A little inaccuracy sometimes saves a ton of explanation.
H. M. Munro (Saki), *The Square Egg*, 1924.

Every decoding is another encoding.
David Lodge, in the voice of Morris Zapp, *Small World*, 1984.

CHAPTER 5

Geostrophic Theory

LARGE-SCALE FLOW IN THE OCEAN AND THE ATMOSPHERE is characterized by an approximate balance in the vertical direction between the pressure gradient and gravity (hydrostatic balance), and in the horizontal direction between the pressure gradient and the Coriolis force (geostrophic balance). In this chapter we exploit these balances to simplify the Navier–Stokes equations and thereby obtain various sets of simplified 'geostrophic equations'. Depending on the precise nature of the assumptions we make, we are led to the *quasi-geostrophic* (QG) system for horizontal scales similar to that on which most synoptic activity takes place and, for very large-scale motion, to the *planetary-geostrophic* (PG) set of equations. By eliminating unwanted or unimportant modes of motion, in particular sound waves and gravity waves, and by building in the important balances between flow fields, these filtered equation sets allow the investigator to better focus on a particular class of phenomena and to potentially achieve a deeper understanding than might otherwise be possible.[1]

Simplifying the equations in this way relies first on scaling the equations. The idea is that we *choose* the scales we wish to describe, typically either on some a-priori basis or by using observations as a guide. We then attempt to derive a set of equations that is simpler than the original set but that consistently describes motion of the chosen scale. An asymptotic method is one way to achieve this, for it systematically tells us which terms we can drop and which we should keep. The combined approach — scaling plus asymptotics — has proven enormously useful, but we should always remember two things: (i) that scaling is a choice; (ii) that the approach does not explain the existence of particular scales of motion, it just describes the motion that might occur on such scales. We have already employed this general approach in deriving the hydrostatic primitive equations, but now we go further.

5.1 GEOSTROPHIC SCALING

5.1.1 Scaling in the Shallow Water Equations

Postponing the complications that come with stratification, we begin with the shallow water equations. With the odd exception, we will denote the scales of variables by capital letters; thus, if L is a typical length scale of the motion we wish to describe, and U is a typical velocity scale, and assuming the scales are horizontally isotropic, we write

$$
\begin{aligned}
(x, y) &\sim L \qquad \text{or} \qquad (x, y) = \mathcal{O}(L) \\
(u, v) &\sim U \qquad \text{or} \qquad (u, v) = \mathcal{O}(U),
\end{aligned}
\tag{5.1}
$$

and similarly for other variables. We may then nondimensionalize the variables by writing

$$
(x, y) = L(\widehat{x}, \widehat{y}), \qquad (u, v) = U(\widehat{u}, \widehat{v}),
\tag{5.2}
$$

where the hatted variables are nondimensional and, by supposition, are $\mathcal{O}(1)$. The various terms in the momentum equation then scale as:

$$
\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \boldsymbol{f} \times \boldsymbol{u} = -g \nabla \eta,
\tag{5.3a}
$$

$$
\frac{U}{T} \qquad \frac{U^2}{L} \qquad fU \;\sim\; g\frac{\mathcal{H}}{L},
\tag{5.3b}
$$

where the ∇ operator acts in the x–y plane and $\mathcal{H}$ is the amplitude of the variations in the surface displacement. (We use η to denote the height of the free surface above some arbitrary reference level, as in Fig. 3.1. Thus, $\eta = H + \Delta\eta$, where $\Delta\eta$ denotes the variation of η about its mean position.)

The ratio of the advective term to the rotational term in the momentum equation (5.3) is $(U^2/L)/(fU) = U/fL$; this is the Rossby number,[2] first encountered in Chapter 2. Using values typical of the large-scale circulation (e.g., from Table 2.1) we find that $Ro \approx 0.1$ for the atmosphere and $Ro \approx 0.01$ for the ocean: small in both cases. If we are interested in motion that has the advective time scale $T = L/U$ then we scale time by L/U so that

$$
t = \frac{L}{U}\hat{t},
\tag{5.4}
$$

and the local time derivative and the advective term then both scale as U^2/L, and both are smaller than the rotation term by a factor of the order of the Rossby number. Then, either the Coriolis term is the dominant term in the equation, in which case we have a state of no motion with $-fv = 0$, or else the Coriolis force is balanced by the pressure force, and the dominant balance is

$$
-fv = -g\frac{\partial \eta}{\partial x},
\tag{5.5}
$$

namely *geostrophic balance*, as encountered in Chapter 2. If we make this non-trivial choice, then the equation informs us that variations in η (i.e., $\Delta\eta$) scale according to

$$
\Delta\eta \sim \mathcal{H} = \frac{fUL}{g}.
\tag{5.6}
$$

We can also write $\mathcal{H}$ as

$$
\mathcal{H} = Ro\,\frac{f^2L^2}{g} = Ro\;H\frac{L^2}{L_d^2},
\tag{5.7}
$$

where $L_d = \sqrt{gH}/f$ is the deformation radius and H is the mean depth of the fluid. The variations in fluid height thus scale as

$$
\frac{\Delta\eta}{H} \sim Ro\,\frac{L^2}{L_d^2},
\tag{5.8}
$$

and the height of the fluid may be written as

$$
\eta = H\left(1 + Ro\,\frac{L^2}{L_d^2}\widehat{\eta}\right) \qquad \text{and} \qquad \Delta\eta = Ro\,\frac{L^2}{L_d^2}H\widehat{\eta},
\tag{5.9}
$$

where $\widehat{\eta}$ is the $\mathcal{O}(1)$ nondimensional value of the surface height deviation.

Nondimensional momentum equation

If we use (5.9) to scale height variations, (5.2) to scale lengths and velocities, and (5.4) to scale time, then the momentum equation (5.3) becomes

$$Ro\left[\frac{\partial\widehat{\boldsymbol{u}}}{\partial\hat{t}}+(\widehat{\boldsymbol{u}}\cdot\nabla)\widehat{\boldsymbol{u}}\right]+\widehat{\boldsymbol{f}}\times\widehat{\boldsymbol{u}}=-\nabla\widehat{\eta}, \tag{5.10}$$

where $\widehat{\boldsymbol{f}} = \mathbf{k}\widehat{f} = \mathbf{k}f/f_0$, where f_0 is a representative value of the Coriolis parameter. (If f is a constant, then $\widehat{f} = 1$, but it is informative to explicitly write $\widehat{f}$ in the equations. Also, where the operator ∇ operates on a nondimensional variable then the differentials are taken with respect to the nondimensional variables $\widehat{x}$, $\widehat{y}$.) All the variables in (5.10) will be assumed to be of order unity, and the Rossby number multiplying the local time derivative and the advective terms indicates the smallness of those terms. By construction, the dominant balance in this equation is the geostrophic balance between the last two terms.

Nondimensional mass continuity (height) equation

The (dimensional) mass continuity equation can be written as

$$\frac{1}{H}\frac{\mathrm{D}\eta}{\mathrm{D}t}+\left(1+\frac{\Delta\eta}{H}\right)\nabla\cdot\boldsymbol{u}=0. \tag{5.11}$$

Using (5.2), (5.4) and (5.9) this equation may be written

$$Ro\left(\frac{L}{L_d}\right)^2\frac{\mathrm{D}\widehat{\eta}}{\mathrm{D}\hat{t}}+\left[1+Ro\left(\frac{L}{L_d}\right)^2\widehat{\eta}\right]\nabla\cdot\widehat{\boldsymbol{u}}=0. \tag{5.12}$$

Equations (5.10) and (5.12) are the nondimensional versions of the full shallow water equations of motion. Evidently, some terms in the equations of motion are small and may be eliminated with little loss of accuracy, and the way this is done will depend on the size of the second nondimensional parameter, $(L/L_d)^2$, which we come to shortly.

Froude and Burger numbers

The Froude number may be generally defined as the ratio of a fluid particle speed to a wave speed. In a shallow water system this gives

$$Fr \equiv \frac{U}{\sqrt{gH}}=\frac{U}{f_0L_d}=Ro\,\frac{L}{L_d}\,. \tag{5.13}$$

The Burger number[3] is a useful measure of the scale of motion of the fluid, relative to the deformation radius, and may be defined by

$$Bu \equiv \left(\frac{L_d}{L}\right)^2=\frac{gH}{f_0^2L^2}=\left(\frac{Ro}{Fr}\right)^2. \tag{5.14}$$

It is also useful to define the parameter $F \equiv Bu^{-1}$, which is like the square of a Froude number but uses the rotational speed fL instead of U in the numerator.

5.1.2 Geostrophic Scaling in the Stratified Equations

We now apply the same scaling ideas, *mutatis mutandis,* to the stratified primitive equations. We use the hydrostatic anelastic equations, which we write as

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{u} = -\nabla_z \phi, \tag{5.15a}$$

$$\frac{\partial \phi}{\partial z} = b, \tag{5.15b}$$

$$\frac{\mathrm{D}b}{\mathrm{D}t} = 0, \tag{5.15c}$$

$$\nabla \cdot (\tilde{\rho}\boldsymbol{v}) = 0, \tag{5.15d}$$

where b is the buoyancy and $\tilde{\rho}$ is a reference density profile. Anticipating that the average stratification may not scale in the same way as the deviation from it, let us separate out the contribution of the advection of a reference stratification in (5.15c) by writing

$$b = \tilde{b}(z) + b'(x, y, z, t). \tag{5.16}$$

The thermodynamic equation then becomes

$$\frac{\mathrm{D}b'}{\mathrm{D}t} + N^2 w = 0, \tag{5.17}$$

where $N^2 \equiv \partial\tilde{b}/\partial z$ (and the advective derivative is still three-dimensional). We then let $\phi = \tilde{\phi}(z) + \phi'$, where $\tilde{\phi}$ is hydrostatically balanced by $\tilde{b}$, and the hydrostatic equation becomes

$$\frac{\partial \phi'}{\partial z} = b'. \tag{5.18}$$

Equations (5.17) and (5.18) replace (5.15c) and (5.15b), and ϕ' is used in (5.15a).

Nondimensional equations

We scale the basic variables by supposing that

$$(x, y) \sim L, \quad (u, v) \sim U, \quad t \sim \frac{L}{U}, \quad z \sim H, \quad f \sim f_0, \quad N \sim N_0 \tag{5.19}$$

where the scaling variables (capitalized, except for f_0) are chosen to be such that the nondimensional variables have magnitudes of the order of unity, and the parameters N_0 and f_0 are representative values of N and f. The scales chosen are such that the Rossby number is small; that is $Ro = U/(f_0 L) \ll 1$. In the momentum equation the pressure term then balances the Coriolis force,

$$|\boldsymbol{f} \times \boldsymbol{u}| \sim |\nabla \phi'|, \tag{5.20}$$

and so the pressure scales as

$$\phi' \sim \Phi = f_o U L. \tag{5.21}$$

Using the hydrostatic relation, (5.21) implies that the buoyancy scales as

$$b' \sim B = \frac{f_0 U L}{H}, \tag{5.22}$$

and from this we obtain

$$\frac{(\partial b'/\partial z)}{N^2} \sim Ro\, \frac{L^2}{L_d^2}, \tag{5.23}$$

where $L_d = N_0 H/f_0$ is the deformation radius in the continuously stratified fluid, analogous to the quantity $\sqrt{gH}/f_0$ in the shallow water system, and we use the same symbol, L_d, for both. In the continuously stratified system, *if the scale of motion is the same as or smaller than the deformation radius, and the Rossby number is small, then the variations in stratification are small.* The choice of scale is the key difference between the planetary-geostrophic and quasi-geostrophic equations.

Finally, we will nondimensionalize the vertical velocity by using the mass conservation equation,

$$\frac{1}{\tilde{\rho}}\frac{\partial \tilde{\rho} w}{\partial z} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right), \tag{5.24}$$

and we suppose that this implies

$$w \sim W = \frac{UH}{L}. \tag{5.25}$$

This is a naïve scaling for rotating flow: if the Coriolis parameter is nearly constant the geostrophic velocity is nearly horizontally non-divergent and the right-hand side of (5.24) is small, and $W \ll UH/L$. We might then estimate w by cross-differentiating geostrophic balance (with $\tilde{\rho}$ constant for simplicity) to obtain the linear geostrophic vorticity equation and corresponding scaling:

$$\beta v \approx f\frac{\partial w}{\partial z}, \qquad w \sim W = \frac{\beta U H}{f_0}. \tag{5.26a,b}$$

However, rather than using (5.26b) from the outset, we will use (5.25) and let the asymptotics guide us to a proper scaling in the fullness of time. Note that if variations in the Coriolis parameter are large and $\beta \sim f_0/L$, then (5.26b) is the same as (5.25).

Given the scalings above (using (5.25) for w) we nondimensionalize by setting

$$\begin{gathered}(\hat{x}, \hat{y}) = L^{-1}(x, y), \qquad \hat{z} = H^{-1}z, \quad (\hat{u}, \hat{v}) = U^{-1}(u, v), \quad \hat{t} = \frac{U}{L}t, \\ \widehat{w} = \frac{L}{UH}w, \quad \hat{f} = \frac{f}{f_0}, \quad \widehat{N} = \frac{N}{N_0}, \quad \hat{\phi} = \frac{\phi'}{f_0 U L}, \quad \hat{b} = \frac{H}{f_0 U L}b',\end{gathered} \tag{5.27}$$

where the hatted variables are nondimensional. The horizontal momentum and hydrostatic equations then become

$$Ro\,\frac{\mathrm{D}\hat{\boldsymbol{u}}}{\mathrm{D}\hat{t}} + \hat{\boldsymbol{f}} \times \hat{\boldsymbol{u}} = -\nabla\hat{\phi}, \tag{5.28}$$

and

$$\frac{\partial \hat{\phi}}{\partial \hat{z}} = \hat{b}. \tag{5.29}$$

The nondimensional mass conservation equation is simply

$$\frac{1}{\tilde{\rho}}\nabla\cdot(\tilde{\rho}\boldsymbol{v}) = \left(\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{1}{\tilde{\rho}}\frac{\partial \tilde{\rho}\widehat{w}}{\partial \hat{z}}\right) = 0, \tag{5.30}$$

and the nondimensional thermodynamic equation is

$$\frac{f_0 U L}{H}\frac{U}{L}\frac{\mathrm{D}\hat{b}}{\mathrm{D}\hat{t}} + \widehat{N}^2 N_0^2 \frac{HU}{L}\widehat{w} = 0, \tag{5.31}$$

or

$$Ro\,\frac{\mathrm{D}\hat{b}}{\mathrm{D}\hat{t}} + \left(\frac{L_d}{L}\right)^2 \widehat{N}^2 \widehat{w} = 0. \tag{5.32}$$

The nondimensional primitive equations are summarized in the box on the following page.

Nondimensional Primitive Equations

Horizontal momentum: $$Ro\frac{\mathrm{D}\widehat{\boldsymbol{u}}}{\mathrm{D}\widehat{t}} + \widehat{\boldsymbol{f}} \times \widehat{\boldsymbol{u}} = -\nabla\widehat{\phi} \qquad \text{(PE.1)}$$

Hydrostatic: $$\frac{\partial\widehat{\phi}}{\partial\widehat{z}} = \widehat{b} \qquad \text{(PE.2)}$$

Mass continuity: $$\left(\frac{\partial\widehat{u}}{\partial\widehat{x}} + \frac{\partial\widehat{v}}{\partial\widehat{y}} + \frac{1}{\widetilde{\rho}}\frac{\partial\widetilde{\rho}\widehat{w}}{\partial\widehat{z}}\right) = 0 \qquad \text{(PE.3)}$$

Thermodynamic: $$Ro\frac{\mathrm{D}\widehat{b}}{\mathrm{D}\widehat{t}} + \left(\frac{L_d}{L}\right)^2\widehat{N}^2\widehat{w} = 0 \qquad \text{(PE.4)}$$

These equations are written for the anelastic equations in a rotating frame of reference. The Boussinesq equations result if we take $\widetilde{\rho} = 1$. The equations in pressure coordinates also have a similar form — see Section 2.6.2.

5.2 THE PLANETARY-GEOSTROPHIC EQUATIONS

We now use the low Rossby number scalings above to derive equation sets that are simpler than the original, 'primitive', ones. The planetary-geostrophic equations are probably the simplest such set of equations, and we derive these equations first for the shallow water equations, and then for the stratified primitive equations.

5.2.1 Using the Shallow Water Equations

Informal derivation

The advection and time derivative terms in the momentum equation (5.10) are order Rossby number smaller than the Coriolis and pressure terms (the term in square brackets is multiplied by Ro), and therefore let us neglect them. The momentum equation straightforwardly becomes

$$\widehat{\boldsymbol{f}} \times \widehat{\boldsymbol{u}} = -\nabla\widehat{\eta}. \tag{5.33}$$

The mass conservation equation (5.12), contains two nondimensional parameters, $Ro = U/(f_0L)$ (the Rossby number), and $F = (L/L_d)^2$ (the ratio of the length scale of the motion to the deformation scale; $F = Bu^{-1}$) and we must make a choice as to the relationship between these two numbers. We will choose

$$F\,Ro = \mathcal{O}(1), \tag{5.34}$$

which implies

$$L^2 \gg L_d^2 \qquad \text{or equivalently} \qquad F \gg 1, \quad Bu \ll 1. \tag{5.35}$$

That is to say, we suppose that the scales of motion are much larger than the deformation scale. Given this choice, all the terms in the mass conservation equation, (5.12), are of roughly the same size, and we retain them all. Thus, the shallow water planetary geostrophic equations are the full mass continuity equation along with geostrophic balance and a geometric relationship between the height field and the fluid thickness, and in dimensional form these are:

$$\begin{gathered}\frac{\mathrm{D}h}{\mathrm{D}t} + h\nabla\cdot\boldsymbol{u} = 0,\\ \boldsymbol{f}\times\boldsymbol{u} = -g\nabla\eta, \qquad \eta = h + \eta_b\,.\end{gathered} \tag{5.36a,b,c}$$

We emphasize that *the planetary-geostrophic equations are only valid for scales of motion much larger than the deformation radius.* The height variations are then as large as the mean height field itself; that is, using (5.8), $\Delta\eta/H = \mathcal{O}(1)$.

Formal derivation

We make the following assumptions:

(i) The Rossby number is small. $Ro = U/f_0 L \ll 1$.

(ii) The scale of the motion is significantly larger than the deformation scale. That is, (5.34) holds or equivalently

$$F = Bu^{-1} = \left(\frac{L}{L_d}\right)^2 \gg 1 \tag{5.37}$$

and in particular

$$F\,Ro = \mathcal{O}(1). \tag{5.38}$$

(iii) Time scales advectively, so that $T = L/U$.

The idea is now to expand the nondimensional velocity and height fields in an asymptotic series with the Rossby number as the small parameter, substitute into the equations of motion and derive a simpler set of equations. It is a nearly trivial exercise in this instance, and so illustrates the methodology well. The expansions are

$$\widehat{\boldsymbol{u}} = \widehat{\boldsymbol{u}}_0 + Ro\,\widehat{\boldsymbol{u}}_1 + Ro^2\,\widehat{\boldsymbol{u}}_2 + \ldots \qquad \text{and} \qquad \widehat{\eta} = \widehat{\eta}_0 + Ro\,\widehat{\eta}_1 + Ro^2\,\widehat{\eta}_2 + \ldots. \tag{5.39a,b}$$

Substituting (5.39) into the momentum equation then gives

$$Ro\left[\frac{\partial \widehat{\boldsymbol{u}}_0}{\partial \hat{t}} + \widehat{\boldsymbol{u}}_0 \cdot \nabla \widehat{\boldsymbol{u}}_0 + \widehat{\boldsymbol{f}} \times \widehat{\boldsymbol{u}}_1\right] + \widehat{\boldsymbol{f}} \times \widehat{\boldsymbol{u}}_0 = -\nabla\widehat{\eta}_0 - Ro\left[\nabla\widehat{\eta}_1\right] + \mathcal{O}(Ro^2). \tag{5.40}$$

The Rossby number is an asymptotic ordering parameter; thus, the sum of all the terms at any particular order in Rossby number must vanish. At lowest order we obtain the simple expression

$$\widehat{\boldsymbol{f}} \times \widehat{\boldsymbol{u}}_0 = -\nabla\widehat{\eta}_0. \tag{5.41}$$

Note that although f_0 is a representative value of f, we have made no assumptions about the constancy of f. In particular, f is allowed to vary by an order one amount, provided that it does not become so small that the Rossby number $U/(fL)$ is not small.

The appropriate height (mass conservation) equation is similarly obtained by substituting (5.39) into the shallow water mass conservation equation. Because $F\,Ro = \mathcal{O}(1)$ at lowest order we simply retain all the terms in the equation to give

$$F\,Ro\left[\frac{\partial \widehat{\eta}_0}{\partial t} + \widehat{\boldsymbol{u}}_0 \cdot \nabla\widehat{\eta}_0\right] + \left[1 + F\,Ro\,\widehat{\eta}\right]\nabla \cdot \widehat{\boldsymbol{u}}_0 = 0. \tag{5.42}$$

Equations (5.41) and (5.42) are a closed set, namely the nondimensional planetary-geostrophic equations. The dimensional forms of these equations are just (5.36).

Variation of the Coriolis parameter

Suppose then that f is a constant (f_0). Then, from the curl of (5.41), $\nabla \cdot \boldsymbol{u}_0 = 0$. This means that we can define a streamfunction for the flow and, from geostrophic balance, the height field is just that streamfunction. That is, in dimensional form,

$$\psi = \frac{g}{f_0}\eta, \qquad \boldsymbol{u} = \mathbf{k} \times \nabla\psi, \tag{5.43a,b}$$

and (5.42) becomes, in dimensional form,

$$\frac{\partial \eta}{\partial t} + \boldsymbol{u} \cdot \nabla \eta = 0 \qquad \text{or} \qquad \frac{\partial \eta}{\partial t} + J(\psi, \eta) = 0, \tag{5.44}$$

where $J(a,b) \equiv a_x b_y - a_y b_x$. But since $\eta \propto \psi$ the advective term is proportional to $J(\psi, \psi)$, which is zero. Thus, the flow does not evolve at this order. The planetary-geostrophic equations are *uninteresting* if the scale of the motion is such that the Coriolis parameter is not variable. On Earth, the scale of motion on which this parameter regime exists is rather limited, since the planetary-geostrophic equations require that the scale of motion also be larger than the deformation radius. In the Earth's atmosphere, any scale that is larger than the deformation radius will be such that the Coriolis parameter varies significantly over it, and we do not encounter this parameter regime. On the other hand, in the Earth's ocean the deformation radius is relatively small and there exists a small parameter regime (called the frontal geostrophic regime) that has scales larger than the deformation radius but smaller than that on which the Coriolis parameter varies.

Potential vorticity

The shallow water PG equations may be written as an evolution equation for an appropriate potential vorticity. A little manipulation reveals that (5.36) are equivalent to:

$$\begin{gathered} \frac{\mathrm{D}Q}{\mathrm{D}t} = 0, \\ Q = \frac{f}{h}, \quad \boldsymbol{f} \times \boldsymbol{u} = -g\nabla\eta, \quad \eta = h + \eta_b \,. \end{gathered} \tag{5.45}$$

Thus, potential vorticity is a material invariant in the approximate equation set, just as it is in the full equations. The other variables — the free surface height and the velocity — are diagnosed from it, a process known as *potential vorticity inversion*. In the planetary geostrophic approximation, the inversion proceeds using the approximate form f/h rather than the full potential vorticity, $(f + \zeta)/h$. Thus, in a strict sense, we do not approximate potential vorticity, because this is the evolving variable. Rather, we approximate the inversion relations from which we derive the height and velocity fields. The simplest way of all to derive the shallow water PG equations is to *begin* with the conservation of potential vorticity, and to note that at small Rossby number the expression $(\zeta + f)/h$ may be approximated by f/h. Then, noting in addition that the flow is geostrophic, (5.45) immediately emerges. *Every* approximate set of equations that we derive in this chapter may be expressed as the evolution of potential vorticity, with the other fields being obtained diagnostically from it.

5.2.2 Planetary-Geostrophic Equations for Stratified Flow

To explore the stratified system we will use the inviscid and adiabatic Boussinesq equations of motion with the hydrostatic approximation. The derivation carries through easily enough using the anelastic or pressure-coordinate equations, but as the PG equations have more oceanographic than atmospheric importance, using the incompressible equations is quite appropriate.

Simplifying the equations

The nondimensional equations we begin with are (5.28)–(5.32). As in the shallow water case we expand these in a series in the Rossby number, so that:

$$\hat{u} = \hat{u}_0 + Ro\ \hat{u}_1 + Ro^2\,\hat{u}_2 + \dots, \qquad \hat{b} = \hat{b}_0 + Ro\ \hat{b}_1 + Ro^2\,\hat{b}_2 + \dots, \tag{5.46}$$

and similarly for $\widehat{v}$, $\widehat{w}$ and $\widehat{\phi}$. Substituting into the nondimensional equations of motion (on page 176) and equating powers of Ro gives the lowest-order momentum, hydrostatic, and mass conservation equations:

$$\widehat{\boldsymbol{f}} \times \widehat{\boldsymbol{u}}_0 = -\nabla\widehat{\phi}_0, \qquad \frac{\partial \widehat{\phi}_0}{\partial \widehat{z}} = \widehat{b}_0, \qquad \nabla \cdot \widehat{\boldsymbol{v}}_0 = 0. \tag{5.47a,b,c}$$

If we also assume that $L_d/L = \mathcal{O}(1)$, then the thermodynamic equation (5.32) becomes

$$\left(\frac{L_d}{L}\right)^2 \widehat{N}^2 \widehat{w}_0 = 0. \tag{5.48}$$

Of course we have neglected any diabatic terms in this equation, which would in general provide a non-zero right-hand side. Nevertheless, this is not a useful equation, because the set of the equations we have derived, (5.47) and (5.48), can no longer evolve: all the time derivatives have been scaled away! Thus, although instructive, these equations are not very useful. If instead we assume that the scale of motion is much larger than the deformation scale then the other terms in the thermodynamic equation will become equally important. Thus, we suppose that $L_d^2 \ll L^2$ or, more formally, that $L^2 = \mathcal{O}(Ro^{-1})L_d^2$, and then all the terms in the thermodynamic equation are retained. A closed set of equations is then given by (5.47) and the thermodynamic equation (5.32).

Dimensional equations

Restoring the dimensions, dropping the asymptotic subscripts, and allowing for the possibility of a source term, denoted by $S_{[b']}$, in the thermodynamic equation, the *planetary-geostrophic* equations of motion are:

$$\begin{gathered} \frac{\mathrm{D}b'}{\mathrm{D}t} + wN^2 = S_{[b']}, \\ \boldsymbol{f} \times \boldsymbol{u} = -\nabla\phi', \qquad \frac{\partial \phi'}{\partial z} = b', \qquad \nabla \cdot \boldsymbol{v} = 0. \end{gathered} \tag{5.49}$$

The thermodynamic equation may also be written simply as

$$\frac{\mathrm{D}b}{\mathrm{D}t} = \dot{b}, \tag{5.50}$$

where b now represents the total stratification. The relevant pressure, ϕ, is then the pressure that is in hydrostatic balance with b, so that geostrophic and hydrostatic balance are most usefully written as

$$\boldsymbol{f} \times \boldsymbol{u} = -\nabla\phi, \qquad \frac{\partial \phi}{\partial z} = b. \tag{5.51a,b}$$

Potential vorticity

Manipulation of (5.49) reveals that we can equivalently write the equations as an evolution equation for potential vorticity. Thus, the evolution equations may be written as

$$\frac{\mathrm{D}Q}{\mathrm{D}t} = \dot{Q}, \qquad Q = f\frac{\partial b}{\partial z}, \tag{5.52}$$

where $\dot{Q} = f\partial\dot{b}/\partial z$, and the inversion — i.e., the diagnosis of velocity, pressure and buoyancy — is carried out using the hydrostatic, geostrophic and mass conservation equations.

Applicability to the ocean and atmosphere

In the atmosphere a typical deformation radius NH/f is about 1000 km. The constraint that the scale of motion be significantly larger than the deformation radius is thus hard to satisfy, since one quickly runs out of room on a planet whose equator-to-pole distance is 10 000 km. Only the largest planetary waves can satisfy the planetary-geostrophic scaling in the atmosphere and we should then also write the equations in spherical coordinates. In the ocean the deformation radius is about 100 km, so there is lots of room for the planetary-geostrophic equations to hold, and much of the theory of the large-scale structure of the ocean involves these equations.

5.3 THE SHALLOW WATER QUASI-GEOSTROPHIC EQUATIONS

We now derive a set of geostrophic equations that is valid (unlike the PG equations) when the horizontal scale of motion is similar to that of the deformation radius. These equations are called the *quasi-geostrophic* equations, and are perhaps the most widely used set of equations for theoretical studies of the atmosphere and ocean. The specific assumptions we make are as follows:

(i) The Rossby number is small, so that the flow is in near-geostrophic balance.

(ii) The scale of the motion is not significantly larger than the deformation scale. Specifically, we shall require that

$$Ro\left(\frac{L}{L_d}\right)^2 = \mathcal{O}(Ro). \tag{5.53}$$

For the shallow water equations, this assumption implies, using (5.9), that the variations in fluid depth are small compared to its total depth. For the continuously stratified system it implies, using (5.23), that the variations in stratification are small compared to the background stratification.

(iii) Variations in the Coriolis parameter are small; that is, $|\beta L| \ll |f_0|$ where L is the length scale of the motion.

(iv) Time scales advectively; that is, the scaling for time is given by $T = L/U$.

The second and third of these differ from the planetary-geostrophic counterparts: we make the second assumption because we wish to explore a different parameter regime, and we then find that the third assumption is necessary to avoid the rather trivial state of $\beta v = 0$ (as we discuss more below). All of the assumptions are the same whether we consider the shallow water equations or a continuously stratified flow, and in this section we consider the former.

5.3.1 Single-layer Shallow Water Quasi-Geostrophic Equations

The algorithm is, again, to expand the variables $\widehat{u}, \widehat{v}, \widehat{\eta}$ in an asymptotic series with the Rossby number as the small parameter, substitute into the equations of motion, and derive a simpler set of equations. Thus we let

$$\widehat{u} = \widehat{u}_o + Ro\, \widehat{u}_1 + Ro^2\, \widehat{u}_2 + \dots, \qquad \widehat{v} = \widehat{v}_o + Ro\, \widehat{v}_1 + Ro^2\, \widehat{v}_2 + \dots, \tag{5.54a}$$

$$\widehat{\eta} = \widehat{\eta}_0 + Ro\, \widehat{\eta}_1 + Ro^2\, \widehat{\eta}_2 \dots. \tag{5.54b}$$

We recognize the smallness of β compared to f_0/L by letting $\beta = \widehat{\beta} U/L^2$, where $\widehat{\beta}$ is assumed to be a parameter of order unity. Then the expression $f = f_0 + \beta y$ becomes

$$\widehat{f} = f/f_0 = \widehat{f}_0 + Ro\, \widehat{\beta}\widehat{y}, \tag{5.55}$$

where $\widehat{f}_0$ is the nondimensional value of f_0; its value is unity, but it is helpful to denote it explicitly. Substitute (5.54) into the nondimensional momentum equation (5.10), and equate powers of Ro. At lowest order we obtain

$$\widehat{f}_0\widehat{u}_0 = -\frac{\partial \widehat{\eta}_0}{\partial \widehat{y}}, \qquad \widehat{f}_0\widehat{v}_0 = \frac{\partial \widehat{\eta}_0}{\partial \widehat{x}}. \tag{5.56}$$

Cross-differentiating gives

$$\nabla \cdot \widehat{\boldsymbol{u}}_0 = 0, \tag{5.57}$$

where, as always, when ∇ operates on a nondimensional variable, the derivatives are taken with respect to the nondimensional coordinates. Evidently the velocity field is divergence-free, with this property arising from the momentum equation rather than the mass conservation equation.

The mass conservation equation is also, at lowest order, $\nabla \cdot \widehat{\boldsymbol{u}}_0 = 0$, and at next order we have

$$F\frac{\partial \widehat{\eta}_0}{\partial \widehat{t}} + F\widehat{\boldsymbol{u}}_0 \cdot \nabla\widehat{\eta}_0 + \nabla \cdot \widehat{\boldsymbol{u}}_1 = 0. \tag{5.58}$$

This equation is not closed, because the evolution of the zeroth-order term involves evaluation of a first-order quantity. For closure, we go to the next order in the momentum equation,

$$\frac{\partial \widehat{\boldsymbol{u}}_0}{\partial \widehat{t}} + (\widehat{\boldsymbol{u}}_0 \cdot \nabla)\widehat{\boldsymbol{u}}_0 + \widehat{\beta}\widehat{y}\mathbf{k} \times \widehat{\boldsymbol{u}}_0 + \widehat{f}_0\mathbf{k} \times \widehat{\boldsymbol{u}}_1 = -\nabla\widehat{\eta}_1, \tag{5.59}$$

and take its curl to give the vorticity equation:

$$\frac{\partial \widehat{\zeta}_0}{\partial \widehat{t}} + (\widehat{\boldsymbol{u}}_0 \cdot \nabla)(\widehat{\zeta}_0 + \widehat{\beta}\widehat{y}) = -\widehat{f}_0\nabla \cdot \widehat{\boldsymbol{u}}_1. \tag{5.60}$$

The term on the right-hand side is the *vortex stretching* term. Only vortex stretching by the background or planetary vorticity is present, because the vortex stretching by the relative vorticity is smaller by a factor of the Rossby number. Equation (5.60) is also not closed; however, we may use (5.58) to eliminate the divergence term to give

$$\frac{\partial \widehat{\zeta}_0}{\partial \widehat{t}} + (\widehat{\boldsymbol{u}}_0 \cdot \nabla)(\widehat{\zeta}_0 + \widehat{\beta}\widehat{y}) = \widehat{f}_0\left(F\frac{\partial \widehat{\eta}_0}{\partial \widehat{t}} + F\widehat{\boldsymbol{u}}_0 \cdot \nabla\widehat{\eta}_0\right), \tag{5.61}$$

or

$$\frac{\partial}{\partial \widehat{t}}(\widehat{\zeta}_0 - \widehat{f}_0F\widehat{\eta}_0) + (\widehat{\boldsymbol{u}}_0 \cdot \nabla)(\widehat{\zeta}_0 + \widehat{\beta}\widehat{y} - F\widehat{f}_0\widehat{\eta}_0) = 0. \tag{5.62}$$

The final step is to note that the lowest-order vorticity and height fields are related through geostrophic balance, so that using (5.56) we can write

$$\widehat{u}_0 = -\frac{\partial \widehat{\psi}_0}{\partial \widehat{y}}, \qquad \widehat{v}_0 = \frac{\partial \widehat{\psi}_0}{\partial \widehat{x}}, \qquad \widehat{\zeta}_0 = \nabla^2\widehat{\psi}_0, \tag{5.63}$$

where $\widehat{\psi}_0 = \widehat{\eta}_0/\widehat{f}_0$ is the streamfunction. Equation (5.62) can thus be written as

$$\frac{\partial}{\partial \widehat{t}}(\nabla^2\widehat{\psi}_0 - \widehat{f}_0^2F\widehat{\psi}_0) + (\widehat{\boldsymbol{u}}_0 \cdot \nabla)(\widehat{\zeta}_0 + \widehat{\beta}\widehat{y} - \widehat{f}_0^2F\widehat{\psi}_0) = 0 \tag{5.64}$$

or

$$\frac{\mathrm{D}_0}{\mathrm{D}\widehat{t}}(\nabla^2\widehat{\psi}_0 + \widehat{\beta}\widehat{y} - \widehat{f}_0^2F\widehat{\psi}_0) = 0, \tag{5.65}$$

where the subscript '0' on the material derivative indicates that the lowest order velocity, the geostrophic velocity, is the advecting velocity. Restoring the dimensions, (5.65) becomes

$$\boxed{\frac{\mathrm{D}}{\mathrm{D}t}\left(\nabla^2\psi + \beta y - \frac{1}{L_d^2}\psi\right) = 0,} \tag{5.66}$$

where $\psi = (g/f_0)\eta$, $L_d^2 = gH/f_0^2$, and the advective derivative is

$$\frac{\mathrm{D}\,\cdot}{\mathrm{D}t} = \frac{\partial\,\cdot}{\partial t} + u_g\frac{\partial\,\cdot}{\partial x} + v_g\frac{\partial\,\cdot}{\partial y} = \frac{\partial\,\cdot}{\partial t} - \frac{\partial\psi}{\partial y}\frac{\partial\,\cdot}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\,\cdot}{\partial y} = \frac{\partial\,\cdot}{\partial t} + J(\psi,\cdot). \tag{5.67}$$

Another form of (5.66) is

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\zeta + \beta y - \frac{f_0}{H}\eta\right) = 0, \tag{5.68}$$

with $\zeta = (g/f_0)\nabla^2\eta$. Equations (5.66) and (5.68) are forms of the shallow water quasi-geostrophic potential vorticity equation. The quantity

$$\boxed{q \equiv \zeta + \beta y - \frac{f_0}{H}\eta = \nabla^2\psi + \beta y - \frac{1}{L_d^2}\psi} \tag{5.69}$$

is the *shallow water quasi-geostrophic potential vorticity.*

Connection to shallow water potential vorticity

The quantity q given by (5.69) is an approximation (except for dynamically unimportant constant additive and multiplicative factors) to the shallow water potential vorticity. To see the truth of this statement, begin with the expression for the shallow water potential vorticity,

$$Q = \frac{f + \zeta}{h}. \tag{5.70}$$

Now let $h = H(1 + \eta'/H)$, where η' is the perturbation of the free-surface height, and assume that η'/H is small to obtain

$$Q = \frac{f+\zeta}{H(1+\eta'/H)} \approx \frac{1}{H}(f+\zeta)\left(1 - \frac{\eta'}{H}\right) \approx \frac{1}{H}\left(f_0 + \beta y + \zeta - f_0\frac{\eta'}{H}\right). \tag{5.71}$$

Because f_0/H is a constant it has no effect in the evolution equation, and the quantity given by

$$q = \beta y + \zeta - f_0\frac{\eta'}{H} \tag{5.72}$$

is materially conserved. Using geostrophic balance we have $\zeta = \nabla^2\psi$ and $\eta' = f_0\psi/g$ so that (5.72) is identical to (5.69). Only the variation in η is important in (5.68) or (5.69).

The approximations needed to go from (5.70) to (5.72) are the same as those used in our earlier, more long-winded, derivation of the quasi-geostrophic equations. That is, we assumed that f itself is nearly constant, and that f_0 is much larger than ζ, equivalent to a low Rossby number assumption. It was also necessary to assume that $H \gg \eta'$ to enable the expansion of the height field which, using assumption *(ii)* on page 180, is equivalent to requiring that the scale of motion not be significantly larger than the deformation scale. The derivation is completed by noting that the advection of the potential vorticity should be by the geostrophic velocity alone, and we recover (5.66) or (5.68).

Two interesting limits

There are two interesting limits to the quasi-geostrophic potential vorticity equation which, taking $\beta = 0$ for simplicity, are as follows:

(i) *Motion on scales much smaller than the deformation radius.* That is, $L \ll L_d$ and thus $Bu \gg 1$ or $F \ll 1$. Then (5.66) becomes

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) = 0, \tag{5.73}$$

where $\zeta = \nabla^2 \psi$ and $J(\psi, \zeta) = \psi_x \zeta_y - \psi_y \zeta_x$. Thus, the motion obeys the two-dimensional vorticity equation. Physically, on small length scales the deviations in the height field are very small and may be neglected.

(ii) *Motion on scales much larger than the deformation radius.* Although scales are not allowed to become so large that $Ro(L/L_d)^2$ is of order unity, we may, a posteriori, still have $L \gg L_d$, whence the potential vorticity equation, (5.66), becomes

$$\frac{\partial \psi}{\partial t} + J(\psi, \psi) = 0 \qquad \text{or} \qquad \frac{\partial \eta}{\partial t} + J(\psi, \eta) = 0, \tag{5.74}$$

because $\psi = g\eta/f_0$. The Jacobian term evidently vanishes. Thus, one is left with a trivial equation that implies there is no advective evolution of the height field. There is nothing wrong with our reasoning; the mathematics has indeed pointed out a limit interesting in its uninterestingness. From a physical point of view, however, such a lack of motion is likely to be rare, because on such large scales the Coriolis parameter varies considerably, and we are led to the planetary-geostrophic equations.

In practice, often the most severe restriction of quasi-geostrophy is that variations in layer thickness are small: what does this have to do with geostrophy? If we scale η assuming geostrophic balance then $\eta \sim fUL/g$ and $\eta/H \sim Ro(L/L_d)^2$. Thus, if Ro is to remain small, η/H can only be of order one if $(L/L_d)^2 \gg 1$. That is, the height variations must occur on a large scale, or we are led to a scaling inconsistency. Put another way, *if there are order-one height variations over a length scale of less than or of the order of the deformation scale, the Rossby number will not be small.* Large height variations are allowed if the scale of motion is large, but this contingency is described by the planetary-geostrophic equations.

Another flow regime

Although perhaps of little terrestrial interest, we can imagine a regime in which the Coriolis parameter varies fully, but the scale of motion remains no larger than the deformation radius. This parameter regime is not quasi-geostrophic, but it gives an interesting result. Because $\eta'/H \sim Ro(L/L_d)^2$ deviations of the height field are at least of order Rossby number smaller than the reference height and $|\eta'| \ll H$. The dominant balance in the height equation is then

$$H\nabla \cdot \boldsymbol{u} = 0, \tag{5.75}$$

presuming that time still scales advectively. This zero horizontal divergence must remain consistent with geostrophic balance,

$$\boldsymbol{f} \times \boldsymbol{u} = -g\nabla\eta, \tag{5.76}$$

where now f is a fully variable Coriolis parameter. Taking the curl of (that is, cross-differentiating) (5.76) gives

$$\beta v + f\nabla \cdot \boldsymbol{u} = 0, \tag{5.77}$$

whence, using (5.75), $v = 0$, and the flow is purely zonal. Although not at all useful as an evolution equation, this illustrates the constraining effect that differential rotation has on meridional velocity. This effect may be the cause of the banded, highly zonal flow on some of the giant planets, and we will revisit this issue in our discussion of geostrophic turbulence.

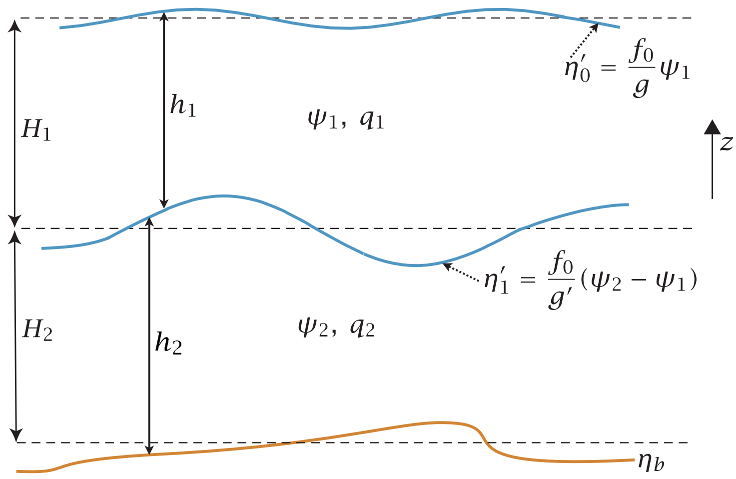

Fig. 5.1 A quasi-geostrophic fluid system consisting of two immiscible fluids of different density. The quantities η' are the interface displacements from the resting basic state, denoted with dashed lines, with η_b being the bottom topography.

5.3.2 Two-layer and Multi-layer Quasi-Geostrophic Systems

Just as for the one-layer case, the multi-layer shallow water equations simplify to a corresponding quasi-geostrophic system in appropriate circumstances. The assumptions are virtually the same as before, although we assume that the variation in the thickness of *each* layer is small compared to its mean thickness. The basic fluid system for a two-layer case is sketched in Fig. 5.1 (and see also Fig. 3.5), and for the multi-layer case in Fig. 5.2.

Let us proceed directly from the potential vorticity equation for each layer. We will also stay in dimensional variables, foregoing a strict asymptotic approach for the sake of informality and insight, and use the Boussinesq approximation. For each layer the potential vorticity equation is just

$$\frac{\mathrm{D}Q_i}{\mathrm{D}t} = 0, \qquad Q_i = \frac{\zeta_i + f}{h_i}. \tag{5.78}$$

Let $h_i = H_i + h_i'$ where $|h_i'| \ll H_i$. The potential vorticity then becomes

$$
\begin{aligned}
Q_i &\approx \frac{1}{H_i}(\zeta_i + f)\left(1 - \frac{h_i'}{H_i}\right) && \text{— variations in layer thickness are small,} && (5.79a)\\
&\approx \frac{1}{H_i}\left(f + \zeta_i - f\frac{h_i'}{H_i}\right) && \text{— the Rossby number is small,} && (5.79b)\\
&\approx \frac{1}{H_i}\left(f + \zeta_i - f_0\frac{h_i'}{H_i}\right) && \text{— variations in Coriolis parameter are small .} && (5.79c)
\end{aligned}
$$

Now, because Q appears in the equations only as an advected quantity, it is only the *variations* in the Coriolis parameter that are important in the first term on the right-hand side of (5.79c), and given this all three terms are of the same approximate magnitude. Then, because mean layer thicknesses are constant, we can define the quasi-geostrophic potential vorticity in each layer by

$$q_i = \left(\beta y + \zeta_i - f_0\frac{h_i'}{H_i}\right), \tag{5.80}$$

and this will evolve according to $\mathrm{D}q_i/\mathrm{D}t = 0$, where the advective derivative is by the geostrophic wind. As in the one-layer case, the quasi-geostrophic potential vorticity has different dimensions from the full shallow water potential vorticity.

Two-layer model

To obtain a closed set of equations we must obtain an advecting field from the potential vorticity. We use geostrophic balance to do this, and neglecting the advective derivative in (3.49) gives

$$\boldsymbol{f}_0 \times \boldsymbol{u}_1 = -g\nabla\eta_0 = -g\nabla(h_1' + h_2' + \eta_b), \tag{5.81a}$$
$$\boldsymbol{f}_0 \times \boldsymbol{u}_2 = -g\nabla\eta_0 - g'\nabla\eta_1 = -g\nabla(h_1' + h_2' + \eta_b) - g'\nabla(h_2' + \eta_b), \tag{5.81b}$$

where $g' = g(\rho_2 - \rho_1)/\rho_1$ and η_b is the height of any bottom topography, and, because variations in the Coriolis parameter are presumptively small, we use a constant value of f (i.e., f_0) on the left-hand side. For each layer there is therefore a streamfunction, given by

$$\psi_1 = \frac{g}{f_0}(h_1' + h_2' + \eta_b), \qquad \psi_2 = \frac{g}{f_0}(h_1' + h_2' + \eta_b) + \frac{g'}{f_0}(h_2' + \eta_b), \tag{5.82a,b}$$

and these two equations may be manipulated to give

$$h_1' = \frac{f_0}{g'}(\psi_1 - \psi_2) + \frac{f_0}{g}\psi_1, \qquad h_2' = \frac{f_0}{g'}(\psi_2 - \psi_1) - \eta_b. \tag{5.83a,b}$$

We note as an aside that the interface displacements are given by

$$\eta_0' = \frac{f_0}{g}\psi_1, \qquad \eta_1' = \frac{f_0}{g'}(\psi_2 - \psi_1). \tag{5.84a,b}$$

Using (5.80) and (5.83) the quasi-geostrophic potential vorticity for each layer becomes

$$\boxed{\begin{aligned} q_1 &= \beta y + \nabla^2\psi_1 + \frac{f_0^2}{g'H_1}(\psi_2 - \psi_1) - \frac{f_0^2}{gH_1}\psi_1, \\ q_2 &= \beta y + \nabla^2\psi_2 + \frac{f_0^2}{g'H_2}(\psi_1 - \psi_2) + f_0\frac{\eta_b}{H_2}. \end{aligned}} \tag{5.85a,b}$$

In the rigid-lid approximation the last term in (5.85a) is neglected. The potential vorticity in each layer is advected by the geostrophic velocity, so that the evolution equation for each layer is just

$$\frac{\partial q_i}{\partial t} + J(\psi_i, q_i) = 0, \qquad i = 1, 2. \tag{5.86}$$

Multi-layer model

A multi-layer quasi-geostrophic model may be constructed by a straightforward extension of the above two-layer procedure (see Fig. 5.2). The quasi-geostrophic potential vorticity for each layer is still given by (5.80). The pressure field in each layer can be expressed in terms of the thickness of each layer using (3.44) and (3.45), and by geostrophic balance the pressure is proportional to the streamfunction, ψ_i, for each layer. Carrying out these steps we obtain, after a little algebra, the following expression for the quasi-geostrophic potential vorticity of an interior layer, in the Boussinesq approximation:

$$\boxed{q_i = \beta y + \nabla^2\psi_i + \frac{f_0^2}{H_i}\left(\frac{\psi_{i-1} - \psi_i}{g_{i-1}'} - \frac{\psi_i - \psi_{i+1}}{g_i'}\right),} \tag{5.87}$$

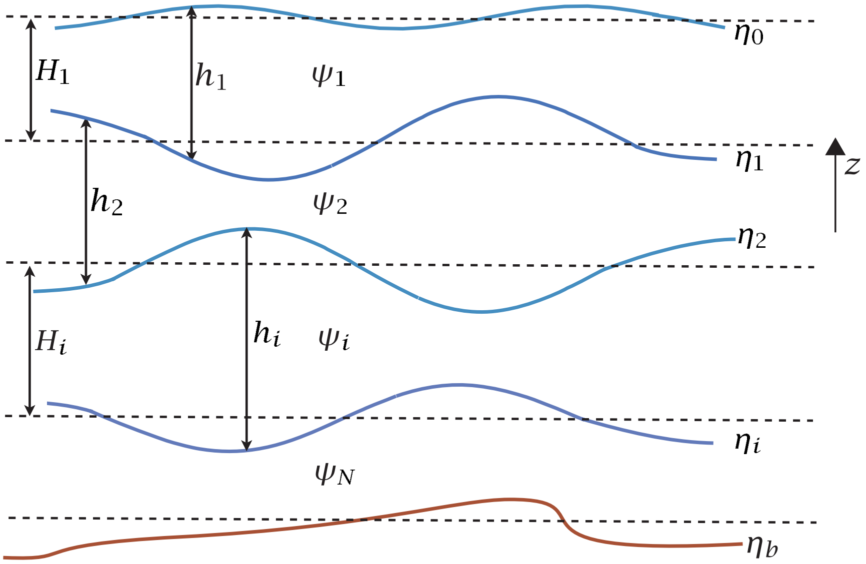

Fig. 5.2 A multi-layer quasi-geostrophic fluid system. Layers are numbered from the top down, i denotes a general interior layer and N denotes the bottom layer.

and for the top and bottom layers,

$$q_1 = \beta y + \nabla^2\psi_1 + \frac{f_0^2}{H_1}\left(\frac{\psi_2 - \psi_1}{g_1'}\right) - \frac{f_0^2}{gH_1}\psi_1, \tag{5.88a}$$

$$q_N = \beta y + \nabla^2\psi_N + \frac{f_0^2}{H_N}\left(\frac{\psi_{N-1} - \psi_N}{g_{N-1}'}\right) + \frac{f_0}{H_N}\eta_b. \tag{5.88b}$$

In these equations H_i is the basic-state thickness of the ith layer, and $g_i' = g(\rho_{i+1} - \rho_i)/\rho_1$. In each layer the evolution equation is (5.86), now for $i = 1 \dots N$. The displacements of each interface are given, similarly to (5.84), by

$$\eta_0' = \frac{f_0}{g}\psi_1, \qquad \eta_i' = \frac{f_0}{g_i'}(\psi_{i+1} - \psi_i). \tag{5.89a,b}$$

5.3.3 † Non-asymptotic and Intermediate Models

The form of the derivation of the previous section suggests that we might be able to improve on the accuracy and the range of applicability of the quasi-geostrophic equations, whilst still filtering gravity waves. For example, a seemingly improved set of geostrophic evolution equations might be

$$\frac{\partial q_i}{\partial t} + \boldsymbol{u}_i \cdot \nabla q_i = 0, \tag{5.90}$$

with

$$q_i = \frac{f + \zeta_i}{h_i}, \qquad \zeta_i = \frac{\partial v_i}{\partial x} - \frac{\partial u_i}{\partial y}, \tag{5.91a,b}$$

and with the velocities given by geostrophic balance, and therefore a function of the layer depths. Thus, the vorticity, height and velocity fields may all be inverted from potential vorticity. Note that the inversion does not involve the linearization of potential vorticity about a resting state — compare (5.91a) with (5.80)] — and we might also choose to keep the full variation of the Coriolis parameter in (5.81). Thus, the model consisting of (5.90) and (5.91) contains both the planetary geostrophic and quasi-geostrophic equations. However, the informality of the derivation hides the fact that this is not an asymptotically consistent set of equations: it mixes asymptotic orders in the same equation, and good conservation properties are not assured. The set above does not, in fact, exactly conserve energy. Models that are either more accurate or more general than the quasi-geostrophic or planetary-geostrophic equations yet that still filter gravity waves are called 'intermediate models'.[4]

A model that is derived asymptotically will, in general, maintain the conservation properties of the original set. To see this, albeit in a rather abstract way, suppose that the original equations (e.g., the primitive equations) may be written in nondimensional form, as

$$\frac{\partial \varphi}{\partial t} = F(\varphi, \epsilon), \tag{5.92}$$

where φ is a set of variables, F is some operator and ϵ is a small parameter, such as the Rossby number. Suppose also that this set of equations has various invariants (such as energy and potential vorticity) that hold for any value of ϵ. The asymptotically derived lowest-order model (such as quasi-geostrophy) is simply a version of this equation set valid in the limit $\epsilon = 0$, and therefore it will preserve the invariants of the original set. These invariants may seem to have a different form in the simplified set: for example, in deriving the hydrostatic primitive equations from the Navier–Stokes equations the small parameter is the aspect ratio, and this multiplies the vertical velocity. Thus, in the limit of zero aspect ratio, and therefore in the primitive equations, the kinetic energy component of the energy invariant has contributions only from the horizontal velocity. In other cases, some invariants may be reduced to trivialities in the simplified set. On the other hand, there is nothing to preclude new invariants emerging that hold only in the limit $\epsilon = 0$, and enstrophy (considered later in this chapter) is one example.

5.4 THE CONTINUOUSLY STRATIFIED QUASI-GEOSTROPHIC SYSTEM

We now consider the quasi-geostrophic equations for the continuously stratified hydrostatic system. The primitive equations of motion are given by (5.15), and we extract the mean stratification so that the thermodynamic equation is given by (5.17). We also stay on the β-plane for simplicity. Readers who wish for a briefer, more informal derivation may peruse the box on page 193; however, it is important to realize that there is a systematic asymptotic derivation of the quasi-geostrophic equations, for it is this that ensures that the resulting equations have good conservation properties, as explained above.

5.4.1 Scaling and Assumptions

The scaling assumptions we make are just those we made for the shallow water system on page 180, with a deformation radius now given by $L_d = NH/f_0$. The nondimensionalization and scaling are initially precisely that of Section 5.1.2, and we obtain the following nondimensional equations:

$$\text{horizontal momentum:} \qquad Ro\,\frac{\mathrm{D}\hat{\boldsymbol{u}}}{\mathrm{D}\hat{t}} + \widehat{\boldsymbol{f}} \times \hat{\boldsymbol{u}} = -\nabla_z \widehat{\phi}, \tag{5.93}$$

$$\text{hydrostatic:} \qquad \frac{\partial \widehat{\phi}}{\partial \hat{z}} = \hat{b}, \tag{5.94}$$

$$\text{mass continuity:} \qquad \frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{1}{\hat{\rho}}\frac{\partial \hat{\rho}\widehat{w}}{\partial \hat{z}} = 0, \tag{5.95}$$

$$\text{thermodynamic:} \qquad Ro\,\frac{\mathrm{D}\hat{b}}{\mathrm{D}\hat{t}} + \left(\frac{L_d}{L}\right)^2 \widehat{N}^2 \widehat{w} = 0. \tag{5.96}$$

In Cartesian coordinates we may express the Coriolis parameter as

$$\boldsymbol{f} = \boldsymbol{f}_0 + \beta y \, \mathbf{k}, \tag{5.97}$$

where $\boldsymbol{f}_0 = f_0 \mathbf{k}$. The variation of the Coriolis parameter is assumed to be small (this is a key difference between the quasi-geostrophic system and the planetary-geostrophic system), and in particular we shall assume that βy is approximately the size of the relative vorticity, and so is much smaller than f_0 itself.[5] Thus,

$$\beta y \sim \frac{U}{L}, \qquad \beta \sim \frac{U}{L^2}, \tag{5.98}$$

and so we define an $\mathcal{O}(1)$ nondimensional beta parameter by

$$\widehat{\beta} = \frac{\beta L^2}{U} = \frac{\beta L}{Ro\, f_0}. \tag{5.99}$$

From this it follows that if $f = f_0 + \beta y$, the corresponding nondimensional version is

$$\widehat{f} = \widehat{f}_0 + Ro\, \widehat{\beta} \widehat{y}. \tag{5.100}$$

where $\widehat{f} = f/f_0$ and $\widehat{f}_0 = f_0/f_0 = 1$.

5.4.2 Asymptotics

We now expand the nondimensional dependent variables in an asymptotic series in Rossby number, and write

$$\widehat{\boldsymbol{u}} = \widehat{\boldsymbol{u}}_0 + Ro\, \widehat{\boldsymbol{u}}_1 + \dots, \qquad \widehat{\phi} = \widehat{\phi}_0 + Ro\, \widehat{\phi}_1 + \dots, \qquad \hat{b} = \hat{b}_0 + Ro\, \hat{b}_1 + \dots. \tag{5.101}$$

Substituting these into the equations of motion, the lowest-order momentum equation is simply geostrophic balance,

$$\widehat{\boldsymbol{f}}_0 \times \widehat{\boldsymbol{u}}_0 = -\nabla \widehat{\phi}_0, \tag{5.102}$$

with a *constant* value of the Coriolis parameter. (Here and for the rest of this chapter we drop the subscript z from the ∇ operator.) From (5.102) it is evident that

$$\nabla \cdot \widehat{\boldsymbol{u}}_0 = 0. \tag{5.103}$$

Thus, the horizontal flow is, to leading order, non-divergent; this is a consequence of geostrophic balance, and is *not* a mass conservation equation. Using (5.103) in the mass conservation equation, (5.95), gives

$$\frac{\partial}{\partial \widehat{z}} (\widetilde{\rho} \widehat{w}_0) = 0, \tag{5.104}$$

which implies that if w_0 is zero somewhere (e.g., at a solid surface) then w_0 is zero everywhere (essentially the Taylor–Proudman effect). A physical way of saying this is that the scaling estimate $W = UH/L$ is an overestimate of the size of the vertical velocity, because even though $\partial w/\partial z \approx -\nabla \cdot \boldsymbol{u}$, the horizontal divergence of the geostrophic flow is small if f is nearly constant and $|\nabla \cdot \boldsymbol{u}| \ll U/L$. We might have anticipated this from the outset, and scaled w differently, perhaps using the geostrophic vorticity balance estimate, $w \sim \beta UH/f_0 = Ro\, UH/L$, as the scaling factor for w, but there is no a-priori guarantee that this would be correct.

At next order the momentum equation is

$$\frac{\mathrm{D}_0 \widehat{\boldsymbol{u}}_0}{\mathrm{D}\hat{t}} + \widehat{\beta} \widehat{y} \mathbf{k} \times \widehat{\boldsymbol{u}}_0 + \widehat{\boldsymbol{f}} \times \widehat{\boldsymbol{u}}_1 = -\nabla \widehat{\phi}_1, \tag{5.105}$$

where $D_0/Dt = \partial/\partial\hat{t} + (\hat{\boldsymbol{u}}_0 \cdot \nabla)$, and the next order mass conservation equation is

$$\nabla_z \cdot (\widetilde{\rho}\hat{\boldsymbol{u}}_1) + \frac{\partial}{\partial z}(\widetilde{\rho}\widehat{w}_1) = 0. \tag{5.106}$$

From (5.96), the lowest-order thermodynamic equation is just

$$\left(\frac{L_d}{L}\right)^2 \widehat{N}^2 \widehat{w}_0 = 0, \tag{5.107}$$

provided that, as we have assumed, the scales of motion are not sufficiently large that $Ro(L/L_d)^2 = \mathcal{O}(1)$. (This is a key difference between quasi-geostrophy and planetary geostrophy.) At next order we obtain an evolution equation for the buoyancy, and this is

$$\frac{\mathrm{D}_0 \hat{b}_0}{\mathrm{D}\hat{t}} + \widehat{w}_1 \widehat{N}^2 \left(\frac{L_d}{L}\right)^2 = 0. \tag{5.108}$$

The potential vorticity equation

To obtain a single evolution equation for lowest-order quantities we eliminate w_1 between the thermodynamic and momentum equations. Cross-differentiating the first-order momentum equation (5.105) gives the vorticity equation,

$$\frac{\partial \widehat{\zeta}_0}{\partial \hat{t}} + (\hat{\boldsymbol{u}}_0 \cdot \nabla)\widehat{\zeta}_0 + \widehat{v}_0 \widehat{\beta} = -\widehat{f}_0 \nabla_z \cdot \hat{u}_1. \tag{5.109}$$

(In dimensional terms, the divergence on the right-hand side is small, but is multiplied by the large term f_0, and their product is of the same order as the terms on the left-hand side.) Using the mass conservation equation (5.106), (5.109) becomes

$$\frac{\mathrm{D}_0}{\mathrm{D}\hat{t}}(\zeta_0 + \widehat{f}) = \frac{\widehat{f}_0}{\widetilde{\rho}} \frac{\partial}{\partial z}(w_1 \widetilde{\rho}). \tag{5.110}$$

Combining (5.110) and (5.108) gives

$$\frac{\mathrm{D}_0}{\mathrm{D}\hat{t}}(\zeta_0 + \widehat{f}) = -\frac{\widehat{f}_0}{\widetilde{\rho}} \frac{\partial}{\partial \widehat{z}} \left[\frac{\mathrm{D}_0}{\mathrm{D}\hat{t}}(F\widetilde{\rho}\hat{b}_0)\right], \tag{5.111}$$

where $F \equiv (L/\widehat{N}L_d)^2$. The right-hand side of this equation is

$$\frac{\partial}{\partial \widehat{z}}\left(\frac{\mathrm{D}_0 \hat{b}_0}{\mathrm{D}\hat{t}}\right) = \frac{\mathrm{D}_0}{\mathrm{D}\hat{t}}\left(\frac{\partial \hat{b}_0}{\partial \widehat{z}}\right) + \frac{\partial \hat{\boldsymbol{u}}_0}{\partial \widehat{z}} \cdot \nabla \hat{b}_0. \tag{5.112}$$

The second term on the right-hand side vanishes identically using the thermal wind equation

$$\mathbf{k} \times \frac{\partial \hat{\boldsymbol{u}}_0}{\partial \widehat{z}} = -\frac{1}{\widehat{f}_0} \nabla \hat{b}_0, \tag{5.113}$$

and so (5.111) becomes

$$\frac{\mathrm{D}_0}{\mathrm{D}\hat{t}}\left[\widehat{\zeta}_0 + \widehat{f} + \frac{\widehat{f}_0}{\widetilde{\rho}} \frac{\partial}{\partial \widehat{z}}\left(\widetilde{\rho} F \hat{b}_0\right)\right] = 0, \tag{5.114}$$

or, after using the hydrostatic equation,

$$\frac{\mathrm{D}_0}{\mathrm{D}\hat{t}}\left[\widehat{\zeta}_0 + \widehat{f} + \frac{\widehat{f}_0}{\widetilde{\rho}} \frac{\partial}{\partial \widehat{z}}\left(\widetilde{\rho} F \frac{\partial \widehat{\phi}_0}{\partial z}\right)\right] = 0. \tag{5.115}$$

Since the lowest-order horizontal velocity is divergence-free, we can define a streamfunction $\widehat{\psi}$ such that

$$\widehat{u}_0 = -\frac{\partial \widehat{\psi}}{\partial \widehat{y}}, \qquad \widehat{v}_0 = \frac{\partial \widehat{\psi}}{\partial \widehat{x}}, \tag{5.116}$$

where also, using (5.102), $\phi_0 = \widehat{f}_0 \widehat{\psi}$. The vorticity is then given by $\widehat{\zeta}_0 = \nabla^2 \widehat{\psi}$ and (5.115) becomes a single equation in a single unknown:

$$\boxed{\frac{\mathrm{D}_0}{\mathrm{D}\widehat{t}}\left[\nabla^2 \widehat{\psi} + \widehat{\beta}\widehat{y} + \frac{\widehat{f}_0^2}{\widetilde{\rho}}\frac{\partial}{\partial \widehat{z}}\left(\widetilde{\rho}F\frac{\partial \widehat{\psi}}{\partial \widehat{z}}\right)\right] = 0,} \tag{5.117}$$

where the material derivative is evaluated using $\widehat{\boldsymbol{u}}_0 = \mathbf{k} \times \nabla \widehat{\psi}$. This is the nondimensional form of the quasi-geostrophic potential vorticity equation, one of the most important equations in dynamical meteorology and oceanography. In deriving it we have reduced the Navier–Stokes equations, which are six coupled nonlinear partial differential equations in six unknowns (u, v, w, T, p, ρ) to a single (albeit nonlinear) first-order partial differential equation in a single unknown.[6]

Dimensional equations

The dimensional version of the quasi-geostrophic potential vorticity equation may be written as

$$\boxed{\frac{\mathrm{D}q}{\mathrm{D}t} = 0, \qquad q = \nabla^2\psi + f + \frac{f_0^2}{\widetilde{\rho}}\frac{\partial}{\partial z}\left(\frac{\widetilde{\rho}}{N^2}\frac{\partial \psi}{\partial z}\right),} \tag{5.118a,b}$$

where only the variable part of f (e.g., βy) is relevant in the second term on the right-hand side of the expression for q. The quantity q is known as the *quasi-geostrophic potential vorticity.* It is analogous to the exact (Ertel) potential vorticity (see Section 5.5 for more about this), and it is conserved when advected by the *horizontal* geostrophic flow. All the other dynamical variables may be obtained from potential vorticity as follows:

(i) Streamfunction, using (5.118b).

(ii) Velocity: $\boldsymbol{u} = \mathbf{k} \times \nabla\psi \; [\equiv \nabla^{\perp}\psi = -\nabla \times (\mathbf{k}\psi)]$.

(iii) Relative vorticity: $\zeta = \nabla^2\psi$.

(iv) Perturbation pressure: $\phi = f_0\psi$.

(v) Perturbation buoyancy: $b' = f_0 \partial\psi/\partial z$.

The length scale, $L_d = NH/f_0$, emerges naturally from the quasi-geostrophic dynamics. It is the scale at which buoyancy and relative vorticity effects contribute equally to the potential vorticity, and is called the *deformation radius;* it is analogous to the quantity $\sqrt{gH}/f_0$ arising in shallow water theory. In the upper ocean, with $N \approx 10^{-2}\,\mathrm{s}^{-1}$, $H \approx 10^3\,\mathrm{m}$ and $f_0 \approx 10^{-4}\,\mathrm{s}^{-1}$, then $L_d \approx 100\,\mathrm{km}$. At high latitudes the ocean is much less stratified and f is somewhat larger, and the deformation radius may be as little as 30 km (see Fig. 12.13 on page 469, where the deformation radius is defined slightly differently). In the atmosphere, with $N \approx 10^{-2}\,\mathrm{s}^{-1}$, $H \approx 10^4\,\mathrm{m}$, then $L_d \approx 1000\,\mathrm{km}$. It is this order of magnitude difference in the deformation scales that accounts for a great deal of the quantitative difference in the dynamics of the ocean and the atmosphere. If we take the limit $L_d \to \infty$ then the stratified quasi-geostrophic equations reduce to

$$\frac{\mathrm{D}q}{\mathrm{D}t} = 0, \qquad q = \nabla^2\psi + f. \tag{5.119}$$

This is the two-dimensional vorticity equation, identical to (4.67). The high stratification of this limit has suppressed all vertical motion, and variations in the flow become confined to the horizontal plane. Finally, we note that it is typical in quasi-geostrophic applications to omit the prime on the buoyancy perturbations, and write $b = f_0 \partial\psi/\partial z$; however, we will keep the prime in this chapter.

5.4.3 Buoyancy Advection at the Surface

The solution of the elliptic equation in (5.118) requires vertical boundary conditions on ψ at the ground and at the top of the atmosphere, and these are given by use of the thermodynamic equation. For a flat, slippery, rigid surface the vertical velocity is zero so that the thermodynamic equation may be written as

$$\frac{\mathrm{D}b'}{\mathrm{D}t} = 0, \qquad b' = f_0 \frac{\partial\psi}{\partial z}. \tag{5.120}$$

We apply this at the ground and at the tropopause, treating the latter as a lid on the lower atmosphere. In the presence of friction and topography the vertical velocity is not zero, but is given by

$$w = r\nabla^2\psi + \boldsymbol{u}\cdot\nabla\eta_b, \tag{5.121}$$

where the first term represents Ekman friction (with the constant r proportional to the thickness of the Ekman layer) and the second term represents topographic forcing. The boundary condition becomes

$$\frac{\partial}{\partial t}\left(f_0\frac{\partial\psi}{\partial z}\right) + \boldsymbol{u}\cdot\nabla\left(f_0\frac{\partial\psi}{\partial z} + N^2\eta_b\right) + N^2 r\nabla^2\psi = 0, \tag{5.122}$$

where all the fields are evaluated at $z = 0$ or at $z = H$, the height of the lid. Thus, the quasi-geostrophic system is characterized by the horizontal advection of potential vorticity in the interior and the advection of buoyancy at the boundary. Instead of a lid at the top, then in a compressible fluid such as the atmosphere we may suppose that all disturbances tend to zero as $z \to \infty$.

✦ *A potential vorticity sheet at the boundary*

Rather than regarding buoyancy advection as providing the boundary condition, it is sometimes useful to think of there being a very thin sheet of potential vorticity just above the ground and another just below the lid, specifically with a vertical distribution proportional to $\delta(z-\epsilon)$ or $\delta(z - H + \epsilon)$, where ϵ is small. The boundary condition (5.120) or (5.122) can be replaced by this, along with the condition that there are no variations of buoyancy at the boundary and $\partial\psi/\partial z = 0$ at $z = 0$ and $z = H$.[7]

To see this, we first note that the differential of a step function is a delta function. Thus, a discontinuity in $\partial\psi/\partial z$ at a level $z = z_1$ is equivalent to a delta function in potential vorticity there:

$$q(z_1) = \left[\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right]_{z_1-}^{z_1+}\delta(z - z_1). \tag{5.123}$$

Now, suppose that the lower boundary condition, given by (5.120), has some arbitrary distribution of buoyancy on it. We can replace this condition by the simpler condition $\partial\psi/\partial z = 0$ at $z = 0$, provided we also add to our definition of potential vorticity a term given by (5.123) with $z_1 = \epsilon$. This term is then advected by the horizontal flow, as are the other contributions. A buoyancy source at the boundary must similarly be treated as a sheet of potential vorticity source in the interior. Any flow with buoyancy variations over a horizontal boundary is thus equivalent to a flow with uniform buoyancy at the boundary, but with a spike in potential vorticity adjacent to the boundary. This approach brings notational and conceptual advantages, in that now everything is expressed in terms of potential vorticity and its advection. However, in practice there may be less to be gained, because the boundary terms must still be included in any particular calculation that is to be performed.

5.4.4 Vertical Velocity and the Omega Equation

The vertical velocity is not needed in order to evolve the quasi-geostrophic equations. However, it is not zero and a relatively simple recipe can be found that is of practical use in diagnosing the vertical velocity in weather charts. When deriving the potential vorticity equation, we eliminated vertical velocity from the vorticity equation and thermodynamic equations to give a single evolution equation. Here our approach is complementary: we begin with the same two equations, but eliminate the time derivatives. We will proceed using dimensional variables and write the vorticity and thermodynamic equations as

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) = \frac{f_0}{\widetilde{\rho}} \frac{\partial(\widetilde{\rho} w)}{\partial z} + \mathcal{Z}, \qquad \frac{\partial b}{\partial z} + J(\psi, b) + wN^2 = Q, \tag{5.124a,b}$$

where $b = f_0 \partial\psi/\partial z$ and $\zeta = \nabla^2\psi$, $\mathcal{Z}$ and Q are friction and heating terms that we can leave unspecified, and $\widetilde{\rho}$ is a reference density profile. If we take $f_0\partial/\partial z$ of the first equation and ∇^2 of the second we can eliminate time derivatives to find

$$\frac{\partial}{\partial z}\left[\frac{f_0^2}{\widetilde{\rho}} \frac{\partial(\widetilde{\rho} w)}{\partial z}\right] + N^2\nabla^2 w = f_0 \frac{\partial}{\partial z}[J(\psi, \zeta)] - \nabla^2[J(\psi, b)] - f_0 \frac{\partial \mathcal{Z}}{\partial z} + \nabla^2 Q. \tag{5.125}$$

The equation is called the *omega equation* because omega (ω) is the vertical velocity in pressure coordinates, which was where the equation first appeared. It is an elliptic equation for w, and is in fact a Poisson equation if $\widetilde{\rho}$ is a constant. It may be easily solved by numerical methods, given the state of the flow at any given time. However, there is rarely a need to solve it exactly, for there is no need to calculate w to step forward the equations. Rather, the equation finds use as an interpretive guide for meteorologists: in the thermodynamic equation both heating itself and warm advection will tend to produce vertical motion, as will the vertical differential of vorticity advection.

5.4.5 Quasi-Geostrophy in Pressure Coordinates

The derivation of the quasi-geostrophic system in pressure coordinates is very similar to that in height coordinates, with the main difference coming at the boundaries, and we give only the results. The starting point is the primitive equations in pressure coordinates, (P.1) on page 81. In pressure coordinates, the quasi-geostrophic potential vorticity is found to be

$$q = f + \nabla^2\psi + \frac{\partial}{\partial p}\left(\frac{f_0^2}{S^2}\frac{\partial \psi}{\partial p}\right), \tag{5.126}$$

where $\psi = \Phi/f_0$ is the streamfunction and Φ the geopotential, and

$$S^2 \equiv -\frac{R}{p}\left(\frac{p}{p_R}\right)^{\kappa} \frac{d\widetilde{\theta}}{dp} = -\frac{1}{\rho\theta}\frac{d\widetilde{\theta}}{dp}, \tag{5.127}$$

where $\widetilde{\theta}$ is a reference profile and a function of pressure only. In log-pressure coordinates, with $Z = -H \ln p$, the potential vorticity may be written as

$$q = f + \nabla^2\psi + \frac{1}{\rho_*}\frac{\partial}{\partial Z}\left(\frac{\rho_* f_0^2}{N_Z^2}\frac{\partial \psi}{\partial Z}\right), \tag{5.128}$$

where

$$N_Z^2 = S^2\left(\frac{p}{H}\right)^2 = -\left(\frac{R}{H}\right)\left(\frac{p}{p_R}\right)^{\kappa}\frac{d\widetilde{\theta}}{dZ} \tag{5.129}$$

Informal Derivation of Stratified QG Equations

We will use the Boussinesq equations, but very similar derivations could be given using the anelastic equations or pressure coordinates. The first ingredient is the vertical component of the vorticity equation, (4.66); in the Boussinesq version there is no baroclinic term and we have

$$\frac{\mathrm{D}_3}{\mathrm{D}t}(\zeta + f) = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial u}{\partial z}\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\frac{\partial w}{\partial x}\right). \tag{QG.1}$$

We now apply the assumptions on page 180. The advection and the vorticity on the left-hand side are geostrophic, but we keep the horizontal divergence (which is small) on the right-hand side where it is multiplied by the big term f. Furthermore, because f is nearly constant we replace it with f_0 except where it is differentiated. The second term (tilting) on the right-hand side is smaller than the advection terms on the left-hand side by the ratio $[UW/(HL)]/[U^2/L^2] = [W/H]/[U/L] \ll 1$, because w is small ($\partial w/\partial z$ equals the divergence of the ageostrophic velocity). We therefore neglect it, and given all this (QG.1) becomes

$$\frac{\mathrm{D}_g}{\mathrm{D}t}(\zeta_g + f) = -f_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = f_0\frac{\partial w}{\partial z}, \tag{QG.2}$$

where the second equality uses mass continuity and $\mathrm{D}_g/\mathrm{D}t = \partial/\partial t + \boldsymbol{u}_g \cdot \nabla$.

The second ingredient is the three-dimensional thermodynamic equation,

$$\frac{\mathrm{D}_3 b}{\mathrm{D}t} = 0. \tag{QG.3}$$

The stratification is assumed to be nearly constant, so we write $b = \tilde{b}(z) + b'(x, y, z, t)$, where $\tilde{b}$ is the basic state buoyancy. Furthermore, because w is small it only advects the basic state, and with $N^2 = \partial\tilde{b}/\partial z$ (QG.3) becomes

$$\frac{\mathrm{D}_g b'}{\mathrm{D}t} + wN^2 = 0. \tag{QG.4}$$

Hydrostatic and geostrophic wind balance enable us to write the geostrophic velocity, vorticity, and buoyancy in terms of streamfunction ψ [$= p/(f_0\rho_0)$]:

$$\boldsymbol{u}_g = \mathbf{k} \times \nabla\psi, \qquad \zeta_g = \nabla^2\psi, \qquad b' = f_0\partial\psi/\partial z. \tag{QG.5}$$

The quasi-geostrophic potential vorticity equation is obtained by eliminating w between (QG.2) and (QG.4), and this gives

$$\frac{\mathrm{D}_g q}{\mathrm{D}t} = 0, \qquad q = \zeta_g + f + \frac{\partial}{\partial z}\left(\frac{f_0 b'}{N^2}\right). \tag{QG.6}$$

This equation is the Boussinesq version of (5.118), and using (QG.5) it may be expressed entirely in terms of the streamfunction, with $\mathrm{D}_g \cdot/\mathrm{D}t = \partial/\partial t + J(\psi, \cdot)$. The vertical boundary conditions, at $z = 0$ and $z = H$ say, are given by (QG.4) with $w = 0$, with straightforward generalizations if topography or friction are present.

is the buoyancy frequency and $\rho_* = \exp(-z/H)$. Temperature and potential temperature are related to the streamfunction by

$$T = -\frac{f_0 p}{R}\frac{\partial \psi}{\partial p} = \frac{H f_0}{R}\frac{\partial \psi}{\partial Z}, \tag{5.130a}$$

$$\theta = -\left(\frac{p_R}{p}\right)^{\kappa}\left(\frac{f_0 p}{R}\right)\frac{\partial \psi}{\partial p} = \left(\frac{p_R}{p}\right)^{\kappa}\left(\frac{H f_0}{R}\right)\frac{\partial \psi}{\partial Z}. \tag{5.130b}$$

In pressure or log-pressure coordinates, potential vorticity is advected along isobaric surfaces, analogously to the horizontal advection in height coordinates.

The surface boundary condition again is derived from the thermodynamic equation. In log-pressure coordinates this is

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial \psi}{\partial Z}\right) + \frac{N_Z^2}{f_0}W = 0, \tag{5.131}$$

where $W = \mathrm{D}Z/\mathrm{D}t$. This is not the real vertical velocity, w, but it is related to it by

$$w = \frac{f_0}{g}\frac{\partial \psi}{\partial t} + \frac{RT}{gH}W. \tag{5.132}$$

Thus, choosing $H = RT(0)/g$, we have, at $Z = 0$,

$$\frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial Z} - \frac{N_Z^2}{g}\psi\right) + \boldsymbol{u}\cdot\nabla\frac{\partial \psi}{\partial Z} = -\frac{N^2}{f_0}w, \tag{5.133}$$

where

$$w = \boldsymbol{u}\cdot\nabla\eta_b + r\nabla^2\psi. \tag{5.134}$$

This differs from the expression in height coordinates only by the second term in the local time derivative. In applications where accuracy is not the main issue the simpler boundary condition $\mathrm{D}(\partial_Z\psi)/\mathrm{D}t = 0$ is sometimes used. Finally, we remark that in pressure coordinates, the equivalent to vertical velocity, $\partial p/\partial t$, is denoted ω (omega), but it need not be evaluated to solve the equations.

5.4.6 The Two-level Quasi-Geostrophic System

The quasi-geostrophic system has, in general, continuous variation in the vertical direction (and horizontal, of course). By finite-differencing the continuous equations we can obtain a *multi-level* model, and a crude but important special case of this is the *two-level* model, also known as the Phillips model.[8] To obtain the equations of motion one way to proceed is to take a crude finite difference of the continuous relation between potential vorticity and streamfunction given in (5.118b). In the Boussinesq case (or in pressure coordinates, with a slight reinterpretation of the meaning of the symbols) the continuous expression for potential vorticity is

$$q = \zeta + f + \frac{\partial}{\partial z}\left(\frac{f_0 b'}{N^2}\right), \tag{5.135}$$

where $b' = f_0\partial\psi/\partial z$. In the case with a flat bottom and rigid lid at the top (and incorporating topography is an easy extension) the boundary condition of $w = 0$ is satisfied by $\mathrm{D}\partial_z\psi/\mathrm{D}t = 0$ at the top and bottom. An obvious finite-differencing of (5.135) in the vertical direction (see Fig. 5.3) then gives

$$q_1 = \zeta_1 + f + \frac{2f_0^2}{N^2 H_1 H}(\psi_2 - \psi_1), \qquad q_2 = \zeta_2 + f + \frac{2f_0^2}{N^2 H_2 H}(\psi_1 - \psi_2). \tag{5.136}$$

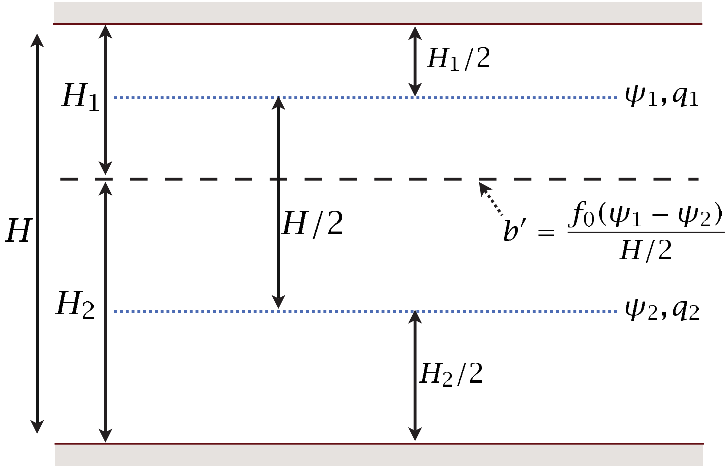

Fig. 5.3 A two-level quasi-geostrophic system with a flat bottom and rigid lid at which $w = 0$.

In atmospheric problems it is common to choose $H_1 = H_2$, whereas in oceanic problems we might choose to have a thinner upper layer, representing the flow above the main thermocline. Note that the boundary conditions of $w = 0$ at the top and bottom are already taken care of in (5.136): *they are incorporated into the definition of the potential vorticity* — a finite-difference analogue of the delta-function construction of Section 5.4.3. At each level the potential vorticity is advected by the streamfunction so that the evolution equation for each level is:

$$\frac{\mathrm{D}q_i}{\mathrm{D}t} = \frac{\partial q_i}{\partial t} + \boldsymbol{u}_i \cdot q_i = \frac{\partial q_i}{\partial t} + J(\psi_i, q_i) = 0, \qquad i = 1, 2. \tag{5.137}$$

Models with more than two levels can be constructed by extending the finite-differencing procedure in a natural way.

Connection to the layered system

The two-level expressions, (5.136), have an obvious similarity to the *two-layer* expressions, (5.85). Noting that $N^2 = \partial\hat{b}/\partial z$ and that $b = -g\delta\rho/\rho_0$ it is natural to let

$$N^2 = -\frac{g}{\rho_0}\frac{\rho_1 - \rho_2}{H/2} = \frac{g'}{H/2}. \tag{5.138}$$

With this identification we find that (5.136) becomes

$$q_1 = \zeta_1 + f + \frac{f_0^2}{g'H_1}(\psi_2 - \psi_1), \qquad q_2 = \zeta_2 + f + \frac{f_0^2}{g'H_2}(\psi_1 - \psi_2). \tag{5.139}$$

These expressions are identical to (5.85) in the flat-bottomed, rigid lid case. Similarly, a multi-layered system with n layers is equivalent to a finite-difference representation with n levels. It should be said, though, that in the pantheon of quasi-geostrophic models the two-level and two-layer models hold distinguished places.

5.5 ✦ QUASI-GEOSTROPHY AND ERTEL POTENTIAL VORTICITY

When using the shallow water equations, quasi-geostrophic theory could be naturally developed beginning with the expression for potential vorticity. Is such an approach possible for the stratified primitive equations? The answer is yes, but with complications.

5.5.1 ✦ Using Height Coordinates

Noting the general expression, (4.117), for potential vorticity in a hydrostatic fluid, the potential vorticity in the Boussinesq hydrostatic equations is given by

$$Q = \left[(v_x - u_y)b_z - v_z b_x + u_z b_y + f b_z\right], \tag{5.140}$$

where the x, y, z subscripts denote derivatives. Without approximation, we write the stratification as $b = \tilde{b}(z) + b'(x, y, z, t)$, and (5.140) becomes

$$Q = [f_0 N^2] + [(\beta y + \zeta)N^2 + f_0 b'_z] + [(\beta y + \zeta)b'_z - (v_z b'_x - u_z b'_y)], \tag{5.141}$$

where, under quasi-geostrophic scaling, the terms in square brackets are in decreasing order of size. Neglecting the third term, and taking the velocity and buoyancy fields to be in geostrophic and thermal wind balance, we can write the potential vorticity as $Q \approx \widetilde{Q} + Q'$, where $\widetilde{Q} = f_0 N^2$ and

$$Q' = (\beta y + \zeta)N^2 + f_0 b'_z = (\beta y + \nabla^2\psi)N^2 + f_0^2 \frac{\partial^2\psi}{\partial z^2}. \tag{5.142}$$

The potential vorticity evolution equation is then

$$\frac{\mathrm{D}Q'}{\mathrm{D}t} + w\frac{\partial\widetilde{Q}}{\partial z} = 0. \tag{5.143}$$

The vertical advection is important only in advecting the basic state potential vorticity $\widetilde{Q}$ and so, neglecting $w\partial Q'/\partial z$ and dividing by N^2, (5.143) becomes

$$\frac{\partial q_*}{\partial t} + \boldsymbol{u}_g \cdot \nabla q_* + \frac{w}{N^2}\frac{\partial\widetilde{Q}}{\partial z} = 0, \tag{5.144}$$

where $\widehat{q}$ is

$$q_* = (\beta y + \zeta) + \frac{f_0}{N^2} b'_z. \tag{5.145}$$

This is the approximation to the (perturbation) Ertel potential vorticity in the quasi-geostrophic limit. However, it is not the same as the expression for the quasi-geostrophic potential vorticity, (5.118b) and, furthermore, (5.144) involves a vertical advection. (Thus, we might refer to the expression in (5.118) as the 'quasi-geostrophic pseudopotential vorticity', but the prefix 'quasi-geostrophic' alone normally suffices.) We can derive (5.118) by eliminating w between (5.144) and the quasi-geostrophic thermodynamic equation $\partial b'/\partial t + \boldsymbol{u}_g \cdot \nabla b' + w\partial\tilde{b}/\partial z = 0$.

5.5.2 Using Isentropic Coordinates

An illuminating and somewhat simpler path from Ertel potential vorticity to the quasi-geostrophic equations goes by way of isentropic coordinates.[9] We begin with the isentropic expression for the Ertel potential vorticity of an ideal gas,

$$Q = \frac{f + \zeta}{\sigma}, \tag{5.146}$$

where $\sigma = -\partial p/\partial\theta$ is the thickness density (which we will just call the thickness), and in adiabatic flow the potential vorticity is advected along isopycnals. We now employ quasi-geostrophic scaling to derive an approximate equation set from this. First, assume that variations in thickness are small compared with the reference state, so that

$$\sigma = \widetilde{\sigma}(\theta) + \sigma', \qquad |\sigma'| \ll |\sigma|, \tag{5.147}$$

and similarly for pressure and density. Assuming also that the variations in the Coriolis parameter are small, then on the β–plane (5.146) becomes

$$Q \approx \left[\frac{f_0}{\widetilde{\sigma}}\right] + \left[\frac{1}{\widetilde{\sigma}}(\zeta + \beta y) - \frac{f_0}{\widetilde{\sigma}}\frac{\sigma'}{\widetilde{\sigma}}\right]. \tag{5.148}$$

We now use geostrophic and hydrostatic balance to express the terms on the right-hand side in terms of a single variable, noting that the first term does not vary along isentropic surfaces. Hydrostatic balance is

$$\frac{\partial M}{\partial \theta} = \Pi, \tag{5.149}$$

where $M = c_p T + gz$ and $\Pi = c_p(p/p_R)^\kappa$. Writing $M = \widetilde{M}(\theta) + M'$ and $\Pi = \widetilde{\Pi}(\theta) + \Pi'$, where $\widetilde{M}$ and $\widetilde{\Pi}$ are hydrostatically balanced reference profiles, we obtain

$$\frac{\partial M'}{\partial \theta} = \Pi' \approx \frac{\mathrm{d}\widetilde{\Pi}}{\mathrm{d}p}p' = \frac{1}{\theta\widetilde{\rho}}p', \tag{5.150}$$

where the last equality follows using the equation of state for an ideal gas and $\widetilde{\rho}$ is a reference profile. The perturbation thickness field may then be written as

$$\sigma' = -\frac{\partial}{\partial \theta}\left(\widetilde{\rho}\theta\frac{\partial M'}{\partial \theta}\right). \tag{5.151}$$

Geostrophic balance is $\boldsymbol{f}_0 \times \boldsymbol{u} = -\nabla_\theta M'$ where the velocity, and the horizontal derivatives, are along isentropic surfaces. This enables us to define a flow streamfunction by

$$\psi \equiv \frac{M'}{f_0}, \tag{5.152}$$

and we can then write all the variables in terms of ψ:

$$u = -\left(\frac{\partial \psi}{\partial y}\right)_\theta, \qquad v = \left(\frac{\partial \psi}{\partial x}\right)_\theta, \qquad \zeta = \nabla_\theta^2\psi, \qquad \sigma' = -f_0\frac{\partial}{\partial \theta}\left(\widetilde{\rho}\theta\frac{\partial \psi'}{\partial \theta}\right). \tag{5.153}$$

Using (5.148), (5.152) and (5.153), the quasi-geostrophic system in isentropic coordinates may be written

$$\boxed{\frac{\mathrm{D}q}{\mathrm{D}t} = 0, \qquad q = f + \nabla_\theta^2\psi + \frac{f_0^2}{\widetilde{\sigma}}\frac{\partial}{\partial \theta}\left(\widetilde{\rho}\theta\frac{\partial \psi}{\partial \theta}\right),} \tag{5.154a,b}$$

where the advection of potential vorticity is by the geostrophically balanced flow, along isentropes. The variable q is an approximation to the second term in square brackets in (5.148), multiplied by $\widetilde{\sigma}$.

Projection back to physical-space coordinates

We can recover the height or pressure coordinate quasi-geostrophic systems by projecting (5.154) on to the appropriate coordinate. This is straightforward because, by assumption, the isentropes in a quasi-geostrophic system are nearly flat. Recall that, from (2.142), a transformation between vertical coordinates may be effected by

$$\left.\frac{\partial}{\partial x}\right|_\theta = \left.\frac{\partial}{\partial x}\right|_p + \left.\frac{\partial p}{\partial x}\right|_\theta \frac{\partial}{\partial p}, \tag{5.155}$$

but the second term is $\mathcal{O}(Ro)$ smaller than the first one because, under quasi-geostrophic scaling, isentropic slopes are small. Thus $\nabla_\theta^2\psi$ in (5.154b) may be replaced by $\nabla_p^2\psi$ or $\nabla_z^2\psi$. The vortex stretching term in (5.154) becomes, in pressure coordinates,

$$\frac{f_0^2}{\widetilde{\sigma}}\frac{\partial}{\partial\theta}\left(\widetilde{\rho}\theta\frac{\partial\psi}{\partial\theta}\right) \approx \frac{f_0^2}{\widetilde{\sigma}}\frac{\mathrm{d}\widetilde{p}}{\mathrm{d}\theta}\frac{\partial}{\partial p}\left(\widetilde{\rho}\theta\frac{\mathrm{d}\widetilde{p}}{\mathrm{d}\theta}\frac{\partial\psi}{\partial p}\right) = \frac{\partial}{\partial p}\left(\frac{f_0^2}{S^2}\frac{\partial\psi}{\partial p}\right), \tag{5.156}$$

where S^2 is given by (5.127). The expression for the quasi-geostrophic potential vorticity in isentropic coordinates is thus approximately equal to the quasi-geostrophic potential vorticity in pressure coordinates. This near-equality holds because the isentropic expression, (5.154b), does not contain a component proportional to the mean stratification: the second square-bracketed term on the right-hand side of (5.148) is the only dynamically relevant one, and its evolution along isentropes is mirrored by the evolution along isobaric surfaces of quasi-geostrophic potential vorticity in pressure coordinates.

5.6 ✦ ENERGETICS OF QUASI-GEOSTROPHY

If the quasi-geostrophic set of equations is to represent a real fluid system in a physically meaningful way then it should have a consistent set of energetics. In particular, the total energy should be conserved, and there should be analogues of kinetic and potential energy and conversion between the two. We now show that such energetic properties do hold, using the Boussinesq set as an example.

Let us write the governing equations as a potential vorticity equation in the interior,

$$\frac{\mathrm{D}}{\mathrm{D}t}\left[\nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right)\right] + \beta\frac{\partial\psi}{\partial x} = 0, \qquad 0 < z < 1, \tag{5.157}$$

and buoyancy advection at the boundary,

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial\psi}{\partial z}\right) = 0, \qquad z = 0, 1. \tag{5.158}$$

For lateral boundary conditions we may assume that ψ = constant, or impose periodic conditions. If we multiply (5.157) by $-\psi$ and integrate over the domain, using the boundary conditions, we easily find

$$\frac{\mathrm{d}\widehat{E}}{\mathrm{d}t} = 0, \qquad \widehat{E} = \frac{1}{2}\int_V\left[(\nabla\psi)^2 + \frac{f_0^2}{N^2}\left(\frac{\partial\psi}{\partial z}\right)^2\right]\mathrm{d}V. \tag{5.159a,b}$$

The term involving β makes no direct contribution to the energy budget. Equation (5.159) is the fundamental energy equation for quasi-geostrophic motion, and it states that in the absence of viscous or diabatic terms the total energy is conserved. The two terms in (5.159b) can be identified as the kinetic energy (KE) and available potential energy (APE) of the flow, where

$$\mathrm{KE} = \frac{1}{2}\int_V(\nabla\psi)^2\,\mathrm{d}V, \qquad \mathrm{APE} = \frac{1}{2}\int_V\frac{f_0^2}{N^2}\left(\frac{\partial\psi}{\partial z}\right)^2\mathrm{d}V. \tag{5.160a,b}$$

The available potential energy may also be written as

$$\mathrm{APE} = \frac{1}{2}\int_V\frac{H^2}{L_d^2}\left(\frac{\partial\psi}{\partial z}\right)^2\mathrm{d}V, \tag{5.161}$$

where L_d is the deformation radius NH/f_0 and we may choose H such that $z \sim H$. At some scale L the ratio of the kinetic energy to the potential energy is thus, roughly,

$$\frac{\mathrm{KE}}{\mathrm{APE}} \sim \frac{L_d^2}{L^2}. \tag{5.162}$$

For scales much larger than L_d the potential energy dominates the kinetic energy, and contrariwise.

5.6.1 Conversion Between APE and KE

Let us return to the vorticity and thermodynamic equations,

$$\frac{D\zeta}{Dt} = f\frac{\partial w}{\partial z}, \qquad \frac{Db'}{Dt} + N^2 w = 0 \tag{5.163a,b}$$

where $\zeta = \nabla^2\psi$, and $b' = f_0 \partial\psi/\partial z$. From (5.163a) we form a kinetic energy equation, namely

$$\frac{1}{2}\frac{d}{dt}\int_V (\nabla\psi)^2 \, dV = -\int_V f_0 \frac{\partial w}{\partial z}\psi \, dV = \int_V f_0 w \frac{\partial \psi}{\partial z}\, dV. \tag{5.164}$$

From (5.163b) we form a potential energy equation, namely

$$\frac{d}{dt}\frac{1}{2}\int_V \frac{f_0^2}{N^2}\left(\frac{\partial\psi}{\partial z}\right)^2 dV = -\int_V f_0 w \frac{\partial\psi}{\partial z}\, dV. \tag{5.165}$$

Thus, the *conversion* from APE to KE is represented by

$$\frac{d}{dt}\mathrm{KE} = -\frac{d}{dt}\mathrm{APE} = \int_v f_0 w \frac{\partial\psi}{\partial z}\, dV. \tag{5.166}$$

Because the buoyancy is proportional to $\partial\psi/\partial z$, when warm fluid rises there is a correlation between w and $\partial\psi/\partial z$ and APE is converted to KE. Whether such a phenomenon occurs depends of course on the dynamics of the flow; however, such a conversion *is,* in fact, a common feature of geophysical flows, and in particular of baroclinic instability, as we shall see in Chapter 9.

5.6.2 Energetics of Two-layer Flows

Two-layer or two-level flows are an important special case. For layers of equal thickness let us write the evolution equations as

$$\frac{D}{Dt}\left[\nabla^2\psi_1 - \frac{1}{2}k_d^2(\psi_1 - \psi_2)\right] + \beta\frac{\partial\psi_1}{\partial x} = 0, \tag{5.167a}$$

$$\frac{D}{Dt}\left[\nabla^2\psi_2 + \frac{1}{2}k_d^2(\psi_1 - \psi_2)\right] + \beta\frac{\partial\psi_2}{\partial x} = 0, \tag{5.167b}$$

where $k_d^2/2 = (2f_0/NH)^2$. On multiplying these two equations by $-\psi_1$ and $-\psi_2$, respectively, and integrating over the horizontal domain, the advective term in the material derivatives and the beta term all vanish, and we obtain

$$\int_A \left[\frac{d}{dt}\frac{1}{2}(\nabla\psi_1)^2 + \frac{1}{2}k_d^2\psi_1\frac{d}{dt}(\psi_1 - \psi_2)\right] dA = 0, \tag{5.168a}$$

$$\int_A \left[\frac{d}{dt}\frac{1}{2}(\nabla\psi_2)^2 - \frac{1}{2}k_d^2\psi_2\frac{d}{dt}(\psi_1 - \psi_2)\right] dA = 0. \tag{5.168b}$$

Adding these gives

$$\frac{d}{dt}\int_A \left[\frac{1}{2}(\nabla\psi_1)^2 + \frac{1}{2}(\nabla\psi_2)^2 + \frac{k_d^2}{4}(\psi_1 - \psi_2)^2\right] dA = 0. \tag{5.169}$$

This is the energy conservation statement for the two layer model. The first two terms represent the kinetic energy and the last term represents the available potential energy.

Energy in the baroclinic and barotropic modes

A useful partitioning of the energy is between the energy in the barotropic and baroclinic modes. The barotropic streamfunction, $\overline{\psi}$, is the vertically averaged streamfunction and the baroclinic mode is the difference between the streamfunctions in the two layers. That is, for equal layer thicknesses,

$$\overline{\psi} \equiv \frac{1}{2}(\psi_1 + \psi_2), \qquad \tau \equiv \frac{1}{2}(\psi_1 - \psi_2). \tag{5.170}$$

Substituting (5.170) into (5.169) reveals that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_A \left[(\nabla\overline{\psi})^2 + (\nabla\tau)^2 + k_d^2\tau^2\right] \mathrm{d}A = 0. \tag{5.171}$$

The energy density in the barotropic mode is thus just $(\nabla\overline{\psi})^2$, and that in the baroclinic mode is $(\nabla\tau)^2 + k_d^2\tau^2$. This partitioning will prove particularly useful when we consider baroclinic turbulence in Chapter 12.

5.6.3 Enstrophy Conservation

Potential vorticity is advected only by the horizontal flow, and thus it is materially conserved on the horizontal surface at every height and

$$\frac{\mathrm{D}q}{\mathrm{D}t} = \frac{\partial q}{\partial t} + \boldsymbol{u}\cdot\nabla q = 0. \tag{5.172}$$

Furthermore, the advecting flow is divergence-free so that $\boldsymbol{u}\cdot\nabla q = \nabla\cdot(\boldsymbol{u}q)$. Thus, on multiplying (5.172) by q and integrating over a horizontal domain A we obtain

$$\frac{\mathrm{d}\widehat{Z}}{\mathrm{d}t} = 0, \qquad \widehat{Z} = \frac{1}{2}\int_A q^2\,\mathrm{d}A. \tag{5.173}$$

The result holds in an enclosed domain, with no-normal flow boundary conditions, or in a channel with periodic boundary conditions in x and no-normal flow conditions in y. The quantity $\widehat{Z}$ is known as the *enstrophy*, and it is conserved at each height as well as, naturally, over the entire volume. (In a doubly-periodic domain, only the relative enstrophy, $\int \zeta^2\,\mathrm{d}A$, is conserved.)

The enstrophy is just one of an infinity of invariants in quasi-geostrophic flow. Because the potential vorticity of a fluid element is conserved, *any* function of the potential vorticity must be a material invariant and we can immediately write

$$\frac{\mathrm{D}}{\mathrm{D}t}F(q) = 0. \tag{5.174}$$

To verify that this is true, simply note that (5.174) implies that $(\mathrm{d}F/\mathrm{d}q)\mathrm{D}q/\mathrm{D}t = 0$, which is true by virtue of (5.172). (However, by virtue of the material advection, the function $F(q)$ need not be differentiable in order for (5.174) to hold.) Each of the material invariants corresponding to different choices of $F(q)$ has a corresponding integral invariant; that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_A F(q)\,\mathrm{d}A = 0, \tag{5.175}$$

with the boundary conditions as before. The enstrophy invariant corresponds to choosing $F(q) = q^2$; it plays a particularly important role because, like energy, it is a quadratic invariant, and its presence profoundly alters the behaviour of two-dimensional and quasi-geostrophic flow compared to three-dimensional flow (see Section 11.3).

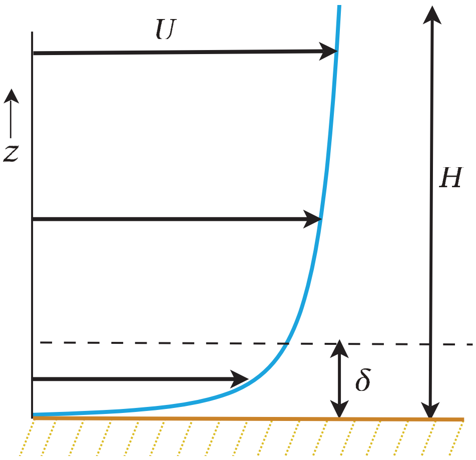

Fig. 5.4 An idealized boundary layer. The values of a field, such as velocity, U, may vary rapidly in a boundary in order to satisfy the boundary conditions at a rigid surface. The parameter δ is a measure of the boundary layer thickness, H is a typical scale of variation away from the boundary, and typically a boundary layer has $\delta \ll H$.

5.7 THE EKMAN LAYER

In the final topic of this chapter we consider the effects of friction. The fluid fields in the interior of a domain are often set by different physical processes from those occurring at a boundary, and consequently often change rapidly in a thin *boundary layer,* as in Fig. 5.4. Such boundary layers nearly always involve one or both of viscosity and diffusion, because these appear in the terms of highest differential order in the equations of motion, and so are responsible for the number and type of boundary conditions that the equations must satisfy — for example, the presence of molecular viscosity leads to the condition that the tangential flow, as well as the normal flow, must vanish at a rigid surface.

In many boundary layers in non-rotating flow the dominant balance in the momentum equation is between the advective and viscous terms. In some contrast, in large-scale atmospheric and oceanic flow the effects of rotation are large, and this results in a boundary layer, known as the *Ekman layer,* in which the dominant balance is between Coriolis and frictional or stress terms.[10] Now, the direct effects of molecular viscosity and diffusion are nearly always negligible at distances more than a few millimetres away from a solid boundary, but it is inconceivable that the entire boundary layer between the free atmosphere (or free ocean) and the surface is only a few millimetres thick. Rather, in practice a balance occurs between the Coriolis terms and the forces due to the stress generated by small-scale turbulent motion, and this gives rise to a boundary layer that has a typical depth of a few tens to several hundreds of metres. Because the stress arises from the turbulence we cannot with confidence determine its precise form; thus, we should try to determine what general properties Ekman layers may have that are *independent* of the precise form of the friction, and these properties turn out to be integral ones such as the total mass flux in the Ekman layer.

The atmospheric Ekman layer occurs near the ground, and the stress at the ground itself is due to the surface wind (and its vertical variation). In the ocean the main Ekman layer is near the surface, and the stress at the ocean surface is largely due to the presence of the overlying wind. There is also a weak Ekman layer at the bottom of the ocean, analogous to the atmospheric Ekman layer. To analyze all these layers, let us assume that:

- The Ekman layer is Boussinesq. This is a very good assumption for the ocean, and a reasonable one for the atmosphere if the boundary layer is not too deep.
- The Ekman layer has a finite depth that is less than the total depth of the fluid, this depth being given by the level at which the frictional stresses essentially vanish. Within the Ekman layer, frictional terms are important, whereas geostrophic balance holds beyond it.
- The nonlinear and time-dependent terms in the equations of motion are negligible, hydro-

static balance holds in the vertical, and buoyancy is constant, not varying in the horizontal.

- As needed, the friction can be parameterized by a viscous term of the form $\rho_0^{-1}\partial\boldsymbol{\tau}/\partial z = A\,\partial^2\boldsymbol{u}/\partial z^2$, where A is constant and $\boldsymbol{\tau}$ is the stress. (In general, stress is a tensor, τ_{ij}, with an associated force given by $F_i = \partial\tau_{ij}/\partial x_j$, summing over the repeated index. It is common in geophysical fluid dynamics that the vertical derivative dominates, and in this case the force is $\boldsymbol{F} = \partial\boldsymbol{\tau}/\partial z$. We still use the word stress for $\boldsymbol{\tau}$, but it now refers to a vector whose derivative in a particular direction (z in this case) is the force on a fluid.) In laboratory settings A may be the molecular viscosity, whereas in the atmosphere and ocean it is a so-called *eddy viscosity*. In turbulent flows momentum is transferred by the near-random motion of small parcels of fluid and, by analogy with the motion of molecules that produces a molecular viscosity, the associated stress is approximately given by using a turbulent or eddy viscosity that may be orders of magnitude larger than the molecular one.

5.7.1 Equations of Motion and Scaling

Frictional–geostrophic balance in the horizontal momentum equation is:

$$\boldsymbol{f} \times \boldsymbol{u} = -\nabla_z \phi + \frac{\partial \widetilde{\boldsymbol{\tau}}}{\partial z}, \tag{5.176}$$

where $\widetilde{\boldsymbol{\tau}} \equiv \boldsymbol{\tau}/\rho_0$ is the kinematic stress and $\boldsymbol{f} = f\mathbf{k}$, where the Coriolis parameter f is allowed to vary with latitude. If we model the stress with an eddy viscosity, (5.176) becomes

$$\boldsymbol{f} \times \boldsymbol{u} = -\nabla_z \phi + A\frac{\partial^2 \boldsymbol{u}}{\partial z^2}. \tag{5.177}$$

The vertical momentum equation is $\partial\phi/\partial z = b$, i.e., hydrostatic balance, and, because buoyancy is constant, we may without loss of generality write this as

$$\frac{\partial \phi}{\partial z} = 0. \tag{5.178}$$

The equation set is completed by the mass continuity equation, $\nabla \cdot \boldsymbol{v} = 0$.

The Ekman number

We nondimensionalize the equations by setting

$$(u, v) = U(\widehat{u}, \widehat{v}), \quad (x, y) = L(\widehat{x}, \widehat{y}), \quad f = f_0 \widehat{f}, \quad z = H\widehat{z}, \quad \phi = \Phi\widehat{\phi}, \tag{5.179}$$

where hatted variables are nondimensional. H is a scaling for the height, and at this stage we will suppose it to be some height scale in the free atmosphere or ocean, not the height of the Ekman layer itself. Geostrophic balance suggests that $\Phi = f_0 UL$. Substituting (5.179) into (5.177) we obtain

$$\widehat{\boldsymbol{f}} \times \widehat{\boldsymbol{u}} = -\widehat{\nabla}\widehat{\phi} + Ek\frac{\partial^2 \widehat{\boldsymbol{u}}}{\partial \widehat{z}^2}, \tag{5.180}$$

where the parameter

$$\boxed{Ek \equiv \left(\frac{A}{f_0 H^2}\right)}, \tag{5.181}$$

is the *Ekman number*, and it determines the importance of frictional terms in the horizontal momentum equation. If $Ek \ll 1$ then the friction is small in the flow interior where $\widehat{z} = \mathcal{O}(1)$. However, the friction term cannot necessarily be neglected in the boundary layer because it is of the highest differential order in the equation, and so determines the boundary conditions; if Ek is small the vertical scales become small and the second term on the right-hand side of (5.180) remains finite. The case when this term is simply omitted from the equation is therefore a *singular limit*, meaning that it differs from the case with $Ek \to 0$. If $Ek \geq 1$ friction is important everywhere, but it is usually the case that Ek is small for atmospheric and oceanic large-scale flow, and the interior flow is very nearly geostrophic. (In part this is because A itself is only large near a rigid surface where the presence of a shear creates turbulence and a significant eddy viscosity.)

Momentum balance in the Ekman layer

For definiteness, suppose the fluid lies above a rigid surface at $z = 0$. Sufficiently far away from the boundary the velocity field is known, and we suppose this flow to be in geostrophic balance. We then write the velocity field and the pressure field as the sum of the interior geostrophic part, plus a boundary layer correction:

$$\widehat{\boldsymbol{u}} = \widehat{\boldsymbol{u}}_g + \widehat{\boldsymbol{u}}_E, \qquad \widehat{\phi} = \widehat{\phi}_g + \widehat{\phi}_E, \tag{5.182}$$

where the Ekman layer corrections, denoted with a subscript E, are negligible away from the boundary layer. Now, in the fluid interior we have, by hydrostatic balance, $\partial\widehat{\phi}_g/\partial\widehat{z} = 0$. In the boundary layer we still have $\partial\widehat{\phi}_g/\partial\widehat{z} = 0$ so that, to satisfy hydrostasy, $\partial\widehat{\phi}_E/\partial\widehat{z} = 0$. But because $\widehat{\phi}_E$ vanishes away from the boundary we have $\widehat{\phi}_E = 0$ everywhere. Thus, *there is no boundary layer in the pressure field.* Note that this is a much stronger result than saying that pressure is continuous, which is nearly always true in fluids; rather, it is a special result for Ekman layers.

Using (5.182) with $\widehat{\phi}_E = 0$, the dimensional horizontal momentum equation (5.176) becomes, in the Ekman layer,

$$\boldsymbol{f} \times \boldsymbol{u}_E = \frac{\partial \widetilde{\boldsymbol{\tau}}}{\partial z}. \tag{5.183}$$

The dominant force balance in the Ekman layer is thus between the Coriolis force and the friction. We can determine the thickness of the Ekman layer if we model the stress with an eddy viscosity so that

$$\boldsymbol{f} \times \boldsymbol{u}_E = A\frac{\partial^2 \boldsymbol{u}_E}{\partial z^2} \qquad \text{or} \qquad \widehat{\boldsymbol{f}} \times \widehat{\boldsymbol{u}}_E = Ek\frac{\partial^2 \widehat{\boldsymbol{u}}_E}{\partial \widehat{z}^2}, \tag{5.184a,b}$$

where the second equation is nondimensional. It is evident that (5.184b) can only be satisfied if $\widehat{z} \neq \mathcal{O}(1)$, implying that H is not a proper scaling for z in the boundary layer. Rather, if the vertical scale in the Ekman layer is $\widehat{\delta}$ (meaning $\widehat{z} \sim \widehat{\delta}$) we must have $\widehat{\delta} \sim Ek^{1/2}$. In dimensional terms this means the thickness of the Ekman layer is

$$\delta = H\widehat{\delta} = HEk^{1/2} \tag{5.185}$$

or, using (5.181),

$$\boxed{\delta = \left(\frac{A}{f_0}\right)^{1/2}.} \tag{5.186}$$

This estimate also emerges directly from (5.184a). Note that (5.185) can be written as

$$Ek = \left(\frac{\delta}{H}\right)^2. \tag{5.187}$$

That is, the Ekman number is equal to the square of the ratio of the depth of the Ekman layer to an interior depth scale of the fluid motion. In laboratory flows where A is the molecular viscosity we can thus estimate the Ekman layer thickness, and if we know the eddy viscosity of the ocean or atmosphere we can estimate their respective Ekman layer thicknesses. We can invert this argument and obtain an estimate of A if we know the Ekman layer depth. In the atmosphere, deviations from geostrophic balance are very small above 1 km, and using this gives $A \approx 10^2\,\mathrm{m^2\,s^{-1}}$. In the ocean Ekman depths are often 50 m or less, and eddy viscosities are about $0.1\,\mathrm{m^2\,s^{-1}}$.

5.7.2 Integral Properties of the Ekman Layer

What can we deduce about the Ekman layer without specifying the detailed form of the frictional term? Using dimensional notation we recall frictional–geostrophic balance,

$$\boldsymbol{f} \times \boldsymbol{u} = -\nabla\phi + \frac{1}{\rho_0}\frac{\partial \boldsymbol{\tau}}{\partial z}, \tag{5.188}$$

where $\boldsymbol{\tau}$ is zero at the edge of the Ekman layer. In the Ekman layer itself we have

$$\boldsymbol{f} \times \boldsymbol{u}_E = \frac{1}{\rho_0}\frac{\partial \boldsymbol{\tau}}{\partial z}. \tag{5.189}$$

Consider either a top or bottom Ekman layer, and integrate over its thickness. From (5.189) we obtain

$$\boldsymbol{f} \times \boldsymbol{M}_E = \boldsymbol{\tau}_T - \boldsymbol{\tau}_B\,, \qquad \text{where} \qquad \boldsymbol{M}_E = \int_{Ek} \rho_0 \boldsymbol{u}_E \,\mathrm{d}z. \tag{5.190}$$

Here $\boldsymbol{M}_E$ is the ageostrophic mass transport in the Ekman layer and $\boldsymbol{\tau}_T$ and $\boldsymbol{\tau}_B$ are the respective stresses at the top and the bottom of the Ekman layer at hand. The stress at the top (bottom) will be zero in a bottom (top) Ekman layer and therefore, from (5.190),

$$\begin{aligned} &\text{top Ekman layer:} && \boldsymbol{M}_E = -\frac{1}{f}\mathbf{k} \times \boldsymbol{\tau}_T, \\ &\text{bottom Ekman layer:} && \boldsymbol{M}_E = \frac{1}{f}\mathbf{k} \times \boldsymbol{\tau}_B. \end{aligned} \tag{5.191a,b}$$

The transport is thus at right angles to the stress at the surface, and proportional to the magnitude of the stress. These properties have a simple physical explanation: integrated over the depth of the Ekman layer the surface stress must be balanced by the Coriolis force, which in turn acts at right angles to the mass transport. A consequence of (5.191) is that the mass transports in adjacent oceanic and atmospheric Ekman layers are equal and opposite, because the stress is continuous across the ocean–atmosphere interface. Equation (5.191a) is particularly useful in the ocean, where the stress at the surface is primarily due to the wind, and is largely independent of the interior oceanic flow. In the atmosphere, the surface stress mainly arises as a result of the interior atmospheric flow, and to calculate it we need to parameterize the stress in terms of the flow.

Finally, we obtain an expression for the vertical velocity induced by an Ekman layer. The mass conservation equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{5.192}$$

Integrating this over an Ekman layer gives

$$\frac{1}{\rho_0}\nabla \cdot \boldsymbol{M}_{To} = -(w_T - w_B), \tag{5.193}$$

where $\boldsymbol{M}_{To}$ is the total (Ekman plus geostrophic) mass transport in the Ekman layer,

$$\boldsymbol{M}_{To} = \int_{Ek} \rho_0 \boldsymbol{u}\,\mathrm{d}z = \int_{Ek} \rho_0(\boldsymbol{u}_g + \boldsymbol{u}_E)\,\mathrm{d}z \equiv \boldsymbol{M}_g + \boldsymbol{M}_E, \tag{5.194}$$

and w_T and w_B are the vertical velocities at the top and bottom of the Ekman layer; the former (latter) is zero in a top (bottom) Ekman layer. Equations (5.194) and (5.190) give

$$\mathbf{k} \times (\boldsymbol{M}_{To} - \boldsymbol{M}_g) = \frac{1}{f}(\boldsymbol{\tau}_T - \boldsymbol{\tau}_B). \tag{5.195}$$

Taking the curl of this (i.e., cross-differentiating) gives

$$\nabla \cdot (\boldsymbol{M}_{To} - \boldsymbol{M}_g) = \mathrm{curl}_z[(\boldsymbol{\tau}_T - \boldsymbol{\tau}_B)/f], \tag{5.196}$$

where the curl_z operator on a vector $\boldsymbol{A}$ is defined by $\mathrm{curl}_z\boldsymbol{A} \equiv \partial_x A_y - \partial_y A_x$. Using (5.193) we obtain, for top and bottom Ekman layers respectively,

$$w_B = \frac{1}{\rho_0}\left(\mathrm{curl}_z \frac{\boldsymbol{\tau}_T}{f} + \nabla \cdot \boldsymbol{M}_g\right), \qquad w_T = \frac{1}{\rho_0}\left(\mathrm{curl}_z \frac{\boldsymbol{\tau}_B}{f} - \nabla \cdot \boldsymbol{M}_g\right), \tag{5.197a,b}$$

where $\nabla \cdot \boldsymbol{M}_g = -(\beta/f)\boldsymbol{M}_g \cdot \mathbf{j}$ is the divergence of the geostrophic transport in the Ekman layer, and this is often small compared to the other terms in these equations. Thus, friction induces a vertical velocity at the edge of the Ekman layer, proportional to the curl of the stress at the surface, and this is perhaps the most used result in Ekman layer theory. Numerical models sometimes do not have the vertical resolution to explicitly resolve an Ekman layer, and (5.197) provides a means of *parameterizing* the layer in terms of resolved or known fields. This is useful for the top Ekman layer in the ocean, where the stress can be regarded as a function of the overlying wind.

5.7.3 Explicit Solutions I: a Bottom Boundary Layer

We now assume that the frictional terms can be parameterized as an eddy viscosity and calculate the explicit form of the solution in the boundary layer. The frictional–geostrophic balance may be written as

$$\boldsymbol{f} \times (\boldsymbol{u} - \boldsymbol{u}_g) = A\frac{\partial^2 \boldsymbol{u}}{\partial z^2}, \tag{5.198a}$$

where

$$f(u_g, v_g) = \left(-\frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial x}\right). \tag{5.198b}$$

We continue to assume there are no horizontal gradients of temperature, so that, via thermal wind, $\partial u_g/\partial z = \partial v_g/\partial z = 0$.

Boundary conditions and solution

Appropriate boundary conditions for a bottom Ekman layer are

$$\text{at } z = 0: \qquad u = 0, \quad v = 0 \qquad \text{(the no slip condition)} \tag{5.199a}$$

$$\text{as } z \to \infty: \qquad u = u_g, \quad v = v_g \qquad \text{(a geostrophic interior).} \tag{5.199b}$$

Let us seek solutions to (5.198a) of the form

$$u = u_g + A_0 \mathrm{e}^{\alpha z}, \qquad v = v_g + B_0 \mathrm{e}^{\alpha z}, \tag{5.200}$$

where A_0 and B_0 are constants. Substituting into (5.198a) gives two homogeneous algebraic equations

$$A_0 f - B_0 A\alpha^2 = 0, \qquad -A_0 A\alpha^2 - B_0 f = 0. \tag{5.201a,b}$$

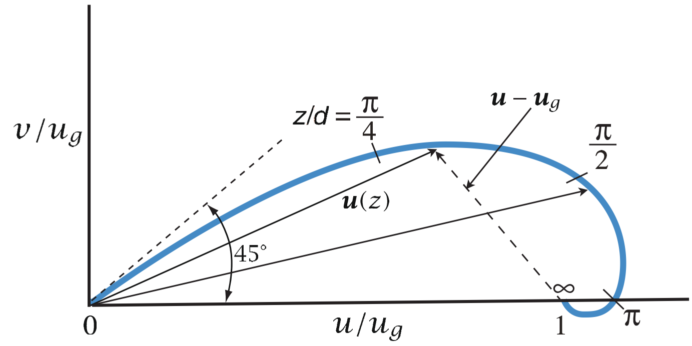

Fig. 5.5 The idealized Ekman layer solution in the lower atmosphere, plotted as a hodograph of the wind components: the arrows show the velocity vectors at particular heights, and the curve traces out the continuous variation of the velocity. The values on the curve are of the nondimensional variable z/d, where $d = (2A/f)^{1/2}$, and v_g is chosen to be zero.

For non-trivial solutions the solvability condition $\alpha^4 = -f^2/A^2$ must hold, from which we find $\alpha = \pm(1 \pm \mathrm{i})\sqrt{f/2A}$. Using the boundary conditions we then obtain the solution

$$u = u_g - \mathrm{e}^{-z/d}\left[u_g \cos(z/d) + v_g \sin(z/d)\right], \tag{5.202a}$$

$$v = v_g + \mathrm{e}^{-z/d}\left[u_g \sin(z/d) - v_g \cos(z/d)\right], \tag{5.202b}$$

where $d = \sqrt{2A/f}$ is, within a constant factor, the depth of the Ekman layer obtained from scaling considerations. The solution decays exponentially from the surface with this e-folding scale, so that d is a good measure of the Ekman layer thickness. Note that the boundary layer correction depends on the interior flow, since the boundary layer serves to bring the flow to zero at the surface.

To illustrate the solution, suppose that the pressure force is directed in the y-direction (northwards), so that the geostrophic current is eastwards. Then the solution, the now-famous *Ekman spiral*, is plotted in Figs. 5.5 and 5.6. The wind falls to zero at the surface, and its direction just above the surface is northeastwards; that is, it is rotated by 45° to the left of its direction in the free atmosphere. Although this result is independent of the value of the frictional coefficients, it is dependent on the form of the friction chosen. The force balance in the Ekman layer is between the Coriolis force, the stress, and the pressure force. At the surface the Coriolis force is zero, and the balance is entirely between the northward pressure force and the southward stress force.

Transport, force balance and vertical velocity

The cross-isobaric flow is given by (for $v_g = 0$)

$$V = \int_0^\infty v \,\mathrm{d}z = \int_0^\infty u_g \mathrm{e}^{-z/d} \sin(z/d)\,\mathrm{d}z = \frac{u_g d}{2}. \tag{5.203}$$

For positive f, this is to the left of the geostrophic flow — that is, down the pressure gradient. In the general case ($v_g \neq 0$) we obtain

$$V = \int_0^\infty (v - v_g)\,\mathrm{d}z = \frac{d}{2}(u_g - v_g). \tag{5.204}$$

Similarly, the additional zonal transport produced by frictional effects is, for $v_g = 0$,

$$U = \int_0^\infty (u - u_g)\,\mathrm{d}z = -\int_0^\infty \mathrm{e}^{-z/d} \sin(z/d)\,\mathrm{d}z = -\frac{u_g d}{2}, \tag{5.205}$$

and in the general case

$$U = \int_0^\infty (u - u_g)\,\mathrm{d}z = -\frac{d}{2}(u_g + v_g). \tag{5.206}$$

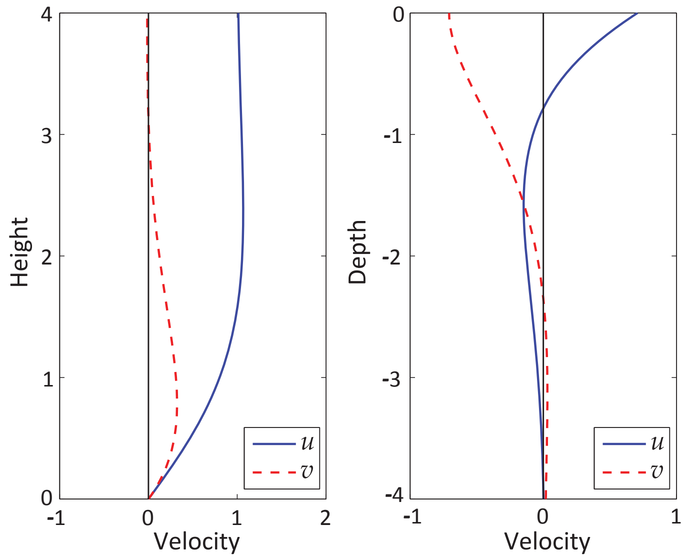

Fig. 5.6 Solutions for a bottom Ekman layer with a given flow in the fluid interior (left), and for a top Ekman layer with a given surface stress (right), both with $d = 1$. On the left we have $u_g = 1$, $v_g = 0$. On the right we have $u_g = v_g = 0$, $\tilde{\tau}_y = 0$ and $\sqrt{2}\tilde{\tau}_x/(fd) = 1$.

Thus, the total transport caused by frictional forces is

$$\boldsymbol{M}_E = \frac{\rho_0 d}{2}\left[-\mathbf{i}(u_g + v_g) + \mathbf{j}(u_g - v_g)\right]. \tag{5.207}$$

The total stress at the bottom surface $z = 0$ induced by frictional forces is

$$\tilde{\boldsymbol{\tau}}_B = A\frac{\partial \boldsymbol{u}}{\partial z}\bigg|_{z=0} = \frac{A}{d}\left[\mathbf{i}(u_g - v_g) + \mathbf{j}(u_g + v_g)\right], \tag{5.208}$$

using the solution (5.202). Thus, using (5.207), (5.208) and $d^2 = 2A/f$, we see that the total frictionally induced transport in the Ekman layer is related to the stress at the surface by $\boldsymbol{M}_E = (\mathbf{k} \times \boldsymbol{\tau}_B)/f$, reprising the result of the previous more general analysis, (5.197). From (5.208), the stress is at an angle of 45° to the left of the velocity at the surface. (However, this result is not generally true for all forms of stress.) These properties are illustrated in Fig. 5.7.

The vertical velocity at the top of the Ekman layer, w_E, is obtained using (5.207) or (5.208). If f is constant we obtain

$$w_E = -\frac{1}{\rho_0}\nabla \cdot \boldsymbol{M}_E = \frac{1}{f_0}\mathrm{curl}_z\tilde{\boldsymbol{\tau}}_B = \frac{d}{2}\zeta_g, \tag{5.209}$$

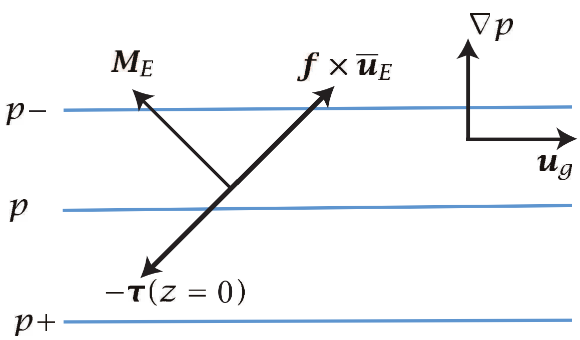

Fig. 5.7 An Ekman layer generated from an eastward geostrophic flow above with associated pressure levels as shown (blue lines). An overbar denotes a vertical integral over the Ekman layer, so that $-\boldsymbol{f} \times \overline{\boldsymbol{u}}_E$ is the Coriolis force on the vertically integrated Ekman velocity. $\boldsymbol{M}_E$ is the frictionally induced boundary layer transport, and $\boldsymbol{\tau}$ is the stress.

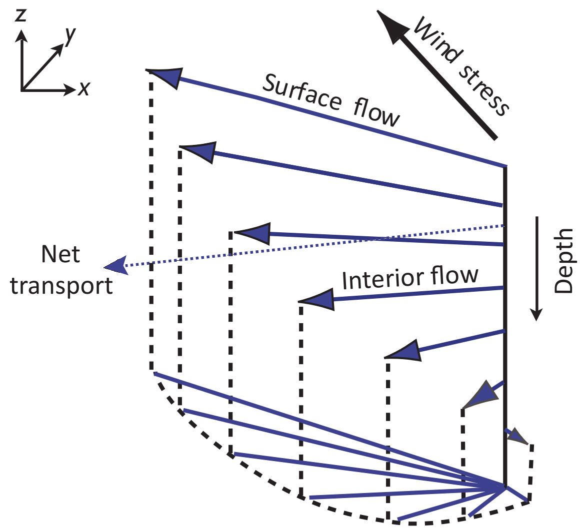

Fig. 5.8 An idealized Ekman spiral in a southern hemisphere ocean, driven by an imposed wind stress. A northern hemisphere spiral would be the reflection of this in the vertical plane. Such a clean spiral is rarely observed in the real ocean. The net transport is at right angles to the wind, independent of the detailed form of the friction. The angle of the surface flow is 45° to the wind only for a Newtonian viscosity.

where ζ_g is the vorticity of the geostrophic flow. Thus, the vertical velocity at the top of the Ekman layer, which arises because of the frictionally-induced divergence of the cross-isobaric flow in the Ekman layer, is proportional to the geostrophic vorticity in the free fluid and is proportional to the Ekman layer height $\sqrt{2A/f_0}$.

Another bottom boundary condition

In the analysis above we assumed a *no slip* condition at the surface, namely that the velocity tangential to the surface vanishes. This is formally appropriate if A is a molecular viscosity, but in a turbulent flow, where A is to be interpreted as an eddy viscosity, the flow close to the surface may be far from zero. Then, unless we wish to explicitly calculate the flow in an additional very thin viscous boundary layer the no-slip condition may be inappropriate. An alternative, slightly more general boundary condition is to suppose that the stress at the surface is given by

$$\boldsymbol{\tau} = \rho_0 C \boldsymbol{u}, \tag{5.210}$$

where C is a constant. The surface boundary condition is then

$$A\frac{\partial \boldsymbol{u}}{\partial z} = C\boldsymbol{u}. \tag{5.211}$$

If C is infinite we recover the no-slip condition. If $C = 0$, we have a condition of no stress at the surface, also known as a *free slip* condition. For intermediate values of C the boundary condition is known as a 'mixed condition'. Evaluating the solution in these cases is left as an exercise for the reader.

5.7.4 Explicit Solutions II: the Upper Ocean

Boundary conditions and solution

The wind provides a stress on the upper ocean, and the Ekman layer serves to communicate this to the oceanic interior. Appropriate boundary conditions are thus:

$$\text{at } z = 0: \qquad A\frac{\partial u}{\partial z} = \tilde{\tau}^x, \quad A\frac{\partial v}{\partial z} = \tilde{\tau}^y, \qquad \text{(a given surface stress)} \tag{5.212a}$$

$$\text{as } z \to -\infty: \qquad u = u_g, \qquad v = v_g, \qquad \text{(a geostrophic interior)} \tag{5.212b}$$

where $\tilde{\boldsymbol{\tau}}$ is the given (kinematic) wind stress at the surface. Solutions to (5.198a) with (5.212) are found by the same methods as before, and are

$$u = u_g + \frac{\sqrt{2}}{fd} \mathrm{e}^{z/d} \left[\tilde{\tau}^x \cos(z/d - \pi/4) - \tilde{\tau}^y \sin(z/d - \pi/4) \right], \tag{5.213}$$

and

$$v = v_g + \frac{\sqrt{2}}{fd} \mathrm{e}^{z/d} \left[\tilde{\tau}^x \sin(z/d - \pi/4) + \tilde{\tau}^y \cos(z/d - \pi/4) \right]. \tag{5.214}$$

Note that the boundary layer correction depends only on the imposed surface stress, and not the interior flow itself. This is a consequence of the type of boundary conditions chosen, for in the absence of an imposed stress the boundary layer correction is zero — the interior flow already satisfies the gradient boundary condition at the top surface. Similarly to the bottom boundary layer, the velocity vectors of the solution trace a diminishing spiral as they descend into the interior (see Fig. 5.8, which is drawn for the Southern Hemisphere).

Transport, surface flow and vertical velocity

The transport induced by the surface stress is obtained by integrating (5.213) and (5.214) from the surface to $-\infty$. We explicitly find

$$U = \int_{-\infty}^{0} (u - u_g)\, \mathrm{d}z = \frac{\tilde{\tau}^y}{f}, \qquad V = \int_{-\infty}^{0} (v - v_g)\, \mathrm{d}z = -\frac{\tilde{\tau}^x}{f}, \tag{5.215}$$

which indicates that the ageostrophic transport is perpendicular to the wind stress, as noted previously from more general considerations. Suppose that the surface wind is eastward; in this case $\tilde{\tau}^y = 0$ and the solutions immediately give

$$u(0) - u_g = \tilde{\tau}^x / fd, \qquad v(0) - v_g = -\tilde{\tau}^x / fd. \tag{5.216}$$

Therefore the magnitudes of the frictional flow in the x and y directions are equal to each other, and the ageostrophic flow is 45° to the right (for $f > 0$) of the wind. This result depends on the form of the frictional parameterization, but not on the size of the viscosity.

At the edge of the Ekman layer the vertical velocity is given by (5.197), and so is proportional to the curl of the wind stress. (The second term on the right-hand side of (5.197) is the vertical velocity due to the divergence of the geostrophic flow, and is usually much smaller than the first term.) The production of a vertical velocity at the edge of the Ekman layer is one of the most important effects of the layer, especially with regard to the large-scale circulation, for it provides an efficient means whereby surface fluxes are communicated to the interior flow (see Fig. 5.9).

5.7.5 † Observations of the Ekman Layer

Ekman layers — and in particular the Ekman spiral — are generally quite hard to observe, in either the ocean or atmosphere, both because of the presence of phenomena that are not included in the theory and because of the technical difficulties of actually measuring the vector velocity profile, especially in the ocean. Ekman-layer theory does not take into account the effects of stratification or of inertial and gravity waves (Section 2.10.4 and Chapter 7), nor does it account for the effects of convection or buoyancy-driven turbulence. If gravity waves are present, the instantaneous flow will be non-geostrophic and so time-averaging will be required to extract the geostrophic flow. If strong convection is present, the simple eddy-viscosity parameterizations used to derive the Ekman spiral will be rendered invalid, and the spiral Ekman profile cannot be expected to be observed in either atmosphere or ocean.

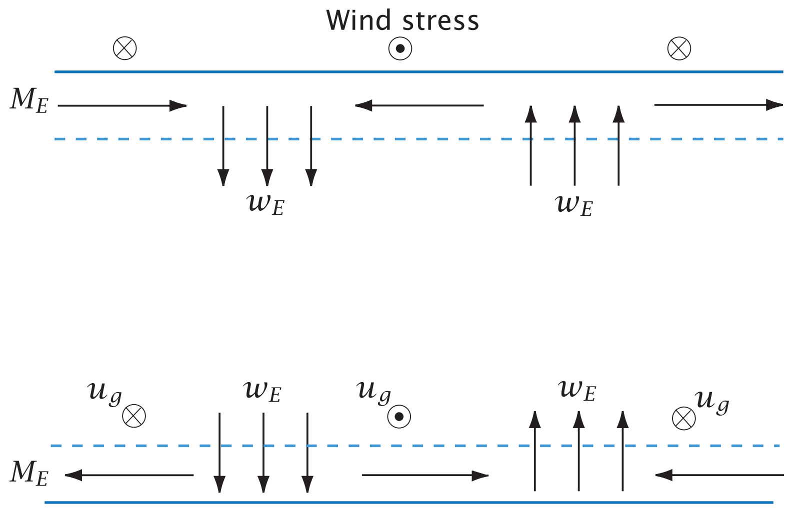

Fig. 5.9 Upper and lower Ekman layers. The upper Ekman layer in the ocean is primarily driven by an imposed wind stress, whereas the lower Ekman layer in the atmosphere or ocean largely results from the interaction of interior geostrophic velocity and a rigid lower surface. The upper part of the figure shows the vertical Ekman 'pumping' velocities that result from the given wind stress, and the lower part of the figure shows the Ekman pumping velocities given the interior geostrophic flow.

In the atmosphere, in convectively neutral cases, the Ekman profile can sometimes be qualitatively discerned. In convectively unstable situations the Ekman profile is generally not observed, but the flow is nevertheless cross-isobaric, from high pressure to low, consistent with the theory. (For most purposes, it is in any case the integral properties of the Ekman layer that is most important.) In the ocean, from about 1980 onwards improved instruments have made it possible to observe the vector current with depth, and to average that current and correlate it with the overlying wind, and a number of observations generally consistent with Ekman dynamics have emerged.[11] There are some differences between observations and theory, and these can be ascribed to the effects of stratification (which causes a shallowing and flattening of the spiral), and to the interaction of the Ekman spiral with turbulence (and the inadequacy of the eddy-diffusivity parameterization). In spite of these differences, Ekman layer theory remains a remarkable and enduring foundation of geophysical fluid dynamics.

5.7.6 ✦ Frictional Parameterization of the Ekman Layer

Suppose that the free atmosphere is described by the quasi-geostrophic vorticity equation,

$$\frac{\mathrm{D}\zeta_g}{\mathrm{D}t} = f_0 \frac{\partial w}{\partial z}, \tag{5.217}$$

where ζ_g is the geostrophic relative vorticity. Let us further model the atmosphere as a single homogeneous layer of thickness H lying above an Ekman layer of thickness $d \ll H$. If the vertical velocity is negligible at the top of the layer (at $z = H + d$) the equation of motion becomes

$$\frac{\mathrm{D}\zeta_g}{\mathrm{D}t} = \frac{f_0[w(H+d) - w(d)]}{H} = -\frac{f_0 d}{2H}\zeta_g\,, \tag{5.218}$$

using (5.209). This equation shows that the Ekman layer acts as a *linear drag* on the interior flow, with a drag coefficient r equal to $f_0 d/2H$ and with associated time scale T_{Ek} given by

$$T_{Ek} = \frac{2H}{f_0 d} = \frac{2H}{\sqrt{2 f_0 A}}. \tag{5.219}$$

In the oceanic case the corresponding vorticity equation for the interior flow is

$$\frac{D\zeta_g}{Dt} = \frac{1}{H} \mathrm{curl}_z \tau_s, \tag{5.220}$$

where τ_s is the surface stress. The surface stress thus acts as if it were a body force on the interior flow, and neither the Coriolis parameter nor the depth of the Ekman layer explicitly appear in this formula.

The Ekman layer is a very efficient way of communicating surface stresses to the interior. To see this, suppose that eddy mixing were the sole mechanism of transferring stress from the surface to the fluid interior, and there were no Ekman layer. Then the time scale of spindown of the fluid would be given by using

$$\frac{d\zeta}{dt} = A \frac{\partial^2 \zeta}{\partial z^2}, \tag{5.221}$$

implying a turbulent spin-down time, T_{turb}, of

$$T_{turb} \sim \frac{H^2}{A}, \tag{5.222}$$

where H is the depth over which we require a spin-down. This is much longer than the spin-down of a fluid that has an Ekman layer, for we have

$$\frac{T_{turb}}{T_{Ek}} = \frac{(H^2/A)}{(2H/f_0 d)} = \frac{H}{d} \gg 1, \tag{5.223}$$

using $d = \sqrt{2A/f_0}$. The effects of friction are enhanced because of the presence of a secondary circulation confined to the Ekman layers (as in Fig. 5.9) in which the vertical scales are much smaller than those in the fluid interior and so where viscous effects become significant; these frictional stresses are then communicated to the fluid interior via the induced vertical velocities at the edge of the Ekman layers.

Notes

1 The phrase 'quasi-geostrophic' seems to have been introduced by Durst & Sutcliffe (1938) and the concept used in Sutcliffe's development theory of baroclinic systems (Sutcliffe 1939, 1947). The first systematic derivation of the quasi-geostrophic equations based on scaling theory was given by Charney (1948). The planetary-geostrophic equations were used by Robinson & Stommel (1959) and Welander (1959) in studies of the thermocline (and were first known as the 'thermocline equations'), and were put in the context of other approximate equation sets by Phillips (1963).

2 Carl-Gustav Rossby (1898–1957) played a dominant role in the development of dynamical meteorology in the early and middle parts of the twentieth century, and his work permeates all aspects of dynamical meteorology today. Perhaps the most fundamental nondimensional number in rotating fluid dynamics, the Rossby number, is named for him, as is the perhaps most fundamental wave, the Rossby wave. He also discovered the conservation of potential vorticity (later generalized by Ertel) and contributed important ideas to atmospheric turbulence and the theory of air masses. Swedish born, he studied first with V. Bjerknes before taking a position in Stockholm in 1922 with

the Swedish Meteorological Hydrologic Service and receiving a 'Licentiat' from the University of Stockholm in 1925. Shortly thereafter he moved to the United States, joining the Government Weather Bureau, a precursor of NOAA's National Weather Service. In 1928 he moved to MIT, playing an important role in developing the meteorology department there, while still maintaining connections with the Weather Bureau. In 1940 he moved to the University of Chicago, where he similarly helped develop meteorology there. In 1947 he became Director of the newly formed Institute of Meteorology in Stockholm, and subsequently divided his time between there and the United States. Thus, as well as his scientific contributions, he played an influential role in the institutional development of the field.

3 Burger (1958).

4 Numerical integrations of the potential vorticity equation using (5.91), and performing the inversion without linearizing potential vorticity, do in fact indicate improved accuracy over either the quasi-geostrophic or planetary-geostrophic equations (Mundt *et al.* 1997). In a similar vein, McIntyre & Norton (2000) show how useful potential vorticity inversion can be, and Allen *et al.* (1990a,b) demonstrate the high accuracy of certain intermediate models. Certainly, asymptotic correctness should not be the only criterion used in constructing a filtered model, because the parameter range in which the model is useful may be too limited. Note that there is a difference between extending the parameter range in which a filtered model is useful, as in the inversion of (5.91), and going to higher asymptotic order accuracy in a given parameter regime, as in Allen (1993) and Warn *et al.* (1995). Using Hamiltonian mechanics it is possible to derive equations that both span different asymptotic regimes and that have good conservation properties (Salmon 1983, Allen *et al.* 2002).

5 There is a difference between the *dynamical* demands of the quasi-geostrophic system in requiring β to be small, and the *geometric* demands of the Cartesian geometry. On Earth the two demands are similar in practice. But without dynamical inconsistency we may imagine a Cartesian system in which $\beta y \sim f$, and indeed this is common in idealized, planetary geostrophic models of the large-scale ocean circulation.

6 The atmospheric and oceanic sciences are sometimes thought of as not being 'beautiful' in the same way as some branches of theoretical physics. Yet surely quasi-geostrophic theory, and the quasi-geostrophic potential vorticity equation, are quite beautiful, combining austerity of description and richness of behaviour.

7 Bretherton (1966b). Schneider *et al.* (2003) look at the non-QG extension. The equivalence between boundary conditions and delta-function sources is a common feature of elliptic and similar problems, and is analogous to the generation of electromagnetic fields by point charges. It is sometimes exploited in the numerical solution of elliptic equations, both as a simple way to include non-homogeneous boundary conditions and, using the so-called capacitance matrix method, to solve problems in irregular domains (e.g., Hockney 1970).

8 Phillips (1954, 1956) used a two-level model for instability studies and to construct a simple general circulation model of the atmosphere.

9 Charney & Stern (1962). See also Berrisford *et al.* (1993) and Vallis (1996).

10 After Ekman (1905). The problem is said to have been posed to V. W. Ekman (1874–1954), a student of Vilhelm Bjerknes, by Fridtjof Nansen, the polar explorer and statesman, who wanted to understand the motion of pack ice and of his ship, the *Fram*, embedded in the ice.

11 For oceanic observations see Davis *et al.* (1981), Price *et al.* (1987), Rudnick & Weller (1993). For the atmosphere see, e.g., Nicholls (1985).