Mid-term Exam

Atmospheric and Oceanic Fluid Dynamics, Spring 2022

Problem 1 (30 pts)

Oceanographers use the geostrophic flow approximation to estimate the speed of the Gulf Stream – the western boundary current in the North Atlantic Ocean. Typical length and velocity scales for the Gulf stream are: the width of the stream Lx=100 km; length Ly=1000 km; cross-stream velocity U = 0.1 m/s; depth H=1 km; along-stream velocity V=1 m/s (maximum values are up to 3 m/s). Take the maximum values for the eddy viscosities $A_H = 10^5$ m^2/s, and $A_V =$ 0.1 m^2/s. Assume that the x-axis is across the stream and the y-axis is along it. Assume steady flow, hydrostatic balance and incompressible fluids.

1) Write down the x momentum equation, estimate the individual terms in the equation, and show that geostrophic balance holds by comparing the size of different terms; obtain values for the Rossby and Ekman numbers (15 pts);

2) Show that the geostrophic flow approximation is not valid for the y momentum equation (This means that the geostrophic flow equations cannot be used to obtain a full solution for the Gulf Stream, 10 pts);

3) Estimate the magnitude of the vertical velocity in the Gulf Stream (5 pts).

Problem 2 (20 pts)

Between 15°N and 45°N, the winds over the North Pacific consist mostly of the easterly trades (15°N to 30°N) and the westerlies (30°N to 45°N). An adequate representation is

$$\tau^x = \tau_0 \, sin\frac{\pi y}{2L}, \;\; \tau^y = 0 \; (-L \le y \le L),$$

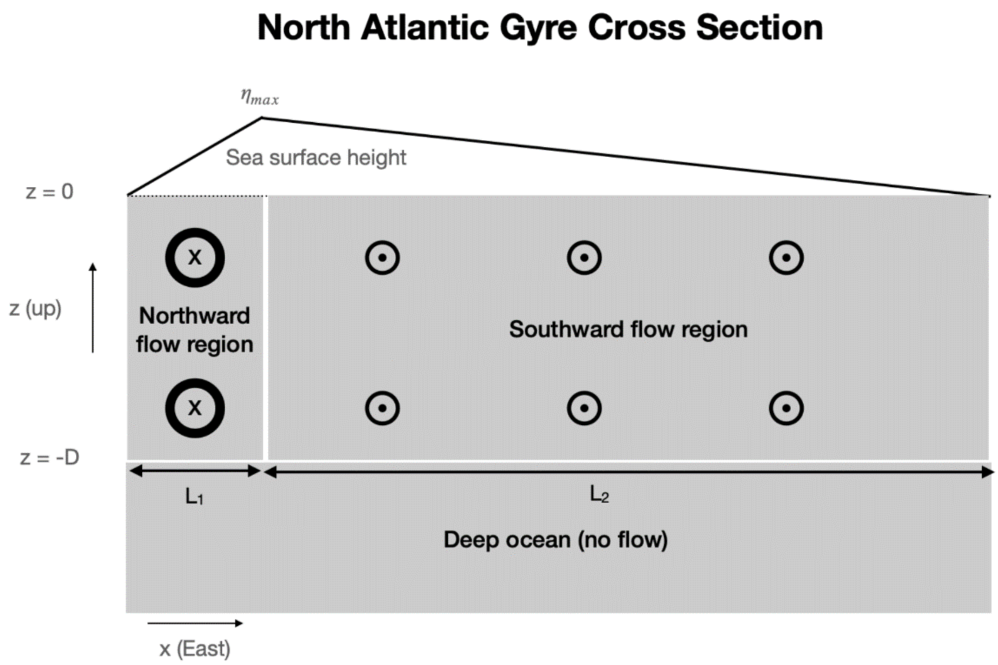
where $\tau_0 = 0.15$ N$/m^2$ is the maximum wind-stress and L=1670 km. Taking $\rho_0 = 1028$ kg/m^3 and the value of the Coriolis parameter corresponding to 30°N.

1) Calculate the Ekman pumping velocity. Which way is it directed (10 pts)?

2) Calculate the vertical volume flux over the entire 15°–45°N strip of the North Pacific (width=8700 km). Express your answer in Sverdrup units (1 Sverdrup = 1 Sv = 10^6 m^3/s, 10 pts).

Problem 3 (30 pts)

The cartoon below shows a zonal cross section across the North Atlantic passing through the Gulf Stream and the subtropical gyre. Going from West to East, the sea-surface height η quickly increases to η_{max} and then decays more slowly back to zero on the Eastern boundary. Assume all these changes are linear; this is a highly simplified representation that makes it easier to make calculations. This SSH pattern is qualitatively similar to the real SSH pattern off the Eastern US. Use the following parameters: L_1 = 500 km, L_2 = 3000 km, η_{max} = 0.5 m.



1) What is the strength of the sea surface height gradient $\partial\eta/\partial x$ in terms of the given parameters in the Northward Flow Region and the Southward Flow Region (5 pts).

2) Using surface geostrophic balance, estimate the speed of the meridional surface current in each of the two regions (5 pts).

3) Calculate the effective depth D over which the meridional current v_g falls to zero (10 pts).

Assume the total northward transport in the Northward Flow region (Gulf Stream) is 50 Sv. Assume equal and opposite transport in the Southward Flow region.

Also assume:

• The current decays linearly with depth until from the surface to z = $-D$

• The current is uniform in the x direction

Note that this part of the problem does not involve geostrophy or anything relating to equations of motion. It is just a volume transport calculation. It does not matter whether you do the calculation for the Northward Flow region or the Southward Flow region. You should get the same answer.

4) Use the thermal wind relation to calculate the implied temperature at the following locations

• $z = -1000\ m, x = L_1$

• $z = -1000\ m, x = L_1 + L_2$ (the Eastern boundary)

Use the linear equation of state $\rho = \rho_0 - \beta_T(T - T_0)$, where β_T=0.15 kg m^{-3} C^{o-1}, and assume that the temperature on the Western boundary at 1000 m depth is 5°C (10 pts).

Problem 4 (40 pts)

Consider an inviscid, homogeneous fluid, bounded above by a flat plate, and bounded below by a moving corrugated surface, given by

$$m(x,t) = -h\cos[k(x+Ut)] \qquad (1)$$

Ignoring variations in the y-direction, the linearized equations describing the flow between these surfaces are:

$$u_t - fv = -p_x/\rho_0 \qquad (2)$$

$$v_t + fu = 0 \qquad (3)$$

$$w_t = -p_z/\rho_0 \qquad (4)$$

$$u_x + w_z = 0 \qquad (5)$$

subject to the boundary conditions

$$w|_{z=m} = hkU\sin[k(x+Ut)] \qquad (6)$$

$$w|_{z=H} = 0 \qquad (7)$$

So long as the amplitude of $m(x,t)$ is small compared to the total water depth ($h << H$), we can apply the bottom boundary condition at $z=0$.

1) (10 pts) Show that equations 1 through 4 can be reduced to a single partial differential equation for w,

$$-w_{zztt} - f^2 w_{zz} = w_{xxtt} \qquad (8)$$

2) (10 pts) We will seek solutions of the form

$$w = \phi(z)\sin(k(x+Ut)) \qquad (9)$$

and solve for $\phi(z)$, the vertical structure function of w. First, show that the equation for $\phi(z)$ can now be written as

$$\varphi_{z'z'} = \varphi\left(\frac{\alpha^2 R_0^2}{R_0^2 - 1}\right) \qquad (10)$$

where $R_0 = Uk/f$ is the Rossby number and $\alpha = kH$ is the aspect ratio, and $z' = z/H$ is the non- dimensional depth.

3) (10 pts) Now, solve the vertical structure function $\phi(z)$, for two limits: first the "shallow water" case in which $R_0 << 1$ and $\alpha << 1$. Second, consider the deep, non-rotating case in

which $R_0 >> 1$ and $\alpha >> 1$ (Hint: It may be simpler to apply the approximations to Equation (10) before solving for each limit).

4) (10 pts) Write a short paragraph describing the differences between this system and the shallow-water system (from the perspective of configurations, assumptions and solutions).