Chapter 5 The Discrete-Time Fourier Transform

5.1 THE DISCRETE-TIME FORURIER TRANSFORM

- **Development**
- **Convergence Issues**
- **Examples**

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

基本信号$e^{j\omega n}$关于ω以2π为周期

$X(e^{j\omega})$ **is periodic with period** 2π

- $X[e^{j(\omega+2\pi k)}] = X(e^{j\omega})$

注：低频出现在 π 的偶数倍，高频出现在 π 的奇数倍

- $$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

注：上式积分区间为任何长度为2π，例：（$-\pi$，π）或（0，2π）

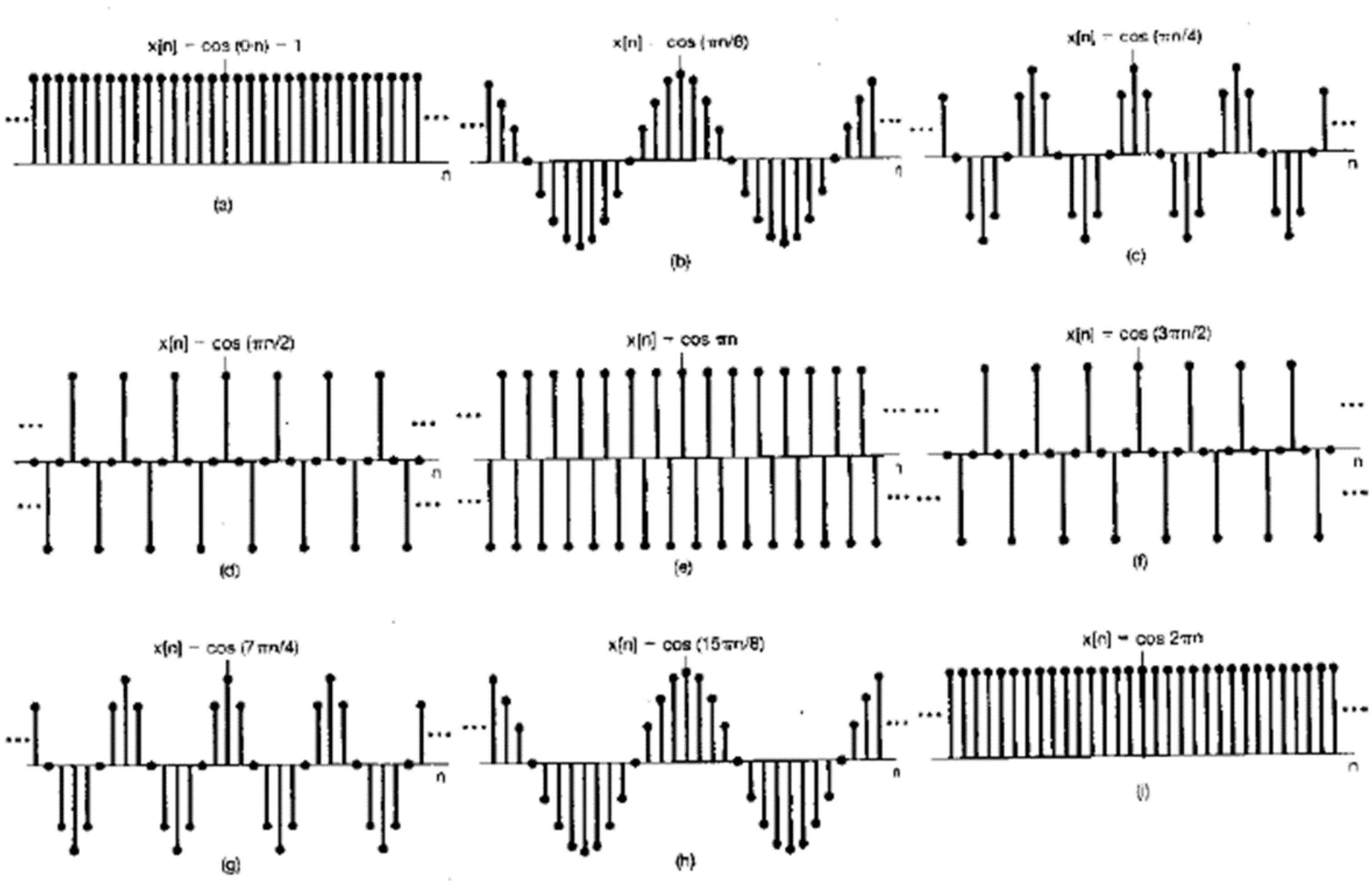


图 1.27 对应于几个不同频率时的离散时间正弦序列

5.1 THE DISCRETE-TIME FORURIER TRANSFORM

- **Development**
- **Convergence Issues**
- **Examples**

$\sum_n x[n]e^{-j\omega n}$ will converge either if x[n] is absolutely summable or has finite energy

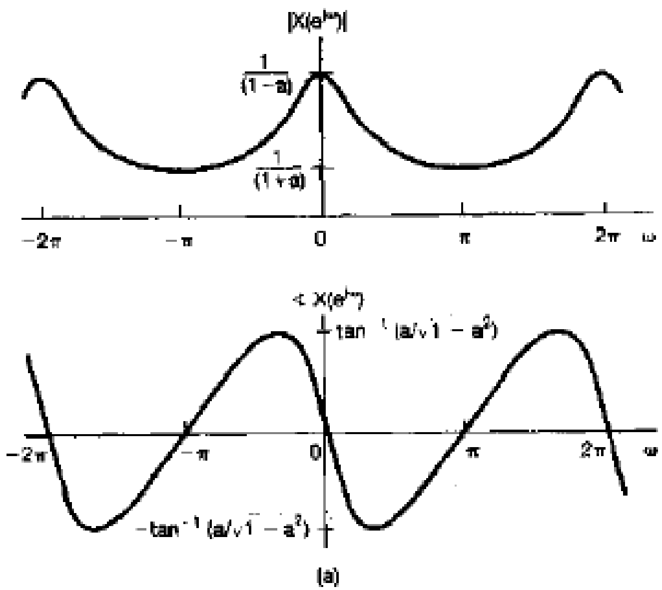
- 能量有限 $\sum_n |x[n]|^2 < \infty$
- 绝对可和 $\sum_n |x[n]| < \infty$

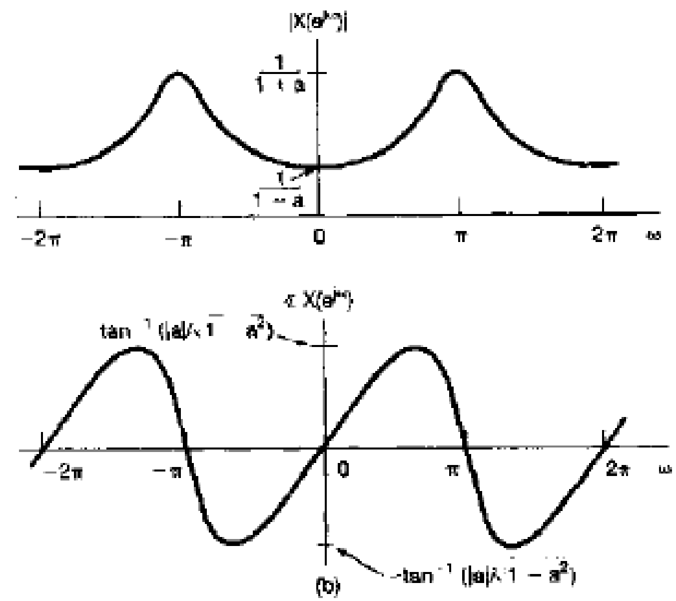
5.1 THE DISCRETE-TIME FORURIER TRANSFORM

- Development
- Convergence Issues
- Examples

①Exponential Signal

$$x[n]=\alpha^n u[n] \leftrightarrow X(e^{j\omega})=\frac{1}{1-\alpha e^{-j\omega}} \qquad |\alpha|<1$$





(a) $\alpha > 0$ (b) $\alpha < 0$

附：等比级数求和公式

$$\sum_{n=n_1}^{n_2} a^n = \begin{cases} \dfrac{a^{n_1}-a^{n_2+1}}{1-a}, & a \neq 1 \\ n_2-n_1+1, & a=1 \end{cases}$$

① $n_1=0, n_2=\infty$

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, |a|<1$$

② $n_1=n_1, n_2=\infty$

$$\sum_{n=n_1}^{\infty} a^n = \frac{a^{n_1}}{1-a}, |a|<1$$

②Unit Impulse

$$x[n]=\delta[n]\leftrightarrow X(e^{j\omega})=1$$

③Dc Signal

$$x[n]=1\leftrightarrow X(e^{j\omega})=2\pi\sum_{l=-\infty}^{\infty}\delta[\omega-2\pi l]$$

注：$X(e^{j\omega})$ 以 2π 为周期

$\mathrm{Pr}\,oof$:

$$\frac{1}{2\pi}\int_{2\pi}X(e^{j\omega})e^{j\omega n}d\omega=\frac{1}{2\pi}\int_{-\pi}^{\pi}2\pi\sum_{l}\delta(\omega-2\pi l)e^{j\omega n}d\omega$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}2\pi\delta(\omega)e^{j\omega n}d\omega=1$$

④Rectangular Pulse

$$x[n]=\begin{cases}1, & |n|\le N_1\\ 0, & |n|>N_1\end{cases}\leftrightarrow\frac{\sin\omega(N_1+\frac{1}{2})}{\sin(\omega/2)}$$

⑤ $$\frac{\sin Wn}{\pi n}\leftrightarrow X(e^{j\omega})=\begin{cases}1, & 0\le|\omega|\le W\\ 0, & W<|\omega|\le\pi\end{cases}$$

注: $W<\pi$

5.2 PROPERTIES OF THE DISCRETE-TIME FORURIER TRANSFORM

- Periodicity
- Linearity
- Time Shifting and Frequency Shifting
- ……

•Periodicity

$$\boxed{X(e^{j(\omega+2\pi)}) = X(e^{j\omega})}$$

Example:

Suppose: $X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} (-1)^k \delta(\omega - \frac{\pi k}{2})$

Determine: $x[n]$

提示：在$(-\pi$，$\pi]$内，$X(e^{j\omega}) = -\delta(\omega + \frac{\pi}{2}) + \delta(\omega) - \delta(\omega - \frac{\pi}{2}) + \delta(\omega - \pi)$

- **Linearity**

$$x_1[n] \leftrightarrow X_1(e^{j\omega}) \quad x_2[n] \leftrightarrow X_2(e^{j\omega})$$

$$\boxed{ax_1[n] + bx_2[n] \leftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega})}$$

- **Time Shifting and Frequency Shifting**

$$x[n] \leftrightarrow X(e^{j\omega})$$

$$\boxed{\begin{aligned} &x[n-n_0] \leftrightarrow e^{-j\omega n_0} X(e^{j\omega}) \\ &e^{j\omega_0 n} x[n] \leftrightarrow X[e^{j(\omega-\omega_0)}] \end{aligned}}$$

Example: $$x[n]=\alpha^n \cos\omega_0 n\cdot u[n],\quad |a|<1$$

$$x[n]=\alpha^n u[n]\cdot\frac{e^{j\omega_0 n}+e^{-j\omega_0 n}}{2}$$

$$\because\ \alpha^n u[n]\leftrightarrow\frac{1}{1-\alpha e^{-j\omega}}$$

$$\therefore\ x[n]\leftrightarrow X(e^{j\omega})=\frac{1}{2}\left[\frac{1}{1-\alpha e^{-j(\omega-\omega_0)}}+\frac{1}{1-\alpha e^{-j(\omega+\omega_0)}}\right]$$

$$=\frac{1-\alpha\cos\omega_0 e^{-j\omega}}{1-2\alpha\cos\omega_0 e^{-j\omega}+\alpha^2 e^{-j2\omega}},\quad |a|<1$$

Example: $$X(e^{j\omega}) = \sin^2 3\omega$$

$$X(e^{j\omega}) = \sin^2 3\omega$$

$$= \frac{1-\cos 6\omega}{2} = \frac{1}{2} - \frac{e^{j6\omega} + e^{-j6\omega}}{4}$$

$$\leftrightarrow \frac{1}{2}\delta[n] - \frac{1}{4}(\delta[n+6] + \delta[n-6])$$

About Time Expansion:

- Continuous-time signal

$$x(t) \leftrightarrow X(j\omega)$$

$$x(at) \leftrightarrow \frac{1}{|a|} X(j\frac{\omega}{a})$$

- Discrete-time signal

$$x[n] \leftrightarrow X(e^{j\omega}) \qquad x[an] \leftrightarrow X_a(e^{j\omega})$$

$X_a(e^{j\omega})$与$X(e^{j\omega})$，则不一定有如上确定关系 !

•**Time Expansion** $x[n] \leftrightarrow X(e^{j\omega})$

Defination:

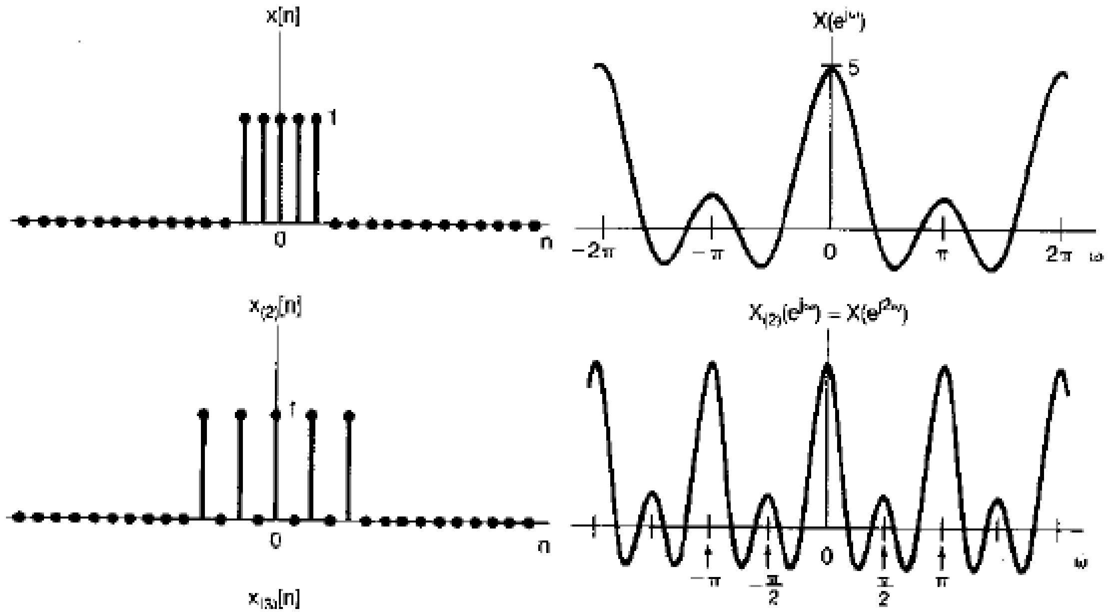
$$x_{(k)}[n] = \begin{cases} x[n/k], & 当n为k的整数倍 \\ 0, & 当n不为k的整数倍 \end{cases} \leftrightarrow X_{(k)}(e^{j\omega})$$

$$\boxed{X_{(k)}(e^{j\omega}) = X(e^{jk\omega})}$$

注：$x_{(k)}[n]$ 是在x[n]的相邻样本点插入(k-1)个零，相对于x[n]来说是“时域展宽”

$X_{(k)}(e^{j\omega})$ 的重复周期变为 $2\pi/k$ ，相对于 $X(e^{j\omega})$ 是“频域压缩”

Example： x[n]与x[n/2]



• **Time Reversal**

$$x[-n] \leftrightarrow X(e^{-j\omega})$$

• **Conjugation and Conjugate Symmetry**

$$x[n] \leftrightarrow X(e^{j\omega})$$

$$\boxed{x^*[n] \leftrightarrow X^*(e^{-j\omega})}$$

If x[n] is real valued

$$X(e^{j\omega}) = X^*(e^{-j\omega})$$ —Conjugate Symmetry

① $$X(e^{j\omega}) = \mathrm{Re}[X(e^{j\omega})] + jI_m[X(e^{j\omega})]$$

$$\boxed{\begin{aligned} \mathrm{Re}[X(e^{j\omega})] &= \mathrm{Re}[X(e^{-j\omega})] \\ I_m[X(e^{j\omega})] &= -I_m[X(e^{-j\omega})] \end{aligned}}$$

② $$X(e^{j\omega}) = |X(e^{j\omega})| e^{j\measuredangle X(e^{j\omega})}$$

$$\boxed{\begin{aligned} \left|X(e^{j\omega})\right| &= \left|X(e^{-j\omega})\right| \\ \angle X(e^{j\omega}) &= -\angle X(e^{-j\omega}) \end{aligned}}$$

③ $x[n]$ real and even $\leftrightarrow X(e^{j\omega})$ real and even

$x[n]$ real and odd $\leftrightarrow X(e^{j\omega})$ purely imaginary and odd

$$x_e[n] = \frac{1}{2}[x[n] + x[-n]] \leftrightarrow \mathrm{Re}[X(e^{j\omega})]$$

$$x_0[n] = \frac{1}{2}[x[n] - x[-n]] \leftrightarrow jI_m[X(e^{j\omega})]$$

•Differencing and Accumulation

$$x[n]-x[n-1]\leftrightarrow(1-e^{-j\omega})X(e^{j\omega})$$

$$\sum_{m=-\infty}^{n}x[m]\leftrightarrow\frac{1}{1-e^{-j\omega}}X(e^{j\omega})+\pi X(e^{j0})\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)$$

Example : $\because u[n]=\sum_{m=-\infty}^{n}\delta[m]$

$$\delta[n]\leftrightarrow 1$$

$$\therefore u[n]\leftrightarrow\frac{1}{1-e^{-j\omega}}+\pi\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)$$

•Differentiation in Frequency

$$\boxed{nx[n] \leftrightarrow j\frac{dX(e^{j\omega})}{d\omega}}$$

Example: 已知$x[n]=n(\frac{1}{2})^{|n|}$，求其傅里叶变换$X(e^{j\omega})$

设$x_1[n]=(\frac{1}{2})^{|n|}$

$$\leftrightarrow X_1(e^{j\omega})=\sum_{n=-\infty}^{-1}(\frac{1}{2})^{-n}e^{-j\omega n}+\sum_{n=0}^{\infty}(\frac{1}{2})^{n}e^{-j\omega n}$$

$$=\frac{\frac{1}{2}e^{j\omega}}{1-\frac{1}{2}e^{j\omega}}+\frac{1}{1-\frac{1}{2}e^{-j\omega}}=\frac{\frac{3}{4}}{(\frac{5}{4}-\cos\omega)}$$

则

$$X(e^{j\omega})=j\frac{dX_1(e^{j\omega})}{d\omega}$$

$$=-j\frac{\frac{3}{4}\sin\omega}{(\frac{5}{4}-\cos\omega)^2}$$

•Parseval's Relation

$$\sum_{n=-\infty}^{\infty}|x[n]|^2=\frac{1}{2\pi}\int_{2\pi}|X(e^{j\omega})|^2\,d\omega$$

$\because$ 在2π区间对每单位频率上的能量 $\dfrac{\left|X(e^{j\omega})\right|^2}{2\pi}$ 积分等于信号$x[n]$的总能量

$\therefore$ $\left|X(e^{j\omega})\right|^2$ 称为能谱密度

5.3 THE CONVOLUTION PROPERTY AND FEQUENCY RESPONSE

- Convolution Property
- Frequency Response

$$x_1[n] \leftrightarrow X_1(e^{j\omega}) \qquad x_2[n] \leftrightarrow X_2(e^{j\omega})$$

$$\boxed{x_1[n] * x_2[n] \leftrightarrow X_1(e^{j\omega}) \cdot X_2(e^{j\omega})}$$

Example: $\sum_{m=-\infty}^{n} x[m] = x[n] * u[n] \leftrightarrow X(e^{j\omega}) \cdot [\frac{1}{1-e^{j\omega}} + \pi \sum_k \delta(\omega - 2\pi k)]$

$$= \frac{X(e^{j\omega})}{1-e^{j\omega}} + \pi \sum_k X(0)\delta(\omega - 2\pi k)$$

Exercise: $x_1[n] = \alpha^n u[n] \qquad x_2[n] = \beta^n u[n]$

$$Determine\ y[n] = x_1[n] * x_2[n]$$

5.3 THE CONVOLUTION PROPERTY AND FEQUENCY RESPONSE

- **Convolution Property**
- **Frequency Response**

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$y[n] = x[n] * h[n]$$

设 $x[n] \leftrightarrow X(e^{j\omega})$ $y[n] \leftrightarrow Y(e^{j\omega})$

$$\boxed{H(e^{j\omega}) = \sum_{n} h[n] e^{-j\omega n}}$$ —frequency response

存在条件： $\sum_{n} |h[n]| < \infty$ —系统稳定

$$\boxed{Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega}) \qquad H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}}$$

1) $H(e^{j\omega})$可完全描述一个（稳定的）LTI系统

2) $H(e^{j\omega})$一般为复数

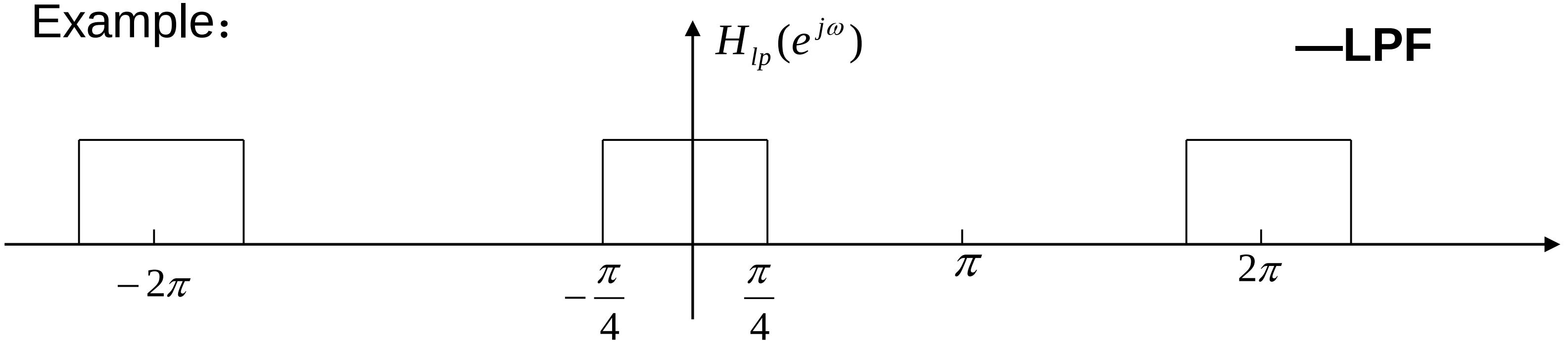
$$H(e^{j\omega}) = |H(e^{j\omega})| e^{j\angle H(e^{j\omega})}$$

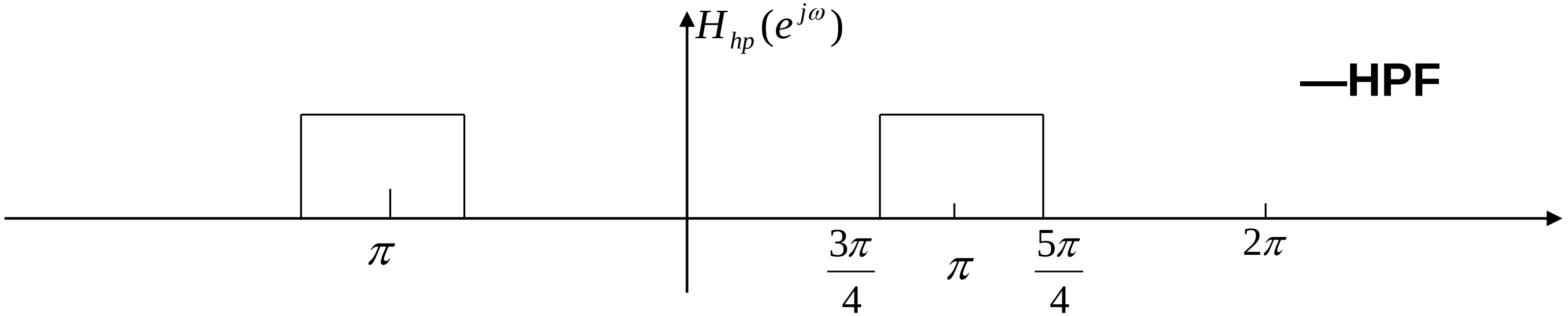
物理意义：给出任意频率分量经过LTI系统后幅度和相位的改变量

注：

- 频谱的周期性
- 低频/高频分量

Example：





$$H_{hp}(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)})$$

$$h_{hp}[n] = e^{jn\pi} h_{lp}[n] = (-1)^n h_{lp}[n]$$

附：**Multiplicaton Property**

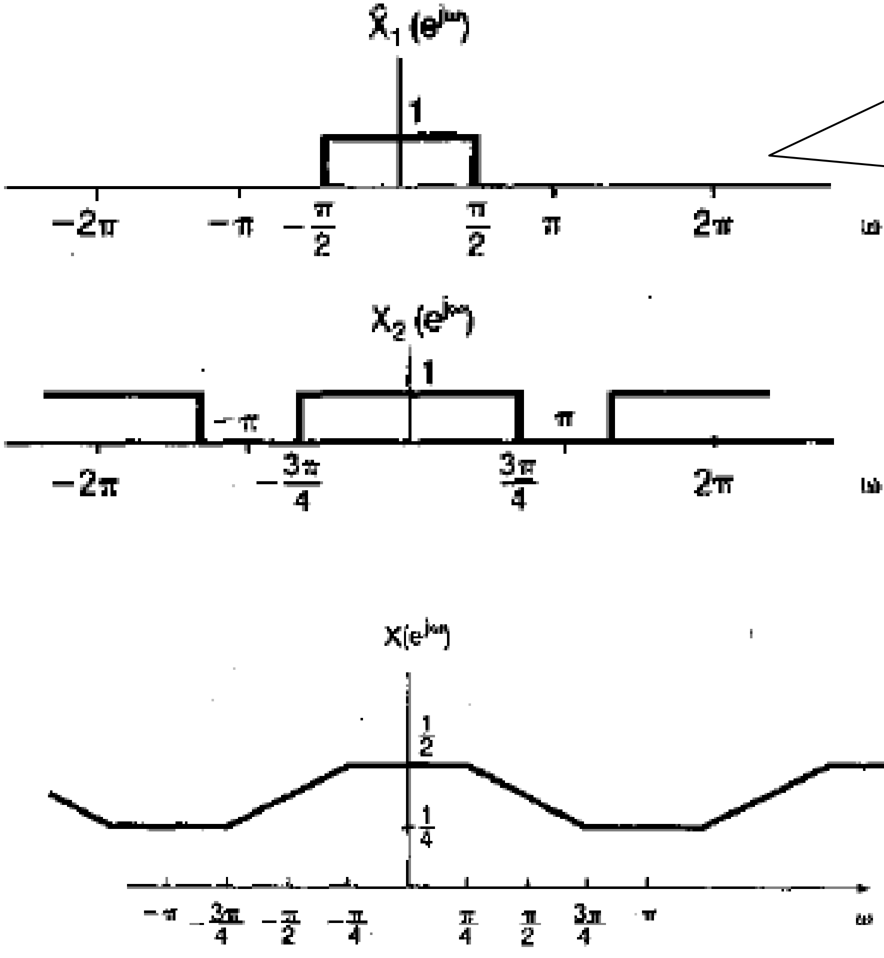
$$x_1[n] \leftrightarrow X_1(e^{j\omega}) \quad x_2[n] \leftrightarrow X_2(e^{j\omega})$$

$$\boxed{x_1[n]\cdot x_2[n] \leftrightarrow \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta}$$

周期卷积！

Example: $x_1[n] = \dfrac{\sin(3\pi n/4)}{\pi n}$ $\quad x_2[n] = \dfrac{\sin(\pi n/2)}{\pi n}$

$$x_1[n]\cdot x_2[n] \leftrightarrow ?$$



取$X_1(e^{j\omega})$的主周期

将周期卷积转换为非周期卷积

5.4 LINEAR CONSTANT-COEFFICENT DIFFERENCE EQUATION

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}$$

Example: $$y[n]-\frac{3}{4}y[n-1]+\frac{1}{8}y[n-2]=2x[n]$$

$$H(j\omega)=\frac{2}{1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-j2\omega}}$$

$$=\frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}$$

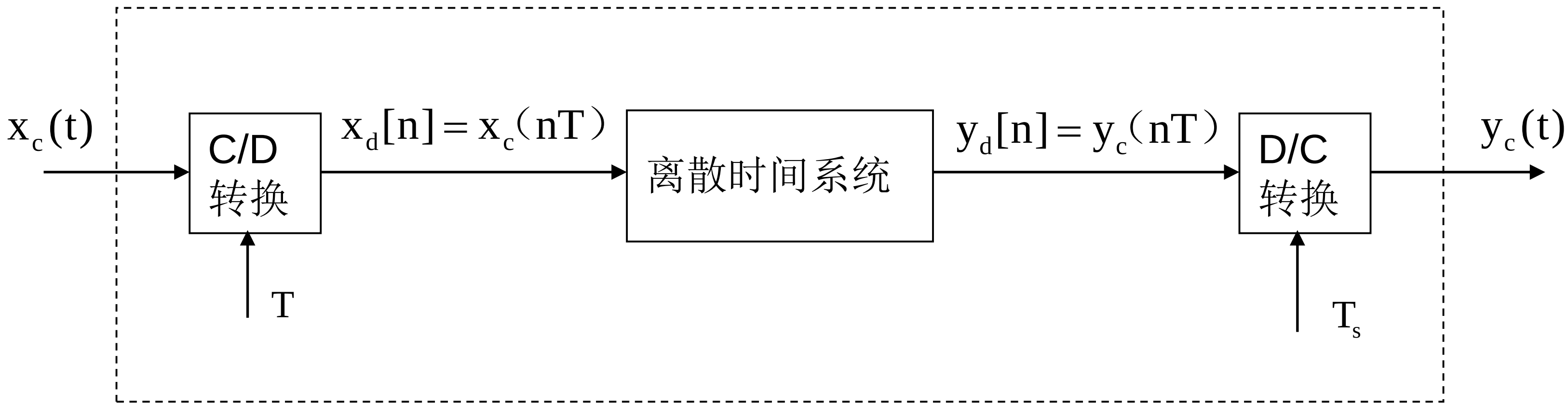
$$H(j\omega)=\frac{4}{1-\frac{1}{2}e^{-j\omega}}-\frac{2}{1-\frac{1}{4}e^{-j\omega}}$$

Expanded by the method of partial fractions

$$h[n]=4(\frac{1}{2})^n u[n]-2(\frac{1}{4})^n u[n]$$

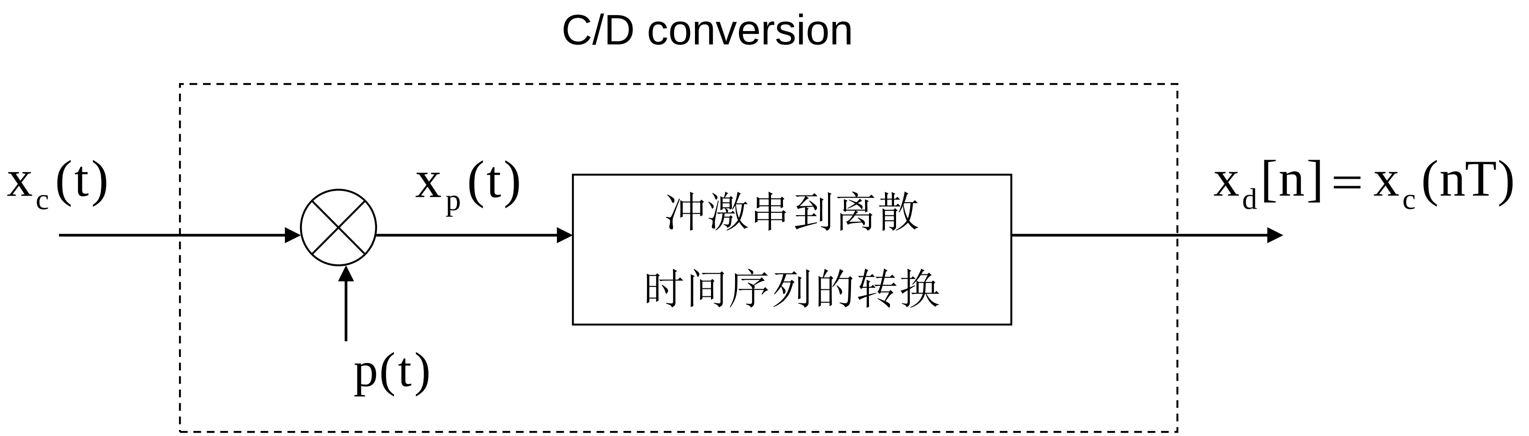
5.5 DISCRETE-TIME PROCESSING OF CONTINUOUS-TIME SIGNALS

——将连续时间信号转换成离散时间信号，经离散时间系统处理后再转为连续时间信号。



■ C/D和D/C转换

1. C/D转换



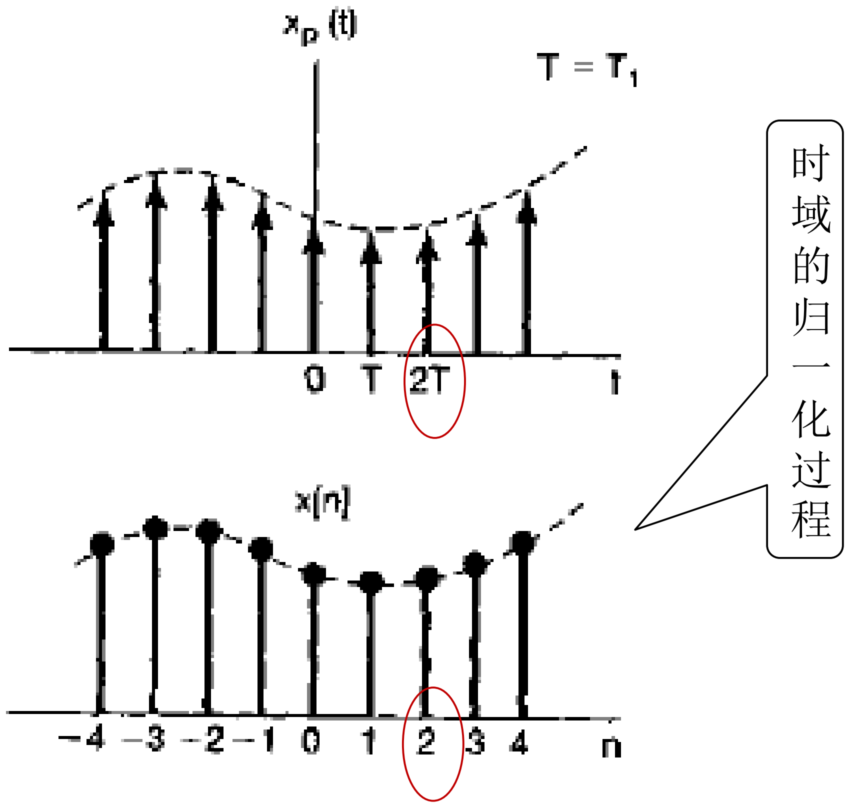
1）取样—满足取样定理，不产生频谱混叠

2）冲激串到离散时间序列的转换

注：用于实现C/D转换的器件——模拟数字（A/D）转换器

✓ **关于冲激串(impulse train)到离散时间序列(D-T sequence)的转换**

- 时域：$x_d[n] = x_c[nT]$，即除以T



- 频域：

连续时间信号的频率为ω，离散时间信号的频率为Ω

$$\because x_p(t)=\sum_{n=-\infty}^{\infty}x_c(nT)\delta(t-nT)$$

$$\therefore X_p(j\omega)=\sum_{n=-\infty}^{\infty}x_c(nT)e^{-j\omega nT}$$

又 $$X_d(e^{j\Omega})=\sum_{n=-\infty}^{\infty}x_d[n]e^{-j\Omega n}$$

$$=\sum_{n=-\infty}^{\infty}x_c[nT]e^{-j\Omega n}$$

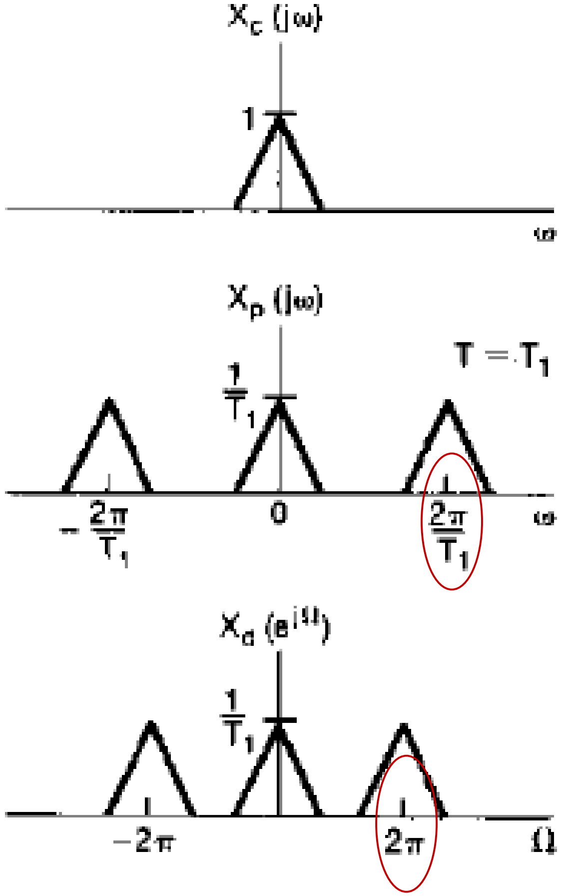
当 $\Omega=\omega T$ 时

注：时域除以T，频域乘以T
——时域和频域的对偶特性！

$$X_p(j\omega)=X_d(e^{j\Omega})$$

$$\because X_p(j\omega)=\frac{1}{T}\sum X_c(j(\omega-k\omega_s)) \quad \omega_s=\frac{2\pi}{T}$$

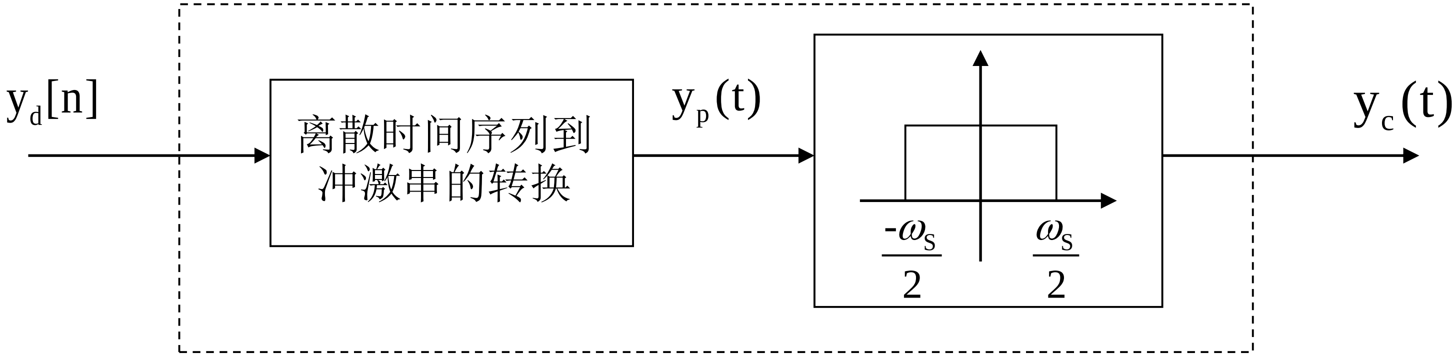
$$\therefore X_d(e^{j\Omega})=\frac{1}{T}\sum X_c(j(\Omega-k2\pi)/T)$$



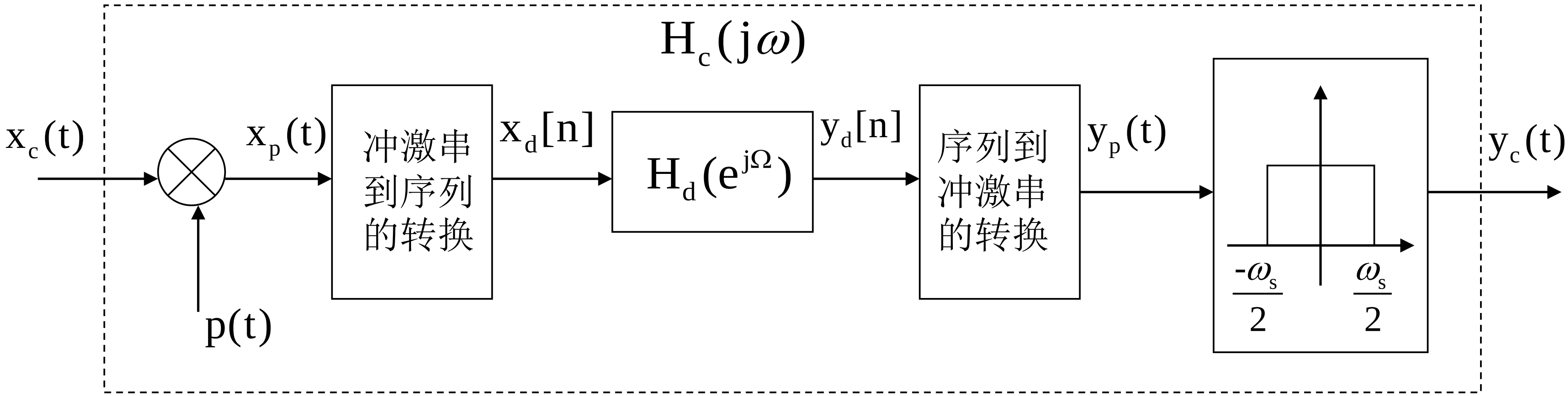
频域的归一归一化过程

2. D/C转换

D/C conversion



■ 冲激响应不变法



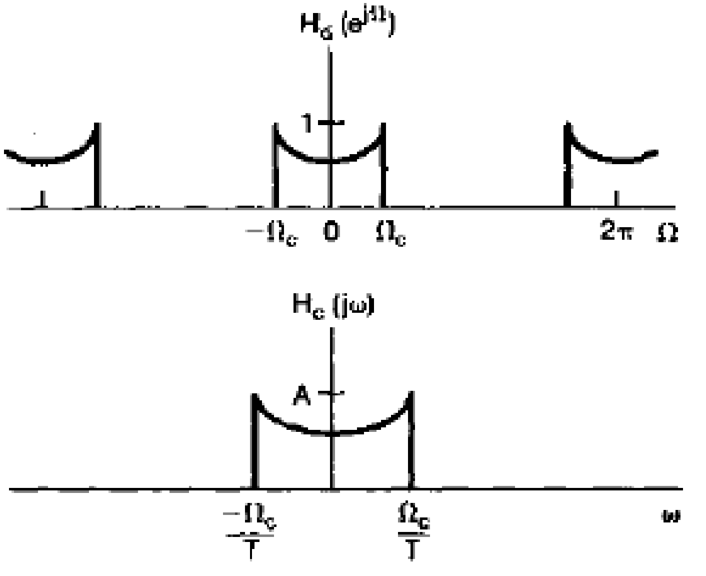
$H_c(j\omega)$
$x_c(t)$
$x_p(t)$
$p(t)$
冲激串到序列的转换
$x_d[n]$
$H_d(e^{j\Omega})$
$y_d[n]$
序列到冲激串的转换
$y_p(t)$
$\frac{-\omega_s}{2}$
$\frac{\omega_s}{2}$
$y_c(t)$

若， 1）离散时间系统是线性和时不变的

2）输入信号是带限的，且采样频率足够高

则，上述整个系统就等效为一个频率响应为 $H_c(j\omega)$ 的连续时间系统

$$H_c(j\omega)=\begin{cases} H_d(e^{j\omega T}), & |\omega|<\dfrac{\omega_s}{2} \\ 0, & |\omega|>\dfrac{\omega_s}{2} \end{cases}$$



$$\because H_d(e^{j\Omega}) = H_c(j\omega), \quad |\omega| < \frac{\omega_s}{2}$$

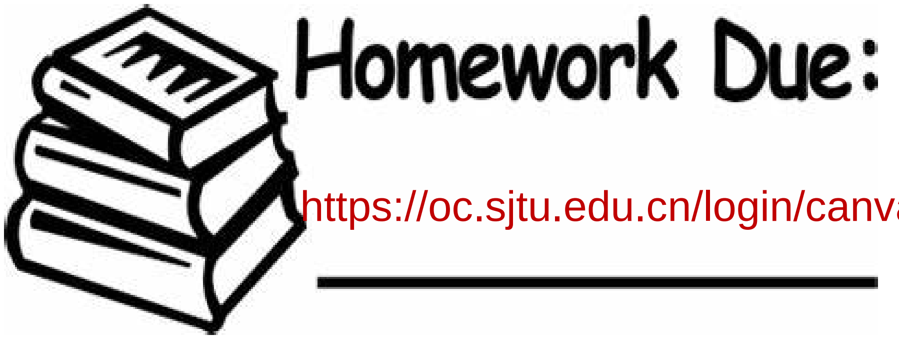
$$H_d(e^{j\Omega}) \leftrightarrow h_d[n] \quad H_c(j\omega) \leftrightarrow h_c(t)$$

若设$h_d[n]$是$h_c(t)$采样获得

考虑到$x_d[n] = x_c(nT)$时，有 $X_d(e^{j\Omega}) = \frac{1}{T}\sum X_c[j(\omega - k\omega_s)]$

$$\therefore \quad \boxed{h_d[n] = Th_c(nT)}$$ ——冲激响应不变法

Q & A



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