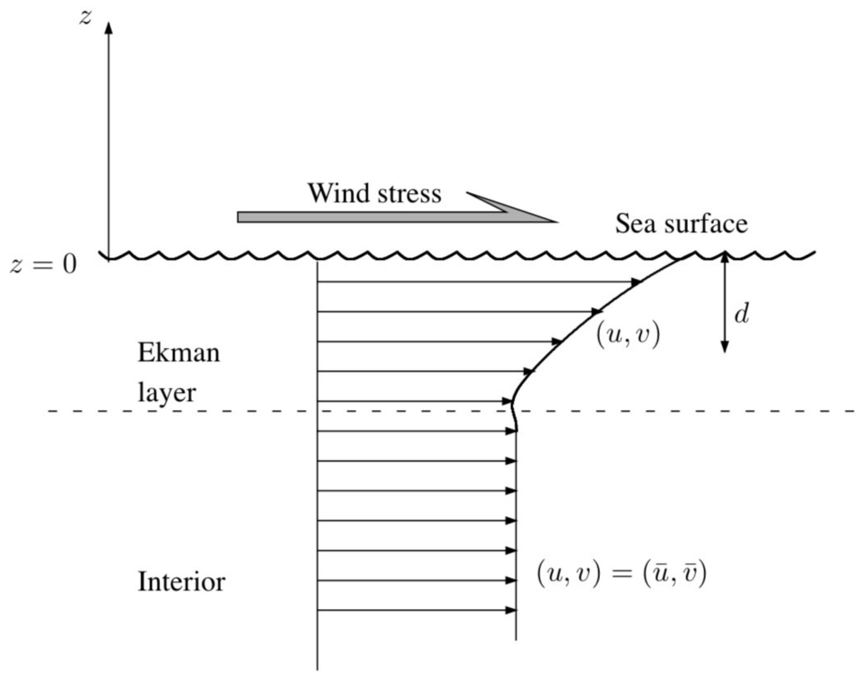
The surface Ekman layer

The interior flow:

$$-f\bar{v} = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial x}, \quad f\bar{u} = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial y}$$



Substitution into the SBL momentum equations:

$$\begin{aligned} -f(v - \bar{v}) &= \nu_E \frac{\partial^2 u}{\partial z^2} \\ f(u - \bar{u}) &= \nu_E \frac{\partial^2 v}{\partial z^2} \end{aligned}$$

Boundary conditions:

$$\begin{aligned} \text{Surface } (z = 0): \quad & \rho_0 \nu_E \frac{\partial u}{\partial z} = \tau^x, \; \rho_0 \nu_E \frac{\partial v}{\partial z} = \tau^y \\ \text{Toward interior } (z \to -\infty): \quad & u = \bar{u}, \quad v = \bar{v}. \end{aligned}$$

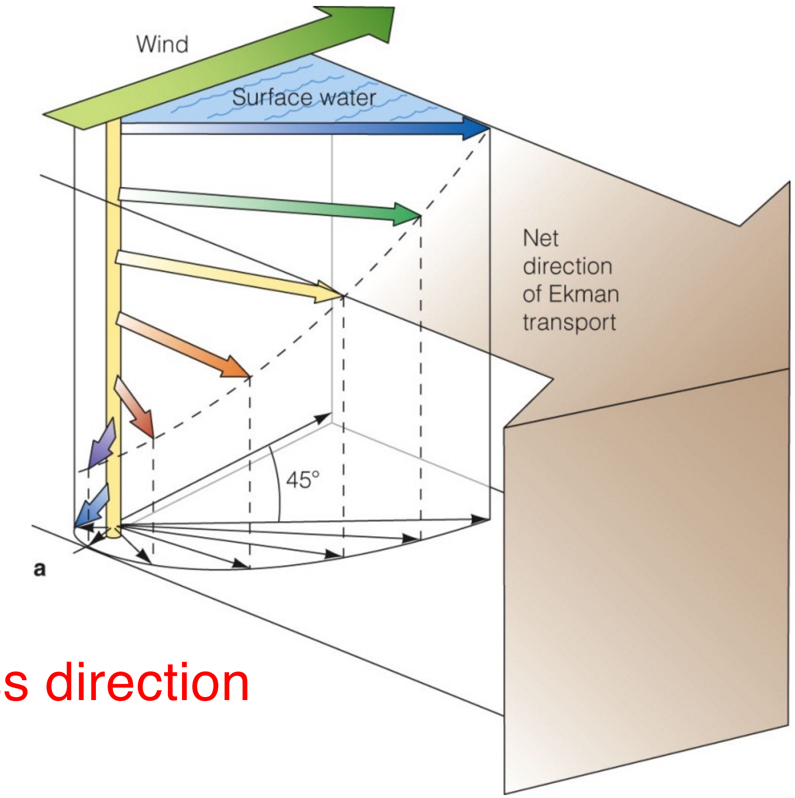
The solutions are:

$$u = \bar{u} + \frac{\sqrt{2}}{\rho_0 f d} e^{z/d} \left[\tau^x \cos\left(\frac{z}{d} - \frac{\pi}{4}\right) - \tau^y \sin\left(\frac{z}{d} - \frac{\pi}{4}\right)\right]$$
$$v = \bar{v} + \frac{\sqrt{2}}{\rho_0 f d} e^{z/d} \left[\tau^x \sin\left(\frac{z}{d} - \frac{\pi}{4}\right) + \tau^y \cos\left(\frac{z}{d} - \frac{\pi}{4}\right)\right]$$

The surface Ekman transport:

$$\mathrm{U} = \int_{-\infty}^{0} (u - \bar{u})\, dz = \frac{1}{\rho_0 f} \tau^y$$
$$\mathrm{V} = \int_{-\infty}^{0} (v - \bar{v})\, dz = \frac{-1}{\rho_0 f} \tau^x.$$

$$\boldsymbol{U} \cdot \boldsymbol{\tau} = 0$$



Ekman transport is perpendicular to the wind stress direction

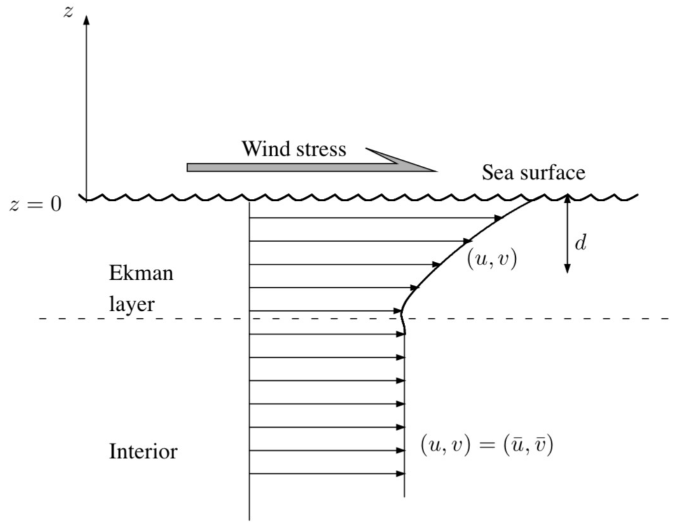
$$
\begin{aligned}
\mathrm{U} &= \int_{-\infty}^{0} (u - \bar{u})\, dz = \frac{1}{\rho_0 f}\, \tau^y \\
\mathrm{V} &= \int_{-\infty}^{0} (v - \bar{v})\, dz = \frac{-1}{\rho_0 f}\, \tau^x .
\end{aligned}
$$

The Ekman transport divergence (equal to the total transport divergence):

$$
\frac{\partial \mathrm{U}}{\partial x} + \frac{\partial \mathrm{V}}{\partial y} = \int_{-\infty}^{0} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dz = \frac{1}{\rho_0} \left[\frac{\partial}{\partial x} \left(\frac{\tau^y}{f} \right) - \frac{\partial}{\partial y} \left(\frac{\tau^x}{f} \right) \right]
$$

$$
-\frac{\partial w}{\partial z}
$$

$$
w|_{z=-\infty} - w|_{z=0}
$$



Ekman pumping velocity:

$$
\bar{w} = \frac{1}{\rho_0} \left[\frac{\partial}{\partial x} \left(\frac{\tau^y}{f} \right) - \frac{\partial}{\partial y} \left(\frac{\tau^x}{f} \right) \right]
$$

$$\bar{w} = \frac{1}{\rho_0} \left[\frac{\partial}{\partial x} \left(\frac{\tau^y}{f} \right) - \frac{\partial}{\partial y} \left(\frac{\tau^x}{f} \right) \right]$$

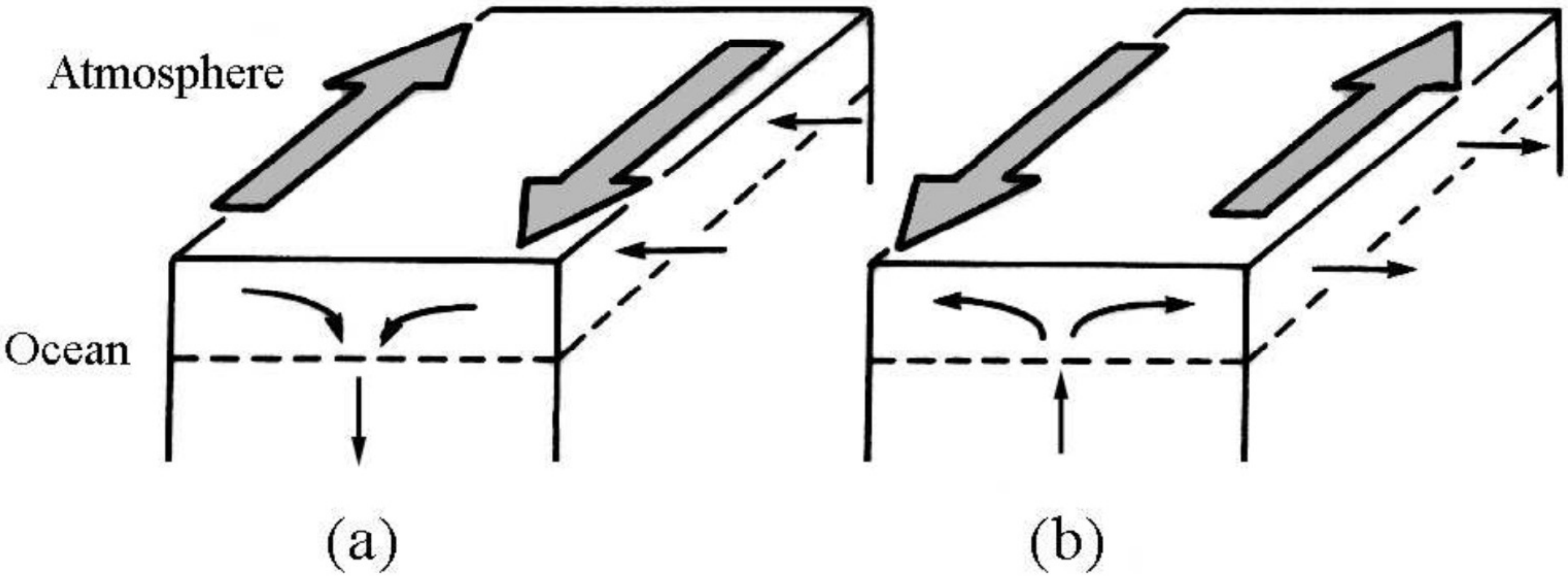
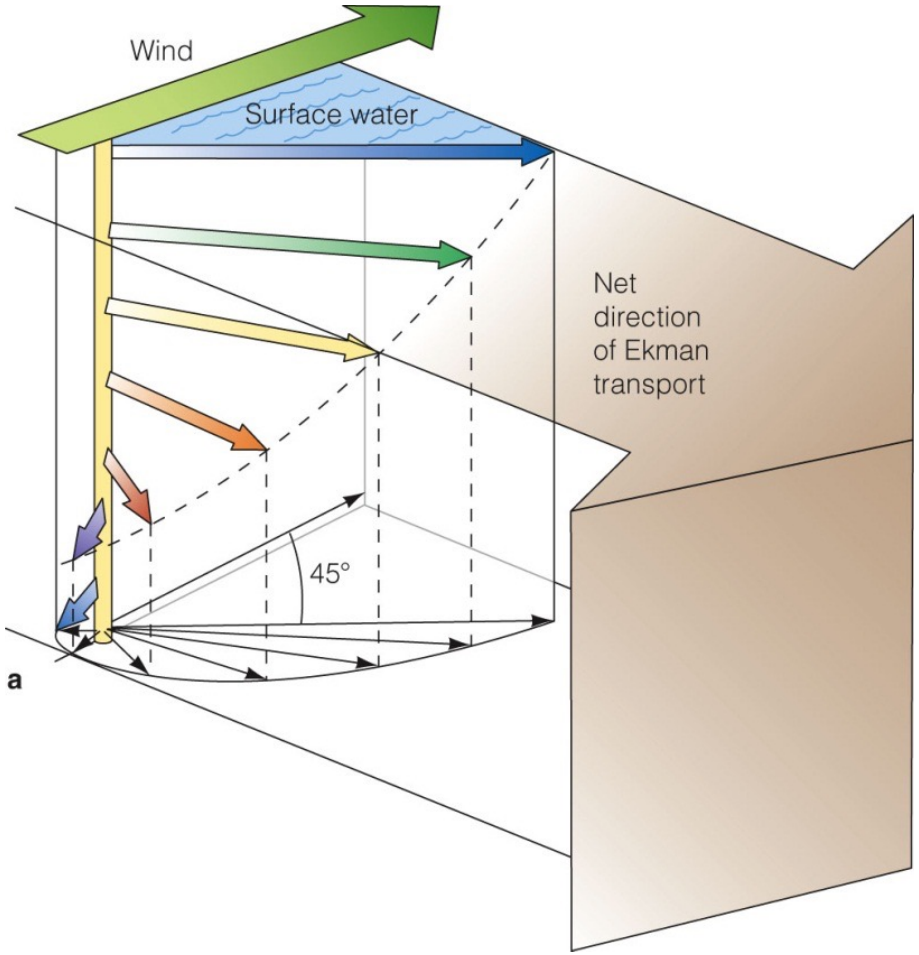
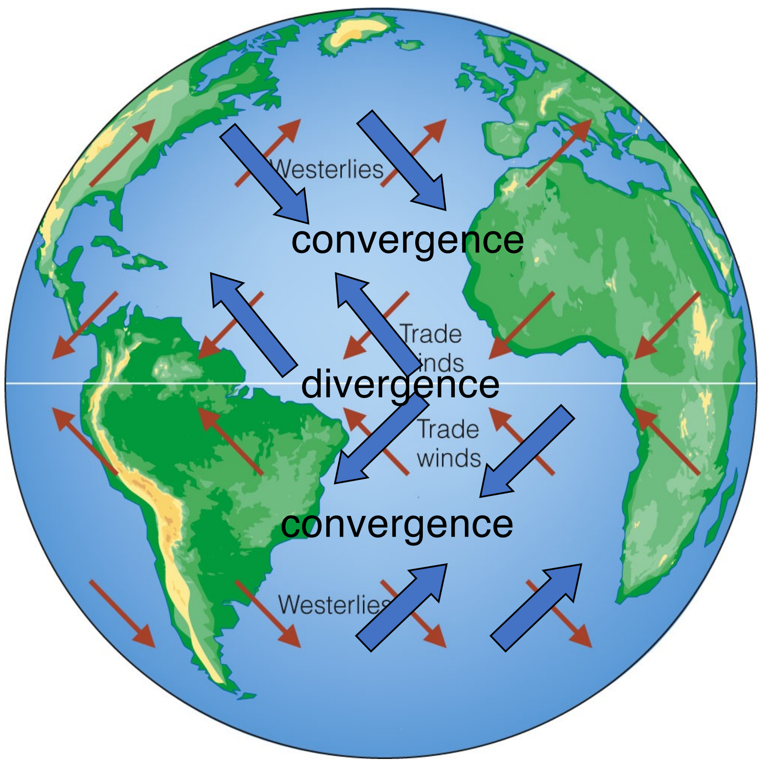
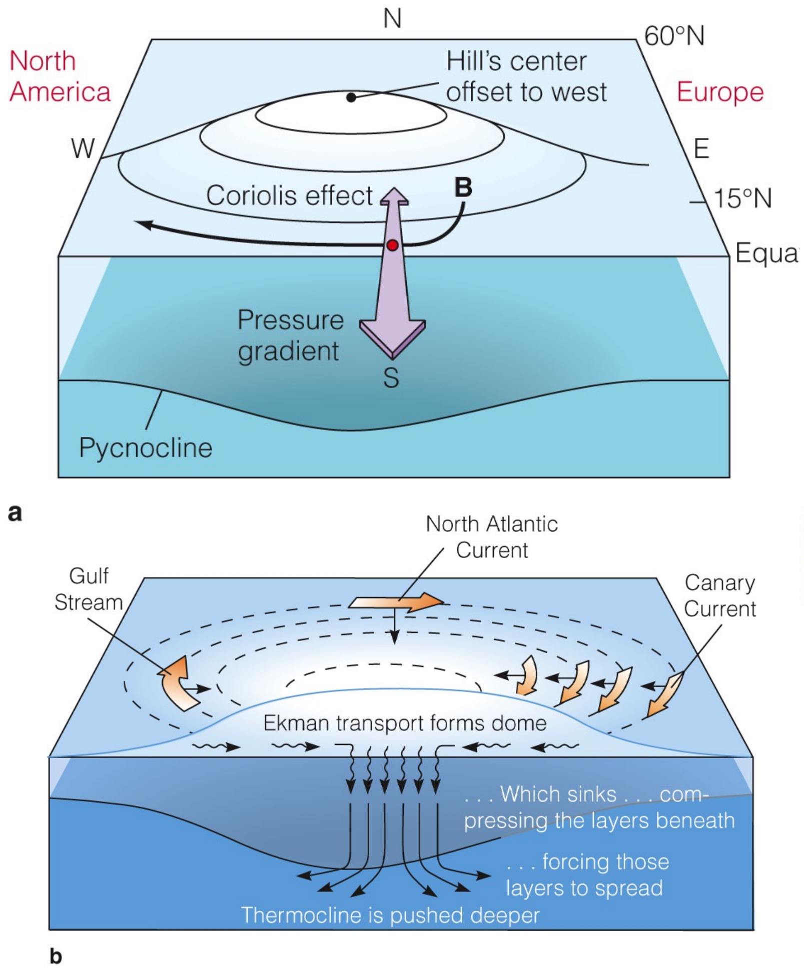


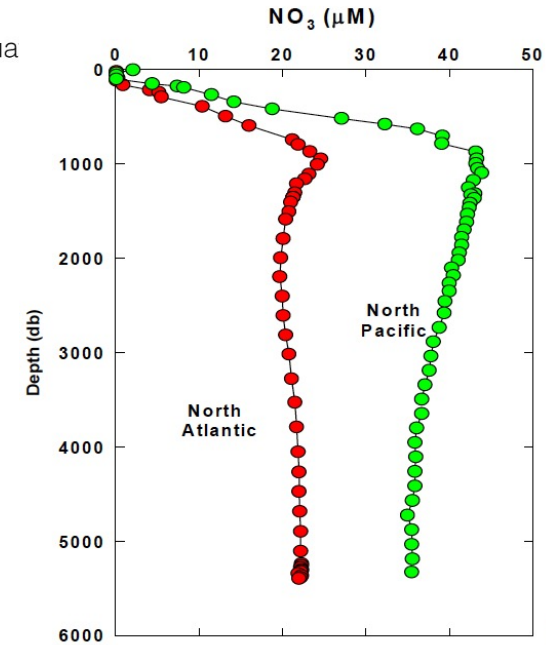
Figure 8-8 Ekman pumping in an ocean subject to sheared winds (case of Northern Hemisphere).

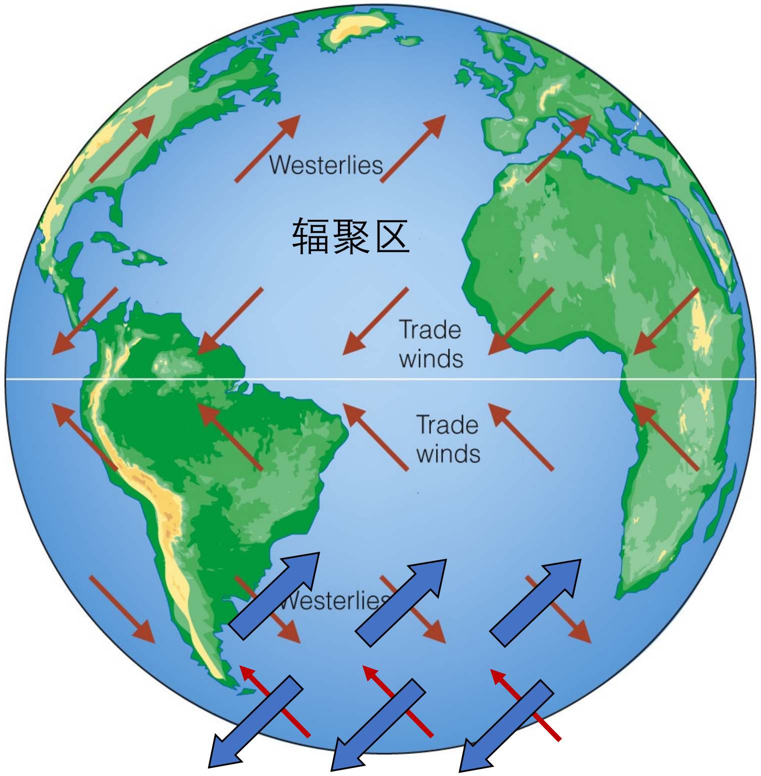
Surface Ekman transport



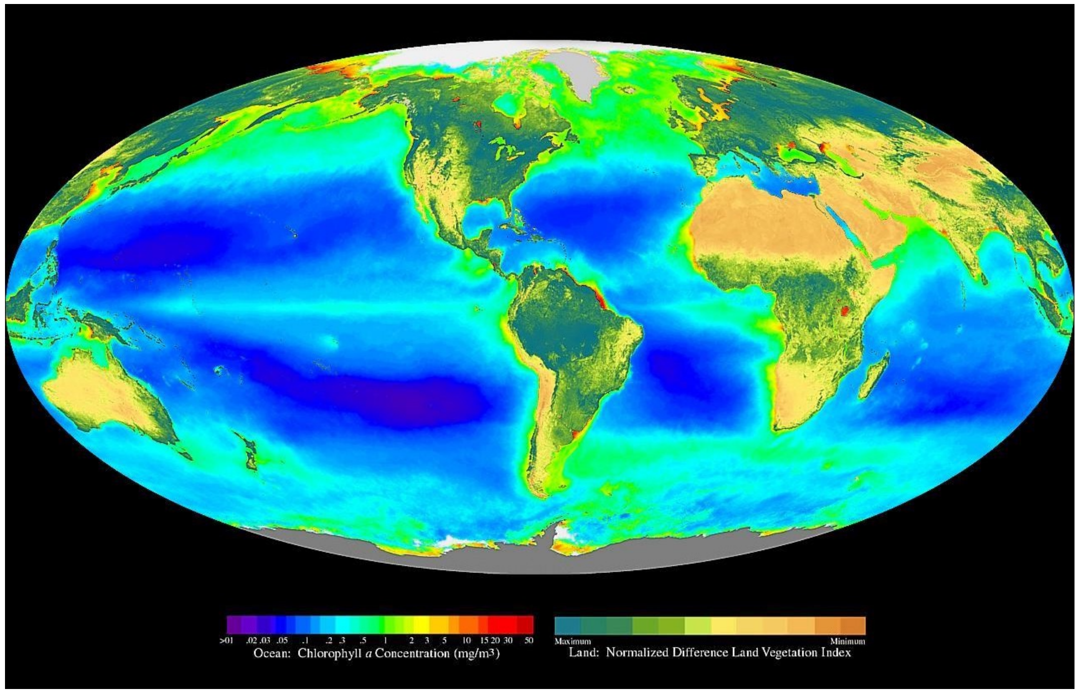

Westerlies
convergence
Trade
winds
divergence
Trade
winds
convergence
Westerlies


N
60°N
North
America
Hill's center
offset to west
Europe
W
E
Coriolis effect
B
15°N
Equa
Pressure
gradient
S
Pycnocline
a
Gulf
Stream
North Atlantic
Current
Canary
Current
Ekman transport forms dome
. . . Which sinks . . . com-
pressing the layers beneath
. . . forcing those
layers to spread
Thermocline is pushed deeper
b


NO3 (μM)
0
10
20
30
40
50
0
1000
2000
3000
4000
5000
6000
Depth (db)
North
Pacific
North
Atlantic

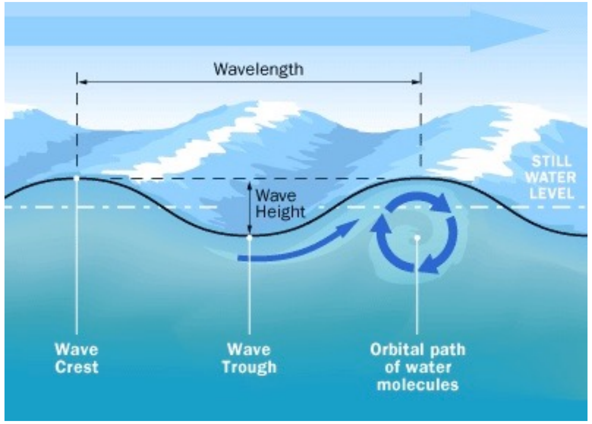


global ocean surface chlorophyll concentration



Surface gravity waves

Wave properties:



wavelength: λ

wave number: $K = 2\pi/\lambda$

wave period: T

wave frequency: $\omega = 2\pi/T$

wave speed: $c = \frac{\lambda}{T} = \frac{\omega}{K}$

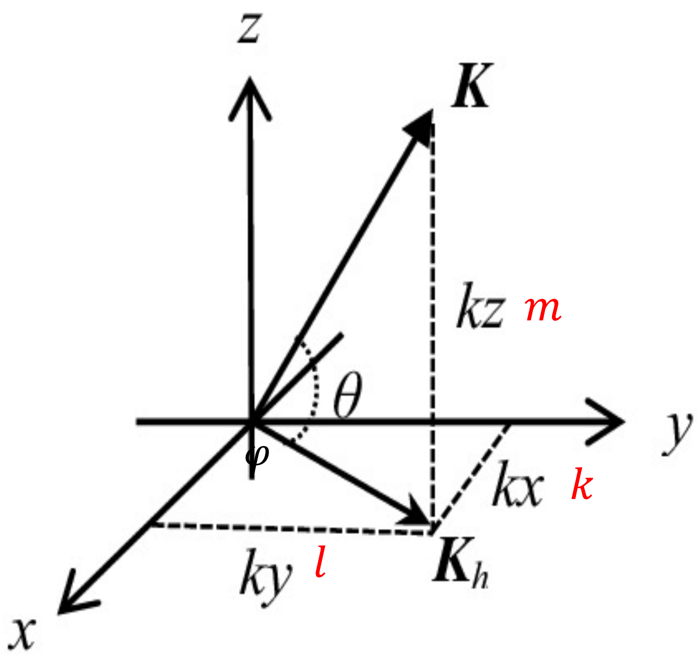
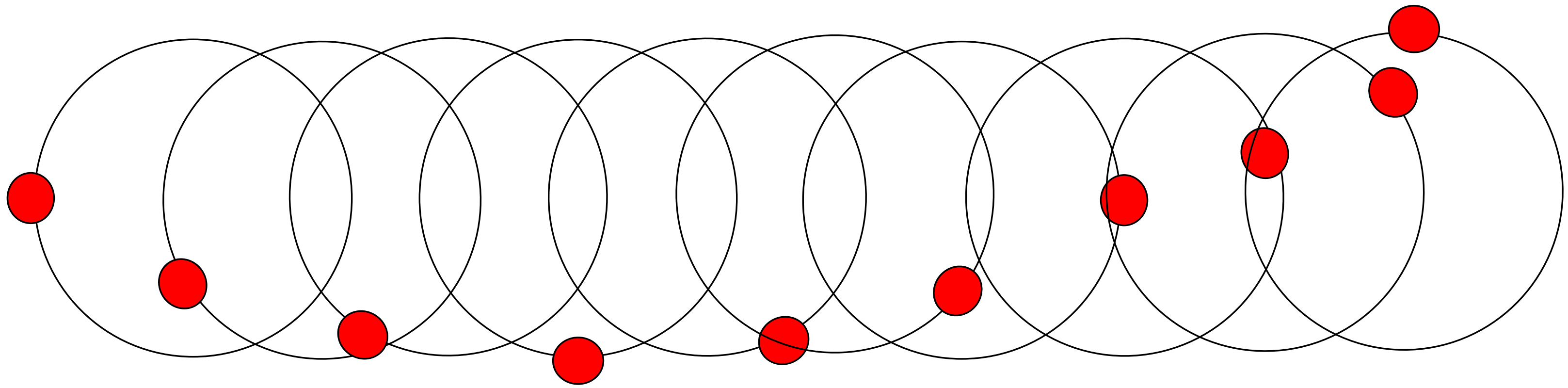


Fig. 1. Wave number vector and its components.

$$\boldsymbol{K} = (k, l, m)$$

$$K^2 = k^2 + l^2 + m^2$$

The direction of wave number denotes the wave propagation direction



Surface gravity waves

Assumptions: incompressible fluid, inviscid motion,

small-scale motion ($R_0 \gg 1$), linear motion (non-linear term neglected)

Governing equation:

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} = -\nabla p - \rho \boldsymbol{g}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

Take the curl of the momentum euqation:

$$\frac{\partial \nabla \times \boldsymbol{u}}{\partial t} = 0$$

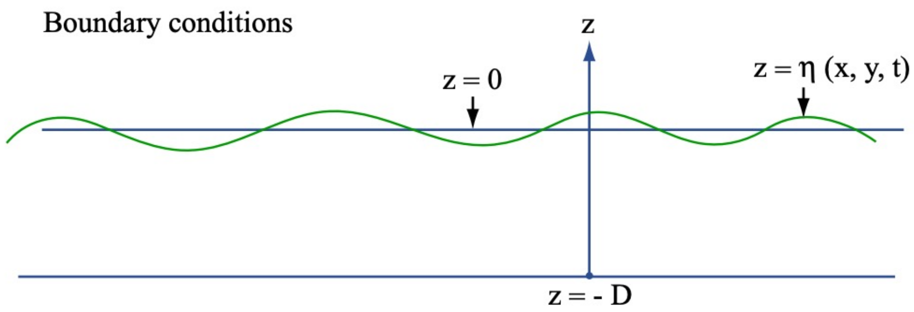
If the curl of velocity is initially 0, then it remains 0 for all time.

If $\boldsymbol{\nabla} \times \boldsymbol{u} = 0$, the velocity can be represented by a velocity potential ϕ:

$$\boldsymbol{u} = \nabla\phi$$

$$\nabla \cdot \boldsymbol{u} = \nabla \cdot \nabla\phi = \boldsymbol{\nabla^2 \phi = 0}$$

Boundary conditions:



bottom boundary ($z = -D$): $\mathrm{w} = \dfrac{\partial \phi}{\partial z} = 0$

surface boundary ($z = \eta$): $p(x, y, z, t) = p_a(x, y, t)$

$$\mathrm{w} = \frac{\partial \phi}{\partial z} = \frac{d\eta}{dt} = \frac{\partial \eta}{\partial t}$$

$$\rho\frac{\partial \boldsymbol{u}}{\partial t} = \frac{\partial \nabla\phi}{\partial t} = -\nabla p - \rho g \nabla z$$

$$\nabla\left(\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + gz\right) = 0$$

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + gz = \overset{0}{F(t)}$$

Apply the surface boundary condition ($z = \eta$): $\dfrac{\partial \phi}{\partial t} + \dfrac{p_a}{\rho} + g\eta = 0$

Take the time derivative:

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \eta}{\partial t} = -\frac{1}{\rho}\frac{\partial p_a}{\partial t} \qquad \mathrm{w} = \frac{\partial \phi}{\partial z} = \frac{d\eta}{dt} = \frac{\partial \eta}{\partial t}$$

$$g\frac{\partial \phi}{\partial z}$$

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z} = -\frac{1}{\rho}\frac{\partial p_a}{\partial t} \quad (z = \eta)$$

Can we change the surface boundary condition to $z = 0$?

Given that η is small:

$$G(x, y, \eta) = G(x, y, 0) + \cancel{\eta\frac{\partial G}{\partial z}|_{z=0}} + \cdots$$

nonlinear term

So the surface boundary condition turns to:

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z} = -\frac{1}{\rho}\frac{\partial p_a}{\partial t} \quad (z = 0)$$

wave problem

Bottom boundary condition ($z = -D$):

$$\mathrm{w} = \frac{\partial\phi}{\partial z} = 0$$

Can we apply a three-dimensional plane wave solution?

$$\phi = \mathrm{R}e^{i(kx+ly+mz-\omega t)}$$

At $z = -D$, $\quad \mathrm{w} = \frac{\partial \phi}{\partial z} \neq 0$, bottom boundary condition cannot be satisfied

So we can only apply a two-dimensional plane (x-y) wave solution:

$$\phi = \mathrm{R(z)}e^{i(kx+ly-\omega t)}$$

Plug the wave solution into the governing equation $\nabla^2\phi = 0$:

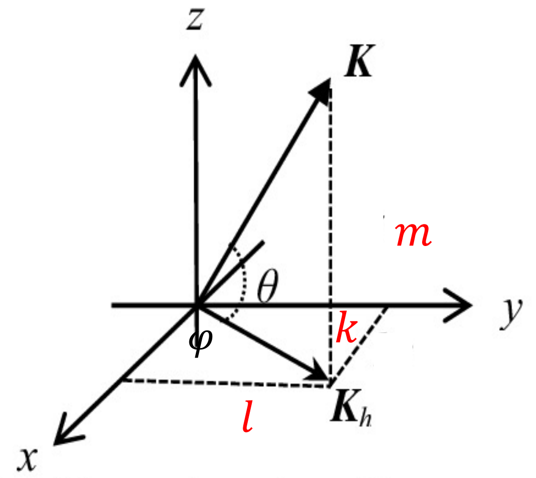


Fig. 1. Wave number vector and its components.

$$-k^2 R - l^2 R + \frac{d^2 R}{\partial z^2} = 0$$

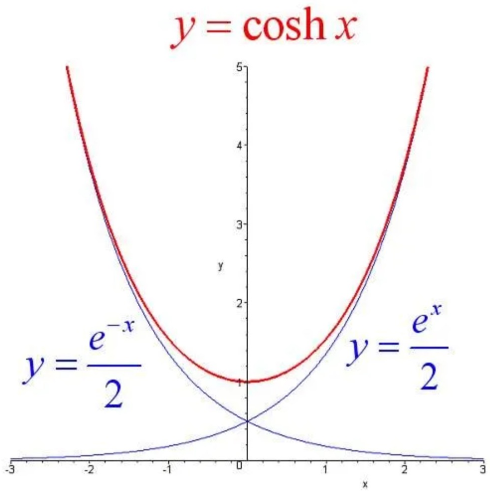
$$\frac{d^2 R}{\partial z^2} - K^2 R = 0 \qquad (K^2 = k^2 + l^2)$$

$$\phi = \mathrm{R(z)}e^{ik(x+ly-\omega t)}$$

Bottom boundary condition ($z = -D$):

$$\mathrm{w} = \frac{\partial \phi}{\partial z} = 0 \Rightarrow \frac{dR}{dz} = 0$$

$$R = AcoshK(z+D)$$



$$coshx = \frac{e^{x} + e^{-x}}{2}$$

$$sinhx = \frac{e^{x} - e^{-x}}{2}$$

Apply the surface boundary condition ($z = 0$):

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} = -\frac{1}{\rho}\frac{\partial p_a}{\partial t}$$

Assume a constant p_a:

$$-\omega^2 R + g\frac{dR}{dz} = 0$$

$$-\omega^2 AcoshKD + gKAsinhKD = 0$$

$$\omega = \pm\sqrt{gKtanhKD}$$

$$\omega = \pm\sqrt{gK tanh KD}$$ dispersion relation

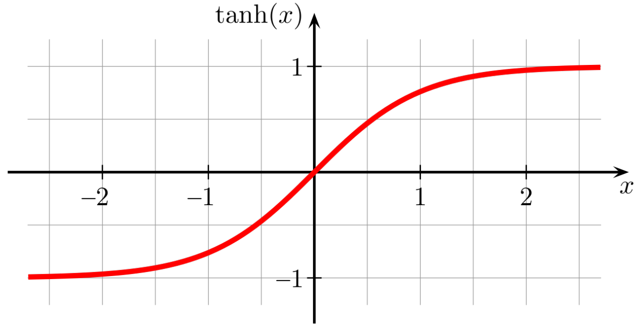
$$c = \frac{\omega}{K} = \pm\sqrt{gD}\left(\frac{tanhKD}{KD}\right)^{1/2}$$

Plane waves with different wavelength have diffferent wave speed, and the pattern will disperse

For $\boldsymbol{KD} \ll \boldsymbol{1}$ ($\lambda \gg$ D, shallow water waves)

(L'Hopital's rule)

$$\lim_{KD\to 0}\frac{tanhKD}{KD} = \lim_{KD\to 0}\frac{(tanhKD)'}{(KD)'} = \lim_{KD\to 0}\frac{1}{cosh^2 KD} = 1$$



$$c = \sqrt{gD}$$

the wave speed does not depend on the wavelength, non-dispersive waves

For $\boldsymbol{KD} \gg \boldsymbol{1}$ ($\lambda \ll$ D, deep water waves): $tanhKD \to 1$

$$c = \sqrt{g/K}$$

the wave speed depends on the wavelength, dispersive waves

$$\omega^2 = gK\tanh KD$$

Group velocity:

$$\boldsymbol{c_g} = \frac{\partial\omega}{\partial K}\frac{\boldsymbol{K}}{K}$$

$$2\omega\partial\omega = g\left[\partial K \tanh KD + K\frac{1}{\cosh^2 KD}D\partial K\right]$$

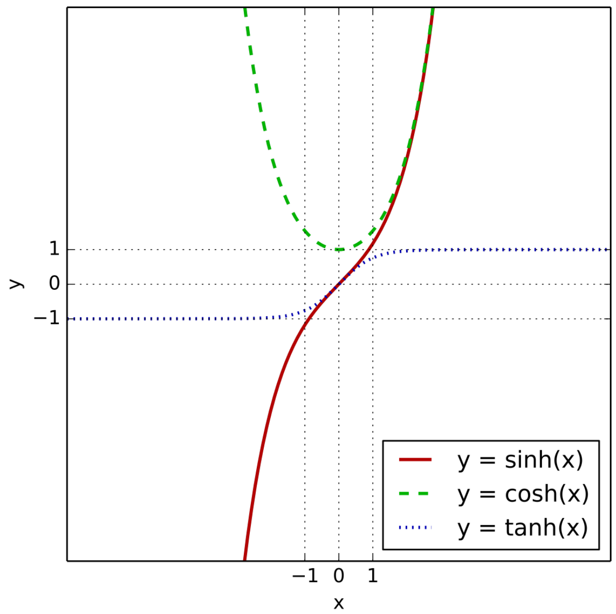
$$\frac{\omega}{K}\frac{\partial\omega}{\partial K} = \frac{1}{2}\frac{g}{K}\left[\tanh KD + KD\frac{1}{\cosh^2 KD}\right]$$

$$c \cdot c_g = \frac{1}{2}\frac{g}{K}\left(\tanh KD + \frac{KD}{\cosh^2 KD}\right)$$

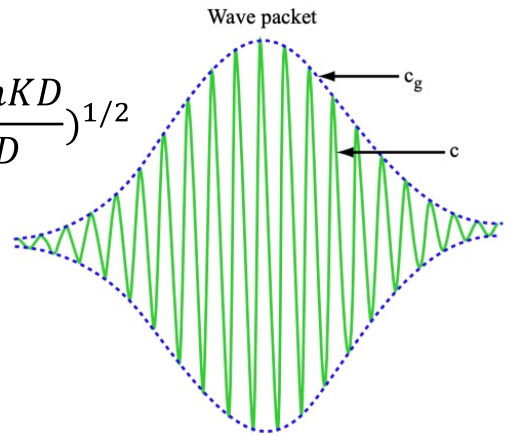
$$\frac{c_g}{c} = \frac{1}{2}\left(1 + \frac{KD}{\sinh KD \cdot \cosh KD}\right)$$

For $\boldsymbol{KD} \ll \mathbf{1}$ (shallow water waves): $c_g = c$

For $\boldsymbol{KD} \gg \mathbf{1}$ (deep water waves): $c_g = \frac{1}{2}c$



$$c = \pm\sqrt{gD}\left(\frac{\tanh KD}{KD}\right)^{1/2}$$



A wave packet propagating with the group velocity carries a plane wave with crest moving with the phase speed