Chapter 9 The Laplace Transform

- 傅里叶变换为我们提供了非常有用的LTI系统的分析方法，例如：滤波、调制、采样。。。
- 但傅里叶变换需要满足狄利赫里条件，例如$\int |h(t)|\, dt < \infty$，因此主要描述稳定的LTI系统，无法分析系统的稳定性和非稳定性；
- 拉普拉斯变换是傅里叶变换的推广，将频率ω推广到复频率$s=\sigma+j\omega$，即将信号表示为e^{st}的线性组合；其可用于非稳定系统的分析，同时提供了更多系统描述和分析的方法。

9.1 DEFINATION OF THE LAPLACE TRANSFORM

9.2 THE REGION OF CONVERGENCE FOR LAPLACE THANSFORMS

9.3 PROPERTIES OF THE LAPLACE TRANSFORM

9.4 THE INVERSE LAPLACE TRANSFORM

9.5 UNILATERAL LAPLACE TRANSFORM

9.6 ANALYSIS OF LTI SYSTEMS USING LAPLACE TRANSFORM

- System Function of LTI System
- System Function and Differential Equation, Causality and Stability , Properties in Time-Domain and Frequency-Domain, Block Diagram

① Fourier Transform of $x(t)$

$$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt$$

此时，傅立叶变换收敛的条件：

- $\int_{-\infty}^{\infty}|x(t)|^2\,dt<\infty$
- $\int_{-\infty}^{\infty}|x(t)|\,dt<\infty$

② 当上述条件不满足时，引入衰减因子 $e^{-\sigma t}$（σ为任意实数）使

$$\int_{-\infty}^{\infty}|x(t)e^{-\sigma t}|dt<\infty$$

③ The Fourier Transform of $x(t)e^{-\sigma t}$

$$x(t)e^{-\sigma t} \leftrightarrow \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt$$

S为复频率

$$= X(\sigma + j\omega)$$

if $s = \sigma + j\omega$ then

$$\boxed{X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt}$$

-Laplace Transform

④ The Inverse Fourier Transform

$$\because x(t)e^{-\sigma t}=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\sigma+j\omega)e^{j\omega t}d\omega$$

$$\therefore x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\sigma+j\omega)e^{(\sigma+j\omega)}d\omega$$

Consider $d\omega=\frac{ds}{j}$ Then

积分区间是s平面
上平行纵轴的直线

$$\boxed{x(t)=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}X(s)e^{st}ds}$$

-Inverse Laplace Transform

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- **收敛域（Region of Converges， ROC)**

—The rang of values of s for which the integral $X(s)=\int_{-\infty}^{\infty}x(t)e^{-st}dt$ converges

即， 使$X(s)=\int_{-\infty}^{\infty}x(t)e^{-st}dt$收敛的$\boldsymbol{\sigma}=\mathrm{Re}[s]$的取值范围

Example: $x(t)=e^{-\alpha t}u(t)\quad \alpha>0$ —因果信号

$$X(s)=\int_0^{\infty}e^{-\alpha t}e^{-st}dt=\left.\frac{e^{-(s+\alpha)t}}{s+\alpha}\right|_0^{\infty}$$

if $R_e[s]+\boldsymbol{\alpha}>0$ then $X(s)=\dfrac{1}{s+\alpha}$

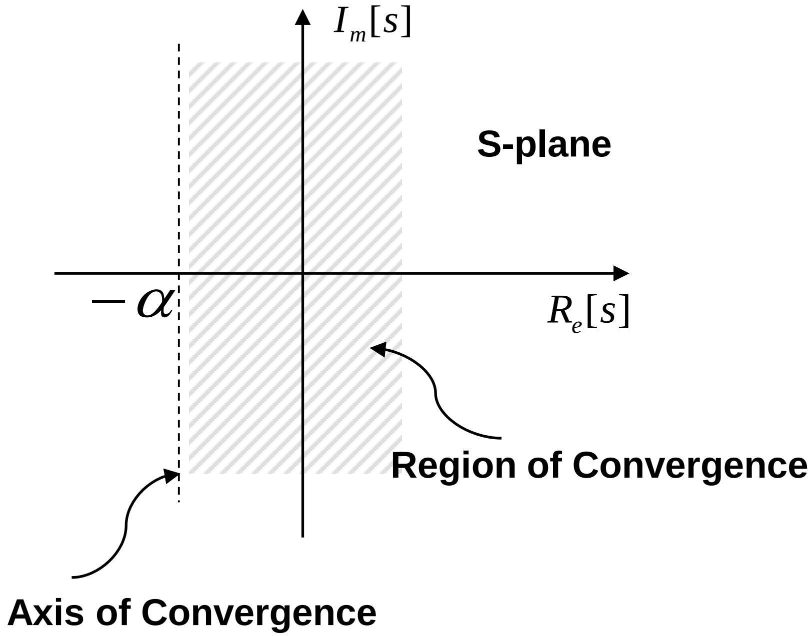
$$\boxed{e^{-\alpha t}u(t)\leftrightarrow\frac{1}{s+\alpha},\quad R_e[s]>-\alpha}$$

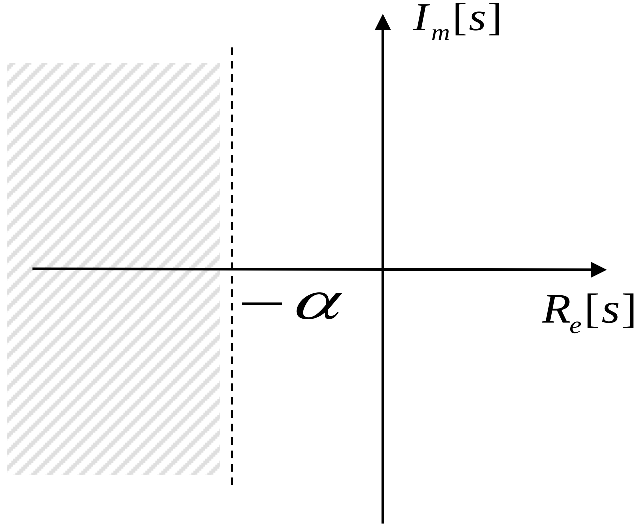
注：信号的X(s) 由其表达式及使表达式成立的s的取值范围——收敛域（ROC)共同决定。

Example: $x(t)=-e^{-\alpha t}u(-t)\quad \alpha>0$ — 反因果信号

$$\boxed{-e^{-\alpha t}u(-t)\leftrightarrow\frac{1}{s+\alpha},\quad R_e[s]<-\alpha}$$

■ ROC的复平面 (s-plane)描述





$$e^{-\alpha t}u(t) \leftrightarrow \frac{1}{s+\alpha}, \quad R_e[s] > -\alpha$$

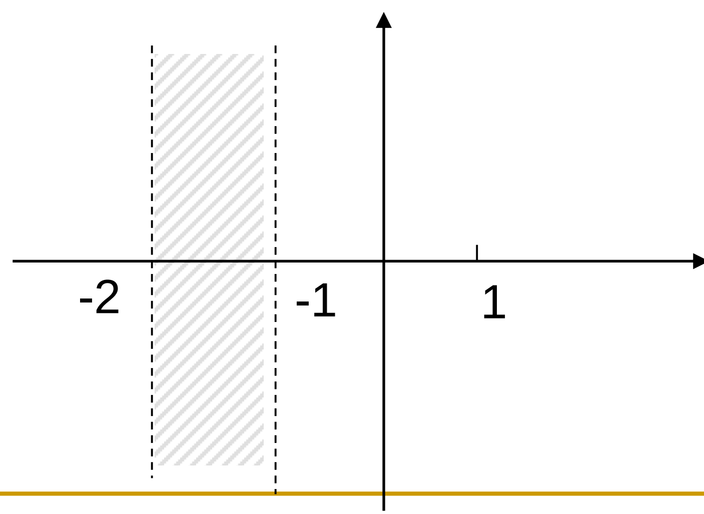
$$-e^{-\alpha t}u(-t) \leftrightarrow \frac{1}{s+\alpha}, \quad R_e[s] < -\alpha$$

Example: $x(t) = 3e^{-2t}u(t) + 2e^{-t}u(-t)$ — 双边信号

$$3e^{-2t}u(t) \leftrightarrow \frac{3}{s+2} \quad R_e[s] > -2$$

$$2e^{-t}u(-t) \leftrightarrow -\frac{2}{s+1} \quad R_e[s] < -1$$

$$X(s) = \frac{3}{s+2} - \frac{2}{s+1} = \frac{s-1}{s^2+3s+2}, \quad -2 < R_e[s] < -1$$



■ 有理拉斯变换 (Rational Laplace Transform)

— 有理拉氏变换X(s)可表示为有理分式，即

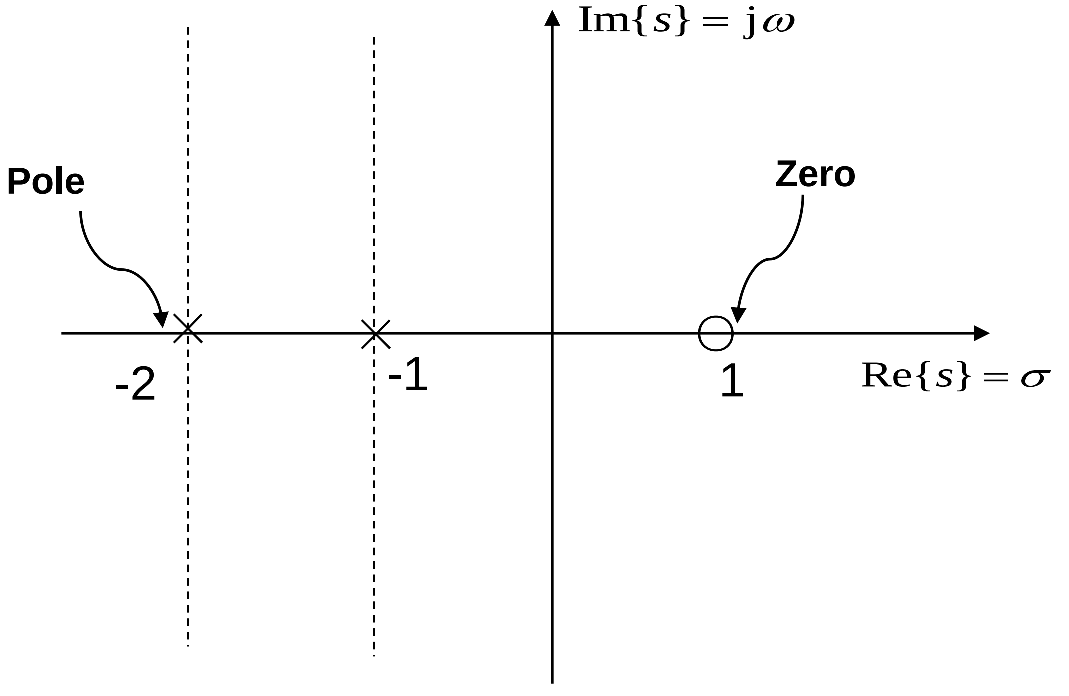
$$X(s)=\frac{N(s)}{D(s)}$$

其中N(s)、D(s)是s的多项式

— 有理拉氏变换X(s)（除常数因子外）可由其零、极点完全表征

- 零点(zero)，使N(s)=0 或 X(s)=0的s — “○”
- 极点(pole)，使D(s)=0 或 X(s)=∞的s —“×”

Example: $X(s)=\dfrac{s-1}{s^2+3s+2}=\dfrac{3}{s+2}-\dfrac{2}{s+1}$



X(s)的零、极点图

$$\Rightarrow X(s)=K\cdot\frac{s-1}{s^2+3s+2}$$

— 有理拉斯变换X(s) 在无穷远处的零极点：如果分母的阶次高出分子k次，则X(s)一定在无穷远处有k阶零点；反之，如果分子的阶次高出分母k次，则X(s)一定在无穷远处有k阶极点；

Example: $$X(s)=\frac{s-1}{s^2+3s+2}$$

两个极点$s_1=-1, s_2=-2$；

一个零点$s=1$, 还有一个零点$s=\infty$

Example: $$X(s)=\frac{s^2+1}{s+1}$$

两个零点$s_1=-j, s_2=j$；

一个极点$s=-1$, 还有一个极点$s=\infty$

Example: $$X(s)=\frac{s}{(s+\alpha)^2}$$

一个2重极点$s=-\alpha$，

一个零点s=0，还有一个零点s=∞

Property 1: **The ROC of X(s) consist of strips parallel to the *jω*-axis in the *s*-plane. (X(s)的ROC在s平面上由平行于jω轴的带状区域组成)**

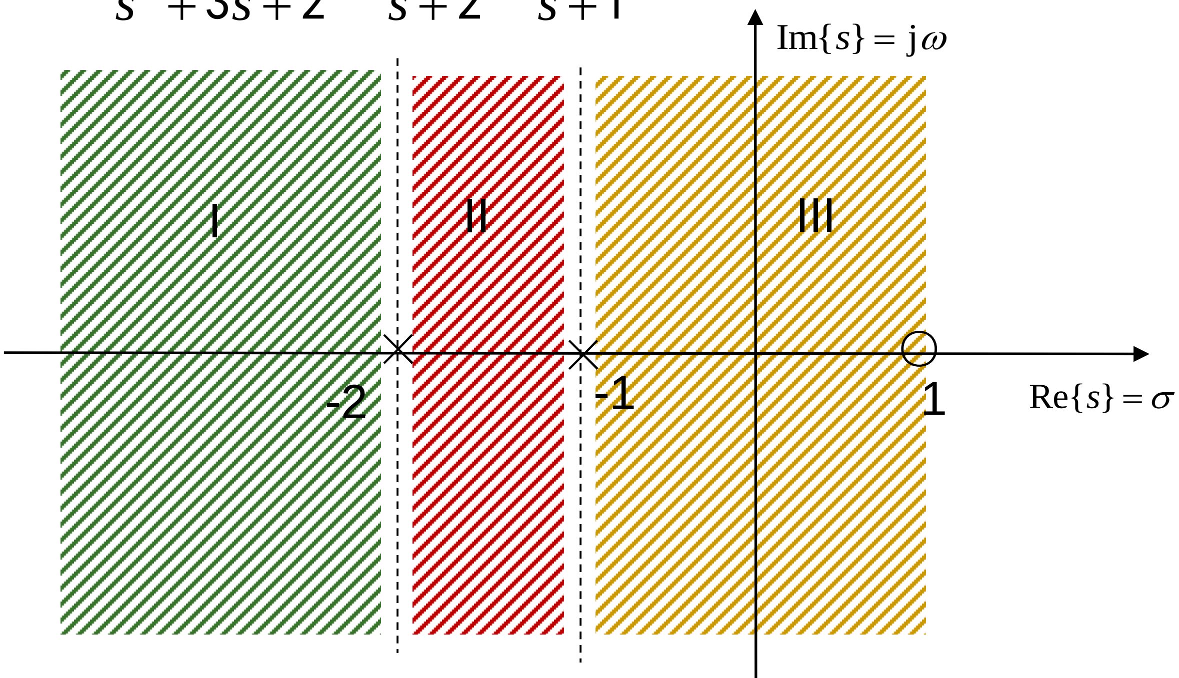
Notes:

$$\int_{-\infty}^{\infty} | x(t)e^{-\sigma t} |dt < \infty$$

Since this condition only depends on the real part of s

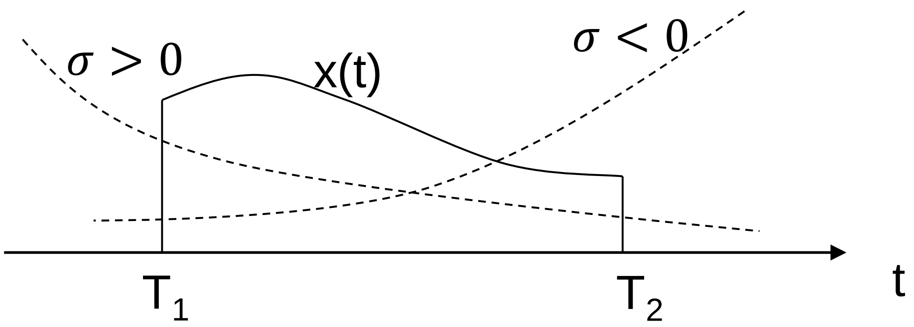
Proerty 2: **If the Laplace transforms X(s) of x(t) is rational, then its ROC is bounded by poles or extend to infinity . In addition, no poles of X(s) are contained in the ROC.(若X(s)是有理的，则其ROC被极点所界定或延伸到无限远,且ROC内不包含任何极点.)**

Example: $X(s)=\dfrac{s-1}{s^2+3s+2}=\dfrac{3}{s+2}-\dfrac{2}{s+1}$ 极点为s1=-1, s2=-2, 可能的ROC有三种：



Property 3: **If x(t) is of finite duration and is absolutely integrable, then the ROC is the entire s-plane. (若x(t)是有限持续时间信号，且绝对可积，则其ROC是整个s平面)**

Example:



$$\Rightarrow \int_{T_1}^{T_2} |x(t)|\, dt < \infty$$

if $\sigma = 0$, $\int_{-\infty}^{\infty} |x(t)| e^{-\sigma t}\, dt = \int_{T_1}^{T_2} |x(t)|\, dt < \infty$

if $\sigma > 0$, $\int_{T_1}^{T_2} |x(t)|\, e^{-\sigma t} dt < e^{-\sigma T_1} \int_{T_1}^{T_2} |x(t)|\, dt < \infty$

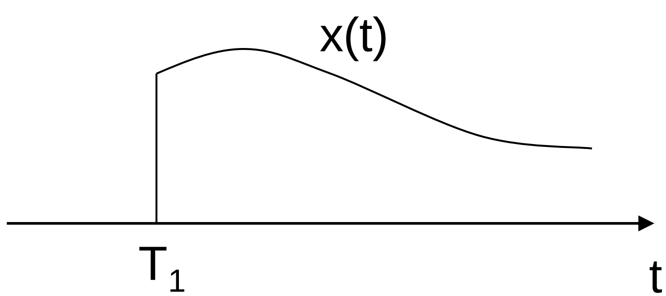
if $\sigma < 0$, $\int_{T_1}^{T_2} |x(t)|\, e^{-\sigma t} dt < e^{-\sigma T_2} \int_{T_1}^{T_2} |x(t)|\, dt < \infty$

Property 4:

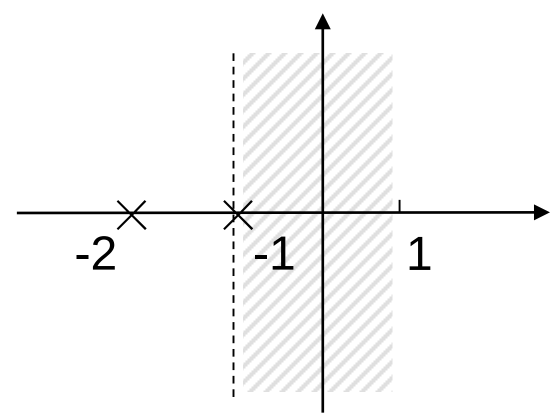
if x(t) is right sided, and if the line Re{s}=σ_0 is in the ROC, then all values of s which Re{s}>σ_0 will also be in the ROC.(若x(t) 为右边信号，则其收敛域将位于某个收敛轴**Re{s}=σ_0**的右边）

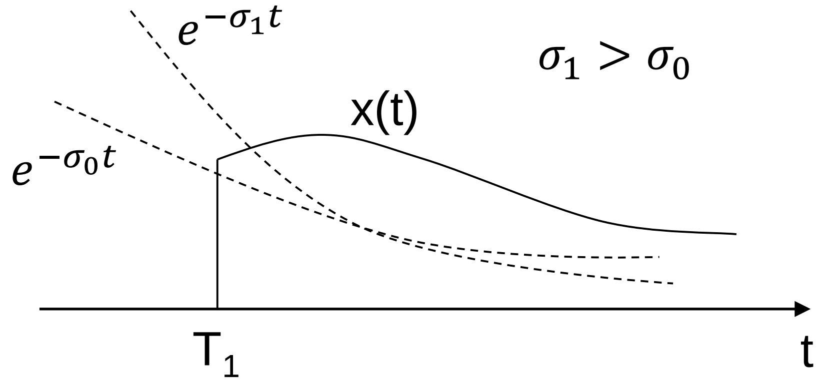
if x(t) is right sided, and the Laplace Transform X(s) of x(t) is rational, then the ROC is the region in the s-plane to the right of the rightmost pole.(若x(t)是右边信号，且**X(s)**是有理的，则其**ROC**位于最右边极点的右边**)**

右边(right-sided)信号：当t<T_1时 x(t)=0



Example: $x(t) = 3e^{-2t}u(t)\text{-}2e^{-t}u(t)$





If $\int_{T_1}^{\infty} |x(t)|\, e^{-\sigma_0 t} dt < \infty$

Then $\int_{T_1}^{\infty} |x(t)|\, e^{-\sigma_1 t} dt = \int_{T_1}^{\infty} |x(t)|\, e^{-\sigma_0 t} e^{-(\sigma_1-\sigma_0)t} dt$

$$\leq e^{-(\sigma_1-\sigma_0)T_1} \int_{T_1}^{\infty} |x(t)|\, e^{-\sigma_0 t} dt$$

即，如果$\sigma_1 > \sigma_0$，那么当t→ +∞时，$e^{-\sigma_1 t}$比$e^{-\sigma_0 t}$衰减得更快；$x(t)e^{-\sigma_0 t}$绝对可积，则$x(t)e^{-\sigma_1 t}$一定绝对可积；或者说，如果$\Re e\{s\} = \sigma_0$位于ROC内，那么$\Re e\{s\} > \sigma_0$的s都在ROC内！

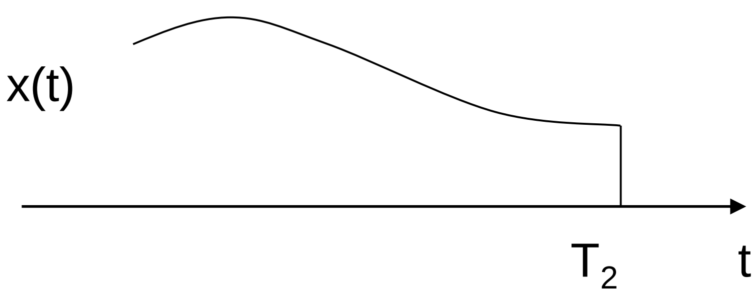
Property 5:

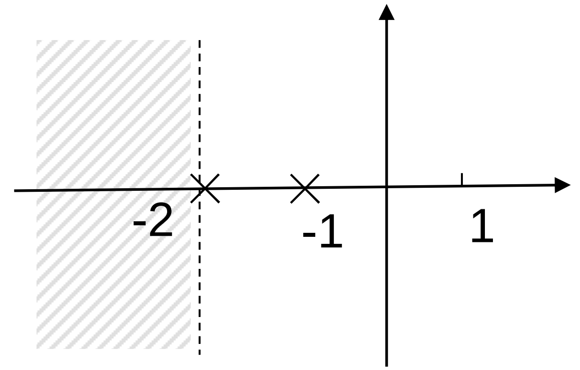
if x(t) is left sided, and if the line Re{s}=σ_0 is in the ROC, then all values of s which Re{s}<σ_0 will also be in the ROC.(若x(t) 为左边信号，则其收敛域将位于某个收敛轴Re{s}=σ_0的左边）

if x(t) is left sided, and the Laplace Transform X(s) of x(t) is rational, then the ROC is the region in the s-plane to the left of the left most pole.(若x(t)是左边信号，且X(s)是有理的，则其ROC位于最左边极点的左边)

左边(lift-sided)信号：当t>T_2时x(t)=0

Example: $x(t) = -3e^{-2t}u(-t) + 2e^{-t}u(-t)$

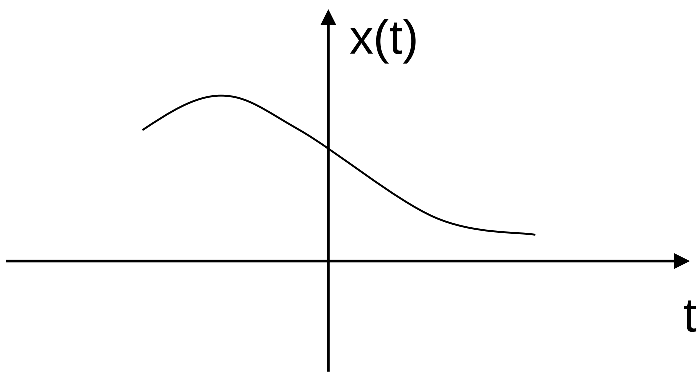




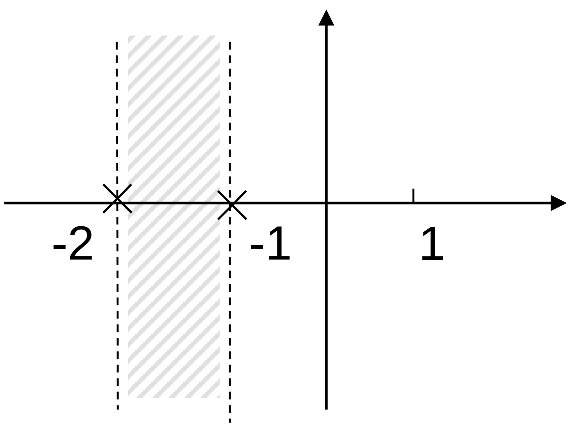
Property 6:

if x(t) is two-sided, and if the line Re{s}=σ_0 is in the ROC, then ROC will consist of a strip in the s-plane that includes the line Re{s}>σ_0.(若x(t) 为双边信号， 且Re{s}=σ_0位于ROC内，则其收敛域是包括收敛轴Re{s}=σ_0的带状区域）

双边(two-sided)信号:



Example: $x(t)=3e^{-2t}u(t)+2e^{-t}u(-t)$



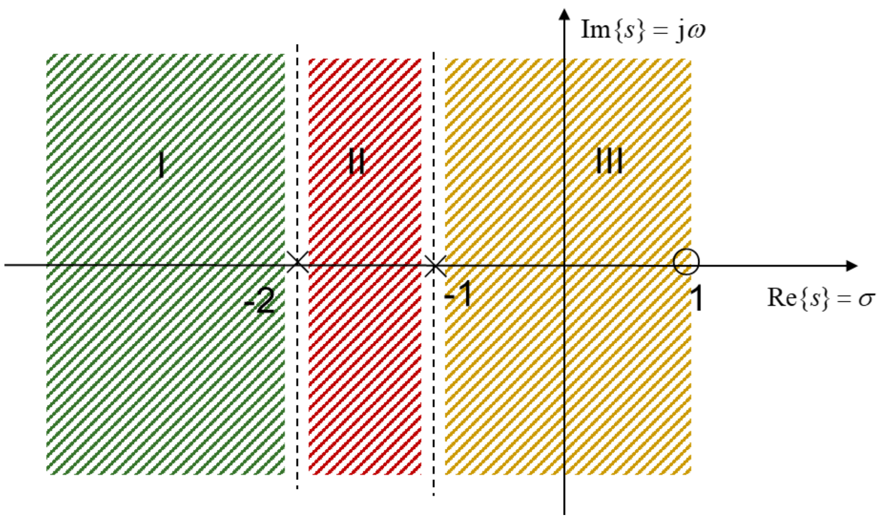
Example: $x(t)=3e^{-2t}u(-t)+2e^{-t}u(t)$

Laplace变换不存在！

Laplace Transform vs Fourier Transform

当ROC包含虚轴（$s = j\omega$）时，$X(j\omega) = X(s)\big|_{s=j\omega}$

Example: $X(s) = \dfrac{s-1}{s^2+3s+2} = \dfrac{3}{s+2} - \dfrac{2}{s+1}$



ROC	x(t)	FT?
I	Left-sided	×
II	Two-sided	×
III	Right-sided	√

Example:

$$\boxed{\delta(t) \leftrightarrow 1, \quad All\ s}$$

$$\boxed{u(t) \leftrightarrow \frac{1}{s}, \quad R_e[s] > 0}$$

- **Linearity**

$$x_1(t) \leftrightarrow X_1(s) \qquad R_1$$

$$x_2(t) \leftrightarrow X_2(s) \qquad R_2$$

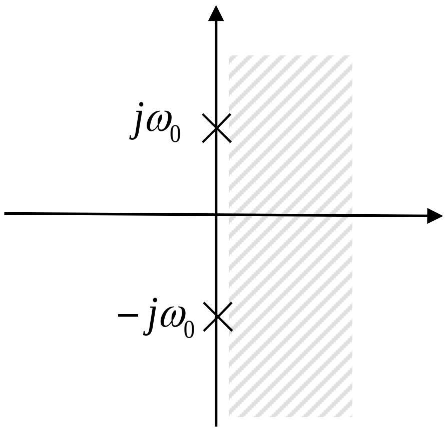
$$\boxed{ax_1(t) + bx_2(t) \leftrightarrow aX_1(s) + bX_2(s)}$$

注：收敛域至少是R1与R2的相交部分

Example:

$$\sin\omega_0 t \cdot u(t) \leftrightarrow \frac{\omega_0}{s^2 + \omega_0^2} \qquad R_e[s] > 0$$

$$\cos\omega_0 t \cdot u(t) \leftrightarrow \frac{s}{s^2 + \omega_0^2} \qquad R_e[s] > 0$$



Example:

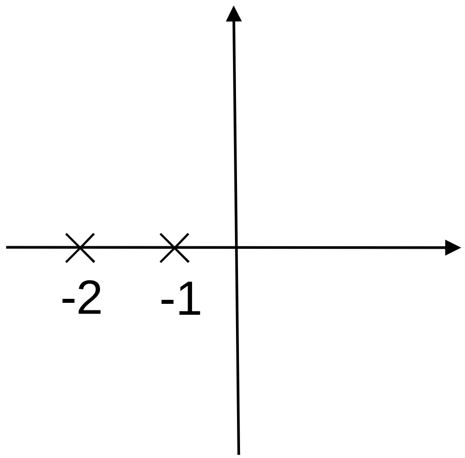
$$x_1(t) = e^{-t}u(t) \leftrightarrow X_1(s) = \frac{1}{s+1} \qquad R_e[s] > -1$$

$$x_2(t) = e^{-t}u(t) - e^{-2t}u(t) \leftrightarrow X_2(s) = \frac{1}{(s+1)(s+2)} \qquad R_e[s] > -1$$

$$X(s) = X_1(s) - X_2(s)$$

$$= \frac{s+1}{(s+1)(s+2)} = \frac{1}{s+2} \qquad R_e[s] > -2$$

$$\leftrightarrow x(t) = x_1(t) - x_2(t) = e^{-2t}u(t)$$



注：ROC扩大，因为s=-1处的零、极点抵消。

- **Time Shifting**

$$x(t) \leftrightarrow X(s) \qquad R$$

$$\boxed{x(t-t_0) \leftrightarrow e^{-st_0}X(s) \qquad R}$$

Example： Consider $e^{-\alpha t}u(-t) \leftrightarrow -\dfrac{1}{s+\alpha}$

Determine the Laplace transform for $e^{-\alpha t}u(-t-t_0)$

Solution:

$$e^{-\alpha t}u(-t-t_0)=e^{-\alpha(t+t_0)}u(-(t+t_0))e^{\alpha t_0}$$

$$\leftrightarrow -\frac{e^{st_0}}{s+\alpha}e^{\alpha t_0} = -\frac{e^{(s+\alpha)t_0}}{s+\alpha}$$

Example： Consider $tu(t) \leftrightarrow \frac{1}{s^2}$

Determine the Laplace transform for:

1) $tu(t-1)$

2) $(t-1)u(t)$

Solution:

1)

$$tu(t-1)=(t-1)u(t-1)+u(t-1)$$

$$\leftrightarrow \frac{e^{-s}}{s^2}+\frac{e^{-s}}{s}$$

2)

$$(t-1)u(t)=tu(t)-u(t)$$

$$\leftrightarrow \frac{1}{s^2}-\frac{1}{s}$$

- **Shifting in the s-Domain**

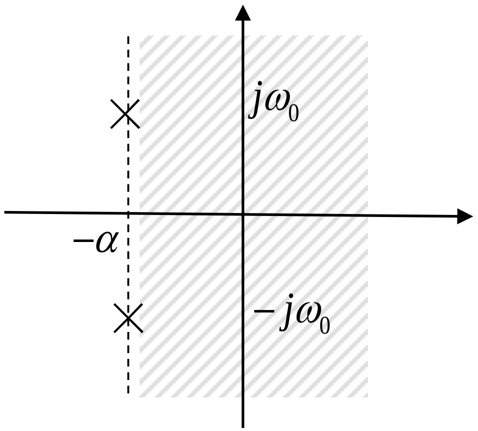
$$x(t) \leftrightarrow X(s) \qquad R$$

$$\boxed{e^{s_0 t}x(t) \leftrightarrow X(s-s_0) \qquad R+\Re_e\{s_0\}}$$

Example:

$$e^{-\alpha t}\sin\omega_0 t \cdot u(t) \leftrightarrow \frac{\omega_0}{(s+\alpha)^2+\omega_0^{\ 2}}, \Re e\{s\} > -\alpha$$

$$e^{-\alpha t}\cos\omega_0 t \cdot u(t) \leftrightarrow \frac{s+\alpha}{(s+\alpha)^2+\omega_0^{\ 2}}, \Re e\{s\} > -\alpha$$



- **Time Scaling**

$$x(t) \leftrightarrow X(s) \qquad R$$

$$\boxed{\begin{array}{l} x(at) \leftrightarrow \dfrac{1}{|\alpha|} X(\dfrac{s}{a}) \qquad aR \\ \text{当} a=-1 \text{时，} x(-t) \leftrightarrow X(-s) \end{array}}$$

Example:

$$x(at-b) = x[a(t-\frac{b}{a})] \leftrightarrow e^{-\frac{b}{a}s} \cdot \frac{1}{|a|} X(\frac{s}{a})$$

- **Conjugation**

$$x(t) \leftrightarrow X(s) \qquad R$$

$$\boxed{x^*(t) \leftrightarrow X^*(s^*) \qquad R}$$

if x(t) is real valued

$$X(s) = X^*(s^*)$$

注： 实信号的零、极点共轭成对出现。即，如果s_0为极点(或零点)，则s_0^*也为极点(或零点).

- **Convolution Property**

$$x_1(t) \leftrightarrow X_1(s) \qquad R_1$$

$$x_2(t) \leftrightarrow X_2(s) \qquad R_2$$

$$\boxed{x_1(t) * x_2(t) \leftrightarrow X_1(s)X_2(s) \qquad R_1 \cap R_2}$$

$$y_{zs}(t) = h(t) * x(t) \leftrightarrow Y_{zs}(s) = H(s)X(s)$$

LTI系统的s域分析

- **Differentiation in the Time-Domain**

$$x(t) \leftrightarrow X(s) \qquad R$$

$$\boxed{\frac{dx(t)}{dt} \leftrightarrow sX(s) \qquad containing\, \mathrm{R}}$$

Proof:

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$

$$\Rightarrow \frac{d}{dt}x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} sX(s)e^{st}ds$$

- **Differentiation in the S-Domain**

$$x(t) \leftrightarrow X(s) \qquad R$$

$$\boxed{-tx(t) \leftrightarrow \frac{dX(s)}{ds} \qquad R}$$

$$tx(t) \leftrightarrow -\frac{dX(s)}{ds}$$

Proof:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \Rightarrow \frac{d}{dt}X(s) = \int_{-\infty}^{\infty} (-tx(t))e^{-st}dt$$

Example:

$$u(t) \leftrightarrow \frac{1}{s}$$

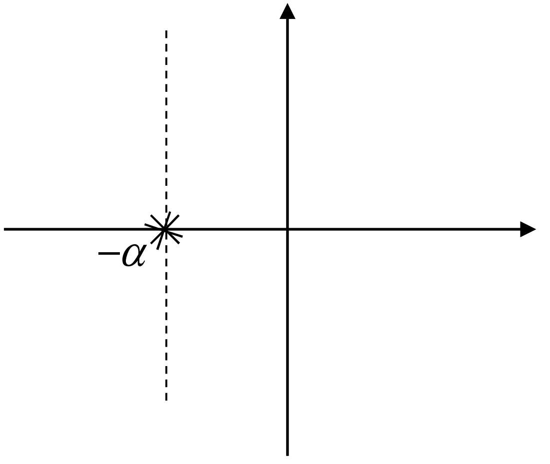
$$tu(t) \leftrightarrow -\frac{d}{ds}(\frac{1}{s}) = \frac{1}{s^2}$$

$$t^2u(t) \leftrightarrow -\frac{d}{ds}(\frac{1}{s^2}) = \frac{2!}{s^3}$$

$$\vdots$$

$$t^n u(t) \leftrightarrow \frac{n!}{s^{n+1}}$$

$$t^n e^{-\alpha t} u(t) \leftrightarrow \frac{n!}{(s+\alpha)^{n+1}}$$



$$\boxed{\frac{t^{n-1}}{(n-1)!} e^{-\alpha t} u(t) \leftrightarrow \frac{1}{(s+\alpha)^n}, \Re e[s] > -\alpha}$$

- **Integration in the Time Domain**

$$x(t) \leftrightarrow X(s) \qquad R$$

$$\boxed{\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{s}X(s) \qquad R\bigcap(\Re e[s]>0)}$$

Proof:

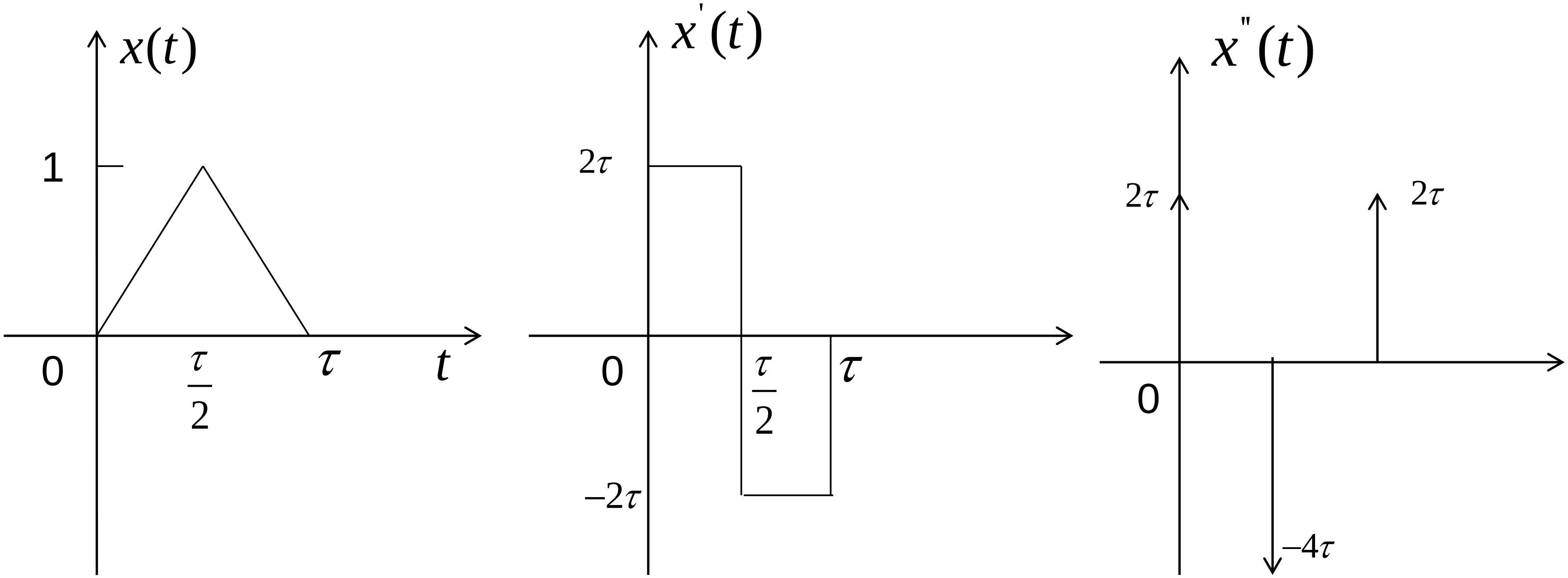
$$\int_{-\infty}^{t} x(\tau)d\tau = x(t) * u(t)$$

and

$$u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s}, \qquad \mathcal{Re}\{s\} > 0$$

$$\Rightarrow \int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{s}\mathrm{X}(\mathrm{s})$$

Example:



$$x''(t) = 2\tau\delta(t) - 4\tau\delta(t-\frac{\tau}{2}) + 2\tau\delta(t-\tau) \leftrightarrow 2\tau - 4\tau e^{-j\frac{\tau}{2}s} + 2\tau e^{-j\tau s} = X_2(s)$$

$$X(s) = X_2(s)/s^2$$

- **Integration in the S-Domain**

$$x(t) \leftrightarrow X(s) \qquad R$$

$$\boxed{\frac{x(t)}{t} \leftrightarrow \int_s^{\infty} X(\lambda)d\lambda}$$

(前提：t<0时x(t)=0, 且 $\lim_{t\to 0}\frac{x(t)}{t}$ 存在)

Proof:

$$\int_s^{\infty} X(\lambda)d\lambda = \int_s^{\infty}\left[\int_0^{\infty} x(t)e^{-\lambda t}dt\right]d\lambda = \int_0^{\infty} x(t)\left[\int_s^{\infty} e^{-\lambda t}d\lambda\right]dt$$

$$= \int_0^{\infty} x(t)\left[\left.\frac{e^{-\lambda t}}{-t}\right|_s^{\infty}\right]dt = \int_0^{\infty}\frac{x(t)}{t}e^{-st}dt$$

Example: $$x(t)=\frac{1}{t}(1-e^{-\alpha t})u(t)$$

$$\because (1-e^{-\alpha t})u(t)\leftrightarrow \frac{1}{s}-\frac{1}{s+\alpha}$$

$$\therefore \frac{1}{t}(1-e^{-\alpha t})u(t)\leftrightarrow \int_s^{\infty}\left(\frac{1}{\lambda}-\frac{1}{\lambda+\alpha}\right)\,\mathrm{d}\lambda=\ln\frac{s+\alpha}{s}$$

- **The Initial and Final-Value Theorems**

1. t<0时，x(t)=0
2. t=0时，x(t)不包含冲激或高阶奇异函数

代入初值定理的**X(s)**须为真分式！

初值定理：

$$\boxed{\lim_{t\to 0_+} x(t) = x(0^+) = \lim_{s\to\infty} s\cdot X(s)}$$

终值定理：

$$\boxed{\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)}$$

Proof:

将$x(t)u(t)$在$t=0^+$展开成泰勒级数：

$$x(t)=[x(0^+)+x^{(1)}(0^+)t+\cdots+x^{(n)}(0^+)\frac{t^n}{n!}+\cdots]u(t)$$

$$\because t^n u(t)\leftrightarrow \frac{n!}{s^{n+1}}$$

$$\therefore X(s)=\sum_{n=0} x^{(n)}(0^+)\frac{1}{s^{n+1}}$$

即 $sX(s)=x(0^+)+x^{(1)}(0^+)/s+x^{(2)}(0^+)/s^2\cdots$

$$\therefore \lim_{s\to\infty} sX(s)=x\ (0^+)$$

Example: 求如下表达式的$x(t)$，并验证初值定理

1）$X(s)=\frac{1}{s+2}$

2）$X(s)=\frac{s+1}{(s+2)(s+3)}$

Solution:

1) Assume the $ROC\ is\ \Re e\{s\}$ >-2,

$$x(t)=e^{-2t}u(t)$$

Therefore $x(0_+)=1$

And $\lim_{s\to\infty} sX(s)=\frac{s}{s+2}=1$

例：(单边)“周期”信号 $x(t)=\sum_{n=0}^{\infty} x_0(t-nT_1)$ 且有 $x(t)=x_0(t),\quad 0<t<T_1$

$$X(s)=\int_0^{\infty} x(t)e^{-st}dt$$

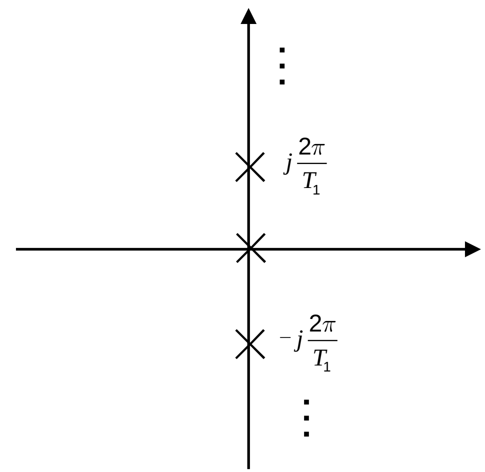
$$=\int_0^{T_1} x_0(t)e^{-st}dt+\int_{T_1}^{2T_1} x_0(t-T_1)e^{-st}dt$$

$$+\cdots+\int_{nT_1}^{(n+1)T_1} x_0(t-nT_1)e^{-st}dt+\cdots$$

$$=X_0(s)\cdot[1+e^{-sT_1}+\cdots+e^{-nsT_1}+\cdots]$$

$$=X_0(s)\cdot\frac{1}{1-e^{-sT_1}}$$

其中，$\int_0^{T_1} x_0(t)e^{-st}dt=X_0(s)$



$s=jk\frac{2\pi}{T_1},\quad k=0,\pm1,\pm2..$

$$\boxed{\sum_{n=0}^{\infty} x_0(t-nT_1)\leftrightarrow\frac{X_0(s)}{1-e^{-sT_1}}}$$

例：(单边)取样信号 $x_p(t) = x(t)\cdot p(t)$

设 $p(t) = \sum_{n=0}^{\infty} \delta(t - nT_1)$

则 $x_p(t) = \sum_{n=0}^{\infty} x(nT)\delta(t - nT_1)$

$$\leftrightarrow \int_0^{\infty} \sum_{n=0}^{\infty} x(nT_1)\delta(t - nT_1)e^{-st}dt$$

$$= \sum_{n=0}^{\infty} x(nT_1)\int_0^{\infty} \delta(t - nT_1)e^{-st}dt$$

$$= \sum_{n=0}^{\infty} x(nT_1)e^{-snT_1}$$

Exercise: Determine the Laplace Transform of the following signal

1） $x(t) = t^2 u(t-2)$

2） $x(t) = 2te^{-2t} u(2t-1)$

通过以上实例，总结零、极点分布对时域波形的影响？

极点：

- 实极点/复极点
- 单极点/复极点

- 左半平面
- 虚轴
- 右半平面

波形：

增长/等幅/衰减? 振荡/非振荡？

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$$\begin{cases}\text{围线积分法} \\ \text{部分分式展开法} \quad \surd\end{cases}$$

$$X(s) = P(s) + X_0(s)$$

① $P(s)$ —多项式 $\leftrightarrow$ 冲激函数及其各阶导数

② $X_0(s)$ —真分式 $\leftrightarrow$ $x_0(t)$

$$X_0(s) = \frac{A(s)}{B(s)} = \frac{a_m s^m + a_{m-1}s^{m+1} + \cdots + a_0}{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_0} \quad (m < n)$$

$$= \frac{a_m (s - z_1)(s - z_2)\cdots(s - z_m)}{b_n (s - p_1)(s - p_2)\cdots(s - p_n)}$$

其中： $z_1 \cdots z_m$ 是 $X_0(s)$ 的零点(zero)

$p_1 \cdots p_n$ 是 $X_0(s)$ 的极点(pole)

设 $b_n = 1$

㈠ $X_0(s)$ 有n个单极点

$$X_0(s) = \frac{A(s)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

$$= \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} \cdots + \frac{k_n}{s-p_n}$$

$$\boxed{\frac{k_i}{s-p_i} \leftrightarrow \begin{cases} k_i e^{p_i t} u(t) & \Re_e(s) > p_i \\ -k_i e^{p_i t} u(-t) & \Re_e(s) < p_i \end{cases}}$$

ROC!

确定系数k_i的方法：

1. 对应项系数平衡相等

2. $k_i = (s - p_i)X_0(s)\big|_{s=p_i} \quad i = 1, 2 \cdots n$

Example:
$$X(s)=\frac{4s^2+11s+10}{2s^2+5s+3}$$

$$X(s)=2+\frac{1}{2}\cdot\frac{s+4}{s^2+\frac{5}{2}s+\frac{3}{2}}$$

$$X_0(s)=\frac{s+4}{s^2+\frac{5}{2}s+\frac{3}{2}}=\frac{s+4}{(s+1)(s+\frac{3}{2})}=\frac{k_1}{s+1}+\frac{k_2}{s+\frac{3}{2}}$$

solution1：

$$\frac{k_1}{s+1}+\frac{k_2}{s+\frac{3}{2}}=\frac{(k_1+k_2)s+\frac{3}{2}k_1+k_2}{(s+1)(s+\frac{3}{2})}=\frac{s+4}{(s+1)(s+\frac{3}{2})}$$

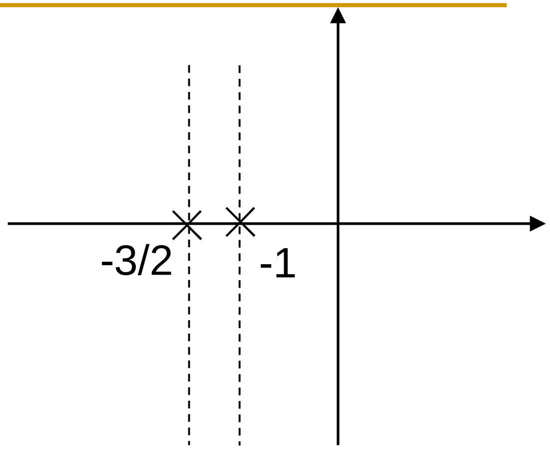
$$\therefore \begin{cases} k_1+k_2=1 \\ \frac{3}{2}k_1+k_2=4 \end{cases} \rightarrow \begin{cases} k_1=6 \\ k_2=-5 \end{cases}$$

solution2：

$$k_1=(s+1)X_0(s)\big|_{s=-1}=\left.\frac{s+4}{s+\frac{3}{2}}\right|_{s=-1}=6$$

$$k_2=(s+\frac{3}{2})X_0(s)\Big|_{s=-\frac{3}{2}}=\left.\frac{s+4}{s+1}\right|_{s=-\frac{3}{2}}=-5$$

$$X_0(s)=\frac{6}{s+1}-\frac{5}{s+\frac{3}{2}}\leftrightarrow x_0(t)$$



For the varied ROC:

$\mathrm{Re}(s)\geq -1$ $\qquad x_0(t)=6e^{-t}u(t)-5e^{-\frac{3}{2}t}u(t)$

$-\frac{3}{2}\leq \mathrm{Re}(s)\leq -1$ $\qquad x_0(t)=-6e^{-t}u(-t)-5e^{-\frac{3}{2}t}u(t)$

$\mathrm{Re}[s]<-\frac{3}{2}$ $\qquad x_0(t)=-6e^{-t}u(-t)+5e^{-\frac{3}{2}t}u(-t)$

又 $2\leftrightarrow \delta(t)$

$$\therefore X(s)=2+\frac{1}{2}X_0(s)\leftrightarrow 2\cdot\delta(t)+\frac{1}{2}x_0(t)$$

（二）$X_0(s)$ 在 $s = p_1$ 处有k重极点

$$X_0(s) = \frac{A(s)}{(s-p_1)^k \cdot D(s)}$$

$$= \frac{k_{11}}{(s-p_1)^k} + \frac{k_{12}}{(s-p_1)^{k-1}} + \cdots + \frac{k_{1k}}{s-p_1} + \frac{E(s)}{D(s)}$$

$$\boxed{\frac{k_{1i}}{(s-p_1)^{k-i+1}} \leftrightarrow k_{1i} \frac{t^{k-i}}{(k-i)!} e^{p_1 t} u(t), \Re e[s] > p_1}$$

$$i = 1,2 \ldots k$$

确定系数k_i的方法：

$$k_{1i}=\frac{1}{(i-1)!}\cdot\frac{d^{i-1}}{ds^{i-1}}X_1(s)\bigg|_{s=p_i}$$

$$where\quad X_1(s)=(s-p_1)^k X_0(s)$$

即：

$$k_{11}=X_1(s)|_{s=p_1}$$

$$k_{12}=\frac{d}{ds}X_1(s)\bigg|_{s=p_1}$$

$$k_{13}=\frac{1}{2}\cdot\frac{d^2}{ds^2}X_1(s)\bigg|_{s=p_1}$$

$\cdots$

Example: $$X(s)=\frac{s+3}{(s+2)(s+1)^3}$$

$$X(s)=\frac{k_{11}}{(s+1)^3}+\frac{k_{12}}{(s+1)^2}+\frac{k_{13}}{(s+1)}+\frac{k_2}{(s+2)}$$

$$\because X_1(s)=(s+1)^3X(s)=\frac{s+3}{s+2}$$

$$\therefore k_{11}=X_1(s)\big|_{s=-1}=\left.\frac{s+3}{s+2}\right|_{s=-1}=2$$

$$k_{12}=\left.\frac{d}{ds}X_1(s)\right|_{s=-1}=\left.\frac{d}{ds}\left(\frac{s+3}{s+2}\right)\right|_{s=-1}=-1$$

$$k_{13}=\left.\frac{1}{2}\frac{d^2}{ds^2}X_1(s)\right|_{s=-1}=1$$

And $$k_2 = (s+2)X(s)|_{s=-2} = \left.\frac{s+3}{(s+1)^3}\right|_{s=-2} = -1$$

thus $$X(s) = \frac{2}{(s+1)^3} - \frac{1}{(s+1)^2} + \frac{1}{s+1} - \frac{1}{s+2}$$

If $\mathrm{Re}(s) > -1$

$$\begin{aligned} X(s) &\leftrightarrow t^2 e^{-t}u(t) - te^{-t}u(t) + e^{-t}u(t) - e^{-2t}u(t) \\ &= (t^2 - t + 1)e^{-t}u(t) - e^{-2t}u(t) \end{aligned}$$

（三）$X_0(s)$ 有共轭复根 ——配方法

$$\frac{\omega_0}{(s+\alpha)^2+\omega_0{}^2} \leftrightarrow e^{-\alpha t}\sin\omega_0 t\cdot u(t), \Re e[s]>-\alpha$$

$$\frac{s+\alpha}{(s+\alpha)^2+\omega_0{}^2} \leftrightarrow e^{-\alpha t}\cos\omega_0 t\cdot u(t), \Re e[s]>-\alpha$$

Example: $$X(s)=\frac{s^3}{s^2+s+1}=s-1+\frac{1}{s^2+s+1}$$

$$=s-1+\frac{\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

If $\mathrm{Re}[s]>-\frac{1}{2}$

$$X(s)\leftrightarrow\delta'(t)-\delta(t)+\frac{2}{\sqrt{3}}\cdot e^{-\frac{1}{2}t}\sin(\frac{\sqrt{3}}{2}t)u(t)$$

Example: $$X(s)=\frac{1-e^{-2s}}{s(s^2+4)}$$

利用拉氏变换的性质！

设 $$X_1(s)=\frac{1}{s(s^2+4)}\leftrightarrow x_1(t)$$

则 $$X(s)=(1-e^{-2s})X_1(s)\leftrightarrow x_1(t)-x_1(t-2)$$

$$\because X_1(s) = \frac{1}{s(s^2+4)} = \frac{k_1}{s} + \frac{k_2 s + k_3}{s^2+4}$$ 其中， $$k_1 = sX_1(s)|_{s=0} = \frac{1}{4}$$

$$\therefore \frac{\frac{1}{4}}{s} + \frac{k_2 s + k_3}{s^2+4} = \frac{(k_2+\frac{1}{4})s^2 + k_3 s + 1}{s(s^2+4)} = \frac{1}{s(s^2+4)}$$

$$\Rightarrow \begin{cases} k_2 + \frac{1}{4} = 0 \\ k_3 = 0 \end{cases} \rightarrow \begin{cases} k_2 = -\frac{1}{4} \\ k_3 = 0 \end{cases}$$

$$\therefore X_1(s) = \frac{\frac{1}{4}}{s} - \frac{\frac{1}{4}s}{s^2+4} \leftrightarrow \frac{1}{4}u(t) - \frac{1}{4}\cos 2t \cdot u(t) = x_1(t)$$

Example: $$X(s)=\frac{1}{1+e^{-s}} \qquad \mathrm{Re}[s]>0$$

注： $$\sum_{n=0}^{\infty} x_0(t-nT_1) \leftrightarrow \frac{X_0(s)}{1-e^{-sT_1}}$$

$$where \quad x_0(t) \leftrightarrow X_0(s)$$

$$X(s)=\frac{1-e^{-s}}{(1+e^{-s})(1-e^{-s})}=\frac{1-e^{-s}}{1-e^{-2s}}$$

$$\because 1-e^{-s}\leftrightarrow\delta(t)-\delta(t-1)=x_0(t)$$

$$\therefore \frac{1-e^{-s}}{1-e^{-2s}}\leftrightarrow\sum_{n=0}^{\infty}x_0(t-2n)$$

$$=\sum_{n=0}^{\infty}[\delta(t-2n)-\delta(t-1-2n)]$$

$$=\sum_{n=0}^{\infty}(-1)^n\delta(t-n)$$

Exercise: Determine the Inverse Laplace Transform of the following signal

1） $X(s) = \dfrac{s}{(s^2+4)^2} \qquad \mathrm{Re}[s] > 0$

2） $X(s) = \dfrac{(s^2+1)+(s^2-1)e^{-s}}{s^2(1+e^{-s})} \qquad \mathrm{Re}(s) > 0$

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- **System Function of LTI System**
- **System Function and Differential Equation , Causality and Stability , Properties in Time-Domain and Frequency-Domain, Block Diagram**

- **Defination**

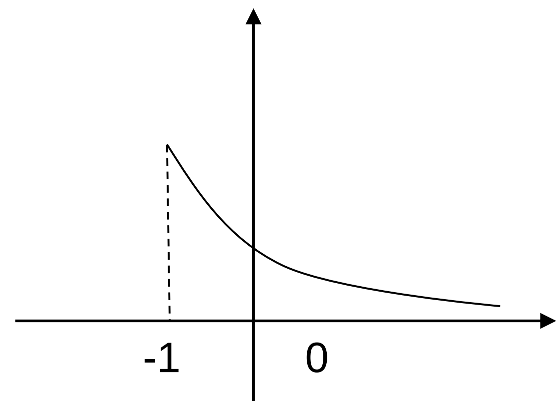
$$X(s)=\int_{0^-}^{\infty} x(t)e^{-st}dt$$ —ROC总在最右边极点的右边

$$x(t)=\frac{1}{2\pi j}\int_{-\infty}^{\infty} X(s)e^{st}ds \qquad t>0^-$$

注：

- 单边拉斯变换只考虑信号 $x(t)$ 在 $t>0^-$ 的情况，但$x(t)$在 $t<0$ 时不一定为0。
- 当x(t)为因果信号，则双边与单边变换相同。

Example: $$x(t)=e^{-\alpha(t+1)}u(t+1)$$



双边： $$X(s)=\frac{e^{s}}{s+\alpha},\quad \Re e[s]>-\alpha$$

单边： $$X(s)=\int_0^{\infty}e^{-\alpha(t+1)}e^{-st}dt=e^{-\alpha}\int_0^{\infty}e^{-(s+\alpha)t}dt$$

$$=\frac{e^{-\alpha}}{s+\alpha},\quad \Re e[s]>-\alpha$$

- **Properties**

1、Convolution

$$x_1(t) = x_2(t) = 0 \quad \text{For all } t<0$$

$$\boxed{x_1(t) * x_2(t) \leftrightarrow X_1(s)X_2(s)}$$

2、Differentiation in the time domain

$$x(t) \leftrightarrow X(s)$$

$$\boxed{\frac{dx(t)}{dt} \leftrightarrow sX(s) - x(0^-)}$$

Proof：

$$\int_{0^-}^{\infty}\frac{dx(t)}{dt}e^{-st}dt = x(t)e^{-st}\Big|_{0^-}^{\infty} + s\int_{0^-}^{\infty}x(t)e^{-st}dt$$

$$= sX(s) - x(0^-)$$

以此类推，得：

$$\boxed{\begin{aligned} x^{(n)}(t) &\leftrightarrow s^n X(s) - \sum_{m=0}^{n-1} s^{n-m-1}x^{(m)}(0^-) \\ &= s^n X(s) - s^{n-1}x(0^-) - s^{n-2}x^{(1)}(0^-) - s^{(n-3)}x^{(2)}(0^-) - \ldots s x^{(n-2)}(0^-) - x^{(n-1)}(0^-) \end{aligned}}$$

求解具有非零初始条件的线性常系数微分方程！
(1)将微分方程转换成代数方程
(2)可直接求解完全响应 ，并同时求出零输入响应和零状态响应

设LTI系统

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

已知：

起始状态为： $\{y(0_-), y'(0_-), \cdots, y^{(N-1)}(0_-)\}$

外加激励在x(t)在t=0时加入， 即t<0时x(t)=0

则：

$$\sum_{k}^{M} a_k [s^k Y(s) - \sum_{p=0}^{k-1} s^{k-1-p} y^{(p)}(0\_)] = \sum_{k=0}^{N} b_k s^k X(s)$$

$$\underbrace{[\sum_k a_k s^k]}_{A(s)} Y(s) - \underbrace{\sum_k a_k [\sum_{p=0}^{k-1} s^{k-1-p} y^{(p)}(0\_)]}_{M(s)} = \underbrace{[\sum_k b_k s^k]}_{B(s)} X(s)$$

$$A(s)Y(s) - M(s) = B(s)X(s)$$

$$\therefore Y(s) = \frac{M(s)}{A(s)} + \frac{B(s)}{A(s)} X(s)$$

零输入响应　　零状态响应

Example: $y''(t)+3y'(t)+2y(t)=2x'(t)+6x(t)$

已知： $x(t)=u(t),\quad y(0\_)=2 \quad y'(0\_)=1$

求： $y(t), y_{zi}(t), y_{zs}(t)$

$$\underline{s^2Y(s)-sy(0\_)-y'(0\_)}+\underline{3sY(s)-3y(0\_)}+\underline{2Y(s)}=2sX(s)+6X(s)$$

即：$(s^2+3s+2)Y(s)-[sy(0\_)+y'(0\_)+3y(0\_)]=(2s+6)X(s)$

$$\therefore Y(s)=\frac{sy(0\_)+y'(0\_)+3y(0\_)}{s^2+3s+2}+\frac{2s+6}{s^2+3s+2}\cdot X(s)$$

将$y(0_)=2, y'(0_)=1$及$X(s)=\dfrac{1}{s}$代入

$$Y_{zi}(s)=\frac{2s+7}{s^2+3s+2}=\frac{5}{s+1}-\frac{3}{s+2}$$

$$\leftrightarrow (5e^{-t}-3e^{-2t})u(t)=y_{zi}(t)$$

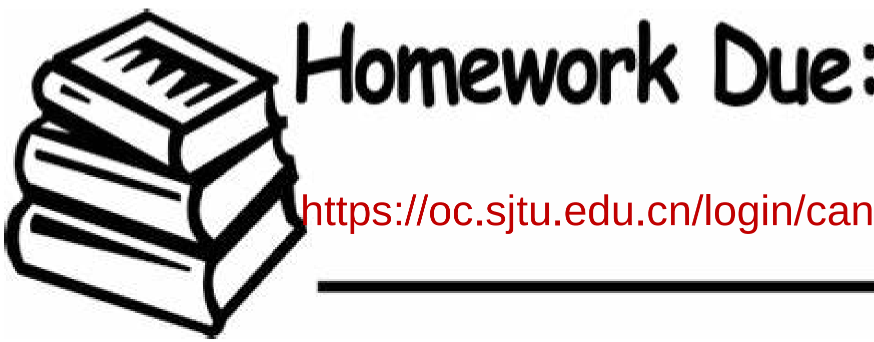
$$Y_{zs}(s)=\frac{2s+6}{s(s^2+3s+2)}=\frac{3}{s}-\frac{4}{s+1}+\frac{1}{s+2}$$

$$\leftrightarrow (3-4e^{-t}+e^{-2t})u(t)=y_{zs}(t)$$

$$\therefore y(t)=y_{zi}(t)+y_{zs}(t)=(3+e^{-t}-2e^{-2t})u(t)$$

Exercise: 若 $x(t)=e^{-t}u(t)$，再解上述方程

Q & A



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