

# Moisture Variables

## D.1 EQUIVALENT POTENTIAL TEMPERATURE

A mathematical expression for  $\theta_e$  can be derived by applying the first law of thermodynamics to a mixture of 1 kg dry air plus  $q$  kg of water vapor ( $q$ , called the *mixing ratio*, is usually expressed as grams of vapor per kilogram of dry air). If the parcel is not saturated, the dry air satisfies the energy equation

$$c_p dT - \frac{d(p - e)}{p - e} RT = 0 \quad (\text{D.1})$$

and the water vapor satisfies

$$c_{pv} dT - \frac{de}{e} \frac{R^*}{m_v} T = 0 \quad (\text{D.2})$$

where the motion is assumed to be adiabatic. Here,  $e$  is the partial pressure of the water vapor,  $c_{pv}$  is the specific heat at constant pressure for the vapor,  $R^*$  is the universal gas constant, and  $m_v$  is the molecular weight of water. If the parcel is saturated, then condensation of  $-dq_s$  kg vapor per kilogram dry air will heat the mixture of air and vapor by an amount of heat that goes into the liquid water, and the saturated parcel must satisfy the energy equation

$$c_p dT + q_s c_{pv} dT - \frac{d(p - e_s)}{p - e_s} RT - q_s \frac{de_s}{e_s} \frac{R^*}{m_v} T = -L_c dq_s \quad (\text{D.3})$$

where  $q_s$  and  $e_s$  are the saturation mixing ratio and vapor pressure, respectively. The quantity  $de_s/e_s$  may be expressed in terms of temperature using the Clausius–Clapeyron equation<sup>1</sup>

$$\frac{de_s}{dT} = \frac{m_v L_c e_s}{R^* T^2} \quad (\text{D.4})$$

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<sup>1</sup>For a derivation, see Curry and Webster (1999, p. 108).

Substituting from (D.4) into (D.3) and rearranging terms, we obtain

$$-L_c d\left(\frac{q_s}{T}\right) = c_p \frac{dT}{T} - \frac{Rd(p - e_s)}{p - e_s} + q_s c_{pv} \frac{dT}{T} \quad (\text{D.5})$$

If we now define the potential temperature of the dry air  $\theta_d$ , according to

$$c_p d \ln \theta_d = c_p d \ln T - Rd \ln (p - e_s)$$

we can rewrite (D.5) as

$$-L_c d\left(\frac{q_s}{T}\right) = c_p d \ln \theta_d + q_s c_{pv} d \ln T \quad (\text{D.6})$$

However, it may be shown that

$$dL_c/dT = c_{pv} - c_w \quad (\text{D.7})$$

where  $c_w$  is the specific heat of liquid water. Combining (D.7) and (D.6) to eliminate  $c_{pv}$  yields

$$-d\left(\frac{L_c q_s}{T}\right) = c_p d \ln \theta_d + q_s c_w d \ln T \quad (\text{D.8})$$

Neglecting the last term in (D.8), we may integrate from the original state ( $p, T, q_s, e_s, \theta_d$ ) to a state where  $q_s \rightarrow 0$ . Therefore, the equivalent potential temperature of a saturated parcel is given by

$$\theta_e = \theta_d \exp(L_c q_s / c_p T) \approx \theta \exp(L_c q_s / c_p T) \quad (\text{D.9})$$

Equation (D.9) may also be applied to an unsaturated parcel provided that the temperature used is the temperature that the parcel would have if brought to saturation by an adiabatic expansion.

## D.2 PSEUDOADIABATIC LAPSE RATE

Section 2.9.2 showed that from the first law of thermodynamics the lapse rate for a saturated parcel undergoing pseudoadiabatic ascent can be obtained from the formula

$$\frac{dT}{dz} + \frac{g}{c_p} = -\frac{L_c}{c_p} \left[ \left( \frac{\partial q_s}{\partial T} \right)_p \frac{dT}{dz} - \left( \frac{\partial q_s}{\partial p} \right)_T \rho g \right] \quad (\text{D.10})$$

Noting that  $q_s \cong \varepsilon e_s/p$ , where  $\varepsilon = 0.622$  is the ratio of the molecular weight of water to that of dry air and utilizing (D.4), we can express the partial derivatives in (D.10) as

$$\left(\frac{\partial q_s}{\partial p}\right)_T \approx -\frac{q_s}{p} \quad \text{and} \quad \left(\frac{\partial q_s}{\partial T}\right)_p \approx \frac{\varepsilon}{p} \frac{\partial e_s}{\partial T} = \frac{\varepsilon^2 L_c e_s}{p R T^2} = \frac{\varepsilon L_c q_s}{R T^2}$$

Substitution into (D.10), and noting that  $g/c_p = \Gamma_d$ , then yields the desired result:

$$\Gamma_s \equiv -\frac{dT}{dz} = \Gamma_d \frac{[1 + L_c q_s / (RT)]}{[1 + \varepsilon L_c^2 q_s / (c_p R T^2)]}$$