第 5-7 周作业

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关键词： 词 1，词 2

Homework (Week 5-7)

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(p.38) 5. 解: (1) $K_4 = \{(1), (12)(34), (13)(24), (14)(23)\} = K_4(1) = K_4$,

$(12)K_4 = \{(12), (34), (1324), (1423)\} = K_4(12)$,

$(13)K_4 = \{(13), (1234), (24), (1432)\} = K_4(13)$,

$(23)K_4 = \{(23), (1342), (1243), (14)\} = K_4(23)$,

$(123)K_4 = \{(123), (134), (243), (142)\} = K_4(123)$,

$(132)K_4 = \{(132), (234), (124), (143)\} = K_4(132)$.

$\Rightarrow S_4 = K_4 \cup (12)K_4 \cup (13)K_4 \cup (23)K_4 \cup (123)K_4 \cup (132)K_4$ (无交并)

$\Rightarrow$ S_3 是 K_4 在 S_4 中的一个左陪集代表元系.

由上面的计算，易得：$\forall \sigma \in S_4$，有 $\sigma K_4 = K_4 \sigma$. $\Leftrightarrow \sigma K_4 \sigma^{-1} \in K_4$.

$\therefore K_4 \lhd S_4$.

6. 证: $HK = \bigcup_{h\in H} hK$, $|hK| \overset{\text{左消去律}}{=\!=\!=} |K|$. 只需求有多少个不同的"$hK, h\in H$".

因 $h_1K = h_2K \Leftrightarrow h_1^{-1}h_2 \in K \overset{h_1,h_2\in H}{\Longleftrightarrow} h_1^{-1}h_2 \in K\cap H$

$\Leftrightarrow h_1(H\cap K) = h_2(H\cap K)$, 故共有 $[H: H\cap K] = \frac{|H|}{|H\cap K|}$ 个"hK".

$\Rightarrow |HK| = [H: H\cap K]\,|hK| = \frac{|H||K|}{|H\cap K|}$. ∎

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(p.47) 1. 证: $g^{o(g)} = e \Rightarrow e' = f(g^{o(g)}) = (f(g))^{o(g)} \Rightarrow o(f(g)) \mid o(g)$.

2. 证: $f^{-1}(f(M)) = \{ g \in G \mid f(g) \in f(M) \}$.

$\forall\, km \in KM$ ($k \in \mathrm{Ker} f$, $m \in M$), 有 $f(km) = f(k)f(m) = f(m) \in f(M)$, $\therefore KM \subseteq f^{-1}(f(M))$.

$\forall\, x \in f^{-1}(f(M))$, 有 $f(x) \in f(M)$, $\overset{\text{i.e.}}{\Longleftrightarrow} \exists\, m \in M : f(x) = f(m)$

$\Rightarrow f(x)(f(m))^{-1} = e_H \Rightarrow xm^{-1} \in \mathrm{Ker} f \Rightarrow x \in MK$.

$\therefore f^{-1}(f(M)) \subseteq MK. \quad \Rightarrow f^{-1}(f(M)) = MK$.

3. 证: $N \lhd G \Longleftrightarrow gNg^{-1} \overset{\subseteq}{\in} N \ (\forall g \in G)$

$\overset{M \leq G}{\Longrightarrow} mNm^{-1} \subseteq N \ (\forall\, m \in M \leq G) \Longleftrightarrow N \lhd M$.

4. 证: 定义 $f: GL_n(\mathbb{R}) \to \mathbb{R}^*$, $x \mapsto \det(x)$ (x的行列式).

显然f是映射. $\forall\, r \in \mathbb{R}^*$, $\exists\, x = \begin{bmatrix} r & 0 & & \\ 0 & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}_{n\times n} \in GL_n(\mathbb{R}) : \overset{f(x)}{\cancel{|x|}} = r$.

$\Rightarrow f$是满射. $\forall\, x, y \in GL_n(\mathbb{R})$: $f(\underset{\sim}{xy}) = \det(xy) = f(x)f(y)$.
矩阵乘法, 是群$(GL_n(\mathbb{R}), \cdot)$的运算

$\Rightarrow f$是同态. $\Rightarrow f$是$GL_n(\mathbb{R}) \to \mathbb{R}^*$的满同态.

因 $\mathrm{Ker} f := \{ x \in GL_n(\mathbb{R}) \mid \det(x) = 1 \} =: SL_n(\mathbb{R})$, 由群同态基本定理得 $GL_n(\mathbb{R}) / SL_n(\mathbb{R}) = GL_n(\mathbb{R}) / \mathrm{Ker} f \cong \mathrm{Im} f = \mathbb{R}^*$.

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(p.47) 5. 证：若 $G/C(G)$ 是循环群，则 $\exists g\in G$: $G/C(G) = \langle gC(G)\rangle$.

$\therefore$ $\forall a,b\in G$, $\exists m,n\in \mathbb{Z}$: $aC(G) = g^m C(G)$, $bC(G) = g^n C(G)$

$\Leftrightarrow$ $\exists c_1, c_2\in C(G)$: $a = g^m c_1$, $b = g^m c_2$

$\Rightarrow$ $ab = (g^m c_1)(g^n c_2) = g^{m+n} c_1c_2 = ba$. $(\forall a,b\in G)$

$\Rightarrow$ G is Abel.

(p.53) 1. 解：定义 f_k : $\mathbb{Z}_{12}\to\mathbb{Z}_{12}$, $\bar{x}\mapsto\overline{kx}$.

则 $\mathrm{Aut}(\mathbb{Z}_{12},+) = \{f_k \mid (k,12)=1\} = \{f_1, f_5, f_7, f_{11}\}$.

$\mathrm{Aut}(\mathbb{Z}_{12},+)$ 的群表：

$\cdot$	f_1	f_5	f_7	f_{11}
f_1	1	5	7	11
f_5	5	1	11	7
f_7	7	11	1	5
f_{11}	11	7	5	1

定义 ϕ : $\mathrm{Aut}(\mathbb{Z}_{12},+)\to K_4$,

$f_1\mapsto e$,

$f_{(5,7,11)}\mapsto (a,b,c$ 的任一三元组$)$

e.g. $f_{(5,7,11)}\mapsto(c,b,a)$. 易知 ϕ 是 $\mathrm{Aut}(\mathbb{Z}_{12},+)\to K_4$ 的同构.

2. $\mathbb{Z}_4=\langle\bar{1}\rangle$, 而 K_4 中除单位元外只有 2 阶元, 不是循环群,

故 $\mathbb{Z}_4$ 与 K_4 不可能同构.

3. 解: $\mathrm{Aut}(\mathbb{Z},+)\cong\mathbb{Z}_2$, $\mathrm{Aut}(\mathbb{Z}_3,+)=\{f_1,f_2\}\cong\mathbb{Z}_2$,

$\Rightarrow$ $\mathrm{Aut}(\mathbb{Z},+)\cong\mathrm{Aut}(\mathbb{Z}_3,+)$, 但 $\mathbb{Z}\ncong\mathbb{Z}_3$.

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(p.53) 4. 证：记 $t := o(g)$, $m := o(g^s)$.

由 $1 = (g^s)^m = g^{sm} \Rightarrow t \mid sm \Rightarrow \frac{t}{(t,s)} \Big| \frac{s}{(t,s)} m$

$\Rightarrow \frac{t}{(t,s)} \Big| m$. [因 $\left(\frac{t}{(t,s)}, \frac{s}{(t,s)}\right) = 1$].

又 $(g^s)^{\frac{t}{(t,s)}} = (g^t)^{\frac{s}{(t,s)}} = 1 \Rightarrow m \Big| \frac{t}{(t,s)}$.

$\therefore\ o(g^s) =: m = \frac{t}{(t,s)} = \frac{o(g)}{(s, o(g))}$.

(i) 由上述结论, $(t,s) = 1 \Rightarrow o(g^s) = \frac{t}{(t,s)} = t$.

$\Rightarrow |\langle g\rangle| = t = |\langle g^s\rangle|$. 而又 $\langle g^s\rangle \subseteq \langle g\rangle$,

故 $\langle g\rangle = \langle g^s\rangle$.

(ii) 已证.

5. 证：(i) $(g_1 g_2)^{[t_1,t_2]} \overset{\text{Abel}}{=\!=} g_1^{[t_1,t_2]} g_2^{[t_1,t_2]} = 1 \Rightarrow o(g_1 g_2) \mid [t_1, t_2]$.

(ii) $(t_1, t_2) = 1 \Rightarrow \exists\ m, n \in \mathbb{Z}:\ mt_1 + nt_2 = 1$, $(n, t_1) = (m, t_2) = 1$.

$\Rightarrow g = g^{mt_1 + nt_2} = \underbrace{g^{mt_1}}_{g_2} \cdot \underbrace{g^{nt_2}}_{g_1} =: g_2 g_1$,

有 $o(g_1) \overset{\text{上题}}{=\!=} \frac{o(g)}{(nt_2, o(g))} = \frac{t_1 t_2}{(nt_2, t_1 t_2)} = \frac{t_1}{(n, t_1)} = t_1$,

$o(g_2) = \frac{o(g)}{(mt_1, o(g))} = \frac{t_1 t_2}{(mt_1, t_1 t_2)} = \frac{t_2}{(m, t_1)} = t_2$.

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(p.53) 6. △ 证：$\forall a^s \in \langle a^{[m,n]}\rangle$，有 $[m,n] \mid s \Rightarrow m \mid s$ 且 $n \mid s$

$\Rightarrow a^s \in \langle a^m\rangle \cap \langle a^n\rangle$. $\therefore \langle a^{[m,n]}\rangle \subseteq \langle a^m\rangle \cap \langle a^n\rangle$.

$\forall a^s \in \langle a^m\rangle \cap \langle a^n\rangle$，有 $m \mid s$ 且 $n \mid s \overset{?}{\Rightarrow} [m,n] \mid s$

[由 (p.53) Q4，$o(a^{[m,n]}) = \dfrac{o(a)}{([m,n], o(a))} = \dfrac{[m,n](m,n)}{([m,n], [m,n](m,n))} = (m,n)$，

$\Rightarrow |\langle a^{[m,n]}\rangle| = (m,n)$.]

(p.57) 1. 证：先证 $Inn(G) \cong G/C(G)$，(prop 7.3, 教材 p.73).

定义 $\psi: G \to Inn(G)$，$g \mapsto \phi_g$，$\forall g \in G$.

其中 $\phi_g: G \to G$，$x \mapsto gxg^{-1}$，$\forall x \in G$. ~~显然~~ 已证 (pp.56-57) ϕ_g 是 G 上的一一变换，Inn(G) 关于变换的乘法作成群，$\phi_{g_1}\phi_{g_2} = \phi_{g_1g_2}$，$(\phi_g)^{-1} = \phi_{g^{-1}}$.

显然 ψ 是满射. 又 $\psi(g_1g_2) = \phi_{g_1g_2} = \phi_{g_1}\phi_{g_2} = \psi(g_1)\psi(g_2)$，$\forall g_1, g_2 \in G$，

$\therefore \psi$ 是 $G \to Inn(G)$ 的满同态. 因 $Ker\,\psi := \{g \in G \mid \phi_g = Id_G\}$

$= \{g \in G \mid gxg^{-1} = x, \forall x \in G\} = \{g \in G \mid gx = xg, \forall x \in G\} = C(G)$.

由群同态基本定理，有 $G/C(G) = G/Ker\,\psi \cong Inn(G)$.

又 (p.47, Q5) 已证："若 $G/C(G)$ 是循环群，则 G 是 Abel 群"

故 Inn(G) 循环 $\Rightarrow G/C(G)$ 循环 $\Rightarrow$ G is Abel. ∎

(p.63) 1. 证: "$\Rightarrow$". G是Abel $\Rightarrow$ G的所有子群都是其正规子群.

[$\forall H \leq G$, $\forall g \in G$: $gH = Hg$, $\Rightarrow$ $H \lhd G$ ($\forall H: H \leq G$)].

~~设~~ $|G|$ 总可分解为 $|G| = p^r m$, 其中 p是素数, ~~r~~, m 是 ~~$r \geq 1$~~ 正整数, $r \in \mathbb{N}_+$, $(p, m) = 1$. 由 Sylow 定理, G有 $p^0, p^1, \cdots, p^r$ 阶子群 (它们都是正规子群, 因 G is Abel). 因 G是单群, 故必有 $m=1$ ~~$r=1$~~, $\Rightarrow$ $r = 0, 1$ ~~$m=1$~~

$\Rightarrow$ $|G| = p$ $\Rightarrow$ G是素数阶群 $\Rightarrow$ G是循环群.

"$\Leftarrow$". $|G| = p$, p素 $\Rightarrow$ G是循环群 $\Rightarrow$ G is Abel.

$\forall H: H \leq G$, 由Lagrange定理 $\Rightarrow$ $|H| \mid |G| = p$ $\Rightarrow$ $|H| = 1$或p

$\Rightarrow$ $H = \{e\}$或G $\Rightarrow$ G无非平凡子群 $\Rightarrow$ G是单群.

2. 证: $\forall g \notin H$: H与gH, Hg是不同的左,右陪集, $\Rightarrow$ $H \cap gH = \emptyset$, $H \cap Hg = \emptyset$. 因$[G:H] = 2$, 故(指数的定义) $G = H \dot{\cup} gH = H \dot{\cup} Hg$.

$\Rightarrow$ $gH = G \backslash H = Hg$ ($\forall g \in G \backslash H$).

显然 $gH = H = Hg$ ($\forall g \in H \leq G$). 故 $\forall g \in G$: $gH = Hg$.

$\overset{\text{def}}{\Longrightarrow}$ $H \lhd G$.

3. 证: $\forall m \in M$, $\forall n \in N$: $\underbrace{mn\overbrace{m^{-1}n^{-1}}^{\in M \lhd G}}_{\in N \lhd G} \in M \cap N = \{e\}$

$\Rightarrow$ $mnm^{-1}n^{-1} = e$ $\Rightarrow$ $mn = nm$ ($\forall m \in M, \forall n \in N$).

为 ?

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References

[1] 刘绍学, 章璞. 近世代数导引 [M]. 1 ed. 北京: 高等教育出版社, 2011.