Chapter4 The Continuous-Time Fourier Transform

4.0 Introduction

4.1 Fourier Series Representation of Periodic Signals

4.2 The Continuous-Time Fourier Transform

4.3 Properties of the Continuous-Time Fourier Transform

4.4 The Fourier Transform for Periodic Signals

4.5 Frequency-Domain Analysis of LTI System

4.6 System Characterized by Linear Constant-Coefficient Differential Equations

傅里叶级数展开

—将周期信号表示成一组成谐波关系复指数信号的线性组合

从周期信号的傅里叶级数表示导出非周期
信号的傅里叶变换!

傅里叶变换

—将非周期信号表示成复指数信号的线性组合

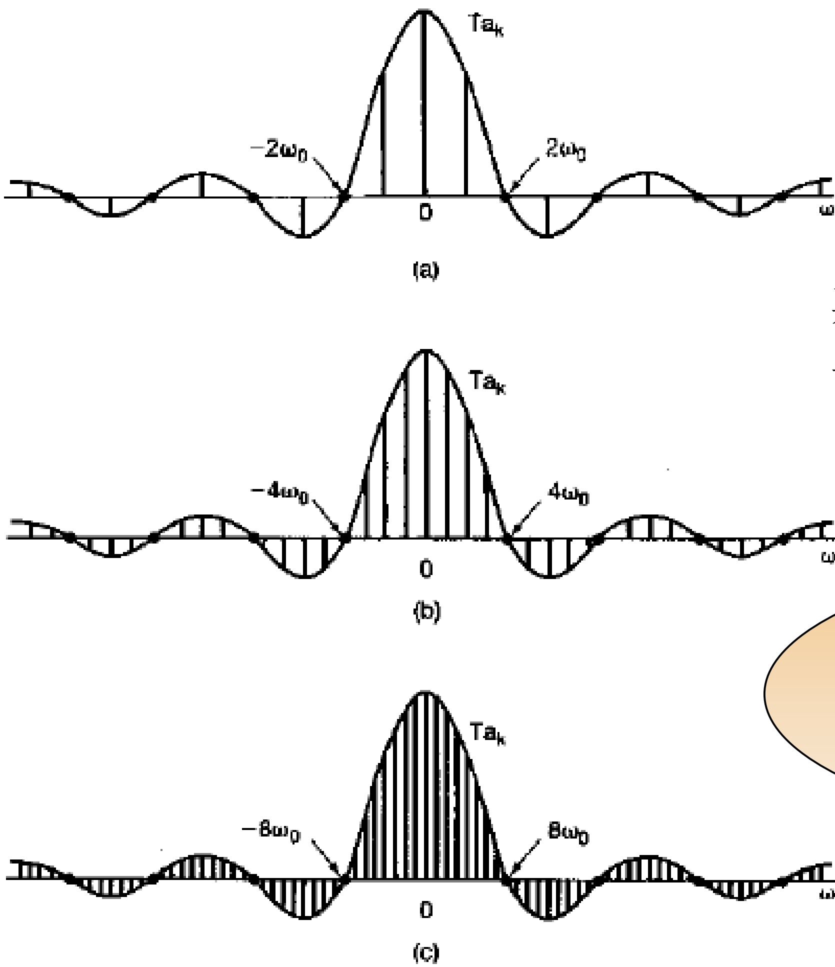
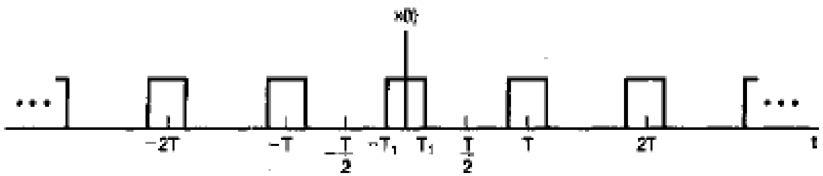


图 4.2 周期方波的傅里叶级数系数及其包络，T_1 固定：
(a) $T=4T_1$；(b) $T=8T_1$；(c) $T=16T_1$

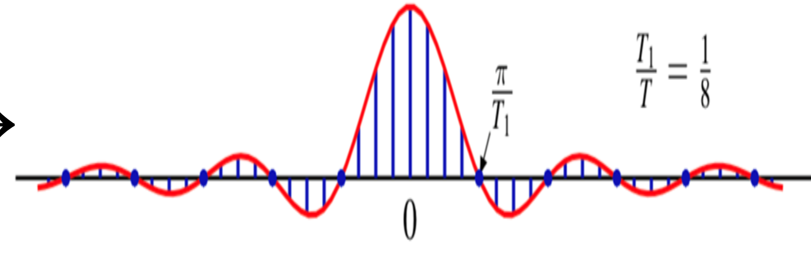
周期信号为离散谱，谱线间隔为$\omega_0=\dfrac{2\pi}{T}$，其中T为信号周期。因此随着周期T的增大，谱线间隔ω_0逐渐减小。

将非周期信号看作$T\to\infty$的周期信号，则在$T\to\infty$时谱线间隔$\omega_0\to 0$，周期信号的离散谱成为对应非周期信号的连续谱。

例：周期矩形脉冲 $\leftrightarrow a_k=\dfrac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$



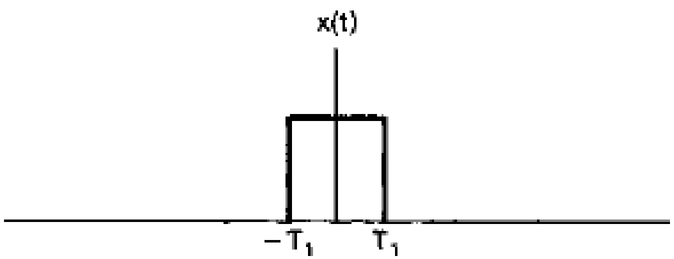
$\leftrightarrow$



周期信号 $\leftrightarrow$ 离散谱，谱线间隔 $\omega_0=\dfrac{2\pi}{T}$

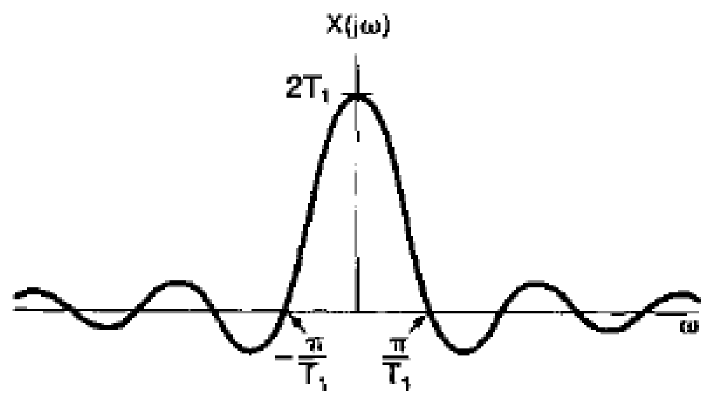
$$a_k=\frac{1}{T}X(j\omega)\Big|_{\omega=k\omega_0}$$

例：矩形脉冲 $\leftrightarrow X(j\omega)=\dfrac{2\sin\omega T_1}{\omega}$



(a)

$\leftrightarrow$



(b)

非周期信号 $\leftrightarrow$ 连续谱，谱线间隔 $\omega_0=\dfrac{2\pi}{T}\to 0$

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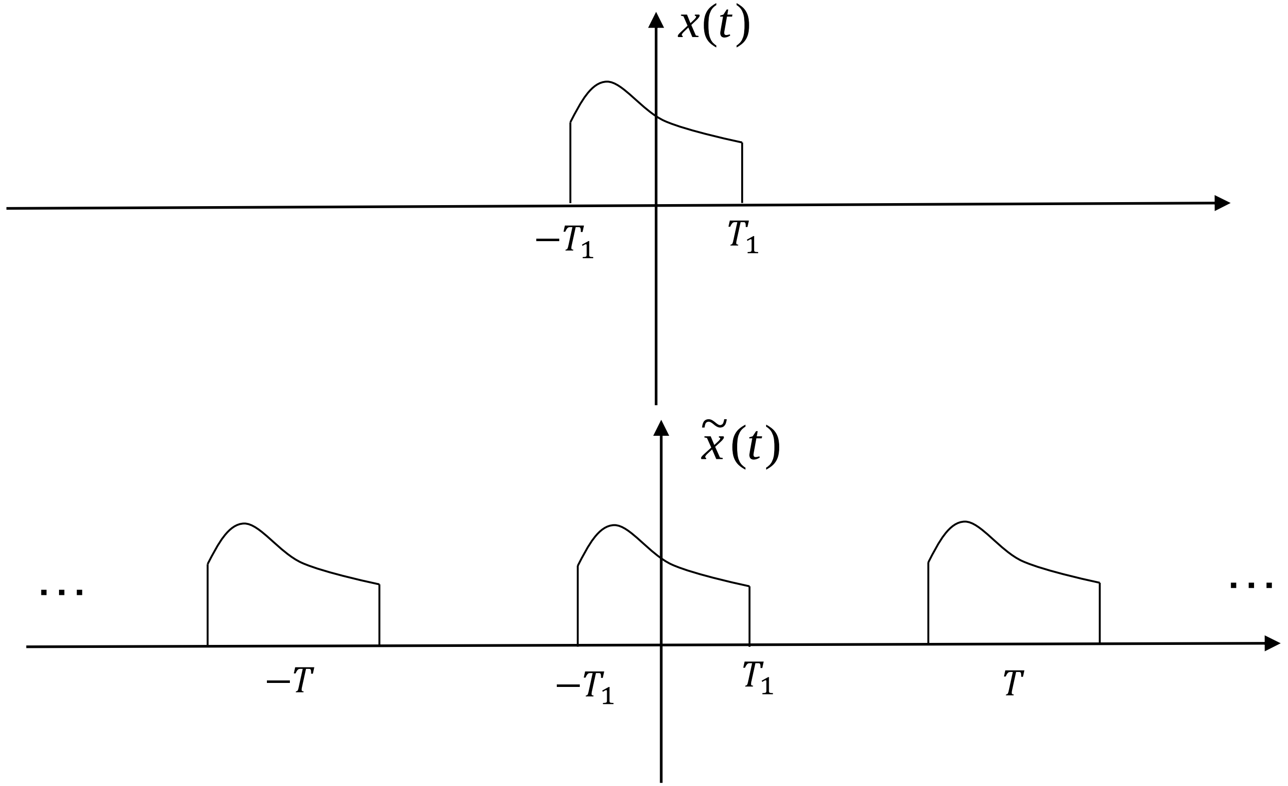
4.2 The Continuous-Time Fourier Transform

- Developmet
- Convergence
- Examples

设$x(t)$为时域有限信号，$\tilde{x}(t)$为$x(t)$的周期拓展



$$\lim_{T\to\infty}\tilde{x}(t)=x(t)$$

$$\tilde{x}(t)=\sum a_k e^{jk\omega_0 t}$$

① $$a_k=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\tilde{x}(t)e^{-jk\omega_0 t}dt=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}x(t)e^{-jk\omega_0 t}dt$$

在$(-\frac{T}{2},\frac{T}{2})$内，$x(t)=\tilde{x}(t)$ 在此之外，$x(t)=0$

设 $$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt$$

则 $$a_k=\frac{1}{T}X(j\omega)\Big|_{\omega=k\omega_0}$$

②

$$\tilde{x}(t) = \sum_k a_k e^{jk\omega_0 t}$$

$$= \sum_k \boxed{\frac{1}{T} X(jk\omega_0)} e^{jk\omega_0 t}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0$$

当$T \to \infty$，即$\omega_0 \to 0$时

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega$$ —综合公式

$$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt$$ —分析公式

$$e^{j\omega t}$$ ——（连续时间）系统频域分析的基本信号

注：

(1) $x(t)$可以表示成基本信号$e^{j\omega t}$的“线性组合”，组合的权值为$\frac{1}{2\pi}X(j\omega)d\omega$;

(2) $X(j\omega)$表示$x(t)$所含各频率分量的幅度和相位，称为$x(t)$的频谱，$X(j\omega)$一般为复数。

4.2 The Continuous-Time Fourier Transform

- Developmet
- Convergence
- Examples

x(t)满足下述条件之一时，其傅里叶变换存在！

1. 能量有限

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty$$

上述条件保证:

① $$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{j\omega t} dt$$ 收敛

② $$\int_{-\infty}^{\infty} |e(t)|^2 \, dt = 0$$

其中

$$e(t) = \hat{x}(t) - x(t) \qquad \hat{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(e^{j\omega}) e^{j\omega t} d\omega$$

2. Dirichlet condition

1）x(t)绝对可积；

2）在任何有限区间内，x(t)只有有限个最大值和最小值；

3）在任何有限区间内，x(t)有有限个不连续点，且在每个不连续点必为有限值。

上述条件保证：

① $X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{j\omega t}dt$ 收敛

② $\hat{x}(t)$除不连续点外与$x(t)$处处相等，且在不连续点处，$\hat{x}(t)$收敛于$x(t)$在不连续点处的均值。

注：引入冲击函数后，对不绝对可积也不平方可积的信号也可给出其傅里叶变换。

4.1 The Continuous-Time Fourier Transform

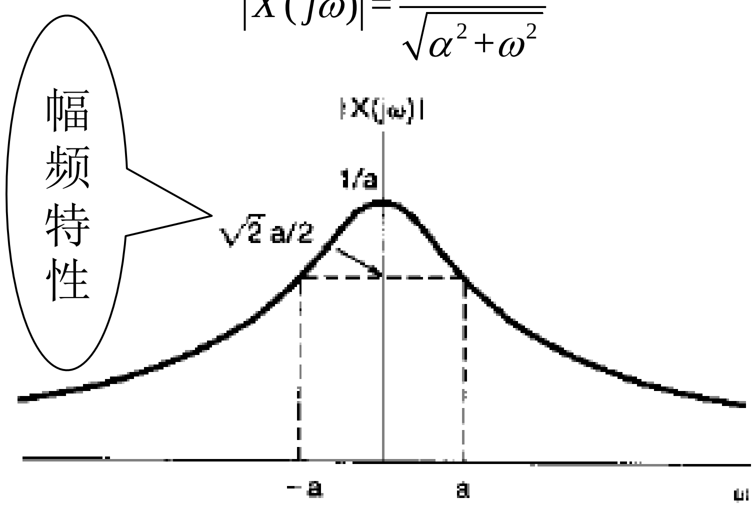
- Developmet
- Convergence
- Examples

•Exponential Signal

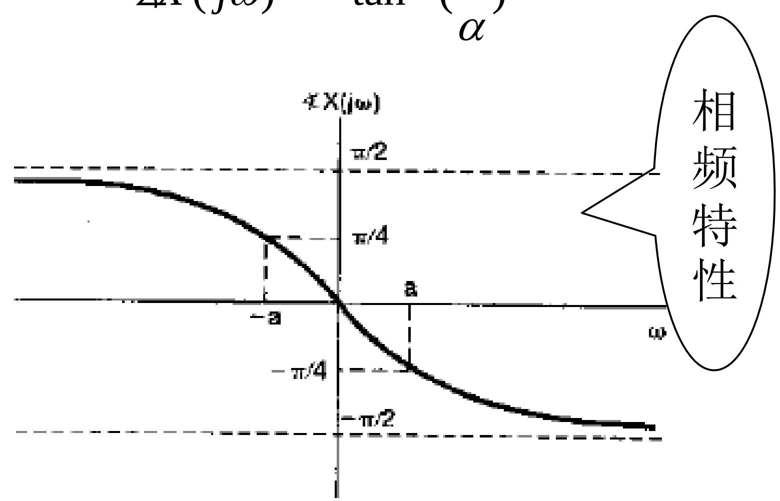
$$x(t)=e^{-\alpha t}u(t)\,,\quad \alpha>0 \quad \leftrightarrow \quad X(j\omega)=\frac{1}{\alpha+j\omega}$$

$$\left|X(j\omega)\right|=\frac{1}{\sqrt{\alpha^2+\omega^2}}$$

$$\measuredangle X(j\omega)=-\tan^{-1}\left(\frac{\omega}{\alpha}\right)$$



(a)

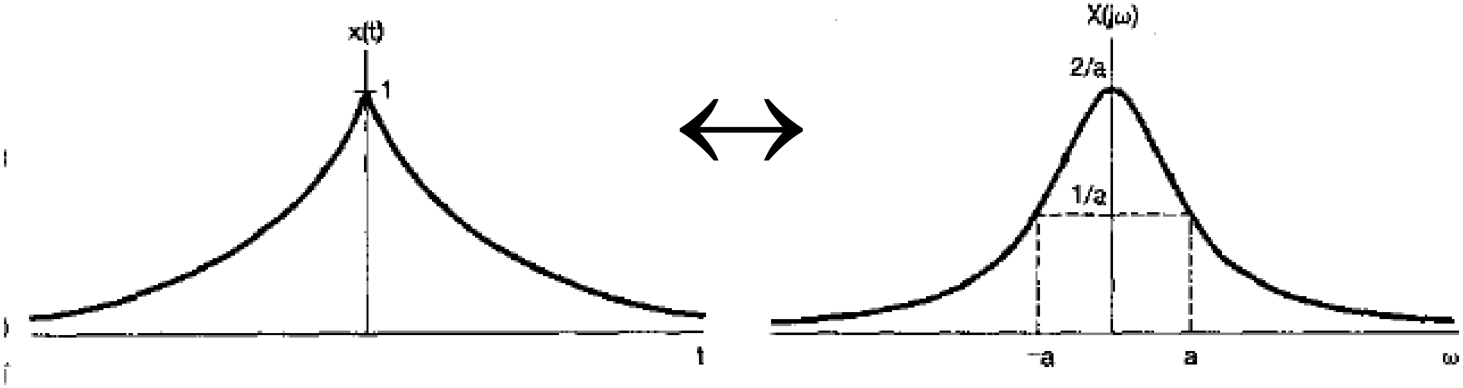


(b)

注：

- 随着频率增大，频谱幅度减小，说明信号能量主要集中在低频部分；
- 时域为实信号，其幅频特性偶对称，相频特性奇对称。

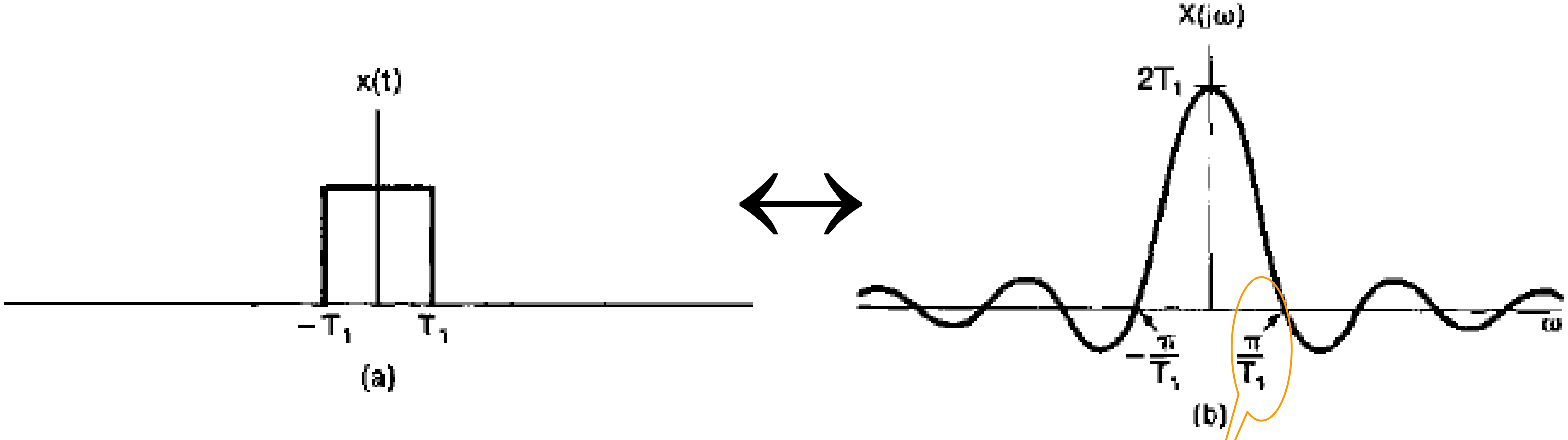
$$x(t)=e^{-\alpha|t|}\,, \quad \alpha>0 \quad \leftrightarrow \quad X(j\omega)=\frac{2\alpha}{\alpha^2+\omega^2}$$



注：时域为实偶信号，其频谱为实的

•Rectangle Pulse

$$x(t)=\begin{cases}1, & |t|<T_1\\ 0, & |t|>T\end{cases} \leftrightarrow X(j\omega)=2\frac{\sin\omega T_1}{\omega}=2T_1S_a(\omega T_1)$$



注：频谱的第一个过零点（也即主瓣带宽）为$\frac{1}{2T_1}(Hz)$，与矩形脉冲宽度成反比

主瓣带宽：信号的频谱从$f=0$到其第一个过零点所占的频率范围

例：矩形脉冲

$$\because X(j\omega)=2\frac{\sin(\omega T_1)}{\omega}$$

$$\therefore \text{当}\,\omega T_1=\pi\text{，即}\,\omega=\pi\cdot\frac{1}{T_1}\,\text{时为第一个过零点}$$

$$\text{则主瓣带宽}=\frac{1}{2T_1}(Hz)=\frac{1}{\text{矩形脉冲宽度}}$$

故，在通信系统中，传输的脉冲宽度越窄，信息传输速率越高，但所需传输带宽增大。

- **Impulse function**

$$\boxed{x(t)=\delta(t)\leftrightarrow X(j\omega)=1}$$

- **Dc signal**

$$\boxed{x(t)=1\leftrightarrow X(j\omega)=2\pi\delta(\omega)}$$

Proof:

$$\frac{1}{2\pi}\int 2\pi\delta(\omega)e^{j\omega t}d\omega=1$$

不满足傅里叶变换收敛的条件

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傅里叶变换的性质！

If: $x(t) \leftrightarrow X(j\omega)$

Then:

$x(-t)$

$x(t-t_0)$

$x(at)$

$\dfrac{dx(t)}{dt}$

$\int x(\tau)d\tau$

…

其频谱与

$X(j\omega)$的关系？

$X(-j\omega)$

$X(j(\omega-\omega_0))$

$X(j\dfrac{\omega}{a})$

$\dfrac{dX(j\omega)}{d\omega}$

$\int X(j\omega)d\omega$

…

其时域表达式与

$x(t)$的关系？

- 化简傅里叶变换（或反变换）的求解
- 指出信号（时域或频域）线性变换的物理含义

- **Lineratiy**

$$x(t) \leftrightarrow X(j\omega) \qquad y(t) \leftrightarrow Y(j\omega)$$

$$\boxed{ax(t)+by(t) \leftrightarrow aX(j\omega)+bY(j\omega)}$$

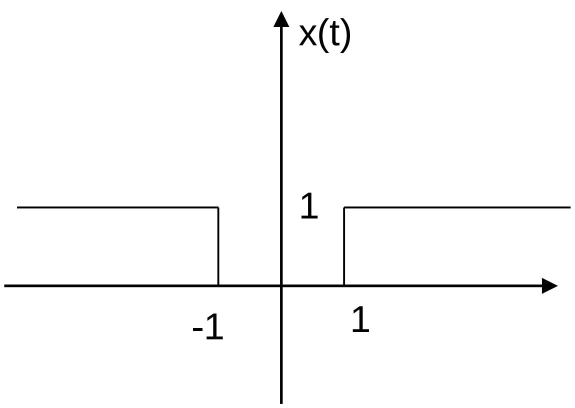
Example：求如图信号x(t)的傅里叶变换

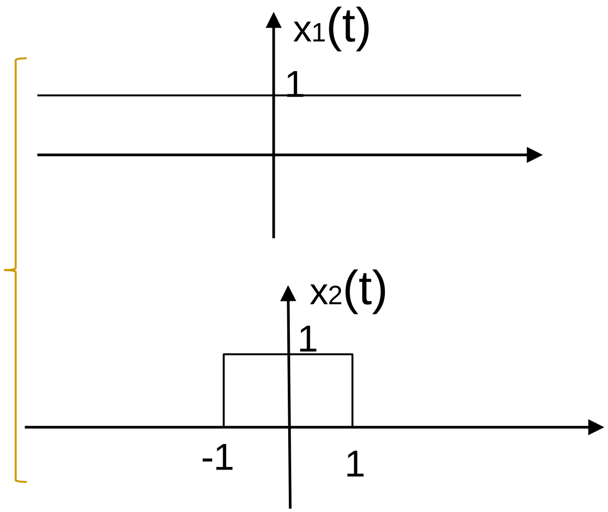
$$\because x(t) = x_1(t) - x_2(t)$$

$$\therefore X(j\omega)$$

$$= X_1(j\omega) - X_2(j\omega)$$

$$= 2\pi\delta(\omega) - 2\frac{\sin(\frac{\omega}{2})}{\omega}$$





Time Shifting

$$x(t) \leftrightarrow X(j\omega)$$

$$\boxed{x(t-t_0) \leftrightarrow e^{-j\omega t_0} X(j\omega)}$$

注：信号在时域发生时移，在频域仅发生相位的超前或滞后，且相移与频率成线性关系。

Example: Determine the Fourier Transform of the following signals

$$e^{-2t}u(t) \leftrightarrow \frac{1}{2+j\omega}$$

$$e^{-2(t-1)}u(t) = e^{2} \cdot e^{-2t}u(t) \leftrightarrow \frac{e^{2}}{2+j\omega}$$

$$e^{-2t}u(t-1) = e^{-2(t-1)}u(t-1) \cdot e^{-2} \leftrightarrow \frac{e^{-(2+j\omega)}}{2+j\omega}$$

Time and Frequency Scaling

$$x(t) \leftrightarrow X(j\omega)$$

$$\boxed{x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)}$$

for $a = -1$ $\qquad x(-t) \leftrightarrow X(-j\omega)$

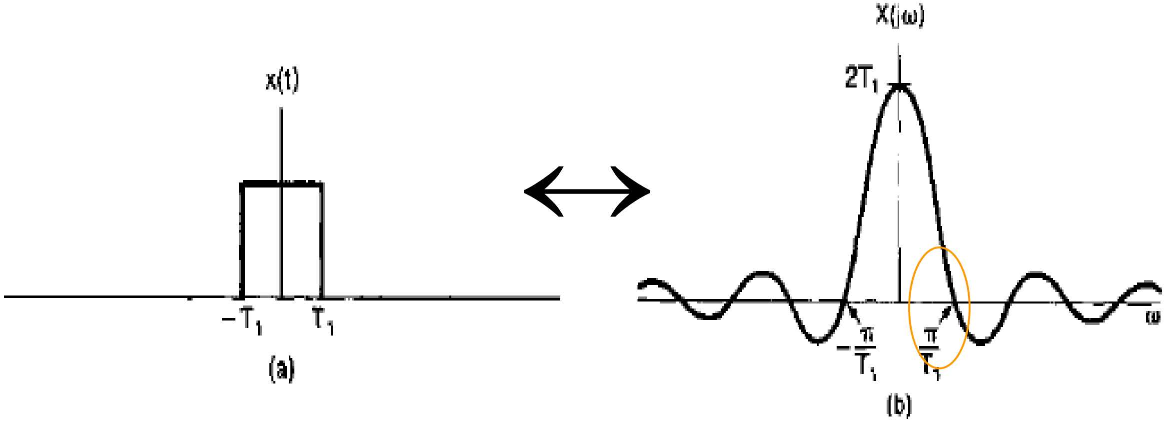
注：信号在时域和频域相反的压缩扩展特性

Exercise： $x(at+b) \leftrightarrow ?$ $\quad \frac{1}{|a|} X\left(j\frac{\omega}{a}\right) e^{j\omega\frac{b}{a}}$

Example: 正弦信号时域的周期增加，频域的频率下降

- 时域压缩↔ 频域扩展
- 时域扩展↔ 频域压缩

Example：矩形脉冲的宽度减小，频谱的主瓣带宽增加。



Differentiation and Integration

$$x(t) \leftrightarrow X(j\omega)$$

$$\boxed{\frac{dx(t)}{dt} \leftrightarrow j\omega X(j\omega)}$$

注： • 微分增强了信号的高频分量

• 微分会消除原信号中的直流分量，因此在未知直流信息的情况下，不能从dx/dt中完全恢复出x(t)

Proof: $\because x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$

$$\therefore \frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} [j\omega X(j\omega)]e^{j\omega t}d\omega$$

$$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$$

注：
- 积分削弱了信号的高频分量
- 积分可能产生直流成分

Proof:

$$let\ y(t)=\int_{-\infty}^{t} x(\tau)d\tau$$

$$\therefore y'(t)=x(t)$$
$$\therefore j\omega Y(j\omega)=X(j\omega) \implies Y(j\omega)$$
$$=\frac{1}{j\omega}X(j\omega), \forall\omega\neq 0$$

consider the DC component of y(t):

$$\bar{y}=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} y(t)dt$$
$$=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau\, dt$$

$$=\int_{-\infty}^{\infty} x(\tau)\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} u(t-\tau)dt\right]d\tau$$
$$=\frac{1}{2}\int_{-\infty}^{\infty} x(\tau)d\tau=\frac{1}{2}X(0)$$

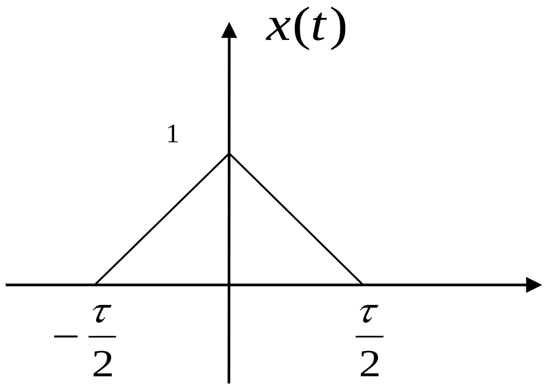
Example：unit step

$$\because \quad u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \qquad \delta(t) \leftrightarrow 1$$

$$\therefore \quad \boxed{u(t) \leftrightarrow \frac{1}{j\omega} + \pi\delta(\omega)}$$

Example: triangle pulse

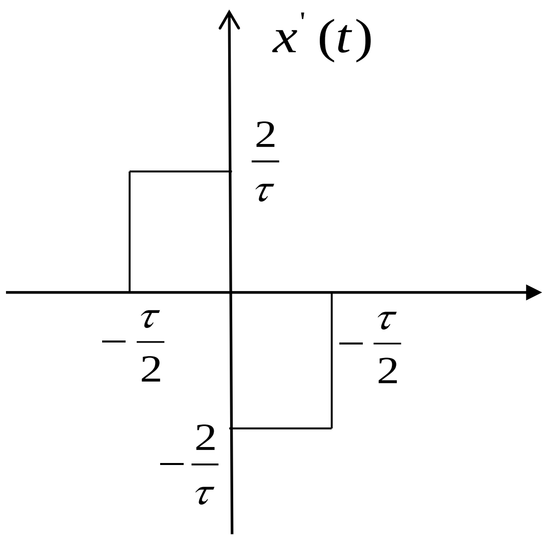
$$x(t)=\begin{cases}1-\dfrac{2}{\tau}\cdot|t|, & |t|\le\dfrac{\tau}{2}\\[2ex] 0, & |t|>\dfrac{\tau}{2}\end{cases}$$



$$x'(t)\leftrightarrow X_1(j\omega)$$

$$X(j\omega)=\frac{X_1(j\omega)}{j\omega}+\pi X_1(0)\delta(\omega)$$

又 $X_1(0)=\int x'(t)dt=0$



$$X(j\omega)=\frac{X_1(j\omega)}{j\omega}=\frac{8\sin^2(\frac{\omega\tau}{4})}{\omega^2\tau}=\frac{\tau}{2}Sa^2(\frac{\omega\tau}{4})$$

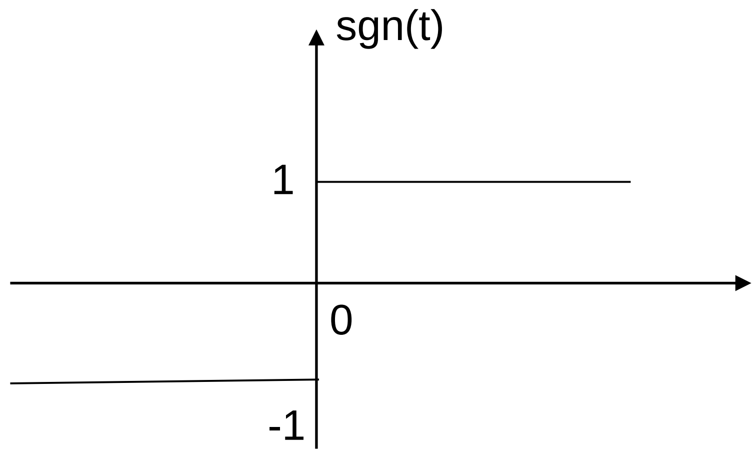
关于X(0)的求解：

$$1.\quad X(0) = X(j\omega)\big|_{\omega=0}$$

$$2.\quad X(0) = \int_{-\infty}^{\infty} x(t)dt$$

Example: sgn()

$$x(t) = \mathrm{sgn}(t) = \begin{cases} 1, & (t > 0) \\ -1, & (t < 0) \end{cases}$$

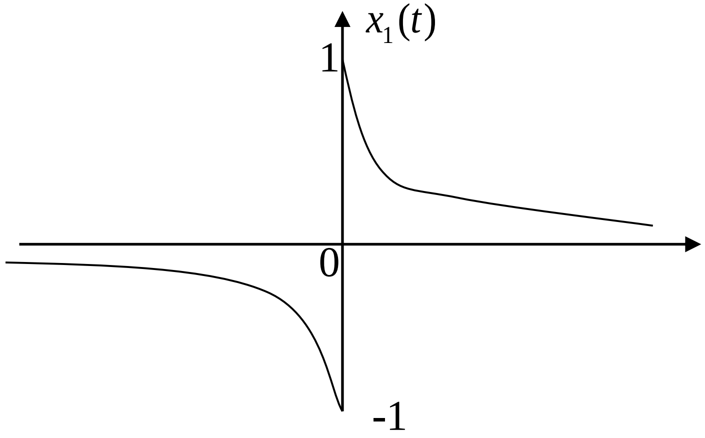


(1)指数信号取极限

$$x_1(t)=\begin{cases} e^{-\alpha t}, & (t>0) \\ -e^{\alpha t}, & (t<0) \end{cases} \quad \alpha>0$$

$$=e^{-\alpha t}u(t)-e^{\alpha t}u(-t)$$

$$\leftrightarrow \frac{1}{\alpha+j\omega}-\frac{1}{\alpha-j\omega}=\frac{-2j\omega}{\alpha^2+\omega^2}$$



$$\because \operatorname{sgn}(t)=\lim_{\alpha\to 0} x_1(t) \qquad \therefore \operatorname{sgn}(t)\leftrightarrow \lim_{\alpha\to 0} X_1(j\omega)=\frac{2}{j\omega}$$

(2)利用阶跃信号

$$\because \operatorname{sgn}(t)=u(t)-u(-t)$$

$$\therefore \operatorname{sgn}(t)\leftrightarrow (\pi\delta(\omega)+\frac{1}{j\omega})-[\pi\delta(-\omega)+\frac{1}{-j\omega}]=\frac{2}{j\omega}$$

(3)利用积分性质

$$\because \operatorname{sgn}(t)=2u(t)-1 \quad \therefore \operatorname{sgn}'(t)=2\delta(t)=x_1(t)$$

设 $x_1(t)=\dfrac{dx(t)}{dt}\leftrightarrow X_1(j\omega)$

当$x(-\infty)\neq 0$时 $\because \int_{-\infty}^{t} x_1(\tau)d\tau = x(t)-x(-\infty)$

$$\therefore x(t)=\int_{-\infty}^{t} x_1(\tau)d\tau + x(-\infty)$$

$$\therefore X(j\omega)=\frac{X_1(j\omega)}{j\omega}+\pi X_1(0)\delta(\omega)+2\pi x(-\infty)\delta(\omega)$$

$$\because X_1(j\omega)=2 \quad \therefore \operatorname{sgn}(t)=\frac{2}{j\omega}+\pi\cdot 2\delta(\omega)+2\pi\cdot(-1)\cdot\delta(\omega)=\frac{2}{j\omega}$$

- **Duality**

$$x(t) \leftrightarrow X(j\omega)$$

$$\boxed{X(t) \leftrightarrow 2\pi x(-\omega)}$$

注：从傅里叶变换的综合和分析公式，可得时域和频域的对偶特性！

Example:

$$\delta(t) \leftrightarrow 1$$

$$1 \leftrightarrow 2\pi\delta(\omega)$$

Proof:

$$\because x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$\therefore x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jt)e^{j\omega t}dt$$

即

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(jt)e^{-j\omega t}dt$$

Example：

$$x(t)=\frac{\sin Wt}{\pi t} \leftrightarrow X(j\omega)=\begin{cases}1, & |\omega|<W \\ 0, & |\omega|>W\end{cases}$$

Proof:

$$X_1(j\omega)=2\frac{\sin(\omega T_1)}{\omega} \leftrightarrow x_{T_1}(t)=\begin{cases}1, & |t|<T_1 \\ 0, & |t|>T_1\end{cases}$$

$$\frac{1}{2\pi}x_W(t) \leftrightarrow \frac{\sin(\omega W)}{\pi\omega}$$

$$\frac{\sin(Wt)}{\pi t} \leftrightarrow 2\pi\cdot\frac{1}{2\pi}x_W(-\omega)=\begin{cases}1, & |\omega|<W \\ 0, & |\omega|>W\end{cases}$$

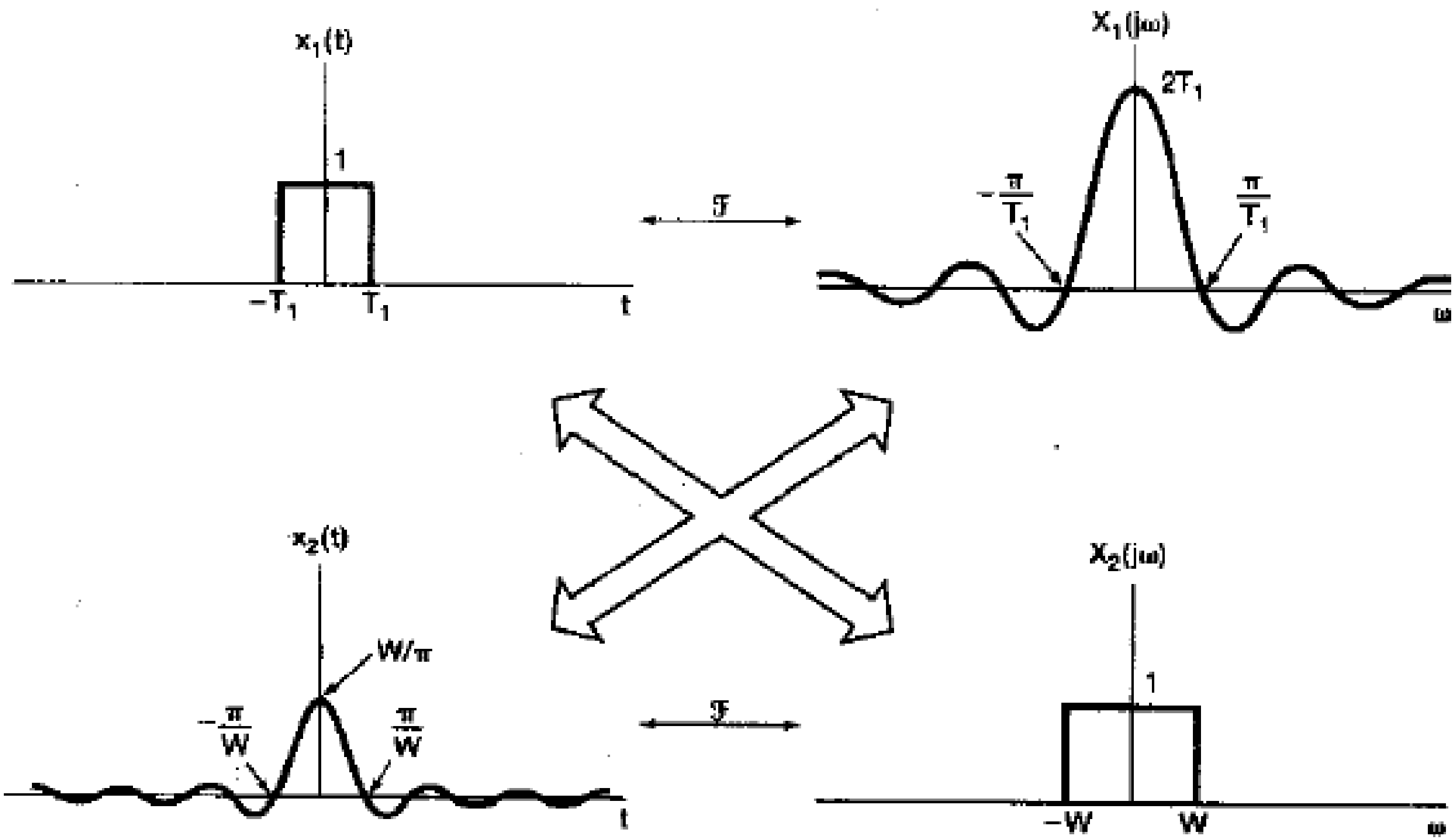


图 4.17 (4.36)式和(4.37)式两对傅里叶变换之间的关系

已知$x(t)$,利用对偶性求$X(j\omega)$

（1）先找到形如$x(t)$的$X_1(j\omega)$

（2）对$X_1(j\omega)\leftrightarrow x_1(t)$做变换，使$X_1(j\omega)$在“形式”上与$x(t)$完全相同

（3）利用对偶性得$X(j\omega)$

Example: $$x(t)=\frac{1}{t}$$

$$\because \operatorname{sgn}(t)\leftrightarrow \frac{2}{j\omega}$$

$$\frac{j}{2}\operatorname{sgn}(t)\leftrightarrow \frac{1}{\omega}$$

$$\therefore \frac{1}{t}\leftrightarrow 2\pi\frac{j}{2}\operatorname{sgn}(-\omega)=-j\pi\operatorname{sgn}(\omega)$$

Exercise: $$x(t)=\frac{1}{1+t^2}$$

已知$X(j\omega)$,利用对偶性求$x(t)$

$$\because \mathfrak{I}[X(t)] = 2\pi x(-\omega)$$

$$\therefore x(-\omega) = \frac{1}{2\pi}\mathfrak{I}[X(t)]$$

将$-\omega$用t代换，得$x(t)$

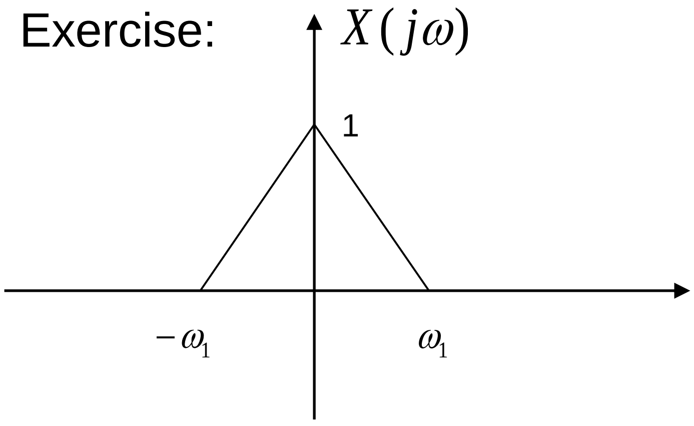
Example: $X(j\omega) = e^{-|\omega|}$

$$\because X(t) = e^{-|t|} \leftrightarrow \frac{2}{1+\omega^2} = 2\pi x(-\omega)$$

$$\therefore x(-\omega) = \frac{1}{2\pi}\cdot\frac{2}{1+\omega^2}$$

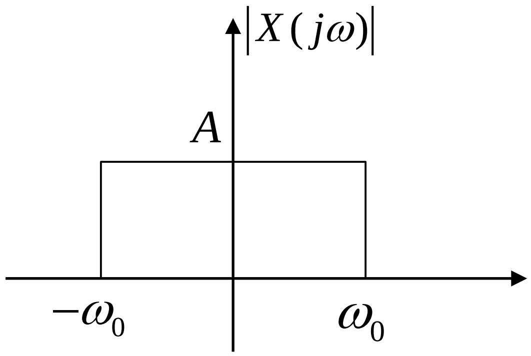
$$\rightarrow x(t) = \frac{1}{\pi}\cdot\frac{1}{1+t^2}$$

Exercise:



$$\leftrightarrow x(t) = \frac{\omega_1}{2\pi} Sa^2(\frac{\omega_1 t}{2})$$

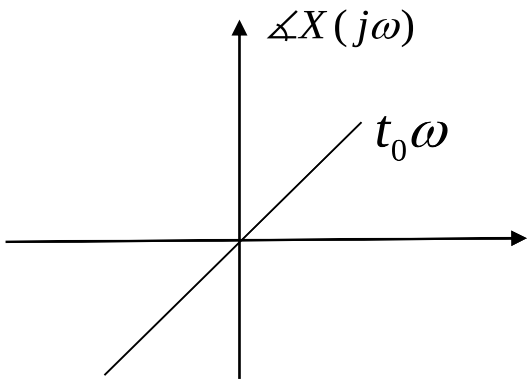
Example：Determine the c-t signal $x(t)$ to the following transform $X(j\omega)$



$$X(j\omega) = |X(j\omega)|\, e^{\measuredangle X(j\omega)}$$

$$\because\ |X(j\omega)| \leftrightarrow A\frac{\sin \omega_0 t}{\pi t}$$

$$\measuredangle X(j\omega) = \omega t_0$$



$$\therefore\ X(j\omega) = A[u(\omega+\omega_0) - u(\omega-\omega_0)]e^{j\omega t_0}$$

$$\leftrightarrow A\frac{\sin \omega_0 (t+t_0)}{\pi (t+t_0)}$$

时域与频域的其它“对偶”特性：

- 频移特性

$$x(t) \leftrightarrow X(j\omega)$$

$$\boxed{e^{j\omega_0 t}x(t) \leftrightarrow X(j(\omega-\omega_0))}$$

Example:

$$\because\ 1 \leftrightarrow 2\pi\delta(\omega)$$

$$\therefore\ e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega-\omega_0)$$

Example:

$$\cos\omega_0 t = \frac{1}{2}[e^{j\omega_0 t} + e^{-j\omega_0 t}]$$

$$\leftrightarrow \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$$

$$\sin\omega_0 t = \frac{1}{2j}[e^{j\omega_0 t} - e^{-j\omega_0 t}]$$

$$\leftrightarrow \frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$$

$$= j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$$

- 频域微分特性

$$\boxed{-jtx(t) \leftrightarrow \frac{dX(j\omega)}{d\omega}}$$

即， $$tx(t) \leftrightarrow j\frac{dX(j\omega)}{d\omega}$$

- 频域积分特性

$$\boxed{-\frac{1}{jt}x(t)+\pi x(0)\delta(t) \leftrightarrow \int_{-\infty}^{\omega} X(\lambda)d\lambda}$$

当$x(0)=0$时 $$\frac{x(t)}{t} \leftrightarrow -j\int_{-\infty}^{t} X(\lambda)d\lambda$$

Example: $$u(t) \leftrightarrow \frac{1}{j\omega} + \pi\delta(\omega)$$

$$tu(t) \leftrightarrow -\frac{1}{\omega^2} + j\pi\delta'(\omega)$$

Example: $$x(t) = te^{-2t}u(t)$$

$$\because e^{-2t}u(t) \leftrightarrow \frac{1}{2+j\omega}$$

$$\therefore te^{-2t}u(t) \leftrightarrow j\frac{d}{d\omega}\left(\frac{1}{2+j\omega}\right) = \frac{1}{(2+j\omega)^2}$$

Exercise: $$x(t) = te^{-2t}u(t-1)$$

Example: $X(j\omega)=\frac{1}{\omega^2}\leftrightarrow x(t)=?$ 注：利用频域的微分性质

设$X_1(j\omega)=-\frac{1}{\omega}\leftrightarrow x_1(t)$

$\because X(j\omega)=X_1^{'}(j\omega)$

$\therefore x(t)=-jtx_1(t)$

又$\because \operatorname{sgn}(t)\leftrightarrow\frac{2}{j\omega}$

$-\frac{j}{2}\operatorname{sgn}(t)\leftrightarrow-\frac{1}{\omega}$

$\therefore x_1(t)=-\frac{j}{2}\operatorname{sgn}(t)$

则$x(t)=-jt\cdot(-\frac{j}{2}\operatorname{sgn}(t))=-\frac{t}{2}\operatorname{sgn}(t)$

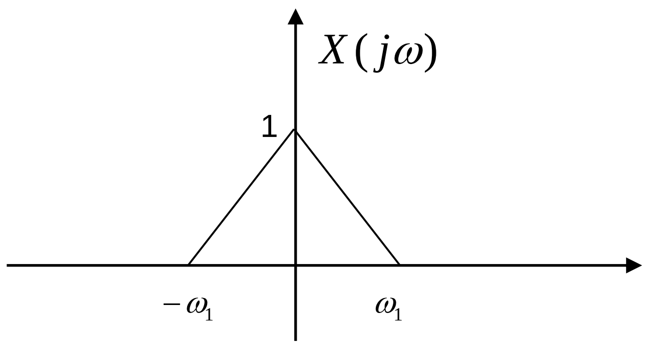
即 $x(t)=-\frac{|t|}{2}$

$$\frac{1}{\omega} \leftrightarrow \frac{j}{2}\operatorname{sgn}(t)$$

$$\frac{1}{\omega^2} \leftrightarrow -\frac{t}{2}\operatorname{sgn}(t) = -\frac{|t|}{2}$$

Exercise: $x(t) = |t| \leftrightarrow X(j\omega) = ?$

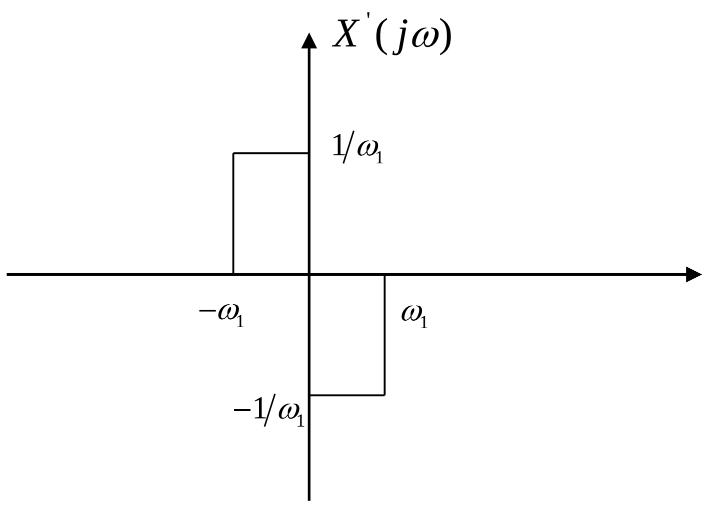
Example: Determine the $x(t)$ of the $X(j\omega)$　注：利用频域的积分性质



$$X'(j\omega) \leftrightarrow x_1(t)$$

$$x(t) = \frac{x_1(t)}{-jt} + \pi x_1(0)\delta(t)$$

又 $x_1(0)=\int X'(j\omega)d\omega = 0$



$$\because x_1(t) = \frac{1}{\omega_1}\frac{\sin(\frac{\omega_1}{2}t)}{\pi t}e^{-j\frac{\omega_1}{2}t} - \frac{1}{\omega_1}\frac{\sin(\frac{\omega_1}{2}t)}{\pi t}e^{j\frac{\omega_1}{2}t}$$

$$= \frac{1}{\omega_1}\frac{\sin(\frac{\omega_1}{2}t)}{\pi t}\cdot[-2j\sin(\frac{\omega_1}{2}t)]$$

$$\therefore x(t) = \frac{x_1(t)}{-jt} = \frac{2}{\pi}\frac{\sin^2(\frac{\omega_1}{2}t)}{\omega_1 t^2}$$

■ Convolution Property

$$x_1(t) \leftrightarrow X_1(j\omega) \quad x_2(t) \leftrightarrow X_2(j\omega)$$

$$\boxed{x_1(t) * x_2(t) \leftrightarrow X_1(j\omega) \cdot X_2(j\omega)}$$

Proof:

$$x_1(t) * x_2(t) \leftrightarrow \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \right] e^{-j\omega t} dt$$

$$= \int_{-\infty}^{+\infty} x_1(\tau) \left[\int_{-\infty}^{+\infty} x_2(t-\tau)\, e^{-j\omega t} dt \right] d\tau$$

$$= \int_{-\infty}^{+\infty} x_1(\tau) e^{-j\omega\tau} X_2(j\omega) d\tau$$

$$= X_2(j\omega) \int_{-\infty}^{+\infty} x_1(\tau) e^{-j\omega\tau} d\tau = X_2(j\omega) X_1(j\omega)$$

LTI系统：

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = x(t) * h(t)$$

设 $x(t) \leftrightarrow X(j\omega) \quad y(t) \leftrightarrow Y(j\omega)$

$$\boxed{h(t) \leftrightarrow H(j\omega)}$$ —frequency response

则

$$Y(j\omega) = X(j\omega)H(j\omega)$$

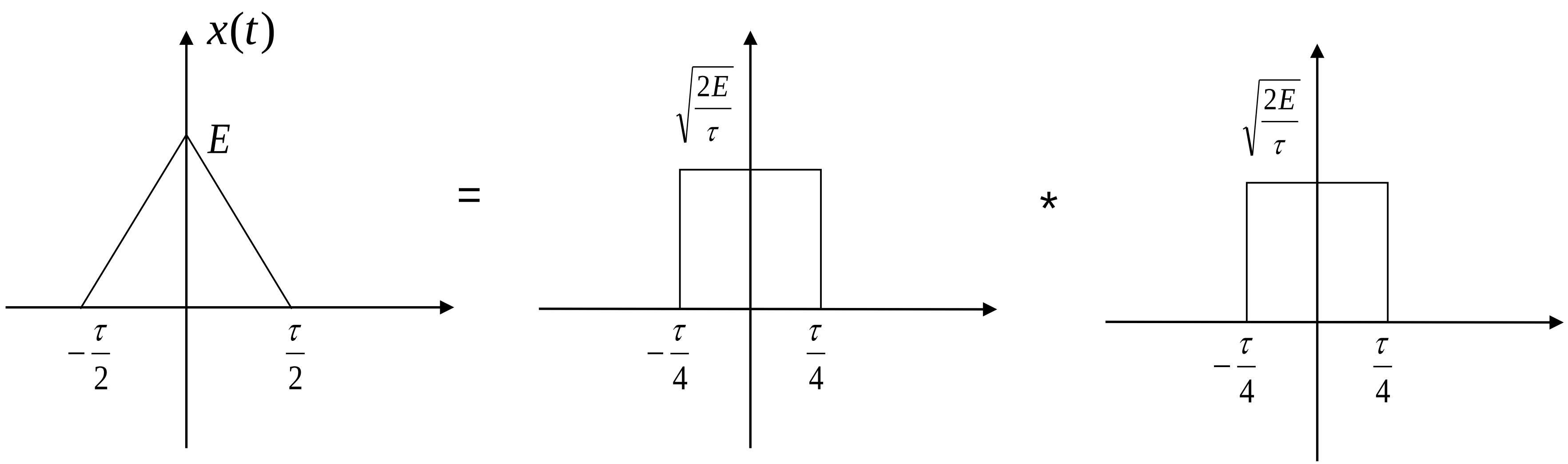
LTI系统的频域分析

例：三角形脉冲

①利用微积分性质

②利用卷积性质

$$x(t)=\begin{cases}E(1-\frac{2|t|}{\tau}), & |t|\le\frac{\tau}{2}\\ 0, & |t|>\frac{\tau}{2}\end{cases}$$



Multiplication Property

$$x_1(t) \leftrightarrow X_1(j\omega) \quad x_2(t) \leftrightarrow X_2(j\omega)$$

$$\boxed{x_1(t)\cdot x_2(t) \leftrightarrow \frac{1}{2\pi}[X_1(j\omega) * X_2(j\omega)]}$$

$$x(t)\cos\omega_0 t \leftrightarrow \frac{1}{2}[X(j(\omega+\omega_0)) + X(j(\omega-\omega_0))]$$

$$x(t)\sin\omega_0 t \leftrightarrow \frac{j}{2}[X(j(\omega+\omega_0)) - X(j(\omega-\omega_0))]$$

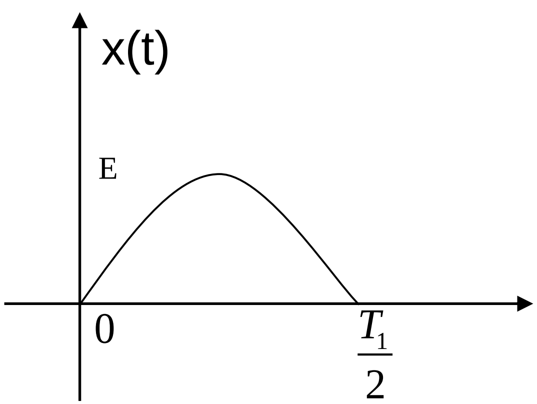
信号的
调制

Example: $x(t)=[te^{-2t}\cos 4t]u(t)$

$$\because te^{-2t}u(t)\leftrightarrow j\frac{d}{d\omega}(\frac{1}{2+j\omega})=\frac{1}{(2+j\omega)^2}$$

$$\therefore (te^{-2t}\cos 4t)u(t)\leftrightarrow \frac{1}{2}[\frac{1}{(2+j(\omega+4))^2}+\frac{1}{(2+j(\omega-4))^2}]$$

Exercise: 求半波正弦信号的傅里叶变换



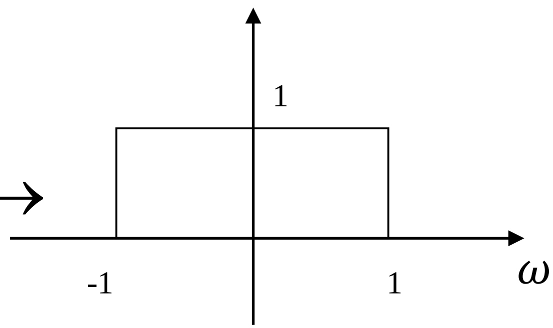
$$x(t)=E\sin\omega_1 t[u(t)-u(t-\frac{T_1}{2})]$$

其中，$\omega_1=\frac{2\pi}{T_1}$

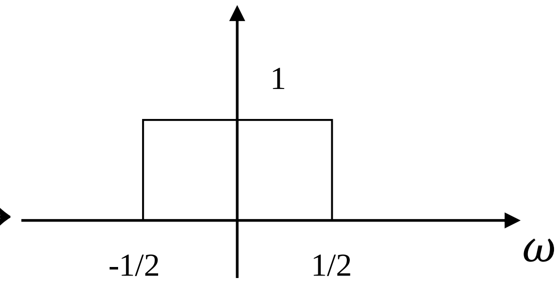
Example: $$x(t)=\frac{\sin t\cdot\sin\frac{t}{2}}{\pi t^{2}}$$

（1）

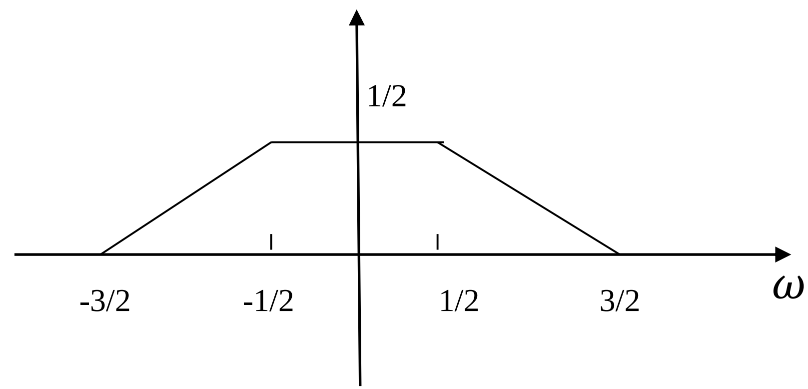
$$\because \frac{\sin t}{\pi t}\leftrightarrow$$



$$\frac{\sin\frac{t}{2}}{\pi t}\leftrightarrow$$



$$\therefore x(t)\leftrightarrow\frac{1}{2\pi}\cdot\pi\{\Im[\frac{\sin t}{\pi t}]*\Im[\frac{\sin\frac{t}{2}}{\pi t}]\}=$$



（2）利用上述结果求

$$\int\frac{\sin t\cdot\sin\frac{t}{2}}{\pi t^{2}}dt=?$$

- **Conjugation and Conjugate Symmetry**

$$x(t) \leftrightarrow X(j\omega)$$

$$x^*(t) \leftrightarrow X^*(-j\omega)$$

If x(t) is real valued

$$\boxed{X(j\omega) = X^*(-j\omega)}$$ —Conjugate Symmetry

(1) $$\mathrm{Re}[X(j\omega)] = \mathrm{Re}[X(-j\omega)]$$

$$\mathrm{Im}[X(j\omega)] = -\mathrm{Im}[X(-j\omega)]$$

(2) $$|X(j\omega)| = |X(-j\omega)|$$

$$\measuredangle X(j\omega) = -\measuredangle X(-j\omega)$$

(3) $x(t)$ *real and even* $\leftrightarrow X(j\omega)$ *real and even*

$x(t)$ *real and odd* $\leftrightarrow X(j\omega)$ *purely imaginary and odd*

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)] \leftrightarrow Re[X(j\omega)]$$

$$x_0(t) = \frac{1}{2}[x(t) - x(-t)] \leftrightarrow jI_m[X(j\omega)]$$

Example:

$$x(t) = e^{-\alpha|t|} = e^{-\alpha t}u(t) + e^{\alpha t}u(-t) = 2E_v\{e^{-\alpha t}u(t)\}$$

$$\because e^{-\alpha t}u(t) \leftrightarrow \frac{1}{\alpha + j\omega}$$

$$\therefore x(t) \leftrightarrow 2\,\mathrm{Re}\{\frac{1}{2 + j\omega}\} = \frac{2\alpha}{\alpha^2 + \omega^2}$$

Parseval's Relation

$$\boxed{\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega}$$

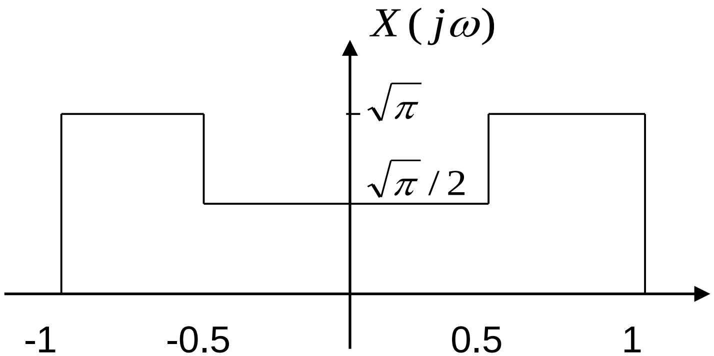
$|X(j\omega)|^2$ ——能谱密度

另： $$\lim_{T\to\infty}\frac{1}{T}\int_{T} |x(t)|^2 \, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty}\frac{|X(j\omega)|^2}{T} d\omega$$

$\lim_{T\to\infty}\dfrac{|X(j\omega)|^2}{T}$ ——功率谱密度

注：信号做傅里叶变换后，在时域和频域能量守恒

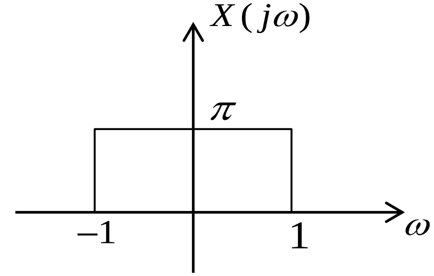
Example1：



$$\leftrightarrow x(t) \qquad 求 \int |x(t)|^2 dt$$

Example2： $x(t) = \dfrac{\sin t}{t}$ 求 $\int_{-\infty}^{+\infty} |x(t)|^2 dt$

$$x(t) = \frac{\sin t}{t} \longleftrightarrow$$



$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega = \pi$$

周期信号$x(t)$，周期为T，基本频率$\omega_0=\dfrac{2\pi}{T}$

$$\because x(t)=\sum_k a_k e^{jk\omega_0 t}$$

$$\therefore \Im[x(t)]=\Im[\sum_k a_k e^{jk\omega_0 t}]=\sum_k a_k \Im[e^{jk\omega_0 t}]$$

又 $e^{jk\omega_0 t} \leftrightarrow 2\pi\delta(\omega-k\omega_0)$ 得

$$\boxed{\text{周期为}T\text{的信号}x(t)\leftrightarrow \sum_{k=-\infty}^{\infty}2\pi a_k\delta(\omega-k\omega_0),\quad \omega_0=\frac{2\pi}{T}}$$

注：周期信号的频谱是离散的，由冲激函数构成，冲激函数的强度取决于a_k

周期为T的信号$x(t)$:

- Fourier Series Representation

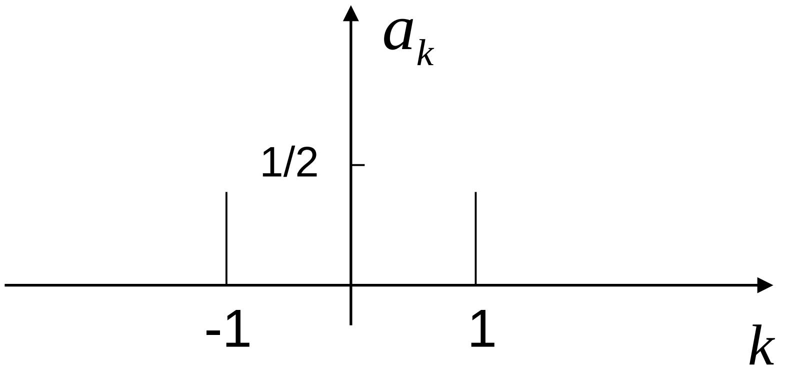
$$x(t)=\sum a_k e^{jk\omega_0 t}$$

$$a_k=\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

- Fourier Transform

$$x(t)\leftrightarrow X(j\omega)=\sum_{k=-\infty}^{\infty}2\pi a_k\delta(\omega-k\omega_0),\quad \omega_0=\frac{2\pi}{T}$$

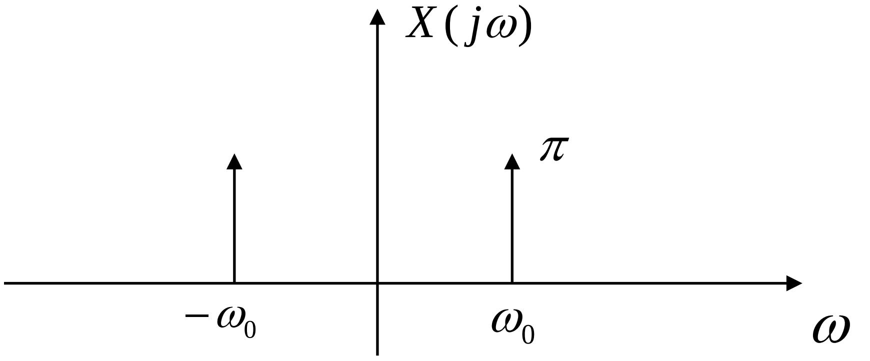
Example: $x(t)=\cos(\omega_0 t)$



① $x(t)=\frac{1}{2}e^{j\omega_0 t}+\frac{1}{2}e^{-j\omega_0 t}$

$\therefore\ a_1=a_{-1}=\frac{1}{2}$

$a_k=0,\quad k\neq\pm1$



② $x(t)\leftrightarrow\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$

Example：

$$\sin\omega_0 t\leftrightarrow j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$$

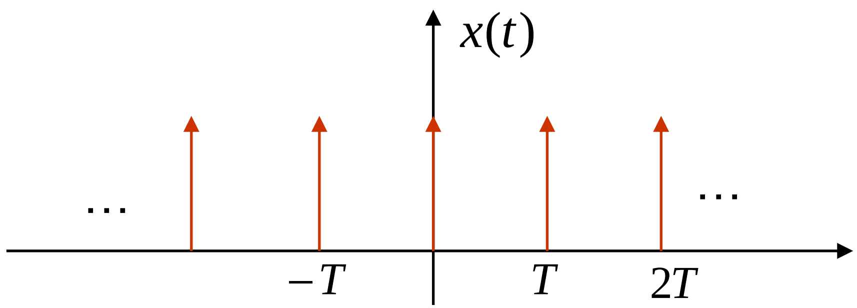
求解a_k的方法：

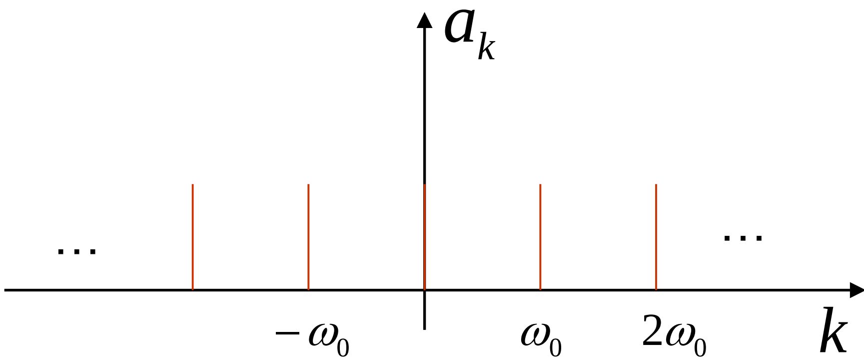
① $$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

② $$a_k = \frac{1}{T}X_0(j\omega)\big|_{\omega=k\omega_0}$$

其中，$x_0(t)$为主周期信号，且$x_0(t) \leftrightarrow X_0(j\omega)$

Example: $x(t)=\sum_{k=-\infty}^{\infty}\delta(t-kT)$



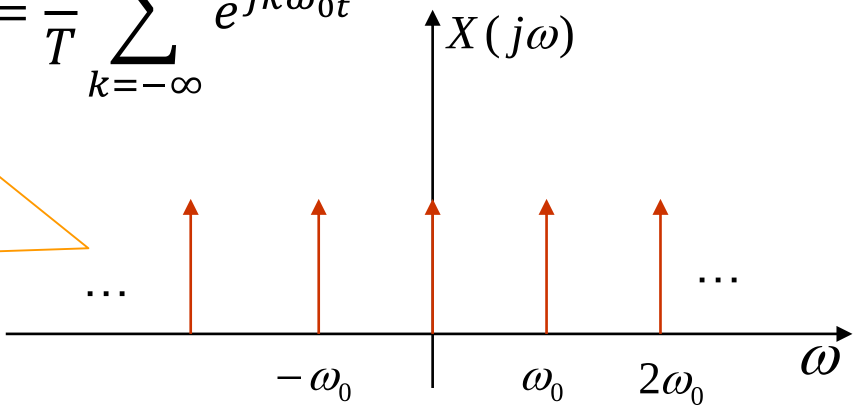


$\because a_k=\frac{1}{T}X_0(j\omega)|_{\omega=k\omega_0}=\frac{1}{T}$

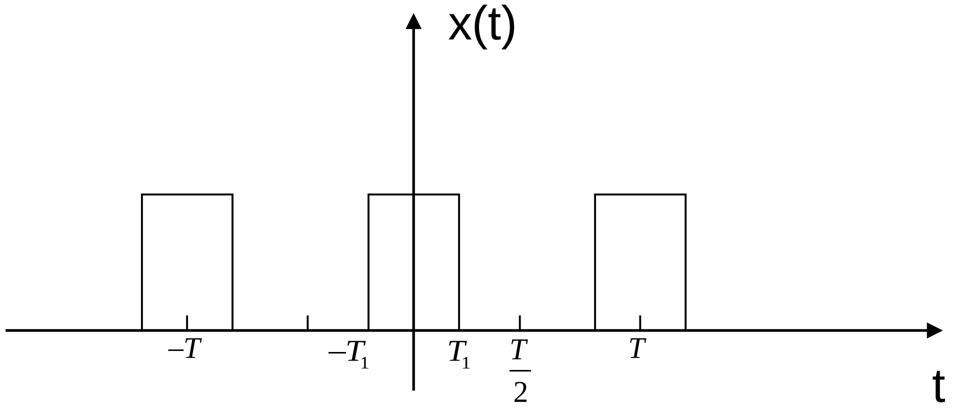
$\therefore x(t)=\frac{1}{T}\sum_{k=-\infty}^{\infty}e^{jk\omega_0 t}$

$$\therefore \sum_k \delta(t-kT)\leftrightarrow \frac{2\pi}{T}\sum_k \delta\left(\omega-k\frac{2\pi}{T}\right)$$

$$=\omega_0\sum_k \delta(\omega-k\omega_0)$$



Example：



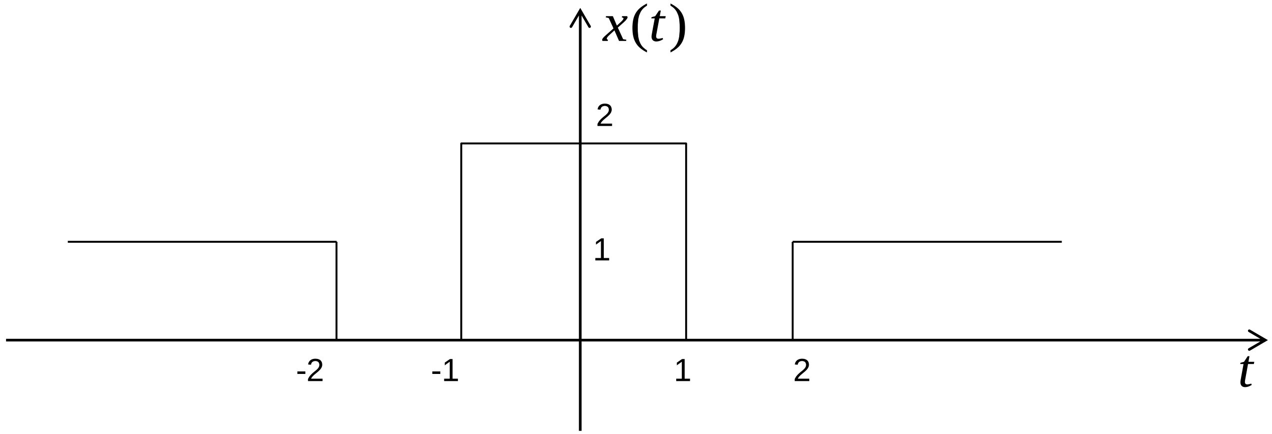
$$X_0(j\omega)=2\frac{\sin(\omega T_1)}{\omega}$$

$$a_k=\frac{1}{T}X_0(j\omega)\Big|_{\omega=k\omega_0}=\frac{\sin(k\omega_0 T_1)}{k\pi}$$

$$x(t)\leftrightarrow 2\pi\sum\frac{\sin(k\omega_0 T_1)}{k\pi}\delta(\omega-k\omega_0)$$

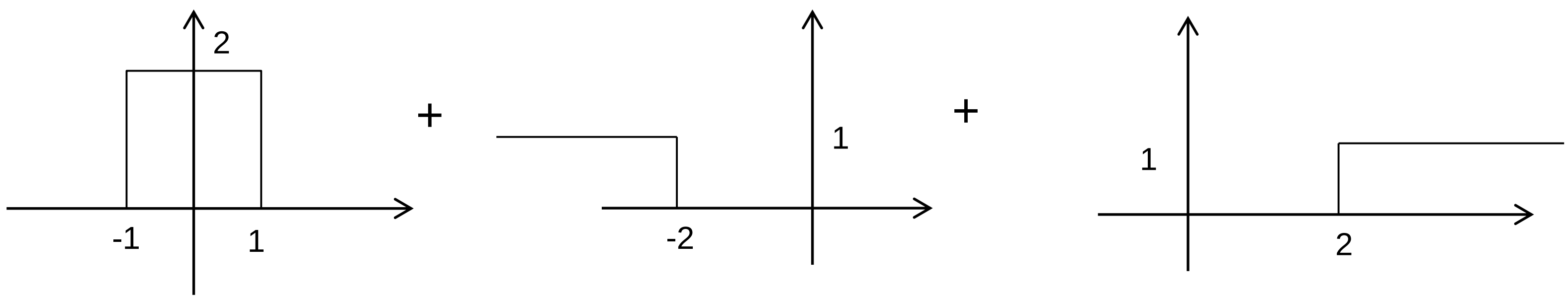
注：利用"常见信号"+"性质"求解傅里叶变换！

Example：求如图 $x(t)$ 对应的$X(j\omega)$



解法1、

$$x(t)=2[u(t+1)-u(t-1)]+u(-t-2)+u(t-2)$$

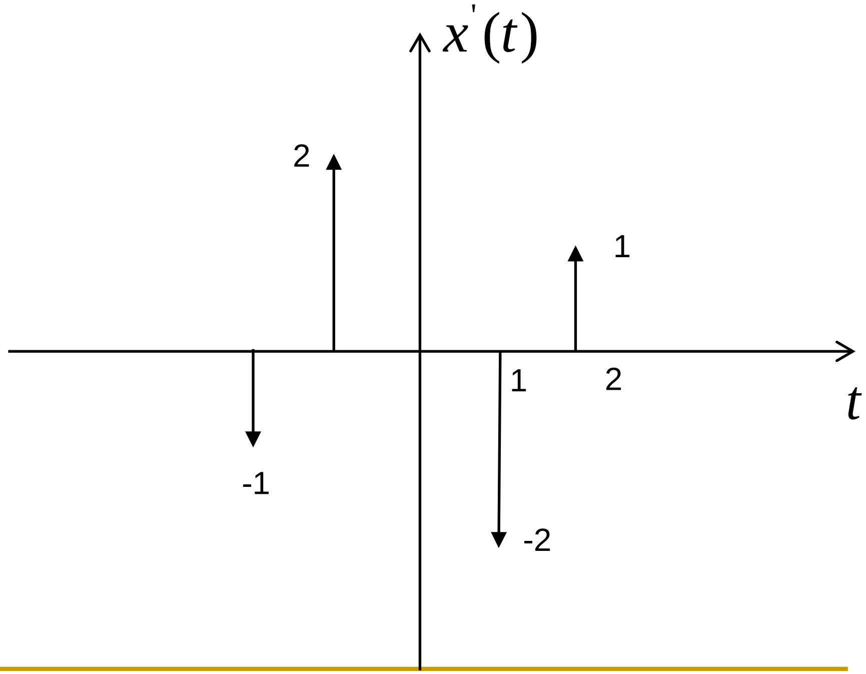


解法2：

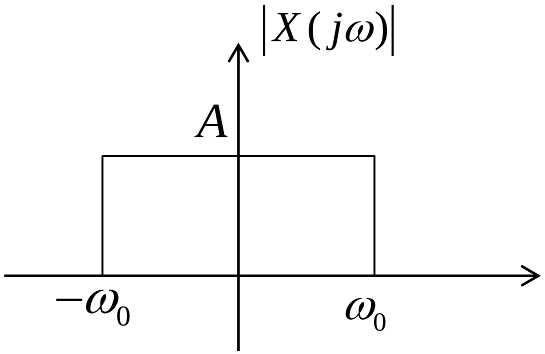
设 $x_1(t) = x^{'}(t)$

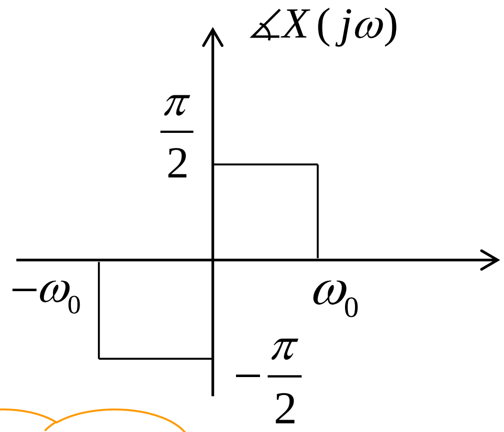
$$x_1(t) \leftrightarrow X_1(j\omega)$$

则 $$X(j\omega) = \frac{X_1(j\omega)}{j\omega} + \pi X_1(0)\delta(\omega) + 2\pi x(-\infty)\delta(\omega)$$



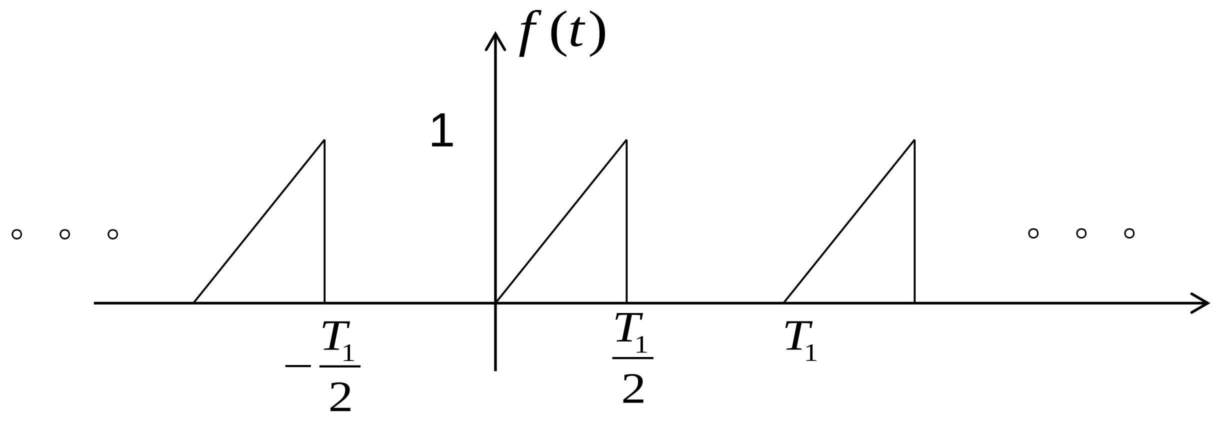
Example：求如图 $X(j\omega)$ 对应的 $x(t)$



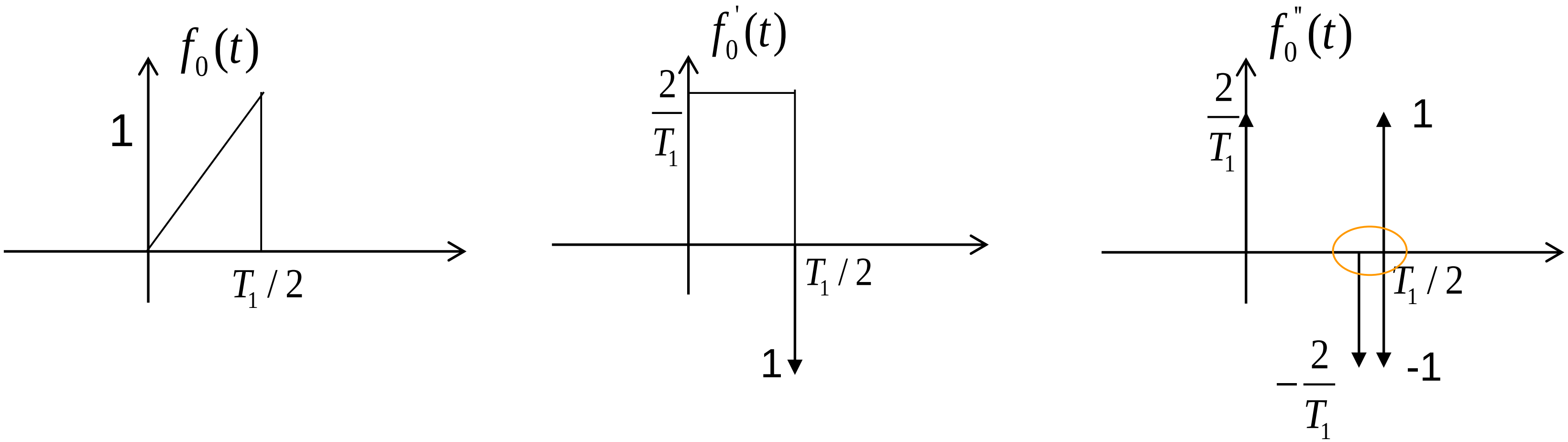


$$X(j\omega) = |X(j\omega)| e^{j\measuredangle X(j\omega)}$$

Example：利用傅里叶变换求周期信号傅里叶级数的系数



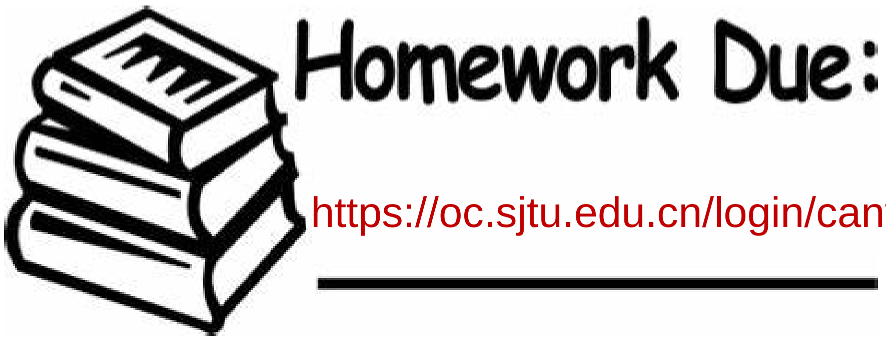
①设 $f_0(t) \leftrightarrow F_0(\omega)$ 为主周期信号



$$f_0^{''}(t) = \frac{2}{T_1}\delta(t) - \frac{2}{T_1}\delta(t - \frac{T_1}{2}) - \delta^{'}(t - \frac{T_1}{2}) \leftrightarrow F_2(\omega)$$

② $F_n = \frac{1}{T_1} F_0(\omega)\big|_{\omega = n\omega_1}$ $\qquad F_0(\omega) = \frac{1}{(j\omega)^2} F_2(\omega)$

Q & A



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