taylor expansion

Consider

$$u(x_j, t^{n+1}) = u(x_j, t^n) + u_t(x_j, t^n)\tau + \frac{1}{2}u_{tt}(x_j, t^n)\tau^2 + O(\tau^3).$$

By the equation, $u_t = -au_x$ and $u_{tt} = -au_{xt} = a^2 u_{xx}$. Hence,

$$u(x_j, t^{n+1}) = u(x_j, t^n) - a\tau u_x + \frac{1}{2}a^2\tau^2 u_{xx} + O(\tau^3).$$

We then obtain the Lax-Wendroff method:

$$u_j^{n+1} = u_j^n - a\tau D_0 u_j^n + \frac{1}{2}a^2\tau^2 D^2 u_j^n.$$

The local truncation error is $O(\tau^2 + h^2)$. Using von-Neumann or MOL eigenvalue approach, the stability condition is

$$|a|\lambda = |a|\tau/h \leq 1.$$

In the Beam-Warming, the upwind idea is used. For $a > 0$, we use $u_{j-2}^n, u_{j-1}^n, u_j^n$ to approximate u_x and u_{xx}.Then, we have

$$u_j^{n+1} = u_j^n - \frac{a\tau}{2h}(3u_j^n - 4u_{j-1}^n + u_{j-2}^n) + \frac{a\tau^2}{2h^2}(u_j^n - 2u_{j-1}^n + u_{j-2}^n).$$

Direct Taylor expansion shows that the local truncation error is $O(\tau^2 + h^2)$. The stability condition is better:

$$\frac{|a|\tau}{h} \leq 2.$$

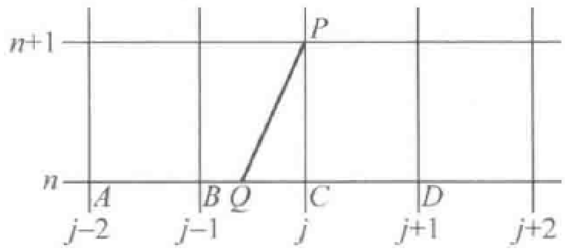

图 1: Illustration of derivation using characteristics

You can use the von-Neumann, or CFL condition to derive this.

Charateristics and interpolation

In Figure 1.2, the solution at P is determined by the solution at Q in the original advection equation. The length of QC is $a\lambda h$.

By linear interpolation,

$$u(P) = u(Q) \approx (1 - a\lambda)u(C) + a\lambda u(B) \Rightarrow u_j^{n+1} = u_j^n - a\lambda(u_j^n - u_{j-1}^n).$$

If one uses B and D to do linear interpolation, then

$$u_j^{n+1} = \frac{1 - a\lambda}{2} u_{j+1}^n + \frac{1}{2}(1 + a\lambda)u_{j-1}^n.$$

Ths is just Lax-Friedrichs.

To achieve better accuracy, one may use second order interpolation. If one uses the BCD to approximate, then one has by Lagrange interpolation

$$\begin{aligned} u_j^{n+1} = \frac{(x_Q - x_C)(x_Q - x_D)}{(x_B - x_C)(x_B - x_D)} u(B) + \frac{(x_Q - x_B)(x_Q - x_D)}{(x_C - x_B)(x_C - x_D)} u(C) \\ + \frac{(x_Q - x_B)(x_Q - x_C)}{(x_D - x_B)(x_D - x_C)} u(D). \end{aligned}$$

Then,

$$\frac{(-a\lambda)(-1 - a\lambda)}{(-1)(-2)} u_{j-1}^n + \frac{(1 - a\lambda)(-1 - a\lambda)}{1 * (-1)} u_j^n + \frac{(1 - a\lambda)(-a\lambda)}{2 * 1} u_{j+1}^n.$$

Hence, one has

$$u_j^{n+1} = u_j^n - \frac{1}{2}a\lambda(u_{j+1}^n - u_{j-1}^n) + \frac{1}{2}a^2\lambda^2(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

This is called the **Lax-Wendroff scheme**. Clearly, there is no difference whether a is positive or negative.

If one considers the flow of information, one may use A, B, C to do the Lagrange interpolation. The resulted formula is

$$u_j^{n+1} = u_j^n - a\lambda(u_j^n - u_{j-1}^n) - \frac{a\lambda}{2}(1 - a\lambda)(u_{j+1}^n - 2u_{j-1}^n + u_{j-2}^n)$$

This is the **Beam-Warming method**.

2 Modified equations: numerical dissipation and dispersion

Code presentation: In this example, we solve $u_t + u_x = 0$ numerically with initial data

$$u_0(x) = \exp(-20(x-2)^2) + \exp(-(x-5)^2).$$

We will compare the Lax-Friedrichs, the upwind scheme and Lax-Wendroff scheme with

$$\lambda = \frac{1}{2}.$$

To solve this problem, we truncate the domain to $[0, 20]$ up to time $t = 10$. Note that for the time interval we considered, $u(0,t) \approx 0$ (for better approximation, you can use $u(0,t) = \exp(-20(t+2)^2) + \exp(-(t+5)^2)$). For the right boundary, we use one-sided approximation for the finite difference.

It is clear that the Lax-Wendroff is more accurate. If we look at the behavior of the schemes more closely, we find that the Lax-Friedrichs and the upwind scheme tend to smooth out the corners and it has some dissipating (diffusion) effect. The diffusion effect in L-F seems bigger. The Lax-Wendroff however causes oscillation near the corners, which suggests that the Lax-Wendroff has the dispersion effect.

Analysis using modified equations

Compared with the original PDE, it is possible to find a PDE that is better satisfied by our numerical method. These PDEs are called the modified equations.

- Consider the upwind scheme for $u_t + au_x = 0$ $a > 0$:

$$\frac{u_j^{n+1} - u_j^n}{\tau} = -\frac{a}{h}(u_j^n - u_{j-1}^n).$$

Suppose $v(x,t)$ is a smooth function that satisfies this numerical method **exactly**. Then, we have

$$\frac{v(x_j, t^{n+1}) - v(x_j, t^n)}{\tau} = -\frac{a}{h}(v(x_j, t^n) - v(x_{j-1}, t^n)).$$

$$\Rightarrow v_t(x_j, t^n) + \frac{1}{2}\tau v_{tt}(x_j, t^n) + O(\tau^2) = -a[v_x(x_j, t^n) - \frac{1}{2}v_{xx}h] + O(h^2)$$

Hence,

$$v_t + av_x = \frac{ah}{2}v_{xx} - \frac{1}{2}\tau v_{tt} + O(h^2 + \tau^2).$$

This suggests

$$v_{tt} = -av_{xt} + O(h + \tau) = a^2 v_{xx} + O(h + \tau).$$

Inserting this into the term above, we have

$$v_t + av_x = \frac{a}{2}(h - a\tau)v_{xx} + O(h^2 + \tau^2)$$

This means that $v(x,t)$ satisfies the equation

$$v_t + av_x = \frac{a}{2}(h - ak)v_{xx}$$

better than $v_t + av_x = 0$. The modified equation is advection-diffusion equation. Clearly, if $h - a\tau > 0$ or $\frac{a\tau}{h} < 1$, there is diffusion effect. This is called the numerical diffusion.

- Similarly, the modified equation for Lax-Wendroff is

$$v_t + av_x = -\frac{1}{6}ah^2(1-(\frac{a\tau}{h})^2)v_{xxx} + O(\tau^3 + h^3)$$

  The main error for the transport equation is v_{xxx}. If you compute the dispersion relation:

$$-i\omega + ai\xi = -\frac{1}{6}ah^2(1-(\frac{a\tau}{h})^2)(-i)\xi^3 \Rightarrow \omega = a\xi - \frac{1}{6}ah^2(1-(\frac{a\tau}{h})^2)\xi^3.$$

  Hence, the main error term is dispersive and that is why the oscillation appears there.

Exercise: do the analysis for Lax-Friedrichs scheme.

3 System of equations with constant coefficients

Consider the first order system of equations

$$\boldsymbol{u}_t + A\boldsymbol{u}_x = 0, \tag{1}$$

where A is a constant matrix, and $\boldsymbol{u} : \mathbb{R}\times[0,\infty) \to \mathbb{R}^p$ is the set of unknown functions.

Definition 1. *The system of equations is called hyperbolic if A is diagonalizable and has real eigenvalues.*

We remark that A has real eigenvalues while $\frac{\partial}{\partial x}$ is a skew operator. Hence, the operator $A\frac{\partial}{\partial x}$ is intrinsically skew symmetry, and the eigenvalues are often imaginary, showing the property of waves.

Recall that the Lax-Friedrichs, Lax-Wendroff and Leapfrog schemes can be applied without knowing the direction for the information propagation. They can be generalized directly. For example, the Lax-Friedrichs scheme is

$$\boldsymbol{u}_j^{n+1} = \frac{1}{2}(\boldsymbol{u}_{j+1}^n + \boldsymbol{u}_{j-1}^n) + \frac{\tau}{2h}A(\boldsymbol{u}_{j+1}^n - \boldsymbol{u}_{j-1}^n).$$

This is a first order scheme. The stability condition can be applied for each mode so that one needs

$$\lambda\rho(A) \le 1.$$

In fact, suppose

$$A = S\Lambda S^{-1}$$

be the eigendecompostion. Then, $\boldsymbol{w} = S^{-1}\boldsymbol{u}$ satisfies

$$\frac{\partial \boldsymbol{w}}{\partial t} + \Lambda \frac{\partial \boldsymbol{w}}{\partial x} = 0. \tag{2}$$

The components are decoupled.

Alternatively, one may find the growth matrix directly to be

$$G(\xi) = (\cos \xi h)I - i\lambda \sin(\xi h)A.$$

Estimating the spectral radius of G will give similar result.

The upwind scheme

For upwind schemes, one cannot apply directly to the original equation (1). Instead, one should focus on (2). For each mode, one needs to apply the upwind scheme.

To avoid the difficulty of determining the signs, we recall the symmtric form of upwind:

$$u_j^{n+1} = u_j^n - \frac{\lambda}{2} a(u_{j+1}^n - u_{j-1}^n) + \frac{\lambda}{2}|a|(u_{j+1}^n - 2u_j^n + u_{j-1}^n).$$

Hence, one has

$$\boldsymbol{w}_j^{n+1} = \boldsymbol{w}_j^n - \frac{\lambda}{2}\Lambda(\boldsymbol{w}_{j+1}^n - \boldsymbol{w}_{j-1}^n) + \frac{\lambda}{2}|\Lambda|(\boldsymbol{w}_{j+1}^n - 2\boldsymbol{w}_j^n + \boldsymbol{w}_{j-1}^n).$$

If we define

$$|A| = S|\Lambda|S^{-1},$$

then the equation method is reduced to

$$\boldsymbol{u}_j^{n+1} = \boldsymbol{u}_j^n - \frac{\lambda}{2}A(\boldsymbol{u}_{j+1}^n - \boldsymbol{u}_{j-1}^n) + \frac{\lambda}{2}|A|(\boldsymbol{u}_{j+1}^n - 2\boldsymbol{u}_j^n + \boldsymbol{u}_{j-1}^n). \tag{3}$$

The condition then becomes

$$\lambda\rho(A) \le 1.$$

4 Upwind scheme for advection equations with variable coefficient

For the scalar equation ($u \in \mathbb{R}$), we give two examples and their analysis. Consider the equation

$$u_t + a(x,t)u_x = 0.$$

Consider applying the Lax-Friedrichs scheme:

$$u_j^{n+1} = \frac{1}{2}(u_{j-1}^n + u_{j+1}^n) - \frac{1}{2}a_j^n\lambda(u_{j+1}^n - u_{j-1}^n) = \frac{1}{2}(1+a_j^n\lambda)u_{j-1}^n + \frac{1}{2}(1-a_j^n\lambda)u_{j+1}^n.$$

For such schemes, we cannot apply the von Neumann analysis. We note that the equation is linear so the stability is again equivalent to

$$\sup_{n:n\tau\le T}\|u^n\| \le C\|u^0\|.$$

- For the ℓ^∞, we note that if $\|a\|_\infty\lambda \le 1$, the right hand side is a convex combination. Hence,

  $$|u_j^{n+1}| \le \|u^n\|_\infty, \forall j. \Rightarrow \|u^{n+1}\|_\infty \le \|u^n\|_\infty.$$

  Hence, it is stable.

- The Lax-Friedrichs scheme can also be estimated using the energy method. This can be found in the book.

Applying the upwind scheme, we have

$$\frac{u_j^{n+1} - u_j^n}{\tau} = -a_j\frac{1}{h}(u_j^n - u_{j-1}^n), \quad a_j \ge 0$$
$$\frac{u_j^{n+1} - u_j^n}{\tau} = -a_j\frac{1}{h}(u_{j+1}^n - u_j^n) \ \ a_j < 0.$$

CFL condition is $\tau\|a\|_\infty/h \le 1$.

The energy method for ℓ^2 stability is so obvious. Here, we can again do ℓ^∞ analysis. The method can be written as

$$u_j^{n+1} = (1 - \frac{k|a_j|}{h})u_j^n + \frac{k|a_j|}{h}u_{j^*}^n, \ \ j^* = j-1, \ a_j > 0 \ and \ j^* = j+1, \ a_j < 0.$$

Hence,

$$|u_j^{n+1}| \le (1 - \frac{\tau|a_j|}{h})\|u^n\|_\infty + \frac{\tau|a_j|}{h}\|u^n\|_\infty = \|u^n\|_\infty.$$

In practice, one may freeze the coefficients and do some simple analysis by regarding them as constant coefficient case.

For the system of equations, some schemes can be complicated, like the Lax-Wendroff scheme. Often, one may consider using the upwind scheme. The resulted method is similar to (3).