To begin at the beginning:
It is spring, moonless night in the small town,
starless and bible-black, the cobblestreets silent
and the hunched, courters'-and-rabbits' wood limping invisible down
to the sloeblack, slow, black, crowblack, fishingboat-bobbing sea.
Dylan Thomas, *Under Milk Wood*, 1954.

CHAPTER 1

Equations of Motion

HAVING NOTHING BUT A BLANK SLATE, we begin by establishing the governing equations of motion for a fluid, with particular attention to the fluids of Earth's atmosphere and ocean. These equations determine how a fluid flows and evolves when forces are applied to it, or when it is heated or cooled, and so involve both dynamics and thermodynamics. And because the equations of motion are nonlinear the two become intertwined and at times inseparable.

1.1   TIME DERIVATIVES FOR FLUIDS

The equations of motion of fluid mechanics differ from those of rigid-body mechanics because fluids form a continuum, and because fluids flow and deform. Thus, even though the same relatively simple physical laws (Newton's laws and the laws of thermodynamics) govern both solid and fluid media, the expression of these laws differs between the two. To determine the equations of motion for fluids we must clearly establish what the time derivative of some property of a fluid actually means, and that is the subject of this section.

1.1.1   Field and Material Viewpoints

In solid-body mechanics one is normally concerned with the position and momentum of identifiable objects — the angular velocity of a spinning top or the motions of the planets around the Sun are two well-worn examples. The position and velocity of a particular object are then computed as a function of time by formulating equations of the form

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = F(\{x_i\}, t), \tag{1.1}$$

where $\{x_i\}$ is the set of positions and velocities of all the interacting objects and the operator F on the right-hand side is formulated using Newton's laws of motion. For example, two massive point objects interacting via their gravitational field obey

$$\frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t} = \boldsymbol{v}_i, \qquad \frac{\mathrm{d}\boldsymbol{v}_i}{\mathrm{d}t} = \frac{Gm_j}{(\boldsymbol{r}_i - \boldsymbol{r}_j)^2}\hat{\boldsymbol{r}}_{i,j}, \qquad i = 1, 2; \ j = 3 - i. \tag{1.2}$$

We thereby predict the positions, $\boldsymbol{r}_i$, and velocities, $\boldsymbol{v}_i$, of the objects given their masses, m_i, and the gravitational constant G, and where $\hat{\boldsymbol{r}}_{i,j}$ is a unit vector directed from $\boldsymbol{r}_i$ to $\boldsymbol{r}_j$.

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In fluid dynamics such a procedure would lead to an analysis of fluid motions in terms of the positions and momenta of different fluid parcels, each identified by some label, which might simply be their position at an initial time. We call this a *material* point of view, because we are concerned with identifiable pieces of material; it is also sometimes called a *Lagrangian* view, after J.-L. Lagrange. The procedure is perfectly acceptable in principle, and if followed would provide a complete description of the fluid dynamical system. However, from a practical point of view it is much more than we need, and it would be extremely complicated to implement. Instead, for most problems we would like to know what the values of velocity, density and so on are at *fixed points* in space as time passes. (A weather forecast we might care about tells us how warm it will be where we live and, if we are given that, we do not particularly care where a fluid parcel comes from, or where it subsequently goes.) Since the fluid is a continuum, this knowledge is equivalent to knowing how the fields of the dynamical variables evolve in space and time, and this is often known as the *field* or *Eulerian* viewpoint, after L. Euler.[1] Thus, whereas in the material view we consider the time evolution of identifiable fluid elements, in the field view we consider the time evolution of the fluid field from a particular frame of reference. That is, we seek evolution equations of the general form

$$\frac{\partial}{\partial t}\varphi(x, y, z, t) = \mathcal{G}(\varphi, x, y, z, t), \tag{1.3}$$

where the field $\varphi(x, y, z, t)$ represents all the dynamical variables (velocity, density, temperature, etc.) and $\mathcal{G}$ is some operator to be determined from Newton's laws of motion and appropriate thermodynamic laws.

Although the field viewpoint will often turn out to be the most practically useful, the material description is invaluable both in deriving the equations and in the subsequent insight it frequently provides. This is because the important quantities from a fundamental point of view are often those which are associated with a given fluid element: it is these which directly enter Newton's laws of motion and the thermodynamic equations. It is thus important to have a relationship between the rate of change of quantities associated with a given fluid element and the local rate of change of a field. The material or advective derivative provides this relationship.

1.1.2   The Material Derivative of a Fluid Property

A *fluid element* is an infinitesimal, indivisible, piece of fluid — effectively a very small fluid parcel of fixed mass. The *material derivative* is the rate of change of a property (such as temperature or momentum) of a particular fluid element or finite mass of fluid; that is, it is the total time derivative of a property of a piece of fluid. It is also known as the 'substantive derivative' (the derivative associated with a parcel of fluid substance), the 'advective derivative' (because the fluid property is being advected), the 'convective derivative' (convection is a slightly old-fashioned name for advection, still used in some fields), or the 'Lagrangian derivative' (after Lagrange).

Let us suppose that a fluid is characterized by a given velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$, which determines its velocity throughout. Let us also suppose that the fluid has another property φ, and let us seek an expression for the rate of change of φ of a fluid element. Since φ is changing in time and in space we use the chain rule,

$$\delta\varphi = \frac{\partial\varphi}{\partial t}\delta t + \frac{\partial\varphi}{\partial x}\delta x + \frac{\partial\varphi}{\partial y}\delta y + \frac{\partial\varphi}{\partial z}\delta z = \frac{\partial\varphi}{\partial t}\delta t + \delta\boldsymbol{x}\cdot\nabla\varphi. \tag{1.4}$$

This is true in general for any δt, δx, etc. The total time derivative is then

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\partial\varphi}{\partial t} + \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\cdot\nabla\varphi. \tag{1.5}$$

If this equation is to represent a material derivative we must identify the time derivative in the second term on the right-hand side with the rate of change of position of a fluid element, namely

its velocity. Hence, the material derivative of the property φ is

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\partial\varphi}{\partial t} + \boldsymbol{v}\cdot\nabla\varphi. \tag{1.6}$$

The right-hand side expresses the material derivative in terms of the local rate of change of φ plus a contribution arising from the spatial variation of φ, experienced only as the fluid parcel moves. Because the material derivative is so common, and to distinguish it from other derivatives, we denote it by the operator D/Dt. Thus, the material derivative of the field φ is

$$\frac{\mathrm{D}\varphi}{\mathrm{D}t} = \frac{\partial\varphi}{\partial t} + (\boldsymbol{v}\cdot\nabla)\varphi. \tag{1.7}$$

The brackets in the last term of this equation are helpful in reminding us that $(\boldsymbol{v}\cdot\nabla)$ is an operator acting on φ. The operator $\partial/\partial t + (\boldsymbol{v}\cdot\nabla)$ is the *Eulerian representation of the Lagrangian derivative as applied to a field.* We use the notation D/Dt rather generally for Lagrangian derivatives, but the operator may take a different form when applied to other objects, such as a fluid volume.

Material derivative of vector field

The material derivative may act on a vector field $\boldsymbol{b}$, in which case

$$\frac{\mathrm{D}\boldsymbol{b}}{\mathrm{D}t} = \frac{\partial\boldsymbol{b}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{b}. \tag{1.8}$$

In Cartesian coordinates this is

$$\frac{\mathrm{D}\boldsymbol{b}}{\mathrm{D}t} = \frac{\partial\boldsymbol{b}}{\partial t} + u\frac{\partial\boldsymbol{b}}{\partial x} + v\frac{\partial\boldsymbol{b}}{\partial y} + w\frac{\partial\boldsymbol{b}}{\partial z}, \tag{1.9}$$

and for a particular component of $\boldsymbol{b}$, b^x say,

$$\frac{\mathrm{D}b^x}{\mathrm{D}t} = \frac{\partial b^x}{\partial t} + u\frac{\partial b^x}{\partial x} + v\frac{\partial b^x}{\partial y} + w\frac{\partial b^x}{\partial z}, \tag{1.10}$$

and similarly for b^y and b^z. In Cartesian tensor notation the expression becomes

$$\frac{\mathrm{D}b_i}{\mathrm{D}t} = \frac{\partial b_i}{\partial t} + v_j\frac{\partial b_i}{\partial x_j} = \frac{\partial b_i}{\partial t} + v_j\partial_j b_i, \tag{1.11}$$

where the subscripts denote the Cartesian components, repeated indices are summed, and $\partial_j b_i \equiv \partial b_i/\partial x_j$. In coordinate systems other than Cartesian the advective derivative of a vector is not simply the sum of the advective derivative of its components, because the coordinate vectors themselves change direction with position; this will be important when we deal with spherical coordinates. Finally, we remark that the advective derivative of the position of a fluid element, $\boldsymbol{r}$ say, is its velocity, and this may easily be checked by explicitly evaluating D$\boldsymbol{r}$/Dt.

1.1.3 Material Derivative of a Volume

The volume that a given, unchanging, mass of fluid occupies is deformed and advected by the fluid motion, and there is no reason why it should remain constant. Rather, the volume will change as a result of the movement of each element of its bounding material surface, and in particular will change if there is a non-zero normal component of the velocity at the fluid surface. That is, if the volume of some fluid is $\int \mathrm{d}V$, then

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \mathrm{d}V = \int_S \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S}, \tag{1.12}$$

where the subscript V indicates that the integral is a definite integral over some finite volume V, although the limits of the integral will be functions of time if the volume is changing. The integral on the right-hand side is over the closed surface, S, bounding the volume. Although intuitively apparent (to some), this expression may be derived more formally using Leibniz's formula for the rate of change of an integral whose limits are changing. Using the divergence theorem on the right-hand side, (1.12) becomes

$$\frac{D}{Dt} \int_V dV = \int_V \nabla \cdot \boldsymbol{v} \, dV. \tag{1.13}$$

The rate of change of the volume of an infinitesimal fluid element of volume ΔV is obtained by taking the limit of this expression as the volume tends to zero, giving

$$\lim_{\Delta V \to 0} \frac{1}{\Delta V} \frac{D \Delta V}{Dt} = \nabla \cdot \boldsymbol{v}. \tag{1.14}$$

We will often write such expressions informally as

$$\frac{D \Delta V}{Dt} = \Delta V \nabla \cdot \boldsymbol{v}, \tag{1.15}$$

with the limit implied.

Consider now the material derivative of some fluid property, ξ say, multiplied by the volume of a fluid element, ΔV. Such a derivative arises when ξ is the amount per unit volume of ξ-substance — the mass density or the amount of a dye per unit volume, for example. Then we have

$$\frac{D}{Dt}(\xi \Delta V) = \xi \frac{D \Delta V}{Dt} + \Delta V \frac{D \xi}{Dt}. \tag{1.16}$$

Using (1.15) this becomes

$$\frac{D}{Dt}(\xi \Delta V) = \Delta V \left(\xi \nabla \cdot \boldsymbol{v} + \frac{D \xi}{Dt} \right), \tag{1.17}$$

and the analogous result for a finite fluid volume is just

$$\frac{D}{Dt} \int_V \xi \, dV = \int_V \left(\xi \nabla \cdot \boldsymbol{v} + \frac{D \xi}{Dt} \right) dV. \tag{1.18}$$

This expression is to be contrasted with the Eulerian derivative for which the volume, and so the limits of integration, are fixed and we have

$$\frac{d}{dt} \int_V \xi \, dV = \int_V \frac{\partial \xi}{\partial t} \, dV. \tag{1.19}$$

Now consider the material derivative of a fluid property φ multiplied by the mass of a fluid element, $\rho \Delta V$, where ρ is the fluid density. Such a derivative arises when φ is the amount of φ-substance per unit mass (note, for example, that the momentum of a fluid element is $\rho \boldsymbol{v} \Delta V$). The material derivative of $\varphi \rho \Delta V$ is given by

$$\frac{D}{Dt}(\varphi \rho \Delta V) = \rho \Delta V \frac{D \varphi}{Dt} + \varphi \frac{D}{Dt}(\rho \Delta V). \tag{1.20}$$

But $\rho \Delta V$ is just the mass of the fluid element, and that is constant — that is how a fluid element is defined. Thus the second term on the right-hand side vanishes and

$$\frac{D}{Dt}(\varphi \rho \Delta V) = \rho \Delta V \frac{D \varphi}{Dt} \quad \text{and} \quad \frac{D}{Dt} \int_V \varphi \rho \, dV = \int_V \rho \frac{D \varphi}{Dt} \, dV, \tag{1.21a,b}$$

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Material and Eulerian Derivatives

The material derivatives of a scalar (φ) and a vector ($\boldsymbol{b}$) field are given by:

$$\frac{D\varphi}{Dt} = \frac{\partial \varphi}{\partial t} + \boldsymbol{v} \cdot \nabla \varphi, \qquad\qquad \frac{D\boldsymbol{b}}{Dt} = \frac{\partial \boldsymbol{b}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{b}. \qquad (D.1)$$

Various material derivatives of integrals are:

$$\frac{D}{Dt} \int_V \varphi \, dV = \int_V \left(\frac{D\varphi}{Dt} + \varphi \nabla \cdot \boldsymbol{v} \right) dV = \int_V \left(\frac{\partial \varphi}{\partial t} + \nabla \cdot (\varphi \boldsymbol{v}) \right) dV, \qquad (D.2)$$

$$\frac{D}{Dt} \int_V dV = \int_V \nabla \cdot \boldsymbol{v} \, dV, \qquad (D.3)$$

$$\frac{D}{Dt} \int_V \rho\varphi \, dV = \int_V \rho \frac{D\varphi}{Dt} \, dV. \qquad (D.4)$$

These formulae also hold if φ is a vector. The Eulerian derivative of an integral is:

$$\frac{d}{dt} \int_V \varphi \, dV = \int_V \frac{\partial \varphi}{\partial t} \, dV, \qquad (D.5)$$

so that

$$\frac{d}{dt} \int_V dV = 0 \quad \text{and} \quad \frac{d}{dt} \int_V \rho\varphi \, dV = \int_V \frac{\partial \rho\varphi}{\partial t} \, dV. \qquad (D.6)$$

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where (1.21b) applies to a finite volume. That expression may also be derived more formally using Leibniz's formula for the material derivative of an integral, and the result also holds when φ is a vector. The result is quite different from the corresponding Eulerian derivative, in which the volume is kept fixed; in that case we have:

$$\frac{d}{dt} \int_V \varphi \rho \, dV = \int_V \frac{\partial}{\partial t} (\varphi \rho) \, dV. \qquad (1.22)$$

Various material and Eulerian derivatives are summarized in the shaded box above.
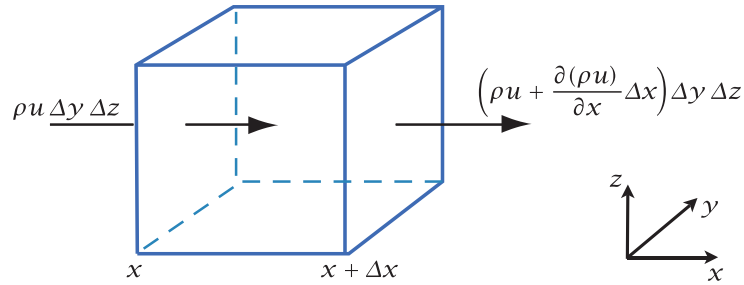
1.2 THE MASS CONTINUITY EQUATION

In classical mechanics mass is absolutely conserved and in solid-body mechanics we normally do not need an explicit equation of mass conservation. However, in fluid mechanics fluid flows into and away from regions, and fluid density may change, and an equation that explicitly accounts for the flow of mass is one of the equations of motion of the fluid.

1.2.1 An Eulerian Derivation

We will first derive the mass conservation equation from an Eulerian point of view; that is to say, our reference frame is fixed in space and the fluid flows through it.

Cartesian derivation

Consider an infinitesimal, rectangular cuboid, control volume, $\Delta V = \Delta x \Delta y \Delta z$ that is fixed in space, as in Fig. 1.1. Fluid moves into or out of the volume through its surface, including through its faces in the y–z plane of area $\Delta A = \Delta y \Delta z$ at coordinates x and $x + \Delta x$. The accumulation of fluid within the control volume due to motion in the x-direction is evidently

$$\Delta y \Delta z [(\rho u)(x, y, z) - (\rho u)(x + \Delta x, y, z)] = -\left.\frac{\partial(\rho u)}{\partial x}\right|_{x,y,z} \Delta x \, \Delta y \, \Delta z. \tag{1.23}$$

To this must be added the effects of motion in the y- and z-directions, namely

$$-\left[\frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right]\Delta x \, \Delta y \, \Delta z. \tag{1.24}$$

This net accumulation of fluid must be accompanied by a corresponding increase of fluid mass within the control volume. This is

$$\frac{\partial}{\partial t}\left(\text{density} \times \text{volume}\right) = \Delta x \, \Delta y \, \Delta z \frac{\partial \rho}{\partial t}, \tag{1.25}$$

because the volume is constant. Thus, because mass is conserved, (1.23), (1.24) and (1.25) give

$$\Delta x \, \Delta y \, \Delta z \left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] = 0. \tag{1.26}$$

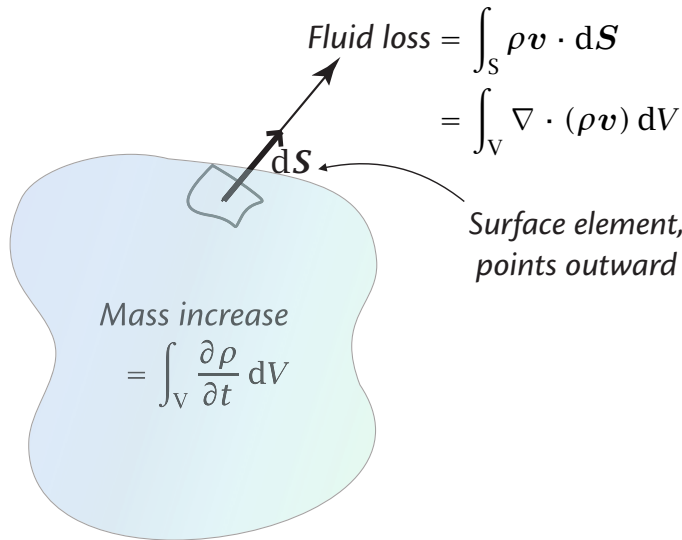The quantity in square brackets must be zero and we therefore have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0. \tag{1.27}$$

This is called the *mass continuity equation* for it recognizes the continuous nature of the mass field in a fluid. There is no diffusion term in (1.27), no term like $\kappa \nabla^2 \rho$. This is because mass is transported by the macroscopic movement of molecules; even if this motion appears diffusion-like any net macroscopic molecular motion constitutes, by definition, a velocity field.

Vector derivation

Consider an arbitrary control volume V bounded by a surface S, fixed in space, with by convention the direction of $\boldsymbol{S}$ being toward the outside of V, as in Fig. 1.2. The rate of fluid loss due to flow through the closed surface S is then given by

$$\text{fluid loss} = \int_S \rho \boldsymbol{v} \cdot d\boldsymbol{S} = \int_V \nabla \cdot (\rho \boldsymbol{v}) \, dV, \tag{1.28}$$

$$\text{Fluid loss} = \int_S \rho \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S}$$

$$= \int_V \nabla \cdot (\rho \boldsymbol{v}) \, \mathrm{d}V$$

Surface element, points outward

$\mathrm{d}\boldsymbol{S}$

Mass increase

$$= \int_V \frac{\partial \rho}{\partial t} \, \mathrm{d}V$$

Fig. 1.2 Mass conservation in an arbitrary Eulerian control volume V bounded by a surface S. The mass increase, $\int_V (\partial \rho/\partial t) \, \mathrm{d}V$ is equal to the mass flowing into the volume, $-\int_S (\rho \boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{S} = -\int_V \nabla \cdot (\rho \boldsymbol{v}) \, \mathrm{d}V$.

using the divergence theorem.

This must be balanced by a change in the mass M of the fluid within the control volume, which, since its volume is fixed, implies a density change. That is

$$\text{fluid loss} = -\frac{\mathrm{d}M}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho \, \mathrm{d}V = -\int_V \frac{\partial \rho}{\partial t} \, \mathrm{d}V. \tag{1.29}$$

Equating (1.28) and (1.29) yields

$$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) \right] \mathrm{d}V = 0. \tag{1.30}$$

Because the volume is arbitrary, the integrand must vanish and we recover (1.27).

1.2.2 Mass Continuity via the Material Derivative

We now derive the mass continuity equation (1.27) from a material perspective. This is the most fundamental approach of all since the principle of mass conservation states simply that the mass of a given element of fluid is, by definition of the element, constant. Thus, consider a small mass of fluid of density ρ and volume ΔV. Then conservation of mass may be represented by

$$\frac{\mathrm{D}}{\mathrm{D}t}(\rho \Delta V) = 0. \tag{1.31}$$

Both the density and the volume of the parcel may change, so

$$\Delta V \frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \frac{\mathrm{D}\Delta V}{\mathrm{D}t} = \Delta V \left(\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot \boldsymbol{v} \right) = 0, \tag{1.32}$$

where the second expression follows using (1.15). Since the volume element is arbitrary, the term in brackets must vanish and

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot \boldsymbol{v} = 0. \tag{1.33}$$

After expansion of the first term this becomes identical to (1.27). This result may be derived more formally by rewriting (1.31) as the integral expression

$$\frac{\mathrm{D}}{\mathrm{D}t} \int_V \rho \, \mathrm{d}V = 0. \tag{1.34}$$

Expanding the derivative using (1.18) gives

$$\frac{D}{Dt} \int_V \rho \, dV = \int_V \left(\frac{D\rho}{Dt} + \rho \nabla \cdot \boldsymbol{v} \right) dV = 0. \tag{1.35}$$

Because the volume over which the integral is taken is arbitrary the integrand itself must vanish and we recover (1.33). Summarizing, equivalent partial differential equations representing conservation of mass are:

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \boldsymbol{v} = 0, \qquad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0. \tag{1.36a,b}$$

1.2.3   A General Continuity Equation

The derivation of a continuity equation for a general scalar property of a fluid is similar to that for density, except that there may be an external source or sink, and potentially a means of transferring the property from one location to another differently than by fluid motion, for example by diffusion. If ξ is the amount of some property of the fluid per unit volume (the volume concentration, sometimes simply called the concentration), and if the net effect per unit volume of all non-conservative processes is denoted by $Q_{[v,\xi]}$, then the continuity equation for concentration may be written:

$$\frac{D}{Dt}(\xi \Delta V) = Q_{[v,\xi]} \Delta V. \tag{1.37}$$

Expanding the left-hand side and using (1.15) we obtain

$$\frac{D\xi}{Dt} + \xi \nabla \cdot \boldsymbol{v} = Q_{[v,\xi]}, \qquad \text{or} \qquad \frac{\partial \xi}{\partial t} + \nabla \cdot (\xi \boldsymbol{v}) = Q_{[v,\xi]}. \tag{1.38}$$

If we are interested in a tracer that is normally measured per unit mass of fluid (which is typical when considering thermodynamic quantities) then the conservation equation would be written

$$\frac{D}{Dt}(\varphi \rho \Delta V) = Q_{[m,\varphi]} \rho \Delta V, \tag{1.39}$$

where φ is the tracer mixing ratio or mass concentration — that is, the amount of tracer per unit fluid mass — and $Q_{[m,\varphi]}$ represents non-conservative sources of φ per unit mass. Then, since $\rho \Delta V$ is constant we obtain

$$\frac{D\varphi}{Dt} = Q_{[m,\varphi]} \qquad \text{or} \qquad \frac{\partial(\rho\varphi)}{\partial t} + \nabla \cdot (\rho\varphi \boldsymbol{v}) = \rho Q_{[m,\varphi]}, \tag{1.40}$$

using the mass continuity equation, (1.36), to obtain the equation on the right. The source term $Q_{[m,\varphi]}$ is evidently equal to the rate of change of φ of a fluid element. When this is so, we often write it simply as $\dot{\varphi}$, so that

$$\frac{D\varphi}{Dt} = \dot{\varphi}. \tag{1.41}$$

A tracer obeying (1.41) with $\dot{\varphi} = 0$ is said to be *materially conserved*. If a tracer is materially conserved except for the effects of sources or sinks, or diffusion terms, then it is sometimes (if rather loosely) said to be an 'adiabatically conserved' variable, although adiabatic properly means with no heat exchange. If those sources and sinks are in the form of the divergence of a flux with φ satisfying $\rho D\varphi/Dt = \nabla \cdot \boldsymbol{F}_\varphi$ or equivalently, using the mass continuity equation, $\partial(\rho\varphi)/\partial t + \nabla \cdot (\rho\boldsymbol{v}\varphi) = \nabla \cdot \boldsymbol{F}_\varphi$, then φ is said to be a *conservative* variable because, with no flux boundary conditions, $\int \rho\varphi \, dV = $ constant. Although momentum as a whole is conserved, momentum is not a materially conserved variable, as we are about to see.

1.3 THE MOMENTUM EQUATION

The momentum equation is a partial differential equation that describes how the velocity or momentum of a fluid responds to internal and imposed forces. We will derive it using material methods, with a very heuristic treatment of the terms representing pressure and viscous forces.

1.3.1 Advection

Let $\boldsymbol{m}(x, y, z, t)$ be the momentum-density field (momentum per unit volume) of the fluid. Thus, $\boldsymbol{m} = \rho \boldsymbol{v}$ and the total momentum of a volume of fluid is given by the volume integral $\int_V \boldsymbol{m} \, dV$. Now, for a fluid the rate of change of a momentum of an identifiable fluid mass is given by the material derivative, and by Newton's second law this is equal to the force acting on it. Thus,

$$\frac{D}{Dt} \int_V \rho \boldsymbol{v} \, dV = \int_V \boldsymbol{F} \, dV, \tag{1.42}$$

where $\boldsymbol{F}$ is the force per unit volume. Now, using (1.21b) (with φ replaced by $\boldsymbol{v}$) to transform the left-hand side of (1.42), we obtain

$$\int_V \left(\rho \frac{D\boldsymbol{v}}{Dt} - \boldsymbol{F} \right) dV = 0. \tag{1.43}$$

Because the volume is arbitrary the integrand itself must vanish and we obtain

$$\rho \frac{D\boldsymbol{v}}{Dt} = \boldsymbol{F} \quad \text{or} \quad \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} = \frac{\boldsymbol{F}}{\rho}, \tag{1.44a,b}$$

having used (1.8) to expand the material derivative.

We have thus obtained an expression for how a fluid accelerates if subject to known forces. These forces are, however, not all external to the fluid itself: a stress arises from the direct contact between one fluid parcel and another, giving rise to pressure and viscous forces, sometimes referred to as *contact* forces. Because a complete treatment of these forces would be very lengthy, and is available elsewhere, we treat them informally and intuitively.

1.3.2 Pressure and Viscous Forces

Pressure

Within or at the boundary of a fluid the pressure is the normal force per unit area due to the collective action of molecular motion. Thus

$$d\widehat{\boldsymbol{F}}_p = -p \, d\boldsymbol{S}, \tag{1.45}$$

where p is the pressure, $\widehat{\boldsymbol{F}}_p$ is the pressure force and $d\boldsymbol{S}$ an infinitesimal surface element. If we grant ourselves this intuitive notion, it is a simple matter to assess the influence of pressure on a fluid, for the pressure force on a volume of fluid is the integral of the pressure over its boundary and so

$$\widehat{\boldsymbol{F}}_p = - \int_S p \, d\boldsymbol{S}. \tag{1.46}$$

The minus sign arises because the pressure force is directed inwards, whereas $\boldsymbol{S}$ is a vector normal to the surface and directed outwards. Applying a form of the divergence theorem to the right-hand side gives

$$\widehat{\boldsymbol{F}}_p = - \int_V \nabla p \, dV, \tag{1.47}$$

where the volume V is bounded by the surface S. The pressure force per unit volume, $\boldsymbol{F}_p$, is therefore just $-\nabla p$, and the force per unit mass is $\nabla p / \rho$. The force is evidently non-zero only if the pressure varies in space and for this reason it is more properly known as the *pressure-gradient* force.

Table 1.1 Experimental values of viscosity for air, water and mercury at room temperature and pressure.

	$\mu\,(\mathrm{kg\,m^{-1}\,s^{-1}})$	$\nu\,(\mathrm{m^2\,s^{-1}})$
Air	1.8×10^{-5}	1.5×10^{-5}
Water	1.1×10^{-3}	1.1×10^{-6}
Mercury	1.6×10^{-3}	1.2×10^{-7}

Viscosity

Viscosity, like pressure, is a force due to the internal motion of molecules. The effects of viscosity are apparent in many situations — the flow of treacle or volcanic lava are obvious examples. In other situations, for example large-scale flow in the atmosphere, viscous effects are negligible. However, for a constant density fluid viscosity is the *only* way that energy may be removed from the fluid, so that if energy is being added in some way viscosity must ultimately become important if the fluid is to reach an equilibrium in which energy input equals energy dissipation. When tea is stirred in a cup, it is viscous effects that cause the fluid to eventually stop spinning after we have removed our spoon.

A number of textbooks show that, for most Newtonian fluids, the viscous force per unit volume is approximately equal to $\mu\nabla^2\boldsymbol{v}$, where μ is the viscosity. Obtaining this expression involves making an incompressibility assumption and is not exact, but it is in fact a good approximation for most liquids and gases. With this term and the pressure-gradient force the momentum equation becomes,

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{v} + \boldsymbol{F}_b, \tag{1.48}$$

where $\nu \equiv \mu/\rho$ is the *kinematic viscosity* and $\boldsymbol{F}_b$ represents external body forces (per unit mass) such as gravity, $\boldsymbol{g}$. Equation (1.48) is sometimes called the Navier–Stokes equation. For gases, dimensional arguments suggest that the magnitude of ν should be given by

$$\nu \sim \text{mean free path} \times \text{mean molecular velocity}, \tag{1.49}$$

which for a typical molecular velocity of $300\,\mathrm{m\,s^{-1}}$ and a mean free path of $7 \times 10^{-8}\,\mathrm{m}$ gives the not unreasonable estimate of $2.1 \times 10^{-5}\,\mathrm{m^2\,s^{-1}}$, within a factor of two of the experimental value (Table 1.1). Interestingly, the kinematic viscosity is smaller for water and mercury than it is for air.

1.3.3 Hydrostatic Balance

The vertical component — the component parallel to the gravitational force, $\boldsymbol{g}$ — of the momentum equation is

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \tag{1.50}$$

where w is the vertical component of the velocity and $\boldsymbol{g} = -g\mathbf{k}$. If the fluid is static the gravitational term is balanced by the pressure term and we have

$$\frac{\partial p}{\partial z} = -\rho g, \tag{1.51}$$

and this relation is known as *hydrostatic balance,* or hydrostasy. It is clear in this case that the pressure at a point is given by the weight of the fluid above it, provided $p = 0$ at the top of the fluid. It might also appear that (1.51) would be a good *approximation* to (1.50) provided that vertical accelerations, $\mathrm{D}w/\mathrm{D}t$, are small compared to gravity, which is nearly always the case in the atmosphere and ocean. While this statement is true if we need only a reasonable approximate value of

the pressure at a point or in a column, the satisfaction of this condition is *not* sufficient to ensure that (1.51) provides an accurate enough pressure to determine the horizontal pressure gradients responsible for producing motion. We return to this point in Section 2.7.

1.4   THE EQUATION OF STATE

In three dimensions the momentum and continuity equations provide four equations, but contain five unknowns — three components of velocity, density and pressure. Obviously other equations are needed, and an *equation of state* is an expression that diagnostically relates the various thermodynamic variables to each other. The *conventional* equation of state, or the *thermal* equation of state, is an expression that relates temperature, pressure, composition (the mass fraction of the various constituents) and density, and we may write it, rather generally, as

$$p = p(\rho, T, \varphi_n),\tag{1.52}$$

where φ_n is mass fraction of the nth constituent. An equation of this form is not the most fundamental equation of state from a thermodynamic perspective, an issue we visit later, but it connects readily measurable quantities.

For an ideal gas (and the air in the Earth's atmosphere is very close to ideal) the thermal equation of state is

$$p = \rho R T,\tag{1.53}$$

where R is the gas constant for the gas in question and T is temperature. R is a specific constant, and is related to the universal gas constant R^* by $R = R^*/\overline{\mu}$, where $\overline{\mu}$ is the mean molar mass (molecular weight in kg/mol) of the constituents of the gas. Equivalently, $R = n_m k_B$, where k_B is Boltzmann's constant and n_m is the number of molecules per unit mass, so that R is proportional to the number of molecules contained in a unit mass. Since $R^* = 8.314\,\mathrm{J\,mol^{-1}\,K^{-1}}$ and, for dry air, $\mu = 29.0 \times 10^{-3}\,\mathrm{kg\,mol^{-1}}$ we obtain $R = 287\,\mathrm{J\,kg^{-1}\,K^{-1}}$. Air has virtually constant composition except for variations in water vapour; these variations make the gas constant, R, in the equation of state for air a weak function of the water vapour content but for now we regard R as a constant.

For a liquid such as seawater no simple expression akin to (1.53) is easily derivable, and semi-empirical equations are usually resorted to. For water in a laboratory setting a reasonable approximation of the equation of state is $\rho = \rho_0[1 - \beta_T(T - T_0)]$, where β_T is a thermal expansion coefficient and ρ_0 and T_0 are constants. In the ocean the density is also significantly affected by pressure and dissolved salts: seawater is a solution of many ions in water — chloride ($\approx 1.9\%$ by weight), sodium (1%), sulfate (0.26%), magnesium (0.13%) and so on, with a total average concentration of about 35‰ (ppt, or parts per thousand). The ratios of the fractions of these salts are almost constant throughout the ocean, and their total concentration may be parameterized by a single measure, the *salinity*, S.[2] Given this, the density of seawater is a function of three variables — pressure, temperature, and salinity — and we may write the conventional equation of state as

$$\rho = \rho(T, S, p) \qquad \text{or} \qquad \alpha = \alpha(T, S, p),\tag{1.54}$$

where $\alpha = 1/\rho$ is the specific volume, or inverse density. For small variations around a reference value we have

$$d\alpha = \left(\frac{\partial \alpha}{\partial T}\right)_{S,p} dT + \left(\frac{\partial \alpha}{\partial S}\right)_{T,p} dS + \left(\frac{\partial \alpha}{\partial p}\right)_{T,S} dp = \alpha\left(\beta_T\, dT - \beta_S\, dS - \beta_p\, dp\right),\tag{1.55}$$

where the rightmost expression serves to define the thermal expansion coefficient β_T, the saline contraction coefficient β_S, and the compressibility coefficient (or inverse bulk modulus) β_p. In general these quantities are not constants, but for small variations around a reference state they may be treated as such and we have

$$\alpha = \alpha_0 \left[1 + \beta_T(T - T_0) - \beta_S(S - S_0) - \beta_p(p - p_0)\right].\tag{1.56}$$

Typical values of these parameters, with variations typically encountered through the ocean, are: $\beta_T \approx 2\,(\pm 1.5) \times 10^{-4}\,\text{K}^{-1}$, $\beta_S \approx 7.6\,(\pm 0.2) \times 10^{-4}\,\text{ppt}^{-1}$, $\beta_p \approx 4.4\,(\pm 0.5) \times 10^{-10}\,\text{Pa}^{-1}$. The value of β_p is also related to the speed of sound, c_s, by $\beta_p = \alpha_0/c_s^2$. Since the variations around the mean density are small, (1.56) implies that

$$\rho = \rho_0 \left[1 - \beta_T(T - T_0) + \beta_S(S - S_0) + \beta_p(p - p_0) \right]. \tag{1.57}$$

In the ocean the pressure term leads to larger density changes than either the salinity or temperature terms but it is not normally as important for the dynamics, because pressure is largely determined by the hydrostatic pressure giving a large vertical density gradient. It is the lateral variations in density that are often more important for the dynamics and these are affected just as much by the saline and temperature terms.

A linear equation of state for seawater is emphatically *not* accurate enough for quantitative oceanography; as mentioned the β parameters in (1.56) themselves vary with pressure, temperature and (more weakly) salinity so introducing nonlinearities to the equation. The most important of these are captured by an equation of state of the form

$$\alpha = \alpha_0 \left[1 + \beta_T(1 + \gamma^* p)(T - T_0) + \frac{\beta_T^*}{2}(T - T_0)^2 - \beta_S(S - S_0) - \beta_p(p - p_0) \right]. \tag{1.58}$$

The starred constants β_T^* and γ^* capture the leading nonlinearities: γ^* is the *thermobaric parameter,* which determines the extent to which the thermal expansion depends on pressure, and β_T^* is the second thermal expansion coefficient.[3] Even this equation of state has some quantitative deficiencies and more complicated empirical formulae are often used if very high accuracy is needed. The variation of density of seawater with temperature, salinity and pressure is illustrated in Fig. 1.3, with more discussion in Section 1.7.2.

Clearly, the equation of state introduces, in general, a sixth unknown, temperature, and we will have to introduce another physical principle — one coming from thermodynamics — to obtain a complete set of equations. However, if the equation of state were such that it linked only density and pressure, without introducing another variable, then the equations would be complete; the simplest case of all is a constant density fluid for which the equation of state is just $\rho = $ constant. A fluid for which the density is a function of pressure alone is called a *barotropic fluid* or a *homentropic fluid*; otherwise, it is a *baroclinic fluid*. (In this context, 'barotropic' is a shortening of the original phrase 'auto-barotropic'.) Equations of state of the form $p = C\rho^\gamma$, where γ is a constant, are called 'polytropic'.

1.5 THERMODYNAMIC RELATIONS

In this section we review a few aspects of thermodynamics. We provide neither a complete nor an a priori development of the subject; rather, we focus on aspects that are particularly relevant to fluid dynamics, and that are needed to derive a 'thermodynamic equation' for fluids. Readers whose interest is solely in an ideal gas or a simple Boussinesq fluid may skim this section and then refer to it later as needed.

1.5.1 A Few Fundamentals

A fundamental postulate of thermodynamics is that the internal energy of a system in equilibrium is a function of its extensive properties: volume, entropy, and the mass of its various constituents. Extensive means that the property value is proportional to the amount of material present, in contrast to an intensive property such as temperature. For our purposes it is more convenient to divide all of these quantities by the mass of fluid present, so expressing the internal energy per unit mass (or the specific internal energy) I, as a function of the specific volume $\alpha = \rho^{-1}$, the specific entropy
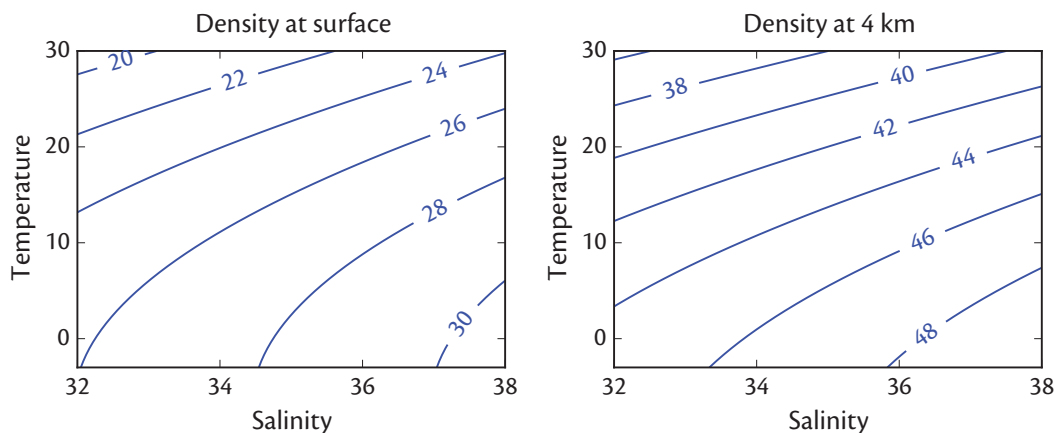
Fig. 1.3   Contours of density as a function of temperature and salinity for seawater. Contour labels are (density − 1000) kg m^{-3}. Left panel: at sea-level ($p =10^5$ Pa, or 1000 mb). Right panel: at $p = 4 \times 10^7$ Pa (about 4 km depth). In both cases the contours are slightly convex, so that if two parcels at the same density but different temperatures and salinities are mixed, the resulting parcel is of higher density. (The average temperature is not exactly conserved on mixing, but it very nearly is.)

η, and the mass fractions of its various components. (However, multiplying by the mass does not turn an intensive variable into a properly extensive one.) Our interest is in two-component fluids (dry air and water vapour, or water and salinity) so that we may parameterize the composition by a single parameter, S. (We follow conventional thermodynamical notation as much as possible, except that we use I instead of u for internal energy, since u is a fluid velocity, and η instead of S for entropy, since S is salinity.) We can also write entropy in terms of internal energy, density and salinity, and thus we have

$$I = I(\alpha, \eta, S) \qquad \text{or} \qquad \eta = \eta(I, \alpha, S). \tag{1.59a,b}$$

Given the functional forms on the right-hand sides, either of these expressions constitutes a complete description of the macroscopic state of a system in equilibrium, and we call either of them the *fundamental equation of state*. The thermal equation of state can be derived from (1.59), but not vice versa. The first differential of (1.59a) gives, formally,

$$dI = \left(\frac{\partial I}{\partial \eta} \right)_{\alpha,S} d\eta + \left(\frac{\partial I}{\partial \alpha} \right)_{\eta,S} d\alpha + \left(\frac{\partial I}{\partial S} \right)_{\alpha,\eta} dS. \tag{1.60}$$

We will now ascribe physical meaning to these differentials.

Conservation of energy states that the internal energy of a body may change because of work done by or on it, or because of a heat input, or because of a change in its chemical composition. We write this as

$$dI = đQ + đW + đC, \tag{1.61}$$

where $đW$ is the work done on the body, $đQ$ is the heat input to the body, and $đC$ is the change in internal energy caused by a change in its chemical composition (e.g., its salinity, or water vapour content), sometimes called the 'chemical work'. The infinitesimal quantities on the right-hand side (denoted with a đ) are so-called imperfect or inexact differentials:[4] Q, W and C are not functions of the state of a body, and the internal energy cannot be regarded as the sum of a 'heat' and a 'work'. We should think of heat and work as having meaning only as fluxes of energy, or rates of energy input,

and not as amounts of energy; their sum changes the internal energy of a body, which *is* a function of its state. Equation (1.61) is sometimes called 'the first law of thermodynamics' (discussed more in Appendix B at the end of the chapter). We first consider the causes of variations of the quantities on the right-hand side, and then make a connection to (1.60).

Heat input: In an infinitesimal quasi-static or reversible process, if an amount of heat đQ (per unit mass) is externally supplied then the specific entropy η will change according to

$$T \, d\eta = đQ. \tag{1.62}$$

The entropy is a function of the state of a body and is, by definition, an adiabatic invariant. As we are dealing with the amount of a quantity per unit mass, η is the specific entropy, although we will often refer to it just as the entropy. We may regard (1.61) as defining the heat input, đQ, by way of a statement of conservation of energy, and (1.62) then says that there is a function of state, the entropy, that changes by an amount equal to the heat input divided by the temperature.

Work done: The work done on a body during a reversible process is equal to the pressure times its change in volume, and the work is positive if the volume change is negative. Thus if an infinitesimal amount of work đW (per unit mass) is applied to a body then its thermodynamic state will change according to

$$- p \, d\alpha = đW, \tag{1.63}$$

where $\alpha = 1/\rho$ is the specific volume of the fluid and p is the pressure.

Composition: The chemical work, which produces a change in internal energy due to a small change in composition, dS, is given by

$$đC = \mu \, dS, \tag{1.64}$$

where μ is the *chemical potential* of the solution. In the ocean, salinity is the compositional variable and changes arise through precipitation and evaporation at the surface and molecular diffusion. When salinity changes the internal energy of a fluid parcel changes by (1.64), but this change is usually small compared to other changes in internal energy. In practice, the most important effect of salinity is that it changes the density of seawater. In the atmosphere the composition of a parcel of air varies mainly according to the amount of water vapour present. Since water vapour and dry air have different chemical potentials these variations cause changes in internal energy, but in the absence of phase-changes the changes are small. An important compositional effect does arise when condensation or evaporation occurs, for then energy is released (or required), as discussed in Chapter 18.

Collecting equations (1.61)–(1.64) together we have

$$dI = T \, d\eta - p \, d\alpha + \mu \, dS. \tag{1.65}$$

We refer to this as *the fundamental thermodynamic relation*. The fundamental equation of state, (1.59), describes the properties of a particular fluid, and the fundamental relation, (1.65), is associated with conservation of energy. Much of classical thermodynamics follows from these two expressions.

1.5.2 Thermodynamic Potentials and Maxwell Relations

Given the fundamental thermodynamic relation, various other 'thermodynamic potentials' and relations between variables can be derived that prove extremely useful. The thermodynamic potentials are, like internal energy and entropy, functions of the state but they have different natural variables by which they are expressed.

If we begin with the internal energy itself, then from (1.60) and (1.65) it follows that

$$T = \left(\frac{\partial I}{\partial \eta}\right)_{\alpha,S}, \qquad p = -\left(\frac{\partial I}{\partial \alpha}\right)_{\eta,S}, \qquad \mu = \left(\frac{\partial I}{\partial S}\right)_{\eta,\alpha}. \tag{1.66a,b,c}$$

These may be regarded as the defining relations for these variables; because of the connection between (1.61) and (1.65) these are not just formal definitions, and the pressure and temperature so defined are indeed related to our intuitive concepts of these variables and to the motion of the fluid molecules. If we write

$$d\eta = \frac{1}{T}\,dI + \frac{p}{T}\,d\alpha - \frac{\mu}{T}\,dS, \tag{1.67}$$

it is also clear that

$$p = T\left(\frac{\partial \eta}{\partial \alpha}\right)_{I,S}, \qquad T^{-1} = \left(\frac{\partial \eta}{\partial I}\right)_{\alpha,S}, \qquad \mu = -T\left(\frac{\partial \eta}{\partial S}\right)_{I,\alpha}. \tag{1.68a,b,c}$$

We also see that I and α (and S) are the natural variables for entropy.

Because the right-hand side of (1.65) is equal to an exact differential, the second derivatives are independent of the order of differentiation. That is,

$$\frac{\partial^2 I}{\partial \eta\, \partial \alpha} = \frac{\partial^2 I}{\partial \alpha\, \partial \eta}, \tag{1.69}$$

and therefore, using (1.66)

$$\left(\frac{\partial T}{\partial \alpha}\right)_\eta = -\left(\frac{\partial p}{\partial \eta}\right)_\alpha. \tag{1.70}$$

This is one of the *Maxwell relations,* which are a collection of four similar relations that follow directly from the fundamental thermodynamic relation (1.65) and simple relations between second derivatives. (Additional Maxwell-like relations exist if we consider chemical effects.) To derive the other Maxwell relations we will introduce thermodynamic potentials enthalpy, h, the Gibbs function, g, and the free energy, f. These are all closely related to the internal energy and they are all extensive functions (and then denoted with an uppercase letter), but for fluid-dynamical purposes it is convenient to divide them by the mass and use their specific forms, denoted with a lowercase letter (with the exception of I itself, the specific internal energy).

Define the *enthalpy* of a fluid by

$$h \equiv I + p\alpha, \tag{1.71}$$

and then (1.65) becomes

$$dh = T\,d\eta + \alpha\,dp + \mu\,dS. \tag{1.72}$$

Evidently, the natural variables for enthalpy are entropy and pressure so that, in general,

$$dh = \left(\frac{\partial h}{\partial \eta}\right)_{p,S}d\eta + \left(\frac{\partial h}{\partial p}\right)_{\eta,S}dp + \left(\frac{\partial h}{\partial S}\right)_{\eta,p}dS. \tag{1.73}$$

Comparing the last two equations we have

$$T = \left(\frac{\partial h}{\partial \eta}\right)_{p,S}, \qquad \alpha = \left(\frac{\partial h}{\partial p}\right)_{\eta,S}, \qquad \mu = \left(\frac{\partial h}{\partial S}\right)_{\eta,p}. \tag{1.74}$$

Noting that

$$\frac{\partial^2 h}{\partial \eta \, \partial p} = \frac{\partial^2 h}{\partial p \, \partial \eta} \tag{1.75}$$

we evidently must have

$$\left(\frac{\partial T}{\partial p} \right)_\eta = \left(\frac{\partial \alpha}{\partial \eta} \right)_p, \tag{1.76}$$

and this is our second Maxwell relation.

To obtain the third, we write

$$dI = T\,d\eta - p\,d\alpha + \mu\,dS = d(T\eta) - \eta\,dT - d(p\alpha) + \alpha\,dp + \mu\,dS, \tag{1.77}$$

or

$$dg = -\eta\,dT + \alpha\,dp + \mu\,dS \quad \text{where} \quad g \equiv I - T\eta + p\alpha = h - TS. \tag{1.78a,b}$$

The quantity g is the *Gibbs function*, also known as the 'Gibbs free energy' or 'Gibbs potential'. (We use g for gravity and g for the specific Gibbs function.) Now, formally, we have,

$$dg = \left(\frac{\partial g}{\partial T} \right)_{p,S} dT + \left(\frac{\partial g}{\partial p} \right)_{T,S} dp + \left(\frac{\partial g}{\partial S} \right)_{T,p} dS. \tag{1.79}$$

Comparing the last two equations we see that

$$\eta = -\left(\frac{\partial g}{\partial T} \right)_{p,S}, \qquad \alpha = \left(\frac{\partial g}{\partial p} \right)_{T,S}, \qquad \mu = \left(\frac{\partial g}{\partial S} \right)_{T,p}. \tag{1.80}$$

Furthermore, because

$$\frac{\partial^2 g}{\partial p \, \partial T} = \frac{\partial^2 g}{\partial T \, \partial p} \tag{1.81}$$

we have our third Maxwell equation,

$$\left(\frac{\partial \eta}{\partial p} \right)_T = -\left(\frac{\partial \alpha}{\partial T} \right)_p. \tag{1.82}$$

The Gibbs function is unique among the thermodynamic potentials in that its natural variables, T and p, are intensive quantities.

The fourth Maxwell equation makes use of the specific free energy or Helmholtz function, f, where

$$f \equiv I - T\eta, \qquad \text{and} \qquad df = -\eta\,dT - p\,d\alpha + \mu\,dS, \tag{1.83}$$

giving

$$\left(\frac{\partial \eta}{\partial \alpha} \right)_T = \left(\frac{\partial p}{\partial T} \right)_\alpha, \tag{1.84}$$

and all four of these Maxwell equations are summarized in the box on the next page. All of them follow from the fundamental thermodynamic relation, (1.65), which is the real silver hammer of thermodynamics. The fundamental relation can also be written in the following form connecting internal energy, enthalpy and entropy,

$$dI + p\,d\alpha = dh - \alpha\,dp = Td\eta + \mu\,dS, \tag{1.85}$$

which turns out to be useful for fluid dynamical applications.

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Thermodynamic Functions and Maxwell Relations

The four canonical Maxwell relations, with their associated potentials, are

Internal energy:

$$I = I,$$

$$dI = T\,d\eta - p\,d\alpha + \mu\,dS,$$

$$\left(\frac{\partial T}{\partial \alpha}\right)_{\eta,S} = -\left(\frac{\partial p}{\partial \eta}\right)_{\alpha,S}.$$

Enthalpy:

$$h = I + p\alpha,$$

$$dh = T\,d\eta + \alpha\,dp + \mu\,dS,$$

$$\left(\frac{\partial T}{\partial p}\right)_{\eta,S} = \left(\frac{\partial \alpha}{\partial \eta}\right)_{p,S}.$$

Gibbs function:

$$g = I - T\eta + p\alpha,$$

$$dg = -\eta\,dT + \alpha\,dp + \mu\,dS,$$

$$\left(\frac{\partial \eta}{\partial p}\right)_{T,S} = -\left(\frac{\partial \alpha}{\partial T}\right)_{p,S}.$$

Helmholtz free energy:

$$f = I - T\eta,$$

$$df = -\eta\,dT - p\,d\alpha + \mu\,dS,$$

$$\left(\frac{\partial \eta}{\partial \alpha}\right)_{T,S} = \left(\frac{\partial p}{\partial T}\right)_{\alpha,S}.$$

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Meaning of the state functions

Internal energy, enthalpy, the Gibbs function, the free energy and the entropy are all state functions from which other thermodynamic quantities can be derived, but with different meanings and uses. The utility of these quantities will become apparent as we proceed, but here is a brief summary.

The internal energy of a body is the total energy within a body, excluding the kinetic energy and the potential energy due to external fields like gravity. It is an invariant if the volume is fixed and there is no heating or chemical change to the body (this is the first law). The other state functions are related to internal energy by Legendre transformations, but they are not necessarily equal to the energy that a body contains. The enthalpy is the Legendre transformation of the internal energy (a function of entropy, density and composition) to a function of entropy, pressure and composition, and it is important in fluids because the energy transfer between a fluid parcel and its environment is associated with a flux of enthalpy, not internal energy. If a parcel is adiabatically displaced its change in potential energy will be balanced by a change in enthalpy, for it is enthalpy that accounts for the work done by pressure forces. When two adjacent parcels at the same pressure mix, their total enthalpy is conserved.

The Gibbs function is useful because it is constant for systems at constant temperature, pressure and composition. Its natural variables, temperature, pressure and composition are all measurable and for that reason it finds use as the fundamental state function from which all other thermodynamic variables may be derived. The Helmholtz free energy is also sometimes used as a fundamental state variable, and is useful for systems at constant temperature, density and composition for then it is constant. For small isothermal and isohaline changes, the increase of free energy is equal to the work done on the system. Free energy is not in fact commonly used in the atmospheric or oceanic sciences. Finally, whole books have been written on entropy (which is not a Legendre transformation of the internal energy and is not a thermodynamic potential in the same sense as the others). Suffice it to say here that entropy is the state function that responds directly to heating and it is a measure of the disorder of a system.[5]

Fundamental equation of state

The fundamental equation of state (1.59) gives complete information about a fluid in thermodynamic equilibrium, and given this we can obtain expressions for the temperature, pressure and chemical potential using (1.66). These expressions are also equations of state; however, each of them, taken individually, contains less information than the fundamental equation because a derivative has been taken. Equivalent to the fundamental equation of state are, using (1.72), an expression for the enthalpy as a function of its natural variables pressure, entropy and composition, or, using (1.78), the Gibbs function as a function of pressure, temperature and composition. Of these, the Gibbs function is particularly useful in practice because the pressure, temperature and composition may all be measured in the laboratory. Given the fundamental equation of state, the thermodynamic state of a body is fully specified by a knowledge of any two of p, ρ, T, η and I, plus its composition. The thermal equation of state, (1.52), is obtained by using (1.59a) to eliminate entropy from (1.66a) and (1.66b).

One simple fundamental equation of state is to take the internal energy to be a function of density and not entropy; that is, $I = I(\alpha)$. Bodies with such a property are called *homentropic*. Using (1.66), temperature and chemical potential have no role in the fluid dynamics and the density is a function of pressure alone — the defining property of a barotropic fluid.[6] Neither water nor air is, in general, homentropic but under some circumstances the flow may be adiabatic and $p = p(\rho)$.

In an ideal gas the molecules do not interact except by elastic collisions, and the volume of the molecules is presumed to be negligible compared to the total volume they occupy. The internal energy of the gas then depends only on temperature, and not on density. A *simple* ideal gas, also called a *perfect* gas (although nomenclature in the literature varies), is an ideal gas for which the heat capacity is constant, so that

$$I = cT, \tag{1.86}$$

where c is a constant. Using this and the conventional ideal gas equation, $p = \rho RT$ (where R is also constant), along with the fundamental thermodynamic relation (1.65), we can infer the fundamental equation of state; however, we will defer that until we discuss potential temperature in Section 1.6.1 — if curious, look ahead to page 25. A general ideal gas also obeys $p = \rho RT$, but has heat capacities that may be a function of temperature (but only of temperature, if composition is fixed), but in this book we only deal with simple ideal gases.

Internal energy and specific heats

We can obtain some useful relations between the internal energy and specific heat capacities, and some useful estimates of their values, by some simple manipulations of the fundamental thermodynamic relation. Assuming that the composition of the fluid is constant, (1.65) is

$$T \, d\eta = dI + p \, d\alpha, \tag{1.87}$$

so that, taking I to be a function of α and T,

$$T \, d\eta = \left(\frac{\partial I}{\partial T} \right)_{\alpha, S} dT + \left[\left(\frac{\partial I}{\partial \alpha} \right)_{T, S} + p \right] d\alpha. \tag{1.88}$$

From this, we see that the heat capacity at constant volume (i.e., constant α) c_v is given by

$$c_v \equiv T \left(\frac{\partial \eta}{\partial T} \right)_{\alpha, S} = \left(\frac{\partial I}{\partial T} \right)_{\alpha, S}. \tag{1.89}$$

Thus, c in (1.86) is equal to c_v.

Similarly, using (1.72) we have

$$T \, d\eta = dh - \alpha \, dp = \left(\frac{\partial h}{\partial T} \right)_{p, S} dT + \left[\left(\frac{\partial h}{\partial p} \right)_{T, S} - \alpha \right] dp. \tag{1.90}$$

The heat capacity at constant pressure, c_p, is then given by

$$c_p \equiv T \left(\frac{\partial \eta}{\partial T} \right)_{p,S} = \left(\frac{\partial h}{\partial T} \right)_{p,S}. \tag{1.91}$$

For an ideal gas $h = I + RT = T(c_v + R)$. But $c_p = (\partial h / \partial T)_p$, and hence $c_p = c_v + R$. For future use we define $\gamma \equiv c_p / c_v$ and $\kappa \equiv R / c_p$, and $(\gamma - 1)/\gamma = \kappa$. Statistical mechanics tells us that for a simple ideal gas the internal energy is equal to $kT/2$ per molecule, or $RT/2$ per unit mass, for each excited degree of freedom, where k is the Boltzmann constant and R the gas constant. The diatomic molecules N_2 and O_2 that form most of our atmosphere have two rotational and three translational degrees of freedom, so that $I \approx 5RT/2$, and so $c_v \approx 5R/2$ and $c_p \approx 7R/2$, both being constants. These are in fact very good approximations to the measured values for the Earth's atmosphere, and give $c_v \approx 714\,\mathrm{J\,kg^{-1}K^{-1}}$ and $c_p \approx 1000\,\mathrm{J\,kg^{-1}K^{-1}}$ (whereas c_p is measured to be $1003\,\mathrm{J\,kg^{-1}K^{-1}}$). The internal energy is simply $c_v T$ and the enthalpy is $c_p T$. For a liquid, especially one like seawater that contains dissolved salts, no such simple relations are possible: the heat capacities are functions of the state of the fluid, and the internal energy is a function of pressure (or density) as well as temperature and composition.

1.6   THERMODYNAMIC EQUATIONS FOR FLUIDS

The thermodynamic relations — for example (1.65) — apply to identifiable bodies or systems; thus, the heat input affects the fluid parcel to which it is applied, and we can apply the material derivative to the above thermodynamic relations to obtain equations of motion for a moving fluid. In doing so we make two assumptions:

(i) That locally the fluid is in thermodynamic equilibrium. This means that, although the thermodynamic quantities like temperature, pressure and density vary in space and time, locally they are related by the thermodynamic relations such as the equation of state and Maxwell's relations.

(ii) That macroscopic fluid motions are reversible and so not entropy producing. Thus, such effects as the viscous dissipation of energy, radiation, and conduction may produce entropy whereas the macroscopic fluid motion itself does not.

The first point requires that the temperature variation on the macroscopic scales must be slow enough that there can exist a volume that is small compared to the scale of macroscopic variations, so that temperature is effectively constant within it, but that is also sufficiently large to contain enough molecules so that macroscopic variables such as temperature have a proper meaning.

Now from (1.61), conservation of energy for an infinitesimal fluid parcel may be written as

$$\mathrm{d}I = -p\,\mathrm{d}\alpha + \text{đ}Q_E, \tag{1.92}$$

where $p\,\mathrm{d}\alpha$ is the work done by the parcel and $\text{đ}Q_E$ is the total energy input to the parcel with contributions from heating and changes in composition. Given the first assumption above, we may form the material derivative of (1.92) to obtain

$$\frac{\mathrm{D}I}{\mathrm{D}t} + p\frac{\mathrm{D}\alpha}{\mathrm{D}t} = \dot{Q}_E, \tag{1.93}$$

where $\dot{Q}_E$ is the rate of total energy input, per unit mass, with (as for $\text{đ}Q_E$) possible contributions from thermal fluxes (including radiative heating, thermal diffusion and heat generated by viscous damping) and fluxes of composition, and note that $\dot{Q}_E$ does not include any mechanical effects. (In general, both the diffusion of heat and composition depend on the gradients of both temperature and composition, although the thermal flux is largely determined by the temperature gradient, and the compositional flux by the gradient of composition.)

Using the mass continuity equation in the form $D\alpha/Dt = \alpha\nabla \cdot \boldsymbol{v}$, we write (1.93) as

$$\frac{DI}{Dt} + p\alpha\nabla \cdot \boldsymbol{v} = \dot{Q}_E. \tag{1.94}$$

This is the *internal energy equation* for a fluid. Internal energy is not a conservative variable because of the compression term involving $\nabla \cdot \boldsymbol{v}$. The internal energy equation may also be written in terms of enthalpy, and using (1.71) and (1.93) we obtain the equivalent equation,

$$\frac{Dh}{Dt} - \alpha\frac{Dp}{Dt} = \dot{Q}_E. \tag{1.95}$$

Because $\dot{Q}_E$ contains, in general, energy fluxes due to changes in composition, we need to know that composition. The composition of a fluid parcel is carried with it as it moves, and changes only if there are non-conservative sources and sinks, such as diffusive fluxes. Thus, and analogously to (1.41), the evolution of composition is determined by

$$\frac{DS}{Dt} = \dot{S}, \tag{1.96}$$

where $\dot{S}$ represents all the non-conservative terms.

Rather than use an internal energy equation, we may use the fundamental thermodynamic relation to infer an evolution equation for entropy. Thus, forming the material derivative from (1.65) and using (1.94) and (1.96), we obtain the *entropy equation*

$$\frac{D\eta}{Dt} = \frac{1}{T}\dot{Q}_E - \frac{\mu}{T}\dot{S} \equiv \frac{1}{T}\dot{Q}, \tag{1.97}$$

where $\dot{Q}$ is the heating rate per unit mass. This equation is simply the material derivative of (1.62), along with assumption *(ii)* above. The heating of a fluid parcel generally needs to be *derived*, since it involves the viscous dissipation of energy as well as radiative and diffusive fluxes. The derivation is the topic of Appendix A at the chapter end, and from now on we assume the heating and energy input are known quantities.

The entropy equation is not independent of the internal energy equation, but is connected via the thermodynamic relations of Sections 1.5.1 and 1.5.2, and by the equation of state. If we use the internal energy equation we can in principle then calculate the entropy using the equation of state in the form $\eta = \eta(I, \alpha, S)$, or if we use the entropy equation we can then calculate the internal energy using $I = I(\eta, \alpha, S)$. Indeed, the internal energy equation and the entropy equation are both commonly referred to as 'the thermodynamic equation'. The heating term, $\dot{Q}$, is, however, not always easy to accurately determine in practice — it is affected by gradients of composition as well as viscosity and radiative fluxes — and the use of the internal energy equation may be more straightforward than the use of an entropy equation, especially in multi-component fluids. On the other hand, internal energy is affected by the $\nabla \cdot \boldsymbol{v}$ term in (1.94) and in a liquid this is small but non-zero and that may cause difficulties. See Section 1.7.3 for more discussion.

In any case, given evolution equations for composition and internal energy or entropy, and given the fundamental equation of state, we have, in principle, a complete set of equations for a fluid, as summarized in the shaded box on the facing page. Let us now look at the special, but very important, case of a dry ideal gas.

Fundamental Equations of Motion of a Fluid

The following equations constitute, in principle, a complete set of equations for a fluid heated at a rate $\dot{Q}$ and whose composition, S, changes at a rate $\dot{S}$.

Evolution equations for velocity, density and composition

$$\frac{D\boldsymbol{v}}{Dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{v} + \boldsymbol{F}, \qquad \frac{D\rho}{Dt} + \rho\nabla\cdot\boldsymbol{v} = 0, \qquad \frac{DS}{Dt} = \dot{S}, \qquad (\text{F.1})$$

where $\boldsymbol{F}$ is some body force per unit mass, such as gravity.

Internal energy equation or entropy equation (the 'thermodynamic' equation)

$$\frac{DI}{Dt} + \frac{p}{\rho}\nabla\cdot\boldsymbol{v} = \dot{Q}_E \qquad \text{or} \qquad \frac{D\eta}{Dt} = \frac{1}{T}\dot{Q}, \qquad (\text{F.2})$$

where $\dot{Q}$ is the heating and $\dot{Q}_E = \dot{Q} + \mu\dot{S}$ is the total rate of energy input.

Fundamental equation of state for internal energy, I, entropy, η, or Gibbs function, g

$$I = I(\rho, \eta, S), \quad \eta = \eta(\rho, I, S) \quad \text{or} \quad g = g(T, p, S). \qquad (\text{F.3})$$

Diagnostic equations for temperature and pressure

$$T = \left(\frac{\partial I}{\partial \eta}\right)_{\alpha,S} \qquad \text{or} \quad T^{-1} = \left(\frac{\partial \eta}{\partial I}\right)_{\alpha,S}. \qquad (\text{F.4})$$

$$p = -\left(\frac{\partial I}{\partial \alpha}\right)_{\eta,S} = \rho^2\left(\frac{\partial I}{\partial \rho}\right)_{\eta,S} \qquad \text{or} \quad p = T\left(\frac{\partial \eta}{\partial \alpha}\right)_{I,S}. \qquad (\text{F.5})$$

The actual method of solution of these equations will depend on the equation of state. For example, for an ideal gas $T = I/c_v$ and the thermal equation of state, $p = \rho RT$, may be used to infer pressure.

The equations describing fluid motion are called the *Euler equations* if the viscous term is omitted, and the *Navier–Stokes equations* if viscosity is included.[7] These appellations are often taken to mean only the momentum and mass conservation equations, and the Euler equations are sometimes taken to be the equations for a fluid of constant density.

1.6.1   Thermodynamic Equation for an Ideal Gas

For a dry ideal gas the internal energy is a function of temperature only and $dI = c_v\,dT$. The first law of thermodynamics becomes

$$đQ = c_v\,dT + p\,d\alpha, \qquad \text{or} \qquad đQ = c_p\,dT - \alpha\,dp, \qquad (1.98\text{a,b})$$

where the second expression is derived using $\alpha = RT/p$ and $c_p - c_v = R$. Forming the material derivative of the above gives two forms of the internal energy equation:

$$c_v\frac{DT}{Dt} + p\frac{D\alpha}{Dt} = \dot{Q} \qquad \text{or} \qquad c_p\frac{DT}{Dt} - \frac{RT}{p}\frac{Dp}{Dt} = \dot{Q}. \qquad (1.99\text{a,b})$$

Using the mass continuity equation, (1.99a) may be written as

$$c_v \frac{\mathrm{D}T}{\mathrm{D}t} + p\alpha\nabla \cdot \boldsymbol{v} = \dot{Q}. \tag{1.100}$$

This is one of the most common and useful forms of the thermodynamic equation for the atmosphere. A less common but equivalent form arises if we use the ideal gas equation to eliminate T in favour of p, giving

$$\frac{\mathrm{D}p}{\mathrm{D}t} + \gamma p\nabla \cdot \boldsymbol{v} = \dot{Q}\frac{\rho R}{c_v}. \tag{1.101}$$

The Earth's atmosphere also contains water vapour with mixing ratio q (commonly referred to as specific humidity), and an evolution equation for it takes the form

$$\frac{\mathrm{D}q}{\mathrm{D}t} = \dot{q}, \tag{1.102}$$

where $\dot{q}$ represents the effects of condensation and evaporation. The main thermodynamic effects of water vapour occur when it condenses and latent heat is released; this heating appears in the thermodynamic equation, with $\dot{Q} = -L\dot{q}$, where L is the latent heat of condensation, as discussed in Chapter 18.

Potential temperature, potential density and entropy

We can use entropy instead of temperature for our thermodynamic equation, and this corresponds to using (1.97) instead of (1.94). However, it is common in meteorology to express entropy in terms of a temperature-like quantity, potential temperature, which has a more intuitive appeal to some.[8] Seawater also has a potential temperature variable, but with a different form.

We begin with the observation that when a fluid parcel changes pressure adiabatically, it will expand or contract and, using (1.98b) with đQ = 0, its temperature change is determined by $c_p\,\mathrm{d}T = \alpha\,\mathrm{d}p$. This temperature change is plainly not caused by heating, but we may construct a temperature-like quantity that changes *only* if diabatic effects are present; specifically *potential temperature*, θ, is defined to be the temperature that a fluid would have if moved adiabatically and at constant composition to some reference pressure (usually taken to be 1000 hPa, which is close to the pressure at the Earth's surface). Thus, in adiabatic flow the potential temperature obeys $\mathrm{D}\theta/\mathrm{D}t = 0$.

In order to relate θ to the other thermodynamic variables we use (1.98b) and the equation of state for an ideal gas to write the fundamental thermodynamic relation as

$$\mathrm{d}\eta = c_p\,\mathrm{d}\ln T - R\,\mathrm{d}\ln p. \tag{1.103}$$

If we move a parcel adiabatically (dη = 0) from p to p_R the temperature changes, by definition, from T to θ, and (1.103) gives

$$\int_T^\theta c_p\,\mathrm{d}\ln T - \int_p^{p_R} R\,\mathrm{d}\ln p = 0. \tag{1.104}$$

For constant c_p and R this equation may be solved to give

$$\theta = T\left(\frac{p_R}{p}\right)^\kappa, \tag{1.105}$$

where p_R is the reference pressure and $\kappa \equiv R/c_p$. (Another derivation of this result is given in Appendix A.) It follows from (1.103) and (1.105) that potential temperature is related to entropy by

$$d\eta = c_p\, d\ln\theta, \tag{1.106}$$

and, if c_p is constant, which it nearly is for Earth's atmosphere,

$$\eta = c_p \ln\theta. \tag{1.107}$$

Forming the material derivative of (1.107) and using (1.97) we obtain

$$c_p \frac{D\theta}{Dt} = \frac{\theta}{T}\dot{Q}, \tag{1.108}$$

with θ given by (1.105). Equations (1.100), (1.101) and (1.108) are all equivalent forms of the thermodynamic equation for an ideal gas.

The *potential density*, ρ_θ, is the density that a fluid parcel would have if moved adiabatically and at constant composition to a reference pressure, p_R. If the equation of state is written as $\rho = f(p, T)$ then the potential density is just $\rho_\theta = f(p_R, \theta)$ and for an ideal gas we therefore have

$$\rho_\theta = \frac{p_R}{R\theta} = \rho\left(\frac{p_R}{p}\right)^{1/\gamma}. \tag{1.109}$$

Finally, for later use, a little manipulation of the equation of state for an ideal gas, for small variations around a reference state, reveals that

$$\frac{\delta\theta}{\theta} = \frac{\delta T}{T} - \kappa\frac{\delta p}{p} = \frac{1}{\gamma}\frac{\delta p}{p} - \frac{\delta\rho}{\rho}. \tag{1.110}$$

The fundamental equation of state for an ideal gas

Equations (1.107) and (1.105) are closely related to the fundamental equation of state, and using $I = c_v T$ and the equation of state $p = \rho RT$, we can express the entropy explicitly in terms of the density and the internal energy. Similarly, we may derive an expression for Gibbs function as a function of pressure and temperature, and we find

$$\eta = c_v \ln I - R\ln\rho + A, \qquad g = c_p T(1 - \ln T) + RT\ln p + BT + C, \tag{1.111a,b}$$

where A, B and C are constants. Either of these two expressions may be regarded as *the fundamental equation of state for a simple ideal gas*. They could in fact be used to define a simple ideal gas (although we have not motivated that approach) and if we were to *begin* with either of them we could derive all the other thermodynamic quantities of interest. For example, using (1.68a) and (1.111a) we immediately recover $p = \rho RT$, and using (1.68b) we obtain $T = c_v/I$. Similarly, from the Gibbs function we obtain $\alpha = (\partial g/\partial p)_T = RT/p$, and the entropy satisfies $\eta = -(\partial g/\partial T)_p$. We provide a more complete set of derivations from the Gibbs function in Appendix A on page 47 and, for moist air, the appendix on page 720.

1.6.2 ✦ Other Forms of the Thermodynamic Equation

For a liquid such as seawater no simple exact equation of state exists and writing down a useful thermodynamic equation is not easy. Thus, although (1.97) holds in general, we need to be able to evaluate the heating and we need an expression relating entropy to the other thermodynamic variables — that is, an equation of state. For quantitative modelling and observational work such an equation of state must be quite accurate and we come back to this in Section 1.7. However, we can gain understanding — for both liquids and gases — by beginning with the entropy equation and making simplifications to it, as follows.

> ### Forms of the Thermodynamic Equation
>
> *General form*
>
> $$\frac{DI}{Dt} + p\frac{D\alpha}{Dt} = \dot{Q}_E \qquad \text{or} \qquad \frac{DI}{Dt} + p\alpha \nabla \cdot \boldsymbol{v} = \dot{Q}_E, \tag{T.1a,b}$$
>
> where I is the internal energy and $\dot{Q}_E$ is the rate of energy input, per unit mass. This may be written in terms of enthalpy, h, or entropy, η
>
> $$\frac{Dh}{Dt} - \alpha\frac{Dp}{Dt} = \dot{Q}_E, \qquad T\frac{D\eta}{Dt} = \dot{Q}_E - \mu\dot{S} = \dot{Q}, \tag{T.2a,b}$$
>
> where $\dot{Q}$ is the heating rate and $\dot{S}$ the rate of change of composition. For a fluid parcel of constant composition, $c_p\, d\ln\theta = d\eta$ and (T.2b) may be written as a potential temperature equation.
>
> *Ideal gas*
>
> For an ideal gas $dI = c_v\, dT$, $dh = c_p\, dT$, $d\eta = c_p\, d\ln\theta$ and the adiabatic thermodynamic equation may be written in the following equivalent, exact, forms:
>
> $$\begin{aligned} c_p\frac{DT}{Dt} - \alpha\frac{Dp}{Dt} &= 0, & \frac{Dp}{Dt} + \gamma p\nabla\cdot\boldsymbol{v} &= 0, \\ c_v\frac{DT}{Dt} + p\alpha\nabla\cdot\boldsymbol{v} &= 0, & \frac{D\theta}{Dt} &= 0, \end{aligned} \tag{T.3}$$
>
> where $\theta = T(p_R/p)^\kappa$, and energy or heating terms in various forms appear on the right-hand sides as needed. The two expressions on the second line are usually the most useful in modelling and theoretical work in meteorology.

I. Thermodynamic equation using pressure and density

If we regard η as a function of pressure and density, and salinity S where appropriate, we obtain

$$\begin{aligned} T\, d\eta &= T\left(\frac{\partial\eta}{\partial\rho}\right)_{p,S} d\rho + T\left(\frac{\partial\eta}{\partial p}\right)_{\rho,S} dp + T\left(\frac{\partial\eta}{\partial S}\right)_{\rho,p} dS \\ &= T\left(\frac{\partial\eta}{\partial\rho}\right)_{p,S} d\rho - T\left(\frac{\partial\eta}{\partial\rho}\right)_{p,S}\left(\frac{\partial\rho}{\partial p}\right)_{\eta,S} dp + T\left(\frac{\partial\eta}{\partial S}\right)_{\rho,p} dS. \end{aligned} \tag{1.112}$$

Forming the material derivative, and using (1.97) and (1.96), we obtain for a moving fluid

$$T\left(\frac{\partial\eta}{\partial\rho}\right)_{p,S}\frac{D\rho}{Dt} - T\left(\frac{\partial\eta}{\partial\rho}\right)_{p,S}\left(\frac{\partial\rho}{\partial p}\right)_{\eta,S}\frac{Dp}{Dt} = \dot{Q} - T\left(\frac{\partial\eta}{\partial S}\right)_{\rho,p}\dot{S}. \tag{1.113}$$

But $(\partial p/\partial\rho)_{\eta,S} = c_s^2$ where c_s is the speed of sound (see Section 1.8). This is a measurable quantity in a fluid, and often nearly constant, and so useful to keep in an equation. The thermodynamic equation may then be written in the form

$$\frac{D\rho}{Dt} - \frac{1}{c_s^2}\frac{Dp}{Dt} = Q_{[\rho,p]}, \tag{1.114}$$

where $Q_{[\rho,p]} \equiv (\partial \rho / \partial \eta)_{p,S} \dot{Q}/T + (\partial \rho / \partial S)_{\eta,p} \dot{S}$ represents the effects of entropy and salinity source terms. This form of the thermodynamic equation is valid for both liquids and gases.

Approximations using pressure and density

The speed of sound in a fluid is related to its compressibility — the less compressible the fluid, the greater the sound speed. In a liquid, sound speed is often sufficiently high that the second term in (1.114) can be neglected, and the thermodynamic equation takes the simple form:

$$\frac{D\rho}{Dt} = Q_{[\rho,p]}. \tag{1.115}$$

The above equation is a very good approximation for many laboratory fluids. It is a *thermodynamic* equation, arising from the principle of conservation of energy for a liquid; it is a very different equation from the mass conservation equation, which for compressible fluids is also an evolution equation for density.

In the ocean the enormous pressures resulting from columns of seawater kilometres deep mean that although the second term in (1.114) may be small, it is not negligible, and a better approximation results if we suppose that the pressure is given by the weight of the fluid above it — the hydrostatic approximation. In this case $dp = -\rho g\, dz$ and (1.114) becomes

$$\frac{D\rho}{Dt} + \frac{\rho g}{c_s^2} \frac{Dz}{Dt} = Q_{[\rho,p]}. \tag{1.116}$$

In the second term the height field varies much more than the density field, so a good approximation is to replace ρ by a constant, ρ_0, in this term only. Taking the speed of sound also to be constant gives

$$\frac{D}{Dt}\left(\rho + \frac{\rho_0 z}{H_\rho}\right) = Q_{[\rho,p]}, \qquad \text{where} \qquad H_\rho = c_s^2/g. \tag{1.117a,b}$$

H_ρ is the *density scale height* of the ocean. In water, $c_s \approx 1500\,\text{m s}^{-1}$ so that $H_\rho \approx 200\,\text{km}$. The quantity in brackets on the left-hand side of (1.117a) is (in this approximation) the *potential density,* this being the density that a parcel would have if moved adiabatically and with constant composition to the reference height $z = 0$. The adiabatic lapse rate of density is the rate at which the density of a parcel changes when undergoing an adiabatic displacement. From (1.117) it is approximately

$$-\left(\frac{\partial \rho}{\partial z}\right)_\eta \approx \frac{\rho_0 g}{c_s^2} \approx 5\,(\text{kg m}^{-3})/\text{km}, \tag{1.118}$$

so that if a parcel is moved adiabatically from the surface to the deep ocean (5 km depth, say) its density will increase by about 25 kg m^{-3}, a fractional change of about 1/40 or 2.5%.

II. Thermodynamic equation using pressure and temperature

Taking entropy to be a function of pressure and temperature (and salinity if appropriate), we have

$$T\, d\eta = T\left(\frac{\partial \eta}{\partial T}\right)_{p,S} dT + T\left(\frac{\partial \eta}{\partial p}\right)_{T,S} dp + T\left(\frac{\partial \eta}{\partial S}\right)_{T,p} dS$$
$$= c_p\, dT + T\left(\frac{\partial \eta}{\partial p}\right)_{T,S} dp + T\left(\frac{\partial \eta}{\partial S}\right)_{T,p} dS. \tag{1.119}$$

For a moving fluid, and using (1.97) and (1.96), this implies,

$$\frac{DT}{Dt} + \frac{T}{c_p}\left(\frac{\partial \eta}{\partial p}\right)_{T,S} \frac{Dp}{Dt} = Q_{[T,p]}, \tag{1.120}$$

where $Q_{[T,p]} \equiv \dot{Q}/c_p - Tc_p^{-1}\dot{S}(\partial\eta/\partial S)$. Now substitute the Maxwell relation (1.82) in the form

$$\left(\frac{\partial\eta}{\partial p}\right)_T = \frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial T}\right)_p \tag{1.121}$$

to give

$$\frac{DT}{Dt} + \frac{T}{c_p\rho^2}\left(\frac{\partial\rho}{\partial T}\right)_p\frac{Dp}{Dt} = Q_{[T,p]}, \qquad \text{or} \qquad \frac{DT}{Dt} - \frac{T}{c_p}\left(\frac{\partial\alpha}{\partial T}\right)_p\frac{Dp}{Dt} = Q_{[T,p]}. \tag{1.122a,b}$$

The density and temperature are related through a coefficient of thermal expansion β_T where

$$\left(\frac{\partial\rho}{\partial T}\right)_p = -\beta_T\rho. \tag{1.123}$$

Equation (1.122) then becomes

$$\frac{DT}{Dt} - \frac{\beta_T T}{c_p\rho}\frac{Dp}{Dt} = Q_{[T,p]} . \tag{1.124}$$

This form of the thermodynamic equation is valid for both liquids and gases. In an ideal gas we have $\beta_T = 1/T$, whereas in a liquid β_T is usually quite small.

Approximations using pressure and temperature

In the hydrostatic approximation we suppose that the pressure in (1.124) varies according only to the weight of the fluid above it. Then $dp = -\rho g\,dz$ and (1.124) becomes

$$\frac{1}{T}\frac{DT}{Dt} + \frac{\beta_T g}{c_p}\frac{Dz}{Dt} = \frac{Q_{[T,p]}}{T}. \tag{1.125}$$

For an ideal gas we have $\beta_T = 1/T$, whence, if c_p is constant,

$$\frac{D}{Dt}\left(c_p T + gz\right) = c_p Q_{[T,p]}. \tag{1.126}$$

The quantity $c_p T + gz$ is known as the dry static energy and we will encounter it again throughout the book. The above equation is closely related to the potential temperature form of the thermodynamic equation, since in hydrostatic balance a little manipulation reveals that

$$\frac{T}{\theta}\frac{\partial\theta}{\partial z} = \frac{\partial T}{\partial z} + \frac{g}{c_p}. \tag{1.127}$$

The quantity $T + gz/c_p$ is a height form of potential temperature for an ideal gas in hydrostatic balance, being the temperature that a fluid at a level z and temperature T would have if moved adiabatically to a reference level of $z = 0$.

If β_T is constant, which is a fair approximation for many liquids, then for small variations of temperature around the value T_0, (1.125) simplifies to

$$\frac{D}{Dt}\left(T + \frac{T_0 z}{H_T}\right) = Q_{[T,p]}, \qquad \text{where} \qquad H_T = \frac{c_p}{\beta_T g}. \tag{1.128a,b}$$

The quantity H_T is the *temperature scale height* of the fluid, and with the oceanic values $\beta_T \approx 2 \times 10^{-4}\,\mathrm{K}^{-1}$ and $c_p \approx 4 \times 10^3\,\mathrm{J\,kg^{-1}\,K^{-1}}$ we obtain $H_T \approx 2000\,\mathrm{km}$. The field $T + T_0 z / H_T$ is a height form of potential temperature for liquids; that is,

$$\theta \approx T + \frac{\beta_T g T_0}{c_p} z. \tag{1.129}$$

The temperature changes because of the work done by or on the fluid parcel as it expands or is compressed. In seawater the expansion coefficient β_T and c_p are functions of pressure and (1.129) is not good enough if high accuracy is required, whereas in a laboratory setting we can often simply neglect the term involving β_T.

The adiabatic lapse rate of temperature is the rate at which the temperature of a parcel changes in the vertical when undergoing an adiabatic displacement. From (1.125) it is

$$\Gamma_z \equiv -\left(\frac{\partial T}{\partial z}\right)_\eta = \frac{T g \beta_T}{c_p}. \tag{1.130}$$

In general Γ_z is a function of temperature, salinity and pressure, but it is a calculable quantity if β_T is known and, with the oceanic values above, it is approximately $0.15\,\mathrm{K\,km^{-1}}$. Equation (1.130) is not accurate enough for quantitative oceanography because the expansion coefficient is a function of pressure; nor is it a good measure of stability, because of the effects of salt. In a dry atmosphere the ideal gas relationship gives $\beta_T = 1/T$ and so

$$\Gamma_z = \frac{g}{c_p}, \tag{1.131}$$

which is approximately $10\,\mathrm{K\,km^{-1}}$. The only approximation involved in deriving this is the use of the hydrostatic relationship.

It is noteworthy that the scale heights given by (1.117b) and (1.128b) differ so much. The first is due to the pressure compressibility of seawater [and so related to c_s^2, or β_p in (1.57)] whereas the second is due to the change of density with temperature [β_T in (1.57)], and is the distance over which the difference between temperature and potential temperature changes by an amount equal to the temperature itself (i.e., by about 273 K). The two heights differ so much because the value of the thermal expansion coefficient is not directly related to the pressure compressibility — for example, fresh water at 4° C has zero thermal expansivity, and so would have an infinite temperature scale height, but its pressure compressibility differs little from water at 20° C.

III. Thermodynamic equation using density and temperature

Taking entropy to be a function of density and temperature (and salinity if appropriate) we have

$$T\,\mathrm{d}\eta = T\left(\frac{\partial \eta}{\partial T}\right)_{\alpha,S}\mathrm{d}T + T\left(\frac{\partial \eta}{\partial \alpha}\right)_{T,S}\mathrm{d}\alpha + T\left(\frac{\partial \eta}{\partial S}\right)_{T,\alpha}\mathrm{d}S$$

$$= c_v\,\mathrm{d}T + T\left(\frac{\partial \eta}{\partial \alpha}\right)_{T,S}\mathrm{d}\alpha + T\left(\frac{\partial \eta}{\partial S}\right)_{T,\alpha}\mathrm{d}S. \tag{1.132}$$

For a moving fluid this implies,

$$\frac{\mathrm{D}T}{\mathrm{D}t} + \frac{T}{c_v}\left(\frac{\partial \eta}{\partial \alpha}\right)_{T,S}\frac{\mathrm{D}\alpha}{\mathrm{D}t} = Q_{[T,\alpha]}, \tag{1.133}$$

where $Q_{[T,\alpha]} \equiv \dot{Q}/c_v - T c_v^{-1}\dot{S}(\partial \eta/\partial S)$.

For an ideal gas, and using (1.84), which is one of Maxwell's relations, (1.133) may be written as

$$c_v \frac{DT}{Dt} + p \frac{D\alpha}{Dt} = \dot{Q}, \tag{1.134}$$

so recovering the internal energy equation (1.99a). On the other hand, for a liquid of nearly constant density, the second term on the left-hand side of (1.133) is small, and $c_p \approx c_v$, and we have to a first approximation $DT/Dt = Q_{[T,\alpha]}$.

Various forms of the thermodynamic equation are summarized in the box on page 26. For an ideal gas, (1.114) and (1.124) are exactly equivalent to (1.100) or (1.101), and numerical models of an ideal gas usually use either a prognostic equation for internal energy $c_v T$ or potential temperature. A discussion of an accurate thermodynamic equation for seawater is given below, with some summary remarks in the box on the next page.

1.7 ✦   THERMODYNAMICS OF SEAWATER

We now discuss the thermodynamics of liquids such as seawater in a little more detail. Readers whose interest is mainly in an ideal gas may skim this section. We begin with a phenomenological discussion of potential temperature and potential density followed by a more accurate treatment of the equation of state.

1.7.1   Potential Temperature, Potential Density and Entropy

Potential temperature and entropy

The potential temperature is defined as the temperature that a parcel would have if moved adiabatically and at constant composition to a given reference pressure p_R, often taken as 10^5 Pa (or 1000 hPa, or 1000 mb, approximately the pressure at the sea-surface). Thus it may be calculated, at least in principle, by the integral

$$\theta(S, T, p, p_R) = T + \int_p^{p_R} \Gamma'_{ad}(S, T, p')\, dp', \tag{1.135}$$

where $\Gamma'_{ad} = (\partial T/\partial p)_{\eta,S}$. Such integrals may be hard to calculate, and if we know the equation of state in the form $\eta = \eta(S, T, p)$ then we can calculate potential temperature more directly because potential temperature must satisfy

$$\eta(S, T, p) = \eta(S, \theta, p_R). \tag{1.136}$$

Solving this equation for θ gives, in principle, $\theta = \theta(\eta, S, p_R) = \theta(T, S, p)$, and examples will be given in (1.152) and in Appendix A. Potential temperature is not a materially conserved variable in the presence of salinity changes.

For a parcel of constant composition, changes in entropy are directly related to changes in potential temperature because, from the right-hand side of (1.136),

$$d\eta = \left(\frac{\partial \eta(S, \theta, p_R)}{\partial \theta} \right)_S d\theta. \tag{1.137}$$

Thus, if a fluid parcel moves adiabatically and at constant composition then $d\eta = 0$ and $d\theta = 0$. Furthermore, if we express entropy as a function of temperature and pressure then

$$T\, d\eta = T \left(\frac{\partial \eta}{\partial T} \right)_{p,S} dT + T \left(\frac{\partial \eta}{\partial p} \right)_{T,S} dp + T \left(\frac{\partial \eta}{\partial S} \right)_{p,T} dS = c_p\, dT + T \left(\frac{\partial \eta}{\partial p} \right)_{T,S} dp + T \left(\frac{\partial \eta}{\partial S} \right)_{p,T} dS, \tag{1.138}$$

Thermodynamics of Liquids

Liquids, unlike ideal gases, do not have a simple equation of state and this has ramifications for the thermodynamic equation. For seawater, a very accurate, albeit equally complex, equation of state is given by TEOS-10. The simpler expression for the Gibbs function, (1.146), is a good approximation in many circumstances and using it we can derive the thermal equation of state and some useful forms of the thermodynamic equation.

Thermal equation of state

A very useful expression for many purposes is given by

$$\alpha = \alpha_0 \left[1 + \beta_T (1 + \gamma^* p)(T - T_0) + \frac{\beta_T^*}{2}(T - T_0)^2 - \beta_S(S - S_0) - \beta_p(p - p_0) \right]. \quad \text{(TL.1)}$$

For laboratory situations where pressure variations are not large a useful approximation is

$$\alpha = \alpha_0 \left[1 + \beta_T(T - T_0) - \beta_S(S - S_0) \right]. \quad \text{(TL.2)}$$

Thermodynamic equation

Using entropy or potential enthalpy, as discussed in Section 1.7.3, as a primary thermodynamic variable is the best way to proceed if high accuracy is required. For idealized or laboratory work we can make further approximations, as follows.

Entropy evolution:
Using the hydrostatic approximation and simplifying (1.148) gives

$$\frac{D\eta}{Dt} = 0, \qquad \eta = c_{p0} \ln \frac{T}{T_0} \left[1 + \beta_S^*(S - S_0) \right] + gz\beta_T. \quad \text{(TL.3)}$$

Given entropy and salinity we can infer temperature and, using (TL.1), density.

Potential temperature and potential density:
Potential temperature or potential density are could also be used as thermodynamic variables. Accurate expressions can be derived, but approximate expressions are

$$\frac{D\theta}{Dt} = 0, \qquad \theta = T + \frac{\beta_T g T_0 z}{c_{p0}}, \quad \text{(TL.4)}$$

$$\text{or} \qquad \frac{D\rho_\theta}{Dt} = 0, \qquad \rho_\theta = \rho + \frac{\rho_0 g z}{c_s^2}. \quad \text{(TL.5)}$$

Given θ or ρ_θ, as well as salinity, we then infer density using an appropriate equation of state as needed.

using the definition of c_p. If we evaluate this expression at the reference pressure, where $T = \theta$ and $dp = 0$, and consider isohaline changes with $dS = 0$, then we have $\theta \, d\eta = c_p(p_R, \theta, S) \, d\theta$, and therefore

$$d\eta = c_p(p_R, \theta, S)\frac{d\theta}{\theta}, \quad \text{(1.139)}$$

and $d\eta/d\theta = c_p(p_R, \theta, S)/\theta$. In the special case of constant c_p integration yields

$$\eta = c_p \ln \theta + \text{constant}, \tag{1.140}$$

as for a simple ideal gas — see (1.107). Given (1.139), the thermodynamic equation can be written

$$c_p \frac{D\theta}{Dt} = \frac{\theta}{T} \dot{Q}, \tag{1.141}$$

where the right-hand side represents heating. We can use (1.141) as a thermodynamic equation instead of (1.97), although we must now relate θ instead of η to the other state variables via an equation of state.

The notion of potential temperature is useful because it is connected to the actual temperature, with which we are familiar; roughly speaking, potential temperature is temperature plus a correction for the effects of thermal expansion. Entropy, on the other hand, may seem alien and unnecessarily exotic. However, the use of potential temperature brings no true simplifications to the equations of motion beyond those already afforded by the use of entropy as a thermodynamic variable.

Potential density

Potential density, ρ_θ, is the density that a parcel would have if it were moved adiabatically and with fixed composition to a given reference pressure, p_R, that is often, but not always, taken as 10^5 Pa, or 1 bar. If the equation of state is of the form $\rho = \rho(S, T, p)$ then by definition we have

$$\rho_\theta = \rho(S, \theta, p_R). \tag{1.142}$$

For a parcel moving adiabatically and at fixed salinity its potential density is therefore conserved, and it is the vertical gradient of potential density that provides the appropriate measure of stability (as we find in Section 2.10.1). Because density of seawater is nearly constant we can obtain an approximate expression for potential density by Taylor-expanding the density around the potential density at the reference level at which $T = \theta$ and $p = p_R$. At first order we then have

$$\rho(S, \theta, p) \approx \rho(S, \theta, p_R) + (p - p_R) \left(\frac{\partial p}{\partial \rho} \right)_{S,\theta}. \tag{1.143}$$

The first term on the right hand side is, by definition, the potential density and the derivative in the second term is the inverse of the square of speed of sound, evaluated at the reference level, and so

$$\rho_\theta \approx \rho - \frac{1}{c_s^2}(p - p_R) \approx \rho + \frac{\rho_0 g z}{c_s^2}. \tag{1.144}$$

To obtain the right-most expression we use hydrostatic balance and take $p = -\rho_0 g z$ and $p_R = 0$ (at $z = 0$), giving the same expression as occurs in (1.117).

Because the density of seawater is nearly constant, it is common in oceanography to subtract the amount $1000 \, \text{kg m}^{-3}$ before quoting its value; then, depending on whether we are referring to *in situ* density or the potential density the results are called σ_T ('sigma-tee') or σ_θ ('sigma-theta') respectively. Thus,

$$\sigma_T = \rho(p, T, S) - 1000, \qquad \sigma_\theta = \rho(p_R, \theta, S) - 1000. \tag{1.145a,b}$$

Instead of the subscript θ, a number can be used to denote the level to which potential density is referenced. Thus, σ_0 is the potential density referenced to the surface and σ_2 is the potential density referenced to 200 bars of pressure, or about 2 kilometres depth.

Parameter	Description	Value
ρ_0	Reference density	1.027×10^3 kg m^{-3}
α_0	Reference specific volume	9.738×10^{-4} m^3 kg^{-1}
T_0	Reference temperature	283 K
S_0	Reference salinity	35 ppt = 35 g kg^{-1}
c_{s0}	Reference sound speed	1490 m s^{-1}
β_T	First thermal expansion coefficient	1.67×10^{-4} K^{-1}
β_T^*	Second thermal expansion coefficient	1.00×10^{-5} K^{-2}
β_S	Haline contraction coefficient	0.78×10^{-3} ppt^{-1}
β_P	Compressibility coefficient ($= \alpha_0/c_{s0}^2$)	4.39×10^{-10} m s^2 kg^{-1}
γ^*	Thermobaric parameter ($\approx \gamma'^*$)	1.1×10^{-8} Pa^{-1}
c_{p0}	Specific heat capacity at constant pressure	3986 J kg^{-1} K^{-1}
β_S^*	Haline heat capacity coefficient	1.5×10^{-3} ppt^{-1}

Table 1.2 Various thermodynamic and equation-of-state parameters appropriate for the seawater equations of state (1.58) and (1.146). The unit ppt (or ‰) is parts per thousand by weight, or g/kg.

We cannot use $p_R = 0$ everywhere and maintain accuracy. Thus, for a parcel near 2 km, σ_2 is more relevant than σ_0. The 'neutral density' (or quasi-neutral density) is a semi-empirical way to avoid this reference-level difficulty. Neutral density is, by construction, a quantity such that the buoyancy force is locally perpendicular to its iso-surfaces, so that a parcel that is displaced adiabatically along a neutral density iso-surface will remain neutrally buoyant.[9] Neutral density is not a thermodynamic state variable and because of form of the seawater equation of state there is no continuous, unique field of neutral density extending through the ocean. Wherever it appears in figures in this book it can be assumed that potential density would look similar.

1.7.2 Equation of State for Seawater

Oceanographers go to great lengths to obtain an accurate equation of state and other physical properties of seawater, and we noted in Section 1.4 that seawater has some nonlinear properties that, although small, are nevertheless important. We need to be able to calculate these properties, and in this section we illustrate how the thermodynamic variables for seawater — the conventional equation of state, an expression for potential temperature, and so on — can be obtained directly from the fundamental equation of state. Writing the fundamental equation in the form $I = I(\eta, S, \alpha)$ is not practically useful, because the variables are not easily measured in the laboratory. However, if we cast the fundamental equation in terms of the Gibbs function, $g = I - T\eta + p\alpha$, then $dg = -\eta\, dT + \alpha\, dp + \mu\, dS$ and the independent variables are the familiar and measurable (T, S, p). A similar, but simpler, procedure is carried out for an ideal gas in Appendix A.

A Gibbs function that reproduces the properties of seawater with high accuracy is very complicated, but we can write down a Gibbs function that, although slightly less accurate, captures the most important properties with a certain degree of economy and transparency. The expression is[10]

$$
\begin{aligned}
g = {}& g_0 - \eta_0(T - T_0) + \mu_0(S - S_0) - c_{p0}T\big[\ln(T/T_0) - 1\big]\big[1 + \beta_S^*(S - S_0)\big] \\
& + \alpha_0(p - p_0)\Big[1 + \beta_T(T - T_0) - \beta_S(S - S_0) - \frac{\beta_P}{2}(p - p_0) \\
& \qquad\qquad + \frac{\beta_T\gamma^*}{2}(p - p_0)(T - T_0) + \frac{\beta_T^*}{2}(T - T_0)^2\Big].
\end{aligned}
\tag{1.146}
$$

In this equation the variables are g, T, S and p. The parameters (which, as in (1.58), all have subscripts or stars, with the starred parameters giving rise to nonlinear effects) are all constants that could in principle be determined in the laboratory with the help of the derived quantities like heat capacity, and their approximate values are given in Table 1.2. We will take $p_0 = 0$ and $\beta_p = \alpha_0/c_{s0}^2$, where c_{s0} is a reference sound speed. Equation (1.146) is in fact quite accurate for most oceanographic situations, and from it we may derive the following quantities of interest:

– The conventional or thermal equation of state, $\alpha = (\partial g/\partial p)_{T,S}$:

$$\alpha = \alpha_0 \left[1 + \beta_T(1 + \gamma^* p)(T - T_0) + \frac{\beta_T^*}{2}(T - T_0)^2 - \beta_S(S - S_0) - \beta_p p \right]. \qquad (1.147)$$

– The entropy, $\eta = -(\partial g/\partial T)_{p,S}$:

$$\eta = \eta_0 + c_{p0} \ln \frac{T}{T_0}\left[1 + \beta_S^*(S - S_0)\right] - \alpha_0 p \left[\beta_T + \beta_T \gamma^* \frac{p}{2} + \beta_T^*(T - T_0)\right]. \qquad (1.148)$$

For temperatures in the range 0°–30° Celsius, entropy increases linearly with temperature to within a few percent.

– The heat capacity, $c_p = T(\partial\eta/\partial T)_{p,S}$:

$$c_p = c_{p0}\left[1 + \beta_S^*(S - S_0)\right] - \alpha_0 p \beta_T^* T. \qquad (1.149)$$

This is to a first approximation constant, varying mildly with salinity and more weakly with temperature and pressure.

– The thermal expansion coefficient, $\widehat{\beta}_T = \alpha^{-1}(\partial\alpha/\partial T)_{S,p}$:

$$\widehat{\beta}_T = (\alpha_0/\alpha)\left[\beta_T + \beta_T\gamma^* p + \beta_T^*(T - T_0)\right], \qquad (1.150)$$

where α is given by (1.147).

– The adiabatic lapse rate, $\Gamma = (\partial T/\partial p)_{\eta,S}$. Using (1.138) gives

$$\Gamma = \left(\frac{\partial T}{\partial p}\right)_{\eta,S} = -\frac{T}{c_p}\left(\frac{\partial\eta}{\partial p}\right)_{T,S} = \frac{T}{c_p}\alpha_0[\beta_T(1 + \gamma^* p) + \beta_T^*(T - T_0)]. \qquad (1.151)$$

where c_p is given by (1.149).

Potential temperature and potential density, revisited

An expression for the potential temperature, θ, may be obtained by solving (1.136) for θ. In general, such an equation must be solved numerically, but for our equation of state, using (1.148) and taking $p_R = 0$, we find

$$\theta = T \exp\left\{-\frac{\alpha_0\beta_T p}{c_p'}\left[1 + \frac{1}{2}\gamma^* p + \frac{\beta_T^*}{\beta_T}(T - T_0)\right]\right\}, \qquad (1.152)$$

where $c_p' = c_{p0}\left[1 + \beta_S^*(S - S_0)\right]$. Equation (1.152) is a relationship between T, θ and p analogous to (1.105) for an ideal gas. The exponent itself is small, the second and third terms in square brackets are small compared to unity, the deviations of both T and θ from T_0 are also presumed to be small,
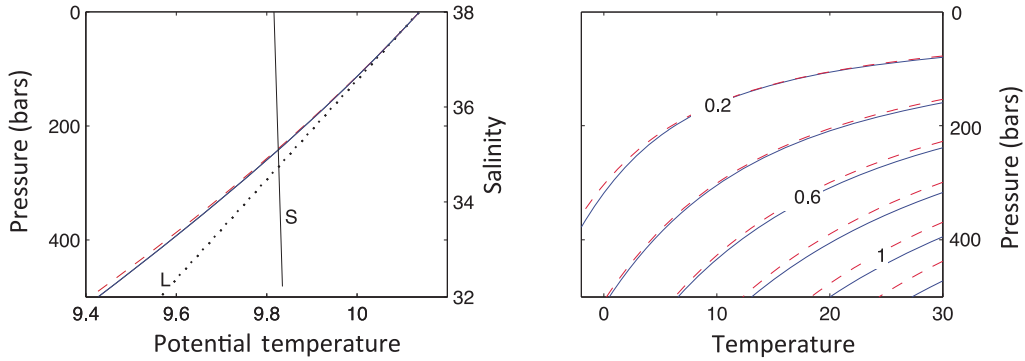
Fig. 1.4 Examples of the variation of potential temperature of seawater with pressure, temperature and salinity. Left panel: the sloping lines show potential temperature as a function of pressure at fixed salinity ($S = 35$ psu) and temperature ($10°$ C). The solid line is computed using an accurate, empirical equation of state, the almost-coincident dashed line uses the simpler expression (1.152) and the dotted line (labelled L) uses the linear expression (1.153c). The near vertical solid line, labelled S, shows the variation of potential temperature with salinity at fixed temperature and pressure. Right panel: Contours of the difference between temperature and potential temperature, $(T − \theta)$ in the pressure–temperature plane, for $S = 35$ psu. The dashed lines use (1.152) and the solid lines use an accurate empirical formula. (Note: 100 bars of pressure (10^7 Pa or 10 MPa) is approximately 1 km depth.)

and c'_p is nearly constant. Taking advantage of all of this enables (1.152) to be rewritten, with increasing levels of approximation, as

$$T' \approx \frac{T_0 \alpha_0 \beta_T}{c_{p0}} p \left(1 + \frac{1}{2} \gamma^* p + T_0 \frac{\alpha_0 \beta_T^*}{c_{p0}} p \right) + \theta' \left(1 + T_0 \frac{\alpha_0 \beta_T^*}{c_{p0}} p \right) \tag{1.153a}$$

$$\approx \frac{T_0 \alpha_0 \beta_T}{c_{p0}} p \left(1 + \frac{1}{2} \gamma^* p \right) + \theta' \left(1 + T_0 \frac{\alpha_0 \beta_T^*}{c_{p0}} p \right) \tag{1.153b}$$

$$\approx \frac{T_0 \alpha_0 \beta_T}{c_{p0}} p + \theta', \tag{1.153c}$$

where $T' = T − T_0$ and $\theta' = \theta − T_0$. The last of the three, (1.153c), holds for a linear equation of state, and is useful for calculating approximate differences between temperature and potential temperature; making use of the hydrostatic approximation reveals that it is essentially the same as (1.129). Plots of the difference between temperature and potential temperature, using both a highly accurate empirical equation of state and using the simplified equation, (1.152), are given in Fig. 1.4, and some examples of the density variation of seawater are given in Fig. 1.5.

To obtain an equation of state that gives density in terms of potential temperature, pressure and salinity we use (1.153b) in the equation of state, (1.147), to give

$$\alpha \approx \alpha_0 \left[1 − \frac{\alpha_0 p}{c_{s0}'^2} + \beta_T (1 + \gamma'^* p) \theta' + \frac{1}{2} \beta_T^* \theta'^2 − \beta_S (S − S_0) \right], \tag{1.154}$$

where $\gamma'^* = \gamma^* + T_0 \beta_T^* \alpha_0/c_{p0}$ and $c_{s0}^{-2} = c_{s0}'^{-2} − \beta_T^2 T_0/c_p$. The parameters γ^* and γ'^* differ by a few percent, and c_s^2 and $c_s'^2$ differ by a few parts in a thousand, and we may neglect the differences. We may further approximate (1.154) by using the hydrostatic pressure instead of the actual pressure; thus, letting $p = −g(z − z_0)/\alpha_0$ where z_0 is the nominal value of z at which $p = 0$, we obtain

$$\alpha \approx \alpha_0 \left[1 + \frac{g(z − z_0)}{c_{s0}^2} + \beta_T \left(1 − \gamma^* \frac{g(z − z_0)}{\alpha_0} \right) \theta' + \frac{\beta_T^*}{2} \theta'^2 − \beta_S (S − S_0) \right]. \tag{1.155}$$
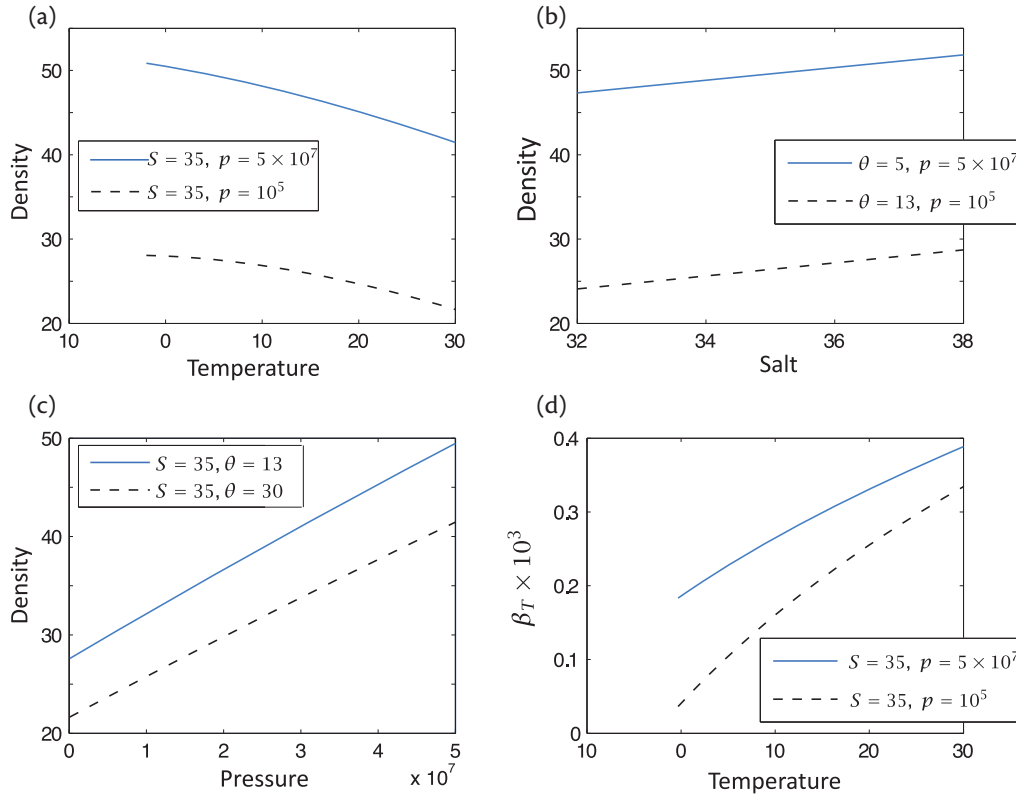
Fig. 1.5 Examples of the variation of density of seawater, $(\rho - 1000)\,\text{kg}\,\text{m}^{-3}$. (a) With potential temperature (°C); (b) with salt (g/kg); and (c) with pressure (Pa), for seawater. Panel (d), shows the thermal expansion coefficient, $\widehat{\beta}_T = -\rho^{-1}(\partial\rho/\partial T)_{p,S}\,\text{K}^{-1}$, for each of the two curves in panel (a).

Using z instead of p in the equation of state entails a slight loss of accuracy, but is necessary to ensure that the Boussinesq equations maintain good conservation properties, as will be discussed in Sections 2.4 and 4.7.1.

Given an expression for density in terms of potential temperature we can write down an expression for potential density, ρ_θ, since by definition it is just the density at a reference pressure, p_R. From (1.147) we have, assuming density variations are small,

$$\rho_\theta \approx \rho_0 \left[1 + \frac{\alpha_0 p_R}{c_{s0}^2} - \beta_T(1 + \gamma^* p_R)\theta' - \frac{1}{2}\beta_T^*\theta'^2 + \beta_S(S - S_0) \right], \tag{1.156a}$$

$$\approx \rho + (p_R - p)\left(\frac{1}{c_{s0}^2} - \beta_T\gamma^*\rho_0\theta' \right). \tag{1.156b}$$

This expression may be compared to the more approximate one, (1.144). The second term in large brackets in (1.156b) is quite small and is a manifestation of the thermobaric effect, that the compressibility of seawater is a weak function of temperature.

1.7.3 Potential Enthalpy as a Thermodynamic Variable

As discussed in Section 1.6, there are various forms of thermodynamic equation that are, in principle, equivalent. However, they are not equivalent in practice, even if the fundamental equation of state of is known. Thus, for example, for a nearly incompressible fluid like seawater the velocity

divergence is small but non-zero and numerically integrating the internal energy equation, (1.94), may be awkward and inaccurate. An ideal choice might be some conservative variable that obeys

$$\rho \frac{D\chi}{Dt} = \nabla \cdot \boldsymbol{F}_\chi \qquad \text{or equivalently} \qquad \frac{\partial}{\partial t}(\rho\chi) + \nabla \cdot (\rho\boldsymbol{v}\chi) = \nabla \cdot \boldsymbol{F}_\chi, \qquad (1.157)$$

where χ is a thermodynamical variable from which all the other variables can be inferred using the fundamental equation of state, and $\boldsymbol{F}_\chi$ represents molecular and radiative fluxes. Unfortunately, there is no variable that exactly has an advective left-hand side and a right-hand side that is the divergence of a flux — but some are nearly so, as we discuss.

Entropy

An obvious choice for a primary thermodynamic variable is entropy, an advected variable satisfying

$$\frac{D\eta}{Dt} = \frac{\dot{Q}}{T}, \qquad (1.158)$$

where $\dot{Q}$ is the heating. The main difficulties with an entropic approach are the determination of the heating and the accurate treatment of that heating once determined. The right-hand side does not have a conservative form even in the case which the heating is due solely to diffusive molecular fluxes and radiation. A further, albeit minor, complication is that heating is affected by irreversible molecular fluxes of salt in the interior. Further difficulties arise in using (1.158) in a turbulent ocean, because heating is affected not only by thermal effects but also by freshwater fluxes at the ocean surface and unresolved fluxes of salinity in the ocean interior. Care must taken to ensure that parameterized fluxes of temperature and salinity do produce (and not reduce) entropy.

Potential temperature, θ, provides no fundamental advantage over entropy, because θ is a function of both entropy and salinity and still requires a computation of heating. Indeed, because $d\eta \approx c_p d \ln \theta$ the evolution equation for potential temperature properly involves the heat capacity, c_p, which is a function (albeit a weak one) of both salinity and pressure. Having said all this, the above problems are not in practice large ones, and entropy and (more commonly) potential temperature have been used very successfully in quantitative ocean models.

Potential enthalpy

An alternative to entropy is to construct a near-conservative variable related to enthalpy. Consider first the fundamental thermodynamic relation, (1.85), in the form

$$\frac{DI}{Dt} + p\frac{D\alpha}{Dt} = T\frac{D\eta}{Dt} + \mu\frac{DS}{Dt} = \frac{Dh}{Dt} - \alpha\frac{Dp}{Dt} = \dot{Q}_E, \qquad (1.159)$$

where $\dot{Q}_E = \dot{Q} + \mu\dot{S}$ is the total rate of non-mechanical energy input per unit mass, $\dot{Q}$ is the heating and $\dot{S}$ represents saline sources and sinks. Now, it is usually easier and more accurate to determine energy input, $\dot{Q}_E$, than the heating, $\dot{Q}$, because $\dot{Q}_E$ is very nearly equal to the divergence of an energy flux. That is to say, $\rho\dot{Q}_E \approx \nabla \cdot \boldsymbol{F}_E$ where $\boldsymbol{F}_E$ is an energy flux, with the small difference arising from the heating due to viscous dissipation of kinetic energy — see Appendix B. This property suggests the use of internal energy or enthalpy as a primary thermodynamical variable, but neither are conservative quantities because of the $D\alpha/Dt$ and Dp/Dt terms in (1.159). However we can form a quantity, *potential enthalpy,* that very nearly is conservative, as follows.[11]

The potential enthalpy, h^0, is defined to be the enthalpy that a fluid parcel has if taken at constant composition and entropy from its current location to a fixed reference pressure, p_R. Thus,

$$h^0(S, \eta, p_R) \equiv h(S, \eta, p) + \int_p^{p_R} \left(\frac{\partial h}{\partial p}\right)_{S,\eta} dp' = h(S, \eta, p) + \int_p^{p_R} \alpha \, dp', \qquad (1.160)$$

using (1.74). That is, $h = h^0 + h^d$ where

$$h^d(\eta, p, S) = - \int_p^{P_R} \left(\frac{\partial h}{\partial p} \right)_{S,\eta} dp' = - \int_p^{P_R} \alpha \, dp' \tag{1.161}$$

is the 'dynamic enthalpy'. The material derivative of the dynamic enthalpy is given by

$$\frac{Dh^d}{Dt} = \frac{D\eta}{Dt}\frac{\partial h^d}{\partial \eta} + \frac{DS}{Dt}\frac{\partial h^d}{\partial S} + \alpha\frac{Dp}{Dt}. \tag{1.162}$$

To obtain this expression we have used $(\partial h^d/\partial p)_{\eta,S} = (\partial h/\partial p)_{\eta,S} = \alpha$, because h^0 is not a function of pressure. Using (1.162) and (1.160) we obtain

$$\frac{Dh^0}{Dt} = \frac{Dh}{Dt} - \frac{Dh^d}{Dt} = \frac{Dh}{Dt} - \left[\alpha\frac{Dp}{Dt} + \frac{D\eta}{Dt}\frac{\partial h^d}{\partial \eta} + \frac{DS}{Dt}\frac{\partial h^d}{\partial S} \right], \tag{1.163}$$

and using the fundamental thermodynamic relation, (1.159), we can write this equation as

$$\frac{Dh^0}{Dt} = \dot{Q}_E - \frac{\partial h^d}{\partial \eta}\dot{\eta} - \frac{\partial h^d}{\partial S}\dot{S}. \tag{1.164}$$

The second and third terms on the right-hand side are much smaller than the other terms and can be neglected in most oceanographic applications. To see this, realize that an approximate size of the first term on the right-hand side is $c_p d\theta/dt$, whereas the approximate size of the second term is $(\partial h^d/\partial \eta) \times (c_p \theta^{-1} d\theta/dt)$. That is, the second term is smaller than the first by the factor

$$\gamma = \frac{1}{\theta}\frac{\partial h^d}{\partial \eta} = \frac{1}{c_p}\frac{\partial h_d}{\partial \theta} \sim \frac{1}{c_p}\frac{\partial \alpha}{\partial \theta}\Delta p, \tag{1.165}$$

using (1.161), where $\Delta p = p - p_R \le 4 \times 10^7$ Pa. Putting in values from the seawater equation of state, (1.154), we find

$$\gamma \sim \frac{1}{c_p}\frac{\partial \alpha}{\partial \theta}\Delta p \approx \frac{\beta_T \Delta p}{\rho_0 c_p} \approx \frac{1.7 \times 10^{-4}}{10^3}\frac{4 \times 10^7}{4 \times 10^3} \approx 1.7 \times 10^{-3}, \tag{1.166}$$

with a similarly small value for the saline term, where the smallness of these terms ultimately stems from the near incompressibility of seawater. These terms may actually be smaller than (1.166) suggests if we choose p_R appropriately. Most of the energy flux into the ocean occurs at the ocean surface and if we choose $p_R = 0$ the fluxes affect enthalpy and potential enthalpy in the same way.

Given the above arguments, an accurate and computable thermodynamic equation for seawater is (1.164) with the last two terms neglected and the source term in flux form, namely

$$\rho\frac{Dh^0}{Dt} = \nabla \cdot \boldsymbol{F}_E \qquad \text{or} \qquad \frac{\partial}{\partial t}(\rho h^0) + \nabla \cdot (\rho \boldsymbol{v} h^0) = \nabla \cdot \boldsymbol{F}_E. \tag{1.167}$$

Given that the right-hand side is a flux divergence, in steady state the interior fluxes of potential enthalpy are (in this approximation) in exact balance with the energy fluxes at the ocean boundaries. The integral of ρh^0 is thus a sensible measure of the total non-kinetic energy, or 'heat content', of the ocean because it responds almost exactly to energy fluxes at the ocean surface.[12]

Why does internal energy not have these same advantages since, from (1.94), it too is conservative apart from a small compression term? The reason is that internal energy can be changed by processes that are entirely internal to the ocean, whereas enthalpy cannot. When two adjacent

fluid parcels mix, the total enthalpy is conserved because of the form of the thermodynamic equation, $\rho Dh/Dt - Dp/Dt = \rho Q_E$. If ρQ_E is the divergence of a flux then the total enthalpy ($\int \rho h \, dV$) is conserved on mixing (because the parcels are at the same pressure), but this conservative property is not shared by internal energy, because of the compression term, $pD\alpha/Dt$, in the internal energy equation. Indeed, when two parcels at different temperatures and salinities are mixed the density of the resulting parcel is higher than the average of the two (see Fig. 1.3); this effect is called cabbeling and the contraction leads to a small increase in internal energy. (The differences in the conservation of enthalpy and internal energy arise from the differences in the equation of motion for enthalpy and internal energy and not directly from the equation of state.) Thus, internal energy is not a good measure of the 'heat content' of the ocean and, furthermore, in practice the very small compression term would be hard to treat accurately. In contrast, internal energy often *is* used as a primary thermodynamic variable in ideal-gas atmospheric models, where $I = c_v T$.

Thus, in short, the practical advantages of potential enthalpy are three-fold: (i) Potential enthalpy is nearly conservative, as in (1.167); (ii) As a consequence, potential enthalpy is itself a useful measure of the non-kinetic energy of the ocean; (iii) Energy flux is a little more easily and more accurately computed than heating. Property (ii) is not shared by internal energy, since that is not conserved when parcels mix in the interior. In practice, potential enthalpy is almost proportional to potential temperature, as we now see.

Conservative temperature and an expression for potential enthalpy

To make a connection to the perhaps more familiar potential temperature it is convenient to define *conservative temperature*, Θ, by

$$\Theta \equiv \frac{h^0(\eta, p_R, S)}{c_p^0}, \tag{1.168}$$

where $c_p^0 = 3991.87 \, \text{J kg}^{-1} \text{K}^{-1}$ is the average heat capacity at the ocean surface and $p_R = 0$. (Potential enthalpy can have any reference pressure; conservative temperature has, by definition $p_R = 0$.) As we see below, conservative temperature is then very similar to potential temperature.

The enthalpy is given in terms of the Gibbs function by

$$h = g + T\eta = g - T\left(\frac{\partial g}{\partial T}\right)_{p,S}. \tag{1.169}$$

The potential enthalpy is therefore given by

$$h^0(S, T, p) = h(S, \theta, p_R) = g(S, \theta, p_R) - T\frac{\partial}{\partial T}g(S, \theta, p_R), \tag{1.170}$$

where θ is the potential temperature referenced to $p = p_R$ (and $p_R = 0$) and the last term is the derivative of the Gibbs function evaluated at $T = \theta$ and $p = p_R$. We can evaluate the derivative using the seawater equation of state, (1.146), giving, to within a constant factor,

$$\begin{aligned} h(T, p, S) = {} & \mu_0 S + c_{p0}T\big[1 + \beta_S^*(S - S_0)\big] \\ & + \alpha_0 p\Big[1 - \beta_T T_0 - \beta_S(S - S_0) - \frac{\beta_P p}{2} - \frac{\beta_T \gamma^*}{2}pT_0 + \frac{\beta_T^*}{2}(T_0^2 - T^2)\Big], \end{aligned} \tag{1.171}$$

and

$$h^0(\theta, 0, S) = h(\theta, 0, S) = \theta c_{p0}\big[1 + \beta_S^*(S - S_0)\big] + \mu_0 S. \tag{1.172}$$

The last term on the right-hand side of the above equation, $\mu_0 S$, is over two orders of magnitude smaller than the first. Also, from (1.149), the factor multiplying θ in (1.172) is just the heat capacity

at $p = 0$, namely c_p^0, which varies only by few percent over the ocean. Thus, using the definition of conservative temperature given in (1.168), we have to a good approximation

$$c_P^0 \Theta \equiv h^0(\theta, 0, S) \approx c_P^0 \theta. \tag{1.173}$$

That is, to this approximation, conservative temperature equals potential temperature. Finally, if we are to use potential enthalpy (or entropy) as a thermodynamic variable we must manipulate the equation of state to obtain variables such as pressure or density from it, analogous to (1.154).

1.8   SOUND WAVES

Full of sound and fury, signifying nothing.
William Shakespeare, *Macbeth*, c. 1606.

We now consider, rather briefly, one of the most common phenomena in fluid dynamics, yet one which is in most circumstances relatively unimportant for geophysical fluid dynamics — sound waves. Their unimportance stems from the fact that the pressure disturbances produced by sound waves are a tiny fraction of the ambient pressure and are too small to affect the circulation. For example, the ambient surface pressure in the atmosphere is about 10^5 Pa and variations due to large-scale weather phenomena are about 10^3 Pa or larger, whereas sound waves of 70 dB (i.e., a loud conversation) produce pressure variations of about 0.06 Pa. (To convert, dBs = $20 \log_{10}(\Delta p / p_r)$ where Δp is the pressure change in Pascals and $p_r = 2 \times 10^{-5}$.)

The smallness of the disturbance produced by sound waves justifies a linearization of the equations of motion and we do so about a spatially uniform basic state that is a time-independent solution to the equations of motion. Thus, we write $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}'$, $\rho = \rho_0 + \rho'$ (where a subscript 0 denotes a basic state and a prime denotes a perturbation) and so on, substitute in the equations of motion, and neglect terms involving products of primed quantities. By choice of our reference frame we will simplify matters further by setting $\boldsymbol{v}_0 = 0$. The linearized momentum and mass conservation equations are then, respectively,

$$\rho_0 \frac{\partial \boldsymbol{v}'}{\partial t} = -\nabla p', \qquad \frac{\partial \rho'}{\partial t} = -\rho_0 \nabla \cdot \boldsymbol{v}'. \tag{1.174a,b}$$

(These linear equations do not in themselves determine the magnitude of the disturbance.) Now, sound waves are largely adiabatic. Thus,

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \left(\frac{\partial p}{\partial \rho} \right)_\eta \frac{\mathrm{d}\rho}{\mathrm{d}t}, \tag{1.175}$$

where $(\partial p / \partial \rho)_\eta$ is the derivative at constant entropy, whose particular form is given by the equation of state for the fluid at hand. Then, from (1.174) and (1.175) we obtain a single equation for pressure,

$$\frac{\partial^2 p'}{\partial t^2} = c_s^2 \nabla^2 p', \tag{1.176}$$

where $c_s^2 = (\partial p / \partial \rho)_\eta$. Equation (1.176) is the classical wave equation; solutions propagate at a speed c_s, which may be identified as the speed of sound. For adiabatic flow in an ideal gas, manipulation of the equation of state leads to $p = C\rho^\gamma$, where $\gamma = c_p / c_v$, whence $c_s^2 = \gamma p / \rho = \gamma R T$. Values of γ typically range from 5/3 for a monatomic gas to 7/5 for a diatomic gas, and so for air, which is almost entirely diatomic, we find $c_s \approx 350 \, \mathrm{m \, s^{-1}}$ at 300 K. In seawater no such theoretical approximation is easily available, but measurements show that $c_s \approx 1500 \, \mathrm{m \, s^{-1}}$.

1.9 COMPRESSIBLE AND INCOMPRESSIBLE FLOW

Although there may be no fluids of truly constant density, in many cases the density of a fluid will vary so little that it is a very good approximation to consider the density effectively constant. The fluid is then said to be *incompressible*. (Some sources take incompressible to mean that density is unaffected by pressure. We take it to mean that density is also unaffected by temperature and composition.) For example, in the Earth's oceans the density varies by less than 5% (usually much less) even though the pressure at the ocean bottom is several hundred times that at the surface. We first consider how the mass continuity equation simplifies when density is truly constant, and then consider conditions under which treating density as constant is a good approximation.

1.9.1 Constant Density Fluids

If a fluid is strictly of constant density then the mass continuity equation, $\mathrm{D}\rho/\mathrm{D}t + \rho\nabla \cdot \boldsymbol{v} = 0$, simplifies to

$$\nabla \cdot \boldsymbol{v} = 0. \qquad (1.177)$$

The *prognostic* equation (1.36) has become a *diagnostic* equation (1.177), or a constraint to be satisfied by the velocity. The volume of each material fluid element is therefore constant; to see this recall that conservation of mass is $\mathrm{D}(\rho\Delta V)/\mathrm{D}t = 0$ and if ρ is constant this becomes $\mathrm{D}\Delta V/\mathrm{D}t = 0$, whence (1.177) is recovered because $\mathrm{D}\Delta V/\mathrm{D}t = \Delta V\nabla \cdot \boldsymbol{v}$.

1.9.2 Incompressible Flows

In reality no fluid is truly incompressible and for (1.177) to approximately hold we just require that

$$\left|\frac{\mathrm{D}\rho}{\mathrm{D}t}\right| \ll \rho\left(\left|\frac{\partial u}{\partial x}\right| + \left|\frac{\partial v}{\partial y}\right| + \left|\frac{\partial w}{\partial z}\right|\right); \qquad (1.178)$$

that is, the material derivative of density is much smaller than the individual terms constituting the divergence. As a working definition we say that *in an incompressible fluid, density changes are so small that they have a negligible effect on the mass balance.* We do not need to assume that the densities of differing fluid elements are similar to each other, but in the ocean (and in most liquids) variations in density, $\delta\rho$, are in fact everywhere small compared to the mean density, ρ_0. A sufficient condition for incompressibility, then, is that

$$\frac{\delta\rho}{\rho_0} \ll 1. \qquad (1.179)$$

The fact that $\nabla\cdot\boldsymbol{v} = 0$ does *not* imply that we may independently use $\mathrm{D}\rho/\mathrm{D}t = 0$. Indeed for a liquid with an equation of state $\rho = \rho_0(1 - \beta_T(T - T_0))$ and a thermodynamic equation $c_p \mathrm{D}T/\mathrm{D}t = \dot{Q}$ we have

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\frac{\beta_T\rho_0}{c_p}\dot{Q}. \qquad (1.180)$$

Furthermore, incompressibility does not necessarily imply the neglect of density variations in the momentum equation — it is only in the mass continuity equation that density variations are neglected, as will become apparent in our discussion of the Boussinesq equations in chapter 2.

Conditions for incompressibility

The conditions under which incompressibility is a good approximation to the full mass continuity equation depend not only on the physical nature of the fluid but also on the flow itself. The condition that density is largely unaffected by pressure gives one necessary condition for the legitimate

use of (1.177), as follows. First assume adiabatic flow, and omit the gravitational term. Then

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \left(\frac{\partial p}{\partial \rho}\right)_{\eta} \frac{\mathrm{d}\rho}{\mathrm{d}t} = c_s^2 \frac{\mathrm{d}\rho}{\mathrm{d}t} \,, \tag{1.181}$$

so that the density and pressure variations of a fluid parcel are related by

$$\delta p \sim c_s^2 \delta \rho. \tag{1.182}$$

From the momentum equation we estimate

$$\frac{U^2}{L} \sim \frac{1}{L} \frac{\delta p}{\rho_0}, \tag{1.183}$$

where U and L are typical velocities and lengths and where ρ_0 is a representative value of the density. Using (1.182) and (1.183) gives $U^2 \sim c_s^2 \delta \rho / \rho_0$. The incompressibility condition (1.179) then becomes

$$\frac{U^2}{c_s^2} \ll 1. \tag{1.184}$$

Thus, for a flow to be incompressible the fluid velocities must be less than the speed of sound; that is, the Mach number, $M \equiv U/c_s$, must be small.

In the Earth's atmosphere it is apparent that density changes with height. To estimate how much density does change, let us first assume hydrostatic balance and an ideal gas, so that $\partial p/\partial z = -\rho g$. If we also assume that atmosphere is isothermal then

$$\frac{\partial p}{\partial z} = \left(\frac{\partial p}{\partial \rho}\right)_T \frac{\partial \rho}{\partial z} = RT_0 \frac{\partial \rho}{\partial z}. \tag{1.185}$$

Using hydrostasy and (1.185) gives

$$\rho = \rho_0 \exp(-z/H_\rho), \tag{1.186}$$

where $H_\rho = RT_0/g$ is the (density) *scale height* of the atmosphere. (It is also the pressure scale height here.) It is easy to see that density changes are negligible only if we concern ourselves with motion less than the scale height, so this is another necessary condition for incompressibility.

In the atmosphere, although the Mach number is small for most flows, vertical displacements often exceed the scale height and the flow cannot then be considered incompressible. In the ocean, density changes from all causes are small and in most circumstances the ocean may be considered to contain an incompressible fluid.

1.10   THE ENERGY BUDGET

The total energy of a fluid includes the kinetic, potential and internal energies. Both fluid flow and pressure forces will, in general, move energy from place to place, but we nevertheless expect, even demand, energy to be conserved in an enclosed volume. Is it?

1.10.1   Constant Density Fluid

For a constant density fluid the momentum equation and the mass continuity equation $\nabla \cdot \boldsymbol{v} = 0$, are sufficient to completely determine the evolution of a system. The momentum equation is

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\nabla\left(\phi + \Phi\right) + \nu\nabla^2\boldsymbol{v}, \tag{1.187}$$

where $\phi = p/\rho_0$ and Φ is the potential for any conservative force per unit mass (e.g., gz for a uniform gravitational field). We can rewrite the advective term on the left-hand side using the identity,

$$(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{\omega} + \nabla(\boldsymbol{v}^2/2), \tag{1.188}$$

where $\boldsymbol{\omega} \equiv \nabla \times \boldsymbol{v}$ is the *vorticity,* discussed more in later chapters. Then, omitting viscosity, we have

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{v} = -\nabla B, \tag{1.189}$$

where $B = (\phi + \Phi + \boldsymbol{v}^2/2)$ is the *Bernoulli function* for constant density flow. Consider for a moment steady flows ($\partial \boldsymbol{v}/\partial t = 0$). Streamlines are, by definition, parallel to $\boldsymbol{v}$ everywhere, and the vector $\boldsymbol{v} \times \boldsymbol{\omega}$ is everywhere orthogonal to the streamlines, so that taking the dot product of the steady version of (1.189) with $\boldsymbol{v}$ gives $\boldsymbol{v} \cdot \nabla B = 0$. That is, for steady flows the Bernoulli function is constant along a streamline, and $DB/Dt = 0$.

Reverting to the time-varying case, take the dot product of (1.189) with $\boldsymbol{v}$ and include the density to yield

$$\frac{1}{2}\frac{\partial \rho_0 \boldsymbol{v}^2}{\partial t} + \rho_0 \boldsymbol{v} \cdot (\boldsymbol{\omega} \times \boldsymbol{v}) = -\rho_0 \boldsymbol{v} \cdot \nabla B. \tag{1.190}$$

The second term on the left-hand side vanishes identically. Defining the kinetic energy density K, or energy per unit volume, by $K = \rho_0 \boldsymbol{v}^2/2$, (1.190) becomes an expression for the rate of change of K,

$$\frac{\partial K}{\partial t} + \nabla \cdot (\rho_0 \boldsymbol{v} B) = 0. \tag{1.191}$$

Because Φ is time-independent this may be written

$$\frac{\partial E}{\partial t} + \nabla \cdot (\rho_0 \boldsymbol{v} B) = 0, \tag{1.192}$$

where $E = K + \rho_0 \Phi$ is the total energy density (i.e, the total energy per unit volume). This has the form of a general conservation equation in which a local change in a quantity is balanced by the divergence of its flux. However, the energy flux, $\rho_0 \boldsymbol{v} B = \rho_0 \boldsymbol{v}(\boldsymbol{v}^2/2 + \Phi + \phi)$, is *not* simply the velocity times the energy density $\rho_0(\boldsymbol{v}^2/2 + \Phi)$; there is an additional term, $\boldsymbol{v}p$, that represents the energy transfer occurring when work is done by the fluid against the pressure force.

Now consider a volume through which there is no mass flux, for example a domain bounded by rigid walls. The rate of change of energy within that volume is then given by the integral of (1.192),

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V E \, \mathrm{d}V = -\int_V \nabla \cdot (\rho_0 \boldsymbol{v} B) \, \mathrm{d}V = -\int_S \rho_0 B \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = 0, \tag{1.193}$$

using the divergence theorem. Thus, the total energy within the volume is conserved. The total kinetic energy is also conserved, total gravitational potential energy is equal to $\int_V \rho_0 gz \, \mathrm{d}V$, and this is a constant, unaffected by a rearrangement of the fluid. Thus, in a constant density fluid there is no exchange between kinetic energy and potential energy.

1.10.2 Variable Density Fluids

We start with the inviscid momentum equation with a time-independent potential Φ,

$$\rho \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\nabla p - \rho \nabla \Phi, \tag{1.194}$$

and take its dot product with $\boldsymbol{v}$ to obtain an equation for the evolution of kinetic energy,

$$\frac{1}{2}\rho \frac{\mathrm{D}\boldsymbol{v}^2}{\mathrm{D}t} = -\boldsymbol{v} \cdot \nabla p - \rho \boldsymbol{v} \cdot \nabla \Phi = -\nabla \cdot (p\boldsymbol{v}) + p\nabla \cdot \boldsymbol{v} - \rho \boldsymbol{v} \cdot \nabla \Phi. \tag{1.195}$$

The internal energy equation for adiabatic flow is

$$\rho \frac{\mathrm{D}I}{\mathrm{D}t} = -p\nabla \cdot \boldsymbol{v}. \tag{1.196}$$

Finally, and somewhat trivially, the potential energy density obeys

$$\rho \frac{\mathrm{D}\Phi}{\mathrm{D}t} = \rho \boldsymbol{v} \cdot \nabla \Phi. \tag{1.197}$$

Adding (1.195), (1.196) and (1.197) we obtain

$$\rho \frac{\mathrm{D}}{\mathrm{D}t} \left(\frac{1}{2} \boldsymbol{v}^2 + I + \Phi \right) = -\nabla \cdot (p\boldsymbol{v}), \tag{1.198}$$

which, on expanding the material derivative and using the mass conservation equation, becomes

$$\frac{\partial}{\partial t} \left[\rho \left(\frac{1}{2} \boldsymbol{v}^2 + I + \Phi \right) \right] + \nabla \cdot \left[\rho \boldsymbol{v} \left(\frac{1}{2} \boldsymbol{v}^2 + I + \Phi + p/\rho \right) \right] = 0. \tag{1.199}$$

This may be written

$$\frac{\partial E}{\partial t} + \nabla \cdot \left[\boldsymbol{v}(E + p) \right] = 0, \tag{1.200}$$

where $E = \rho(\boldsymbol{v}^2/2 + I + \Phi)$ is the total energy per unit volume of the fluid. This is the energy equation for an unforced, inviscid and adiabatic, compressible fluid. The energy flux term vanishes when integrated over a closed domain with rigid boundaries, implying that the total energy is conserved. However, there can be an exchange of energy between kinetic, potential and internal components. It is the divergent term, $\nabla \cdot \boldsymbol{v}$, that connects the kinetic energy equation, (1.195), and the internal energy equation, (1.196). In an incompressible fluid this term is absent, and the internal energy is divorced from the other components of energy. This consideration will be important when we consider the Boussinesq equations in Section 2.4. Note finally that the flux of energy, $\boldsymbol{F}_E = \boldsymbol{v}(E + p)$ is not equal to the velocity times the energy; rather, energy is also transferred by pressure. We may write the energy flux as

$$\boldsymbol{F}_E = \rho \boldsymbol{v} \left(\frac{\boldsymbol{v}^2}{2} + \Phi + h \right), \tag{1.201}$$

where $h = I + p/\rho$ is the enthalpy. That is, the local rate of change of energy is effected by the fluxes of kinetic and potential energy and *enthalpy*, not internal energy.

Bernoulli's theorem

The quantity

$$B = \left(E + \frac{p}{\rho} \right) = \left(\frac{1}{2} \boldsymbol{v}^2 + I + \Phi + p\alpha \right) = \left(\frac{1}{2} \boldsymbol{v}^2 + h + \Phi \right), \tag{1.202}$$

is the general form of the Bernoulli function, equal to the sum of the kinetic energy, the potential energy and the enthalpy. Equation (1.200) may be written as

$$\frac{\partial E}{\partial t} + \nabla \cdot (\rho \boldsymbol{v} B) = 0. \tag{1.203}$$

This equation may also be written $\partial(\rho B)/\partial t + \nabla \cdot (\rho \boldsymbol{v} B) = \partial p/\partial t$, so obviously the Bernoulli function itself is not conserved, even for adiabatic flow. For steady flow $\nabla \cdot (\rho \boldsymbol{v}) = 0$, and the $\partial/\partial t$ terms

vanish so that (1.203) may be written $\boldsymbol{v}\cdot\nabla B = 0$, or even $DB/Dt = 0$. The Bernoulli function is then a constant along streamlines, a result commonly known as Bernoulli's theorem.[13] For adiabatic flow at constant composition we also have $D\theta/Dt = 0$. Thus, steady flow is both along surfaces of constant θ and along surfaces of constant B, and the vector

$$\boldsymbol{l} = \nabla\theta \times \nabla B \tag{1.204}$$

is parallel to streamlines. A related result for unsteady flow is given in Section 4.8.

Viscous effects

We might expect that viscosity will always act to reduce the kinetic energy of a flow, and we will demonstrate this for a constant density fluid. Retaining the viscous term in (1.187), the energy equation becomes

$$\frac{d\widehat{E}}{dt} \equiv \frac{d}{dt}\int_V E\,dV = \mu\int_V \boldsymbol{v}\cdot\nabla^2\boldsymbol{v}\,dV. \tag{1.205}$$

The right-hand side is negative definite. To see this we use the vector identity

$$\nabla\times(\nabla\times\boldsymbol{v}) = \nabla(\nabla\cdot\boldsymbol{v}) - \nabla^2\boldsymbol{v}, \tag{1.206}$$

and because $\nabla\cdot\boldsymbol{v} = 0$ we have $\nabla^2\boldsymbol{v} = -\nabla\times\boldsymbol{\omega}$, where $\boldsymbol{\omega} \equiv \nabla\times\boldsymbol{v}$. Thus,

$$\frac{d\widehat{E}}{dt} = -\mu\int_V \boldsymbol{v}\cdot(\nabla\times\boldsymbol{\omega})\,dV = -\mu\int_V \boldsymbol{\omega}\cdot(\nabla\times\boldsymbol{v})\,dV = -\mu\int_V \boldsymbol{\omega}^2\,dV, \tag{1.207}$$

after integrating by parts, providing $\boldsymbol{v}\times\boldsymbol{\omega}$ vanishes at the boundary. Thus, viscosity acts to extract kinetic energy from the flow. The loss of kinetic energy reappears as an irreversible warming of the fluid (called 'Joule heating'), and the total energy of the fluid is conserved, but this effect plays no role in a constant density fluid. The heating is normally locally small, at least in the Earth's ocean and atmosphere, but it is needed to preserve total energy.

1.10.3 Enthalpy, Static Energy and Energy Flux

We saw above that it is the quantity

$$B = \text{kinetic energy} + \text{potential energy} + \text{enthalpy}, \tag{1.208}$$

that, when multiplied by $\rho\boldsymbol{v}$, fluxes the energy. However, the conserved energy contains only the kinetic and potential energies plus the internal energy. The difference between B and energy is the fluid dynamical analogue of the thermodynamical principle that it is the flux of *enthalpy* that changes the energy of a system, because this accounts for work done by the pressure.

For an example, consider an ideal gas in a uniform gravitational field for which $I = c_vT$ and $p\alpha = RT$ so that $B = c_vT + RT + \text{KE} + gz = c_pT + \text{KE} + gz$. The sum of the enthalpy and the potential energy,

$$h_d^* \equiv c_pT + gz, \tag{1.209}$$

is known as the *dry static energy*. Dry static energy is not an integral conserved quantity nor is it a measure of the total energy itself. To see the importance of enthalpy fluxes, consider an adiabatic rearrangement of a fluid with $d\eta = 0$. From (1.87) we then have $dI = -p\,d\alpha$, meaning that the internal energy changes because of work done. At the same time, from (1.72) with $d\eta = 0$ we also have $dh = \alpha\,dp$. For a fluid in hydrostatic balance, (1.51), we have that $dp = -g\,dz/\alpha$ and thus

$$d(h + gz) = 0. \tag{1.210}$$

The change in potential energy of a parcel arises from a change in the enthalpy, not the internal energy, because of the work that must be done by the pressure force to move the parcel.

The form of the conservation law (1.210) is different from the total energy conservation previously derived. Equation (1.200) is an *integral* conservation law, whereas (1.210) is a *parcel* conservation law, which we might write in fluid dynamical form as

$$\frac{\mathrm{D}}{\mathrm{D}t}(h + gz) = 0. \tag{1.211}$$

Equation (1.211) is a form of the thermodynamic equation for adiabatic changes to an ideal gas in hydrostatic balance, similar to (1.126) or (1.141). The equivalence arises because changes in h_d^* and θ are related by $\mathrm{d}h_d^* = c_p\,\mathrm{d}T + g\,\mathrm{d}z = c_p(T/\theta)\,\mathrm{d}\theta$, as in (1.127).

Potential enthalpy (see Section 1.7.3 for an oceanographic discussion) is the enthalpy that a parcel would have if adiabatically moved to a reference pressure, and since the enthalpy for an ideal gas is c_pT, the potential enthalpy is just $c_p\theta$. The dry static energy, $c_pT + gz$, is the enthalpy that a parcel at a height z and temperature T would have if moved adiabatically and hydrostatically to $z = 0$, and is thus a special form of potential enthalpy (and sometimes called 'generalized enthalpy'). Energy and enthalpy both play major roles in the chapters ahead, but let's round off this chapter with an introduction to scaling in a pure fluid-dynamical setting.

1.11   AN INTRODUCTION TO NONDIMENSIONALIZATION AND SCALING

The units we use to measure length, velocity and so on are irrelevant to the dynamics and it is useful to express the equations of motion in terms of 'nondimensional' variables, by which we mean expressing every variable as the ratio of its value to some reference value. We choose the reference as a natural one for a given flow so that, as far as possible, the nondimensional variables are order-unity quantities, and doing this is called *scaling the equations*. There is no reference that is universally appropriate, and much of the art of fluid dynamics lies in choosing sensible scaling factors for the problem at hand. We introduce the methodology here with a simple example.

1.11.1   The Reynolds Number

Consider the constant-density momentum equation in Cartesian coordinates. If a typical velocity is U, a typical length is L, a typical time scale is T, and a typical value of the pressure deviation is Φ, then the approximate sizes of the various terms in the momentum equation are given by

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\nabla\phi + \nu\nabla^2\boldsymbol{v}, \tag{1.212a}$$

$$\frac{U}{T} \qquad \frac{U^2}{L} \quad \sim \quad \frac{\Phi}{L} \qquad \nu\frac{U}{L^2}. \tag{1.212b}$$

The ratio of the inertial terms to the viscous terms is $(U^2/L)/(\nu U/L^2) = UL/\nu$, and this is the *Reynolds number*.[14] More formally, we can nondimensionalize the momentum equation by writing

$$\widehat{\boldsymbol{v}} = \frac{\boldsymbol{v}}{U}, \qquad \widehat{\boldsymbol{x}} = \frac{\boldsymbol{x}}{L}, \qquad \widehat{t} = \frac{t}{T}, \qquad \widehat{\phi} = \frac{\phi}{\Phi}, \tag{1.213}$$

where the terms with hats on are *nondimensional* values of the variables and the capitalized quantities are known as *scaling values*, and these are the approximate magnitudes of the variables. We now choose the scaling values so that the nondimensional variables are of order unity, or $\widehat{u} = \mathcal{O}(1)$. Thus, for example, we choose U so that $u = \mathcal{O}(U)$, where this notation should be taken to mean that the magnitude of the variable u is of order U, or that $u \sim U$, and we say that 'u scales like U'.

Because there are no external forces in this problem, appropriate scaling values for time and pressure are

$$T = \frac{L}{U}, \qquad \Phi = U^2. \tag{1.214}$$

Substituting (1.213) and (1.214) into the momentum equation gives

$$\frac{U^2}{L} \left[\frac{\partial \hat{\boldsymbol{v}}}{\partial \hat{t}} + (\hat{\boldsymbol{v}} \cdot \nabla) \hat{\boldsymbol{v}} \right] = -\frac{U^2}{L} \nabla \hat{\phi} + \frac{\nu U}{L^2} \nabla^2 \hat{\boldsymbol{v}}, \tag{1.215}$$

where we use the convention that when ∇ operates on a nondimensional variable it is a nondimensional operator. Equation (1.215) then simplifies to

$$\frac{\partial \hat{\boldsymbol{v}}}{\partial \hat{t}} + (\hat{\boldsymbol{v}} \cdot \nabla) \hat{\boldsymbol{v}} = -\nabla \hat{\phi} + \frac{1}{Re} \nabla^2 \hat{\boldsymbol{v}}, \qquad \text{where} \qquad Re \equiv \frac{UL}{\nu}. \tag{1.216a,b}$$

The parameter Re is, as before, the Reynolds number. If we have chosen our length and velocity scales sensibly — that is, if we have scaled them properly — each variable in (1.216a) is order unity, with the viscous term being multiplied by $1/Re$. There are two important conclusions:

 (i) The ratio of the importance of the inertial terms to the viscous terms is given by the Reynolds number, defined above. In the absence of other forces, such as those due to gravity and rotation, the Reynolds number is the only nondimensional parameter explicitly appearing in the momentum equation. Hence its value, along with the boundary conditions and geometry, controls the behaviour of the system.

 (ii) More generally, by scaling the equations of motion appropriately the parameters determining the behaviour of the system become explicit. *Scaling the equations is intelligent nondimensionalization.*

Nondimensionalizing the equations does not, however, absolve the investigator from the responsibility of producing dimensionally correct equations. One should regard 'nondimensional' equations as dimensional equations in units appropriate for the problem at hand.

APPENDIX A: THERMODYNAMICS OF AN IDEAL GAS FROM THE GIBBS FUNCTION

All the thermodynamic quantities of interest for a simple ideal gas, sometimes called a perfect gas, may be derived in a straightforward way from the fundamental equation of state. (An analogous treatment for moist air is given in Appendix A of Chapter 18.) To show this we begin with the specific Gibbs function, $g = I - T\eta + p\alpha$, which for an ideal gas is given by

$$g = \mathcal{C}_p T (1 - \ln T) + \mathcal{R} T \ln p + B T + C, \tag{1.217}$$

where $\mathcal{C}_p$, $\mathcal{R}$, B and C are constants (and our notation anticipates what $\mathcal{C}_p$ and $\mathcal{R}$ really are). The procedure below is especially useful when the Gibbs function is more complex than (1.217), with the main difficulty then lying in obtaining the Gibbs function in the first instance.

Density and the thermal equation of state

From the fundamental relation involving the Gibbs function, (1.78a), specific volume is given by

$$\alpha = \left(\frac{\partial g}{\partial p} \right)_T = \frac{\mathcal{R} T}{p}, \qquad \text{or} \qquad p = \rho \mathcal{R} T. \tag{1.218}$$

Thus, R, the gas constant used elsewhere in this chapter, is equal to $\mathcal{R}$.

Entropy

$$\eta = -\left(\frac{\partial g}{\partial T}\right)_p = \mathcal{C}_p \ln T - \mathcal{R} \ln p - B \tag{1.219}$$

Using (1.218) and (1.221) in (1.219), the entropy may be expressed in terms of I and α, giving $\eta = (\mathcal{C}_p - R) \ln I - \mathcal{R} \ln \rho + \text{constant}$, as in (1.111).

The internal energy

Using (1.218) and (1.219) in the definition of the Gibbs function, $g = I - T\eta + p\alpha$, gives

$$I = g + T\eta - p\alpha = g - T\left(\frac{\partial g}{\partial T}\right)_p - p\left(\frac{\partial g}{\partial p}\right)_T, \tag{1.220}$$

and so

$$I = (\mathcal{C}_p - \mathcal{R})T + C = \mathcal{C}_v T + C, \tag{1.221}$$

where $\mathcal{C}_v \equiv \mathcal{C}_p - \mathcal{R}$. Equation (1.221) also suggests we may sensibly take $C = 0$, but no physical result in classical mechanics depends on this choice.

Heat capacities

Let c_p be the heat capacity at constant pressure. We have

$$c_p \equiv T\left(\frac{\partial \eta}{\partial T}\right)_p = \mathcal{C}_p. \tag{1.222}$$

To obtain the heat capacity at constant volume, c_v, first rewrite the entropy using the thermal equation of state as

$$\eta = (\mathcal{C}_p \ln T - \mathcal{R} \ln T + \mathcal{R} \ln \alpha) + \text{constant}. \tag{1.223}$$

We then have

$$c_v \equiv T\left(\frac{\partial \eta}{\partial T}\right)_\alpha = \mathcal{C}_p - \mathcal{R}. \tag{1.224}$$

Adiabatic lapse rate

The adiabatic lapse rate, Γ_p, is the rate of change of temperature with pressure at constant entropy. Thus

$$\Gamma_p \equiv \left(\frac{\partial T}{\partial p}\right)_\eta, \tag{1.225}$$

Now, from (1.138) we have

$$\left(\frac{\partial T}{\partial p}\right)_\eta = -\frac{T}{\mathcal{C}_p}\left(\frac{\partial \eta}{\partial p}\right)_T, \tag{1.226}$$

and using (1.219) to evaluate the right-hand side gives

$$\Gamma_p = -\frac{T}{\mathcal{C}_p}\left(\frac{\partial \eta}{\partial p}\right)_T = \frac{T}{\mathcal{C}_p}\frac{\mathcal{R}}{p} = \frac{1}{\mathcal{C}_p \rho}, \tag{1.227}$$

using the thermal equation of state. It is common to write this expression in terms of a rate of change of temperature with respect to height by using the hydrostatic approximation, $dp = -\rho g\, dz$, whence

$$\Gamma_z \equiv -\left(\frac{\partial T}{\partial z}\right)_\eta = \frac{g}{\mathcal{C}_p}. \tag{1.228}$$

The physical significance of this quantity is explored in Chapter 2.

Potential temperature

As in (1.136), potential temperature, θ, satisfies

$$\eta(T, p) = \eta(\theta, p_R).$$ (1.229)

That is to say, θ is the temperature that a parcel will have if moved at constant entropy from a pressure p to a reference pressure p_R. Using (1.219) gives

$$\mathcal{C}_p \ln T - \mathcal{R} \ln p = \mathcal{C}_p \ln \theta - \mathcal{R} \ln p_R, \qquad \text{or} \qquad \ln(T/\theta)^{\mathcal{C}_p} = \ln(p/p_R)^{\mathcal{R}}.$$ (1.230)

Re-arranging gives

$$\theta = T \left(\frac{p_R}{p} \right)^{\mathcal{R}/\mathcal{C}_p}.$$ (1.231)

We can derive the same result by noting that, by definition, potential temperature satisfies

$$\theta \equiv T(p_R) = T(p) + \int_p^{p_R} \left(\frac{\partial T}{\partial p'} \right)_\eta dp' = T(p) + \int_p^{p_R} \frac{\mathcal{R}T}{\mathcal{C}_p p'} dp',$$ (1.232)

where the rightmost expression uses (1.227). It is easy to verify that the solution to this integral equation is $T = \theta(p/p_R)^{\mathcal{R}/\mathcal{C}_p}$, although solving the equation *ab initio* is a little more difficult. Finally, in an ideal gas potential temperature can be related to entropy using (1.219) and (1.231), giving $\eta = \mathcal{C}_p \ln \theta + \text{constant}$.

Enthalpy and potential enthalpy

Enthalpy is related to the Gibbs function by

$$h = g + T\eta = g - T \left(\frac{\partial g}{\partial T} \right)_p = \mathcal{C}_p T.$$ (1.233)

Potential enthalpy, h^0, is the enthalpy that a parcel would have if moved adiabatically to a reference pressure. It therefore satisfies an equation similar to (1.229), namely $\eta(h, p) = \eta(h^0, p_R)$. Since $h = \mathcal{C}_p T$ we immediately obtain

$$h^0 = \mathcal{C}_p \theta.$$ (1.234)

That is, the potential enthalpy is equal to the potential temperature times the heat capacity at constant pressure.

APPENDIX B: THE FIRST LAW OF THERMODYNAMICS FOR FLUIDS

In its usual form the first law states that changes in the internal energy of a body are equal to the sum of heat supplied, the work done, and changes due to composition (the chemical work), and in a reversible process the entropy of a body then changes according to the heat supplied. However, the heating is not known a priori and the first law may be regarded as a definition of heating by way of energy conservation; heating should then be considered a derived quantity, as we noted in our discussion of (1.61) and (1.62). The real problem is to determine what the heating actually is, and this is not wholly trivial for fluids that also have mechanical energy and viscosity, as well as thermal and compositional diffusion. The first law is not a statement of total energy conservation, since it does not involve kinetic energy.

To obtain an unambiguous prescription for the heating — and hence a useful thermodynamic equation — we *begin* with total energy conservation and work backwards to obtain the heating, and since energy conservation is the more fundamental physical law this procedure is natural.[15]

Effectively, we subtract off the evolution of mechanical energy from an equation for total energy conservation, and the remainder is the thermodynamic equation. Equivalently (and this is how we proceed below) we demand consistency between total energy conservation and an energy equation derived in a forward fashion using a thermodynamic equation representing the first law but with the heating unspecified, and thereby deduce an explicit expression for that heating. Underlying this procedure is the fundamental notion that the work done and the generation of mechanical energy are less ambiguous than heating because they can be measured and/or arise through well-defined forces in the momentum equation. We will assume that energy and energy fluxes are knowable or calculable, but we will not discuss the questions of what energy fundamentally is or why it is conserved.

For reference, we first write down the fundamental thermodynamic relation, (1.85), and its inputs,

$$\frac{DI}{Dt} + p\frac{D\alpha}{Dt} = T\frac{D\eta}{Dt} + \mu\frac{DS}{Dt} = \frac{Dh}{Dt} - \alpha\frac{Dp}{Dt} = \dot{Q}_E, \tag{1.235}$$

where $\dot{Q}_E = \dot{Q} + \mu\dot{S}$ accounts for the total energy input to a fluid parcel from both heating, $\dot{Q}$, and compositional changes, $\mu\dot{S}$. The inclusion of the term $\dot{Q}_E$ connects the above equation to the first law, but the first law is not useful until we know what the heating is.

B.1 Single Component Fluid

Consider the energetics of a fluid as in Section 1.10.2, with two additional effects: the fluid is viscous, and there is an additional energy source in the fluid, for example radiation or thermal conduction, that does not appear in the momentum equation. We write the momentum equation, (1.194), in Cartesian tensor notation (where repeated indices are summed) as

$$\rho\frac{Dv_i}{Dt} = -\frac{\partial p}{\partial x_i} - \rho\frac{\partial \Phi}{\partial x_i} + \mu_v\frac{\partial^2 v_i}{\partial x_j \partial x_j}, \tag{1.236}$$

where μ_v is the coefficient of viscosity and Φ is a time-independent external potential field, such as gz. The form of the viscous term is exact only for incompressible fluids with constant viscosity but it is usually an excellent approximation in the atmosphere and ocean, because the Mach number is small and the scale on which dissipation occurs is very much smaller than the density scale height. (In any case those restrictions can be relaxed.) We also write the entropy equation as

$$T\frac{D\eta}{Dt} = \dot{Q}, \tag{1.237}$$

where $\dot{Q}$ is the heating, whose form we do not yet know. In a single component fluid we can use the fundamental thermodynamic relation, (1.235), to write (1.237) as an internal energy equation,

$$\rho\frac{DI}{Dt} + p\nabla \cdot \boldsymbol{v} = \rho\dot{Q}, \tag{1.238}$$

also having used $D\alpha/Dt = \alpha\nabla \cdot \boldsymbol{v}$.

We obtain a kinetic energy equation by multiplying (1.236) by v_i to give

$$\frac{1}{2}\rho\frac{Dv_i^2}{Dt} = -\partial_i(pv_i) + p\partial_i v_i - \rho v_i\partial_i\Phi + \mu_v\, v_i\partial_j(\partial_j v_i), \tag{1.239}$$

where $\partial_i \equiv \partial/\partial x_i$. The viscous term may be written as

$$\mu_v\, v_i\partial_j(\partial_j v_i) = \mu_v\left[\partial_j(v_i\partial_j v_i) - (\partial_j v_i)^2\right]. \tag{1.240}$$

The first term on the right-hand side is the divergence of a flux, and so is energy conserving, and the second term is negative definite, representing kinetic energy dissipation.

We now proceed, just as in Section 1.10.2, to obtain a total energy equation from (1.238) and (1.239) and the result is

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}v_i^2 + I + \Phi\right)\right] + \partial_i\left[\rho v_i\left(\frac{1}{2}v_j^2 + I + \Phi + p/\rho\right)\right] = \mu_v\left[\partial_j(v_i\partial_j v_i) - (\partial_j v_i)^2\right] + \rho\dot{Q}. \quad (1.241)$$

Now, the general form of the energy conservation law for a fluid takes the form

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}v_i^2 + I + \Phi\right)\right] + \partial_i\left[\rho v_i\left(\frac{1}{2}v_j^2 + I + \Phi + p/\rho\right)\right] = \partial_j\left[\mu_v v_i\partial_j v_i + F_{Ej}\right], \quad (1.242)$$

where F_{Ej} (or $\boldsymbol{F}_E$, and $\partial_j F_{Ej} = \nabla \cdot \boldsymbol{F}_E$) is the total energy flux due to radiation, conduction and any other effects — the important point being that the right-hand side of (1.242) must be the divergence of a flux in order to guarantee energy conservation. We have not used the first law to obtain (1.242), just the fundamental thermodynamic relation, the momentum equation and energy conservation. The above two equations are consistent only if the heating term has the form $\rho\dot{Q} = \partial_j F_{Ej} + \mu_v(\partial_j v_i)^2$, and the internal energy and entropy equations are then

$$\rho\frac{\mathrm{D}I}{\mathrm{D}t} + p\nabla \cdot \boldsymbol{v} = \nabla \cdot \boldsymbol{F}_E + \mu_v(\partial_j v_i)^2, \qquad \rho T\frac{\mathrm{D}\eta}{\mathrm{D}t} = \nabla \cdot \boldsymbol{F}_E + \mu_v(\partial_j v_i)^2. \quad (1.243a,b)$$

Evidently, the 'heating' of a fluid is given by the sum of the energy fluxes and a positive definite term due to viscous dissipation. Either of the above equivalent equations may be considered to be statements of the first law with explicit expressions for energy input and heating, and either of them then provides a useful predictive thermodynamic equation for a fluid.

B.2 Multi-Component Fluids

Consider now a two component fluid, such as dry air and water vapour or water and salinity. We refer to the second component as concentration and we assume it obeys

$$\rho\frac{\mathrm{D}S}{\mathrm{D}t} = \nabla \cdot \boldsymbol{F}_S. \quad (1.244)$$

The only difference from the previous derivation is that, using the fundamental relation, the internal energy equation is now

$$\rho\frac{\mathrm{D}I}{\mathrm{D}t} + p\nabla \cdot \boldsymbol{v} = \dot{Q} + \mu\nabla \cdot \boldsymbol{F}_S, \quad (1.245)$$

where μ is the chemical potential and the second term on the right-hand side accounts for the effects of concentration fluxes on internal energy (i.e., the chemical work). Proceeding to calculate the energy equation as before, we find that (1.241) then has an additional term $\mu\nabla \cdot \boldsymbol{F}_S$ on the right-hand side. This is consistent with (1.242) only if the internal energy and entropy equations obey

$$\rho\frac{\mathrm{D}I}{\mathrm{D}t} + p\nabla \cdot \boldsymbol{v} = \nabla \cdot \boldsymbol{F}_E + \mu_v(\partial_j v_i)^2, \qquad \rho T\frac{\mathrm{D}\eta}{\mathrm{D}t} = \nabla \cdot \boldsymbol{F}_E - \mu\nabla \cdot \boldsymbol{F}_S + \mu_v(\partial_j v_i)^2, \quad (1.246a,b)$$

where the energy flux, $\boldsymbol{F}_E$, now includes the effects of any fluxes of composition. Again, either of the above two equations is a statement of the first law, and the right-hand side of (1.246b) is the heating. Additional terms may appear on the right-hand sides of the above equations if there are additional source or sink terms, for example a source of concentration.

Equation (1.246) differs from the single component case by the addition of a concentration flux. However, in a two component fluid the diffusion of temperature and of concentration are affected by gradients of both temperature and concentration, so that the heat flux itself differs from the single component case. Thermodynamics provides constraints on these fluxes, but the reader must look elsewhere to learn about them.[16] Finally, in both oceanography and meteorology the viscous heating term is small, at least on Earth, but it must be included if energy balance is desired — for example if incoming solar radiation is to balance outgoing infrared radiation.

Notes

1 Joseph-Louis Lagrange (1736–1813) was a Franco-Italian, born and raised in Turin who then lived and worked mainly in Germany and France. He made notable contributions in analysis, number theory and mechanics and was recognized as one of the greatest mathematicians of the eighteenth century. He laid the foundations of the calculus of variations (to wit, the 'Lagrange multiplier') and first formulated the principle of least action, and his treatise *Mécanique Analytique* (1788) provides a unified analytic framework (it contains no diagrams, a feature emulated in Whittaker's *Treatise on Analytical Dynamics*, 1927) for all Newtonian mechanics.

Leonard Euler (1707–1783), a Swiss mathematician who lived and worked for extended periods in Berlin and St. Petersburg, made important contributions in many areas of mathematics and mechanics, including the analytical treatment of algebra, the theory of equations, calculus, number theory and classical mechanics. He was the first to establish the form of the equations of motion of fluid mechanics, writing down both the field description of fluids *and* what we now call the material or advective derivative.

Truesdell (1954) points out that 'Eulerian' and 'Lagrangian', especially the latter, are inappropriate eponyms. The Eulerian description was introduced by d'Alembert in 1749 and generalized by Euler in 1752, and the so-called Lagrangian description was introduced by Euler in 1759. (It is sometimes said that advances in mathematics are named after the next person to discover them after Euler — the Coriolis effect is another example.) The modern confusion evidently stems from a monograph by Dirichlet in 1860 that credits Euler in 1757 and Lagrange in 1788 for the respective methods.

Clifford Truesdell (1919–2000) was a remarkable figure himself, known both for his own contributions to many areas of continuum mechanics and for his scholarly investigations on the history of mathematics and science. He also had a trenchant and at times pungent writing style. Ball & James (2002) provide a biography.

2 Salinity is a mass fraction and thus is nondimensional, but it is commonly referred to in units of g/kg. For many years the measure of salinity of seawater that was used in oceanography was based on electrical conductivity and referred to as 'practical salinity', S_P, since this was (and still is) more easily measured. In thermodynamical calculations practical salinity is now largely dropped in favour of the true salinity, generally referred to as absolute salinity and denoted S_A. Differences between practical and absolute salinity are small but not negligible (Millero *et al.* 2008, IOC *et al.* 2010).

3 See also de Szoeke (2004). Nycander & Roquet (2015) and Roquet *et al.* (2015) show that this equation of state can, in fact, be used to give a quantitatively accurate simulation of the ocean.

4 The use of inexact differentials in thermodynamics is questionable for they do not have a straightforward mathematical foundation. Their use can be avoided and, were this a rigorous treatise on thermodynamics, probably should be avoided, but here they are useful artifacts. Reif (1965) and Callen (1985) both make use of them but Truesdell (1969) is particularly scathing on the matter.

5 It is said that the early students of ideas related to entropy were unusually prone to suicide, Ludwig Boltzmann being a tragic example. Thankfully there are many counter-examples, such as William Thompson (Lord Kelvin). He did foundational work early in his career on thermodynamics and among other achievements put forward a formulation of the second law. Neither this nor his much less successful later work seems to have caused him too much distress, and he lived for 83 years. Perhaps Truesdell (1969) gets it right when he says that entropy gives 'intense headaches to those who have studied thermodynamics'.

6 Because the word barotropic has other meanings — sometimes it is just taken to mean the vertical average — it might be better to always refer to fluids for which density is a function only of pressure as homentropic. Unfortunately the current usage is deeply ingrained and to insist on homentropic would be tilting at windmills.

7 Claude-Louis-Marie-Henri Navier (1785–1836) was a French civil engineer, professor at the École Polytechnique and later at the École des Ponts et Chaussée. He was an expert in road and bridge building (he developed the theory of suspension bridges) and, relatedly, made lasting theoretical contributions to the theory of elasticity, being the first to publish a set of general equations for the dynamics of an elastic solid. In fluid mechanics, he laid down the now-called *Navier–Stokes equations*, including the viscous terms, in 1822.

George Gabriel Stokes (1819–1903). Irish born (in Skreen, County Sligo), he was a professor of mathematics at Cambridge from 1849 until his retirement. As well as having a role in the development of fluid mechanics, especially through his considerations of viscous effects, Stokes worked on the dynamics of elasticity, fluorescence, the wave theory of light, and was (perhaps rather ill-advisedly in hindsight) a proponent of the idea of an ether permeating all space.

8 Potential temperature was known to William Thomson in 1857.

9 Jackett & McDougall (1997), extending McDougall (1987). A similar quantity was described by Eden & Willebrand (1999). de Szoeke (2000), Nycander (2011) and Tailleux (2016) provide more discussion.

10 Building from de Szoeke (2004) and with input from W. R. Young. See also Fofonoff (1959) and Warren (2006) for some historical background. A very accurate semi-empirical formula for the Gibbs function is given by Feistel (2008). Using this as a basis, seawater equations of state are now available in the form of the TEOS-10 standard (IOC *et al.* 2010) and from Roquet *et al.* (2015), and these fit laboratory measurements close to the accuracy of the measurements themselves.

11 Potential enthalpy was introduced to oceanography by McDougall (2003) and its use is advocated in IOC *et al.* (2010). The advantages and disadvantages of various thermodynamic variables, including entropy and potential enthalpy, are discussed there and in Graham & McDougall (2013). Useful discussion is also to be found in Warren (1999), Young (2010) and Nycander (2011). I am very grateful to T. McDougall for discussions on these and other thermodynamic matters.

12 Referring to the 'heat content' of a fluid borders on dangerous language, because heat itself is a type of energy transfer, like work, and not a state variable. On the other hand, heat content has an intuitive appeal and, provided it is properly understood, conveys a useful meaning in oceanography, since the compression work done on the ocean is small (as water is almost incompressible) and kinetic energy is small compared to internal and potential energy.

13 Bernoulli's theorem was developed mainly by Daniel Bernoulli (1700–1782). It was based on earlier work on the conservation of energy that Daniel had done with his father, Johann Bernoulli (1667–1748), and so perhaps should be known as Bernoullis' theorem. The two men fell out when Daniel was a young man, reputedly because of Johann's jealousy of Daniel's abilities, and subsequently had a very strained relationship. The Bernoulli family produced several (at least eight) talented mathematicians over three generations in the seventeenth and eighteenth centuries, and is often regarded as the most mathematically distinguished family of all time.

14 Osborne Reynolds (1842–1912) was an Irish born (Belfast) physicist who was professor of engineering at Manchester University from 1868–1905. His early work was in electricity and magnetism, but he is now most famous for his work in hydrodynamics. The 'Reynolds number', which determines the ratio of inertial to viscous forces, and the 'Reynolds stress', which is the stress on the mean flow due to the fluctuating components, are both named after him. He was also one of the first scientists to think about the concept of group velocity.

15 See also Landau & Lifshitz (1987) and IOC *et al.* (2010). For an interesting and somewhat idiosyncratic view of the first law applied to the ocean, read Warren (2006).

16 Onsager (1931), Salmon (1998). For example, in a single component fluid, total entropy increases if the heat flux is proportional to a downgradient temperature flux.

Further Reading

General fluid dynamics

There are numerous books on hydrodynamics, an early one being

Lamb, H., 1932. *Hydrodynamics.*
    Lamb's book is a classic in the field, although now too dated to make it useful as an introduction.

Two somewhat more modern references, at a fairly advanced level, are

Batchelor, G. K., 1967. *An Introduction to Fluid Dynamics.*
Landau, L. D. & Lifshitz, E. M., 1987. *Fluid Mechanics.*
    These two books both contain a detailed derivation of the equations of motion, including viscous and pressure forces.

At a more elementary level we have

Kundu, P., Cohen, I. &. Dowling, D., 2015. *Fluid Mechanics.*
    This book is written at the advanced undergraduate/beginning graduate level, is easier-going than Batchelor or Landau & Lifshitz, and contains material on geophysical fluid dynamics.

For the connoisseur, a more specialized treatment is

Truesdell, C., 1954. *The Kinematics of Vorticity.*
    Written in Truesdell's inimitable style, this book discusses many aspects of vorticity with numerous historical references. Truesdell's books are all gems in their own way.

Thermodynamics

There are many books on thermodynamics, and two that I have found particularly useful are

Reif, F., 1965. *Fundamentals of Statistical and Thermal Physics.*
Callen, H. B., 1985. *Thermodynamics and an Introduction to Thermostatistics.*
    Reif's book has become something of a classic, and Callen provides an axiomatic approach that will be an antidote for those who feel that thermodynamic reasoning is mysterious or even circular.

For the subtopic of atmospheric thermodynamics see the further reading section at the end of Chapter 14.

Geophysical fluid dynamics

Gill, A. E., 1982. *Atmosphere–Ocean Dynamics.*
    A richly textured book, especially strong on equatorial dynamics and gravity wave motion.
Pedlosky, J., 1987. *Geophysical Fluid Dynamics.*
    A primary reference for flow at low Rossby number. Although the book requires some effort, there is a handsome pay-off for those who study it closely.
Holton, J. R. & Hakim, G., 2012. *An Introduction to Dynamical Meteorology.*
    A very well-known textbook at the undergraduate/beginning graduate level.
Salmon, R., 1998. *Lectures on Geophysical Fluid Dynamics.*
    Covers the fundamentals as well as Hamiltonian fluid dynamics, geostrophic turbulence and oceanic circulation.