Place me on Sunium's marbled steep,
Where nothing, save the waves and I,
May hear our mutual murmurs sweep.
George Gordon Byron (Lord Byron), *The Isles of Greece*, 1820.

CHAPTER 16

Planetary Waves and Zonal Asymmetries

PLANETARY WAVES ARE LARGE-SCALE ROSSBY WAVES in which the potential vorticity gradient is provided by differential rotation (i.e., the beta-effect). They are ubiquitous in Earth's atmosphere and almost certainly in other planetary atmospheres. They propagate horizontally over the two Poles, and they propagate vertically into the stratosphere and beyond. In the previous chapter we saw that it is the propagation of Rossby waves away from their mid-latitude source that gives rise to the mean eastward eddy-driven jet. In this chapter we will see that the dynamics of such waves also largely determines the large-scale *zonally asymmetric* circulation of the mid-latitude atmosphere. In the first few sections we discuss the properties and propagation of planetary waves themselves, and in many ways these sections are a continuation of Chapter 6. We then look more specifically at planetary waves forced by surface variations in topography and thermal properties, for it is these waves that give rise to the zonally asymmetric circulation.

In proceeding this way we are dividing our task of constructing a theory of the general circulation of the extratropical atmosphere into two. The first task (Chapters 14 and 15) was to understand the zonally averaged circulation and the transient zonal asymmetries by supposing that, to a first approximation, this circulation is qualitatively the same as it would be if the boundary conditions were zonally symmetric, with no mountains or land–sea contrasts. Given the statistically zonally symmetric circulation, the second task is to understand the zonally asymmetric circulation. We may do this by supposing that the latter is a perturbation on the former, and using a theory linearized about the zonally symmetric state. It is by no means obvious that such a procedure will be successful, for it depends on the nonlinear interactions among the zonal asymmetries being weak. We might make some a priori estimates that suggest that this might be the case, but the ultimate justification for the approach lies in its a posteriori success. In our discussion of stationary waves we will focus first on the response to orography at the lower boundary, and then consider thermodynamic forcing — arising, for example, from an inhomogeneous surface temperature field. Our focus throughout this chapter is the mid-latitudes.

16.1 ROSSBY WAVE PROPAGATION IN A SLOWLY VARYING MEDIUM

In Chapters 6 and 7 we looked at wave propagation using linearized equations of motion. We now focus and extend this discussion by looking at Rossby wave propagation in a medium in which the parameters (such as the zonal wind and the stratification) vary spatially — as occurs in the real atmosphere. If the parameters do vary then waves may propagate into a region in which they amplify, perhaps violating the initial assumption of linearity, so let us first look at what the conditions for linearity are.[1]

16.1.1 Linear Dynamics

If the linear equations are to be an accurate representation of the dynamics then the perturbation quantities need to be small compared to the background state, or at least the nonlinear terms must be small. In reality this is not always the case and indeed it may be that in course of propagation the waves amplify and may even *break*. Wave breaking is familiar to anyone who has been to the beach and watched water waves move toward the shore and crash in the 'surf zone' as the mean depth becomes too shallow to support laminar surface waves. Manifestly, the linear approximation breaks down at this point. More generally, wave breaking simply refers to an irreversible deformation of material surfaces, generally leading to dissipation. Since Rossby waves generally grow in amplitude as they propagate up (because density falls) we can expect Rossby wave breaking to occur somewhere in the atmosphere, but waves can also break as they propagate laterally, if and when they grow in size to such an extent that the nonlinear terms in the equations of motion become important.

To examine this consider the quasi-geostrophic potential vorticity equation,

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right) q = 0, \qquad q = \beta y + \nabla^2\psi' + \frac{f_0^2}{\rho_R}\frac{\partial}{\partial z}\left(\frac{\rho_R}{N^2}\frac{\partial \psi}{\partial z}\right). \tag{16.1a,b}$$

The derivation of this equation was given in Chapter 5 and all the terms are defined there. In brief, q is the quasi-geostrophic potential vorticity and ψ the streamfunction, f_0 is the Coriolis parameter and ρ_R is a density profile, a function of z only. Breaking the above equation up into mean and perturbation quantities in the usual way we obtain

$$\left(\frac{\partial}{\partial t} + \overline{u}(y,z)\frac{\partial}{\partial x}\right) q' + v'\frac{\partial \overline{q}}{\partial y} = -\left(\frac{\partial u'q'}{\partial x} + \frac{\partial v'q'}{\partial y}\right). \tag{16.2}$$

In the linear approximation we neglect the terms on the right-hand side and, seeking wave-like solutions of the form $\psi = F(x - ct)$, we obtain

$$(\overline{u} - c)\frac{\partial q'}{\partial x} + v'\frac{\partial \overline{q}}{\partial y} = 0. \tag{16.3}$$

For the linear approximation to be valid the terms in this equation must be larger than the nonlinear terms in (16.2), and this will be the case if

$$|\overline{u} - c| \gg |u'| \qquad \text{and} \qquad \left|\frac{\partial \overline{q}}{\partial y}\right| \gg \left|\frac{\partial q'}{\partial y}\right|. \tag{16.4a,b}$$

Although it is common to only treat the case in which $\overline{u}$ is a constant, we may also consider the case in which $\overline{u}$ varies slowly, either in latitude or height or both, and (16.3) then approximately holds locally. But if a wave propagates into a region in which $\overline{u} = c$ then the linear criterion *must* break down. Regions where $\overline{u} = c$ are called *critical lines, critical surfaces, critical heights* or *critical latitudes,* depending on context, and in many circumstances a *critical layer* of finite width will surround the critical line, in which frictional and/or nonlinear effects are important. The location of a critical surface does not depend on the frame of reference used to measure the velocities.

For reference we first write down a few results for the simplest case when $\partial\overline{q}/\partial y$, $\overline{u}$, N^2 and ρ_R are all constant, referring to Section 6.5 as needed. We look for solutions of the form

$$\psi' = \operatorname{Re}\widetilde{\psi}\mathrm{e}^{\mathrm{i}(kx+ly+mz-\omega t)}, \tag{16.5}$$

and obtain the dispersion relation

$$\omega = \overline{u}k - \frac{k\beta}{k^2 + l^2 + Pr^2 m^2}, \tag{16.6}$$

where $Pr = f_0/N$ is the Prandtl ratio. The components of the group velocity are given by

$$c_g^x = \overline{u} + \frac{(k^2 - l^2 - Pr^2 m^2)\beta}{(k^2 + l^2 + Pr^2 m^2)^2}, \qquad c_g^y = \frac{2kl\beta}{(k^2 + l^2 + Pr^2 m^2)^2}, \qquad c_g^z = \frac{2kmPr^2\beta}{(k^2 + l^2 + Pr^2 m^2)^2}. \tag{16.7a,b,c}$$

16.1.2 Conditions for Wave Propagation

Suppose that the zonal wind varies slowly with latitude and height, but that, for simplicity, the density, ρ_R, is a constant. The equation of motion is

$$\left(\frac{\partial}{\partial t} + \overline{u}(y,z)\frac{\partial}{\partial x}\right)q' + v'\frac{\partial \overline{q}}{\partial y} = 0. \tag{16.8}$$

Because the coefficients of the equation are not constant we cannot assume harmonic solutions in the y and z directions; rather, we seek solutions of the form

$$\psi' = \widetilde{\psi}(y,z)\mathrm{e}^{\mathrm{i}k(x-ct)}. \tag{16.9}$$

If the parameters in (16.8) are varying slowly compared to the wavelength of the waves then a dispersion relation still exists (as discussed in Section 6.3), but the relation will be of the form $\omega = \Omega(\boldsymbol{k}; \boldsymbol{x}, t)$; where the function Ω varies slowly in space. Now, if the medium is not an explicit function of x or of time the x-wavenumber and the frequency will be a constant, and hence c is constant too, and we can use the dispersion relation to find what are effectively the other wavenumbers in the problem. Using (16.9) in (16.8) we find (with N^2 constant)

$$\frac{\partial^2 \widetilde{\psi}}{\partial y^2} + \frac{f_0^2}{N^2}\frac{\partial^2 \widetilde{\psi}}{\partial z^2} + n^2(y,z)\widetilde{\psi} = 0, \qquad \text{where} \qquad n^2(y,z) = \frac{\partial \overline{q}/\partial y}{\overline{u} - c} - k^2. \tag{16.10a,b}$$

Equation (16.10a) is similar to the Rayleigh or Rayleigh–Kuo equation encountered in Chapter 9, but now c is given and is not an eigenvalue; rather, the frequency is known and the dispersion relation gives the quantity n. The quantity n is the *refractive index* and it greatly affects how the waves propagate: solutions are wavelike when n^2 is positive and evanescent when n^2 is negative. To see this in a simple case, suppose there is no z-variation so that $\partial^2\widetilde{\psi}/\partial y^2 + n^2\widetilde{\psi} = 0$, whereas if n is constant and real we have harmonic solutions in the y-direction of the form $\exp(\mathrm{i}ny)$. If $n^2 < 0$ the solutions will evanesce. Waves tend to propagate toward regions of large n^2 and turn away from regions of negative n^2, as we will see in the examples to follow.

The value of n^2 will become very large if and as $\overline{u}$ approaches c from above and the waves, being very short, will tend to break. If $\overline{u}$ continues to diminish and becomes smaller than c then n^2 switches from being large and positive to large and negative. If n^2 diminishes because $\partial\overline{q}/\partial y$ diminishes then it will transition smoothly to a negative value. The location where $\overline{u} = c$ is called a critical surface (or line). The location where n^2 passes through zero is called a turning surface (or line).

The bounds on n^2 can be translated into bounds on the zonal phase speed c. Given a zonal wind $\overline{u}$, wave propagation requires that c is bounded by

$$\overline{u} - \frac{\partial \overline{q}/\partial y}{k^2 + \gamma^2} < c < \overline{u}. \tag{16.11}$$

At the upper bound (a critical surface) the wavelength is small and wave breaking is likely to occur. At the lower bound (a turning surface) the refractive index tends to zero and the wavelength tends to infinity. Waves will tend to propagate away from regions with a small n and be refracted toward regions of large n. The bounds can also be expressed in terms of the zonal velocity:

$$0 < \overline{u} - c < \frac{\partial \overline{q}/\partial y}{k^2 + \gamma^2}. \tag{16.12}$$

This form is useful when considering a situation in which the wave speed is given, for example by boundary conditions; Equation (16.12) then tells us under what configurations of zonal velocity wave propagation can occur. The lower bound corresponds to a critical surface and the upper bound to a turning surface.

It is algebraically complicated to continue our analysis in the three-dimensional case, so let us consider the cases in which the inhomogeneities in the medium occur separately in the horizontal and vertical. A summary of some key concepts is provided on page 590.

16.2 HORIZONTAL PROPAGATION OF ROSSBY WAVES

Consider the purely horizontal problem for which the linearized equation of motion is

$$\left(\frac{\partial}{\partial t} + \overline{u}(y)\frac{\partial}{\partial x}\right) q' + v' \frac{\partial \overline{q}}{\partial y} = 0, \tag{16.13}$$

where $q' = \nabla^2 \psi'$, $v' = \partial\psi'/\partial x$ and $\partial\overline{q}/\partial y = \beta - \overline{u}_{yy}$, which we will denote β^*. If $\overline{u}$ and β^* do not vary in space then we may obtain wavelike solutions in the usual way and obtain the dispersion relation

$$\omega \equiv ck = \overline{u}k - \frac{\partial \overline{q}/\partial y}{k^2 + l^2}, \tag{16.14}$$

where k and l are the x- and y-wavenumbers.

If the parameters do vary in the y-direction then we seek a solution $\psi' = \widetilde{\psi}(y)\exp[\mathrm{i}k(x - ct)]$ and obtain, analogous to (16.10),

$$\frac{\partial^2 \widetilde{\psi}}{\partial y^2} + l^2(y)\widetilde{\psi} = 0, \quad \text{where} \quad l^2(y) = \frac{\beta^*}{\overline{u} - c} - k^2. \tag{16.15a,b}$$

If the parameter variation is sufficiently small, occurring on a spatial scale longer than the wavelength of the waves, then we may expect that the disturbance will propagate locally as a plane wave. The solution is then of WKB form (see Appendix A to Chapter 6), namely

$$\widetilde{\psi}(y) = A_0 l^{-1/2} \exp\left(\mathrm{i}\int l \,\mathrm{d}y\right), \tag{16.16}$$

where A_0 is a constant. The phase of the wave in the y-direction, θ, is evidently given by $\theta = \int l\,\mathrm{d}y$, so that the local wavenumber is given by $\mathrm{d}\theta/\mathrm{d}y = l$. The group velocity is calculated in the normal way using the dispersion relation (16.14) and we obtain

$$c_g^x = \overline{u} + \frac{(k^2 - l^2)\beta^*}{(k^2 + l^2)^2}, \qquad c_g^y = \frac{2kl\,\beta^*}{(k^2 + l^2)^2}, \tag{16.17a,b}$$

where l is given by (16.15b), with both quantities varying slowly in the y-direction.

16.2.1 Wave Amplitude

As a Rossby wave propagates its amplitude is not necessarily constant because, in the presence of a shear, the wave may exchange energy with the background state, and the WKB solution, (16.15), tells us that the variation goes like $l^{-1/2}(y)$. This variation can be understood from somewhat more general considerations. As discussed in Chapter 10 (specifically Section 10.2.1) an inviscid, adiabatic wave will conserve its wave activity, and specifically its pseudomomentum, meaning that

$$\frac{\partial \mathcal{P}}{\partial t} + \nabla \cdot \boldsymbol{\mathcal{F}} = 0, \tag{16.18}$$

where $\mathcal{P}$ is quadratic in the wave amplitude and $\boldsymbol{\mathcal{F}}$ is the flux of $\mathcal{P}$, and the two are related by the group velocity property $\boldsymbol{\mathcal{F}} = \mathbf{c}_g \mathcal{P}$. In the zonally-averaged case the pseudomomentum and flux for the stratified quasi-geostrophic equations are given by

$$\mathcal{P} = \frac{\overline{q'^2}}{2\beta^*}, \qquad \boldsymbol{\mathcal{F}} = -\overline{u'v'}\,\mathbf{j} + \frac{f_0}{N^2}\overline{v'b'}\,\mathbf{k}, \tag{16.19}$$

with $\boldsymbol{\mathcal{F}}$ being the Eliassen–Palm (EP) flux. If the waves are steady then $\nabla \cdot \boldsymbol{\mathcal{F}} = 0$, and in the two-dimensional case under consideration $b' = 0$ and $\partial \overline{u'v'}/\partial y = 0$. Thus $u'v' = kl|\widetilde{\psi}|^2 =$ constant, and since k is constant along a ray the amplitude of a wave varies like

$$|\widetilde{\psi}| = \frac{A_0}{\sqrt{l(y)}}\,, \tag{16.20}$$

as in the WKB solution. The energy of the wave then varies like

$$\text{Energy} = (k^2 + l^2)\frac{A_0^2}{l}. \tag{16.21}$$

16.2.2 Two Examples

To illustrate the above ideas in a concrete fashion we consider two examples, one with a turning line and one with a critical line. Very close to the turning line and critical line more detailed analysis is needed to obtain a complete solution, but we can obtain a sense of the behaviour with an elementary treatment.

Waves with a turning latitude

A turning line arises where $l = 0$ and it corresponds to the lower bound of c in (16.11). The line arises if the potential vorticity gradient diminishes to such an extent that $l^2 < 0$ and the waves then cease to propagate in the y-direction. This may happen even in unsheared flow as a wave propagates polewards and the magnitude of beta diminishes.

As a wave packet approaches a turning latitude then n goes to zero so the amplitude and the energy of the wave approach infinity. However, the wave will never reach the turning latitude because the meridional component of the group velocity is zero, as can be seen from the expressions for the group velocity, (16.17). As a wave approaches the turning latitude $c_g^x \to (\beta - \overline{u}_{yy})/k^2$ and $c_g^y \to 0$, so the group velocity is purely zonal and indeed, as $l \to 0$,

$$\frac{c_g^x - \overline{u}}{c_g^y} = \frac{k}{2l} \to \infty. \tag{16.22}$$

Because the meridional wavenumber is small the wavelength is large, so we do not expect the waves to break. Rather, we intuitively expect that a wave packet will turn — hence the eponym 'turning latitude' — and be reflected.

Rossby Wave Propagation in a Slowly Varying Medium

The linear equation of motion is, in terms of streamfunction,

$$\left(\frac{\partial}{\partial t}+\overline{u}(y,z)\frac{\partial}{\partial x}\right)\left[\nabla^2\psi'+\frac{f_0^2}{\rho_R}\frac{\partial}{\partial z}\left(\frac{\rho_R}{N^2}\frac{\partial\psi'}{\partial z}\right)\right]+\frac{\partial\psi'}{\partial x}\frac{\partial\overline{q}}{\partial y}=0. \tag{RP.1}$$

We suppose that the parameters of the problem vary slowly in y and/or z but are uniform in x and t. The frequency and zonal wavenumber are therefore constant. We seek solutions of the form $\psi'=\widetilde{\psi}(y,z)\mathrm{e}^{\mathrm{i}k(x-ct)}$ and find (if, for simplicity, N^2 and ρ_R are constant)

$$\frac{\partial^2\widetilde{\psi}}{\partial y^2}+\frac{f_0^2}{N^2}\frac{\partial^2\widetilde{\psi}}{\partial z^2}+n^2(y,z)\widetilde{\psi}=0, \qquad \text{where} \qquad n^2(y,z)=\frac{\partial\overline{q}/\partial y}{\overline{u}-c}-k^2. \tag{RP.2}$$

The value of n^2 must be positive in order that waves can propagate, and so waves cease to propagate when they encounter either

(i) *A turning surface,* where $n^2=0$, or
(ii) *A critical surface,* where $\overline{u}=c$ and n^2 becomes infinite.

For a given wave speed, the location of the turning surface, but not that of the critical surface, depends on wavenumber. The condition for wave propagation may be expressed as bounds on the zonal flow, to wit

$$0<\overline{u}-c<\frac{\partial\overline{q}/\partial y}{k^2}. \tag{RP.3}$$

If the length scale over which the parameters of the problem vary is much longer than the wavelengths themselves we can expect the solution to look locally like a plane wave and a WKB analysis can be employed. In the purely horizontal problem we assume a solution of the form $\psi'=\widetilde{\psi}(y)\mathrm{e}^{\mathrm{i}k(x-ct)}$ and find

$$\frac{\partial^2\widetilde{\psi}}{\partial y^2}+l^2(y)\widetilde{\psi}=0, \qquad l^2(y)=\frac{\partial\overline{q}/\partial y}{\overline{u}-c}-k^2. \tag{RP.4}$$

The WKB solution is of the form

$$\widetilde{\psi}(y)=Al^{-1/2}\exp\left(\pm\,\mathrm{i}\int l\,\mathrm{d}y\right). \tag{RP.5}$$

Thus, $l(y)$ is the local y-wavenumber, and the amplitude of the solution varies like $l^{-1/2}$. However, the WKB condition fails at both a critical line and a turning line.

Approaching a critical line the amplitude of the wave diminishes (in the WKB approximation) because l is large. In the critical layer wave amplitude is in fact nearly constant but the vorticity becomes very large, and either nonlinearity and/or dissipation become important, and since wavelength is small the waves may break. At a turning line the amplitude and energy will both be large, but since the wavelength is long the waves will not necessarily break; rather, they are reflected.

A similar analysis may be employed for vertically propagating Rossby waves, and either a turning level or a critical level will prevent the upward propagation of waves into the stratosphere — this is the 'Charney–Drazin' condition, discussed in Section 16.5.3.

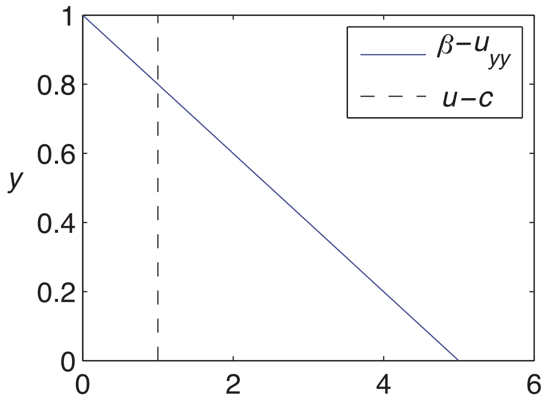

Fig. 16.1 Parameters for the first example considered in Section 16.2.2, with all variables nondimensional. The zonal flow is uniform with $u = 1$ and $c = 0$ (so that $\overline{u}_{yy} = 0$) and β diminishes linearly as y increases polewards as shown. With zonal wavenumber $k = 1$ there is a turning latitude at $y = 0.8$, and the wave properties are illustrated in Fig. 16.2.

To illustrate this, consider waves propagating in a background state that has no horizontal shear but with a beta effect that diminishes polewards. To be concrete suppose that $\beta = 5$ at $y = 0$, diminishing linearly to $\beta = 0$ at $y = 0$, and that $\overline{u} - c = 1$ everywhere. There is no critical line but depending on the x-wavenumber there may be a turning line, and if we choose $k = 1$ then the turning line occurs when $\beta = 1$ and so at $y = 0.8$. The turning latitude depends on the value of the x-wavenumber — if the zonal wavenumber is larger then waves will turn further south. The parameters are illustrated in Fig. 16.1.

For a given zonal wavenumber ($k = 1$ in this example) the value of l^2 is computed using (16.15b), and the components of the group velocity using (16.17), and these are illustrated in Fig. 16.2. We may choose either a positive or a negative value of l, corresponding to northward or southward oriented waves, and we illustrate both in the figure. The value of l^2 becomes zero at $y = 0.8$, and this corresponds to a turning latitude. The values of the wave amplitude and energy are computed using (16.20) and (16.21) (with an arbitrary amplitude at $y = 0$) and these both become infinite at the turning latitude.

What is happening physically? We may suppose that at some location in the domain there is a source of waves — baroclinic disturbances for example. Waves propagate away from the source (since the waves must carry energy away), and this determines the sign of n of any particular wave packet. The disturbance may in general consist of many zonal wavenumbers and many frequencies (or phase speeds, c), but the dispersion relation must be satisfied for each pair and this determines the meridional wavenumber via an equation like (16.15). As the wave packet propagates away from the source then, as we noted in Section 6.3 on ray theory, if the medium is zonally symmetric the x-wavenumber, k, is preserved. If the medium is not time-varying then the frequency, and therefore the wave speed c, are also preserved. We may approximately construct a ray by following the arrows in Fig. 16.2, and we see that a ray propagating polewards will bend eastward as it approaches the turning latitude Although its amplitude will become large it will not necessarily break because the wavelength is large; in fact, the packet may be reflected southward. We may heuristically construct a ray trajectory by drawing a line that is always parallel to the arrows marking the group velocity. Indeed, the entire procedure might be thought of as an Eulerian analogue of ray theory; rather than following a wave packet we just evaluate the field of group velocity, and if there is no explicit time dependence in the problem a ray follows the arrows.

The above argument suggests but does not demonstrate reflection at the turning latitude. The arrows of Fig. 16.2 are all zonal at the turning line and do not actually turn back. One might imagine using a WKB analysis but the WKB approximation fails in the vicinity of a turning latitude: the meridional wavenumber l tends to zero but $\mathrm{d}l/\mathrm{d}y$ does not, and the WKB condition (6.164)

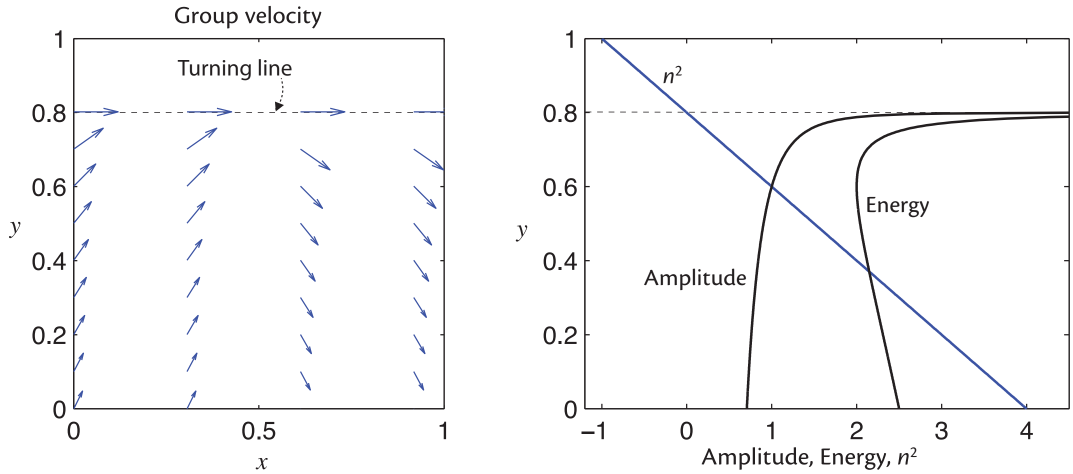

Fig. 16.2 Left: The group velocity evaluated using (16.17) for the parameters illustrated in Fig. 16.1, which give a turning latitude at $y = 0.8$. For $x < 0.5$ we choose positive values of n, and a northward group velocity, whereas for $x > 0.5$ we choose negative values of n. Right panel: Values of refractive index squared (n^2), the energy and the amplitude of a wave. n^2 is negative for $y > 0.8$. See text for more description.

cannot be satisfied (even though the wave equation itself is not singular). However, a momentum flux argument shows that the reflection is in fact perfect in the absence of dissipation. Suppose there is a wave source in mid- or low latitudes (say at $y = 0$ in Fig. 16.2) producing poleward propagating waves. Poleward of the turning line the waves evanesce and the zonally-averaged polewards momentum flux is zero at large y. However, this decay has occurred in the absence of friction, and therefore that momentum flux is zero *everywhere*. To see this with equations, away from forcing regions the inviscid barotropic pseudomomentum conservation equation, (16.18), becomes

$$\frac{\partial \mathcal{P}}{\partial t} + \overline{v'\zeta'} = 0, \qquad \text{or} \qquad \frac{\partial \mathcal{P}}{\partial t} - \frac{\partial}{\partial y}\overline{u'v'} = 0. \tag{16.23}$$

Thus, in a statistically steady state, $\overline{u'v'}$ is a constant, and that constant is zero if the flux is zero at large y. Since the forcing is producing a poleward propagating Rossby wave (with $\overline{u'v'} < 0$ in the Northern Hemisphere) there must be a reflected wave with $\overline{u'v'} > 0$, and that reflected wave must come from the vicinity of the turning line.

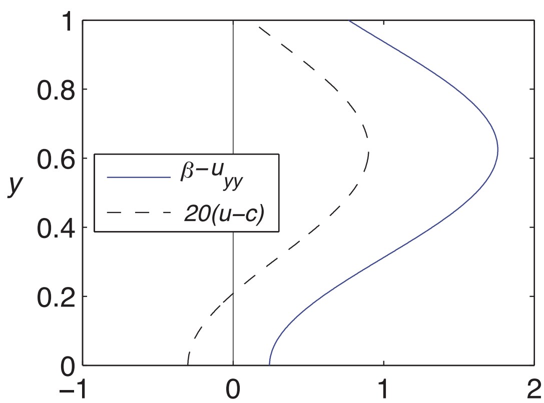

Fig. 16.3 Parameters for the second example considered in Section 16.2.2, with all variables nondimensional. The zonal flow has a broad eastward jet and β is constant. There is a critical line at $y = 0.2$, and with zonal wavenumber $k = 5$ the wave properties are illustrated in Fig. 16.4.

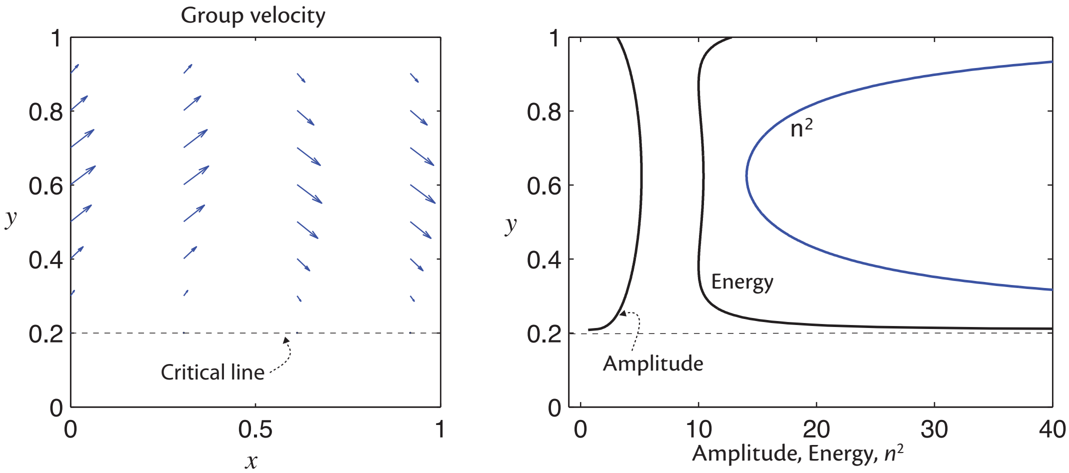

Fig. 16.4 Left: The group velocity evaluated using (16.17) for the parameters illustrated in Fig. 16.1, which give a critical line at $y = 0.2$. For $x < 0.5$ we choose positive values of n, and a northward group velocity, whereas for $x > 0.5$ we choose negative values of n. Right panel: Values of refractive index squared, the energy and the amplitude of a wave. The value of n^2 becomes infinite at the critical line and the linear theory breaks down in its vicinity. See text for more description.

Waves with a critical latitude

A critical line occurs when $\overline{u} = c$, corresponding to the upper bound of c in (16.11), and from (16.15) we see that at a critical line the meridional wavenumber approaches infinity. From (16.17) we see that both the x- and y-components of the group velocity are zero — a wave packet approaching a critical line just stops (at least according to ray theory). Specifically, as l becomes large

$$c_g^x - \overline{u} \to 0, \qquad c_g^y \to 0, \qquad \frac{c_g^x - \overline{u}}{c_g^y} \to -\frac{l}{k} \to -\infty. \tag{16.24}$$

From (16.20) the amplitude of the wave packet also approaches zero, but its energy approaches infinity. Since the wavelength is very small we expect the waves to *break* and deposit their momentum, and this situation commonly arises when Rossby waves excited in mid-latitudes propagate equatorward and encounter a critical latitude in the subtropics.

To illustrate this let us construct a background state that has an eastward jet in mid-latitudes becoming westward at low latitudes, with β constant chosen to be large enough so that $\beta - \overline{u}_{yy}$ is positive everywhere. (Specifically, we choose $\beta = 1$ and $\overline{u} = -0.03\sin(8\pi y/5 + \pi/2) - 0.5$, but the precise form is not important.) If $c = 0$ then there is a critical line when $\overline{u}$ passes through zero, which in this example occurs at $x = 0.2$. (The value of $\overline{u} - c$ is small at $y = 1$, but no critical line is actually reached.) These parameters are illustrated in Fig. 16.3. We also choose $k = 5$, which results in a positive value for l^2 everywhere.

As in the previous example we compute the value of l^2 using (16.15b) and the components of the group velocity using (16.17), and these are illustrated in Fig. 16.4, with northward propagating waves shown for $x < 0.5$ and southward propagating waves for $x > 0.5$. The value of n^2 increases considerably at the northern and southern edges of the domain, and is actually infinite at the critical line at $y = 0.2$. Using (16.20) the amplitude of the wave diminishes as the critical line approaches, but the energy increases rapidly, suggesting that the linear approximation will break down. The waves (in the linear approximation) tend to stall before reaching the critical line, because both the x and the y components of the group velocity become very small. This is a little misleading because ray theory breaks down and a disturbance can reach the critical line, and in the region

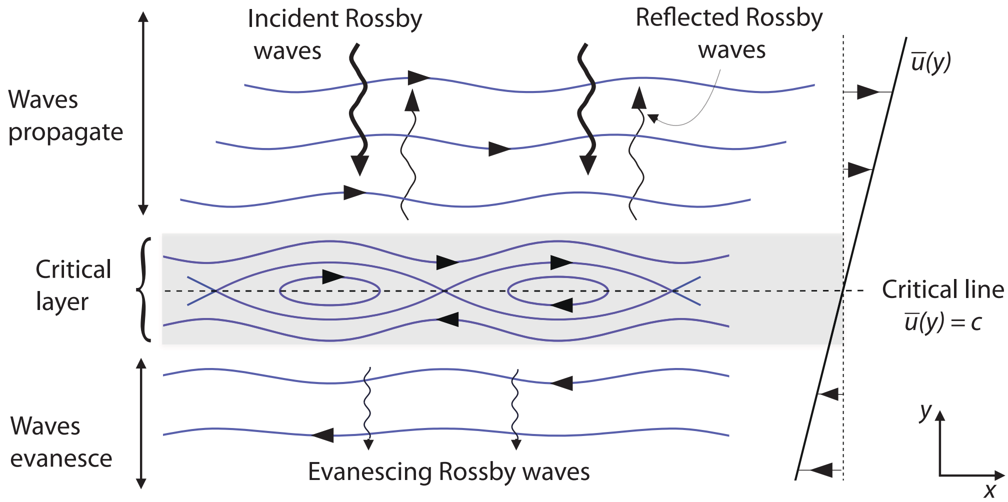

Fig. 16.5 Sketch of a Rossby-wave critical layer. Incident Rossby waves from mid-latitudes propagate in a horizontally sheared flow toward a critical line at which $\overline{u}(y) = c$. Surrounding the critical line is a critical layer, in which either nonlinear or frictional effects, or both, are important. If nonlinear effects are important then the critical layer may reflect, but there is still likely to be dissipation in the critical layer. Equatorward of the critical layer the waves evanesce.[3]

of the critical line — that is, in the critical layer — either frictional or nonlinear effects, or both, become important, as we see in the next section. The situation illustrated in this example is of particular relevance to the maintenance of the zonal wind structure in the troposphere: waves are generated in mid-latitudes and propagate equatorward, and as they approach a critical line in the subtropics they break, deposit westward momentum and retard the flow.

16.3 ✦ CRITICAL LINES AND CRITICAL LAYERS

We now look a little more closely at the behaviour of Rossby waves near a critical line and then, more briefly, at gravity wave behaviour. Critical layer theory is extensive and technical, and our discussion only scratches the surface and is mostly linear; readers wishing for more should go to the literature.[2] We use Northern Hemisphere conventions, envisioning waves propagating southward toward a subtropical critical line.

16.3.1 Preliminaries

Consider horizontally propagating Rossby waves obeying the linear barotropic vorticity equation on the beta-plane (vertically propagating waves may be considered using similar techniques). The equation of motion is

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\nabla^2\psi' + \beta^*\frac{\partial \psi'}{\partial x} = -r\nabla^2\psi', \tag{16.25}$$

where $\beta^* = \beta - \partial_y^2\overline{u}$. The parameter r is a drag coefficient that acts directly on the relative vorticity, and we shall assume that it is small compared to the Doppler-shifted frequency of the waves, except possibly near a critical line. It is not a particularly realistic form of dissipation but it is simple and captures the essential process. We seek solutions of the form

$$\psi'(x, y, t) = \widetilde{\psi}(y)\mathrm{e}^{\mathrm{i}(k(x-ct))}. \tag{16.26}$$

Substituting into (16.25) we find, after a couple of lines of algebra, that $\widetilde{\psi}$ satisfies, analogously to (16.15),

$$\frac{\mathrm{d}^2\widetilde{\psi}}{\mathrm{d}y^2} + l^2(y)\widetilde{\psi} = 0, \quad \text{where} \quad l^2(y) = \frac{\beta^*}{\overline{u} - c - \mathrm{i}r/k} - k^2. \tag{16.27a,b}$$

If the zonal wind has a lateral shear then β^* may vary with y, and thus so does l. If r is non-zero then l has an imaginary component so that the wave decays away from its source region, and as $\overline{u} \to c$ the decay will be particularly strong.

Suppose that the friction is zero. Near the critical line the y-wavenumber will be much larger than k and the streamfunction will obey an equation of the form

$$\frac{\mathrm{d}^2\widetilde{\psi}}{\mathrm{d}y^2} \approx -\frac{\beta^*}{\overline{u} - c}\widetilde{\psi} = -\frac{A}{y - y_c}\widetilde{\psi}, \tag{16.28}$$

where A is a constant, and by shifting the origin we will take $y_c = 0$. By inspection an approximate solution to this equation for small y is

$$\frac{\mathrm{d}\widetilde{\psi}}{\mathrm{d}y} = B \ln y, \qquad \widetilde{\psi} = B\left(y \ln y - y - \frac{1}{A}\right) \approx -\frac{B}{A}. \tag{16.29}$$

where B is a constant. (Equation (16.28) actually has Bessel function solutions.) The vorticity, $\widetilde{\psi}_{yy}$ then goes as $1/y$ and the velocity goes as $\ln y$ near the critical line. Both quantities blow up, *but the streamfunction itself does not*. The pseudomomentum also blows up as the critical layer. To see this, multiply (16.25) by ζ/β^* and zonally average to give

$$\frac{\partial \mathcal{P}}{\partial t} + \frac{\partial \mathcal{F}}{\partial y} = -\alpha \mathcal{P}, \tag{16.30}$$

where $\mathcal{P} = \overline{\zeta'^2}/2\beta^*$ is the pseudomomentum, $\partial \mathcal{F}/\partial y = -\partial_y(\overline{u'v'}) = \overline{v'\zeta'}$ is its flux divergence, and $\alpha = 2r$. If $\alpha = 0$ then $\mathcal{P}$ must blow up at the critical line because ζ does. Finally, the WKB approximation fails approaching a critical line: equations (16.27) and (16.28) tell us that

$$l^2 \sim \frac{1}{y} \qquad \text{and} \qquad \frac{\mathrm{d}l}{\mathrm{d}y} \sim \frac{1}{y^{3/2}}\,. \tag{16.31}$$

Thus, for small y, $\mathrm{d}l/\mathrm{d}y > l^2$ because $y^{-3/2} > y^{-1}$ for small y. The WKB condition that the wavenumber, l, varies more slowly than l^2 (requiring, as in (6.164), that $\mathrm{d}l/\mathrm{d}y \ll l^2$) is not satisfied.

Behaviour near a critical line

A detailed analysis is required to determine what does happen at the critical line, but in the linear problem we can make some useful headway. As in Fig. 16.5 we imagine there is a small region surrounding the critical line — the critical layer — in which either the nonlinear terms or the frictional terms, or both, are important. In the critical layer y derivatives are much larger than x derivatives so that $\zeta' \approx -\partial u'/\partial y = \partial^2\psi'/\partial y^2$. Furthermore, since $\overline{u}-c$ is small, a Taylor expansion gives

$$\overline{u} - c \approx y \left.\frac{\partial \overline{u}}{\partial y}\right|_{\overline{u}=c}, \tag{16.32}$$

and we will denote the derivative as $\partial_y \overline{u}_c$. Using (16.27) with $l^2 \gg k^2$ then gives

$$\tilde{\zeta} = \frac{-\beta^*\widetilde{\psi}}{\overline{u} - c - \mathrm{i}r/k} \approx \frac{-\beta^*\widetilde{\psi}}{y\partial_y\overline{u}_c - \mathrm{i}r/k} = \frac{-\widehat{\beta}\widetilde{\psi}}{y - \mathrm{i}\hat{r}} = \frac{-\widehat{\beta}(y + \mathrm{i}\hat{r})\widetilde{\psi}}{y^2 + \hat{r}^2}, \tag{16.33}$$

where $\widehat{\beta} = \beta^*/\partial_y \overline{u}_c$ and $\hat{r} = r/(k\partial_y \overline{u}_c)$. The vorticity flux in the critical layer is given by

$$\overline{v'\zeta'} = \frac{1}{2}\mathrm{Re}\,\overline{\mathrm{i}k\widetilde{\psi}\tilde{\zeta}^*} = \frac{-k\widehat{\beta}\hat{r}}{2(y^2+\hat{r}^2)}|\widetilde{\psi}^2|. \tag{16.34}$$

The factor of 2 comes from the averaging, and only the part of $\tilde{\zeta}$ proportional to $\mathrm{i}\hat{r}$ contributes because of its phase relative to $\mathrm{i}\widetilde{\psi}$. There are two interesting aspects to (16.34):

(i) The vorticity flux is negative. (All the individual terms, including k, are positive.)

(ii) The stream function is almost constant in the critical layer, so that for small friction (16.34) is sharply peaked around $y = 0$. It tends to a delta-function as $\hat{r} \to 0$, because

$$\delta(y) = \frac{1}{\pi}\lim_{\hat{r}\to 0}\frac{\hat{r}}{y^2+\hat{r}^2}\,. \tag{16.35}$$

We can see that the thickness of the critical layer, δ_r is just $\hat{r}$.

The first property above tells us that the critical layer is dissipative. The second property tells us that the eddy momentum flux has a finite jump across the critical line. For small r we have

$$\left[\overline{u'v'}\right]_-^+ = -\int_-^+ \overline{\zeta'v'}\,\mathrm{d}y = \int_-^+ \frac{-k\widehat{\beta}\hat{r}}{2(y^2+\hat{r}^2)}|\widetilde{\psi}^2|\,\mathrm{d}y = -\int_-^+ \frac{1}{2}k\widehat{\beta}\pi|\widetilde{\psi}^2|\delta(y)\,\mathrm{d}y = -\frac{1}{2}k\widehat{\beta}\pi|\widetilde{\psi}^2|, \tag{16.36}$$

where the integrals are over the critical layer, and the momentum flux therefore diminishes across it. The last term on the right-hand side has no dependence on r, which means *there is finite absorption, even as friction tends to zero.* There will be no transmission through the critical layer, as we can see using a wave activity argument. For $y < 0$ (south of the critical line) the waves evanesce, but if (as we assume) friction there is negligible then in a statistically steady $\partial_y\overline{u'v'} = 0$. Thus $\overline{u'v'} = 0$ at the southern edge of the critical line, and therefore pseudomomentum must fall to zero across the critical line. Put simply, the eddy momentum fluxes are zero far from the critical layer (because waves evanesce), but the flux does not vary with y so it is zero at the southern edge of the critical layer. In fact, in such a layer there is also no reflection, and absorption is complete.

The zonal velocity itself also jumps across the layer. Since the streamfunction, and hence $v' = \partial\psi'/\partial x$, vary only weakly across the critical layer, $\zeta' \approx -\partial u'/\partial y$. The jump in velocity across the critical layer is then given by

$$[u']_-^+ = -\int_-^+ \zeta'\,\mathrm{d}y, \tag{16.37a,b}$$

which also may be calculated.

Nonlinear effects

We will say only a few words about this problem. First consider the critical layer thickness. Another way to evaluate the thickness in the linear problem is to directly note that the relative sizes of the advection term and the frictional term in the equation of motion is $y\partial_y\overline{u}_c/r$ so that an estimate for the frictional critical layer thickness is $\delta_r \sim r/(k\partial_y\overline{u}_c)$, as before. We can use a similar argument in the nonlinear case, and if we suppose a balance between the nonlinear terms and advection of the mean flow then the critical layer thickness, δ_{nl}, can be estimated from

$$y\frac{\partial\overline{u}_c}{\partial y}\frac{\partial\zeta'}{\partial x} \sim v'\frac{\partial\zeta'}{\partial y} \sim k\psi'\frac{\zeta'}{\delta} \qquad \text{whence} \qquad \delta_{nl} \sim \left|\frac{v'}{k\partial_y\overline{u}_c}\right|^{1/2} \sim \left|\frac{\psi'}{\partial_y\overline{u}_c}\right|^{1/2}, \tag{16.38}$$

where v' is the meridional velocity near the critical layer. In Earth's atmosphere the nonlinear terms typically *are* important (and $\delta_{nl} > \delta_r$). Nevertheless, dissipation can and normally will occur in a nonlinear critical layer, either directly by the drag term in (16.25) or because the nonlinear interactions lead to wave breaking and a nonlinear cascade to dissipation within the layer itself, and that dissipation will retard the mean flow (i.e., make it more westward). This effect is important in the subtropics, where Rossby waves propagating equatorward from mid-latitudes in the upper troposphere encounter a critical level, and generate a critical layer. Rossby waves propagating upwards toward and into the stratosphere can also encounter a critical level, contributing to the QBO phenomenon described in Section 17.6.

If the critical layer is thin, then the total flow is a superposition of the background flow, $\overline{u}(y) \approx y\partial\overline{u}_c/\partial y$ and the disturbance field. The disturbance streamfunction, ψ' varies like $\exp(ikx)$, with no y variation because it is continuous across the critical line, so that the total streamfunction varies like

$$\psi \approx -\frac{1}{2}\frac{\partial\overline{u}_c}{\partial y}y^2 + \operatorname{Re}\psi'(0)\mathrm{e}^{ikx}. \tag{16.39}$$

This gives a pattern like that illustrated in Fig. 16.5, known as a Kelvin cat's eye pattern.

The dynamics of the pattern determines the value of $\int \overline{v'\zeta'}$ which in turn determines the value of $\overline{u'v'}$ at the edges, as in (16.36). If this value is reduced from the value of the incoming waves then the critical layer is reflecting, but determining whether this is so requires a more detailed analysis than we can provide here.

16.3.2 Internal Gravity Wave Critical Layers

Gravity wave critical layers have a rather different character than Rossby wave critical layers because of the nature of the dispersion relation. Referring back to (7.53) on page 259, we linearize the Boussinesq equations about a sheared mean flow $\overline{u}(z)$. Confining attention to two dimensions, x and z, then a little algebra results in the 'Taylor–Goldstein' equation,

$$\frac{\mathrm{d}^2\widetilde{\Psi}}{\mathrm{d}z^2} + \left[\frac{N^2}{(\overline{u}-c)^2} - \frac{\overline{u}_{zz}}{\overline{u}-c} - k^2\right]\widetilde{\Psi} = 0. \tag{16.40}$$

Here, $\widetilde{\Psi}$ is the amplitude of the streamfunction in the vertical plane, with $(u, w) = (-\partial\Psi/\partial z, \partial\Psi/\partial x)$. The presence of the first term in square brackets gives the equation a different nature than the Rossby problem, since if we omit the second term the equation has the form

$$\frac{\mathrm{d}^2\widetilde{\Psi}}{\mathrm{d}z^2} \sim -\frac{Ri\,\widetilde{\Psi}}{z^2}, \tag{16.41}$$

where $Ri = N^2/(\partial_z\overline{u}_c)^2$ is the Richardson number. This means that, if the flow is stable, the waves oscillate *extremely* rapidly as the critical line is approached — in fact an infinite number of times — and the group velocity also diminishes rapidly. (Solutions to (16.41) have the form $\widetilde{\psi} = z^\alpha$ where $\alpha = (1 \pm \sqrt{1-4\,Ri})/2$ and are oscillatory if $Ri > 1/4$, which is the condition for flow stability.) The slowdown gives dissipation more time to act, and commonly a wave will completely dissipate before the critical line is reached. If nonlinearity is allowed, the flow will break down into turbulence near the critical layer with a rapid cascade to dissipative scales.

The WKB approximation and notions of group velocity remain valid to a much greater degree than in the Rossby wave case. From (16.41) the vertical wavenumber, m, obeys

$$m^2 \sim \frac{1}{z^2}, \qquad \frac{\mathrm{d}m}{\mathrm{d}z} \sim \frac{1}{z^2} \sim m^2, \tag{16.42}$$

Thus, the WKB condition is neither obviously well satisfied nor badly violated, and we can expect group velocity and WKB theory to provide useful information much closer to the critical line than in the Rossby wave case. In particular, a Rossby wave will typically reach a critical layer even as the group velocity stalls, whereas a gravity wave may be absorbed before that. Aspects of the gravity wave analysis are continued in the discussion of the stratosphere in Section 17.3.3.

16.4 ✦ A WKB WAVE–MEAN-FLOW PROBLEM FOR ROSSBY WAVES

Let us now assume that the background properties do vary slowly and see how far we can get with a WKB approximation. As we saw above, WKB theory fails approaching a critical line, so we cannot determine what happens when $\overline{u} - c$ is very small, but the analysis is nevertheless instructive. If the friction is small and $r \ll k(\overline{u} - c)$, and if the meridional wavenumber l is larger than the zonal wavenumber k then l is given by

$$l^2(y) \approx \left[\frac{\beta^*(\overline{u} - c + \mathrm{i}r/k)}{(\overline{u} - c)^2 + r^2/k^2}\right] \approx \frac{\beta^*}{\overline{u} - c}\left[1 + \frac{\mathrm{i}r}{k(\overline{u} - c)}\right], \tag{16.43}$$

whence

$$l(y) \approx \left(\frac{\beta^*}{\overline{u} - c}\right)^{1/2}\left[1 + \frac{\mathrm{i}r}{2k(\overline{u} - c)}\right]. \tag{16.44}$$

The solution for the streamfunction is given by, in the WKB approximation,

$$\widetilde{\psi} = A l^{-1/2} \exp\left(\pm \mathrm{i} \int^y l \, \mathrm{d}y'\right), \tag{16.45}$$

just as in (16.16), but now the wave will decay as it moves away from its source and deposit momentum into the mean flow. Let us calculate this.

The momentum flux, F_k, associated with an x-wavenumber of k is given by

$$F_k(y) = \overline{u'v'} = -\mathrm{i}k\left(\psi \frac{\partial \psi^*}{\partial y} - \psi^* \frac{\partial \psi}{\partial y}\right), \tag{16.46}$$

and using (16.44) and (16.45) in (16.46) we obtain

$$F_k(y) = F_0 \exp\left(\int_0^y \frac{\pm r \beta^{*1/2}}{k(\overline{u} - c)^{3/2}} \, \mathrm{d}y'\right). \tag{16.47}$$

In deriving this expression we use the fact that the amplitude of $\widetilde{\psi}$ (i.e., $l^{-1/2}$) varies only slowly with y so that when calculating $\partial \widetilde{\psi}/\partial y$ the derivative of l may be ignored. In (16.47) F_0 is the value of the flux at $y = 0$ and the sign of the exponent must be chosen so that the group velocity is directed away from the wave source region. Clearly, if $r = 0$ then the momentum flux is constant.

The integrand in (16.47) is the attenuation rate of the wave and it has a straightforward physical interpretation. Using the real part of (16.44) in (16.17b), and assuming $|l| \gg |k|$, the meridional component of the group velocity is given by

$$c_g^y = \frac{2kl\,\beta^*}{(k^2 + l^2)^2} \approx \frac{2k\,\beta^*}{l^3} = \frac{2k(\overline{u} - c)^{3/2}}{\beta^{*1/2}}. \tag{16.48a,b}$$

Thus we have

$$\text{Wave attenuation rate} = \frac{r\beta^{*1/2}}{k(\overline{u} - c)^{3/2}} = \frac{2 \times \text{Dissipation rate } (2r)}{\text{Meridional group velocity } (c_g^y)}. \tag{16.49}$$

This result is of some generality in wave dynamics, and a simple interpretation is that as the group velocity diminishes the dissipation has more time to act and the wave is preferentially attenuated.

How does this attenuation affect the mean flow? The mean flow is subject to many waves and so obeys the equation

$$\frac{\partial \overline{u}}{\partial t} = -\sum_k \frac{\partial F_k}{\partial y} + \text{viscous terms}. \tag{16.50}$$

Because the amplitude varies only slowly compared to the phase, the amplitude of $\partial F_k/\partial y$ varies mainly with the attenuation rate (16.49). Consider a Rossby wave propagating away from some source region with a given frequency and x-wavenumber. Because k is negative a Rossby wave always carries westward (or negative) momentum with it. That is, F_k is always negative and increases (becomes more positive) as the wave is attenuated; that is to say, if $r \neq 0$ then $\partial F_k/\partial y$ is positive and from (16.50) the mean flow is accelerated *westward* as the wave dissipates. The dissipation, and attendant acceleration, will be particularly strong as the wave approaches a critical line where $\overline{u} = c$, although here the quantitative aspects of the analysis begin to fail.

The situation arises when Rossby waves, generated in mid-latitudes, propagate equatorward. As the waves enter the subtropics $\overline{u} - c$ becomes smaller and the waves dissipate, producing a westward force on the mean flow. Globally, momentum is conserved because there is an equal and opposite (and therefore eastward) wave force at the wave source producing an eastward eddy-driven jet, as discussed in the previous chapter.

Interpretation using wave activity

We can derive and interpret the above results by thinking about the propagation of wave activity, specifically the pseudomomentum given by (16.30). Referring as needed to the discussion in Sections 10.2.1 and 10.2.2, the flux obeys the group velocity property so that

$$\frac{\partial \mathcal{P}}{\partial t} + \frac{\partial}{\partial y}(\boldsymbol{c}_g \mathcal{P}) = -2r\mathcal{P}. \tag{16.51}$$

Let us suppose that the wave is in a statistical steady state and that the spatial variation of the group velocity occurs on a longer spatial scale than the variations in wave activity, consistent with the WKB assumption that group velocity varies slowly. We then have

$$c_g^y \frac{\partial \mathcal{P}}{\partial y} = -2r\mathcal{P}. \tag{16.52}$$

which integrates to give

$$\mathcal{P}(y) = \mathcal{P}_0 \exp\left(-\int^y \frac{2r}{c_g^y}\,\mathrm{d}y'\right). \tag{16.53}$$

That is, the attenuation rate of the wave activity is the dissipation rate of wave activity divided by the group velocity, as in (16.47) and (16.49). The wave-activity method of derivation suggests that this result is a general one, not restricted to Rossby waves, and indeed in Section 17.3.2 we will find that the attenuation rate of vertically propagating gravity waves is given by a similar expression.

The divergence of wave activity will lead to a force on the mean zonal flow, much as discussed in Section 15.1. For definiteness, suppose that waves propagate away from a mid-latitude source in the Northern Hemisphere. South of the source c_g^y is negative and north of the source c_g^y is positive. In either case, from (16.53) the wave activity density decreases away from the source and, with reference to (15.34a), the ensuing force on the mean flow is negative, or westward.

16.5 VERTICAL PROPAGATION OF ROSSBY WAVES

We now consider the vertical propagation of Rossby waves in a stratified atmosphere. The vertical propagation is important both because it must be taken into account to obtain an accurate picture of the tropospheric response to topographic and thermal forcing, and because it can excite motion in the stratosphere, as considered in Chapter 17. We will continue to use the stratified quasi-geostrophic equations, but we now allow the model to be compressible and semi-infinite, extending from $z = 0$ to $z = \infty$. It is simplest to first consider the problem slightly generally, without regard to boundary conditions; in Section 16.5.2 we will consider the lower boundary conditions and the requirements for waves to propagate vertically into the stratosphere. Our governing equation is the quasi-geostrophic potential vorticity equation, and with applications to the stratosphere in mind we will use log-pressure coordinates so that the equation of motion is

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0, \qquad q = \nabla^2\psi + \beta y + \frac{f_0^2}{\rho_R}\frac{\partial}{\partial z}\left(\frac{\rho_R}{N^2}\frac{\partial \psi}{\partial z}\right), \tag{16.54}$$

where $z = H\ln(p/p_0)$ and $\rho_R = \rho_0 \mathrm{e}^{-z/H}$ with H being a specified density scale height, typically $RT(0)/g$.

16.5.1 Conditions for Wave Propagation

Let us linearize (16.54) about a zonal wind that depends only on z; that is, we let

$$\psi = -\overline{u}(z)y + \psi', \tag{16.55}$$

and obtain

$$\frac{\partial q'}{\partial t} + \overline{u}\frac{\partial q'}{\partial x} + v'\frac{\partial \overline{q}}{\partial y} = 0, \qquad \frac{\partial \overline{q}}{\partial y} = \beta - \frac{f_0^2}{\rho_R}\frac{\partial}{\partial z}\left(\frac{\rho_R}{N^2}\frac{\partial \overline{u}}{\partial z}\right), \tag{16.56}$$

or equivalently, in terms of streamfunction,

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\left[\nabla^2\psi' + \frac{f_0^2}{\rho_R}\frac{\partial}{\partial z}\left(\frac{\rho_R}{N^2}\frac{\partial \psi'}{\partial z}\right)\right] + \frac{\partial \psi'}{\partial x}\left[\beta - \frac{f_0^2}{\rho_R}\frac{\partial}{\partial z}\left(\frac{\rho_R}{N^2}\frac{\partial \overline{u}}{\partial z}\right)\right] = 0. \tag{16.57}$$

The first term in square brackets is the perturbation potential vorticity, q' and the second term equals $\partial\overline{q}/\partial y$. Seeking solutions of the form $\psi' = \mathrm{Re}\,\widetilde{\psi}(z)\exp[\mathrm{i}(kx + ly - kct)]$ gives

$$\left[\frac{f_0^2}{\rho_R}\frac{\partial}{\partial z}\left(\frac{\rho_R}{N^2}\frac{\partial \widetilde{\psi}}{\partial z}\right)\right] = \widetilde{\psi}\left(K^2 - \frac{\partial\overline{q}/\partial y}{\overline{u} - c}\right). \tag{16.58}$$

Let us simplify by assuming that both $\overline{u}$ and N^2 are constants so that $\partial\overline{q}/\partial y = \beta$. Equation (16.58) further simplifies if we define

$$\Phi(z) = \widetilde{\psi}(z)\left(\frac{\rho_R}{\rho_R(0)}\right)^{1/2} = \widetilde{\psi}(z)\mathrm{e}^{-z/2H} \tag{16.59}$$

whence we obtain

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}z^2} + m^2\Phi = 0, \quad \text{where} \quad m^2 = \frac{N^2}{f_0^2}\left(\frac{\beta}{\overline{u} - c} - K^2 - \gamma^2\right), \tag{16.60a,b}$$

where $\gamma^2 = f_0^2/(4N^2H^2) = 1/(2L_d)^2$ and where L_d is the deformation radius as sometimes defined (i.e., $L_d = NH/f_0$). The above equation has the same form as (16.10b). If the parameters on the right-hand side of (16.60b) are constant then so is m, and (16.60) has solutions of the form $\Phi(z) = \Phi_0\mathrm{e}^{\mathrm{i}mz}$, so that the streamfunction itself varies as

$$\psi' = \mathrm{Re}\,\Phi_0\exp\left[\mathrm{i}(kx + ly + mz - kct) + z/2H\right]. \tag{16.61}$$

In the (more realistic) case in which m varies with height then, if the variation is slow enough, the solution looks locally like a plane wave with m being a slowly varying vertical wavenumber, and WKB techniques may be used to find a solution, as we discuss further in Section 16.6. But

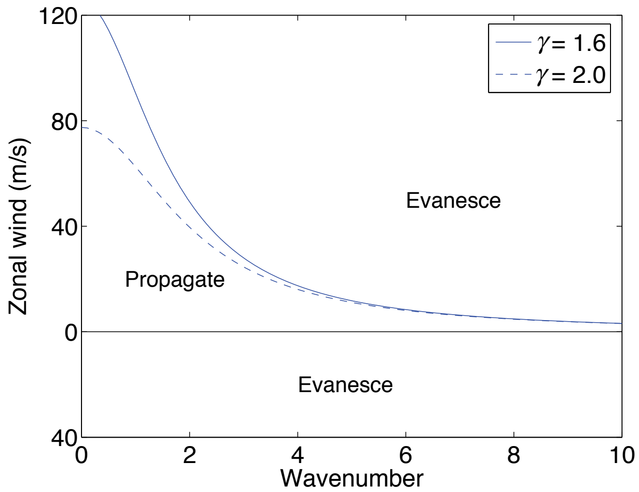

Fig. 16.6 The boundary between propagating waves and evanescent waves as a function of zonal wind and wavenumber, using (16.63), for a couple of values of γ. With $N = 2 \times 10^{-2}\,\mathrm{s}^{-1}$, $\gamma = 1.6$ ($\gamma = 2$) corresponds to a scale height of 7.0 km (5.5 km) and a deformation radius NH/f of 1400 km (1100 km).

even then essentially the same condition for propagation applies, namely that $m^2 > 0$ and, using (16.60b), this condition is satisfied if

$$\boxed{0 < \overline{u} - c < \frac{\beta}{K^2 + \gamma^2}.} \tag{16.62}$$

This condition is obviously similar to (16.12). For waves of some given frequency ($\omega = kc$) the above expression provides a condition on $\overline{u}$ for the vertical propagation of planetary waves. For stationary waves, $c = 0$, the condition becomes

$$\boxed{0 < \overline{u} < \frac{\beta}{K^2 + \gamma^2}.} \tag{16.63}$$

That is, in words, stationary, vertically oscillatory modes can exist only for zonal flows that are eastwards and that are less than the critical velocity $U_c = \beta/(K^2 + \gamma^2)$. This criterion, known as the *Charney–Drazin condition*,[4] is illustrated in Fig. 16.6 and we return to it in Section 16.5.3. The critical velocity for stationary waves evidently depends on the scale of the wave. For waves of a non-zero frequency the criterion is less severe, but stationary waves have a particular importance because they can be readily generated by surface topography.

Dispersion relation and group velocity

Noting that $\omega = ck$ and rearranging (16.60b) we obtain the dispersion relation for three-dimensional Rossby waves, namely

$$\omega = \overline{u}k - \frac{\beta k}{K^2 + \gamma^2 + m^2 f_0^2/N^2}. \tag{16.64}$$

The three components of the group velocity for these waves are then:

$$c_g^x = \overline{u} + \frac{\beta[k^2 - (l^2 + m^2 f_0^2/N^2 + \gamma^2)]}{(K^2 + m^2 f_0^2/N^2 + \gamma^2)^2}\,, \tag{16.65a}$$

$$c_g^y = \frac{2\beta kl}{(K^2 + m^2 f_0^2/N^2 + \gamma^2)^2}\,, \qquad c_g^z = \frac{2\beta km f_0^2/N^2}{(K^2 + m^2 f_0^2/N^2 + \gamma^2)^2}\,. \tag{16.65b,c}$$

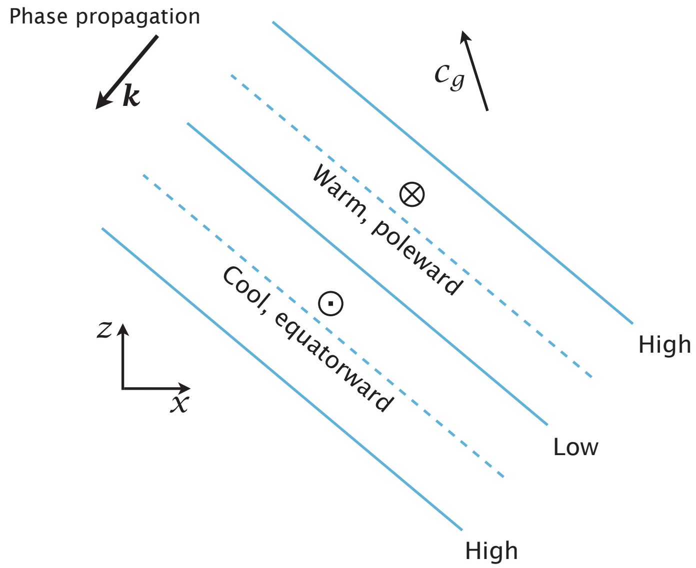

Fig. 16.7 East–west section of an upwardly propagating Rossby wave. The slanting lines are lines of constant phase and 'high' and 'low' refer to the pressure or streamfunction values. Both k and m are negative so the phase lines are oriented up and to the west. The phase propagates westward and downward, but the group velocity is upward.

The propagation in the horizontal is analogous to the propagation in a shallow water model, as in (6.66b); we see that higher baroclinic modes (bigger m) will have a more westward group velocity. The vertical group velocity is proportional to m, and for waves that propagate signals upward we must choose m to have the same sign as k so that c_g^z is positive. If there is no mean flow then the zonal wavenumber k is negative (in order that frequency is positive) and m must then also be negative. Energy then propagates upward but the phase propagates downward.

16.5.2 ✦ A Solution for Topographically-excited Waves

We now derive some explicit solutions for Rossby waves excited at a lower boundary by topography. Rossby waves may also be excited by thermal anomalies at the lower boundary, although in Earth's atmosphere their amplitude is somewhat smaller, and the treatment of such waves is left to the reader.[5] The lower boundary is obtained using the thermodynamic equation,

$$\frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial z}\right) + J\left(\psi, \frac{\partial \psi}{\partial z}\right) + \frac{N^2}{f_0} w = 0, \tag{16.66}$$

along with an equation for the vertical velocity, w, at the lower boundary. This is

$$w = \boldsymbol{u} \cdot \nabla h_b + r\zeta, \tag{16.67}$$

where the two terms respectively represent the kinematic contribution to vertical velocity due to flow over topography and the contribution from Ekman pumping, with r a constant, and the effects are taken to be additive. Linearizing the thermodynamic equation about the zonal flow and using (16.67) gives the boundary condition at $z - 0$,

$$\frac{\partial}{\partial t}\left(\frac{\partial \psi'}{\partial z}\right) + \overline{u}\frac{\partial}{\partial x}\frac{\partial \psi'}{\partial z} - v'\frac{\partial \overline{u}}{\partial z} = -\frac{N^2}{f_0}\left(\overline{u}\frac{\partial h_b}{\partial x} + r\nabla^2\psi'\right), \tag{16.68}$$

Solution

We look for solutions of (16.56) and (16.68) in the form

$$\psi' = \operatorname{Re}\widetilde{\psi}(z)\sin ly\,\mathrm{e}^{\mathrm{i}k(x-ct)} \qquad \text{with} \qquad h_b = \operatorname{Re}\widetilde{h}_b \sin ly\,\mathrm{e}^{\mathrm{i}kx}, \tag{16.69}$$

with $\widetilde{h}_b$ being purely real. Solutions must then satisfy

$$\left[\frac{f_0^2}{\rho_R}\frac{\partial}{\partial z}\left(\frac{\rho_R}{N^2}\frac{\partial\widetilde{\psi}}{\partial z}\right)\right] = \widetilde{\psi}\left(K^2 - \frac{\partial\overline{q}/\partial y}{\overline{u}-c}\right) \tag{16.70}$$

in the interior, and the boundary condition

$$(\overline{u}-c)\frac{\partial\widetilde{\psi}}{\partial z} - \widetilde{\psi}\frac{\partial\overline{u}}{\partial z} + \frac{\mathrm{i}rN^2K^2}{kf_0}\widetilde{\psi} = -\frac{N^2\overline{u}\widetilde{h}_b}{f_0}, \qquad \text{at } z=0, \tag{16.71}$$

as well as a radiation condition at plus infinity (and we must have that $\rho_0\widetilde{\psi}^2$ be finite). Let us simplify by considering the case of constant $\overline{u}$ and N^2 and with $r = 0$. As before we let $\Phi(z) = \widetilde{\psi}(z)\exp(-z/2H)$ and obtain the interior equation

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}z^2} + m^2\Phi = 0, \qquad \text{where} \quad m^2 = \frac{N^2}{f_0^2}\left(\frac{\beta}{\overline{u}-c} - K^2 - \gamma^2\right), \tag{16.72a,b}$$

and $\gamma^2 = f_0^2/(4N^2H^2) = 1/(2L_d)^2$, where L_d is the deformation radius, which is essentially the same as (16.60). The conditions for wave propagagation, and in particular the Charney–Drazin condition, are just as previously derived. The surface boundary condition is now

$$(\overline{u}-c)\left(\frac{\mathrm{d}\Phi}{\mathrm{d}z} + \frac{\Phi}{2H}\right) = -\frac{N^2\overline{u}\widetilde{h}_b}{f_0}, \qquad \text{at } z=0. \tag{16.73}$$

This leads to an expression for the streamfunction amplitude, namely

$$\Phi_0 = \frac{N^2\widetilde{h}_b/f_0}{(\alpha, -\mathrm{i}m) - (2H)^{-1}}, \qquad \text{where} \qquad \alpha = +\frac{N}{f_0}\left(K^2 + \gamma^2 - \frac{\beta}{\overline{u}}\right)^{1/2}, \tag{16.74a,b}$$

and $(\alpha, -\mathrm{i}m)$ refers to the (trapped, oscillatory) case. A little algebra gives the solutions in the form

$$\psi'(x,y,z) = \text{Re } \exp[\mathrm{i}(kx+mz) + z/2H] \sin ly \frac{f_0\widetilde{h}_b\left[\mathrm{i}m - (2H)^{-1}\right]}{K_s^2 - K^2}, \qquad m^2 > 0 \tag{16.75a}$$

$$\psi'(x,y,z) = \text{Re } \exp[(2H)^{-1} - \alpha z + \mathrm{i}kx] \sin ly \frac{N^2\widetilde{h}_b}{f_0\left[\alpha - (2H)^{-1}\right]}, \qquad m^2 < 0 \tag{16.75b}$$

where $K_s^2 = \beta/\overline{u}$.

Resonance is possible when $\alpha = 1/(2H)$ or $K^2 = K_s^2$ and this condition obtains when barotropic Rossby waves are stationary. The wave resonates because the wave is a solution of the unforced, inviscid equations for the barotropic wave. If $K > K_s$ then $\alpha > 1/(2H)$ and the forced wave (i.e., the amplitude of ψ) decays with height with no phase variation. If $\alpha < 1/(2H)$ then $\widetilde{\psi}$ increases with height (although $\rho_R|\widetilde{\psi}|^2$ decreases with height), and this occurs when $(K_s^2 - \gamma^2)^{1/2} < K < K_s$. If $(K_s^2 - \gamma^2)^{1/2} > K$ then the amplitude of ϕ, (i.e., $\rho_R|\widetilde{\psi}|^2$) is independent of height; their vertical structure is oscillatory, like $\exp(\mathrm{i}mz)$.

16.5.3 Properties of the Solutions

Upward propagation and the Charney–Drazin condition

Let us return to the criterion for upward propagation given in (16.63) and illustrated in Fig. 16.6. One way to interpret this condition is to note that in a resting medium the Rossby wave frequency has a minimum value (and maximum absolute value), when $m = 0$, of

$$\omega = -\frac{\beta k}{K^2 + \gamma^2}. \tag{16.76}$$

Suppose that the waves are generated by bottom topography, and that $\overline{u}$ is uniform. In a frame moving with speed $\overline{u}$ our Rossby waves (stationary in the Earth's frame) have frequency $-\overline{u}k$, and this is the forcing frequency arising from the now-moving bottom topography. Thus, (16.63) is equivalent to saying that for oscillatory waves to exist *the forcing frequency must lie within the frequency range of vertically propagating Rossby waves.*

For westward flow, or for sufficiently strong eastward flow, the waves decay exponentially as $\Phi = \Phi_0 \exp(-\alpha z)$ where α is given by (16.74b). The critical velocity $u_c = (\beta/K^2 + \gamma^2)$ is a function of wavenumber, increasing with horizontal wavelength. Thus, for a given eastward flow long waves may penetrate vertically when short waves are trapped, an effect sometimes referred to as 'Charney–Drazin filtering'. There are three important consequences of this:

(i) Stratospheric motion is typically of larger horizontal scale than that of the troposphere, because Rossby waves tend to be excited first in the troposphere (by both baroclinic instability and flow over topography), but the shorter waves are trapped and only the longer ones reach the stratosphere.

(ii) The Rossby waves more commonly reach the stratosphere in the Northern Hemisphere than in the Southern Hemisphere. In both hemispheres the shorter, baroclinically forced Rossby waves are filtered, but the Northern Hemisphere also generates long Rossby waves through topographic interactions. The overturning circulation in the stratosphere (that is, the Brewer–Dobson circulation) is therefore stronger in the Northern Hemisphere.

(iii) Rossby waves find it hard to reach the stratosphere in summer, for then the stratospheric winds are often westwards (because the pole is warmer than the equator) and all waves are trapped in the troposphere. The eastward stratospheric winds that favour vertical penetration occur in the other three seasons, although very strong eastward winds can suppress penetration in mid-winter.

Other properties of the solution

Various other properties of the solution are described below, and a summary is given in the box on the next page.

Amplitudes and phases. The decaying solutions have no vertical phase variations (they are 'equivalent barotropic') and the streamfunction is exactly in phase or out of phase with the topography according as $K > K_s$ and $\alpha > (2H)^{-1}$, or $K < K_s$ and $\alpha < (2H)^{-1}$. In the latter case the amplitude of the streamfunction actually increases with height, but the energy, proportional to $\rho_R|\psi'^2|$ falls. The oscillatory solutions have (if there is no shear) constant energy with height but a shifting phase. The phase of the streamfunction at the surface may be in or out of phase with the topography, depending on m, but the potential temperature, $\partial\psi/\partial z$ is always out of phase with the topography. That is, positive values of h_b are associated with cool fluid parcels.

Vertical energy propagation. As noted, the energy propagates upwards for the oscillatory waves. This may be verified by calculating $\overline{p'w'}$ (the vertical component of the energy flux), where p' is the pressure perturbation, proportional to ψ', and w' is the vertical velocity perturbation. To this end, linearize the thermodynamic equation (16.66) to give

$$\frac{\partial}{\partial t}\left(\frac{\partial\psi'}{\partial z}\right) + \overline{u}\frac{\partial}{\partial x}\frac{\partial\psi'}{\partial z} - \frac{\partial\overline{u}}{\partial z}\frac{\partial\psi'}{\partial x} + \frac{N^2}{f_0}w' = 0. \tag{16.77}$$

Then, multiplying by ψ' and integrating by parts gives a balance between the second and fourth terms,

$$N^2\overline{\psi'w'} = \overline{u}\overline{b'v'}, \tag{16.78}$$

Upward Propagating Rossby Waves

In general Rossby waves satisfy

$$\psi' = \mathrm{Re}\, \Phi_0 \exp\left[\mathrm{i}(kx + ly + mz - kct) + z/2H\right], \tag{T.1}$$

where Φ_0 is a constant determined by the lower boundary conditions and

$$m = \pm\frac{N}{f_0}\left(\frac{\beta}{\overline{u}} - K^2 - \gamma^2\right)^{1/2}, \quad \text{with} \quad \gamma = \frac{f_0}{2NH}. \tag{T.2}$$

If $m^2 > 0$ the solutions are propagating, or radiating, waves in the vertical direction. If $m^2 < 0$ the energy of the solution, $|\rho_R \psi'^2|$, is vertically evanescent. The condition $m^2 > 0$ is equivalent to

$$0 < \overline{u} < \frac{\beta}{k^2 + l^2 + (f_0/2NH)^2}, \tag{T.3}$$

which is known as the Charney–Drazin condition. Vertical penetration is favoured when the winds are weakly eastwards, and the range of $\overline{u}$-values that allows this is larger for longer waves. Some other properties of the solution are:

- In order that the energy propagate upwards the vertical component of the group velocity must be positive, and hence k and m must have the same sign.
- The meridional heat flux is proportional to km, and thus upward propagation of waves is associated with a poleward heat transport.
- In an atmosphere in which density falls exponentially with height the amplitude of the streamfunction grows exponentially, so eventually nonlinear terms will become important. The waves may break, even in the absence of a critical layer.

where $b' = f_0 \partial\psi'/\partial z$ and $v' = \partial\psi'/\partial x$. Thus, the upward transfer of energy is proportional to the poleward heat flux. Evidently, the transfer of energy is upward when $km > 0$, and from (16.65), this corresponds to the condition that the vertical component of group velocity is positive, which has to be the case from general arguments. For Rossby waves $k < 0$ so that upward energy propagation requires $m < 0$ and therefore downward phase propagation.

Meridional heat transport. The meridional heat transport associated with a wave is

$$\rho_R \overline{v'b'} = \rho_R f_0 \overline{\frac{\partial\psi'}{\partial x}\frac{\partial\psi'}{\partial z}}. \tag{16.79}$$

For an oscillatory wave the heat flux is proportional to km. Now, the condition that energy is directed upward is that km is positive, for then c_g^z is positive. Thus, upward propagation is associated with a polewards heat flux. The meridional transport associated with a trapped solution is identically zero.

Form drag. If the waves propagate energy upwards, there must be a surface interaction to supply that energy. There is a force due to *form drag* (see Section 3.6) associated with this interaction given by

$$\text{form drag} = \overline{p' \frac{\partial h_b}{\partial x}} . \tag{16.80}$$

In the trapped case, the streamfunction is either exactly in or out of phase with the topography, so this interaction is zero. In the oscillatory case

$$\overline{\psi' \frac{\partial h_b}{\partial x}} = \frac{f_0 \widetilde{h}_b^2 k m}{4(K_s^2 - K^2)} , \tag{16.81}$$

where the factor of 4 arises from the x and y averages of the squares of sines and cosines. The rate of doing work is $\overline{u}$ times (16.81).

16.6 ♦ VERTICAL PROPAGATION OF ROSSBY WAVES IN SHEAR

In the real atmosphere the zonal wind and the stratification change with height and there may be regions in which propagation occurs and regions where it does not, and in this section we illustrate that phenomenon with two examples. In one example the zonal wind increases sufficiently with height that wave propagation ceases because the wind is too strong, and in the other the zonal wind decreases aloft and becomes negative (westward), again causing wave propagation to cease. If the zonal wind and the stratification both vary sufficiently slowly with height — meaning that the scale of the variation is much greater than a vertical wavelength — then *locally* the solution will look like a plane wave and the analysis is straightforward, very similar to that performed in Section 7.5 (where we looked at internal waves with varying stratification) and Section 16.2.2 (where we looked at horizontally propagating Rossby waves).

For simplicity we consider Rossby waves in a flow with vertical shear but no horizontal shear, with constant stratification and constant density. With reference to Section 16.1.2, the equation of motion is

$$\left(\frac{\partial}{\partial t} + \overline{u}(z) \frac{\partial}{\partial x} \right) q' + \beta v' = 0. \tag{16.82}$$

We seek solutions of the form

$$\psi' = \widetilde{\psi}(z) \mathrm{e}^{\mathrm{i}k(x-ct)+ly}, \tag{16.83}$$

obtaining

$$\frac{\partial^2 \widetilde{\psi}}{\partial z^2} + m^2(z)\widetilde{\psi} = 0, \qquad \text{where} \qquad m^2(z) = \frac{N^2}{f_0^2} \left[\frac{\beta}{\overline{u} - c} - (k^2 + l^2) \right]. \tag{16.84a,b}$$

The WKB solution to this equation (see Appendix A to Chapter 6) is

$$\widetilde{\psi}(z) = A m^{-1/2} \, \mathrm{e}^{\mathrm{i} \int m \, \mathrm{d}z}. \tag{16.85}$$

where A is a constant. The local vertical wavenumber is just m itself (for this is the derivative of the phase), and the amplitude varies like $m^{-1/2}$. This variation of amplitude is consistent with the conservation of wave activity, which in this case means that the Eliassen–Palm flux is constant. As there is no horizontal divergence in this problem, the constancy of $\boldsymbol{\mathcal{F}}$ in (16.19) implies $\partial_z \overline{v'b'} = 0$ and therefore

$$km|\widetilde{\psi}|^2 = \text{constant}. \tag{16.86}$$

Since the horizontal wavenumber is constant the dependence of the amplitude on $m^{-1/2}$ immediately follows. The energy of the wave is not constant unless there is no shear, since it may be extracted or given up to the mean flow.

As discussed in earlier sections, wave propagation requires than m^2 be positive. For stationary waves ($c = 0$) this gives the condition that $0 < \overline{u} < \beta/(k^2 + l^2)$ At the lower bound there is a critical line and $m^2 \to \infty$. At the upper bound $m^2 = 0$ and this is a turning line. Let us illustrate the behaviour approaching these regions with two examples.

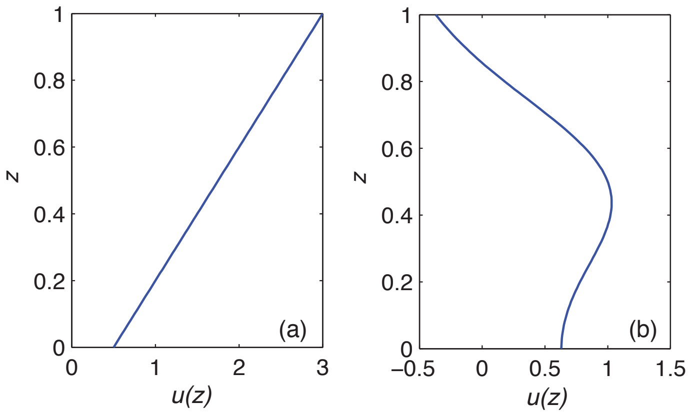

Fig. 16.8 Two profiles of nondimensional zonal wind used in the calculations illustrated in Fig. 16.9 and Fig. 16.10. (a) is a uniform shear that gives rise to a turning latitude, and (b) shows a profile in which the zonal wind diminishes to zero aloft, giving rise to a critical line.

16.6.1 Two Examples

In much the same way as we illustrated horizontal propagation in Section 16.2 we'll calculate the group velocity and wave amplitudes in two cases, one with a turning line and the other with a critical line. The same caveats apply — that the slowly-varying assumption fails at theses lines, and we cannot use WKB or ray theory to properly calculate reflection or absorption.

Waves with a turning line

Consider Rossby waves propagating in a background state in which the zonal wind increases uniformly with height, as in Fig. 16.8a, but in which all other parameters are constant. Specifically, we choose (nondimensional) values of $\beta = 5$, $k = l = 1$ and $c = 0$ (the reader may re-dimensionalize). We also scale the vertical coordinate so that $Pr = 1$. For the profile chosen m^2 is positive for $\overline{u} < 2.5$ and so for $0 < z < 0.8$, as shown in Fig. 16.9. For l fixed and m given by (16.84b) we calculate the group velocity using (16.7) and these are displayed in Fig. 16.9. We choose upwardly propagating waves (i.e. $m > 0$); in any physical situation the group velocity will be directed away from the source, and we are assuming this occurs at the surface. We also show equatorward moving waves for $y < 0.5$ and poleward moving waves for $y > 0.5$, but this is for illustrative purposes. The right-hand panel of the figure shows the value of m^2 diminishing with height, along with the vertical profiles of the amplitude (which goes like $m^{-1/2}$) and the energy (which goes like $(k^2 + l^2 + m^2)m$).

We see from Fig. 16.9 that the group velocity turns away from the turning line, and we can understand this from the ratio of the group velocities given in (16.7), namely

$$\frac{c_g^z}{c_g^y} = \frac{Pr^2 m}{l}, \tag{16.87}$$

where $Pr = f/N$. The group velocity is horizontal at the turning line. The amplitude of the waves is infinite, but the waves do not necessarily break because the vertical wavelength is very large. In fact, for the reasons given in the horizontally propagating case, we expect the waves to reflect.

Waves with a critical line

Now consider waves in a zonal wind that initially increases with height and then decreases and becomes negative, as illustrated in Fig. 16.8b. There is a critical line where $\overline{u}$ passes through zero, but all the other parameters are the same as in the previous example. The value of m^2 now generally increases with height, as illustrated in the right-hand panel of Fig. 16.10, becoming infinite at the critical line and negative above it. The amplitude of the wave, being proportional to $m^{-1/2}$ actually goes to zero at the critical line but the energy increases without bound (in the linear, inviscid approximation).

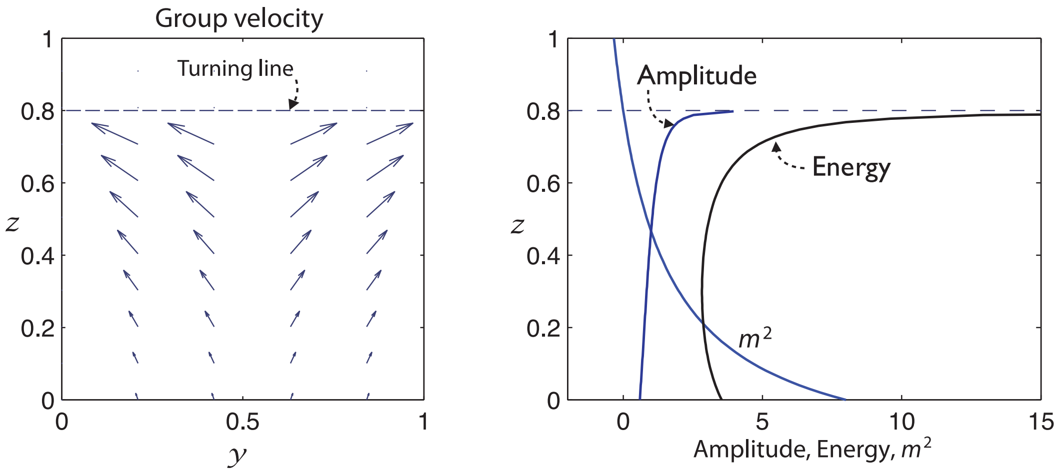

Fig. 16.9 Vertically propagating Rossby waves approaching a turning line. Left panel: group velocity vectors calculated using (16.7) for the parameters shown in Fig. 16.8a and with a source at $y = 0.5$. Right: profiles of m^2, wave amplitude and energy. The horizontal line at $z = 0.8$ marks a turning line: the group velocity turns away from it and the amplitude and energy both tend to infinity there.

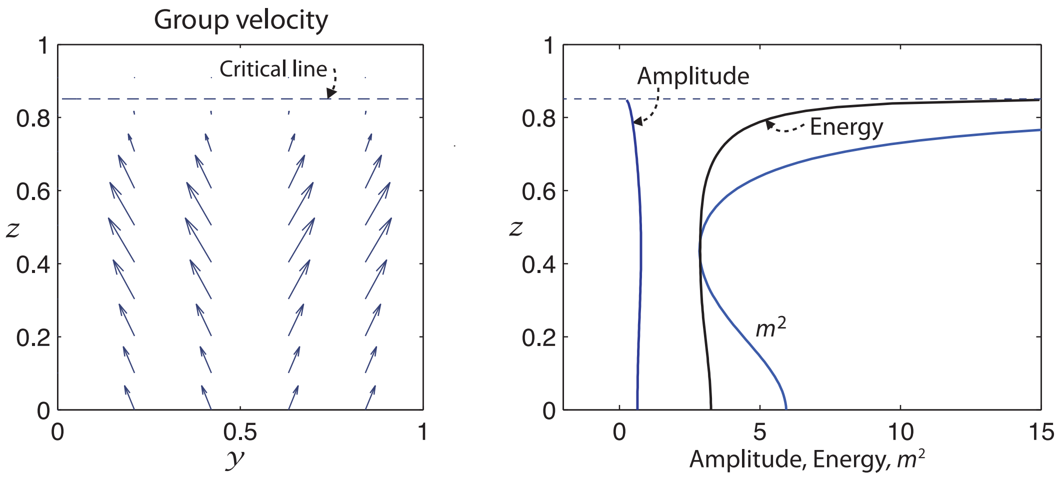

Fig. 16.10 Vertically propagating Rossby waves approaching a critical line. Left panel: group velocity vectors calculated using (16.7) for the parameters shown in Fig. 16.8b. Right: profiles of m^2, wave amplitude and energy. The horizontal line at $z \approx 0.85$ marks a critical line; the group velocity turns toward it but its amplitude diminishes as the critical line is approached.

The group velocity, shown in the left panel of Fig. 16.10, turns upward and toward the critical line and, from (16.87), is purely vertical at the critical line. The amplitude of the group velocity also diminishes, frictional and nonlinear effects become more important, and the notion of group velocity itself ceases to be meaningful close to the critical line where the WKB approximation breaks down.

As a final remark, we note that the condition that there be neither a critical line or a turning line is essentially a version of the Charney–Drazin criterion for wave propagation.

16.7 FORCED AND STATIONARY ROSSBY WAVES

We now turn our attention to understanding the large-scale zonally asymmetric circulation of the atmosphere, much of which is determined by the presence of stationary Rossby waves forced by topographic and thermal anomalies at the surface.[6]

16.7.1 A Simple One-layer Case

Many of the essential ideas can be illustrated by a one-layer quasi-geostrophic model, with potential vorticity equation

$$\frac{\mathrm{D}q}{\mathrm{D}t} = 0, \qquad q = \zeta + \beta y - \frac{f_0}{H}(\eta - h_b), \tag{16.88}$$

where H is the mean thickness of the layer, η is the height of the free surface, h_b is the bottom topography, and the velocity and vorticity are given by $\boldsymbol{u} = (g/f_0)\nabla^\perp\eta \equiv (g/f_0)\,\mathbf{k}\times\nabla\eta$ and $\zeta = (\partial v/\partial x - \partial u/\partial y) = (g/f_0)\nabla^2\eta$. Linearizing (16.88) about a flat-bottomed state with zonal flow $\overline{u}(y) = -(g/f_0)\partial\overline{\eta}/\partial y$ gives

$$\frac{\partial q'}{\partial t} + \overline{u}\frac{\partial q'}{\partial x} + v'\frac{\partial \overline{q}}{\partial y} = 0, \tag{16.89}$$

where $q' = \zeta' - (f_0/H)(\eta' - h_b)$ and $\partial\overline{q}/\partial y = \beta + \overline{u}/L_d^2$ with $L_d = \sqrt{gH}/f_0$, the radius of deformation. Equation (16.89) may be written, after the cancellation of a term proportional to $\overline{u}\partial\eta'/\partial x$, as

$$\frac{\partial}{\partial t}\left(\zeta' - \frac{\psi'}{L_d^2}\right) + \overline{u}\frac{\partial \zeta'}{\partial x} + \beta v' = -\overline{u}\frac{\partial \widehat{h}}{\partial x}, \tag{16.90}$$

where $\psi' = (g/f_0)\eta'$ and $\widehat{h} = h_b f_0/H = h_b g/(L_d^2 f_0)$.

The solution of this equation consists of the solution to the homogeneous problem (with the right-hand side equal to zero, as considered in section 6.4 on Rossby waves) and the particular solution. We proceed by decomposing the variables into their Fourier components

$$(\zeta', \psi', \widehat{h}) = \mathrm{Re}\,(\tilde{\zeta}, \widetilde{\psi}, \widetilde{h}_b)\sin ly\,\mathrm{e}^{\mathrm{i}kx}, \tag{16.91}$$

where such a decomposition is appropriate for a channel, periodic in the x-direction and with no variation at the meridional boundaries, $y = (0, L)$. The full solution will be a superposition of such Fourier modes and, because the problem is linear, these modes do not interact. The free Rossby waves, the solution to the homogeneous problem, evolve according to

$$\psi = \mathrm{Re}\,\widetilde{\psi}\sin ly\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}, \tag{16.92}$$

where ω is given by the dispersion relation,

$$\omega = k\overline{u} - \frac{k\partial_y\overline{q}}{K^2 + k_d^2} = \frac{k(\overline{u}K^2 - \beta)}{K^2 + k_d^2}, \tag{16.93a,b}$$

where $K^2 = k^2 + l^2$ and $k_d = 1/L_d$. Stationary waves occur at the wavenumbers for which $K = K_S \equiv \sqrt{\beta/\overline{u}}$. To the free waves we add the solution to the steady problem,

$$\overline{u}\frac{\partial \zeta'}{\partial x} + \beta v' = -\overline{u}\frac{\partial \widehat{h}}{\partial x}, \tag{16.94}$$

which is, using the notation of (16.91)

$$\widetilde{\psi} = \frac{\widetilde{h}_b}{(K^2 - K_s^2)}. \tag{16.95}$$

Now, $\widetilde{h}_b$ is a complex amplitude; thus, for $K > K_s$ the streamfunction response is *in phase* with the topography. For $K^2 \gg K_s^2$ the steady equation of motion is

$$\overline{u}\frac{\partial \zeta'}{\partial x} \approx -\overline{u}\frac{\partial \widehat{h}}{\partial x}, \tag{16.96}$$

and the topographic vorticity source is balanced by zonal advection of relative vorticity. For $K^2 < K_s^2$ the streamfunction response is *out of phase* with the topography, and the dominant balance for very large scales is between the meridional advection of planetary vorticity, $v\partial f/\partial y$ or βv, and the topographic source. For $K = K_s$ the response is infinite, with the stationary wave resonating with the topography. Now, any realistic topography can be expected to have contributions from *all* Fourier components. Thus, for *any* given zonal wind there will be a resonant wavenumber and an infinite response. This, of course, is not observed, and one reason is that the real system contains friction. The simplest way to include this is by adding a linear damping to the right-hand side of (16.90), giving

$$\frac{\partial}{\partial t}\left(\zeta' - \frac{\psi'}{L_d^2}\right) + \overline{u}\frac{\partial \zeta'}{\partial x} + \beta v' = -r\zeta' - \overline{u}\frac{\partial \widehat{h}}{\partial x}. \tag{16.97}$$

The free Rossby waves all decay monotonically to zero. However, the steady problem, obtained by omitting the first term on the left-hand side, has solutions

$$\widetilde{\psi} = \frac{\widetilde{h}_b}{(K^2 - K_s^2 - \mathrm{i}R)}, \tag{16.98}$$

where $R = (rK^2/\overline{u}k)$, and the singularity has been removed. The amplitude of the response is still a maximum for the stationary wave, and for this wave the phase of the response is shifted by $\pi/2$ with respect to the topography. The solution is shown in Fig. 16.11.[7] It is typical that for a mountain range whose Fourier composition contains all wavenumbers, there is a minimum in the streamfunction a little downstream of the mountain ridge.

16.7.2 Application to Earth's Atmosphere

With three parameters, I can fit an elephant.
William Thomson, Lord Kelvin (1824–1907).

Perhaps surprisingly, given the complexity of the real system and the simplicity of the model, when used with realistic topography a one-layer model can give reasonably realistic answers for the Earth's atmosphere (although we must always be careful of just fitting a model to observations). Thus, we calculate the stationary response to the Earth's topography using (16.97), using a reasonably realistic representation of the Earth's topography and, with qualification, the zonal wind. The zonal wind on the left-hand side of (16.97) is interpreted as the wind in the mid-troposphere, whereas the wind on the right-hand side is better interpreted as the surface wind, and so perhaps is about 0.4 times the mid-troposphere wind. Since the problem is linear, this amounts to tuning the amplitude of the response. The results, obtained using a rather crude representation of the Earth's topography, are plotted in Fig. 16.12. Also plotted is the observed time averaged response of the real atmosphere (the 500 hPa height field at 45° N). The agreement between model and observation is quite good, but this must be regarded as somewhat fortuitous if only because the other main source of the stationary wave field — thermal forcing — has been completely omitted from the calculation. Quantitative agreement is thus a consequence of the aforesaid tuning. Nevertheless, the calculation does suggest that the stationary, zonally asymmetric, features of the Earth's atmosphere arise via the interaction of the zonally symmetric wind field and the zonally asymmetric lower boundary, and that these may be calculated to a reasonable approximation with a linear model.

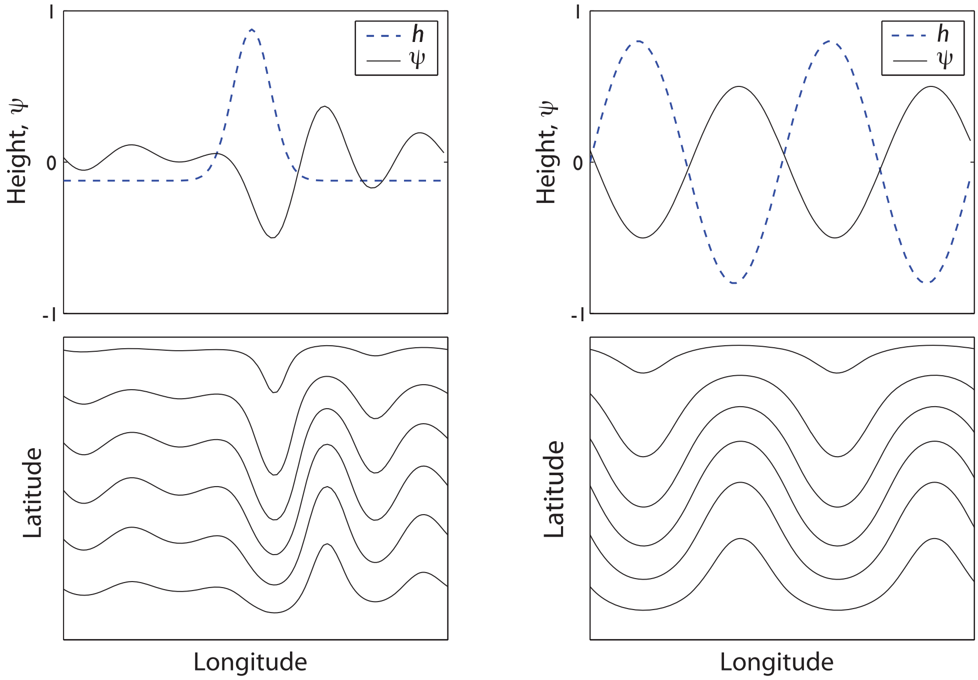

Fig. 16.11 The response to topographic forcing, i.e., the solution to the steady version of (16.97), for topography consisting of an isolated Gaussian ridge (left panels) and a pure sinusoid (right panels). The wavenumber of the stationary wave is about 4 and $r/(\overline{u}k) = 1$. The upper panels show the amplitude of the topography (dashed curve) and the perturbation streamfunction response (solid curve). The lower panels are contour plots of the streamfunction, including the mean flow. With the ridge, the response is dominated by the resonant wave and there is a streamfunction minimum, a 'trough', just downstream of the ridge. In the case on the right, the flow cannot resonate with the topography, which consists only of wavenumber 2, and the response is exactly out of phase with the topography.

16.7.3 ✦ One-dimensional Rossby Wave Trains

Although the Fourier analysis above gives exact (linear) results, it is not particularly revealing of the underlying dynamics. We see from Fig. 16.11 that the response to the Gaussian ridge is largely downstream of the ridge, and this suggests that it will be useful to consider the response as being due to Rossby *wavetrains* being excited by local features. This is also suggested by Fig. 16.13, which shows that the response to realistic topography is relatively local, and may be considered to arise from two relatively well-defined wavetrains, each of finite extent, one coming from the Rockies and the other from the Himalayas.

One way to analyse these wavetrains, and one which also brings up the concept of group velocity in a natural way, is to exploit (as in Section 15.1.2) a connection between changes in wavenumber and changes in frequency. Consider the linear barotropic vorticity equation in the form

$$\frac{\partial}{\partial t}(\zeta - k_d^2\psi) + \overline{u}\frac{\partial \zeta}{\partial x} + \beta\frac{\partial \psi}{\partial x} = -r\zeta, \tag{16.99}$$

where r is a frictional coefficient, which we presume to be small. Setting $k_d = 0$ for simplicity, the linear dispersion relation is

$$\omega = \overline{u}k - \frac{\beta k}{K^2} - \mathrm{i}r \equiv \omega_R(k,l) - \mathrm{i}r, \tag{16.100}$$

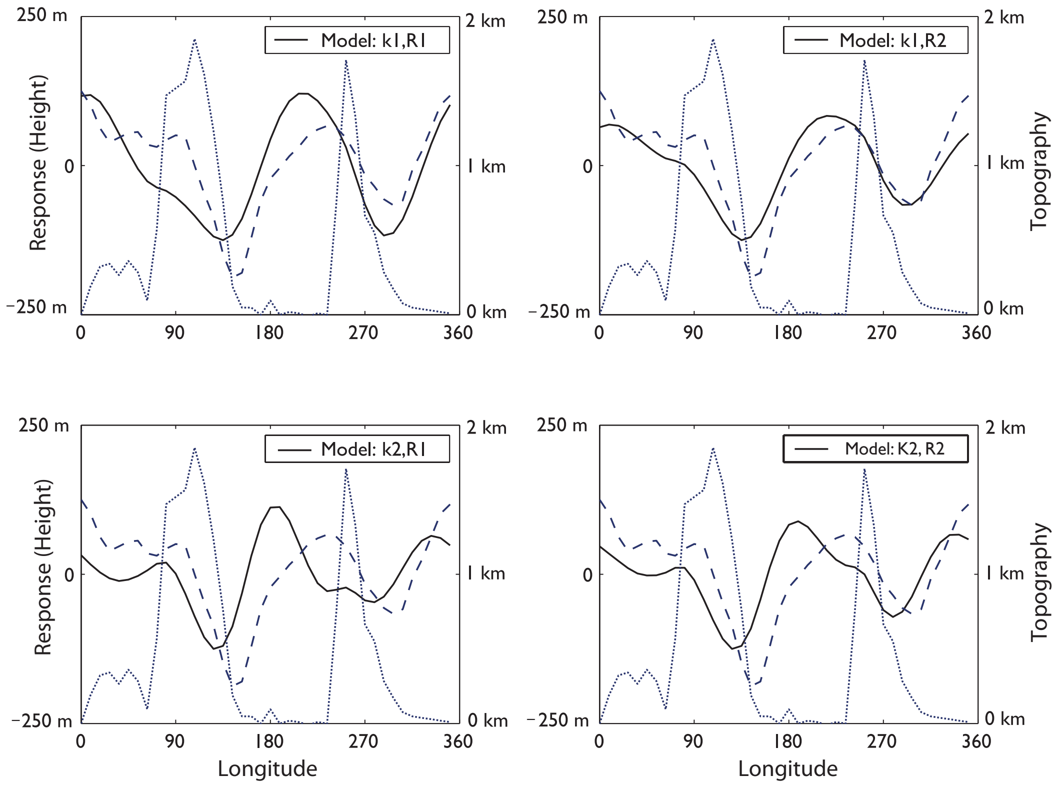

Fig. 16.12 Solutions of the Charney–Eliassen model. The solid lines are the steady solution of (16.97) using the Earth's topography at 45° N with two values of friction (R1 ≈ 6 days, R2 ≈ 3 days) and two values of resonant zonal wavenumber (2.5 for k1, 3.5 for k2), corresponding to zonal winds of approximately 17 and 13 m s^{-1}. The solutions are given in terms of height, η', where $\eta' = f_0\psi'/g$, with the scale on the left of each panel. The dashed line in each panel is the observed average height field at 500 hPa at 45° N in January. The dotted line is the topography used in the calculations, with the scale on the right of each panel.

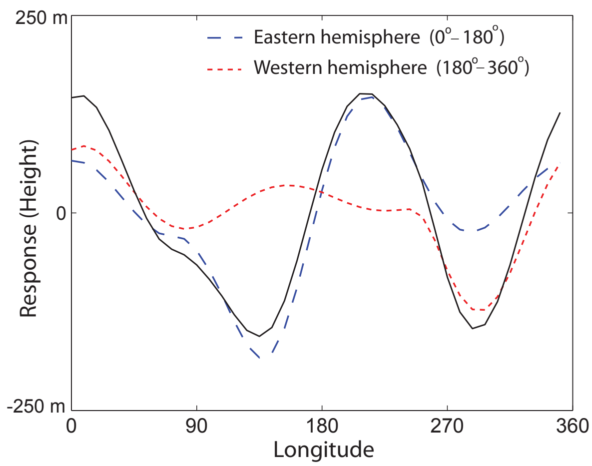

Fig. 16.13 The solution of the upper left-hand panel of Fig. 16.12 (solid line), and the solution divided into two contributions (dashed lines), one due to the topography only of the western hemisphere (i.e., with the topography in the east set to zero) and the other due to the topography only of the eastern hemisphere.

where $K^2 = k^2 + l^2$ and $\omega_R(k,l)$ is the inviscid dispersion relation for Rossby waves. Now, if there is a local source of the waves, for example an isolated mountain, we may expect to see a *spatial* attenuation of the wave as it moves away from the source. We may then regard the system as having a fixed, real frequency, but a changing, possibly complex, wavenumber. To determine this wavenumber for stationary waves (and so with $\omega = 0$), for small friction we expand the dispersion relation in a Taylor series about the inviscid value of ω_R at the real stationary wavenumber k_s, where $k_s = (K_s^2 - l^2)^{1/2}$ and $K_s = \sqrt{\beta/\overline{u}}$. This gives

$$\omega + \mathrm{i}r = \omega_R(k,l) \approx \omega_R(k_s,l) + \left.\frac{\partial \omega_R}{\partial k}\right|_{k=k_s} k' + \cdots . \tag{16.101}$$

Thus, $k' \approx \mathrm{i}r/c_g^x$, where c_g^x is the zonal component of the group velocity evaluated at a fixed position and at the stationary wavenumber; using (6.61b) this is given by

$$c_g^x = \left.\frac{\partial \omega_R}{\partial k}\right|_{k=k_s} = \frac{2\overline{u}k_s^2}{k_s^2 + l^2} . \tag{16.102}$$

The solution therefore decays away from a source at $x = 0$ according to

$$\psi \sim \exp(\mathrm{i}kx) = \exp\left[\mathrm{i}(k_s + k')x\right] \approx \exp(\mathrm{i}k_s x - rx/c_g^x), \tag{16.103}$$

and, because $c_g^x > 0$, the response is east of the source. The approximate solution for the streamfunction (denoted ψ_δ) of (16.97) in an infinite channel, with the topography being a δ-function mountain ridge at $x = x'$, and with all fields varying meridionally like $\sin ly$, is thus

$$\psi_\delta(x - x', y) \sim \begin{cases} 0 & x \le x' \\ -\dfrac{1}{k_s} \sin ly \sin[k_s(x - x')] \exp[-r(x - x')/c_g^x] & x \ge x'. \end{cases} \tag{16.104}$$

In the more general problem in which the topography is a general function of space, every location constitutes a separate source of wavetrains, and the complete (approximate) solution is given by the integral

$$\psi'(x,y) = \int_{-\infty}^{\infty} \widehat{h}(x)\psi_\delta(x - x', y)\,\mathrm{d}x' . \tag{16.105}$$

The field $\psi_\delta(x - x', y)$, is the *Green function* for the problem, sometimes denoted $G(x - x', y)$.

Example solutions calculated using both the Fourier and Green function methods are illustrated in Fig. 16.14. As in Fig. 16.11 there is a trough immediately downstream of the mountain, a result that holds for a broad range of parameters. In these solutions, the streamfunction decays almost completely in one circumnavigation of the channel, and thus, downstream of the mountain, both methods give virtually identical results. Such a correspondence will not hold if the wave can circumnavigate the globe with little attenuation, for then resonance will occur and the Green function method will be inaccurate; thus, whether the resonant picture or the wavetrain picture is more appropriate depends largely on the frictional parameter. A frictional time scale of about 10 days is often considered to approximately represent the Earth's atmosphere, in which case waves are only slightly damped on a global circumnavigation, and the Fourier picture is natural with the possibility of resonance. However, the smaller (more frictional) value of 5 days seems to give quantitatively better results in the barotropic problem, and the solution is more evocative of wavetrains. The larger friction may perform better because it is crudely parameterizing the meridional propagation and dispersion of Rossby waves that is neglected in the one-dimensional model.[8]

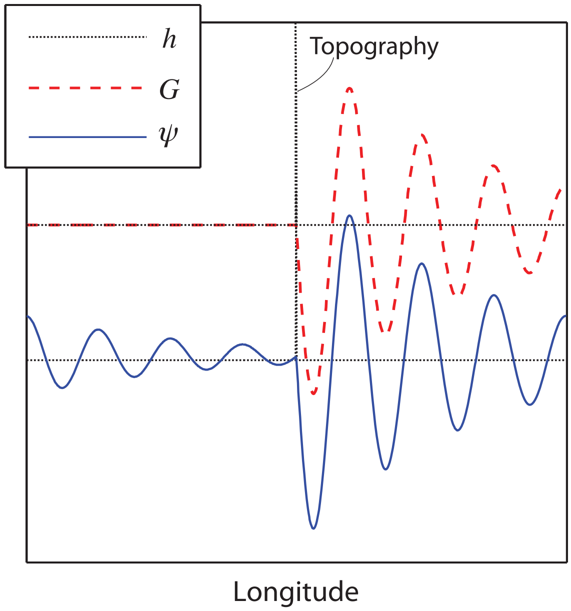

Fig. 16.14 A one-dimensional Rossby wave train excited by uniform eastward flow over a δ-function mountain ridge (h) in the centre of the domain. The upper curve, G, shows the Green's function (16.104), whereas the lower curve shows the exact (linear) response, ψ, in a re-entrant channel calculated numerically using the Fourier method.

The two solutions are both centred around zero and offset for clarity; the only noticeable difference is upstream of the ridge, where there is a finite response in the Fourier case because of the progression of the wave-train around the channel. The stationary wavenumber is 7.5.

16.7.4 The Adequacy of Linear Theory

Having calculated some solutions, we are in a position to estimate, *post facto,* the adequacy of the linear theory by calculating the magnitude of the omitted nonlinear terms. The linear problem here differs in kind from that which arises when using linear theory to evaluate the stability of a flow, as in Chapter 9. In that case, we assume a small initial perturbation and the initial evolution of that perturbation is then accurately described using linear equations. In this case the amplitude of the perturbation is arbitrary, for it may grow exponentially and its size at any given time is proportional to the magnitude of the initial perturbation, which is assumed small but which is otherwise unconstrained. In contrast, when we are calculating the stationary linear response to flow over topography or to a thermal source, the amplitude of the solution is *not* arbitrary; rather, it is determined by the parameters of the problem, including the size of the topography, and represents a real quantity that might be compared to observations. Of course, because the problem *is* linear, the amplitude of the solution is directly proportional to the magnitude of the topography or thermal perturbation.

From (16.98), and recalling that the amplitude of $\widetilde{h}$ is scaled relative to the real topography by the factor $(g/L_d^2 f_0)$, we crudely estimate the amplitude of the response to topography to be

$$|\psi'| \sim \frac{\alpha g h_b}{f_0} \approx \alpha \times 10^8\,\mathrm{m^2\,s^{-1}} = 2\times 10^7\,\mathrm{m^2\,s^{-1}}, \qquad |\eta'| = \frac{f_0|\psi'|}{g} \sim \alpha h_b \approx 0.2\,\mathrm{km}, \tag{16.106}$$

where the nondimensional parameter α accounts for the distance of the response from resonance and the ratio of the length scale to the deformation scale. Choosing $\alpha = 0.2$ and $h_b = 1\,\mathrm{km}$ gives the numerical values above, which are similar to those calculated more carefully, or observed (Fig. 16.12).

If linear theory is to be accurate, we must demand that the self-advection of the response is much smaller than the advection by the basic state, and so that

$$|J(\psi', \nabla^2\psi')| \ll |\overline{u}\frac{\partial}{\partial x}\nabla^2\psi'|, \tag{16.107}$$

or, again rather crudely, that $|\psi'/L| \ll \overline{u}$. For $L = 5000\,\mathrm{km}$ we have $\psi'/L = 4\,\mathrm{m\,s^{-1}}$, which is a few times smaller than a typical mid-troposphere zonal flow of 20 m s^{-1}, suggesting that the linear approximation may hold water. However, the inequality is not a large one, especially as a different choice of numerical factors would give a different answer, and the use of a simple barotropic model also implies inaccuracies. Rather, we conclude that we must carefully calculate the linear response, and compare it with the observations and the implied nonlinear terms, before concluding that linear theory is appropriate, although it certainly does give qualitative insight.

16.8 ✦ EFFECTS OF THERMAL FORCING

How does thermal forcing influence the stationary waves? To give an accurate answer for the real atmosphere is a little more difficult than for the orographic case where the forcing can be included reasonably accurately in a quasi-geostrophic model with a term $\overline{u} \cdot \nabla h_b$ at the lower boundary. Anomalous (i.e., variations from a zonal or temporal mean) thermodynamic forcing typically also arises initially at the lower boundary through, for example, variations in the surface temperature. However, such anomalies may be felt throughout the lower troposphere on a relatively short time scale by way of such non-geostrophic phenomena as convection, so that the effective thermodynamic source that should be applied in a quasi-geostrophic calculation has a finite vertical extent. However, an accurate parameterization of this may depend on the structure of the atmospheric boundary layer and this cannot always be represented in a simple way. Because of such uncertainties our treatment concentrates on the fundamental and qualitative aspects of thermal forcing, and the reader should look to the literature for more complete derivations.[9]

The quasi-geostrophic potential vorticity equation, linearized around a uniform zonal flow, is (cf. (16.57))

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\left[\nabla^2\psi' + \frac{f_0^2}{\rho_R}\frac{\partial}{\partial z}\left(\frac{\rho_R}{N^2}\frac{\partial \psi'}{\partial z}\right)\right] + \frac{\partial \psi'}{\partial x}\left[\beta - \frac{f_0^2}{\rho_R}\frac{\partial}{\partial z}\left(\frac{\rho_R}{N^2}\frac{\partial \overline{u}}{\partial z}\right)\right] = \frac{f_0}{N^2}\frac{\partial Q}{\partial z} \equiv T, \tag{16.108}$$

where T is defined for convenience and Q is the source term in the (linear) thermodynamic equation,

$$\frac{\partial}{\partial t}\left(\frac{\partial \psi'}{\partial z}\right) + \overline{u}\frac{\partial}{\partial x}\frac{\partial \psi'}{\partial z} - v'\frac{\partial \overline{u}}{\partial z} + \frac{N^2}{f_0}w' = \frac{Q}{f_0}. \tag{16.109}$$

A particular solution to (16.108) may be constructed if $\overline{u}$ and N^2 are constant, and if Q has a simple vertical structure. If we again write $\psi' = \mathrm{Re}\,\widetilde{\psi}(z)\sin ly \exp(\mathrm{i}kx)$ and let $\Phi(z) = \widetilde{\psi}(z)\exp(-z/2H)$ we obtain

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}z^2} + m^2\Phi = \frac{T}{\mathrm{i}k\overline{u}}\mathrm{e}^{-z/2H}, \qquad \text{where} \quad m^2 = \frac{N^2}{f_0^2}\left(\frac{\beta}{\overline{u}} - K^2 - \gamma^2\right). \tag{16.110}$$

If we let $T = T_0\exp(-z/H_Q)$, so that the heating decays exponentially away from the Earth's surface, then the particular solution to the stationary problem is found to be

$$\widetilde{\psi} = \mathrm{Re}\,\frac{\mathrm{i}\widehat{T}\mathrm{e}^{-z/H_Q}}{k\overline{u}\left[(N/f_0)^2(K_s^2 - K^2) + H_Q^{-2}(1 + H_Q/H)\right]}, \tag{16.111}$$

where $\widehat{T}$ is a constant proportional to T_0. This solution does not satisfy the boundary condition at $z = 0$, which in the absence of topography and friction is

$$\overline{u}\frac{\partial}{\partial x}\frac{\partial \psi'}{\partial z} - v'\frac{\partial \overline{u}}{\partial z} = \frac{Q(0)}{f_0}. \tag{16.112}$$

A homogeneous solution must therefore be added, and just as in the topographic case this leads to a vertically radiating or a surface trapped response, depending on the sign of m^2. One way to calculate the homogeneous solution is to first use the linearized thermodynamic equation (16.109), or the linearized vorticity equation (16.114), to calculate the vertical velocity at the surface implied by (16.111), $w_p(0)$ say. We then notice that the homogeneous solution is effectively forced by an equivalent topography given by $h_e = -w_p(0)/(\mathrm{i}k\overline{u(0)})$, and so proceed as in the topographic case. The complete solution is rather hard to interpret, and is in any case available only in special cases, so it is useful to take a more qualitative approach.

16.8.1 Thermodynamic Balances

It is the properties of the particular solution that mainly distinguish the response to thermodynamic forcing from that due to topography, because the homogeneous solutions of the two cases are similar. Far from the source region the homogeneous solution will dominate, giving rise to wavetrains as discussed previously.

We can determine many of the properties of the response to thermodynamic forcing by considering the balance of terms in the steady linear thermodynamic equation, which we write as

$$\overline{u}\frac{\partial}{\partial x}\frac{\partial \psi'}{\partial z} - \frac{\partial \psi'}{\partial x}\frac{\partial \overline{u}}{\partial z} + \frac{N^2}{f_0}w' = \frac{Q}{f_0} \equiv R \tag{16.113a}$$

or

$$f_0\overline{u}\frac{\partial v'}{\partial z} - f_0 v'\frac{\partial \overline{u}}{\partial z} + N^2 w' = Q. \tag{16.113b}$$

The vorticity equation is

$$\overline{u}\frac{\partial \zeta'}{\partial x} + \beta v' = \frac{f_0}{\rho_R}\frac{\partial \rho_R w'}{\partial z}. \tag{16.114}$$

Assuming that the diabatic forcing is significant, we may imagine three possible simple balances in the thermodynamic equation:

(i) zonal advection dominates, and $v' = \partial\psi'/\partial x \sim QH_Q/(f_0\overline{u})$;

(ii) meridional advection dominates, and $v' \sim QH_u/(f_0\overline{u})$;

(iii) vertical advection dominates, and $w' \sim Q/N^2$. Then, for large enough horizontal scales the balance in the vorticity equation is $\beta v' \sim f_0 w'_z$ and $v' \sim f_0 Q/(\beta N^2 H_Q)$. For smaller horizontal scales advection of relative vorticity may dominate that of planetary vorticity, and β is replaced by $\overline{u}K^2$.

Here, H_Q is the vertical scale of the source (so that $\partial Q/\partial z \sim Q/H_Q$) and H_u is the vertical scale of the zonal flow (so that $\partial\overline{u}/\partial z \sim \overline{u}/H_u$). We also assume that the vertical scale of the solution is H_Q, so that $\partial v'/\partial z \sim v'/H_Q$. Which of the above three balances is likely to hold? Heuristically, we might suppose that the balance with the smallest v' will dominate, if only because meridional motion is suppressed on the β-plane. Then, zonal advection dominates meridional advection if $H_u > H_Q$, and vice versa. Defining $\widehat{H} = \min(H_u, H_Q)$, then horizontal advection will dominate vertical advection if

$$\mu_1 = \frac{\beta N^2 H_Q \widehat{H}}{\overline{u} f_0^2} \ll 1. \tag{16.115}$$

More systematically, we can proceed in *reductio ad absurdum* fashion by first neglecting the vertical advection term in (16.113), and seeing if we can construct a self-consistent solution. If $\psi' = \mathrm{Re}\,\widetilde{\psi}_p(z)\mathrm{e}^{\mathrm{i}kx}$, and noting that $\overline{u}\partial\widetilde{\psi}_p/\partial z - \widetilde{\psi}_p\partial\overline{u}/\partial z = \overline{u}^2(\partial/\partial z)(\widetilde{\psi}_p/\overline{u})$, we obtain

$$\widetilde{\psi}_p = \frac{\mathrm{i}\overline{u}}{kf_0}\int_z^\infty \frac{\widetilde{Q}}{\overline{u}^2}\,\mathrm{d}z, \tag{16.116}$$

where $\widetilde{Q}$ denotes the Fourier amplitude of Q. Then, from the vorticity equation (16.114), we obtain the (Fourier amplitude of the) vertical velocity

$$\widetilde{w}_p = \frac{-\mathrm{i}k}{f_0\rho_R}\int_z^\infty \rho_R\overline{u}(K_s^2 - K^2)\widetilde{\psi}_p\,\mathrm{d}z. \tag{16.117}$$

Using this one may, at least in principle, check whether the vertical advection in (16.113) is indeed negligible. If $\overline{u}$ is uniform (and so $H_u \gg H_Q$), then we find

$$\widetilde{\psi}_p \propto \frac{\mathrm{i}QH_Q}{kf_0\overline{u}} \qquad \text{and} \qquad \widetilde{w}_p \propto \frac{QH_Q^2(K_s^2 - K^2)}{f_0^2}. \tag{16.118a,b}$$

Using this, vertical advection indeed makes a small contribution to the thermodynamic equation provided that

$$\mu_2 = \frac{N^2H_Q^2|K_s^2 - K^2|}{f_0^2} \ll 1. \tag{16.119}$$

If $K_s^2 \gg K^2$ and $\widehat{H} = H_Q$ then (16.119) is equivalent to (16.115). If $\overline{u}$ is not constant and if $H_u \ll H_Q$ then H_u replaces H_Q and the criterion for the dominance of horizontal advection becomes

$$\mu = \frac{N^2\widehat{H}H_Q|K_s^2 - K^2|}{f_0^2} \ll 1. \tag{16.120}$$

This is the condition that the first term in the denominator of (16.111) is negligible compared with the second. For a typical tropospheric value of $N^2 = 10^{-4}\,\mathrm{s}^{-1}$ and for $K > K_S$ we find that $\mu \approx (H_Q/7\,\mathrm{km})^2$, and so we can expect $\mu < 1$ in extra-equatorial regions where the heating is shallow. At low latitudes f_0 is smaller, β is bigger and $\mu \approx (H_Q/1\,\mathrm{km})^2$, and we can expect $\mu > 1$. However, there is both uncertainty and variation in these values.

Equivalent topography

In the case in which zonal advection dominates, the equivalent topography is given by

$$h_e = \frac{-\widetilde{w}_p(0)}{\mathrm{i}k\overline{u(0)}} = \frac{1}{\overline{u(0)}f_0\rho_R(0)}\int_0^\infty \rho_R\overline{u}(K_s^2 - K^2)\widetilde{\psi}_p\,\mathrm{d}z, \tag{16.121}$$

where $\widetilde{\psi}_p$ is given by (16.116). The point to notice here is that if $K < K_s$ the equivalent topography is in phase with ψ_p.

16.8.2 Properties of the Solution

In the tropics μ may be large for H_Q greater than a kilometre or so. Heating close to the surface cannot produce a large vertical velocity and will therefore produce a meridional velocity. However, away from the surface the heat source will be balanced by vertical advection. For scales such that $K < K_S$, a criterion that might apply at low latitudes for wavelengths longer than a few thousand kilometres, the associated vortex stretching $f\partial w/\partial z > 0$ is balanced by βv and a poleward meridional motion occurs. This implies a trough west of the heating and/or a ridge east of the heating, although the use of quasi-geostrophic theory to draw tropical inferences may be a little suspect.

In mid-latitudes μ is typically small and horizontal advection locally balances diabatic heating. In this case there is a trough a quarter-wavelength downstream from the heating, and equatorward motion at the longitude of the source. (To see this, note that if the heating has a structure like $\cos kx$ then from either (16.111) or (16.116) the solution goes like $\psi_p \propto -\sin kx$.) The trough may be warm or cold, but is often warm. If $H_Q \ll H_u$, as is assumed in obtaining (16.111), then θ is

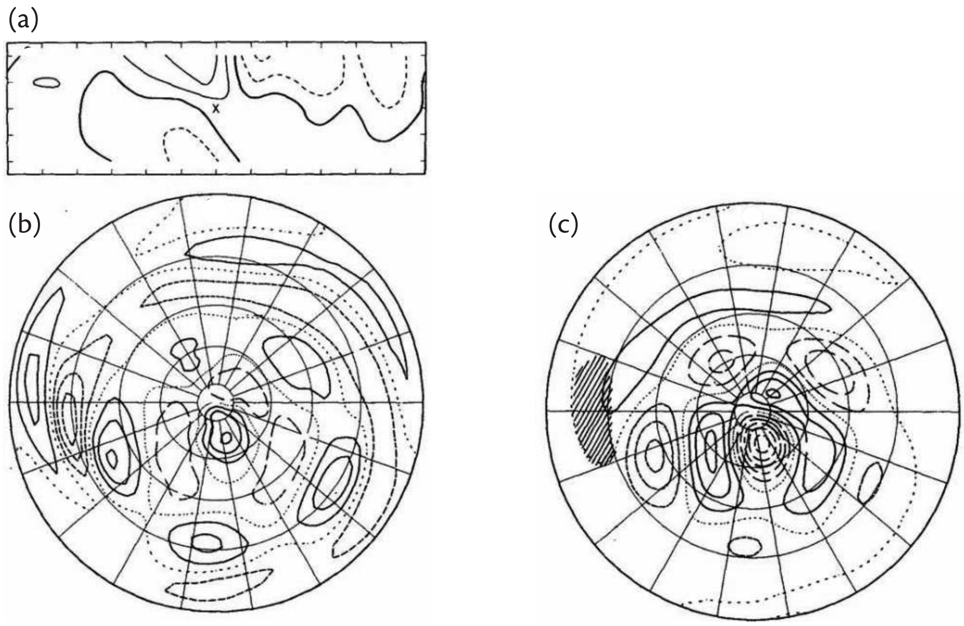

Fig. 16.15 Numerical solution of a baroclinic primitive equation model with a deep heat source at 15° N and a zonal flow similar to that of northern hemisphere winter. (a) Height field in a longitude height at 18° N (the tick marks on the vertical axis are at 100, 300, 500, 700 and 900 hPa); (b) 300 hPa vorticity field; (c) 300 hPa height field. The cross in (a) and the hatched region in (c) indicate the location of the heating.[10]

positive and warm. This is because zonal advection dominates and so the effect of the heating is advected downstream. If $H_Q \gg H_u$ and meridional advection is dominant, then the trough is still warm provided Q decreases with height. The vertical velocity can be inferred from the vorticity balance. If $f_0 \partial w/\partial z \approx \beta v$ and if $w = 0$ at the surface (in the absence of Ekman pumping and any topographic effects) there is *descent* in the neighbourhood of a heat source. This counter-intuitive result arises because it is the horizontal advection that is balancing the diabatic heating. (This result cannot be inferred from the particular solution alone.) If the advection of relative vorticity balances vortex stretching, the opposite may hold.

The homogeneous solution can be inferred from (16.121) and (16.75). Consider, for example, waves that are trapped ($m^2 < 0$) but still have $K < K_S$; that is $K^2 < K_S^2 < K^2 + \gamma^2$. The homogeneous solution forced by the equivalent topography is out of phase with that topography, and so out of phase with ψ_p, using (16.121). For still shorter waves, $K > K_s$, the homogeneous solution is in phase with the equivalent topography, and so again out of phase with ψ_p. Thermal sources produced by large-scale continental land masses may have $K^2 < K_s^2$ and, if $K^2 + \gamma^2 < K_S^2$ they will produce waves that penetrate up into the stratosphere and typically these solutions will dominate far from the source. Evidently though, the precise relationship between the particular and homogeneous solution is best dealt with on a case-by-case basis. A few more general points are summarized in the box on page 619.

16.8.3 Numerical Solutions

The numerically calculated response to an isolated heat source is illustrated in Figs. 16.15 and 16.16. The first figure shows the response to a 'deep' heating at 15° N. As the reasoning above would suggest, the vertical velocity field (not shown) is upwards in the vicinity of the source. Away from

Thermal Forcing of Stationary Waves

(i) The solution is composed of a particular solution and a homogeneous solution.

(ii) The homogeneous solution may be thought of as being forced by an 'equivalent topography', chosen so that the complete solution satisfies the boundary condition on vertical velocity at the surface.

(iii) For a localized source, the far field is dominated by the homogeneous solution. This solution has the same properties as a solution forced by real topography. Thus, it may include waves that penetrate vertically into the stratosphere as well as wavetrains propagating around the globe with an equivalent barotropic structure.

(iv) In the extratropics a heating is typically balanced by horizontal advection, producing a trough a quarter wavelength east (downstream) of a localized heat source. The heat source is balanced by advection of cooler air from higher latitudes, and there may be sinking air over the heat source. This can occur when $\mu \ll 1$; see (16.120).

(v) In the tropics, a heat source may be locally balanced by vertical advection, that is adiabatic cooling as air ascends. This can occur when $\mu \gg 1$.

(vi) In the real atmosphere, the stationary solutions must coexist with the chaos of time-dependent, nonlinear flows. Thus, they are likely to manifest themselves only in time averaged fields and in a modified form.

the source, the solution is dominated by the homogeneous solutions in the form of wavetrains, as described in Section 16.7.3, with a simple vertical structure. (In fact, the pattern is quite similar to that obtained with a suitably forced barotropic model, as was found in the topographically forced case.)

Figure 16.16 shows the response to a perturbation at 45° N, and again the solutions are qualitatively in agreement with the reasoning above. The local heating is balanced by an equatorward wind, and there is a surface trough about 20° east of the source, and an upper-level pressure maximum, or ridge, about 60° east. The scale height of the wind field, H_u is about 8 km, greater than that of the source, and the balance in the thermodynamic equation is between the zonal advection of the temperature anomaly $\overline{u}\theta'_x$ and the heat source, so producing a temperature maximum downstream. Again, the far field is dominated by the wavetrain of the homogeneous solution.

Finally, we show a calculation (Fig. 16.17) that, although linear, includes realistic forcing from topography, heat sources and observed transient eddy flux convergences, and uses a realistic zonally averaged zonal flow, although some physical parameters representing friction and diffusion in the calculation must be changed in order that a steady solution can be achieved. Such a calculation is likely to be the most accurate achievable by a linear model, and discrepancies from observations indicate the presence of nonlinearities that are neglected in the calculation. In fact, a generally good agreement with the observed fields is found, and provides some *post facto* justification for the use of linear, stationary wave models.[13]

In such realistic calculations it is virtually impossible to see the wavetrains emerging from iso-

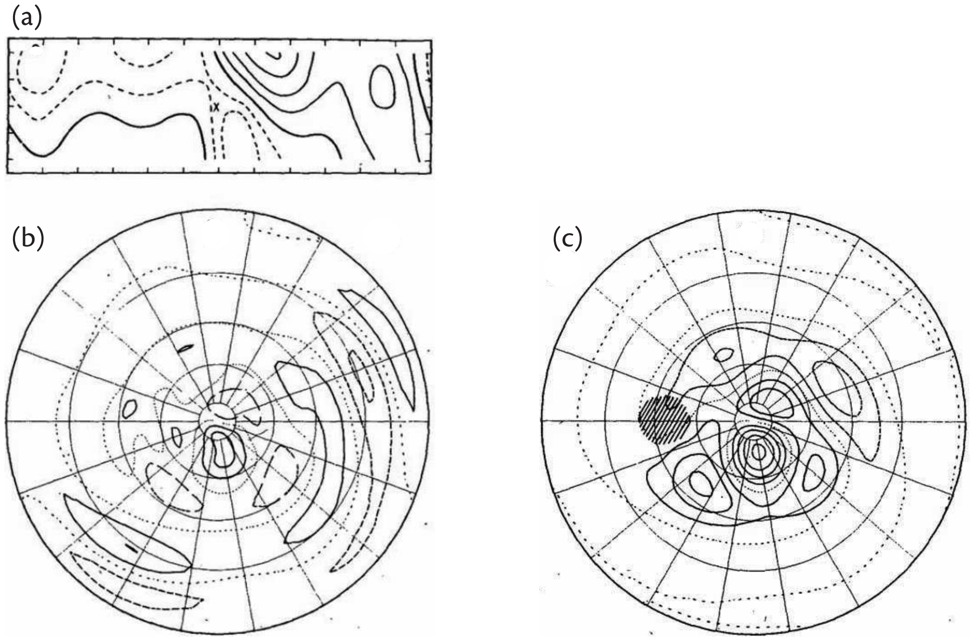

Fig. 16.16 As for Fig. 16.15, but now the solution of a baroclinic primitive equation model with a deep heat source at 45° N. (a) Height field in a longitude height at 18° N; (b) 300 hPa vorticity field; (c) 300 hPa height field. The cross in (a) and the hatched region in (c) indicate the location of the heating.[11]

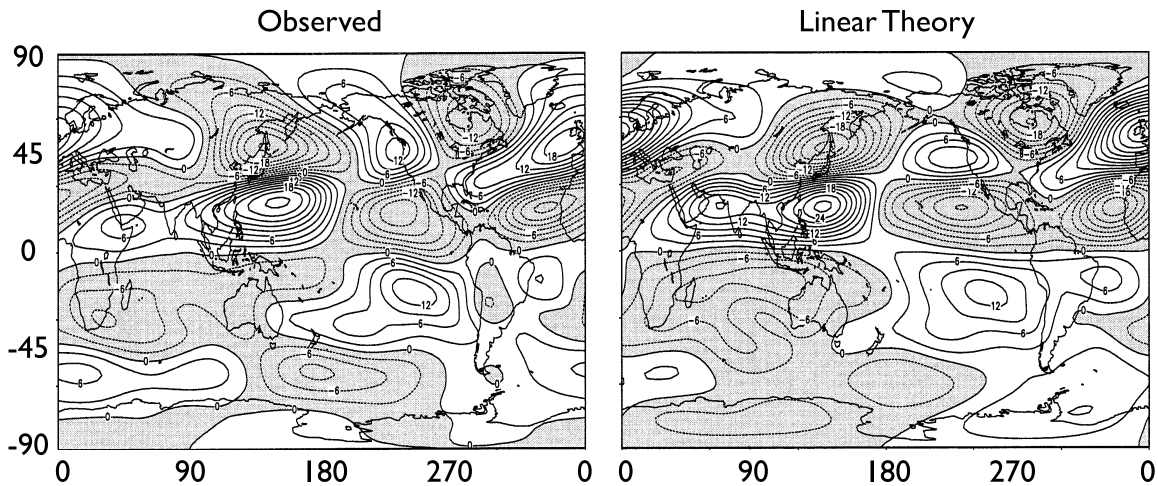

Fig. 16.17 Left: the observed stationary (i.e., time-averaged) streamfunction at 300 hPa (about 8.5 km altitude) in northern hemisphere winter. Right: the steady, linear response to forcing by orography, heat sources and transient eddy flux convergences, calculated using a linear model with the observed height-varying zonally averaged zonal wind. Contour interval is 3×10^6 m^2 s^{-1}, and negative values are shaded. Note the generally good agreement, and also the much weaker zonal asymmetries in the southern hemisphere.[12]

lated features like the Rockies or Himalayas, because they are combined with the responses from all the other sources included in the calculation. Breaking up the forcing into separate contributions from orographic forcing, heating, and the time averaged momentum and heat fluxes from transient eddies reveals that all of these separate contributions have a non-neglible influence. We should also remember that the effects of the fluxes from the transient eddies are not explained by such a calculation, merely included in a diagnostic sense. Nevertheless, the agreement does reveal the extent to which we might understand the steady zonally asymmetric circulation of the real atmosphere as the response due to the interaction of a zonally uniform zonal wind with the asymmetric features of the Earth's geography and transient eddy field. The quasi-stationary response of the planetary waves to surface anomalies, and the interaction of transient eddies with the large-scale planetary wave field, are important factors in the natural variability of climate, and their understanding remains a challenge for dynamical meteorologists.

16.9 ♦ WAVE PROPAGATION USING RAY THEORY

Catch a wave and you're sitting on top of the world.
Brian Wilson and Mike Love (The Beach Boys), *Catch a Wave*, 1963.

Rossby waves propagate meridionally as well as zonally, and we can expect the major mountain ranges on Earth, as well as thermal anomalies, to generate Rossby waves that propagate both zonally and meridionally. The coefficients of the linear equations that determine this propagation will vary with space: on the sphere β is a function of latitude and in general topography is a function of both latitude and longitude; thus calculating the trajectories of the waves will be difficult, and we cannot expect to solve the full problem except numerically, but a few ideas from wave tracing illustrate many of the features of the response, and indeed of the stationary wave pattern in the Earth's atmosphere.[14]

Let us first recall a few results about rays and ray tracing that we encountered in Section 6.3. Most of the important properties of a wave, such as the energy (if conserved) and the wave activity, propagate along rays at the group velocity. Rays themselves are lines that are parallel to the group velocity, generally emanating from some wave source. A ray is perpendicular to the local wave front, and in a homogeneous medium a wave propagates in a straight line. In non-homogeneous media the group velocity varies with position; however, if the medium varies only slowly, on a scale much larger than that of the wavelength of the waves, the wave activity still propagates along rays at the group velocity.

The variation of wavenumber and frequency vary according to, respectively,

$$\frac{\partial k_i}{\partial t} + c_{gj}\frac{\partial k_i}{\partial x_j} = -\frac{\partial \Omega}{\partial x_i}, \qquad \frac{\partial \omega}{\partial t} + c_{gj}\frac{\partial \omega}{\partial x_j} = \frac{\partial \Omega}{\partial t}. \tag{16.122}$$

If the frequency is not an explicit function of space then the wavenumber is constant along a ray, and similarly for time and frequency. Thus, in problems of the form

$$\frac{\partial}{\partial t}\nabla^2\psi + \beta(y)\frac{\partial \psi}{\partial x} = 0, \tag{16.123}$$

then the frequency and the x-wavenumber, but not the y-wavenumber, are constant along a ray. The wavenumber is not constant in the y-direction because the frequency is a function of y.

16.9.1 Rossby Waves and Rossby Rays

If the wave source is localized, then ray theory provides a useful way of calculating and interpreting the response. On the β-plane and away from the orographic source the steady linear response to a zonally uniform but meridionally varying zonal wind will obey

$$\overline{u}(y)\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi+\beta\frac{\partial\psi'}{\partial x}=0. \tag{16.124}$$

In fact, an equation of this form applies on the sphere. To see this, we transform the spherical coordinates (λ, ϑ) into Mercator coordinates with the mapping[15]

$$x' = a\lambda, \quad \frac{1}{a}\frac{\partial}{\partial\lambda}=\frac{\partial}{\partial x'}, \qquad y'=\frac{a}{2}\ln\left(\frac{1+\sin\vartheta}{1-\sin\vartheta}\right), \quad \frac{1}{a}\frac{\partial}{\partial\vartheta}=\frac{1}{\cos\vartheta}\frac{\partial}{\partial y'}. \tag{16.125}$$

The spherical-coordinate vorticity equation then becomes

$$\overline{u}_M\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}\right)\psi+\beta_M\frac{\partial\psi}{\partial x'}=0, \tag{16.126}$$

where $\overline{u}_M = \overline{u}/\cos\vartheta$ and

$$\beta_M=\frac{2\Omega}{a}\cos^2\vartheta-\frac{\mathrm{d}}{\mathrm{d}y'}\left[\frac{1}{\cos^2\vartheta}\frac{\mathrm{d}}{\mathrm{d}y}(\overline{u}_M\cos^2\vartheta)\right]=\cos\vartheta\left(\beta_s+\frac{1}{a}\frac{\partial\overline{\zeta}}{\partial\vartheta}\right), \tag{16.127}$$

where $\beta_s = 2a^{-1}\Omega\cos\vartheta$. Thus, β_M is the meridional gradient of the absolute vorticity, multiplied by the cosine of latitude. An advantage of Mercator coordinates over their spherical counterparts is that (16.126) has a Cartesian flavour to it, with the metric coefficients being absorbed into the parameters $\overline{u}_M$ and β_M. Of course, unlike the case on the true β-plane, the parameter β_M is not a constant, but this is not a particular disadvantage if $\overline{u}_{yy}$ is also varying with y.

Having noted the spherical relevance we revert to the Cartesian β-plane and seek solutions of (16.124) with the form $\psi' = \widetilde{\psi}(y)\exp(\mathrm{i}kx)$, whence

$$\frac{\mathrm{d}^2\widetilde{\psi}}{\mathrm{d}y^2}=\left(k^2-\frac{\beta}{\overline{u}}\right)\widetilde{\psi}=\left(k^2-K_s^2\right)\widetilde{\psi}, \tag{16.128}$$

where $K_s = (\beta/\overline{u})^{1/2}$. From this equation it is apparent that if $k < K_s$ the solution is harmonic in y and Rossby waves may propagate away from their source. On the other hand, wavenumbers $k > K_s$ are trapped near their source; that is, short waves are meridionally trapped by eastward flow. Without solving (16.128), we can expect an isolated mountain to produce two wavetrains, one for each meridional wavenumber $l = \pm(K_s^2 - k^2)^{1/2}$. These wavetrains will then propagate along a ray, and given the dispersion relation this trajectory can be calculated (usually numerically) using the expressions of the previous section. The local dispersion relation of Rossby waves is

$$\omega=\overline{u}k-\frac{\beta k}{k^2+l^2}, \tag{16.129}$$

so that their group velocity is

$$c_g^x=\frac{\partial\omega}{\partial k}=\overline{u}-\frac{\beta(l^2-k^2)}{(k^2+l^2)^2}=\frac{\omega}{k}+\frac{2\beta k^2}{(k^2+l^2)^2}, \qquad c_g^y=\frac{\partial\omega}{\partial l}=\frac{2\beta kl}{(k^2+l^2)^2}. \tag{16.130a,b}$$

The sign of the meridional wavenumber thus determines whether the waves propagate polewards (positive l) or equatorwards (negative l). Also, because the dispersion relation (16.130) is independent of x and t, the zonal wavenumber and frequency in the wave group are constant along the ray, and the *meridional wavenumber* must adjust to satisfy the local dispersion relation (16.129). Thus, from (16.128), the meridional scale becomes larger as K_s approaches k from above and an

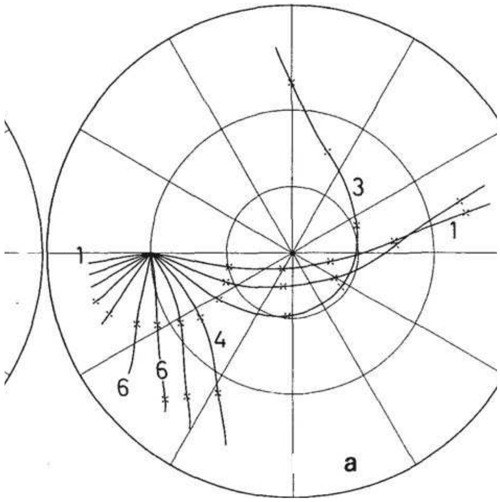

Fig. 16.18 The rays emanating from a point source at 30° N and 180° (nine o'clock), calculated using the observed value of the wind at 300 hPa.[16] The crosses mark every 180° of phase, and mark the positions of successive positive and negative extrema. The numbers indicate the zonal wavenumber of the ray. The ray paths may be compared with the full linear calculation shown in Fig. 16.19.

incident wavetrain is reflected, its meridional wavenumber changes sign, and it continues to propagate eastwards.

Stationary waves have $\omega = 0$, and the trajectory of a ray is parameterized by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{c_g^y}{c_g^x} = \frac{l}{k}. \tag{16.131}$$

For a given zonal wavenumber the trajectory is then fully determined by this condition and that for the local meridional wavenumber, which from (16.129) is

$$l^2 = K_s^2 - k^2. \tag{16.132}$$

Finally, from (16.130) the magnitude of the group velocity is

$$|c_g| = [(c_g^x)^2 + (c_g^y)^2]^{1/2} = 2\frac{k}{K_s}\overline{u}, \tag{16.133}$$

which is double the speed of the projection of the basic flow, $\overline{u}$, onto the wave direction. Given the above relations, and the zonal wind field, we can compute rays emanating from a given source, although the calculation must still be done numerically. One example is given in Fig. 16.18.

✦ *A WKB solution*

The wave amplitudes along a ray can be obtained using a WKB approach. We write (16.128) as

$$\frac{\mathrm{d}^2\widetilde{\psi}}{\mathrm{d}y^2} + l^2(y)\widetilde{\psi} = 0, \qquad \text{where} \qquad l^2(y) = K_s^2 - k^2, \tag{16.134a,b}$$

and if $l(y)$ is varying sufficiently slowly in y then the WKB solution for the stationary streamfunction is

$$\psi(x, y) = A l^{-1/2} \exp\left[\mathrm{i}\left(kx + \int^y l(y)\,\mathrm{d}y\right)\right], \tag{16.135}$$

where A is a constant. Consider, for example, the disturbance excited by an isolated low-latitude peak, with $\overline{u}$ increasing, and so K_s decreasing, polewards of the source. Assuming that initially there exists a zonal wavenumber k less than K_s then two eastward propagating wavetrains are excited. The meridional wavenumber of the poleward wavetrain diminishes according to (16.134b),

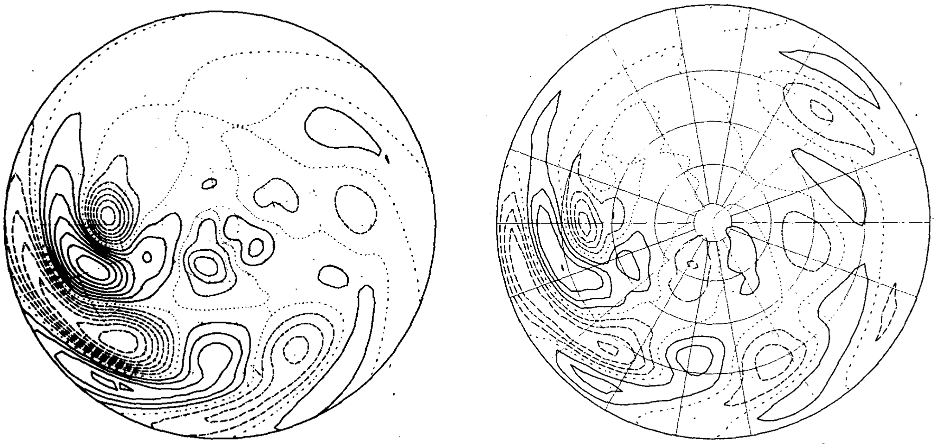

Fig. 16.19 The linear stationary response induced by a circular mountain at 30° N and at 180° longitude (nine o'clock). The figure on the left uses a barotropic model, whereas the figure on the right uses a multi-layer baroclinic model.[18] In both cases the mountain excites a low-wavenumber polar wavetrain and a higher-wavenumber subtropical train.

so that, using (16.131), the ray becomes more zonal. At the latitude where $k = K_s$, the 'turning latitude' the wave is reflected but continues propagating eastwards. The southward propagating wavetrain is propagating into a medium with smaller $\overline{u}$ and larger K_s. At the critical latitude, where $\overline{u} = 0$, $l \to \infty$ but c_g^x and c_g^y both tend to zero, but [using (16.130)] in such a way that $c_g^x / c_g^y \to 0$. That is, the rays become meridionally oriented and their speed tends to zero, and the waves may be absorbed.[17] Finally, we mention without derivation that for zonal flows with constant angular velocity the trajectories are great circles.

16.9.2 Application to an Idealized Atmosphere

Given the complexity of the real atmosphere, and the availability of computers, it is probably best to think of the remarks above as helping us interpret more complete numerical, but still linear, calculations of stationary Rossby waves — for example, numerical solutions of the stationary barotropic vorticity equation in spherical coordinates,

$$\frac{\overline{u}}{a\cos\vartheta}\frac{\partial\zeta'}{\partial\lambda} + v'\left(\frac{1}{a}\frac{\partial\overline{\zeta}}{\partial\vartheta} + \beta\right) = -\frac{\overline{u} f_0}{aH\cos\vartheta}\frac{\partial h_b}{\partial\lambda} - r\zeta', \tag{16.136}$$

where $[u, v] = a^{-1}[-\partial\psi/\partial\vartheta, (\partial\psi/\partial\lambda)/\cos\vartheta]$, $\beta = 2\Omega a^{-1}\cos\vartheta$ and $\zeta = \nabla^2\psi$. The last term in (16.136) crudely represents the effects of friction and generally reduces the sensitivity of the solutions to resonances. Solutions to (16.136) may be obtained first by discretizing and then numerically inverting a matrix, and although the actual procedure is quite involved it is analogous to the Fourier methods used earlier for the simpler one-dimensional problem. Such linear calculations, in turn, help us interpret the stationary wave pattern from more comprehensive models and in the Earth's atmosphere.

Figure 16.19 shows the stationary solution to the problem with a realistic northern hemisphere zonal flow and an isolated, circular mountain at 30° N. The topography excites two wavetrains, both of which slowly decay downstream because of frictional effects, rather like the one-dimensional wavetrain in Fig. 16.14. The polewards propagating wavetrain develops a more meridional orientation, corresponding to a smaller meridional wavenumber l, before moving southwards again, developing a much more zonal orientation eventually to decay completely as it meets the equatorial westward flow. The equatorially propagating train decays a little more rapidly than its polewards moving counterpart because of its proximity to the critical latitude. More complicated patterns naturally result if a realistic distribution of topography is used, as we see in Fig. 16.17. We can see wavetrains emanating from both the Rockies and the Himalayas, but distinct poleward and equatorward wavetrains are hard to discern.

Notes

1 Thanks to Isaac Held and Peter Haynes for various comments on Rossby waves and critical lines.

2 Early discussions of Rossby wave critical layers include those of Warn & Warn (1976), Stewartson (1977) and Killworth & McIntyre (1985). Booker & Bretherton (1967) consider gravity wave critical layers. Haynes (2015) provides an accessible review of both.

3 Modelled after a figure in Haynes (2015).

4 After Charney & Drazin (1961).

5 A quite extensive discussion of the thermal (and the topographically forced) problem is given by Pedlosky (1987a).

6 Much of our basic understanding in this area stems from conceptual and numerical work on forced Rossby waves by Charney & Eliassen (1949), who looked at the response to orography using a barotropic model. This was followed by a study by Smagorinsky (1953) on the response to thermodynamic forcing using a baroclinic, quasi-geostrophic model. Seeking more realism later studies have employed the primitive equations and spherical coordinates in studies that are at least partly numerical (e.g., Egger 1976, and a host of others), although most theoretical studies perforce still use the quasi-geostrophic equations. We also draw from various review articles, among them Smith (1979), Dickinson (1980), Held (1983), particularly for Sections 16.7.3 and 16.8, and Wallace (1983). See also the collection in the *Journal of Climate*, vol. 15, no. 16, 2002.

7 To obtain the solutions shown in Fig. 16.11 and Fig. 16.12, the topography is first specified in physical space. Its Fourier transform is taken and the streamfunction in wavenumber space is calculated using (16.98). The inverse Fourier transform of this gives the streamfunction in physical space.

8 The difference between wavetrains emanating from an isolated topographic feature and a global resonant response is relevant for intra-seasonal variability, which might be considered a quasi-stationary response to slowly changing boundary conditions like the sea-surface temperature. If resonance is important, we might expect to see global-scale anomalies, whereas the viewpoint of damped wave-trains is more local. This whole area is one of continuing, active, research with deep roots going back to Namias (1959) and Bjerknes (1959) and beyond.

A different point of view, one that we do not explore in this book, is that the zonally asymmetric features of the Earth's atmosphere are predominantly due to *nonlinear* effects. One possibility is that eddies might significantly modify (and perhaps amplify and sustain) stationary patterns through their large-scale turbulent transfers; see, for example, Green (1977) and Shutts (1983). We could incorporate such effects into a linear model by including the eddy effects as a forcing term on the right-hand side of a linear equation such as (16.90), or its two- or three-dimensional analogue, although the forcing term would have to be calculated using a nonlinear theory or taken from observations. Different again is the notion, inspired by models of low-order dynamical systems, that the atmosphere might have *regimes* of behaviour, and that the zonally asymmetric patterns are manifestations of the time spent in a particular regime before transiting to another. See for example Kimoto & Ghil (1993) and Palmer (1997).

9 See endnote 6 above for references. Because of these difficulties, understanding the effects of sea-surface temperature anomalies on the atmosphere has become largely the subject of GCM experiments, and one plagued with ambiguous results that depend in part on the particular configuration of the GCM. Some of the modelling issues are reviewed by Kushnir *et al.* (2002).

10 From Hoskins & Karoly (1981).

11 From Hoskins & Karoly (1981).

12 Adapted from Held *et al.* (2002).

13 Such solutions are nearly always most easily obtained numerically. One way is to use a Fourier method described earlier. A related method is to write the equations in finite difference form, schematically as $\mathbf{A}\boldsymbol{X} = \boldsymbol{F}$, where $\boldsymbol{X}$ is the vector of all the model fields, $\boldsymbol{F}$ represents the known forcing and $\mathbf{A}$ is a matrix obtained from the equations of motion and boundary conditions, and solve for $\boldsymbol{X}$. A quite different method is to use a nonlinear time-dependent model, such as a GCM: prescribe or hold steady the zonally averaged zonal flow as well as all the zonally asymmetric forcing terms, but multiply the asymmetric terms by a small number (e.g., 0.01) to ensure the response is linear; then calculate the steady response by forward time integration, and then divide that solution by the small number to obtain the final solution.

14 The description of the stationary waves in terms of wavetrains comes from Hoskins & Karoly (1981), with some earlier theoretical results having been derived by Longuet-Higgins (1964).

15 Steers (1962) and Phillips (1973).

16 From Hoskins & Karoly (1981).

17 At the critical latitude the WKB analysis fails and both dissipative and nonlinear effects are likely to play a role, as discussed by Dickinson (1968), Tung (1979) and others.

18 From Grose & Hoskins (1979) and Hoskins & Karoly (1981).