Planetary Rossby Waves (Barotropic)

Assumptions: shallow-water model

flat bottom

β-plane approximation: $f = f_0 + \beta_0 y$ $\quad \boxed{\beta_0 = 2(\Omega/a)\cos\varphi_0}$

$$\frac{\beta_0 L}{f_0} \ll 1$$

The linearized governing equations:

$$\frac{\partial u}{\partial t} - (f_0 + \beta_0 y)v = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + (f_0 + \beta_0 y)u = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0,$$

large terms

small terms

To 1st order approximation – geostrophic balance:

$$-f_0 v = -g\frac{\partial \eta}{\partial x}$$
$$f_0 u = -g\frac{\partial \eta}{\partial y}$$

$$\boxed{\begin{aligned} \frac{\partial u}{\partial t} - (f_0 + \beta_0 y)v &= -g\frac{\partial \eta}{\partial x} \\ \frac{\partial v}{\partial t} + (f_0 + \beta_0 y)u &= -g\frac{\partial \eta}{\partial y} \end{aligned}}$$

Subsitute the solutions into the small terms of the governing equations:

$$\begin{aligned} -\frac{g}{f_0}\frac{\partial^2 \eta}{\partial y \partial t} - f_0 v - \frac{\beta_0 g}{f_0} y \frac{\partial \eta}{\partial x} &= -g\frac{\partial \eta}{\partial x} \\ +\frac{g}{f_0}\frac{\partial^2 \eta}{\partial x \partial t} + f_0 u - \frac{\beta_0 g}{f_0} y \frac{\partial \eta}{\partial y} &= -g\frac{\partial \eta}{\partial y} \end{aligned}$$

ageostrophic flow

The solutions for u and v are:

$$\begin{aligned} u &= -\frac{g}{f_0}\frac{\partial \eta}{\partial y} \underline{- \frac{g}{f_0^2}\frac{\partial^2 \eta}{\partial x \partial t} + \frac{\beta_0 g}{f_0^2} y \frac{\partial \eta}{\partial y}} \\ v &= +\frac{g}{f_0}\frac{\partial \eta}{\partial x} \underline{- \frac{g}{f_0^2}\frac{\partial^2 \eta}{\partial y \partial t} - \frac{\beta_0 g}{f_0^2} y \frac{\partial \eta}{\partial x}} \end{aligned}$$

$$u = -\frac{g}{f_0}\frac{\partial \eta}{\partial y} - \frac{g}{f_0^2}\frac{\partial^2 \eta}{\partial x \partial t} + \frac{\beta_0 g}{f_0^2} y \frac{\partial \eta}{\partial y}$$

$$v = +\frac{g}{f_0}\frac{\partial \eta}{\partial x} - \frac{g}{f_0^2}\frac{\partial^2 \eta}{\partial y \partial t} - \frac{\beta_0 g}{f_0^2} y \frac{\partial \eta}{\partial x}$$

Substitution into the continuity equation: $\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$

$$\frac{\partial \eta}{\partial t} - R^2 \frac{\partial}{\partial t}\nabla^2 \eta - \beta_0 R^2 \frac{\partial \eta}{\partial x} = 0 \qquad \boxed{R = \sqrt{gH}/f_0}$$

Apply a plane wave solution $\eta = Ae^{i(kx+ly-\omega t)}$, and the dispersion relation is:

$$\omega = -\beta_0 R^2 \frac{k}{1 + R^2(k^2 + l^2)}$$

$$\omega = -\beta_0 R^2 \frac{k}{1 + R^2(k^2 + l^2)}$$

If $\beta_0 = 0, \quad \omega = 0,$ geostrophic flow

If $R^2K^2 \ll 1, \; L \gg R,$ long wave:

$$\omega \sim \frac{\beta_0 R^2}{L} \ll \beta_0 \mathrm{L} \ll f_0$$

$$\boxed{\frac{\beta_0 L}{f_0} \ll 1}$$

If $R^2K^2 \geq 1, \; L \leq R,$ short wave:

$$\omega \sim \beta_0 \mathrm{L} \ll f_0$$

Planetary Rossby waves are subinertial (low frequency) waves

The zonal wave speed:

$$c = \frac{\omega}{k} = \frac{-\beta_0 R^2}{1 + R^2(k^2 + l^2)} \qquad < 0$$

Planetary Rossby waves always propagate westward in the zonal direction

For very long waves ($R^2 K^2 \ll 1$):

$$c_x = -\beta_0 R^2$$

maximum zonal wave speed

The dispersion relation can be reorganized as:

$$\omega = -\beta_0 R^2 \frac{k}{1 + R^2(k^2 + l^2)}$$

$$(k + \frac{\beta_0}{2\omega})^2 + l^2 = \frac{\beta_0^2}{4\omega^2} - \frac{1}{R^2}$$

$$\omega \leq \frac{\beta_0 R}{2}$$

maximum frequency

The dispersion relation diagram

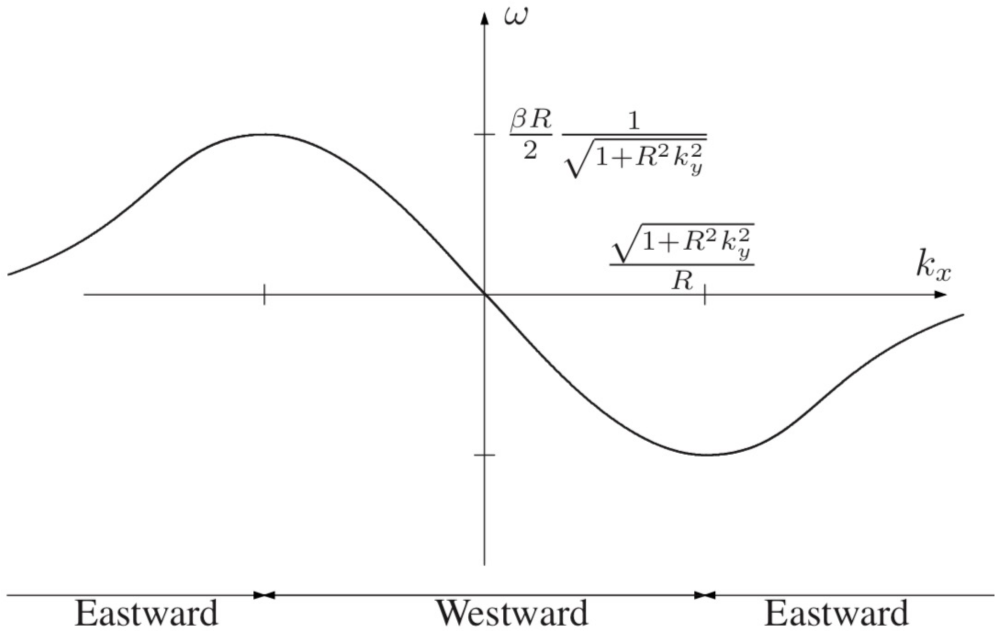
$$\omega = -\beta_0 R^2 \frac{k}{1 + R^2(k^2 + l^2)}$$

$$k = 0, \;\; \omega = 0$$

$$k > 0, \;\; \omega < 0; \quad k < 0, \quad \omega > 0$$

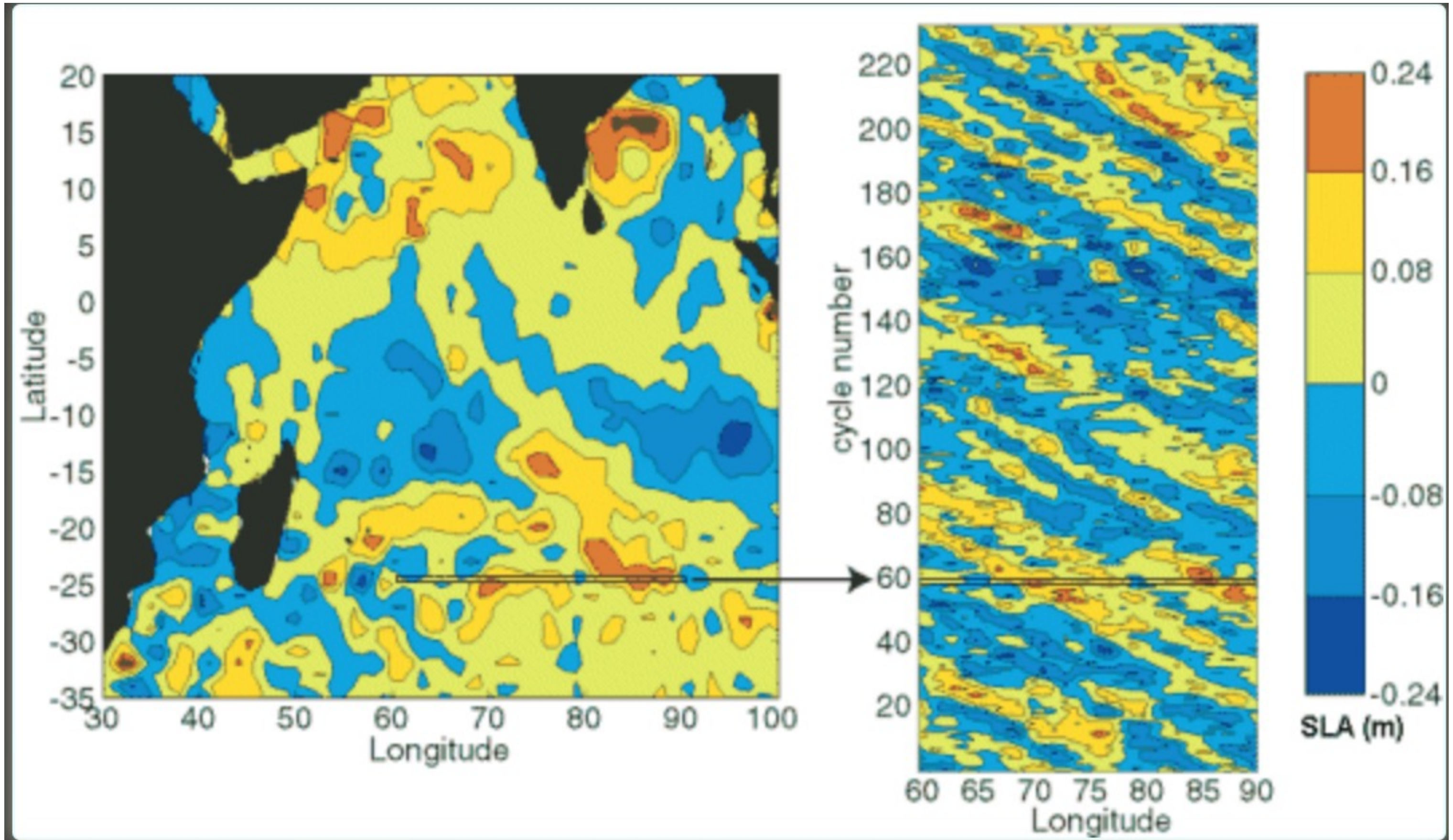
$$k \to \infty, \omega \to 0$$

$$\frac{d\omega}{dk} = 0, \;\; k = \pm \frac{\sqrt{1+R^2 l^2}}{R}, \qquad |\omega| = \frac{\beta R}{2} \frac{1}{\sqrt{1+R^2 l^2}}$$



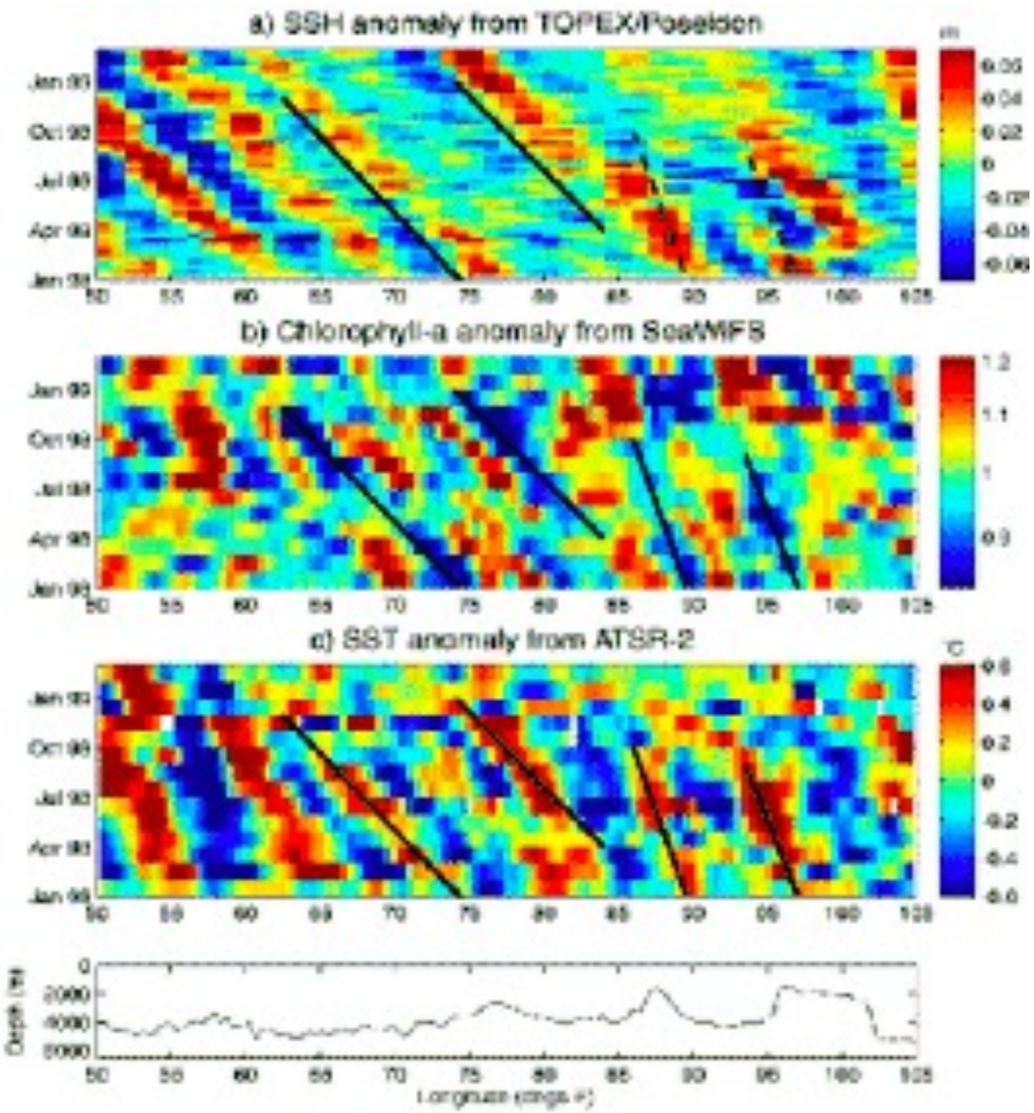
for long Rossby waves, energy propagates westward

for short Rossby waves, energy propagates eastward (opposite to wave propagation)



Map of sea level anomalies, on the left, and longitude-time diagram on the right (the time is indicated in Topex/Poseidon cycles). Diagrams like these bring out the variations of sea level over time along a particular parallel. The elevation at 90°E in cycle 20 (in yellow) can be found about 3 months (10 cycles) later at 85°E, 6 months later at 80°E, and so on. The westward motion of this elevation can be seen on the diagram as a sloped line. Another elevation follows the same path with a 3 month offset, creating a parallel trace. (Credits Southampton Oceanography Center).

https://www.aviso.altimetry.fr/news/idm/2002/may-2002-planetary-waves-small-amplitudes-large-effects.html



SSH

Chl-a

SST

Comparison of longitude-time diagrams from three different sensors (altimeter, water colour, surface temperature). (Credits Southampton Oceanography Center).

Topographic Rossby Waves

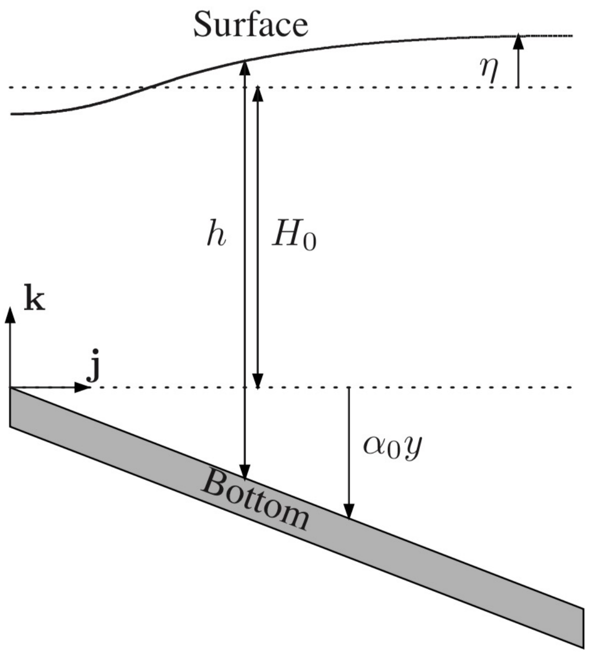
Assumptions: shallow water model

sloping topography (small angle)

$$H = H_0 + \alpha_0 y$$

$$\alpha = \frac{\alpha_0 L}{H_0} \ll 1$$

$$h(x,\, y,\, t) = H_0 + \alpha_0 y + \eta(x,\, y,\, t)$$



The continuity equation: $$\frac{\partial \eta}{\partial t} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} = 0$$

$$\frac{\partial \eta}{\partial t} + \cancel{\left(u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}\right)} + (H_0 + \cancel{\alpha_0 y})\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \eta\cancel{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)} + \alpha_0 v = 0.$$

small-amplitude (linear) waves:

$$\Delta H \ll H$$

The governing equations:

$$\begin{aligned}
\frac{\partial \eta}{\partial t} + H_0 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \alpha_0 v &= 0 \\
\frac{\partial u}{\partial t} - f v &= -g \frac{\partial \eta}{\partial x} \\
\frac{\partial v}{\partial t} + f u &= -g \frac{\partial \eta}{\partial y}
\end{aligned}$$

To 1st order approximation – geostrophic balance:

$$-fv = -g\frac{\partial \eta}{\partial x}$$

$$fu = -g\frac{\partial \eta}{\partial y}$$

Subsitute the solutions into the small terms of the governing equations:

$$\begin{aligned}
u &= -\frac{g}{f}\frac{\partial \eta}{\partial y} - \underline{\frac{g}{f^2}\frac{\partial^2 \eta}{\partial x \partial t}} \\
v &= +\frac{g}{f}\frac{\partial \eta}{\partial x} - \underline{\frac{g}{f^2}\frac{\partial^2 \eta}{\partial y \partial t}}
\end{aligned}$$

ageostrophic flow

$$u = -\frac{g}{f}\frac{\partial \eta}{\partial y} - \frac{g}{f^2}\frac{\partial^2 \eta}{\partial x \partial t}$$
$$v = +\frac{g}{f}\frac{\partial \eta}{\partial x} - \frac{g}{f^2}\frac{\partial^2 \eta}{\partial y \partial t}$$

Substitution in continuity equation: $\frac{\partial \eta}{\partial t} + H_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \alpha_0 v = 0$

$$\frac{\partial \eta}{\partial t} - R^2\frac{\partial}{\partial t}\nabla^2\eta + \frac{\alpha_0 g}{f}\frac{\partial \eta}{\partial x} = 0.$$

Apply a wave solution $\eta = Ae^{i(kx+ly-\omega t)}$:

$$\omega = \frac{\alpha_0 g}{f}\frac{k}{1+R^2(k^2+l^2)}$$

$$\omega = \frac{\alpha_0 g}{f} \frac{k}{1 + R^2(k^2 + l^2)}$$

If $\alpha_0 = 0, \quad \omega = 0,$ gesotrophic flow

$$\alpha = \frac{\alpha_0 L}{H_0} \ll 1$$

If $R^2K^2 \ll 1, \ L \gg R,$ long wave:

$$\omega = \frac{\alpha_0 g k}{f} \sim \frac{\alpha_0 L g}{f L^2} \ll \frac{\alpha_0 L g}{f R^2} (\sim \frac{\alpha_0 L g f^2}{f g H}) \ll f$$

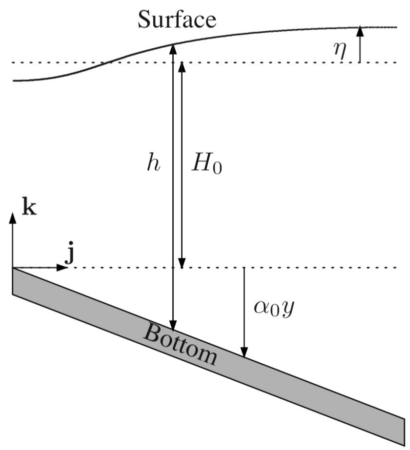
If $R^2K^2 \geq 1, \ L \leq R,$ short wave:

$$\omega = \frac{\alpha_0 g}{f} \frac{k}{R^2K^2} \sim \frac{\alpha_0 g}{f} \frac{L}{R^2} \sim \frac{\alpha_0 L}{f} \frac{g f^2}{g H} \quad \ll f$$

Topographic Rossby waves are also subinertial (low frequency) waves

The zonal wave speed:

$$c = \frac{\omega}{k} = \frac{\alpha_0 g}{f} \frac{1}{1 + R^2(k^2 + l^2)} \quad >0$$



topographic Rossby waves propogate with shallow water on the right (left) in the northern (southern) hemisphere

For very long waves ($R^2K^2 \ll 1$):

$$c_x = \frac{\alpha_0 g}{f} \qquad \text{maximum zonal wave speed}$$

The dispersion relation can be reorganized as:

$$\omega = \frac{\alpha_0 g}{f} \frac{k}{1 + R^2(k^2 + l^2)}$$

$$(k - \frac{\alpha_0 g}{2f\omega R^2})^2 + l^2 = (\frac{\alpha_0^2 g^2}{4f^2R^4\omega^2} - \frac{1}{R^2})$$

$$\omega < \frac{\alpha_0 g}{2fR} \qquad \text{maximum frequency}$$

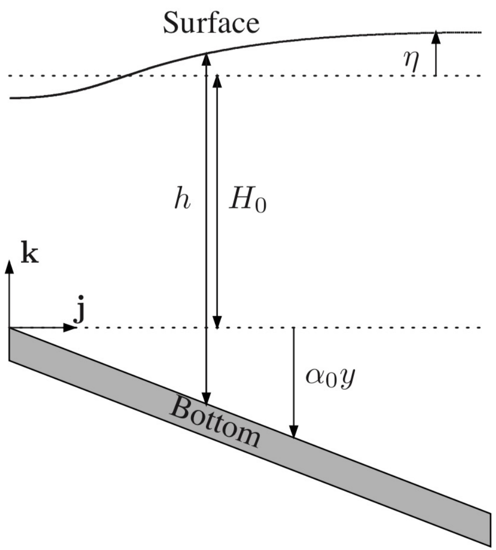
Analogy between planetary and topographic Rossby waves

"North-Shallow" analogy (northern hemisphere)

The potential vorticity:

planetary wave
$$q = \frac{f_0 + \beta_0 y + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}{H_0}$$

toward the north, $y \uparrow$, q increases

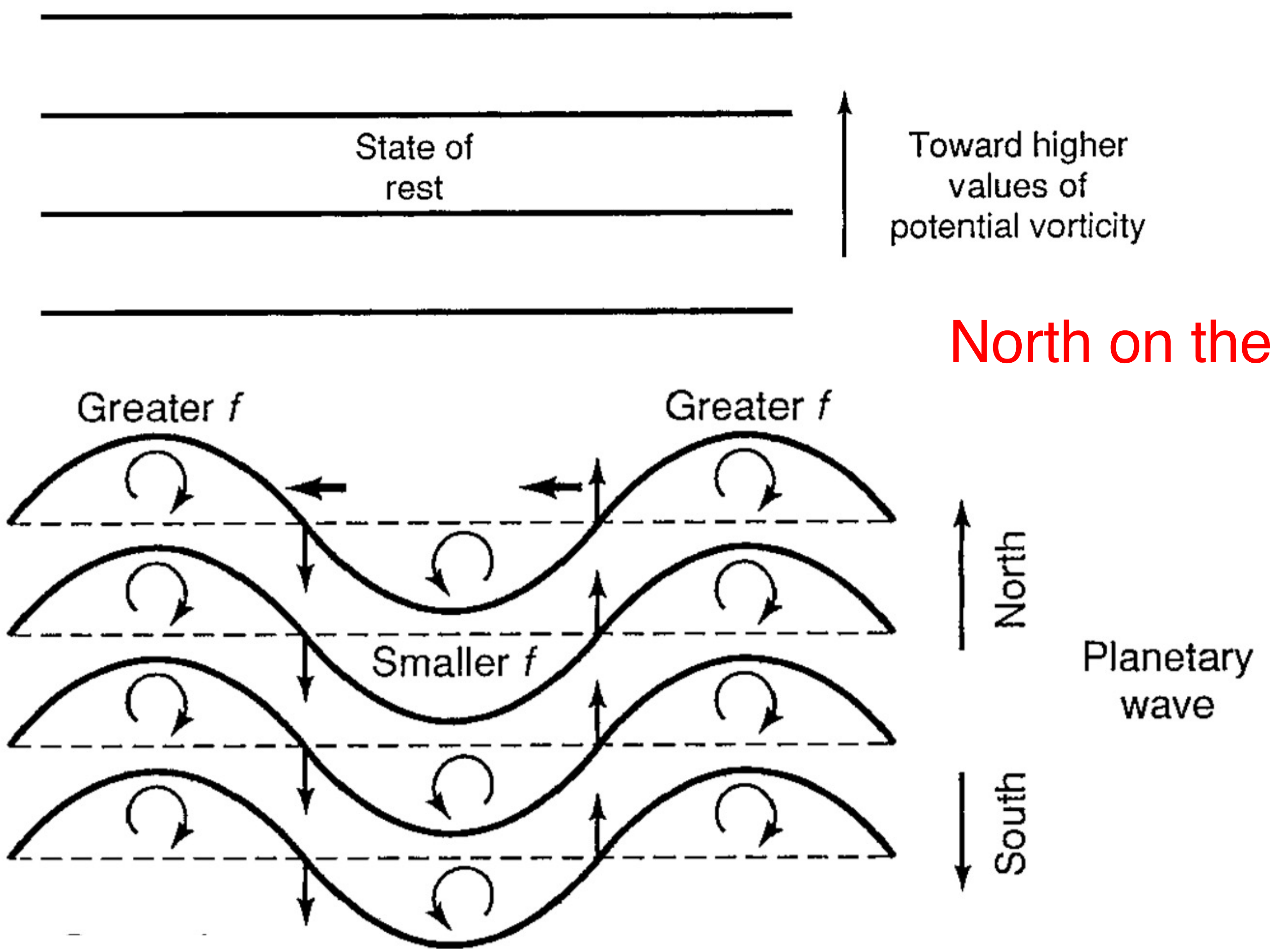


topographic wave
$$q = \frac{f_0 + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}{H_0 + \alpha_0 y + \eta}$$

toward the shallow water, $y \downarrow$, q increases

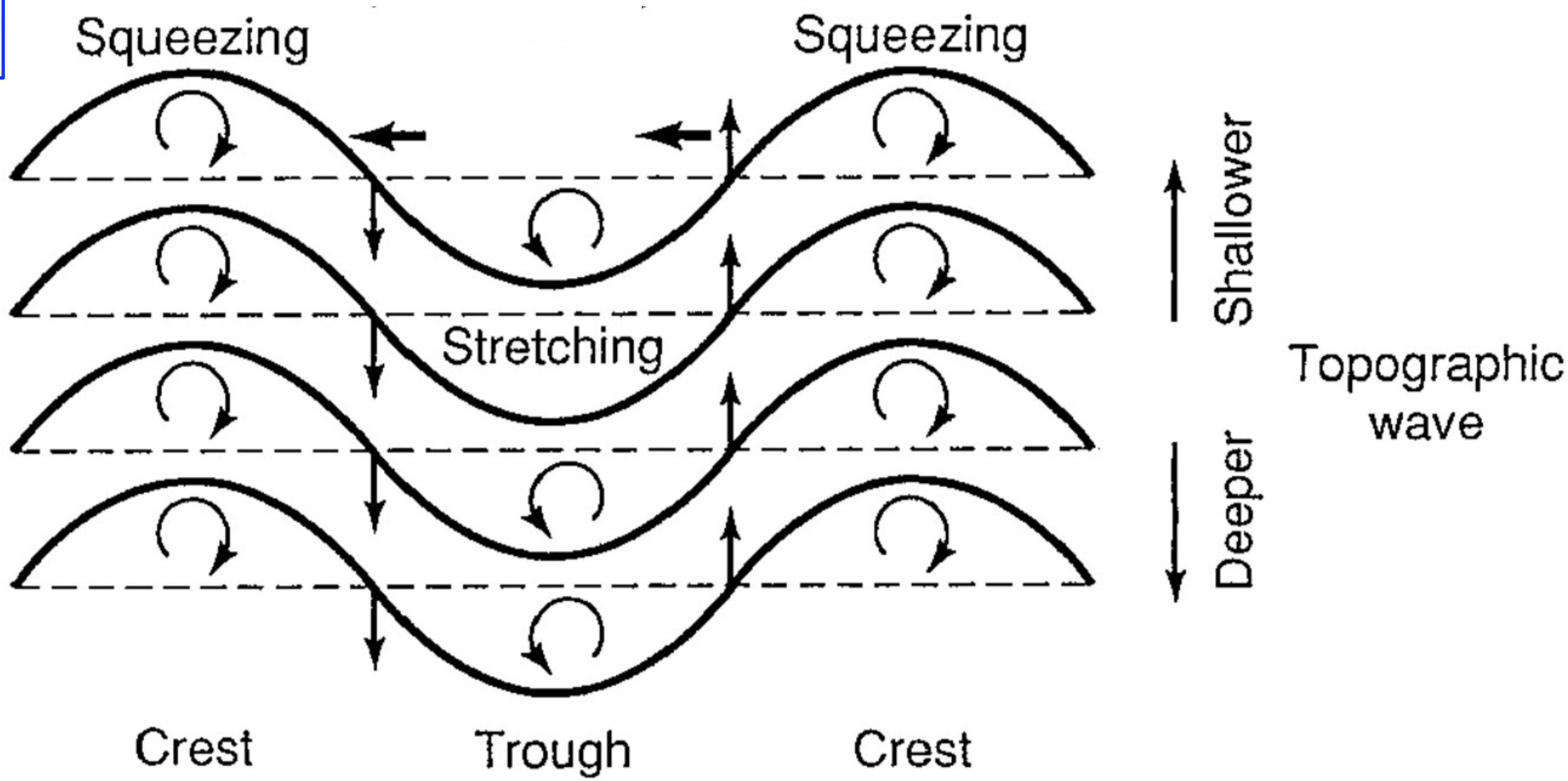
Mechanisms for Rossby Wave propagation

$$\boxed{\frac{d}{dt}\left(\frac{f+\zeta}{h}\right) = 0}$$



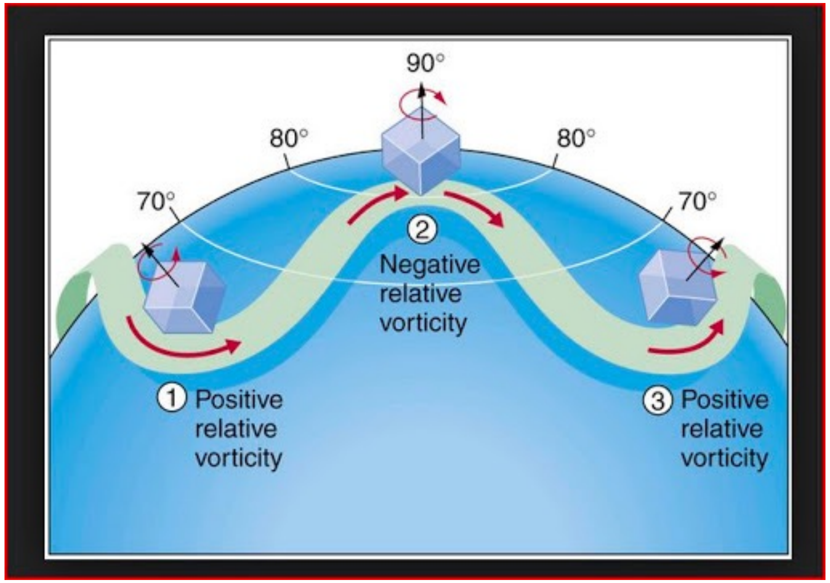
North on the right

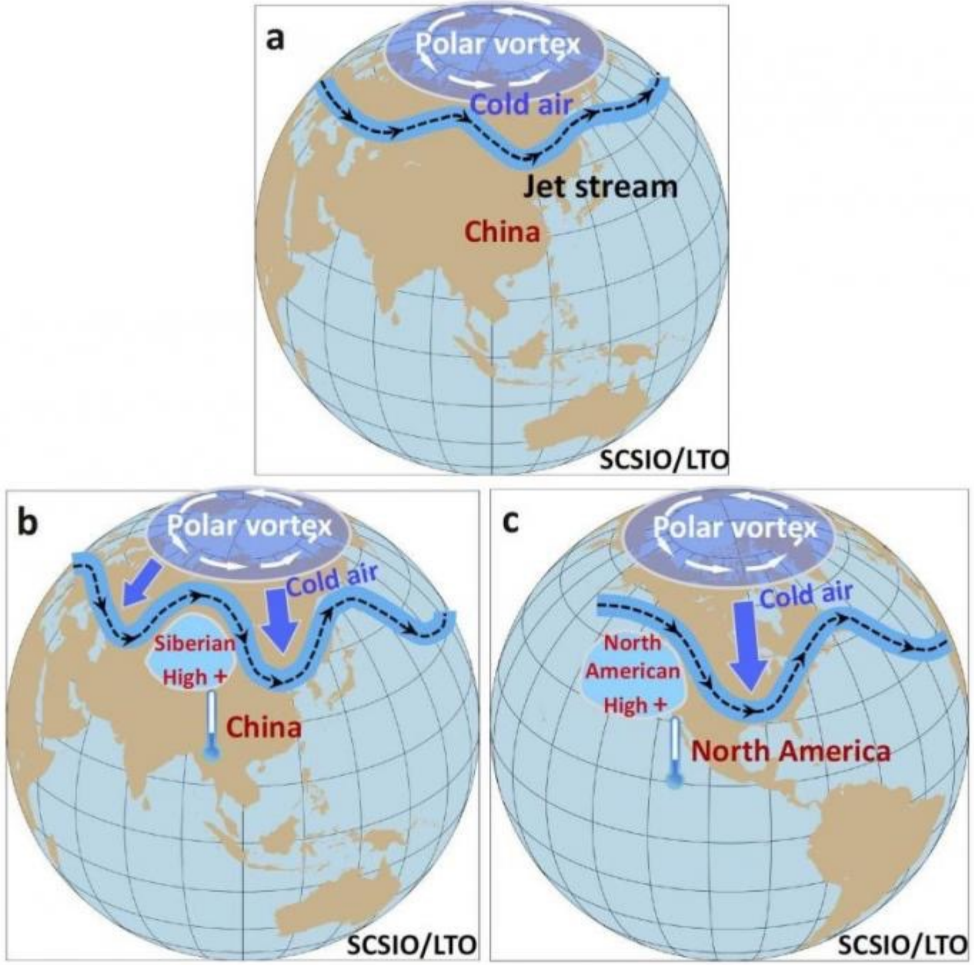
$$\boxed{\frac{d}{dt}\left(\frac{f+\zeta}{h}\right)=0}$$



Shallow-water on the right

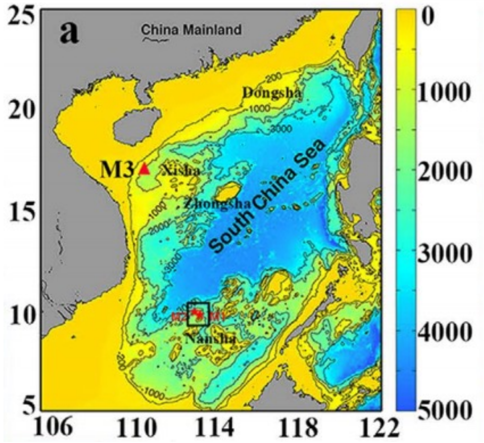
Rossby waves in the atmosphere





‘Bomb cyclone’

Topographic Rossby waves in the South China Sea



Shu et al. (2015)

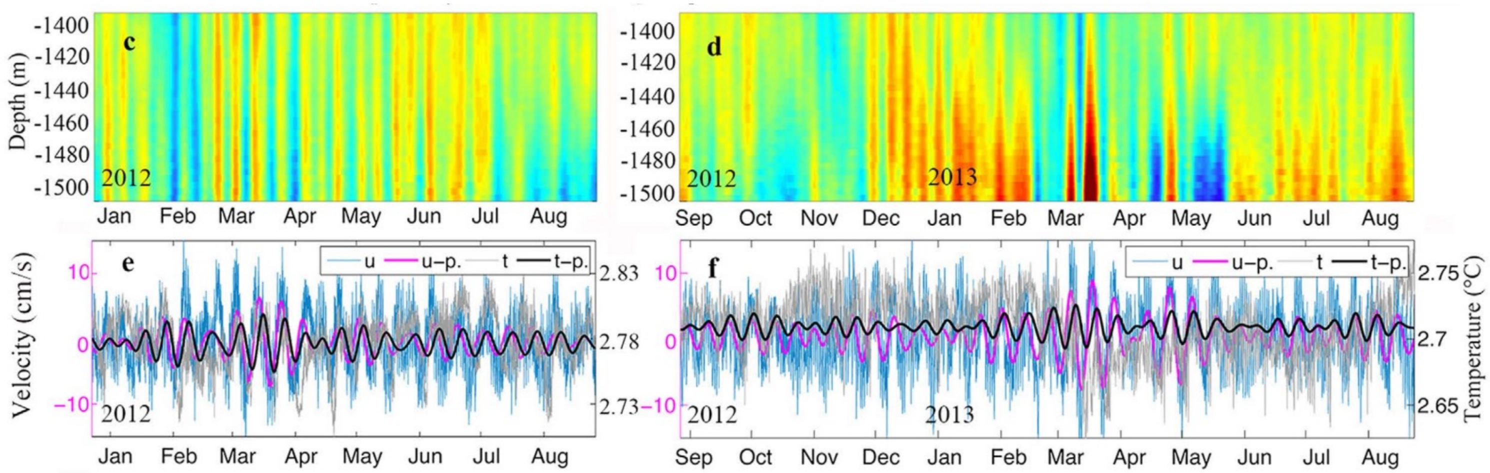


Figure 2. Deep-layer current observations. Observed near-bottom zonal velocity at M1 (unit: cm/s). (**a**–**d**) Time series of 3 day low-pass filtered zonal current profiles from the four segments of ADCP observations. Time series of the raw zonal velocity (cyan line) and temperature (grey line) obtained by the Aanderaa current meter at 1730 m (20 m above the bottom) (**e**) from December 23, 2011 to August 26, 2012 and (**f**) from August 27, 2012 to August 23, 2013. The solid heavy black and magenta lines represent the 9–14 day band-pass filtered zonal velocity and temperature, respectively. Figures were plotted using MATLAB.

Quasi-geostrophic dynamics for stratified fluids

$$\rho = \bar{\rho}(z) + \rho'(x,y,z,t) \quad \text{with} \quad |\rho'| \ll |\bar{\rho}|$$

$$p = \bar{p}(z) + p'(x,y,z,t)$$

Governing equations:

$$\underset{\color{blue}{S}}{\frac{du}{dt}} - \underset{\color{red}{L}}{f_0 v} - \underset{\color{blue}{S}}{\beta_0 y v} = \underset{\color{red}{L}}{-\frac{1}{\rho_0}\frac{\partial p'}{\partial x}}$$

$$\frac{dv}{dt} + f_0 u + \beta_0 y u = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}$$

$$0 = -\frac{\partial p'}{\partial z} - \rho' g$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial \rho'}{\partial t} + u\frac{\partial \rho'}{\partial x} + v\frac{\partial \rho'}{\partial y} + w\frac{d\bar{\rho}}{dz} = 0.$$

Balance of large terms:

$$-f_0 v = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$

$$+f_0 u = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}$$

$$\boxed{\begin{aligned} u_g &= -\frac{1}{f_0\rho_0}\frac{\partial p'}{\partial y} \\ v_g &= +\frac{1}{f_0\rho_0}\frac{\partial p'}{\partial x} \end{aligned}}$$

Plug u_g and v_g into the small terms of the momentum equations, and neglect the vertical advection term:

$$\begin{aligned} \frac{du}{dt} - f_0 v - \beta_0 y v &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} \\ \frac{dv}{dt} + f_0 u + \beta_0 y u &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial y} \end{aligned}$$

$$J(a,b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial b}{\partial x}\frac{\partial a}{\partial y}$$

$$\begin{aligned} -\frac{1}{\rho_0 f_0}\frac{\partial^2 p'}{\partial y \partial t} - \frac{1}{\rho_0^2 f_0^2} J\left(p', \frac{\partial p'}{\partial y}\right) - f_0 v - \frac{\beta_0}{\rho_0 f_0} y \frac{\partial p'}{\partial x} &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} \\ +\frac{1}{\rho_0 f_0}\frac{\partial^2 p'}{\partial x \partial t} + \frac{1}{\rho_0^2 f_0^2} J\left(p', \frac{\partial p'}{\partial x}\right) + f_0 u - \frac{\beta_0}{\rho_0 f_0} y \frac{\partial p'}{\partial y} &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial y} \end{aligned}$$

$$\boxed{\begin{aligned} u_g &= -\frac{1}{f_0 \rho_0}\frac{\partial p'}{\partial y} \\ v_g &= +\frac{1}{f_0 \rho_0}\frac{\partial p'}{\partial x} \end{aligned}}$$

$$\begin{aligned} u = u_g + u_a &= -\frac{1}{\rho_0 f_0}\frac{\partial p'}{\partial y} - \frac{1}{\rho_0 f_0^2}\frac{\partial^2 p'}{\partial t \partial x} \\ &\quad - \frac{1}{\rho_0^2 f_0^3} J\left(p', \frac{\partial p'}{\partial x}\right) + \frac{\beta_0}{\rho_0 f_0^2} y \frac{\partial p'}{\partial y} \\ v = v_g + v_a &= +\frac{1}{\rho_0 f_0}\frac{\partial p'}{\partial x} - \frac{1}{\rho_0 f_0^2}\frac{\partial^2 p'}{\partial t \partial y} \\ &\quad - \frac{1}{\rho_0^2 f_0^3} J\left(p', \frac{\partial p'}{\partial y}\right) - \frac{\beta_0}{\rho_0 f_0^2} y \frac{\partial p'}{\partial x} \end{aligned}$$

Substitution into the continuity equation:

$$\frac{\partial w}{\partial z} = \frac{1}{\rho_0 f_0^2}\left[\frac{\partial}{\partial t}\nabla^2 p' + \frac{1}{\rho_0 f_0} J(p', \nabla^2 p') + \beta_0 \frac{\partial p'}{\partial x}\right]$$

The density equation: $$\frac{\partial \rho'}{\partial t} + u\frac{\partial \rho'}{\partial x} + v\frac{\partial \rho'}{\partial y} + w\frac{\partial \bar{\rho}}{\partial z} = 0$$

$$\boxed{\begin{aligned} u_g &= -\frac{1}{f_0\rho_0}\frac{\partial p'}{\partial y} \\ v_g &= +\frac{1}{f_0\rho_0}\frac{\partial p'}{\partial x} \end{aligned}}$$

Plug u_g and v_g into the density equation:

$$\frac{\partial \rho'}{\partial t} + \frac{1}{\rho_0 f_0}J(p',\rho') - \frac{\rho_0 N^2}{g}w = 0$$

Divide the equation by $\frac{N^2}{g}$, and take the z-derivative:

$$\boxed{0 = -\frac{\partial p'}{\partial z} - \rho' g}$$

$$\frac{\partial}{\partial t\partial z}\left(\frac{g}{N^2}\rho'\right) + \frac{1}{\rho_0 f_0}\left[\frac{\partial p'}{\partial x}\frac{\partial}{\partial z}\left(\frac{g}{N^2}\frac{\partial \rho'}{\partial y}\right) - \frac{\partial p'}{\partial y}\frac{\partial}{\partial z}\left(\frac{g}{N^2}\frac{\partial \rho'}{\partial x}\right)\right] - \rho_0\frac{\partial w}{\partial z} = 0$$

$$-\frac{\partial}{\partial t\partial z}\left(\frac{1}{N^2}\frac{\partial p'}{\partial z}\right) + \frac{1}{\rho_0 f_0}\left\{\frac{\partial p'}{\partial x}\frac{\partial}{\partial y}\left[\frac{\partial}{\partial z}(-\frac{1}{N^2}\frac{\partial p'}{\partial z})\right] - \frac{\partial p'}{\partial y}\frac{\partial}{\partial x}\left[\frac{\partial}{\partial z}(-\frac{1}{N^2}\frac{\partial p'}{\partial z})\right]\right\} - \rho_0\frac{\partial w}{\partial z} = 0$$

$$-\frac{\partial}{\partial t}\left[\frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial p'}{\partial z}\right)\right] - \frac{1}{\rho_0 f_0}J\left(p',\frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial p'}{\partial z}\right)\right) - \rho_0\frac{\partial w}{\partial z} = 0$$

$$\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial}{\partial t}\left[\frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial p'}{\partial z}\right)\right] - \frac{1}{{\rho_0}^2 f_0}J\left(p',\frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial p'}{\partial z}\right)\right)$$

$$\frac{\partial w}{\partial z}=-\frac{1}{\rho_0}\frac{\partial}{\partial t}\left[\frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial p'}{\partial z}\right)\right]-\frac{1}{\rho_0{}^2 f_0}J\left(p',\frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial p'}{\partial z}\right)\right)$$

From the derivations based on the momentum equations:

$$\frac{\partial w}{\partial z}=\frac{1}{\rho_0 f_0{}^2}\left[\frac{\partial}{\partial t}\nabla^2 p'+\frac{1}{\rho_0 f_0}J(p',\nabla^2 p')+\beta_0\frac{\partial p'}{\partial x}\right]$$

$$\frac{\partial}{\partial t}\left[\nabla^2 p'+\frac{\partial}{\partial z}\left(\frac{f_0{}^2}{N^2}\frac{\partial p'}{\partial z}\right)\right]+\frac{1}{\rho_0 f_0}J\left(p',\nabla^2 p'+\frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial p'}{\partial z}\right)\right)+\beta_0\frac{\partial p'}{\partial x}=0$$

The geostrophic flows have streamfunction ψ:

$$u_g=-\frac{1}{\rho_0 f_0}\frac{\partial p'}{\partial y}=-\frac{\partial\psi}{\partial y}$$

$$v_g=\frac{1}{\rho_0 f_0}\frac{\partial p'}{\partial x}=\frac{\partial\psi}{\partial x}$$

$$\Longrightarrow \quad p'=\rho_0 f_0\psi$$

$+\beta_0 y$ $\qquad\qquad$ $J(\psi,\beta_0 y)$

$$\rho_0 f_0\frac{\partial}{\partial t}\left[\nabla^2\psi+\frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right)\right]+\rho_0 f_0 J\left(\psi,\nabla^2\psi+\frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right)\right)+\rho_0 f_0\underline{\beta_0\frac{\partial\psi}{\partial x}}=0$$

$$\rho_0 f_0 \frac{\partial}{\partial t}\left[\nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right) + \beta_0 y\right] + \rho_0 f_0 J\left(\psi, \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right)\right) + \rho_0 f_0 J(\psi, \beta_0 y) = 0$$

ζ planetary vorticity

$$q = \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right) + \beta_0 y$$ potential vorticity

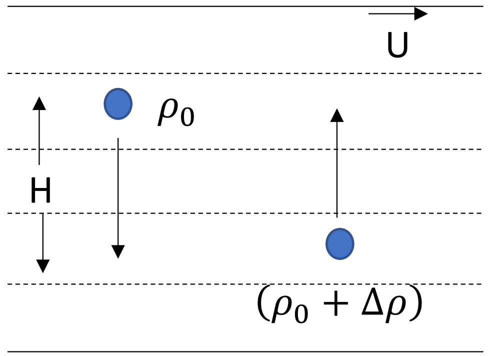
$$\frac{\partial q}{\partial t} + J(\psi, q) = 0$$

$$\frac{\partial q}{\partial t} + \frac{\partial\psi}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial q}{\partial x} = 0$$

$$\frac{\partial q}{\partial t} + u_g\frac{\partial q}{\partial x} + v_g\frac{\partial q}{\partial y} = 0$$

$$\frac{dq}{dt} = 0$$ potential vorticity conservation

The importance of stratification: the Froude number



For per unit volume,

Potential energy change:

$$\Delta PE = (\rho_0 + \Delta\rho) gH - \rho_0 gH = \Delta\rho gH$$

Kinetic energy:

$$KE = \frac{1}{2}\rho_0 U^2 + \frac{1}{2}(\rho_0 + \Delta\rho)\, U^2 \approx \rho_0 U^2$$

$$\sigma = \frac{KE}{\Delta PE} = \frac{\rho_0 U^2}{\Delta\rho gH} \sim \frac{U^2}{N^2 H^2}$$

Froude number: $Fr = \frac{U}{NH}$

$\sigma > 1$, PE change consumes a small portion of the KE of the system, so it takes little cost to break stratification, stratification is unimportant

$\sigma \leq 1$, PE change consumes all KE of the system, or KE is not sufficient to supply ΔPE, stratification cannot be broken and is important relative vorticity planetary vorticity

$$q = \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right) + \beta_0 y$$

$$-\frac{g}{\rho_0 f_0}\frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\rho'\right)$$ vertical stretching

$$0 = -\frac{\partial p'}{\partial z} - \rho' g \qquad p' = \rho_0 f_0 \psi \qquad \frac{\partial\psi}{\partial z} = -\frac{g}{\rho_0 f_0}\rho'$$

relative vorticity: $\frac{U}{L}$ vertical stretching: $\frac{f_0^2 UL}{N^2H^2}$

$$\frac{\frac{U}{L}}{\frac{f_0^2 UL}{N^2H^2}} = \frac{N^2H^2}{f_0^2L^2} = \frac{\frac{U^2}{f_0^2L^2}}{\frac{U^2}{N^2H^2}} = \left(\frac{R_0}{Fr}\right)^2$$

Bu: Burger number

$Bu < 1$, rotation is more important, vertical stretching dominates the PV

$Bu > 1$, stratification is more important, relative vorticity dominates the PV