It mounts at sea, a concave wall,
Down-ribbed with shine,
And pushes forward, building tall,
Its steep incline.

Thom Gunn, *From the Wave*, 1971.

CHAPTER 7

Gravity Waves

GRAVITY WAVES ARE, UNSURPRISINGLY, waves in a fluid in which gravity provides the restoring force. (Gravitational waves are a prediction of general relativity theory.) For gravity to have an effect the fluid density must vary, and thus the waves must either exist at a fluid interface or stratification must be present — and a fluid interface is just an abrupt form of stratification. It is thus common to think of gravity waves as being either internal waves or surface waves: the former being in the interior of a fluid where the density changes may be continuous, and the latter at a fluid interface. Naturally enough the two waves have many similarities — indeed surface waves (also called interfacial waves) are a limiting form of internal waves, when the density variations in the vertical become discontinuous. We considered such interfacial waves in the hydrostatic, shallow water case in Chapter 3, and we will first extend that to the nonhydrostatic case. We then consider internal waves in the continuously stratified equations and that constitutes the bulk of the chapter.[1]

In most of the chapter we will restrict attention to the Boussinesq equations, because in making the incompressibility approximation sound waves are eliminated, greatly simplifying the treatment. In the atmosphere the Boussinesq equations are not a quantitatively good approximation except for motions of a small vertical extent; the anelastic equations improve matters in allowing for a vertical variation of the basic state density, an effect important when considering the vertical propagation of gravity waves high into the atmosphere. Nevertheless, no truly new types of waves are introduced in this way, and so we leave the details to the original literature. If, on the other hand, the fluid is truly compressible then sound waves make themselves heard, and we consider the algebraically complex case of *acoustic-gravity* waves at the end of this chapter. We begin with the simpler case of surface gravity waves atop a constant density fluid.

7.1 SURFACE GRAVITY WAVES

Let us consider an incompressible fluid with a free surface and a flat bottom that obeys the three-dimensional momentum and mass continuity equations, namely

$$\frac{D\boldsymbol{v}}{Dt} = -\nabla_3\phi - g\mathbf{k}, \qquad \nabla\cdot\boldsymbol{v} = 0, \tag{7.1}$$

using our standard notation where $\phi = p/\rho_0$. The free surface at the top of the fluid is at $z = \eta(x, y, t)$, the mean position of the free surface is at $z = 0$ and the bottom of the fluid, assumed flat, is at $z = -H$ — refer to Fig. 3.1 on on page 106. In this chapter we use a subscript 3 on ∇ to denote a three-dimensional operator, and the unsubscripted operator is horizontal.

In a state of rest the pressure, ϕ_0 say, is given by hydrostatic balance and so $\phi_0 = -gz$. If we write $\phi = -gz + \phi'$ the momentum equation becomes, without approximation,

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\nabla_3 \phi'. \tag{7.2}$$

Linearizing the equations of motion about such a resting state straightforwardly yields

$$\frac{\partial \boldsymbol{v}'}{\partial t} = -\nabla_3 \phi', \qquad \nabla_3 \cdot \boldsymbol{v}' = 0, \tag{7.3a,b}$$

where a prime denotes a perturbation quantity in the usual way. We now proceed by expressing the problem solely in terms of pressure. (An equivalent alternative would be to use a velocity potential, ξ say, such that $\boldsymbol{v} = \nabla_3 \xi$, which is possible because, from (7.3a) the flow is irrotational.) Taking the divergence of (7.3a) and using (7.3b) gives us Laplace's equation for the pressure, namely

$$\nabla_3^2 \phi' = 0. \tag{7.4}$$

This equation has no explicit time dependence, but the boundary conditions are time dependent and that is how we will obtain the dispersion relation.

7.1.1 Boundary Conditions

Since (7.4) is an equation for pressure we seek boundary conditions on pressure. At the bottom of the fluid ($z = -H$) the condition that $w = 0$ may be turned into a condition on pressure using (7.3a), namely that

$$\frac{\partial \phi'}{\partial z} = 0 \qquad \text{at } z = -H. \tag{7.5}$$

At the top surface, $z = \eta$, the pressure must equal that of the atmosphere above. We will take this to be a constant, and in particular zero, so that $\phi = 0$ at $z = \eta$. Now, the perturbation pressure is given by $\phi = -gz + \phi'$, so that at $z = \eta$ we obtain

$$\phi' = g\eta. \tag{7.6}$$

A second boundary condition at the top is the kinematic condition that a fluid parcel in the free surface must remain within the fluid, and therefore that (with full nonlinearity)

$$\frac{\mathrm{D}}{\mathrm{D}t}(z - \eta) = 0. \tag{7.7}$$

If we linearize this equation and use the definition of w we obtain $w' = \partial\eta/\partial t$ at $z = \eta$, which using (7.6) becomes $w' = g^{-1}\partial\phi'/\partial t$. Using the vertical component of the momentum equation, (7.3a), we obtain the pressure boundary condition

$$\frac{1}{g}\frac{\partial^2 \phi'}{\partial t^2} = -\frac{\partial \phi'}{\partial z} \qquad \text{at } z = \eta. \tag{7.8}$$

The value of η is in fact unknown without solving the problem itself, and in the general (nonlinear) case we have to solve the whole problem in a self-consistent fashion. However, in the linear problem η is presumptively small (we are linearizing the free surface about $z = 0$) and we will apply this boundary condition at $z = 0$ rather than at $z = \eta$, for the error will only be second order.

Having established the equations and the boundary conditions, and noting that we will be dealing exclusively with linear equations in the rest of this section (and for most of this chapter), we'll now drop the primes on perturbation quantities unless needed.

7.1.2 Wave Solutions

We now seek solutions to (7.4) in the form

$$\phi = \mathrm{Re}\, \Phi(z) \exp(\mathrm{i}[\boldsymbol{k} \cdot \boldsymbol{x} - \omega t]) \tag{7.9}$$

where $\boldsymbol{x} = \mathbf{i}x + \mathbf{j}y$ and $\boldsymbol{k} = \mathbf{i}k + \mathbf{j}l$ and Re denotes that the real part is to be taken, a notation that we drop unless it causes ambiguity. Equation (7.4) becomes

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}z^2} - K^2\Phi = 0, \tag{7.10}$$

where $K^2 = k^2 + l^2$ and the boundary conditions are that $\mathrm{d}\Phi/\mathrm{d}z = 0$ at $z = -H$ and $\mathrm{d}^2\Phi/\mathrm{d}z^2 = -g\mathrm{d}\Phi/\mathrm{d}z$ at $z = 0$. The bottom boundary condition is satisfied by a solution of the form

$$\Phi = A \cosh K(z + H). \tag{7.11}$$

Substituting into the top boundary condition, (7.8) at $z = 0$, we obtain

$$-\omega^2 \cosh KH = gK \sinh KH = 0, \tag{7.12}$$

or

$$\boxed{\omega = \pm\sqrt{gK \tanh KH}.} \tag{7.13}$$

This is the dispersion relation for surface gravity waves. The corresponding phase speed is given by

$$c_p = \frac{\omega}{K} = \pm\sqrt{gH}\left(\frac{\tanh KH}{KH}\right)^{1/2}. \tag{7.14}$$

Using (7.9) and (7.11) the full solution for the pressure field is

$$\phi = \mathrm{Re}\, \Phi_0 \cosh K(z + H) \exp(\mathrm{i}[\boldsymbol{k} \cdot \boldsymbol{x} - \omega t]) \tag{7.15}$$

with ω given by (7.13) and the amplitude Φ_0 being set by the initial conditions. It is convenient to write the amplitude Φ_0 in terms of the amplitude of the free surface elevation, η_0, using the upper boundary condition that $\phi = g\eta$, so that $\eta_0 = \Phi_0/g$. The other field variables may be found from (7.3a) and are given by

$$u = \eta_0 \frac{k}{\omega} g\, C \cosh K(z + H), \quad v = \eta_0 \frac{l}{\omega} g\, C \cosh K(z + H), \quad w = -i\eta_0 \frac{K}{\omega} g\, C \sinh K(z + H), \tag{7.16a,b,c}$$

where $C = \exp(\mathrm{i}[\boldsymbol{k} \cdot \boldsymbol{x} - \omega t])/\cosh KH$, and only the real parts of each expression should be taken. Thus, if we take η_0 to be real then u and v vary like $\cos(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)$ and w varies as $\sin(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)$, and this is what we will assume.

7.1.3 Properties of the Solution

First, from (7.13) we see that for each wavevector amplitude there are two waves propagating in opposite directions, with a frequency and phase speed that depend only on the wavelength K and not the orientation of the wave vector. Second, the waves are *dispersive*. That is, similary to Rossby waves but unlike light waves in a vacuum or shallow water waves, the phase speed is different for waves of different wavelengths. Since the frequency is a function only of K (and not of k or l individually) the group velocity is parallel to the wave vector itself and is given by

$$\boldsymbol{c}_g = \nabla_{\boldsymbol{k}}\omega = \frac{\partial\omega}{\partial K}\frac{\boldsymbol{k}}{K}, \tag{7.17}$$

where $\boldsymbol{k} = k\mathbf{i} + l\mathbf{j}$ and $\boldsymbol{k}/K$ is the unit vector in the direction of propagation. Using the dispersion relation $\omega^2 = gK \tanh KH$ we obtain

$$2\omega\frac{\partial\omega}{\partial K} = g\left(\tanh KH + \frac{KH}{\cosh^2 KH}\right), \tag{7.18}$$

so that

$$\boldsymbol{c}_g = \frac{g}{2c_pK}\left(\tanh KH + \frac{KH}{\cosh^2 KH}\right)\boldsymbol{k}, \tag{7.19}$$

and, now using (7.14), the ratio of the group speed (i.e., the magnitude of the group velocity) to the phase speed is given by

$$\frac{c_g}{c_p} = \frac{1}{2}\left(1 + \frac{2KH}{\sinh 2KH}\right), \tag{7.20}$$

having used the relation $2\sinh x\,\cosh x = \sinh 2x$.

We note two important limiting cases:

(i) The long wavelength or shallow water limit, $KH \ll 1$. It is this limit that is relevant to large-scale flow in the ocean and atmosphere. In this limit the wavelength is much greater than the depth of the fluid and the dispersion relation (7.13) reduces to $\omega = K\sqrt{gH}$ (since for small x, $\tanh x \to x$) and $c_p = c_g = \sqrt{gH}$, and the waves are nondispersive. This result is apparent from (7.20) in the limit of $KH \ll 1$. As expected, this is the same dispersion relation as was previously derived *ab initio* for shallow water waves in Chapter 3. This limit is appropriate as water waves approach the shore and start feeling the bottom, and for long waves such as tides and tsunamis.

The pressure field in this limit is given, using (7.15),

$$\phi = \eta_0 g \exp(\mathrm{i}[\boldsymbol{k}\cdot\boldsymbol{x} - \omega t]). \tag{7.21}$$

This is the *perturbation* pressure associated with the wave, and evidently it does not depend on depth. The total pressure at a given point in the fluid is given by the static pressure plus perturbation pressure and this is, including the density ρ_0,

$$p = -\rho_0 g z + \rho_0\phi = \rho_0 g(\eta - z). \tag{7.22}$$

Evidently, the pressure in the shallow water limit is hydrostatic. If $1/k > 20H$ the error in this approximation is less than 3%.

(ii) The short wavelength or deep water limit, $KH \gg 1$. For large KH, $\tanh KH \to 1$ so that the dispersion relation becomes $\omega^2 = gK$ and $c_p^2 = g/K$. These waves are dispersive, with long waves travelling faster than short waves. A familiar manifestation of this arises when a rock is thrown into a pool; initially, waves of all wavelengths are excited (for the initial disturbance is like a delta function), but the long waves propagate away faster than the short waves and reach distant objects first. The group speed in this case is given by

$$c_g = \frac{\partial\omega}{\partial k} = \frac{g}{2\omega} = \frac{1}{2}\sqrt{\frac{g}{K}} = \frac{c_p}{2}. \tag{7.23}$$

This result is also apparent from (7.20) in the limit of short waves, $KH \gg 1$, and it has an interesting consequence for wave packets. Consider a packet of short waves moving in the positive x-direction; the envelope moves with the group speed and the individual crests move with the phase speed, so that individual crests enter the packet from the rear and travel through the packet, exiting at the front.

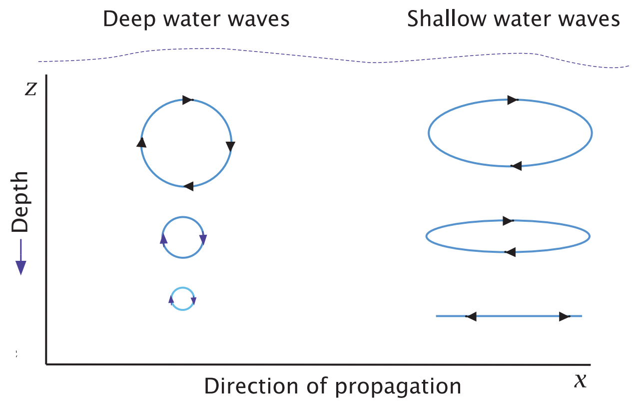

Fig. 7.1 Parcel motion for deep and shallow water waves. The motion is circular for deep water waves, with an amplitude that decreases exponentially with depth. The motion is elliptical for shallow water waves, but the horizontal excursion is independent of depth and the vertical excursion decays linearly with depth.

Parcel motion

The trajectories of water parcels are rather interesting in water waves. It turns out that in deep water the parcels make circular orbits with an amplitude diminishing with depth, whereas shallow water waves trace elliptic paths, as illustrated in Fig. 7.1 and as we now explain.

We obtain the parcel excursions using the expressions for velocity (7.16), taking $v = 0$ without loss of generality. For shallow water waves ($KH \ll 1$) u is depth independent and the velocity and the excursion in the x direction, which we denote as X, are given by

$$u = \eta_0 \frac{kg}{\omega} \cos(kx - \omega t), \qquad X = -\eta_0 \frac{gk}{\omega^2} \sin(kx - \omega t), \tag{7.24a}$$

and this is independent of z. The excursion in the z direction, Z, is given by

$$w = \eta_0 \frac{k^2}{\omega}(z + H) \sin(kx - \omega t), \qquad Z = \eta_0 \frac{gk^2}{\omega^2}(z + H) \cos(kx - \omega t), \tag{7.24b}$$

where $\omega = k\sqrt{gH}$. We see that $Z = \eta_0$ at $z = 0$, as expected. The above expressions for X and Z are, at some fixed location x and z, parametric representations of an ellipse. As z varies the horizontal amplitude of the ellipses remains constant whereas the vertical amplitude decreases linearly from the top $z = 0$ to a zero amplitude at the bottom, $z = -H$. The vertical amplitude is also generally much less than the horizontal amplitude, by the ratio

$$\frac{|Z|}{|X|} = \frac{|w|}{|u|} \sim kH \ll 1. \tag{7.25}$$

That is, the fluid motion is mostly horizontal.

In the deep water limit, $kH \gg 1$, the horizontal and vertical velocities and excursions are given by

$$u = \eta_0 \frac{kg}{\omega} \exp kz \cos(kx - \omega t), \qquad X = -\eta_0 \frac{kg}{\omega^2} \exp kz \sin(kx - \omega t), \tag{7.26a}$$

$$w = \eta_0 \frac{kg}{\omega} \exp kz \sin(kx - \omega t), \qquad Z = \eta_0 \frac{kg}{\omega^2} \exp kz \cos(kx - \omega t), \tag{7.26b}$$

where $\omega^2 = gk$, and again we have that $Z = \eta$ at $z = 0$. The expressions for X and Z, having the same amplitude, are now parametric representations of circles whose amplitudes diminish exponentially with depth. Evidently, all the dynamical variables decrease exponentially with depth, with an e-folding scale of the wavelength itself. The wave field cannot feel the bottom of the fluid container and all the expressions become independent of the water depth H.

✦ *Energy propagation*

For our final discussion on this topic we look at the energy propagation of surface waves. The kinetic energy per unit horizontal area is given by

$$\mathrm{KE} = \int_{-H}^{0} \frac{1}{2}\rho_0 \boldsymbol{v}^2 \,\mathrm{d}z. \tag{7.27}$$

The upper limit on the integration is taken to be $z = 0$, rather than $z = \eta$, because using the latter would lead to a term of order $\eta\boldsymbol{v}^2$, which is third order in perturbation quantities. The potential energy per unit horizontal area is

$$\mathrm{PE} = \int_{-H}^{\eta} \rho_0 g z \,\mathrm{d}z = \frac{\rho_0 g}{2}(\eta^2 - H^2). \tag{7.28}$$

The integral now must be over the complete depth of the fluid in order to calculate the potential energy to quadratic order. The term in H^2 is a constant and so is largely irrelevant to the problem of energy propagation. Also, since ρ_0 is a constant we will set its value to unity.

The kinetic energy equation is obtained by taking the dot product of the linearized momentum equation, (7.3a) with $\boldsymbol{v}$ and integrating over the depth of the fluid to give

$$\int_{-H}^{0} \mathrm{d}z \left[\frac{\partial}{\partial t}\frac{\boldsymbol{v}^2}{2} + \nabla\cdot(\boldsymbol{u}\phi) + \frac{\partial w\phi}{\partial z}\right] = 0, \tag{7.29}$$

noting that $\boldsymbol{v} = \boldsymbol{u} + w\mathbf{k}$ and $\nabla\cdot\boldsymbol{v} = 0$. The boundary conditions on w are that $w = 0$ at $Z = -H$ and $w = \partial\eta/\partial t$ at $z = 0$. Further, at $z = 0$ $\phi = g\eta$, and using these results (7.29) becomes

$$\int_{-H}^{0} \mathrm{d}z \left[\frac{\partial}{\partial t}\frac{\boldsymbol{v}^2}{2} + \nabla\cdot(\boldsymbol{u}\phi)\right] + g\frac{\partial}{\partial t}\frac{\eta^2}{2} = 0, \tag{7.30}$$

which, using (7.27) and (7.28), is just

$$\boxed{\frac{\partial}{\partial t}(\mathrm{KE} + \mathrm{PE}) + \nabla\cdot\boldsymbol{F} = 0,} \tag{7.31}$$

where $\boldsymbol{F} = \int_{-H}^{z} \boldsymbol{u}\phi \,\mathrm{d}z$ is the energy flux, a vector with only horizontal components. (Thus, the divergence term in (7.31) is just a horizontal divergence.)

Equation (7.31) is an energy conservation equation for the linearized equations. It is fairly general at the moment, for we have not specialized to the case of *wave* motion. Let's do that now, by using the properties of the waves derived above and averaging over a wave period. Without loss of generality we'll assume the waves are propagating in the x direction so that $v = 0$ and $K = k$; nevertheless, the calculation is rather algebraic and the trusting reader may skim it.

The kinetic energy averaged over a wave period, $\overline{\mathrm{KE}}$ is given by

$$\begin{aligned}\overline{\mathrm{KE}} &= \frac{\omega}{2\pi}\int \mathrm{d}t \left(\int \frac{1}{2}\boldsymbol{v}^2 \,\mathrm{d}z\right) \\ &= \frac{k^2\eta_0^2 g^2}{2\omega^2\cosh^2 kH}\frac{\omega}{2\pi}\int \mathrm{d}t \int \mathrm{d}z \left[\cosh^2 k(z+H)\cos^2(kx-\omega t) + \sinh^2 k(z+H)\sin^2(kx-\omega t)\right].\end{aligned} \tag{7.32}$$

In this expression the time integrals range from 0 to $2\pi/\omega$ and the vertical integrals range from $-H$ to 0. The time averages of $\sin^2$ and $\cos^2$ produce a factor of $1/2$, and noting that $\cosh^2 x + \sinh^2 x = \cosh 2x$ we obtain

$$\overline{\mathrm{KE}} = \frac{k^2\eta_0^2 g^2}{2\omega^2\cosh^2 kH}\frac{1}{2}\frac{\sinh(2kH)}{2k}. \tag{7.33}$$

Using the dispersion relation $\omega^2 = gk \tanh kH$ we finally obtain the simple expression

$$\overline{\mathrm{KE}} = \frac{g\eta_0^2}{4}. \tag{7.34}$$

The perturbation potential energy is given by

$$\overline{\mathrm{PE}} = \frac{\omega}{2\pi}\int \frac{1}{2} g\eta^2 \,\mathrm{d}t = \frac{g\eta_0^2}{2}\frac{\omega}{2\pi}\int \cos^2(kx - \omega t)\,\mathrm{d}t = \frac{g\eta_0^2}{4}. \tag{7.35}$$

Evidently, from (7.34) and (7.35) there is equipartitioning of energy time-averaged potential and kinetic energy components. Such equipartitioning is not, however, a universal property of wave motion.

The time averaged energy flux, which is in the x direction, is given by

$$\overline{F} = \frac{\omega}{2\pi}\int \mathrm{d}t \int u'\phi' \,\mathrm{d}z. \tag{7.36}$$

Using the wave expressions (7.15) we obtain, after a couple of lines of algebra,

$$\overline{F} = \frac{1}{2}\eta_0^2 \frac{g^2}{2c}\frac{1}{\cosh^2 kH}\left[\frac{\sinh 2kH}{2k} + H\right]. \tag{7.37}$$

Using (7.19) and the fact that $\sinh 2hK = 2\sinh kH \,\cosh kH$ we obtain

$$\overline{F} = \frac{\eta_0^2 g}{2} c_g = (\overline{\mathrm{KE}} + \overline{\mathrm{PE}})c_g. \tag{7.38}$$

Thus, using (7.34), (7.35) and (7.38), and generalizing the direction of propagation, we have that

$$\boxed{\frac{\partial \overline{E}}{\partial t} + \nabla\cdot \boldsymbol{c}_g \overline{E} = 0,} \tag{7.39}$$

where $\overline{E} = \overline{\mathrm{KE}} + \overline{\mathrm{PE}}$. Thus, the flux of energy is equal to the energy times the group velocity, or equivalently the energy in the wave propagates with the group velocity. As we established in Chapter 6, this property is a rather general one for wave motion and it is satisfying to see how it applies to surface waves.

7.2 SHALLOW WATER WAVES ON FLUID INTERFACES

Let us now generalize our treatment of surface gravity waves to those waves that exist on the interface between *two* moving fluids of different densities. The ensuing waves are a simple model of gravity waves that exist in the interior of the atmosphere and, perhaps especially, the ocean, in which we idealize the continuous stratification of the real fluid by supposing that the fluid comprises two (or conceivably more) layers of immiscible fluids of different densities stacked on top of each other. We will consider only the hydrostatic case in which case the layers form a 'stacked shallow water' system. We further limit ourselves to two moving layers; an extension to multiple layers is conceptually if not algebraically straightforward, but it soon becomes easier to treat the continuously stratified case, which we do in later sections.

7.2.1 Equations of Motion

Consider a two-layer shallow water model as illustrated in Fig. 3.5 on page 113. From Section 3.3 the nonlinear momentum equations are, for the upper layer,

$$\frac{\mathrm{D}\boldsymbol{u}_1}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{u}_1 = -g\nabla\eta_0, \tag{7.40a}$$

and in the lower layer

$$\frac{\mathrm{D}\boldsymbol{u}_2}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{u}_2 = -\frac{\rho_1}{\rho_2}\left(g\nabla\eta_0 + g_1'\nabla\eta_1\right). \tag{7.40b}$$

where $g_1' = g(\rho_2 - \rho_1)/\rho_1$ (we will henceforth drop the subscript 1 and denote this as g'), and in the Boussinesq case we take $\rho_1/\rho_2 = 1$. We will only consider the non-rotating case, and after linearization about a resting state we have for the upper and lower layers respectively

$$\frac{\partial \boldsymbol{u}_1'}{\partial t} = -g\nabla\eta_0', \qquad \frac{\partial \boldsymbol{u}_2'}{\partial t} = -g\nabla\eta_0' - g'\nabla\eta_1'. \tag{7.41}$$

The equations of motion are completed by the mass continuity equations for each layer, namely

$$\frac{\mathrm{D}}{\mathrm{D}t}(\eta_0 - \eta_1) + h_1\nabla\cdot\boldsymbol{u}_1 = 0 \quad\Longrightarrow\quad \frac{\partial}{\partial t}(\eta_0' - \eta_1') + H_1\nabla\cdot\boldsymbol{u}_1' = 0 \tag{7.42a,b}$$

and

$$\frac{\mathrm{D}\eta_1}{\mathrm{D}t} + h_2\nabla\cdot\boldsymbol{u}_2 = 0 \quad\Longrightarrow\quad \frac{\partial\eta_1'}{\partial t} + H_2\nabla\cdot\boldsymbol{u}_2' = 0, \tag{7.43a,b}$$

where the two rightmost expressions follow after linearization and we assume that the bottom is flat; that is $\eta_b = 0$. Henceforth we will also omit any primes on the perturbed quantities.

7.2.2 Dispersion Relation

We first eliminate the velocity from (7.41a) and (7.42b) to give

$$\frac{\partial^2}{\partial t^2}(\eta_0 - \eta_1) - gH_1\nabla^2\eta_0 = 0, \tag{7.44}$$

and similarly for the lower layer:

$$\frac{\partial^2\eta_1}{\partial t^2} - H_2(g\nabla^2\eta_0 + g'\nabla^2\eta_1) = 0. \tag{7.45}$$

Equations (7.44), and (7.45) form a complete set and in the usual fashion we may look for solutions of the form $\eta_i = \mathrm{Re}\,\tilde{\eta}_i \exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)]$. We obtain

$$(\omega^2 - gH_1K^2)\tilde{\eta}_0 - \omega^2\tilde{\eta}_1 = 0, \tag{7.46a}$$

$$-gH_2K^2\tilde{\eta}_0 + (\omega^2 - g'H_2K^2)\tilde{\eta}_1 = 0, \tag{7.46b}$$

where $K^2 = k^2 + l^2$. For these equations to have non-trivial solutions we must have

$$(\omega^2 - gH_1K^2)(\omega^2 - g'H_2K^2) - \omega^2 gH_2K^2 = 0, \tag{7.47}$$

which, for small $g'/g \ll 1$ gives, after a couple of lines of algebra,

$$\omega^2 = \frac{1}{2}K^2gH \pm \frac{1}{2}K^2gH\sqrt{1 - 4\frac{g'}{g}\frac{H_1H_2}{H^2}} \approx \frac{1}{2}K^2gH \pm \frac{1}{2}K^2gH\left(1 - 2\frac{g'}{g}\frac{H_1H_2}{H^2}\right), \tag{7.48}$$

where $H = H_1 + H_2$. If $g' = 0$ we recover the familiar single-layer dispersion relation, $\omega = K\sqrt{gH}$ (as well as $\omega = 0$). In the more general case there are two distinct modes:

(i) A fast mode with phase speed given by

$$c_p^2 = \left(\frac{\omega}{k}\right)^2 = gH\left(1 - \frac{g'}{g}\frac{H_1 H_2}{H^2}\,,\right), \tag{7.49}$$

where, for simplicity (and, in fact, without loss of generality, since it amounts only to an alignment of our coordinate system), we take $l = 0$. Using (7.46a) we then find that

$$\frac{\eta_0}{\eta_1} \approx \frac{H}{H_2}. \tag{7.50}$$

That is, since $H > H_2$, the displacement of the upper surface is larger than that of the lower. This mode is sometimes called the 'barotropic' mode, for the oscillations are vertically coherent (the phase on the interior surface is the same as that at the surface), and virtually the same oscillation would exist even in the absence of a density jump in the interior.

(ii) A slower mode with phase speed given by

$$c_p^2 \approx g'\frac{H_1 H_2}{H}, \tag{7.51}$$

and vertical structure

$$\frac{\eta_0}{\eta_1} \approx \frac{g' H_2}{gH} \ll 1. \tag{7.52}$$

In this case the displacement of the upper surface is smaller than the interior displacement by the ratio of g' to g; in the ocean, where density differences are small, the ratio might well be of order 1/100. Furthermore, the internal displacement is *out of phase* with that at the surface. Often, in oceanic situations the interface may be taken as representing the thermocline, in which case $H_2 \gg H_1$ (i.e., the abyss has a greater depth than the thermocline) and $H \approx H_2$. In this case $c_p^2 \approx g' H_1$, and internal waves on the thermocline behave rather like surface waves, but with a weaker restoring force (and consequently a larger amplitude) because the density difference between the two layers of seawater is much smaller than the density difference between the seawater and air above it.

7.3 INTERNAL WAVES IN A CONTINUOUSLY STRATIFIED FLUID

We now turn our attention to *internal gravity waves,* namely waves that are internal to a given fluid and that owe their existence to the restoring force of gravity. Interfacial waves are, of course, a model of internal waves with a discontinuous jump in density within the fluid. Surface waves might even be thought of as internal waves if one supposes that part of the fluid has zero density, although this stretches the definition of the word internal somewhat. In this section we will consider the simplest and most fundamental case, that of internal waves in a Boussinesq fluid with constant stratification and no background rotation.

Reprising and extending the material of Section 2.10.4, let us consider a continuously stratified Boussinesq fluid, initially at rest, in which the background buoyancy varies only with height and so the buoyancy frequency, N, is a function only of z. Linearizing the equations of motion about this basic state gives the linear momentum equations,

$$\frac{\partial \boldsymbol{u}'}{\partial t} = -\nabla\phi', \qquad \frac{\partial w'}{\partial t} = -\frac{\partial \phi'}{\partial z} + b', \tag{7.53a,b}$$

the mass continuity and thermodynamic equations,

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0, \qquad \frac{\partial b'}{\partial t} + w' N^2 = 0. \tag{7.53c,d}$$

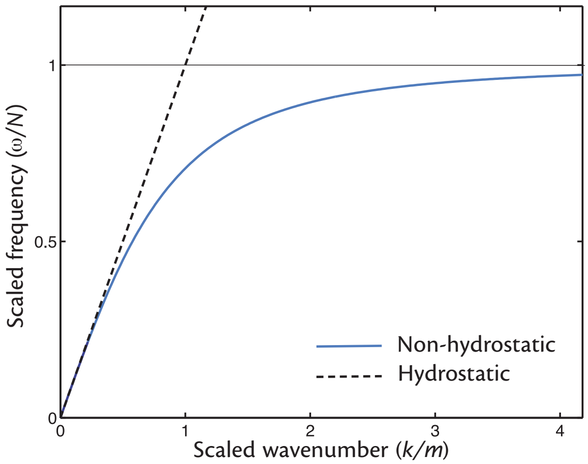

Fig. 7.2 Scaled frequency, ω/N, plotted as a function of scaled horizontal wavenumber, k/m, using the full dispersion relation of (7.56) with $l = 0$ (solid line, asymptoting to unit value for large k/m), and with the hydrostatic dispersion relation (7.60) (dashed line, tending to ∞ for large k/m).

Our notation is such that $\boldsymbol{u} \equiv u\mathbf{i}+v\mathbf{j}$, $\boldsymbol{v} \equiv u\mathbf{i}+v\mathbf{j}+w\mathbf{k}$, and the gradient operator is two-dimensional unless noted. Thus, $\nabla \equiv \mathbf{i}\,\partial_x + \mathbf{j}\,\partial_y$ and $\nabla_3 \equiv \mathbf{i}\,\partial_x + \mathbf{j}\,\partial_y + \mathbf{k}\,\partial_z$.

A little algebra gives a single equation for w',

$$\left[\frac{\partial^2}{\partial t^2}\left(\nabla^2 + \frac{\partial^2}{\partial z^2}\right) + N^2\nabla^2\right] w' = 0. \tag{7.54}$$

This equation is evidently *not* isotropic. If N^2 is a constant — that is, if the background buoyancy varies linearly with z — then the coefficients of each term are constant, and we may then seek solutions of the form

$$w' = \mathrm{Re}\ \widetilde{w}\mathrm{e}^{\mathrm{i}(kx+ly+mz-\omega t)}, \tag{7.55}$$

where Re denotes the real part, a denotion that will frequently be dropped unless ambiguity arises, and other variables oscillate in a similar fashion. Using (7.55) in (7.54) yields the dispersion relation:

$$\boxed{\omega^2 = \frac{(k^2+l^2)N^2}{k^2+l^2+m^2} = \frac{K^2N^2}{K_3^2}}\,, \tag{7.56}$$

where $K^2 = k^2 + l^2$ and $K_3^2 = k^2 + l^2 + m^2$. The frequency (see Fig. 7.2) is thus always less than N, approaching N for small horizontal scales, $K^2 \gg m^2$. If we neglect pressure perturbations, as in the parcel argument, then the two equations,

$$\frac{\partial w'}{\partial t} = b', \qquad \frac{\partial b'}{\partial t} + w'N^2 = 0, \tag{7.57}$$

form a closed set, and give $\omega^2 = N^2$.

If the basic state density increases with height then $N^2 < 0$ and we expect this state to be unstable. Indeed, the disturbance grows exponentially according to $\exp(\sigma t)$ where

$$\sigma = \mathrm{i}\omega = \pm\frac{K\widetilde{N}}{K_3}, \tag{7.58}$$

where $\widetilde{N}^2 \equiv -N^2$ and $K_3 = \sqrt{K_3^2}$. Most convective activity in the ocean and atmosphere is, ultimately, related to an instability of this form, although of course there are many complicating issues — water vapour in the atmosphere, salt in the ocean, the effects of rotation and so forth.

7.3.1 Hydrostatic Internal Waves

Let us now suppose that the fluid satisfies the hydrostatic Boussinesq equations, and for simplicity assume that $l = 0$. The linearized two-dimensional equations of motion become

$$\frac{\partial \boldsymbol{u}'}{\partial t} = -\nabla\phi', \qquad 0 = -\frac{\partial \phi'}{\partial z} + b', \tag{7.59a}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0, \qquad \frac{\partial b'}{\partial t} + w' N^2 = 0, \tag{7.59b}$$

where these are the horizontal and vertical momentum equations, the mass continuity equation and the thermodynamic equation respectively. A little algebra gives the dispersion relation,

$$\boxed{\omega^2 = \frac{(k^2 + l^2)N^2}{m^2}\,.} \tag{7.60}$$

The frequency and, if N^2 is negative, the growth rate, are unbounded as $K^2/m^2 \to \infty$, and the hydrostatic approximation thus has quite unphysical behaviour for small horizontal scales. Many numerical models of the large-scale circulation in the atmosphere and ocean do make the hydrostatic approximation. In these models convection must be *parameterized;* otherwise, it would simply occur at the smallest scale available, namely the size of the numerical grid, and this type of unphysical behaviour should be avoided. In nonhydrostatic models convection must also be parameterized if the horizontal resolution of the model is too coarse to properly resolve the convective scales.

7.3.2 Some Properties of Internal Waves

Internal waves have a number of interesting and counter-intuitive properties — let's discuss them.

The dispersion relation

We can write the dispersion relation, (7.56), as

$$\boxed{\omega = \pm N \cos\vartheta,} \tag{7.61}$$

where $\cos^2\vartheta = K^2/(K^2 + m^2)$ so that ϑ is the angle between the three-dimensional wave-vector, $\boldsymbol{k} = k\mathbf{i} + l\mathbf{j} + m\mathbf{k}$, and the horizontal. The frequency is evidently a function only of N and ϑ, and, if this is given, the frequency is not a function of wavelength. This has some interesting consequences for wave reflection, as we see below.

We can also write the dispersion relation, (7.56), as

$$\frac{\omega^2}{N^2 - \omega^2} = \frac{K^2}{m^2}. \tag{7.62}$$

Thus, and consistently with our first point, given the wave frequency the ratio of the vertical to the horizontal wavenumber is fixed.

Polarization relations

The oscillations of pressure, velocity and buoyancy are, naturally, connected, and we can obtain the relations between them with some simple manipulations. If the pressure field is oscillating like $\phi' = \widetilde{\phi}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)] = \widetilde{\phi}\exp[\mathrm{i}(kx + ly + mz - \omega t)]$ then, using (7.53a), the horizontal velocity components have the phases

$$(\widetilde{u}, \widetilde{v}) = (k, l)\,\omega^{-1}\widetilde{\phi}. \tag{7.63a}$$

As the frequency is real, the velocities are in phase with the pressure. A little algebra also reveals that the buoyancy perturbation is related to the pressure perturbation by

$$\tilde{b} = \frac{\mathrm{i}mN^2}{N^2 - \omega^2}\tilde{\phi} = \frac{\mathrm{i}N^2K^2}{m\omega^2}\tilde{\phi} = \frac{\mathrm{i}K_3^2}{m}\tilde{\phi}, \tag{7.63b}$$

using the dispersion relation, so that the buoyancy and pressure perturbations are $\pi/2$ out of phase.

The vertical velocity is related to the pressure perturbation by

$$\tilde{w} = \frac{-\omega m}{N^2 - \omega^2}\tilde{\phi} = \frac{-K^2}{m\omega}\tilde{\phi}, \tag{7.63c}$$

where the second expression uses (7.62). The vertical velocity is in phase with the pressure perturbation, and for regions of positive m (and so with upward phase propagation) regions of high relative pressure are associated with downward fluid motion.

The pressure, buoyancy and velocity fields are all real fields and we can write the above phase relationships in terms of sines and cosines as follows:

$$\phi = \Phi_0 \cos(kx + ly + mz - \omega t), \tag{7.64a}$$

$$(u, v) = (k, l)\frac{\Phi_0}{\omega}\cos(kx + ly + mz - \omega t), \tag{7.64b}$$

$$w = \left(\frac{-\omega m}{N^2 - \omega^2} = \frac{-K^2}{m\omega}\right)\Phi_0 \cos(kx + ly + mz - \omega t). \tag{7.64c}$$

$$b = \left(\frac{mN^2}{N^2 - \omega^2} = \frac{N^2K^2}{m\omega^2}\right)\Phi_0 \sin(kx + ly + mz - \omega t), \tag{7.64d}$$

where Φ_0 is a constant. We might equally well have chosen ϕ to have a sine dependence in (7.64a); nothing of substance differs, but (7.64b,c,d) should be changed appropriately. The relations between pressure, buoyancy and velocity in (7.63) and (7.64) are known as *polarization relations.*

Relation between wave vector and velocity

On multiplying (7.64b) and (7.64c) by (k, l) and m, respectively, we see that

$$\boldsymbol{k} \cdot \tilde{\boldsymbol{v}} = 0, \tag{7.65}$$

where $\boldsymbol{k}$ and $\tilde{\boldsymbol{v}}$ are three-dimensional vectors. This means that, at any instant, the wave vector is perpendicular to the velocity vector, and the velocity is therefore aligned *along* the direction of the troughs and crests, along which there is no pressure gradient. If the wave vector is purely horizontal (i.e., $m = 0$), then the motion is purely vertical and $\omega = N$.

The vertical and horizontal velocities are related to the wavenumbers. If (for simplicity, and with no loss of generality) the motion is in the x–y plane with $v = l = 0$, then it is a corollary of (7.65) that

$$\frac{\tilde{u}}{\tilde{w}} = -\frac{m}{k}. \tag{7.66}$$

Furthermore, from (7.55) with $l = 0$, at any given instant all of the perturbation quantities in the wave are constant along the lines $kx+mz$ = constant. Thus, *all fluid parcel motions are parallel to the wave fronts.* Now, since the wave frequency is related to the background buoyancy frequency by $\omega = \pm N \cos\vartheta$, it follows that the fluid parcels oscillate along lines that are at an angle $\vartheta = \cos^{-1}(\omega/N)$ to the vertical. The polarization relations and the group and phase velocities are illustrated in Fig. 7.3. Let us now discuss the wave properties in a little more detail.

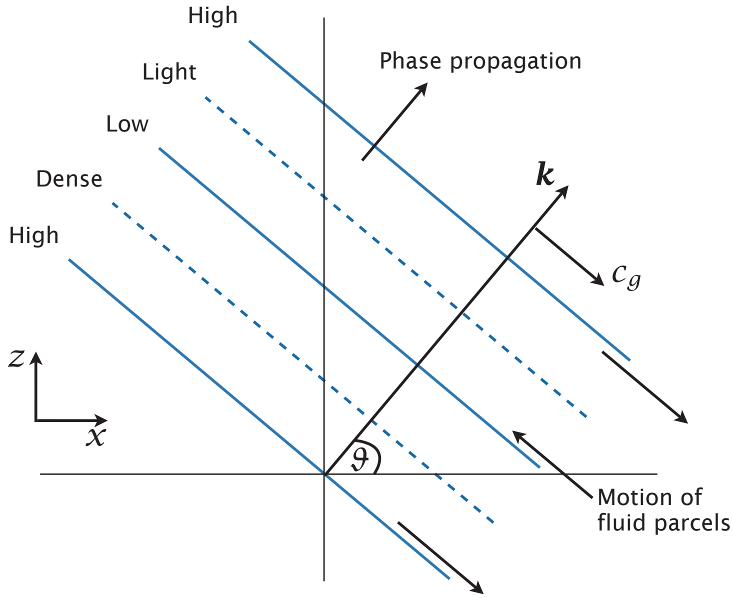

Fig. 7.3 An internal wave propagating in the direction $\boldsymbol{k}$. Both k and m are positive for the wave shown. The solid lines show crests and troughs of constant pressure, and the dashed lines the corresponding crests and troughs of buoyancy (or density). The motion of the fluid parcels is along the lines of constant phase, as shown, and is parallel to the group velocity and perpendicular to the phase speed.

7.3.3 A Parcel Argument and Physical Interpretation

Let us consider first the dispersion relation itself and try to derive it more physically, or at least heuristically. Let us suppose there is a wave propagating in the (x, z) plane at some angle ϑ to the horizontal, with fluid parcels moving parallel to the troughs and crests, as in Fig. 7.3. In general the restoring force on a parcel is due to both the pressure gradient and gravity, but along the crests there is no pressure gradient. Referring to Fig. 7.4, for a total displacement Δs the restoring force, F_{res}, in the direction of the particle displacement is

$$F_{res} = g\cos\vartheta \times \Delta\rho = g\cos\vartheta \times \frac{\partial\rho}{\partial z}\Delta z = g\cos\vartheta \times \frac{\partial\rho}{\partial z}\Delta s\cos\vartheta = \rho_0\frac{\partial b}{\partial z}\cos^2\vartheta\,\Delta s, \tag{7.67}$$

noting that $\Delta z = \cos\vartheta\,\Delta s$. The equation of motion of a parcel moving along a trough or crest is therefore

$$\rho_0\frac{\mathrm{d}^2\Delta s}{\mathrm{d}t^2} = -\rho_0 N^2\cos^2\vartheta\,\Delta s, \tag{7.68}$$

which implies a frequency $\omega = N\cos\vartheta$, as in (7.61). One of the $\cos\vartheta$ factors in (7.68) comes from the fact that the parcel displacement is at an angle to the direction of gravity, and the other comes from the fact that the restoring force that a parcel experiences is proportional to $N\cos\vartheta$. (The reader may also wish to refer ahead to Fig. 7.14 and Section 7.6.1 for a similar argument.)

Now consider the wave illustrated in Fig. 7.3. For this wave both k and m are positive, and the frequency is assumed positive by convention to avoid duplicative solutions. The slanting solid and dashed lines are lines of constant phase, and from (7.63b) the buoyancy and pressure are 1/4 of a wavelength out of phase. When k and m are both positive the extrema in the buoyancy field lag the extrema in the vertical velocity by $\pi/2$, as illustrated. The perturbation velocities are zero along the lines of extreme buoyancy. This follows because the velocities are in phase with the pressure, which as we noted is out of phase with the buoyancy.

Given the direction of the fluid parcel displacement in Fig. 7.3, the direction of the phase propagation c_p up and to the right may be deduced from the following argument. Buoyancy perturbations arise because of vertical advection of the background stratification, $w'\partial b_0/\partial z = w'N^2$. A local maximum in rising motion, and therefore a tendency to increase the fluid density, is present

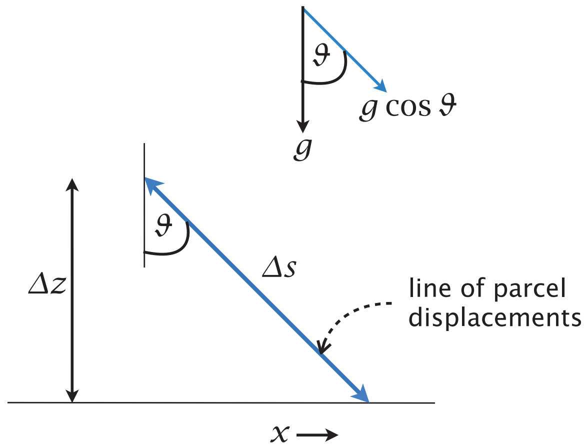

Fig. 7.4 Parcel displacements and associated forces in an internal gravity wave in which the parcel displacements are occurring at an angle ϑ to the vertical, as in Fig. 7.3.

along the 'Low' line 1/4 wavelength upward and to the right of the 'Dense' phase line. Thus, the density of fluid along the 'Low' phase line increases and the 'Dense' phase line moves upward and to the right. If the fluid parcel motion were reversed the pattern of 'High–Dense–Low–Light–High' in Fig. 7.3 would remain the same. However, the downward fluid motion along the 'Low' line would cause the fluid to lose density, and so the phase lines would propagate downward and to the left. Evidently, the wave fronts, or the lines of constant phase, move at right angles to the fluid-parcel trajectories. In the figure we see that the group velocity is denoted as being at right angles to the phase speed, so let's discuss this.

7.3.4 Group Velocity and Phase Speed

As we noted above, the frequency of internal waves is given by $\omega = N\cos\vartheta$, where ϑ is the angle the wave vector makes with the horizontal. This means that the surfaces of constant frequency are *cones*, as illustrated in Fig. 7.5.

To evaluate phase and group velocities in a useful way it is convenient to use spherical polar coordinates, as in Fig. 7.6, in which

$$k = K_3 \cos\vartheta\cos\lambda, \qquad l = K_3\cos\vartheta\sin\lambda, \qquad m = K_3\sin\vartheta, \tag{7.69}$$

so that $\boldsymbol{k} = K_3(\cos\vartheta\cos\lambda, \cos\vartheta\sin\lambda, \sin\vartheta)$. The angles are ϑ, the angle of the wave vector with the horizontal and λ, which determines the orientation in the horizontal plane. (The notation is similar to the spherical coordinates of Chapter 2 — see Fig. 2.3 — although here ϑ is the angle with the horizontal, not the angle with the equatorial plane.) We also note that

$$\sin^2\vartheta = \frac{m^2}{k^2+l^2+m^2}, \qquad \cos^2\vartheta = \frac{K^2}{K_3^2} = \frac{k^2+l^2}{k^2+l^2+m^2}, \qquad \tan\lambda = \frac{l}{k}. \tag{7.70}$$

In many problems we can align the direction of the wave propagation with the x-axis and take $l = 0$ and $\tan\lambda = 0$.

The phase speed of the internal waves in the direction of the wave vector (sometimes referred to as the phase velocity) is given by

$$c_p = \frac{\omega}{K_3} = \frac{N}{K_3}\cos\vartheta = \frac{NK}{K_3^2}. \tag{7.71}$$

The phase speeds (as conventionally-defined) in the x, y and z directions are

$$c_p^x \equiv \frac{\omega}{k} = \frac{N}{k}\cos\vartheta, \qquad c_p^y \equiv \frac{\omega}{l} = \frac{N}{l}\cos\vartheta, \qquad c_p^z \equiv \frac{\omega}{m} = \frac{N}{m}\cos\vartheta. \tag{7.72a,b,c}$$

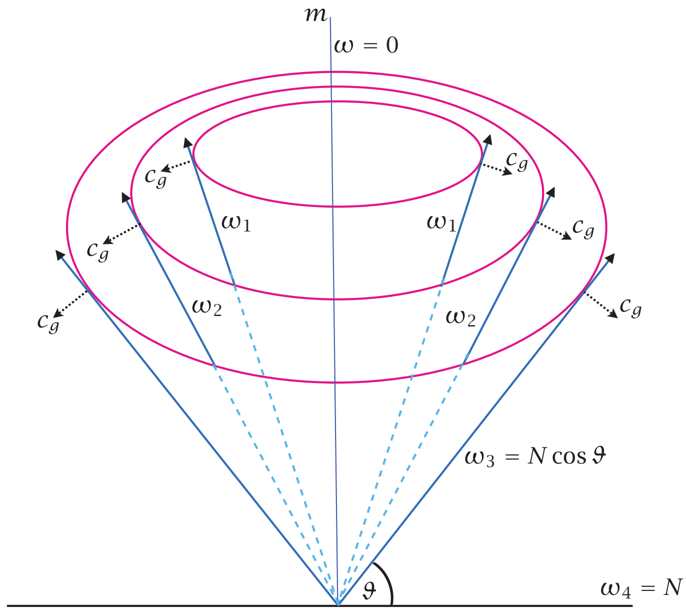

Fig. 7.5 Internal wave cones. The surfaces of constant frequency are cones, defined by the surface that has a constant angle to the horizontal. The wave vector, and so the phase velocity, point along the cone away from the origin, and the frequency of any wave with a wave vector in the cone is $N\cos\vartheta$. The group velocity is at right angles to the cone and pointed in the direction of increasing frequency, as indicated by the arrows on the dotted lines. In the vertical direction the phase speed and group velocity have opposite signs.

As noted in Section 6.1.2, these quantities are the speed of propagation of the wave crests in the respective directions. In general, each speed is *larger* than the phase speed in the direction perpendicular to the wave crests (that is, in the direction of the wave vector), but no information is transmitted at these speeds.

The group velocity is given by

$$\boldsymbol{c}_g = \left(\frac{\partial\omega}{\partial k}, \frac{\partial\omega}{\partial l}, \frac{\partial\omega}{\partial m}\right). \tag{7.73}$$

Using (7.56) we find

$$c_g^x = \frac{\partial\omega}{\partial k} = \frac{Nm}{K_3^2}\frac{km}{KK_3} = \left(\frac{N}{K_3}\sin\vartheta\right)\cos\lambda\sin\vartheta, \tag{7.74a}$$

$$c_g^y = \frac{\partial\omega}{\partial l} = \frac{Nm}{K_3^2}\frac{lm}{KK_3} = \left(\frac{N}{K_3}\sin\vartheta\right)\sin\lambda\sin\vartheta, \tag{7.74b}$$

$$c_g^z = \frac{\partial\omega}{\partial m} = -\frac{Nm}{K_3^2}\frac{K}{K_3} = -\left(\frac{N}{K_3}\sin\vartheta\right)\cos\vartheta. \tag{7.74c}$$

The magnitude of the group velocity is evidently

$$|c_g| = \frac{N}{K_3}\sin\vartheta, \tag{7.75}$$

and the group velocity vector is directed at an angle ϑ to the vertical, as in Fig. 7.5. This angle is perpendicular to the cone itself; that is, the group velocity is perpendicular to the wave vector, as may be verified by taking the dot product of (7.69) and (7.74) which gives

$$\boldsymbol{k}\cdot\boldsymbol{c}_g = 0. \tag{7.76}$$

The group velocity is therefore parallel to the motion of the fluid parcels, as illustrated in Fig. 7.3. Furthermore, because energy propagates with the group velocity, and the latter is *parallel* to lines

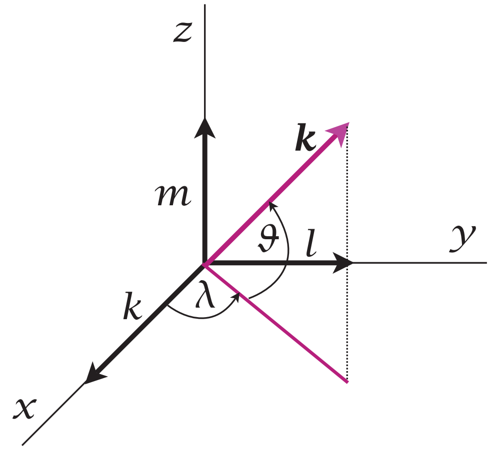

Fig. 7.6 The spherical coordinates used to describe internal waves, as in (7.69). The angle ϑ is the angle of the wave vector with the horizontal, and λ determines the orientation in the horizontal plane. The wave vector $\boldsymbol{k}$ is given by $\boldsymbol{k} = (k, l, m)$, these being the wavenumbers in the direction of increasing (x, y, z), respectively.

of constant phase, energy propagates perpendicular to the direction of phase propagation — very different from the case of acoustic waves or even shallow water waves. In the vertical direction we see from (7.72c) and (7.74c) that

$$\frac{\omega}{m}\frac{\partial \omega}{\partial m} = -\frac{N^2}{K_3^2}\cos^2\vartheta < 0. \tag{7.77}$$

That is, the phase speed and the group velocity have opposite signs, meaning that if the wave crests move downward the group moves upward!

Effect of a mean flow

Suppose that there is a mean flow, U, in the x-direction, as is common in both atmosphere and ocean. The dispersion relation, (7.56), simply becomes

$$(\omega - Uk)^2 = \frac{K^2N^2}{K^2 + m^2}. \tag{7.78}$$

The frequency is Doppler shifted, as expected, but the upward propagation of waves is affected in an interesting way. From (7.78) we find that the vertical component of the group velocity may be written as

$$\frac{\partial \omega}{\partial m} = \frac{-m(\omega - Uk)}{K^2 + m^2} = \frac{-mk(c - U)}{K^2 + m^2}, \tag{7.79}$$

where $c = \omega/k$ is the phase speed in the x-direction. If U is not constant but is varying slowly with z then (7.79) still holds, although m itself will also vary slowly with z. The point to note is that the group velocity goes to zero at the location where $U = c$, that is at a critical line, and the wave stalls. Of course m itself may become large near a critical line (as we consider in more detail in Section 17.3). In this case — which is essentially the hydrostatic one, with $m^2 \gg K^2$ — we obtain

$$\frac{\partial \omega}{\partial m} = \frac{-k(c - U)}{m} = \frac{-k^2(c - U)^2}{KN}. \tag{7.80}$$

The physical consequence of group velocity going to zero as the wave approaches a critical line is that any dissipation that may be present has more time to act. That is, we can expect a wave to be preferentially dissipated near a critical line, giving up its momentum to the mean flow and its energy to create mixing — the former being important in the atmosphere (for this is the mechanism producing the quasi-biennial oscillation) and the latter in the ocean.

7.3.5 Energetics of Internal Waves

In this section we explore the energetics of internal waves, and we first show that the linearized equations conserve a sensible form of energy. Linearized equations do not, of course, automatically conserve energy even if the original nonlinear equations from which they derive do: an unstable wave will draw energy from the background state and grow in amplitude, as we will see in Chapter 9 on baroclinic instability.

Energy conservation

From (7.53a,b) we obtain an equation for the evolution of kinetic energy, namely

$$\frac{\partial}{\partial t}\left(\frac{\boldsymbol{v}'^2}{2}\right) = b'w' - \nabla_3 \cdot (\phi' \boldsymbol{v}'), \tag{7.81}$$

where $\boldsymbol{v}'^2 = u'^2 + v'^2 + w'^2$, and from (7.53c,d) we obtain

$$\frac{1}{N^2}\frac{\partial}{\partial t}\frac{b'^2}{2} + w'b' = 0. \tag{7.82}$$

Adding the above two equations gives

$$\boxed{\frac{\partial}{\partial t}\frac{1}{2}\left(\boldsymbol{v}'^2 + \frac{b'^2}{N^2}\right) + \nabla_3 \cdot (\phi' \boldsymbol{v}') = 0.} \tag{7.83}$$

This is the linear version of the energy conservation equation for Boussinesq flow, as in (2.112) on page 74.

Two differences are apparent: (i) the transport of energy is only by way of the pressure term and the advective transport is absent, as expected in a linear model; (ii) the potential energy term bz of the linear model is replaced by b'^2/N^2. It is less obvious why this should be so. However, the quantity

$$A = \frac{1}{2}\int \frac{\overline{b'^2}}{\partial \overline{b}/\partial z}\,\mathrm{d}z\,\mathrm{d}A = \frac{1}{2}\int \frac{\overline{b'^2}}{N^2}\,\mathrm{d}z\,\mathrm{d}A \tag{7.84}$$

is just the *available potential energy* (APE) of a Boussinesq fluid in which the isopycnal surfaces vary only slightly from a stable, purely horizontal, resting state, and it is only the APE that participates in the linear system.

If we integrate (7.83) over a volume such that the normal component of the velocity vanishes at the boundaries (for example, we integrate over a volume enclosed by rigid walls), then the divergence term vanishes and we obtain the integral conservation statement:

$$\widehat{E} = \frac{1}{2}\int \left(\boldsymbol{v}'^2 + \frac{b'^2}{N^2}\right)\mathrm{d}V, \qquad \frac{\mathrm{d}\widehat{E}}{\mathrm{d}t} = 0. \tag{7.85}$$

The quantity $\widehat{E}$ is an example of a *wave activity:* a conserved quantity that is quadratic in wave amplitude. This conservation statement (7.83) is true whether or not the basic state is stably stratified; that is, whether or not N^2 is positive. However, (7.85) only provides a bound on growing perturbations if N^2 is positive, in which case all the terms that constitute $\widehat{E}$ are positive definite. If $N^2 < 0$ then both $\boldsymbol{v}'^2$ and b'^2 can grow without bound even as $\widehat{E}$ itself remains constant.

Consider now the energy in a *wave*, and we will denote by $\overline{E}$ the energy density, meaning the mean perturbation energy per unit volume, averaged over a wavelength. Thus

$$2\overline{E} = \overline{\boldsymbol{v}'^2} + \frac{\overline{b'^2}}{N^2}. \tag{7.86}$$

If we use the polarization relations of Section 7.3.2 then the kinetic and potential energy densities may be written in terms of the pressure amplitude as

$$2\overline{KE} = \left(\frac{k^2}{\omega^2} + \frac{l^2}{\omega^2} + \frac{(k^2+l^2)^2}{m^2\omega^2}\right)|\widetilde{\phi}|^2 = \frac{K^2K_3^2}{m^2\omega^2}|\widetilde{\phi}|^2, \tag{7.87a}$$

$$2\overline{PE} = \frac{N^2K^4}{m^2\omega^4} = \frac{K^2K_3^2}{m^2\omega^2}|\widetilde{\phi}|^2, \tag{7.87b}$$

using also the dispersion relation, $\omega^2K_3^2 = K^2N^2$. Thus, there is *equipartition* between the kinetic and potential energies, a common feature of waves in non-rotating systems (although not a universal feature of waves). The total energy density is thus

$$\overline{E} = \frac{K^2K_3^2}{m^2\omega^2}|\widetilde{\phi}|^2 = \frac{K_3^2}{K^2}|\widetilde{w}|^2 = \frac{|\widetilde{w}|^2}{\cos^2\vartheta}, \tag{7.88}$$

using (7.64c), where $\widetilde{w}$ is the amplitude of the vertical component of the velocity perturbation.

Energy propagation and the group velocity property

In Section 6.7 we derived, from rather general considerations, the 'group velocity property' for wave activity. We showed that if a wave activity, $\mathcal{A}$, and its flux, $\boldsymbol{\mathcal{F}}$ obeyed a conservation law of the form $\partial\mathcal{A}/\partial t + \nabla\cdot\boldsymbol{\mathcal{F}} = 0$, and if the wave activity and its flux were both quadratic functions of the wave amplitude, then the flux is related to the wave activity by $\boldsymbol{\mathcal{F}} = \boldsymbol{c}_g\mathcal{A}$. The internal wave energy density and its flux do have these properties — see (7.83) — so we should expect the group velocity property to hold, and we now demonstrate that explicitly, albeit briefly.

The energy flux vector for internal waves is $\boldsymbol{\mathcal{F}} = \overline{\phi'\boldsymbol{v}'}$ and using (7.63a) and (7.63c) this is

$$\boldsymbol{\mathcal{F}} = \left(\frac{k}{\omega}, \frac{l}{\omega}, -\frac{K^2}{m\omega}\right)|\widetilde{\phi}|^2. \tag{7.89}$$

Using (7.74) and (7.88) the group velocity times the energy density is

$$c_g^x \times \overline{E} = \left[\frac{Nm^2}{K_3^3}\frac{k}{K}\right] \times \left[\frac{K^2K_3^2}{m^2\omega^2}|\widetilde{\phi}|^2\right] = \frac{k}{\omega}|\widetilde{\phi}|^2, \tag{7.90a}$$

$$c_g^y \times \overline{E} = \left[\frac{Nm^2}{K_3^3}\frac{l}{K}\right] \times \left[\frac{K^2K_3^2}{m^2\omega^2}|\widetilde{\phi}|^2\right] = \frac{l}{\omega}|\widetilde{\phi}|^2, \tag{7.90b}$$

$$c_g^z \times \overline{E} = \left[\frac{NmK}{K_3^3}\right] \times \left[\frac{K^2K_3^2}{m^2\omega^2}|\widetilde{\phi}|^2\right] = -\frac{K^2}{m\omega}|\widetilde{\phi}|^2, \tag{7.90c}$$

which evidently is the same as (7.89), completing our demonstration.

7.4 ✦ INTERNAL WAVE REFLECTION

Suppose a propagating internal wave encounters a solid boundary — sloping topography, for example. The boundary effectively acts as a source of waves and so the original wave is reflected in some fashion. However, because of the nature of the dispersion relation for internal waves the reflection occurs in a rather peculiar way, as we now discuss.

For algebraic simplicity let us initially suppose that the wave is propagating in the x–z plane, and the equation of mass continuity $\partial_x u + \partial_z w = 0$ is then satisfied by introducing a streamfunction ψ such that

$$u = -\frac{\partial\psi}{\partial z}, \qquad w = \frac{\partial\psi}{\partial x}. \tag{7.91}$$

If the incident wave is denoted ψ_1 and the reflected wave ψ_2 then the total wave field is

$$\psi = \widetilde{\psi}_1 \exp\{i(k_1 x + m_1 z - \omega_1 t)\} + \widetilde{\psi}_2 \exp\{i(k_2 x + m_2 z - \omega_2 t)\}, \tag{7.92}$$

where as usual a tilde denotes a complex wave amplitude and the real part of the expression is implied. The total streamfunction must be constant *at the boundary* — in fact without loss of generality we may suppose that $\psi = 0$ at the boundary — and this can only be achieved if

$$k_1 x + m_1 z - \omega_1 t = k_2 x + m_2 z - \omega_2 t \tag{7.93}$$

for all t and for all x and z along the boundary. This implies that

$$\omega_1 = \omega_2 \tag{7.94}$$

and

$$k_1 x + m_1 z_b(x) = k_2 x + m_2 z_b(x), \tag{7.95}$$

where $z_b(x)$ parameterizes the height of the reflecting boundary. We can view this another way: suppose that the boundary slopes at an angle γ to the horizontal, as in Fig. 7.7 or Fig. 7.8. We then have $z_b = x \tan\gamma$ and a unit vector along the boundary satisfies $\boldsymbol{j}_\gamma = \mathbf{i}\cos\gamma + \mathbf{j}\sin\gamma$. Equation (7.92) may be written as

$$\psi = \widetilde{\psi}_1 \exp\left\{i\left[(k_1 + m_1 \tan\gamma)x - \omega_1 t\right]\right\} + \widetilde{\psi}_2 \exp\left\{i\left[(k_2 + m_1 \tan\gamma)x - \omega_2 t\right]\right\}, \tag{7.96}$$

from which the wavenumber condition that must be satisfied is

$$k_1 + m_1 \tan\gamma = k_2 + m_2 \tan\gamma \tag{7.97}$$

or, and as may also be seen from (7.95),

$$\boldsymbol{k}_1 \cdot \boldsymbol{j} = \boldsymbol{k}_2 \cdot \boldsymbol{j}. \tag{7.98}$$

This means that the components of the wave vector parallel to the boundary for the incoming and outgoing wave are equal to each other. This, and the conservation of frequency expressed by (7.94), are *general* results about linear wave reflection; they apply to light waves, for example. However, the dispersion relation of internal waves gives rise to rather unintuitive and decidedly non-specular properties of reflection.

7.4.1 Properties of Internal Wave Reflection

Suppose an internal wave is incident on a solid boundary, sloping at an angle γ to the horizontal, as in Fig. 7.7 or Fig. 7.8. The incident and reflected waves must satisfy the following conditions:

(i) The frequency of the reflected wave is equal to that of the incident wave. Because the frequency is given by $\omega = N\cos\vartheta$, the angle of the reflected wave *with respect to the horizontal* is equal to that of the incident wave.

(ii) The components of the wave vector along the slope of the reflected wave and incident wave are equal.

(iii) The group velocity of the reflected wave must be directed away from the slope.

We did not derive the third of these conditions, but the reflected wave must carry energy and information away from the slope, and these are carried by the group velocity. Similarly, a wave incident on a boundary is one in which the group velocity is directed toward the slope.

Consider a wave approaching a slope as in Fig. 7.7, such that the incoming wave vector makes an angle of ϑ_1 with the horizontal, and the boundary slope is γ. The condition (7.98) states that the

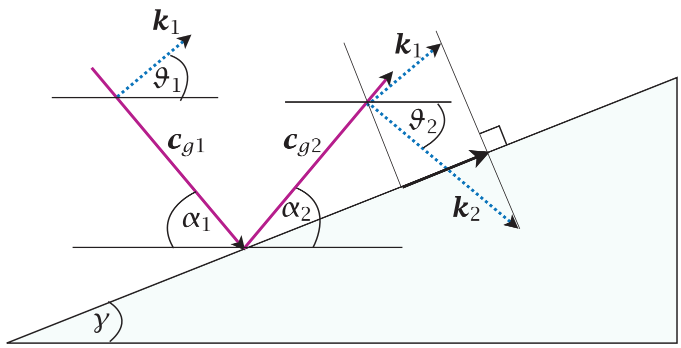

Fig. 7.7 Internal wave reflection from a shallow sloping boundary. The incoming wave vector, $\boldsymbol{k}_1$, makes an angle ϑ_1 with the horizontal, and the incoming group velocity, $\boldsymbol{c}_{g1}$ makes an angle $\alpha_1 = \pi/2 - \vartheta_1$.

The group velocity of the reflected wave, $\boldsymbol{c}_{g2}$ is directed away from the slope, and to satisfy the frequency condition $\alpha_2 = \alpha_1$. The projection along the slope of the reflected wave vector, $\boldsymbol{k}_2$ must be equal to that of the incoming wave vector (the projection is the short thick arrow along the slope), and so the magnitude of the reflected wave vector is larger than that of the incoming wave.

projections along the boundary of the the incoming and outgoing wave vectors are equal to each other, and so

$$\kappa_1 \cos(\vartheta - \gamma) = \kappa_2 \cos(\vartheta + \gamma), \tag{7.99}$$

where κ_1 and κ_2 are the magnitudes of the incoming and reflected wave vectors and $\vartheta = \vartheta_1 = \vartheta_2$, because the outgoing wave makes the same angle with the horizontal as does the incoming wave. The group velocity is perpendicular to the wave vector and makes an angle $\alpha = \pi/2 - \vartheta$ to the horizontal, and in terms of this (7.99) may be written, provided $\alpha > \gamma$,

$$\kappa_1 \sin(\alpha + \gamma) = \kappa_2 \sin(\alpha - \gamma). \tag{7.100}$$

For a sufficiently steep boundary slope we may have $\alpha < \gamma$, and in this case the wave will be back reflected down the slope, as in Fig. 7.8. A little geometry reveals that the condition (7.100) should be replaced by

$$\kappa_1 \sin(\alpha + \gamma) = \kappa_2 \sin(\gamma - \alpha). \tag{7.101}$$

The case with $\alpha = \gamma$ is plainly a critical one. In this case the group velocity of the reflected wave is directed along the slope, and the wave vector is perpendicular to the slope. The magnitude of the reflected wave vector is infinite; that is, the waves have zero wavelength, and so would in reality be subject to viscous dissipation and diffusion. Reflection of internal waves is in fact an important mechanism leading to mixing in the ocean.

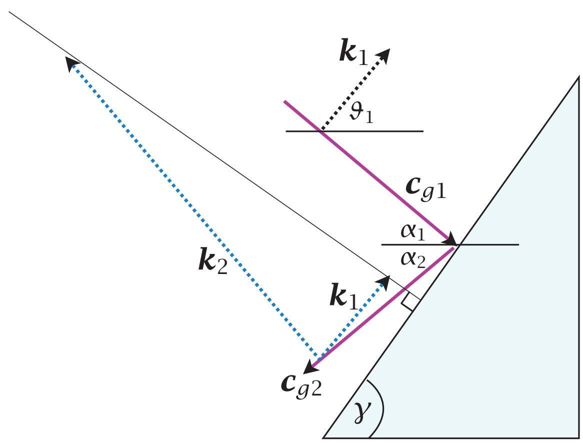

Fig. 7.8 As for Fig. 7.7, but now showing reflection from a steep slope. The wave is back-reflected down the slope, and in this example the magnitude of the reflected wave is again larger than that of the incoming wave.

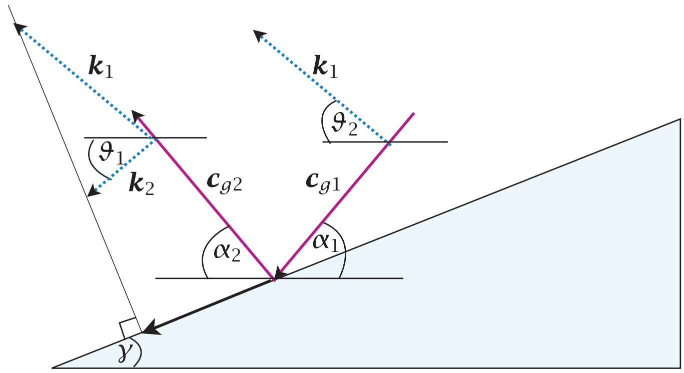

Fig. 7.9 As for Fig. 7.7, but now showing the production of a reflected wave with a longer wavelength than the incident wave. The wavevector of the reflected wave is more nearly parallel to the sloping boundary than is the wave vector of the incident wave.

The reflected wave need not, of course, always have a wavenumber that is higher than that of the incident wave: it is a matter of whether the incoming wave vector is more nearly aligned with the slope of the boundary than is the reflected wave, and if it is the reflected wave will have a higher wavenumber, and contrariwise. An example of reflection producing a longer wave is illustrated in Fig. 7.9. Still, the process whereby waves are reflected to produce waves of a shorter wavelength that are then dissipated is an irreversible one, and the net effect of many quasi-random wave reflections is likely to be the dissipation of short waves.

Finally, one might ask why the reflected wave could not simply be back along the track of the incident wave — for example, why could we not have $\boldsymbol{c}_{g1} = -\boldsymbol{c}_{g2}$? If this were so then we would have $\boldsymbol{k}_2 = -\boldsymbol{k}_1$, and it would be impossible for the two wave vectors to project equally on the sloping boundary.

7.5 ✦ INTERNAL WAVES IN A FLUID WITH VARYING STRATIFICATION

In most realistic situations the stratification N^2 is not constant. In the ocean the stratification is largest in the upper ocean (in the 'pycnocline') diminishing with depth in the weakly stratified abyss. In the atmosphere the stratification tends to be fairly constant in the troposphere but increases fairly abruptly as we pass into the stratosphere. In such circumstances the wave equation (7.54) no longer has constant coefficients and we cannot easily obtain wavelike solutions. However, if the stratification varies *slowly* in the vertical direction, meaning that its variations occur on a larger space scale than the vertical wavelength, while remaining constant in the horizontal direction, then we expect the solution to look locally like plane waves and we can obtain approximate solutions. This is the territory of the WKB approximation, as described in Appendix A of Chapter 6, and we will employ this technology.

7.5.1 Obtaining a WKB Solution

No assumptions are made about the uniformity of N when deriving (7.54), so the equation of motion is again

$$\left[\frac{\partial^2}{\partial t^2}\left(\nabla^2 + \frac{\partial^2}{\partial z^2}\right) + N^2\nabla^2\right] w' = 0, \tag{7.102}$$

where $N^2 = N^2(z)$. Let us seek solutions in the form

$$w' = \operatorname{Re} W(z) \mathrm{e}^{i(kx+ly-\omega t)}, \tag{7.103}$$

whence we obtain

$$\frac{\mathrm{d}^2 W}{\mathrm{d}z^2} + m^2(z) W = 0, \tag{7.104}$$

where

$$m^2 \equiv \frac{(N^2 - \omega^2)K^2}{\omega^2} . \tag{7.105}$$

This is closely related to the dispersion relation for the gravity waves, with m being the vertical wavenumber. The WKB solution to (7.104) is

$$W(z) = Cm^{-1/2} \exp\left(\pm i \int^z m \, dz'\right), \tag{7.106}$$

where C is a constant. The phase of the wave, $\theta(z)$, is given by

$$\theta(z) = \int^z m \, dz' = \int^z \pm K \left(\frac{N^2 - \omega^2}{\omega^2}\right)^{1/2} dz'. \tag{7.107}$$

Since $m = d\theta/dz$, locally the flow behaves like a plane wave with vertical wavenumber m and with amplitude varying as $m^{-1/2}$. Re-arranging (7.105) we obtain the dispersion relation in a familiar form,

$$\omega^2 = \frac{N^2 K^2}{K^2 + m^2} = N^2 \cos^2 \vartheta(z), \tag{7.108}$$

where $\cos^2 \vartheta = K^2/(K^2 + m^2)$. Given this, we can interpret (7.105) as giving the vertical wavenumber in a medium in which the stratification is varying and the frequency and horizontal wavenumber are known. We see that N, ϑ and m are functions of z, but ω is not, because the medium is time independent (cf., the discussion in Section 6.3).

7.5.2 Properties of the Solution

The WKB solution above is *almost* that of a plane wave with slowly varying wavenumber. Thus, it seems that the solution (7.106) might be further approximated as

$$w \approx Cm^{-1/2} \exp\left(\pm i m(z) z\right), \tag{7.109}$$

where $m(z)$ is given by (7.105). The accuracy of this solution increases as the variation of m diminishes, and in many circumstances (7.109) may be used to infer the qualitative behaviour of a wave. Nonetheless, it is an integral that appears in the phase in the solution (7.106), so the solution is not truly local.

From (7.106) the amplitude varies with height as $m^{-1/2}$, so that if the stratification (N^2) increases m will increase and the amplitude will decrease. Here we have derived this result directly by solving the wave equations of motion, but the result is a consequence of the conservation of energy in internal waves: energy here is a 'wave activity' — namely a conserved quantity, quadratic in the wave amplitude — in this problem. As discussed in Section 7.3.5, the vertical component of the energy flux, F^z, is $c_g^z \overline{E}$, where $\overline{E}$ is the energy density and c_g^z is the vertical component of the group velocity, and for a wave propagating vertically this energy flux must be constant. Now, manipulating (7.74c) and using (7.88) we have

$$c_g^z = -\frac{\omega m}{K_3^2}, \qquad \overline{E} = \left(\frac{W}{\cos \vartheta}\right)^2, \tag{7.110a,b}$$

so that

$$F^z = c_g^z \overline{E} = -\frac{W^2 \omega m}{K^2} = \text{constant}. \tag{7.111}$$

Thus, because the horizontal wavenumber K is preserved (since there are no inhomogeneities in the horizontal) and the frequency is constant (because the medium itself is not time varying), we must have $W \propto m^{-1/2}$, as in (7.106).

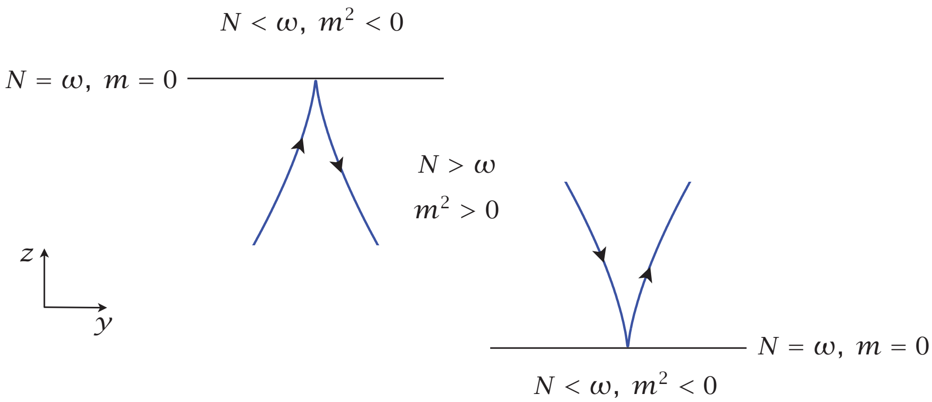

Fig. 7.10 Trajectories of internal waves approaching a turning height where $N = \omega$. The trajectory makes a cusp, as given by (7.115). If a region of high stratification is sandwiched between two regions of lower stratification then the waves may be vertically confined to a waveguide.

7.5.3 Wave Trajectories and an Idealized Example

Rays

As we discussed in Section 6.3, a wave packet will follow a *ray*, where a ray is a trajectory following the group velocity. Restricting attention to two dimensions and using (7.74) the horizontal and vertical components of the group velocity are (for $l > 0$),

$$c_g^y = \frac{Nm^2}{(l^2 + m^2)^{3/2}}, \qquad c_g^z = \frac{-Nlm}{(l^2 + m^2)^{3/2}}. \tag{7.112a,b}$$

The path of a ray may thus be parameterized by the expression

$$\frac{\mathrm{d}z}{\mathrm{d}y} = \frac{c_g^z}{c_g^y} = -\frac{l}{m} = \frac{-\omega}{\sqrt{N^2 - \omega^2}}, \tag{7.113}$$

where the rightmost expression follows from the dispersion relation (7.56) with $k = 0$. The above expressions hold even when N varies in the vertical. Now, for there to be vertical propagation the vertical wavenumber must be positive and the wave frequency must be less than N. Suppose a wave is generated in a strongly stratified region and propagates vertically to a more weakly stratified region (with smaller N). The vertical wavenumber m becomes smaller and smaller, both the vertical and horizontal components of the group velocity tend to zero and the wave packet will stall. However, c_g^y goes to zero faster than c_g^z and the ray path turns toward the region of lower stratification.

This behaviour may be interpreted in terms of the dispersion relation $\omega = N\cos\vartheta$, where $\vartheta = \cos^{-1}[l^2/(l^2 + m^2)]$ is the angle between the three-dimensional wavevector and the horizontal (see Section 7.3.2). If N decreases as we move vertically then ϑ must decrease until we reach the maximum value of $\cos\vartheta = 1$ and the wave vector is purely horizontal. The group velocity is perpendicular to the wave vector and so is then purely vertical. The wave cannot propagate into the region in which $N^2 < \omega^2$ for then m is imaginary and the disturbance will decay. Rather, the wave will tend to reflect, and the region where $N = \omega$ is often called a turning level. The trajectory can be obtained analytically in the region of the turning level as follows. Suppose that $N = \omega$ at $z = z^*$ so that, expanding N^2 around that point, we have $N^2(z) \approx N^2(z^*) + (z - z^*)\mathrm{d}N^2(z^*)/\mathrm{d}z$. Equation (7.113) becomes

$$\frac{\mathrm{d}z}{\mathrm{d}y} = \frac{-\omega}{\sqrt{(z - z^*)\mathrm{d}N^2/\mathrm{d}z}}, \tag{7.114}$$

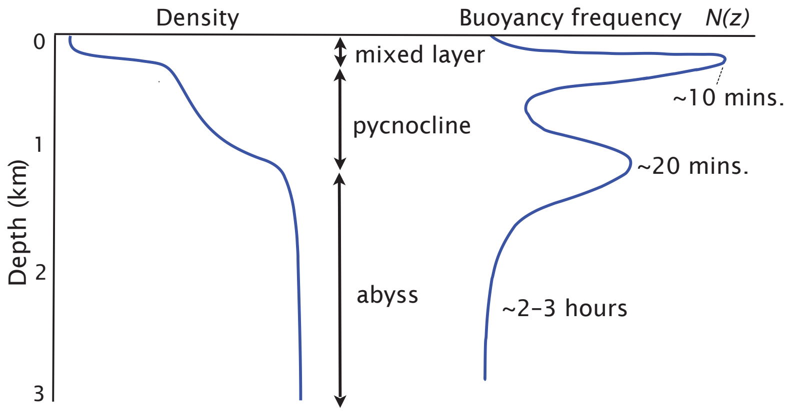

Fig. 7.11 Sketch of an ocean density profile, left, and buoyancy frequency, right, labelled with the approximate period. The pycnocline is sandwiched between two weakly stratified regions. The double peak in the buoyancy frequency is exaggerated, but the pycnocline is generally the region of highest frequency internal waves.

which, upon integrating, yields

$$z - z^* = \frac{\omega(y^* - y)^{2/3}}{\sqrt{\mathrm{d}N^2/\mathrm{d}z}}. \tag{7.115}$$

This cusp-like trajectory is illustrated in Fig. 7.10.

An idealized oceanic waveguide

The stratification of the ocean is decidedly nonuniform in the vertical, as schematically illustrated in Fig. 7.11. The density is almost uniform in a layer at the top of the ocean about 50–100 m deep known as the mixed layer. The density then increases fairly rapidly over a region 500–1000 m deep known as the pycnocline, and is then fairly uniform in the abyss. The weak stratification in the abyss and in the mixed layer will inhibit the propagation of internal waves generated in the thermocline. For example, consider a wave of frequency ω propagating downwards from the oceanic thermocline with and into the weakly stratified abyss. As soon as $N(z) < \omega$ the vertical wavenumber becomes imaginary and the disturbance will vary like $\mathrm{e}^{\pm mz}$. On physical grounds we must choose the solution that evanesces with depth. Similar behaviour will occur for a wave propagating up from the thermocline into the weakly stratified mixed layer. Thus, waves are trapped in a region where $N^2 > \omega^2$, and this region forms a *wave guide*, as sketched in Fig. 7.12. Similar dynamics are described again in an atmospheric context below.

The profile of N^2 is a simple exponential and the corresponding value of m^2 is calculated using (7.105) with $K = \omega = 1$ (the values are nondimensional). The value of m goes to zero near the top and the bottom of the domain, as illustrated. The corresponding group velocities are illustrated in Fig. 7.13, and can be seen to be purely vertical at the two turning heights. The amplitude of a wave becomes very large near the turning heights, but the wave itself need not break because its energy is constant and its vertical wavelength is very large. Rather, the wave will be reflected (following the trajectory illustrated in Fig. 7.10), and the wave is confined in the waveguide.

7.5.4 Atmospheric Considerations

The atmosphere differs from the ocean in many ways, but for the purposes of internal waves two of these are particularly important: (i) the density diminishes in the vertical and so the Boussinesq approximation is not valid, except for small vertical displacements; (ii) there is no upper surface, so we must consider radiation conditions for large z, or require that the solutions remain bounded for $z \to \infty$, rather than conventional boundary conditions.

There are two commonly-used ways to deal with density variations — through the use of pressure coordinates or the anelastic equations. We will use pressure coordinates in Chapter 17, but

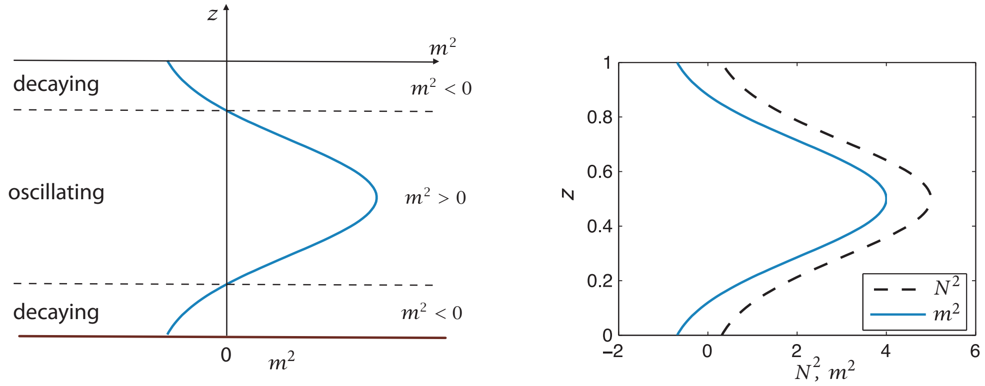

Fig. 7.12 An oceanic wave guide. The left panel shows m^2 and the regions of oscillation and decay, and the right panel also shows the value of N^2 from which m^2 is calculated using (7.105). Waves generated in the central region will propagate before evanescing in the region of negative m^2, so confining the waves to the central wave guide.

here we briefly consider the anelastic equations. These (see Section 2.5) differ from the Boussinesq primarily in the mass continuity equation, which becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\rho_0}\frac{\partial}{\partial z}(w\rho_0) = 0, \tag{7.116}$$

where $\rho_0 = \rho_0(z)$ is a specified profile of density. Using (7.116) instead of (7.53c) gives the equation of motion

$$\frac{\partial^2}{\partial t^2}\left(\nabla^2 w' + \frac{\partial}{\partial z}\frac{1}{\rho_0}\frac{\partial \rho_0 w'}{\partial z}\right) + N^2\nabla^2 w' = 0, \tag{7.117}$$

in place of (7.54). Because ρ_0 is a function of z we cannot find plane wave solutions without additional approximation — for example unless we assume that ρ_0 changes only slowly with z. For this reason the Boussinesq approximation is often imposed from the outset in theoretical work, even for the atmosphere; the approximation is quantitatively poor, but the qualitative character of the waves is captured.

The second factor (the lack of an upper surface) becomes an issue when considering gravity waves propagating high into the atmosphere, a phenomenon we look at in Chapter 17 and in Section 7.7, where we consider the generation of internal waves by flow over topography. To finish this section off, let us consider an atmospheric waveguide. The dynamics are very similar to those of the oceanic waveguide discussed above but, for the sake of variety, we will treat it in a slightly different way.

An atmospheric waveguide

We suppose the atmosphere to be a semi-infinite region from the ground at $z = 0$ to infinity. If N^2 is constant then solutions, as in the bounded case, vary sinusoidally in z, for example $w' \sim \sin mz$, where m is the vertical wavenumber. These solutions remain bounded as $z \to \infty$, although they do not decay. If N varies, then other possibilities exist. Suppose that a region of small stratification, N_1 overlies a region of larger stratification, N_2; that is

$$N = \begin{cases} N_1 & z > H, \\ N_2 & 0 < z < H, \end{cases} \tag{7.118}$$

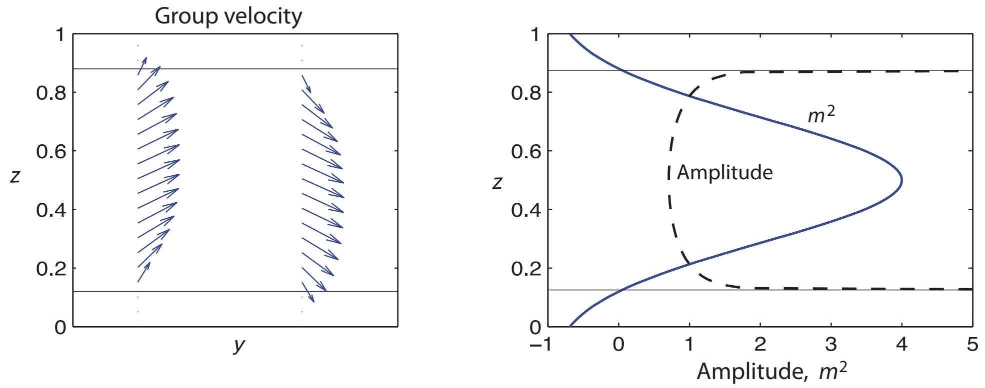

Fig. 7.13 Left panel: Group velocity vectors for upward and downward propagating gravity waves in a stratification illustrated in Fig. 7.12, calculated using (7.112). Right panel: The values of m^2 and the amplitude of the wave, the latter varying as $m^{-1/2}$. The thin horizontal lines in both panels indicate the height at which $m^2 = 0$.

where $N_2 > N_1$. (This is *not* a model of the stratosphere overlying the troposphere, because the stratosphere is highly stratified. If anything, it is a model of the mesosphere overlying the stratosphere and troposphere.) The frequency in the two regions must be the same and if $\omega < N_1 < N_2$ then

$$\omega^2 = \frac{N_1^2}{K^2 + m_1^2} = \frac{N_2^2}{K^2 + m_2^2}, \tag{7.119}$$

whence

$$m_1 = m_2 \left(\frac{N_1^2 - \omega^2}{N_2^2 - \omega^2} \right)^{1/2}. \tag{7.120}$$

In contrast, if $N_1 < \omega < N_2$ then wave-like solutions are not allowed in the upper region, because the frequency must always be less than the local value of N. Rather, solutions in the upper region evanesce according to

$$w_1' = \widetilde{w}_1 \mathrm{e}^{-\mu z} \mathrm{e}^{\mathrm{i}(kx+ly-\omega t)}, \tag{7.121}$$

where

$$\mu^2 = \frac{\omega^2 - N_1^2}{\omega^2} K^2. \tag{7.122}$$

The solutions still vary sinusoidally in the lower layer, according to

$$w_2' = \widetilde{w}_2 \sin m_1 z \mathrm{e}^{\mathrm{i}(kx+ly-\omega t)}, \tag{7.123}$$

where m now takes on only discrete values in order to satisfy the boundary conditions that w and ϕ are continuous $z = H$, and that w vanishes at $z = 0$.

7.6 INTERNAL WAVES IN A ROTATING FRAME OF REFERENCE

In the presence of both a Coriolis force and stratification a displaced fluid will feel two restoring forces — one due to gravity and the other to rotation. The first gives rise to gravity waves, as we have discussed, and the second to inertial waves. When the two forces both occur the resulting waves are called *inertia-gravity waves.* The algebra describing them can be complicated so we begin with a simple parcel argument to lay bare the basic dynamics; refer back to Section 7.3.3 as needed.

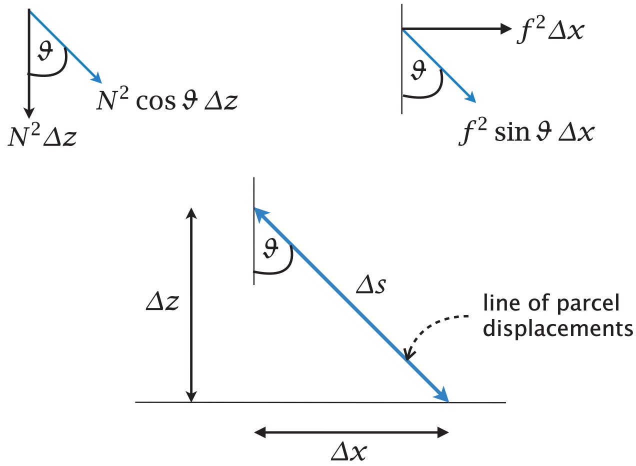

Fig. 7.14 Parcel displacements and associated forces in an inertia-gravity wave in which the parcel displacements are occurring at an angle ϑ to the vertical. Both Coriolis and buoyancy forces are present, and $\Delta s = \Delta z/\cos\vartheta = \Delta x/\sin\vartheta$.

7.6.1 A Parcel Argument

Consider a parcel that is displaced along a slantwise path in the x–z plane, as shown in Fig. 7.14, with a horizontal displacement of Δx and a vertical displacement of Δz. Let us suppose that the fluid is Boussinesq and that there is a stable and uniform stratification given by $N^2 = -g\rho_0^{-1}\partial\rho_0/\partial z = \partial b/\partial z$. Referring to (7.67) as needed, the component of the restoring buoyancy force, F_b say, in the direction of the parcel oscillation is given by (7.67),

$$F_b = -N^2 \cos\vartheta\,\Delta z = -N^2 \cos^2\vartheta\,\Delta s. \tag{7.124}$$

The parcel will also experience a restoring Coriolis force, F_C, and the component of this in the direction of the parcel displacement is

$$F_C = -f^2 \sin\vartheta\,\Delta x = -f^2 \sin^2\vartheta\,\Delta s. \tag{7.125}$$

Here, and for the rest of the chapter, we denote the Coriolis parameter by f. It should be regarded as a constant in any given problem (so there are no Rossby waves), but its value varies with latitude. Using (7.124) and (7.125) the (Lagrangian) equation of motion for a displaced parcel is

$$\frac{\mathrm{d}^2\Delta s}{\mathrm{d}t^2} = -(N^2\cos^2\vartheta + f^2\sin^2\vartheta)\Delta s, \tag{7.126}$$

and hence the frequency is given by

$$\omega^2 = N^2\cos^2\vartheta + f^2\sin^2\vartheta. \tag{7.127}$$

Now, nearly everywhere in both atmosphere and ocean, $N^2 > f^2$. From (7.127) we then see that the frequency lies in the interval $N^2 > \omega^2 > f^2$. (To see this, put $N = f$ or $f = N$ in (7.127), and use $\sin^2\vartheta + \cos^2\vartheta = 1$.) If the parcel displacements approach the vertical then the Coriolis force diminishes and $\omega \to N$, and similarly $\omega \to f$ as the displacements become horizontal. The ensuing waves are then pure inertial waves.

We can write (7.127) in terms of wavenumbers since, for motion in the x-z plane,

$$\cos^2\vartheta = \frac{k^2}{k^2+m^2}, \qquad \sin^2\vartheta = \frac{m^2}{k^2+m^2}, \tag{7.128}$$

where k and m are the horizontal and vertical wavenumbers and $l = 0$. The dispersion relation becomes

$$\boxed{\omega^2 = \frac{N^2k^2 + f^2m^2}{k^2+m^2}.} \tag{7.129}$$

Let's now move on to a discussion using the linearized equations of motion.

7.6.2 Equations of Motion

In a rotating frame of reference, specifically on an f-plane, the linearized equations of motion are the momentum equations

$$\frac{\partial \boldsymbol{u}'}{\partial t} + \boldsymbol{f}_0 \times \boldsymbol{u}' = -\nabla\phi', \qquad \frac{\partial w'}{\partial t} = -\frac{\partial \phi'}{\partial z} + b', \tag{7.130a,b}$$

and the mass continuity and thermodynamic equations,

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0, \qquad \frac{\partial b'}{\partial t} + w' N^2 = 0. \tag{7.130c,d}$$

These are similar to (7.53), with the addition of a Coriolis term in the horizontal momentum equations.

To obtain a single equation for w' we take the horizontal divergence of (7.130a) and use the continuity equation to give

$$\frac{\partial}{\partial t}\left(\frac{\partial w'}{\partial z}\right) + f\zeta' = \nabla^2\phi', \tag{7.131}$$

where $\zeta' \equiv (\partial v'/\partial x - \partial u'/\partial y)$ is the vertical component of the vorticity. We may obtain an evolution equation for that vorticity by taking the curl of (7.130a), giving

$$\frac{\partial \zeta'}{\partial t} = f\frac{\partial w'}{\partial z}. \tag{7.132}$$

Eliminating vorticity between these equations gives

$$\left(\frac{\partial^2}{\partial t^2} + f^2\right)\frac{\partial w'}{\partial z} = \frac{\partial}{\partial t}\nabla^2\phi'. \tag{7.133}$$

We may obtain another equation linking pressure and vertical velocity by eliminating the buoyancy between (7.130b) and (7.130d), so giving

$$\frac{\partial^2 w'}{\partial t^2} + N^2 w' = -\frac{\partial}{\partial t}\frac{\partial \phi'}{\partial z}. \tag{7.134}$$

Eliminating ϕ' between (7.133) and (7.134) gives a single equation for w' analogous to (7.54), namely

$$\boxed{\left[\frac{\partial^2}{\partial t^2}\left(\nabla^2 + \frac{\partial^2}{\partial z^2}\right) + f^2\frac{\partial^2}{\partial z^2} + N^2\nabla^2\right] w' = 0.} \tag{7.135}$$

If we assume a time dependence of the form $w' = \widehat{w}\mathrm{e}^{-\mathrm{i}\omega t}$, this equation may be written in the sometimes useful form,

$$\frac{\partial^2 \widehat{w}}{\partial z^2} = \left(\frac{N^2 - \omega^2}{\omega^2 - f^2}\right)\nabla^2\widehat{w}. \tag{7.136}$$

7.6.3 Dispersion Relation

Assuming wave solutions to (7.135) of the form $w' = \tilde{w} \exp[\mathrm{i}(kx + ly + mz - \omega t)]$ we readily obtain the dispersion relation

$$\omega^2 = \frac{f^2 m^2 + (k^2 + l^2) N^2}{k^2 + l^2 + m^2}, \tag{7.137}$$

which is a minor generalization of (7.129). We can also write the dispersion relation as

$$\omega^2 = f^2 \sin^2 \vartheta + N^2 \cos^2 \vartheta, \tag{7.138}$$

or

$$\omega^2 = f^2 + (N^2 - f^2) \cos^2 \vartheta, \qquad \text{or} \qquad \omega^2 = N^2 - (N^2 - f^2) \sin^2 \vartheta, \tag{7.139}$$

where ϑ is the angle of the wavevector with the horizontal. The frequency therefore lies between N and f. The waves satisfying (7.137) are called inertia-gravity waves and are analogous to surface gravity waves in a rotating frame — that is, Poincaré waves — discussed in Section 3.8.2.

In many atmospheric and oceanic situations $f \ll N$ (in fact typically $N/f \sim 100$, the main exception being weakly stratified near-surface mixed layers and the deep abyss in the ocean) and $f < \omega < N$. From (7.138) the frequency is dependent only on the angle the wavevector makes with the horizontal, and the surfaces of constant frequency again form cones in wavenumber space, although depending on the values of f and ω the frequency does not necessarily decrease monotonically with ϑ as in the non-rotating case. For reference, the group velocity is

$$c_g^x = \left[\frac{N^2 - f^2}{\omega K_3^4} Km \right] \frac{km}{K} = \left[\frac{N^2 - f^2}{\omega K_3} \cos \vartheta \sin \vartheta \right] \cos \lambda \sin \vartheta, \tag{7.140a}$$

$$c_g^y = \left[\frac{N^2 - f^2}{\omega K_3^4} Km \right] \frac{lm}{K} = \left[\frac{N^2 - f^2}{\omega K_3} \cos \vartheta \sin \vartheta \right] \sin \lambda \sin \vartheta, \tag{7.140b}$$

$$c_g^z = -\left[\frac{N^2 - f^2}{\omega K_3^4} Km \right] K = -\left[\frac{N^2 - f^2}{\omega K_3} \cos \vartheta \sin \vartheta \right] \cos \vartheta, \tag{7.140c}$$

using (7.69), and where $K_3^4 \equiv (k^2 + l^2 + m^2)^2$. These expressions reduce to (7.74) if $f = 0$, in which case $\omega = N \cos \vartheta$. Notice that the directional factors — the sin and cos terms outside of the square brackets — are the same as those in (7.74). Thus, the group velocity is, as in the non-rotating case, at an angle ϑ to the vertical, or $\alpha = \pi/2 - \vartheta$ to the horizontal. The magnitude of the group velocity is now given by

$$|\boldsymbol{c}_g| = \frac{N^2 - f^2}{\omega K_3^3} Km = \frac{N^2 - f^2}{\omega K_3} \cos \vartheta \sin \vartheta. \tag{7.141}$$

There are a few notable limits:

1. A purely horizontal wave vector. In this case $m = 0$ and $\omega = N$. The waves are then unaffected by the Earth's rotation. This is because the Coriolis force is (in the f-plane approximation) due to the product of the Coriolis parameter and the horizontal component of the velocity. If the wave vector is horizontal, the fluid velocities are purely vertical and so the Coriolis force vanishes.

2. A purely vertical wave vector. In this case $\omega = f$, the fluid velocities are horizontal and the fluid parcels do not feel the stratification. The oscillations are then known as *inertial waves*, although they are not inertial in the sense of there being no implied force in an inertial frame of reference. This case and the previous one show that when $\omega = N$ or $\omega = f$ the group velocity is zero.

3. In the limit $N \to 0$ we have pure inertial waves with a frequency $0 < \omega < f$, and specifically $\omega = f \sin\vartheta$. Similarly, as $f \to 0$ we have pure internal waves, as discussed previously, with $\omega = N\cos\vartheta$.
4. The hydrostatic limit, which we discuss below.

The hydrostatic limit

Hydrostasy occurs in the limit of large horizontal scales, $k, l \ll m$. If we therefore neglect k^2 and l^2 where they appear with m^2 in (7.137) we obtain

$$\omega^2 = f^2 + N^2\frac{k^2 + l^2}{m^2} = f^2 + N^2\cos^2\vartheta, \tag{7.142}$$

where the rightmost expression arises from (7.138) if we take

$$\sin^2\vartheta = \frac{m^2}{k^2 + l^2 + m^2} \to 1, \qquad \cos^2\vartheta = \frac{K^2}{k^2 + m^2} \to \frac{K^2}{m^2} \ll 1, \tag{7.143}$$

with $K^2 = k^2 + l^2$.

If we make the hydrostatic approximation from the outset in the rotating, linearized, equations of motion then we have

$$\frac{\partial u'}{\partial t} - fv = -\frac{\partial \phi'}{\partial x}, \qquad \frac{\partial v'}{\partial t} + fu = -\frac{\partial \phi'}{\partial y}, \qquad 0 = -\frac{\partial \phi'}{\partial z} + b', \tag{7.144a}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0, \qquad \frac{\partial b'}{\partial t} + w'N^2 = 0. \tag{7.144b}$$

This reduces to the single equation

$$\left[\frac{\partial^2}{\partial t^2}\frac{\partial^2}{\partial z^2} + f^2\frac{\partial^2}{\partial z^2} + N^2\nabla^2\right] w' = 0, \tag{7.145}$$

and corresponding dispersion relation

$$\omega^2 = \frac{f^2 m^2 + K^2 N^2}{m^2} = f^2 + N^2\frac{K^2}{m^2}, \tag{7.146}$$

so recovering (7.142). This limit is sometimes known as the *rapidly rotating regime*. The Coriolis parameter f now appears in isolation, and simply provides inertial oscillations that are independent of the wavenumber and the stratification.

Another way to think about the small aspect ratio limit follows if we define

$$\alpha' \equiv \frac{\text{vertical scale}}{\text{horizontal scale}} = \frac{K}{m} = \frac{1}{\tan\vartheta} \ll 1. \tag{7.147}$$

From the nonhydrostatic dispersion relation, (7.137), a line or two of algebra gives

$$\alpha'^2 = \left(\frac{\omega^2 - f^2}{N^2 - \omega^2}\right). \tag{7.148}$$

The hydrostatic limit requires that this aspect ratio is small, and therefore that $N^2 \gg \omega^2$. Put differently, low frequencies will tend to have a small aspect ratio and be hydrostatic. Using (7.140) and (7.139) we find that the ratio of the vertical to the horizontal group velocities scales as the aspect ratio; that is

$$\frac{c_g^z}{c_g^h} = \alpha' = \left(\frac{\omega^2 - f^2}{N^2 - \omega^2}\right)^{1/2}. \tag{7.149}$$

where $c_g^h = (c_g^{x2} + c_g^{y2})^{1/2}$, and the above ratio is small in the hydrostatic limit. We return to this limit in Section 17.2 on gravity waves in the stratosphere.

7.6.4 Polarization Relations

Just as in the non-rotating case, we can derive phase relations between the various fields, useful if we are trying to identify internal waves from observations. As for all waves in an incompressible fluid, the condition $\nabla_3 \cdot \boldsymbol{v} = 0$ gives

$$\boldsymbol{k} \cdot \boldsymbol{v}' = 0, \tag{7.150}$$

so that the fluid motion is in the plane that is perpendicular to the wave vector. The derivations of the other polarization relations are left as exercises for the reader, and the relations are found to be

$$\tilde{u} = \frac{k\omega + \mathrm{i}lf}{\omega^2 - f^2}\tilde{\phi}, \qquad \tilde{v} = \frac{l\omega - \mathrm{i}kf}{\omega^2 - f^2}\tilde{\phi}, \tag{7.151a,b}$$

which should be compared with (7.63a). We also have a relation between buoyancy and pressure,

$$\tilde{b} = \frac{\mathrm{i}mN^2}{N^2 - \omega^2}\tilde{\phi}, \tag{7.152}$$

and one between vertical velocity and pressure,

$$\tilde{w} = \frac{-m\omega}{N^2 - \omega^2}\tilde{\phi} = \frac{-\omega K_3^2}{(N^2 - f^2)m}\tilde{\phi}, \tag{7.153}$$

with the second equality following with use of the dispersion relation.

7.6.5 Geostrophic Motion and Vortical Modes

If we seek *steady* solutions to (7.130), the equations of motion become

$$-fv = -\frac{\partial \phi'}{\partial x}, \qquad fu = -\frac{\partial \phi'}{\partial y}, \qquad 0 = -\frac{\partial \phi'}{\partial z} + b', \tag{7.154a,b}$$

and

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0, \qquad w'N^2 = 0. \tag{7.155a,b}$$

These are the equations of geostrophic and hydrostatic balance, with zero vertical velocity. What can we say about this solution? If instead of eliminating pressure between (7.133) and (7.134) we eliminate vertical velocity we obtain

$$\frac{\partial}{\partial t}\left[\frac{\partial^2}{\partial t^2}\left(\nabla^2 + \frac{\partial^2}{\partial z^2}\right) + f^2\frac{\partial^2}{\partial z^2} + N^2\nabla^2\right]\phi' = 0, \tag{7.156}$$

which is similar to (7.135), except for the extra time derivative, which allows for the possibility of a solution with $\omega = 0$. If $\omega \neq 0$ then

$$\left[\frac{\partial^2}{\partial t^2}\left(\nabla^2 + \frac{\partial^2}{\partial z^2}\right) + f^2\frac{\partial^2}{\partial z^2} + N^2\nabla^2\right]\phi' = 0, \tag{7.157}$$

and the dispersion relation is given by (7.137). If $\omega = 0$, then the quantity in square brackets in (7.156) may not be a function of time; that is

$$\left[\frac{\partial^2}{\partial t^2}\left(\nabla^2 + \frac{\partial^2}{\partial z^2}\right) + f^2\frac{\partial^2}{\partial z^2} + N^2\nabla^2\right]\phi' = \chi(x, y, z), \tag{7.158}$$

where χ is a function of space, but not time, and so determined by the initial conditions of ϕ'. When $\omega \neq 0$, then $\chi = 0$. What is χ? We shall see that it is nothing but the potential vorticity of the flow!

Potential vorticity

Recall the vorticity equation and the buoyancy equation, namely

$$\frac{\partial \zeta'}{\partial t} = f\frac{\partial w'}{\partial z}, \qquad \frac{\partial b'}{\partial t} + w'N^2 = 0. \tag{7.159a,b}$$

If we eliminate w' from these equations we obtain

$$\frac{\partial q}{\partial t} = 0, \qquad \text{where} \qquad q = \left[\zeta' + f\frac{\partial}{\partial z}\left(\frac{b'}{N^2}\right)\right] \tag{7.160a,b}$$

and q is the potential vorticity for this problem. In general, for adiabatic flow, potential vorticity is conserved on fluid parcels and $\mathrm{D}Q/\mathrm{D}t = 0$ where for a Boussinesq fluid $Q = \omega_a \cdot \nabla b$. There are two differences between this general case and ours; first, because we have linearized the dynamics the advective term is omitted, and $\partial q/\partial t = 0$. Second, q is not exactly the same as Q, but it is an approximation to it valid when the stratification is dominated by its background value, N^2. Very informally, we have then, for constant N,

$$Q = (\boldsymbol{\omega} + \boldsymbol{f}_0)\cdot\nabla b \approx (\zeta + f)\left(N^2 + \frac{\partial b'}{\partial z}\right) \approx fN^2 + f\frac{\partial b'}{\partial z} + \zeta N^2 = N^2\left[f + \zeta + f\frac{\partial}{\partial z}\left(\frac{b'}{N^2}\right)\right]. \tag{7.161}$$

The first term on the right-hand side of this expression, fN^2, is a constant and so dynamically unimportant, and the remaining terms are equal to q as given by (7.160b).

Another way to see that (7.160b) is the potential vorticity is to note that the displacement of an isentropic surface, η say, is related to the change in buoyancy by

$$\eta \approx -\frac{b'}{\partial\bar{b}/\partial z} = -\frac{b'}{N^2}, \tag{7.162}$$

as illustrated in Fig. 3.13 on page 138. The thickness of an isentropic layer is the difference between the heights of two neighbouring isentropic surfaces, and so is given by

$$h = -\frac{b_1'}{N^2} + \frac{b_2'}{N^2} \approx -H\frac{\partial}{\partial z}\left(\frac{b'}{N^2}\right), \tag{7.163}$$

where H is the mean separation between the surfaces, or the mean thickness. Thus, the expression (7.160b) may be written

$$q = \left[\zeta' - \frac{fh}{H}\right], \tag{7.164}$$

which is the shallow water expression for the potential vorticity of a fluid layer, linearized about a mean thickness H and a state of rest (with $|\zeta'| \ll f$).

Let us now relate q to χ, and we do this by expressing ζ' and b' in terms of ϕ' and w'. From (7.131) and (7.130b) respectively we have,

$$f\zeta' = \nabla^2\phi' - w_{zt}', \tag{7.165a}$$

$$\frac{f^2}{N^2}b_z' = \frac{f^2}{N^2}w_{zt} + \frac{f^2}{N^2}\phi_{zz}', \tag{7.165b}$$

using subscripts to denote derivatives. Thus, f times the potential vorticity is

$$fq = \nabla^2\phi' + \frac{f^2}{N^2}\phi_{zz}' + \frac{f^2}{N^2}w_{zt} - w_{zt}. \tag{7.166}$$

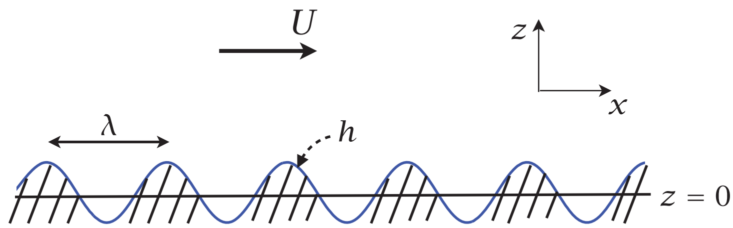

Fig. 7.15 Uniform flow, U, in the x-direction flowing over sinusoidal topography, h. The vertical co-ordinate is stretched, and in reality $|h| \ll \lambda$.

We now use (7.134) to express the second w_{zt} term in terms of ϕ', giving

$$fq = \nabla^2\phi' + \frac{f^2}{N^2}\phi'_{zz} + \frac{f^2}{N^2}w'_{zt} + \frac{1}{N^2}\left(w'_{zttt} + \phi'_{zztt}\right), \tag{7.167}$$

and we then use (7.133) to eliminate w', giving

$$fq = \nabla^2\phi' + \frac{f^2}{N^2}\phi'_{zz} + \frac{1}{N^2}\left(\nabla^2\phi'_{tt} + \phi'_{zztt}\right), \tag{7.168}$$

or, re-arranging,

$$fq = \frac{1}{N^2}\left[\frac{\partial^2}{\partial t^2}\left(\nabla^2\phi' + \frac{\partial^2\phi'}{\partial z^2}\right) + N^2\nabla^2\phi' + f^2\frac{\partial^2\phi'}{\partial z^2}\right]. \tag{7.169}$$

Comparing this with (7.158), we can see that

$$\chi = fN^2q. \tag{7.170}$$

That is to say, the conserved quantity for motions with $\omega = 0$ is nothing but a constant multiple of the potential vorticity. When $\omega \neq 0$, then χ and hence the potential vorticity are zero. In other words, *oscillating linear gravity waves, even in a rotating reference frame, have zero potential vorticity.* This is an important result, because large-scale balanced dynamics is characterized by the advection of potential vorticity, so that (in the linear approximation at least) internal waves play no direct role in the potential vorticity budget. However, they *do* play an important role in transporting and dissipating energy, as we will see later on.

7.7 TOPOGRAPHIC GENERATION OF INTERNAL WAVES

How are internal waves generated? One way that is important in both the ocean and atmosphere is by way of a horizontal flow, such as a mean wind or, in the ocean, a tide or a mesoscale eddy, passing over a topographic feature. This forces the fluid to move up and/or down, so generating an internal wave, commonly known as a mountain wave. In this section we illustrate the mechanism with simple examples of steady, uniform, flow with constant stratification over idealized topography.[2]

7.7.1 Sinusoidal Mountain Waves

For simplicity we ignore the effects of the Earth's rotation and pose the problem in two dimensions, x and z, using the Boussinesq approximation. Our goal is to calculate the response to a steady, uniform flow of magnitude U over a sinusoidally varying boundary $h = \widetilde{h}\cos kx$ at $z = 0$, as in Fig. 7.15 with $k = 2\pi/\lambda$. The topographic variations are assumed small, so allowing the dynamics to be linearized, which will enable an arbitrarily shaped boundary to be considered by appropriately summing over Fourier modes.

Equation of motion and dispersion relation

The momentum equations, the buoyancy equation, and the mass continuity equation are, respectively,

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)u' = -\frac{\partial \phi}{\partial x}, \qquad \left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)w' = -\frac{\partial \phi}{\partial z}+b', \tag{7.171a,b}$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)u' + w'N^2 = 0, \qquad \frac{\partial u'}{\partial x}+\frac{\partial w'}{\partial z} = 0. \tag{7.171c,d}$$

We henceforth drop the primes on the perturbation quantities.

The dispersion relation is obtained by noting that the equations are the same is with no mean flow, save that $\partial/\partial t$ is replaced by $\partial/\partial t + U\partial/\partial x$. Thus, similar to (7.78), the dispersion relation is

$$(\omega - Uk)^2 = \frac{k^2N^2}{k^2+m^2} \qquad \text{or} \qquad \omega = Uk \pm \frac{k^2N^2}{(k^2+m^2)^{1/2}}\,. \tag{7.172a,b}$$

The horizontal and vertical components of the group velocity are then given by

$$c_g^z = \pm\frac{Nmk}{(k^2+m^2)^{3/2}}, \qquad c_g^x = U \pm \frac{Nm^2}{(k^2+m^2)^{3/2}}\,. \tag{7.173a,b}$$

Steady waves have $\omega = 0$ and if $U > 0$ such waves are associated with the negative root in (7.172b), the positive root in (7.173a) and the negative root in (7.173b). Energy will propagate upwards, and away from the mountain, if c_g^z is positive.

The solution

If we are looking for steady solutions we can follow the recipe of Section 7.5 with $U\partial/\partial x$ replacing the time derivative. By analogy to (7.54) we find a single equation for w namely

$$\left[U\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)+N^2\frac{\partial^2}{\partial x^2}\right]w = 0. \tag{7.174}$$

We take the topography to have the form $h = \operatorname{Re} h_0 \exp(\mathrm{i}kx)$ and its presence is felt via a lower boundary condition, $w = U\partial h/\partial x = \operatorname{Re} Uh_0 \mathrm{i}k \exp(\mathrm{i}kx)$ at $z = 0$. We thus seek solutions to (7.173) of the form

$$w = \operatorname{Re} Uh_0 \mathrm{i}k e^{\mathrm{i}(kx+mz)}. \tag{7.175}$$

The harmonic dependence in z is valid only because we take N to be constant. The value of m is given by the dispersion relation (7.172) with $\omega = 0$, which gives

$$\boxed{m^2 = \left(\frac{N}{U}\right)^2 - k^2.} \tag{7.176}$$

Equation (7.176) is just the dispersion relation for internal gravity waves, but here we are using it to determine the vertical wavenumber since the frequency is given (it is zero). We see that m^2 may be negative, and so m imaginary, if $N^2 < k^2U^2$, so evidently there will be a qualitative difference between short waves, with $k > (N/U)^2$, and long waves, with $k < (N/U)^2$. In the atmosphere we might take $U \sim 10\,\mathrm{m\,s^{-1}}$ and $N \sim 10^{-2}\,\mathrm{s^{-1}}$, in which case $U/N \sim 1\,\mathrm{km}$. In the ocean below the thermocline we might take $U \sim 0.1\,\mathrm{m\,s^{-1}}$ and $N \sim 10^{-3}\,\mathrm{s^{-1}}$, whence $U/N \sim 100\,\mathrm{m}$.

A useful interpretation of (7.176) arises if the mean flow is constant and we consider the problem in the frame of reference of that mean flow. In this case the topography has the form $h = \operatorname{Re} h_0 \exp \mathrm{i}k(x - Ut)$, and the solution in the moving frame is

$$w = \operatorname{Re} Uh_0 \mathrm{i}k e^{\mathrm{i};(kx+mz+Ukt)}. \tag{7.177}$$

That is, the fluid now oscillates with a frequency $\omega = -Uk$. The condition for propagation, $N^2 < k^2U^2$ is just the same as $\omega^2 < N^2$, which is just the condition for gravity waves to oscillate in a stratified fluid.

Given the solution for w we can use the polarization relations of section 7.3.2 to obtain the solutions for perturbation horizontal velocity and pressure. One way to proceed is to pose the problem in the moving frame, with $\omega = -Uk$, and directly use (7.64). Then back in the stationary frame we obtain the solutions

$$w = w_0 e^{i(kx+mz)} = iUkh_0 e^{imz} e^{ikx}, \tag{7.178a}$$
$$u = u_0 e^{i(kx+mz)} = -imUh_0 e^{imz} e^{ikx}, \tag{7.178b}$$
$$\phi = \phi_0 e^{i(kx+mz)} = imU^2 h_0 e^{imz} e^{ikx}, \tag{7.178c}$$

where m is given by (7.176). Let us see what the solutions mean, and if and how waves propagate.

7.7.2 Energy Propagation

The direction of energy propagation is given by the group velocity. For steady waves ($\omega = 0$) we have, using (7.173a),

$$c_g^z = \frac{Nkm}{(k^2+m^2)^{3/2}} = \frac{Ukm}{k^2+m^2}\,. \tag{7.179a,b}$$

This means that for positive U an upward group velocity, and hence upward energy propagation, occur when k and m have the same sign. This property also may be deduced by evaluating the vertical energy flux, $\overline{w\phi}$ using (7.178), with appropriate care to take the real part of the fields. The phase speed, $c_p^z = \omega/m$, is zero in the stationary frame and it is $-Uk/m$ in the translating frame.

Short, trapped waves

If the undulations on the boundary are sufficiently short then $k^2 > (N/U)^2$ and m^2 is negative and m is pure imaginary. Writing $m = is$, so that $s^2 = k^2 - (N/U)^2$, the solutions have the form

$$w = \mathrm{Re}\, w_0 e^{ikx-sz}. \tag{7.180}$$

We must choose the solution with $s > 0$ in order that the solution decays away from the mountain, and internal waves are not propagated into the interior. (If there were a rigid lid or a density discontinuity at the top of the fluid, as at the top of the ocean, then the possibility of reflection would arise and we would seek to satisfy the upper boundary condition with a combination of decaying and amplifying modes.) The above result is entirely consistent with the dispersion relation for internal waves, namely $\omega = N\cos\vartheta$: because $\cos\vartheta < 1$ the frequency ω must be less than N so that if the forcing frequency is higher than N no internal waves will be generated.

Because the waves are trapped waves we do not expect energy to propagate away from the mountains. To verify this, from the polarization relation (7.178) we have

$$w = \frac{k}{mU}\phi = \frac{-ik}{sU}\phi. \tag{7.181}$$

The pressure and the vertical velocity are therefore out of phase by $\pi/2$, and the vertical energy flux, $\overline{w\phi}$ (see (7.83) and Section 7.3.5) is identically zero. This is consistent with the fact that the energy flux is in the direction of the group velocity; the group velocity is given by (7.179a) and for an imaginary m the real part is zero. A solution to the problem in the short wave limit is shown in top panels of Fig. 7.16 with $s = 1$ ($m = i$).

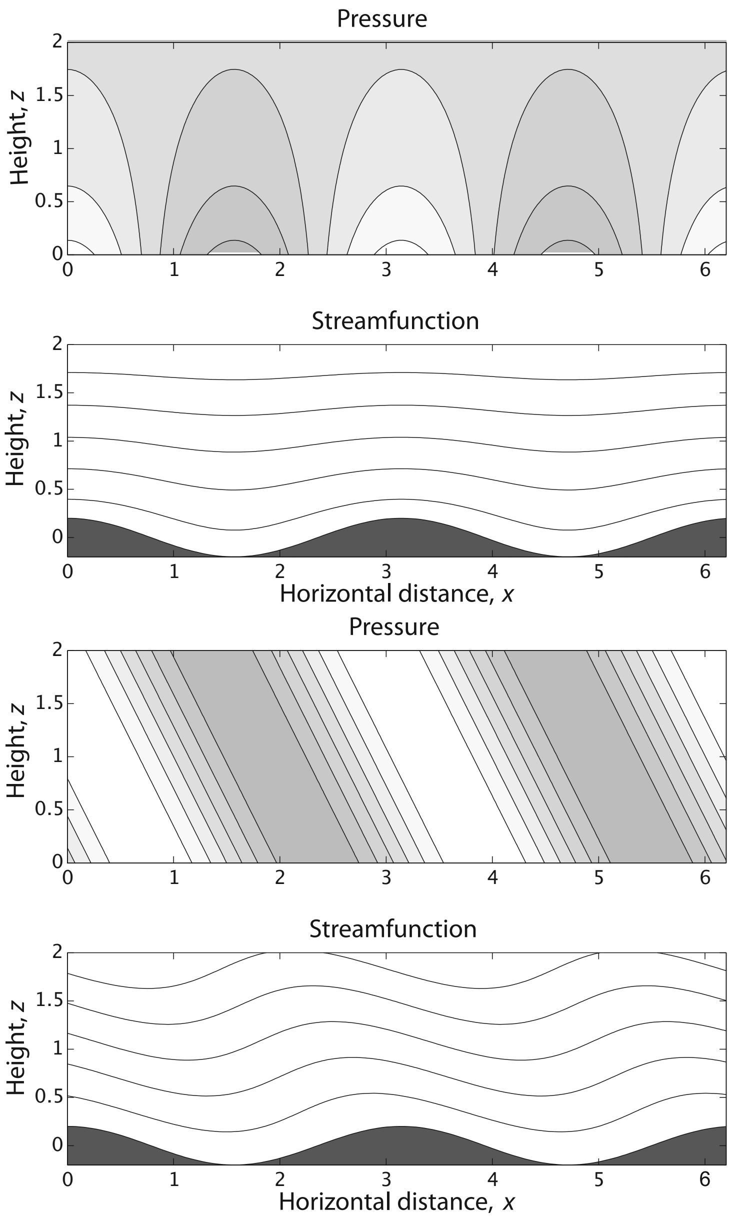

Fig. 7.16 Top two panels. Solutions for the flow over a sinusoidal ridge, using (7.178), in the short wave limit ($Uk > N$) and with $m = \mathrm{i}$. The top panel shows the pressure, with darker gray indicating higher pressure. The second panel shows contours of the total streamfunction, $\psi - Uz$, with flow coming in from the left, and the topography itself (solid). The perturbation amplitude decreases exponentially with height.

Bottom two panels. The same as above, but in the long wave limit ($Uk < N$) with $m = 1$. The pressure is high on the windward side of the topography, and phase lines tilt upstream with height for both pressure and streamfunction.

Long, propagating waves

Suppose now that $k^2 < (N/U)^2$ so that $\omega^2 < N^2$. From (7.176) m is now real and the solution has propagating waves of the form

$$w = w_0 \mathrm{e}^{\mathrm{i}(kx+mz)}, \qquad m^2 = \left(\frac{N}{U}\right)^2 - k^2. \tag{7.182}$$

Vertical propagation is occurring because the forcing frequency is less than the buoyancy frequency. The angle at which fluid parcel oscillations occur is then slanted off the vertical at an angle ϑ such

Topographically Generated Gravity Waves (Mountain Waves)

- In both atmosphere and ocean an important mechanism for the generation of gravity waves is flow over bottom topography, and the ensuing waves are sometimes called mountain waves. A canonical case is that of a uniform flow over a sinusoidal topography, with constant stratification. If the flow is in the x-direction and there is no y-variation then the boundary condition is

$$w(x, z = 0) = U\frac{\partial h}{\partial x} = -\mathrm{i}Uk\widetilde{h}. \tag{MW.1}$$

  Solutions of the problem may be found in the form $w(x, z, t) = w_0 \exp[i(kx + mz - \omega t)]$, where the boundary condition at $z = 0$ is given by (MW.1), the frequency is given by the internal wave dispersion relation, and the other dynamical fields are obtained using the polarization relations.

- One way to easily solve the problem is to transform into a frame moving with the background flow, U. The topography then appears to oscillate with a frequency $-Uk$, and this in turn becomes the frequency of the gravity waves.

- Propagating gravity waves can only be supported if the frequency is less than N, meaning that $Uk < N$. That is, the waves must be sufficiently long and therefore the topography must be of sufficiently large scale.

- When propagating waves exist, energy is propagated upward away from the topography. The topography also exerts a drag on the background flow.

- If the waves are too short they are evanescent, decaying exponentially with height. That is, they are trapped near the topography

- In the presence of rotation the wave frequency must lie between the buoyancy frequency N and the inertial frequency f. If the flow is constant, waves can thus radiate upward if

$$f < Uk < N. \tag{MW.2}$$

  Thus, both very long waves and very short waves are evanescent.

that the forcing frequency is equal to the natural frequency of oscillations at that angle, namely

$$\vartheta = \cos^{-1}\left(\frac{Uk}{N}\right). \tag{7.183}$$

The angle ϑ is also the angle between the wavevector $\boldsymbol{k}$ and the horizontal, as in (7.61), because the wavevector is at right angles to the parcel oscillations. If $Uk = N$ then the fluid parcel oscillations are vertical and, using (7.176), $m = 0$. Thus, although the group velocity is directed vertically, parallel to the fluid parcel oscillations, its magnitude is zero, from (7.179).

Our intuition suggests that if there is vertical propagation there must be an upwards energy flux, since the energy source is at the ground. Let's confirm this. Using the polarization relations (7.178a,c) we obtain

$$w_0 = \frac{k}{mU}\phi_0. \tag{7.184}$$

and the energy flux in the vertical direction is, from (7.89)

$$F^z = \frac{k}{2mU}|\phi_0|^2 = \frac{mU}{2k}|w_0|^2, \tag{7.185}$$

which is evidently non-zero. This energy flux must be upward, away from the source (the topography), and this determines the sign of m that must be chosen by the solution. Specifically, for positive U, the group velocity must be positive so from (7.179) m must be positive. If U were negative the sign of m would be negative, and if $m = 0$ there is no vertical energy propagation.

Because energy is propagating upward and away from the topography there must be a drag at the lower boundary. The stress at the boundary, τ, is the rate at which horizontal momentum is transported upwards and so is given by

$$\tau = -\rho_0 \overline{uw}, \tag{7.186}$$

where the overbar denotes averaging over a wavelength and ρ_0 is the density, which is constant. From (7.178)

$$u_0 = -\mathrm{i}mUh_0, \qquad w_0 = \mathrm{i}Ukh_0, \tag{7.187}$$

so that

$$\widetilde{\tau} \equiv \tau/\rho_0 = -\overline{uw} = \frac{1}{2}kmU^2h_0^2, \tag{7.188}$$

where the factor of 1/2 comes from the averaging, and we take the product $u_0 w_0^*$ where w_0^* is the complex conjugate of w_0. The sign of the stress depends on the sign of m, and thus on the sign of U. For positive U, m is positive and so the stress is positive at the surface.

Solutions for flow over topography in the long wave limit and $m = 1$ are shown in the lower panels Fig. 7.16. The flow is coming in from the left, and the phase lines evidently tilt upstream with height. Lines of constant phase follow $kx + mz =$ constant, and in the solution shown both k and m are positive ($k = m = 1$). Thus, the lines slope back at a slope $x/z = -m/k$, and energy propagates up and to the left. The phase propagation is actually downward in this example, as the reader may confirm. The pressure is high on the upstream side of the mountain, and this provides a drag on the flow — a topographic form drag.

7.7.3 Flow over an Isolated Ridge

Most mountains are of course not perfect sinusoids, but we can construct a solution for any given topography using a superposition of Fourier modes. In this section we will illustrate the solution for a mountain consisting of a single ridge; the actual solution must usually be obtained numerically, and we will sketch the method and show some results.

Sketch of the methodology

The methodology to compute a solution is as follows. Consider a topographic profile, $h(x)$, and let us suppose that it is periodic in x over some distance L. Such a profile can (nearly always) be decomposed into a sum of Fourier coefficients, meaning that we can write

$$h(x) = \sum_k \widetilde{h}_k \mathrm{e}^{\mathrm{i}kx}, \tag{7.189}$$

where $\widetilde{h}_k$ are the Fourier coefficients. We can obtain the set of $\widetilde{h}_k$ by multiplying (7.189) by e^{-ikx} and integrating over the domain from $x = 0$ to $x = L$, a procedure known as taking the discrete Fourier transform of $h(x)$, and there are standard computer algorithms for doing this efficiently. Once we have obtained the values of $\widetilde{h}_k$ we essentially solve the problem separately for each k in precisely the same manner as we did in the previous section. For *each* k there will be a vertical wavenumber

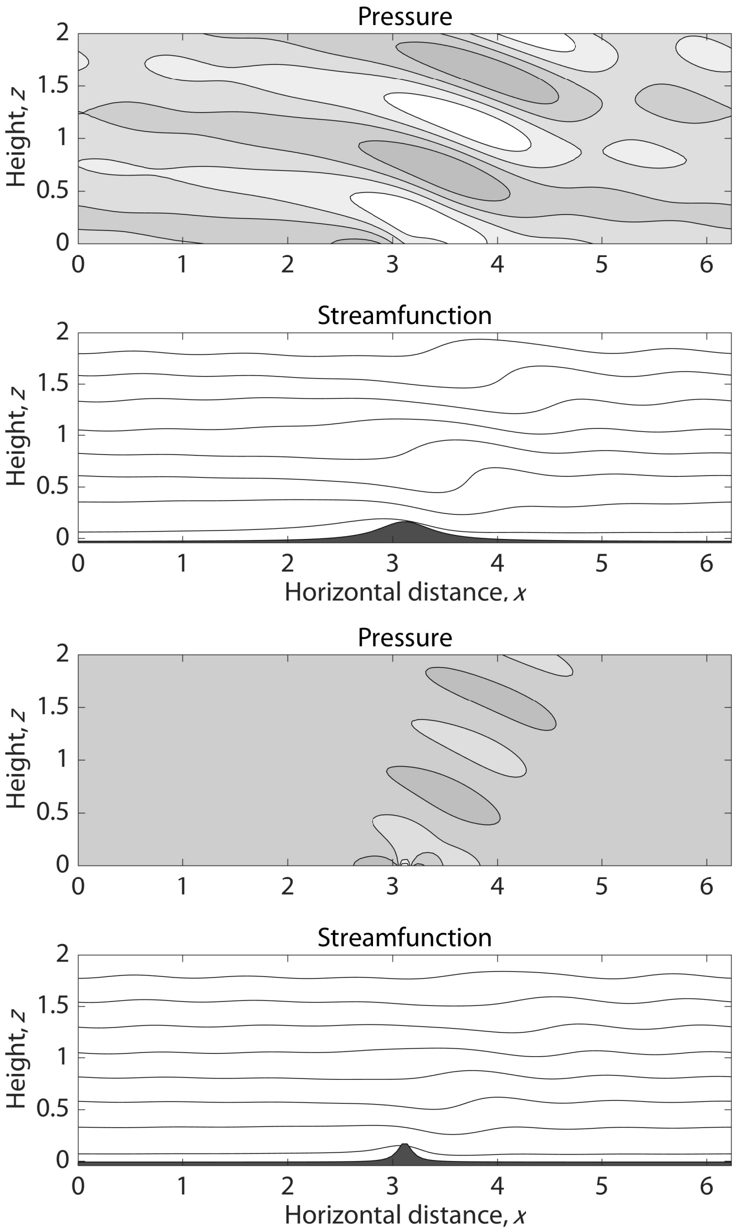

Fig. 7.17 Upper two panels. Solutions for the flow over a bell-shaped ridge (7.191), with $a^2 = 4U^2/N^2$. High pressure is shaded darker, and the flow comes in from the left.

Lower two panels. Same as above, but now for a narrow ridge, of the same height but with $a^2 = U^2/4N^2$.

(The dark shading for the mountain is graphically superimposed on the solution, and the streamfunction does not actually intersect the topography.)

given by (7.176), so that for each wavenumber we obtain a solution for pressure of the form $\widetilde{\phi}_k(z)$, and similarly for the other variables. Once we have the solution for each wavenumber, then at each level we sum over all the wavenumbers to obtain the solution in real space; that is, we evaluate

$$\phi(x,z) = \sum_k \widetilde{\phi}_z(z)\mathrm{e}^{\mathrm{i}kx}; \tag{7.190}$$

that is to say, we take the inverse discrete Fourier transform.

A witches' brew

For specificity let us consider the bell-shaped topographic profile

$$h(x) = \frac{h_0 a^2}{a^2 + x^2}, \tag{7.191}$$

sometimes called the Witch of Agnesi.[3] (Results with a Gaussian profile are quite similar.) Such a profile is composed of *many* (in fact an infinite number of) Fourier coefficients of differing amplitudes. If the profile is narrow (meaning a is small, in a sense made clearer below) then there will be a great many significant coefficients at high wavenumbers. In fact, in the limiting case of an infinitely thin ridge (a delta function) all wavenumbers are present with equal weight, so there are certainly more large wavenumbers than small wavenumbers. However, if a is large, then the contributing wavenumbers will predominantly be small.

In the problem of flow over topography the natural horizontal scale is U/N. If $a \gg U/N$ then the dominant wavenumbers are small and the solution will consist of waves propagating upward with little loss of amplitude and phase lines tilting upstream, as illustrated in the upper two panels of Fig. 7.17. (The influence of the mountain is downstream, but the solutions are obtained on a finite, periodic domain and some influence returns upstream.) In the case of a narrow ridge, as illustrated in the lower two panels of Fig. 7.17, the perturbation is largely trapped near to the mountain and the perturbation fields largely decay exponentially with height. Nevertheless, because the ridge *does* contain some small wavenumbers, some weak, propagating large-scale disturbances are generated. The fluid acts as a low-pass filter, and the perturbation aloft consists only of large scales.

A hydrostatic solution

If the ridge is sufficiently wide then the solution is essentially hydrostatic, with little dependence of the vertical structure on the horizontal wavenumber; that is, using (7.176) at large scales, $m^2 \approx (N/U)^2$. Furthermore, the x-component of the group velocity is zero, which can be seen from (7.173b) using $m = N/U$ (since m is positive for upward wave propagation and $m \gg k$ by hydrostasy). Explicitly we have

$$c_g^x = U - \frac{Nm^2}{(k^2 + m^2)^{3/2}} \approx U - \frac{N}{m} = U - U = 0. \tag{7.192}$$

Thus, the disturbance appears directly over the mountain, with no downstream propagation, as in Fig. 7.18. The pattern therefore repeats itself in the vertical at intervals of $2\pi/m$ or $2\pi U/N$, with neither a downnstream nor upstream influence. This solution is in fact the most atmospherically relevant one, for it is that produced by atmospheric flow over mountains that have horizontal scales larger than a few kilometres (i.e. $a \gg U/N$). It is also oceanographically relevant for flow over features of greater than a few kilometers, except possibly in regions of very weak stratification and large abyssal currents. Scales that are much larger are filtered by the Coriolis effect, as we now see.

7.7.4 Effects of Rotation

General considerations

We now briefly consider the effects of a Coriolis force on mountain waves. The problem is in many ways similar to the non-rotating case but the dispersion relation and so the criteria for upward propagation differ accordingly, for it transpires that the Coriolis effect filters the waves at very large scales. To proceed, we first note that the steady flow must be in geostrophic balance, so that if the flow is zonal there is a background meridional pressure gradient that satisfies

$$fU = -\frac{\partial \Phi}{\partial y}, \tag{7.193}$$

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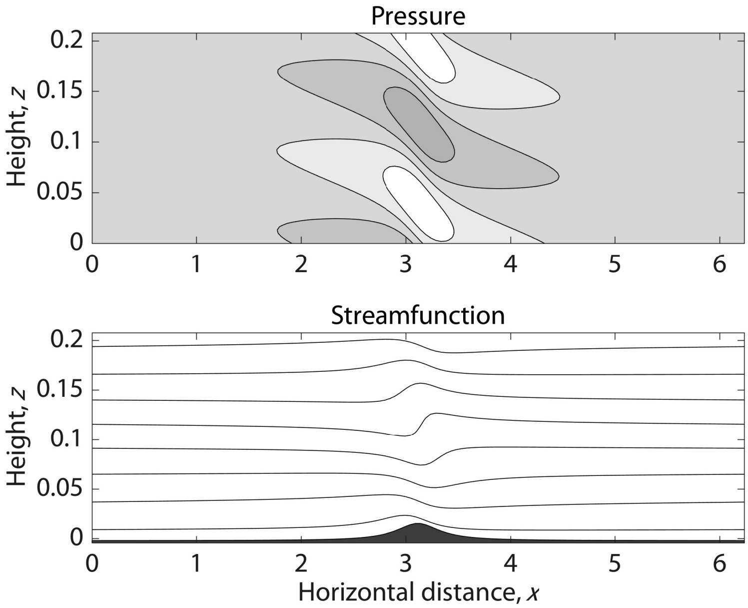

Fig. 7.18 Similar to the upper two panels of Fig. 7.17, but with a smaller U. Thus, the ridge is of the same height and width, but the flow is in a hydrostatic regime with $a^2 = 10U^2/N^2$. This is a very relevant regime for flow over mountains in the atmosphere.

The group velocity is almost purely vertical and the disturbance is confined to a region directly above the mountain. The vertical extent of the domain shown contains about one wavenumber, and the streamfunction is a mirror image of the mountain at about $z = 0.1$.

where as before we take f to be constant in any given problem. The main difference arises from the fact that the waves now obey the dispersion relation with rotation, namely (7.137) or, restricting attention to the x–z plane,

$$\omega^2 = \frac{f^2m^2 + k^2N^2}{k^2 + m^2}. \tag{7.194}$$

As in the non-rotating case we can obtain an expression for the vertical wavenumber in steady flow by letting $\omega = -Uk$, whence

$$\boxed{m^2 = \frac{k^2(N^2 - U^2k^2)}{U^2k^2 - f^2}.} \tag{7.195}$$

Evanescent solutions arise when m is imaginary and, as before, such solutions arise for small scales for which $k > N/U$. However, from (7.195), evanescent solutions also arise for very large scales for which $k < f/U$. Propagating waves thus exist in the wavenumber and lengthscale ($L = 2\pi/k$) intervals

$$\frac{N}{U} > k > \frac{f}{U} \qquad \text{or} \qquad \frac{U}{N} < \frac{L}{2\pi} < \frac{U}{f}, \tag{7.196}$$

and these waves have frequencies between N and f. In an atmosphere with $U = 10\,\mathrm{m\,s^{-1}}$ and $f = 10^{-4}\,\mathrm{s^{-1}}$ the large scale at which evanescence reappears is $L = 2\pi U/f \approx 600\,\mathrm{km}$, which of course is not very large at all relative to global scales (and still smaller if we take $U = 5\,\mathrm{m\,s^{-1}}$). Thus, upward propagating gravity waves exist between scales of a few kilometres and several hundred kilometres. For the deep ocean, let us take $N = 10^{-3}\,\mathrm{s^{-1}}$, $f = 10^{-4}\,\mathrm{s^{-1}}$ and $U = 1\,\mathrm{cm\,s^{-1}}$. Then, very roughly, propagating waves exist between scales of a few tens of meters to a few hundred metres, and the hydrostatic regime (which requires $L \gg U/N$ and $L < 2\pi U/f$)) may be quite limited.

Wave solutions and energy propagation

Obtaining a wave solution in the rotating case follows a similar path to the non-rotating case. In the resting frame vertical velocity satisfies the boundary condition $w = U\partial h/\partial x$, and in the moving frame $w = \partial h/\partial t$. If U is constant we may use the polarization relations of Section 7.6.4, with $l = 0$ and $\omega = -Uk$, to obtain relations analogous to (7.178), and we find

$$w = \widetilde{w}(z)e^{ikx} = w_0 e^{i(kx+mz)} = iUkh_0 e^{imz}e^{ikx}, \tag{7.197a}$$

$$u = \widetilde{u}(z)e^{ikx} = u_0 e^{i(kx+mz)} = -imUh_0 e^{imz}e^{ikx}, \tag{7.197b}$$

$$\phi = \widetilde{\phi}(z)e^{ikx} = \phi_0 e^{i(kx+mz)} = \frac{im(U^2k^2 - f^2)}{k^2}h_0 e^{imz}e^{ikx}, \tag{7.197c}$$

$$v = \widetilde{v}(z)e^{ikx} = v_0 e^{i(kx+mz)} = -if\frac{m}{k}h_0 e^{imz}e^{ikx}, \tag{7.197d}$$

Of these, the expressions for w and u are no different from the non-rotating case, because w is set by the same boundary condition and u is given by mass continuity, $\partial u/\partial x + \partial w/\partial z = 0$, in both rotating and non-rotating cases. However, the solution now produces a meridional velocity, (7.197d), even when there is no variation in the topography in the y-direction, and to obtain it we use (7.151) with $l = 0$, giving $\widetilde{v} = -i\widetilde{u}f\omega = i\widetilde{u}f/Uk$.

As in the non-rotating case, when there are propagating waves there is high pressure on the windward (upstream) side of the topography and low pressure on the leeward side, and the phase lines tilt upstream with height. The drag on the flow is equal to the rate of upward momentum transport and using (7.197a,b) we obtain

$$\overline{uw} = -\frac{1}{2}kmU^2h_0^2 < 0. \tag{7.198}$$

There are some subtleties associated with the horizontal force across a wavy boundary that we shall not go into.[4] In any case, as in the non-rotating case a momentum flux divergence will only arise in the free atmosphere if dissipation occurs, for example if the waves break and/or if viscous effects become important. The vertical flux of energy density is given by

$$\overline{\phi w} = \frac{1}{2}U\frac{m}{k}h_0^2(U^2k^2 - f^2) > 0. \tag{7.199}$$

If m and k have the same sign then energy propagates away from the mountain, which as the reader may verify is consistent with the group velocity being directed upward.

Atmospheric and oceanic parameters

Are evanescent or propagating gravity waves more likely to be excited for typical atmospheric and oceanic parameters? Consider the atmosphere with a surface flow of $U = 10\,\mathrm{m\,s^{-1}}$ and $N = 10^{-2}\,\mathrm{s^{-1}}$. The critical wavenumber separating evanescent and propagating waves is then $k = N/U = 2 \times 10^{-3}$, corresponding to a wavelength of about 6000 m. Topographic features like the Rockies, Andes and Himalayas certainly contain such large wavelengths and so we can expect them to excite upward propagating gravity waves. For larger horizontal scales the flow is hydrostatic and the influence is felt directly above the mountain, as in Fig. 7.18. At still larger scales the waves are inhibited by the Coriolis effect, which causes evanescence for waves with a horizontal wavelength larger than a few hundred kilometres.

In the ocean the abyssal stratification is quite weak, typically with $N \sim 10^{-3}\,\mathrm{s^{-1}}$, and the velocities are also weak compared to those of the upper ocean, although they can be of order $1\,\mathrm{cm\,s^{-1}}$ in eddying regions. With $N = 10^{-3}$ and $U = 1\,\mathrm{cm\,s^{-1}}$ we find a critical wavelength of about 60 m. The ocean bathymetry obviously has many scales larger than this (and for smaller values of U the critical scales are correspondingly smaller) meaning that it is relatively easy for abyssal flow to generate gravity waves that propagate upward into the ocean interior, and as in the atmospheric case in many circumstances the flow will be hydrostatic and the influence will be directly above the bathymetry. However, as noted earlier, the Coriolis parameter will inhibit gravity waves of too large a wavelength — for $U = 1\,\mathrm{cm\,s^{-1}}$ the horizontal scales must be less than $2\pi U/f$ or about 600 m. The upper ocean is much more greatly stratified, with $N \approx 10^{-2}\,\mathrm{s^{-1}}$. Gravity waves are no longer generated by flow over topography but by the stirring effects of winds making a turbulent mixed layer. The forcing frequency must still be less than N in order to efficiently generate gravity waves, and using a velocity of $10\,\mathrm{cm\,s^{-1}}$ we might heuristically estimate that propagating gravity waves can be generated with scales of tens of metres, with the Coriolis cut off now occurring at about 6 km. One message of these very rough calculations is that, in both atmosphere and ocean, gravity waves tend to have a smaller horizontal scale than Rossby waves.

7.8 ✦ ACOUSTIC-GRAVITY WAVES IN AN IDEAL GAS

In the final section of this chapter we consider wave motion in a stratified, *compressible* fluid such as the Earth's atmosphere. The stratification allows gravity waves to exist, and the compressibility allows sound waves to exist. The resulting problem is, not surprisingly, complicated and arcane and to make it as tractable as possible we will specialize to the case of an isothermal, stationary atmosphere and ignore the effects of rotation and sphericity. The results are not without interest, both in themselves and in illustrating the importance of simplifying the equations of motion from the outset, for example by making the Boussinesq or hydrostatic approximation, in order to isolate phenomena of interest.

In what follows we denote the unperturbed state with a subscript 0 and the perturbed state with a prime ($'$), and we omit some of the algebraic details. Because it is at rest, the basic state is in hydrostatic balance,

$$\frac{\partial p_0}{\partial z} = -\rho_0(z)g. \tag{7.200}$$

Ignoring variations in the y-direction for algebraic simplicity (and without loss of generality, in fact) the linearized equations of motion are:

$$u \text{ momentum:} \qquad \rho_0 \frac{\partial u'}{\partial t} = -\frac{\partial p'}{\partial x}, \tag{7.201a}$$

$$w \text{ momentum:} \qquad \rho_0 \frac{\partial w'}{\partial t} = -\frac{\partial p'}{\partial z} - \rho' g, \tag{7.201b}$$

$$\text{mass conservation:} \qquad \frac{\partial \rho'}{\partial t} + w'\frac{\partial \rho_0}{\partial z} = -\rho_0\left(\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z}\right), \tag{7.201c}$$

$$\text{thermodynamic:} \qquad \frac{\partial p'}{\partial t} - w'\frac{p_0}{H} = -\gamma p_0\left(\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z}\right). \tag{7.201d}$$

where $\gamma = c_p/c_v = 1/(1-\kappa)$. For an isothermal basic state we have $p_0 = \rho_0 R T_0$ where T_0 is a constant, so that $\rho_0 = \rho_s e^{-z/H}$ and $p_0 = p_s e^{-z/H}$ where $H = RT_0/g$. The thermodynamic equation, (7.201d), is the linear form of (1.101) on page 24, which has the equation of state for an ideal gas built-in, and (7.201a,b,c,d) thus form a complete set with variables u', w', ρ' and p'.

Differentiating (7.201a) with respect to time and using (7.201d) leads to

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2\frac{\partial^2}{\partial x^2}\right)u' = c_s^2\left(\frac{\partial}{\partial z} - \frac{1}{\gamma H}\right)\frac{\partial}{\partial x}w', \tag{7.202a}$$

where $c_s^2 = \gamma p_0/\rho_0$ is the square of the speed of sound (equal to $\partial p/\partial \rho$ or $\gamma R T_0$). Similarly, differentiating (7.201b) with respect to time and using (7.201c) and (7.201d) leads to

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2\left[\frac{\partial^2}{\partial z^2} - \frac{1}{H}\frac{\partial}{\partial z}\right]\right)w' = c_s^2\left(\frac{\partial}{\partial z} - \frac{\kappa}{H}\right)\frac{\partial u'}{\partial x}. \tag{7.202b}$$

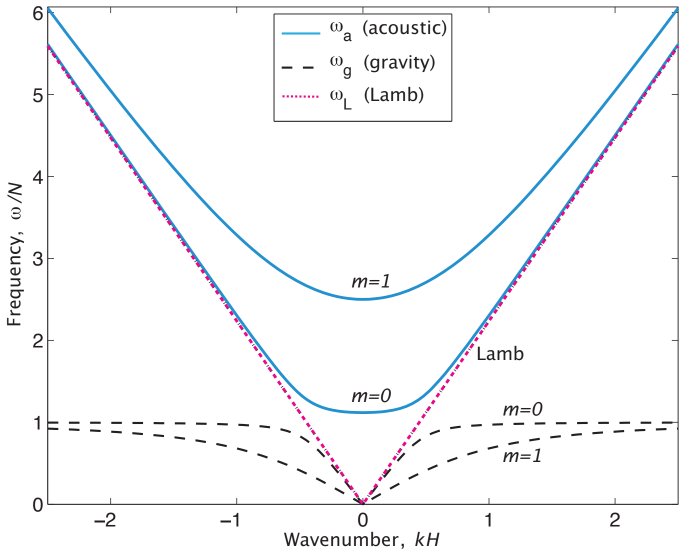

Fig. 7.19 Dispersion diagram for acoustic-gravity waves in an isothermal atmosphere, calculated using (7.208). The frequency is in units of buoyancy frequency N, and the wavenumbers are nondimensionalized by the inverse of the scale height, H. Solid curves indicate acoustic waves, whose frequency is always higher than that of the corresponding Lamb wave (i.e., ck), and of the base acoustic frequency $\approx 1.12N$. The dashed curves are internal gravity waves, whose frequency asymptotes to N at small horizontal scales.

Equations (7.202a) and (7.202b) combine to give, after some cancellation,

$$\frac{\partial^4 w'}{\partial t^4} - c_s^2 \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{H}\frac{\partial}{\partial z}\right) w' - c_s^2 \frac{\kappa g}{H}\frac{\partial^2 w'}{\partial x^2} = 0. \tag{7.203}$$

If we set $w' = W(x,z,t)e^{z/(2H)}$, so that $W = (\rho_0/\rho_s)^{1/2} w'$, then the term with the single z-derivative is eliminated, giving

$$\frac{\partial^4 W}{\partial t^4} - c_s^2 \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{4H^2}\right) W - c_s^2 \frac{\kappa g}{H}\frac{\partial^2 W}{\partial x^2} = 0. \tag{7.204}$$

Although superficially complicated, this equation has constant coefficients and we may seek wave-like solutions of the form

$$W = \mathrm{Re}\, \widetilde{W} e^{i(kx+mz-\omega t)}, \tag{7.205}$$

where $\widetilde{W}$ is the complex wave amplitude. Using (7.205) in (7.204) leads to the dispersion relation for acoustic-gravity waves, namely

$$\omega^4 - c_s^2\omega^2\left(k^2 + m^2 + \frac{1}{4H^2}\right) + c_s^2 N^2 k^2 = 0, \tag{7.206}$$

with solution

$$\boxed{\omega^2 = \frac{1}{2}c_s^2 K^2\left[1 \pm \left(1 - \frac{4N^2k^2}{c_s^2 K^4}\right)^{1/2}\right],} \tag{7.207}$$

where $K^2 = k^2 + m^2 + 1/(4H^2)$. (The reader may investigate as to whether the factor $[1 - 4N^2k^2/(c_s^2K^4)]$ is positive.) For an isothermal, ideal-gas atmosphere $4N^2H^2/c_s^2 \approx 0.8$ and so this may be written

$$\frac{\omega^2}{N^2} \approx 2.5\widehat{K}^2\left[1 \pm \left(1 - \frac{0.8\widehat{k}^2}{\widehat{K}^4}\right)^{1/2}\right], \tag{7.208}$$

where $\widehat{K}^2 = \widehat{k}^2 + \widehat{m}^2 + 1/4$, and $(\widehat{k}, \widehat{m}) = (kH, mH)$.

7.8.1 Interpretation

Acoustic and gravity waves

There are two branches of roots in (7.207), corresponding to acoustic waves (using the plus sign in the dispersion relation) and internal gravity waves (using the minus sign). These (and the Lamb wave, described below) are plotted in Fig. 7.19. If $4N^2k^2/c_s^2K^4 \ll 1$ then the two sets of waves are well separated. From (7.208) this is satisfied when

$$\frac{4\kappa}{\gamma}(kH)^2 \approx 0.8(kH)^2 \ll \left[(kH)^2 + (mH)^2 + \frac{1}{4}\right]^2; \tag{7.209}$$

that is, when *either* $mH \gg 1$ or $kH \gg 1$. The two roots of the dispersion relation are then

$$\omega_a^2 \approx c_s^2K^2 = c_s^2\left(k^2 + m^2 + \frac{1}{4H^2}\right) \tag{7.210}$$

and

$$\omega_g^2 \approx \frac{N^2k^2}{k^2 + m^2 + 1/(4H^2)}, \tag{7.211}$$

corresponding to acoustic and gravity waves, respectively. The acoustic waves owe their existence to the presence of compressibility in the fluid, and they have no counterpart in the Boussinesq system. On the other hand, the internal gravity waves are just modified forms of those found in the Boussinesq system, and if we take the limit $(kH, mH) \to \infty$ then the gravity wave branch reduces to $\omega_g^2 = N^2k^2/(k^2 + m^2)$, which is the dispersion relationship for gravity waves in the Boussinesq approximation. We may consider this to be the limit of infinite scale height or (equivalently) the case in which wavelengths of the internal waves are sufficiently small that the fluid is essentially incompressible.

Vertical structure

Recall that $w' = W(x, z, t)\mathrm{e}^{z/(2H)}$ and, by inspection of (7.202), u' has the same vertical structure. That is,

$$w' \propto \mathrm{e}^{z/(2H)}, \qquad u' \propto \mathrm{e}^{z/(2H)}, \tag{7.212}$$

and the amplitude of the velocity field of the internal waves increases with height. The pressure and density perturbation amplitudes fall off with height, varying like

$$p' \propto \mathrm{e}^{-z/(2H)}, \qquad \rho' \propto \mathrm{e}^{-z/(2H)}. \tag{7.213}$$

The kinetic energy of the perturbation, $\rho_0(u'^2 + w'^2)$ is *constant* with height, because $\rho_0 = \rho_s e^{-z/H}$.

Hydrostatic approximation and Lamb waves

Equations (7.202) also admit to a solution with $w' = 0$. We then have

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2\frac{\partial^2}{\partial x^2}\right)u' = 0 \qquad \text{and} \qquad \left(\frac{\partial}{\partial z} - \frac{\kappa}{H}\right)\frac{\partial u'}{\partial x} = 0, \tag{7.214}$$

and these have solutions of the form

$$u' = \mathrm{Re}\,\widetilde{U}\mathrm{e}^{\kappa z/H}\mathrm{e}^{\mathrm{i}(kx-\omega t)}, \qquad \omega = ck, \tag{7.215}$$

where $\widetilde{U}$ is the wave amplitude. These are horizontally propagating sound waves, known as *Lamb waves* after the hydrodynamicist Horace Lamb. Their velocity perturbation amplitude increases with height, but the pressure perturbation falls with height; that is

$$u' \propto \mathrm{e}^{\kappa z/H} \approx \mathrm{e}^{2z/(7H)}, \qquad p' \propto \mathrm{e}^{(\kappa-1)z/H} \approx \mathrm{e}^{-5z/(7H)}. \tag{7.216}$$

Their kinetic energy density, $\rho_0 u'^2$, varies as

$$\mathrm{KE} \propto \mathrm{e}^{-z/H+2\kappa z/H} = \mathrm{e}^{(2R-c_p)z/(c_pH)]} = \mathrm{e}^{(R-c_v)z/(c_pH)} \approx \mathrm{e}^{-3z/(7H)}, \tag{7.217}$$

for an ideal gas. (In a simple ideal gas, $c_v = nR/2$ where n is the number of excited degrees of freedom, 5 for a diatomic molecule.) The kinetic energy density thus falls away exponentially from the surface, and in this sense Lamb waves are an example of edge waves or surface-trapped waves.

Now consider the case in which we make the hydrostatic approximation ab initio, but without restricting the perturbation to have $w' = 0$. The linearized equations are identical to (7.201), except that (7.201b) is replaced by

$$\frac{\partial p'}{\partial z} = -\rho' g. \tag{7.218}$$

The consequence of this is that first term $(\partial^2 w'/\partial t^2)$ in (7.202b) disappears, as do the first two terms in (7.203) (i.e., the terms $\partial^4 w'/\partial t^4 - c^2(\partial^2/\partial t^2)(\partial^2 w'/\partial x^2)$). It is a simple matter to show that the dispersion relation is then

$$\omega^2 = \frac{N^2 k^2}{m^2 + 1/(4H^2)}. \tag{7.219}$$

These are long gravity waves, and may be compared with the corresponding Boussinesq result (7.60). Again, the frequency increases without bound as the horizontal wavelength diminishes. The Lamb wave, of course, still exists in the hydrostatic model, because (7.214) is still a valid solution. Thus, horizontally propagating sound waves still exist in hydrostatic (primitive equation) models, but vertically propagating sound waves do not — essentially because the term $\partial w/\partial t$ is absent from the vertical momentum equation.

Notes

1 The book by Sutherland (2010) treats internal waves in some detail. I would like to thank S. Legg for encouraging me to add material on gravity waves and for her unpublished lecture notes.

2 See Durran (1990) for more discussion and, for a review, Durran (2015). I am also grateful to Dale Durran for many comments and corrections on the chapters on waves.

3 A treatment of this rather canonical profile was given by Queney (1948). The profile is named for Maria Agnesi, 1718–1799, an Italian mathematician and later a theologian, who had discussed the properties of the curve, as had Pierre de Fermat and Guido Grandi (a professor at the University of Pisa) somewhat earlier. The term 'witch' (in 'Witch of Agnesi') seems to be a mistranslation, inadvertent or otherwise, from Italian of versiera, which refers to a curve and not an adversary of God or a she-devil (which would be avversiera). Maria Agnesi may have been the first woman appointed to a professorship of mathematics, at the University of Bologna. The curve is similar to a Cauchy distribution and to a Lorentzian, and has some useful Fourier-transform properties that enable an analytic approach, although our solution is numerical.

4 Jones (1967) and Bretherton (1969) show that it is the quantity $\overline{\rho(u - f\eta)w}$, where η is the particle displacement parallel to the y-axis, that represents the force across a wavy boundary and that is constant with height, with an extra form drag arising because of the Coriolis force.