Changes in latitude, changes in attitude,
Nothing remains quite the same.
Jimmy Buffett, *Changes in Latitude*, 1997.

CHAPTER 8

Linear Dynamics at Low Latitudes

AT LOW LATITUDES THE ATMOSPHERE AND OCEAN take on rather different characters than in mid-latitudes, and this chapter is our first taste of that. The tasting will be rather anodyne and mathematical, focusing on the linear dynamics of wave motion. We won't get into the real *phenomenology* of low latitudes: the tropical atmosphere with its humidity, its convection, and its towering cumulonimbus clouds, or the equatorial ocean with its undercurrents and countercurrents. And most certainly we don't get into low latitude atmosphere-ocean interaction and the wonderful phenomenon of El Niño — these all come later. Rather, this chapter is really just about the linear geophysical fluid dynamics of the shallow water equations at low latitudes, when the beta effect is important and the flow is not completely geostrophically balanced. Still, let us not be too deprecatory about these dynamics — they are important both in their own right and as prerequisites for these more complex phenomena that we encounter later.[1]

Wave motion at low latitudes can be more complicated than its mid-latitude counterpart. In mid-latitudes there is a fairly clear separation in the time and space scales between balanced and unbalanced motion, and it is useful to recognize this by explicitly filtering out gravity wave motion and considering purely balanced motion, using for example the quasi-geostrophic equations. In equatorial regions, where the Coriolis parameter can become very small and is zero at the equator, the Rossby number may be order unity or larger and such a separation is less useful. However, even as f becomes small, β becomes large and Rossby waves remain important but the frequency separation between Rossby and gravity waves is smaller. The reader may then readily imagine the complications arising even from linear wave problems in equatorial regions and determining the dispersion relation for combined Rossby and gravity waves in a continuously stratified fluid is an algebraically complex task. The task is much simplified by posing the problem in the context of the shallow water equations, and these arise both as a physical model (e.g., of the thermocline in the ocean) or via a modal expansion, as in Section 3.4.1.

Before diving into the details, ask why do we talk about the 'tropical' atmosphere but the 'equatorial' ocean? It is because an essential demarcation in the atmosphere lies at the edge of the Hadley Cell, at about 25°–30° latitude, and the dynamics are rather different poleward and equatorward of this edge. In some contrast, the dynamics of the ocean do not change their essential character until we approach quite close to the equator. At 10° latitude ocean dynamics has many of the characteristics of the mid-latitudes — the Rossby number is still very small, for example. Only when we get to within a very few degrees of the equator do the dynamics change in a qualitative way.

8.1 CO-EXISTENCE OF ROSSBY AND GRAVITY WAVES

To see how Rossby waves and gravity waves co-exist, consider the linear, single layer, rotating shallow water equations,

$$\frac{\partial u}{\partial t} - fv = -\frac{\partial \phi}{\partial x}, \qquad \frac{\partial v}{\partial t} + fu = -\frac{\partial \phi}{\partial y}, \qquad \frac{\partial \phi}{\partial t} + c^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0, \tag{8.1a,b,c}$$

where, in terms of the familiar shallow water variables, $\phi = g'\eta$ and $c^2 = g'H$, where ϕ is the kinematic pressure, η is the free surface height, H is the reference depth of the fluid and g' is the reduced gravity. After some manipulation (described in Section 8.2), but with no additional approximation, these equations reduce to a single equation for v, namely

$$\frac{1}{c^2}\frac{\partial^3 v}{\partial t^3} + \frac{f^2}{c^2}\frac{\partial v}{\partial t} - \frac{\partial}{\partial t}\nabla^2 v - \beta\frac{\partial v}{\partial x} = 0. \tag{8.2}$$

On the beta plane the Coriolis parameter is given by $f = f_0 + \beta y$; thus, (8.2) has a non-constant coefficient, entailing considerable algebraic difficulties. We will address these difficulties in Section 8.2, but for now let us suppose that both β and f are constants in (8.2). Effectively, we are assuming that Coriolis parameter, f, is constant except where differentiated, an approximation common in a mid-latitude setting in the quasi-geostrophic equations.[2] This approximation provides a useful introduction to the more complex problem.

Equation (8.2) then has constant coefficients and we may look for plane wave solutions of the form $v = \widetilde{v} \exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)]$, whence

$$\frac{\omega^2 - f_0^2}{c^2} - (k^2 + l^2) - \frac{\beta k}{\omega} = 0. \tag{8.3a}$$

This is a cubic equation in ω, as expected given (8.1). Written differently it becomes

$$\left(k + \frac{\beta}{2\omega}\right)^2 + l^2 = \left(\frac{\beta}{2\omega}\right)^2 + \frac{\omega^2 - f_0^2}{c^2}, \tag{8.3b}$$

which may be compared to (6.107). Noting that $k_d^2 = f_0^2/g'H = f_0^2/c^2$, the two equations are identical except for the appearance of a term involving frequency in the last term on the right-hand side of (8.3b). The wave propagation diagram is illustrated in Fig. 8.1. The wave vectors at a given frequency all lie on a circle centred at $(-\beta/2\omega, 0)$ and with radius R given by

$$R = \left[\left(\frac{\beta}{2\omega}\right)^2 + \frac{\omega^2 - f_0^2}{c^2}\right]^{1/2}, \tag{8.4}$$

and the radius must be positive in order for the waves to exist. In the low frequency case the diagram is essentially the same as that shown in Fig. 6.8, but is quantitatively significantly different in the high frequency case. These limiting cases are discussed further in Section 8.1.1 below.

To plot the full dispersion relation it is useful to nondimensionalize using the following scales for time (T), distance (L) and velocity (U):

$$T = f_0^{-1}, \qquad L = L_d = k_d^{-1} = c/f_0, \qquad U = L/T = c. \tag{8.5a,b,c}$$

Denoting nondimensional quantities with a hat we then have

$$\omega = \widehat{\omega} f_0, \qquad (k, l) = (\widehat{k}, \widehat{l})k_d, \qquad \beta = \widehat{\beta}\frac{f_0^2}{c} = \widehat{\beta}\frac{f_0}{L_d} = \widehat{\beta} f_0 k_d. \tag{8.6}$$

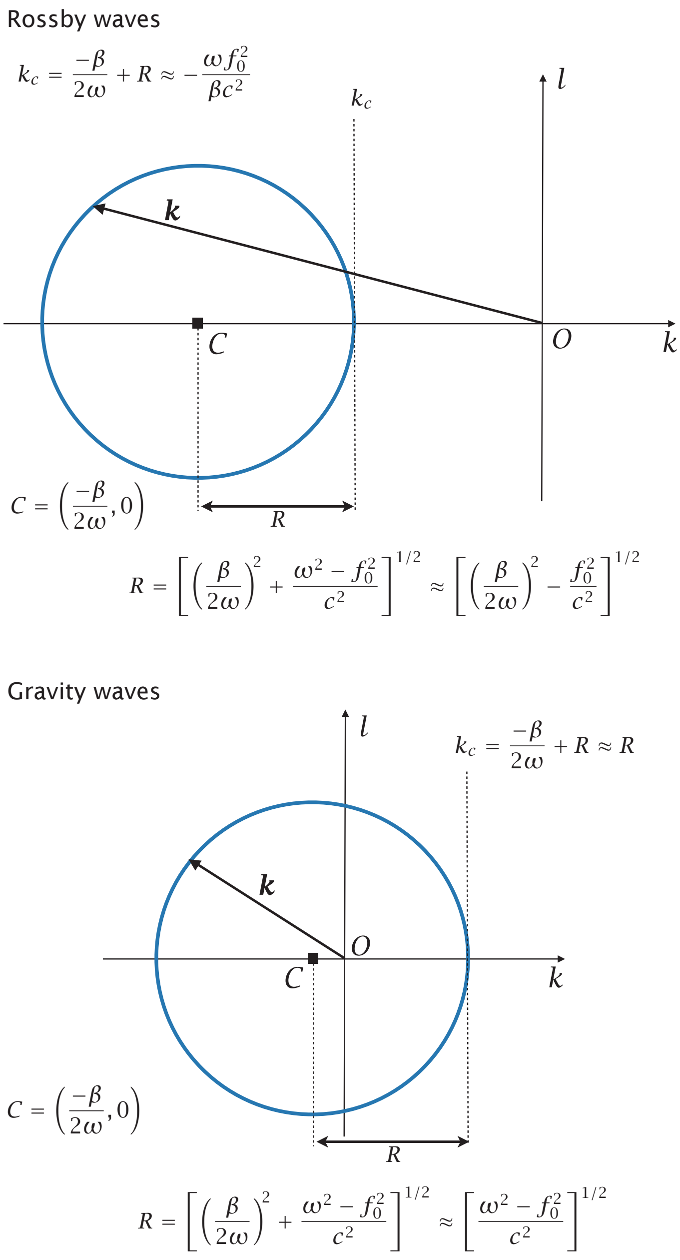

Fig. 8.1 Wave propagation diagrams for Rossby-gravity waves, obtained using (8.3). The top figure shows the diagram in the low frequency, Rossby wave limit, and the bottom figure shows the high frequency, gravity wave limit.

In each case the the locus of wavenumbers for a given frequency is a circle centred at $C = (-\beta/2\omega, 0)$ with a radius R given by (8.4), but the approximate expressions differ significantly at high and low frequency.

The dispersion relation (8.3) may then be written as

$$\widehat{\omega}^2 - 1 - (\widehat{k}^2 + \widehat{l}^2) - \widehat{\beta}\frac{\widehat{k}}{\widehat{\omega}} = 0. \tag{8.7}$$

We may expect that two of the roots correspond to gravity waves and the third to Rossby waves. The only parameter in the dispersion relation is $\widehat{\beta} = \beta c/f_0^2 = \beta L_d/f_0$. In the atmosphere a representative value for L_d is 1000 km, whence $\widehat{\beta} = 0.1$. In the ocean $L_d \sim 100$ km, whence $\widehat{\beta} = 0.01$. If we allow ourselves to consider 'external' Rossby waves (which are of some oceanographic relevance) then $c = \sqrt{gH} = 200\,\mathrm{m\,s^{-1}}$ and $L_d = 2000$ km, whence $\widehat{\beta} = 0.2$.

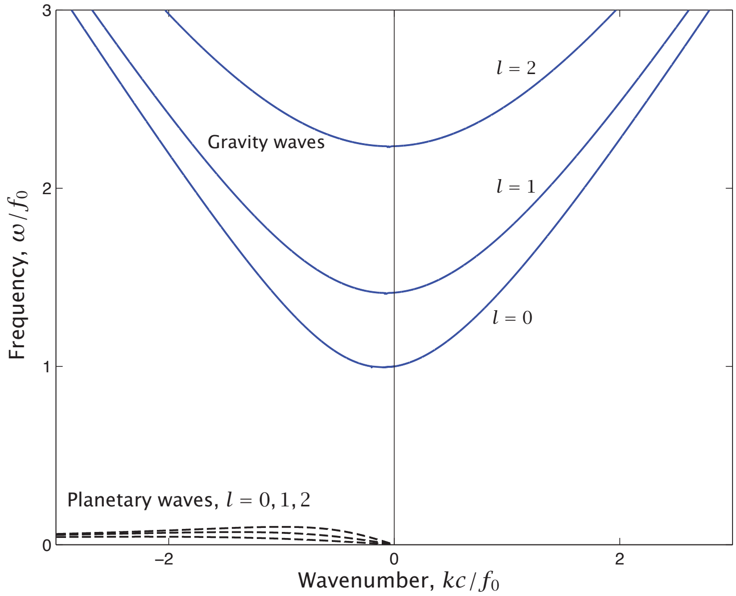

Fig. 8.2 Dispersion relation for Rossby-gravity waves, obtained from (8.8) with $\widehat{\beta} = 0.2$ for three values of l. There is a frequency gap between the Rossby or planetary waves and the gravity waves. For the stratified mid-latitude atmosphere or ocean the frequency gap is in reality even larger.

To actually obtain a solution we regard the equation as a quadratic in k and solve in terms of the frequency, giving

$$\widehat{k} = -\frac{\widehat{\beta}}{2\widehat{\omega}} \pm \frac{1}{2}\left[\frac{\widehat{\beta}^2}{\widehat{\omega}^2} + 4(\widehat{\omega}^2 - \widehat{l}^2 - 1)\right]^{1/2}. \tag{8.8}$$

The solutions are plotted in Fig. 8.2, with $\widehat{\beta} = 0.2$, and we see that the waves fall into two groups, labelled gravity waves and planetary waves in the figure. The gap between the two groups of waves is in fact still larger if a smaller (and generally more relevant) value of $\widehat{\beta}$ is used. To interpret all this let us consider some limiting cases.

8.1.1 Special Cases and Properties of the Waves

We now consider a few special cases of the dispersion relation.

(i) Constant Coriolis parameter

If $\beta = 0$ then the dispersion relation becomes

$$\omega\left[\omega^2 - f_0^2 - (k^2 + l^2)c^2\right] = 0, \tag{8.9}$$

with the roots

$$\omega = 0, \qquad \omega^2 = f_0^2 + c^2(k^2 + l^2). \tag{8.10a,b}$$

The root $\omega = 0$ corresponds to geostrophic motion (and, since $\beta = 0$, Rossby waves are absent), with the other root corresponding to Poincaré waves, considered in Chapter 3. The frequency is higher than the inertial frequency f; that is, $\omega^2 > f_0^2$.

(ii) High frequency waves

If we take the limit of $\omega \gg f_0$ then (8.3a) gives

$$\frac{\omega^2}{c^2} - (k^2 + l^2) - \frac{\beta k}{\omega} = 0. \tag{8.11}$$

Rossby and Gravity Waves

- Generically speaking, Rossby-gravity waves are waves that arise under the combined effects of a potential vorticity gradient and stratification. Sometimes the definition is restricted to a wave on a single branch of the dispersion curve connecting Rossby and gravity waves. In mid-latitudes on Earth Rossby waves and gravity waves are well separated and have distinct physical mechanisms.

- The simplest setting in which such waves occur is the linearized shallow water equations which may be written as a single equation for v, namely

$$\frac{1}{c^2}\frac{\partial^3 v}{\partial t^3} + \frac{f^2}{c^2}\frac{\partial v}{\partial t} - \frac{\partial}{\partial t}\nabla^2 v - \beta\frac{\partial v}{\partial x} = 0. \tag{RG.1}$$

- If we take both f and β to be constants then the equation above admits of plane-wave solutions with dispersion relation

$$\omega^2 - \frac{\beta k c^2}{\omega} = f_0^2 + c^2(k^2 + l^2). \tag{RG.2}$$

- In Earth's atmosphere and ocean it is common, especially in mid-latitudes, for there to be a frequency separation between two classes of solution. To a good approximation, high frequency waves satisfy

$$\omega^2 = f_0^2 + c^2(k^2 + l^2). \tag{RG.3}$$

  These are inertio-gravity waves, also known as Poincaré waves. The low frequency waves satisfy

$$\omega = \frac{-\beta k c^2}{f_0^2 + c^2(k^2 + l^2)} = \frac{-\beta k}{k_d^2 + k^2 + l^2}, \tag{RG.4}$$

  where $k_d^2 = f_0^2/c^2$, and these are called Rossby waves or planetary waves.

- Rossby-gravity waves also exist in the stratified equations. Solutions may be found by decomposing the vertical structure into a series of orthogonal modes, and a sequence of shallow water equations for each mode results, with a different c for each mode. Solutions may also be found if f is allowed to vary in (RG.1), at the price of some algebraic complexity, as in Section 8.2.

To be physically realistic we should also now eliminate the β term, because if $\omega \gg f_0$ then, from geometric considerations on a sphere, $k^2 \gg \beta k/\omega$. Thus, the dispersion relation is simply $\omega^2 = c^2(k^2 + l^2)$. These waves are just gravity waves uninfluenced by rotation, and are a special case of Poincaré waves.

(iii) Low frequency waves

Consider the limit of $\omega \ll f_0$. The dispersion relation reduces to

$$\omega = \frac{-\beta k}{k^2 + l^2 + k_d^2}. \tag{8.12}$$

This is just the dispersion relation for quasi-geostrophic Rossby waves as previously obtained — see (6.65) or (6.92). In this limit, the requirement that the radius of the circle be positive becomes

$$\omega^2 < \frac{\beta^2}{4k_d^2}. \tag{8.13}$$

That is to say, the Rossby waves have a maximum frequency, and directly from (8.12) this occurs when $k = k_d$ and $l = 0$.

The frequency gap

The maximum frequency of Rossby waves is usually much less than the frequency of the Poincaré waves: the lowest frequency of the Poincaré waves is f_0 and the highest frequency of the Rossby waves is $\beta/2k_d$. Thus,

$$\frac{\text{Low gravity wave frequency}}{\text{High Rossby wave frequency}} = \frac{f_0}{\beta/2k_d} = \frac{f_0^2}{2\beta c}. \tag{8.14}$$

If $f_0 = 10^{-4}\,\mathrm{s}^{-1}$, $\beta = 10^{-11}\,\mathrm{m}^{-1}\,\mathrm{s}^{-1}$ and $k_d = 1/100\,\mathrm{km}^{-1}$ (a representative oceanic baroclinic deformation radius) then $f_0/(\beta/2k_d) = 200$. If $L_d = 1000\,\mathrm{km}$ (an atmospheric baroclinic radius) then the ratio is 20. If we use a barotropic deformation radius of $L_d = 2000\,\mathrm{km}$ then the ratio is 10. Evidently, for most mid-latitude applications there is a large gap between the Rossby wave frequency and the gravity wave frequency. Because of this frequency gap, to a good approximation Fig. 8.2 may be obtained by separately plotting (8.10b) for the gravity waves, and (8.12) for the Rossby or planetary waves. The differences between these and the exact results become smaller as $\widehat{\beta}$ gets smaller, virtually indistinguishable in the plots shown.

Finally, we remark that a 'Rossby-gravity wave' is sometimes defined to be the wave on a single branch of the dispersion curve that connects Rossby waves and gravity waves across a range of wavenumbers. The equatorial beta plane does support such a wave — the 'Yanai wave' that will be derived in Section 8.2 and shown in Fig. 8.6, and this wave bridges the frequency gap near the equator. However, in the mid-latitude system above there is no such wave; rather, there are what are essentially separate Rossby waves and gravity waves.

8.1.2 Planetary-Geostrophic Rossby waves

A good approximation for the large-scale ocean circulation involves ignoring the time-derivatives and nonlinear terms in the momentum equation, allowing evolution only to occur in the thermodynamic equation. This is the planetary-geostrophic approximation, introduced in Section 5.2, and it is interesting to see to what extent that system supports Rossby waves.[3] It is easiest just to begin with the linear shallow water equations themselves, and omitting time derivatives in the momentum equation gives

$$-fv = -\frac{\partial\phi}{\partial x}, \qquad fu = -\frac{\partial\phi}{\partial y}, \tag{8.15a,b}$$

$$\frac{\partial\phi}{\partial t} + c^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0. \tag{8.15c}$$

From these equations we straightforwardly obtain

$$\frac{\partial\phi}{\partial t} - \frac{c^2\beta}{f^2}\frac{\partial\phi}{\partial x} = 0. \tag{8.16}$$

Again we will treat both f and β as constants so that we may look for solutions in the form $\phi = \widetilde{\phi}\exp[\mathrm{i}(\boldsymbol{k}\cdot x - \omega t)]$. The ensuing dispersion relation is

$$\omega = -\frac{c^2\beta}{f_0^2}k = -\frac{\beta k}{k_d^2}, \tag{8.17}$$

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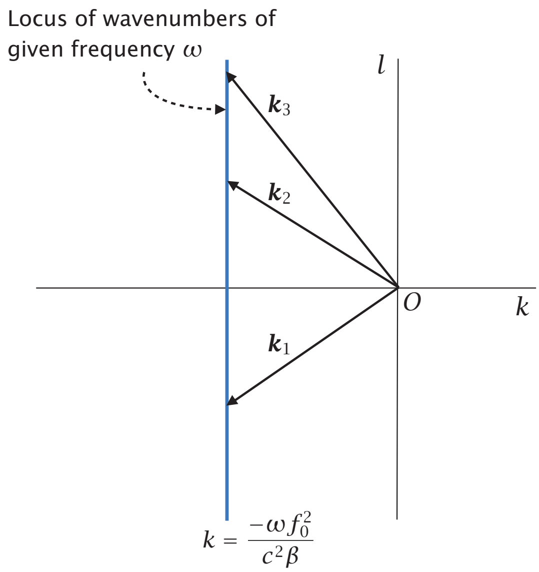

Fig. 8.3 The locus of points on planetary-geostrophic Rossby waves. Waves of a given frequency all have the same x-wavenumber, given by (8.17).

which is a limiting case of (8.12) with $k^2, l^2 \ll k_d^2$. The waves are a form of Rossby waves with phase and group speeds given by

$$c_p = -\frac{c^2\beta}{f_0^2}, \qquad c_g^x = -\frac{c^2\beta}{f_0^2}. \tag{8.18}$$

That is, the waves are non-dispersive and propagate westward. Equation (8.16) has the general solution $\phi = G(x + \beta c^2/f^2 t)$, where G is any function, so an initial disturbance will just propagate westward at a speed given by (8.18), without any change in form.

Note finally that the locus of wavenumbers in k–l space is no longer a circle, as it is for the usual Rossby waves. Rather, since the frequency does not depend on the y-wavenumber, the locus is a straight line, parallel to the y-axis, as in Fig. 8.3. Waves of a given frequency all have the same x-wavenumber, given by $k = -\omega f_0^2/(c^2\beta) = -\omega k_d^2/\beta$, as shown in Fig. 8.3.

Physical mechanism

Because the waves *are* a form of Rossby wave their physical mechanism is related to that discussed in Section 6.4.3, but with an important difference: relative vorticity is no longer important, but the flow divergence is. Thus, consider flow round a region of high pressure, as illustrated in Fig. 8.4. If the pressure is circularly symmetric as shown, the flow to the south of H in the left-hand sketch, and to the south of L in the right-hand sketch, is larger than that to the north. Hence, in the left sketch the flow converges at W and diverges at E, and the flow pattern moves westward. In the flow depicted in the right sketch the low pressure propagates westward in a similar fashion.

8.2 WAVES ON THE EQUATORIAL BETA PLANE

We now discuss the properties of shallow water waves at low latitudes, allowing the Coriolis parameter to properly vary in all terms, albeit using the β-plane approximation.[4] Thus, Taylor-expanding the Coriolis parameter around a latitude ϑ_0 we obtain

$$f = 2\Omega \sin\vartheta \approx 2\Omega \sin\vartheta_0 + 2\Omega(\vartheta - \vartheta_0)\cos\vartheta_0 = f_0 + \beta y, \tag{8.19}$$

where $f_0 = 2\Omega \sin\vartheta_0$, $\beta = 2\Omega\cos\vartheta_0/a$ and $y = a(\vartheta - \vartheta_0)$ where a is the radius of the Earth. For motions at low latitudes we take $\vartheta_0 = 0$, giving the *equatorial beta-plane approximation* in which

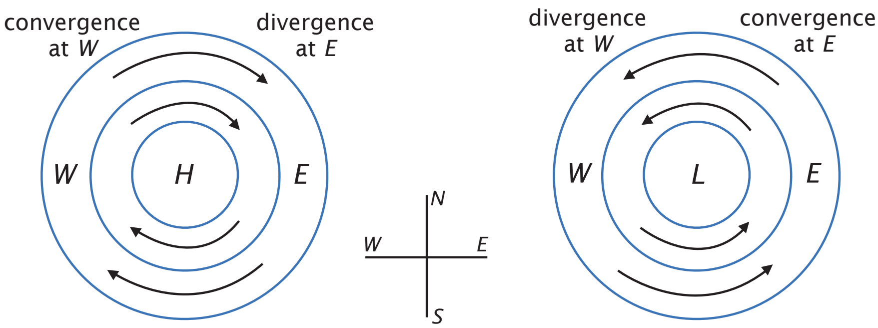

Fig. 8.4 The westward propagation of planetary-geostrophic Rossby waves. The circular lines are isobars centred around high and low pressure centres. Because of the variation of the Coriolis force, the mass flux between two isobars is greater to the south of a pressure centre than it is to the north. Hence, in the left-hand sketch there is convergence to the west of the high pressure and the pattern propagates westward. Similarly, if the pressure centre is a low, as in the right-hand sketch, there is divergence to the west of the pressure centre and the pattern still propagates westward.

$\sin\vartheta \approx \vartheta$, $\cos\vartheta \approx 1$ and $f = 2\Omega\vartheta = \beta y$. The linearized momentum and mass conservation equations are then

$$\frac{\partial u}{\partial t} - fv = -\frac{\partial \phi}{\partial x}, \qquad \frac{\partial v}{\partial t} + fu = -\frac{\partial \phi}{\partial y}, \qquad \frac{\partial \phi}{\partial t} + c^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0. \tag{8.20a,b,c}$$

These are exactly the same as (8.1) except that now $f = \beta y$. There are just two dimensional parameters in the above equations, c and β, and from these we may form the scales

$$T_{eq} = (c\beta)^{-1/2}, \qquad L_{eq} = \left(\frac{c}{\beta}\right)^{1/2}. \tag{8.21a,b}$$

These are the fundamental time and length scales for equatorial dynamics (although some definitions differ by a factor of $\sqrt{2}$, as discussed later). The length scale L_{eq} is known as the equatorial radius of deformation. If we regard the above shallow water equations as coming from a modal decomposition of the primitive equations, as in Section 3.4, then there is a wave speed, $c_m = \sqrt{gH_m}$ and deformation radius for each mode. For the first baroclinic mode and for the atmosphere, if $c_1 = 25\,\mathrm{m\,s^{-1}}$ and $\beta = 2.3\times10^{-11}\,\mathrm{m^{-1}\,s^{-1}}$, then $L_{eq} \approx 1000\,\mathrm{km}$ and $T_{eq} \approx 0.5$ days; for the equatorial ocean with $c = 2\,\mathrm{m\,s^{-1}}$ then $L_{eq} \approx 300\,\mathrm{km}$ and $T_{eq} \approx 1.7$ days.

Cross-differentiating (8.20a) and (8.20b) and using (8.20c) to eliminate the divergence we may also derive the linearized potential vorticity equation, namely

$$\frac{\partial}{\partial t}\left(\zeta - \frac{f\phi}{c^2}\right) + \beta v = 0. \tag{8.22}$$

This is the same as the familiar linearized potential vorticity equation on the f-plane, with the addition of the term $\mathrm{D}f/\mathrm{D}t = \beta v$. Equation (8.22) is not independent of (8.20) but it will be convenient to use it sometimes.

To obtain a single equation for a single unknown, operate on (8.20a) with $(f/c^2)\partial_t$, on (8.20b) with $(1/c^2)\partial_{tt}$, on (8.20c) with $(1/c^2)\partial_{ty})$ and on (8.22) with ∂_x. Using subscripts to denote derivatives the resulting equations are

$$\frac{f}{c^2}u_{tt} - \frac{f^2}{c^2}v_t = -\frac{f}{c^2}\phi_{xt}, \qquad \frac{1}{c^2}v_{ttt} + \frac{f}{c^2}u_{tt} = -\frac{1}{c^2}\phi_{ytt}, \tag{8.23a,b}$$

$$\frac{1}{c^2}\phi_{tty} + (u_{xyt} + v_{yyt}) = 0, \qquad v_{xxt} - u_{xyt} - \frac{f}{c^2}\phi_{xt} + \beta v_x = 0. \tag{8.23c,d}$$

These equations linearly combine (a - (b+c+d)) to give a single equation for v, namely

$$\boxed{\frac{1}{c^2}\frac{\partial^3 v}{\partial t^3} + \frac{f^2}{c^2}\frac{\partial v}{\partial t} - \frac{\partial}{\partial t}\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2}\right) - \beta\frac{\partial v}{\partial x} = 0.} \tag{8.24}$$

This equation is third order and has non-constant coefficients, and is thus somewhat complicated. Before proceeding, note one common approximation, sometimes called the *longwave approximation*. If zonal scales are much greater than meridional scales then we expect the zonal wind to be in geostrophic balance with the meridional pressure gradient. In this case we replace (8.20b) by

$$fu = -\frac{\partial \phi}{\partial y}, \tag{8.25}$$

and (8.22) and (8.23b,d) are modified accordingly. Then, in instead of (8.24), we obtain

$$\frac{f^2}{c^2}\frac{\partial v}{\partial t} - \frac{\partial}{\partial t}\left(\frac{\partial^2 v}{\partial y^2}\right) - \beta\frac{\partial v}{\partial x} = 0. \tag{8.26}$$

This equation is first order in time and the dispersion relation may be obtained reasonably straightforwardly. This approximation is particularly useful in the forced-dissipative problem as we will see in Section 8.4. In the free problem the dispersion equation can in fact be obtained easily enough in the general case, that is from (8.24), allowing us to make the longwave approximation at a later stage.

8.2.1 Dispersion Relations

In this section we explore the properties of (8.24), in particular obtaining a dispersion relation. The coefficients of (8.24) vary in the meridional direction but are constant in the zonal direction. We thus search for solutions in the form of a plane wave in the zonal direction only and we let

$$v = \widetilde{v}(y)\mathrm{e}^{\mathrm{i}(kx-\omega t)}, \tag{8.27}$$

and assume boundary conditions of $\widetilde{v}(y) \to 0$ as $y \to \pm\infty$. Substituting (8.27) into (8.24) gives

$$\frac{\mathrm{d}^2\widetilde{v}}{\mathrm{d}y^2} + \left(\frac{\omega^2}{c^2} - k^2 - \frac{\beta k}{\omega} - \frac{\beta^2 y^2}{c^2}\right)\widetilde{v} = 0. \tag{8.28}$$

Given the velocity, c, and the presence of the beta effect there is a rather obvious way to nondimensionalize the equations. However, it turns out that by introducing an additional factor of $\sqrt{2}$ into the scaling, the mathematics of one of the problems that we address later is simplified. At the risk of discussing a trivial difference, let's do both — the confident and impatient reader may choose one and skim the other.

Nondimensionalization I

Let us scale time and distance with the quantities $T_{eq} = (c\beta)^{-1/2}$, $L_{eq} = (c/\beta)^{1/2}$ as in (8.21). The nondimensional frequency, lengthscale and wavenumber are then given by

$$\widehat{\omega} = \frac{\omega}{(\beta c)^{1/2}}, \qquad \widehat{y} = y\left(\frac{\beta}{c}\right)^{1/2}, \qquad \widehat{k} = k\left(\frac{c}{\beta}\right)^{1/2}. \tag{8.29}$$

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If we take $\delta\rho/\rho_0 = 0.002$, $H = 100\,\mathrm{m}$ and $\beta = 2\Omega/a = 2.3 \times 10^{-11}\,\mathrm{m^{-1}\,s^{-1}}$ we find

$$g' \approx 0.02\,\mathrm{m\,s^{-2}}, \quad c \approx 1.4\,\mathrm{m\,s^{-1}}, \quad L_{eq} \approx 250\,\mathrm{km}, \quad T_\beta = 1.7 \times 10^5\,\mathrm{s} \approx 2\,\mathrm{days}. \tag{8.30}$$

The mid-latitude shallow-water deformation radius, L_d is usually defined as $L_d = c/f$ which differs from (8.21b) most notably in the power of f. However, if in the mid-latitude expression we take $f = \beta y$, as if near the equator, and $y = L_d$, then $L_d = c/(\beta L_d)$, which is the same as (8.21b).

Substituting (8.29) into (8.28) gives the slightly simpler-looking equation

$$\frac{\mathrm{d}^2\widetilde{v}}{\mathrm{d}\widehat{y}^2} + \left(\widehat{\omega}^2 - \widehat{k}^2 - \frac{\widehat{k}}{\widehat{\omega}} - \widehat{y}^2\right)\widetilde{v} = 0. \tag{8.31}$$

This equation may be put into a standard form[5] by writing $v(\widehat{y}) = \Psi(\widehat{y})\exp(-\widehat{y}^2/2)$, whence (8.31) becomes

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}\widehat{y}^2} - 2\widehat{y}\frac{\mathrm{d}\Psi}{\mathrm{d}\widehat{y}} + \lambda\Psi = 0, \tag{8.32}$$

where $\lambda = \widehat{\omega}^2 - \widehat{k}^2 - \widehat{k}/\widehat{\omega} - 1$. Equation (8.32) is known as *Hermite's equation,* and it is an eigenvalue equation, with solutions if and only if $\lambda = 2m$, for $m = 0, 1, 2, \ldots$. The solutions are Hermite polynomials, $\Psi(\widehat{y}) = H_m(\widehat{y})$, where the first few polynomials are given by

$$\begin{gathered} H_0 = 1, \qquad H_1 = 2\widehat{y}, \qquad H_2 = 4\widehat{y}^2 - 2, \\ H_3 = 8\widehat{y}^3 - 12\widehat{y}, \qquad H_4 = 16\widehat{y}^4 - 48\widehat{y}^2 + 12. \end{gathered} \tag{8.33}$$

A Hermite polynomial is even or odd when m is even or odd, respectively; that is $H_m(-\widehat{y}) = (-1)^m H_m(\widehat{y})$. Note that we are using Hermite polynomials to describe the v field, so that mirror symmetry across the equator occurs when m is odd. The v field is then odd, but the u and ϕ fields are even, as shown in Appendix A to this chapter.

Hermite polynomials multiplied by a Gaussian are a form of *parabolic cylinder function*,

$$V_m(y) = H_m(y)\exp(-y^2/2). \tag{8.34}$$

These functions are also orthogonal in the interval $[-\infty, +\infty]$; that is

$$\int_{-\infty}^{\infty} V_n V_m\,\mathrm{d}y = \int_{-\infty}^{\infty} H_n(y)H_m(y)\exp(-y^2)\,\mathrm{d}y = \sqrt{\pi}2^n n!\,\delta_{nm}, \tag{8.35}$$

Appendix A gives additional details. Given the Hermite solution for Ψ, the solutions for v are given by

$$v(\widehat{y}) = V_m(\widehat{y}) = H_m(\widehat{y})\mathrm{e}^{-\widehat{y}^2/2}, \qquad m = 0, 1, 2\ldots, \tag{8.36}$$

and so decay exponentially as $\widehat{y} \to \pm\infty$ (as we require) with a decay scale of the equatorial deformation radius $\sqrt{c/\beta}$. The functions V_m are plotted in Fig. 8.5 for $m = 0$ to 3.

The dispersion relation follows from the quantization condition $\lambda = 2m$, which implies

$$\widehat{\omega}^2 - \widehat{k}^2 - \frac{\widehat{k}}{\widehat{\omega}} = 2m + 1, \tag{8.37a}$$

or, using (8.29), the dimensional form,

$$\boxed{\omega^2 - c^2k^2 - \beta\frac{kc^2}{\omega} = (2m+1)\beta c.,} \tag{8.37b}$$

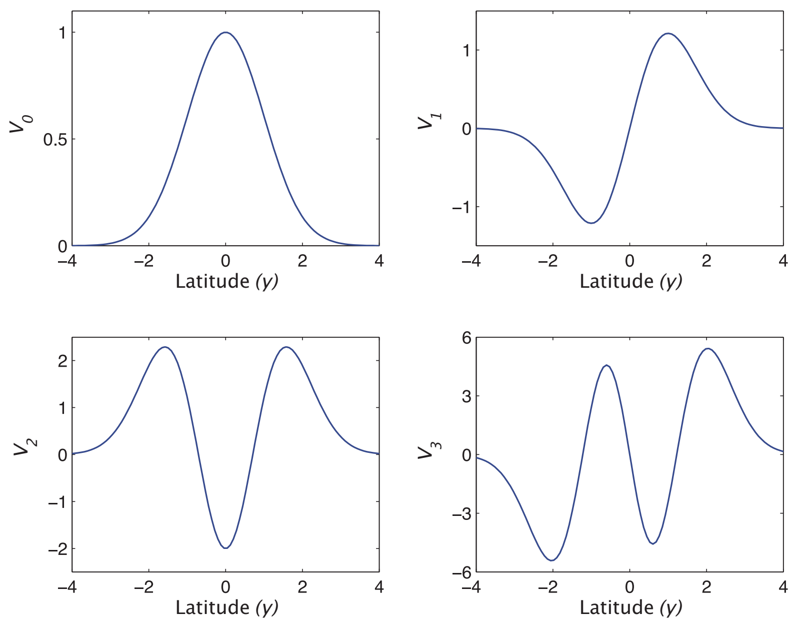

Fig. 8.5 Latitudinal variation of the wave amplitudes, the parabolic cylinder functions $V_m(y)$, given by (8.34), for $m = 0, 1, 2, 3$. The parameter m is analogous to a meridional wavenumber. The parabolic cylinder functions given by (8.43) have a similar but not identical form.

This is a cubic equation in ω, and although a solution is possible, it is easier to solve the quadratic equation for the wavenumber in terms of the frequency giving

$$\widehat{k} = -\frac{1}{2\widehat{\omega}} \pm \frac{1}{2}\left[\left(\frac{1}{\widehat{\omega}} - 2\widehat{\omega}\right)^2 - 8m\right]^{1/2}, \tag{8.37c}$$

or, in dimensional form,

$$k = -\frac{\beta}{2\omega} \pm \frac{1}{2}\left[\left(\frac{\beta}{\omega} - \frac{2\omega}{c}\right)^2 - \frac{8m\beta}{c}\right]^{1/2}. \tag{8.37d}$$

Equations (8.37) are forms of the dispersion relation for the shallow water equations on an equatorial beta-plane. Before exploring their properties we nondimensionalize in a different way.

Nondimensionalization II

We now scale time and distance with the quantities

$$T_{eq} = (2c\beta)^{-1/2}, \qquad L_{eq} = (c/2\beta)^{1/2}. \tag{8.38a,b}$$

Velocity is still nondimensionalized by c. The nondimensional version of (8.28) becomes

$$\frac{\mathrm{d}^2\widetilde{v}}{\mathrm{d}\widehat{y}^2} + \left(\widehat{\omega}^2 - \widehat{k}^2 - \frac{\widehat{k}}{2\widehat{\omega}} - \frac{\widehat{y}^2}{4}\right)\widetilde{v} = 0, \tag{8.39}$$

which may be compared with (8.31). We now make the substitution

$$\widetilde{v}(\widehat{y}) = \Phi \exp(-\widehat{y}^2/4), \tag{8.40}$$

which leads to

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\widehat{y}^2} - \widehat{y}\frac{\mathrm{d}\Phi}{\mathrm{d}\widehat{y}} + \gamma\Phi = 0, \tag{8.41}$$

where $\gamma = \widehat{\omega}^2 - \widehat{k}^2 - \widehat{k}/2\widehat{\omega} - 1/2$. Equation (8.41) could be transformed into (8.32) by changing to the independent variable $y' = \widehat{y}/\sqrt{2}$, and the dispersion relation then follows in the same way. More directly, solutions of (8.41) are given by the modified Hermite polynomials $\Phi(\widehat{y}) = G_m(\widehat{y})$ where

$$(G_0, G_1, G_2, G_3, G_4) = (1, \widehat{y}, \widehat{y}^2 - 1, \widehat{y}^3 - 3\widehat{y}, \widehat{y}^4 - 6\widehat{y}^2 + 3). \tag{8.42}$$

These are also known as the probabilists' Hermite polynomials, with (8.33) or (8.151) being the physicists' Hermite polynomials, reflecting historical use; the two sets of polynomials are connected by $H_n(y) = 2^{n/2}G_n(y\sqrt{2})$. The corresponding parabolic cylinder functions are given by

$$D_n(\widehat{y}) = G_n(\widehat{y})\exp(-\widehat{y}^2/4), \tag{8.43}$$

and these functions are solutions of (8.39). The orthonormality condition on the modified polynomials is that

$$\int_{-\infty}^{\infty} D_n(y)D_m(y)\,\mathrm{d}y = \int_{-\infty}^{\infty} G_n(y)G_m(y)\exp(-y^2/2)\,\mathrm{d}y = \sqrt{2\pi}\,n!\;\delta_{nm}, \tag{8.44}$$

which may be compared to (8.35). The quantization condition on γ is that $\gamma = m$, where $m = 0, 1, 2, \ldots$. Thus, the nondimensional dispersion relation is

$$\widehat{\omega}^2 - \widehat{k}^2 - \frac{\widehat{k}}{2\widehat{\omega}} - \frac{1}{2} = m, \tag{8.45}$$

and restoring the dimensions using (8.38) gives (8.37b). Later on, when dealing with the steady, forced-dissipative problem, the use of the probabilists' polynomials is a little more convenient.

8.2.2 Limiting and Special Cases

To further explore the wave case we stay with our first nondimensionalization, namely (8.21), and with the goal of figuring out what's going on we'll consider various special cases of the dispersion relations (8.37). It is convenient to first partition the waves by frequency, and consider separately high frequency gravity waves and low frequency planetary waves. We need do this only for the case $m \geq 1$ because the $m = 0$ case (mixed Rossby-gravity waves) may be treated exactly. Then finally we look at the so-called $m = -1$ case, namely Kelvin waves.

High and low frequency waves

1. *High frequency waves.* The term $\beta k c^2/\omega$ in (8.37) is small and may be neglected. The dispersion relation becomes

$$\widehat{\omega}^2 = \widehat{k}^2 + 2m + 1 \qquad \text{or} \qquad \omega^2 = c^2k^2 + \beta c(2m + 1). \tag{8.46a,b}$$

   This dispersion relation is similar to that of mid-latitude Poincaré waves, with βc replacing f_0^2: recall the form of (3.121), namely $\omega^2 = c^2(k^2 + l^2) + f_0^2$. Waves satisfying (8.46) are thus sometimes called equatorially trapped Poincaré waves or equatorially trapped gravity waves.

   The approximation requires that $\omega \gg \beta/|k|$, and is somewhat inaccurate for small k: note that (8.46) is symmetric around $k = 0$, whereas the full dispersion relation, plotted in Fig. 8.6, is offset. (Formally, the limit is valid for $\widehat{k} \to \infty$, $\widehat{\omega} \to \infty$ and $\widehat{k}/\widehat{\omega} =$ constant.)

   For finite m the limiting case at high wavenumbers is just $\widehat{\omega} = \pm\widehat{k}$, or, in dimensional form, $\omega = \pm ck$. This is just the dispersion relation for familiar conventional shallow water gravity waves, unaffected by rotation and the β-effect. However, in the rotating case the waves are trapped at the equator and propagate only in the zonal direction, albeit both eastward and westward.

2. *Low frequency waves.* For low frequency waves we neglect the term involving ω^2 in (8.37) and the dispersion relation becomes

$$\widehat{\omega} = \frac{-\widehat{k}}{2m+1+\widehat{k}^2}, \qquad \omega = \frac{-\beta k}{(2m+1)\beta/c + k^2}, \tag{8.47}$$

nondimensionally and dimensionally, respectively. This is recognizable as the dispersion relation for a zonally propagating Rossby wave with large x-wavenumber, and these waves are called equatorially trapped Rossby waves, or equatorially trapped planetary waves. We may further consider two limits of these waves, as follows.

(a) *Short, low frequency waves, with* $\widehat{k} \to \infty$, $\widehat{\omega} \to 0$. The dispersion relation becomes

$$\widehat{\omega} = -\frac{1}{\widehat{k}}, \qquad \omega = -\frac{\beta}{k}. \tag{8.48}$$

The phase speed and group velocity in this limit are given by, dimensionally,

$$c_p = -\frac{\beta}{k^2}, \qquad c_g = \frac{\beta}{k^2}. \tag{8.49}$$

Thus, the phase speed is westward but the group velocity, and so the direction of energy propagation, is eastward.

(b) *Long low frequency waves, with* $\widehat{k} \to 0$, $\widehat{\omega} \to 0$. The dispersion relation (8.37) becomes, in nondimensional and dimensional form,

$$\widehat{\omega} = \frac{-\widehat{k}}{2m+1}, \qquad \omega = \frac{-ck}{2m+1}. \tag{8.50}$$

These represent westward propagating waves whose speed is given by $c/(2m+1)$. For $m = 1$ (the smallest allowable value for planetary waves) they have one-third the speed of a non-rotating gravity wave or of a Kelvin wave (discussed below). However, these waves propagate only westward, and they match with the westward propagating planetary waves derived above as wavenumber increases. They are conveniently nondispersive, and are also important near western boundaries where they superpose to create western boundary currents. The longwave approximation may be made from the outset, and is equivalent to assuming that the zonal flow is in geostrophic balance; that is, (8.20b) is replaced by $fu = -g'\partial\eta/\partial y$. Then, instead of solving (8.24) we solve (8.26). The only difference is in the value of λ in (8.32) — we find $\lambda = -\widehat{k}/\widehat{\omega} - 1$ — and so (8.50) immediately emerges. Short waves are filtered out of the system. This approximation will turn out to be particularly important when we consider the steady problem in Section 8.5.

There is a distinct gap in frequencies between the minimum frequency of the gravity waves, given by (8.46), and the maximum frequency of the planetary waves, given by (8.49) also with m small. The minimum gravity wave frequency occurs when $k = 0$ and is $\omega^2_{gmin} = \beta c(2m+1)$. From (8.47) the maximum planetary wave frequency occurs when $k^2 = (2m+1)\beta/c$ and gives $\omega^2_{pmax} = \beta c/[4(2m+1)]$. The ratio of these two frequencies is

$$\frac{\omega_{gmin}}{\omega_{pmax}} = 2(2m+1), \tag{8.51}$$

giving a value of six for $m = 1$ and two for $m = 0$ (a case we consider more below). Note that this ratio is *independent* of the values of the physical parameters β and c. Although the gap is distinct, it is not as large as the corresponding gap at mid-latitudes, which may be an order of magnitude or more.

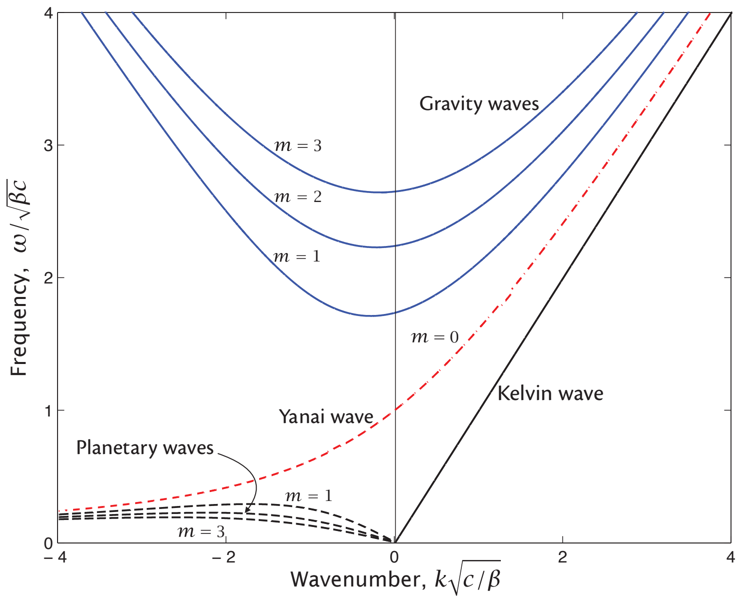

Fig. 8.6 Dispersion relation for equatorial waves, as given by (8.37), for $m = 0, 1, 2, 3$. The upper group of curves are gravity waves, given approximately by (8.46). The lower group with $k < 0$ are westward propagating planetary waves, given approximately by (8.47). Also shown are the Yanai wave with $m = 0$, satisfying (8.54), and the eastward propagating Kelvin wave (the '$m = -1$' wave) satisfying $\omega = ck$ for $k \geq 0$.

Special values of m

In addition to the limiting cases at low and high frequency, there are two other cases in which we can readily solve the dispersion relation, namely when $m = 0$ and the Kelvin wave case, as follows:

1. *The case with $m = 0$.* The resulting waves are known as *Yanai waves*,[6] or *Rossby-gravity* waves, since they span the two types of waves. They are antisymmetric across the equator. From (8.37a) the dispersion relation simplifies to

$$\widehat{k} = -\widehat{\omega} \qquad \text{or} \qquad \widehat{k} = -\frac{1}{\widehat{\omega}} + \widehat{\omega}. \tag{8.52a,b}$$

or dimensionally

$$k = -\frac{\omega}{c}, \qquad k = -\frac{\beta}{\omega} + \frac{\omega}{c}. \tag{8.53a,b}$$

The case $k = -\omega/c$ is non-physical, for it represents a gravity wave moving westward. Such a wave grows without bound as $|y|$ increases away from the equator, as we demonstrate explicitly in the discussion on Kelvin waves below. The physically realizable case, (8.53b), has the explicit dispersion relation

$$\omega = \frac{kc}{2} \pm \frac{1}{2}\sqrt{k^2c^2 + 4\beta c}. \tag{8.54}$$

Again it is useful to consider various limiting cases:

- $k = 0$. In this case (8.54) gives $\omega = \sqrt{\beta c}$ and there is a balance between the two terms on the right-hand side of (8.53b). Note that in Fig. 8.6 the Yanai wave at $k = 0$ intercepts the ordinate at a value of nondimensional frequency of 1.
- $k \to +\infty$. In this case $\omega = ck$, with a balance between the left-hand side and the second term on the right-side of (8.53b). Evidently, this corresponds to eastward propagating gravity waves.
- $k \to -\infty$. In this case, because ω must be positive, we have $\omega = -\beta/k$, and a balance between the left-hand side and the first term on the right-side of (8.53b). The waves are westward propagating Rossby or planetary waves.

Yanai waves, therefore, are mixed Rossby-gravity waves: the phase of the Rossby wave propagates westward (like all Rossby waves) and has a low frequency, and the gravity wave propagates eastward (and only eastward, unlike conventional gravity waves). The group velocity of Yanai waves is positive in all cases, being given by, from (8.53b),

$$c_g^x \equiv \frac{\partial \omega}{\partial k} = \frac{\omega^2 c}{\beta c + \omega^2}. \tag{8.55}$$

The group velocity of the full problem is, from (8.37),

$$c_g^x = \frac{c^2 \omega(\beta + 2\omega k)}{2\omega^3 + \beta k c^2}. \tag{8.56}$$

This may be positive or negative, and vanishes when $\omega = -\beta/2k$.

2. *Kelvin waves, or the '$m = -1$' case.* (This section may be considered to be an extension of Section 3.8.3.) In general, Hermite's equation, (8.32), has solutions when m is a positive integer or zero. However, there is a class of waves that happens to satisfy the dispersion relation (8.37) with $m = -1$, namely equatorial Kelvin waves. These waves have identically zero meridional velocity and so their equations of motion are

$$\frac{\partial u}{\partial t} = -g' \frac{\partial \eta}{\partial x}, \qquad f u = -g' \frac{\partial \eta}{\partial y}, \qquad \frac{\partial \eta}{\partial t} + H \frac{\partial u}{\partial x} = 0, \tag{8.57a,b,c}$$

where $f = \beta y$. The zonal velocity is in geostrophic balance with the meridional pressure gradient, and (8.57a) and (8.57c) give the classic wave equation,

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \tag{8.58}$$

where $c = \sqrt{g'H}$ as before, and so the dispersion relation $\omega = \pm ck$. This is, in fact, a solution of (8.37) with $m = -1$, as may easily be checked.

The solution to (8.58), and the corresponding solution for η, is

$$u = F_1(x + ct, y) + F_2(x - ct, y), \qquad \eta = \left(\frac{H}{g'}\right)^{1/2} \left[-F_1(x + ct, y) + F_2(x - ct, y)\right], \tag{8.59}$$

where F_1 and F_2 are arbitrary functions, representing waves travelling westwards and eastwards, respectively. We obtain the y-dependence of these functions by using (8.57b) giving

$$\beta y F_1 = c \frac{\partial F_1}{\partial y}, \qquad \beta y F_2 = -c \frac{\partial F_2}{\partial y}. \tag{8.60}$$

The solutions of these equations are

$$F_1 = F(x + ct) \exp[y^2/(2L_{eq}^2)], \qquad F_2 = G(x - ct) \exp[-y^2/(2L_{eq}^2)], \tag{8.61a,b}$$

where F and G are the amplitudes at $y = 0$. Evidently, F_1 increases without bound away from the equator, and so this solution must be eliminated. The complete solution is thus:

$$u = G(x - ct) \exp[-y^2/(2L_{eq}^2)], \qquad \eta = \frac{H}{c} u, \qquad v = 0, \tag{8.62}$$

with dispersion relation

$$\omega = ck. \tag{8.63}$$

These waves are equatorially trapped Kelvin waves. They propagate eastward only, without dispersion, and their amplitude decays away from the equator in precisely the same way as the other equatorial waves considered above, and in a slightly different way from the Kelvin waves on the f-plane given by (3.135).

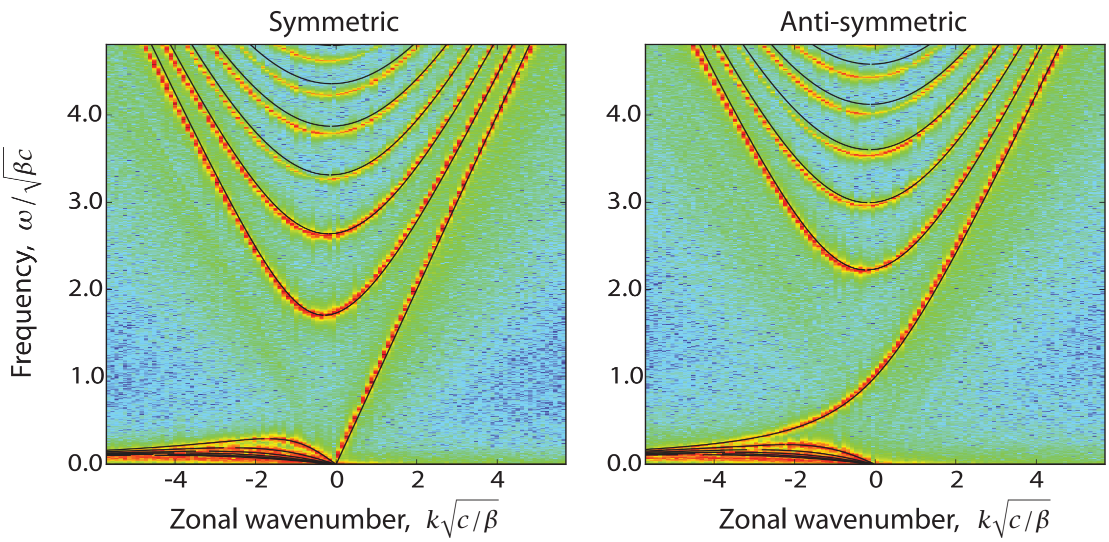

Fig. 8.7 Power spectra from a numerical simulation of the shallow water equations on the sphere(colour shading, with red the most intense), with the analytic dispersion relation for equatorial Rossby and gravity waves overlaid (solid black lines, as in Fig. 8.6). The left panel shows the symmetric component, obtained by adding Northern and Southern Hemispheres and with only the odd values of m plotted analytically, and the right panel plots the antisymmetric component and the even values of m.

8.2.3 A Numerical Illustration

After the many mathematical manipulations above, the reader, if like the author, may well be sceptical that such waves do actually exist, especially on a sphere where Kelvin waves are not exactly realizable. To assuage this doubt, Fig. 8.7 shows the power spectrum from a numerical simulation of the nonlinear shallow water equations over the full sphere. The height field is initialized with small random perturbations everywhere and the system allowed to freely evolve, with no damping except for that residing in the numerical scheme.[7] The figure shows the resulting power spectrum, over a region from 15° S to 15° N, from the near-statistical equilibrium state that emerges. The equatorial waves emerge with beautiful clarity above the noise, with only small deviations for the highest modes due to resolution issues with the numerics that slow the waves. To see a simulation showing Rossby and Kelvin waves in physical space look ahead to Fig. 22.18. The real world is never quite so limpid, but Fig. 18.23 and Fig. 18.24 will suggest that the waves are not merely figments of the imathination.

8.2.4 Why do Kelvin Waves have a Preferred Direction of Travel?

Both equatorial and coastal Kelvin waves have a preferred direction of travel: equatorial Kelvin waves move eastward and, consistently, coastal Kelvin waves travel such that they have a wall to their right in the Northern Hemisphere and to their left in the Southern Hemisphere. Why?

Consider the linear zonal momentum and mass continuity equations,

$$\frac{\partial u}{\partial t} = -g'\frac{\partial \eta}{\partial x}, \qquad \frac{\partial \eta}{\partial t} = -H\frac{\partial u}{\partial x}. \tag{8.64}$$

Looking for wavelike solutions of the form $(u, h) = (\tilde{u}, \tilde{\eta})e^{i(kx-\omega t)}$ we obtain $\tilde{u} = g'\tilde{\eta}/c$ and $c\tilde{\eta} = H\tilde{u}$. This means that under the crests of fluid (i.e., positive values of η) u has the same sign as c; the parcels of fluid are moving in the same direction as the phase of the wave. This property is also apparent if one considers how the fluid must move in order that the troughs and crests progress in

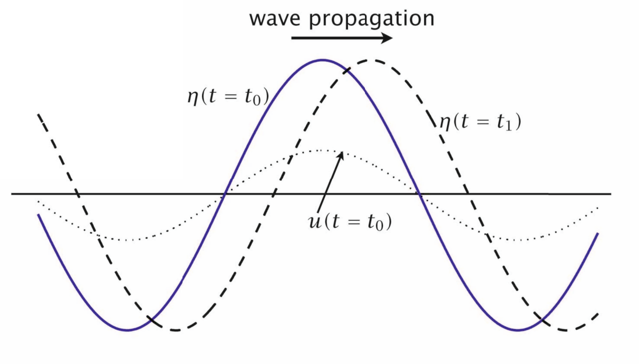

Fig. 8.8 A shallow water gravity wave, showing the fluid interface at an initial and later time $\eta(t_0)$ and $\eta(t_1)$, and the fluid velocity at the initial time, $u(t_0)$. The fluid flow is in the same direction as the phase speed (positive in this example) under the fluid crests, and is in the opposite direction under the troughs.

a a particular direction, as illustrated in Fig. 8.8. This property holds for shallow water waves quite generally, and is not restricted to Kelvin waves.

Now we add rotation and restrict attention to Kelvin waves. In the direction perpendicular to the direction of travel of the wave, the flow is in geostrophic balance:

$$fu = -g' \frac{\partial \eta}{\partial y}. \tag{8.65}$$

Consider the flow under a fluid crest in an equatorial Kelvin wave, as illustrated in Fig. 8.9. The pressure gradient force is directed away from the equator and, if the wave is travelling eastward the pressure force can balanced by the Coriolis force directed toward the equator. Under a trough the fluid is flowing in the opposite direction to the wave itself, and both the pressure gradient force and the Coriolis force are reversed and geostrophic balance still holds. If the wave were to travel westwards, no such balances could be achieved.

Very similar reasoning holds for coastal Kelvin waves, with a cross-wave pressure gradient supported by a wall. Geostrophic balance can now be maintained only if the wall is to the right of the direction of travel in the Northern Hemisphere (where $f > 0$) and to the left in the Southern Hemisphere (where $f < 0$).

8.2.5 Potential Vorticity Dynamics of Equatorial Rossby Waves

The Rossby waves and Rossby-gravity waves derived above are rather similar to their mid- latitude counterparts, which can be derived from a balanced potential vorticity equation without involving unbalanced dynamics at all. Can we do something similar for equatorial Rossby waves? The answer is yes, although the method is a little ad hoc.[8] Kelvin waves and inertia-gravity waves are filtered out, but Rossby waves and Rossby-gravity waves are reproduced in a way that transparently illuminates their dynamics.

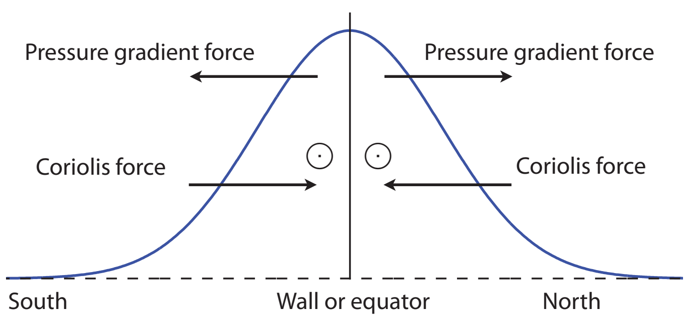

Fig. 8.9 Balance of forces across a Kelvin wave. The solid line is the fluid surface and the phase speed is directed out of the page. Beneath a crest the fluid flow is in the direction of the phase speed and produces Coriolis forces as shown, so balancing the pressure gradient forces. If the wave were travelling in the opposite direction no such geostrophic balance could be achieved.

Let us begin with the unforced linearized potential vorticity equation, which, to remind ourselves, is

$$\frac{\partial}{\partial t}\left(\zeta - \frac{f\phi}{c^2}\right) + \beta v = 0, \tag{8.66}$$

where, as before, $f = \beta y$. Let us now suppose that the divergence is small and the flow close to geostrophic balance so that the velocity, vorticity and height fields can all be written in terms of a streamfunction,

$$u = -\frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x}, \qquad \zeta = \nabla^2 \psi, \qquad \phi = f\psi. \tag{8.67}$$

This is similar to what is done in the quasi-geostrophic approximation, except that here the Coriolis parameter is allowed to vary, with $f = \beta y$. Equation (8.67) is best regarded as an ansatz — an approximation or assumption made for convenience — for it has not been rigorously justified.

Using (8.67) in (8.66) gives

$$\frac{\partial}{\partial t}\left(\nabla^2 \psi - \frac{f^2 \psi}{c^2}\right) + \beta \frac{\partial \psi}{\partial x} = 0. \tag{8.68}$$

We can seek wavelike solutions of this in the form

$$\psi = \widetilde{\psi}(y) \mathrm{e}^{\mathrm{i}(kx - \omega t)}, \tag{8.69}$$

and (8.68) becomes

$$\frac{\mathrm{d}^2 \widetilde{\psi}}{\mathrm{d}y^2} - \left(k^2 + \frac{\beta k}{\omega} + \frac{\beta^2 y^2}{c^2}\right)\widetilde{\psi} = 0. \tag{8.70}$$

This is almost the same as (8.28) except for the replacement of $\widetilde{v}$ by $\widetilde{\psi}$ and the absence of the ω^2 term in the bracketed expression. Since meridional velocity is just $\partial\psi/\partial x \propto k\psi$ the meridional velocity obeys the same equation as $\widetilde{\psi}$, and the absence of ω^2 arises because we are in the low-frequency limit. We thus simply repeat the development following (8.28) and obtain a dispersion relation similar to (8.37b) but without the ω^2 term, to wit

$$\omega = \frac{-\beta k}{(2m+1)\beta/c + k^2}. \tag{8.71}$$

This is the same as the dispersion relation for low frequency waves discussed in Section 8.2.2. The balanced system (8.68) thus *exactly* reproduces the Rossby waves and Rossby-gravity waves in the low frequency limit. We are not able to recover the behaviour of Kelvin waves by this methodology because such waves are essentially non-balanced: in the meridional direction the Coriolis force balances the height field, as in (8.65), but in the zonal direction there is a balance between the zonal acceleration and the pressure gradient.

8.3 RAY TRACING AND EQUATORIAL TRAPPING

We have seen that equatorial waves are trapped near the equator. What then happens to a wave that initially propagates in a direction away from the equator? The waves must either change their character completely, or be refracted back toward the equator. The former can only happen if there exists a class of mid-latitude waves with similar frequency and wavenumber; otherwise no such waves can be excited and the waves must, if they are not absorbed, bend back if energy is to be conserved. Let us explore this using some ideas from ray theory, as discussed in Section 6.3.

8.3.1 Dispersion Relation and Ray Equations

Consider again the wave equation of motion for the meridional velocity, (8.24). We seek solutions of the form $v = \widetilde{v}(y)\mathrm{e}^{\mathrm{i}(kx-\omega t)}$, giving

$$\frac{\mathrm{d}^2\widetilde{v}}{\mathrm{d}y^2} + \left(\frac{\omega^2}{c^2} - k^2 - \frac{\beta k}{\omega} - \frac{\beta^2 y^2}{c^2}\right)\widetilde{v} = 0. \tag{8.72}$$

If the term in brackets is positive then sinusoidal-like solutions in y are possible, but if the term is negative, which will occur for y larger than some critical value y_c, then the physically realizable solutions decay exponentially with y; that is, wavelike solutions are trapped between two critical latitudes. Using the dispersion relation (8.37), Equation (8.72) becomes

$$\frac{\mathrm{d}^2\widetilde{v}}{\mathrm{d}y^2} + \left(\frac{(2m+1)\beta}{c} - \frac{\beta^2 y^2}{c^2}\right)\widetilde{v} = 0, \tag{8.73}$$

and therefore the critical latitudes are given by

$$y_c = \pm\left(\frac{\omega^2}{\beta^2} - \frac{c^2k^2}{\beta^2} - \frac{c^2k}{\beta\omega}\right)^{1/2} = \left((2m+1)\frac{c}{\beta}\right)^{1/2}, \tag{8.74}$$

For $k = 0$, and so for meridionally propagating waves, the critical latitudes are given by $y_c = \omega/\beta$, and at the critical latitude $\omega = f$. The waves are therefore trapped within their *inertial latitudes*, the latitudes at which their frequency is f. For larger k the critical latitudes are correspondingly smaller.

To explore this phenomenon using ray theory we assume that the medium is varying sufficiently slowly that it is possible to find wavelike solutions with spatially varying y-wavenumbers. We write (8.72) as

$$\frac{\mathrm{d}^2\widetilde{v}}{\mathrm{d}y^2} + l^2(y)\widetilde{v} = 0, \tag{8.75}$$

$$l^2(y) = \frac{\omega^2}{c^2} - k^2 - \frac{\beta k}{\omega} - \frac{\beta^2 y^2}{c^2} = \frac{\beta^2}{c^2}(y_c^2 - y^2)m \tag{8.76}$$

where $l(y)$ is assumed to vary slowly in the WKB sense (see Appendix A on page 247). The WKB solution is

$$\widetilde{v}(y) = l^{-1/2}\exp\left(\pm\mathrm{i}\int l\,\mathrm{d}y\right). \tag{8.77}$$

The trajectory of the waves is determined by the ray paths — paths that are parallel to the direction of the group velocity — so that their trajectory, $x(t)$, $y(t)$ is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c_g^x, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = c_g^y \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{c_g^y}{c_g^x}. \tag{8.78}$$

Using the dispersion relation (8.76) gives

$$\frac{\partial\omega}{\partial l} = \frac{2\omega^2 lc^2}{2\omega^3 + \beta kc^2}, \tag{8.79}$$

and using this and (8.56) gives the slope of the ray in the x–y plane,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{c_g^y}{c_g^x} = \frac{l}{k + \beta/(2\omega)}. \tag{8.80}$$

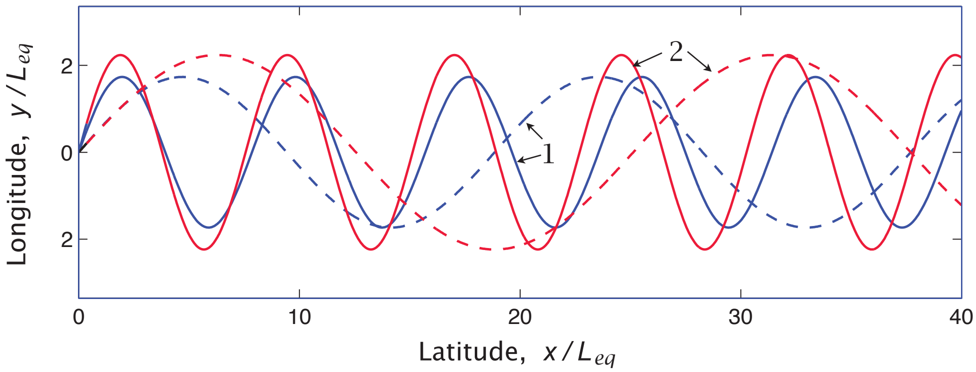

Fig. 8.10 Rays in the equatorial waveguide calculated using (8.83). The dashed lines show planetary wave trajectories and the solid lines are gravity wave trajectories, with $m = 1, 2$ (numbers marked on the graph) and $\widehat{k} = 1$. The turning latitude for each wave is $(2m+1)^{1/2}L_{eq}$, where $L_{eq} = \sqrt{c/\beta}$.

Using the expression for l given by (8.76) we can write this in terms of y instead of l, so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\beta(y_c^2 - y^2)^{1/2}}{kc + \beta c/(2\omega)}. \tag{8.81}$$

Using the standard result that

$$\int \frac{\mathrm{d}y}{(y_c^2 - y^2)^{1/2}} = \sin^{-1}\frac{y}{y_c}, \tag{8.82}$$

we finally obtain

$$y = y_c \sin\left[\frac{\beta x}{ck + \beta c/(2\omega)}\right], \qquad \widehat{y} = (2m+1)^{1/2}\sin\left[\frac{\widehat{x}}{\widehat{k} + 1/(2\widehat{\omega})}\right]. \tag{8.83}$$

where the second expression is the nondimensional form. The ray path is therefore a sinusoid moving along the equator; the waves are confined to a *waveguide* centred at the equator and with a polewards extent of $y = \pm y_c$, as in Fig. 8.10. Equation (8.83) holds for both planetary and gravity waves, and for the latter the term $\beta c/(2\omega)$ may be neglected.

8.4 ✦ FORCED-DISSIPATIVE WAVELIKE FLOW

We now consider linear equatorial dynamics in the presence of forcing and damping. (A somewhat simpler treatment of this issue is given in Section 22.7 and some readers may wish to delay considering the problem until then. The important special case of the forced, steady problem is treated in Section 8.5, and it is also possible to skip to that section and refer back here as needed.) If forcing is present then damping is needed so that a steady state can be reached, and the simplest form is a linear drag. From a physical perspective the presence of such a drag is the most unsatisfactory aspect of our treatment, for it has no real physical justification especially as, for mathematical reasons, the drag must be the same for momentum and height (implying a frictional spindown time equal to a radiative spindown time). Nevertheless, unresolved small scale processes often do act as some form of damping and a linear damping is the simplest form. We consider the full problem initially and then special cases.[9]

The dimensional linear forced-dissipative equations of motion are

$$\frac{\partial u}{\partial t} + \alpha u - fv + \frac{\partial \phi}{\partial x} = F^x, \qquad \frac{\partial v}{\partial t} + \alpha v + fu + \frac{\partial \phi}{\partial y} = F^y, \tag{8.84a,b}$$

 https://doi.org/10.1017/9781107588417.009

$$\frac{\partial\phi}{\partial t} + \alpha\phi + c^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -Q, \tag{8.84c}$$

where $F^x$and F^y are the x and y components of the imposed forces, Q is a thermal or mass source and α is a damping coefficient, assumed the same for all three variables. If we interpret $\boldsymbol{F} = (F^x, F^y)$ as wind stress, $\boldsymbol{\tau}$, acting on a layer of fluid we might make the association of $\boldsymbol{F} = \boldsymbol{\tau}/\rho_0 H$. Still, for now we will treat this system simply as a problem in geophysical fluid dynamics. The potential vorticity equation corresponding to (8.84), obtained by cross-differentiating (8.84a) and (8.84b), is

$$\left[\frac{\partial}{\partial t} + \alpha\right]\left(\zeta - \frac{f}{c^2}\phi\right) + \beta v = \mathrm{curl}_z \boldsymbol{F} + \frac{fQ}{c^2}. \tag{8.85}$$

In much the same way as we derived (8.24) we can derive a single partial differential equation for v, namely

$$\begin{aligned}\frac{1}{c^2}\left[\frac{\partial}{\partial t} + \alpha\right]^3 v + \frac{f^2}{c^2}\left[\frac{\partial}{\partial t} + \alpha\right] v - \left[\frac{\partial}{\partial t} + \alpha\right]\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2}\right) - \beta\frac{\partial v}{\partial x} \\ = \frac{1}{c^2}\left[\frac{\partial}{\partial t} + \alpha\right]\frac{\partial Q}{\partial y} - \frac{f}{c^2}\frac{\partial Q}{\partial x} \\ + \frac{1}{c^2}\left[\frac{\partial}{\partial t} + \alpha\right]^2 F^y - \frac{f}{c^2}\left[\frac{\partial}{\partial t} + \alpha\right] F^x - \frac{\partial}{\partial x}\left(\frac{\partial F^y}{\partial x} - \frac{\partial F^x}{\partial y}\right).\end{aligned} \tag{8.86}$$

The left-hand side is a minor variation of that of (8.24). This equation is obviously very complicated and perhaps not very attractive. Although the equation might be solved by similar methods to those used on (8.24) (or solved numerically) we will proceed in a slightly more informative way, with two differences:

(i) We consider only special cases of (8.84). For example, we simplify (8.84b) to geostrophic balance, $fu = -\partial\phi/\partial y$, and in Section 8.5 we will pay particular attention to the steady version of the equations.

(ii) We will change variables from (u, v, ϕ) to a set denoted (q, r, v), defined below, that allow an easier connection to be made between v and the variables u and ϕ.

8.4.1 Mathematical Development

For algebraic convenience we introduce the following linear combinations of u and ϕ,

$$q \equiv \frac{\phi}{c} + u, \qquad r \equiv \frac{\phi}{c} - u. \tag{8.87}$$

Note that u and ϕ have the same natural symmetry across the equator, with both symmetric unless forcing deems otherwise, whereas v tends to be antisymmetric. The u-momentum and the height (ϕ) equations may be written as equations for q and r, namely

$$\left(\frac{\partial}{\partial t} + \alpha\right) q + c\frac{\partial q}{\partial x} + c\frac{\partial v}{\partial y} - fv = F^x - Q/c, \tag{8.88a}$$

$$\left(\frac{\partial r}{\partial t} + \alpha\right) r - c\frac{\partial r}{\partial x} + c\frac{\partial v}{\partial y} + fv = -F^x - Q/c, \tag{8.88b}$$

and the v-momentum equation becomes

$$\left(\frac{\partial}{\partial t} + \alpha\right) v + \frac{f}{2}(q - r) = -\frac{c}{2}\frac{\partial}{\partial y}(q + r) + F^y. \tag{8.88c}$$

Nondimensionalization

We scale velocity by c and time and distance by

$$T_{eq} = (c2\beta)^{-1/2}, \qquad L_{eq} = (c/2\beta)^{1/2}. \tag{8.89a,b}$$

The nondimensional equations of motion are then

$$\left(\frac{\partial}{\partial t} + \alpha\right) q + \frac{\partial q}{\partial x} + \frac{\partial v}{\partial y} - \frac{1}{2} y v = F^x - Q, \tag{8.90a}$$

$$\left(\frac{\partial}{\partial t} + \alpha\right) r - \frac{\partial r}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2} y v = -F^x - Q, \tag{8.90b}$$

$$\left(\frac{\partial}{\partial t} + \alpha\right) v + \frac{y}{4}(q - r) = -\frac{1}{2}\frac{\partial}{\partial y}(q + r) + F^y. \tag{8.90c}$$

To avoid clutter here we do not use special notation for nondimensional variables.

The solutions of these equations may be expressed in terms of parabolic cylinder functions, $D_n(y)$. That is, we seek solutions of the form

$$(v, q, r) = \sum_{n=0}^{\infty} \left(v_n(x,t), q_n(x,t), r_n(x,t)\right) D_n(y). \tag{8.91}$$

with the forcing terms expanded in a similar fashion. The parabolic cylinder functions themselves have the form

$$(D_0, D_1, D_2, D_3) = (1, y, y^2 - 1, y^3 - 3y) \exp(-y^2/4), \tag{8.92}$$

and so on. The polynomial terms are just the modified Hermite polynomials $G_n(y)$ given by (8.42). The parabolic cylinder functions obey the ladder properties that

$$\frac{\mathrm{d}D_n}{\mathrm{d}y} + \frac{1}{2} y D_n = n D_{n-1}, \qquad \frac{\mathrm{d}D_n}{\mathrm{d}y} - \frac{1}{2} y D_n = -D_{n+1}. \tag{8.93a,b}$$

If we substitute (8.91) into (8.90) we obtain ordinary differential equations for the amplitudes. From the q equation, (8.90a), we obtain, after a little algebra,

$$\left(\frac{\partial}{\partial t} + \alpha\right) q_0 + \frac{\partial q_0}{\partial x} = F_0^x - Q_0, \tag{8.94a}$$

$$\left(\frac{\partial}{\partial t} + \alpha\right) q_{n+1} + \frac{\partial q_{n+1}}{\partial x} - v_n = F_{n+1}^x - Q_{n+1}, \qquad n = 0, 1, 2, 3, \ldots. \tag{8.94b}$$

From the r equation, (8.90b), we find

$$\left(\frac{\partial}{\partial t} + \alpha\right) r_{n-1} - \frac{\partial r_{n-1}}{\partial x} + n v_n = -\left(F_{n-1}^x + Q_{n-1}\right), \qquad n = 1, 2, 3, \ldots, \tag{8.95}$$

and from the v equation, (8.90c), we find

$$\left(\frac{\partial}{\partial t} + \alpha\right) v_0 + \frac{q_1}{2} = F_0^y, \tag{8.96a}$$

$$\left(\frac{\partial}{\partial t} + \alpha\right) v_n + \frac{(n+1)}{2} q_{n+1} - \frac{r_{n-1}}{2} = F_n^y, \qquad n = 1, 2, 3, \ldots. \tag{8.96b}$$

Finally, we note without derivation that these equations may be combined to give

$$\left[\frac{\partial}{\partial t} + \alpha\right]^3 v_n + \left[\frac{\partial}{\partial t} + \alpha\right]\left((2n+1) v_n - \frac{\partial^2 v_n}{\partial x^2}\right) - \frac{\partial v_n}{\partial x} = G, \tag{8.97}$$

where G is a combination of the various forcing terms. This equation can be most easily derived by substituting (8.91) into the nondimensional form of (8.86).

In principle, the above equations provide a means of solving the problem for almost any forcing. The equations have constant coefficients and may be solved by a superposition of harmonic functions in the x-direction, in conjunction with the variation in the y-direction given by the parabolic cylinder functions. In general this procedure would be somewhat tedious and uninformative. Thus, and to avoid being asphyxiated by an avalanche of algebra, we will consider some special cases. Enthusiasts may continue with the general development by themselves. (We might also note that modern geophysical fluid dynamics has advanced by way of using numerical methods to find solutions to complicated equations, in conjunction with using analytic methods to find solutions of simplified cases or to find general relations.) We will discuss two particularly important problems later, in Sections 8.5 and 22.7,and for the rest of this section we content ourselves with some general comments about forced waves.[10]

8.4.2 ✦ Forced Waves

Consider the problem of forced waves in which we retain some of the forcing terms but neglect the damping. Our purpose is to show what kinds of waves might be excited and to help interpret (8.94)–(8.97). When $\alpha = 0$ Equation (8.97) becomes, in dimensional form,

$$\frac{\partial}{\partial t}\left[\frac{1}{c^2}\frac{\partial^2 v_n}{\partial t^2} - \frac{\partial^2 v_n}{\partial x^2} + (2n+1)\frac{\beta}{c}v_n\right] - \beta\frac{\partial v_n}{\partial x} = G. \tag{8.98}$$

From (8.98) the dispersion relation for free waves, that is, (8.37), follows if we let $G = 0$ and seek harmonic solutions of the form $\exp(\mathrm{i}kx - \mathrm{i}\omega t)$.

Consider now a forcing, $\widetilde{F}$ say, that projects only onto the zeroth order parabolic function, D_0, which is a constant. Equation (8.94a) becomes, in dimensional form,

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)q_0 = \widetilde{G}. \tag{8.99}$$

The free solutions of this are Kelvin waves propagating eastwards at speeds $c = \omega/k$ for each k that might be excited; that is $q_0 = \mathrm{Re}\, C\exp[\mathrm{i}k(x - ct)]$, where C is a constant. Suppose that the forcing is harmonic in x and time,

$$\widetilde{F} = \mathrm{Re}\, A\left\{\exp[\mathrm{i}(k_1x - \omega_1 t)] + \exp[\mathrm{i}(k_1 x + \omega_1 t)]\right\} = A\left[\cos(k_1x - \omega_1 t) + \cos(k_1x + \omega_1 t)\right], \tag{8.100}$$

and A is real. The solution to (8.99) with this forcing is given by

$$q_0 = -\frac{A\sin(k_1x - \omega_1 t)}{\omega_1 - ck_1} + \frac{A\sin(k_1x + \omega_1 t)}{\omega_1 + ck_1}. \tag{8.101}$$

All the parameters in the above equation, c, k_1, ω_1, are positive. If the forcing is just one harmonic then, in general, $c \neq \omega_1/k_1$. However, if the forcing is a superposition of many harmonics then there may be one that is in resonance with the free mode, and this wave, an eastward propagating Kelvin wave represented by an expression like the first term on the right-hand side of (8.101), will be preferentially excited. Similar considerations apply to other waves too; that is, the forcing will excite waves that most resemble the forcing and can resonate with it. Sometimes, a forcing will resemble a delta function in both space and time: for example, a sudden and localized burst of wind over the ocean because of intense storm activity, will give rise to a forcing that contains nearly all space and time scales, since a Dirac delta function has equal representation of all Fourier modes. In this case, both eastward propagating Kelvin waves and westward propagating planetary waves will be excited, and to look at some of these it is useful to make a longwave approximation, as we now discuss.

Planetary waves, revisited

In the planetary wave, or longwave, approximation the highest time derivative in (8.98) is omitted, leaving

$$\frac{\partial}{\partial t}\left[\frac{\partial^2 v_n}{\partial x^2} - (2n+1)\frac{\beta}{c}v_n\right] + \beta\frac{\partial v_n}{\partial x} = G. \tag{8.102}$$

If $G = 0$ this equation gives the dispersion relation

$$\omega = \frac{-\beta k}{(2n+1)\beta/c + k^2}, \tag{8.103}$$

as in (8.47). Planetary waves will be excited when the forcing itself has a low frequency.

The longwave approximation, revisited

Many situations in low latitudes are characterized by having a longer zonal scale than meridional scale; thus, $|\partial\phi/\partial y| \gg |\partial\phi/\partial x|$. When this is the case, geostrophic balance will hold to a good approximation for the zonal flow even in the presence of forcing and dissipation, but not for the meridional flow, and to a good approximation the meridional momentum equation (8.84b) may be replaced by

$$fu = -\frac{\partial\phi}{\partial y}. \tag{8.104}$$

In this limit (8.98) simplifies to

$$\frac{\partial}{\partial t}\left[(2n+1)\frac{\beta}{c}v_n\right] - \beta\frac{\partial v_n}{\partial x} = G, \tag{8.105}$$

from which, with $G = 0$, the dispersion relation,

$$\omega = \frac{-kc}{2n+1}, \tag{8.106}$$

immediately follows. This equation is the small k limit of (8.103) and the wave is non-dispersive. When $n = -1$ we have eastward propagating Kelvin waves and when $n \geq 0$ we have westward propagating long Rossby waves.

The amplitude equations, (8.94)–(8.96) then simplify as follows, also taking $\alpha = 0$. The q equations become

$$\frac{\partial q_0}{\partial t} + \frac{\partial q_0}{\partial x} = F_0^x - Q_0, \tag{8.107a}$$

$$\frac{\partial q_{n+1}}{\partial t} + \frac{\partial q_{n+1}}{\partial x} - v_n = F_{n+1}^x - Q_{n+1}, \qquad n = 0, 1, 2, 3, \ldots. \tag{8.107b}$$

The r equation becomes

$$\frac{\partial r_{n-1}}{\partial t} - \frac{\partial r_{n-1}}{\partial x} + nv_n = -\left(F_{n-1}^x + Q_{n-1}\right), \qquad n = 1, 2, 3, \ldots, \tag{8.108}$$

and from the v equation (geostrophic balance) we find

$$q_1 = 0, \tag{8.109a}$$

$$(n+1)q_{n+1} = r_{n-1}, \qquad n = 1, 2, 3, \ldots \tag{8.109b}$$

If we use (8.109b) to eliminate r_{n-1} in (8.108), and then use (8.107b) to eliminate v_n we obtain

$$(2n+1)\frac{\partial q_{n+1}}{\partial t} - \frac{\partial q_{n+1}}{\partial x} = n\,(F^x_{n+1} - Q_{n+1}) - (F^x_{n-1} + Q_{n-1})\,. \tag{8.110}$$

The above set of equations provide, in principle, a means for studying the response of the system to an imposed forcing, such as winds blowing over the ocean or a diabatic source in the atmosphere. Having neglected dissipation, wavelike solutions of constant amplitude will be found only if the forcing is oscillatory rather than steady. Solutions are found by solving the first-order wave equations (8.107a) and (8.110) for q_n, and then using (8.109b) to obtain r_n. A simple expression for v_n results if we add (8.107b) and (8.110).

Waves and adjustment

The wave described by (8.107a) is a Kelvin wave, moving eastwards with nondimensional speed unity, or dimensional speed c. The speed also follows from the dispersion relation, $\omega = -kc/(2n+1)$, with $n = -1$. In contrast, the waves described by (8.110) are westwards propagating, long, low frequency planetary waves. In dimensional form (8.110) becomes

$$(2n+1)\frac{\partial q_{n+1}}{\partial t} - c\frac{\partial q_{n+1}}{\partial x} = n\,(F^x_{n+1} - Q_{n+1}) - (F^x_{n-1} + Q_{n-1})\,, \tag{8.111}$$

and hence the waves have a speed $-c/(2n+1)$, just as in (8.50). There are no short low frequency waves in this approximation.

As we noted above, an arbitrary forcing will in general excite both gravity waves and planetary waves and the initial flow will be out of geostrophic balance. In the mid-latitude case (Section 3.9) the gravity waves radiate to infinity (in the idealized problem) leaving behind an adjusted flow in geostrophic balance, determined by potential vorticity conservation. The process of adjustment is less efficient at low latitudes, because the waves are trapped between their inertial latitudes (Section 8.3) and in the absence of dissipation the fluid will oscillate endlessly. In the zonal direction both planetary and Kelvin waves propagate. A gravity wave front moves away more quickly, with the eventual adjustment occurring by way of planetary waves.

Let us now turn our attention to a rather concrete problem, that of the steady response to a localised thermal forcing.

8.5 FORCED, STEADY FLOW: THE MATSUNO–GILL PROBLEM

We now consider the forced, steady version of the equatorial wave problem; that is to say, we seek steady solutions of (8.84), but with a mechanical or thermal forcing on the right-hand side.[11] Because of its importance to the tropical circulation of the atmosphere this problem has become somewhat iconic and some readers may be tempted to begin reading this chapter here. However, the problem is really just the forced, steady version of the wave problems studied in Sections 8.2 and 8.4, and the reader should have at least a passing familiarity with that material before proceeding. Those readers who have followed the previous sections closely will find the material that follows, namely the *Matsuno–Gill* problem, a pleasant stroll in the park.

8.5.1 Mathematical Development

We begin with (8.84) and make two additional simplifications. First, that the flow is steady and second that the zonal wind is in geostrophic balance with the meridional pressure gradient. This 'semi-geostrophic' approximation is similar to the longwave approximation discussed in previous sections, and requires that αv is smaller than fu. The equations of motion become

$$\alpha u - fv + \frac{\partial \phi}{\partial x} = F^x, \qquad fu + \frac{\partial \phi}{\partial y} = 0, \qquad \alpha\phi + c^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -Q. \tag{8.112a,b,c}$$

From these equations we may derive a single equation for v, namely

$$\frac{f^2}{c^2}\alpha v - \alpha\frac{\partial^2 v}{\partial y^2} - \beta\frac{\partial v}{\partial x} = \frac{\alpha}{c^2}\frac{\partial Q}{\partial y} - \frac{f}{c^2}\frac{\partial Q}{\partial x} - \frac{f}{c^2}\alpha F^x + \frac{\partial^2 F^x}{\partial x\,\partial y}. \tag{8.113}$$

This is just a simplification of (8.86) appropriate for a steady system with the zonal wind in geostrophic balance, obtained by omitting all the time derivatives, the term involving α^3, and the F^yterm on the right-hand side. We nondimensionalize all the variables using the time and length scales

$$T_{eq} = (2c\beta)^{-1/2}, \qquad L_{eq} = \left(\frac{c}{2\beta}\right)^{1/2}, \tag{8.114a,b}$$

so that the various dimensional and nondimensional (hatted) variables are related by

$$\begin{gathered}(u, v) = c\,(\widehat{u}, \widehat{v}), \qquad \phi = c^2\widehat{\phi}, \qquad (x, y) = L_{eq}(\widehat{x}, \widehat{y}), \\ \alpha = \frac{\widehat{\alpha}}{T_{eq}}, \qquad Q = \frac{c^3}{L_{eq}}\widehat{Q}, \qquad F^x = \frac{c^2}{L_{eq}}\widehat{F}^x, \qquad \widehat{f} = \frac{1}{2}\widehat{y}, \qquad \widehat{\beta} = \frac{1}{2}\,.\end{gathered} \tag{8.115}$$

Unless specifically noted the variables from now on are nondimensional and we drop the hats. The equations of motion, (8.116), become

$$\alpha u - \frac{y}{2}v + \frac{\partial\phi}{\partial x} = F^x, \qquad \frac{y}{2}u + \frac{\partial\phi}{\partial y} = 0, \qquad \alpha\phi + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -Q. \tag{8.116a,b,c}$$

and the v equation becomes

$$\frac{y^2}{4}\alpha v - \alpha\frac{\partial^2 v}{\partial y^2} - \frac{1}{2}\frac{\partial v}{\partial x} = \alpha\frac{\partial Q}{\partial y} - \frac{y}{2}\frac{\partial Q}{\partial x} - \frac{\alpha y}{2}\alpha F^x + \frac{\partial^2 F^x}{\partial x\partial y}. \tag{8.117}$$

As before when dealing with wave-like problems it is convenient to change variables to p and q where

$$q = \phi + u, \qquad r = \phi - u. \tag{8.118a,b}$$

The equations of motion, (8.116), become

$$\begin{gathered}\alpha q + \frac{\partial q}{\partial x} + \frac{\partial v}{\partial y} - \frac{1}{2}yv = F^x - Q, \qquad \alpha r - \frac{\partial r}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2}yv = -F^x - Q, \\ \frac{y}{4}(q - r) + \frac{1}{2}\frac{\partial}{\partial y}(q + r) = 0.\end{gathered} \tag{8.119a,b,c}$$

These are special cases of (8.90), the first two equations being combinations of the u-momentum and pressure equations and the last one being the v-momentum equation (zonal geostrophic balance).

As in the general treatment given earlier we expand the variables and the forcing in terms of parabolic cylinder functions. Thus, for example,

$$Q(x) = \sum_{n=0}^{\infty} Q_n(x)D_n(y), \tag{8.120}$$

and similarly for the other variables. The resulting ordinary differential equations are special cases of (8.94)–(8.97), specifically

$$\alpha q_0 + \frac{\partial q_0}{\partial x} = F_0^x - Q_0, \tag{8.121a}$$

$$\alpha q_{n+1} + \frac{\partial q_{n+1}}{\partial x} - v_n = F^x_{n+1} - Q_{n+1}, \qquad n = 0, 1, 2, 3, \ldots \tag{8.121b}$$

$$\alpha r_{n-1} - \frac{\partial r_{n-1}}{\partial x} + n v_n = -\left(F^x_{n-1} + Q_{n-1}\right), \qquad n = 1, 2, 3, \ldots, \tag{8.122}$$

$$q_1 = 0, \tag{8.123a}$$

$$(n+1) q_{n+1} - r_{n-1} = 0, \qquad n = 1, 2, 3, \ldots, \tag{8.123b}$$

Using (8.121b), (8.122) and (8.123b) we obtain

$$\alpha (2n+1) q_{n+1} - \frac{\partial q_{n+1}}{\partial x} = n\left(F^x_{n+1} - Q_{n+1}\right) - \left(F^x_{n-1} + Q_{n-1}\right) \qquad n = 1, 2, 3, \ldots. \tag{8.124}$$

Finally, although we shall not use it, the v equation (8.113) becomes

$$\alpha\left((2n+1) v_n - \frac{\partial^2 v_n}{\partial x^2}\right) - \frac{\partial v_n}{\partial x} = G, \tag{8.125}$$

where G represents the various forcing terms.

As in the wavelike case, the above equations provide, at least in principle, a means of solving for the response for any particular forcing. The procedure is to project the forcing onto parabolic cylinder functions, and then solve the amplitude equations (8.121)–(8.123) for the zonal dependence, and then finally to reconstruct the solutions using the $q_n(x)$, $r_n(x)$ and $v_n(x)$ and the parabolic cylinder functions. Naturally enough, this is easier said than done and we will go through the procedure in detail for just one important case.

8.5.2 Symmetric Heating

A canonical case is that in which the system is forced by a heating that is confined in both the x- and y-directions, and is symmetric across the equator. Confinement in the y-direction may be achieved by supposing that the heating projects solely onto the first parabolic function, so that

$$Q(x) = Q_0(x) D_0(y) = G(x) \exp(-y^2/4), \tag{8.126}$$

and confinement in the x-direction may be achieved by supposing that the heating is of the form

$$G(x) = \begin{cases} A \cos kx & |x| < L \\ 0 & |x| > L, \end{cases} \tag{8.127}$$

where $k = \pi/2L$. This may seem an odd form to choose, but the harmonic variation for $|x| < L$ enables an analytic solution to be found in that region, and the absence of any forcing at all in the far field enables solutions to be found there in the form of decaying wavelike disturbances. Although this problem is clearly a special case the qualitative form of the solution transcends its precise details.

Kelvin wave contribution

We noted in Section 8.4.2 that the equation for q_0 represents an eastwards propagating Kelvin wave, and this holds in the damped case also. That is to say, there will be a non-zero solution of (8.121) only in the forced region and eastward of it, where it will be progressively damped. Using this insight we can easily derive the solution in all three regions. First, for $X < -L$, we have

$$q_0 = 0, \qquad x < -L. \tag{8.128a}$$

In the forcing region we have to solve (8.121) with a boundary condition of $q_0 = 0$ at $x = -L$. The solution is

$$q_0 = \frac{-A}{\alpha^2 + k^2}\left\{\alpha \cos kx + k\left[\sin kx + e^{-\alpha(x+L)}\right]\right\} \qquad |x| < L. \tag{8.128b}$$

For $x > L$ we solve (8.121), but with a right-hand side equal to zero, with a boundary condition at $x = L$ given by (8.128b), namely $q_0 = -Ak(\alpha^2 + k^2)^{-1}[1 + \exp(-2\alpha L)]$. The solution is

$$q_0 = \frac{-Ak}{\alpha^2 + k^2}\left(1 + e^{-2\alpha L}\right) e^{\alpha(L-x)}, \qquad x > L. \tag{8.128c}$$

Because the motion is a decaying Kelvin wave $v = 0$ and the nondimensional u and ϕ fields are equal to each other, with $r = 0$. Thus, from (8.118) and (8.128),

$$u = \phi = \frac{1}{2} q_0(x) \exp(-y^2/4), \qquad v = 0. \tag{8.129}$$

This does not mean r_0 is zero; rather, it is associated with the planetary wave solution discussed below. The vertical velocity may be reconstructed from

$$w = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = \alpha\phi + Q, \tag{8.130}$$

whence

$$w = \frac{1}{2}\left[\alpha q_0(x) + Q_0(x)\right] \exp(-y^2/4). \tag{8.131}$$

We now complete the solution by finding a planetary wave contribution.

Planetary wave contribution

We now find the solution associated with q_2 and r_0. From (8.124) we have

$$\frac{dq_2}{dx} - 3\alpha q_2 = Q_0. \tag{8.132a}$$

From (8.123b) and (8.121b) we have

$$r_0 = 2q_2, \qquad v_1 = \alpha q_2 + \frac{dq_2}{dx}. \tag{8.132b,c}$$

These are planetary waves propagating westwards at a dimensional speed of $c/(2n+1) = c/3$. Thus, no signal is transmitted eastwards and we can find a solution to the above equations in an analogous fashion to the way we found a solution for the Kelvin wave problem. After just a little algebra, the solution is found to be:

$$q_2 = 0, \qquad x > L, \tag{8.133a}$$

$$q_2 = \frac{A}{(3\alpha)^2 + k^2}\left[-3\alpha \cos kx + k\left(\sin kx - e^{3\alpha(x-L)}\right)\right], \qquad |x| < L, \tag{8.133b}$$

$$q_2 = \frac{-Ak}{(3\alpha)^2 + k^2}\left[1 + e^{-6\alpha L}\right] e^{3\alpha(x+L)}, \qquad x < -L. \tag{8.133c}$$

The corresponding solutions for the pressure and velocity fields are

$$u = \frac{e^{-y^2/4}}{2} q_2(x)(y^2 - 3), \qquad v = y e^{-y^2/4}\left[Q_0(x) + 4\alpha q_2(x)\right], \tag{8.134a,b}$$

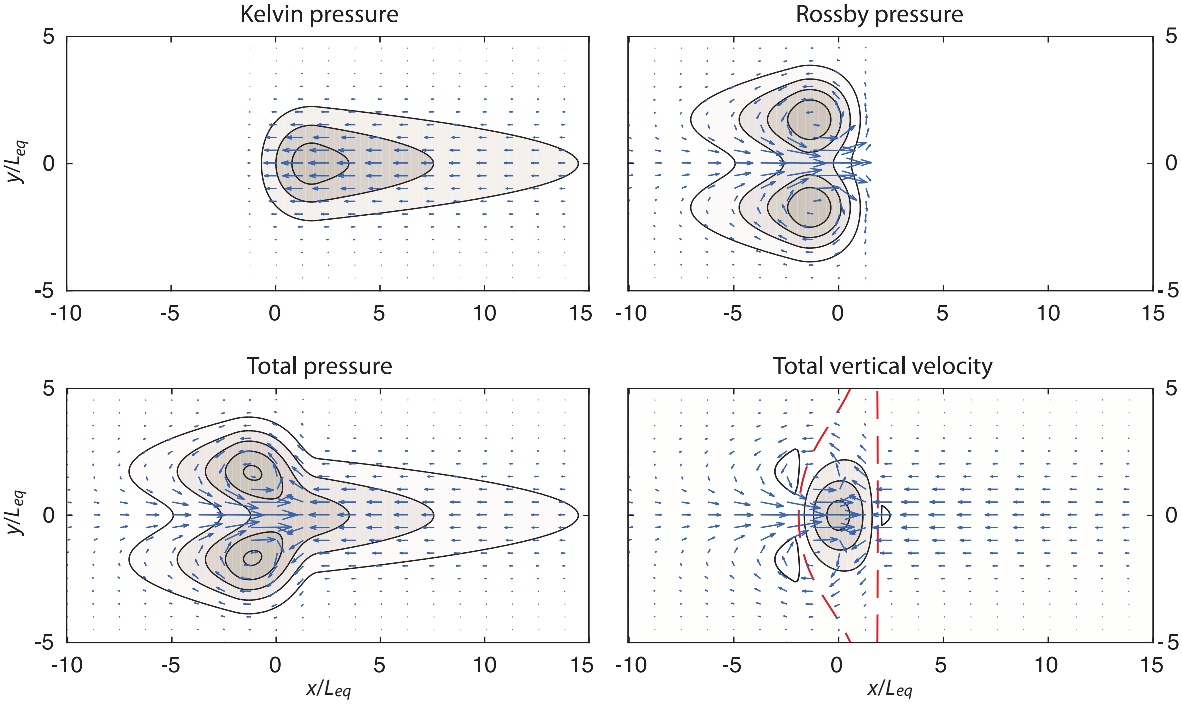

Fig. 8.11 Nondimensional solutions of the Matsuno–Gill model, with heating close to the origin and given by (8.127) with $L = 2$ and $\alpha = 0.1$. The shaded contours show the fields as indicated, and the arrows show the associated horizontal velocities. The 'Kelvin' and 'Rossby' designations indicate that just the Kelvin wave or Rossby (planetary) wave contributions are plotted as given by (8.129)–(8.131) and (8.134)–(8.135), respectively. For the pressure fields the contour interval is 0.3 and all fields are negative (so dark shading is low pressure) with the zero contour omitted. For vertical velocity the contour interval is 0.3 beginning at -0.1, and so is -0.1, 0.2, 0.5..., with an additional zero contour (red dashed) with upward motion within it.

$$\phi = \frac{e^{-y^2/4}}{2} q_2(x)(1 + y^2), \qquad w = \frac{e^{-y^2/4}}{2} \left[Q_0(x) + \alpha q_2(x)(1 + y^2) \right]. \tag{8.135a,b}$$

The solutions appear complicated (they are complicated!), although still amenable to interpretation. But first we combine the Kelvin and planetary wave contributions and restore the dimensions, to give

$$u = \frac{c}{2} \left[q_0(x) + q_2(x)(2\beta y^2/c - 3) \right] e^{-\beta y^2/2c}, \tag{8.136a}$$

$$v = cy \Big[Q_0(x) + (4\alpha/c) q_2(x) \Big] e^{-\beta y^2/2c}, \tag{8.136b}$$

$$\phi = \frac{c^2}{2} \left[q_0(x) + q_2(x)(2\beta y^2/c + 1) \right] e^{-\beta y^2/2c}, \tag{8.136c}$$

$$w = w_0 \frac{e^{-\beta y^2/2c}}{2} \left[2Q_0(x) + \alpha q_0(x) + \alpha q_2(x)(1 + 2\beta y^2/c) \right], \tag{8.136d}$$

where $w_0 = (2c\beta)^{1/2} H$, and q_0, q_2 and Q_0 are still nondimensional functions. The nondimensional forms are recovered by setting $w_0 = c = 1$ and $\beta = 1/2$. The solutions above are specific to the form of the forcing function we chose. However, a similar methodology could in principle be applied to forcing of any form, including forcing in the momentum equations, and, because the equations are linear, the solutions could be superposed. The solution above represents the physically important case of a localized heating, and the gross structure of the far field is largely independent of the details of the forcing: there is a rapidly decaying disturbance west and polewards of the forcing and a more slowly decaying disturbance east of the forcing close to the equator (Fig. 8.11).

Interpretation

Let's now try to figure out what's going on. A solution is illustrated in Fig. 8.11. The heating is confined to a region from $-2 < x < 2$ and exponentially falls away from the equator with an e-folding distance of 2, more-or-less corresponding to the shaded region of vertical velocity in the lower right panel, as intuitively expected and discussed more below.

Consider first the flow in the forcing region. Here the vertical velocity is positive, with the associated horizontal convergence being that of the zonal flow: the meridional flow is polewards, *away* from the maximum of the heating. To understand this, consider the limit $\alpha \to 0$. From (8.136) the vertical velocity field coincides with the heating and the (nondimensional) meridional velocity is given by

$$v = yQ_0 \exp(-y^2/4) = yw. \tag{8.137}$$

Thus, vertical motion is associated with poleward motion. To understand this, consider the inviscid vorticity equation

$$\beta v + f\nabla \cdot \boldsymbol{u} = 0, \qquad \text{or} \qquad \beta v = fw, \tag{8.138a,b}$$

which in nondimensional form is

$$v + y\nabla \cdot \boldsymbol{u} = 0, \qquad \text{or} \qquad v = yw. \tag{8.139a,b}$$

Evidently, (8.137) and (8.139b) are equivalent. Another way to think about this is to note that the rising motion in the region of the forcing causes vortex stretching, as discussed in Chapter 4, and hence the generation of cyclonic vertical vorticity and a polewards migration. From the perspective of potential vorticity, then to the extent that the flow is adiabatic the quantity $(f + \zeta)/h$ is conserved following the flow. The heating increases the value of h (the stretching), so that $f + \zeta$ also tends to increase in magnitude. The flow finds it easier to migrate polewards to increase its value of f than to increase its relative vorticity alone, for the latter would require more energy. If we interpret these equations as the lower layer of a two-layer system, then the flow in the lower layer is away from the source, and toward the source in the upper layer.

Consider now the flow to the west of the heating, associated with $q_2(x)$. The disturbance here is produced by a decaying westwards propagating Rossby wave — a form of 'Rossby plume' that we will also encounter in Chapter 19 (see Fig. 19.14 on page 754 and the associated discussion). The vertical velocity is negative, and the horizontal velocity is almost geostrophically balanced: the pressure perturbation is negative everywhere, and so circulating cyclonically around the centres of low pressure just to the west of heating. The flow converges to the equator, producing an eastward flow along the equator, converging in the heating zone. We may be tempted to interpret this in terms of the inviscid vorticity equation, as we did in the forcing region. This would suggest that, away from the forcing region, because the flow is divergent ($\nabla \cdot \boldsymbol{u} > 0,\ w < 0$) then from (8.138) the meridional velocity should be toward the equator in both hemispheres. However, this explanation is at best qualitative, because the vorticity equation above is not exactly satisfied by the solution (8.138), because non-zero solutions away from the forcing region depend entirely on the presence of dissipation.

The flow east of the forcing region motion is induced by an eastward propagating Kelvin wave, or more precisely the steady, eastward-decaying analogue of such a wave. Evidently, from Fig. 8.11, the pressure field extends further east of the source than west of the source, and this is because

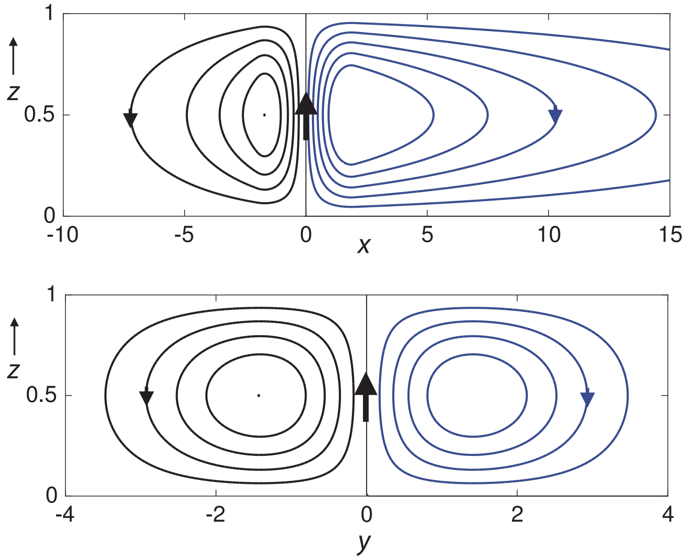

Fig. 8.12 The zonal overturning streamfunction of the meridionally averaged flow (top), and the meridional overturning streamfunction of the zonally-averaged flow (bottom) in the Matsuno–Gill problem with symmetric heating with a maximum at $y = 0$ and $x = 0$, the same as in Fig. 8.11.

There is rising motion around the location of the heating, sinking elsewhere. The contour interval in the top plot is about four times that of the bottom plot; that is, the Walker circulation here is stronger than the Hadley circulation.

Kelvin waves decay more slowly than Rossby waves. Keeping both the time derivative and the damping, the unforced Kelvin wave satisfies, from (8.94),

$$\left[\alpha + \frac{\partial}{\partial t}\right] q_0 + \frac{\partial q_0}{\partial x} = 0, \tag{8.140}$$

whereas the unforced Rossby wave satisfies, from (8.110) and (8.124) for $n = 1$,

$$3\left[\alpha + \frac{\partial}{\partial t}\right] q_1 - \frac{\partial q_1}{\partial x} = 0. \tag{8.141}$$

Thus, the effective damping rate of the Rossby wave is three times that of the Kelvin wave. Put another way, the Kelvin wave travels three times as fast as the Rossby wave so that if the damping rate, α, is the same the influence of the Kelvin wave spreads three times further east. The horizontal velocity in the Kelvin wave is purely zonal, and near the surface it is directed toward the heating source.

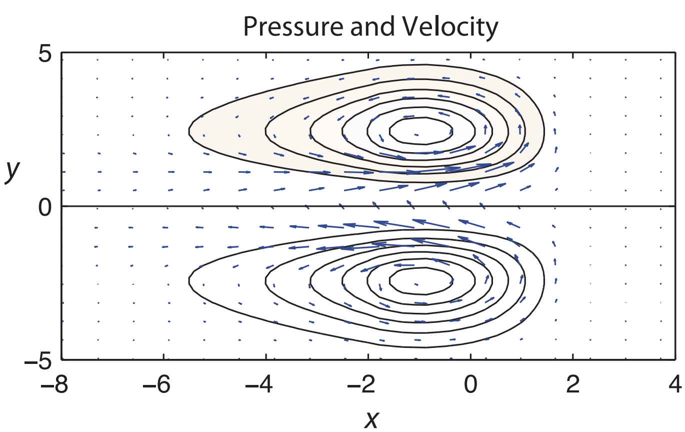

Fig. 8.13 Pressure (contours) and horizontal velocity (arrows) in the Matsuno–Gill problem with an antisymmetric heating given by (8.143) and nondimensional decay factor $\alpha = 0.1$. The heating is in the Northern Hemisphere generating a low pressure region (shaded) with inflow and ascent, and cooling is in the Southern Hemisphere. The contour interval is 0.3 and the zero contour is along $y = 0$.

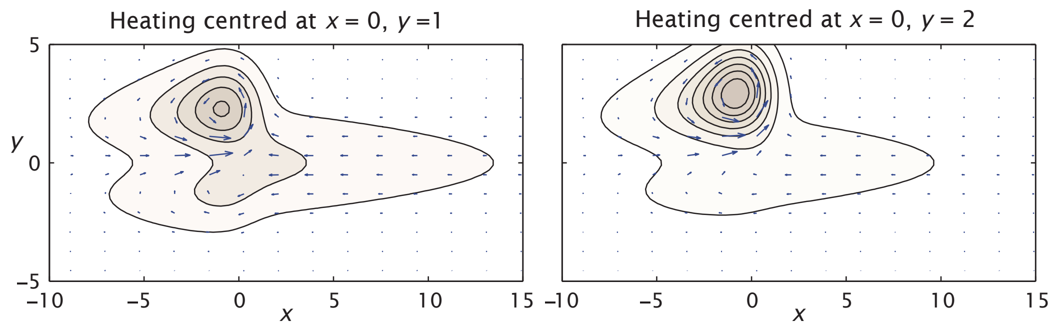

Fig. 8.14 Pressure (contours) and horizontal velocity (arrows) in the Matsuno–Gill model with the heating centred off the equator, as labelled, but otherwise similar to that of Fig. 8.11. As the heating moves to higher latitudes the Kelvin wave response weakens but the magnitude of the local response increases (the contour interval is the same in both panels).

Vertical structure

The zonal structure of the solution is a coarse representation of the Walker circulation in the equatorial Pacific. Here, the sea-surface temperature is high in the west, near Indonesia, and low in the east, near South America, because of the upwelling that brings deep, cold water to the surface. This distribution of sea-surface temperature effectively provides a heating in the western Pacific and induces westward winds along the equator, enhancing the westward trade winds that already exist as part of the general circulation. The overturning circulation in the zonal and meridional planes is illustrated in Fig. 8.12. This solution is obtained by supposing that the fields represent the first vertical mode, as discussed in Section 3.4. If the stratification is uniform then the modes are just sines and cosines and so we have

$$(u, v, \phi) = (\widetilde{u}, \widetilde{v}, \widetilde{\phi}) \cos(\pi z/D), \qquad w = \widetilde{w} \sin(\pi z/D). \tag{8.142}$$

Now the modal form of the mass continuity equation, (3.66), is $\widetilde{w} = -(c^2/g)\nabla \cdot \widetilde{\boldsymbol{u}}$. If this is to be consistent with the usual form of $\partial w/\partial z = -\nabla\cdot \boldsymbol{u}$ then we make the association $\pi/D = g/c^2 = 1/H_1^*$, where H_1^* is the equivalent depth of the first mode. Given this vertical structure, we integrate the solutions meridionally so enabling a streamfunction to be defined (because $v = 0$ as $y \to \pm\infty$, so $\partial \overline{u}^y/\partial x + \partial \overline{w}^y/\partial z = 0$, with the overbar denoting meridional integration). The expressions for the meridional integrals are given in Appendix B to this chapter, with the streamfunctions given by (8.175) and (8.175).

8.5.3 Antisymmetric Forcing

An analytic solution with asymmetric forcing may be obtained by using a forcing of the form

$$Q(x, y) = Q_1(x) D_1(y) = y \cos kx \exp(-y^2/4), \tag{8.143}$$

using the same form of zonal localization as before. The algebra needed to obtain a solution is somewhat tedious but straightforward, of a very similar nature to that described above. One finds that there are, again, two parts to the response. One part corresponds to a long planetary wave with $n = 0$ and using (8.121)–(8.123) we find

$$q_1 = 0, \qquad v_0 = Q_1. \tag{8.144}$$

There is no response outside the forcing region because long mixed waves have zero propagation velocity. The other part of the solution is obtained, again using (8.121)–(8.123), from

$$v_2 = \frac{\mathrm{d}q_3}{\mathrm{d}x} + \alpha q_3, \qquad r_1 = 3q_3, \qquad \frac{\mathrm{d}q_3}{\mathrm{d}x} - 5\alpha q_3 = Q_1. \tag{8.145a,b,c}$$

The solution of these equations is left as an exercise for the reader and is illustrated in Fig. 8.13. The solutions are zero east of the forcing region because there is no long wave so propagating. West of the forcing region there is eastward inflow into the heating region in the Nortern Hemisphere (which is being heated), as well as a tendency for poleward flow for the reasons described earlier. Thus, there is a cyclone with upward motion somewhat west of the main heating region, and a corresponding anti-cyclone in the cooled region, as illustrated in Fig. 8.13. The zonally averaged solutions (not shown) resemble an asymmetric Hadley Cell, with the air rising in the Northern (summer) Hemisphere, moving southwards aloft into the winter hemisphere before sinking.

8.5.4 Other Forcings

The solution to more general forcings can be constructed by using other forcing coefficients, or a superposition of forcing coefficients, and many solutions of interest to the tropical atmosphere and ocean may be so constructed. Solutions may also be constructed (sometimes more easily) numerically, either by time-stepping the linear shallow water equations to equilibrium or by solving the elliptic equation (8.113) using standard techniques.[12] We will present the solutions to two such cases: (i) a heating source centred off the equator in the Northern Hemisphere; and (ii) a line source of heating, either centred at the equator or just north of it, mimicking the Inter-Tropical Convergence Zone (ITCZ).

Heating off the equator

Solutions for heating off the equator may be constructed by adding the solutions for antisymmetric and symmetric heating presented above. In Fig. 8.14, we present a solution that has heating of a very similar form to that of the symmetric heating shown in Fig. 8.11, but centred off the equator at $y = 1$ and $y = 2$. The pattern is dominated by a low pressure region just to the west of the heating, with convergence and upward motion within it, and an eastward inflow between the equator and the centre of the heating. In the solution with the heating centred at $y = 1$ there is also a response east of the heating region, largest at the equator, produced by the eastward propagating, damped Kelvin wave. As the heating moves further from the equator (in the right panel of Fig. 8.14), the pressure response becomes stronger but the flow around the heating is in near geostrophic balance.

A line of heating

Finally, let us consider the solutions when the heating is independent of x, and the solutions themselves are then independent of x. Two such solutions are presented in Fig. 8.15 and in Fig. 8.16, for a line of heating at the equator and at $y = 1$. As we noted above, these solutions might be thought of as rather idealized versions of the ITCZ (although in the real ITCZ the location of the convective region is determined as part of the solution for the overall flow).

Consider first the solution with heating at the equator. A low pressure region develops over the heating and the flow converges there, producing equatorward and westward 'trade winds' and consequent upward motion at the equator, with the zonal velocity rapidly decreasing actually at the equator. Now consider what happens when the heating is off-equator, noting that the real ITCZ is generally situated a little north of the equator, especially in the Pacific Ocean. A low pressure region is formed along the line of the heating and the meridional velocity converges sharply there, with more inflow coming from the equatorial side of the line of heating (as can be seen in the right-hand panels of both Fig. 8.15 and Fig. 8.16). As regards the zonal velocity, there is an *eastward* jet along the line of the heating, with westward flow to either side. That is to say, there is a splitting of the westward trades caused by the line of sharp heating.

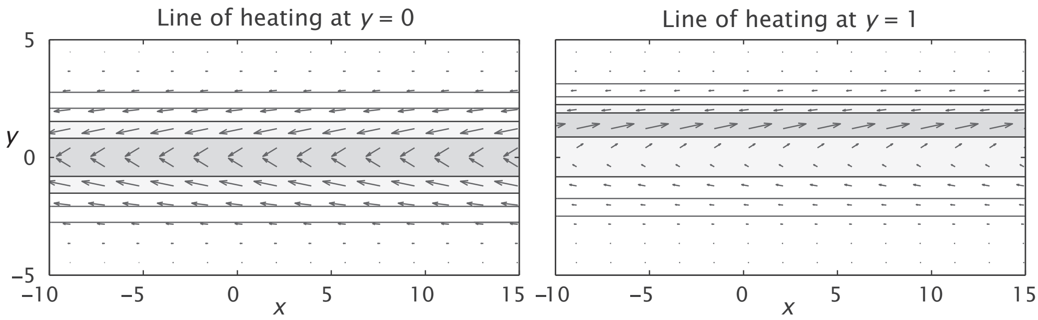

Fig. 8.15 As for Fig. 8.14 but with a line of heating at the equator (left panel) and at $y = 1$ (right panel). The heating generates a region of low pressure (shaded) where the flow converges. In the right panel the meridional velocity is larger on the equatorward side of the line than on the poleward side. See also Fig. 8.16.

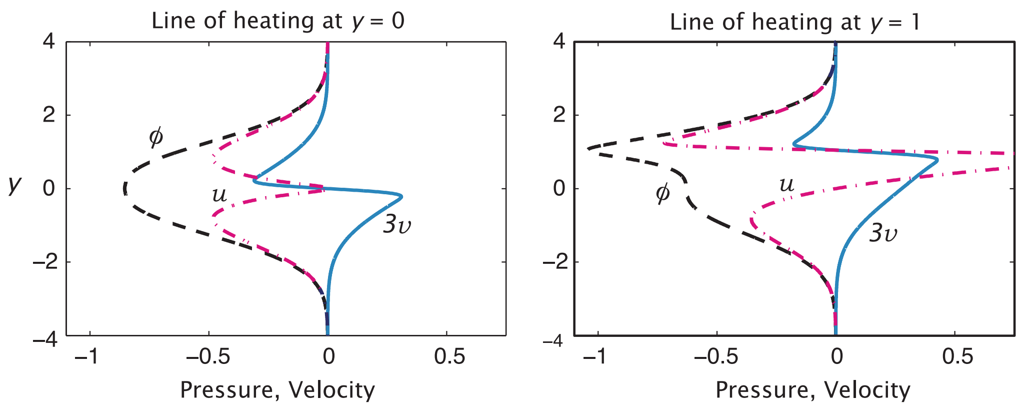

Fig. 8.16 As for Fig. 8.15, but showing line plots of pressure, ϕ, zonal velocity, u and three times (for presentational purposes) the meridional velocity v. Left panel is for a line of heating at the equator and the right panel for heating at $y = 1$. The heating creates a region of low pressure where the flow converges. Note that in the right panel the meridional velocity is larger on the equatorward side of the line than on the poleward side.

APPENDIX A: NONDIMENSIONALIZATION AND PARABOLIC CYLINDER FUNCTIONS

This appendix provides a brief discussion of the nondimensionalization used to derive the various dispersion relations in this chapter and some of the properties of the associated Hermite polynomials and parabolic cylinder functions. We do not provide proofs or detailed derivations.[13]

In discussions of equatorial waves and their steady counterparts, one of two slightly different nondimensionalizations is often employed. They lead to the use of parabolic cylinder functions in two slightly different forms; they are essentially equivalent but one may be more convenient than the other depending on the setting. For definiteness, we begin with (8.28), namely

$$\frac{\mathrm{d}^2\tilde{v}}{\mathrm{d}y^2} + \left(\frac{\omega^2}{c^2} - k^2 - \frac{\beta k}{\omega} - \frac{\beta^2 y^2}{c^2}\right)\tilde{v} = 0. \tag{8.146}$$

If we nondimensionalize time and distance using

$$T_{eq} = (c\beta)^{-1/2}, \qquad L_{eq} = (c/\beta)^{1/2}, \tag{8.147a,b}$$

we obtain

$$\frac{\mathrm{d}^2 v}{\mathrm{d}\hat{y}^2} + \left(\hat{\omega}^2 - \hat{k}^2 - \frac{\hat{k}}{\hat{\omega}} - \hat{y}^2\right) v = 0. \tag{8.148}$$

The substitution

$$v(\hat{y}) = \Psi(\hat{y}) \mathrm{e}^{-\hat{y}^2/2}, \tag{8.149}$$

leads to

$$\frac{\mathrm{d}^2 \Psi}{\mathrm{d}\hat{y}^2} - 2\hat{y}\frac{\mathrm{d}\Psi}{\mathrm{d}\hat{y}} + \lambda\Psi = 0, \tag{8.150}$$

where $\lambda = \hat{\omega}^2 - \hat{k}^2 - \hat{k}/\hat{\omega} - 1$. This is Hermite's equation with solutions if and only if $\lambda = 2m$ for $m = 0, 1, 2, \ldots$, and it is this quantization condition that gives the dispersion relation. The solutions are Hermite polynomials; that is, $\Psi(\hat{y}) = H_m(\hat{y})$, where

$$(H_0, H_1, H_2, H_3, H_4) = (1,\ 2\hat{y},\ 4\hat{y}^2 - 2,\ 8\hat{y}^3 - 12\hat{y},\ 16\hat{y}^4 - 48\hat{y}^2 + 12). \tag{8.151}$$

The Hermite polynomial multiplied by a Gaussian is a form of parabolic cylinder function, $V_m(y)$; that is

$$V_m(y) = H_m(y)\exp(-y^2/2). \tag{8.152}$$

The function $V_m(y)$ satisfies

$$\frac{\mathrm{d}^2 V_m}{\mathrm{d}y^2} + (2m + 1 - y^2)V_m = 0. \tag{8.153}$$

It is often useful to include the normalization coefficient in the definition of the cylinder function; that is, if

$$P_m = \frac{V_m}{\sqrt{2^m m!\,\sqrt{\pi}}}, \qquad \text{then} \qquad \int_{-\infty}^{\infty} P_m P_n = \delta_{mn}. \tag{8.154a,b}$$

As may be verified by direct manipulation, these forms of parabolic cylinder functions obey certain recurrence relations, namely

$$\frac{\mathrm{d}P_m}{\mathrm{d}y} = -\frac{(m+1)^{1/2}}{\sqrt{2}}P_{m+1} + \frac{m^{1/2}}{\sqrt{2}}P_{m-1} \quad \text{and} \quad yP_m = \frac{m^{1/2}}{\sqrt{2}}P_{m-1} + \frac{(m+1)^{1/2}}{\sqrt{2}}P_{m+1}, \tag{8.155a,b}$$

or equivalently

$$\frac{\mathrm{d}P_m}{\mathrm{d}y} + yP_m = (2m)^{1/2}P_{m-1} \qquad \text{and} \qquad \frac{\mathrm{d}P_m}{\mathrm{d}y} - yP_m = -\sqrt{2}(m+1)^{1/2}P_{m+1}. \tag{8.156}$$

When $m = 0$ the recurrence relations are

$$\frac{\mathrm{d}P_0}{\mathrm{d}y} = \frac{-1}{\sqrt{2}}P_1 \qquad \text{and} \qquad yP_0 = \frac{1}{\sqrt{2}}P_1. \tag{8.157a,b}$$

Had we developed the forced-dissipative problem using this form of cylinder functions these relations would have been used instead of (8.93).

The above relations may be used in conjunction with (8.20) and (8.36) to obtain the relations between u, v and ϕ in the equatorial wave problem. Using (8.20) the v and u, and the v and ϕ, fields are related by

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = \beta y\frac{\partial v}{\partial t} + c^2\frac{\partial^2 v}{\partial x\,\partial y}, \qquad \frac{\partial^2 \phi}{\partial x^2} - c^{-2}\frac{\partial^2 \phi}{\partial t^2} = \beta y\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial y\,\partial t}, \tag{8.158}$$

which in nondimensional form have $\beta = c = 1$. Suppose that the v field is a single Hermite mode of unit amplitude, meaning that

$$\widehat{v}(x,y,t) = \widetilde{v}(y)\mathrm{e}^{\mathrm{i}(kx-\omega t)} = P_m(y)\mathrm{e}^{\mathrm{i}(kx-\omega t)}. \tag{8.159}$$

If $u(x,y,t) = \widetilde{u}(y)\mathrm{e}^{\mathrm{i}(kx-\omega t)}$ then, using (8.158a) we obtain

$$\widetilde{u} = \frac{-\mathrm{i}}{(k^2-\omega^2)}\left[\widehat{y}\omega P_m - k\frac{\mathrm{d}P_m}{\mathrm{d}\widehat{y}}\right] = \frac{\mathrm{i}}{\sqrt{2}}\left[\frac{m^{1/2}P_{m-1}}{k+\omega} - \frac{(m+1)^{1/2}P_{m+1}}{k-\omega}\right], \tag{8.160}$$

where the rightmost expression uses the recurrence relations (8.155). Evidently, if m and $\widetilde{v}$ are odd (even) then $\widetilde{u}$ is an even (odd) function of y. The height field $\widetilde{\phi}$ may similarly be related to P_m using (8.158b), giving

$$\widetilde{\phi} = \frac{-\mathrm{i}}{k^2-\omega^2/c^2}\left[ykP_m + \mathrm{i}\omega\frac{\mathrm{d}P_m}{\mathrm{d}y}\right] = \frac{-\mathrm{i}}{\sqrt{2}}\left[\frac{m^{1/2}P_{m-1}}{k+\omega} + \frac{(m+1)^{1/2}P_{m+1}}{k-\omega}\right]. \tag{8.161}$$

Thus, for a given m, ϕ has the same symmetry across the equator as does u, the opposite of v.

Implications

As discussed in the main text, the gravest mode in y is the Kelvin wave, which has $v = 0$, and u and ϕ fields centred on the equator that decay away exponentially in y^2, as for P_0. The Yanai, or mixed Rossby-gravity mode, has $m = 0$ and the v field is even around the equator, meaning it is an antisymmetric mode. Using the recurrence relation (8.157) we see that this mode only generates the P_1 mode in u and ϕ. The gravest Rossby mode has $m = 1$, and from (8.160) and (8.161) we see that this generates $m = 0$ and $m = 2$ modes in u and ϕ. Thus, a symmetric disturbance centred at the equator will in general generate an eastward propagating Kelvin mode and a Rossby mode with both an equatorial signal and off equatorial modes with a structure similar to those of the stationary pattern in Fig. 8.11, and as we will see again in Fig. 22.18 when we study El Niño.

Other parabolic cylinder functions

The other commonly used form of parabolic cylinder functions, denoted $D_n(y)$, are the modified Hermite polynomials (8.42) multiplied by a Gaussian; that is

$$D_n(y) = G_n(y)\exp(-y^2/4), \tag{8.162}$$

and these functions are solutions of (8.39) which arises when we use the nondimensionalization

$$T_{eq} = (2c\beta)^{-1/2}, \qquad L_{eq} = (c/2\beta)^{1/2}. \tag{8.163a,b}$$

These parabolic cylinder functions satisfy

$$\frac{\mathrm{d}^2 D_m}{\mathrm{d}y^2} + \frac{1}{2}(2m+1-\frac{1}{2}y^2)D_m = 0, \tag{8.164}$$

which is sometimes called the Weber differential equation. The functions have the property that

$$\frac{\mathrm{d}D_n}{\mathrm{d}y} + \frac{1}{2}yD_n = nD_{n-1}, \qquad \frac{\mathrm{d}D_n}{\mathrm{d}y} - \frac{1}{2}yD_n = -D_{n+1}. \tag{8.165a,b}$$

The above two equations may be combined to give (8.164), and by subtracting them we see that

$$D_{n+1} - yD_n + nD_{n-1} = 0. \tag{8.166}$$

The form of these particular ladder operators makes these parabolic cylinder functions convenient in our development of the forced, steady (i.e., Matsuno–Gill) problem, although the use of (8.156) would be equivalent.

APPENDIX B: MATHEMATICAL RELATIONS IN THE MATSUNO–GILL PROBLEM

Here we provide various zonal and meridional integrals of the solutions given in Section 8.5.2. The zonal integral of the forcing is given by

$$I = \int_{-\infty}^{\infty} Q_0(x)\,\mathrm{d}x = \int_{-L}^{L} \cos(kx)\,\mathrm{d}x = \frac{4L}{\pi}, \tag{8.167}$$

using $k = \pi/2L$. The zonal integrals of the various q, r and v fields are given as follows. Using (8.121) with $F_0 = 0$ we see that

$$\int_{-\infty}^{\infty} \alpha q_0\,\mathrm{d}x = -\Big[q_0\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} Q_0(x)\,\mathrm{d}x = -I. \tag{8.168}$$

Using (8.132) we obtain similar results for q_2, r_0 and v_1, to wit

$$\int_{-\infty}^{\infty} (q_0, q_2, r_0, v_1)\,\mathrm{d}x = \left(-1, -\frac{1}{3}, -\frac{2}{3}, -\frac{\alpha}{3}\right)\frac{I}{\alpha}. \tag{8.169}$$

The zonally integrated pressure and velocity fields are obtained using (8.167), (8.169) and the nondimensional form of (8.136), giving

$$\int_{-\infty}^{\infty} (u, v, w, \phi)\,\mathrm{d}x = \left(\frac{-y^2}{6\alpha}, \frac{-y}{3}, \frac{2-y^2}{6}, \frac{-4-y^2}{6\alpha}\right)\left(\frac{4L}{\pi}\right)\exp(-y^2/4). \tag{8.170}$$

The meridional integrals of the velocity fields may also be calculated. To do this we first note the integrals

$$\int_{-\infty}^{\infty} (1, y, y^2)\exp(-y^2/4)\,\mathrm{d}y = (2, 0, 4)\sqrt{\pi}. \tag{8.171}$$

The first of these is a standard result, the second follows from considerations of symmetry and the third follows on integration by parts. Using (8.171) and the nondimensional form of (8.136) we obtain

$$\int_{-\infty}^{\infty} u\,\mathrm{d}y = \sqrt{\pi}\left[q_0(x) - q_2(x)\right], \qquad \int_{-\infty}^{\infty} v\,\mathrm{d}y = 0, \tag{8.172a,b}$$

$$\int_{-\infty}^{\infty} w\,\mathrm{d}y = \sqrt{\pi}\left[\alpha q_0(x) + 3\alpha q_2(x) + 2Q_0(x)\right], \qquad \int_{-\infty}^{\infty} \phi\,\mathrm{d}y = \sqrt{\pi}\left[q_0(x) + 3q_2(x)\right]. \tag{8.172c,d}$$

Equations (8.170) and (8.172) are useful because they allow us to define streamfunctions for the overturning circulation in the zonal and meridional plane, respectively. From (8.170) and (8.172), and using (8.121) and (8.132a), we find that

$$\overline{w}^x + \frac{\partial \overline{v}^x}{\partial y} = 0, \qquad \overline{w}^y + \frac{\partial \overline{u}^y}{\partial x} = 0, \tag{8.173}$$

with the overbar denoting a zonal or meridional average, as indicated. These results are to be expected from the mass continuity equation, $w = -(\partial_x u + \partial_y v)$, on zonal and meridional integration, respectively, but the fact that the solutions show it so explicitly is a demonstration of the karma of mathematics.

A streamfunction may be constructed by supposing that, in a fluid of depth H, the horizontal and vertical velocities vary as

$$(u, v) = (\tilde{u}, \tilde{v})\cos(\pi z/H), \qquad w = \tilde{w}\sin(\pi z/H). \tag{8.174a,b}$$

Using (8.170) the streamfunction in the meridional plane, Ψ_M is given by

$$\Psi_M(y, z) = \frac{IH}{\pi}\frac{-y}{3}\exp(-y^2/4)\sin \pi z/H. \tag{8.175}$$

Using (8.172) the streamfunction in the zonal plane, Ψ_Z, is given by

$$\Psi_Z(x,z) = \frac{\sqrt{\pi}H}{\pi}\left[q_0(x) - q_2(x)\right]\sin \pi z/H. \tag{8.176}$$

Notes

1 My thanks to Jacob Wenegrat for many comments on this chapter and others, and to Peter Gent for various comments on equatorial dynamics.

2 Drawing from unpublished lecture notes of M. Hendershott in Chapman *et al.* (1989). For more, see Paldor *et al.* (2007) and Heifetz & Caballero (2014).

3 Waves of this type were deduced by Bjerknes (1937).

4 The first complete treatment of this problem seems to have been given by Matsuno (1966), with special cases to be found in Stern (1963) and Bretherton (1964). An analysis of the rotating linear shallow water equations on the sphere (as opposed to β-plane) was given by Longuet-Higgins (1968) with Paldor & Sigalov (2011) providing an extension to any rotating, smooth surface. A review of equatorial waves in an oceanic context was provided by McCreary (1985).

5 Standard forms are in the eye of the beholder.

6 After M. Yanai. See Yanai & Maruyama (1966).

7 A semi-implicit, semi-Lagrangian scheme. I am grateful to James Penn for the simulation and the figure.

8 Verkley & van der Velde (2010).

9 For more on this type of problem see Gill & Clarke (1974).

10 Readers who wish to study the forced problem in more detail might start with Lighthill (1969), McCreary (1981) or Clarke (2008).

11 This problem was considered by Matsuno (1966) and revisited by Gill (1980) in the context of understanding the response of the tropical atmosphere to diabatic heating. It is now commonly referred to as the *Matsuno–Gill* problem. The treatment given here is similar to that of Gill.

Adrian Gill (1937–1986) was an Australian who spent his career in the U.K., first at Cambridge University and then, all too briefly, at Oxford as part of the U.K. Meteorological Office. He is known both for his marvellous book (*Atmosphere–Ocean Dynamics*, 1982) and for his insightful work on, to name but a few topics, equatorial dynamics, internal waves, the Antarctic Circumpolar Current and adjustment processes. Gill is admired for his scientific style, for he was somehow able to distil complex problems to an austere essence that he was often able to solve analytically. He also communicated clearly and concisely, and is said to have had a rather understated sense of humour that those close to him greatly appreciated. The field suffered an untimely loss when he died, of natural causes, decades before his time.

12 A code that solves the elliptic problem using Fourier transforms and a tridiagonal inversion was graciously provided by Chris Bretherton and Adam Sobel. Timestepping can also be a particularly simple way to obtain some solutions, as noted by Matthew Barlow. Both numerical and (where possible) analytic methods were used to obtain the solutions shown.

13 For more information about Hermite polynomials and parabolic cylinder functions see, for example, Jeffreys & Jeffreys (1946), Abramowitz & Stegun (1965) or mathematical software such as Maple or Python.