Appendix F

Symmetric Baroclinic Oscillations

A variant of the derivation of the Sawyer–Eliassen equation for forced transverse circulations in baroclinic zones (9.15) can be used to obtain an equation for free symmetric transverse oscillations, which can be used to derive an expression for the growth rate of unstable symmetric oscillations or the frequency of stable symmetric oscillations.

Suppose that the flow field is zonally symmetric so that $u_g = u_g(y, z)$ and $b = b(y, z)$. The ageostrophic (transverse) flow is given by the streamfunction $\psi(y, z)$ so that $v_a = -\partial\psi/\partial z;\ w_a = \partial\psi/\partial y$. Then from (9.12) $Q_2 = 0$ so that any transverse circulation is unforced and must arise from a departure from exact geostrophic balance. This can be simply represented by adding an acceleration term in the y-momentum equation so that (9.10) becomes

$$\frac{\partial}{\partial t}\left(-\frac{\partial^2\psi}{\partial z^2}\right) + f\frac{\partial u_g}{\partial z} + \frac{\partial b}{\partial y} = 0 \tag{F.1}$$

Then, combining (9.11) and (9.13) as before and applying (F.1), we obtain

$$\frac{D}{Dt}\left[\frac{\partial}{\partial t}\left(\frac{\partial^2\psi}{\partial z^2}\right)\right] + N_s^2\frac{\partial^2\psi}{\partial y^2} + F^2\frac{\partial^2\psi}{\partial z^2} + 2S^2\frac{\partial^2\psi}{\partial y\partial z} = 0 \tag{F.2}$$

Neglecting terms quadratic in the streamfunction,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_a\frac{\partial}{\partial y} + w_a\frac{\partial}{\partial z} \approx \frac{\partial}{\partial t}$$

then yields the desired result:

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2\psi}{\partial z^2}\right) + N_s^2\frac{\partial^2\psi}{\partial y^2} + F^2\frac{\partial^2\psi}{\partial z^2} + 2S^2\frac{\partial^2\psi}{\partial y\partial z} = 0 \tag{F.3}$$