

Equations of Fluid Motion

ABSTRACT

The objective of this chapter is to establish the equations governing the movement of a stratified fluid in a rotating environment. These equations are then simplified somewhat by taking advantage of the so-called Boussinesq approximation. This chapter concludes by introducing finite-volume discretizations and showing their relation with the budget calculations used to establish the mathematical equations of motion.

3.1 MASS BUDGET

A necessary statement in fluid mechanics is that mass be conserved. That is, any imbalance between convergence and divergence in the three spatial directions must create a local compression or expansion of the fluid. Mathematically, the statement takes the following form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0, \quad (3.1)$$

where ρ is the density of the fluid (in kg/m^3), and (u, v, w) are the three components of velocity (in m/s). All four variables generally vary in the three spatial directions, x and y in the horizontal, z in the vertical, as well as time t .

This equation, often called the *continuity equation*, is classical in traditional fluid mechanics. Sturm (2001, page 4) reports that Leonardo da Vinci (1452–1519) had derived a simplified form of the statement of mass conservation for a stream with narrowing width. However, the three-dimensional differential form provided here was most likely written much later and credit ought probably to go to Leonhard Euler (1707–1783). For a detailed derivation, the reader is referred to Batchelor (1967), Fox and McDonald (1992), or Appendix A of the present text.

Note that spherical geometry introduces additional curvature terms, which we neglect to be consistent with our previous restriction to length scales substantially shorter than the global scale.

3.2 MOMENTUM BUDGET

For a fluid, Isaac Newton's second law "*mass times acceleration equals the sum of forces*" is better stated per unit volume with density replacing mass and, in the absence of rotation ($\Omega = 0$), the resulting equations are called the Navier-Stokes equations. For geophysical flows, rotation is important, and acceleration terms must be augmented as done in (2.20):

$$x: \quad \rho \left(\frac{du}{dt} + f_* w - f v \right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau^{xx}}{\partial x} + \frac{\partial \tau^{xy}}{\partial y} + \frac{\partial \tau^{xz}}{\partial z} \quad (3.2a)$$

$$y: \quad \rho \left(\frac{dv}{dt} + f u \right) = -\frac{\partial p}{\partial y} + \frac{\partial \tau^{xy}}{\partial x} + \frac{\partial \tau^{yy}}{\partial y} + \frac{\partial \tau^{yz}}{\partial z} \quad (3.2b)$$

$$z: \quad \rho \left(\frac{dw}{dt} - f_* u \right) = -\frac{\partial p}{\partial z} - \rho g + \frac{\partial \tau^{xz}}{\partial x} + \frac{\partial \tau^{yz}}{\partial y} + \frac{\partial \tau^{zz}}{\partial z}, \quad (3.2c)$$

where the x -, y -, and z -axes are directed eastward, northward, and upward, respectively, $f = 2\Omega \sin \varphi$ is the Coriolis parameter, $f_* = 2\Omega \cos \varphi$ is the reciprocal Coriolis parameter, ρ is the density, p is the pressure, g is the gravitational acceleration, and the τ terms represent the normal and shear stresses due to friction.

That the pressure force is equal and opposite to the pressure gradient, and that the viscous force involves the derivatives of a stress tensor should be familiar to the student who has had an introductory course in fluid mechanics. Appendix A retraces the formulation of those terms for the student new to fluid mechanics.

The effective gravitational force (sum of true gravitational force and the centrifugal force; see Section 2.2) is ρg per unit volume and is directed vertically downward. So, the corresponding term occurs only in the third equation for the vertical direction.

Because the acceleration in a fluid is not counted as the rate of change in velocity at a fixed location but as the change in velocity of a fluid particle as it moves along with the flow, the time derivatives in the acceleration components, du/dt , dv/dt and dw/dt , consist of both the local time rate of change and the so-called advective terms:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}. \quad (3.3)$$

This derivative is called the *material derivative*.

The preceding equations assume a Cartesian system of coordinates and thus hold only if the dimension of the domain under consideration is much shorter than the earth's radius. On Earth, a length scale not exceeding 1000 km is usually acceptable. The neglect of the curvature terms is in some ways analogous to the distortion introduced by mapping the curved earth's surface onto a plane.

Should the dimensions of the domain under consideration be comparable with the size of the planet, the x -, y -, and z -axes need to be replaced by spherical

coordinates, and curvature terms enter all equations. See Appendix A for those equations. For simplicity in the exposition of the basic principles of geophysical fluid dynamics, we shall neglect throughout this book the extraneous curvature terms and use Cartesian coordinates exclusively.

Equations (3.2a) through (3.2c) can be viewed as three equations providing the three velocity components, u , v , and w . They implicate, however, two additional quantities, namely, the pressure p and the density ρ . An equation for ρ is provided by the conservation of mass (3.1), and one additional equation is still required.

3.3 EQUATION OF STATE

The description of the fluid system is not complete until we also provide a relation between density and pressure. This relation is called the *equation of state* and tells us about the nature of the fluid. To go further, we need to distinguish between air and water.

For an incompressible fluid such as pure water at ordinary pressures and temperatures, the statement can be as simple as $\rho = \text{constant}$. In this case, the preceding set of equations is complete. In the ocean, however, water density is a complicated function of pressure, temperature, and salinity. Details can be found in Gill (1982, Appendix 3), but for most applications, it can be assumed that the density of seawater is independent of pressure (incompressibility) and linearly dependent upon both temperature (warmer waters are lighter) and salinity (saltier waters are denser) according to:

$$\rho = \rho_0[1 - \alpha(T - T_0) + \beta(S - S_0)], \quad (3.4)$$

where T is the temperature (in degrees Celsius or Kelvin), and S is the salinity (defined in the past as grams of salt per kilogram of seawater, i.e., in parts per thousand, denoted by ‰, and more recently by the so-called practical salinity unit “psu,” derived from measurements of conductivity and having no units). The constants ρ_0 , T_0 , and S_0 are reference values of density, temperature, and salinity, respectively, whereas α is the coefficient of thermal expansion, and β is called, by analogy, the coefficient of saline contraction¹. Typical seawater values are $\rho_0 = 1028 \text{ kg/m}^3$, $T_0 = 10^\circ\text{C} = 283 \text{ K}$, $S_0 = 35$, $\alpha = 1.7 \times 10^{-4} \text{ K}^{-1}$, and $\beta = 7.6 \times 10^{-4}$.

For air, which is compressible, the situation is quite different. Dry air in the atmosphere behaves approximately as an ideal gas, and so we write:

$$\rho = \frac{p}{RT}, \quad (3.5)$$

¹The latter expression is a misnomer, since salinity increases density not by contraction of the water but by the added mass of dissolved salt.

where R is a constant, equal to $287 \text{ m}^2 \text{ s}^{-2} \text{ K}^{-1}$ at ordinary temperatures and pressures. In the preceding equation, T is the absolute temperature (temperature in degrees Celsius + 273.15).

Air in the atmosphere most often contains water vapor. For moist air, the preceding equation is generalized by introducing a factor that varies with the *specific humidity* q :

$$\rho = \frac{p}{RT(1 + 0.608q)} \quad (3.6)$$

The specific humidity q is defined as

$$q = \frac{\text{mass of water vapor}}{\text{mass of air}} = \frac{\text{mass of water vapor}}{\text{mass of dry air} + \text{mass of water vapor}} \quad (3.7)$$

For details, the reader is referred to Curry and Webster (1999).

Unfortunately, our set of governing equations is not yet complete. Although we have added one equation, by doing so, we have also introduced additional variables, namely temperature and, depending on the nature of the fluid, either salinity or specific humidity. Additional equations are clearly necessary.

3.4 ENERGY BUDGET

The equation governing temperature arises from conservation of energy. The principle of energy conservation, also known as the first law of thermodynamics, states that the internal energy gained by a parcel of matter is equal to the heat it receives minus the mechanical work it performs. Per unit mass and unit time, we have

$$\frac{de}{dt} = Q - W, \quad (3.8)$$

where d/dt is the material derivative introduced in (3.3), e is the internal energy, Q is the rate of heat gain, and W is the rate of work done by the pressure force onto the surrounding fluid, all per unit mass. The internal energy, a measure of the thermal agitation of the molecules inside the fluid parcel, is proportional to the temperature:

$$e = C_v T$$

where C_v is the heat capacity at constant volume, and T is the absolute temperature. For air at sea-level pressure and ambient temperatures, $C_v = 718 \text{ J kg}^{-1} \text{ K}^{-1}$, whereas for seawater, $C_v = 3990 \text{ J kg}^{-1} \text{ K}^{-1}$.

In the ocean, there is no internal heat source², whereas in the atmosphere release of latent heat by water-vapor condensation or, conversely,

²In most cases, the absorption of solar radiation in the first meters of the upper ocean is treated as a surface flux, though occasionally it must be taken into account as a radiative absorption.

uptake of latent heat by evaporation constitute internal sources. Leaving such complication for more advanced textbooks in dynamical and physical meteorology (Curry & Webster, 1999), the Q term in (3.8) includes only the heat gained by a parcel through its contact with its neighbors through the process of diffusion. Using the Fourier law of heat conduction, we write

$$Q = \frac{k_T}{\rho} \nabla^2 T,$$

where k_T is the thermal conductivity of the fluid, and the Laplace operator ∇^2 is defined as the sum of double derivatives:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

The work done by the fluid is the pressure force (= pressure \times area) multiplied by the displacement in the direction of the force. Counting area times displacement as volume, the work is pressure multiplied by the change in volume and on a per-mass and per-time basis:

$$W = p \frac{dv}{dt},$$

where v is the volume per mass, i.e., $v = 1/\rho$.

With its pieces assembled, Eq. (3.8) becomes

$$\begin{aligned} C_v \frac{dT}{dt} &= \frac{k_T}{\rho} \nabla^2 T - p \frac{dv}{dt} \\ &= \frac{k_T}{\rho} \nabla^2 T + \frac{p}{\rho^2} \frac{d\rho}{dt}. \end{aligned} \quad (3.9)$$

Elimination of $d\rho/dt$ with the continuity Eq. (3.1) leads to:

$$\rho C_v \frac{dT}{dt} + p \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = k_T \nabla^2 T. \quad (3.10)$$

This is the energy equation, which governs the evolution of temperature.

For water, which is nearly incompressible, the divergence term ($\partial u/\partial x + \partial v/\partial y + \partial w/\partial z$) can be neglected (to be shown later), whereas for air, one may introduce the *potential temperature* θ defined as

$$\theta = T \left(\frac{\rho_0}{\rho} \right)^{R/C_v}, \quad (3.11)$$

for which, the physical interpretation will be given later (Section 11.3). Taking its material derivative and using Eqs. (3.5) and (3.9), lead

successively to

$$\begin{aligned}
 C_v \frac{d\theta}{dt} &= \left(\frac{\rho_0}{\rho} \right)^{R/C_v} \left(C_v \frac{dT}{dt} - \frac{RT}{\rho} \frac{d\rho}{dt} \right) \\
 C_v \frac{d\theta}{dt} &= \frac{\theta}{T} \left(C_v \frac{dT}{dt} - \frac{p}{\rho^2} \frac{d\rho}{dt} \right) \\
 \rho C_v \frac{d\theta}{dt} &= k_T \frac{\theta}{T} \nabla^2 T.
 \end{aligned} \tag{3.12}$$

The net effect of this transformation of variables is the elimination of the divergence term.

When k_T is zero or negligible, the right-hand side of the equation vanishes, leaving only

$$\frac{d\theta}{dt} = 0. \tag{3.13}$$

Unlike the actual temperature T , which is subject to the compressibility effect (through the divergence term), the potential temperature θ of an air parcel is conserved in the absence of heat diffusion.

3.5 SALT AND MOISTURE BUDGETS

The set of equations is not yet complete because there is a remaining variable for which a last equation is required: salinity in the ocean and specific humidity in the atmosphere.

For seawater, density varies with salinity as stated in [Eq. \(3.4\)](#). Its evolution is governed by the salt budget:

$$\frac{dS}{dt} = \kappa_S \nabla^2 S, \tag{3.14}$$

which simply states that a seawater parcel conserves its salt content, except for redistribution by diffusion. The coefficient κ_S is the coefficient of salt diffusion, which plays a role analogous to the heat diffusivity k_T .

For air, the remaining variable is specific humidity and, because of the possibility of evaporation and condensation, its budget is complicated. Leaving this matter for more advanced texts in meteorology, we simply write an equation similar to that of salinity:

$$\frac{dq}{dt} = \kappa_q \nabla^2 q, \tag{3.15}$$

which states that specific humidity is redistributed by contact with neighboring parcels of different moisture contents, and in which the diffusion coefficient κ_q is the analog of k_T and κ_S .

3.6 SUMMARY OF GOVERNING EQUATIONS

Our set of governing equations is now complete. For air (or any ideal gas), there are seven variables (u , v , w , p , ρ , T , and q), for which we have a continuity equation (3.1), three momentum equations (3.2a) through (3.2c), an equation of state (3.5), an energy equation (3.10), and a humidity equation (3.15). Vilhelm Bjerknes (see biography at the end of this chapter) is credited for having been the first to recognize that atmospheric physics can, in theory, be fully described by a set of equations governing the evolution of the seven aforementioned variables (Bjerknes, 1904; see also Nebeker, 1995, chapter 5).

For seawater, the situation is similar. There are again seven variables (u , v , w , p , ρ , T , and S), for which we have the same continuity, momentum and energy equations, the equation of state (3.4), and the salt equation (3.14). No particular person is credited with this set of equations.

3.7 BOUSSINESQ APPROXIMATION

Although the equations established in the previous sections already contain numerous simplifying approximations, they are still too complicated for the purpose of geophysical fluid dynamics. Additional simplifications can be obtained by the so-called *Boussinesq approximation* without appreciable loss of accuracy.

In most geophysical systems, the fluid density varies, but not greatly, around a mean value. For example, the average temperature and salinity in the ocean are $T = 4^\circ\text{C}$ and $S = 34.7$, respectively, to which corresponds a density $\rho = 1028 \text{ kg/m}^3$ at surface pressure. Variations in density within one ocean basin rarely exceed 3 kg/m^3 . Even in estuaries where fresh river waters ($S = 0$) ultimately turn into salty seawaters ($S = 34.7$), the relative density difference is less than 3%.

By contrast, the air of the atmosphere becomes gradually more rarefied with altitude, and its density varies from a maximum at ground level to nearly zero at great heights, thus covering a 100% range of variations. Most of the density changes, however, can be attributed to hydrostatic pressure effects, leaving only a moderate variability caused by other factors. Furthermore, weather patterns are confined to the lowest layer, the troposphere (approximately 10 km thick), within which the density variations responsible for the winds are usually no more than 5%.

As it appears justifiable in most instances³ to assume that the fluid density, ρ , does not depart much from a mean reference value, ρ_0 , we take the liberty to write the following:

$$\rho = \rho_0 + \rho'(x, y, z, t) \quad \text{with} \quad |\rho'| \ll \rho_0, \quad (3.16)$$

³The situation is obviously somewhat uncertain on other planets that are known to possess a fluid layer (Jupiter and Neptune, for example), and on the sun.

where the variation ρ' caused by the existing stratification and/or fluid motions is small compared with the reference value ρ_0 . Armed with this assumption, we proceed to simplify the governing equations.

The continuity equation, (3.1), can be expanded as follows:

$$\rho_0 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + \rho' \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + \left(\frac{\partial \rho'}{\partial t} + u \frac{\partial \rho'}{\partial x} + v \frac{\partial \rho'}{\partial y} + w \frac{\partial \rho'}{\partial z} \right) = 0.$$

Geophysical flows indicate that relative variations of density in time and space are not larger than—and usually much less than—the relative variations of the velocity field. This implies that the terms in the third group are on the same order as—if not much less than—those in the second. But, terms in this second group are always much less than those in the first because $|\rho'| \ll \rho_0$. Therefore, only that first group of terms needs to be retained, and we write

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \quad (3.17)$$

Physically, this statement means that conservation of mass has become conservation of volume. The reduction is to be expected because volume is a good proxy for mass when mass per volume (= density) is nearly constant. A hidden implication of this simplification is the elimination of sound waves, which rely on compressibility for their propagation.

The x - and y -momentum equations (3.2a) and (3.2b), being similar to each other, can be treated simultaneously. There, ρ occurs as a factor only in front of the left-hand side. So, wherever ρ' occurs, ρ_0 is there to dominate. It is thus safe to neglect ρ' next to ρ_0 in that pair of equations. Further, the assumption of a Newtonian fluid (viscous stresses proportional to velocity gradients), with the use of the reduced continuity equation, (3.17), permits us to write the components of the stress tensor as

$$\begin{aligned} \tau^{xx} &= \mu \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} \right), & \tau^{xy} &= \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), & \tau^{xz} &= \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \\ \tau^{yy} &= \mu \left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y} \right), & \tau^{yz} &= \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \\ \tau^{zz} &= \mu \left(\frac{\partial w}{\partial z} + \frac{\partial w}{\partial z} \right), \end{aligned} \quad (3.18)$$

where μ is called the coefficient of dynamic viscosity. A subsequent division by ρ_0 and the introduction of the *kinematic viscosity* $\nu = \mu/\rho_0$ yield

$$\frac{du}{dt} + f_* w - f v = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \nabla^2 u \quad (3.19)$$

$$\frac{dv}{dt} + f u = -\frac{1}{\rho_0} \frac{\partial p}{\partial y} + \nu \nabla^2 v. \quad (3.20)$$

The next candidate for simplification is the z -momentum equation, (3.2c). There, ρ appears as a factor not only in front of the left-hand side but also in a product with g on the right. On the left, it is safe to neglect ρ' in front of ρ_0 for the same reason as previously, but on the right, it is not. Indeed, the term ρg accounts for the weight of the fluid, which, as we know, causes an increase of pressure with depth (or, a decrease of pressure with height, depending on whether we think of the ocean or atmosphere). With the ρ_0 part of the density goes a hydrostatic pressure p_0 , which is a function of z only:

$$p = p_0(z) + p'(x, y, z, t) \quad \text{with} \quad p_0(z) = P_0 - \rho_0 g z, \quad (3.21)$$

so that $dp_0/dz = -\rho_0 g$, and the vertical-momentum equation at this stage reduces to

$$\frac{dw}{dt} - f_* u = -\frac{1}{\rho_0} \frac{\partial p'}{\partial z} - \frac{\rho' g}{\rho_0} + \nu \nabla^2 w, \quad (3.22)$$

after a division by ρ_0 for convenience. No further simplification is possible because the remaining ρ' term no longer falls in the shadow of a neighboring term proportional to ρ_0 . Actually, as we will see later, the term $\rho' g$ is the one responsible for the buoyancy forces that are such a crucial ingredient of geophysical fluid dynamics.

Note that the hydrostatic pressure $p_0(z)$ can be subtracted from p in the reduced momentum equations, (3.19) and (3.20), because it has no derivatives with respect to x and y , and is dynamically inactive.

For water, the treatment of the energy equation, (3.8), is straightforward. First, continuity of volume, (3.17), eliminates the middle term, leaving

$$\rho C_v \frac{dT}{dt} = k_T \nabla^2 T.$$

Next, the factor ρ in front of the first term can be replaced once again by ρ_0 , for the same reason as it was done in the momentum equations. Defining the heat kinematic diffusivity $\kappa_T = k_T/\rho_0 C_v$, we then obtain

$$\frac{dT}{dt} = \kappa_T \nabla^2 T, \quad (3.23)$$

which is isomorphic to the salt equation, (3.14).

For seawater, the pair of Eqs. (3.14) and (3.23) for salinity and temperature, respectively, combine to determine the evolution of density. A simplification results if it may be assumed that the salt and heat diffusivities, κ_S and κ_T , can be considered equal. If diffusion is primarily governed by molecular processes, this assumption is invalid. In fact, a substantial difference between the rates of salt and heat diffusion is responsible for peculiar small-scale features, such as salt fingers, which are studied in the discipline called *double diffusion* (Turner, 1973, Chapter 8). However, molecular diffusion generally affects only small-scale processes, up to a meter or so, whereas turbulence regulates diffusion on larger scales. In turbulence, efficient diffusion is accomplished by eddies, which mix salt and heat at equal rates. The values of diffusivity coefficients in most geophysical applications may not be taken as those of molecular diffusion; instead, they should be taken much larger and equal to each other. The corresponding turbulent diffusion coefficient, also called *eddy diffusivity*, is typically expressed as the product of a turbulent eddy velocity with a mixing length (Pope, 2000; Tennekes & Lumley, 1972) and, although there exists no single value applicable to all situations, the value $\kappa_S = \kappa_T = 10^{-2} \text{ m}^2/\text{s}$ is frequently adopted. Noting $\kappa = \kappa_S = \kappa_T$ and combining Eqs. (3.14) and (3.23) with the equation of state (3.4), we obtain

$$\frac{d\rho'}{dt} = \kappa \nabla^2 \rho', \quad (3.24)$$

where $\rho' = \rho - \rho_0$ is the density variation. In sum, the energy and salt conservation equations have been merged into a density equation, which is not to be confused with mass conservation (3.1).

For air, the treatment of the energy equation (3.10) is more subtle, and the reader interested in a rigorous discussion is referred to the article by Spiegel and Spiegel and Veronis (1960). Here, for the sake of simplicity, we limit ourselves to suggestive arguments. First, the change of variable in Eq. (3.11) from actual temperature to potential temperature eliminates the divergence term in Eq. (3.10) and takes care of the compressibility effect. Then, for weak departures from a reference state, the relation between actual and potential temperatures and the equation of state can both be linearized. Finally, assuming that heat and moisture are diffused by turbulent motions at the same rate, we can combine their respective budget into a single equation, (3.24).

In summary, the Boussinesq approximation, rooted in the assumption that the density does not depart much from a mean value, has allowed the replacement of the actual density ρ by its reference value ρ_0 everywhere, except in front of the gravitational acceleration and in the energy equation, which has become an equation governing density variations.

At this point, since the original variables ρ and p no longer appear in the equations, it is customary to drop the primes from ρ' and p' without risk of

ambiguity. So, from here on, the variables ρ and p will be used exclusively to denote the perturbation density and perturbation pressure, respectively. This perturbation pressure is sometimes called the *dynamic pressure* because it is usually a main contributor to the flow field. The only place where total pressure comes into play is the equation of state.

3.8 FLUX FORMULATION AND CONSERVATIVE FORM

The preceding equations form a complete set of equations, and there is no need to invoke further physical laws. Nevertheless, we can manipulate the equations to write them in another form, which, though mathematically equivalent, has some practical advantages. Consider, for example, the equation for temperature (3.23), which was deduced from the energy equation using the Boussinesq approximation. If we now expand its material derivative using Eq. (3.3)

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = \kappa_T \nabla^2 T, \quad (3.25)$$

and then use volume conservation (3.17), which was obtained under the same Boussinesq approximation, we obtain

$$\begin{aligned} \frac{\partial T}{\partial t} + \frac{\partial}{\partial x}(uT) + \frac{\partial}{\partial y}(vT) + \frac{\partial}{\partial z}(wT) \\ - \frac{\partial}{\partial x} \left(\kappa_T \frac{\partial T}{\partial x} \right) - \frac{\partial}{\partial y} \left(\kappa_T \frac{\partial T}{\partial y} \right) - \frac{\partial}{\partial z} \left(\kappa_T \frac{\partial T}{\partial z} \right) = 0. \end{aligned} \quad (3.26)$$

The latter form is called a *conservative formulation*, the reason for which will become clear upon applying the divergence theorem. This theorem, also known as Gauss's Theorem, states that for any vector (q_x, q_y, q_z) , the volume integral of its divergence is equal to the integral of the flux over the enclosing surface:

$$\int_{\mathcal{V}} \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right) dx dy dz = \int_S (q_x n_x + q_y n_y + q_z n_z) dS \quad (3.27)$$

where the vector (n_x, n_y, n_z) is the outward unit vector normal to the surface S delimiting the volume \mathcal{V} (Fig. 3.1). Integrating the conservative form (3.26) over a fixed volume is then particularly simple and leads to an expression for the evolution of the heat content in the volume as a function of the fluxes entering and leaving the volume:

$$\frac{d}{dt} \int_{\mathcal{V}} T dt + \int_S \mathbf{q} \cdot \mathbf{n} dS = 0. \quad (3.28)$$

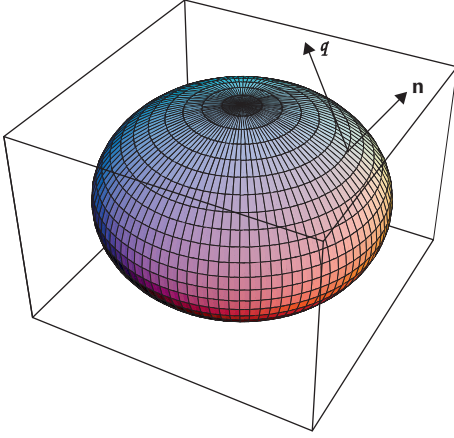


FIGURE 3.1 The divergence theorem allows the replacement of the integral over the volume \mathcal{V} of the divergence $\partial q_x/\partial x + \partial q_y/\partial y + \partial q_z/\partial z$ of a flux vector $\mathbf{q} = (q_x, q_y, q_z)$ by the integral, over the surface \mathcal{S} containing the volume, of the scalar product of the flux vector and the normal vector \mathbf{n} to this surface.

The flux \mathbf{q} of temperature is composed of an advective flux (uT, vT, wT) due to flow across the surface and a diffusive (conductive) flux $-\kappa_T(\partial T/\partial x, \partial T/\partial y, \partial T/\partial z)$. If the value of each flux is known on a closed surface, the evolution of the average temperature inside the volume can be calculated without knowing the detailed distribution of temperature. This property will be used now for the development of a particular discretization method.

3.9 FINITE-VOLUME DISCRETIZATION

The conservative form (3.26) naturally leads to a numerical method with a clear physical interpretation, the so-called *finite-volume* approach. To illustrate the concept, we consider the equation for temperature in a one-dimensional (1D) case

$$\frac{\partial T}{\partial t} + \frac{\partial q}{\partial x} = 0, \quad (3.29)$$

in which the flux q for temperature $T = T(x, t)$ includes both advection uT and diffusion $-\kappa_T \partial T/\partial x$:

$$q = uT - \kappa_T \frac{\partial T}{\partial x}. \quad (3.30)$$

We can integrate (3.29) over a given interval (labeled by index i) with boundaries noted by indices $i - 1/2$ and $i + 1/2$, so that we integrate over x in the range $x_{i-1/2} < x < x_{i+1/2}$. Though the interval of integration is of finite size

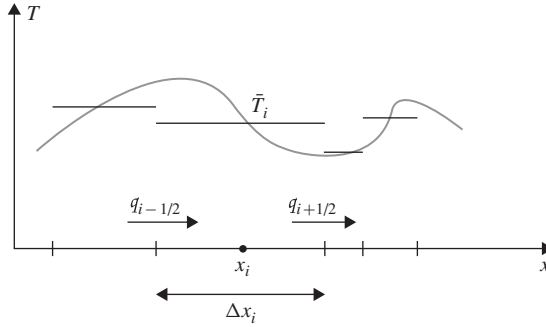


FIGURE 3.2 Replacement of the continuous function T by its cell-averaged discrete values \bar{T}_i . The evolution of the finite-volume averaged temperature is given by the difference of the flux q between the surrounding interfaces at $x_{i-1/2}$ and $x_{i+1/2}$.

$\Delta x_i = x_{i+1/2} - x_{i-1/2}$ (Fig. 3.2), the integration is performed exactly:

$$\frac{d}{dt} \int_{x_{i-1/2}}^{x_{i+1/2}} T dx + q_{i+1/2} - q_{i-1/2} = 0. \quad (3.31)$$

By defining the cell-average temperature \bar{T}_i for cell i as

$$\bar{T}_i = \frac{1}{\Delta x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} T dx, \quad (3.32)$$

we obtain the evolution equation of the discrete field \bar{T}_i :

$$\frac{d\bar{T}_i}{dt} + \frac{q_{i+1/2} - q_{i-1/2}}{\Delta x_i} = 0. \quad (3.33)$$

Although we seem to have fallen back on a *discretization* of the spatial derivative, of q in this instance, the equation we just obtained is *exact*. This seems to be paradoxical compared with our previous discussions on inevitable errors associated with discretization. At first sight, it appears that we found a discretization method without errors, but we must realize that (3.33) is still incomplete in the sense that two different variables appear in a single equation, the discretized average \bar{T}_i and the discretized flux, $q_{i-1/2}$ and $q_{i+1/2}$. These two, however, are related to the *local* value of the continuous temperature field [the advective flux at $x_{i\pm 1/2}$ is $uT(x_{i\pm 1/2}, t)$], whereas the integrated equation is written for the *average* value of temperature. The averaging of the equation prevents us from retrieving information at the local level, and only average values (over the spatial scale Δx_i) can be determined. Therefore, we have to find an

approximate way of assessing the local value of fluxes based solely on average temperature values. We also observe that with the grid size Δx_i we only retain information at scales longer than Δx_i , a property we have already mentioned in the context of aliasing (Section 1.12). The shorter spatial scales have simply been eliminated by the spatial averaging (Fig. 3.2).

A further exact time-integration of (3.33) yields

$$\bar{T}_i^{n+1} - \bar{T}_i^n + \frac{\int_{t^n}^{t^{n+1}} q_{i+1/2} dt - \int_{t^n}^{t^{n+1}} q_{i-1/2} dt}{\Delta x_i} = 0,$$

expressing that the difference in average temperature (i.e., heat content) is given by the net flux entering the finite cell during the given time interval. Again, to this stage, no approximation is needed, and the equation is *exact* and can be formulated in terms of time-averaged fluxes \hat{q}

$$\hat{q} = \frac{1}{\Delta t_n} \int_{t^n}^{t^{n+1}} q dt \quad (3.34)$$

to yield an equation for discrete averaged quantities:

$$\frac{\bar{T}_i^{n+1} - \bar{T}_i^n}{\Delta t_n} + \frac{\hat{q}_{i+1/2} - \hat{q}_{i-1/2}}{\Delta x_i} = 0. \quad (3.35)$$

This equation is still *exact* but to be useful needs to be supplemented with a scheme to calculate the average fluxes \hat{q} as functions of average temperatures \bar{T} . Only at that point are discretization approximations required, and discretization errors introduced.

It is noteworthy also to realize how easy the introduction of nonuniform grid spacing and timestepping has been up to this point. Though we refer the interfaces by index $i \pm 1/2$, the position of an interface does not need to lie at mid-distance between consecutive grid nodes x_i . Only their logical, topological position must be ordered in the sense that grid nodes and interfaces must be interleaved.

Without further investigation of the way average fluxes can be computed, we interpret different discretization methods in relation to the mathematical budget formulation used to establish the governing equations (Fig. 3.3). From brute-force replacement of differential operators by finite differences to the establishment of equations for finite volumes and subsequent discretization of fluxes, all methods aim at replacing the continuous problem by a finite set of discrete equations.

One of the main advantages of the finite-volume approach presented here is its conservation property. Consider the set of integrated equations for

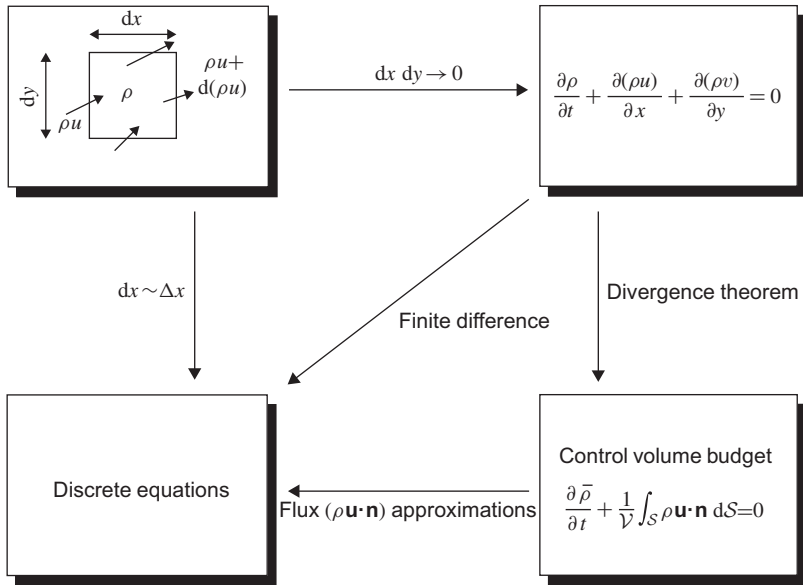


FIGURE 3.3 Schematic representation of several discretization methods. From the budget calculations (upper-left box), the limit to infinitesimal values of dx , dy leads to the continuous equations (upper-right box), whereas keeping differentials formally at finite values lead to crude finite differencing (downward path from upper left to bottom left). If the operators in the continuous equations are discretized using Taylor expansions, higher-quality finite-difference methods are obtained (diagonal path from upper right to lower left). Finally, by preliminary integration of the continuous equations over a finite volume and then discretization of fluxes (path from upper right to lower right and then to lower left), discrete equations satisfying conservation properties can be designed.

consecutive cells:

$$\begin{aligned}
 \Delta x_1 \bar{T}_1^{n+1} &= \Delta x_1 \bar{T}_1^n + \Delta t_n \hat{q}_{1/2} - \Delta t_n \hat{q}_{1+1/2} \\
 &\dots \\
 \Delta x_{i-1} \bar{T}_{i-1}^{n+1} &= \Delta x_{i-1} \bar{T}_{i-1}^n + \Delta t_n \hat{q}_{i-1-1/2} - \Delta t_n \hat{q}_{i-1/2} \\
 \Delta x_i \bar{T}_i^{n+1} &= \Delta x_i \bar{T}_i^n + \Delta t_n \hat{q}_{i-1/2} - \Delta t_n \hat{q}_{i+1/2} \\
 \Delta x_{i+1} \bar{T}_{i+1}^{n+1} &= \Delta x_{i+1} \bar{T}_{i+1}^n + \Delta t_n \hat{q}_{i+1/2} - \Delta t_n \hat{q}_{i+1+1/2} \\
 &\dots \\
 \Delta x_m \bar{T}_m^{n+1} &= \Delta x_m \bar{T}_m^n + \Delta t_n \hat{q}_{m-1/2} - \Delta t_n \hat{q}_{m+1/2}.
 \end{aligned}$$

Since every flux appears in two consecutive equations with opposite sign, the flux leaving a cell enters its neighbor, and there is no loss or gain of the quantity

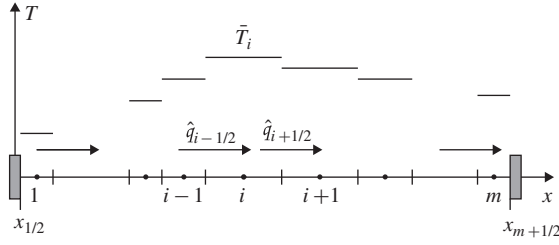


FIGURE 3.4 Within a domain covered by m finite volumes, fluxes at the interface ensure conservation of the relevant property (heat in the case of temperature) between finite volumes, since the fluxes are uniquely defined at interfaces. Fluxes therefore redistribute the property from cell to cell across the domain, without actually changing their total content, except for import and export at the end points. The finite-volume approach easily ensures both local and global conservation.

being transported across cells (heat in the case of temperature). This is an expression of *local conservation* between grid cells (Fig. 3.4).

Furthermore, summation of all equations leads to complete cancellation of the fluxes except for the very first and last ones. What we obtain is none other than the exact expression for evolution of the total quantity. In the case of temperature, this is a global heat budget:

$$\frac{d}{dt} \int_{x_{1/2}}^{x_{m+1/2}} T dx = q_{1/2} - q_{m+1/2} \quad (3.36)$$

which states that the total heat content of the system increases or decreases over time according to the import or export of heat at the extremities of the domain. In particular, if the domain is insulated ($q = 0$ at both boundaries), the total heat content is conserved in the numerical scheme and in the original mathematical model. Moreover, this holds irrespectively of the way by which the fluxes are evaluated from the cell-averaged temperatures, provided that they are uniquely defined at every cell interface $x_{i+1/2}$. Therefore, the finite-volume approach also ensures *global conservation*.

We will show later how advective and diffusive fluxes can be approximated using the cell-averaged discrete values \bar{T}_i , but will have to remember then that the conservative character of the finite-volume approach is ensured simply by using a unique flux estimate at each volume interface.

ANALYTICAL PROBLEMS

3.1. Derive the energy equation (3.10) from Eqs. (3.1) and (3.9).

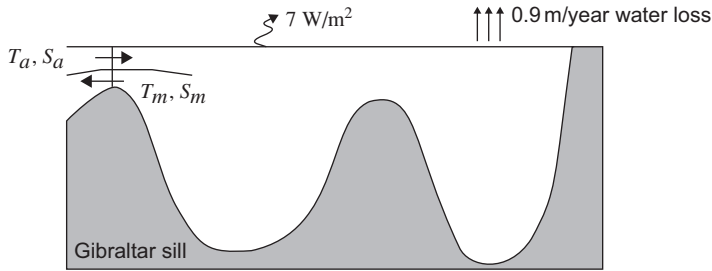


FIGURE 3.5 Schematic representation of the Mediterranean basin and its exchanges across the Strait of Gibraltar in order to perform budget calculations, relating Atlantic Water characteristics T_a, S_a , and losses over the basin, to Mediterranean outflow characteristics T_m, S_m .

- 3.2.** Derive the continuity equation (3.17) from first principles by invoking conservation of volume. (*Hint:* State that the volume in a cube of dimensions $\Delta x \Delta y \Delta z$ is unchanged as fluid is imported and exported through all six sides.)
- 3.3.** A laboratory tank consists of a cylindrical container 30 cm in diameter, filled while at rest with 20 cm of fresh water and then spun at 30 rpm. After a state of solid-body rotation is achieved, what is the difference in water level between the rim and the center? How does this difference compares with the minimum depth at the center?
- 3.4.** Consider the Mediterranean Sea of surface $S = 2.5 \times 10^{12} \text{ m}^2$ over which an average heat loss of 7 W/m^2 is observed. Because of an average surface water loss of 0.9 m/year (evaporation being more important than rain and river runoff combined), salinity would increase, water level would drop, and temperature would decrease, if it were not for a compensation by exchange with the Atlantic Ocean through the Strait of Gibraltar. Assuming that water, salt, and heat contents of the Mediterranean do not change over time and that exchange across Gibraltar is accomplished by a two-layer process (Fig. 3.5), establish sea-wide budgets of water, salt, and heat. Given that the Atlantic inflow is characterized by $T_a = 15.5^\circ\text{C}$, $S_a = 36.2$, and a volume flow of 1.4 Sv ($1 \text{ Sv} = 10^6 \text{ m}^3/\text{s}$), what are the outflow characteristics? Is the outflow at the surface or at the bottom?
- 3.5.** Within the Boussinesq approximation and for negligible diffusion in (3.24), show that for an ocean at rest, density can only be a function of depth: $\rho = \rho(z)$. (*Hint:* The situation at rest is characterized by the absence of movement and temporal variations.)
- 3.6.** Neglecting atmospheric pressure, calculate the pressure $p_0(z) = -\rho_0 g z$ at 500 m depth in the ocean. Compare it with the dynamic pressure of an

ocean at rest of density $\rho = \rho_0 - \rho' e^{z/h}$, where $\rho' = 5 \text{ kg/m}^3$ and $h = 30 \text{ m}$. What do you conclude? Do you think measurements of absolute pressure could be useful in determining the depth of observation?

- 3.7.** In a dry atmosphere where the potential temperature is constant over the vertical, the density $\rho(z)$ can be expressed in terms of the actual temperature $T(z)$ as

$$\rho(z) = \rho_0 \left(\frac{T(z)}{\theta} \right)^{C_v/R} \quad (3.37)$$

according to (3.11). This allows the equation of state (3.5) to be expressed in terms of only pressure $p(z)$ and temperature. By taking the vertical derivative of this expression and using the hydrostatic balance ($dp/dz = -\rho g$), show that the vertical temperature gradient dT/dz is constant. Of what sign is this constant?

NUMERICAL EXERCISES

- 3.1.** Compare values of density obtained with the full equation of state for sea-water found in MATLAB™ file `ies80.m` with values obtained from the linearized version (3.4), for various trial values of T and S . Then, compare density differences between two different water masses, calculated again with both state equations. Finally, using numerical derivatives of the full equation of state with the help of MATLAB™ file `ies80.m`, can you provide a numerical estimate for the expansion coefficients α and β introduced in (3.4) for a Mediterranean water mass of $T_0 = 12.8^\circ\text{C}$, $S_0 = 38.4$?
- 3.2.** Generalize the finite-volume method to a two-dimensional system. In particular, what kind of fluxes do you have to define and how do you interpret them? Is local and global conservation still ensured?
- 3.3.** Derive a conservative form of the momentum equations without friction in spherical coordinates and outline a finite-volume discretization. (*Hint:* Use volume conservation expressed in spherical coordinates and volume integrals in spherical coordinates according to $\int_V u dV = \int_r \int_\lambda \int_\varphi u r^2 \cos \varphi d\varphi d\lambda dr$.)
- 3.4.** Using the finite-volume approach of the one-dimensional temperature evolution assuming that only advection is present, with a flow directed towards increasing x ($u > 0$), discretize the average fluxes. What kind of hypothesis do you need to make to obtain an algorithm allowing you to calculate \bar{T}_i^{n+1} knowing the values of \bar{T} at the preceding time-step?
- 3.5.** For flux calculations, interpolations at the cell interface are generally used. Analyze how a linear interpolation using the two neighbor points behaves

compared with a cubic interpolation using four points. To do so, sample the function e^x at $x = -1.5, -0.5, 0.5, 1.5$ and interpolate at $x = 0$. Compare with the exact value. What happens if you calculated the interpolation not at the center but at $x = 3$ or $x = -3$ (extrapolation)? Redo the exercise but add an alternating error of $+0.1$ and -0.1 to the four sampled values.

Joseph Valentin Boussinesq 1842–1929



Perhaps not as well known as he deserves, Joseph Boussinesq was a French physicist who made significant contributions to the theory of hydrodynamics, vibration, light, and heat. One possible reason for this relative obscurity is the ponderous style of his writings. Among his subjects of study was hydraulics, which led to his research on turbulent flow. In 1896, the work of Osborne Reynolds (see biography at the end of the following chapter) was barely a year old when it was picked up by Boussinesq, who applied the partitioning between average and fluctuating quantities to observations of pipe and river flows. This led him to identify correctly that the cause of turbulence in those instances is friction against boundaries. This paved the way for Ludwig Prandtl's theory of boundary layers (see biography at end of Chapter 8).

It can almost be claimed that the word *turbulence* itself is owed in large part to Boussinesq. Indeed, although Osborne Reynolds spoke of “sinuous motion,” Boussinesq used the more expressive phrase “écoulement tourbillonnant et tumultueux,” which was reduced by one of his followers to “régime turbulent,” hence turbulence. (*Photo from Ambassade de France au Canada*)

Vilhelm Frimann Koren Bjerknes
1862–1951



Early in his career, Bjerknes became interested in applying the then-recent work of Lord Kelvin and Hermann von Helmholtz on energy and vorticity dynamics to motions in the atmosphere and ocean. He argued that the dynamics of air and water flows on geophysical scales could be framed as a problem of physics and that, given a particular state of the atmosphere, one should be able to compute its future states. In other words, weather forecasting is reducible to seek the solution of a mathematical problem. This statement, self-evident today, was quite revolutionary at the time (1904).

When in 1917, he was offered a professorship at the University of Bergen in Norway, Bjerknes founded the Bergen Geophysical Institute and began systematic efforts at developing a self-contained mathematical model for the evolution of weather based on measurable quantities. Faced by the complexity of these equations, he gradually shifted his efforts toward more qualitative aspects of weather description, and out of this work came the now familiar concepts of air masses, cyclones, and fronts.

Throughout his work, Bjerknes projected enthusiasm for his ideas and was able to attract and stimulate young scientists to follow in his footsteps, including his son Jacob Bjerknes. (*Photo courtesy of the Bergen Geophysical Institute*)