Chapter4 The Continuous-Time Fourier Transform

4.5 Frequency-Domain Analysis of LTI System

- **Frequency Response**
- **Filtering**
- **Modulating**
- **Sampling**

■ 频率响应

1. 定义

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = x(t) * h(t)$$

设 $\quad x(t) \leftrightarrow X(j\omega) \quad y(t) \leftrightarrow Y(j\omega)$

$$\boxed{H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt}$$

—frequency response

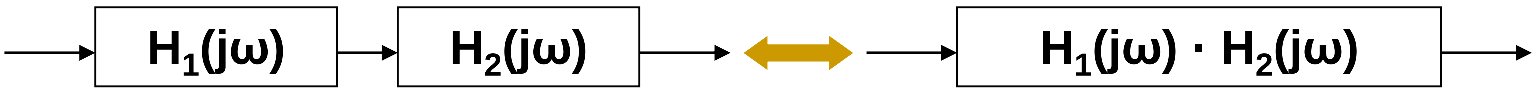
（频率响应）

则

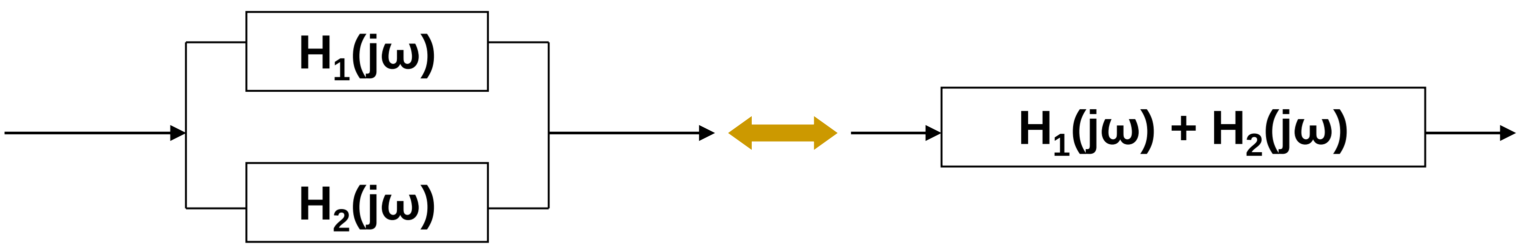
$$\boxed{Y(j\omega) = X(j\omega) H(j\omega) \qquad H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}}$$

注：$H(j\omega)$存在条件 $\int_{-\infty}^{\infty} |h(t)| dt < \infty$ 此时所描述的LTI系统是稳定的

2. $H(j\omega)$可完全描述一个（稳定的）LTI系统



注：级联系统的频率响应等于各自系统频率响应的乘积



注：并联系统的频率响应等于各自系统频率响应之和

3. $H(j\omega)$一般为复数

$$\boxed{H(j\omega)=\left|H(j\omega)\right|e^{j\measuredangle H(j\omega)}}$$

For LTI system

$$\because Y(j\omega)=X(j\omega)\cdot H(j\omega)$$

$$\therefore \left|Y(j\omega)\right|=\left|X(j\omega)\right|\cdot\left|H(j\omega)\right|$$

$$\measuredangle Y(j\omega)=\measuredangle X(j\omega)+\measuredangle H(j\omega)$$

- $\boldsymbol{H(j\omega)}$**给出输入的任意频率分量经过LTI系统后幅度和相位的改变量**
- $|\boldsymbol{H(j\omega}|$**为系统增益(gain of system)**

$\measuredangle\boldsymbol{H(j\omega)}$**为系统相移(phase shift of system)**

① $x(t)$为复指数信号

$$x(t)=e^{j\omega_0 t} \rightarrow y(t)=H(j\omega)\big|_{\omega=\omega_0} e^{j\omega_0 t}$$

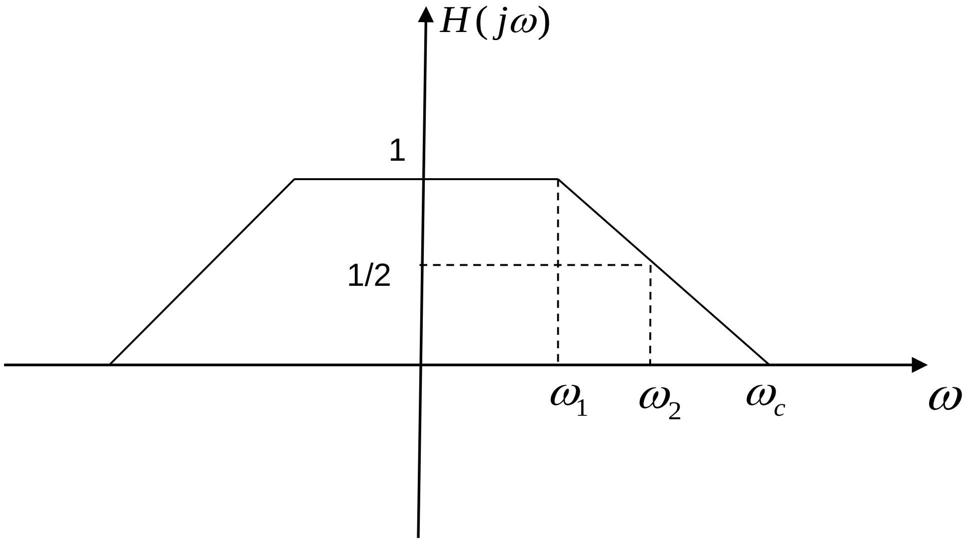
Proof: $x(t)=e^{j\omega t} \Longrightarrow y(t)=\int_{-\infty}^{\infty} h(\tau)e^{j\omega(t-\tau)}d\tau$

$$=\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau \cdot e^{j\omega t}=H(j\omega)e^{j\omega t}$$

LTI对复指数的响应仍是同频率的复指数信号，只是幅度和相位发生改变，改变量取决于$H(j\omega)$在该频率的模值和相位。

$e^{j\omega t}$ ——特征函数(eigenfunction)，$H(j\omega)$为其特征值(eigenvalue)

Example:



if $\quad x(t)=1+e^{j\omega_1 t}+e^{j\omega_2 t}+e^{j\omega_c t}$

根据 $\quad x(t)=e^{j\omega_0 t}\rightarrow y(t)=H(j\omega)|_{\omega=\omega_0}\, e^{j\omega_0 t}$

then $\quad y(t)=1+1\cdot e^{j\omega_1 t}+\dfrac{1}{2}\cdot e^{j\omega_2 t}+0\cdot e^{j\omega_c t}$

Example: Consider a LTI system with $h(t) = e^{-t}u(t)$, If the input $x(t) = \sin(t)$, determine the output $y(t)$

$$x(t) = \sin(\omega_0 t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

$$\Rightarrow y(t) = \frac{1}{2j} H(j\omega_0) e^{j\omega_0 t} - \frac{1}{2j} H(-j\omega_0) e^{-j\omega_0 t}$$

$$= \frac{1}{2j} |H(j\omega_0)| e^{j\omega_0 t + j\measuredangle H(j\omega_0)} - \frac{1}{2j} |H(-j\omega_0)| e^{-j\omega_0 t + j\measuredangle H(-j\omega_0)}$$

$$\because h(t)\ real$$

$$\therefore y(t) = \frac{1}{2j} |H(j\omega_0)| e^{j\omega_0 t + j\measuredangle H(j\omega_0)}$$

$$- \frac{1}{2j} |H(j\omega_0)| e^{-j\omega_0 t - j\measuredangle H(j\omega_0)}$$

$$= |H(j\omega_0)| \sin(\omega_0 t + \measuredangle H(j\omega_0))$$

$$\because h(t) = e^{-t}u(t) \leftrightarrow H(j\omega) = \frac{1}{1 + j\omega}$$

$$\Rightarrow |H(j\omega)| = \frac{1}{\sqrt{1+\omega^2}} \quad \measuredangle H(j\omega) = \tan^{-1}(-\omega)$$

$$\therefore y(t) = |H(j\omega)|_{\omega=1} \sin(t + \measuredangle H(j\omega)|_{\omega=1})$$

$$= \frac{1}{\sqrt{2}} \sin(t - 45^\circ)$$

② $x(t)$为任意信号

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega=\lim_{\omega_0\to 0}\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}X(jk\omega_0)e^{jk\omega_0 t}\omega_0$$

$$\Rightarrow y(t)=\lim_{\omega_0\to 0}\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}X(jk\omega_0)H(jk\omega_0)e^{jk\omega_0 t}\omega_0$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)H(j\omega)e^{j\omega t}d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}Y(j\omega)e^{j\omega t}d\omega$$

$$x(t) = \frac{1}{2\pi}\int X(j\omega)e^{j\omega t}d\omega \longrightarrow \boxed{h(t) \leftrightarrow H(j\omega)} \longrightarrow y(t) = \frac{1}{2\pi}\int X(j\omega)H(j\omega)e^{j\omega t}d\omega$$

$x(t) = \dfrac{1}{2\pi}\displaystyle\int X(j\omega)e^{j\omega t}d\omega$，就是将$x(t)$分解成$e^{j\omega t}$的“线性组合，$LTI$系统就是对$x(t)$的各频率分量的幅度加权($|H(j\omega)|$)，同时对各频率分量产生各自的相移($\measuredangle H(j\omega)$)。在输出端再对改变后的各频率分量“合成”，就得到系统的响应$y(t) = \dfrac{1}{2\pi}\displaystyle\int X(j\omega)H(j\omega)e^{j\omega t}d\omega$

$$= \frac{1}{2\pi}\int Y(j\omega)e^{j\omega t}d\omega$$

■ 无失真传输

- 无失真传输(Distortionless Transmission)
  ——输出与输入相比，只有幅度大小和出现时间先后的不同，波形形状未发生改变

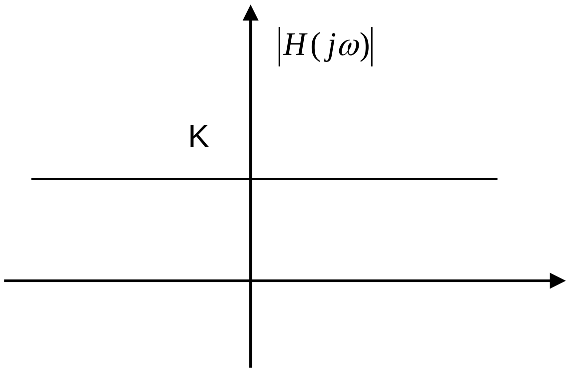
$$\because y(t) = Kx(t - t_0)$$

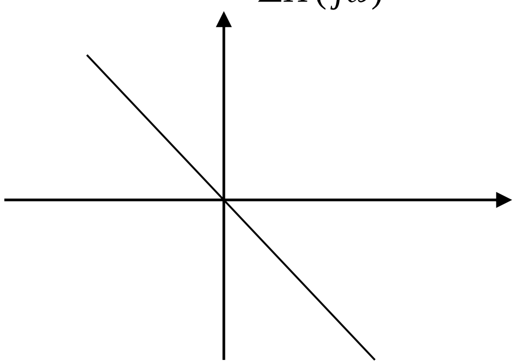
$$\therefore H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = Ke^{-j\omega t_0}$$

$$\leftrightarrow h(t) = K\delta(t - t_0)$$

无失真传输的条件：

$$\begin{cases} |H(j\omega)| = K \\ \measuredangle H(j\omega) = -\omega t_0 \end{cases}$$





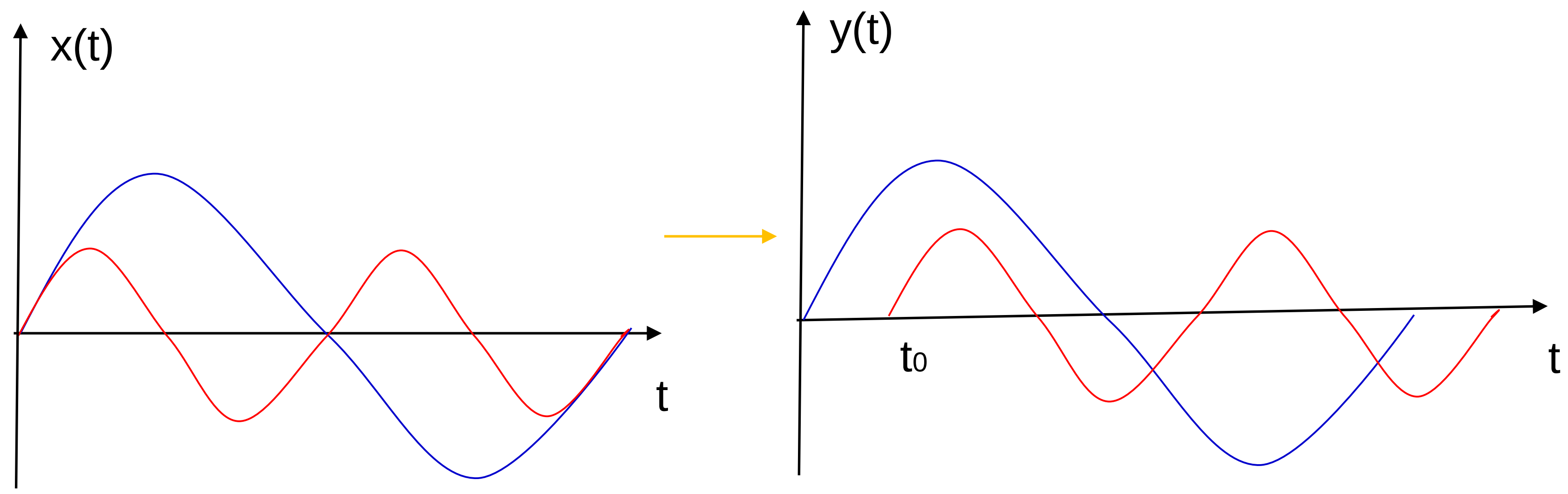
即，无失真传输就是传输信号的各频率分量的幅度获得相同的增益，且各频率分量的相移是频率的线性函数。

此时，群延时(group delay)为：

$$\tau = -\frac{d}{d\omega}\{\measuredangle H(j\omega)\} = t_0$$

即，输入的所有频率分量经过无失真传输后具有相同的时移！

- 失真
  - 幅度失真，系统对各频率分量的幅度产生不同程度衰减，使各频率分量的相对幅度发生改变。
  - 相位失真，系统对各频率分量的相移不是频率的线性函数，使输出的各频率分量在时间轴上的相对位置发生改变。



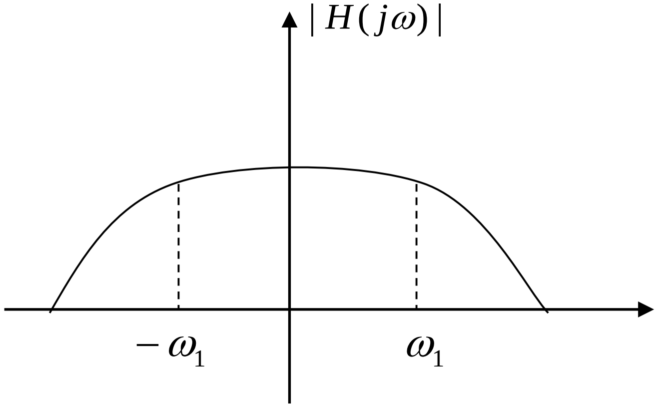
$$x(t)=\cos(\omega_1 t)+\frac{1}{2}\cos(\omega_2 t)$$

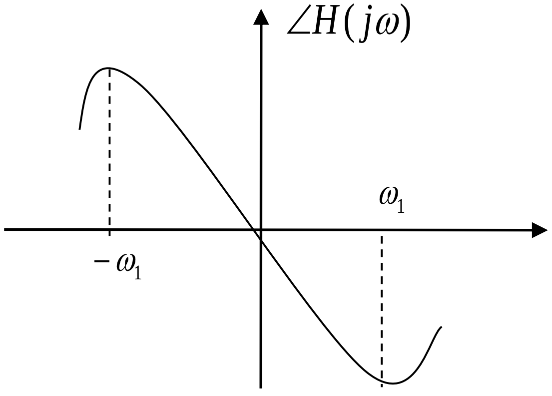
$$y(t)=\cos(\omega_1 t)+\frac{1}{2}\cos(\omega_2 (t-t_0))$$

e.g. 相位失真

- 实际系统不要求在整个频率轴上满足无失真传输条件，只要在信号频率范围内近似满足即可。

例：





■ 其它常见系统的频率响应

① 延时器

$$\because y(t) = x(t - t_0)$$

$$\therefore Y(j\omega) = X(j\omega)e^{-j\omega t_0}$$

$$\boxed{H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = e^{-j\omega t_0}}$$

② 微分器

$$\because y(t) = \frac{dx(t)}{dt}$$

$$\therefore Y(j\omega) = j\omega \cdot X(j\omega)$$

$$\boxed{H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = j\omega}$$

③ 积分器

$$\because\ y(t)=\int_{-\infty}^{t}x(\tau)d\tau$$

$$\therefore Y(j\omega)=\frac{X(j\omega)}{j\omega}+\pi X(0)\delta(\omega)$$

$$\boxed{H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{1}{j\omega}+\pi\delta(\omega)}$$

$$if\ \ X(0)=\int x(t)dt=0\ then$$

$$H(j\omega)=\frac{1}{j\omega}$$

一些常见系统的$h(t)$和$H(j\omega)$

	$h(t)$	$H(j\omega)$
id	$\delta(t)$	1
τ_{t_0}	$\delta(t-t_0)$	$e^{-j\omega t_0}$
$\frac{d}{dt}$	$\delta'(t)$	$j\omega$
$\int_{-\infty}^{t}$	$u(t)$	$\frac{1}{j\omega}+\pi\delta(\omega)$
ideal lowpass	$\frac{\sin(\omega_c t)}{\pi t}$	$u(\omega+\omega_c)-u(\omega-\omega_c)$
1st order lowpass	$\frac{1}{\tau}e^{-t/\tau}u(t)$	$\frac{1}{1+j\tau\omega}$

4.5 Frequency-Domain Analysis of LTI System

- Frequency Response
- Filtering
- Modulating
- Sampling

- **Filtering（滤波）**

  －改变信号中各频率分量的相对大小或完全消除某些频率分量

  - **Frequency-shaping Filters（频率成形滤波器）**

    －改变信号频谱形状的LTI系统

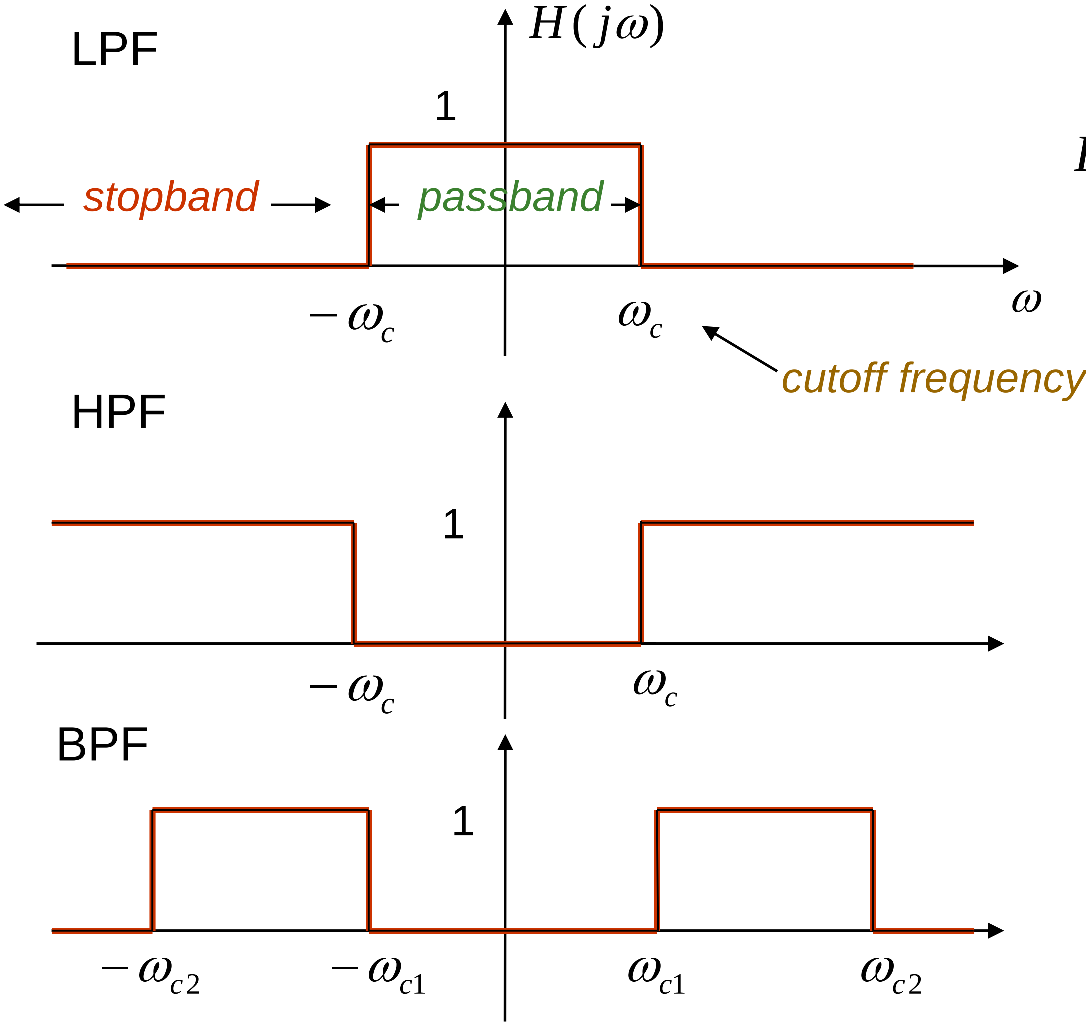
    例：音响系统

- **Frequency-Selective Filters（频率选择滤波器）**

—让某些频率分量无失真通过，消除另一些频率分量的LTI系统

具有理想特性的频率选择性滤波器：

- LPF(Lowpass Filter)
- HPF(Highpass Filter)
- BPF(Bandpass Filter)

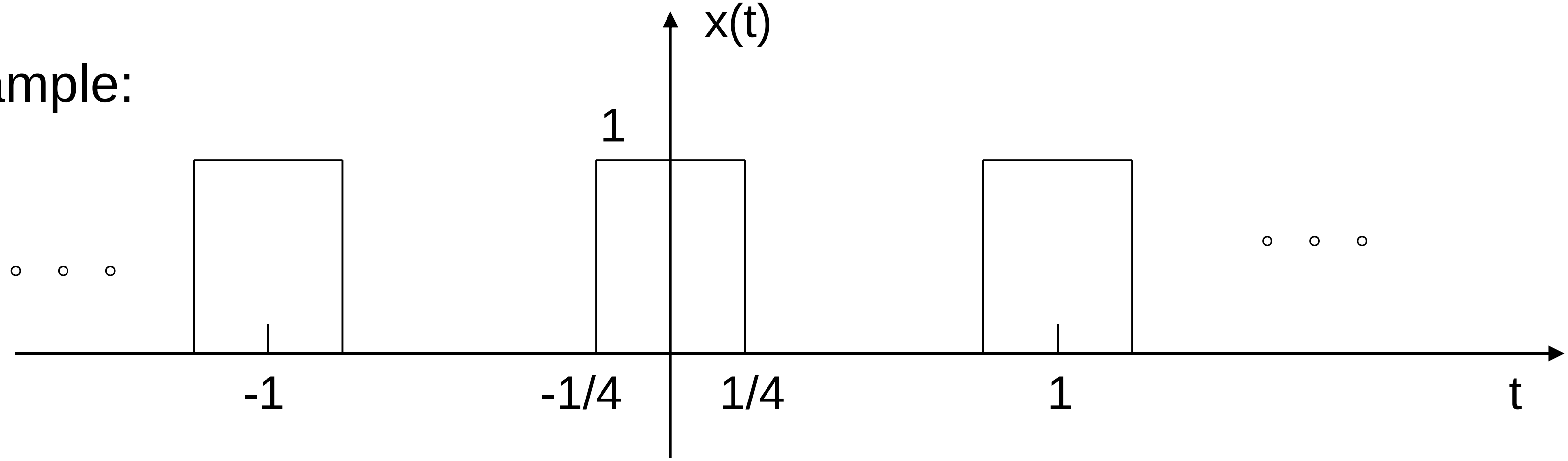


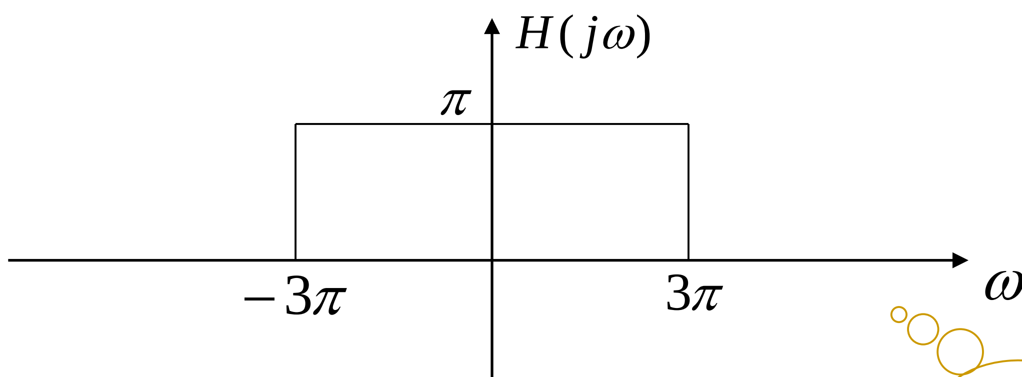
$$H(j\omega)=\begin{cases}1, & |\omega|\le\omega_c\\ 0, & |\omega|>\omega_c\end{cases}$$

$$H(j\omega)=\begin{cases}1, & |\omega|>\omega_c\\ 0, & |\omega|\le\omega_c\end{cases}$$

$$H(j\omega)=\begin{cases}1, & \omega_{c1}<|\omega|<\omega_{c2}\\ 0, & |\omega|\le\omega_{c1},|\omega|\ge\omega_{c2}\end{cases}$$

Example:





求信号x(t)经过如图LPF的输出y(t)

注：$H(j\omega)$的物理意义！

Solution: $\because \omega_0 = 2\pi$

$\therefore$只有$x(t)$的直流和一次谐波经过$H(j\omega)$后有输出

$$\because a_k = \frac{\sin(k\omega_0 T_1)}{k\pi} = \frac{\sin(k\frac{\pi}{2})}{k\pi}$$

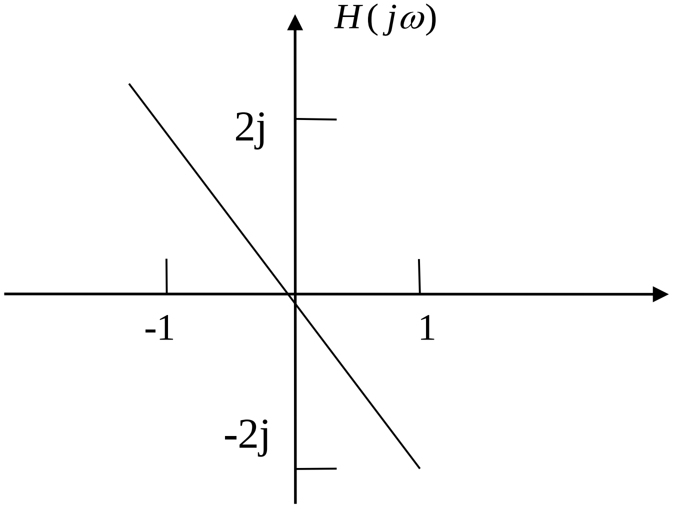
$$\therefore a_1 = a_{-1} = \frac{1}{\pi} \qquad a_0 = \frac{1}{2}$$

$$x(t) = \frac{1}{2} + \frac{1}{\pi}e^{j2\pi t} + \frac{1}{\pi}e^{-j2\pi t} + \ldots\ldots$$

经过$H(j\omega)$后

$$y(t) = \pi \cdot \frac{1}{2} + \pi \cdot \frac{1}{\pi}e^{j2\pi t} + \pi \cdot \frac{1}{\pi}e^{-j2\pi t} = \frac{\pi}{2} + 2\cos 2\pi t$$

Example：求信号经过如图H(jω)的输出



$H(j\omega) = -2j\omega$

1、 $x(t) = e^{jt}$

$\because H(j\omega)$为微分器

$\therefore y(t) = -2\dfrac{dx(t)}{dt} = -2je^{jt}$

2、 $X(j\omega) = \dfrac{1}{(j\omega)(6 + j\omega)}$

$\because Y(j\omega) = X(j\omega)H(j\omega) = \dfrac{1}{6 + j\omega}$

$\therefore y(t) = -2e^{-6t}u(t)$

注：时域or频域求解！

Example：设$x(t)$为周期等于1，经半波整流的正弦波，即

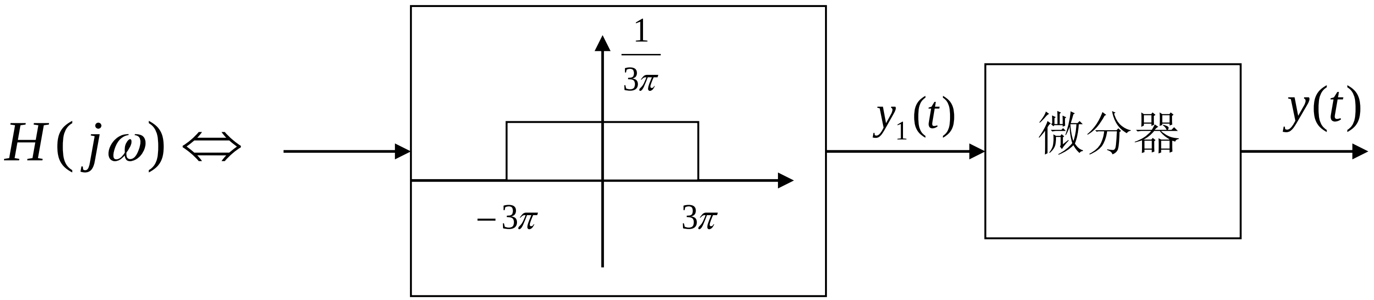
$$x(t)=\begin{cases}\sin 2\pi t, m\le t\le (m+\frac{1}{2})\\ 0,(m+\frac{1}{2})\le t\le m+1\end{cases}$$ ，m为整数

求$x(t)$经过$x(t)=\begin{cases}\sin 2\pi t, m\le t\le (m+\frac{1}{2})\\ 0,(m+\frac{1}{2})\le t\le m+1\end{cases}$的输出$y(t)$

Solution:

$$x(t) \leftrightarrow 2\pi \sum_k a_k \delta(\omega - k\omega_0),\ \omega_0 = \frac{2\pi}{T} = 2\pi$$

所以 $x(t)$ 包含的频率分量 $0, \pm 2\pi, \pm 4\pi, \cdots$



仅 $x(t)$ 的直流和一次谐波分量在LPF的通带内，考虑到微分器，只需求一次谐波经过LPF的输出。

$$\because a_1 = \int_0^{1/2} \sin 2\pi t e^{-j2\pi t} dt = \frac{1}{4j}$$

$$a_{-1} = \int_0^{1/2} \sin 2\pi t e^{+j2\pi t} dt = -\frac{1}{4j}$$

$\therefore LPF$的输出

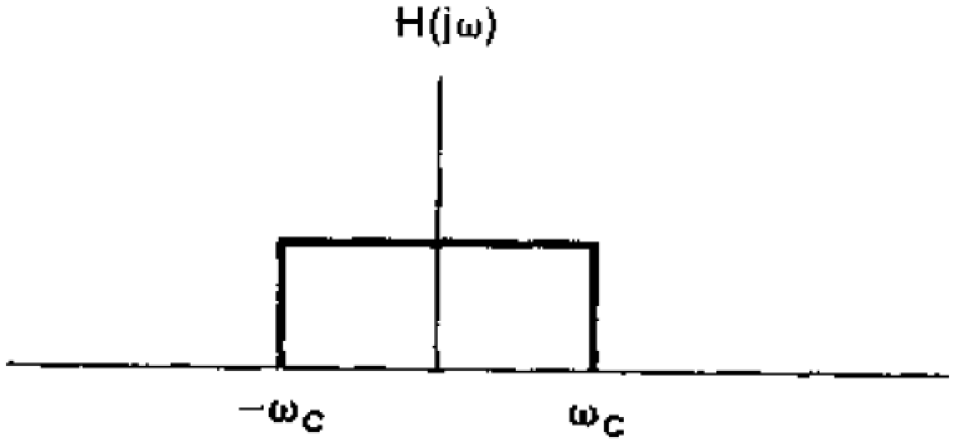
$$y_1(t) = \frac{1}{3\pi}\left[\frac{1}{4j} e^{j2\pi t} - \frac{1}{4j} e^{-j2\pi t}\right] = \frac{1}{6\pi} \sin 2\pi t$$

经过微分后

$$y(t) = \frac{dy_1(t)}{dt} = \frac{1}{3} \cos 2\pi t$$

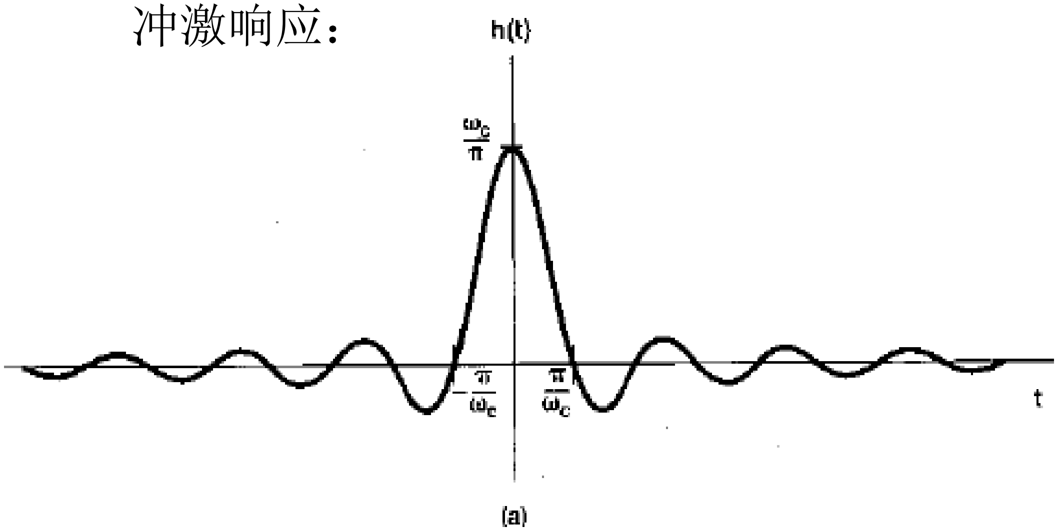
Time- and Frequency- Domain Aspects of Filter

- 具有零相位的理想特性的LPF

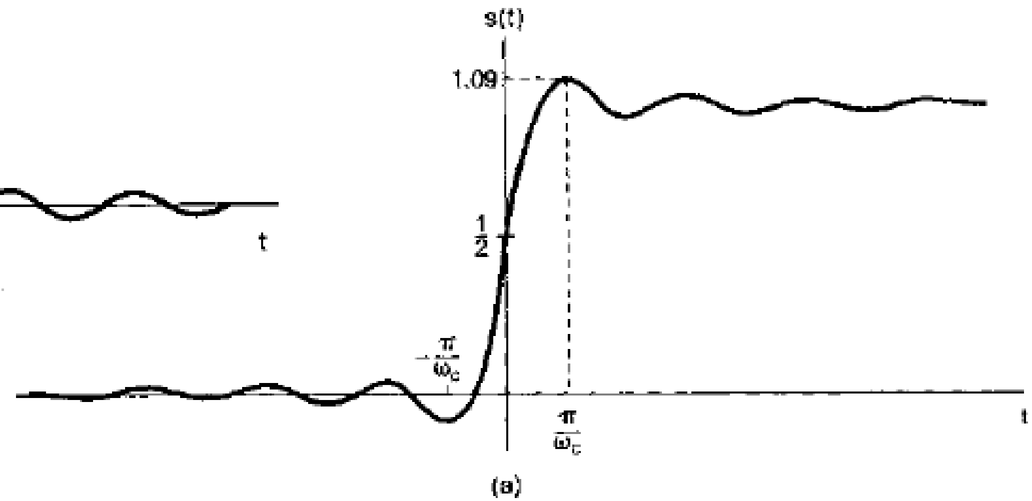


$$H(j\omega)=\begin{cases}1, & |\omega|<\omega_c\\ 0, & |\omega|>\omega_c\end{cases}$$

冲激响应：



阶跃响应：



$$h(t)=\frac{\sin\omega_c t}{\pi t}$$

$$=\frac{\omega_c}{\pi}Sa(\omega_c t)$$

$$s(t)=\frac{1}{2}+\frac{1}{\pi}Si(\omega_c t)$$

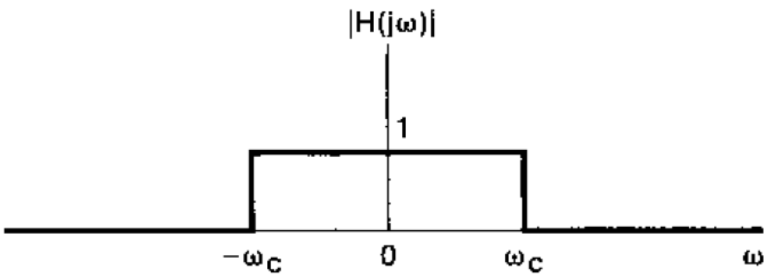
注： $Si(y)=\int_0^y \frac{\sin x}{x}dx$ —正弦积分

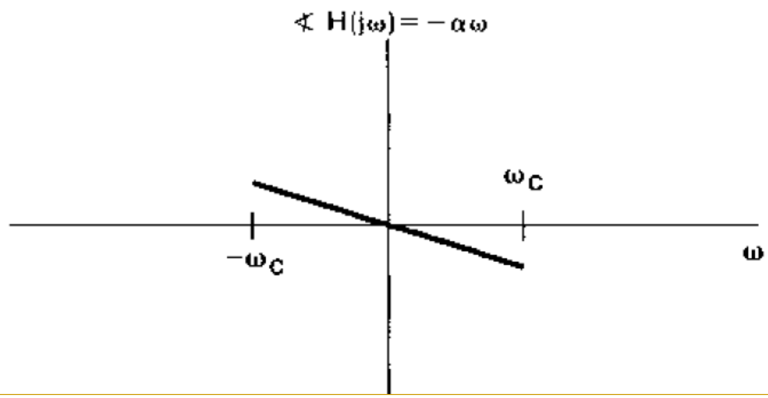
理想的LPF具有完美的频率选择特性，但存在如下问题：

- h(t)非因果，存在振荡
- s(t)存在过冲和振荡

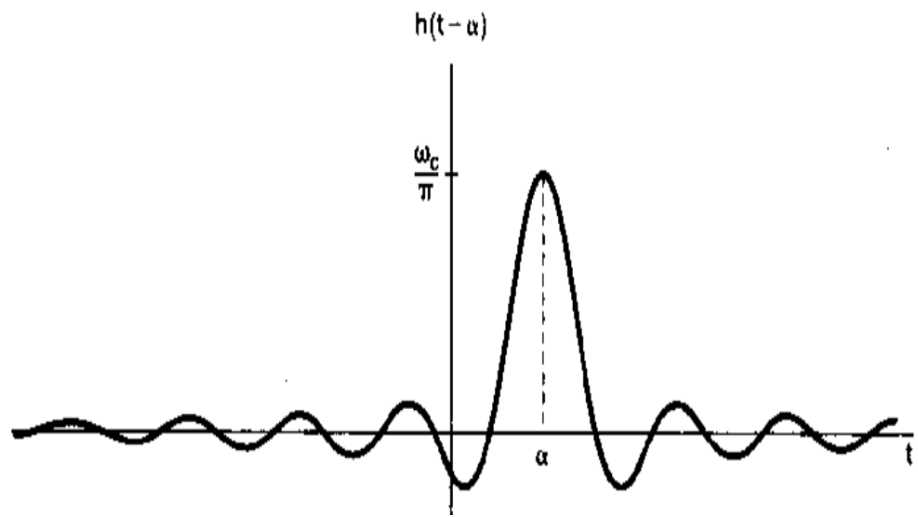
- 具有线性相位的理想特性的LPF

$$H(j\omega)=e^{-j\omega\alpha},\quad |\omega|\le\omega_c$$



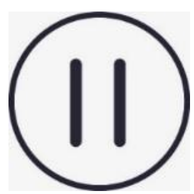


冲激响应：



$$h(t)=\frac{\sin\omega_c(t-\alpha)}{\pi(t-\alpha)}$$

- 理想特性实现的可能性？
- 是否需要理想特性的滤波器？



- 非理想特性的LPF

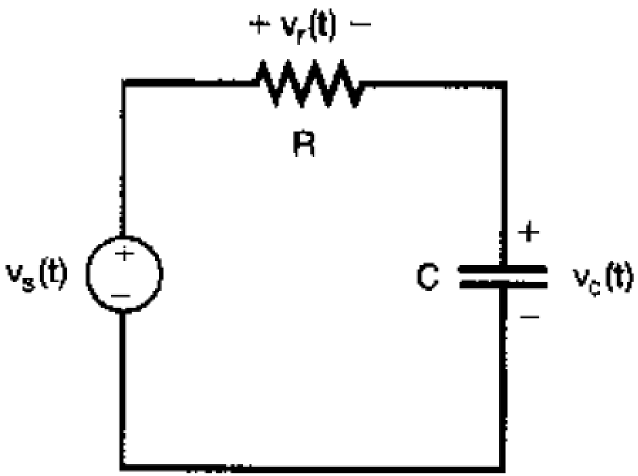
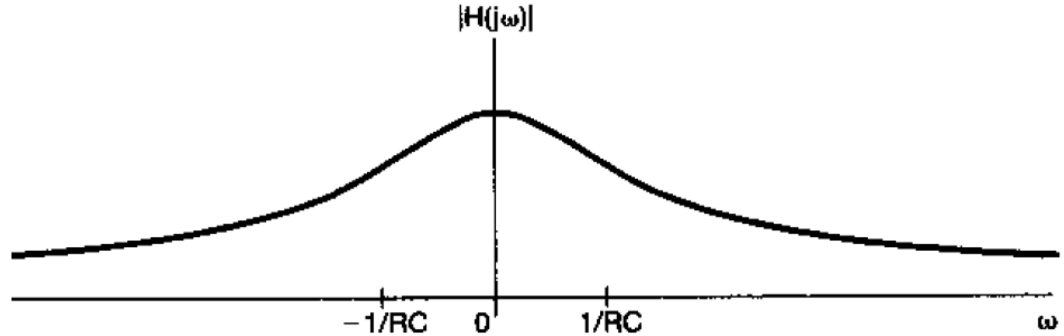


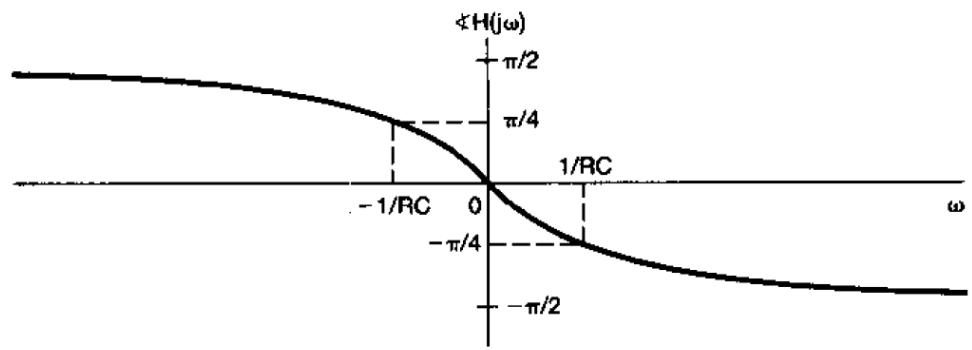
图 3.29 一阶 *RC* 滤波器

$$H(j\omega)=\frac{\alpha}{\alpha+j\omega},\quad \alpha=\frac{1}{RC}$$



(a)

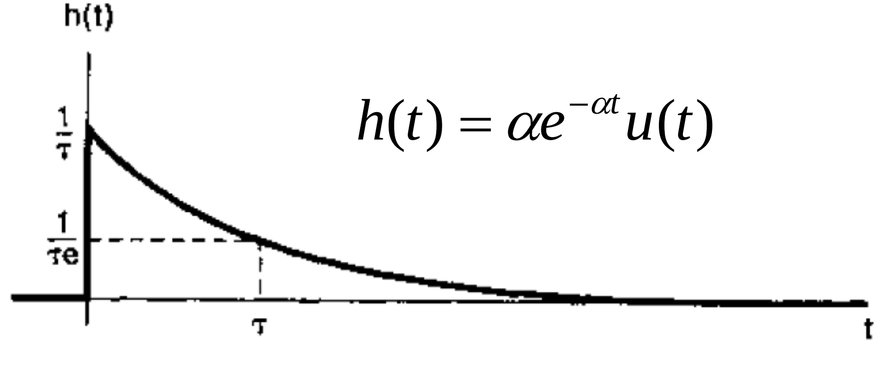
$$|H(j\omega)|=\frac{\alpha}{\sqrt{\alpha^2+\omega^2}}$$



(b)

$$\measuredangle H(j\omega)=\tan^{-1}(\frac{\omega}{\alpha})$$

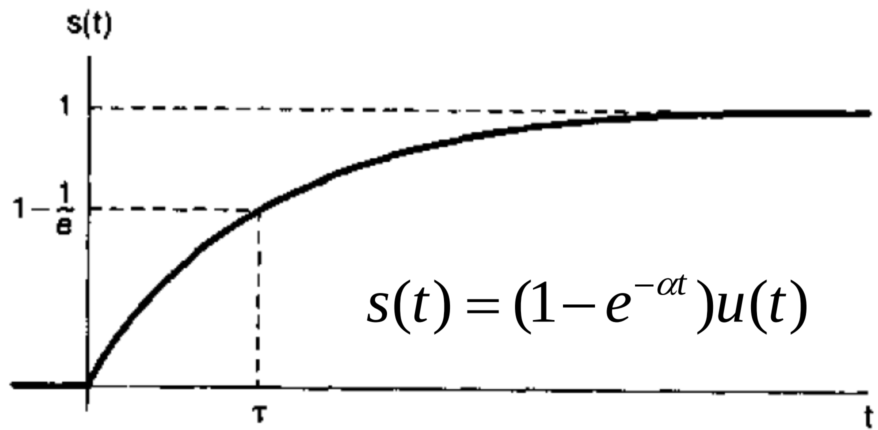
冲激响应：



$$h(t) = \alpha e^{-\alpha t} u(t)$$

(a)

阶跃响应：



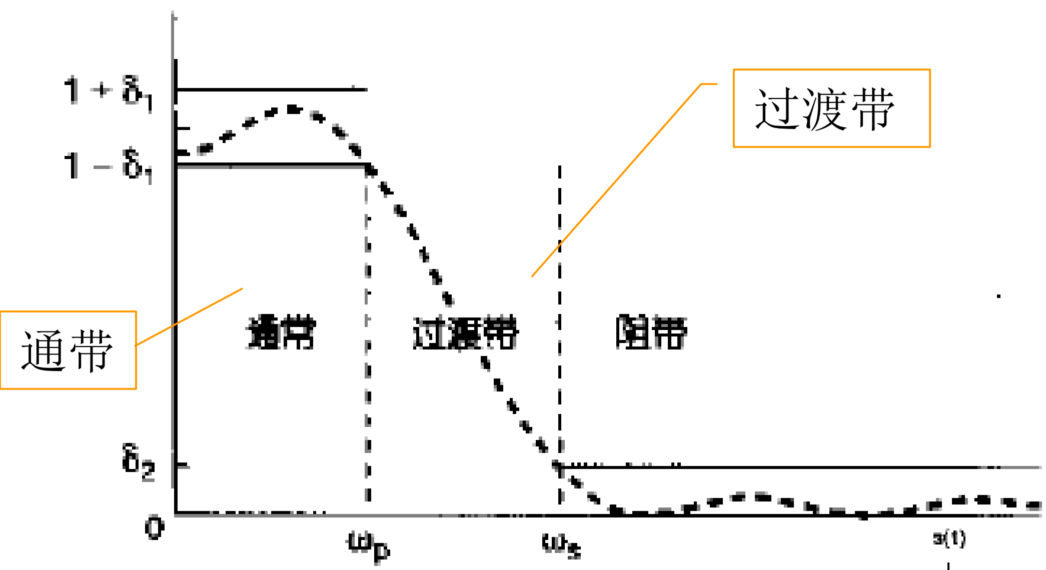
$$s(t) = (1 - e^{-\alpha t})u(t)$$

(b)

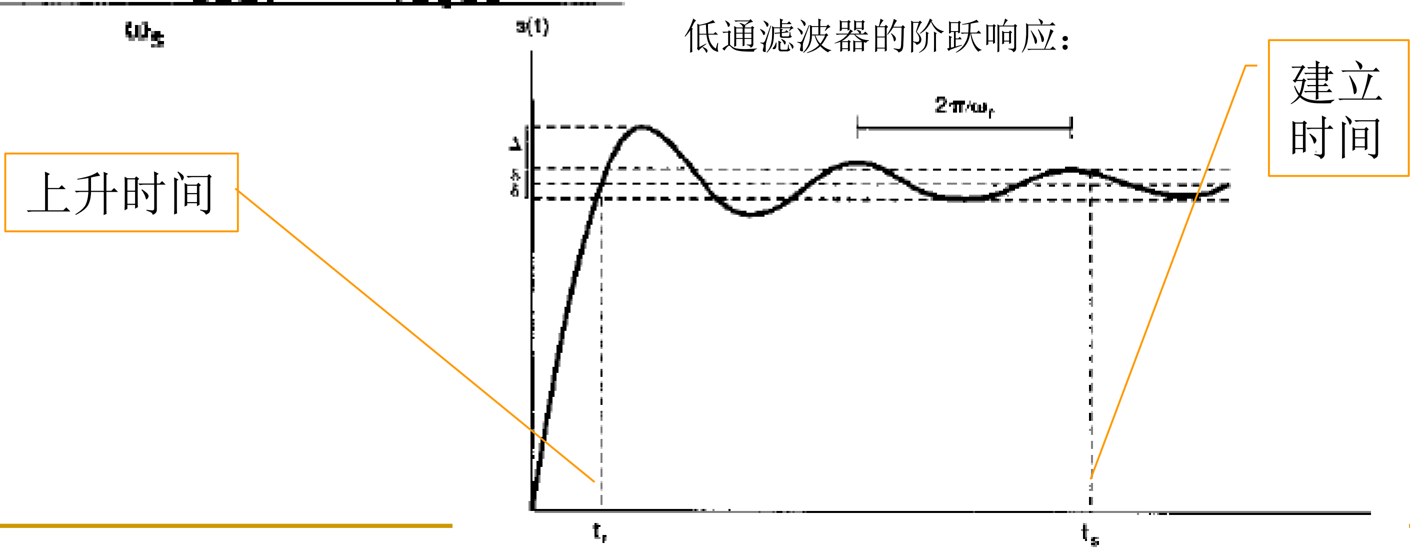
- h(t)是因果的，单调衰减
- s(t)无过冲和振荡

- 滤波器的时域和频域特性

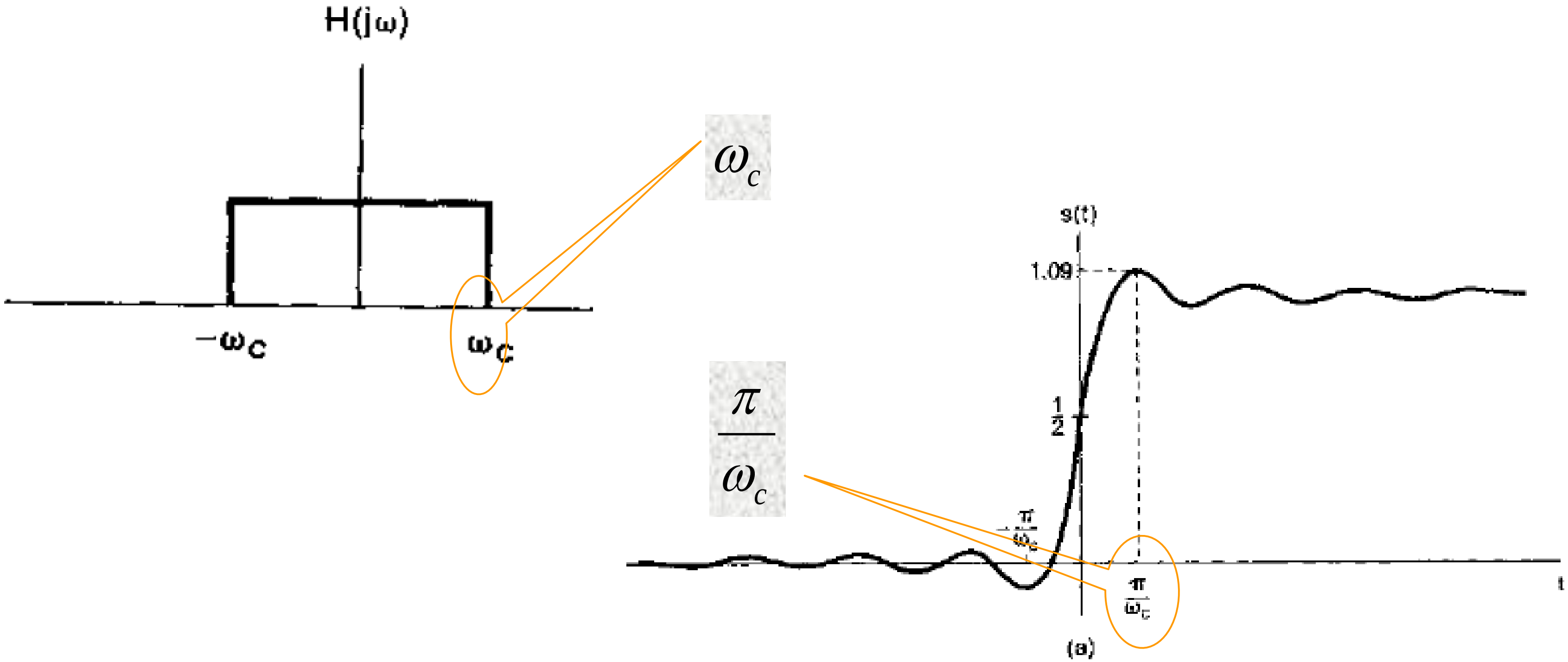
低通滤波器的幅频特性：



低通滤波器的阶跃响应：



注：设计滤波器时要综合考虑系统的时域和频域特性

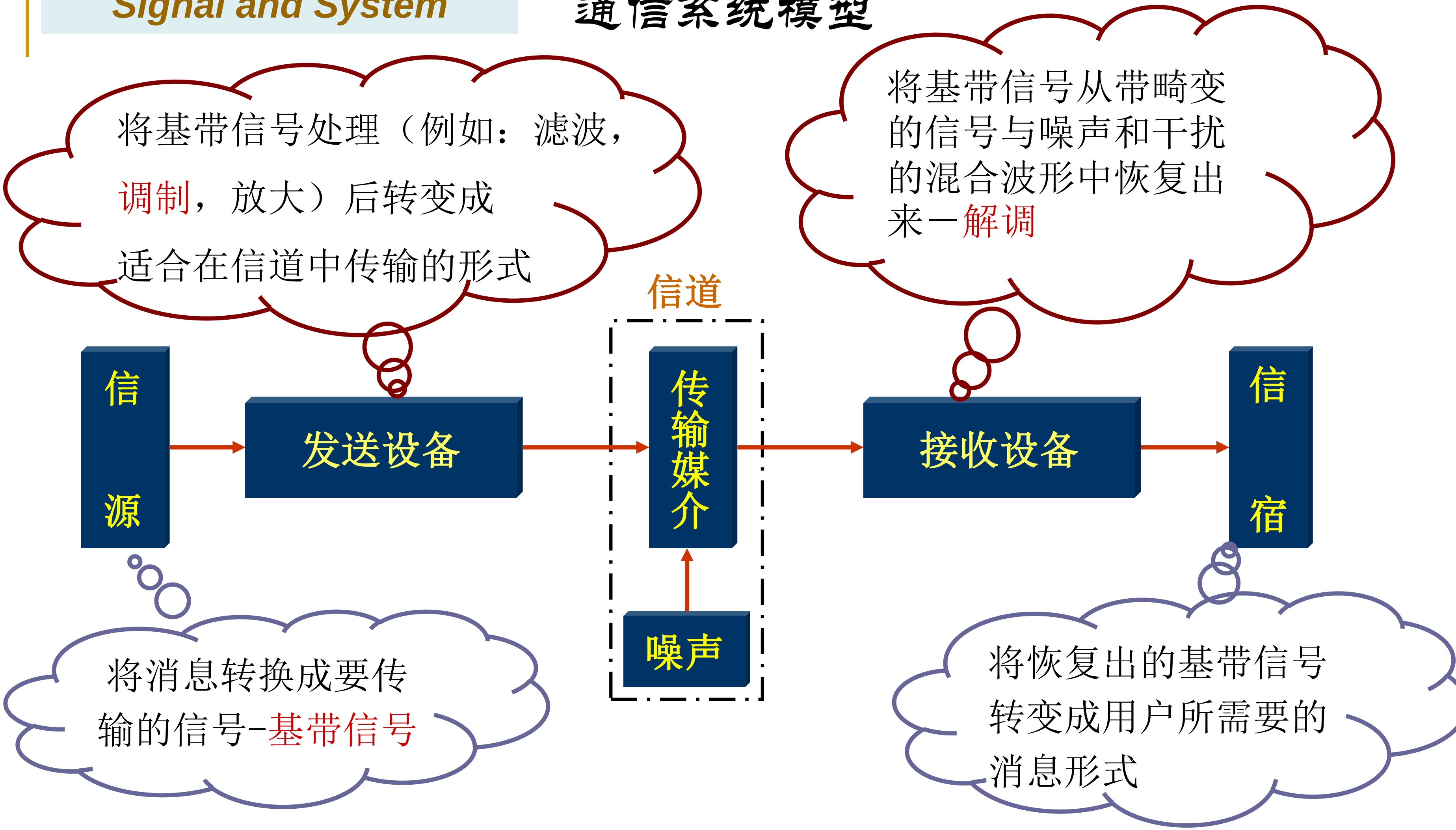


例如：上升时间与通带宽度的折衷，建立时间与过渡带宽度的折衷

4.5 Frequency-Domain Analysis of LTI System

- Frequency Response
- Filtering
- Modulating
- Sampling

通信系统模型



注：
- 不同的传输媒质具有不同的传输特性（例如：同轴电缆、光纤、无线空间）
- 信道带宽有限会使传输信号发生畸变，同时叠加干扰和噪声

关于调制/解调：

- 调制的作用：将要传输的基带信号转换成适合在信道中传输的信号
- 调制的实现：使载波的某个参量随基带信号的改变而改变
- 调制的分类：
  - 幅度调制-载波幅度随基带信号改变
  - 角度调制-载波的频率/相位随基带信号改变
- 解调是调制的逆过程，将基带信号从接收到的信号中恢复出来

说明：

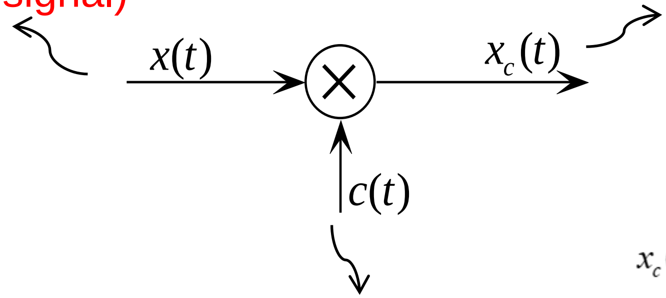
- 基带信号：由信源产生的信号，一般为低频信号（例如：语音）
- 载波：正/余弦信号，周期脉冲信号等，一般具有较高的频率（例如：$Acos(\omega_c t+\theta)$ ）

- 双边带（DSB）调制

  —double-sidebands(DSB) sinusoidal amplitude modulation

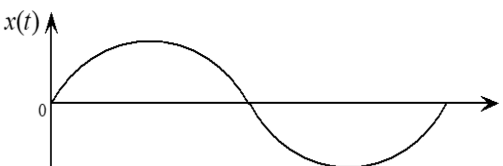
调制信号（modulating signal)

已调信号（modulated signal)

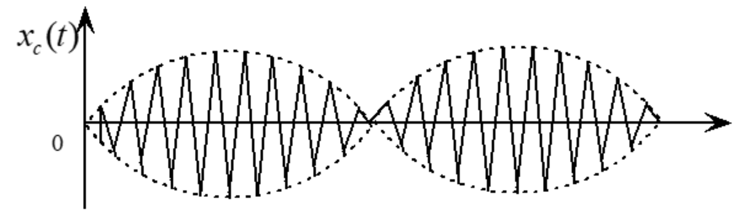


$$x_c(t) = x(t)\cos\omega_c t$$

载波(carrier signal)



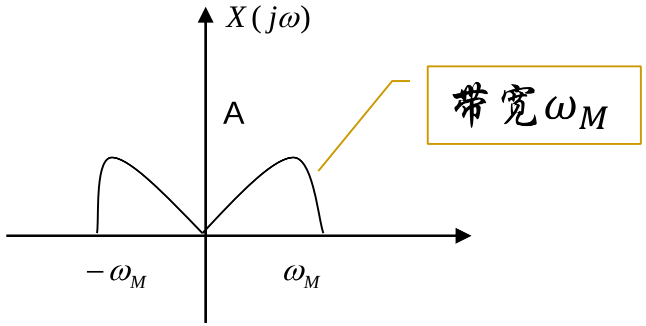
(b)



(c)

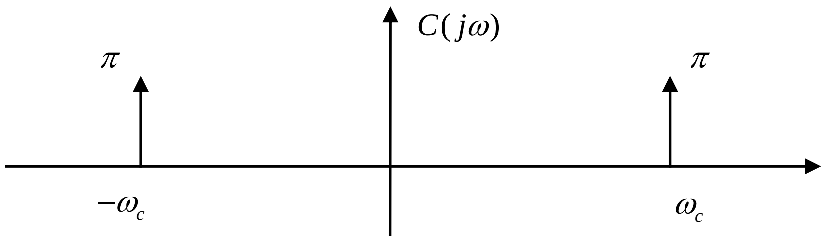
- **Modulation** $\boxed{x_c(t) = x(t)\cdot c(t)}$

$$x(t) \leftrightarrow X(j\omega)$$



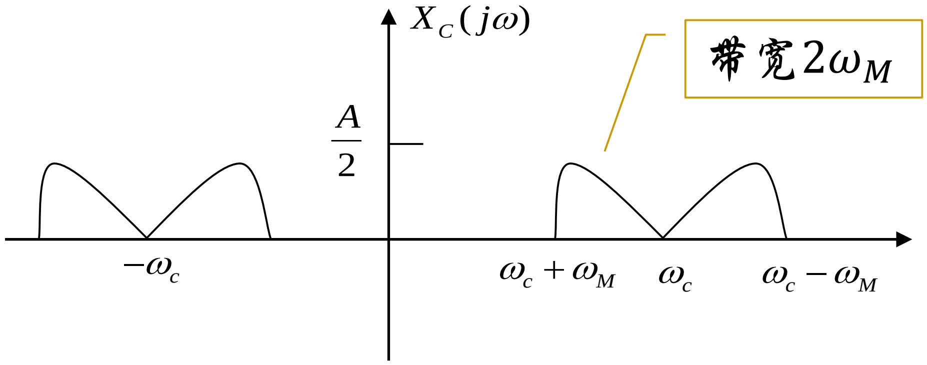
$$c(t) = \cos\omega_c t \leftrightarrow$$

$$C(j\omega) = \pi[\delta(\omega+\omega_c) + \delta(\omega-\omega_c)]$$

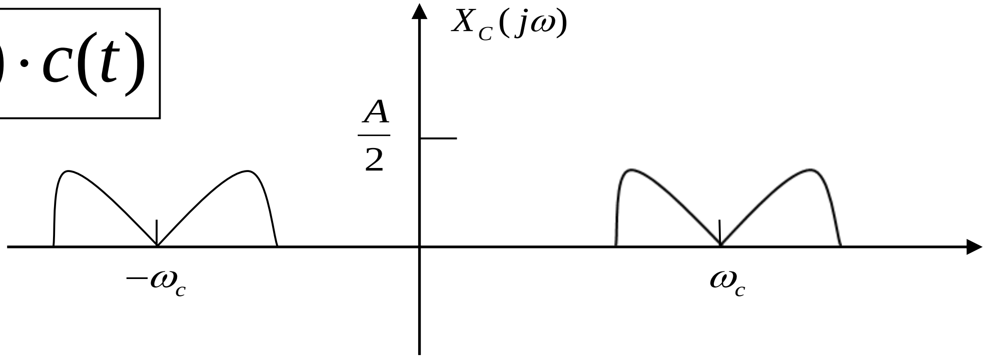


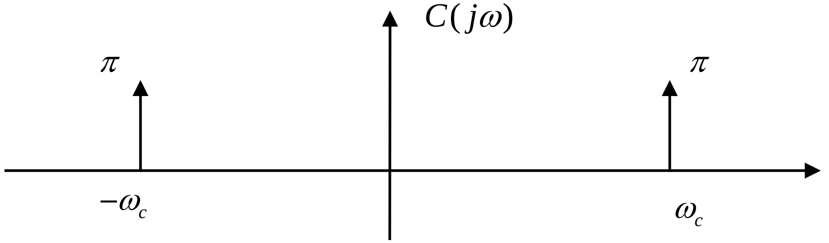
$$X_C(j\omega) = \frac{1}{2\pi}[X(j\omega) * C(j\omega)]$$

$$= \frac{1}{2}[X(j(\omega-\omega_c)) + X(j(\omega+\omega_c))]$$



- **Demodulation** $\boxed{y(t) = x_c(t) \cdot c(t)}$



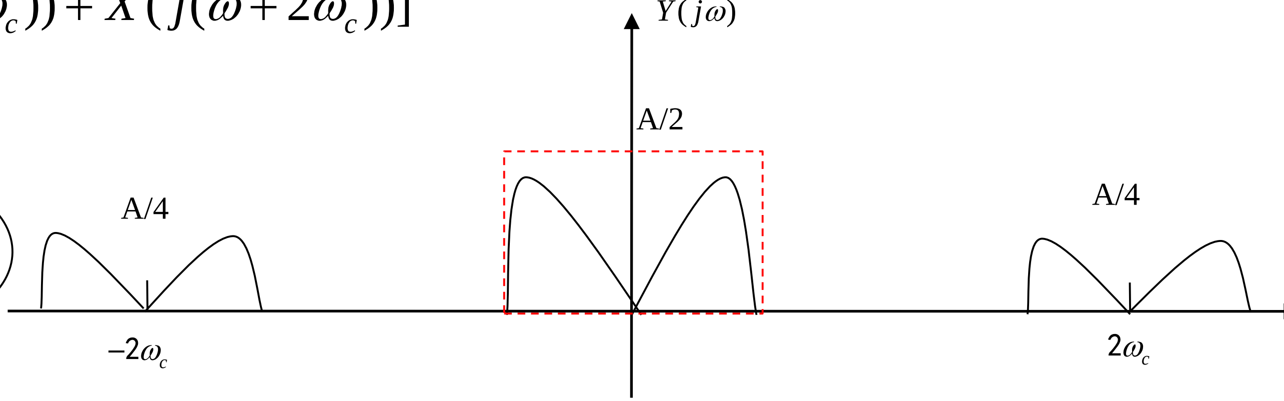


$$Y(j\omega) = \frac{1}{2\pi}[X_c(j\omega) * C(j\omega)]$$

$$= \frac{1}{2}[X_c(j(\omega-\omega_c)) + X_c(j(\omega+\omega_c))]$$

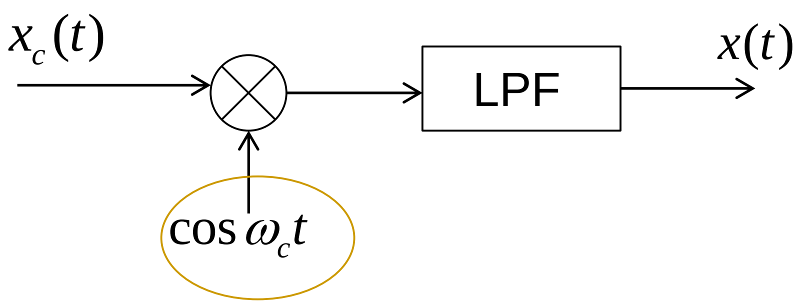
$$= \frac{1}{2}X(j\omega) + \frac{1}{4}[X(j(\omega-2\omega_c)) + X(j(\omega+2\omega_c))]$$

幅度调制和解调
的过程就是频谱
的“搬移”！
——线性调制



接收端的载波必须和发送端同频同相!

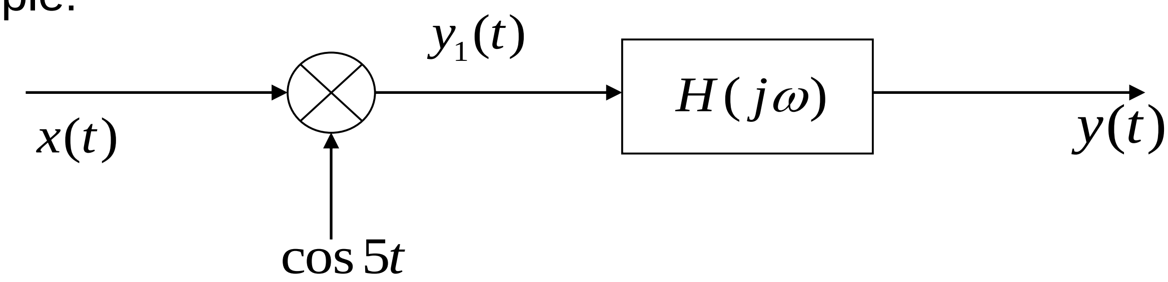
——同步解调/相干解调（synchronous demodulation）



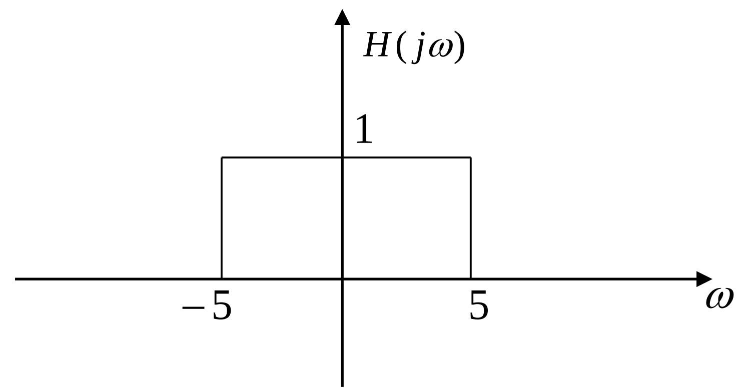
$$
\begin{aligned}
& x_c(t)\cdot\cos\omega_c t \\
&= x(t)\cos\omega_c t\cdot\cos\omega_c t \\
&= x(t)\left(\frac{1+\cos 2\omega_c t}{2}\right) \\
&= \frac{1}{2}x(t)+\frac{1}{2}x(t)\cos 2\omega_c t
\end{aligned}
$$

思考：当本地载波和发送端的载波存在相差，即本地载波为$\cos[\omega_c t+\theta(t)]$ 时，对解调的影响？

Example:



已知： $x(t) = \dfrac{\sin 2t}{\pi t}$



求： $y(t)$

Solution:

注：画频谱图！

$$y_1(t) = x(t)\cdot\cos(5t) \leftrightarrow$$

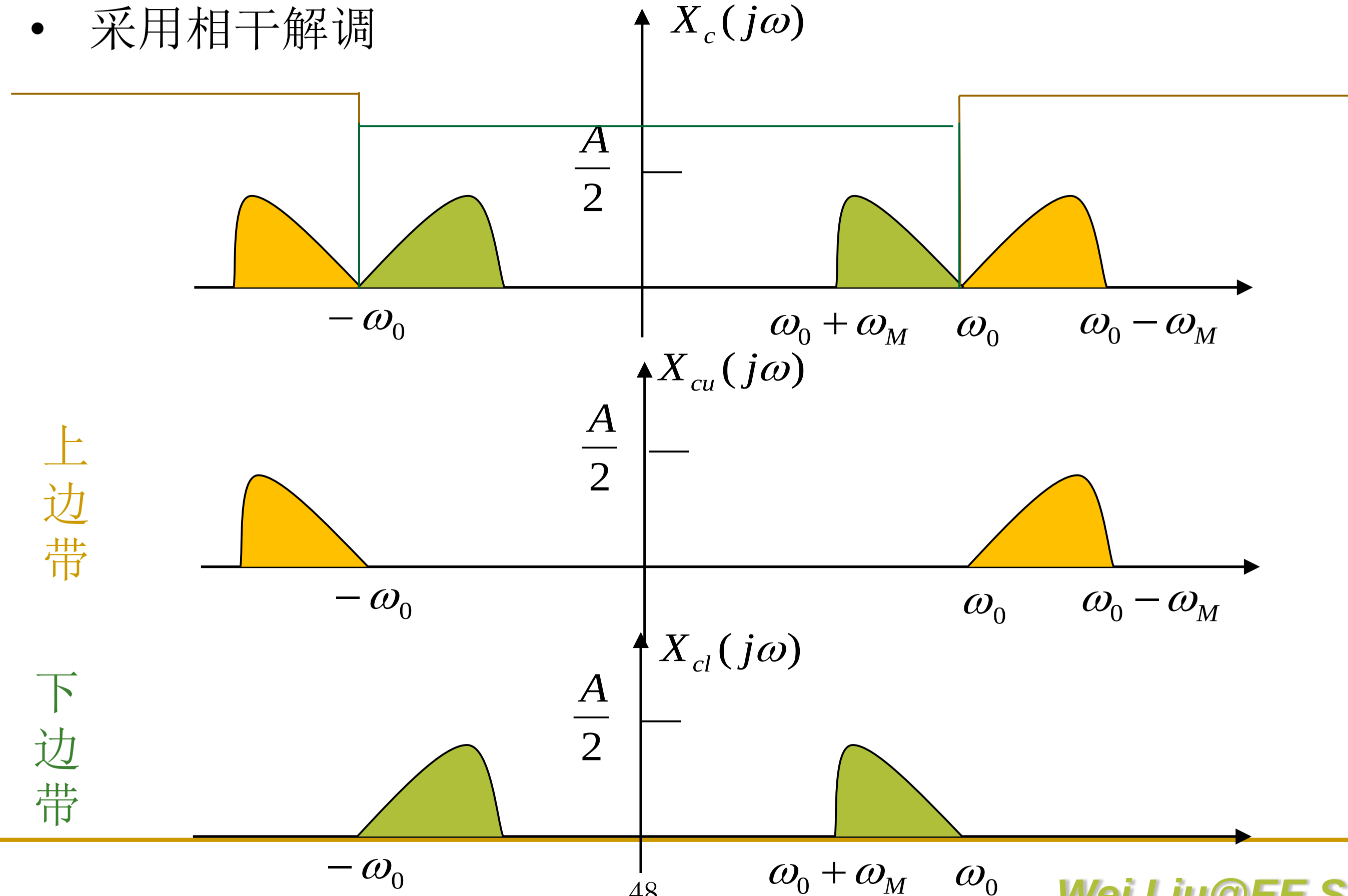
$$Y_1(j\omega) = \frac{1}{2}[X(j(\omega+5)) + X(j(\omega-5))]$$

$$Y(j\omega) = Y_1(j\omega)\cdot H(j\omega) \leftrightarrow ?$$

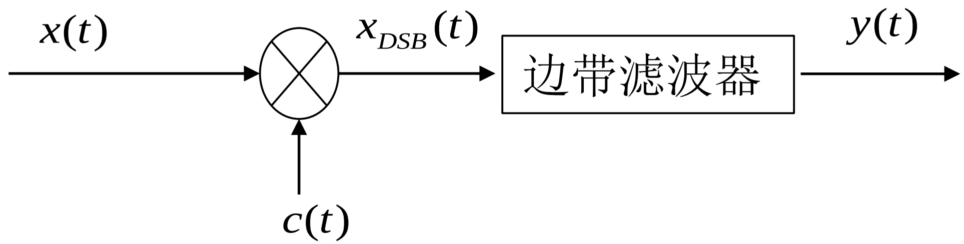
$$y(t) = \frac{\sin t}{\pi t}\cdot\cos(4t)$$

其它幅度调制：

- 单边带(SSB)调制 —single-sideband sinusoidal amplitude modulation
  - 所需传输带宽为DSB调制的一半
  - 采用相干解调

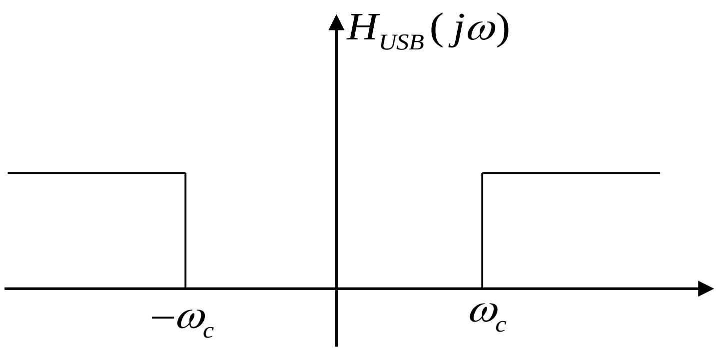


SSB调制原理图①



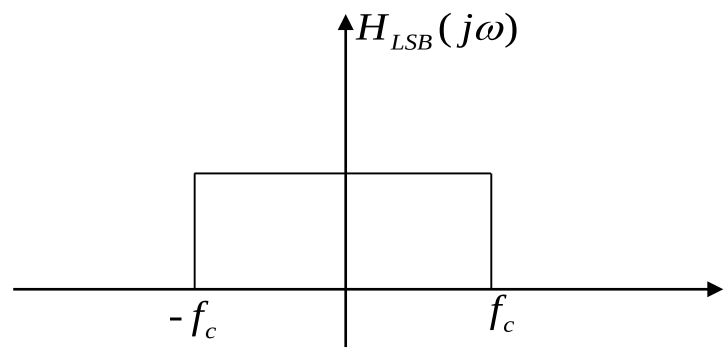
$$Y(j\omega) = X_{DSB}(j\omega) \cdot H_{SSB}(j\omega)$$

上边带滤波器



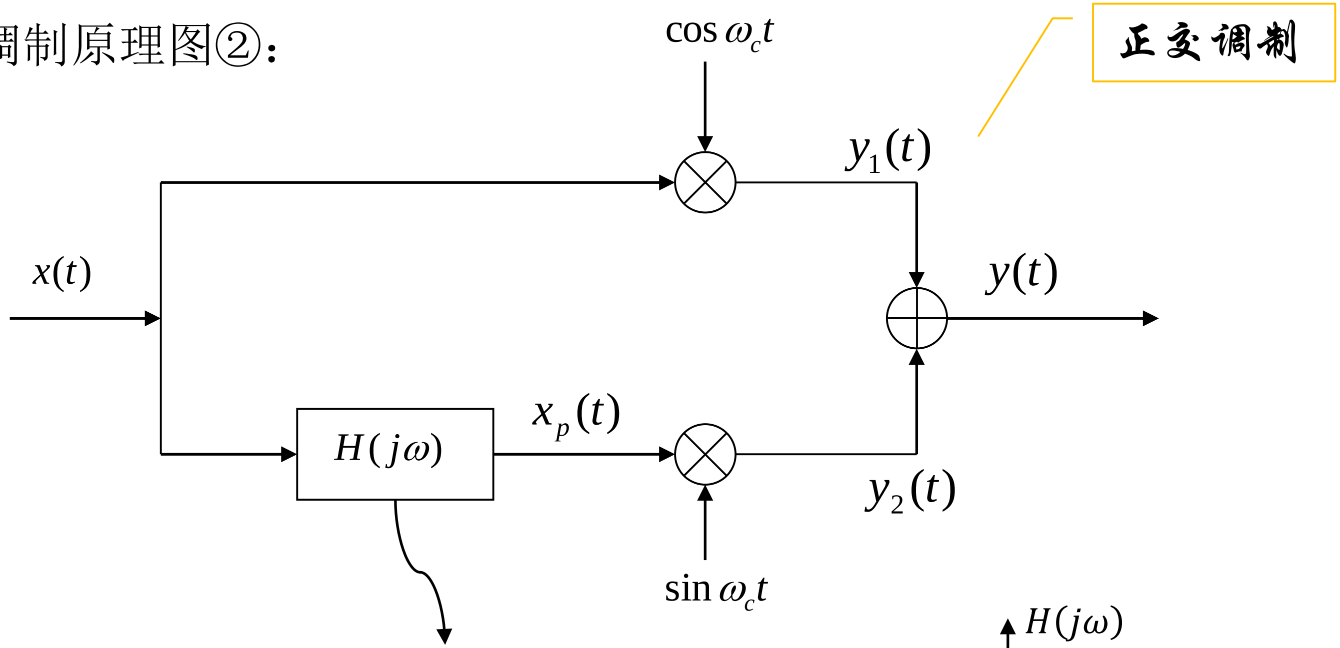
$$H_{USB}(j\omega) = \begin{cases} 1 & |\omega| > \omega_c \\ 0 & |\omega| \le \omega_c \end{cases}$$

下边带滤波器

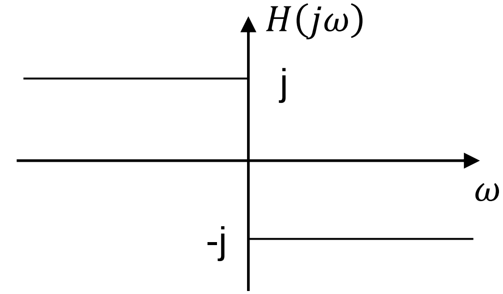


$$H_{LSB}(j\omega) = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

SSB调制原理图②：



$$H(j\omega)=\begin{cases}-j, & \omega>0\\ +j, & \omega<0\end{cases}$$

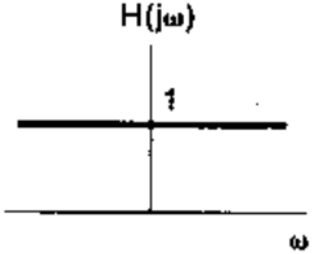


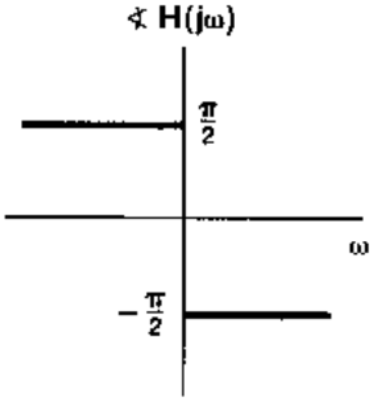
$$y(t)=x(t)\cos(\omega_0 t)\pm x_p(t)\sin(\omega_0 t)$$

关于Hilbert变换器：

$$H(j\omega)=\begin{cases}-j, & \omega>0\\ +j, & \omega<0\end{cases}=-jsgn(\omega)\longleftrightarrow \quad h(t)=\frac{1}{\pi t}$$

$$\boxed{x_p(t)=x(t)*h(t)=\int\frac{x(\tau)}{\pi(t-\tau)}d\tau}$$ —Hilbert变换





注：Hilbert变换器是一个“ 90^o 移相网络”

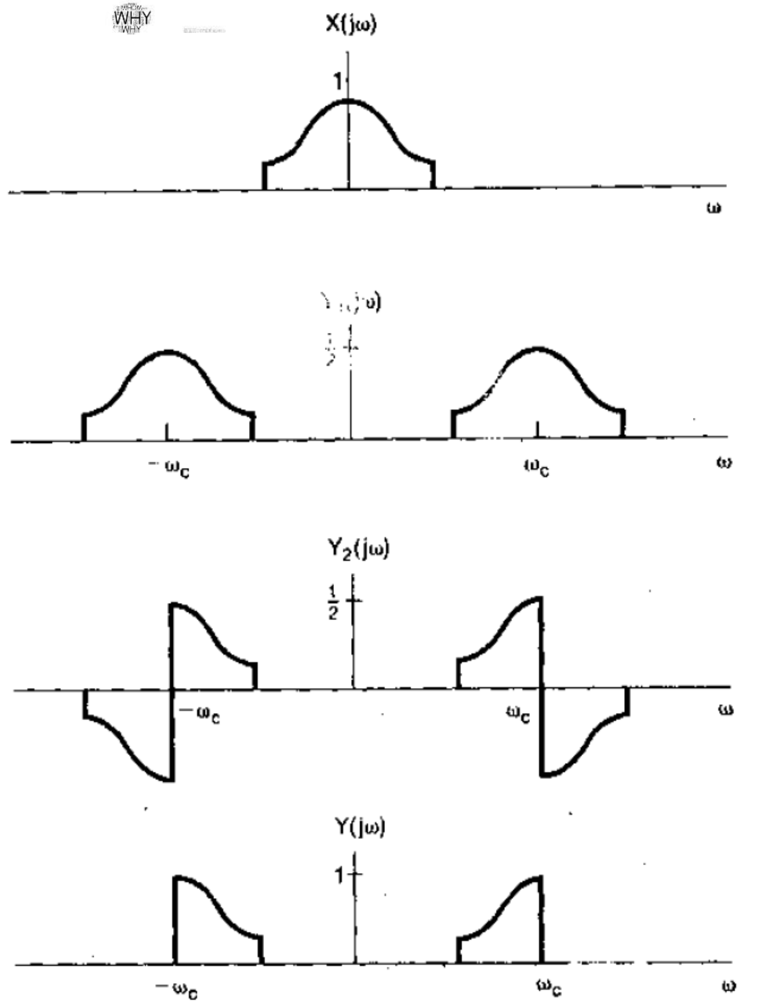
原理图②为什么可以实现SSB调制？

$x(t) \leftrightarrow$

$x(t) \cdot \cos(\omega_c\ \mathrm{t}) \leftrightarrow$

$x_p(t) \cdot \sin(\omega_c\ \mathrm{t}) \leftrightarrow$

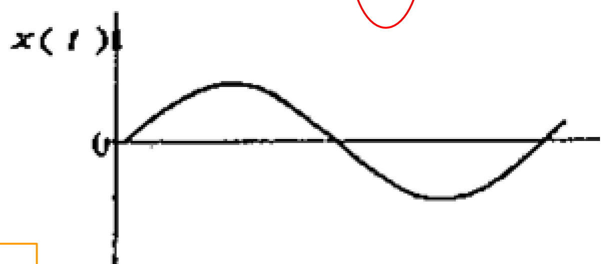
$x(t) \cdot \cos(\omega_c\ \mathrm{t})$
$+x_p(t) \cdot \sin(\omega_c\ \mathrm{t}) \leftrightarrow$



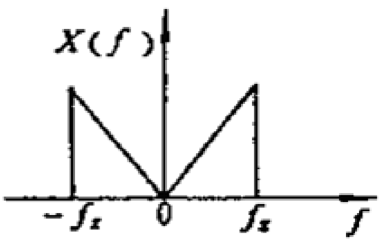
SSB调制图例

- 振幅调制(AM) —sinusoidal amplitude modulation
  - 所需传输带宽与DSB调制相同
  - 可采用非相干解调 (asynchronous demodulation)

$$x_c(t) = [A_0 + x(t)]\cos\omega_c t = x(t)\cos\omega_c t + A_0\cos\omega_c t$$

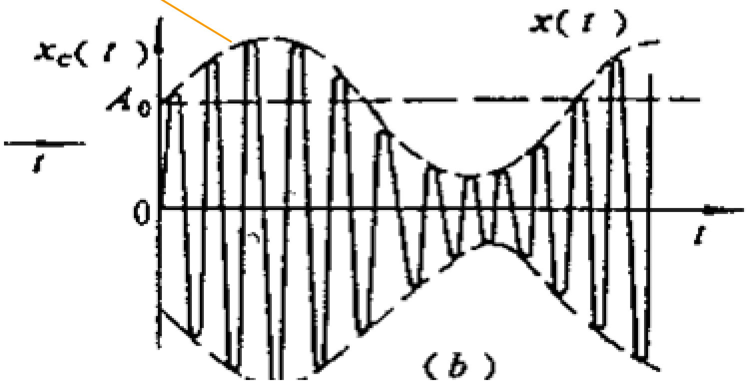


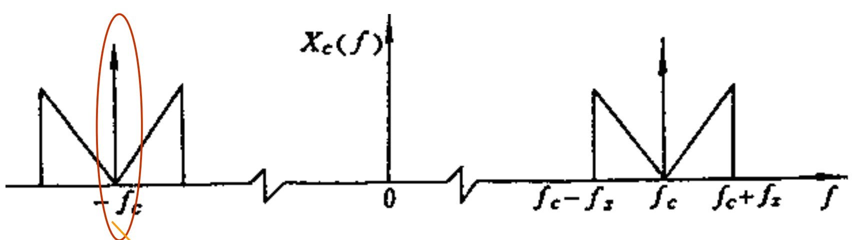
(a)



(d)

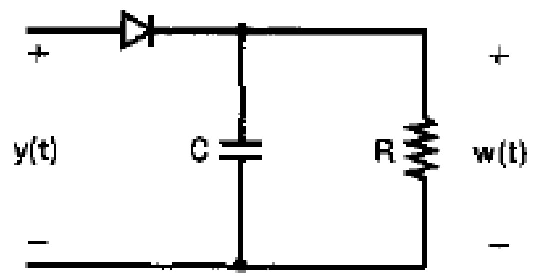
包络



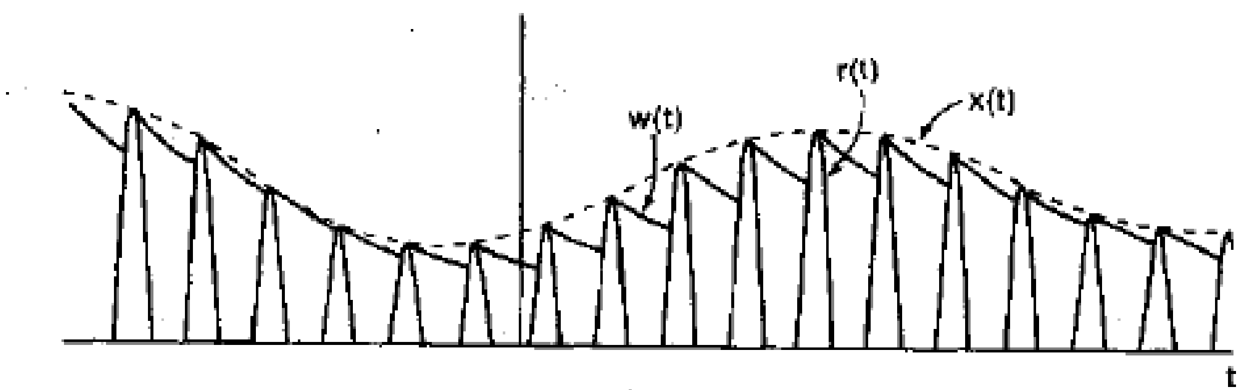


载波分量

附：包络检波



(a)



(b)

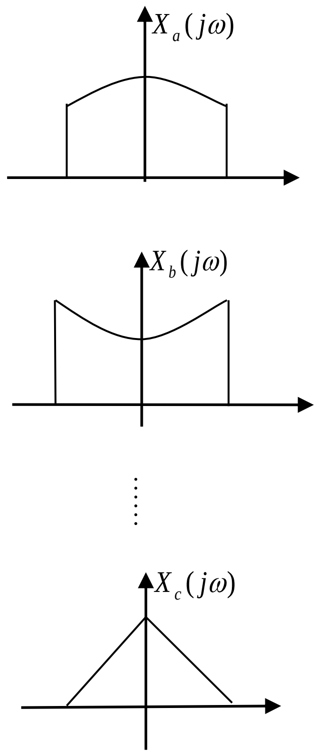
采用包络检波的条件：

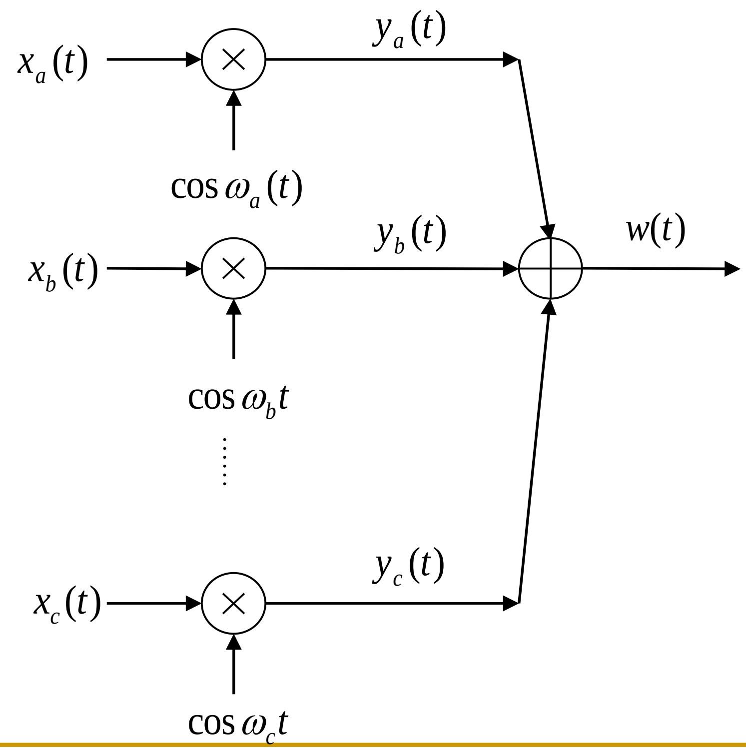
- 调制系数 $\mathrm{m}=\frac{|x(t)|_{max}}{A_0}\leq 1$
- $x(t)$的变化远慢于ω_c

■ 频分多路复用(**FDM— Frequency-Division Multiplexing)**

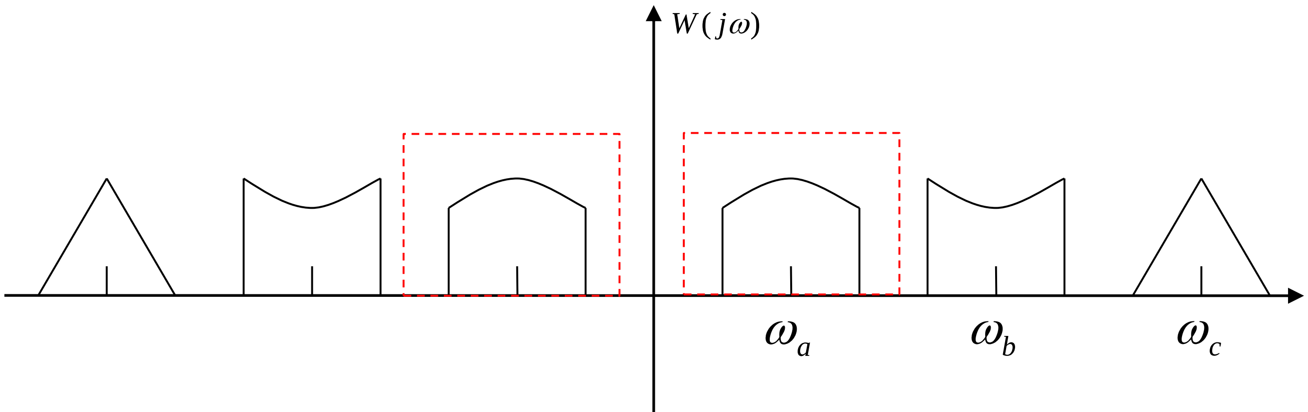
——传输信号的信道带宽>>一路信号的传输带宽时，可在同一信道上同时传输多路信号。此时需将各路信号搬到不同的载波频率上，使其频谱不重叠。

- 发端：





- 收端：先用中心频率不同的BPF分离各路信号，再分别解调。



例：

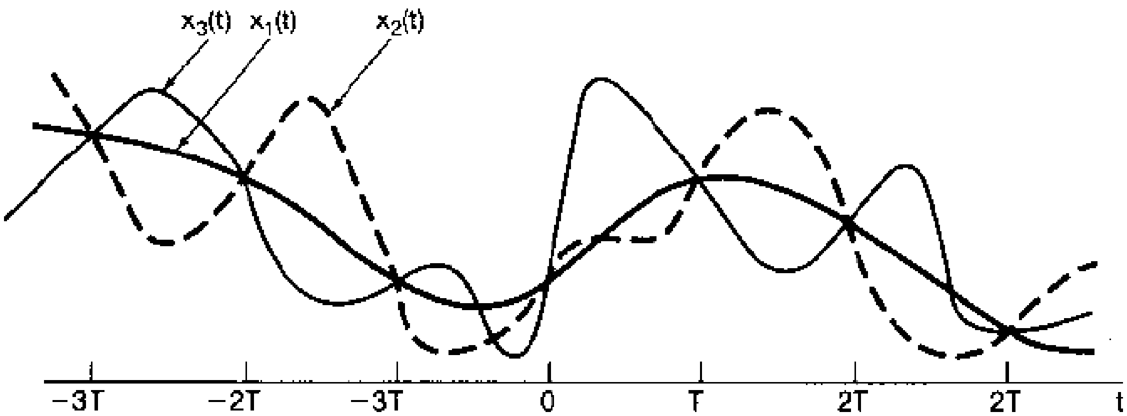
- 话音通信中的PSTN（公用电话交换网）
- 由 CATV 网演变而来的HFC（混合光纤同轴电缆）网

传统的FDM需要在相邻载波间设置保护频带，现在的正交频分复用（OFDM-Orthogonal FDM）是FDM的改进，其各子载波相互正交，子载波的频谱可以相互重叠，从而提高频谱的利用率。

4.5 Frequency-Domain Analysis of LTI System

- **Frequency Response**
- **Filtering**
- **Modulating**
- **Sampling**

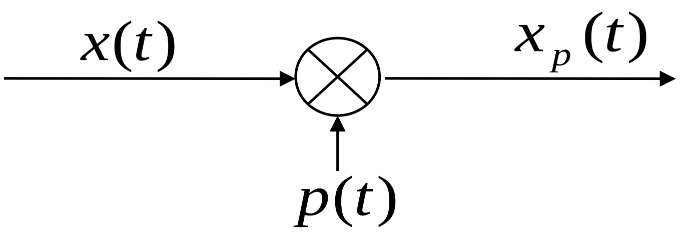
随着数字技术的发展，需要通过采样，将连续时间信号转换为离散时间信号，经过离散时间系统处理后，再转换回连续时间信号。



注：图中x1(t)、x2(t)和x3(t)在T的整数倍上具有相同的值！

在什么条件下，连续时间信号可用其在等时间间隔的样本表示，并可用这些样本将该信号恢复出来？

■ 采样(Sampling)



$x(t)$ —模拟信号，

当$|\omega|>\omega_m$时，$X(j\omega)=0$ —带限信号(band-limited signal)

$p(t)$ —取样函数(sampling function)

$p(t)$为周期信号，T_s称为取样周期，$\omega_s=\dfrac{2\pi}{T_s}$称为取样频率

$$\because x_p(t) = x(t) \cdot p(t)$$

$$\therefore X_p(j\omega) \leftrightarrow \frac{1}{2\pi}[X(j\omega) * P(j\omega)]$$

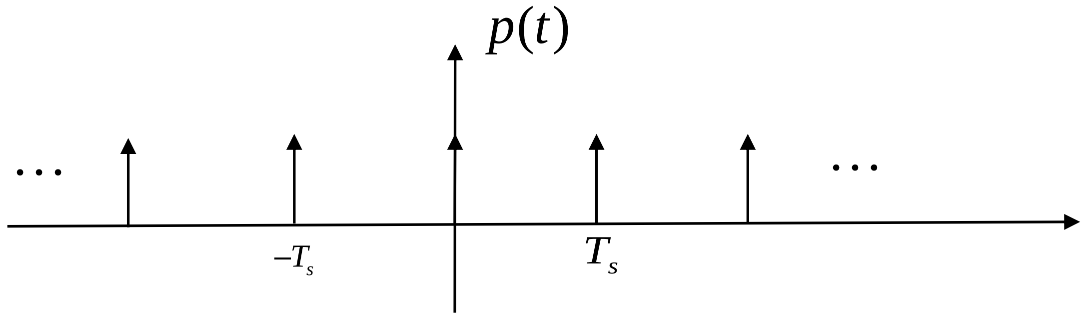
又 $P(j\omega) = 2\pi \sum P_k \delta(\omega - k\omega_s)$

其中，P_k为傅立叶级数的系数，则

$$\boxed{X_p(j\omega) = \sum_{k=-\infty}^{\infty} P_k X[j(\omega - k\omega_s)]}$$

注：$X_p(j\omega)$是$X(j\omega)$的周期拓展，且拓展后的频谱幅度被P_k加权

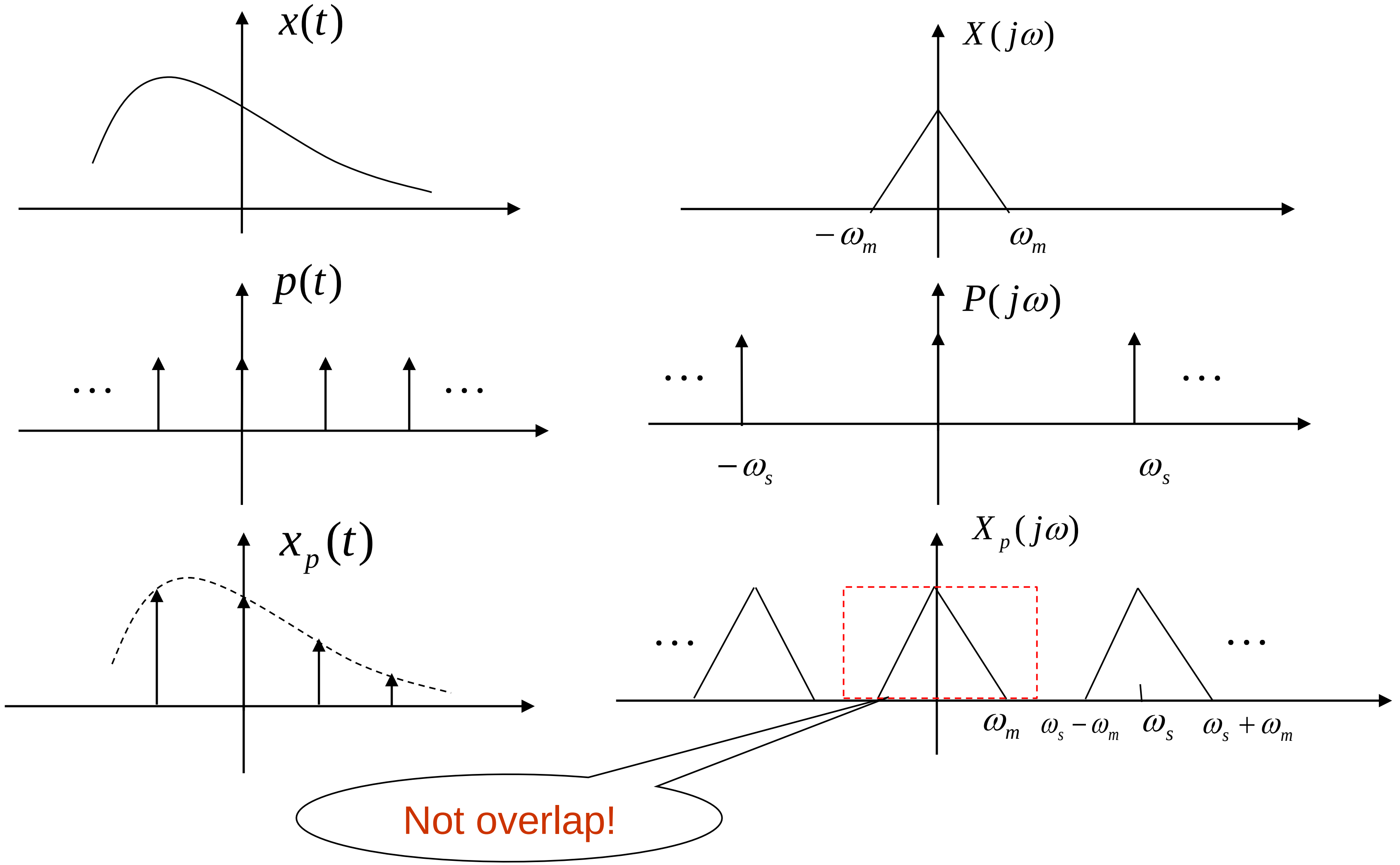
①冲激串取样



$$\because\ p(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT_s)\quad\Rightarrow P_k=\frac{1}{T_s}$$

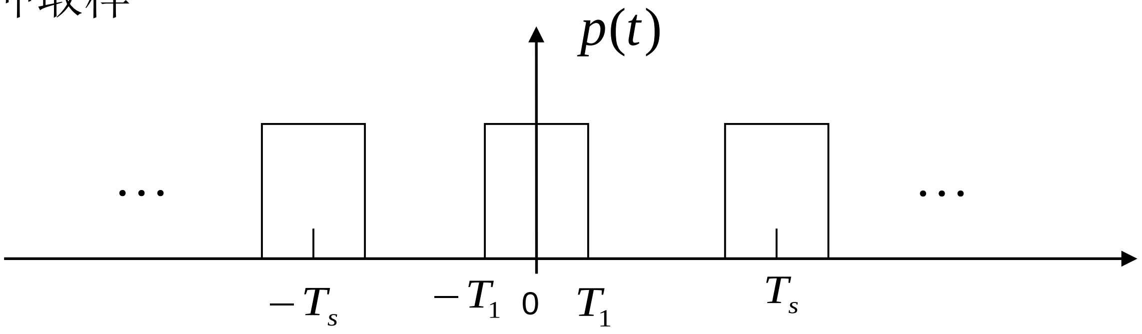
$$\therefore\ X_p(j\omega)=\frac{1}{T_s}\sum_k X[j(\omega-k\omega_s)]$$

注：$X_p(j\omega)$是$X(j\omega)$ 做等幅的周期拓展



$x(t)$
$X(j\omega)$
$-\omega_m$
ω_m
$p(t)$
$P(j\omega)$
$\cdots$
$-\omega_s$
ω_s
$x_p(t)$
$X_p(j\omega)$
ω_m
$\omega_s-\omega_m$
ω_s
$\omega_s+\omega_m$
Not overlap!

②矩形脉冲取样



$$\because P_k = \frac{1}{T_s} \cdot \frac{2\sin \omega T_1}{\omega} \Big|_{\omega = k\omega_s} = \frac{2T_1}{T_s} \cdot Sa(k\omega_s T_1)$$

$$\therefore X_p(j\omega) = \frac{2T_1}{T_s} \sum_k Sa(k\omega_s T_1) X[j(\omega - k\omega_s)]$$

注：$X_p(j\omega)$的包络按$Sa(.)$改变

Sampling Theorem:

Let us be a band – limited signal with $X(j\omega)=0$ *for* $|\omega|>\omega_m$*. Then* $x(t)$ *is uniquely determined by its samples* $x(nT_s)$, $n=0,\pm1,\pm2,\ldots$, *if*

$$\omega_s > 2\omega_m$$

where $\omega_s = \dfrac{2\pi}{T_s}$

Nyquist rate

Given these samples, we can reconstruct $x(t)$ *by generating a periodic impulse train in which successive impulse have amplitudes that are successive sample values. This impulse train is then processed through an ideal lowpass filter with gain* T *and cutoff frequency* ω_c*. if*

$$\omega_s - \omega_m > \omega_c > \omega_m$$

the resulting output signal will exactly equal $x(t)$.

Example:

$$Consider\ a\ band-limited\ signal\ x(t)\ with\ X(j\omega)=0\ \ for\ |\omega|>\omega_m.$$

$$Determine\ the\ Nyquist\ rate\ for\ the\ following\ signals:$$

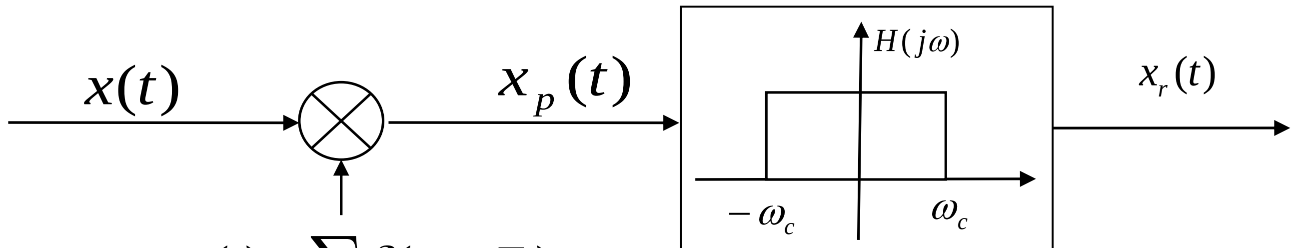
1) $2x(t)+1$ $\qquad \omega_s > 2\omega_m$

2) $x^2(t)$ $\qquad \omega_s > 4\omega_m$

3) $\dfrac{dx(t)}{dt}$ $\qquad \omega_s > 2\omega_m$

■ 重建(Reconstruction)

①



$$p(t)=\sum_n \delta(t-nT_s)$$

$$\omega_m<\omega_c<\omega_s-\omega_m$$

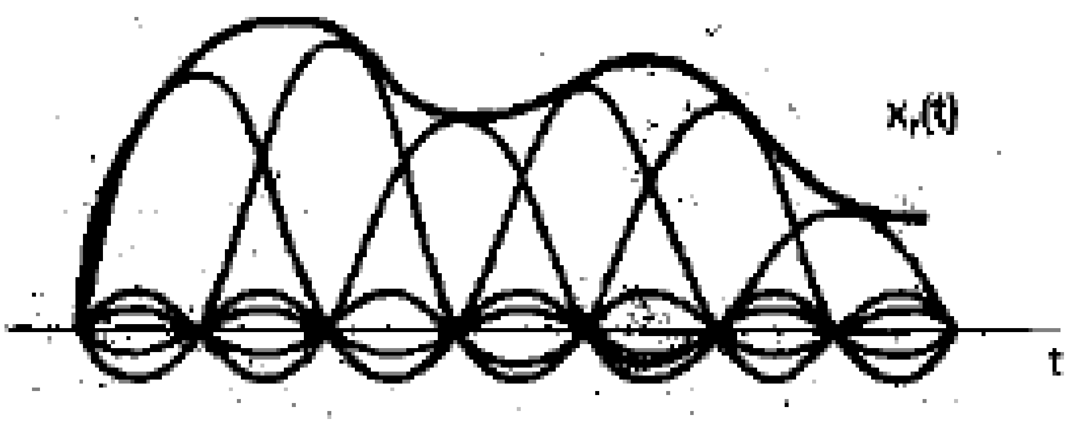
$$\because x_r(t)=x_p(t)*h(t)$$

and $x_p(t)=x(t)\cdot\sum_n \delta(t-nT_s)=\sum_n x(nT_s)\delta(t-nT_s)$

$$h(t)=\Im^{-1}\{H(j\omega)\}=\frac{\sin\omega_c t}{\pi t}=\frac{\omega_c}{\pi}S_a\left(\omega_c t\right)$$

$$\therefore x_r(t) = \sum_n x(nT_s)\cdot\delta(t-nT_s) * \frac{\omega_c}{\pi} Sa(\omega_c \tau)$$

$$= \sum_n \frac{\omega_c}{\pi} x(nT_s)\cdot Sa[\omega_c(t-nT_s)]$$



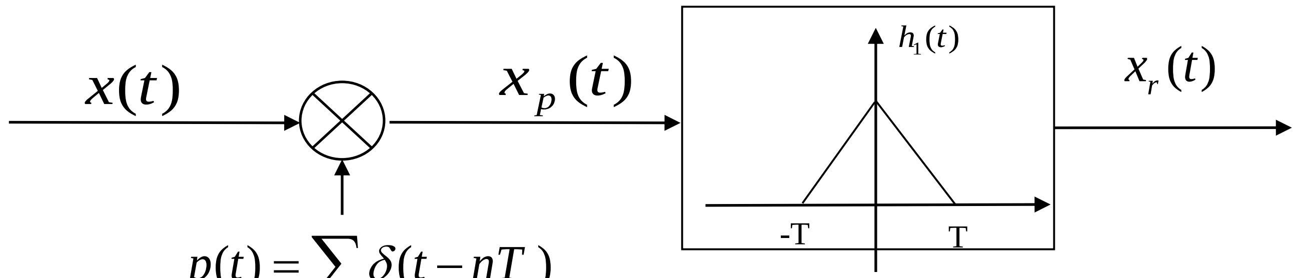
(c)

带限内插 / 高阶保持

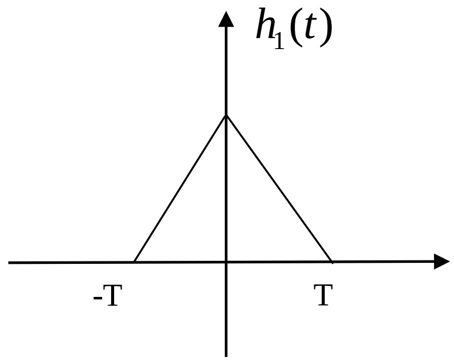
内插（Interpolation）
—用连续信号对一组样本值拟合

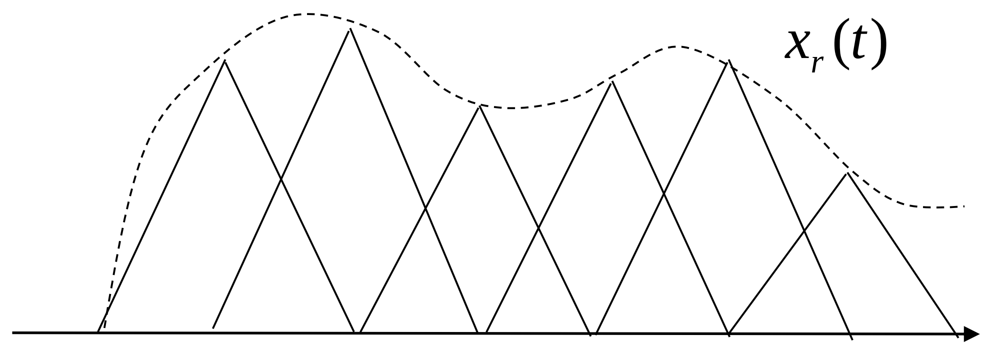
- 带限内插(band-limited interpolation) / 高阶保持
- 线性内插(linear interpolation) / 一阶保持
- 零阶保持

②线性内插/一阶保持



$$p(t)=\sum_n \delta(t-nT_s)$$

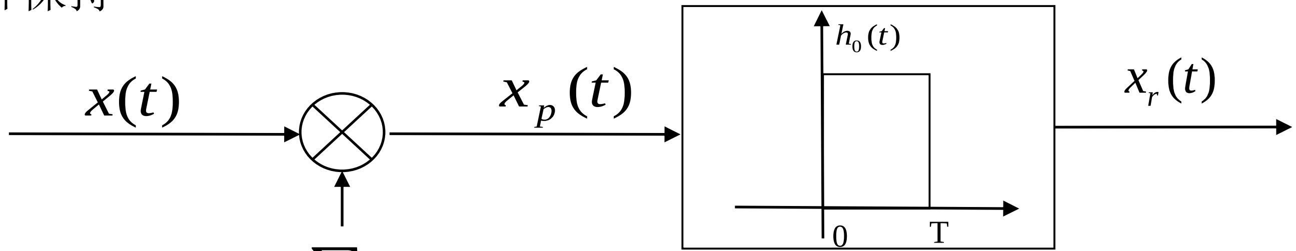




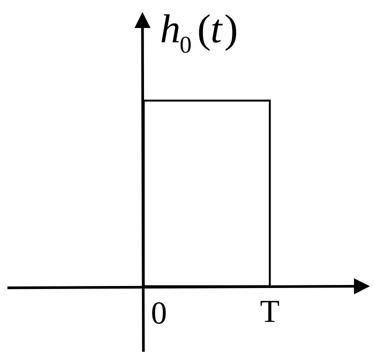
$$\leftrightarrow H_1(j\omega)=\frac{1}{T}\left[\frac{\sin(\omega T/2)}{\omega/2}\right]^2$$

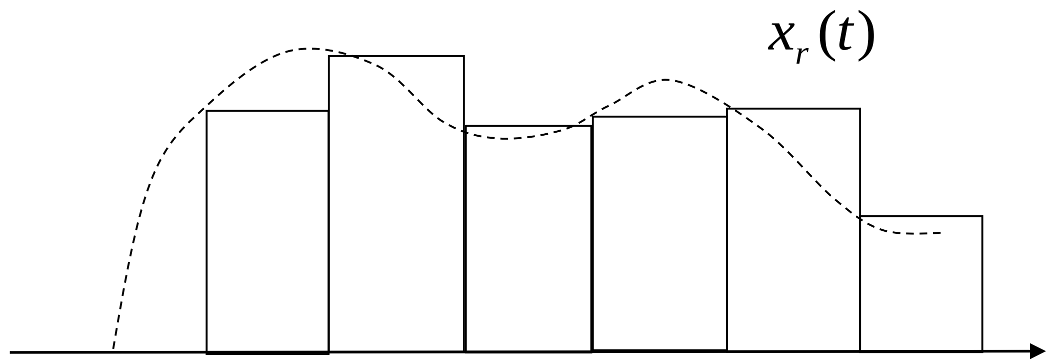
$$x_r(t)=\sum_n x(nT_s)\cdot h_1(t-nT_s)]$$

③零阶保持



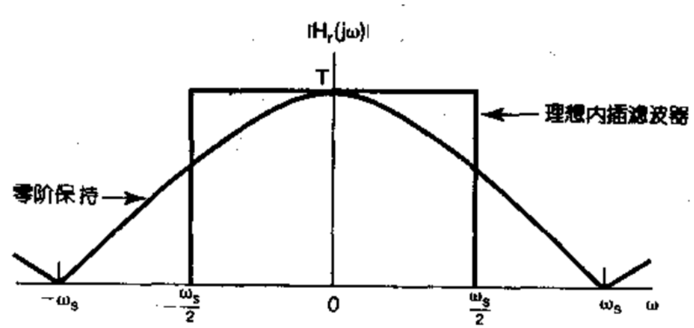
$$p(t)=\sum_n \delta(t-nT_s)$$

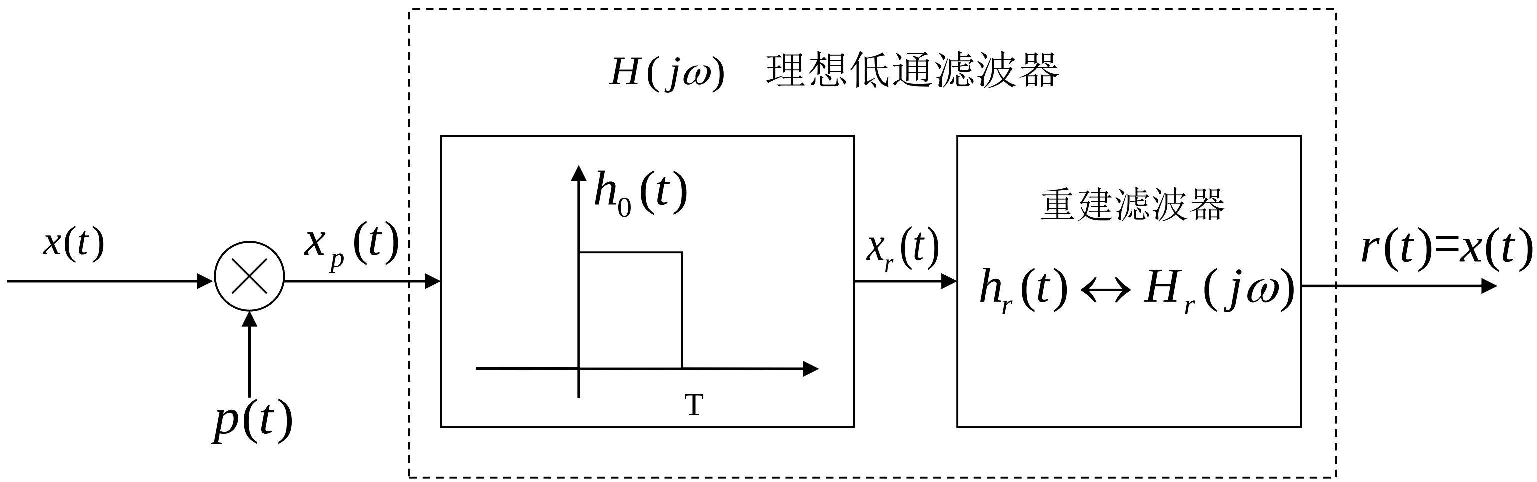




$$\leftrightarrow H_0(j\omega)=e^{-j\omega T/2}[\frac{2\sin(\omega T/2)}{\omega}]$$

$$x_r(t)=\sum_n x(nT_s)\cdot h_0(t-nT_s)]$$

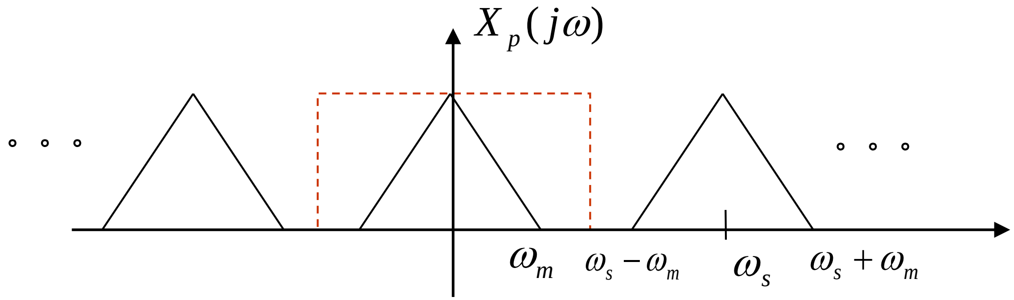


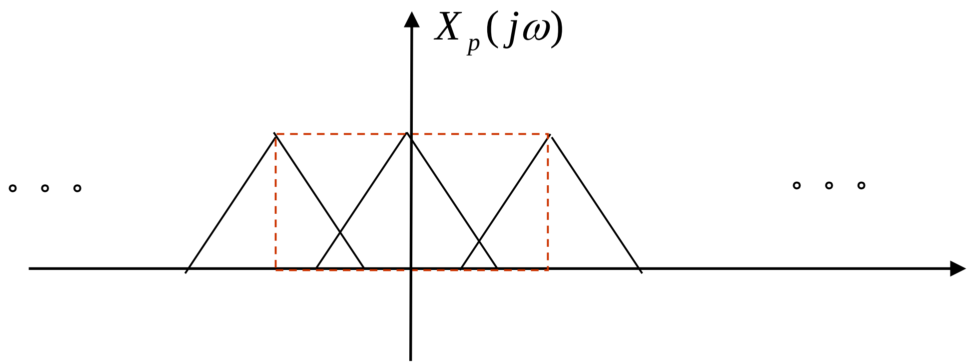


要使$r(t)=x(t)$，须 $H_r(j\omega)=\frac{H(j\omega)}{H_0(j\omega)}=\frac{e^{j\omega\frac{T}{2}}\cdot H(j\omega)}{\frac{2\sin(\omega T/2)}{\omega}}$

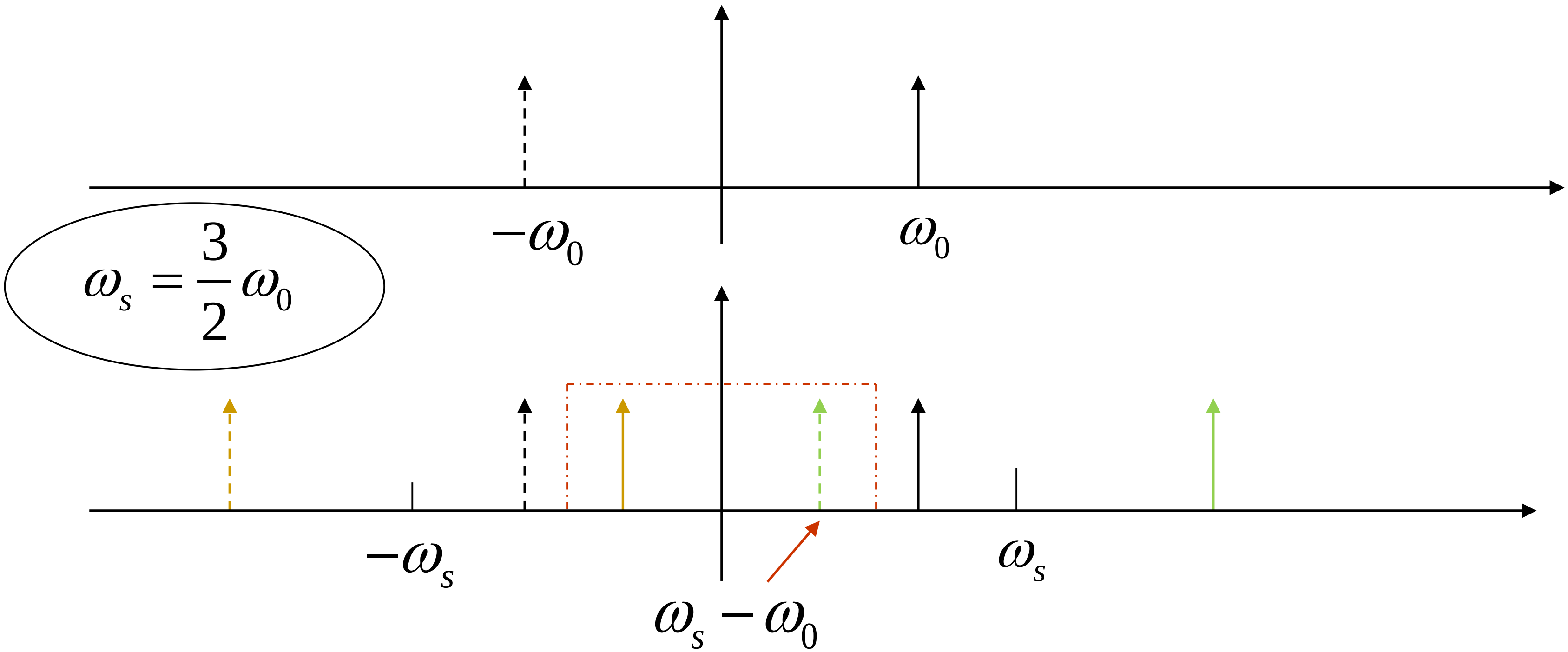
■ 欠采样

当$\omega_s < 2\omega_m$时，取样后的频谱$X_p(j\omega)$将不是模拟信号频谱$X(j\omega)$的周期拓展，而是会发生频谱混叠 (Aliasing)



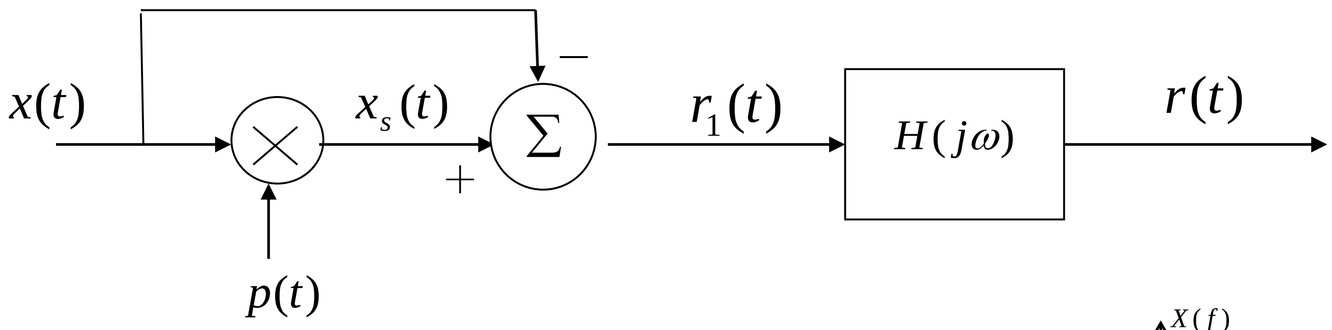


Example: $x(t) = \cos \omega_0 t$

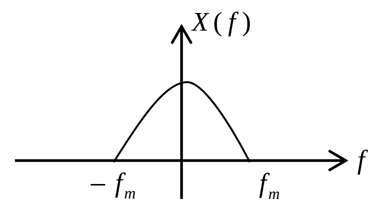


此时，频率为ω_0的信号被“混叠”成较低的频率$\omega_s - \omega_0$。采用LPF恢复出的信号是$x_r(t) = \cos[(\omega_s - \omega_0)t]$

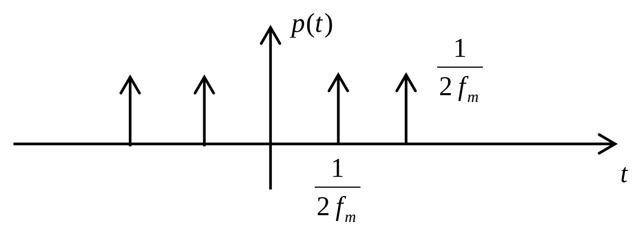
Exercise：



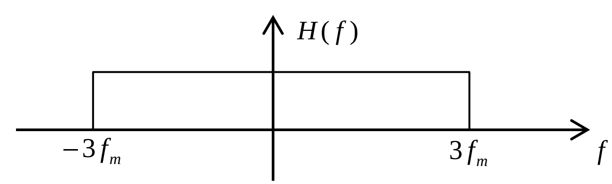
$x(t)$为带限实信号，带宽为f_m;



$p(t)$为周期冲激序列，且周 期$T_s = \dfrac{1}{2f_m}$;



$H(j\omega)$为理想LPF,且带宽$B = 3f_m$;



求 $r(t)$=？

- 微分方程与频率响应

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

根据微分特性，将方程两边作傅里叶变换得

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{M} a_k (j\omega)^k}$$

Example： $$\frac{d^2 y(t)}{dt^2}+4\frac{dy(t)}{dt}+3y(t)=\frac{dx(t)}{dt}+2x(t)$$

$$(j\omega)^2 Y(j\omega)+4(j\omega)Y(j\omega)+3Y(j\omega)=(j\omega)X(j\omega)+2X(j\omega)$$

$$H(j\omega)=\frac{j\omega+2}{(j\omega)^2+4(j\omega)+3}$$

注：根据方程两边的系数可直接写出系统的频率响应

■ 系统的零状态响应—部分分式展开法

Example：

$$H(j\omega)=\frac{j\omega+2}{(j\omega)^2+4(j\omega)+3}=\frac{j\omega+2}{(j\omega+1)(j\omega+3)}=\frac{A_1}{j\omega+1}+\frac{A_2}{j\omega+3}$$

设 $j\omega=v$

则 $A_1=(v+1)H(v)|_{v=-1}=\frac{v+2}{v+3}|_{v=-1}=\frac{1}{2}$

$A_2=(v+3)H(v)|_{v=-3}=\frac{v+2}{v+1}|_{v=-3}=\frac{1}{2}$

$$\therefore H(j\omega)=\frac{\frac{1}{2}}{j\omega+1}+\frac{\frac{1}{2}}{j\omega+3}$$

又 $\frac{1}{\alpha+j\omega}\leftrightarrow e^{-\alpha t}u(t)$ $\quad\therefore h(t)=\frac{1}{2}e^{-t}u(t)+\frac{1}{2}e^{-3t}u(t)$

Example：设$x(t)=e^{-t}u(t)$，对上例的$H(j\omega)$，求系统的$y_{zs}(t)$

$$Y(j\omega)=H(j\omega)X(j\omega)=\frac{j\omega+2}{(j\omega+1)^2(j\omega+3)}=\frac{A_{11}}{j\omega+1}+\frac{A_{12}}{(j\omega+1)^2}+\frac{A_2}{j\omega+3}$$

$$A_{11}=\frac{1}{(2-1)!}\frac{d}{dv}[(v+1)^2Y(v)]|_{v=-1}=\frac{d}{dv}[\frac{v+2}{v+3}]|_{v=-1}=\frac{1}{(v+3)^2}|_{v=-1}=\frac{1}{4}$$

$$A_{12}=(v+1)^2Y(v)|_{v=-1}=\frac{1}{2}$$ ——多重极点

$$A_2=(v+3)Y(v)|_{v=-3}=\frac{v+2}{(v+1)^2}|_{v=-3}=-\frac{1}{4}$$

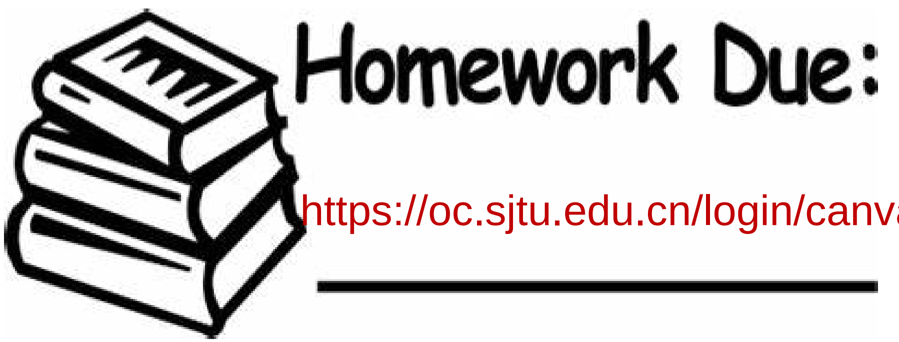
$$\therefore Y(j\omega)=\frac{\frac{1}{4}}{j\omega+1}+\frac{\frac{1}{2}}{(j\omega+1)^2}-\frac{\frac{1}{4}}{j\omega+3}$$

$$\because e^{-\alpha t}u(t)\leftrightarrow\frac{1}{\alpha+j\omega}$$

$$te^{-\alpha t}u(t)\leftrightarrow j\cdot\frac{d}{d\omega}[\frac{1}{\alpha+j\omega}]=\frac{1}{(\alpha+j\omega)^2}$$

$$\therefore y(t)=[\frac{1}{4}e^{-t}+\frac{1}{2}te^{-t}-\frac{1}{4}e^{-3t}]u(t)$$

Q & A



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