Layered models

In a stratified system, as density increases monotonically downward, it can be used as the vertical coordinate

$$\rho(x, y, z, t) \Rightarrow z(x, y, \rho, t)$$

$$a = a(x, y, \rho, t)$$

Let $a = z$:

$$0 = z_x + z_\rho \rho_x$$

$$0 = z_y + z_\rho \rho_y$$

$$1 = z_\rho \rho_z$$

$$0 = z_t + z_\rho \rho_t$$

The derivative transformation:

$$\frac{\partial a}{\partial x}|_z = \frac{\partial a}{\partial x}|_\rho + \frac{\partial a}{\partial \rho}|_\rho \frac{\partial \rho}{\partial x}|_z$$

$$\frac{\partial a}{\partial y}|_z = \frac{\partial a}{\partial y}|_\rho + \frac{\partial a}{\partial \rho}|_\rho \frac{\partial \rho}{\partial y}|_z$$

$$\frac{\partial a}{\partial z}|_z = \frac{\partial a}{\partial \rho}|_\rho \frac{\partial \rho}{\partial z}|_z$$

$$\frac{\partial a}{\partial t}|_z = \frac{\partial a}{\partial t}|_\rho + \frac{\partial a}{\partial \rho}|_\rho \frac{\partial \rho}{\partial t}|_z$$

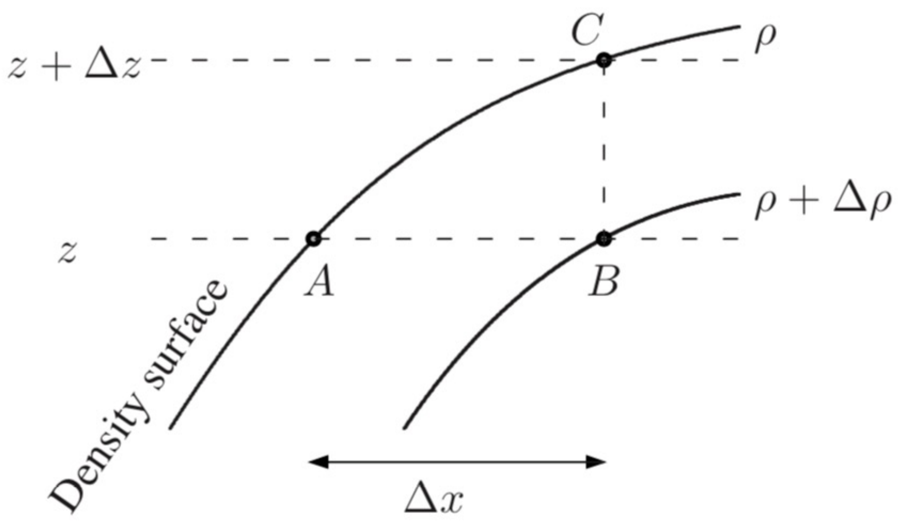
$$\frac{\partial a}{\partial x}|_z = \frac{\partial a}{\partial x}|_\rho - \frac{z_x}{z_\rho}\frac{\partial a}{\partial \rho}|_\rho$$

$$\frac{\partial a}{\partial y}|_z = \frac{\partial a}{\partial y}|_\rho - \frac{z_y}{z_\rho}\frac{\partial a}{\partial \rho}|_\rho$$

$$\frac{\partial a}{\partial z}|_z = \frac{1}{z_\rho}\frac{\partial a}{\partial \rho}|_\rho$$

$$\frac{\partial a}{\partial t}|_z = \frac{\partial a}{\partial t}|_\rho - \frac{z_t}{z_\rho}\frac{\partial a}{\partial \rho}|_\rho$$

$$\frac{\partial a}{\partial x}\Big|_z = \frac{\partial a}{\partial x}\Big|_\rho - \frac{z_x}{z_\rho}\frac{\partial a}{\partial \rho}\Big|_\rho$$



constant z: $\dfrac{a(B)-a(A)}{\Delta x}$ $\quad \dfrac{\partial a}{\partial x}\Big|_z$

constant ρ: $\dfrac{a(C)-a(A)}{\Delta x}$ $\quad \dfrac{\partial a}{\partial x}\Big|_\rho$

$$\frac{\partial a}{\partial x}\Big|_\rho - \frac{\partial a}{\partial x}\Big|_z = \frac{a(C)-a(A)}{\Delta x} = \frac{a(C)-a(A)}{\Delta z}\frac{\Delta z}{\Delta x}$$

$$= \frac{a(C)-a(A)}{\Delta\rho\frac{\Delta z}{\Delta\rho}}\frac{\Delta z}{\Delta x}$$

$$= \frac{z_x}{z_\rho}\frac{\partial a}{\partial \rho}\Big|_\rho$$

$$\frac{\partial a}{\partial x}\Big|_z = \frac{\partial a}{\partial x}\Big|_\rho - \frac{z_x}{z_\rho}\frac{\partial a}{\partial \rho}\Big|_\rho$$

$$\frac{\partial a}{\partial x}|_z = \frac{\partial a}{\partial x}|_\rho - \frac{z_x}{z_\rho}\frac{\partial a}{\partial \rho}|_\rho \qquad \frac{\partial a}{\partial z}|_z = \frac{1}{z_\rho}\frac{\partial a}{\partial \rho}|_\rho \qquad \frac{\partial a}{\partial t}|_z = \frac{\partial a}{\partial t}|_\rho - \frac{z_t}{z_\rho}\frac{\partial a}{\partial \rho}|_\rho$$

The pressure gradient term:

Hydrostatic balance:

$$\frac{\partial p}{\partial z} = -\rho g$$

$$\frac{\partial p}{\partial \rho} = -\rho g \frac{\partial z}{\partial \rho}$$

$$\boxed{P = p + \rho g z}$$

$$\frac{\partial p}{\partial x}|_z = \frac{\partial p}{\partial x}|_\rho - \frac{z_x}{z_\rho}\frac{\partial p}{\partial \rho}|_\rho = \frac{\partial p}{\partial x}|_\rho + \rho g \frac{\partial z}{\partial x}|_\rho = \frac{\partial P}{\partial x}|_\rho$$

$$\frac{\partial P}{\partial \rho} = \frac{\partial p}{\partial \rho} + gz + \rho g \frac{\partial z}{\partial \rho} = gz$$

$$\frac{\partial a}{\partial x}|_z = \frac{\partial a}{\partial x}|_\rho - \frac{z_x}{z_\rho}\frac{\partial a}{\partial \rho}|_\rho \qquad \frac{\partial a}{\partial z}|_z = \frac{1}{z_\rho}\frac{\partial a}{\partial \rho}|_\rho \qquad \frac{\partial a}{\partial t}|_z = \frac{\partial a}{\partial t}|_\rho - \frac{z_t}{z_\rho}\frac{\partial a}{\partial \rho}|_\rho$$

The full derivative ($\frac{d}{dt}$) in the LHS of the governing equation:

Let $a = \rho$:

$$\frac{\partial \rho}{\partial x}|_z = -\frac{z_x}{z_\rho} \qquad \frac{\partial \rho}{\partial y}|_z = -\frac{z_y}{z_\rho} \qquad \frac{\partial \rho}{\partial z}|_z = \frac{1}{z_\rho} \qquad \frac{\partial \rho}{\partial t}|_z = -\frac{z_t}{z_\rho}$$

For incompressible fluids: $\frac{d\rho}{dt} = 0$

ρ-coordinate

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = 0 \Longrightarrow -z_t - uz_x - vz_y + w = 0 \Longrightarrow \frac{dz}{dt} = z_t + uz_x + vz_y$$

$$\boldsymbol{\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}}$$

There is no across-isopycnal advection, and the model is ideal for studying along-isopycnal processes

Without friction, the governing equations in density coordinate become:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -\frac{1}{\rho_0}\frac{\partial P}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -\frac{1}{\rho_0}\frac{\partial P}{\partial y}$$

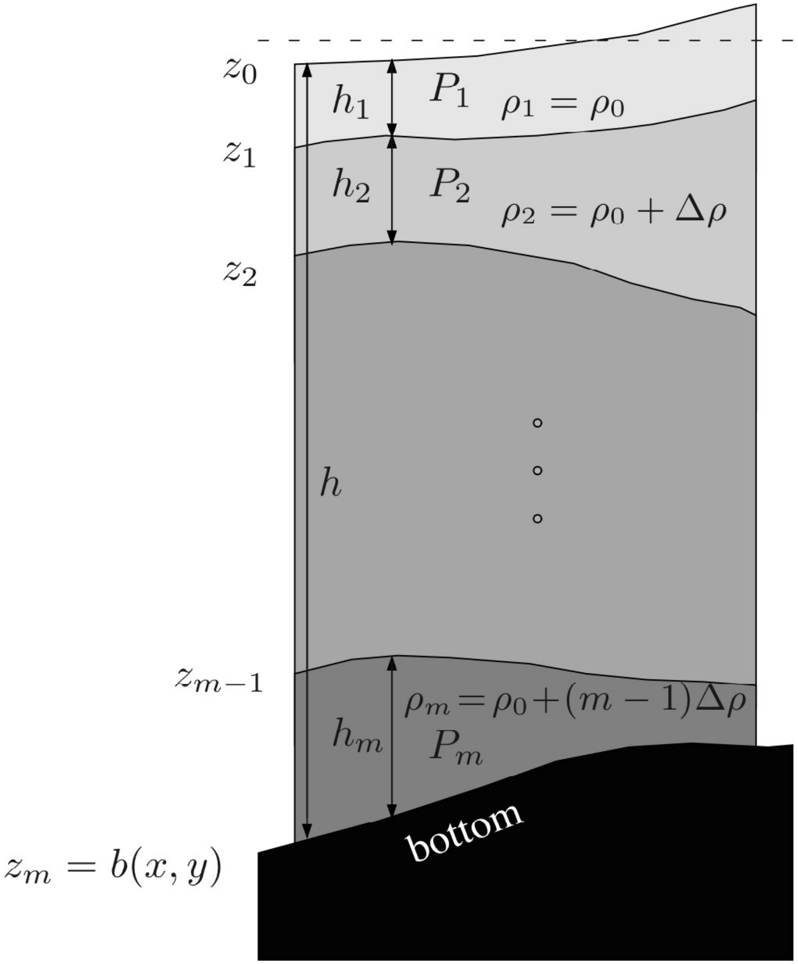
$$\frac{\partial P}{\partial \rho} = gz$$

$$\frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} + \frac{\partial hv}{\partial y} = 0$$

$$h = -\Delta\rho\frac{\partial z}{\partial \rho}$$

the thickness of a fluid layer between ρ and $\rho + \Delta\rho$

Layered models



(a)

We need to determine ***P*** and ***z*** for different layers, and we can obtain the equations for these layers

For ***z***:

$$z_m = b$$

upward:

$$z_{k-1} = z_k + h_k, \quad k = m \text{ to } 1.$$

For ***P***:

$$\boxed{P = p + \rho g z}$$

$$P_1 = p_{\text{atm}} + \rho_0 g z_0$$

downward:

$$\boxed{\frac{\partial P}{\partial \rho} = gz}$$

$$P_{k+1} = P_k + \Delta\rho g z_k, \quad k = 1 \text{ to } m-1.$$

$\boxed{z_m = b}$ $\boxed{z_{k-1} = z_k + h_k}$ $\boxed{P_1 = p_{\mathrm{atm}} + \rho_0 g z_0}$ $\boxed{P_{k+1} = P_k + \Delta\rho g z_k}$

One layer:

$$z_0 = h_1 + b \qquad P_1 = \rho_0 g(h_1 + b)$$
$$z_1 = b$$

$$\boxed{g' = \frac{\Delta\rho}{\rho_0} g}$$

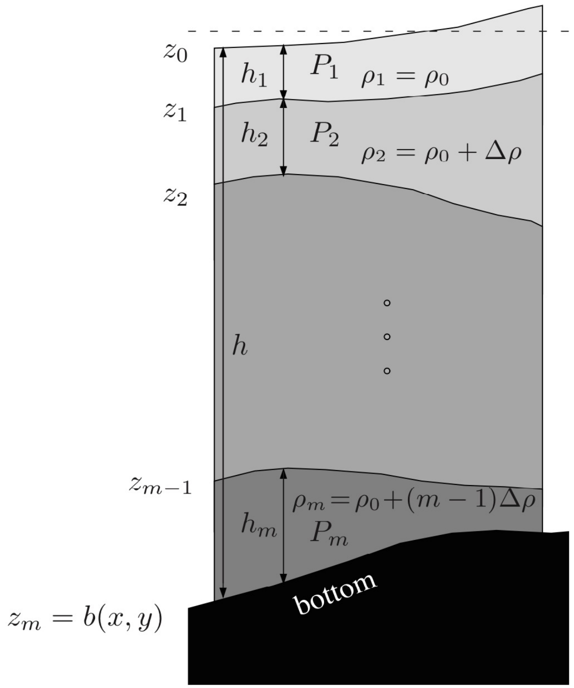
reduced gravity

Two layers:

$$z_0 = h_1 + h_2 + b \qquad P_1 = \rho_0 g(h_1 + h_2 + b)$$
$$z_1 = h_2 + b \qquad P_2 = \rho_0 g h_1 + \rho_0(g + g')(h_2 + b)$$
$$z_2 = b$$

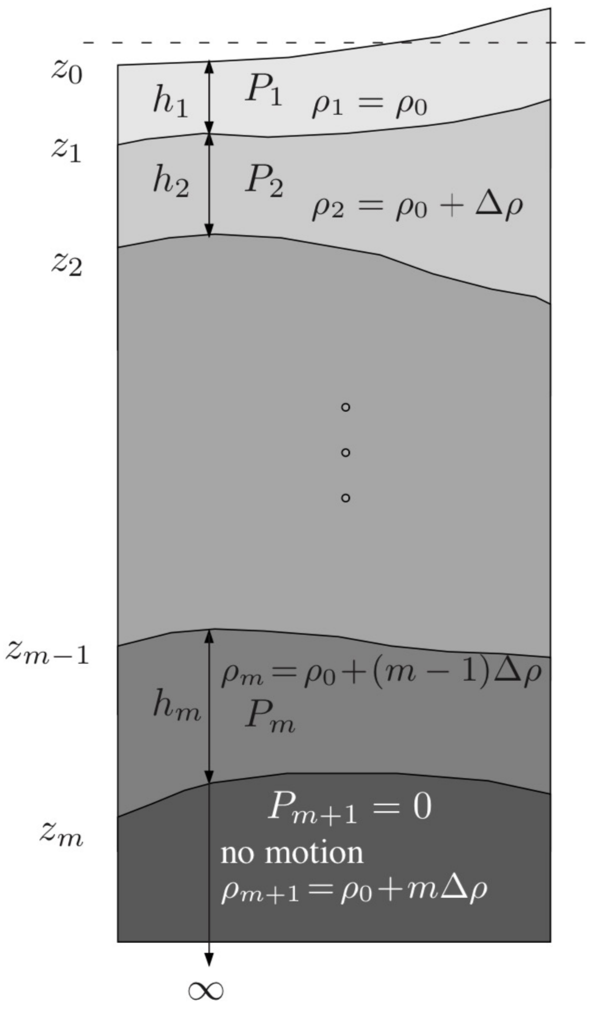
Three layers:

$$z_0 = h_1 + h_2 + h_3 + b \qquad P_1 = \rho_0 g(h_1 + h_2 + h_3 + b)$$
$$z_1 = h_2 + h_3 + b \qquad P_2 = \rho_0 g h_1 + \rho_0(g + g')(h_2 + h_3 + b)$$
$$z_2 = h_3 + b \qquad P_3 = \rho_0 g h_1 + \rho_0(g + g')h_2$$
$$+\rho_0(g + 2g')(h_3 + b)$$
$$z_3 = b$$



(a)

Reduced gravity model



The lowest layer may be imagined to be infinitely deep and at rest

$$P_{m+1} = 0$$

Rigid-lid approximation: $z_0 = 0$

upward calculation

$$\boxed{P_{k+1} = P_k + \Delta\rho g z_k}$$

One layer:

$$z_1 = -h_1 \qquad P_1 = \rho_0 g' h_1$$

Two layers:

$$z_1 = -h_1 \qquad P_1 = \rho_0 g'(2h_1 + h_2)$$
$$z_2 = -h_1 - h_2 \qquad P_2 = \rho_0 g'(h_1 + h_2)$$

Three layers:

$$z_1 = -h_1 \qquad P_1 = \rho_0 g'(3h_1 + 2h_2 + h_3)$$
$$z_2 = -h_1 - h_2 \qquad P_2 = \rho_0 g'(2h_1 + 2h_2 + h_3)$$
$$z_3 = -h_1 - h_2 - h_3 \qquad P_3 = \rho_0 g'(h_1 + h_2 + h_3)$$

shallow-water reduced gravity model – one layer

One layer:

$$z_1 = -h_1 \qquad P_1 = \rho_0 g' h_1$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g'\frac{\partial h}{\partial x} \quad (1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g'\frac{\partial h}{\partial y} \quad (2)$$

$$\frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} + \frac{\partial hv}{\partial y} = 0 \Longrightarrow \frac{dh}{dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

Review for homogeneous fluids:

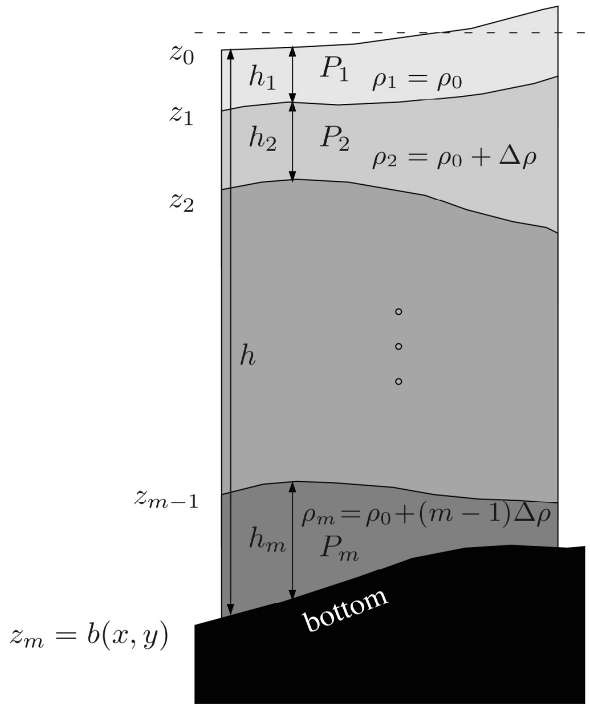
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial \eta}{\partial y}$$

shallow-water model (general) – two layers

Two layers:

$$z_0 = h_1 + h_2 + b \qquad P_1 = \rho_0 g(h_1 + h_2 + b)$$
$$z_1 = h_2 + b \qquad P_2 = \rho_0 g h_1 + \rho_0 (g + g')(h_2 + b)$$
$$z_2 = b$$



(a)

Governing equations:

$$\begin{aligned}
\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y} - f v_1 &= -g \frac{\partial (h_1 + h_2 + b)}{\partial x} \\
\frac{\partial v_1}{\partial t} + u_1 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_1}{\partial y} + f u_1 &= -g \frac{\partial (h_1 + h_2 + b)}{\partial y} \\
\frac{\partial h_1}{\partial t} + \frac{\partial (h_1 u_1)}{\partial x} + \frac{\partial (h_1 v_1)}{\partial y} &= 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} + v_2 \frac{\partial u_2}{\partial y} - f v_2 &= -g \frac{\partial h_1}{\partial x} - (g + g') \frac{\partial (h_2 + b)}{\partial x} \\
\frac{\partial v_2}{\partial t} + u_2 \frac{\partial v_2}{\partial x} + v_2 \frac{\partial v_2}{\partial y} + f u_2 &= -g \frac{\partial h_1}{\partial y} - (g + g') \frac{\partial (h_2 + b)}{\partial y} \\
\frac{\partial h_2}{\partial t} + \frac{\partial (h_2 u_2)}{\partial x} + \frac{\partial (h_2 v_2)}{\partial y} &= 0
\end{aligned}$$

$$- g \, \frac{\partial (h_1 + h_2 + b)}{\partial x} \;=\; - g \, \frac{\partial \eta}{\partial x}$$

$$- g' \, \frac{\partial (h_2 + b)}{\partial x} \;=\; - g' \, \frac{\partial a}{\partial x}$$

$$\boxed{-g \frac{\partial (h_1 + h_2 + b)}{\partial x}}$$

$$\boxed{-g \frac{\partial h_1}{\partial x} - (g + g') \frac{\partial (h_2 + b)}{\partial x}}$$

$$= -g \frac{\partial (h_1 + h_2 + b)}{\partial x} - g' \, \frac{\partial (h_2 + b)}{\partial x}$$

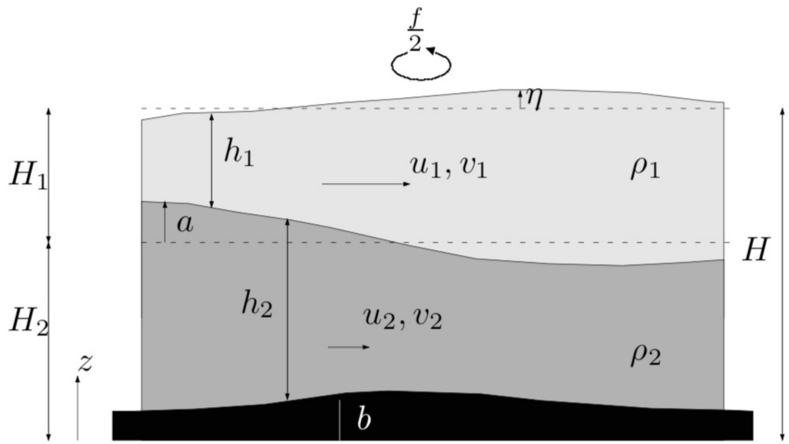
The linearized equations become:

$$\begin{aligned}
\frac{\partial u_1}{\partial t} - f v_1 &= -g \frac{\partial \eta}{\partial x} \\
\frac{\partial v_1}{\partial t} + f u_1 &= -g \frac{\partial \eta}{\partial y}
\end{aligned}$$

$$\frac{\partial (\eta - a)}{\partial t} + \frac{\partial (H_1 u_1)}{\partial x} + \frac{\partial (H_1 v_1)}{\partial y} = 0$$

$$\begin{aligned}
\frac{\partial u_2}{\partial t} - f v_2 &= -g \frac{\partial \eta}{\partial x} - g' \frac{\partial a}{\partial x} \\
\frac{\partial v_2}{\partial t} + f u_2 &= -g \frac{\partial \eta}{\partial y} - g' \frac{\partial a}{\partial y}
\end{aligned}$$

$$\frac{\partial a}{\partial t} + \frac{\partial (H_2 u_2)}{\partial x} + \frac{\partial (H_2 v_2)}{\partial y} = 0$$



$$h_1 + h_2 + b = H + \eta$$

$$h_2 + b = H_2 + a$$

$$h_1 = H_1 + \eta - a$$

Let

$$u_2 = \lambda u_1, v_2 = \lambda v_1 \qquad \eta = \mu a$$

$$\frac{\partial u_1}{\partial t} - f v_1 = -g \frac{\partial \eta}{\partial x} \qquad \frac{\partial u_2}{\partial t} - f v_2 = -g \frac{\partial \eta}{\partial x} - g' \frac{\partial a}{\partial x}$$

$$\lambda(\frac{\partial u_1}{\partial t} - f v_1) = -(\mu g + g')\frac{\partial a}{\partial x} \qquad \lambda \frac{\partial u_1}{\partial t} - \lambda f v_1 = -\mu g \frac{\partial a}{\partial x} - g' \frac{\partial a}{\partial x}$$

The horizontal momentum equations can be reduced into one set if $\frac{\lambda}{1} = \frac{g\mu + g'}{g\mu}$

$$\frac{\partial(\eta - a)}{\partial t} + \frac{\partial(H_1 u_1)}{\partial x} + \frac{\partial(H_1 v_1)}{\partial y} = 0 \qquad \frac{\partial a}{\partial t} + \frac{\partial(H_2 u_2)}{\partial x} + \frac{\partial(H_2 v_2)}{\partial y} = 0$$

$$(\mu - 1)\frac{\partial a}{\partial t} + H_1(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}) = 0 \qquad \frac{\partial a}{\partial t} + \lambda H_2(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}) = 0$$

The continuity equation can be reduced into one set if $\frac{1}{\mu - 1} = \frac{H_2 \lambda}{H_1}$

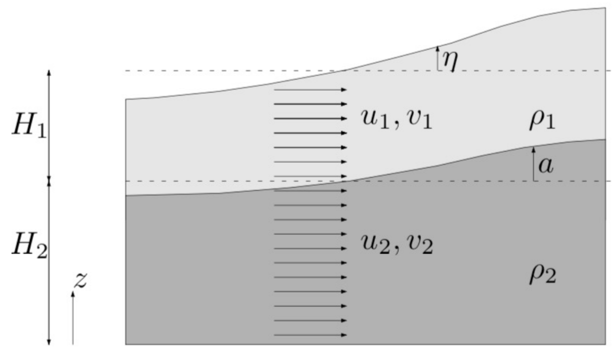
$$\frac{\lambda}{1} = \frac{g\mu + g'}{g\mu} \qquad\qquad \frac{1}{\mu - 1} = \frac{H_2\lambda}{H_1}$$

$$H_2\lambda^2 + \left(H_1 - H_2 - \cancel{\frac{g'}{g}} H_2\right)\lambda - H_1 = 0 \qquad\qquad g'/g = \Delta\rho/\rho_0 \ll 1$$

$$\lambda = \frac{(H_2 - H_1) \pm (H_2 + H_1)}{2H_2}$$

If $\lambda = 1$: $u_1 = u_2$ $\quad v_1 = v_2$ **barotropic mode**

$$\mu = \frac{H}{H_2} \qquad a = \eta/\mu = H_2\eta/H$$



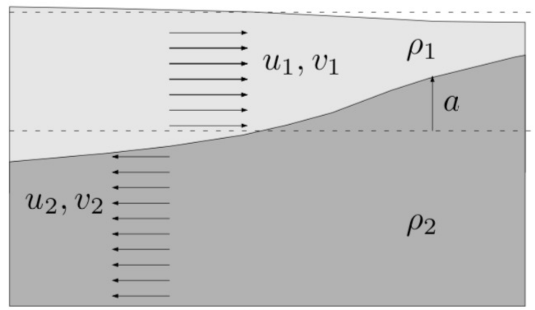
If $\lambda = -\frac{H_1}{H_2}$: $\quad u_2 = -\frac{H_1}{H_2}u_1 \quad H_2u_2 = -H_1u_1$

$$v_2 = -\frac{H_1}{H_2}v_1 \quad H_2v_2 = -H_1v_1$$

baroclinic mode

the vertical integration of transport is zero

$\mu = -g'H_2/gH \sim 0 \qquad \eta \ll a \qquad$ nearly rigid lid



Barotropic mode: $\frac{\lambda}{1} = \frac{g\mu + g'}{g\mu}$ $\lambda = 1$

$$\frac{\partial u_1}{\partial t} - f v_1 = -g\frac{\partial \eta}{\partial x}$$
$$\frac{\partial v_1}{\partial t} + f u_1 = -g\frac{\partial \eta}{\partial y}$$
$$\frac{\partial (\eta - a)}{\partial t} + \frac{\partial (H_1 u_1)}{\partial x} + \frac{\partial (H_1 v_1)}{\partial y} = 0$$

$$\frac{\partial u_2}{\partial t} - f v_2 = -g\frac{\partial \eta}{\partial x} - g'\frac{\partial a}{\partial x}$$
$$\frac{\partial v_2}{\partial t} + f u_2 = -g\frac{\partial \eta}{\partial y} - g'\frac{\partial a}{\partial y}$$
$$\frac{\partial a}{\partial t} + \frac{\partial (H_2 u_2)}{\partial x} + \frac{\partial (H_2 v_2)}{\partial y} = 0$$

$$u_T = u_1 = u_2,\ v_T = v_1 = v_2$$

Summation of the two sets of equations:

$$\frac{\partial u_T}{\partial t} - f v_T = -g\frac{\partial \eta}{\partial x}$$
$$\frac{\partial v_T}{\partial t} + f u_T = -g\frac{\partial \eta}{\partial y}$$
$$\frac{\partial \eta}{\partial t} + H\frac{\partial u_T}{\partial x} + H\frac{\partial v_T}{\partial y} = 0$$

Baroclinic mode: $\frac{\lambda}{1} = \frac{g\mu + g'}{g\mu}$ $\lambda = -\frac{H_1}{H_2}$

$$\begin{aligned} \frac{\partial u_1}{\partial t} - f v_1 &= -g\frac{\partial \eta}{\partial x} \\ \frac{\partial v_1}{\partial t} + f u_1 &= -g\frac{\partial \eta}{\partial y} \end{aligned}$$

$$\frac{\partial(\eta - a)}{\partial t} + \frac{\partial(H_1 u_1)}{\partial x} + \frac{\partial(H_1 v_1)}{\partial y} = 0$$

$$\begin{aligned} \frac{\partial u_2}{\partial t} - f v_2 &= -g\frac{\partial \eta}{\partial x} - g'\frac{\partial a}{\partial x} \\ \frac{\partial v_2}{\partial t} + f u_2 &= -g\frac{\partial \eta}{\partial y} - g'\frac{\partial a}{\partial y} \end{aligned}$$

$$\frac{\partial a}{\partial t} + \frac{\partial(H_2 u_2)}{\partial x} + \frac{\partial(H_2 v_2)}{\partial y} = 0$$

$$u_B = u_1 - u_2 \qquad v_B = v_1 - v_2$$

$$\eta = -(g' H_2 / g H) a$$

Difference between the two sets of equations:

$$\begin{aligned} \frac{\partial u_B}{\partial t} - f v_B &= +g'\frac{\partial a}{\partial x} \\ \frac{\partial v_B}{\partial t} + f u_B &= +g'\frac{\partial a}{\partial y} \end{aligned}$$

$$-\frac{\partial a}{\partial t} + \frac{H_1 H_2}{H}\frac{\partial u_B}{\partial x} + \frac{H_1 H_2}{H}\frac{\partial v_B}{\partial y} = 0$$