Chapter 4

Circulation, Vorticity, and Potential Vorticity

In classical mechanics the principle of conservation of angular momentum is often invoked in the analysis of motions that involve rotation. This principle provides a powerful constraint on the behavior of rotating objects. Analogous conservation laws also apply to the rotational field of a fluid. However, it should be obvious that in a continuous medium, such as the atmosphere, the definition of "rotation" is subtler than that for rotation of a solid object.

Circulation and vorticity are the two primary measures of rotation in a fluid. Circulation, which is a scalar integral quantity, is a *macroscopic* measure of rotation for a finite area of the fluid. Vorticity, however, is a vector field that gives a *microscopic* measure of the rotation at any point in the fluid. Potential vorticity extends the concept of vorticity to include thermodynamic constraints on the motion, yielding a powerful framework for interpreting atmospheric dynamics.

4.1 THE CIRCULATION THEOREM

The *circulation*, C, about a closed contour in a fluid is defined as the line integral evaluated along the contour of the component of the velocity vector that is locally tangent to the contour:

$$C \equiv \oint \mathbf{U} \cdot d\mathbf{l} = \oint |\mathbf{U}| \cos\, \alpha\, dl$$

where $\mathbf{l}(s)$ is a position vector extending from the origin to the point $s(x, y, z)$ on the contour C, and $d\mathbf{l}$ represents the limit of $\delta\mathbf{l} = \mathbf{l}(s + \delta s) - \mathbf{l}(s)$ as $\delta s \to 0$. Hence, as indicated in Figure 4.1, $d\mathbf{l}$ is a displacement vector locally tangent to the contour. By convention the circulation is taken to be positive if $C > 0$ for counterclockwise integration around the contour.

That circulation is a measure of rotation is demonstrated readily by considering a circular ring of fluid of radius R in solid-body rotation at angular velocity $\boldsymbol{\Omega}$ about the z axis. In this case, $\mathbf{U} = \boldsymbol{\Omega} \times \mathbf{R}$, where $\mathbf{R}$ is the distance from the axis of rotation to the ring of fluid. Thus, the circulation about the ring is

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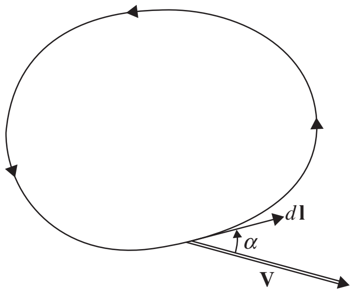

FIGURE 4.1 Circulation about a closed contour.

given by

$$C \equiv \oint \mathbf{U} \cdot d\mathbf{l} = \int_0^{2\pi} \Omega R^2 d\lambda = 2\Omega\pi R^2$$

In this case the circulation is just 2π times the angular momentum of the fluid ring about the axis of rotation. Alternatively, note that $C/(\pi R^2) = 2\Omega$, so that the circulation divided by the area enclosed by the loop is just twice the angular speed of rotation of the ring. Unlike angular momentum or angular velocity, circulation can be computed without reference to an axis of rotation; it can thus be used to characterize fluid rotation in situations where "angular velocity" is not easily defined.

The circulation theorem is obtained by taking the line integral of Newton's second law for a closed chain of fluid particles. In the absolute coordinate system, the result (neglecting viscous forces) is

$$\oint \frac{D_a\mathbf{U}_a}{Dt} \cdot d\mathbf{l} = -\oint \frac{\nabla p \cdot d\mathbf{l}}{\rho} - \oint \nabla\Phi \cdot d\mathbf{l} \tag{4.1}$$

where the gravitational force is represented as the gradient of geopotential Φ^*, defined in terms of true gravity, so that $\nabla\Phi^* = -\mathbf{g}^* = g^*\mathbf{k}$. The integrand on the left side can be rewritten as[1]

$$\frac{D_a\mathbf{U}_a}{Dt} \cdot d\mathbf{l} = \frac{D}{Dt}(\mathbf{U}_a \cdot d\mathbf{l}) - \mathbf{U}_a \cdot \frac{D_a}{Dt}(d\mathbf{l})$$

or after observing that since $\mathbf{l}$ is a position vector, $D_a\mathbf{l}/Dt \equiv \mathbf{U}_a$,

$$\frac{D_a\mathbf{U}_a}{Dt} \cdot d\mathbf{l} = \frac{D}{Dt}(\mathbf{U}_a \cdot d\mathbf{l}) - \mathbf{U}_a \cdot d\mathbf{U}_a \tag{4.2}$$

[1]Note that for a scalar, $D_a/Dt = D/Dt$ (i.e., the rate of change following the motion does not depend on the reference system). For a vector, however, this is not the case, as was shown in Section 2.1.1.

Substituting (4.2) into (4.1) and using the fact that the line integral about a closed loop of a perfect differential is zero, so that

$$\oint \nabla\Phi \cdot d\mathbf{l} = \oint d\Phi = 0$$

and noting that

$$\oint \mathbf{U}_a \cdot d\mathbf{U}_a = \frac{1}{2}\oint d(\mathbf{U}_a \cdot \mathbf{U}_a) = 0$$

we obtain the circulation theorem:

$$\frac{DC_a}{Dt} = \frac{D}{Dt}\oint \mathbf{U}_a \cdot d\mathbf{l} = -\oint \rho^{-1} dp \tag{4.3}$$

The term that is on the right side in (4.3) is called the solenoidal term, where $dp = \nabla p \cdot d\mathbf{l}$ is the pressure increment along an increment of arc length. For a barotropic fluid, the density is a function only of pressure, and the solenoidal term is zero. Thus, in a barotropic fluid the absolute circulation is conserved following the motion. This result, called *Kelvin's circulation theorem*, is a fluid analog of angular momentum conservation in solid-body mechanics.

For meteorological analysis, it is more convenient to work with the relative circulation C rather than the absolute circulation; a portion of the absolute circulation, C_e, is due to the rotation of Earth about its axis. To compute C_e, we apply Stokes's theorem to the vector $\mathbf{U}_e$, where $\mathbf{U}_e = \mathbf{\Omega} \times \mathbf{r}$ is the velocity of Earth at position $\mathbf{r}$:

$$C_e = \oint \mathbf{U}_e \cdot d\mathbf{l} = \int_A\int (\nabla \times \mathbf{U}_e) \cdot \mathbf{n}\, dA$$

where A is the area enclosed by the contour and the unit normal $\mathbf{n}$ is defined by the counterclockwise sense of the line integration using the "right rule." Thus, for the contour of Figure 4.1, $\mathbf{n}$ would be directed out of the page. If the line integral is computed in the horizontal plane, $\mathbf{n}$ is directed along the local vertical (Figure 4.2). Now, by a vector identity (see Appendix C),

$$\nabla \times \mathbf{U}_e = \nabla \times (\mathbf{\Omega} \times \mathbf{r}) = \nabla \times (\mathbf{\Omega} \times \mathbf{R}) = \mathbf{\Omega}\nabla \cdot \mathbf{R} = 2\mathbf{\Omega}$$

so that $(\nabla \times \mathbf{U}_e) \cdot \mathbf{n} = 2\Omega \sin\phi \equiv f$ is just the Coriolis parameter. Hence, the circulation in the horizontal plane due to the rotation of Earth is

$$C_e = 2\Omega\langle\sin\phi\rangle\, A = 2\Omega A_e$$

where $\langle\sin\phi\rangle$ denotes an average over the area element A, and A_e is the projection of A in the equatorial plane, as illustrated in Figure 4.2. Thus, the relative circulation may be expressed as

$$C = C_a - C_e = C_a - 2\Omega A_e \tag{4.4}$$

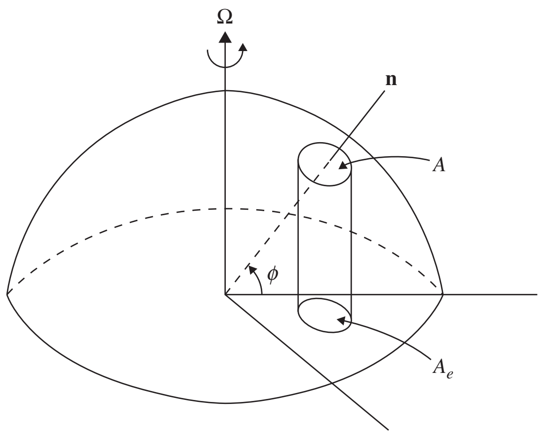

FIGURE 4.2 Area A_e subtended on the equatorial plane by horizontal area A centered at latitude ϕ.

Differentiating (4.4) following the motion and substituting from (4.3), we obtain the *Bjerknes circulation theorem*:

$$\frac{DC}{Dt} = -\oint \frac{dp}{\rho} - 2\Omega \frac{DA_e}{Dt} \tag{4.5}$$

For a barotropic fluid, (4.5) can be integrated following the motion from an initial state (designated by subscript 1) to a final state (designated by subscript 2), yielding the circulation change

$$C_2 - C_1 = -2\Omega\,(A_2 \sin\phi_2 - A_1 \sin\phi_1) \tag{4.6}$$

Equation (4.6) indicates that in a barotropic fluid the relative circulation for a closed chain of fluid particles will be changed if either the horizontal area enclosed by the loop changes or the latitude changes. Furthermore, a negative absolute circulation in the Northern Hemisphere can develop only if a closed chain of fluid particles is advected across the equator from the Southern Hemisphere. The anomalous gradient wind balances discussed in Section 3.2.5 are examples of systems with negative absolute circulations (see Problem 4.6).

Examples

Suppose that the air within a circular region of radius 100 km centered at the equator is initially motionless with respect to Earth. If this circular air mass were moved to the North Pole along an isobaric surface preserving its area, the circulation about the circumference would be

$$C = -2\Omega\pi r^2 \sin(\pi/2) - \sin(0)$$

Thus, the mean tangential velocity at the radius $r = 100$ km would be

$$V = C/(2\pi r) = -\Omega r \approx -7 \text{ m s}^{-1}$$

The negative sign here indicates that the air has acquired anticyclonic relative circulation.

In a baroclinic fluid, circulation may be generated by the pressure-density solenoid term in (4.3). This process can be illustrated effectively by considering the development of a sea breeze circulation, as shown in Figure 4.3. For the situation depicted, the mean temperature in the air over the ocean is colder than the mean temperature over the adjoining land. Thus, if the pressure is uniform at ground level, the isobaric surfaces above the ground will slope downward toward the ocean, while the isopycnic surfaces (surfaces of constant density) will slope downward toward the land.

To compute the acceleration as a result of the intersection of the pressure-density surfaces, we apply the circulation theorem by integrating around a circuit in a vertical plane perpendicular to the coastline. Substituting the ideal gas law into (4.3), we obtain

$$\frac{DC_a}{Dt} = -\oint RTd \ln p$$

For the circuit shown in Figure 4.3, there is a contribution to the line integral only for the vertical segments of the loop, since the horizontal segments are taken at constant pressure. The resulting rate of increase in the circulation is

$$\frac{DC_a}{Dt} = R \ln\left(\frac{p_0}{p_1}\right)(\overline{T}_2 - \overline{T}_1) > 0$$

Letting $\langle v \rangle$ be the mean tangential velocity along the circuit, we find that

$$\frac{D\langle v \rangle}{Dt} = \frac{R \ln (p_0/p_1)}{2(h+L)}(\overline{T}_2 - \overline{T}_1) \tag{4.7}$$

If we let $p_0 = 1000$ hPa, $p_1 = 900$ hPa, $\overline{T}_2 - \overline{T}_1 = 10^\circ$ C, $L = 20$ km, and $h = 1$ km, (4.7) yields an acceleration of about 7×10^{-3} m s^{-2}. In the absence of frictional retarding forces, this would produce a wind speed of 25 m s^{-1} in about 1 h. In reality, as the wind speed increases, the frictional force reduces the acceleration rate, and temperature advection reduces the land–sea temperature contrast so that a balance is obtained between the generation of kinetic energy by the pressure-density solenoids and frictional dissipation.

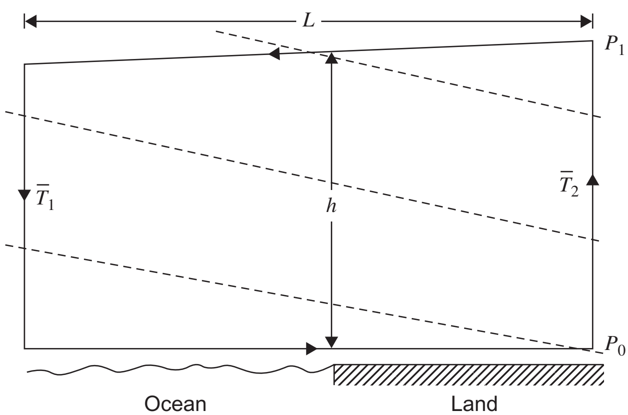

FIGURE 4.3 Application of the circulation theorem to the sea breeze problem. The closed *heavy solid line* is the loop about which the circulation is to be evaluated. *Dashed lines* indicate surfaces of constant density.

4.2 VORTICITY

Vorticity, the microscopic measure of rotation in a fluid, is a vector field defined as the curl of velocity. The absolute vorticity $\boldsymbol{\omega}_a$ is the curl of the absolute velocity, whereas the relative vorticity $\boldsymbol{\omega}$ is the curl of the relative velocity:

$$\boldsymbol{\omega}_a \equiv \nabla \times \mathbf{U}_a, \qquad \boldsymbol{\omega} \equiv \nabla \times \mathbf{U}$$

so that in Cartesian coordinates,

$$\boldsymbol{\omega} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

A scale analysis of the scalar components of the vorticity vector for synoptic-scale motions reveals that the dominant contributions to the horizontal components, $\frac{\partial u}{\partial z}$ and $\frac{\partial v}{\partial z}$,[2] scale like $\frac{U}{H}$, whereas the vertical components, $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial x}$, scale like $\frac{U}{L}$. Therefore, the vertical component of vorticity is of size $\frac{H}{L}$ relative to the horizontal component, or about $10\,\text{km}/1000\,\text{km} = 0.01$. We see that the vorticity vector points mainly in the horizontal direction, and this is true even if we include the planetary contribution, which has size $\frac{fH}{U} = \text{Ro}^{-1}\frac{H}{L} \sim 0.1$, where Ro is the Rossby number. For a westerly jet stream with winds increasing upward in the troposphere, the vorticity vector points mostly toward the north.

As a result of this analysis, it may seem strange, but in fact for large-scale dynamic meteorology we are in general concerned only with the vertical components of absolute and relative vorticity, which are designated by η and ζ, respectively:

$$\eta \equiv \mathbf{k} \cdot (\nabla \times \mathbf{U}_a), \qquad \zeta \equiv \mathbf{k} \cdot (\nabla \times \mathbf{U})$$

In the remainder of this book, η and ζ are referred to as absolute and relative vorticities, respectively, without adding the explicit modifier "vertical component of." We defer a full explanation for why we focus on the relatively small vertical component of vorticity until after we have discussed potential vorticity, but the essential point is that, on synoptic and larger scales, the vertical component of vorticity is tightly coupled to the vertical gradient of potential temperature, which is very large compared to the horizontal gradient.

Regions of positive ζ are associated with cyclonic storms in the Northern Hemisphere; regions of negative ζ are associated with cyclonic storms in the Southern Hemisphere. In both cases ζ has the same sign as f, the local value of the planetary rotation, so that we may uniformly define cyclonic by $f\zeta > 0$. Thus, the distribution of relative vorticity is an excellent diagnostic for weather analysis.

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[2]Terms involving w are negligible by comparison for synoptic-scale weather systems.

The difference between absolute and relative vorticity is *planetary vorticity*, which is just the local vertical component of the vorticity of Earth due to its rotation; $\mathbf{k} \cdot \nabla \times \mathbf{U}_e = 2\Omega \sin\phi \equiv f$. Thus, $\eta = \zeta + f$ or, in Cartesian coordinates,

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \qquad \eta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f$$

The relationship between relative vorticity and relative circulation C discussed in the previous section can be clearly seen by considering an alternative approach in which the vertical component of vorticity is defined as the circulation about a closed contour in the horizontal plane divided by the area enclosed, in the limit where the area approaches zero:

$$\zeta \equiv \lim_{A \to 0} \left(\oint \mathbf{V} \cdot d\mathbf{l} \right) A^{-1} \tag{4.8}$$

This latter definition makes explicit the relationship between circulation and vorticity discussed in the introduction to this chapter. The equivalence of these two definitions of ζ is demonstrated by considering the circulation about a rectangular element of area $\delta x \delta y$ in the (x, y) plane, as shown in Figure 4.4. Evaluating $\mathbf{V} \cdot \mathbf{dl}$ for each side of the rectangle in the figure yields the circulation

$$\begin{aligned}\delta C &= u\delta x + \left(v + \frac{\partial v}{\partial x}\delta x \right)\delta y - \left(u + \frac{\partial u}{\partial y}\delta y \right)\delta x - v\delta y \\ &= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)\delta x \delta y\end{aligned}$$

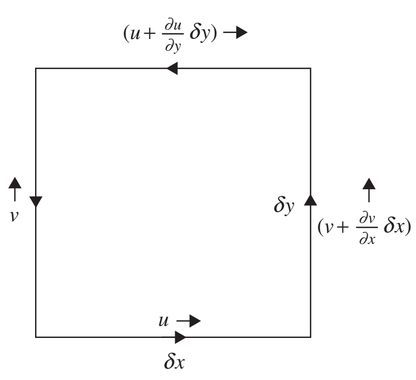

FIGURE 4.4 Relationship between circulation and vorticity for an area element in the horizontal plane.

Dividing through by the area $\delta A = \delta x \delta y$ gives

$$\frac{\delta C}{\delta A} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \equiv \zeta$$

In more general terms the relationship between vorticity and circulation is given simply by Stokes's theorem applied to the velocity vector:

$$\oint \mathbf{U} \cdot d\mathbf{l} = \iint_A (\nabla \times \mathbf{U}) \cdot \mathbf{n} \, dA$$

Here A is the area enclosed by the contour and $\mathbf{n}$ is a unit normal to the area element dA (positive in the right sense). Thus, Stokes's theorem states that the circulation about any closed loop is equal to the integral of the normal component of vorticity over the area enclosed by the contour. Hence, for a finite area, circulation divided by area gives the *average* normal component of vorticity in the region. As a consequence, the vorticity of a fluid in solid-body rotation is just twice the angular velocity of rotation. Vorticity may thus be regarded as a measure of the local angular velocity of the fluid.

4.2.1 Vorticity in Natural Coordinates

Physical interpretation of vorticity is facilitated by considering the vertical component of vorticity in the natural coordinate system (see Section 3.2.1). If we compute the circulation about the infinitesimal contour shown in Figure 4.5, we obtain[3]

$$\delta C = V\left[\delta s + d(\delta s)\right] - \left(V + \frac{\partial V}{\partial n}\delta n\right)\delta s$$

However, from this figure, $d(\delta s) = \delta\beta\delta n$, where $\delta\beta$ is the angular change in the wind direction in the distance δs. Hence,

$$\delta C = \left(-\frac{\partial V}{\partial n} + V\frac{\delta\beta}{\delta s}\right)\delta n \, \delta s$$

or, in the limit $\delta n, \delta s \to 0$

$$\zeta = \lim_{\delta n, \delta s \to 0} \frac{\delta C}{(\delta n \, \delta s)} = -\frac{\partial V}{\partial n} + \frac{V}{R_s} \tag{4.9}$$

[3] Recall that n is a coordinate in the horizontal plane perpendicular to the local flow direction with positive values to the left of an observer facing downstream.

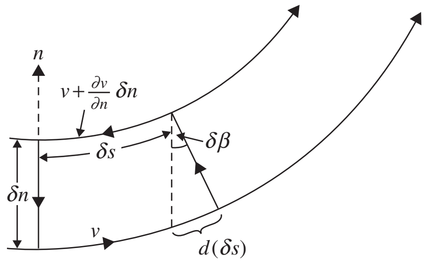

FIGURE 4.5 Circulation for an infinitesimal loop in the natural coordinate system.

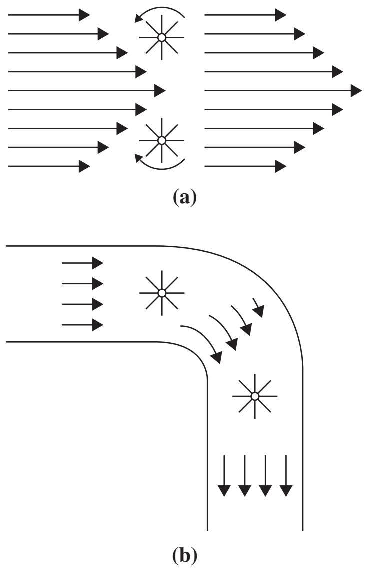

FIGURE 4.6 Two types of two-dimensional flow: (a) linear shear flow with vorticity and (b) curved flow with zero vorticity.

where R_s is the radius of curvature of the streamlines [Eq. (3.20)]. It is now apparent that the net vertical vorticity component is the result of the sum of two parts: (1) the rate of change of wind speed normal to the direction of flow $-\partial V/\partial n$, called the *shear* vorticity; and (2) the turning of the wind along a streamline V/R_s, called the *curvature* vorticity. Thus, even straight-line motion may have vorticity if the speed changes normal to the flow axis. For example, in the jet stream shown schematically in Figure 4.6a, there will be cyclonic relative vorticity north of the velocity maximum and anticyclonic relative vorticity to the south (Northern Hemisphere conditions), as is recognized easily when the turning of a small paddle wheel placed in the flow is considered. The lower of the two paddle wheels in Figure 4.6a will turn in a clockwise direction (anticyclonically) because the wind force on the blades north of its axis of rotation is stronger than the force on the blades to the south of the axis. The upper wheel will, of course, experience a counterclockwise (cyclonic) turning. Thus, the poleward and equatorward sides of a westerly jet stream are referred to as the cyclonic and anticyclonic shear sides, respectively.

Conversely, curved flow may have zero vorticity provided that the shear vorticity is equal and opposite to the curvature vorticity. This is the case in the example shown in Figure 4.6b, where a frictionless fluid with zero relative vorticity upstream flows around a bend in a canal. The fluid along the inner boundary on the curve flows faster in just the right proportion so that the paddle wheel does not turn.

4.3 THE VORTICITY EQUATION

The previous section discussed kinematic properties of vorticity. This section addresses vorticity dynamics using the equations of motion to determine contributions to the time rate of change of vorticity.

4.3.1 Cartesian Coordinate Form

For motions of synoptic scale, the vorticity equation can be derived using the approximate horizontal momentum equations (2.24) and (2.25). We differentiate the zonal component equation with respect to y and the meridional component equation with respect to x:

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-fv=-\frac{1}{\rho}\frac{\partial p}{\partial x}\right) \tag{4.10}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+fu=-\frac{1}{\rho}\frac{\partial p}{\partial y}\right) \tag{4.11}$$

Subtracting (4.10) from (4.11) and recalling that $\zeta = \partial v/\partial x - \partial u/\partial y$, we obtain the vorticity equation

$$\begin{aligned}\frac{\partial \zeta}{\partial t}+u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}+w\frac{\partial \zeta}{\partial z}+(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\\ +\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)+v\frac{df}{dy}=\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)\end{aligned} \tag{4.12}$$

Using the fact that the Coriolis parameter depends only on y so that $Df/Dt = v(df/dy)$, (4.12) may be rewritten in the form

$$\frac{D}{Dt}(\zeta+f) = -(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)+\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right) \tag{4.13}$$

Equation (4.13) states that the rate of change of the absolute vorticity following the motion is given by the sum of the three terms on the right, called the divergence (or vortex stretching) term, the tilting or twisting term, and the solenoidal term, respectively.

The concentration or dilution of vorticity by the divergence field—the first term on the right in (4.13)—is the fluid analog of the change in angular velocity resulting from a change in the moment of inertia of a solid body when angular momentum is conserved. If the horizontal flow is divergent, the area enclosed by a chain of fluid parcels will increase with time, and if circulation is to be conserved, the average absolute vorticity of the enclosed fluid must decrease (i.e., the vorticity will be diluted). If, however, the flow is convergent, the area enclosed by a chain of fluid parcels will decrease with time and the vorticity will be concentrated. This mechanism for changing vorticity following the motion is very important in synoptic-scale disturbances.

The second term on the right in (4.13) represents vertical vorticity generated by the tilting of horizontally oriented components of vorticity into the vertical by a nonuniform vertical motion field. This mechanism is illustrated in Figure 4.7,

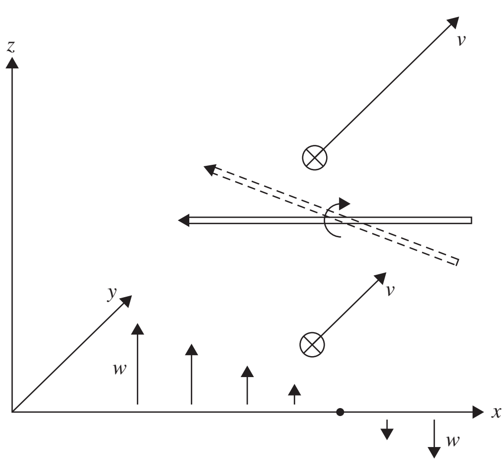

FIGURE 4.7 Vorticity generation by the tilting of a horizontal vorticity vector (*double arrow*).

which shows a region where the y component of velocity is increasing with height so that there is a component of shear vorticity oriented in the negative x direction as indicated by the double arrow. If at the same time there is a vertical motion field in which w decreases with increasing x, advection by the vertical motion will tend to tilt the vorticity vector initially oriented parallel to x so that it has a component in the vertical. Thus, if $\partial v/\partial z > 0$ and $\partial w/\partial x < 0$, there will be a generation of positive vertical vorticity.

Finally, the third term on the right in (4.13) is just the microscopic equivalent of the solenoidal term in the circulation theorem (4.5). To show this equivalence, we may apply Stokes's theorem to the solenoidal term to get

$$-\oint \alpha dp \equiv -\oint \alpha \nabla p \cdot d\mathbf{l} = -\iint_A \nabla \times (\alpha \nabla p) \cdot \mathbf{k} dA$$

where A is the horizontal area bounded by the curve $\mathbf{l}$. Applying the vector identity $\nabla \times (\alpha \nabla p) \equiv \nabla \alpha \times \nabla p$, the equation becomes

$$-\oint \alpha dp = -\iint_A (\nabla \alpha \times \nabla p) \cdot \mathbf{k} dA$$

However, the solenoidal term in the vorticity equation can be written

$$-\left(\frac{\partial \alpha}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \alpha}{\partial y}\frac{\partial p}{\partial x}\right) = -(\nabla \alpha \times \nabla p) \cdot \mathbf{k}$$

Thus, the solenoidal term in the vorticity equation is just the limit of the solenoidal term in the circulation theorem divided by the area when the area goes to zero.

4.3.2 The Vorticity Equation in Isobaric Coordinates

A somewhat simpler form of the vorticity equation arises when the motion is referred to the isobaric coordinate system. This equation can be derived in vector form by operating on the momentum equation (3.2) with the vector operator $\mathbf{k} \cdot \nabla \times$, where ∇ now indicates the horizontal gradient on a surface of constant pressure. However, to facilitate this process it is desirable to first use the vector identity

$$(\mathbf{V} \cdot \nabla)\mathbf{V} = \nabla \left(\frac{\mathbf{V} \cdot \mathbf{V}}{2}\right) + \zeta\, \mathbf{k} \times \mathbf{V} \tag{4.14}$$

where $\zeta = \mathbf{k} \cdot (\nabla \times \mathbf{V})$, to rewrite (3.2) as

$$\frac{\partial \mathbf{V}}{\partial t} = -\nabla \left(\frac{\mathbf{V} \cdot \mathbf{V}}{2} + \Phi\right) - (\zeta + f)\mathbf{k} \times \mathbf{V} - \omega \frac{\partial \mathbf{V}}{\partial p} \tag{4.15}$$

We now apply the operator $\mathbf{k}\cdot\nabla\times$ to (4.15). Using the facts that for any scalar A, $\nabla\times\nabla A=0$ and for any vectors $\mathbf{a}$, $\mathbf{b}$,

$$\nabla\times(\mathbf{a}\times\mathbf{b})=(\nabla\cdot\mathbf{b})\,\mathbf{a}-(\mathbf{a}\cdot\nabla)\,\mathbf{b}-(\nabla\cdot\mathbf{a})\,\mathbf{b}+(\mathbf{b}\cdot\nabla)\,\mathbf{a} \tag{4.16}$$

we can eliminate the first term on the right and simplify the second term so that the resulting vorticity equation becomes

$$\frac{\partial\zeta}{\partial t}=-\mathbf{V}\cdot\nabla(\zeta+f)-\omega\frac{\partial\zeta}{\partial p}-(\zeta+f)\,\nabla\cdot\mathbf{V}+\mathbf{k}\cdot\left(\frac{\partial\mathbf{V}}{\partial p}\times\nabla\omega\right) \tag{4.17}$$

Comparing (4.13) and (4.17), we see that in the isobaric system there is no vorticity generation by pressure-density solenoids. This difference arises because in the isobaric system, horizontal partial derivatives are computed with p held constant so that the vertical component of vorticity is $\zeta=(\partial v/\partial x-\partial u/\partial y)_p$, whereas in height coordinates it is $\zeta=(\partial v/\partial x-\partial u/\partial y)_z$. In practice the difference is generally unimportant because, as shown in the next section, the solenoidal term is usually sufficiently small so that it can be neglected for synoptic-scale motions.

4.3.3 Scale Analysis of the Vorticity Equation

In Section 2.4 the equations of motion were simplified for synoptic-scale motions by evaluating the order of magnitude of various terms. The same technique can be applied to the vorticity equation. Characteristic scales for the field variables based on typical observed magnitudes for synoptic-scale motions are chosen as follows:

$U\sim 10\ \mathrm{m\ s^{-1}}$	horizontal scale
$W\sim 1\ \mathrm{cm\ s^{-1}}$	vertical scale
$L\sim 10^6\ \mathrm{m}$	length scale
$H\sim 10^4\ \mathrm{m}$	depth scale
$\delta p\sim 10\ \mathrm{hPa}$	horizontal pressure scale
$\rho\sim 1\ \mathrm{kg\ m^{-3}}$	mean density
$\delta\rho/\rho\sim 10^{-2}$	fractional density fluctuation
$L/U\sim 10^5\ \mathrm{s}$	time scale
$f_0\sim 10^{-4}\ \mathrm{s^{-1}}$	Coriolis parameter
$\beta\sim 10^{-11}\ \mathrm{m^{-1}\ s^{-1}}$	"beta" parameter

Again we have chosen an advective time scale because the vorticity pattern, like the pressure pattern, tends to move at a speed comparable to the horizontal wind speed. Using these scales to evaluate the magnitude of the terms in (4.12), we first note that

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \lesssim \frac{U}{L} \sim 10^{-5}\mathrm{s}^{-1}$$

where the inequality in this expression means less than or equal to in order of magnitude. Thus,

$$\zeta / f_0 \lesssim U/\,(f_0\,L) \equiv \mathrm{Ro} \sim 10^{-1}$$

For midlatitude synoptic-scale systems, the relative vorticity is often small (order Rossby number) compared to the planetary vorticity. For such systems, ζ may be neglected compared to f in the divergence term in the vorticity equation

$$(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \approx f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

This approximation does not apply near the center of intense cyclonic storms. In such systems $|\zeta/f| \sim 1$, and the relative vorticity should be retained.

The magnitudes of the various terms in (4.12) can now be estimated as

$$\begin{aligned}
\frac{\partial \zeta}{\partial t}, u\frac{\partial \zeta}{\partial x}, v\frac{\partial \zeta}{\partial y} &\sim \frac{U^2}{L^2} \sim 10^{-10}\ \mathrm{s}^{-2}\\
w\frac{\partial \zeta}{\partial z} &\sim \frac{WU}{HL} \sim 10^{-11}\ \mathrm{s}^{-2}\\
v\frac{df}{dy} &\sim U\beta \sim 10^{-10}\ \mathrm{s}^{-2}\\
f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &\lesssim \frac{f_0 U}{L} \sim 10^{-9}\ \mathrm{s}^{-2}\\
\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) &\lesssim \frac{WU}{HL} \sim 10^{-11}\ \mathrm{s}^{-2}\\
\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right) &\lesssim \frac{\delta\rho\delta p}{\rho^2 L^2} \sim 10^{-11}\ \mathrm{s}^{-2}
\end{aligned}$$

The inequality is used in the last three terms because in each case it is possible that the two parts of the expression might partially cancel so that the actual magnitude would be less than indicated. In fact, this must be the case for the divergence term (the fourth in the list) because if $\partial u/\partial x$ and $\partial v/\partial y$ were not nearly equal and opposite, the divergence term would be an order of magnitude greater than any other term and the equation could not be satisfied. Therefore, scale analysis of the vorticity equation indicates that synoptic-scale motions must be quasi-nondivergent. The divergence term will be small enough to be balanced by the vorticity advection terms only if

$$\left|\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right| \lesssim 10^{-6}\ \mathrm{s}^{-1}$$

so that the horizontal divergence must be small compared to the vorticity in synoptic-scale systems. From the aforementioned scalings and the definition of the Rossby number, we see that

$$\left|\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\Big/ f_0\right| \lesssim \mathrm{Ro}^2$$

and

$$\left|\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\Big/ \zeta\right| \lesssim \mathrm{Ro}$$

Thus, the ratio of the horizontal divergence to the relative vorticity is the same magnitude as the ratio of relative vorticity to planetary vorticity.

Retaining only the terms of order $10^{-10}\ \mathrm{s}^{-2}$ in the vorticity equation yields the approximate form valid for synoptic-scale motions,

$$\frac{D_h(\zeta+f)}{Dt} = -f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) \tag{4.18}$$

where

$$\frac{D_h}{Dt} \equiv \frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}$$

As mentioned earlier, (4.18) is not accurate in intense cyclonic storms. For these the relative vorticity should be retained in the divergence term:

$$\frac{D_h(\zeta+f)}{Dt} = -(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) \tag{4.19}$$

Equation (4.18) states that the change of absolute vorticity following the horizontal motion on the synoptic scale is given approximately by the concentration or dilution of planetary vorticity caused by the convergence or divergence of the horizontal flow, respectively. In (4.19), however, it is the concentration or dilution of absolute vorticity that leads to changes in absolute vorticity following the motion.

The form of the vorticity equation given in (4.19) also indicates why cyclonic disturbances can be much more intense than anticyclones. For a fixed amplitude of convergence, relative vorticity will increase, and the factor $(\zeta+f)$ becomes larger, which leads to even higher rates of increase in the relative vorticity. For a fixed rate of divergence, however, relative vorticity will decrease, but when $\zeta \to -f$, the divergence term on the right approaches zero and the relative vorticity cannot become more negative no matter how strong the divergence. (This difference in the potential intensity of cyclones and anticyclones was discussed in Section 3.2.5 in connection with the gradient wind approximation.)

The approximate forms given in (4.18) and (4.19) do not remain valid, however, in the vicinity of atmospheric fronts. The horizontal scale of variation in frontal zones is only ~100 km, and the vertical velocity scale is ~10 cm s^{-1}. For these scales, vertical advection, tilting, and solenoidal terms all may become as large as the divergence term.

4.4 POTENTIAL VORTICITY

We return now to circulation to show that Kelvin's theorem applies to baroclinic dynamics for certain integration contours, which has very deep implications for dynamic meteorology. Rather than arbitrary closed contours, we now restrict the possibilities to those that fall on isentropic surfaces, for which potential temperature is constant. We prove that for these contours, the solenoidal term is exactly zero.

With the aid of the ideal gas law (1.25), the definition of potential temperature (2.44) can be expressed as a relationship between pressure and density for a surface of constant θ:

$$\rho = p^{c_v/c_p}(R\theta)^{-1}(p_s)^{R/c_p}$$

Hence, on an isentropic surface, density is a function of pressure alone, and the solenoidal term in the circulation theorem (4.3) vanishes;

$$\oint \frac{dp}{\rho} \propto \oint dp^{(1-c_v/c_p)} = 0$$

For adiabatic and frictionless flow, then, the circulation computed for a closed chain of fluid parcels on a constant θ surface reduces to the same form as in a barotropic fluid; that is, it *satisfies Kelvin's circulation theorem* even though the fluid is baroclinic!

From Stokes's theorem, we may replace the circulation in favor of the component of vorticity normal to the surface

$$C_a = \oint \mathbf{U}_a \cdot d\mathbf{l} = \iint_A \omega_a \cdot \mathbf{n}\, dA \tag{4.20}$$

where $\mathbf{n}$ is a unit vector normal to the surface. Now we replace $\mathbf{n}$ and dA in favor of conserved quantities. Consider an infinitesimal cylinder (Figure 4.8) that is bounded above and below by constant potential temperature surfaces; for adiabatic flow, both the potential temperature and the mass of air in this cylinder, dm, are conserved. A leading-order Taylor approximation across the cylinder gives $d\theta \approx |\nabla\theta| dh$, where dh is the "height" of the cylinder. The mass

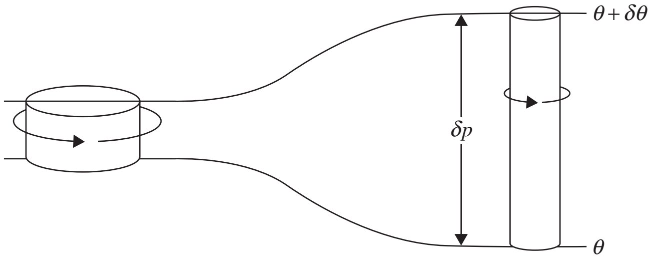

FIGURE 4.8 A cylindrical column of air moving adiabatically, conserving potential vorticity.

of air in the cylinder is given by $dm = \rho dAdh$ so that

$$dA = \frac{dm}{\rho}\frac{|\nabla\theta|}{d\theta} \tag{4.21}$$

Taking the infinitesimal cylinder to be sufficiently small so that the vorticity normal to the surface is nearly constant,

$$\frac{DC_a}{Dt} \approx \frac{D}{Dt}\left[\boldsymbol{\omega}_a \cdot \mathbf{n}\, dA\right] = 0 \tag{4.22}$$

where

$$\mathbf{n} = \frac{\nabla\theta}{|\nabla\theta|} \tag{4.23}$$

Using (4.21) and (4.23) in (4.22), and noting that both dm and $d\theta$ are conserved following the motion, gives

$$\frac{D}{Dt}\left[\frac{\boldsymbol{\omega}_a \cdot \nabla\theta}{\rho}\right] = 0 \tag{4.24}$$

This celebrated equation, the **Ertel potential vorticity theorem**, is one of the most important theoretical results in dynamic meteorology. It says that the potential vorticity, or PV,

$$\Pi = \frac{\boldsymbol{\omega}_a \cdot \nabla\theta}{\rho} \tag{4.25}$$

is conserved following the motion.

What makes this result so profound is that it links all of the basic physical conservation laws into a single expression. Although it may appear, as one studies a fluid, that the momentum field fluctuates independently of the thermodynamic fields, the potential vorticity places a powerful constraint on those fluctuations: They must evolve in a way that preserves the PV following the motion. Take, for example, the situation where the absolute vorticity vector has no horizontal components so that the PV is given by

$$\frac{1}{\rho}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f\right)\frac{\partial \theta}{\partial z} \tag{4.26}$$

In this case we see that the PV is the product of the absolute vorticity and the static stability so that if one increases, the other must decrease. This is illustrated in Figure 4.8: As the air parcel moves to the right, the static stability decreases, and as a result the vorticity must increase to conserve PV; the connections to angular momentum conservation and to vortex stretching are particularly clear.

The vertical contribution to the PV in (4.25) scales like $\frac{U\theta^*}{\rho HL} + \frac{f\theta^*}{H}$, where θ^* and ρ^* are typical potential temperature and density scale values; the second term represents the contribution from planetary rotation. The horizontal contribution—that is, from expanding (4.25)—scales like $\frac{U\theta^*}{\rho^* HL}$ (neglecting the small contribution from w), so that the ratio of vertical to horizontal contributions is

$$1 + \mathrm{Ro}^{-1} \tag{4.27}$$

On synoptic and larger scales, where the Rossby number is small (roughly 0.1), this analysis shows that the dominant contribution to the PV comes from the vertical direction. This serves to *explain why we focus on the vertical component of vorticity in dynamic meteorology*: Although the vorticity vector lies primarily in the horizontal direction, the gradient of potential temperature is mainly in the vertical and this promotes the importance of vertical vorticity. A leading-order approximation to the PV is

$$\Pi \approx \frac{f}{\rho}\frac{\partial \theta}{\partial z} \tag{4.28}$$

Picking typical values for the troposphere, we get an estimate of characteristic PV values of

$$\begin{aligned}\Pi_{trop} &\cong \left(\frac{10^{-4}\mathrm{s}^{-1}}{1\mathrm{kg\ m}^{-3}}\right)\left(5\ \mathrm{K\ km}^{-1}\right)\\ &= 0.5 \times 10^{-6}\ \mathrm{K\ m^2\ s^{-1}\ kg^{-1}} \equiv 0.5\,\mathrm{PVU}\end{aligned} \tag{4.29}$$

The definition "PVU" is a commonly adopted scaling of "PV Units" ($1\,\mathrm{PVU} = 10^{-6}\ \mathrm{K\ m^2\ s^{-1}\ kg^{-1}}$). In the lower stratosphere, where $\frac{\partial\theta}{\partial z}$ is an order of magnitude larger, characteristic values of PV are larger as well.

Because there is an abrupt jump in PV values at the troposphere PV provides a useful dynamical definition for that interface. In fact, for adiabatic and frictionless conditions, when both PV and potential temperature are conserved, maps of potential temperature on the "dynamical" tropopause (defined as a PV surface) provide a particularly useful summary of extratropical weather systems,

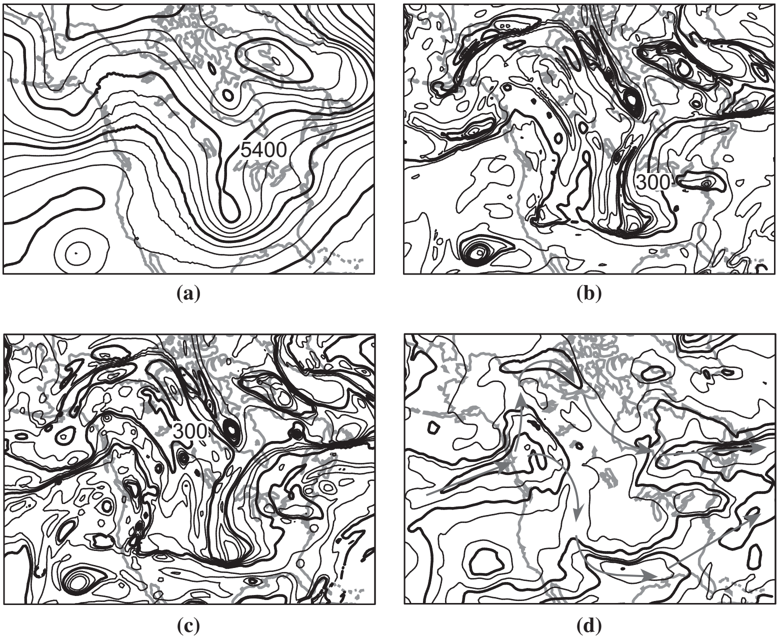

FIGURE 4.9 Comparison of 500-hPa geopotential height (a) and tropopause maps of pressure (b) potential temperature (c) and wind speed (d) on January 12, 2012. Geopotential height is shown every 60 m, potential temperature every 5 K, pressure every 50 hPa, and wind speed every 10 m s^{-1}. Every fourth contour is shown *bold* except every second contour for wind speed. *Gray arrows* in (d) illustrate the path of the main branches of jet streams.

because the potential temperature contours are simply advected by the wind on the tropopause. Moreover, compared to the alternative of PV contours on potential temperature surfaces, a single map suffices to describe the state of the tropopause.

An example comparing the structure of the dynamical tropopause with a conventional map of 500-hPa geopotential height is shown in Figure 4.9. Familiar features on the 500 hPa chart include a long-wave pattern with a ridge over western North America and a trough over the center of the continent, and short-wave troughs near New York and approaching the Pacific coastline near the Alaska panhandle. In addition to the wave-like features, cyclonic vortices are located southwest of California, over Hudson Bay, and near Baffin Island.

Tropopause pressure (Figure 4.9b) reveals that troughs are regions where the tropopause is depressed to lower altitudes; in fact, the tropopause is below 500 hPa in several of the troughs and vortices. Therefore, troughs are associated with large values of stratospheric potential vorticity that have been locally lowered to elevations normally considered tropospheric. In ridges and subtropical latitudes, the tropopause is elevated, with pressure values of 200 to 250 hPa, and therefore the PV is anomalously low compared to the surroundings.

Tropopause potential temperature (Figure 4.9c) shows that most of the troughs apparent in the 500-hPa chart are in fact vortical near their core. We may infer this by the appearance of closed potential temperature contours, since, if potential temperature and potential vorticity are conserved, air is trapped within closed tropopause potential temperature contours. For this reason, these features are sometimes called material eddies, since the only means of moving the disturbance is to displace the material comprising it. This stands in contrasts to waves, which may propagate information without net transport of the medium. In addition to the disturbances, notice the long ribbons where the horizontal gradient of tropopause potential temperature is concentrated into fronts, separated by zones that are mixed and filled with small-scale noise. Along some of these fronts—for example, the base of the trough in the central United States—the pressure field reveals that the tropopause is essentially vertical.

Finally, since jet streams are located near the tropopause, this perspective is much more useful than isobaric surfaces, which require several maps to depict jet streams located at different elevations. We see that a midlatitude jet reaches the west coast of Canada before splitting into northern and southern branches; the southern branch also has a subtropical connection that is apparent near the southern edge of the domain (Figure 4.9d).

For completeness, we note that a form of the Ertel potential vorticity equation may be derived that includes a source of momentum, $\mathcal{F}$, such as frictional dissipation, and a source of entropy, $\mathcal{H}$, such as latent heating. The result is

$$\frac{D\Pi}{Dt} = \frac{\boldsymbol{\omega}_a}{\rho} \boldsymbol{\cdot} \nabla \mathcal{H} + \frac{\nabla\theta}{\rho} \boldsymbol{\cdot} \left(\nabla \times \frac{\mathcal{F}}{\rho} \right) \tag{4.30}$$

where Π is the Ertel PV defined by (4.25). The first right-side term is positive, where the vorticity vector points in the direction of a local maximum in the entropy source. This often occurs near extratropical cyclones, where clouds and precipitation are found in the lower and midtroposphere (Figure 4.10a).

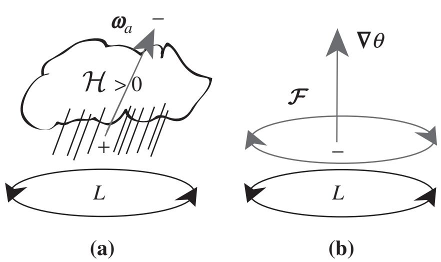

FIGURE 4.10 Cartoon illustrating the role of sources of (a) entropy and (b) momentum on changing the potential vorticity. (a) Depicts a precipitating cloud above a surface cyclone, where the vorticity vector, $\boldsymbol{\omega}$, points upward and to the right. Latent heating reaches a maximum in the cloud, which produces a positive (negative) PV tendency below (above) the cloud. (b) Depicts the role of surface friction (*gray arrow* opposite surface circulation). The curl of surface friction points in the opposite direction of the vorticity vector, producing a negative PV tendency.

In this case, near the surface the vertical component of vorticity points upward toward a maximum in latent heat release where condensation takes place; above the maximum in heating, potential vorticity decreases because the vorticity vector and gradient in heating point in opposite directions. The second right-side term is positive where the curl of the frictional force points in the same direction as the gradient in entropy. Again, appealing to extratropical cyclones, assuming that surface friction acts opposite to the motion, the curl of friction is a vector that points in the opposite direction from the gradient in entropy, which produces a negative tendency to the Ertel PV (Figure 4.10b).

4.5 SHALLOW WATER EQUATIONS

In a homogeneous incompressible fluid, potential vorticity conservation takes a somewhat simpler form. To see this, we first derive the shallow water equations, which provide a very useful simplification that we will use later to isolate essential aspects of atmospheric dynamics. Beyond constant density, the shallow approximation implies that the depth of the fluid is small compared to the the horizontal scale of the features of interest. As a result of this small aspect ratio, we may use the hydrostatic approximation

$$\frac{\partial p}{\partial z} = \rho_0 g \tag{4.31}$$

where ρ_0 is a constant density. Integrating the hydrostatic equation over the depth of the fluid, $h(x, y)$, gives

$$p(z) = \rho_0 g(h - z) + p(h) \tag{4.32}$$

where $p(h)$ is the pressure at the top of the layer of shallow water due to the layer above, which we take to be a constant. Using (4.32) to replace pressure in the momentum equation gives

$$\frac{D_h \mathbf{V}}{Dt} = -g\nabla_h h \; - f\mathbf{k} \times \mathbf{V} \tag{4.33}$$

We assume that $\mathbf{V}$ is initially a function of (x, y) only, and since h is a function of (x, y), (4.33) indicates that $\mathbf{V}$ will remain two-dimensional for all time.

Mass conservation for a constant density fluid has the simple form

$$\nabla \cdot (u, v, w) = 0 \tag{4.34}$$

Incompressibility also simplifies the first law of thermodynamics, since the fluid can no longer do work. Therefore, if no heat is added, the temperature is constant following the motion (see 2.41). Water pressure is a function of density and temperature, $p = f(T, \rho)$, but following the motion incompressibility implies

$dp = \frac{\partial f}{\partial T} dT + \frac{\partial f}{\partial \rho} d\rho = 0$, so that

$$\frac{Dp}{Dt} = 0 \tag{4.35}$$

Applying this fact to (4.32) yields

$$\frac{D_h h}{Dt} = w(h) \tag{4.36}$$

where $w = \frac{Dz}{Dt}$ is the vertical motion, which *is* a function of (x, y, z). Integrating (4.34) over depth h gives $w(h) = -h\nabla_h \boldsymbol{\cdot} \mathbf{V}$, which may be used to eliminate w from (4.36)

$$\frac{D_h h}{Dt} = -h\nabla_h \boldsymbol{\cdot} \mathbf{V} \tag{4.37}$$

The shallow water equations consist of (4.33) and (4.37), which describe the evolution of three unknowns (u, v, h). Following a derivation similar to that in Section 4.3.1, the shallow water vorticity equation may be derived from (4.33) and yields

$$\frac{D_h}{Dt}(\zeta + f) = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{4.38}$$

In the shallow water system, absolute vorticity increases following the motion through vortex stretching.

Shallow water potential vorticity conservation is obtained by replacing divergence on the right side of (4.38) using (4.37) to give

$$\frac{D_h}{Dt}\left[\frac{\zeta + f}{h}\right] = 0 \tag{4.39}$$

Thus, the shallow water potential vorticity,$(\zeta + f)/h$, is given by the absolute vorticity divided by the fluid depth. Following the motion, if the absolute vorticity increases, then the depth of the fluid must as well. One may think of the inverse depth as an analog of static stability in shallow water by noting that the layer top and bottom surfaces are isentropic surfaces. When the layer depth, h, becomes smaller, the isentropes are closer together.

Since the potential vorticity depends on x and y, only (4.39) provides a dramatic simplification to the Ertel PV, which is very useful for gaining insight into large-scale dynamics. For example, consider westerly flow impinging on an infinitely long topographic barrier as shown in Figure 4.11. We suppose that flow upstream of the mountain barrier is uniform with $\zeta = 0$. Each fluid column of depth h is confined between constant entropy surfaces θ_0 and $\theta_0 + \delta\theta$ and remains between those surfaces as it crosses the barrier. Air that previously crossed the barrier leaves the upper surface higher near the mountain both upstream and downstream, for reasons that will be explored in Chapter 5.

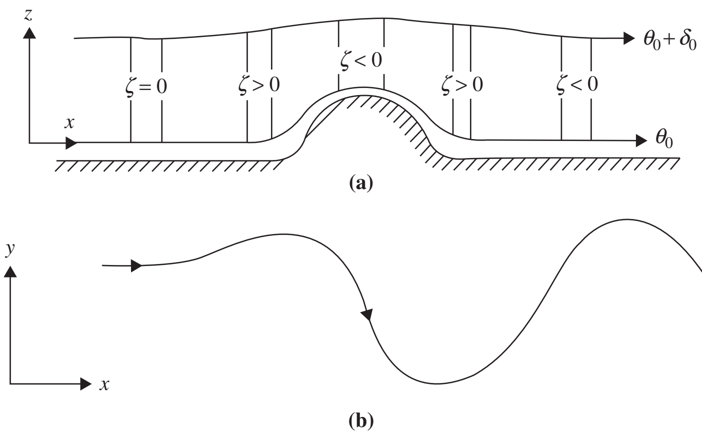

FIGURE 4.11 Schematic view of westerly flow over a topographic barrier: (a) the depth of a fluid column as a function of x and (b) the trajectory of a parcel in the $(x, \ y)$ plane.

As fluid columns approach the topographic barrier, they stretch vertically, which causes ζ to increase in order to conserve potential vorticity, $(\zeta + f)/h$. Positive cyclonic vorticity is associated with cyclonic curvature in the flow, which causes a poleward drift so that f also increases; this in turn reduces the change in ζ required for potential vorticity conservation. As the column begins to cross the barrier, its vertical extent decreases, and the relative vorticity must then become negative. Thus, the air column will acquire anticyclonic vorticity and move southward.

When the air column has passed over the barrier and returned to its original depth, it will be south of its original latitude so that f will be smaller and the relative vorticity must be positive. Thus, the trajectory must have cyclonic curvature and the column will be deflected poleward. When the parcel returns again to its original latitude, it will still have a poleward velocity component and will continue poleward gradually, acquiring anticyclonic curvature until its direction is again reversed. The parcel will then move downstream, conserving potential vorticity by following a wave-like trajectory in the horizontal plane. Therefore, steady westerly flow over a large-scale ridge will result in an anticyclonic flow over the mountain, a cyclonic flow pattern to the east of the barrier, followed by a wavetrain downstream.

The situation for easterly flow impinging on a mountain barrier is quite different (Figure 4.12). Upstream stretching leads to a cyclonic turning of the flow, which results in an equatorward component of motion. As the column moves westward and equatorward over the barrier, its depth contracts and its absolute vorticity must then decrease so that potential vorticity can be conserved. This reduction in absolute vorticity arises both from development of anticyclonic relative vorticity and from a decrease in f due to the equatorward motion. The anticyclonic relative vorticity gradually turns the column so that when it reaches

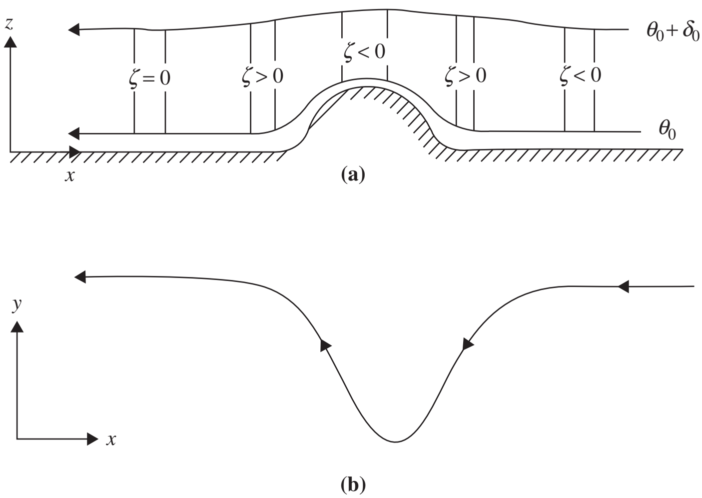

FIGURE 4.12 As in Figure 4.11, but for easterly flow.

the top of the barrier, it is headed westward. As it continues westward down the barrier, conserving potential vorticity, the process is simply reversed, with the result that some distance downstream from the mountain barrier, the air column is again moving westward at its original latitude.

Thus, the dependence of the Coriolis parameter on latitude creates a dramatic difference between westerly and easterly flow over large-scale topographic barriers. In the case of a westerly wind, the barrier generates a wave-like disturbance in the streamlines that extends far downstream. However, in the case of an easterly wind, the disturbance in the streamlines damps out away from the barrier.

4.5.1 Barotropic Potential Vorticity

Further simplification of potential vorticity is possible beyond the shallow water equations by making the barotropic assumption. Recall that a barotropic fluid is defined as one for which pressure depends only on density. For shallow water, (4.32) implies that in this case h and z are constant, so that $w = 0$ and the fluid is horizontally nondivergent

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4.40}$$

Potential vorticity conservation simplifies to

$$\frac{D_h}{Dt}(\zeta + f) = 0 \tag{4.41}$$

which states that absolute vorticity is conserved following the motion. Equation (4.41) is called the barotropic vorticity equation and has been widely used in theoretical studies of large-scale dynamical meteorology.

For nondivergent horizontal motion, the flow field can be represented by a *streamfunction* $\psi\,(x, y)$ defined so that the velocity components are given as $u = -\partial\psi/\partial y,\ v = +\partial\psi/\partial x$. The vorticity is then given by

$$\zeta = \partial v/\partial x - \partial u/\partial y = \partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 \equiv \nabla_h^2\psi$$

Thus, the velocity field and the vorticity can both be represented in terms of the variation of the single scalar field $\psi\,(x, y)$, and (4.41) can be written as a prognostic equation for vorticity in the form

$$\frac{\partial}{\partial t}\nabla^2\psi = -\mathbf{V}_\psi \cdot \nabla\left(\nabla^2\psi + f\right) \tag{4.42}$$

where $\mathbf{V}_\psi \equiv \mathbf{k} \times \nabla\psi$ is a nondivergent horizontal wind. Equation (4.42) states that the local tendency of relative vorticity is given by the advection of absolute vorticity. This equation can be solved numerically to predict the evolution of the streamfunction and thus of the vorticity and wind fields. Because the flow in the midtroposphere is often nearly nondivergent on the synoptic scale, (4.42) provides a surprisingly good model for short-term forecasts of the synoptic-scale 500-hPa flow field.

Conservation of absolute vorticity following the motion provides a strong constraint on the flow, as can be shown by a simple example that again illustrates an asymmetry between westerly and easterly flow. Suppose that at a certain point (x_0, y_0) the flow is in the zonal direction and the relative vorticity vanishes so that $\eta(x_0, y_0) = f_0$. Then, if absolute vorticity is conserved, the motion at any point along a parcel trajectory that passes through (x_0, y_0) must satisfy $\zeta + f = f_0$. Because f increases toward the north, trajectories that curve northward in the downstream direction must have $\zeta = f_0 - f < 0$, whereas trajectories that curve southward must have $\zeta = f_0 - f > 0$. However, as indicated in Figure 4.13, if the flow is westerly, northward curvature downstream implies $\zeta > 0$, whereas southward curvature implies $\zeta < 0$. Thus, westerly zonal flow must remain purely zonal if absolute vorticity is to be conserved following the motion. The easterly flow case, also shown in Figure 4.13, is just the opposite. Northward and southward curvatures are associated with negative and positive

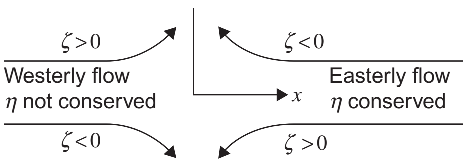

FIGURE 4.13 Absolute vorticity conservation for curved flow trajectories.

relative vorticities, respectively. Hence, an easterly current can curve either to the north or to the south and still conserve absolute vorticity.

4.6 ERTEL POTENTIAL VORTICITY IN ISENTROPIC COORDINATES

We consider here a more detailed treatment of the Ertel potential vorticity in isentropic coordinates, including nonconservative effects due to sources of momentum and entropy. We begin with a description of the basic conservation laws in isentropic coordinates.

4.6.1 Equations of Motion in Isentropic Coordinates

If the atmosphere is stably stratified so that potential temperature θ is a monotonically increasing function of height, θ may be used as an independent vertical coordinate. The vertical "velocity" in this coordinate system is just $\dot{\theta} \equiv D\theta/Dt$. Thus, adiabatic motions are two-dimensional when viewed in an isentropic coordinate frame. An infinitesimal control volume in isentropic coordinates with cross-sectional area δA and vertical extent $\delta\theta$ has a mass

$$\delta M = \rho \delta A \delta z = \delta A \left(-\frac{\delta p}{g} \right) = \frac{\delta A}{g} \left(-\frac{\partial p}{\partial \theta} \right) \delta\theta = \sigma \delta A \delta\theta \tag{4.43}$$

Here the "density" in $(x,\ y,\ \theta)$ space is defined as

$$\sigma \equiv -g^{-1} \partial p / \partial \theta \tag{4.44}$$

The horizontal momentum equation in isentropic coordinates may be obtained by transforming the isobaric form (4.15) to yield

$$\frac{\partial \mathbf{V}}{\partial t} + \nabla_\theta \left(\frac{\mathbf{V} \cdot \mathbf{V}}{2} + \Psi \right) + (\zeta_\theta + f)\,\mathbf{k} \times \mathbf{V} = -\dot{\theta} \frac{\partial \mathbf{V}}{\partial \theta} + \mathbf{F}_r \tag{4.45}$$

where ∇_θ is the gradient on an isentropic surface, $\zeta_\theta \equiv \mathbf{k} \cdot \nabla_\theta \times \mathbf{V}$ is the isentropic relative vorticity, and $\Psi \equiv c_p T + \Phi$ is the Montgomery streamfunction (see Problem 2.11 in Chapter 2). We have included a frictional term $\mathbf{F}_r$ on the right side, along with the diabatic vertical advection term. The continuity equation can be derived with the aid of (4.43) in a manner analogous to that used for the isobaric system in Section 3.1.2. The result is

$$\frac{\partial \sigma}{\partial t} + \nabla_\theta \cdot (\sigma \mathbf{V}) = -\frac{\partial}{\partial \theta} \left(\sigma \dot{\theta} \right) \tag{4.46}$$

The Ψ and σ fields are linked through the pressure field by the hydrostatic equation, which in the isentropic system takes the form

$$\frac{\partial \Psi}{\partial \theta} = \Pi(p) \equiv c_p \left(\frac{p}{p_s} \right)^{R/c_p} = c_p \frac{T}{\theta} \tag{4.47}$$

where Π is called the *Exner function*. Equations (4.44) through (4.47) form a closed set for prediction of $\mathbf{V}$, σ, Ψ, and p, provided that $\dot{\theta}$ and $\mathbf{F}_r$ are known.

4.6.2 The Potential Vorticity Equation

If we take $\mathbf{k}\cdot\nabla_\theta\times$(4.45) and rearrange the resulting terms, we obtain this isentropic vorticity equation:

$$\frac{\widetilde{D}}{Dt}(\zeta_\theta+f)+(\zeta_\theta+f)\,\nabla_\theta\cdot\mathbf{V}=\mathbf{k}\cdot\nabla_\theta\times\left(\mathbf{F}_r-\dot{\theta}\frac{\partial\mathbf{V}}{\partial\theta}\right) \tag{4.48}$$

where

$$\frac{\widetilde{D}}{Dt}=\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla_\theta$$

is the total derivative following the horizontal motion on an isentropic surface.

Noting that $\sigma^{-2}\partial\sigma/\partial t=-\partial\sigma^{-1}/\partial t$, we can rewrite (4.46) in the form

$$\frac{\widetilde{D}}{Dt}\left(\sigma^{-1}\right)-\left(\sigma^{-1}\right)\nabla_\theta\cdot\mathbf{V}=\sigma^{-2}\frac{\partial}{\partial\theta}\left(\sigma\dot{\theta}\right) \tag{4.49}$$

Multiplying each term in (4.48) by σ^{-1} and in (4.49) by $(\zeta_\theta+f)$ and adding, we obtain the desired conservation law:

$$\frac{\widetilde{D}\Pi}{Dt}=\frac{\partial\Pi}{\partial t}+\mathbf{V}\cdot\nabla_\theta\Pi=\frac{\Pi}{\sigma}\frac{\partial}{\partial\theta}\left(\sigma\dot{\theta}\right)+\sigma^{-1}\mathbf{k}\cdot\nabla_\theta\times\left(\mathbf{F}_r-\dot{\theta}\frac{\partial\mathbf{V}}{\partial\theta}\right) \tag{4.50}$$

where $\Pi\equiv(\zeta_\theta+f)/\sigma$ is the Ertel potential vorticity. If the diabatic and frictional terms on the right side of (4.50) can be evaluated, it is possible to determine the evolution of Π following the *horizontal* motion on an isentropic surface. When the diabatic and frictional terms are small, potential vorticity is approximately conserved following the motion on isentropic surfaces.

Weather disturbances that have sharp gradients in dynamical fields, such as jets and fronts, are associated with large anomalies in the Ertel potential vorticity. In the upper troposphere such anomalies tend to be advected rapidly under nearly adiabatic conditions. Thus, the potential vorticity anomaly patterns are conserved materially on isentropic surfaces. This material conservation property makes potential vorticity anomalies particularly useful in identifying and tracing the evolution of meteorological disturbances.

4.6.3 Integral Constraints on Isentropic Vorticity

The isentropic vorticity equation (4.48) can be written in the form

$$\frac{\partial\zeta_\theta}{\partial t}=-\nabla_\theta\cdot\left[(\zeta_\theta+f)\mathbf{V}\right]+\mathbf{k}\cdot\nabla_\theta\times\left(\mathbf{F}_r-\dot{\theta}\frac{\partial\mathbf{V}}{\partial\theta}\right) \tag{4.51}$$

Using the fact that any vector $\mathbf{A}$ satisfies the relationship

$$\mathbf{k} \cdot (\nabla_\theta \times \mathbf{A}) = \nabla_\theta \cdot (\mathbf{A} \times \mathbf{k})$$

we can rewrite (4.51) in the form

$$\frac{\partial \zeta_\theta}{\partial t} = -\nabla_\theta \cdot \left[(\zeta_\theta + f)\mathbf{V} - \left(\mathbf{F}_r - \dot{\theta} \frac{\partial \mathbf{V}}{\partial \theta} \right) \times \mathbf{k} \right] \tag{4.52}$$

Equation (4.52) expresses the remarkable fact that isentropic vorticity can only be changed by the divergence or convergence of the horizontal flux vector in brackets on the right side. The vorticity cannot be changed by vertical transfer across the isentropes. Furthermore, integration of (4.52) over the area of an isentropic surface and application of the divergence theorem (Appendix C.2) show that for an isentrope that does not intersect the surface of Earth, the global average of ζ_θ is constant. Furthermore, integration of ζ_θ over the sphere shows that the global average ζ_θ is exactly zero. Vorticity on such an isentrope is neither created nor destroyed; it is merely concentrated or diluted by horizontal fluxes along the isentropes.

SUGGESTED REFERENCES

Acheson, *Elementary Fluid Dynamics*, provides a good introduction to vorticity at a graduate level.

Hoskins et al. provide an advanced discussion of Ertel potential vorticity and its uses in diagnosis and prediction of synoptic-scale disturbances.

Pedlosky, *Geophysical Fluid Dynamics*, provides a thorough treatment of circulation, vorticity, and potential vorticity in Chapter 2.

Vallis, *Atmospheric and Oceanic Fluid Dynamics*, discusses vorticity and potential vorticity in Chapter 4.

Williams and Elder, *Fluid Physics for Oceanographers and Physicists*, provides an introduction to vorticity dynamics at an elementary level. This book also provides a good general introduction to fluid dynamics.

PROBLEMS

4.1. What is the circulation about a square of 1000 km on a side for an easterly (i.e., westward flowing) wind that decreases in magnitude toward the north at a rate of 10 m s^{-1} per 500 km? What is the mean relative vorticity in the square?

4.2. A cylindrical column of air at 30°N with radius 100 km expands to twice its original radius. If the air is initially at rest, what is the mean tangential velocity at the perimeter after expansion?

4.3. An air parcel at 30°N moves northward, conserving absolute vorticity. If its initial relative vorticity is 5×10^{-5} s^{-1}, what is its relative vorticity upon reaching 90°N?

4.4. An air column at 60°N with $\zeta = 0$ initially stretches from the surface to a fixed tropopause at 10 km height. If the air column moves until it is over a mountain barrier 2.5 km high at 45°N, what is its absolute vorticity and relative vorticity as it passes the mountain top, assuming that the flow satisfies the barotropic potential vorticity equation?

4.5. Find the average vorticity within a cylindrical annulus of inner radius 200 km and outer radius 400 km if the tangential velocity distribution is given by $V = A/r$, where $A = 10^6\ \mathrm{m^2\ s^{-1}}$ and r is in meters. What is the average vorticity within the inner circle of radius 200 km?

4.6. Show that the anomalous gradient wind cases discussed in Section 3.2.5 have negative absolute circulation in the Northern Hemisphere and thus have negative average absolute vorticity.

4.7. Compute the rate of change of circulation about a square in the (x, y) plane with corners at $(0, 0)$, $(0, L)$, (L, L), and $(L, 0)$ if temperature increases eastward at a rate of 1°C per 200 km and presure increases northward at a rate of 1 hPa per 200 km. Let $L = 1000$ km and the pressure at the point $(0, 0)$ be 1000 hPa.

4.8. Verify the identity (4.14) by expanding the vectors in Cartesian components.

4.9. Derive a formula for the dependence of depth on radius for an incompressible fluid in solid-body rotation in a cylindrical tank with a flat bottom and free surface at the upper boundary. Let H be the depth at the center of the tank, Ω be the angular velocity of rotation of the tank, and a be the radius of the tank.

4.10. By how much does the relative vorticity change for a column of fluid in a rotating cylinder if the column is moved from the center of the tank to a distance 50 cm from the center? The tank is rotating at the rate of 20 revolutions per minute, the depth of the fluid at the center is 10 cm, and the fluid is initially in solid-body rotation.

4.11. A cyclonic vortex is in cyclostrophic balance with a tangential velocity profile given by the expression $V = V_0(r/r_0)^n$, where V_0 is the tangential velocity component at the distance r_0 from the vortex center. Compute the circulation about a streamline at radius r, the vorticity at radius r, and the pressure at radius r. (Let p_0 be the pressure at r_0 and assume that density is a constant.)

4.12. A westerly zonal flow at 45° is forced to rise adiabatically over a north–south-oriented mountain barrier. Before striking the mountain, the westerly wind increases linearly toward the south at a rate of 10 m s^{-1} per 1000 km. The crest of the mountain range is at 800 hPa and the tropopause, located at 300 hPa, remains undisturbed. What is the initial relative vorticity of the air? What is its relative vorticity when it reaches the crest if it is deflected 5° latitude toward the south during the forced ascent? If the current assumes a uniform speed of 20 m s^{-1} during its ascent to the crest, what is the radius of curvature of the streamlines at the crest?

4.13. A cylindrical vessel of radius a and constant depth H rotating at an angular velocity Ω about its vertical axis of symmetry is filled with a homogeneous, incompressible fluid that is initially at rest with respect to the vessel. A volume of fluid V is then withdrawn through a point sink at the center of the cylinder, thus creating a vortex. Neglecting friction, derive an expression for the resulting relative azimuthal velocity as a function of radius (i.e., the velocity in a coordinate system rotating with the tank). Assume that the motion is independent of depth and that $V \ll \pi a^2 H$. Also compute the relative vorticity and the relative circulation.

4.14. **(a)** How far must a zonal ring of air initially at rest with respect to Earth's surface at 60° latitude and 100-km height be displaced latitudinally to acquire an easterly (east to west) component of 10 m s^{-1} with respect to Earth's surface?

(b) To what height must it be displaced vertically in order to acquire the same velocity? Assume a frictionless atmosphere.

4.15. The horizontal motion within a cylindrical annulus with permeable walls of inner radius 10 cm, outer radius 20 cm, and 10-cm depth is independent of height and azimuth and is represented by the expressions $u = 7 - 0.2r$, $v = 40 + 2r$, where u and v are the radial and tangential velocity components in cm s^{-1}, positive outward and counterclockwise, respectively, and r is distance from the center of the annulus in centimeters. Assuming an incompressible fluid, find

(a) Circulation about the annular ring

(b) Average vorticity within the annular ring

(c) Average divergence within the annular ring

(d) Average vertical velocity at the top of the annulus if it is zero at the base

4.16. Prove that, as stated in Eq. (4.52), the globally averaged isentropic vorticity on an isentropic surface that does not intersect the ground must be zero. Show that the same result holds for the isobaric vorticity on an isobaric surface.

MATLAB Exercises

M4.1. Equation 4.41 showed that for nondivergent horizontal motion, the flow field can be represented by a *streamfunction* $\psi\,(x, y)$, and the vorticity is then given by $\zeta = \partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 \equiv \nabla^2\psi$. Thus, if the vorticity is represented by a single sinusoidal wave distribution in both x and y, the streamfunction has the same spatial distribution as the vorticity and the opposite sign, as can be verified easily from the fact that the second derivative of a sine is proportional to minus the same sine function. An example is shown in the MATLAB script **vorticity_1.m**. However, when the vorticity pattern is localized in space, the space scales of the streamfunction and vorticity are much different. This latter situation is illustrated in the MATLAB script **vorticity_demo.m**, which shows the streamfunction corresponding to a point source of vorticity at (x, y) = (0, 0). For this problem you must modify the code in **vorticity_1.m** by specifying $\zeta\,(x, y) = \exp\left[-b\left(x^2 + y^2\right)\right]$ where b is a constant. Run the model for several values of b from $b = 1 \times 10^{-4}$ km^{-2} to 4×10^{-7} km^{-2}. Show in a table or line plot the dependence of the ratio of the horizontal scales on which vorticity and the streamfunction decay to one-half of their maximum values as a function of the parameter b. Note that for geostrophic motions (with constant Coriolis parameter), the streamfunction defined here is proportional to geopotential height. What can you conclude from this exercise about the information content of a 500-hPa height map versus that of a map of the 500-hPa vorticity field?

M4.2. The MATLAB scripts **geowinds_1.m** (to be used when the mapping toolbox is available) and **geowinds_2.m** (to be used if no mapping toolbox is available) contain contour plots showing the observed 500-hPa height and horizontal wind fields for November 10, 1998, in the North American sector. Also shown is a color plot of the magnitude of the 500-hPa wind with height contours superposed. Using centered difference formulas (see Section 13.2.1), compute the geostrophic wind components, the magnitude of the geostrophic wind, the relative vorticity, the vorticity of the geostrophic wind, and the vorticity minus the vorticity of the geostrophic wind. Following the models of Figures 4.1 and 4.2, superpose these fields on the maps of the 500-hPa height field. Explain the distribution and the sign of regions with large differences between vorticity and geostrophic vorticity in terms of the force balances that were studied in Chapter 3.