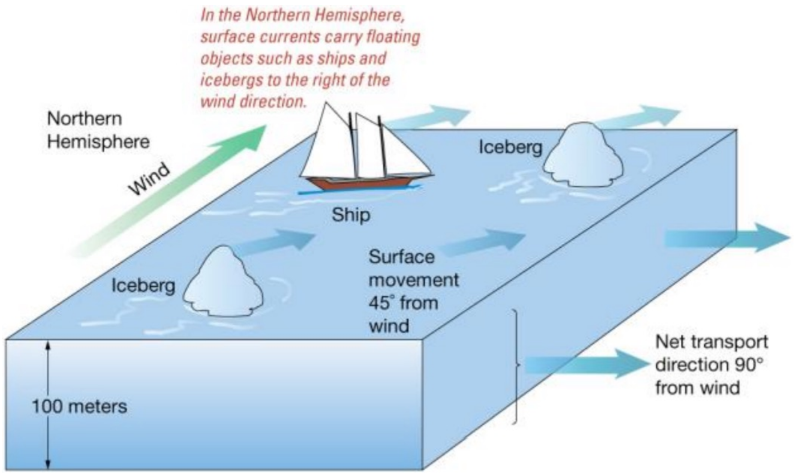
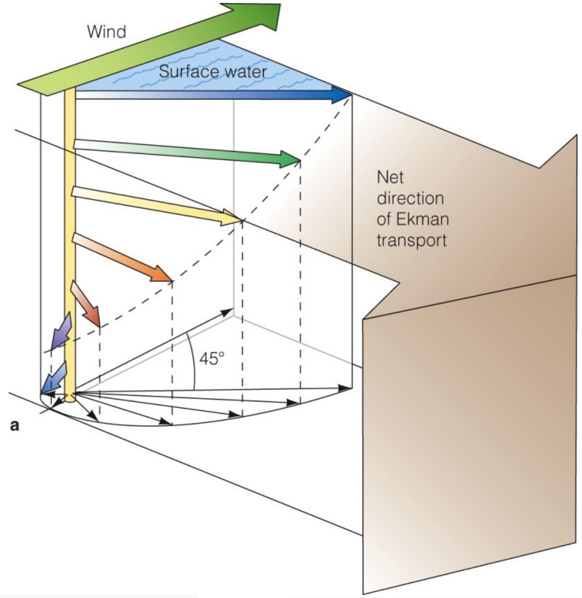
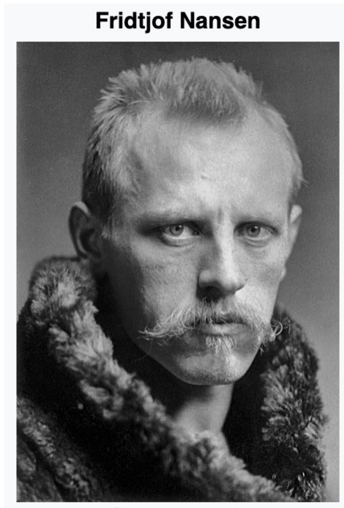
Ekman (boundary) layer dynamics







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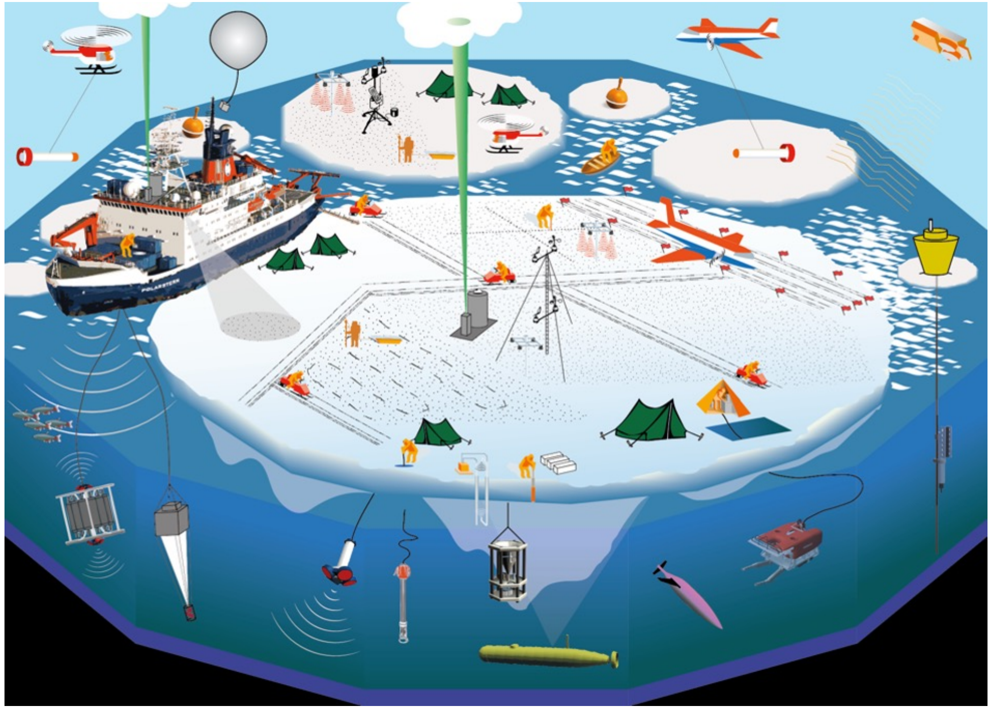


Ekman (boundary) layer dynamics

The viscosity term:

$$\frac{\partial}{\partial x}\left(A_H \frac{\partial \mathrm{u}}{\partial \mathrm{x}}\right)+\frac{\partial}{\partial y}\left(A_H \frac{\partial \mathrm{u}}{\partial \mathrm{y}}\right)+\frac{\partial}{\partial \mathrm{z}}\left(\overset{\nu_E}{A_V} \frac{\partial \mathrm{u}}{\partial \mathrm{z}}\right)$$

Ekman number (E_k) = scale of vertical viscosity term / scale of Coriolis term

$$= \nu_E \frac{U}{H^2} / fU = \frac{\nu_E}{fH^2}$$

For ocean interior: $f \sim 10^{-4}\, s^{-1}$, $\nu_E = 10^{-2}\, m^2\, s^{-1}$, $H \sim 100 - 1000m$

$E_k \sim 10^{-4} - 10^{-2}$ friction can be neglected

In the boundary layers, friction should be dominant, $E_k \sim 1$:

d: boundary layer thickness $\frac{\nu_E}{fd^2} \sim 1 \Longrightarrow d \sim \sqrt{\nu_E / f}$

The bottom Ekman layer

Assumptions: large-scale motion ($R_0 \ll 1$)

steady flow ($\frac{\partial}{\partial t} = 0$)

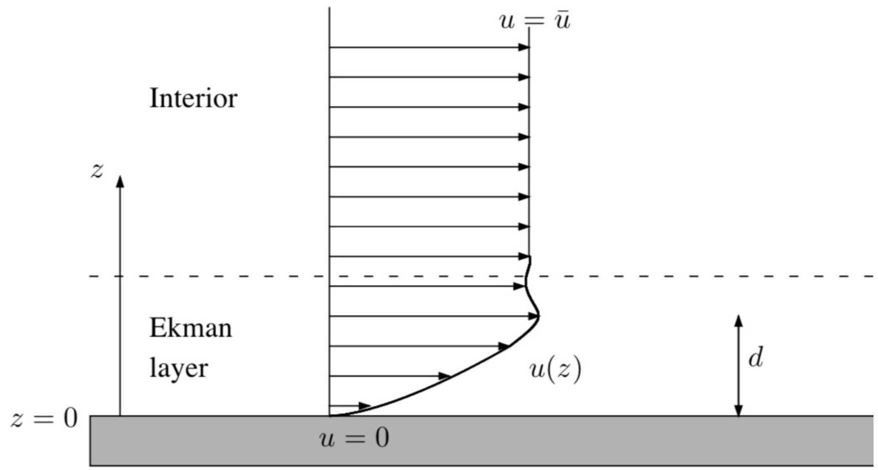
geostrophic flow in the interior

$$u = \bar{u}, \qquad v = 0$$

homogeneous fluid ($\rho = \rho_0$)

$$\frac{\partial p'}{\partial z} + \rho' g = 0 \Rightarrow \frac{\partial p'}{\partial z} = 0$$

flat bottom



Governing equations in the bottom boundary layer (BBL):

p' (dynamic pressure)

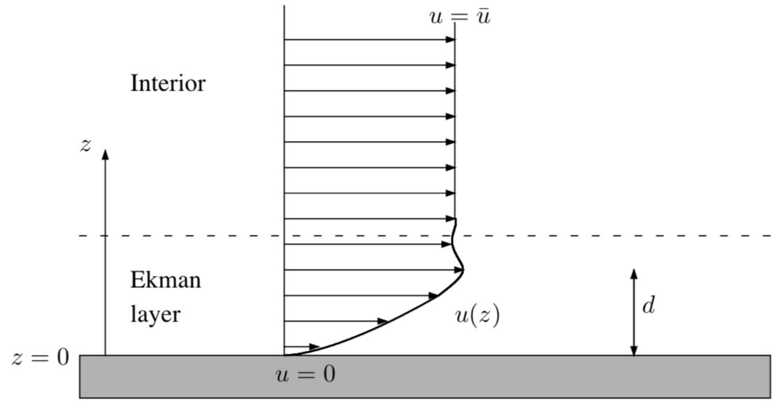
$$-fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu_E \frac{\partial^2 u}{\partial z^2}$$

p'

$$+fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \nu_E \frac{\partial^2 v}{\partial z^2}$$

Boundary conditions:

$$\text{Bottom } (z=0): \qquad u = 0, \quad v = 0,$$



$$\text{Toward the interior } (z \gg d): \qquad u = \bar{u}, \quad v = 0$$

For the interior (geostrophic) flow:

$$
\begin{aligned}
0 &= -\frac{1}{\rho_0}\frac{\partial \bar{p}'}{\partial x}, \\
f\bar{u} &= -\frac{1}{\rho_0}\frac{\partial \bar{p}'}{\partial y} = \text{constant.}
\end{aligned}
$$

$$\frac{\partial p'}{\partial z} = 0$$

Substitution of the pressure gradient force into the boundary layer:

$$
\begin{aligned}
-fv &= \nu_E \frac{d^2 u}{dz^2} \\
f(u - \bar{u}) &= \nu_E \frac{d^2 v}{dz^2}
\end{aligned}
$$

$$-fv = \nu_E \frac{d^2u}{dz^2} \qquad (1)$$

$$f(u - \bar{u}) = \nu_E \frac{d^2v}{dz^2} \qquad (2)$$

From (2), $u = \bar{u} + \frac{\nu_E}{f}\frac{d^2v}{dz^2}$, and substitution into (1):

$$\frac{{\nu_E}^2}{f^2}\frac{d^4v}{dz^4} + v = 0$$

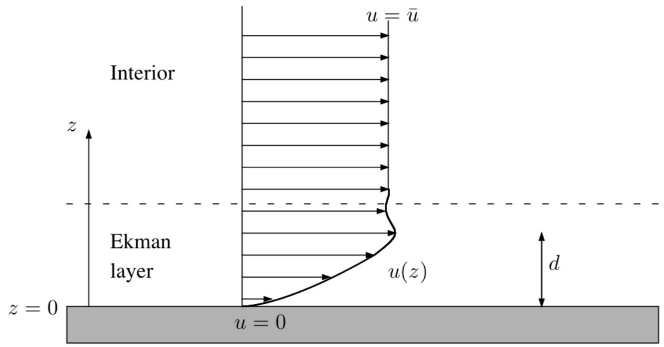
Let $v = e^{\lambda z}$,

$$\frac{{\nu_E}^2}{f^2}\lambda^4 + 1 = 0 \Longrightarrow \lambda^2 = \pm i\frac{f}{\nu_E} = \frac{(1 \pm i)^2}{2}\frac{f}{\nu_E}$$

$$d = \sqrt{\frac{2\nu_E}{f}} \qquad \lambda = \pm\sqrt{\frac{f}{2\nu_E}}(1 \pm i) = \pm\frac{1}{d}(1 \pm i)$$

scale of boundary layer thickness

$$\lambda = \pm\frac{1}{d}(1 \pm i)$$



If $\lambda = \frac{1}{d}(1 \pm i)$:

$$v = e^{\lambda z} = e^{\frac{z}{d}} e^{\pm\frac{z}{d}i}$$

cannot satisfy the upper boudary condition $v \rightarrow 0$, not considered

So

$\lambda = -\frac{1}{d}(1 \pm i)$:

$$v = Ae^{-\frac{1}{d}(1+i)z} + Be^{-\frac{1}{d}(1-i)z}$$

$$= e^{-\frac{z}{d}}(Ae^{-\frac{z}{d}i} + Be^{\frac{z}{d}i})$$

$$\boxed{e^{i\theta} = cos\theta + isin\theta}$$

$$= e^{-\frac{z}{d}}\,(Ecos\frac{z}{d} + Fsin\frac{z}{d})$$

$$u = \bar{u} + \frac{\nu_E}{f}\frac{d^2 v}{dz^2} \quad = \bar{u} + e^{-\frac{z}{d}}\,(Esin\frac{z}{d} - Fcos\frac{z}{d})$$

$$u = \bar{u} + e^{-\frac{z}{d}}(E sin\frac{z}{d} - F cos\frac{z}{d})$$

$$v = e^{-\frac{z}{d}}(E cos\frac{z}{d} + F sin\frac{z}{d})$$

Lower boundary condition:

z = 0: $u = \bar{u} - F = 0,\quad F = \bar{u}$

$v = E = 0$

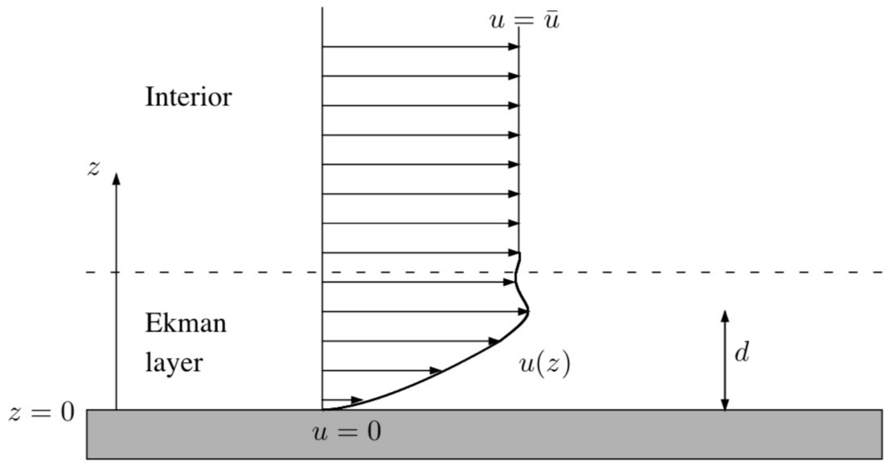
The solutions for u and v are:

$$u = \bar{u}(\overset{u_g}{1} - \overset{u_E}{e^{-\frac{z}{d}}cos\frac{z}{d}})$$

$$v = \bar{u}e^{-\frac{z}{d}}sin\frac{z}{d}$$

$$u = \bar{u}(1 - e^{-\frac{z}{d}} cos\frac{z}{d})$$

$$v = \bar{u}e^{-\frac{z}{d}} sin\frac{z}{d}$$



Properties:

In the BBL, $v \neq 0$: friction induces a traverse flow

$z \to 0$: $$u = \bar{u} - \bar{u}\lim_{z\to 0} e^{-\frac{z}{d}} cos\frac{z}{d}$$ $$\lim_{z\to 0}\frac{cos\frac{z}{d}}{e^{\frac{z}{d}}} = \frac{cos\frac{0}{d} - \frac{z}{d}sin\frac{z}{d}|_{z=0}}{e^{\frac{0}{d}} + \frac{1}{d}e^{\frac{z}{d}}|_{z=0}} = \frac{1}{1+\frac{z}{d}} = 1 - \frac{z}{d}$$

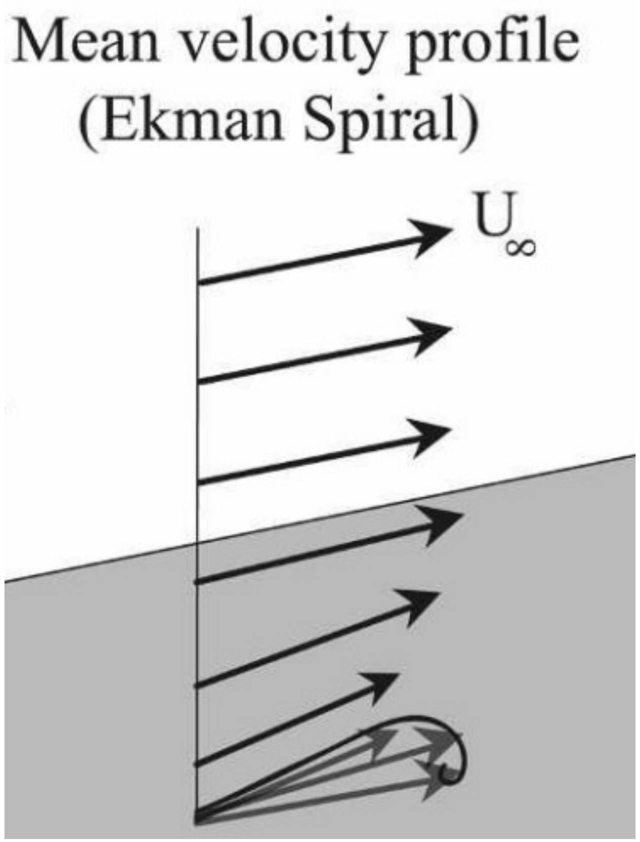
$$u = \frac{\bar{u}}{d}z$$

$$v = \bar{u}\lim_{z\to 0} e^{-\frac{z}{d}} sin\frac{z}{d} = \frac{\bar{u}}{d}z$$

near the bottom the flow is 45 degrees to the left of the interior flow

$$\frac{\partial u}{\partial z} = 0, \quad cos\frac{z}{d} + \sin\frac{z}{d} = 0, \quad z = \frac{3\pi}{4}d$$

maximum velocity in the interior flow direction



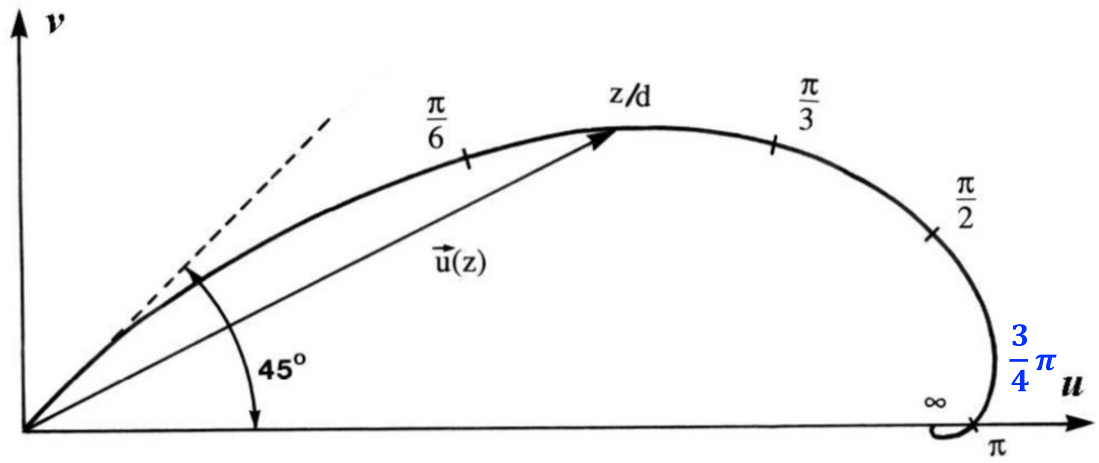


Figure 8-4 The velocity spiral in the bottom Ekman layer. The figure is drawn for the Northern Hemisphere ($f > 0$), and the deflection is to the left of the current above the layer. The reverse holds for the Southern Hemisphere.

Non-uniform currents

The interior flow:

$$-f\bar{v} = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial x}, \qquad f\bar{u} = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial y}$$

Subsitution into the BBL momentum equations:

$$\begin{aligned} -f(v-\bar{v}) &= \nu_E \frac{\partial^2 u}{\partial z^2} \\ f(u-\bar{u}) &= \nu_E \frac{\partial^2 v}{\partial z^2} \end{aligned}$$

The solutions are:

$$\begin{aligned} u &= \bar{u}\left(\underbrace{1 -}_{u_g} \underbrace{e^{-z/d}\cos\frac{z}{d}\right) - \bar{v}\,e^{-z/d}\sin\frac{z}{d}}_{u_E} \\ v &= \underbrace{\bar{u}\,e^{-z/d}\sin\frac{z}{d}}_{} + \bar{v}\left(\underbrace{1 -}_{v_g} \underbrace{e^{-z/d}\cos\frac{z}{d}}_{v_E}\right). \end{aligned}$$

Ekman transport:

$$\mathrm{U} = \int_0^\infty \overbrace{(u - \bar{u})}^{u_E}\, dz = -\frac{d}{2}\,(\bar{u} + \bar{v})$$

$$\mathrm{V} = \int_0^\infty \overbrace{(v - \bar{v})}^{v_E}\, dz = \frac{d}{2}\,(\bar{u} - \bar{v})\,.$$

Ekman transport divergence:

$$\frac{\partial \mathrm{U}}{\partial x} + \frac{\partial \mathrm{V}}{\partial y} = \int_0^\infty \underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}_{-\frac{\partial w}{\partial z}} dz = -\frac{d}{2}\left(\underline{\frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y}}\right)$$

$\bar{\zeta}$: relative vorticity of interior flow

$$w|_{z=\infty} - \cancel{w|_{z=0}} = \frac{d}{2}\bar{\zeta}$$

Ekman pumping velocity

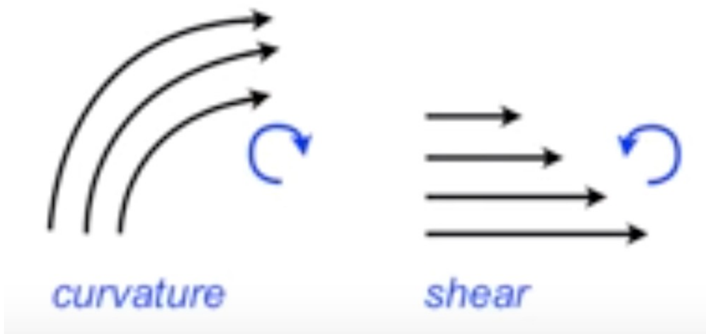
Vorticity

Vorticity: curl of velocity (a measure of spin)

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$$
$$= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\boldsymbol{k}$$

For 2-D flow on the horizontal plane: $\boldsymbol{u} = u\mathbf{i} + v\mathbf{j}$

$$\boldsymbol{\omega} = \mathbf{k}\left(\underbrace{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}_{\zeta}\right)$$



$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = \int_0^\infty \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dz = -\frac{d}{2} \underbrace{\left(\frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y} \right)}_{\bar{\zeta}} > 0 \quad \text{divergence}$$

$$w|_{z=\infty} = \frac{d}{2}\bar{\zeta} < 0$$

Divergence in the bottom

boundary layer induces

downwelling from the interior

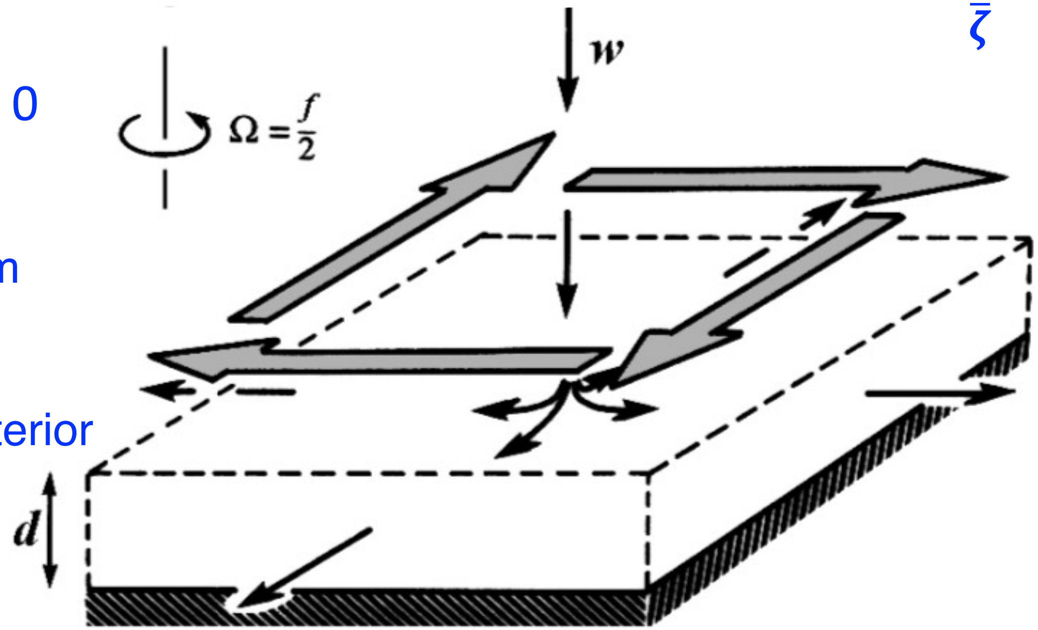
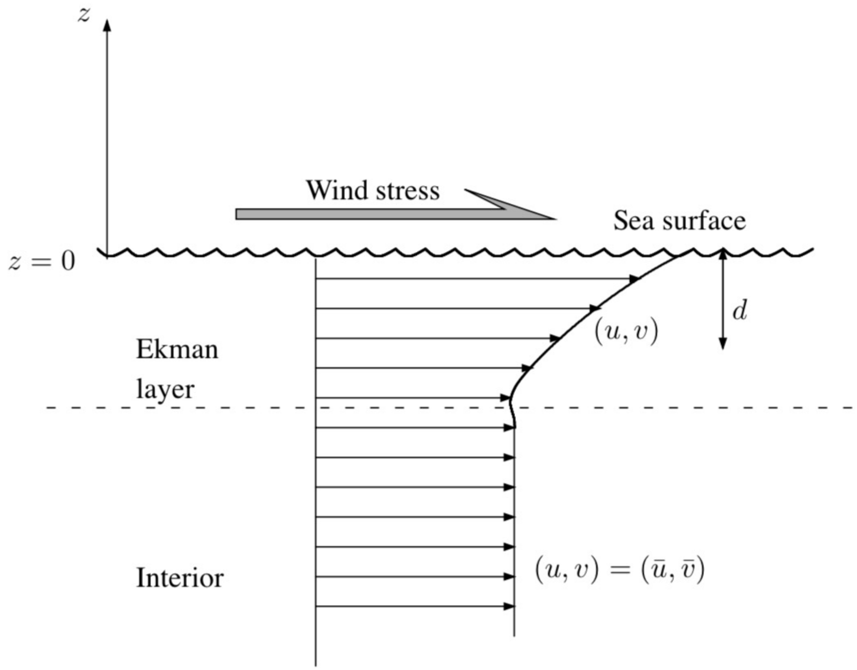


Figure 8-5 Divergence in the bottom Ekman layer and compensating downwelling in the interior. Such a situation arises in the presence of an anticyclonic gyre in the interior, as depicted by the large horizontal arrows. Similarly, interior cyclonic motion causes convergence in the Ekman layer and upwelling in the interior.

The surface Ekman layer

The interior flow:

$$-f\bar{v} = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial x}, \quad f\bar{u} = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial y}$$



Substitution into the SBL momentum equations:

$$\begin{aligned} -f(v - \bar{v}) &= \nu_E \frac{\partial^2 u}{\partial z^2} \\ f(u - \bar{u}) &= \nu_E \frac{\partial^2 v}{\partial z^2} \end{aligned}$$

Boundary conditions:

$$\begin{aligned} \text{Surface } (z = 0): &\quad \rho_0 \nu_E \frac{\partial u}{\partial z} = \tau^x, \quad \rho_0 \nu_E \frac{\partial v}{\partial z} = \tau^y \\ \text{Toward interior } (z \to -\infty): &\quad u = \bar{u}, \quad v = \bar{v}. \end{aligned}$$

The solutions are:

$$u = \bar{u} + \frac{\sqrt{2}}{\rho_0 f d} e^{z/d} \left[\tau^x \cos\left(\frac{z}{d} - \frac{\pi}{4}\right) - \tau^y \sin\left(\frac{z}{d} - \frac{\pi}{4}\right) \right]$$

$$v = \bar{v} + \frac{\sqrt{2}}{\rho_0 f d} e^{z/d} \left[\tau^x \sin\left(\frac{z}{d} - \frac{\pi}{4}\right) + \tau^y \cos\left(\frac{z}{d} - \frac{\pi}{4}\right) \right]$$

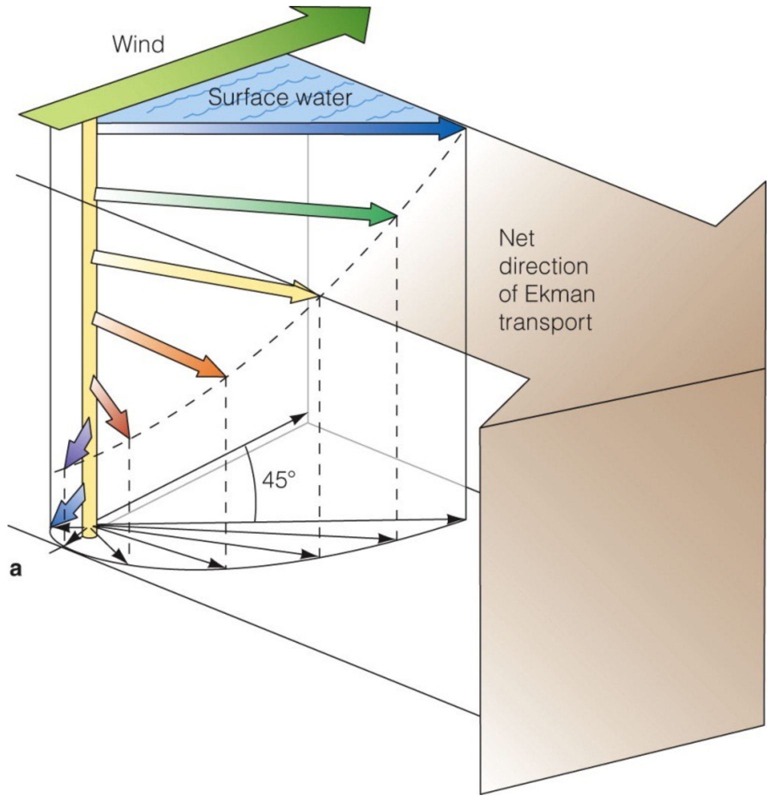
The surface Ekman transport:

$$\mathbf{U} = \int_{-\infty}^{0} (u - \bar{u})\, dz = \frac{1}{\rho_0 f} \tau^y$$

$$\mathbf{V} = \int_{-\infty}^{0} (v - \bar{v})\, dz = \frac{-1}{\rho_0 f} \tau^x.$$

$$\int_{-\infty}^{0} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dz = \frac{1}{\rho_0} \left[\frac{\partial}{\partial x} \left(\frac{\tau^y}{f} \right) - \frac{\partial}{\partial y} \left(\frac{\tau^x}{f} \right) \right]$$

$$\bar{w} = \frac{1}{\rho_0} \left[\frac{\partial}{\partial x} \left(\frac{\tau^y}{f} \right) - \frac{\partial}{\partial y} \left(\frac{\tau^x}{f} \right) \right]$$



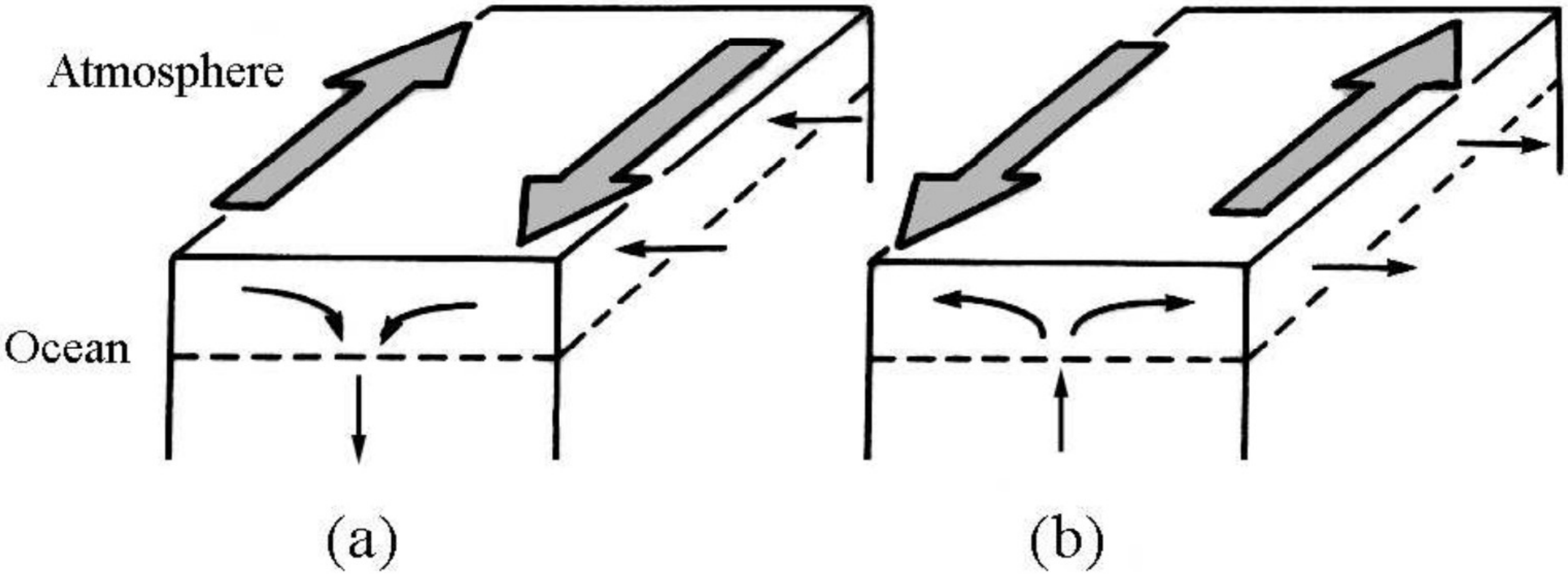


Figure 8-8 Ekman pumping in an ocean subject to sheared winds (case of Northern Hemisphere).