

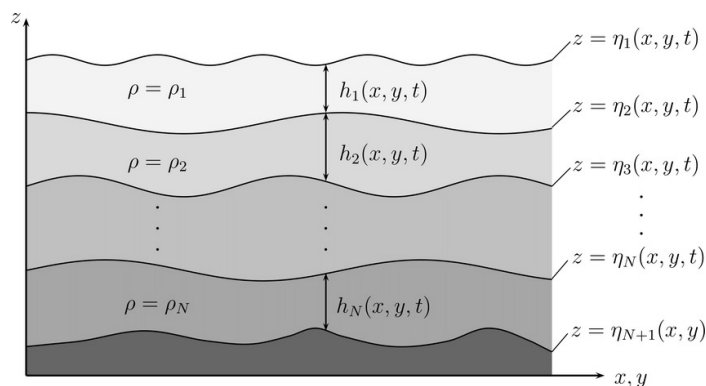
Shallow water equation – layered models

In a stratified system, as density increases monotonically downward, it can be used as the vertical coordinate

$$\rho(x, y, z, t) \longrightarrow z(x, y, \rho, t)$$

$$a = a(x, y, \rho, t)$$

The derivative transformation:



Let $a = z$:

$$0 = z_x + z_\rho \rho_x$$

$$0 = z_y + z_\rho \rho_y$$

$$1 = z_\rho \rho_z$$

$$0 = z_t + z_\rho \rho_t$$

$$\frac{\partial a}{\partial x} \Big|_z = \frac{\partial a}{\partial x} \Big|_\rho + \frac{\partial a}{\partial \rho} \Big|_\rho \frac{\partial \rho}{\partial x} \Big|_z$$

$$\frac{\partial a}{\partial y} \Big|_z = \frac{\partial a}{\partial y} \Big|_\rho + \frac{\partial a}{\partial \rho} \Big|_\rho \frac{\partial \rho}{\partial y} \Big|_z$$

$$\frac{\partial a}{\partial z} \Big|_z = \frac{\partial a}{\partial \rho} \Big|_\rho \frac{\partial \rho}{\partial z} \Big|_z$$

$$\frac{\partial a}{\partial t} \Big|_z = \frac{\partial a}{\partial t} \Big|_\rho + \frac{\partial a}{\partial \rho} \Big|_\rho \frac{\partial \rho}{\partial t} \Big|_z$$

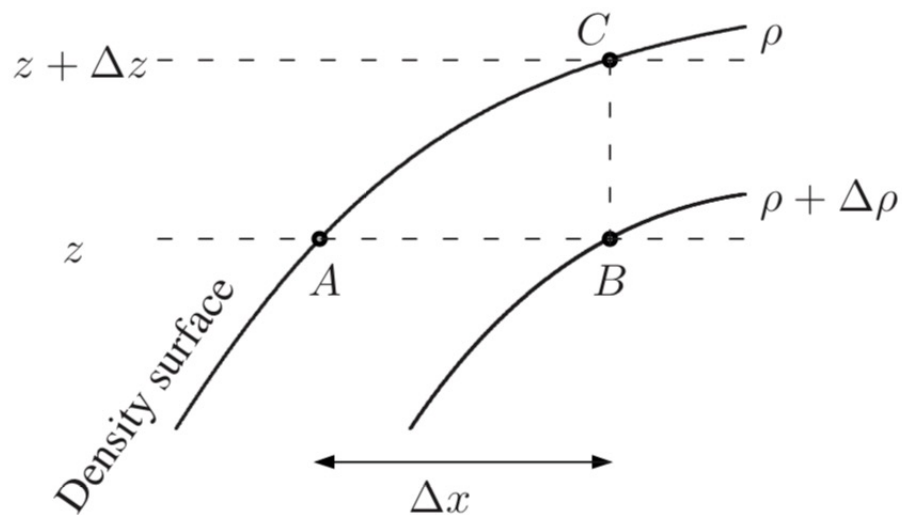
$$\frac{\partial a}{\partial x} \Big|_z = \frac{\partial a}{\partial x} \Big|_\rho - \frac{z_x}{z_\rho} \frac{\partial a}{\partial \rho} \Big|_\rho$$

$$\frac{\partial a}{\partial y} \Big|_z = \frac{\partial a}{\partial y} \Big|_\rho - \frac{z_y}{z_\rho} \frac{\partial a}{\partial \rho} \Big|_\rho$$

$$\frac{\partial a}{\partial z} \Big|_z = \frac{1}{z_\rho} \frac{\partial a}{\partial \rho} \Big|_\rho$$

$$\frac{\partial a}{\partial t} \Big|_z = \frac{\partial a}{\partial t} \Big|_\rho - \frac{z_t}{z_\rho} \frac{\partial a}{\partial \rho} \Big|_\rho$$

$$\left. \frac{\partial a}{\partial x} \right|_z = \left. \frac{\partial a}{\partial x} \right|_\rho - \frac{z_x}{z_\rho} \left. \frac{\partial a}{\partial \rho} \right|_\rho$$



$$\text{constant } z: \frac{a(B) - a(A)}{\Delta x} \quad \left. \frac{\partial a}{\partial x} \right|_z$$

$$\text{constant } \rho: \frac{a(C) - a(A)}{\Delta x} \quad \left. \frac{\partial a}{\partial x} \right|_\rho$$

$$\left. \frac{\partial a}{\partial x} \right|_\rho - \left. \frac{\partial a}{\partial x} \right|_z = \frac{a(C) - a(B)}{\Delta x} = \frac{a(C) - a(B)}{\Delta z} \frac{\Delta z}{\Delta x}$$

$$= \frac{a(C) - a(B)}{\Delta \rho \frac{\Delta z}{\Delta \rho}} \frac{\Delta z}{\Delta x}$$

$$= \frac{z_x}{z_\rho} \left. \frac{\partial a}{\partial \rho} \right|_\rho$$

$$\left. \frac{\partial a}{\partial x} \right|_z = \left. \frac{\partial a}{\partial x} \right|_\rho - \frac{z_x}{z_\rho} \left. \frac{\partial a}{\partial \rho} \right|_\rho$$

$$\frac{\partial a}{\partial x}|_z = \frac{\partial a}{\partial x}|_\rho - \frac{z_x}{z_\rho} \frac{\partial a}{\partial \rho}|_\rho$$

$$\frac{\partial a}{\partial z}|_z = \frac{1}{z_\rho} \frac{\partial a}{\partial \rho}|_\rho$$

$$\frac{\partial a}{\partial t}|_z = \frac{\partial a}{\partial t}|_\rho - \frac{z_t}{z_\rho} \frac{\partial a}{\partial \rho}|_\rho$$

The pressure gradient term:

Hydrostatic balance: $\frac{\partial p}{\partial z} = -\rho g$

$$\frac{\partial p}{\partial \rho} = -\rho g \frac{\partial z}{\partial \rho}$$

$$P = p + \rho g z$$

$$\frac{\partial p}{\partial x}|_z = \frac{\partial p}{\partial x}|_\rho - \frac{z_x}{z_\rho} \frac{\partial p}{\partial \rho}|_\rho = \frac{\partial p}{\partial x}|_\rho + \rho g \frac{\partial z}{\partial x}|_\rho = \frac{\partial P}{\partial x}|_\rho$$

$$\frac{\partial P}{\partial \rho} = \frac{\partial p}{\partial \rho} + g z + \rho g \frac{\partial z}{\partial \rho} = g z$$

$$\left. \frac{\partial a}{\partial x} \right|_z = \left. \frac{\partial a}{\partial x} \right|_\rho - \frac{z_x}{z_\rho} \left. \frac{\partial a}{\partial \rho} \right|_\rho$$

$$\left. \frac{\partial a}{\partial z} \right|_z = \frac{1}{z_\rho} \left. \frac{\partial a}{\partial \rho} \right|_\rho$$

$$\left. \frac{\partial a}{\partial t} \right|_z = \left. \frac{\partial a}{\partial t} \right|_\rho - \frac{z_t}{z_\rho} \left. \frac{\partial a}{\partial \rho} \right|_\rho$$

The full derivative ($\frac{d}{dt}$) in the LHS of the governing equation:

Let $a = \rho$:

$$\left. \frac{\partial \rho}{\partial x} \right|_z = -\frac{z_x}{z_\rho}$$

$$\left. \frac{\partial \rho}{\partial y} \right|_z = -\frac{z_y}{z_\rho}$$

$$\left. \frac{\partial \rho}{\partial z} \right|_z = \frac{1}{z_\rho}$$

$$\left. \frac{\partial \rho}{\partial t} \right|_z = -\frac{z_t}{z_\rho}$$

For incompressible fluids: $\frac{d\rho}{dt} = 0$

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} = 0 \quad \xrightarrow{\text{\textit{\rho-coordinate}}} \quad -z_t - uz_x - vz_y + w = 0 \quad \xrightarrow{\quad} \quad \frac{dz}{dt} = z_t + uz_x + vz_y$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}$$

There is no across-isopycnal advection, and the model is ideal for studying along-isopycnal processes

Without friction, the governing equations in **density coordinate** become:

$$\frac{\partial \mathbf{u}}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - f v = -\frac{1}{\rho_0} \frac{\partial P}{\partial x}$$

$$\frac{\partial \mathbf{v}}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + f u = -\frac{1}{\rho_0} \frac{\partial P}{\partial y}$$

$$\frac{\partial P}{\partial \rho} = g \mathbf{z}$$

$$\frac{\partial \mathbf{h}}{\partial t} + \frac{\partial h u}{\partial x} + \frac{\partial h v}{\partial y} = 0$$

$$h = -\Delta \rho \frac{\partial z}{\partial \rho}$$

the thickness of a fluid layer between
 ρ and $\rho + \Delta \rho$

Layered models

We need to determine **P** and **z** for different layers, and we can obtain the equations for these layers

For **z** :

$$z_m = b$$

upward:

$$z_{k-1} = z_k + h_k, \quad k = m \text{ to } 1.$$

For **P** :

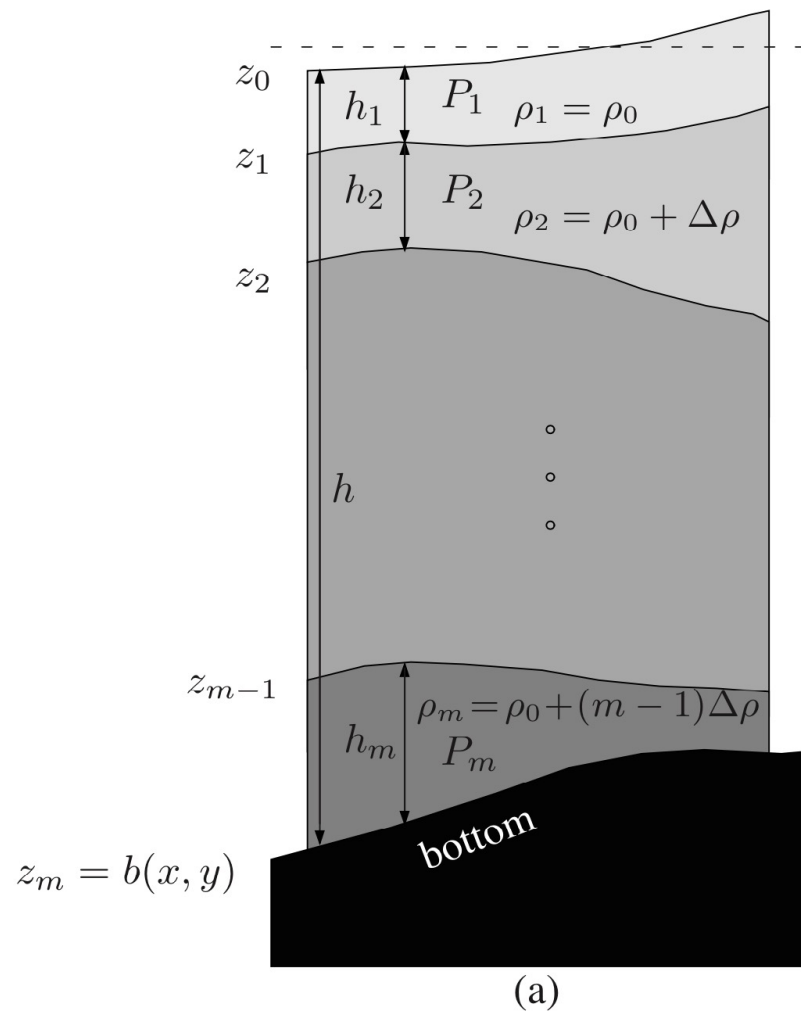
$$P = p + \rho g z$$

$$P_1 = p_{\text{atm}} + \rho_0 g z_0$$

downward:

$$\frac{\partial P}{\partial \rho} = g z$$

$$P_{k+1} = P_k + \Delta \rho g z_k, \quad k = 1 \text{ to } m - 1.$$



$$z_m = b$$

$$z_{k-1} = z_k + h_k$$

$$P_1 = p_{\text{atm}} + \rho_0 g z_0$$

$$P_{k+1} = P_k + \Delta \rho g z_k$$

One layer:

$$z_0 = h_1 + b$$

$$z_1 = b$$

$$P_1 = \rho_0 g (h_1 + b)$$

$$g' = \frac{\Delta \rho}{\rho_0} g$$

reduced gravity

Two layers:

$$z_0 = h_1 + h_2 + b$$

$$z_1 = h_2 + b$$

$$z_2 = b$$

$$P_1 = \rho_0 g (h_1 + h_2 + b)$$

$$P_2 = \rho_0 g h_1 + \rho_0 (g + g') (h_2 + b)$$

Three layers:

$$z_0 = h_1 + h_2 + h_3 + b \quad P_1 = \rho_0 g (h_1 + h_2 + h_3 + b)$$

$$z_1 = h_2 + h_3 + b$$

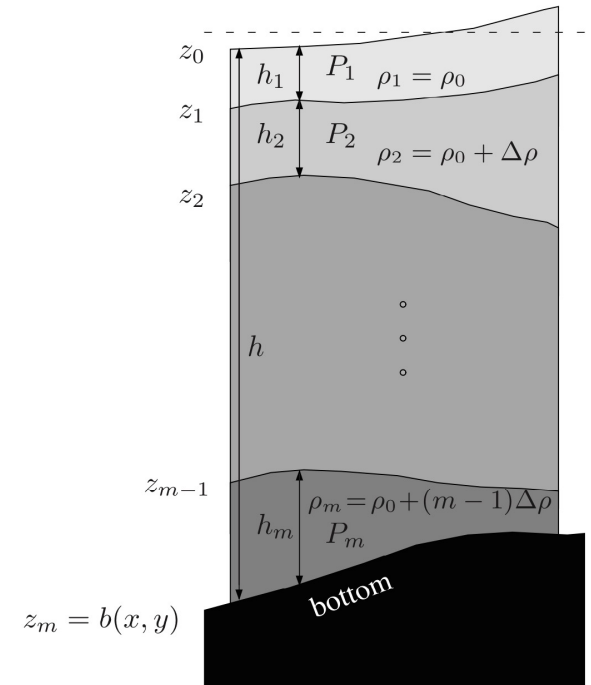
$$P_2 = \rho_0 g h_1 + \rho_0 (g + g') (h_2 + h_3 + b)$$

$$z_2 = h_3 + b$$

$$P_3 = \rho_0 g h_1 + \rho_0 (g + g') h_2$$

$$z_3 = b$$

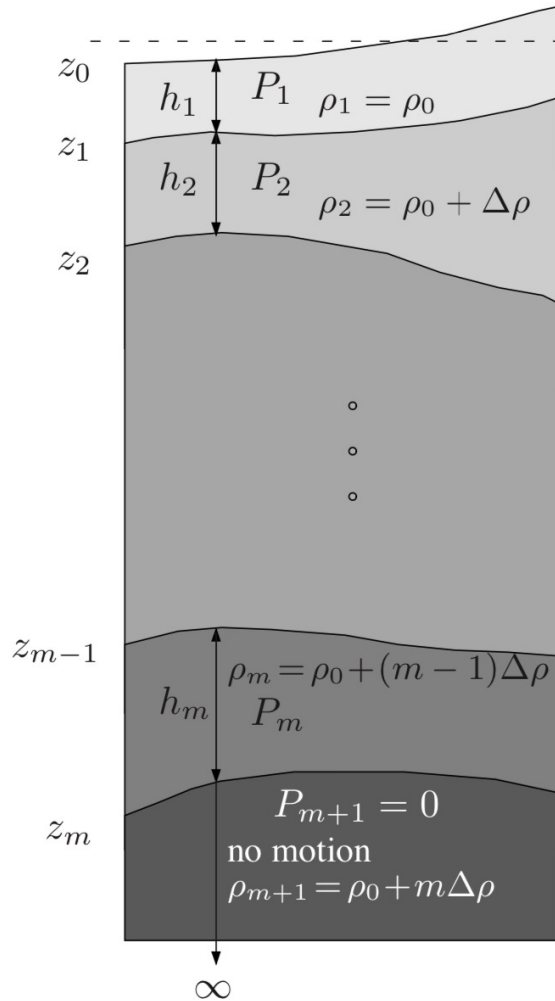
$$+ \rho_0 (g + 2g') (h_3 + b)$$



(a)

Reduced gravity model

The lowest layer may be imagined to be infinitely deep and at rest



$$P_{m+1} = 0$$

Rigid-lid approximation: $z_0 = 0$

One layer:

$$z_1 = -h_1$$

$$P_1 = \rho_0 g' h_1$$

upward calculation

$$P_{k+1} = P_k + \Delta \rho g z_k$$

Two layers:

$$z_1 = -h_1$$

$$P_1 = \rho_0 g' (2h_1 + h_2)$$

$$z_2 = -h_1 - h_2$$

$$P_2 = \rho_0 g' (h_1 + h_2)$$

Three layers:

$$z_1 = -h_1$$

$$P_1 = \rho_0 g' (3h_1 + 2h_2 + h_3)$$

$$z_2 = -h_1 - h_2$$

$$P_2 = \rho_0 g' (2h_1 + 2h_2 + h_3)$$

$$z_3 = -h_1 - h_2 - h_3$$

$$P_3 = \rho_0 g' (h_1 + h_2 + h_3)$$

shallow-water reduced gravity model – one layer

One layer:

$$z_1 = -h_1$$

$$P_1 = \rho_0 g' h_1$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - f v = -g' \frac{\partial h}{\partial x} \quad (1)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + f u = -g' \frac{\partial h}{\partial y} \quad (2)$$

$$\frac{\partial h}{\partial t} + \frac{\partial h u}{\partial x} + \frac{\partial h v}{\partial y} = 0 \quad \longrightarrow \quad \frac{dh}{dt} + h \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0$$

Review for homogeneous fluids:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - f v = -g \frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + f u = -g \frac{\partial \eta}{\partial y}$$

shallow-water model (general) – two layers

Two layers:

$$z_0 = h_1 + h_2 + b$$

$$z_1 = h_2 + b$$

$$z_2 = b$$

$$P_1 = \rho_0 g(h_1 + h_2 + b)$$

$$P_2 = \rho_0 g h_1 + \rho_0 (g + g')(h_2 + b)$$

Governing equations:

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y} - f v_1 = -g \frac{\partial(h_1 + h_2 + b)}{\partial x}$$

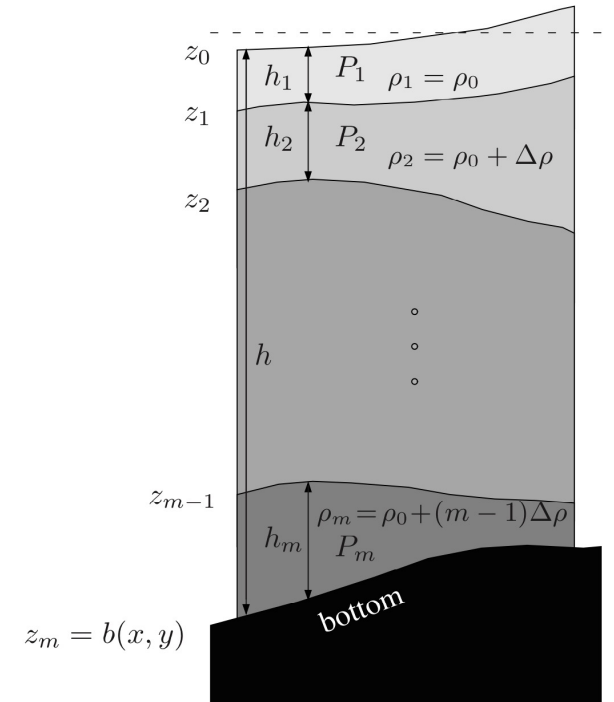
$$\frac{\partial v_1}{\partial t} + u_1 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_1}{\partial y} + f u_1 = -g \frac{\partial(h_1 + h_2 + b)}{\partial y}$$

$$\frac{\partial h_1}{\partial t} + \frac{\partial(h_1 u_1)}{\partial x} + \frac{\partial(h_1 v_1)}{\partial y} = 0$$

$$\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} + v_2 \frac{\partial u_2}{\partial y} - f v_2 = -g \frac{\partial h_1}{\partial x} - (g + g') \frac{\partial(h_2 + b)}{\partial x}$$

$$\frac{\partial v_2}{\partial t} + u_2 \frac{\partial v_2}{\partial x} + v_2 \frac{\partial v_2}{\partial y} + f u_2 = -g \frac{\partial h_1}{\partial y} - (g + g') \frac{\partial(h_2 + b)}{\partial y}$$

$$\frac{\partial h_2}{\partial t} + \frac{\partial(h_2 u_2)}{\partial x} + \frac{\partial(h_2 v_2)}{\partial y} = 0$$



(a)

$$-g \frac{\partial(h_1 + h_2 + b)}{\partial x} = -g \frac{\partial \eta}{\partial x}$$

$$-g' \frac{\partial(h_2 + b)}{\partial x} = -g' \frac{\partial a}{\partial x}$$

The **linearized** equations become:

$$\frac{\partial u_1}{\partial t} - f v_1 = -g \frac{\partial \eta}{\partial x}$$

$$\frac{\partial v_1}{\partial t} + f u_1 = -g \frac{\partial \eta}{\partial y}$$

$$\frac{\partial(\eta - a)}{\partial t} + \frac{\partial(H_1 u_1)}{\partial x} + \frac{\partial(H_1 v_1)}{\partial y} = 0$$

$$\frac{\partial u_2}{\partial t} - f v_2 = -g \frac{\partial \eta}{\partial x} - g' \frac{\partial a}{\partial x}$$

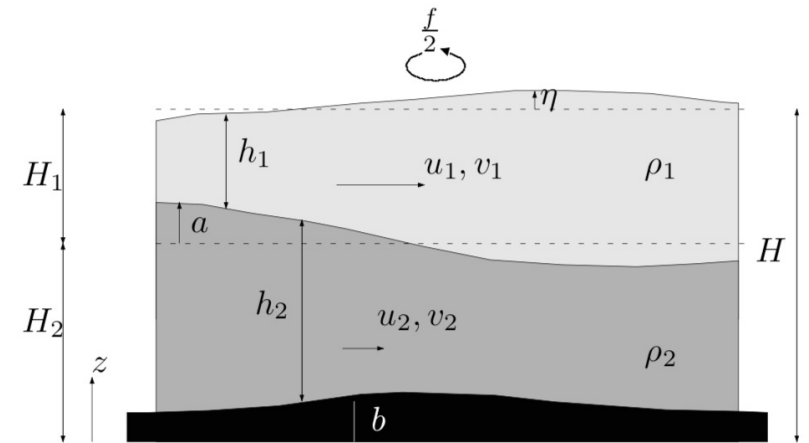
$$\frac{\partial v_2}{\partial t} + f u_2 = -g \frac{\partial \eta}{\partial y} - g' \frac{\partial a}{\partial y}$$

$$\frac{\partial a}{\partial t} + \frac{\partial(H_2 u_2)}{\partial x} + \frac{\partial(H_2 v_2)}{\partial y} = 0$$

$$-g \frac{\partial(h_1 + h_2 + b)}{\partial x}$$

$$-g \frac{\partial h_1}{\partial x} - (g + g') \frac{\partial(h_2 + b)}{\partial x}$$

$$= -g \frac{\partial(h_1 + h_2 + b)}{\partial x} - g' \frac{\partial(h_2 + b)}{\partial x}$$



$$h_1 + h_2 + b = H + \eta$$

$$h_2 + b = H_2 + a$$

$$h_1 = H_1 + \eta - a$$

Let

$$u_2 = \lambda u_1, v_2 = \lambda v_1 \quad \eta = \mu a$$

$$\frac{\partial u_1}{\partial t} - f v_1 = -g \frac{\partial \eta}{\partial x}$$

$$\frac{\partial u_2}{\partial t} - f v_2 = -g \frac{\partial \eta}{\partial x} - g' \frac{\partial a}{\partial x}$$

$$\lambda \left(\frac{\partial u_1}{\partial t} - f v_1 \right) = -(\mu g + g') \frac{\partial a}{\partial x}$$

$$\lambda \frac{\partial u_1}{\partial t} - \lambda f v_1 = -\mu g \frac{\partial a}{\partial x} - g' \frac{\partial a}{\partial x}$$

The horizontal momentum equations can be reduced into one set if $\frac{\lambda}{1} = \frac{g\mu + g'}{g\mu}$

$$\frac{\partial(\eta - a)}{\partial t} + \frac{\partial(H_1 u_1)}{\partial x} + \frac{\partial(H_1 v_1)}{\partial y} = 0$$

$$\frac{\partial a}{\partial t} + \frac{\partial(H_2 u_2)}{\partial x} + \frac{\partial(H_2 v_2)}{\partial y} = 0$$

$$(\mu - 1) \frac{\partial a}{\partial t} + H_1 \left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \right) = 0$$

$$\frac{\partial a}{\partial t} + \lambda H_2 \left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \right) = 0$$

The continuity equation can be reduced into one set if $\frac{1}{\mu - 1} = \frac{H_2 \lambda}{H_1}$

$$\frac{\lambda}{1} = \frac{g\mu + g'}{g\mu} \qquad \frac{1}{\mu - 1} = \frac{H_2\lambda}{H_1}$$

$$H_2\lambda^2 + \left(H_1 - H_2 - \cancel{\frac{g'}{g}} H_2 \right) \lambda - H_1 = 0$$

$$\lambda = \frac{(H_2 - H_1) \pm (H_2 + H_1)}{2H_2}$$

$$g'/g = \Delta\rho/\rho_0 \ll 1$$

If $\lambda = 1$: $u_1 = u_2$ $v_1 = v_2$ **barotropic mode**

$$\mu = \frac{H}{H_2} \qquad a = \eta/\mu = H_2\eta/H$$

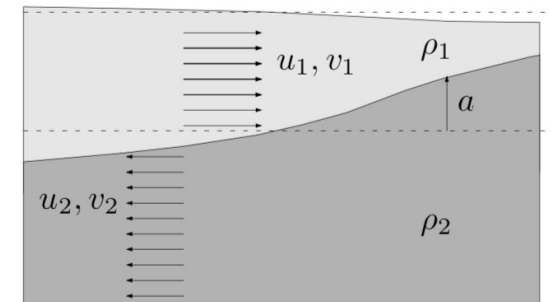
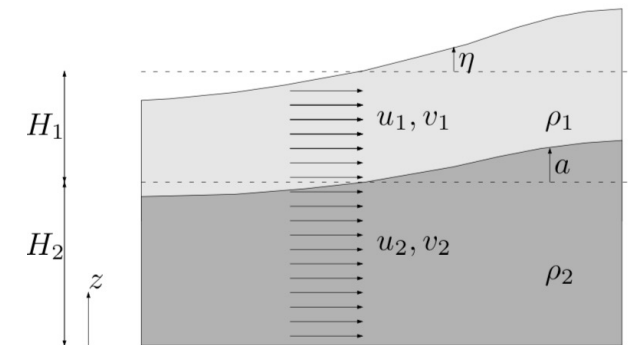
If $\lambda = -\frac{H_1}{H_2}$: $u_2 = -\frac{H_1}{H_2}u_1$ $H_2u_2 = -H_1u_1$

$$v_2 = -\frac{H_1}{H_2}v_1 \qquad H_2v_2 = -H_1v_1$$

baroclinic mode

the vertical integration of transport is zero

$$\mu = -g'H_2/gH \sim 0 \qquad \eta \ll a \qquad \text{nearly rigid lid}$$



Barotropic mode: $\frac{\lambda}{1} = \frac{g\mu + g'}{g\mu} \quad \lambda = 1$

$$\frac{\partial u_1}{\partial t} - f v_1 = -g \frac{\partial \eta}{\partial x}$$

$$\frac{\partial v_1}{\partial t} + f u_1 = -g \frac{\partial \eta}{\partial y}$$

$$\frac{\partial(\eta - a)}{\partial t} + \frac{\partial(H_1 u_1)}{\partial x} + \frac{\partial(H_1 v_1)}{\partial y} = 0$$

$$\frac{\partial u_2}{\partial t} - f v_2 = -g \frac{\partial \eta}{\partial x} - g' \frac{\partial a}{\partial x}$$

$$\frac{\partial v_2}{\partial t} + f u_2 = -g \frac{\partial \eta}{\partial y} - g' \frac{\partial a}{\partial y}$$

$$\frac{\partial a}{\partial t} + \frac{\partial(H_2 u_2)}{\partial x} + \frac{\partial(H_2 v_2)}{\partial y} = 0$$

$$u_T = u_1 = u_2, v_T = v_1 = v_2$$

Summation of the two sets of equations:

$$\frac{\partial u_T}{\partial t} - f v_T = -g \frac{\partial \eta}{\partial x}$$

$$\frac{\partial v_T}{\partial t} + f u_T = -g \frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + H \frac{\partial u_T}{\partial x} + H \frac{\partial v_T}{\partial y} = 0$$

Baroclinic mode: $\frac{\lambda}{1} = \frac{g\mu + g'}{g\mu} \quad \lambda = -\frac{H_1}{H_2}$

$$\frac{\partial u_1}{\partial t} - f v_1 = -g \frac{\partial \eta}{\partial x}$$

$$\frac{\partial v_1}{\partial t} + f u_1 = -g \frac{\partial \eta}{\partial y}$$

$$\frac{\partial(\eta - a)}{\partial t} + \frac{\partial(H_1 u_1)}{\partial x} + \frac{\partial(H_1 v_1)}{\partial y} = 0$$

$$\frac{\partial u_2}{\partial t} - f v_2 = -g \frac{\partial \eta}{\partial x} - g' \frac{\partial a}{\partial x}$$

$$\frac{\partial v_2}{\partial t} + f u_2 = -g \frac{\partial \eta}{\partial y} - g' \frac{\partial a}{\partial y}$$

$$\frac{\partial a}{\partial t} + \frac{\partial(H_2 u_2)}{\partial x} + \frac{\partial(H_2 v_2)}{\partial y} = 0$$

$$u_B = u_1 - u_2 \quad v_B = v_1 - v_2$$

$$\eta = -(g' H_2 / g H) a$$

Difference between the two sets of equations:

$$\frac{\partial u_B}{\partial t} - f v_B = +g' \frac{\partial a}{\partial x}$$

$$\frac{\partial v_B}{\partial t} + f u_B = +g' \frac{\partial a}{\partial y}$$

$$- \frac{\partial a}{\partial t} + \frac{H_1 H_2}{H} \frac{\partial u_B}{\partial x} + \frac{H_1 H_2}{H} \frac{\partial v_B}{\partial y} = 0$$