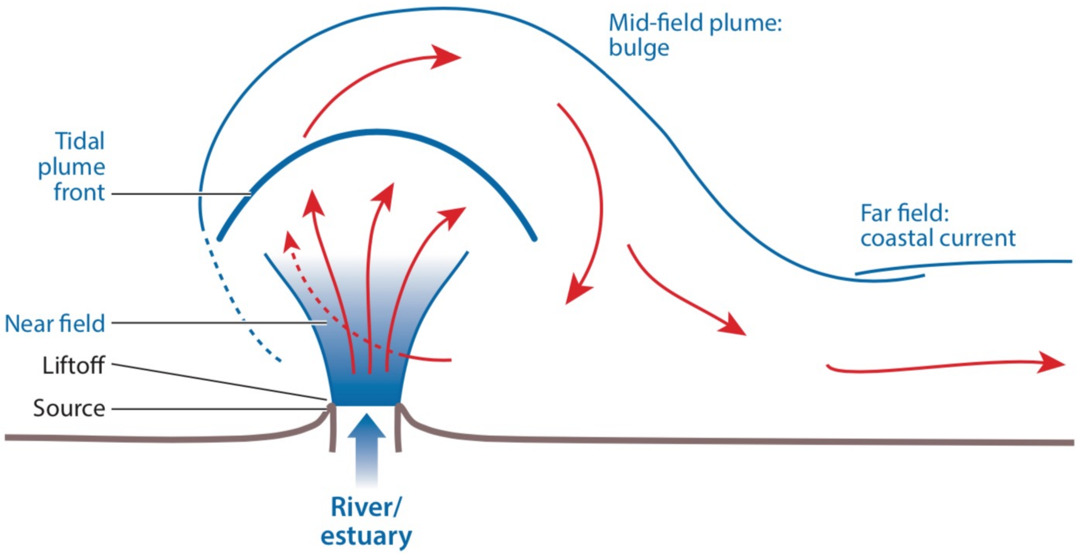
河口近海动力过程

河流冲淡水



Horner-Devine et al. (2014)

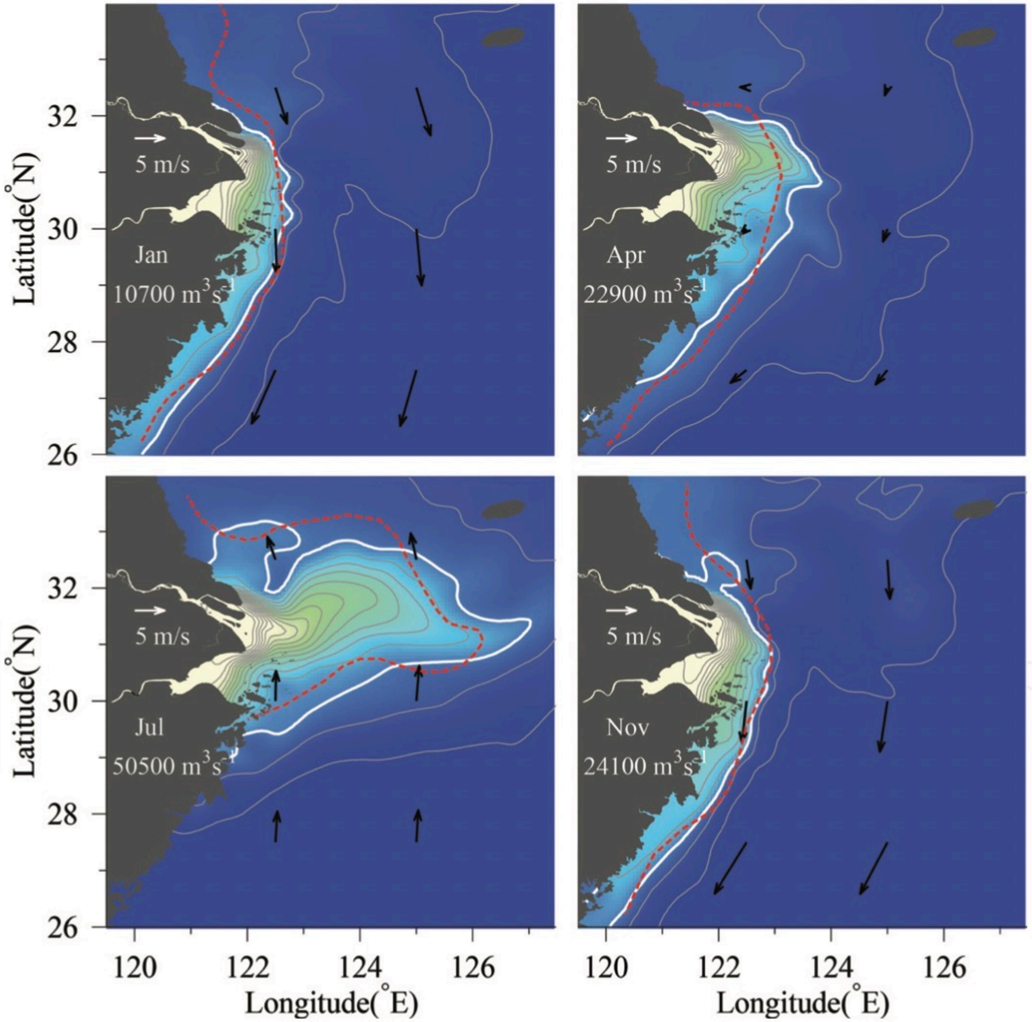
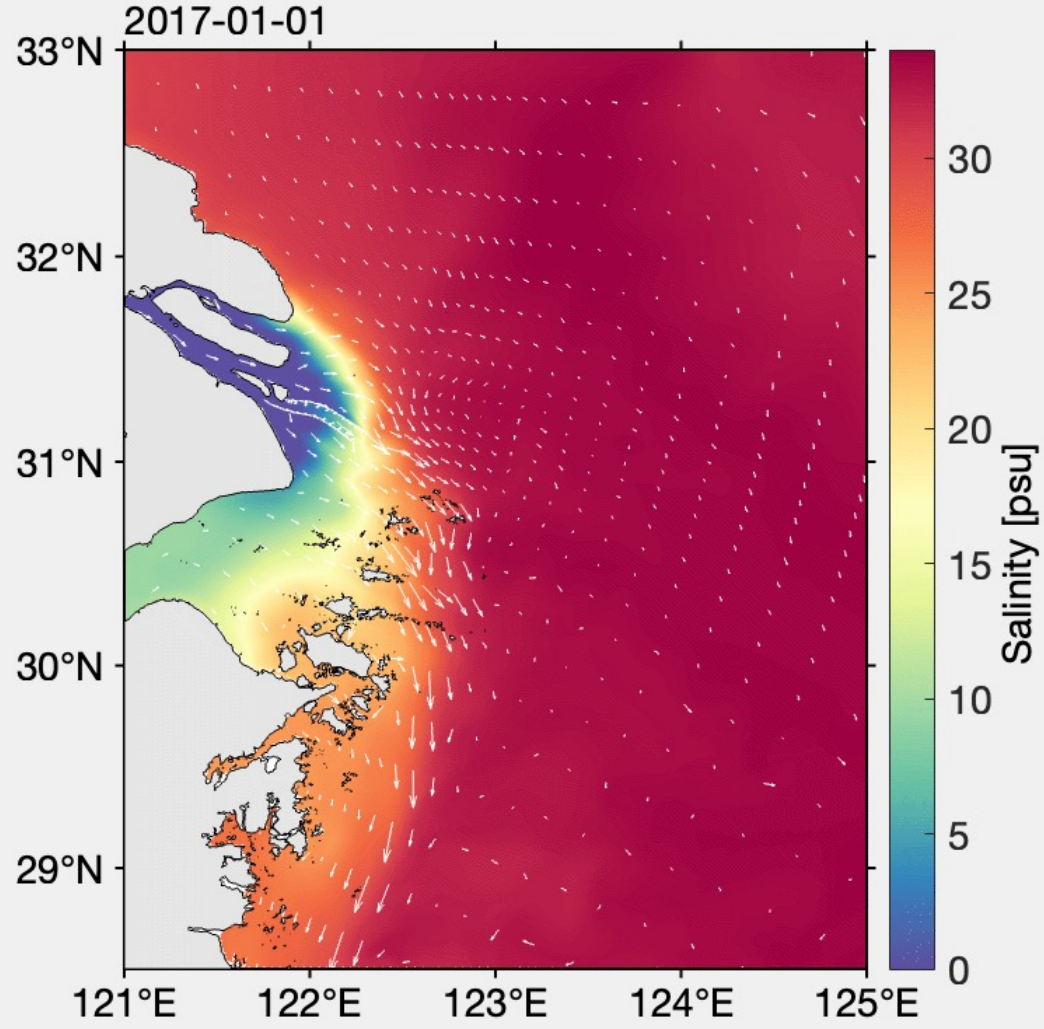
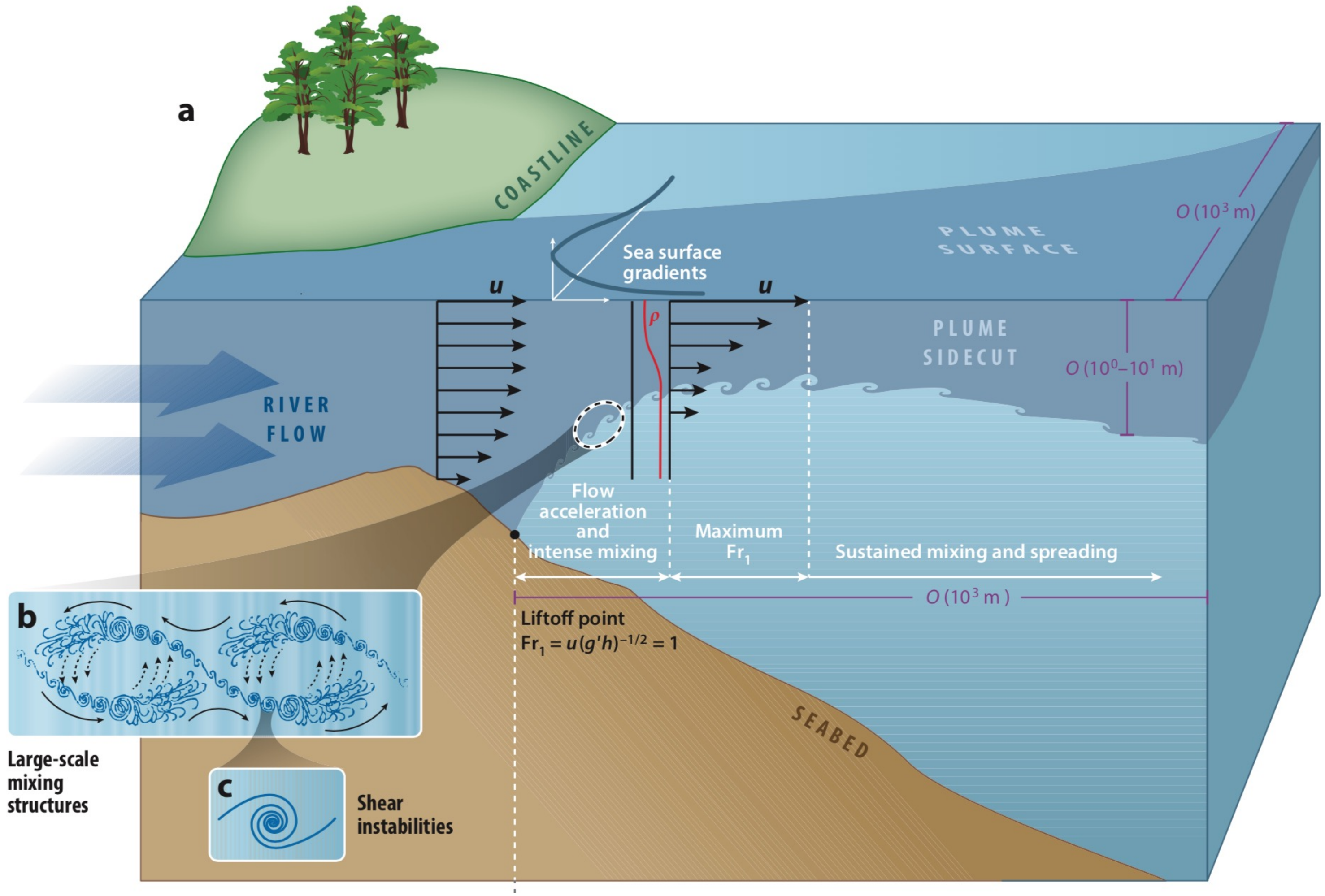


Figure 4. Sea surface salinity resulting from the climatological run in four selected months. The arrows are the monthly climatological wind. Monthly Changjiang River discharges are labeled. The contour interval of surface salinity is 2 psu, and the 30-psu contour is highlighted with thick white line. Dashed red line shows the 30-psu isohaline digitized from Editorial Board for Marine Atlas [2].

Wu et al. (2014)


2017-01-01
33°N
32°N
31°N
30°N
29°N
121°E
122°E
123°E
124°E
125°E
Salinity [psu]
0
5
10
15
20
25
30



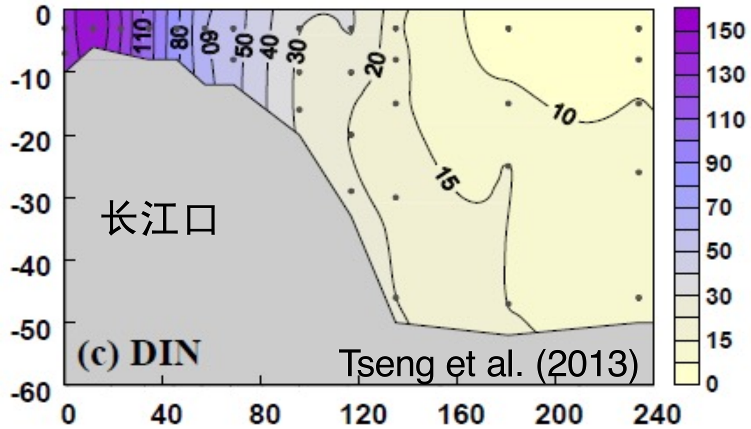
Horner-Devine et al. (2014)

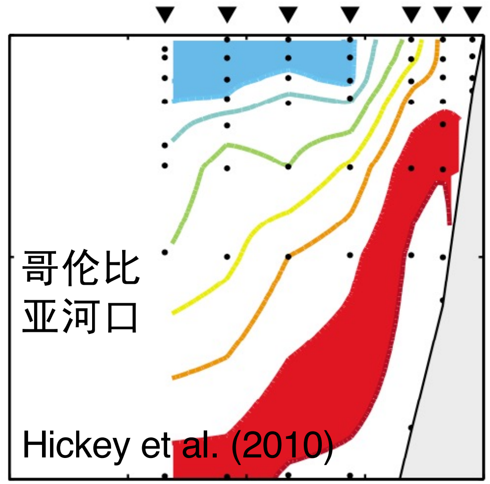
冲淡水关键生态动力学问题

- **营养盐浓度的快速下降**

  物理混合？

  生物吸收？

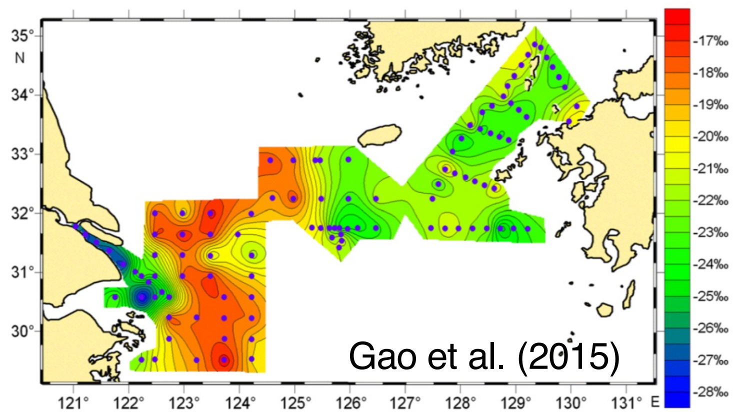




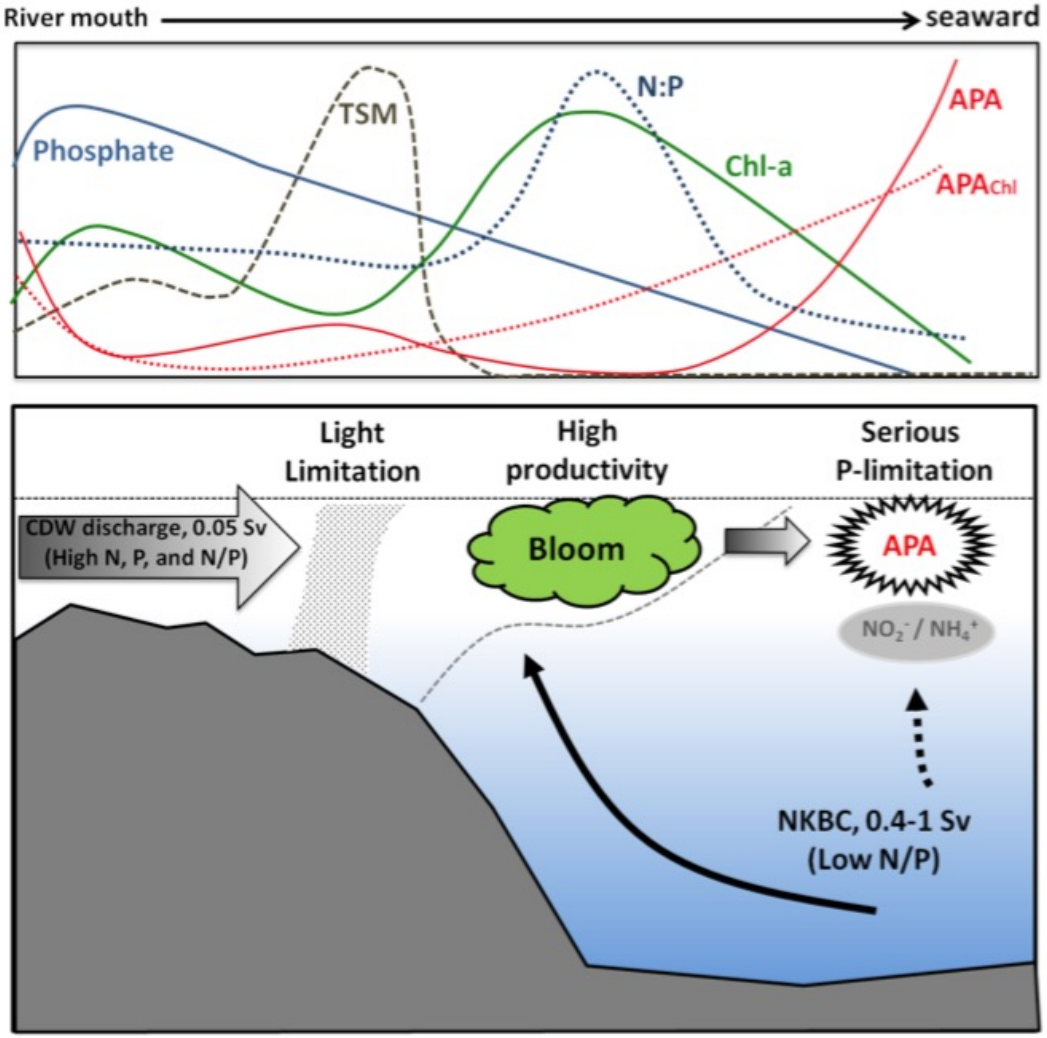
- **冲淡水外缘的藻华爆发**

  触发机制是什么？内边界由何决定？

  冲淡水动力过程在其中发挥怎样的作用？



长江口生化过程从河口到外海的变化与机制－大面站观测

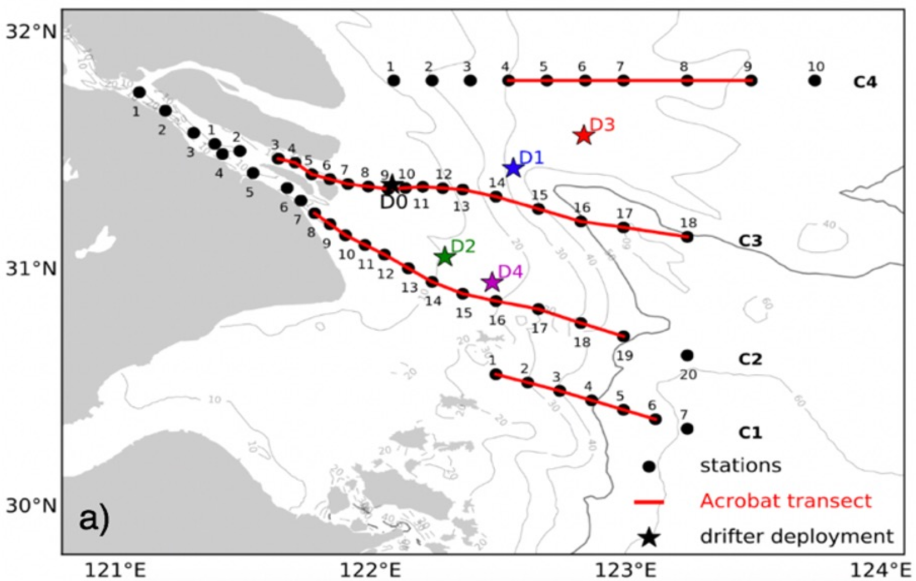


如何准确定义生态现象的边界，

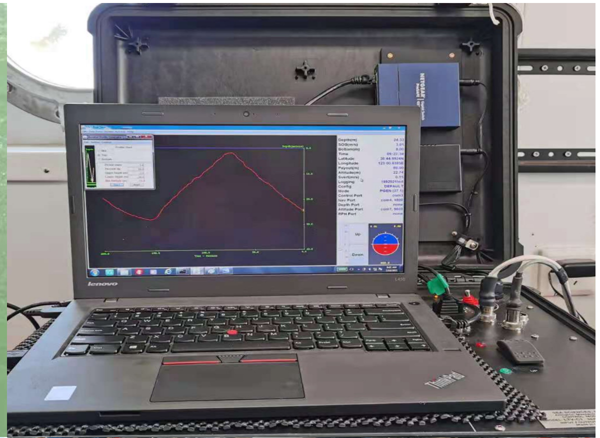
从而决定其触发机制？

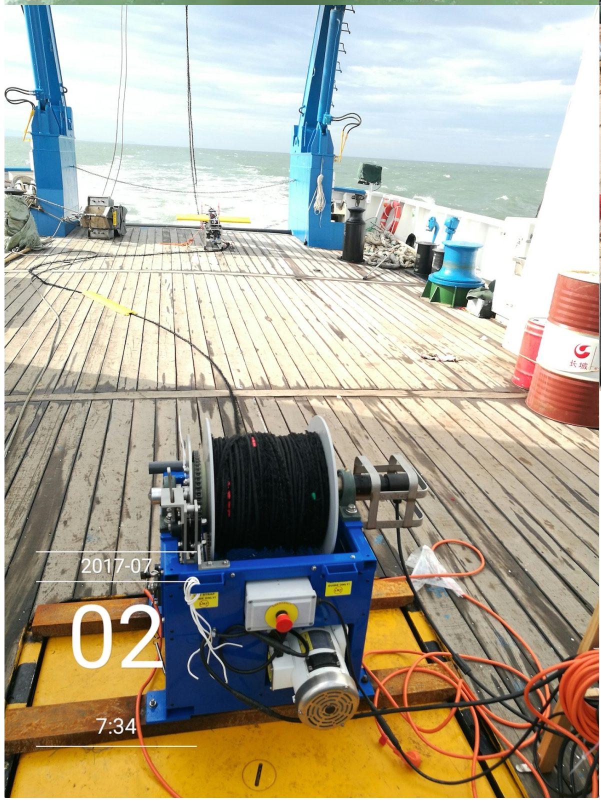
Tseng et al. (2014)

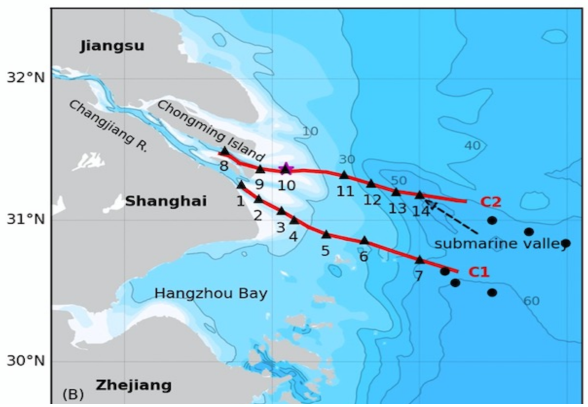
2017年夏季长江冲淡水综合调查航次



观测仪器	测量参数
近海拖曳式走航观测系统	水文、浊度、CDOM、PAR、叶绿素
表层水走航系统	溶解氧、O2/Ar（NCP）
CTD和多参数生化传感器集成系统	水文、叶绿素、CDOM、浊度、PAR、溶解氧、颗粒物等
采水器	分析营养盐、同位素、DOC、POC、 微生物、浮游生物
ADCP	流速
漂流浮标	冲淡水扩散路径







近海拖曳式走航观测系统

❑ **Acrobat (Sea Sciences)**

观测平台

❑ **AML-MVP CTD**

测量频率： 25 Hz

❑ **RBR Concerto logger**

测量频率： 12 Hz

- CDOM
- Turbitidy
- PAR
- Fluorescence

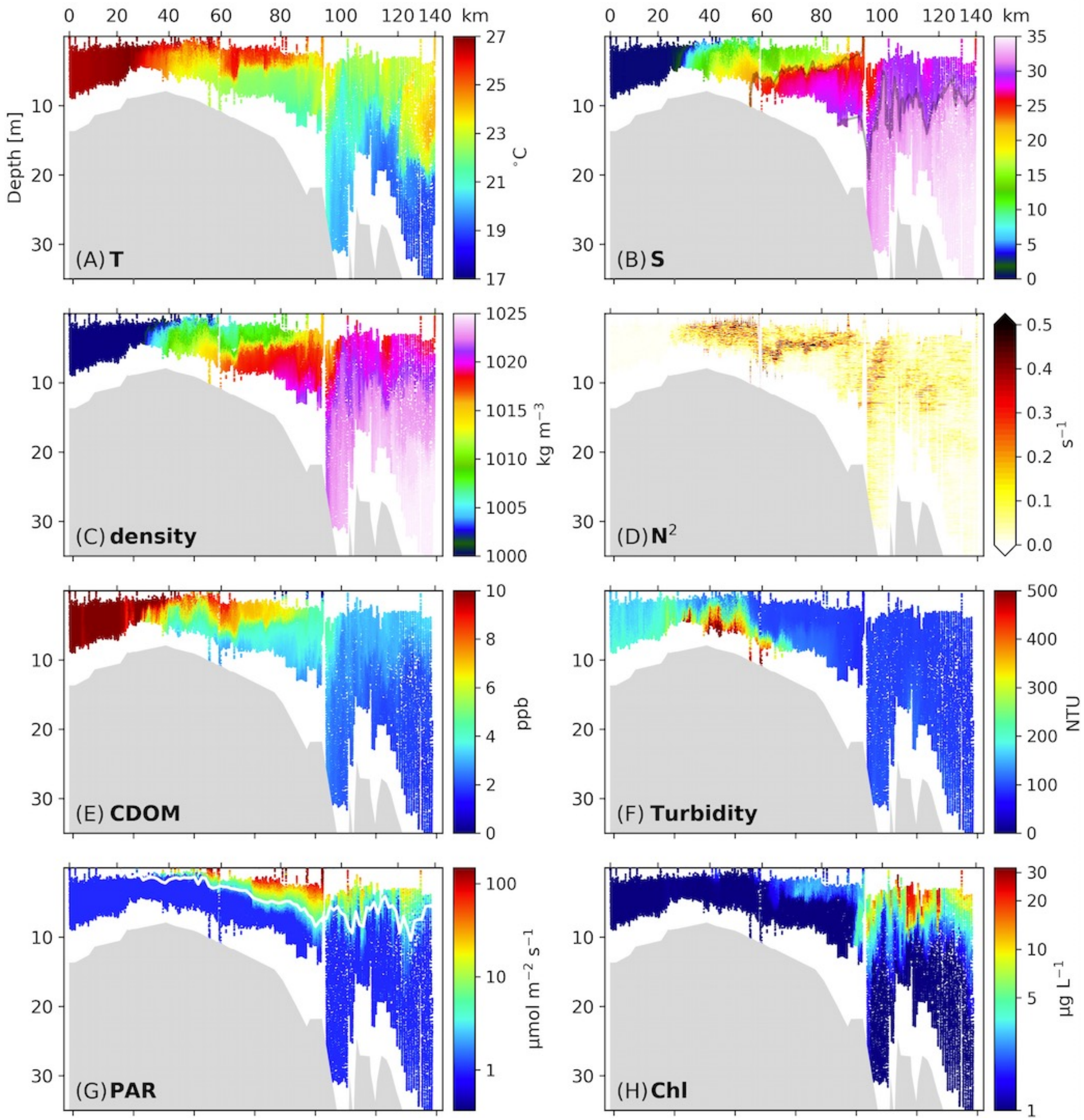
水平分辨率：

- 200 -1000 m

混合与层结

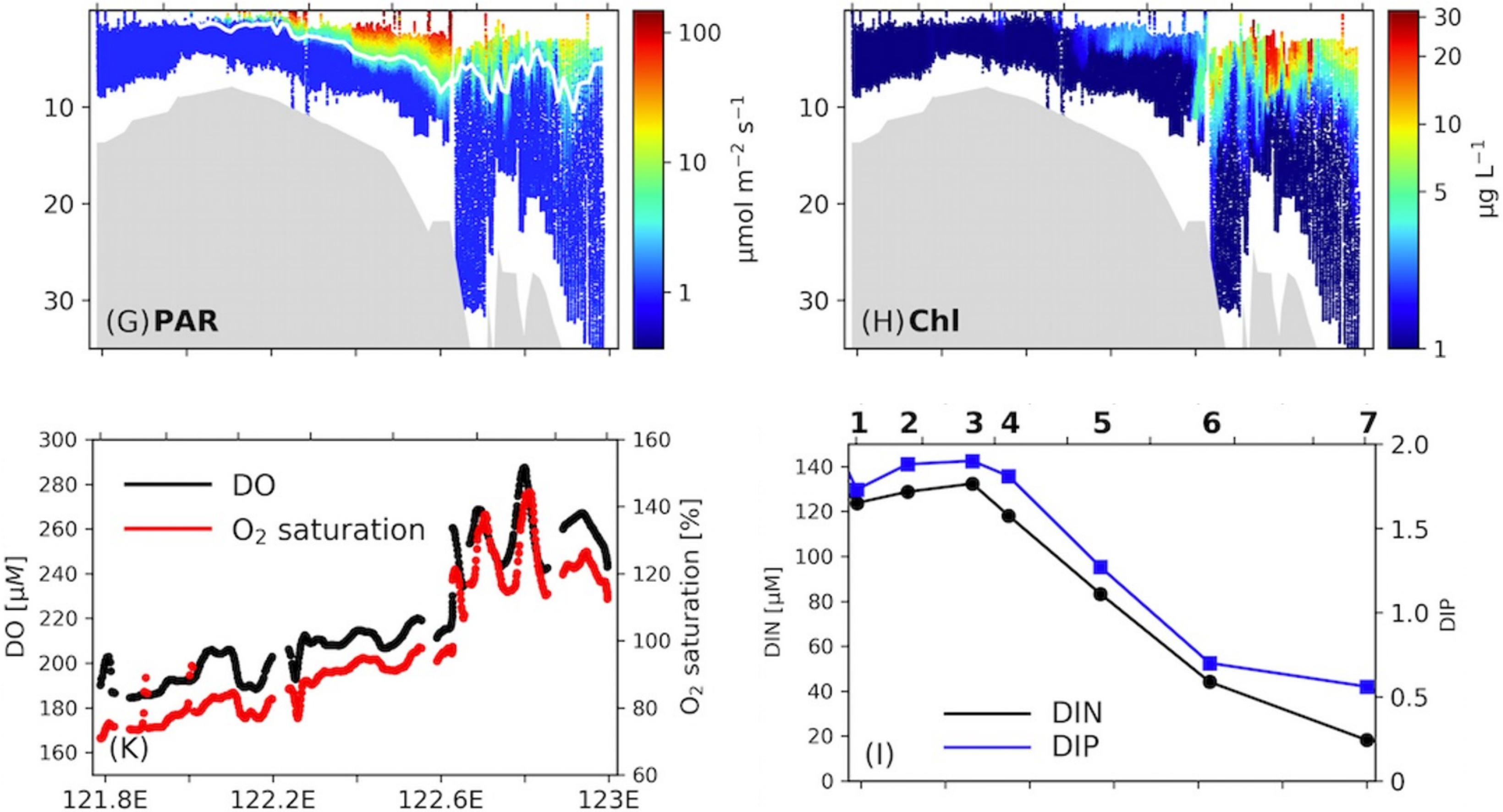
- 最大混浊带发生在强混合区；
- 层结增强后，浊度迅速降低，真光层变深，初级生产力开始增加；
- 叶绿素的显著高值区发生于冲淡水的外缘。

Zhang et al. (*Front. Mar. Sci.*, 2020)

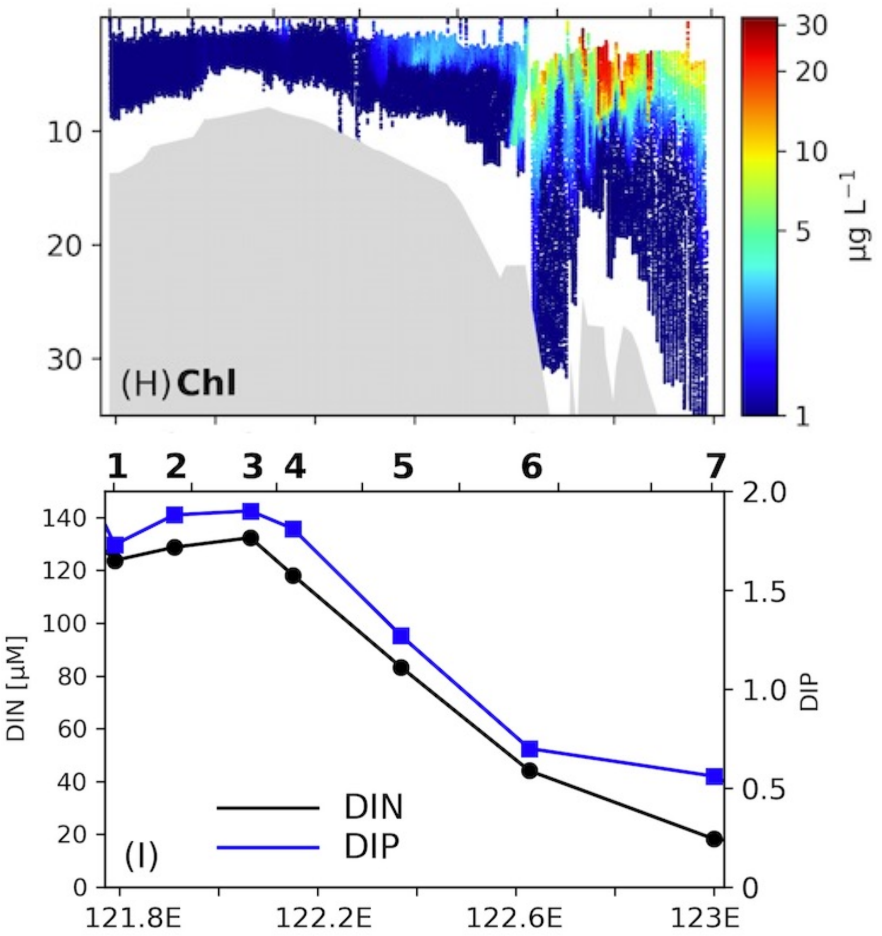


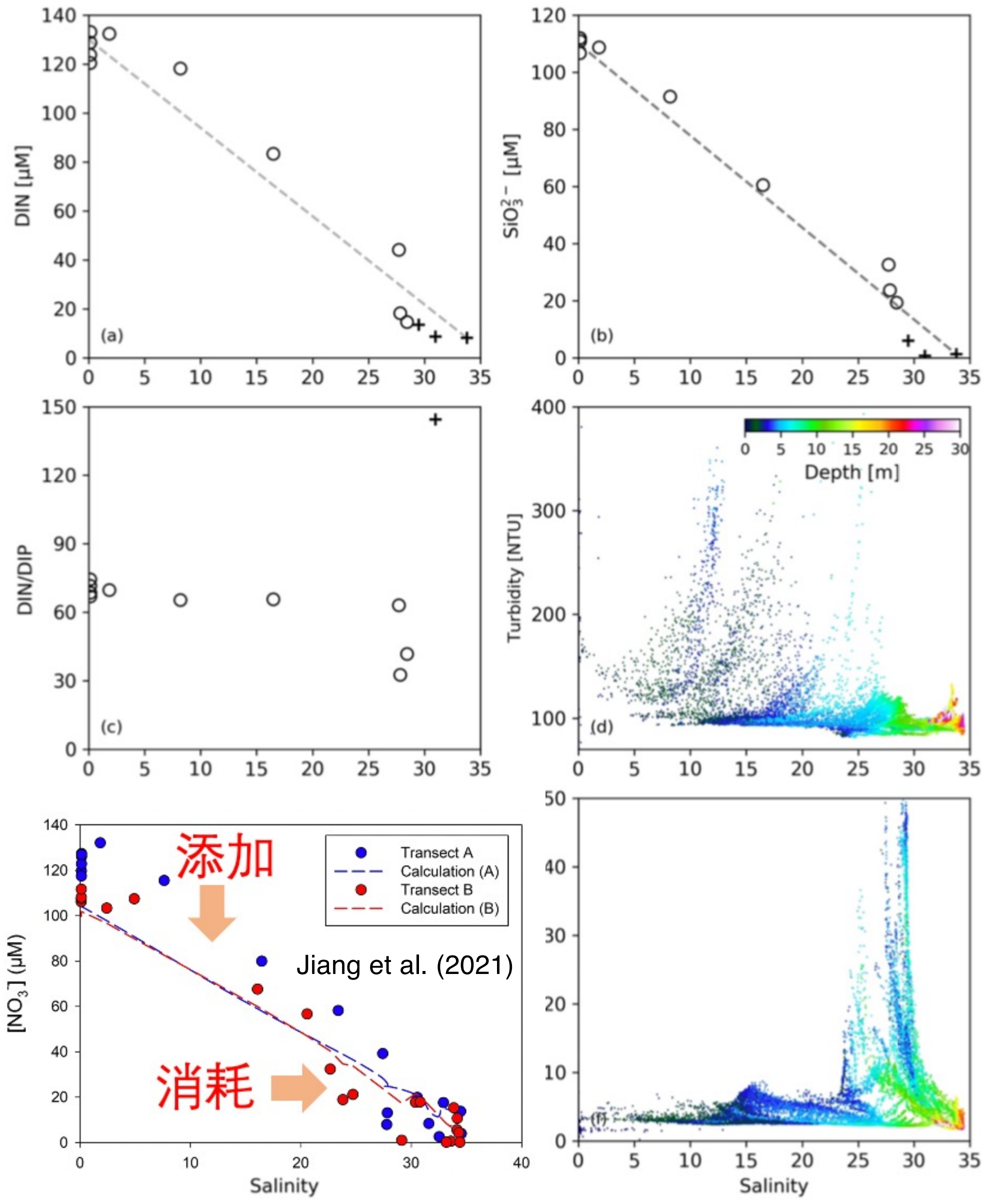
层结区溶解氧与营养盐的变化

- 表层溶解氧含量缓慢上升；
- 营养盐浓度迅速下降



物理混合？生物吸收？

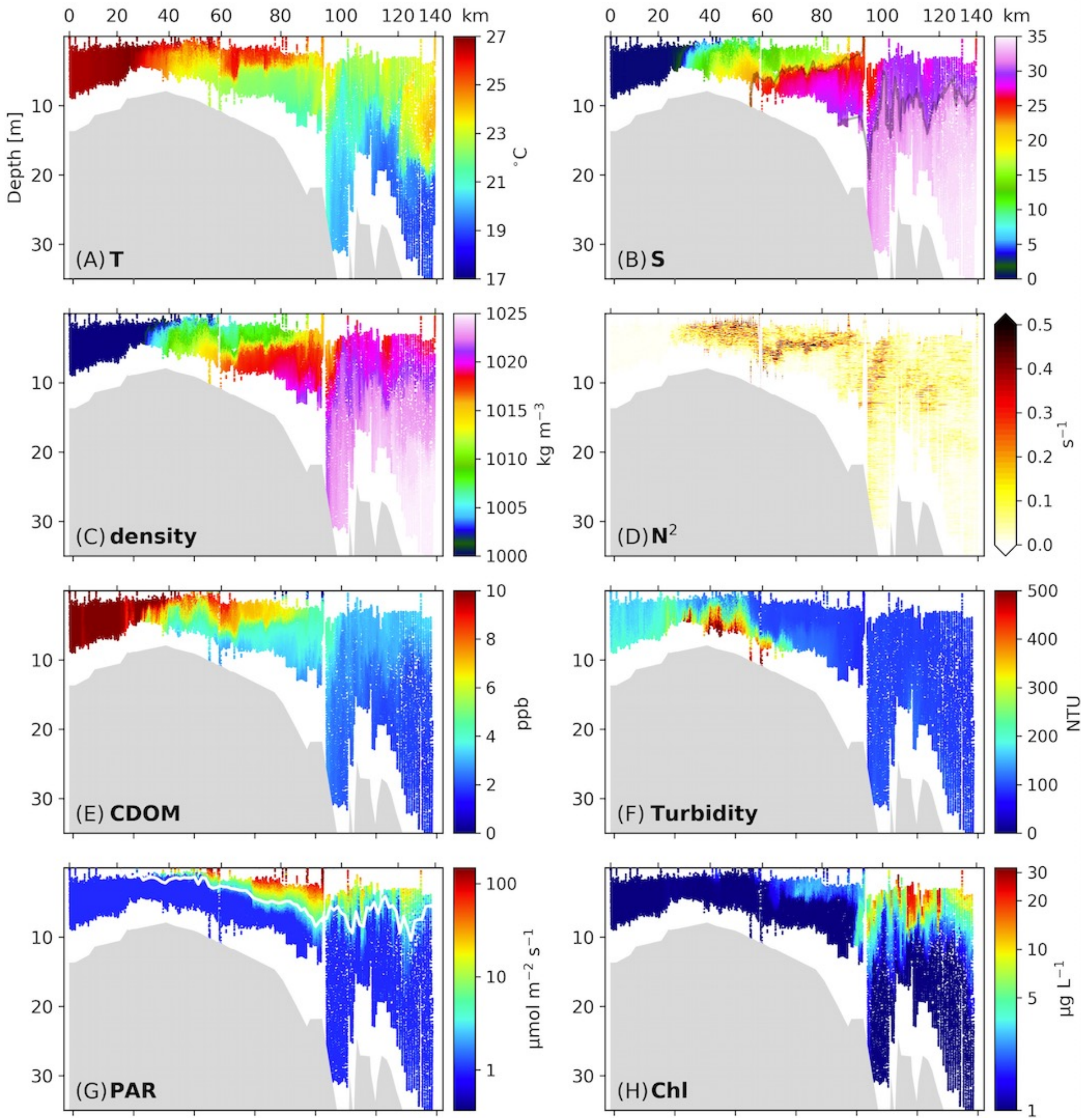




- 营养盐浓度的快速下降主要由物理混合所致
- 有机物分解会导致营养盐一定程度的添加

冲淡水表锋面：藻华爆发

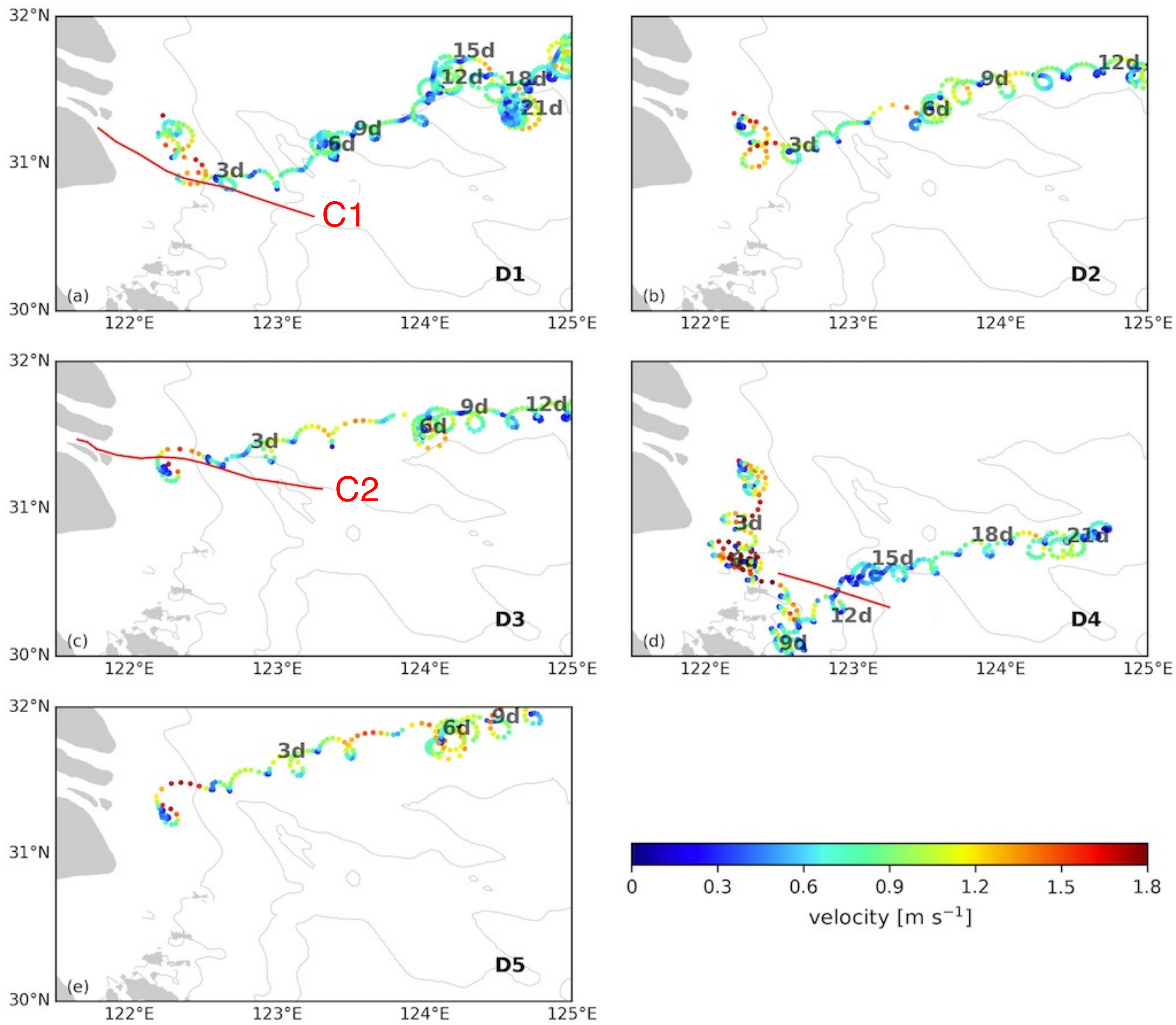
- 藻华开始出现于冲淡水表锋面处
- 下降流现象表征存在表层物质辐聚效应，是导致藻华产生的重要原因之一
- 下降流进一步抑制沉积物的悬浮，增加真光层深度



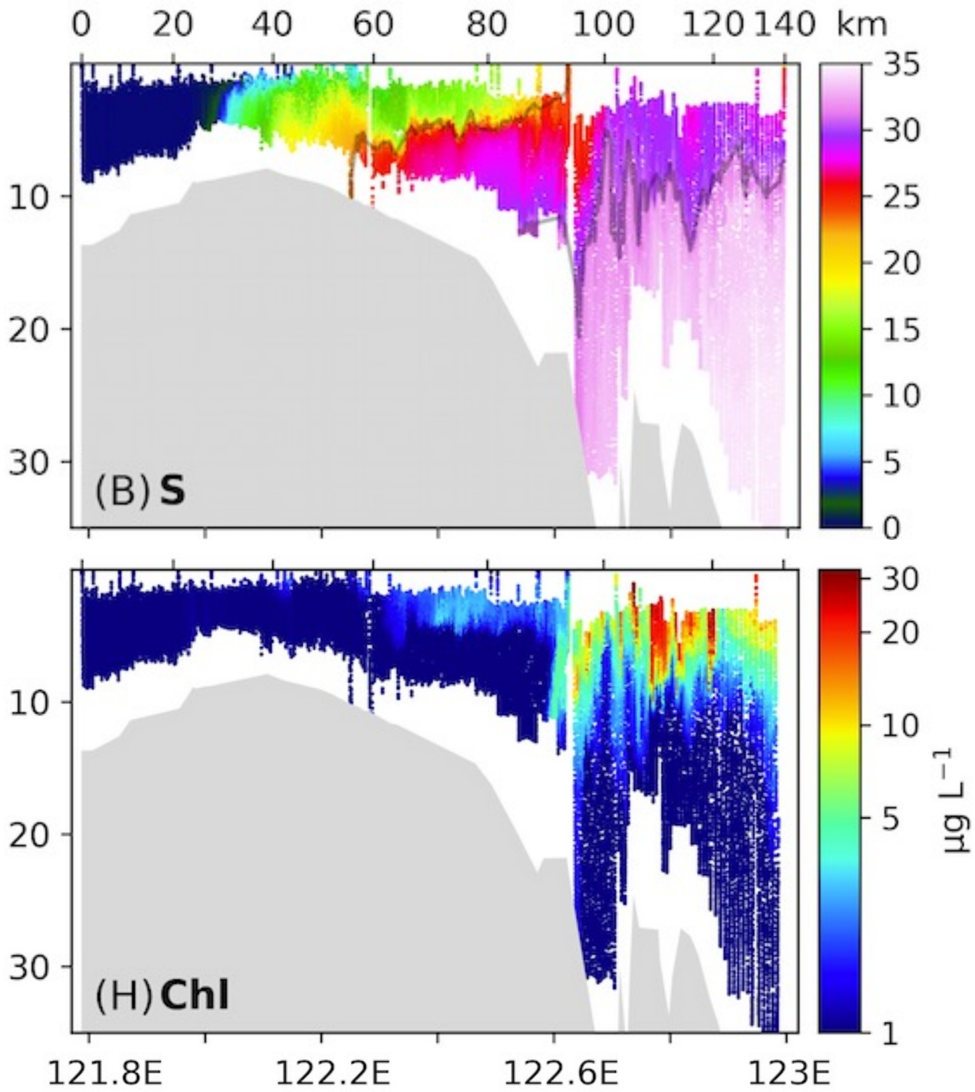
冲淡水扩散路径

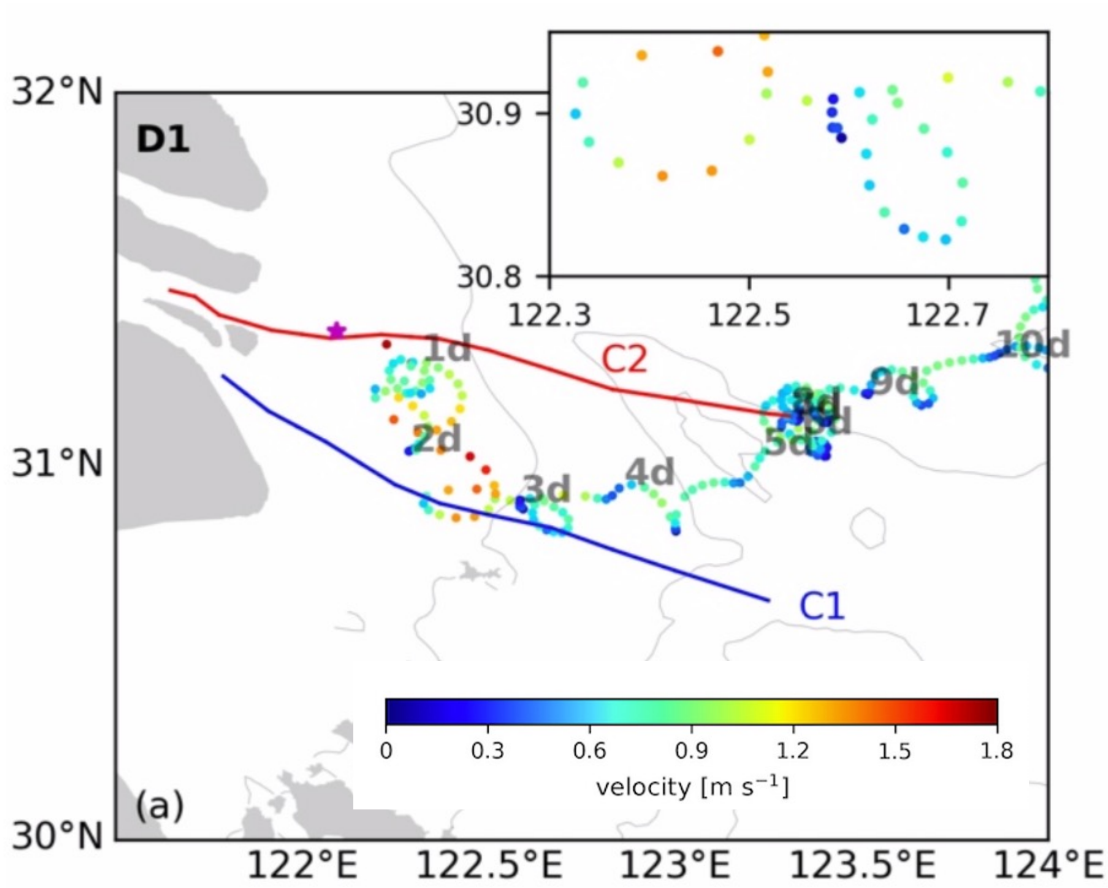
124E以西区域的停留时间

- 东北向扩散：5-6 天
- 东南-东北向扩散：9-12 天
- 南向-东北向扩散：～18天



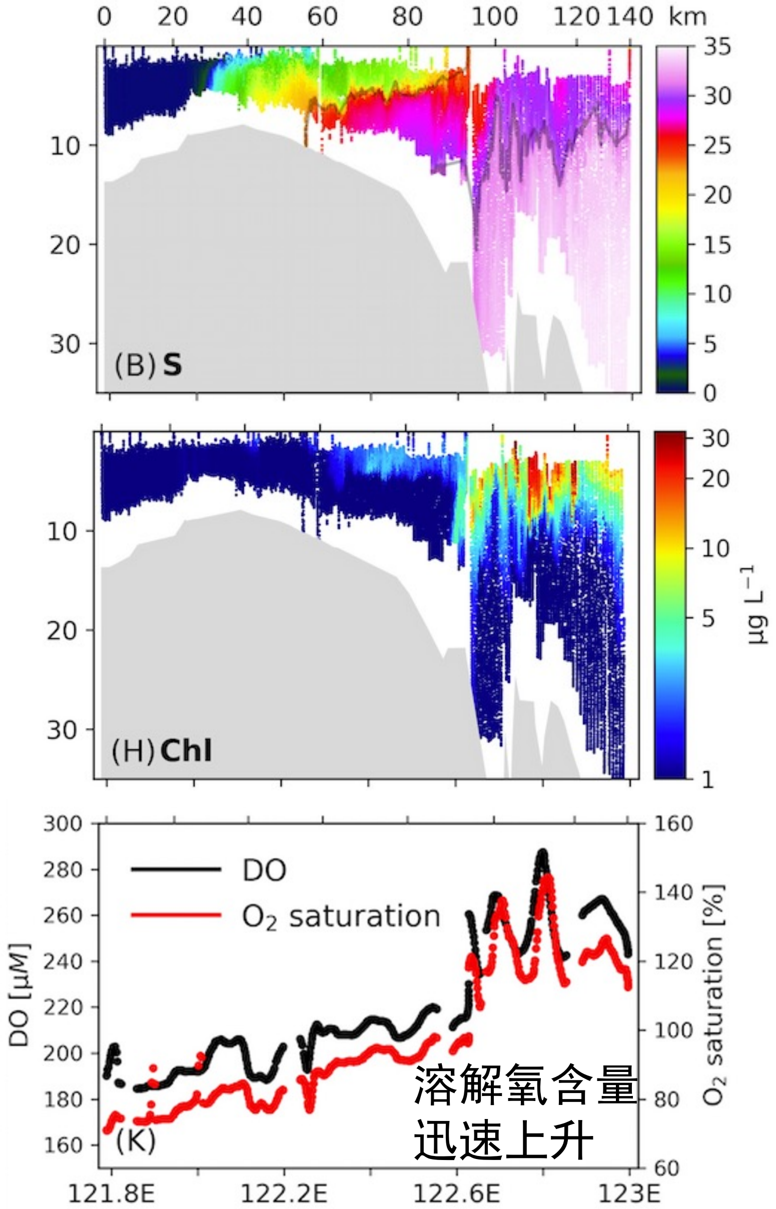
表锋面处的辐聚现象

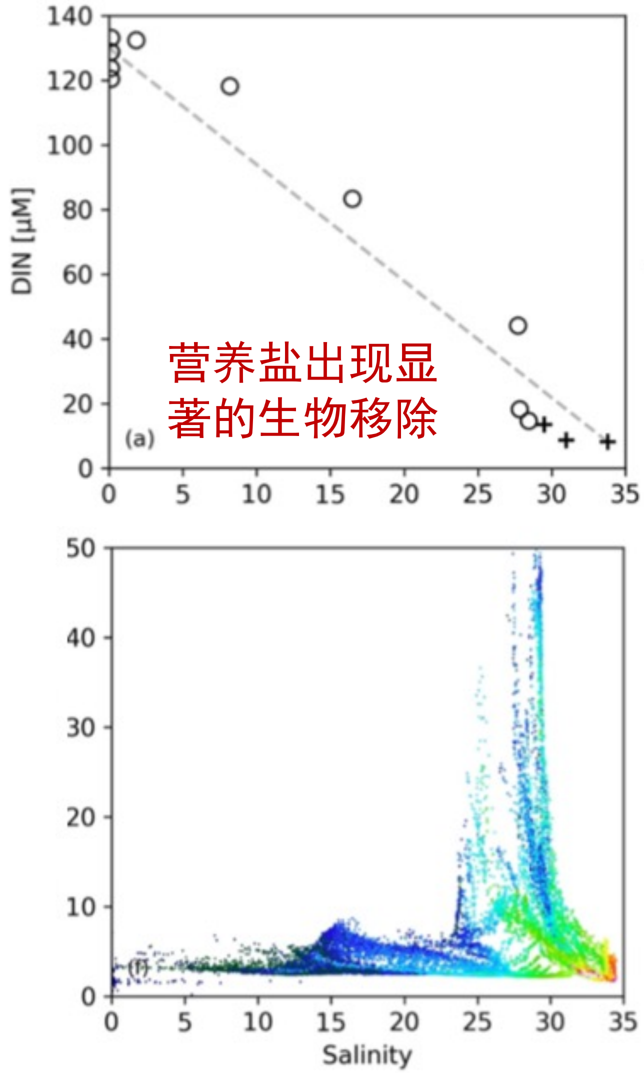




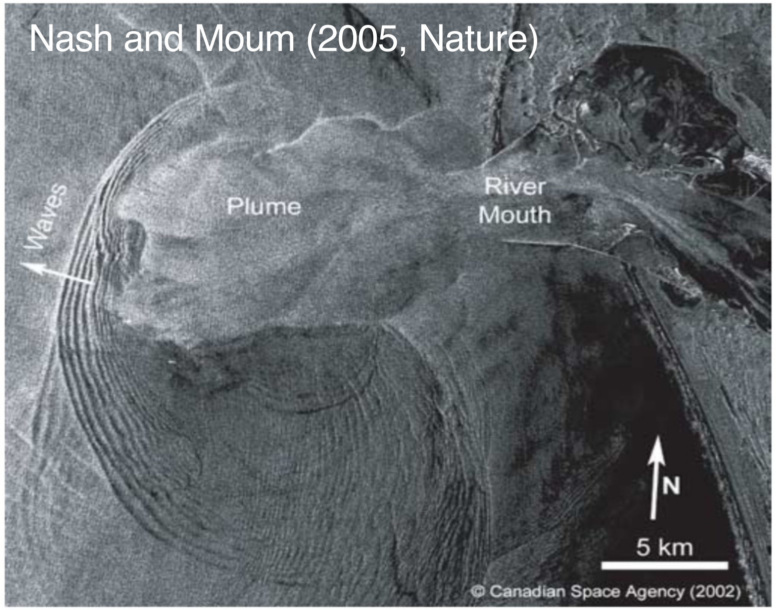
表锋面的辐聚效应和停留时间的延长是藻华爆发的重要触发机制

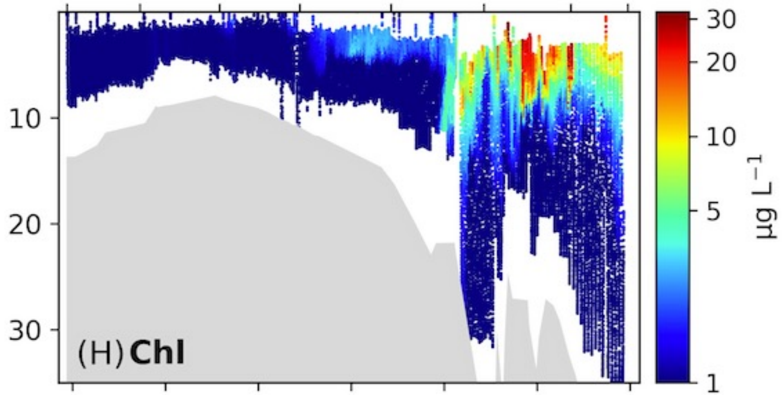
表锋面区溶解氧与营养盐的变化





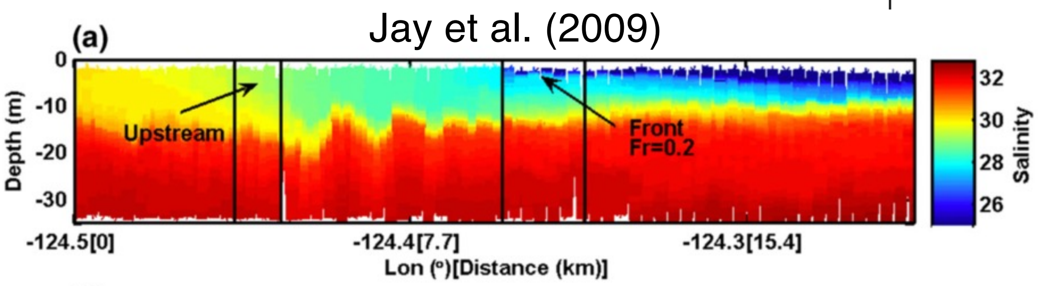
锋面外海侧的波动信号

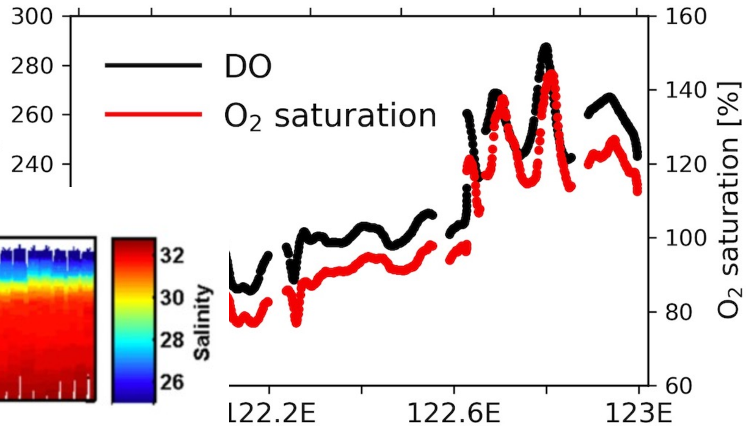




Froude number:

$$\boxed{Fr = u/\sqrt{g'h}}$$

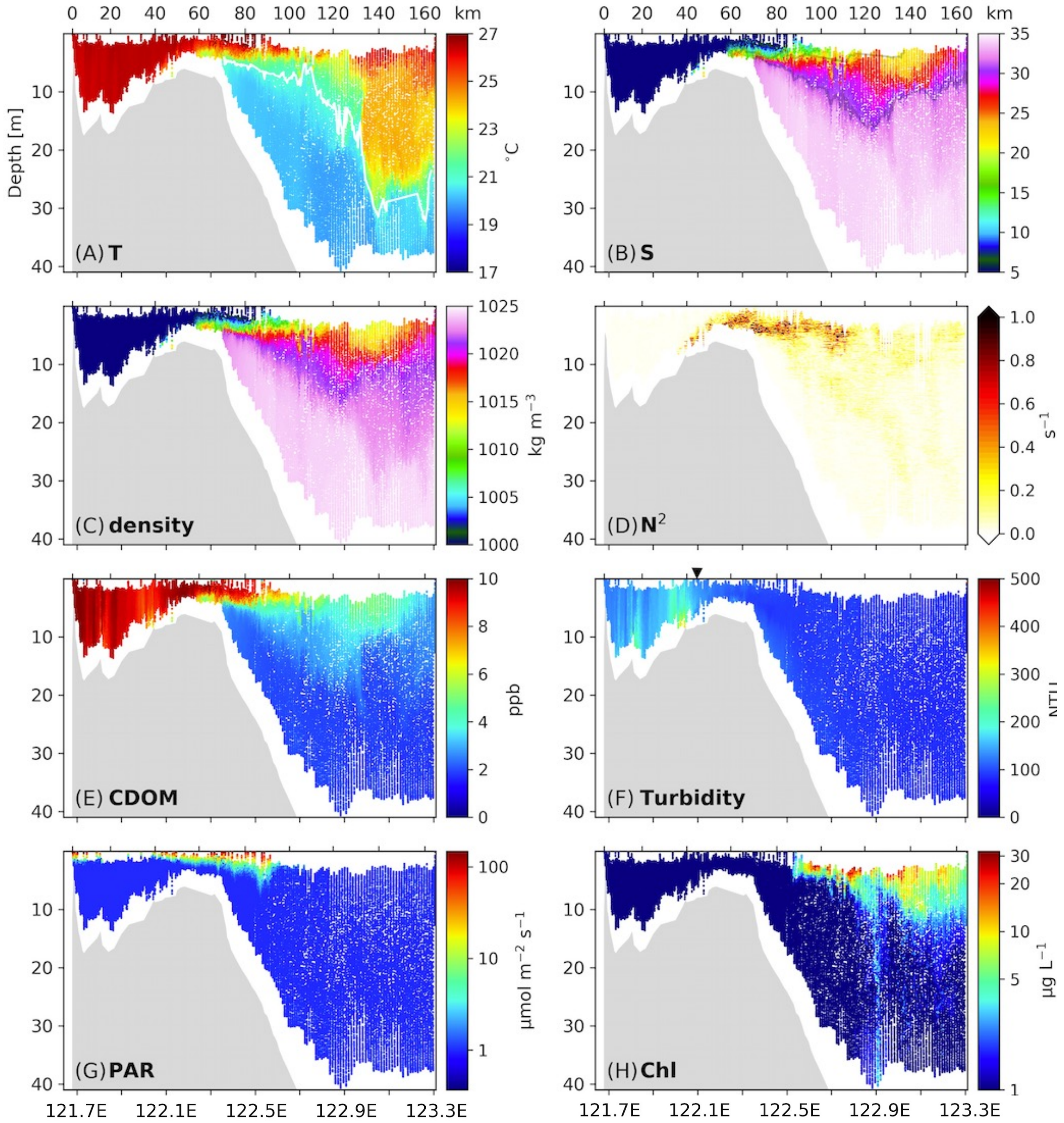


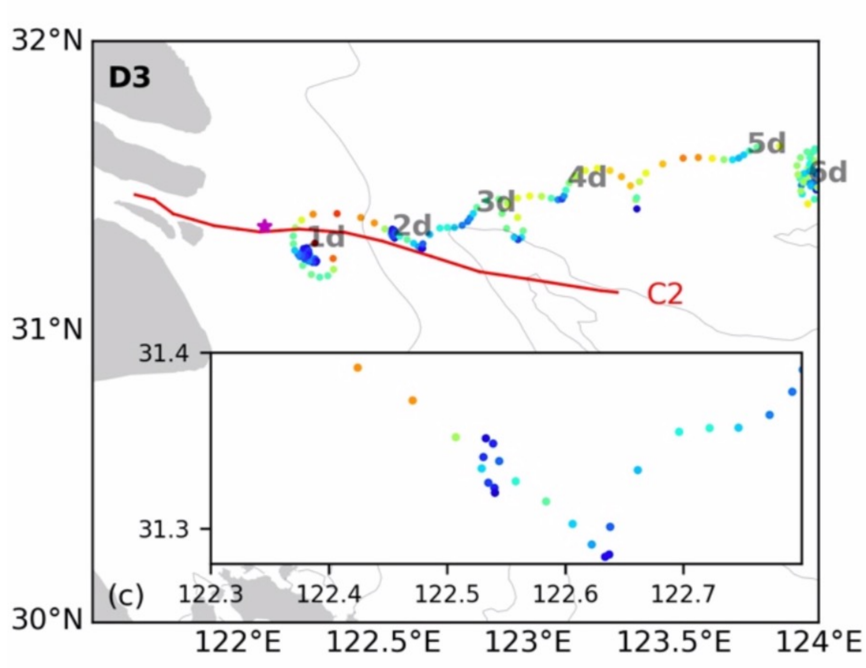


	Inshore of 122.5°E	Offshore of 122.5°E
u **[m s⁻¹]**	1.2	0.7
Δρ [kg m⁻³]	10	10 [a]
h **[m]**	5	10 [b]
Fr	1.7	0.7
	1.7	**0.7**

锋面向岸一侧：超临界流动　　锋面外海一侧：亚临界流动

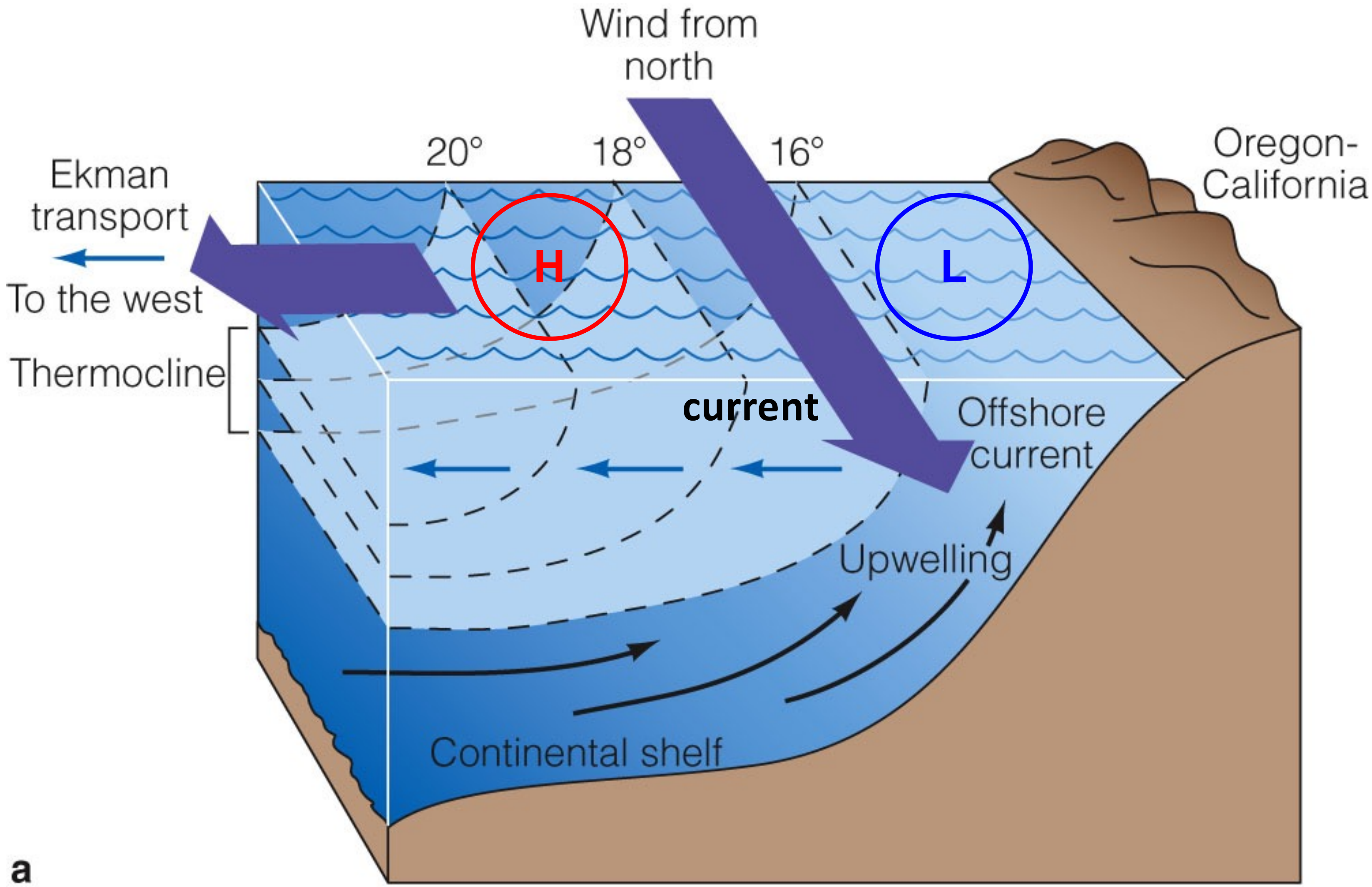
长江口北港外断面 (C2)



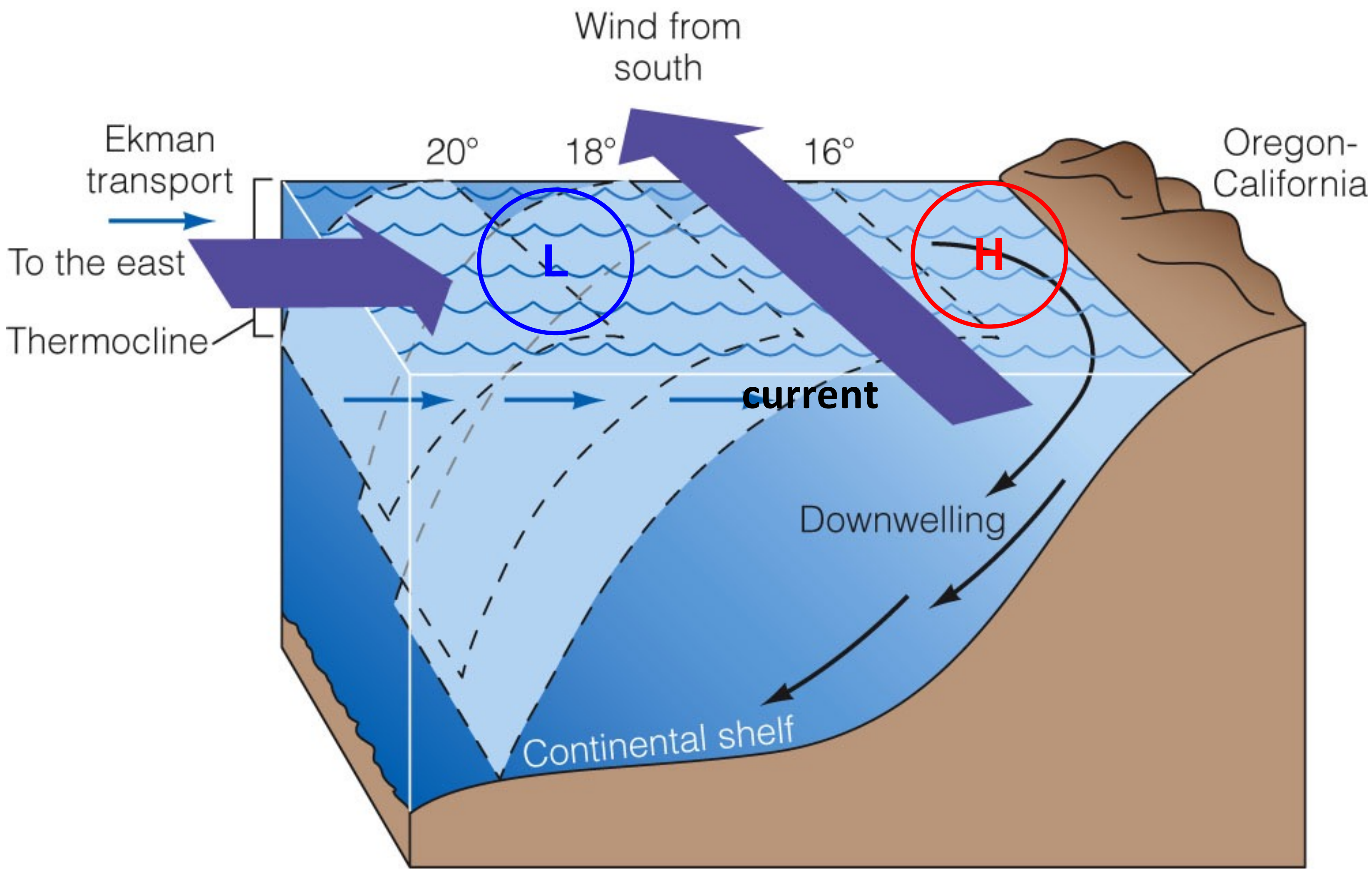


锋面过程是藻华发生的关键影响因素

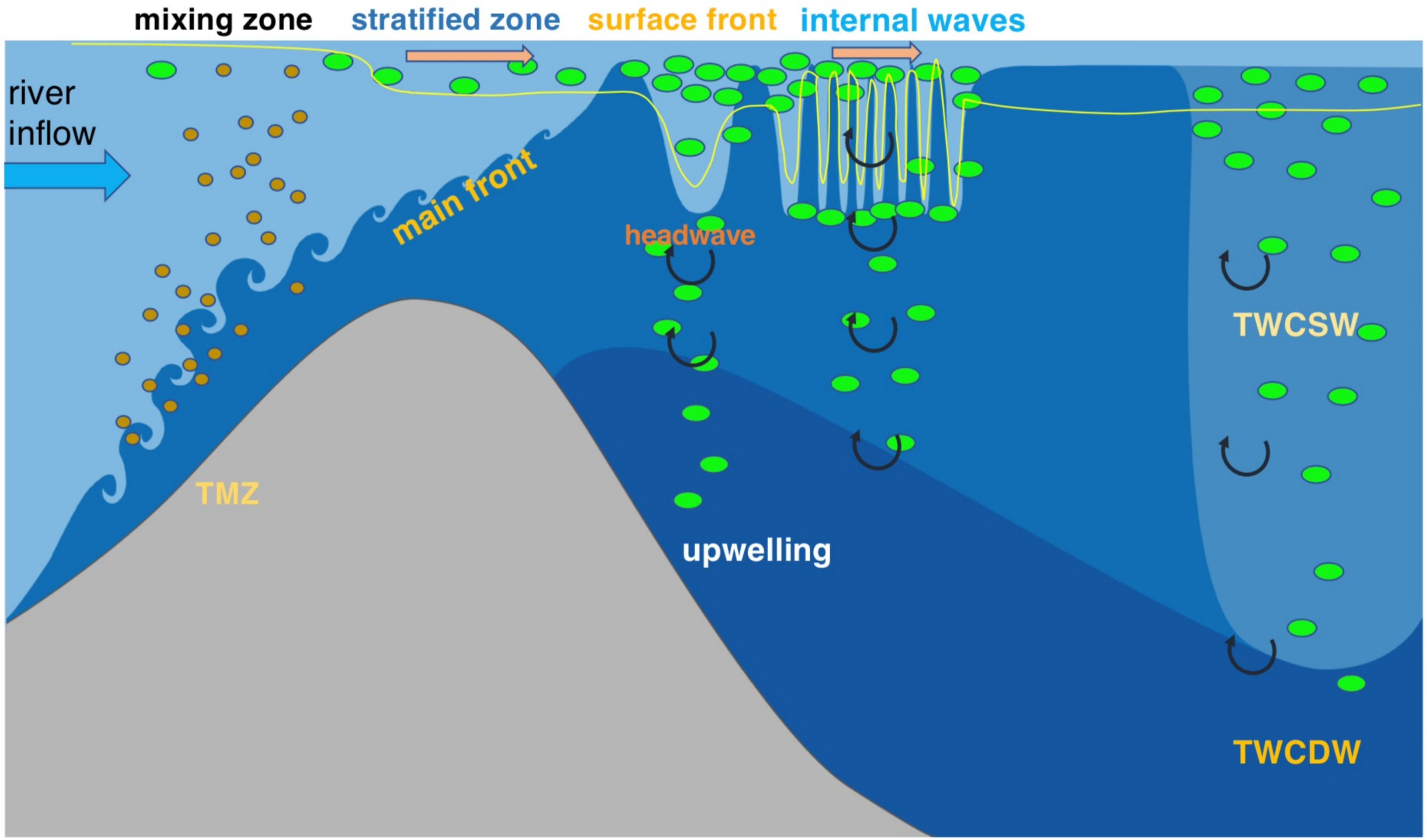
近岸上升流 / 下降流对初级生产力的影响



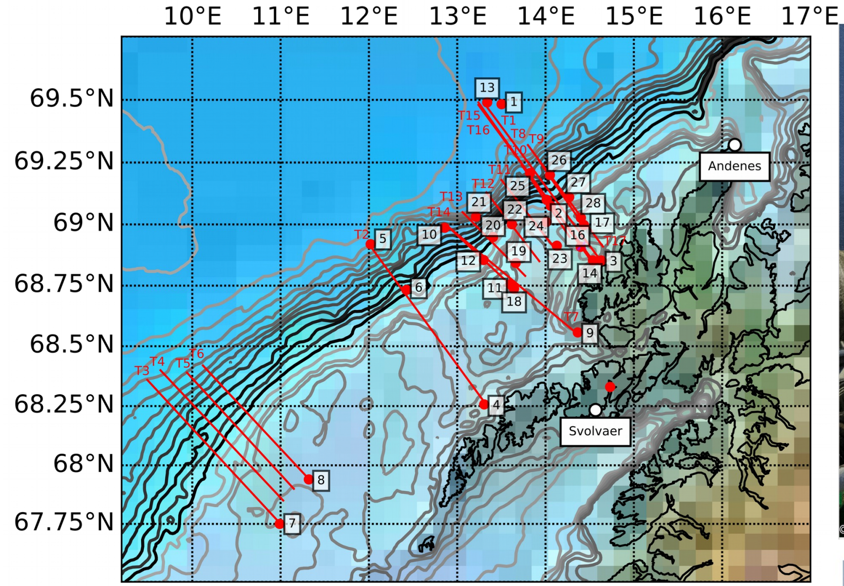
近岸上升流 / 下降流对初级生产力的影响

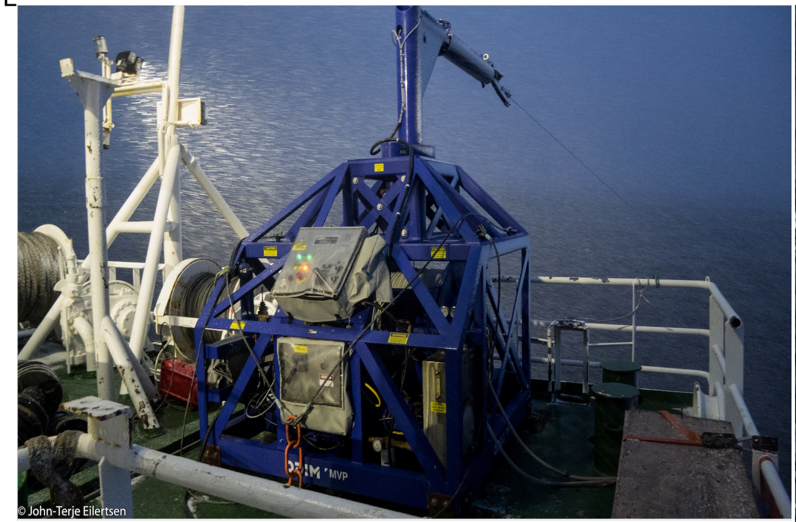


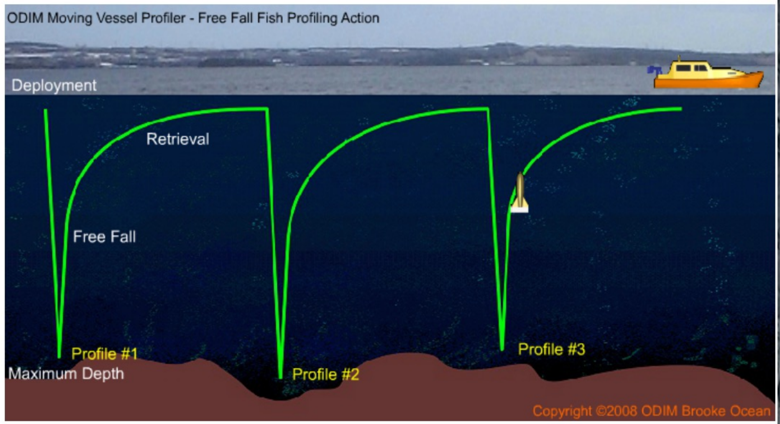
小结：长江口浮游植物量分布的多重物理调控机制



挪威海陆架和陆坡区高分辨率观测 – Moving Vessel Profiler (MVP)







Down cast (free fall) and up cast (low recovery)

Summary of the governing equations

Momentum equation:
$$\frac{d\boldsymbol{v}}{dt} + f\boldsymbol{k}\times\boldsymbol{v} = -\frac{1}{\rho}\nabla p + \nu_E\nabla^2\boldsymbol{v} + \boldsymbol{g}$$

Continuity equation:
$$\frac{d\rho}{dt} + \rho\nabla\cdot\boldsymbol{v} = 0$$

Tracer equations:
$$\frac{dT}{dt} = K\nabla^2 T$$

$$\frac{dS}{dt} = K\nabla^2 S$$

Equation of state:

$$\rho = \rho(\mathrm{T}, \mathrm{S}, \mathrm{P})$$

Then the z-momentum equation becomes:

$b = -\Delta\rho g$: buoyancy force

$$\frac{\mathrm{dw}}{\mathrm{dt}} + f_* u = -\frac{1}{\rho}\frac{\partial p'}{\partial z} - \frac{\Delta\rho}{\rho} g + \nu\nabla^2 \mathrm{w}$$

ρ_0 ρ_0

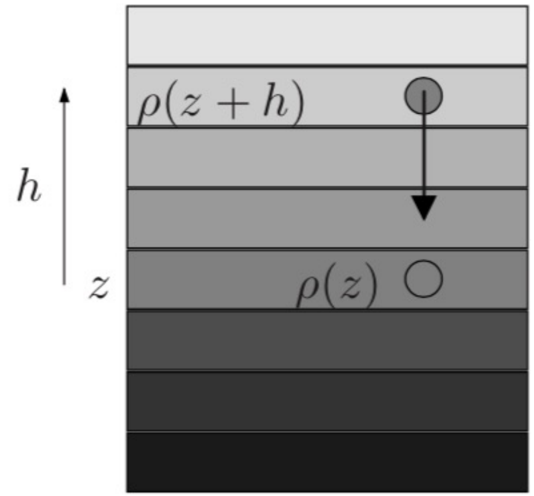
Boussinesq approximation: ρ can be replaced by ρ_0 (or $\Delta\rho$ can be neglected) everywhere except in the gravity term

Horizontal momentum equations

$$\frac{du}{dt} + f_* w - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\nabla^2 u$$

$$\frac{dv}{dt} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \nu\nabla^2 v.$$

$$\frac{\mathrm{dw}}{\mathrm{dt}} + f_* u = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{\rho}{\rho_o} g + \nu\nabla^2 \mathrm{w}$$



Reynolds-averaged equations

Momentum

$$\text{x: } \frac{du}{dt} - fv + f_* w = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(A_H \frac{\partial \mathrm{u}}{\partial \mathrm{x}}\right) + \frac{\partial}{\partial y}\left(A_H \frac{\partial \mathrm{u}}{\partial \mathrm{y}}\right) + \frac{\partial}{\partial \mathrm{z}}\left(A_V \frac{\partial \mathrm{u}}{\partial \mathrm{z}}\right)$$

$$\text{y:} \frac{dv}{dt} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(A_H \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_H \frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_V \frac{\partial v}{\partial z}\right)$$

$$\text{z: } \frac{dw}{dt} + f_* u = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left(A_H \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_H \frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_V \frac{\partial w}{\partial z}\right) - \frac{\rho}{\rho_0} g$$

Tracers

$$\frac{dT}{dt} = \frac{\partial}{\partial x}\left(K_H \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_H \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_V \frac{\partial T}{\partial z}\right)$$

$$\frac{dS}{dt} = \frac{\partial}{\partial x}\left(K_H \frac{\partial S}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_H \frac{\partial S}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_V \frac{\partial S}{\partial z}\right)$$

The momentum equation

z direction:

$$\cancel{\frac{\partial \mathrm{w}}{\partial \mathrm{t}}} + \cancel{\mathrm{u}\frac{\partial \mathrm{w}}{\partial \mathrm{x}}} + \cancel{\mathrm{v}\frac{\partial \mathrm{w}}{\partial \mathrm{y}}} + \cancel{\mathrm{w}\frac{\partial \mathrm{w}}{\partial \mathrm{z}}} + \cancel{f_* u} =$$

$$\cancel{\frac{W}{T}} \quad \cancel{\frac{UW}{L}} \quad \cancel{\frac{UW}{L}} \quad \cancel{W\frac{W}{H}} \quad \cancel{fU}$$

$$-\frac{1}{\rho_0}\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left(\cancel{A_H\frac{\partial \mathrm{w}}{\partial \mathrm{x}}}\right) + \frac{\partial}{\partial y}\left(\cancel{A_H\frac{\partial \mathrm{w}}{\partial \mathrm{y}}}\right) + \frac{\partial}{\partial \mathrm{z}}\left(\cancel{A_V\frac{\partial \mathrm{w}}{\partial \mathrm{z}}}\right) - \frac{\rho}{\rho_0}g$$

$$\frac{P}{\rho_0 H} \quad \cancel{A_H\frac{W}{L^2}} \quad \cancel{A_H\frac{W}{L^2}} \quad \cancel{A_V\frac{W}{H^2}} \quad \frac{\Delta\rho}{\rho_0}\mathrm{g}$$

From the x-direction equation:

$$\frac{P}{\rho_0 L} \sim \mathrm{fU} \sim \frac{U^2}{L} \sim A_H\frac{U}{L^2} \sim A_V\frac{U}{H^2}$$

As $H \ll L, \quad W \ll U$

$$\frac{P}{\rho_0 H} \gg \mathrm{fU}$$

$$\frac{P}{\rho_0 H} \gg \frac{UW}{L}$$

$$\frac{P}{\rho_0 H} \gg A_H\frac{W}{L^2}$$

$$\frac{P}{\rho_0 H} \gg A_V\frac{W}{H^2}$$

$$\frac{\partial p}{\partial z} + \rho g = 0$$

Hydrostatic equation (balance)

Kinematic boundary conditions — velocity

Principle: flow cannot penetrate a solid wall

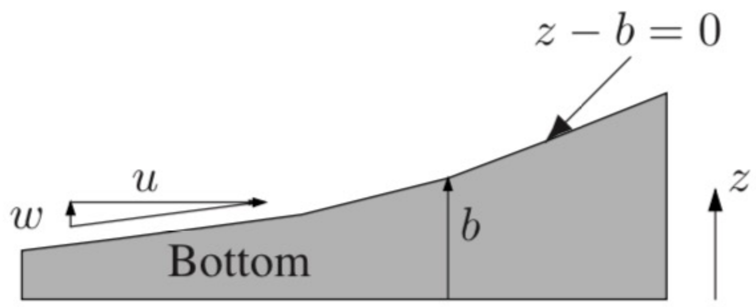
Bottom boudary: $\overset{G}{z - b(x,y)} = 0$

$$\nabla G = \left(-\frac{\partial b}{\partial x}, -\frac{\partial b}{\partial y}, 1\right)$$

Considering impermeability:

$$\boldsymbol{u} \cdot \nabla G = 0$$

$$\boxed{w = u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y}}$$



For flat bottom:

$$w = 0$$

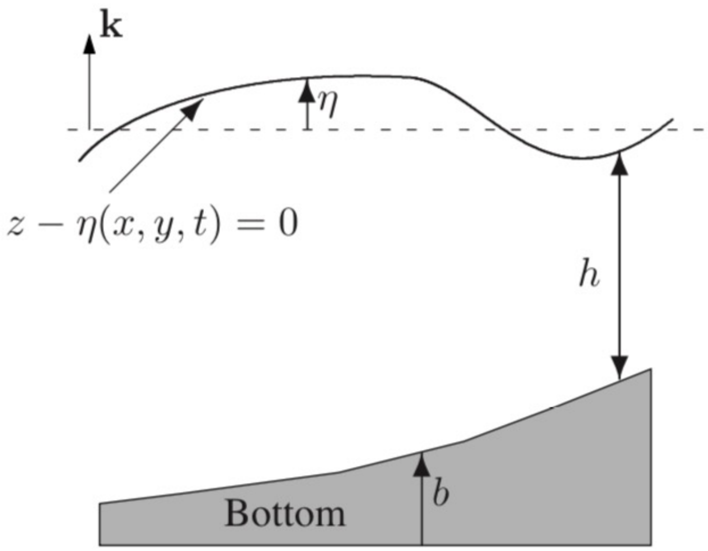
Surface boundary: the boundary is moving with the fluid (free surface)

$$z - \eta(x, y, t) = 0$$

Without precipication and evaporation, it can be considered as a material surface:

$$\frac{d}{dt}(z - \eta) = 0$$

$$\boxed{w = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}}$$

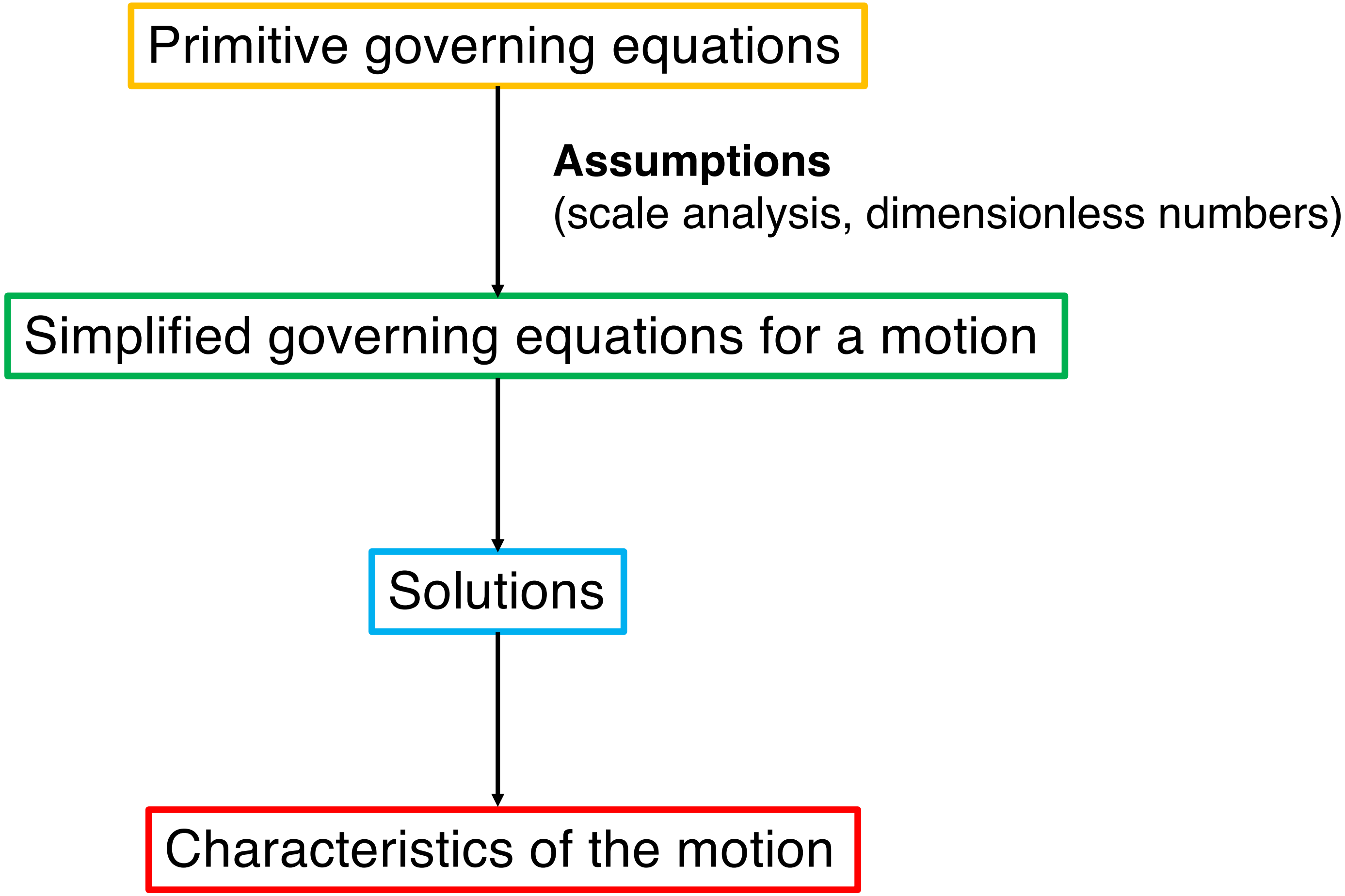


Rigid-lid approximation:

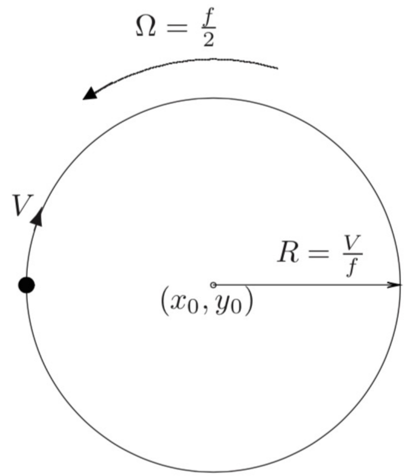
$$\eta = C$$

$$w = 0$$

Part III. Different types of motions



Free motions – Inertial Oscillations



Not subject to real force:

$$\frac{du}{dt} - fv = 0$$

$$\frac{dv}{dt} + fu = 0$$

$$V = \sqrt{u^2 + v^2} \qquad \frac{dx}{dt} \; u = V sin(ft + \phi) \qquad \frac{dy}{dt} \; v = V cos(ft + \phi)$$

f : inertial frequency

$$T = \frac{2\pi}{f}$$

$$x = x_0 - \frac{V}{f}\cos(ft + \phi)$$

$$y = y_0 + \frac{V}{f}\sin(ft + \phi)$$

$$(x - x_0)^2 + (y - y_0)^2 = (\frac{V}{f})^2$$

Table 9.1 Inertial Oscillations

Latitude (φ)	T_i (hr)	D (km) for V = 20 cm/s
90°	11.97	2.7
35°	20.87	4.8
10°	68.93	15.8

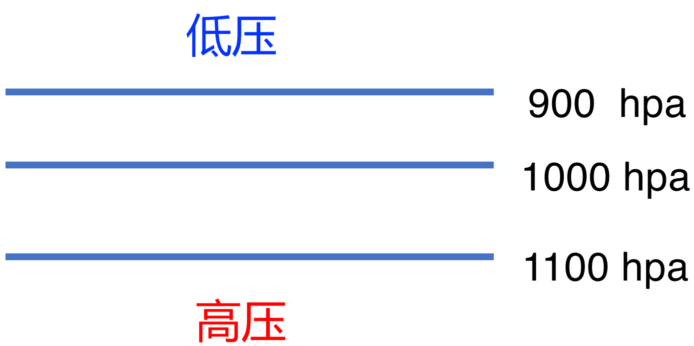
Geostrophic flows

Assumptions:

$R_o \ll 1$ and $E_k \ll 1$ (ocean interior)

Inertial acceleration, nonlinear advection and viscosity terms are neglected

Governing equations



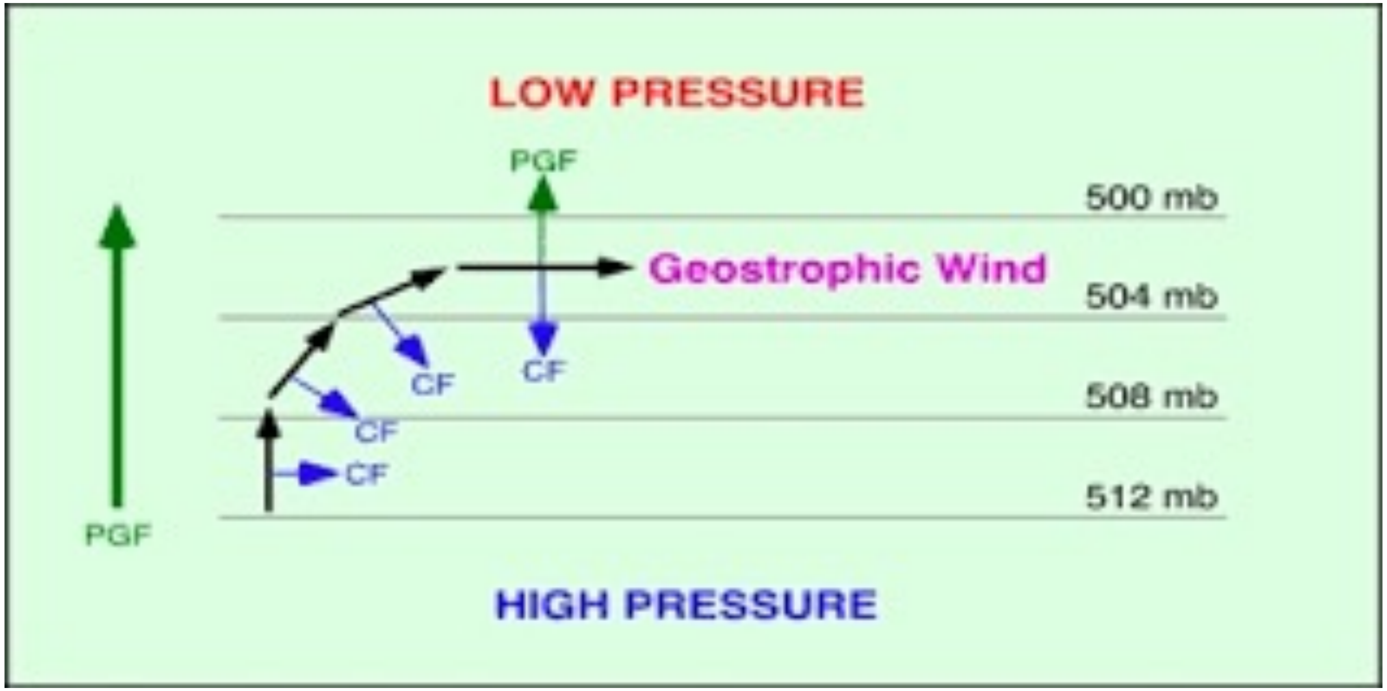
$$\boxed{\begin{aligned} -fv &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \\ +fu &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \end{aligned}}$$

Geostrophic balance

balance between the pressure gradient term and the Coriolis term

$$\boldsymbol{u} \cdot \nabla p = 0$$

Geostrophic flow is perpendicular to the pressure gradient (force).



High pressure is to the right (left) of the geostrophic flow in the northern (southern) hemisphere

So for $\rho=\rho_0$:

$$-fv = -g\frac{\partial \eta}{\partial x}$$

$$fu = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial u}{\partial z} = 0$$

$$\frac{\partial v}{\partial z} = 0$$

The flow has no vertical shear and the fluid moves like a slab – **Taylor column**

$$\boxed{\begin{aligned} -fv &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \\ +fu &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \end{aligned}}$$

For f-plane:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial}{\partial x}\left(\frac{1}{\rho_0 f}\frac{\partial p}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\rho_0 f}\frac{\partial p}{\partial x}\right) = 0.$$

Geostrophic flows are horizontally non-divergent

From the continuity equation: $\frac{\partial w}{\partial z} = 0$

For flat surface or bottom, $w = 0$ through the water column

Streamfunction ψ

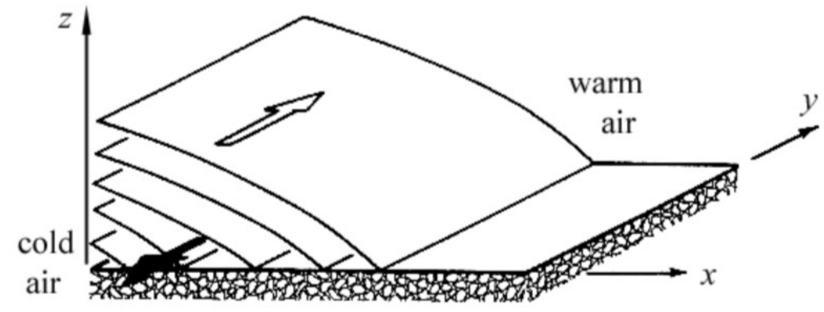
$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}$$

Thermal wind balance

$$-fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$+fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

atmospheric front

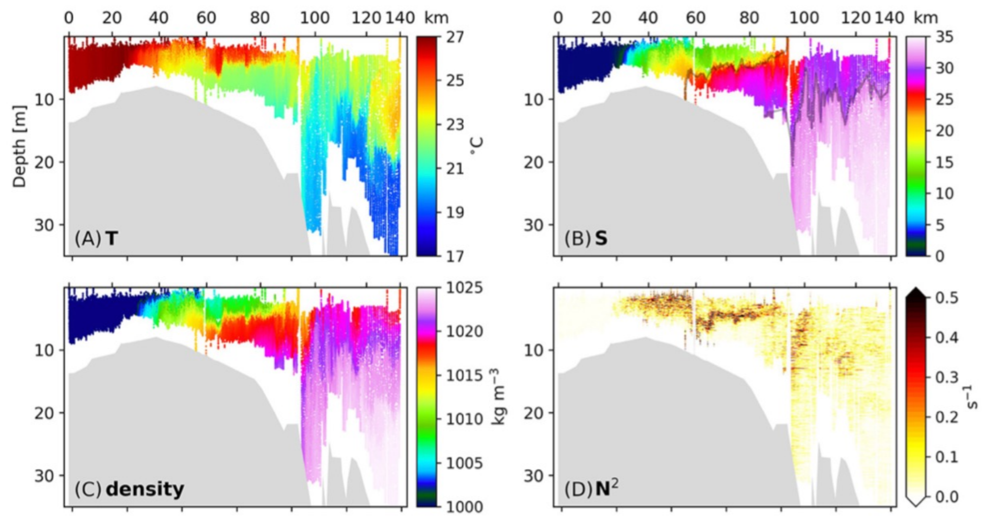


Hydrostatic balance: $$\frac{\partial p}{\partial z} + \rho g = 0$$

$$\frac{\partial v}{\partial z} = -\frac{g}{\rho_0 f}\frac{\partial \rho}{\partial x}$$

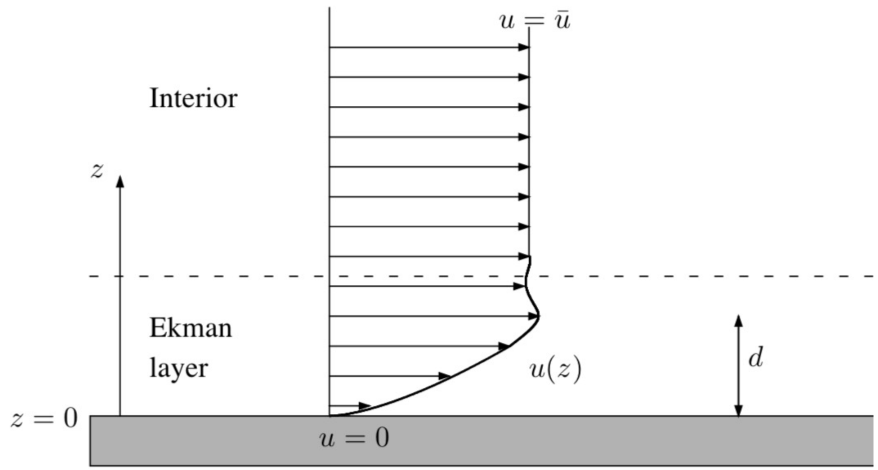
$$\frac{\partial u}{\partial z} = +\frac{g}{\rho_0 f}\frac{\partial \rho}{\partial y}$$

oceanic front



Zhang et al. (2020)

The bottom Ekman layer



Assumptions: large-scale motion ($R_0 \ll 1$)

steady flow ($\frac{\partial}{\partial t} = 0$)

geostrophic flow in the interior

$$u = \bar{u}, \qquad v = 0$$

homogeneous fluid ($\rho = \rho_0$)

$$\frac{\partial p'}{\partial z} + \rho' g = 0 \quad \Rightarrow \quad \frac{\partial p'}{\partial z} = 0$$

flat bottom

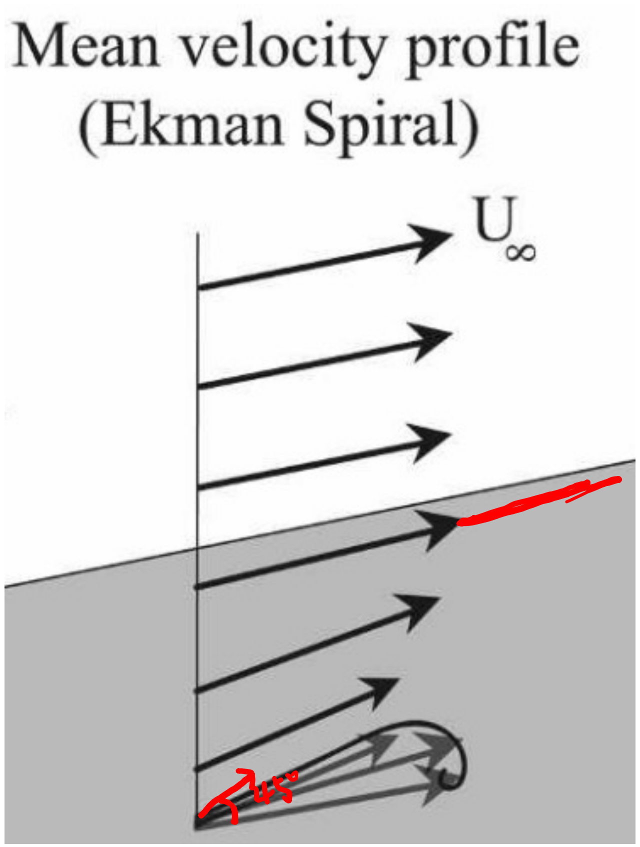
Governing equations in the bottom boundary layer (BBL):

p' (dynamic pressure)

$$-fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu_E \frac{\partial^2 u}{\partial z^2}$$

p'

$$+fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \nu_E \frac{\partial^2 v}{\partial z^2}$$



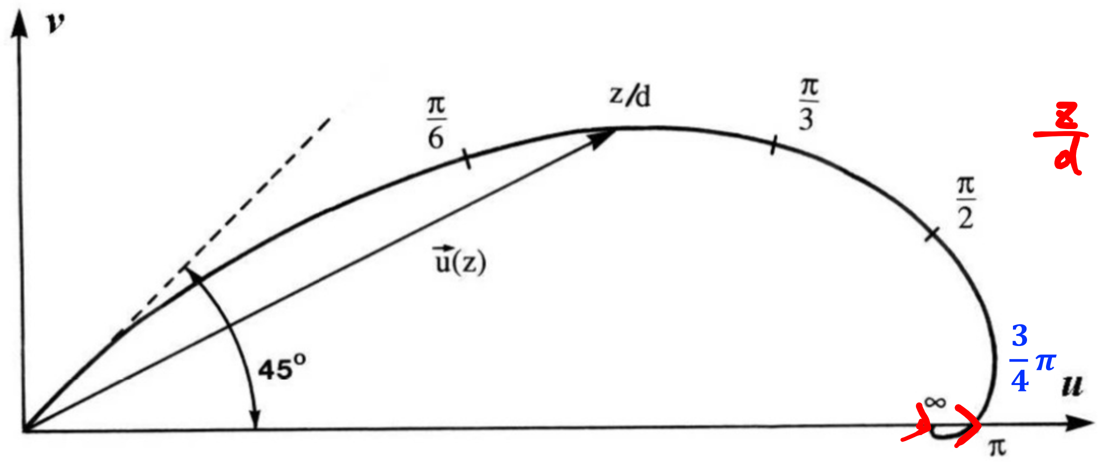


Figure 8-4 The velocity spiral in the bottom Ekman layer. The figure is drawn for the Northern Hemisphere ($f > 0$), and the deflection is to the left of the current above the layer. The reverse holds for the Southern Hemisphere.

$$\frac{\partial \mathrm{U}}{\partial x} + \frac{\partial \mathrm{V}}{\partial y} = \int_0^\infty \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz = -\frac{d}{2}\left(\underbrace{\frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y}}_{\bar{\zeta}}\right) > 0$$ divergence

$$w|_{z=\infty} = \frac{d}{2}\bar{\zeta} < 0$$

Divergence in the bottom

boundary layer induces

downwelling from the interior

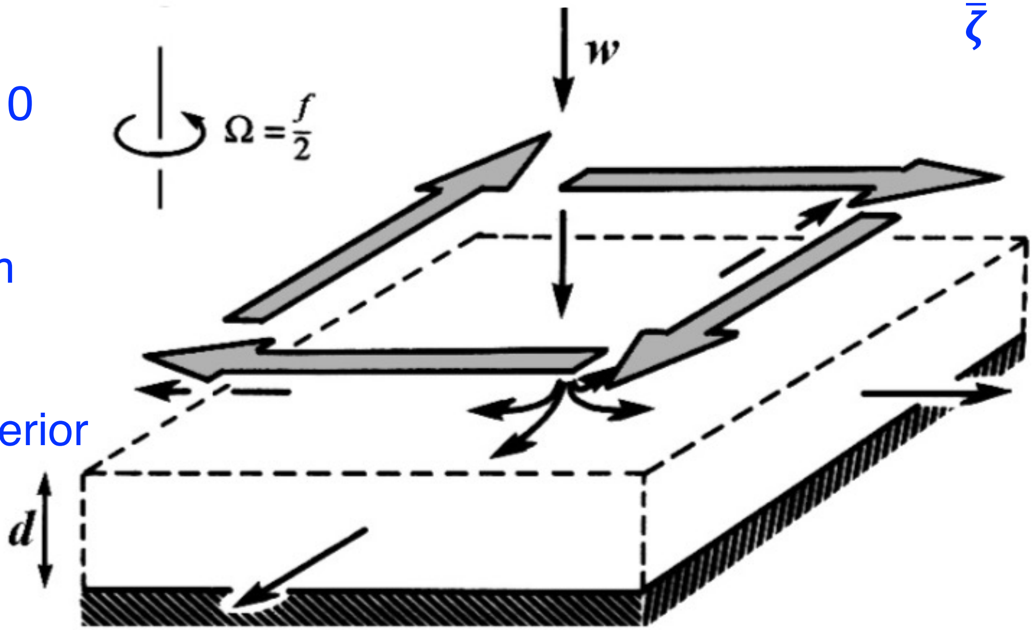


Figure 8-5 Divergence in the bottom Ekman layer and compensating downwelling in the interior. Such a situation arises in the presence of an anticyclonic gyre in the interior, as depicted by the large horizontal arrows. Similarly, interior cyclonic motion causes convergence in the Ekman layer and upwelling in the interior.

The solutions are:

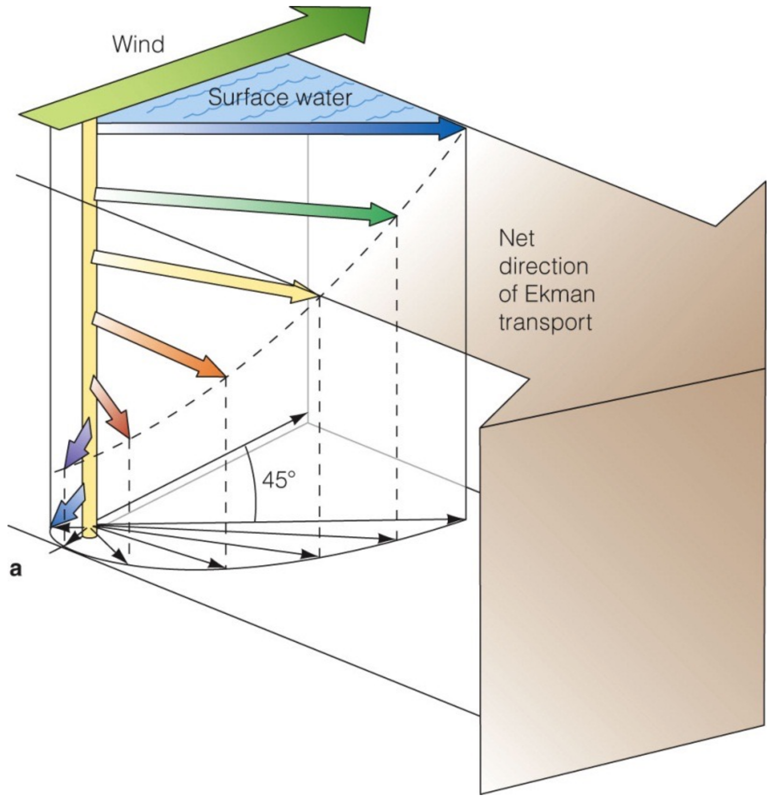
$$u = \bar{u} + \frac{\sqrt{2}}{\rho_0 f d} e^{z/d} \left[\tau^x \cos\left(\frac{z}{d} - \frac{\pi}{4}\right) - \tau^y \sin\left(\frac{z}{d} - \frac{\pi}{4}\right) \right]$$
$$v = \bar{v} + \frac{\sqrt{2}}{\rho_0 f d} e^{z/d} \left[\tau^x \sin\left(\frac{z}{d} - \frac{\pi}{4}\right) + \tau^y \cos\left(\frac{z}{d} - \frac{\pi}{4}\right) \right]$$

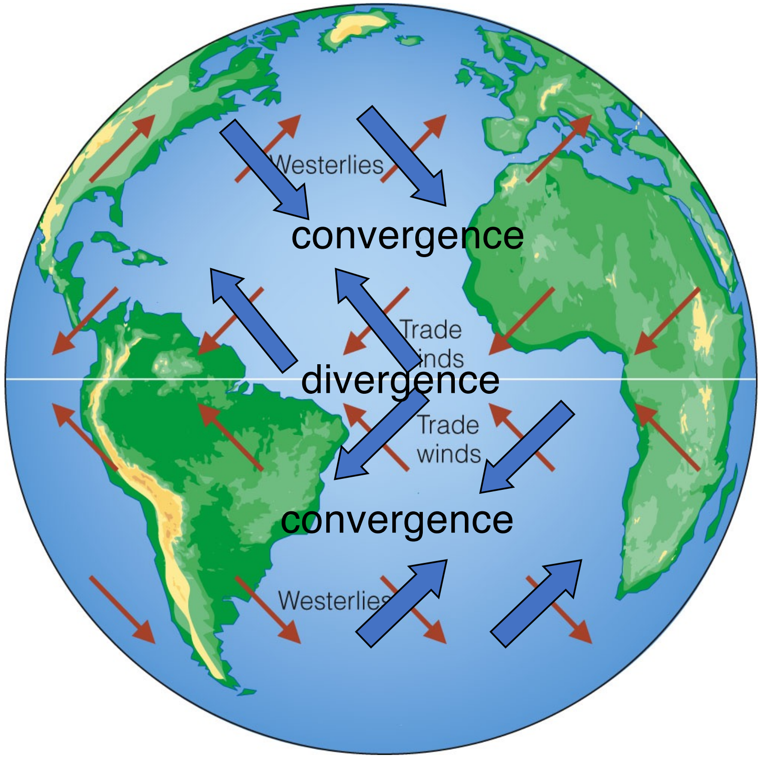
The surface Ekman transport:

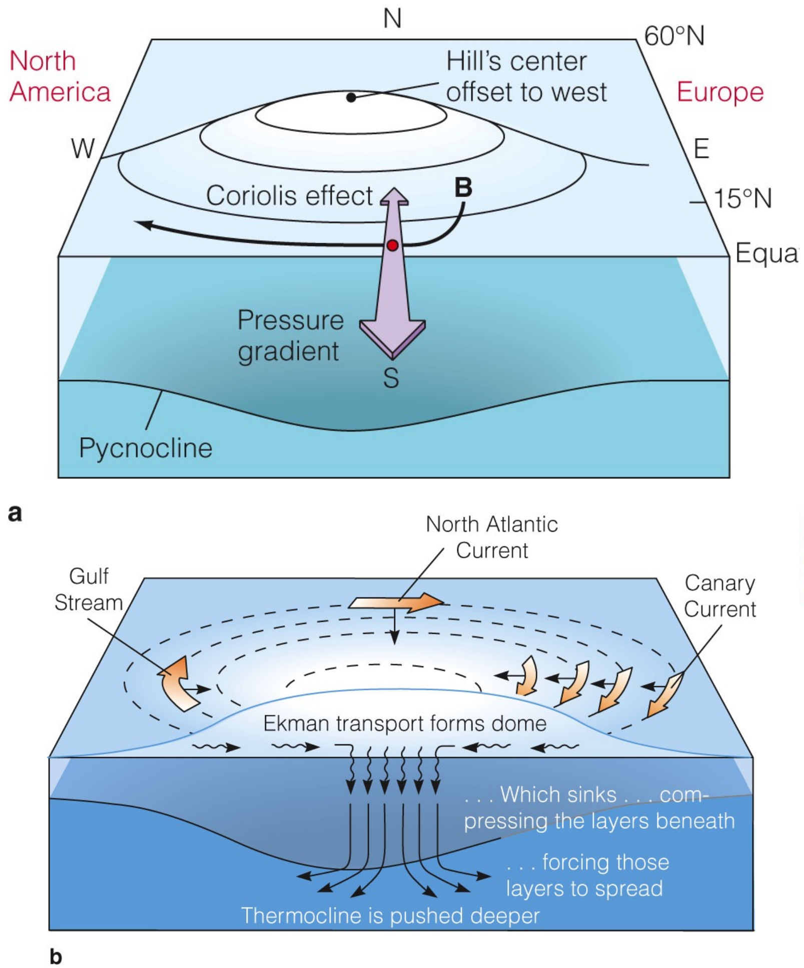
$$\mathbf{U} = \int_{-\infty}^{0} (u - \bar{u})\, dz = \frac{1}{\rho_0 f} \tau^y$$
$$\mathbf{V} = \int_{-\infty}^{0} (v - \bar{v})\, dz = \frac{-1}{\rho_0 f} \tau^x.$$

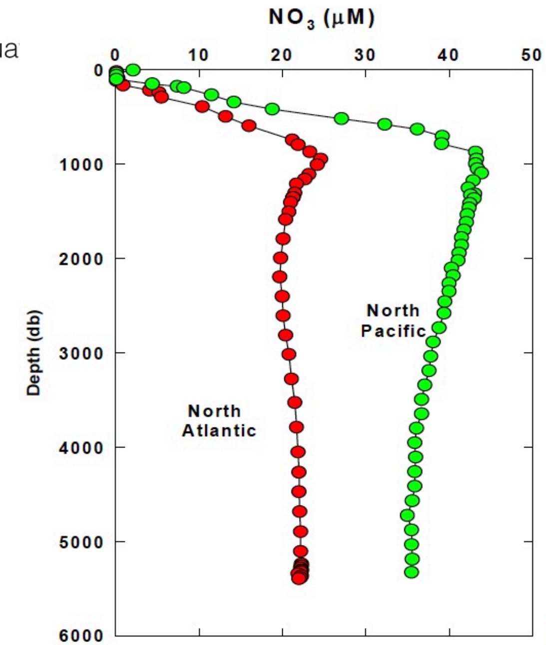
$$\int_{-\infty}^{0} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dz = \frac{1}{\rho_0} \left[\frac{\partial}{\partial x} \left(\frac{\tau^y}{f} \right) - \frac{\partial}{\partial y} \left(\frac{\tau^x}{f} \right) \right]$$

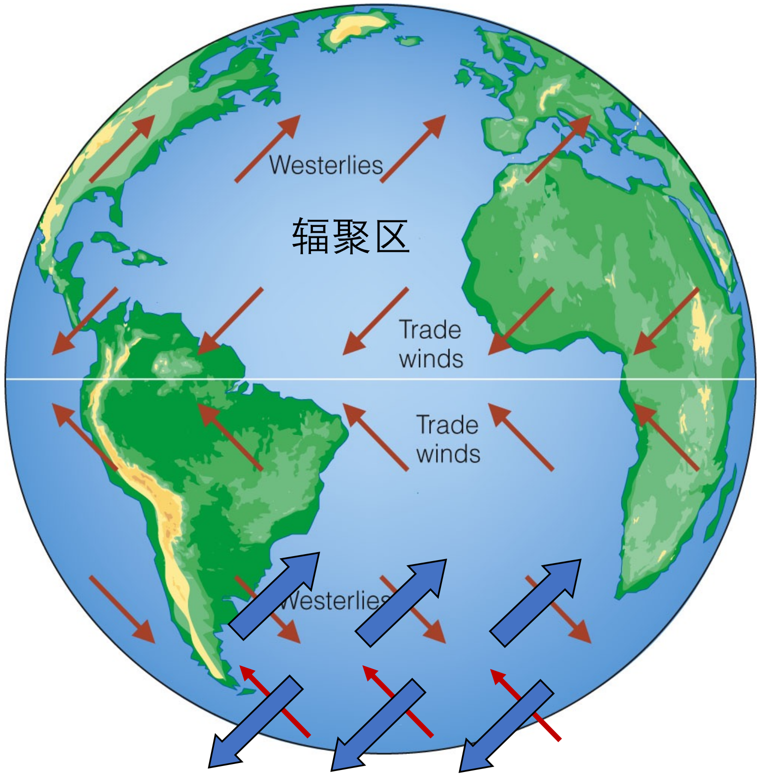
$$\bar{w} = \frac{1}{\rho_0} \left[\frac{\partial}{\partial x} \left(\frac{\tau^y}{f} \right) - \frac{\partial}{\partial y} \left(\frac{\tau^x}{f} \right) \right]$$

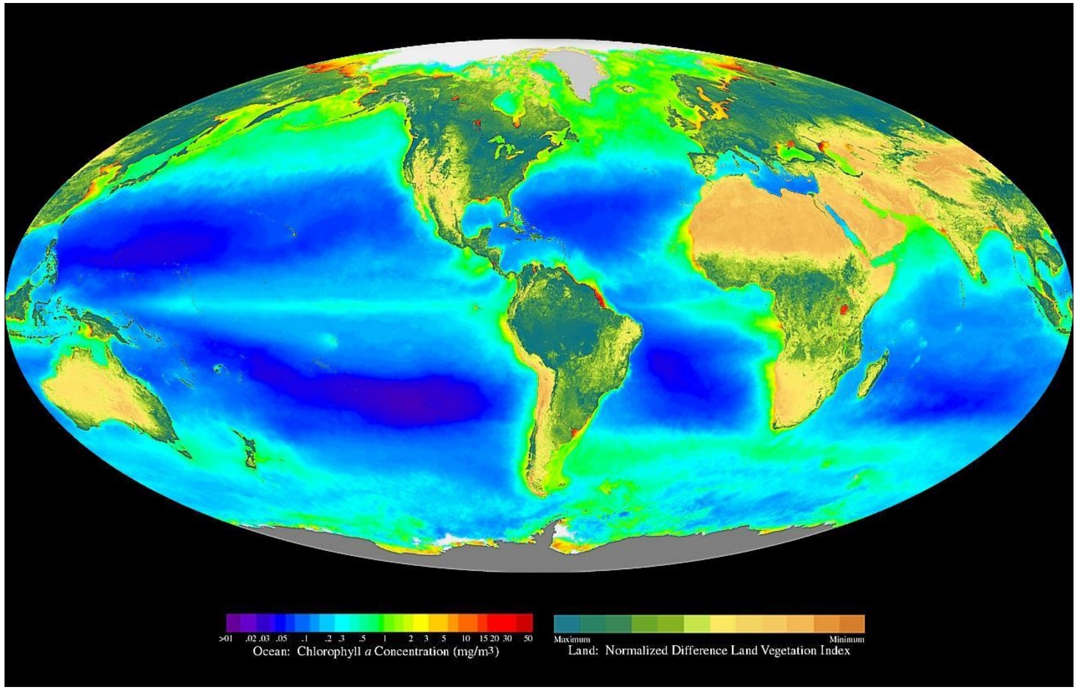



Westerlies
convergence
Trade
divergence
Trade
winds
convergence
Westerlies


N
60°N
North
America
Hill's center
offset to west
Europe
W
E
Coriolis effect
B
15°N
Equa
Pressure
gradient
S
Pycnocline
a
North Atlantic
Current
Gulf
Stream
Canary
Current
Ekman transport forms dome
. . . Which sinks . . . com-
pressing the layers beneath
. . . forcing those
layers to spread
Thermocline is pushed deeper
b


NO_3 (µM)
0
10
20
30
40
50
Depth (db)
0
1000
2000
3000
4000
5000
6000
North
Pacific
North
Atlantic


Westerlies
辐聚区
Trade
winds
Trade
winds
Westerlies


>01 .02 .03 .05 .1 .2 .3 .5 1 2 3 5 10 15 20 30 50
Ocean: Chlorophyll a Concentration (mg/m3)
Maximum
Minimum
Land: Normalized Difference Land Vegetation Index

Surface gravity waves

Assumptions: incompressible fluid, inviscid motion,

small-scale motion ($R_0 \gg 1$), linear motion (non-linear term neglected)

Governing equation:

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} = -\nabla p - \rho \boldsymbol{g}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

Take the curl of the momentum euqation:

$$\frac{\partial \nabla \times \boldsymbol{u}}{\partial t} = 0$$

If the curl of velocity is initially 0, then it remains 0 for all time.

$$\omega = \pm\sqrt{gK tanh KD}$$ dispersion relation

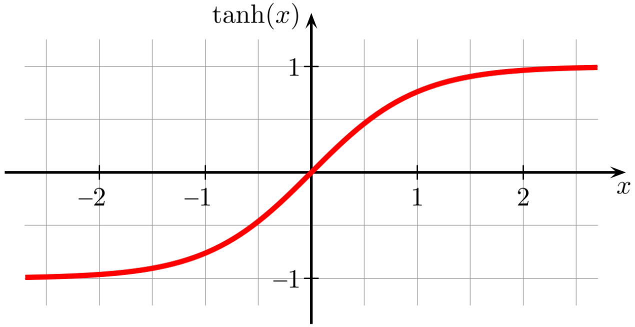
$$c = \frac{\omega}{K} = \pm\sqrt{gD}\left(\frac{tanh KD}{KD}\right)^{1/2}$$

Plane waves with different wavelength have diffferent wave speed, and the pattern will disperse

For $\boldsymbol{KD} \ll \mathbf{1}$ ($\lambda \gg$ D, shallow water waves)

(L'Hopital's rule)

$$\lim_{KD\to 0}\frac{tanh KD}{KD} = \lim_{KD\to 0}\frac{(tanh KD)'}{(KD)'} = \lim_{KD\to 0}\frac{1}{cosh^2 KD} = 1$$



$$c = \sqrt{gD}$$

The wave speed does not depend on the wavelength, non-dispersive waves

For $\boldsymbol{KD} \gg \mathbf{1}$ ($\lambda \ll$ D, deep water waves): $tanh KD \to 1$

$$c = \sqrt{g/K}$$

the wave speed depends on the wavelength, dispersive waves

$$\omega^2 = gK\tanh KD$$

Group velocity:

$$\boldsymbol{c_g} = \frac{\partial\omega}{\partial K}\frac{\boldsymbol{K}}{K}$$

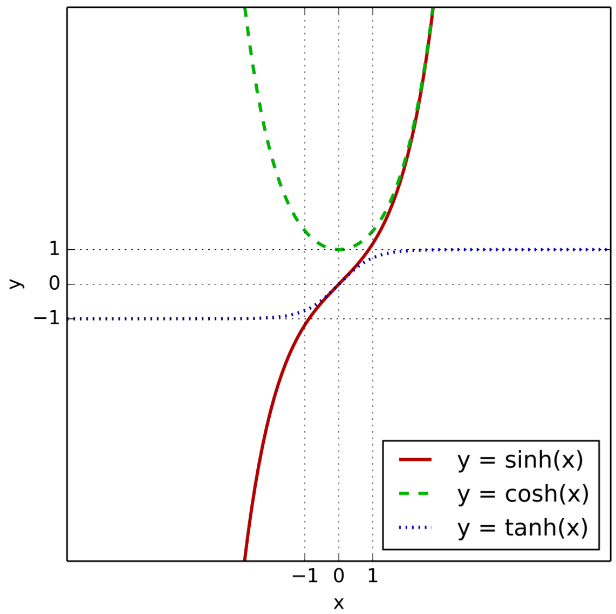
$$2\omega\partial\omega = g[\partial K\tanh KD + K\frac{1}{\cosh^2 KD}D\partial K]$$

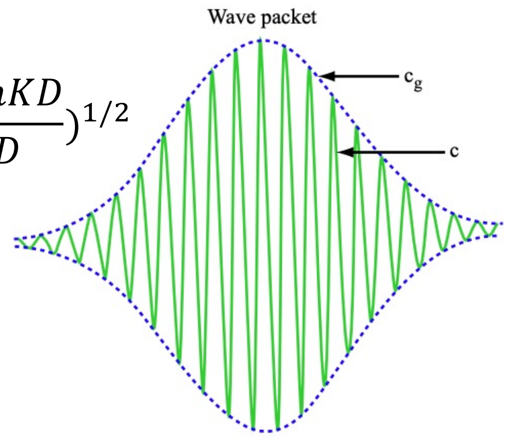
$$\frac{\omega}{K}\frac{\partial\omega}{\partial K} = \frac{1}{2}\frac{g}{K}\left[\tanh KD + KD\frac{1}{\cosh^2 KD}\right]$$

$$c \cdot c_g = \frac{1}{2}\frac{g}{K}(\tanh KD + \frac{KD}{\cosh^2 KD})$$

$$\frac{c_g}{c} = \frac{1}{2}(1 + \frac{KD}{\sinh KD \cdot \cosh KD})$$

$$c = \pm\sqrt{gD}(\frac{\tanh KD}{KD})^{1/2}$$





A wave packet propagating with the group velocity carries a plane wave with crest moving with the phase speed

For $\boldsymbol{KD} \ll \mathbf{1}$ (shallow water waves): $c_g = c$

For $\boldsymbol{KD} \gg \mathbf{1}$ (deep water waves): $c_g = \frac{1}{2}c$

Internal waves – with rotation

Assumptions: stratified, rotational, inviscid, incompressible, small perturbation

The governing equations are:

$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{\rho}{\rho_0}g$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial \rho}{\partial t} + w\frac{d\bar{\rho}}{dz} = 0$$

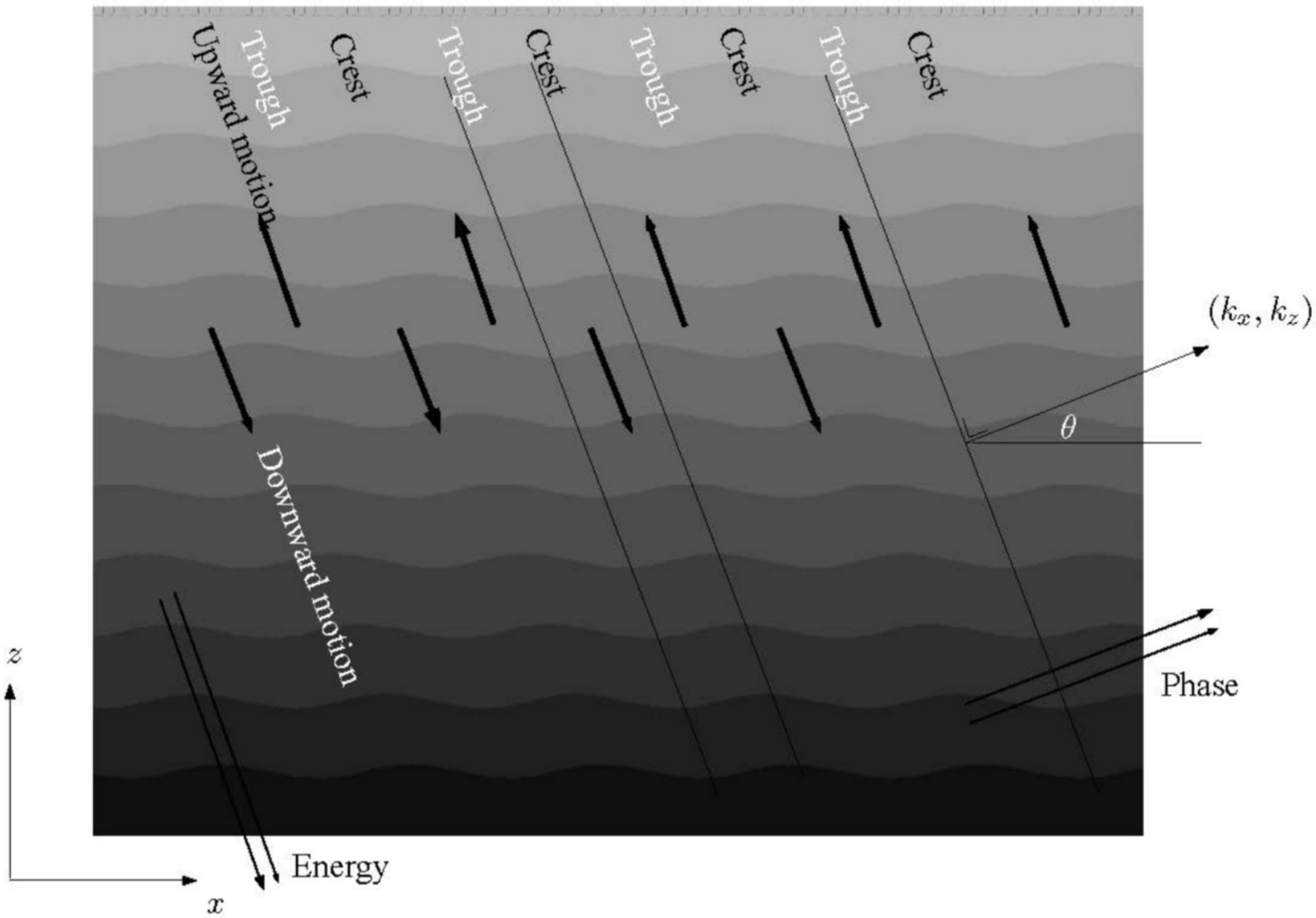


Figure 13-3 Vertical structure of an internal wave.

Shallow water model

Assumptions:

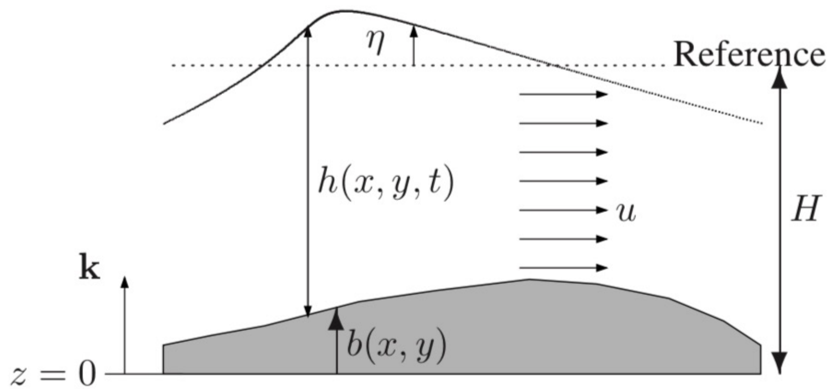
thin layer ($H \ll L$)

inviscid

motions initially independent of z

$$\frac{\partial}{\partial z} = 0 \text{ always}$$

homogeneous (constant density)



$$-g\frac{\partial \eta}{\partial x}$$

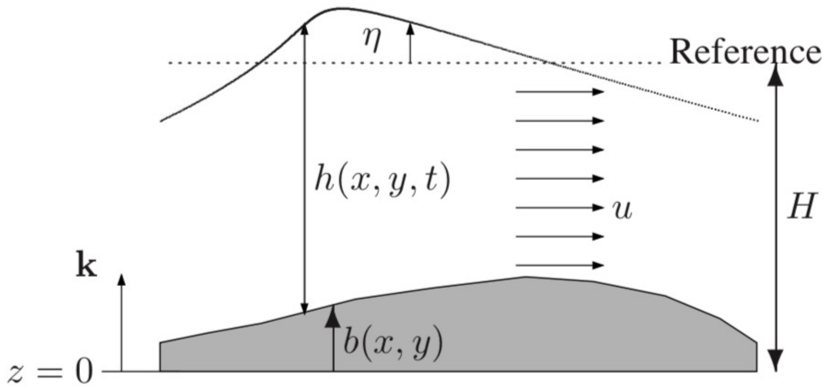
$$\text{x:}\quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \cancel{w\frac{\partial u}{\partial z}} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$\text{y:}\quad \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \cancel{w\frac{\partial v}{\partial z}} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

$$-g\frac{\partial \eta}{\partial y}$$

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$



Integrate vertically from b to η:

$$\int_{z=b}^{z=b+h} \frac{\partial w}{\partial z} dz = -\int_{z=b}^{z=b+h} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz$$

$$w|_{b+h} - w|_{b}$$

$$\frac{\partial \eta}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} = -h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$\boxed{\frac{\partial \eta}{\partial t} + \frac{\partial hu}{\partial x} + \frac{\partial hv}{\partial y} = 0}$$

Boudary conditions

$$w|_{b+h} = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}$$

$$w|_{b} = u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y}$$

$$\boxed{\eta = h + b - H}$$

Inertia-gravity waves (Poincaré waves)

Assumption: flat bottom

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

Give a wave solution:

$$u = Ue^{i(kx+ly-\omega t)}$$

$$v = Ve^{i(kx+ly-\omega t)}$$

$$\eta = Ae^{i(kx+ly-\omega t)}$$

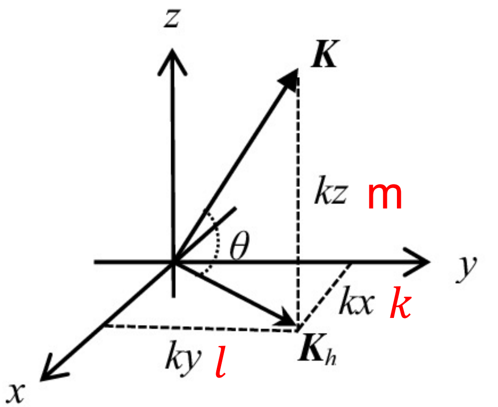


Fig. 1. Wave number vector and its components.

$$-i\omega U - fV = -gikA$$

$$-i\omega V + fU = -gilA$$

$$-i\omega A + ikHU + ilHV = 0$$

$$\begin{pmatrix} -i\omega & -f & gik \\ f & -i\omega & gil \\ ikH & ilH & -i\omega \end{pmatrix} \begin{bmatrix} U \\ V \\ A \end{bmatrix} = 0$$

This homogeneous equation has non-trivial solutions only if the determinant of the coefficient matrix vanishes:

$$\omega[\omega^2 - f^2 - gH(k^2 + l^2)] = 0$$ dispersion relation

1. $\omega = 0, \ \dfrac{\partial}{\partial t} = 0,$ geostrophic flow

$$\boxed{R_d = \frac{\sqrt{gH}}{f}}$$

Rossby deformation radius (barotropic)

2. $\omega = \sqrt{f^2 + gH\underbrace{(k^2 + l^2)}_{K^2}}$

a. rotation is weak, $f^2 \ll gHK^2, \ \lambda \ll R_d$ (short-wave limit)

$$\omega = \sqrt{gH}K, \quad c = \sqrt{gH},$$ gravity waves

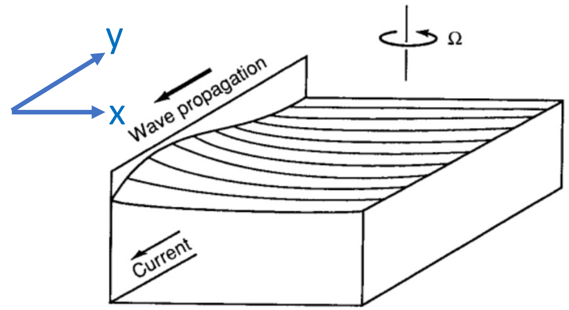
b. rotation is important, $f^2 \gg gHK^2, \ \lambda \gg R_d$ (long-wave limit)

$K\ (k, l)\ is\ small$

pressure gradient term is negliable, equations reduced to inertial-motion

$\omega \sim f,$ inertial oscillations

The Kelvin wave



Assumptions: flat bottom

one side boundary (y-axis)

velocity normal to the boundary is zero everywhere (u=0)

The momentum equations:

geostrophic flow

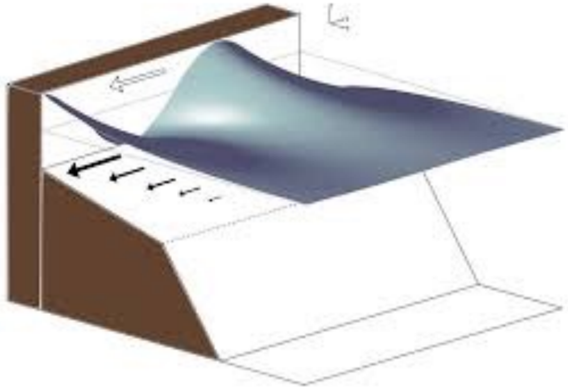
$$\cancel{\frac{\partial u}{\partial t}} - fv = -g\frac{\partial \eta}{\partial x} \quad (1)$$

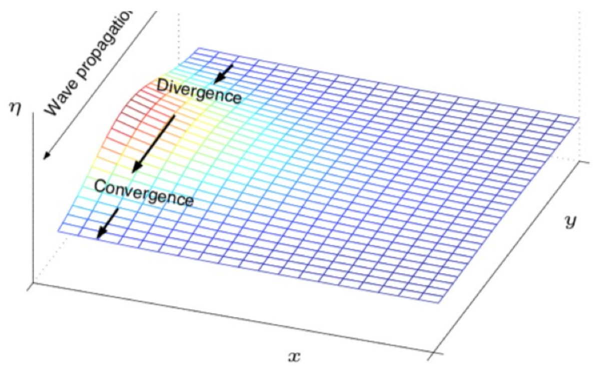
$$\frac{\partial v}{\partial t} + \cancel{fu} = -g\frac{\partial \eta}{\partial y} \quad (2) \quad \Rightarrow \quad \frac{\partial^2 v}{\partial t^2} = -g\frac{\partial}{\partial y}\left(\frac{\partial \eta}{\partial t}\right) = \overset{c^2}{\underline{gH}}\,\frac{\partial^2 v}{\partial y^2}$$

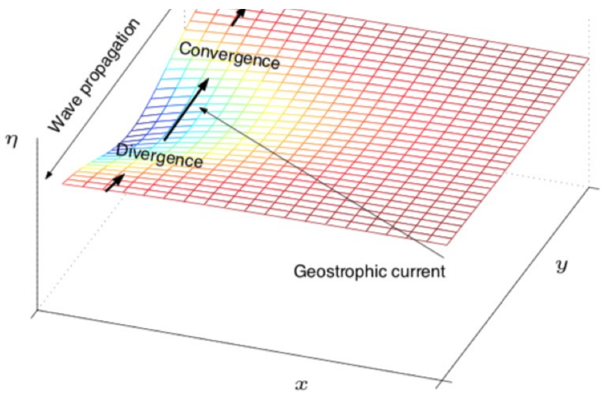
The continuity equation:

$$v = F_1(x)e^{i(ly-\omega t)} + F_2(x)e^{i(ly+\omega t)}$$

$$\frac{\partial \eta}{\partial t} + H\left(\cancel{\frac{\partial u}{\partial x}} + \frac{\partial v}{\partial y}\right) = 0 \quad (3)$$







$$u = 0$$

$$v = Ae^{-x/R_d}e^{i(ly+\omega t)}$$

$$\eta = -A\sqrt{\frac{H}{g}}\,e^{-x/R_d}e^{i(ly+\omega t)}$$

- Kelvin waves propagate with the boundary on the right (left) in the Northern (Southern) Hemisphere
- The wave speed is gravity wave speed ($c = \sqrt{gH}$)
- Velocity perpendicular to the boundary is nail; along-boundary flow is geostrophic
- Surface elevation and along-boundary velocity decay from the boundary to the interior ocean, and the decay scale is R_d (trapped wave)

An upwelling wave ($\eta > 0$) has currents flowing in the direction of wave propagation ($v < 0$)

Shallow water equation – layered models

In a stratified system, as density increases monotonically downward, it can be used as the vertical coordinate

Let $a = z$:

$$\rho(x, y, z, t) \longrightarrow z(x, y, \rho, t)$$

$$0 = z_x + z_\rho \rho_x$$

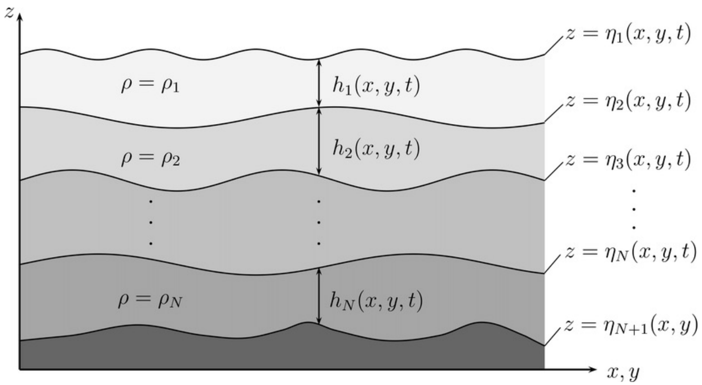
$$0 = z_y + z_\rho \rho_y$$

$$a = a(x, y, \rho, t)$$

$$1 = z_\rho \rho_z$$

$$0 = z_t + z_\rho \rho_t$$

The derivative transformation:



$$\frac{\partial a}{\partial x}\Big|_z = \frac{\partial a}{\partial x}\Big|_\rho + \frac{\partial a}{\partial \rho}\Big|_\rho \frac{\partial \rho}{\partial x}\Big|_z \qquad \frac{\partial a}{\partial x}\Big|_z = \frac{\partial a}{\partial x}\Big|_\rho - \frac{z_x}{z_\rho}\frac{\partial a}{\partial \rho}\Big|_\rho$$

$$\frac{\partial a}{\partial y}\Big|_z = \frac{\partial a}{\partial y}\Big|_\rho + \frac{\partial a}{\partial \rho}\Big|_\rho \frac{\partial \rho}{\partial y}\Big|_z \qquad \frac{\partial a}{\partial y}\Big|_z = \frac{\partial a}{\partial y}\Big|_\rho - \frac{z_y}{z_\rho}\frac{\partial a}{\partial \rho}\Big|_\rho$$

$$\frac{\partial a}{\partial z}\Big|_z = \frac{\partial a}{\partial \rho}\Big|_\rho \frac{\partial \rho}{\partial z}\Big|_z \qquad \frac{\partial a}{\partial z}\Big|_z = \frac{1}{z_\rho}\frac{\partial a}{\partial \rho}\Big|_\rho$$

$$\frac{\partial a}{\partial t}\Big|_z = \frac{\partial a}{\partial t}\Big|_\rho + \frac{\partial a}{\partial \rho}\Big|_\rho \frac{\partial \rho}{\partial t}\Big|_z \qquad \frac{\partial a}{\partial t}\Big|_z = \frac{\partial a}{\partial t}\Big|_\rho - \frac{z_t}{z_\rho}\frac{\partial a}{\partial \rho}\Big|_\rho$$

Without friction, the governing equations in density coordinate become:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -\frac{1}{\rho_0}\frac{\partial P}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -\frac{1}{\rho_0}\frac{\partial P}{\partial y}$$

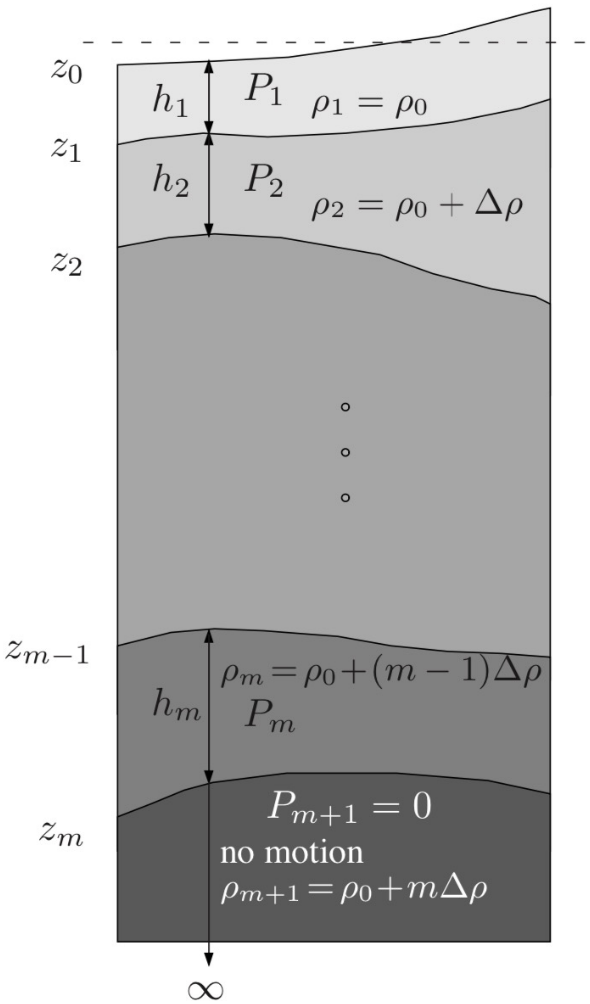
$$\frac{\partial P}{\partial \rho} = gz$$

$$\frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} + \frac{\partial hv}{\partial y} = 0$$

$$h = -\Delta\rho\frac{\partial z}{\partial \rho}$$

the thickness of a fluid layer between ρ and $\rho + \Delta\rho$

Reduced gravity model



The lowest layer may be imagined to be infinitely deep and at rest

$$P_{m+1} = 0$$

Rigid-lid approximation: $z_0 = 0$

upward calculation

$$\boxed{P_{k+1} = P_k + \Delta\rho g z_k}$$

One layer:

$$z_1 = -h_1 \qquad P_1 = \rho_0 g' h_1$$

Two layers:

$$z_1 = -h_1 \qquad P_1 = \rho_0 g'(2h_1 + h_2)$$
$$z_2 = -h_1 - h_2 \qquad P_2 = \rho_0 g'(h_1 + h_2)$$

Three layers:

$$z_1 = -h_1 \qquad P_1 = \rho_0 g'(3h_1 + 2h_2 + h_3)$$
$$z_2 = -h_1 - h_2 \qquad P_2 = \rho_0 g'(2h_1 + 2h_2 + h_3)$$
$$z_3 = -h_1 - h_2 - h_3 \qquad P_3 = \rho_0 g'(h_1 + h_2 + h_3)$$

shallow-water reduced gravity model – one layer

One layer:

$$z_1 = -h_1 \qquad\qquad P_1 = \rho_0 g' h_1$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g'\frac{\partial h}{\partial x} \quad (1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g'\frac{\partial h}{\partial y} \quad (2)$$

$$\frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} + \frac{\partial hv}{\partial y} = 0 \quad \Longrightarrow \quad \frac{dh}{dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

Review for homogeneous fluids:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial \eta}{\partial y}$$

Potential vorticity conservation for the shallow water model

The horizontal momentum equations (**homogeneous, invisicd,** $\boldsymbol{\frac{\partial}{\partial z} = 0}$):

$$\frac{\partial \mathrm{u}}{\partial \mathrm{t}} + \mathrm{u}\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \mathrm{v}\frac{\partial \mathrm{u}}{\partial \mathrm{y}} - fv = -g\frac{\partial \eta}{\partial x} \quad (1)$$

$$\frac{\partial \mathrm{v}}{\partial \mathrm{t}} + \mathrm{u}\frac{\partial \mathrm{v}}{\partial \mathrm{x}} + \mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{y}} + fu = -g\frac{\partial \eta}{\partial y} \quad (2)$$

Taking the curl of the momentum equations by doing $\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(1)$:

$$\frac{\partial}{\partial \mathrm{t}}\left(\frac{\partial \mathrm{v}}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{\partial u}{\partial x}\frac{\partial \mathrm{v}}{\partial x} + u\frac{\partial^2 v}{\partial x^2} - \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + v\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} - v\frac{\partial^2 u}{\partial y^2} + f\frac{\partial u}{\partial x} + \frac{\partial f}{\partial y}v + f\frac{\partial v}{\partial y} = 0$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial u}{\partial x}\zeta + u\frac{\partial \zeta}{\partial x} + \frac{\partial v}{\partial y}\zeta + v\frac{\partial \zeta}{\partial y} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v = 0$$

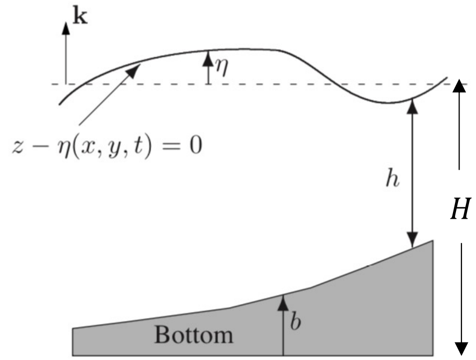
$$\frac{d\zeta}{dt} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(f + \zeta) + \beta v = 0$$

$$\boxed{\frac{df}{dt} = \frac{\partial f}{dt} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} = \beta v}$$

$$\frac{d(f + \zeta)}{dt} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(f + \zeta) = 0$$

$$\frac{d(f+\zeta)}{dt} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(f+\zeta) = 0 \ (1)$$

The continuity equation:



$$\frac{\partial \eta}{\partial t} + \frac{\partial hu}{\partial x} + \frac{\partial hv}{\partial y} = 0$$

$$\boxed{\eta = h + b - H}$$

$$\frac{dh}{dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \ \ (2)$$

Combining (1) and (2):

$$\frac{d(f+\zeta)}{dt} - \frac{f+\zeta}{h}\frac{dh}{dt} = 0$$

f: planetary vorticity

ζ: relative vorticity

$f+\zeta$: absolute vorticity

$$\boxed{\frac{d}{dt}\left(\frac{f+\zeta}{h}\right) = 0}$$

potential vorticity

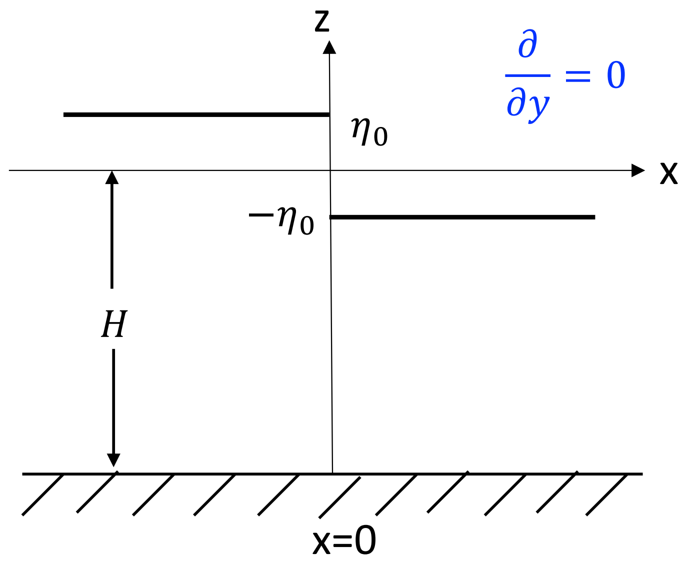
$$\frac{\zeta}{f} \sim \frac{U/L}{f} \sim \frac{U}{fL}$$ Rossby number

Geostrophic (Rossby) adjustment - barotropic

Initial state (unbalanced):

$$\eta = \begin{cases} \eta_0, & x < 0 \\ -\eta_0, & x > 0 \end{cases} \qquad \eta_0 << H$$

$$u = v = 0$$



For a steady final state (based on shallow water model):

$$\cancel{\frac{\partial u}{\partial t}} + u\cancel{\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}} - fv = -g\frac{\partial \eta}{\partial x}$$

$$\cancel{\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial x}} + \cancel{v\frac{\partial v}{\partial y}} + \cancel{fu} = \cancel{-g\frac{\partial \eta}{\partial y}}$$

geostrophic flow

$$\cancel{\frac{\partial \eta}{\partial t}} + \frac{\partial hu}{\partial x} + \cancel{\frac{\partial hv}{\partial y}} = 0$$

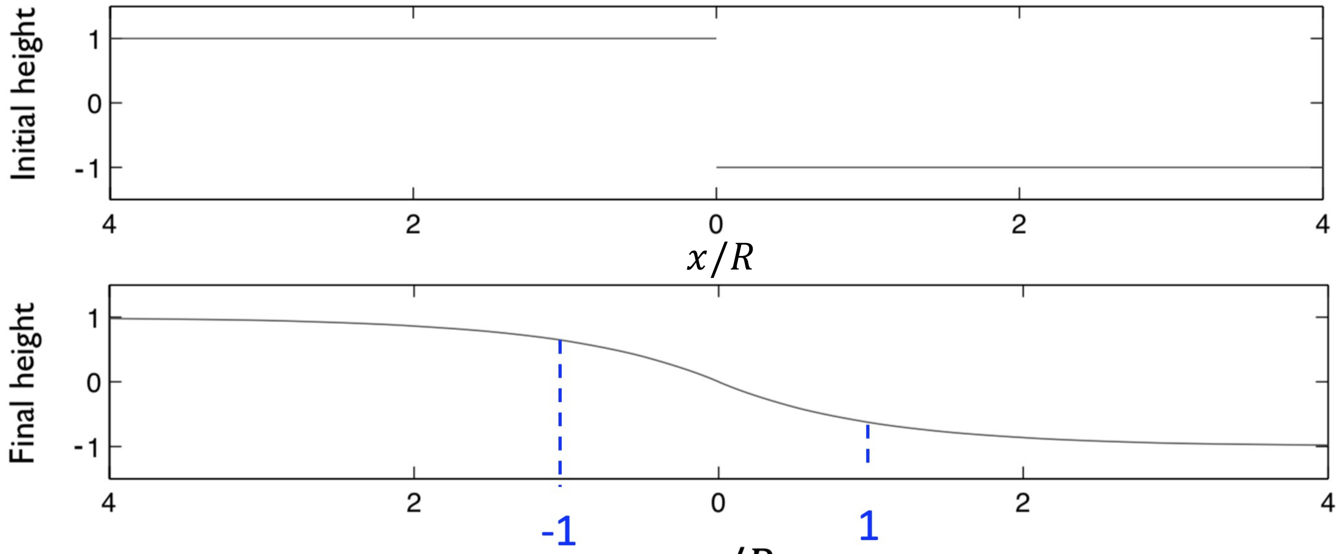
sufficient condition: $u = 0$ everywhere

$x \to -R, \quad \eta \to \eta_0$

$$\eta = \begin{cases} -\eta_0 e^{\frac{x}{R}} + \eta_0, & x < 0 \\ \eta_0 e^{-\frac{x}{R}} - \eta_0, & x > 0 \end{cases}$$

$x \to R, \quad \eta \to -\eta_0$

η/η_0



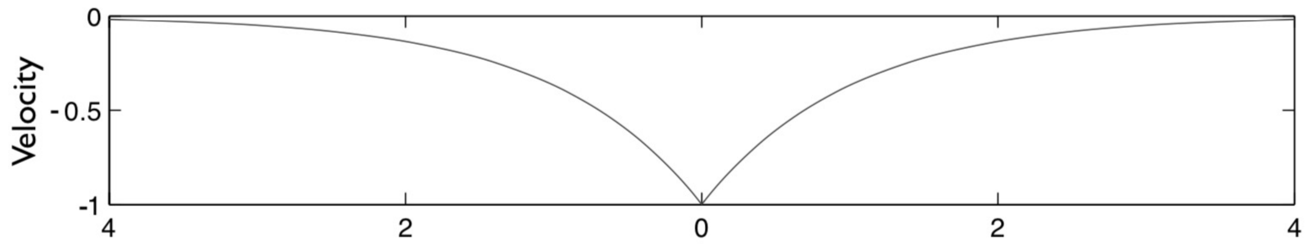
x/R

The adjustment spatial scale is the Rossby deformation radius $R = \frac{\sqrt{gH}}{f}$

$x \to -R, \quad v \to 0$

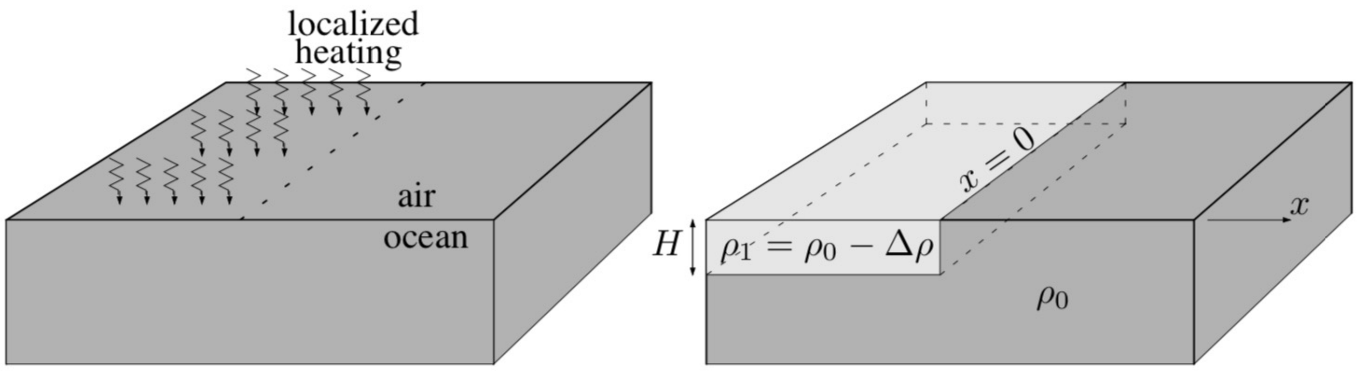
$$v = \frac{g}{f}\frac{\partial \eta}{\partial x} = \begin{cases} -\sqrt{\frac{g}{H}}\,\eta_0 e^{\frac{x}{R}}, & x < 0 \\ -\sqrt{\frac{g}{H}}\,\eta_0 e^{-\frac{x}{R}}, & x > 0 \end{cases}$$

$x \to R, \quad v \to 0$



x/R

Geostrophic adjustment - baroclinic



(a) Initial state (b) Immediately after heating event

$$\frac{\partial}{\partial y} = 0$$

For the lighter layer:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - fv = -g'\frac{\partial h}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + fu = 0$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0.$$

$$f(x) = Ae^{x/R}$$

$x \to d, \qquad h \to 0: \qquad f(x) \to -\mathrm{H}$

$\mathrm{h} = \mathrm{f(x)} + H$

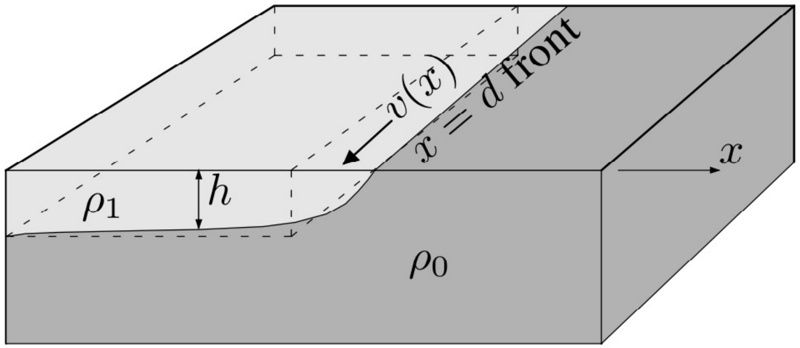
$$f(x) = Be^{(x-d)/R} \qquad B = -H$$

$$h = H(1 - e^{\frac{x-d}{R}})$$

$$-fv = -g'\frac{dh}{dx}$$

$$v = -\sqrt{g'H}\, e^{\frac{x-d}{R}}$$

The depleted volume of light water in x < 0 should be equal to the volume of light water in x > 0:



(c) After adjustment

$$\int_{-\infty}^{0}(H - h)dx = \int_{0}^{d} h dx$$

$$H\int_{-\infty}^{0} e^{\frac{x-d}{R}} dx = Hd - H\int_{0}^{d} e^{\frac{x-d}{R}} dx$$

$$Re^{\frac{x-d}{R}}|_{-\infty}^{0} = d - Re^{\frac{x-d}{R}}|_{0}^{d}$$

$d = R \qquad$ adjustment spatial scale is R

Planetary Rossby Waves (Barotropic)

Assumptions: shallow-water model

flat bottom

β-plane approximation: $f = f_0 + \beta_0 y$ $\boxed{\beta_0 = 2(\Omega/a)\cos\varphi_0}$

$$\frac{\beta_0 L}{f_0} \ll 1$$

The linearized governing equations:

$$\frac{\partial u}{\partial t} - (f_0 + \beta_0 y)v = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + (f_0 + \beta_0 y)u = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0,$$

large terms

small terms

$$\omega = -\beta_0 R^2 \frac{k}{1 + R^2(k^2 + l^2)}$$

If $\beta_0 = 0$, $\omega = 0$, geostrophic flow

If $R^2 K^2 \ll 1$, $L \gg R$, long wave:

$$\omega \sim \frac{\beta_0 R^2}{L} \ll \beta_0 \mathrm{L} \ll f_0$$

$$\boxed{\frac{\beta_0 L}{f_0} \ll 1}$$

If $R^2 K^2 \geq 1$, $L \leq R$, short wave:

$$\omega \sim \beta_0 \mathrm{L} \ll f_0$$

Planetary Rossby waves are subinertial (low frequency) waves

The zonal wave speed:

$$c = \frac{\omega}{k} = \frac{-\beta_0 R^2}{1 + R^2(k^2 + l^2)} \qquad < 0$$

Planetary Rossby waves always propagate westward in the zonal direction

For very long waves ($R^2 K^2 \ll 1$):

$$c_x = -\beta_0 R^2 \qquad \text{maximum zonal wave speed}$$

The dispersion relation can be reorganized as:

$$\omega = -\beta_0 R^2 \frac{k}{1 + R^2(k^2 + l^2)}$$

$$(k + \frac{\beta_0}{2\omega})^2 + l^2 = \frac{\beta_0^2}{4\omega^2} - \frac{1}{R^2}$$

$$\omega \leq \frac{\beta_0 R}{2} \qquad \text{maximum frequency}$$

The dispersion relation diagram

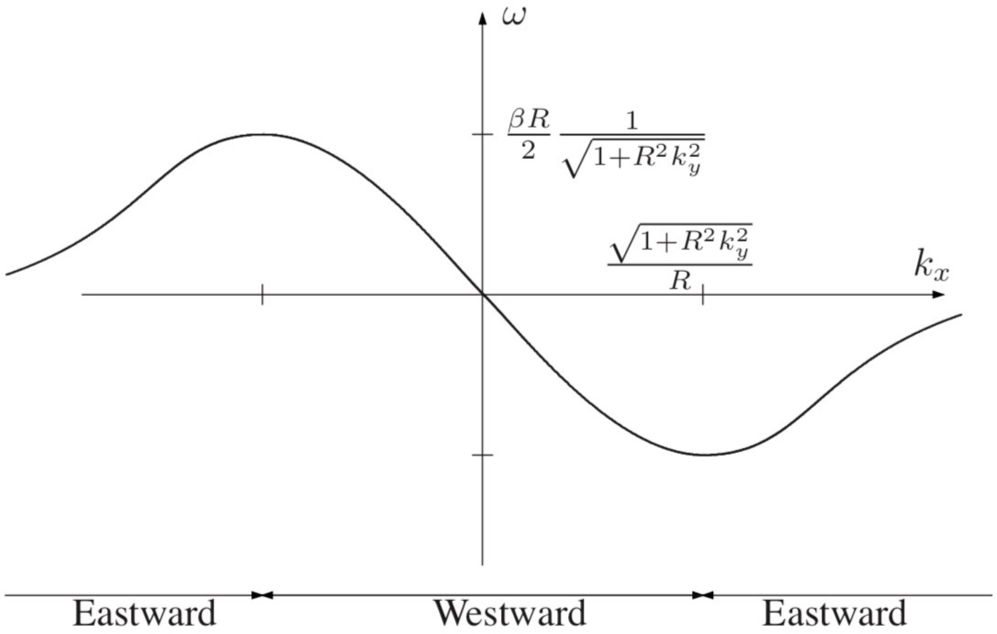
$$\omega = -\beta_0 R^2 \frac{k}{1 + R^2(k^2 + l^2)}$$

$$k = 0, \;\; \omega = 0$$

$$k > 0, \;\; \omega < 0; \quad k < 0, \quad \omega > 0$$

$$k \to \infty, \omega \to 0$$

$$\frac{d\omega}{dk} = 0, \;\; k = \pm \frac{\sqrt{1+R^2 l^2}}{R}, \quad |\omega| = \frac{\beta R}{2} \frac{1}{\sqrt{1+R^2 l^2}}$$



for long Rossby waves, energy propagates westward

for short Rossby waves, energy propagates eastward (opposite to wave propagation)

Topographic Rossby Waves

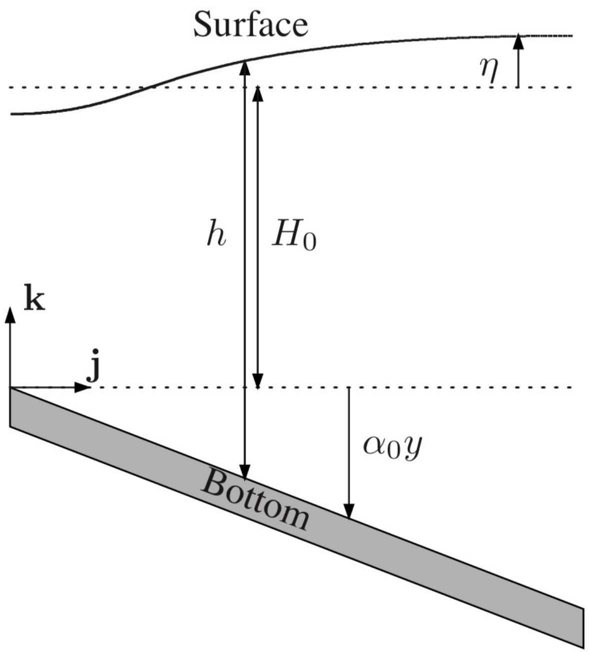
Assumptions: shallow water model

sloping topography (small angle)

$$H = H_0 + \alpha_0 y$$

$$\alpha = \frac{\alpha_0 L}{H_0} \ll 1$$

$$h(x,\, y,\, t) = H_0 + \alpha_0 y + \eta(x,\, y,\, t)$$



The continuity equation: $$\frac{\partial \eta}{\partial t} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} = 0$$

$$\frac{\partial \eta}{\partial t} + \cancel{\left(u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}\right)} + (H_0 + \cancel{\alpha_0 y})\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$
$$+ \eta\cancel{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)} + \alpha_0 v = 0.$$

small-amplitude (linear) waves:

$$\Delta H \ll H$$

$$\omega = \frac{\alpha_0 g}{f} \frac{k}{1 + R^2(k^2 + l^2)}$$

If $\alpha_0 = 0, \quad \omega = 0,$ gesotrophic flow

$$\alpha = \frac{\alpha_0 L}{H_0} \ll 1$$

If $R^2K^2 \ll 1, \; L \gg R,$ long wave:

$$\omega = \frac{\alpha_0 g k}{f} \sim \frac{\alpha_0 L g}{f L^2} \ll \frac{\alpha_0 L g}{f R^2} (\sim \frac{\alpha_0 L g f^2}{f g H}) \ll f$$

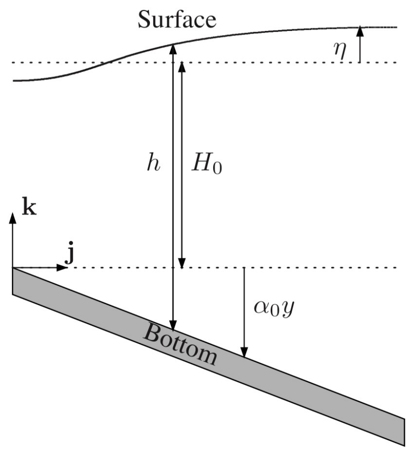
If $R^2K^2 \geq 1, \; L \leq R,$ short wave:

$$\omega = \frac{\alpha_0 g}{f} \frac{k}{R^2 K^2} \sim \frac{\alpha_0 g}{f} \frac{L}{R^2} \sim \frac{\alpha_0 L}{f} \frac{g f^2}{g H} \quad \ll f$$

Topographic Rossby waves are also subinertial (low frequency) waves

The zonal wave speed:

$$c = \frac{\omega}{k} = \frac{\alpha_0 g}{f} \frac{1}{1 + R^2(k^2 + l^2)} \quad >0$$



topographic Rossby waves propogate with shallow water on the right (left) in the northern (southern) hemisphere

For very long waves ($R^2K^2 \ll 1$):

$$c_x = \frac{\alpha_0 g}{f} \qquad \text{maximum zonal wave speed}$$

The dispersion relation can be reorganized as:

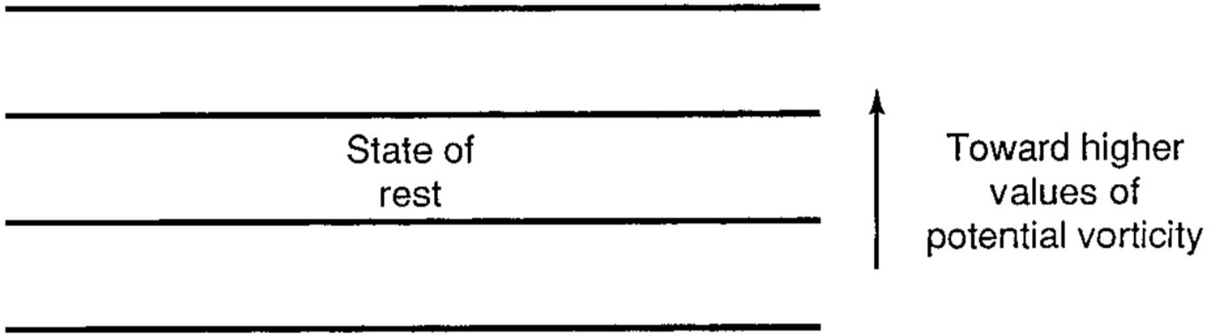
$$\omega = \frac{\alpha_0 g}{f} \frac{k}{1 + R^2(k^2 + l^2)}$$

$$(k - \frac{\alpha_0 g}{2f\omega R^2})^2 + l^2 = (\frac{\alpha_0^2 g^2}{4f^2R^4\omega^2} - \frac{1}{R^2})$$

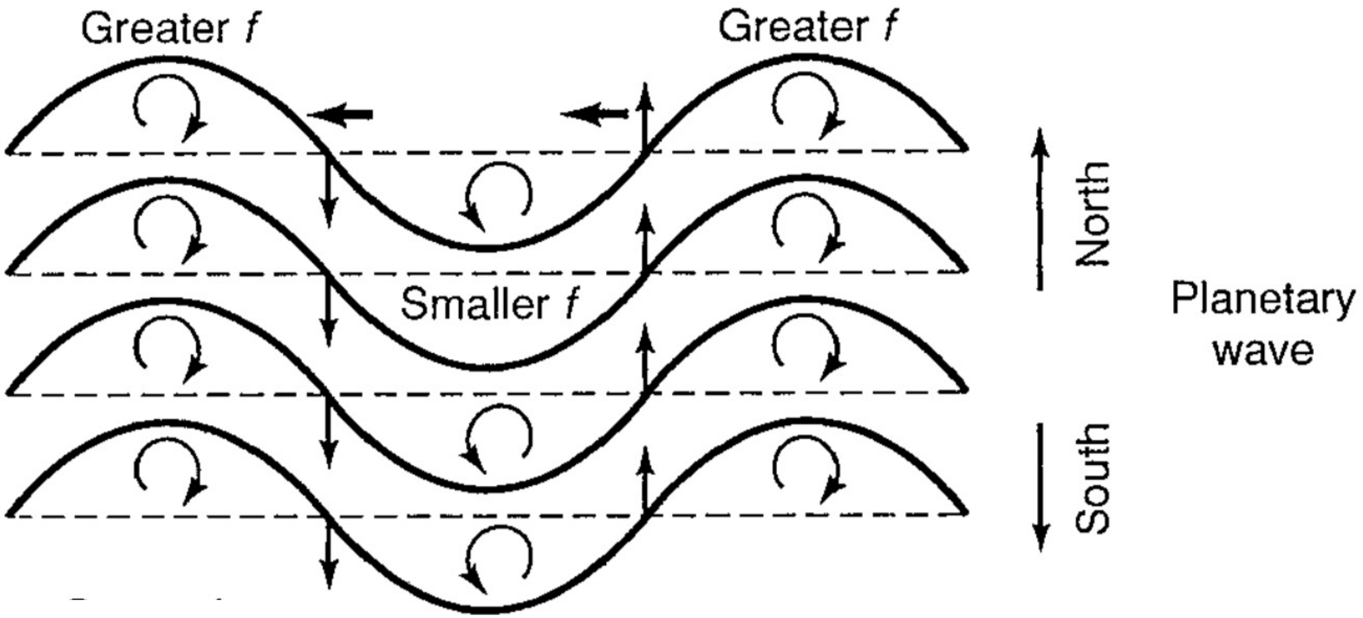
$$\omega < \frac{\alpha_0 g}{2fR} \qquad \text{maximum frequency}$$

Mechanisms for Rossby Wave propagation

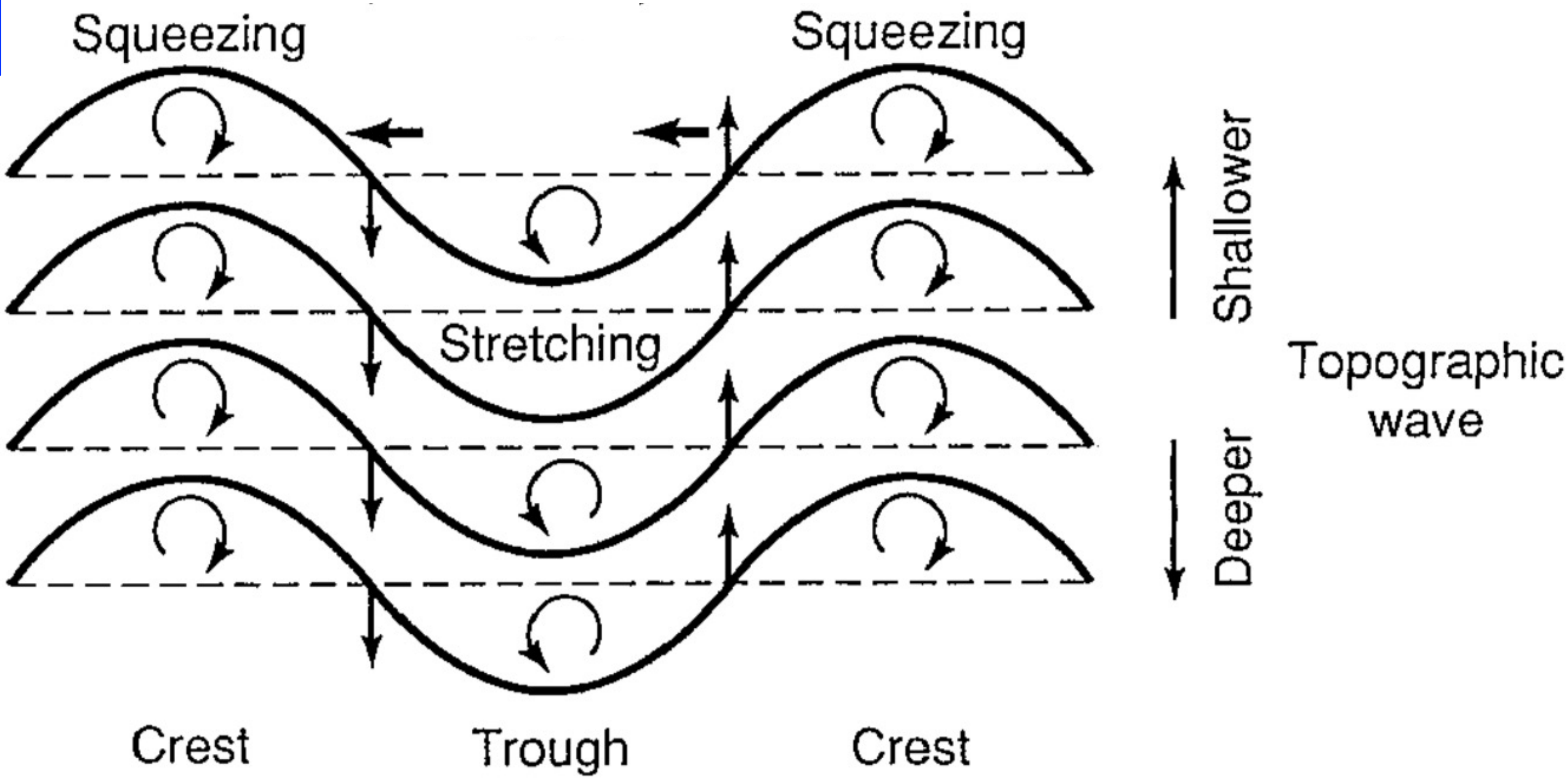
$$\boxed{\frac{d}{dt}\left(\frac{f + \zeta}{h}\right) = 0}$$



North on the right



$$\boxed{\frac{d}{dt}\left(\frac{f + \zeta}{h}\right) = 0}$$



Shallow-water on the right

Quasi-geostrophic dynamics for stratified fluids

$$\rho = \bar{\rho}(z) + \rho'(x,y,z,t) \quad \text{with} \quad |\rho'| \ll |\bar{\rho}|$$

$$p = \bar{p}(z) + p'(x,y,z,t)$$

Governing equations:

$$\overset{S}{\frac{du}{dt}} - \overset{L}{f_0 v} - \overset{S}{\beta_0 y v} = \overset{L}{-\frac{1}{\rho_0}\frac{\partial p'}{\partial x}}$$

$$\frac{dv}{dt} + f_0 u + \beta_0 y u = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}$$

$$0 = -\frac{\partial p'}{\partial z} - \rho' g$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial \rho'}{\partial t} + u\frac{\partial \rho'}{\partial x} + v\frac{\partial \rho'}{\partial y} + w\frac{d\bar{\rho}}{dz} = 0$$

Balance of large terms:

$$-f_0 v = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$

$$+f_0 u = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}$$

$$\boxed{\begin{aligned} u_g &= -\frac{1}{f_0\rho_0}\frac{\partial p'}{\partial y} \\ v_g &= +\frac{1}{f_0\rho_0}\frac{\partial p'}{\partial x} \end{aligned}}$$

$$\frac{\partial}{\partial t}\left[\nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right) + \beta_0 y\right] + J\left(\psi, \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right)\right) + J(\psi, \beta_0 y) = 0$$

ζ planetary vorticity

$$q = \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right) + \beta_0 y$$

potential vorticity

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0$$

$$\frac{\partial q}{\partial t} + \frac{\partial\psi}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial q}{\partial x} = 0$$

$$\frac{\partial q}{\partial t} + u_g\frac{\partial q}{\partial x} + v_g\frac{\partial q}{\partial y} = 0$$

$$\frac{dq}{dt} = 0$$

potential vorticity conservation

relative vorticity　　　planetary vorticity

$$q = \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right) + \beta_0 y$$

$$-\frac{g}{\rho_0 f_0}\frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\rho'\right)$$ vertical stretching

$$0 = -\frac{\partial p'}{\partial z} - \rho' g \qquad p' = \rho_0 f_0 \psi \qquad \frac{\partial\psi}{\partial z} = -\frac{g}{\rho_0 f_0}\rho'$$

relative vorticity: $\frac{U}{L}$　　　vertical stretching: $\frac{f_0^2 UL}{N^2H^2}$

$$\frac{\frac{U}{L}}{\frac{f_0^2 UL}{N^2H^2}} = \frac{N^2H^2}{f_0^2L^2} = \frac{\frac{U^2}{f_0^2L^2}}{\frac{U^2}{N^2H^2}} = \left(\frac{R_0}{Fr}\right)^2$$

Bu: Burger number

$Bu < 1$, rotation is more important, vertical stretching dominates the PV

$Bu > 1$, stratification is more important, relative vorticity dominates the PV

Burger number:

$$Bu = \left(\frac{R_0}{Fr}\right)^2$$

$$= \frac{U^2}{f^2L^2} \Big/ \frac{U^2}{N^2H^2}$$

$$= \frac{N^2H^2}{f^2L^2} = \frac{g'H}{f^2L^2} = \frac{R^2}{L^2}$$

a measure of relative importance of rotation and stratification

$$\boxed{N^2 = -\frac{g}{\rho_0}\frac{d\rho}{dz} \simeq \frac{g}{\rho_0}\frac{\Delta\rho}{H} = \frac{g'}{H}}$$

$L < R$, $Bu > 1$, $Fr < R_0$, motion is more affected by stratification

$L > R$, $Bu < 1$, $R_0 < Fr$, motion is more affected by rotation

$$Fr^2 = \frac{U^2}{N^2H^2} = \frac{U^2}{g'H} \qquad Fr = \frac{U}{\sqrt{g'H}}$$

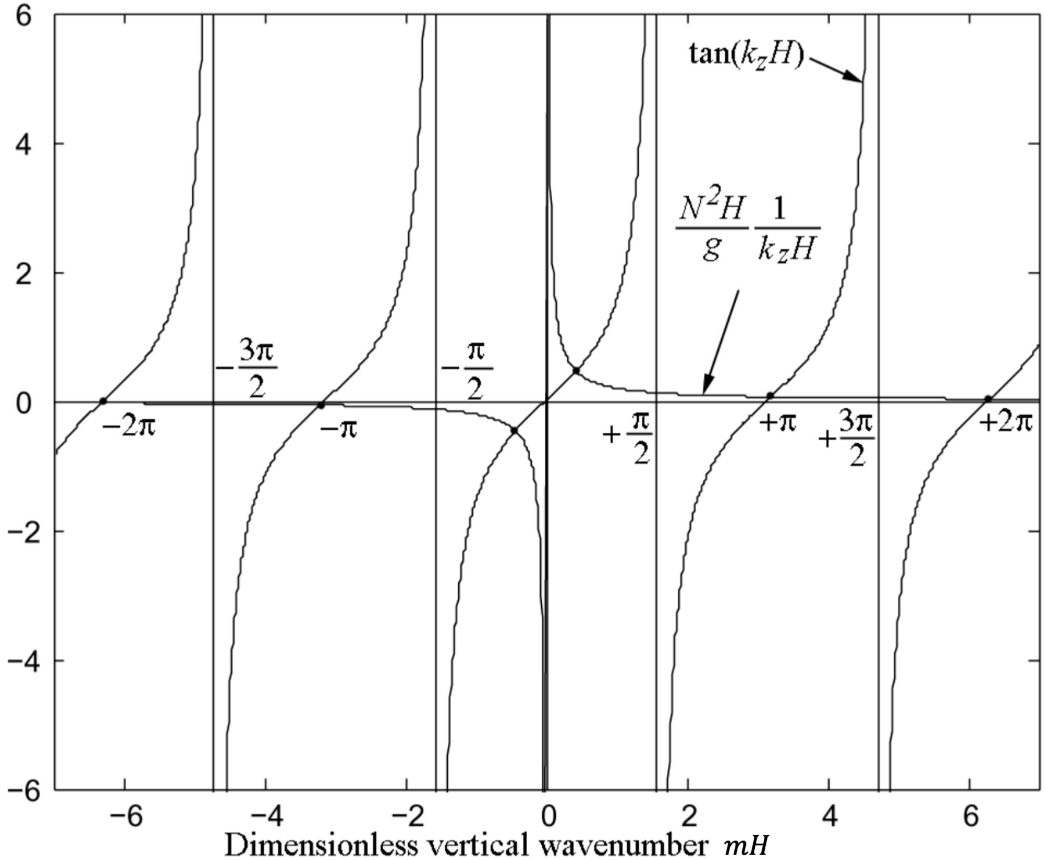
$\sqrt{g'H}$: internal gravity wave speed

• Planetary waves in stratified fluids

$$tanmH = \frac{N^2 H}{g}\frac{1}{mH}$$

$$\boxed{\omega = -\frac{\beta_0 k}{k^2 + l^2 + m^2 f_0^2/N^2}}$$

$$\frac{N^2 H}{g} \sim \frac{g}{\rho_0}\frac{\Delta\rho}{H}\frac{H}{g} \sim \frac{\Delta\rho}{\rho_0}$$



For the first solution: mH is small

$$mH = \frac{N^2 H}{g}\frac{1}{mH}$$

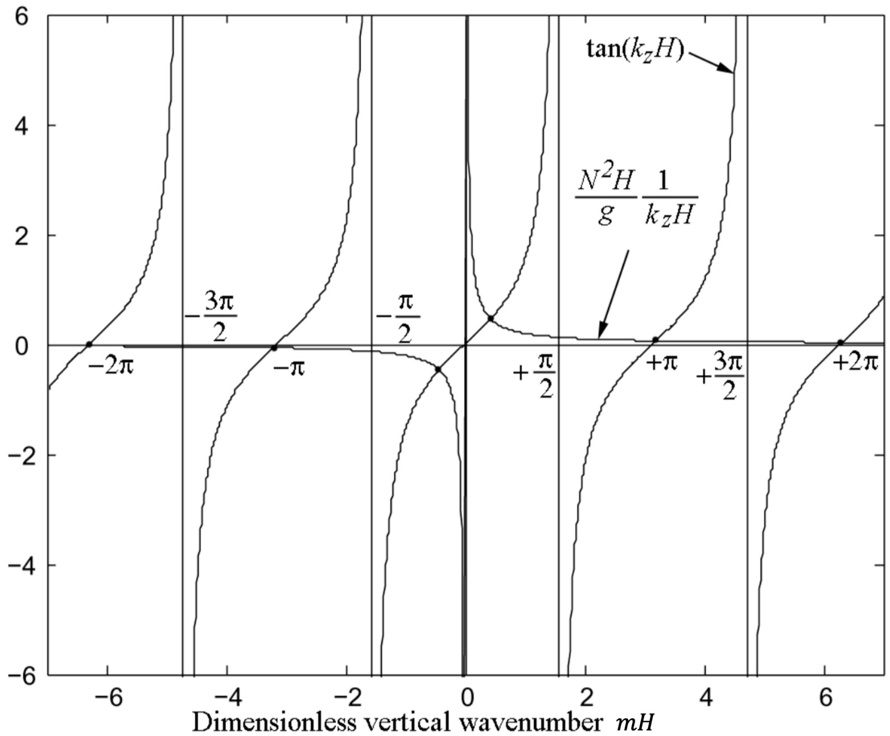
$$m = \frac{N}{\sqrt{gH}}$$

$$\omega = -\frac{\beta_0 k}{k^2 + l^2 + f_0^2/gH}$$

$$= -\beta_0 R^2 \frac{k}{1 + R^2(k^2 + l^2)}$$

barotropic Rossby wave

$$\boxed{\omega = -\frac{\beta_0 k}{k^2 + l^2 + m^2 f_0^2 / N^2}}$$



For larger solutions:

$$tanmH \sim 0$$

$$m_j = j\frac{\pi}{H}, j = 1, 2, 3, \ldots$$

baroclinic modes

$$\omega_j = -\frac{\beta_0 k}{k^2 + l^2 + (j\pi f_0 / NH)^2}$$

$$c_x < 0$$ westward propagation

$$R_j = \frac{1}{j}\frac{NH}{\pi f_0}$$

The maximum wave speed in the x direction is when

$$k^2 + l^2 \rightarrow 0$$ long waves

$$|c_x|_{max} = \beta_0 R_j^2$$

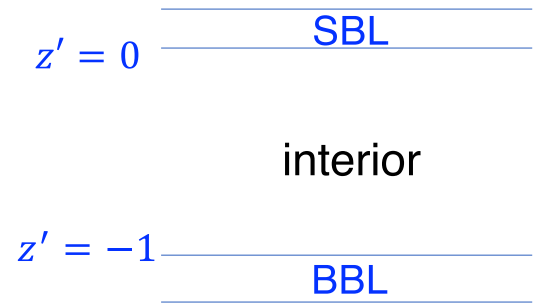
Quasi-geostrophic dynamics for frictional flows

Assumptions: homegeneous, rotational, visicid fluids, flat surface and bottom

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-fv=-\frac{1}{\rho_0}\frac{\partial p}{\partial x}+\frac{\partial}{\partial z}(A_V\frac{\partial u}{\partial z})$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+fu=-\frac{1}{\rho_0}\frac{\partial p}{\partial y}+\frac{\partial}{\partial z}(A_V\frac{\partial v}{\partial z})$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$



Define the non-dimensional variables:

$$(u,v)=U(u',v') \qquad w=Ww' \qquad (x,y)=L(x',y') \qquad z=Hz' \qquad p=Pp'$$

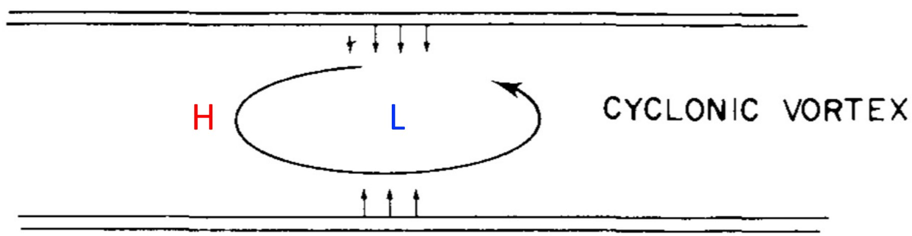
$$\frac{U}{T}\frac{\partial u'}{\partial t'}+\frac{U^2}{L}u'\frac{\partial u'}{\partial x'}+\frac{U^2}{L}v'\frac{\partial u'}{\partial y'}+\frac{U^2}{L}w'\frac{\partial u'}{\partial z'}-fUv'=-\frac{P}{\rho_0 L}\frac{\partial p'}{\partial x'}+\frac{A_V U}{H^2}\frac{\partial^2 u'}{\partial z'^2}$$

Spindown time (decay timescale of the flow or vorticity):

$$\zeta_0 = \zeta_c e^{-rt} = \zeta_c e^{-\frac{t}{1/r}}$$

The e-folding decay timescale: $\tau = \frac{1}{r} = \frac{\varepsilon}{E_v^{\frac{1}{2}}} = \frac{U/fL}{E_v^{\frac{1}{2}}} = \frac{1/fT}{E_v^{\frac{1}{2}}} \sim O(1)$ $\quad T \sim \frac{1}{f E_v^{\frac{1}{2}}}$

larger E_v, smaller T, faster disspitation of vorticity and kinetic energy

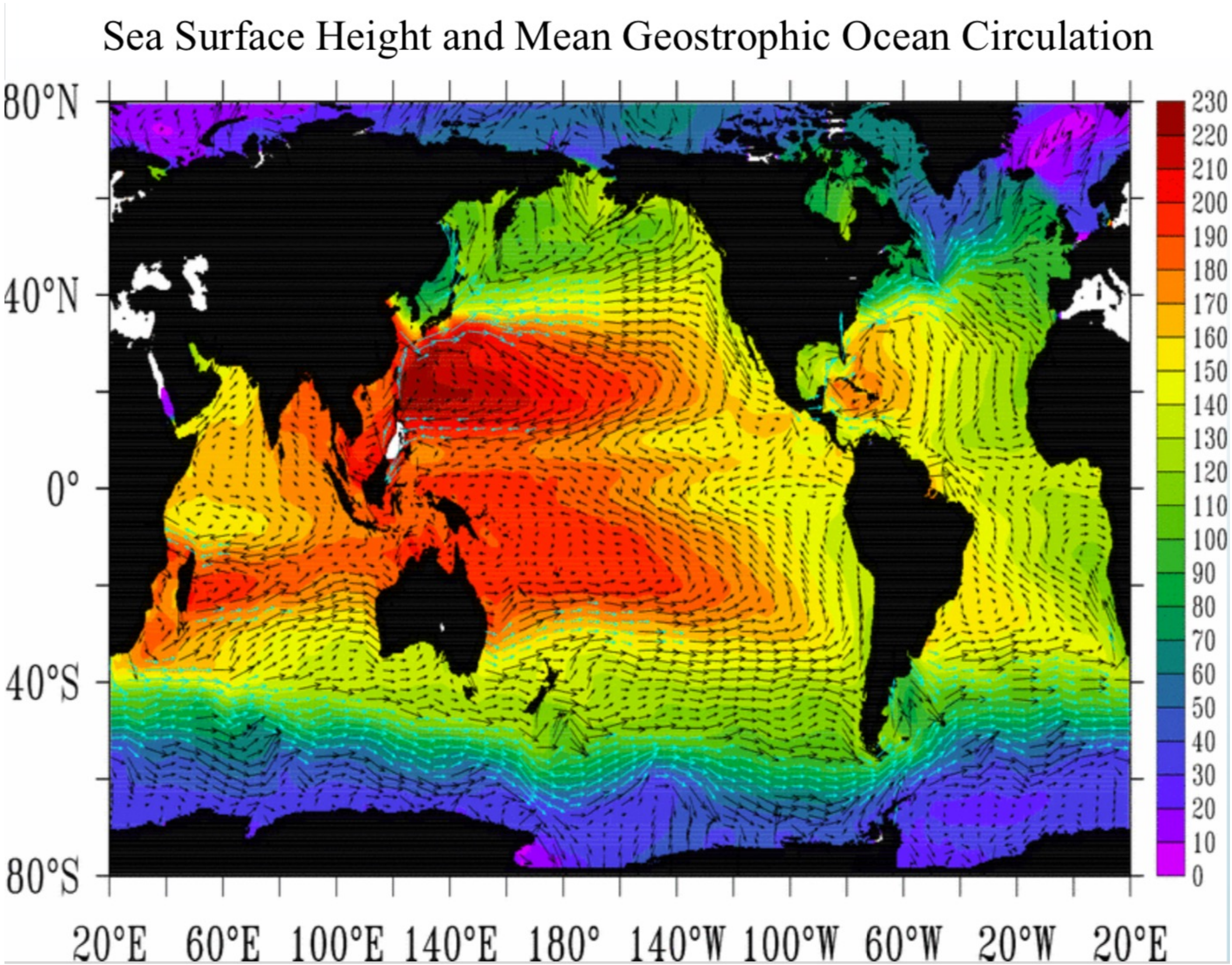


$$w_1|_{z=-1} = \frac{E_v^{\frac{1}{2}}}{2\varepsilon}\zeta_0 > 0 \qquad w_1|_{z=0} = -\frac{E_v^{\frac{1}{2}}}{2\varepsilon}\zeta_0 < 0$$

The water column is squeezed ($h \downarrow$), and for PV conservation, $\zeta_0 \downarrow$

The pressure gradient force does negative work, flow deaccelerates

Dynamics for the subtropical gyre



$$H\left(\frac{U}{LT}\frac{d\zeta'}{dt}+\beta_0 U v'\right)=\frac{1}{\rho_0}\frac{\tau_0}{L}curl\tau'+\frac{\tau_0}{\rho_0 f}\beta_0\tau^{x\prime}-\frac{d}{2}f\frac{U}{L}\zeta'$$

$$E_k=\frac{\nu_E}{fH^2}$$

$$H\frac{U^2}{L^2}\left(\frac{d\zeta'}{dt}+\frac{\beta_0 L^2}{U}v'\right)=\frac{\tau_0}{\rho_0 L}\left(curl\tau'+\frac{\beta_0 L}{f}\tau^{x\prime}\right)-\frac{d}{2}f\frac{U}{L}\zeta'$$

$$d=\sqrt{\frac{\nu_E}{f}}=E_k{}^{1/2}\mathrm{H}$$

<<1

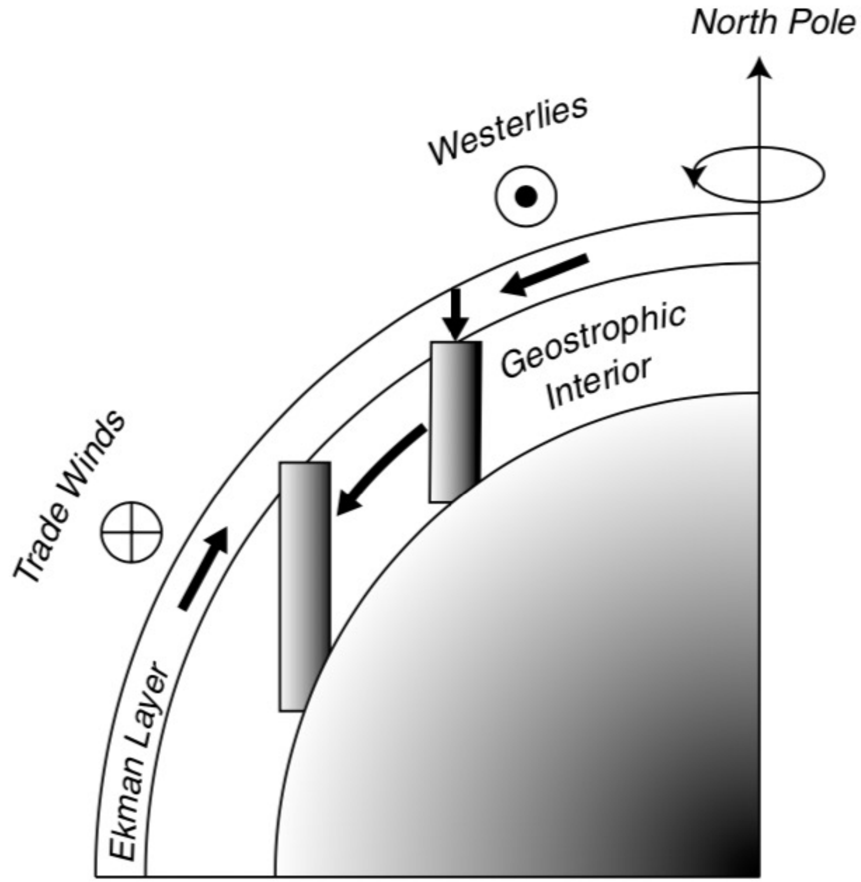
$$\frac{d\zeta'}{dt}+\underbrace{\frac{\beta_0 L^2}{U}}_{\beta}v' =\frac{\tau_0 L}{\rho_0 H U^2}\left(curl\tau'+\cancel{\frac{\beta_0 L}{f}}\tau^{x\prime}\right)-\underbrace{\frac{1}{2}\frac{fL}{U}\frac{d}{H}\zeta'}_{-\frac{1}{2}\frac{1}{R_o}E_k{}^{1/2}\zeta'}$$

Remove the prime *symbol* ′

$$\cancel{\frac{d\zeta}{dt}}+\beta v=\frac{\tau_0 L}{\rho_0 H U^2}curl\tau-\cancel{r\zeta}$$

For steady, large-scale flow, and assuming the bottom friction is negligible:

$v = curl\tau$ Sverdrup balance

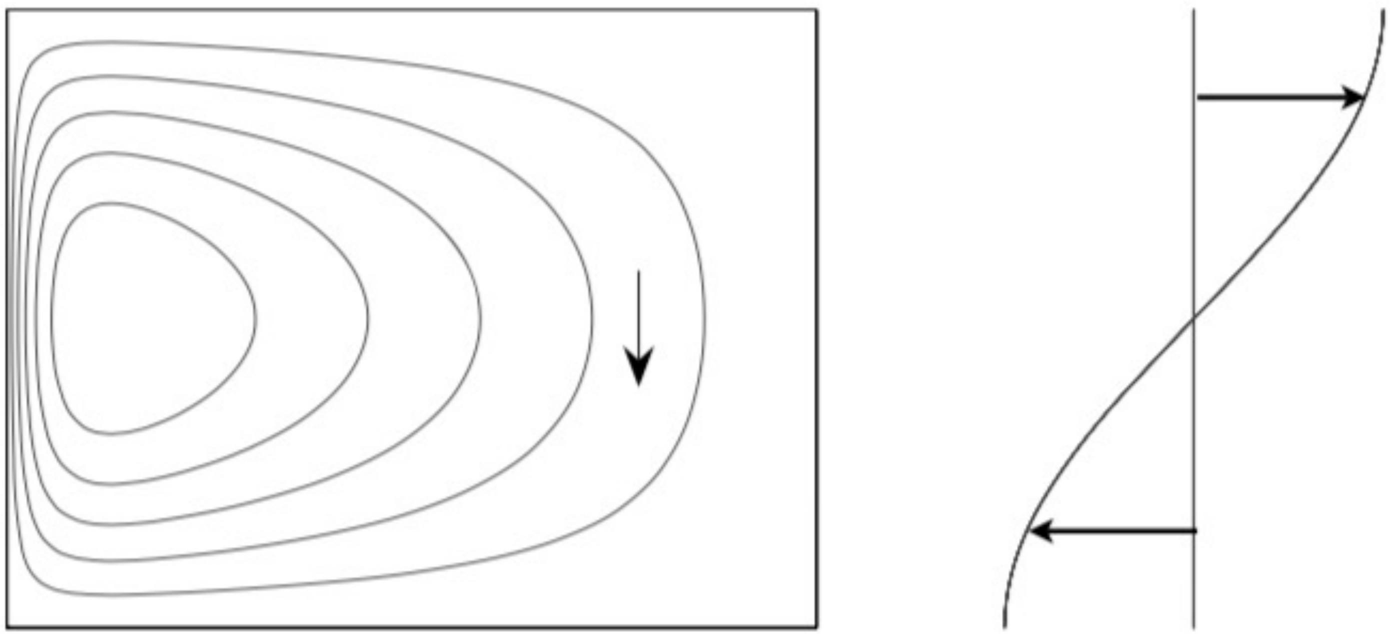


$$\frac{d}{dt}\left(\frac{f\ +\ \zeta}{h}\right)\ =\ 0$$

Figure 12.7 Ekman pumping that produces a downward velocity at the base of the Ekman layer forces the fluid in the interior of the ocean to move southward. (From Niiler, 1987)

Streamfunction

Wind stress



Vorticity dynamics

$$\boxed{\frac{d}{dt}\left(\frac{f\ +\ \zeta}{h}\right)\ =\ 0}$$

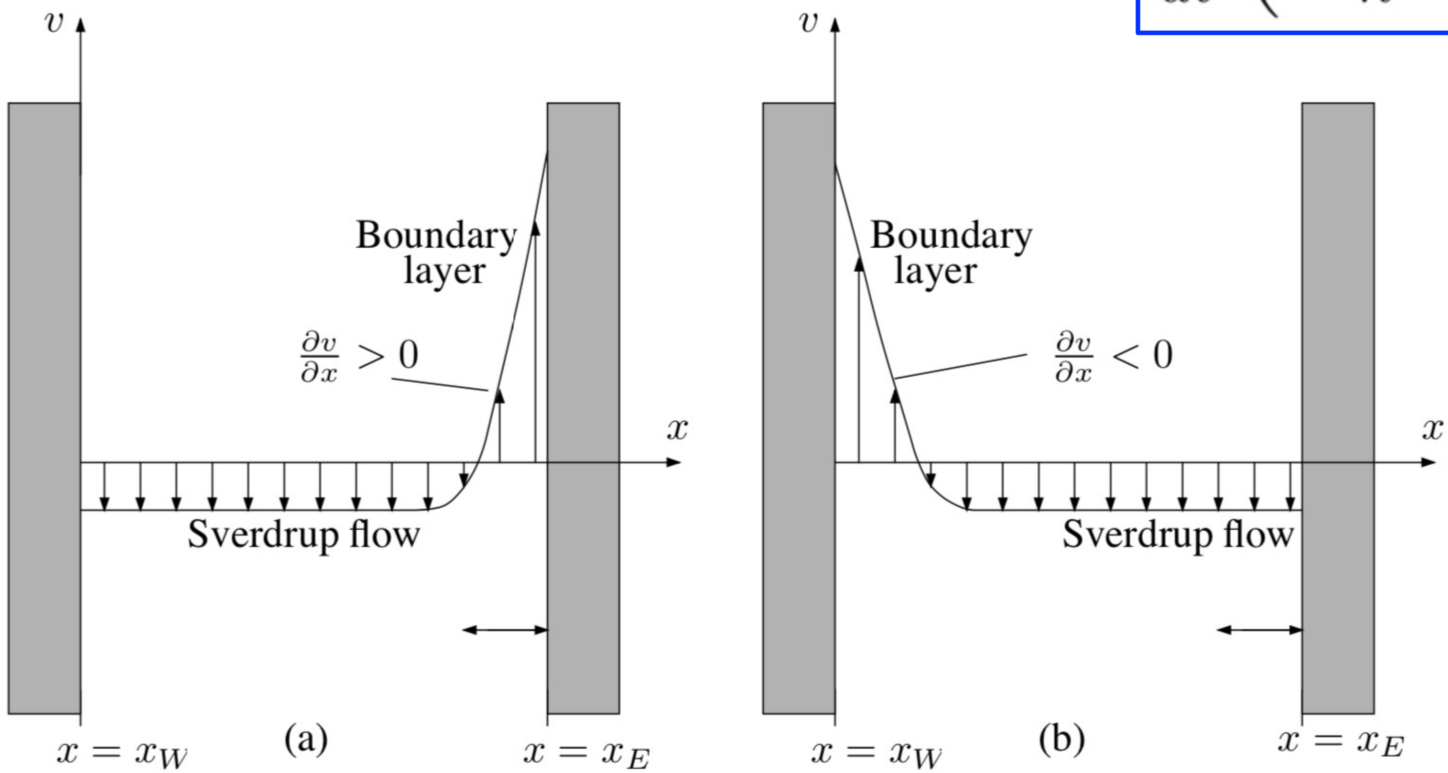


Figure 20-7 The two possible configurations for a northward boundary current to return the southward Sverdrup flow that exists across most of an ocean basin in the mid-latitudes of the Northern Hemisphere: (a) boundary current on the eastern side, (b) boundary current on the western side. The former is to be rejected on dynamic grounds, leaving the latter as the correct configuration.