Shallow water equation – layered models

In a stratified system, as density increases monotonically downward, it can be used as the vertical coordinate

$$\rho(x,y,z,t) \longrightarrow z(x,y,\rho,t)$$

$$a = a(x,y,\rho,t)$$

Let $a = z$:

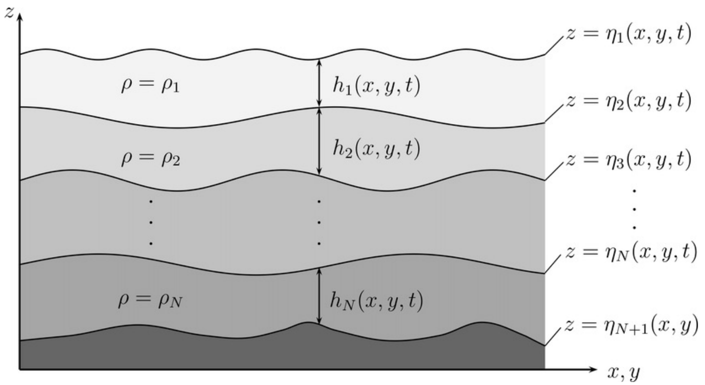
$$0 = z_x + z_\rho \rho_x$$

$$0 = z_y + z_\rho \rho_y$$

$$1 = z_\rho \rho_z$$

$$0 = z_t + z_\rho \rho_t$$

The derivative transformation:



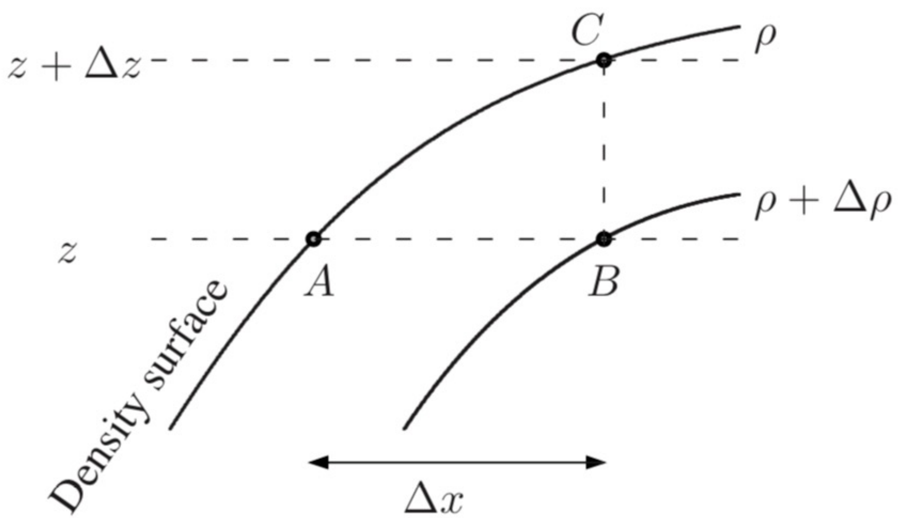
$$\frac{\partial a}{\partial x}\Big|_z = \frac{\partial a}{\partial x}\Big|_\rho + \frac{\partial a}{\partial \rho}\Big|_\rho \frac{\partial \rho}{\partial x}\Big|_z \qquad \frac{\partial a}{\partial x}\Big|_z = \frac{\partial a}{\partial x}\Big|_\rho - \frac{z_x}{z_\rho}\frac{\partial a}{\partial \rho}\Big|_\rho$$

$$\frac{\partial a}{\partial y}\Big|_z = \frac{\partial a}{\partial y}\Big|_\rho + \frac{\partial a}{\partial \rho}\Big|_\rho \frac{\partial \rho}{\partial y}\Big|_z \qquad \frac{\partial a}{\partial y}\Big|_z = \frac{\partial a}{\partial y}\Big|_\rho - \frac{z_y}{z_\rho}\frac{\partial a}{\partial \rho}\Big|_\rho$$

$$\frac{\partial a}{\partial z}\Big|_z = \frac{\partial a}{\partial \rho}\Big|_\rho \frac{\partial \rho}{\partial z}\Big|_z \qquad \frac{\partial a}{\partial z}\Big|_z = \frac{1}{z_\rho}\frac{\partial a}{\partial \rho}\Big|_\rho$$

$$\frac{\partial a}{\partial t}\Big|_z = \frac{\partial a}{\partial t}\Big|_\rho + \frac{\partial a}{\partial \rho}\Big|_\rho \frac{\partial \rho}{\partial t}\Big|_z \qquad \frac{\partial a}{\partial t}\Big|_z = \frac{\partial a}{\partial t}\Big|_\rho - \frac{z_t}{z_\rho}\frac{\partial a}{\partial \rho}\Big|_\rho$$

$$\frac{\partial a}{\partial x}\Big|_z = \frac{\partial a}{\partial x}\Big|_\rho - \frac{z_x}{z_\rho}\frac{\partial a}{\partial \rho}\Big|_\rho$$



constant z: $\frac{a(B)-a(A)}{\Delta x}$ $\qquad \frac{\partial a}{\partial x}\Big|_z$

constant ρ: $\frac{a(C)-a(A)}{\Delta x}$ $\qquad \frac{\partial a}{\partial x}\Big|_\rho$

$$\frac{\partial a}{\partial x}\Big|_\rho - \frac{\partial a}{\partial x}\Big|_z = \frac{a(C)-a(B)}{\Delta x} = \frac{a(C)-a(B)}{\Delta z}\frac{\Delta z}{\Delta x}$$

$$= \frac{a(C)-a(B)}{\Delta\rho\frac{\Delta z}{\Delta\rho}}\frac{\Delta z}{\Delta x}$$

$$= \frac{z_x}{z_\rho}\frac{\partial a}{\partial \rho}\Big|_\rho$$

$$\frac{\partial a}{\partial x}\Big|_z = \frac{\partial a}{\partial x}\Big|_\rho - \frac{z_x}{z_\rho}\frac{\partial a}{\partial \rho}\Big|_\rho$$

$$\frac{\partial a}{\partial x}|_z = \frac{\partial a}{\partial x}|_\rho - \frac{z_x}{z_\rho}\frac{\partial a}{\partial \rho}|_\rho \qquad \frac{\partial a}{\partial z}|_z = \frac{1}{z_\rho}\frac{\partial a}{\partial \rho}|_\rho \qquad \frac{\partial a}{\partial t}|_z = \frac{\partial a}{\partial t}|_\rho - \frac{z_t}{z_\rho}\frac{\partial a}{\partial \rho}|_\rho$$

The pressure gradient term:

Hydrostatic balance: $$\frac{\partial p}{\partial z} = -\rho g$$

$$\frac{\partial p}{\partial \rho} = -\rho g \frac{\partial z}{\partial \rho}$$

$$\boxed{P = p + \rho g z}$$

$$\frac{\partial p}{\partial x}|_z = \frac{\partial p}{\partial x}|_\rho - \frac{z_x}{z_\rho}\frac{\partial p}{\partial \rho}|_\rho = \frac{\partial p}{\partial x}|_\rho + \rho g \frac{\partial z}{\partial x}|_\rho = \frac{\partial P}{\partial x}|_\rho$$

$$\frac{\partial P}{\partial \rho} = \frac{\partial p}{\partial \rho} + gz + \rho g \frac{\partial z}{\partial \rho} = gz$$

$$\frac{\partial a}{\partial x}|_z = \frac{\partial a}{\partial x}|_\rho - \frac{z_x}{z_\rho}\frac{\partial a}{\partial \rho}|_\rho \qquad \frac{\partial a}{\partial z}|_z = \frac{1}{z_\rho}\frac{\partial a}{\partial \rho}|_\rho \qquad \frac{\partial a}{\partial t}|_z = \frac{\partial a}{\partial t}|_\rho - \frac{z_t}{z_\rho}\frac{\partial a}{\partial \rho}|_\rho$$

The full derivative ($\frac{d}{dt}$) in the LHS of the governing equation:

Let $a = \rho$:

$$\frac{\partial \rho}{\partial x}|_z = -\frac{z_x}{z_\rho} \qquad \frac{\partial \rho}{\partial y}|_z = -\frac{z_y}{z_\rho} \qquad \frac{\partial \rho}{\partial z}|_z = \frac{1}{z_\rho} \qquad \frac{\partial \rho}{\partial t}|_z = -\frac{z_t}{z_\rho}$$

For incompressible fluids: $\frac{d\rho}{dt} = 0$

ρ-coordinate

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = 0 \Rightarrow -z_t - uz_x - vz_y + w = 0 \Rightarrow \frac{dz}{dt} = z_t + uz_x + vz_y$$

$$\boldsymbol{\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}}$$

There is no across-isopycnal advection, and the model is ideal for studying along-isopycnal processes

Without friction, the governing equations in density coordinate become:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -\frac{1}{\rho_0}\frac{\partial P}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -\frac{1}{\rho_0}\frac{\partial P}{\partial y}$$

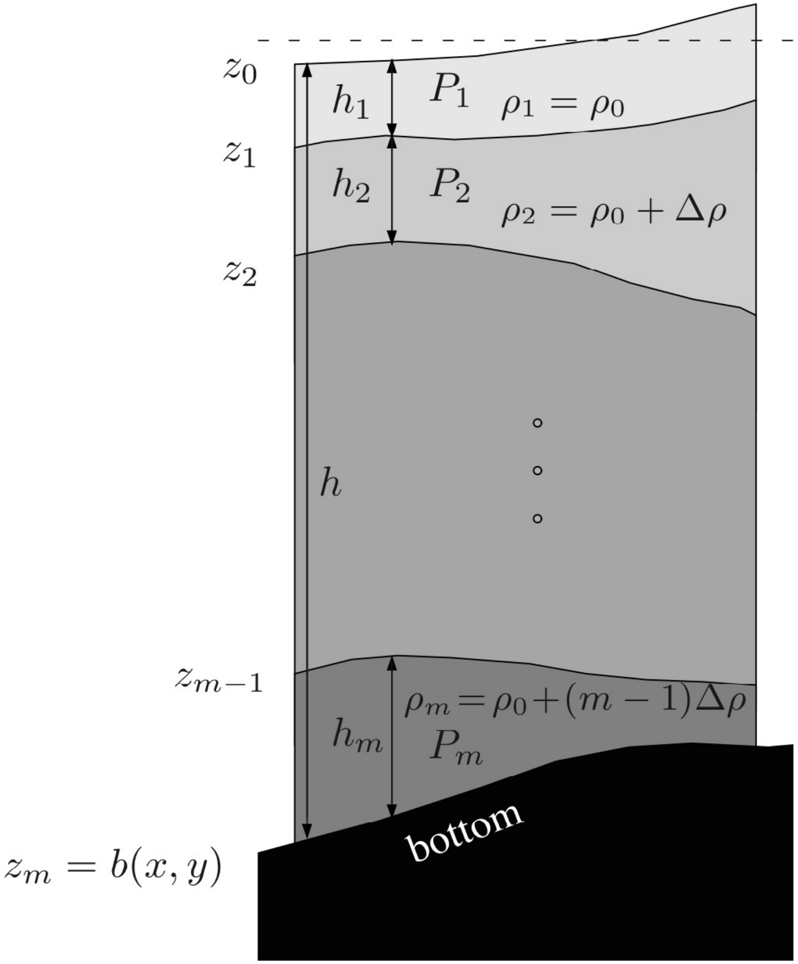
$$\frac{\partial P}{\partial \rho} = gz$$

$$\frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} + \frac{\partial hv}{\partial y} = 0$$

$$h = -\Delta\rho\frac{\partial z}{\partial \rho}$$

the thickness of a fluid layer between ρ and $\rho + \Delta\rho$

Layered models



(a)

We need to determine P and z for different layers, and we can obtain the equations for these layers

For z:

$$z_m = b$$

upward:

$$z_{k-1} = z_k + h_k, \quad k = m \text{ to } 1.$$

For P:

$$\boxed{P = p + \rho g z}$$

$$P_1 = p_{\text{atm}} + \rho_0 g z_0$$

downward:

$$\boxed{\frac{\partial P}{\partial \rho} = g z}$$

$$P_{k+1} = P_k + \Delta\rho g z_k, \quad k = 1 \text{ to } m-1.$$

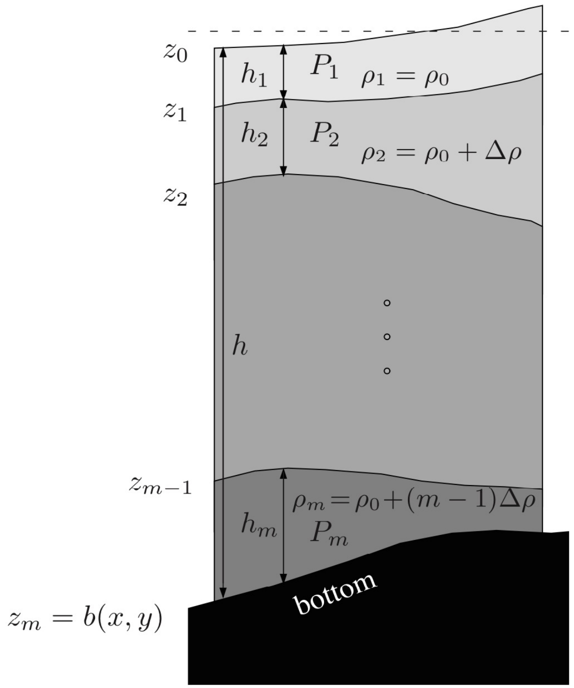
$$\boxed{z_m = b} \qquad \boxed{z_{k-1} = z_k + h_k} \qquad \boxed{P_1 = p_{\mathrm{atm}} + \rho_0 g z_0} \qquad \boxed{P_{k+1} = P_k + \Delta\rho g z_k}$$

One layer:

$$z_0 = h_1 + b \qquad P_1 = \rho_0 g(h_1 + b)$$
$$z_1 = b$$

$$\boxed{g' = \frac{\Delta\rho}{\rho_0} g}$$

reduced gravity



(a)

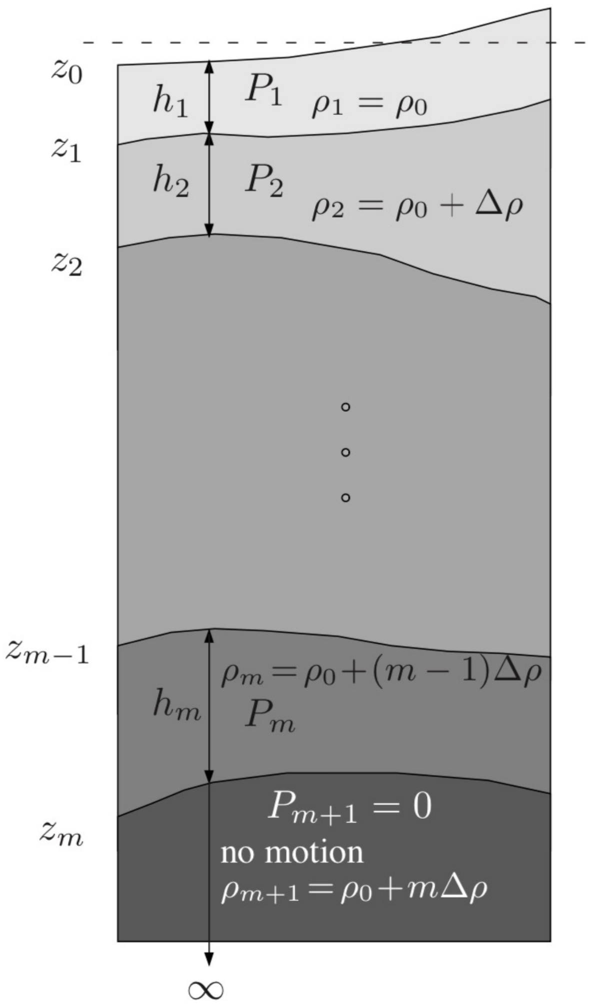
Two layers:

$$z_0 = h_1 + h_2 + b \qquad P_1 = \rho_0 g(h_1 + h_2 + b)$$
$$z_1 = h_2 + b \qquad P_2 = \rho_0 g h_1 + \rho_0 (g + g')(h_2 + b)$$
$$z_2 = b$$

Three layers:

$$z_0 = h_1 + h_2 + h_3 + b \qquad P_1 = \rho_0 g(h_1 + h_2 + h_3 + b)$$
$$z_1 = h_2 + h_3 + b \qquad P_2 = \rho_0 g h_1 + \rho_0 (g + g')(h_2 + h_3 + b)$$
$$z_2 = h_3 + b \qquad P_3 = \rho_0 g h_1 + \rho_0 (g + g') h_2$$
$$z_3 = b \qquad \qquad + \rho_0 (g + 2g')(h_3 + b)$$

Reduced gravity model



The lowest layer may be imagined to be infinitely deep and at rest

$$P_{m+1} = 0$$

Rigid-lid approximation: $z_0 = 0$

upward calculation

$$\boxed{P_{k+1} = P_k + \Delta\rho g z_k}$$

One layer:

$$z_1 = -h_1 \qquad P_1 = \rho_0 g' h_1$$

Two layers:

$$z_1 = -h_1 \qquad P_1 = \rho_0 g'(2h_1 + h_2)$$
$$z_2 = -h_1 - h_2 \qquad P_2 = \rho_0 g'(h_1 + h_2)$$

Three layers:

$$z_1 = -h_1 \qquad P_1 = \rho_0 g'(3h_1 + 2h_2 + h_3)$$
$$z_2 = -h_1 - h_2 \qquad P_2 = \rho_0 g'(2h_1 + 2h_2 + h_3)$$
$$z_3 = -h_1 - h_2 - h_3 \qquad P_3 = \rho_0 g'(h_1 + h_2 + h_3)$$

shallow-water reduced gravity model – one layer

One layer:

$$z_1 = -h_1 \qquad\qquad P_1 = \rho_0 g' h_1$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g'\frac{\partial h}{\partial x} \quad (1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g'\frac{\partial h}{\partial y} \quad (2)$$

$$\frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} + \frac{\partial hv}{\partial y} = 0 \Longrightarrow \frac{dh}{dt} + h(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = 0$$

Review for homogeneous fluids:

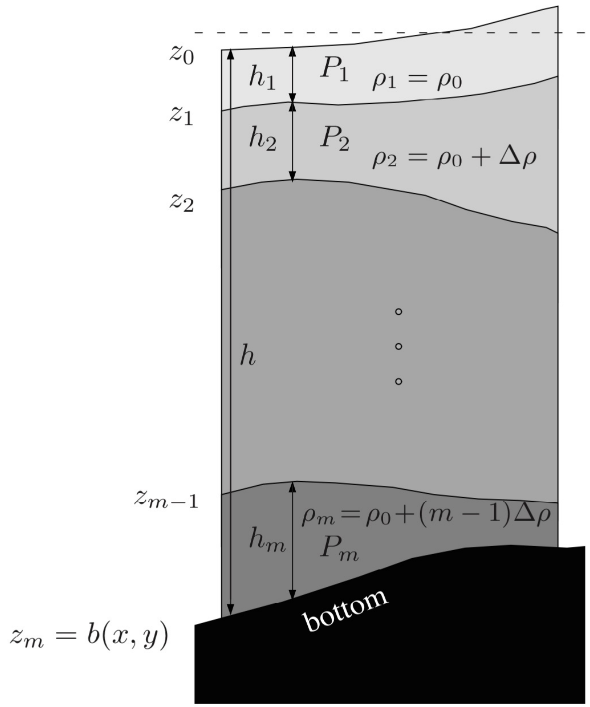
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial \eta}{\partial y}$$

shallow-water model (general) – two layers

Two layers:

$$z_0 = h_1 + h_2 + b \qquad P_1 = \rho_0 g(h_1 + h_2 + b)$$
$$z_1 = h_2 + b \qquad P_2 = \rho_0 g h_1 + \rho_0 (g + g')(h_2 + b)$$
$$z_2 = b$$



(a)

Governing equations:

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y} - f v_1 = -g \frac{\partial (h_1 + h_2 + b)}{\partial x}$$
$$\frac{\partial v_1}{\partial t} + u_1 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_1}{\partial y} + f u_1 = -g \frac{\partial (h_1 + h_2 + b)}{\partial y}$$
$$\frac{\partial h_1}{\partial t} + \frac{\partial (h_1 u_1)}{\partial x} + \frac{\partial (h_1 v_1)}{\partial y} = 0$$

$$\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} + v_2 \frac{\partial u_2}{\partial y} - f v_2 = -g \frac{\partial h_1}{\partial x} - (g + g') \frac{\partial (h_2 + b)}{\partial x}$$
$$\frac{\partial v_2}{\partial t} + u_2 \frac{\partial v_2}{\partial x} + v_2 \frac{\partial v_2}{\partial y} + f u_2 = -g \frac{\partial h_1}{\partial y} - (g + g') \frac{\partial (h_2 + b)}{\partial y}$$

$$\frac{\partial h_2}{\partial t} + \frac{\partial (h_2 u_2)}{\partial x} + \frac{\partial (h_2 v_2)}{\partial y} = 0$$

$$- g\,\frac{\partial(h_1+h_2+b)}{\partial x} = -g\,\frac{\partial \eta}{\partial x}$$

$$- g'\,\frac{\partial(h_2+b)}{\partial x} = -g'\,\frac{\partial a}{\partial x}$$

$$\boxed{-g\frac{\partial(h_1+h_2+b)}{\partial x}}$$

$$\boxed{-g\frac{\partial h_1}{\partial x} - (g+g')\frac{\partial(h_2+b)}{\partial x}}$$

$$= -g\frac{\partial(h_1+h_2+b)}{\partial x} - g'\,\frac{\partial(h_2+b)}{\partial x}$$

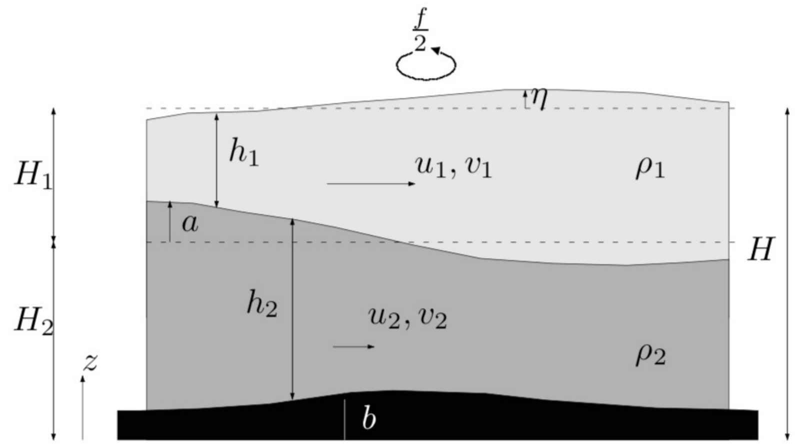
The linearized equations become:

$$\begin{aligned}
\frac{\partial u_1}{\partial t} - f v_1 &= -g\frac{\partial \eta}{\partial x} \\
\frac{\partial v_1}{\partial t} + f u_1 &= -g\frac{\partial \eta}{\partial y}
\end{aligned}$$

$$\frac{\partial(\eta - a)}{\partial t} + \frac{\partial(H_1 u_1)}{\partial x} + \frac{\partial(H_1 v_1)}{\partial y} = 0$$

$$\begin{aligned}
\frac{\partial u_2}{\partial t} - f v_2 &= -g\frac{\partial \eta}{\partial x} - g'\frac{\partial a}{\partial x} \\
\frac{\partial v_2}{\partial t} + f u_2 &= -g\frac{\partial \eta}{\partial y} - g'\frac{\partial a}{\partial y}
\end{aligned}$$

$$\frac{\partial a}{\partial t} + \frac{\partial(H_2 u_2)}{\partial x} + \frac{\partial(H_2 v_2)}{\partial y} = 0$$



$$h_1 + h_2 + b = H + \eta$$

$$h_2 + b = H_2 + a$$

$$h_1 = H_1 + \eta - a$$

Let

$$u_2 = \lambda u_1, v_2 = \lambda v_1 \qquad \eta = \mu a$$

$$\frac{\partial u_1}{\partial t} - f v_1 = -g\frac{\partial \eta}{\partial x} \qquad \frac{\partial u_2}{\partial t} - f v_2 = -g\frac{\partial \eta}{\partial x} - g'\frac{\partial a}{\partial x}$$

$$\lambda(\frac{\partial u_1}{\partial t} - f v_1) = -(\mu g + g')\frac{\partial a}{\partial x} \qquad \lambda\frac{\partial u_1}{\partial t} - \lambda f v_1 = -\mu g\frac{\partial a}{\partial x} - g'\frac{\partial a}{\partial x}$$

The horizontal momentum equations can be reduced into one set if $\frac{\lambda}{1} = \frac{g\mu + g'}{g\mu}$

$$\frac{\partial(\eta - a)}{\partial t} + \frac{\partial(H_1 u_1)}{\partial x} + \frac{\partial(H_1 v_1)}{\partial y} = 0 \qquad \frac{\partial a}{\partial t} + \frac{\partial(H_2 u_2)}{\partial x} + \frac{\partial(H_2 v_2)}{\partial y} = 0$$

$$(\mu - 1)\frac{\partial a}{\partial t} + H_1(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}) = 0 \qquad \frac{\partial a}{\partial t} + \lambda H_2(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}) = 0$$

The continuity equation can be reduced into one set if $\frac{1}{\mu - 1} = \frac{H_2 \lambda}{H_1}$

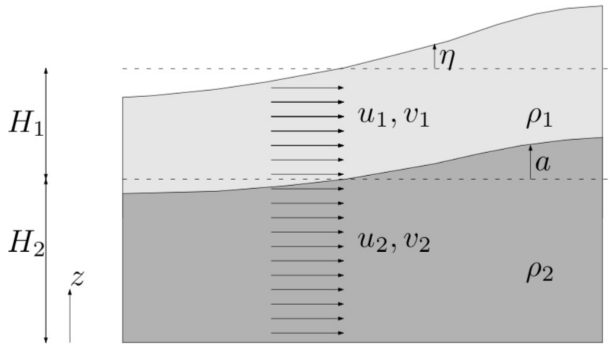
$$\frac{\lambda}{1} = \frac{g\mu + g'}{g\mu} \qquad \frac{1}{\mu - 1} = \frac{H_2\lambda}{H_1}$$

$$H_2\lambda^2 + \left(H_1 - H_2 - \cancel{\frac{g'}{g}} H_2\right)\lambda - H_1 = 0 \qquad g'/g = \Delta\rho/\rho_0 \ll 1$$

$$\lambda = \frac{(H_2 - H_1) \pm (H_2 + H_1)}{2H_2}$$

If $\lambda = 1$: $u_1 = u_2$ $\quad v_1 = v_2$ barotropic mode

$$\mu = \frac{H}{H_2} \qquad a = \eta/\mu = H_2\eta/H$$

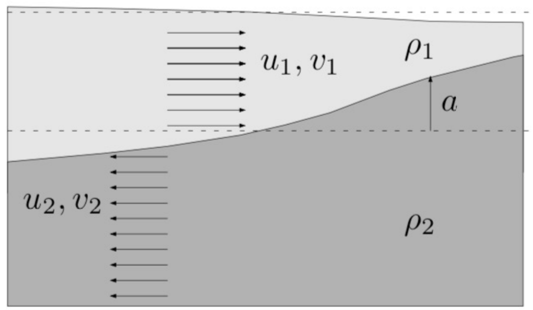


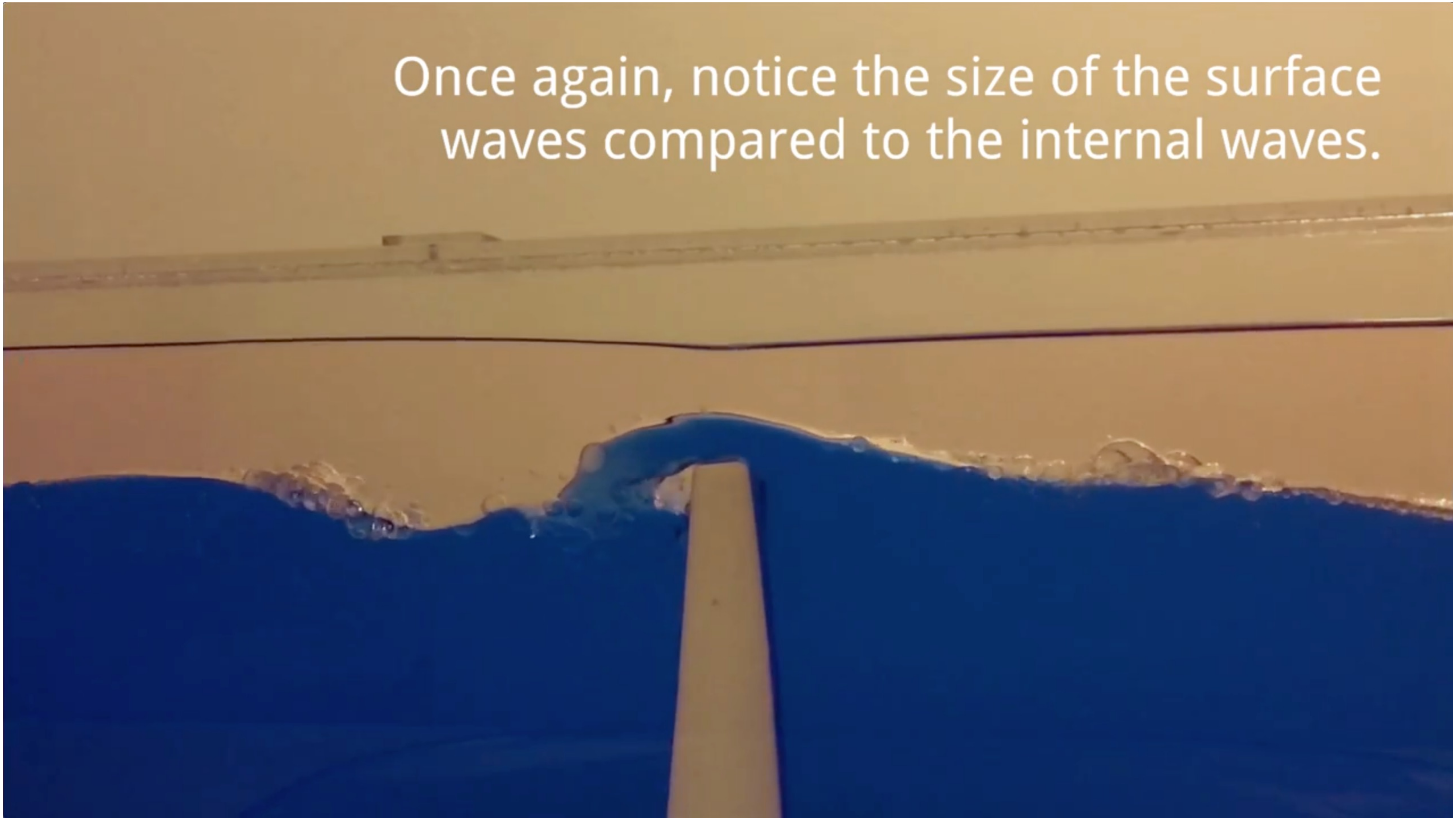
If $\lambda = -\frac{H_1}{H_2}$: $u_2 = -\frac{H_1}{H_2}u_1 \quad H_2u_2 = -H_1u_1$

$v_2 = -\frac{H_1}{H_2}v_1 \quad H_2v_2 = -H_1v_1$ baroclinic mode

the vertical integration of transport is zero

$\mu = -g'H_2/gH \sim 0 \qquad \eta \ll a \qquad$ nearly rigid lid




Once again, notice the size of the surface
waves compared to the internal waves.

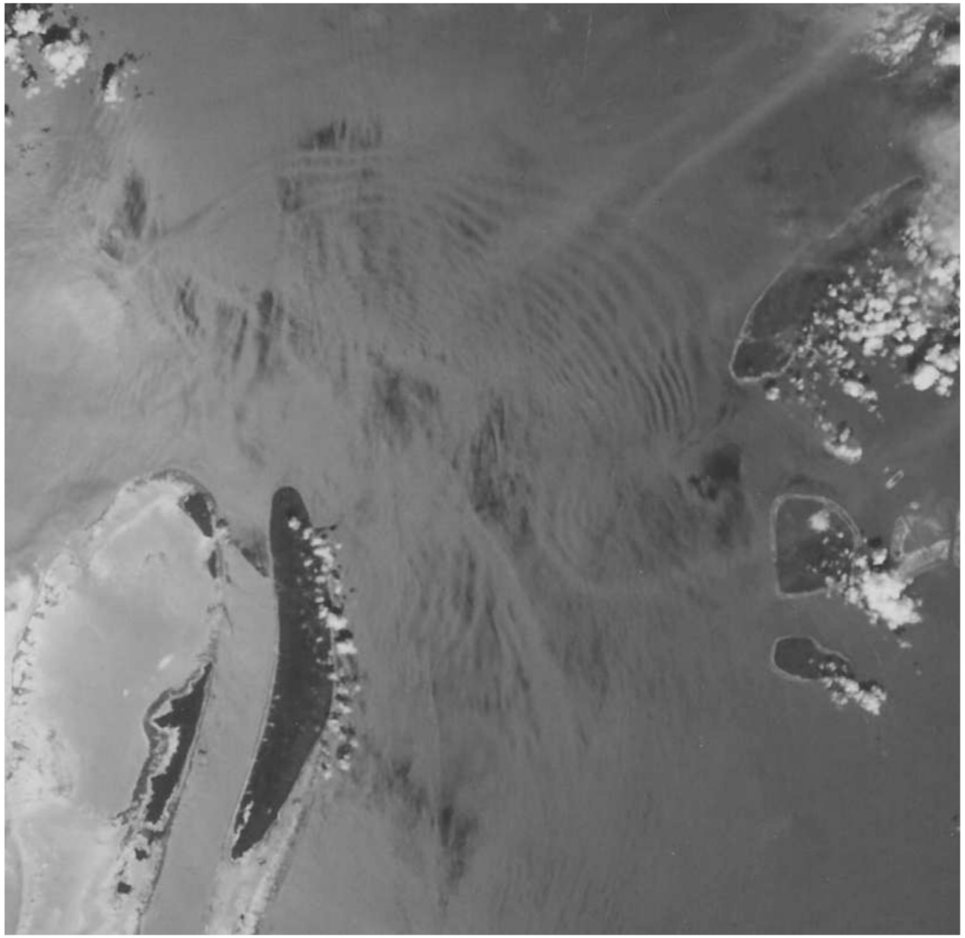


Figure 13-2 Surface manifestation of oceanic internal waves. The upward energy propagation of internal waves modifies the properties of surface waves rendering them visible from space. In this sunglint photograph taken from the space shuttle *Atlantis* on 19 November 1990 over Sibutu Passage in the Philippines (5°N, 119.5°E), a large group of tidally-generated internal waves is seen to propagate northward into the Sulu Sea. (NASA Photo STS-38-084-060)

Barotropic mode: $\frac{\lambda}{1} = \frac{g\mu + g'}{g\mu}$ $\lambda = 1$

$$\begin{aligned}\frac{\partial u_1}{\partial t} - f v_1 &= -g \frac{\partial \eta}{\partial x} \\ \frac{\partial v_1}{\partial t} + f u_1 &= -g \frac{\partial \eta}{\partial y}\end{aligned}$$

$$\frac{\partial (\eta - a)}{\partial t} + \frac{\partial (H_1 u_1)}{\partial x} + \frac{\partial (H_1 v_1)}{\partial y} = 0$$

$$\begin{aligned}\frac{\partial u_2}{\partial t} - f v_2 &= -g \frac{\partial \eta}{\partial x} - g' \frac{\partial a}{\partial x} \\ \frac{\partial v_2}{\partial t} + f u_2 &= -g \frac{\partial \eta}{\partial y} - g' \frac{\partial a}{\partial y}\end{aligned}$$

$$\frac{\partial a}{\partial t} + \frac{\partial (H_2 u_2)}{\partial x} + \frac{\partial (H_2 v_2)}{\partial y} = 0$$

$$u_T = u_1 = u_2, v_T = v_1 = v_2$$

Summation of the two sets of equations:

$$\begin{aligned}\frac{\partial u_T}{\partial t} - f v_T &= -g \frac{\partial \eta}{\partial x} \\ \frac{\partial v_T}{\partial t} + f u_T &= -g \frac{\partial \eta}{\partial y}\end{aligned}$$

$$\frac{\partial \eta}{\partial t} + H \frac{\partial u_T}{\partial x} + H \frac{\partial v_T}{\partial y} = 0$$

Baroclinic mode: $\frac{\lambda}{1} = \frac{g\mu + g'}{g\mu}$ $\lambda = -\frac{H_1}{H_2}$

$$\begin{aligned}\frac{\partial u_1}{\partial t} - f v_1 &= -g\frac{\partial \eta}{\partial x} \\ \frac{\partial v_1}{\partial t} + f u_1 &= -g\frac{\partial \eta}{\partial y}\end{aligned}$$

$$\frac{\partial(\eta - a)}{\partial t} + \frac{\partial(H_1 u_1)}{\partial x} + \frac{\partial(H_1 v_1)}{\partial y} = 0$$

$$\begin{aligned}\frac{\partial u_2}{\partial t} - f v_2 &= -g\frac{\partial \eta}{\partial x} - g'\frac{\partial a}{\partial x} \\ \frac{\partial v_2}{\partial t} + f u_2 &= -g\frac{\partial \eta}{\partial y} - g'\frac{\partial a}{\partial y}\end{aligned}$$

$$\frac{\partial a}{\partial t} + \frac{\partial(H_2 u_2)}{\partial x} + \frac{\partial(H_2 v_2)}{\partial y} = 0$$

$$u_B = u_1 - u_2 \qquad v_B = v_1 - v_2$$

$$\eta = -(g' H_2 / g H) a$$

Difference between the two sets of equations:

$$\begin{aligned}\frac{\partial u_B}{\partial t} - f v_B &= +g'\frac{\partial a}{\partial x} \\ \frac{\partial v_B}{\partial t} + f u_B &= +g'\frac{\partial a}{\partial y}\end{aligned}$$

$$-\frac{\partial a}{\partial t} + \frac{H_1 H_2}{H}\frac{\partial u_B}{\partial x} + \frac{H_1 H_2}{H}\frac{\partial v_B}{\partial y} = 0$$