The heavens themselves, the planets, and this centre
Observe degree, priority, and place,
Insisture, course, proportion, season, form,
Office, and custom, in all line of order.
And therefore is the glorious planet Sol
In noble eminence enthroned and sphered.
William Shakespeare, *Troilus and Cressida*, c. 1602.

Eppur si muove. (And yet it does move.)
Galileo Galilei, apocryphal, 1633.

CHAPTER 2

Effects of Rotation and Stratification

THE ATMOSPHERE AND OCEAN are shallow layers of fluid on a sphere, 'shallow' because their thickness is much less than their horizontal extent. Their motion is strongly influenced by two effects: rotation and stratification, the latter meaning that there is a mean vertical gradient of (potential) density that is often large compared with the horizontal gradient. Here we consider how the equations of motion are affected by these effects. First, we consider some elementary effects of rotation on a fluid and derive the Coriolis and centrifugal forces, and write down the equations of motion appropriate for motion on a sphere. Then we discuss some approximations to the equations of motion that are appropriate for large-scale flow in the ocean and atmosphere, in particular the hydrostatic and geostrophic approximations, and finally we look at the possible static instability of stratified flows.

2.1 EQUATIONS OF MOTION IN A ROTATING FRAME

Newton's second law of motion, that the acceleration of a body is proportional to the imposed force divided by the body's mass, applies in so-called inertial frames of reference; that is, frames that are stationary or moving only with a constant rectilinear velocity relative to the distant galaxies. Now Earth spins round its own axis with a period of almost 24 hours (23h 56m, the difference due to Earth's rotation around the Sun) and so the surface of the Earth manifestly is not an inertial frame. Nevertheless, it is very convenient to describe the flow relative to Earth's surface (which in fact is moving at speeds of up to a few hundreds of metres per second), rather than in some inertial frame.[1] This necessitates recasting the equations into a form appropriate in a rotating frame of reference, and that is the subject of this section.

2.1.1 Rate of Change of a Vector

Consider first a vector $\boldsymbol{C}$ of constant length rotating relative to an inertial frame at a constant angular velocity $\boldsymbol{\Omega}$. Then, in a frame rotating with that same angular velocity it appears stationary and constant. If in a small interval of time δt the vector $\boldsymbol{C}$ rotates through a small angle $\delta\lambda$ then the change in $\boldsymbol{C}$, as perceived in the inertial frame, is given by (see Fig. 2.1)

$$\delta \boldsymbol{C} = |\boldsymbol{C}| \cos\vartheta \, \delta\lambda \, \boldsymbol{m}, \tag{2.1}$$

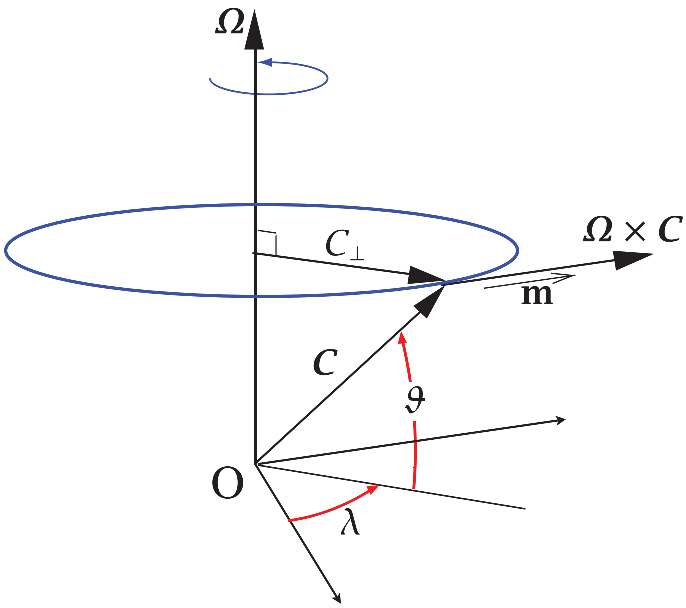

Fig. 2.1 A vector C rotating at an angular velocity Ω. It appears to be a constant vector in the rotating frame, whereas in the inertial frame it evolves according to $(\mathrm{d}C/\mathrm{d}t)_I = \Omega \times C$.

where the vector $\boldsymbol{m}$ is the unit vector in the direction of change of $\boldsymbol{C}$, which is perpendicular to both $\boldsymbol{C}$ and $\boldsymbol{\Omega}$. But the rate of change of the angle λ is just, by definition, the angular velocity so that $\delta\lambda = |\boldsymbol{\Omega}|\delta t$ and

$$\delta \boldsymbol{C} = |\boldsymbol{C}||\boldsymbol{\Omega}| \sin \hat{\vartheta}\, \boldsymbol{m}\, \delta t = \boldsymbol{\Omega} \times \boldsymbol{C}\, \delta t, \tag{2.2}$$

using the definition of the vector cross product, where $\hat{\vartheta} = (\pi/2 - \vartheta)$ is the angle between $\boldsymbol{\Omega}$ and $\boldsymbol{C}$. Thus

$$\left(\frac{\mathrm{d}\boldsymbol{C}}{\mathrm{d}t}\right)_I = \boldsymbol{\Omega} \times \boldsymbol{C}, \tag{2.3}$$

where the left-hand side is the rate of change of $\boldsymbol{C}$ as perceived in the inertial frame.

Now consider a vector $\boldsymbol{B}$ that changes in the inertial frame. In a small time δt the change in $\boldsymbol{B}$ as seen in the rotating frame is related to the change seen in the inertial frame by

$$(\delta \boldsymbol{B})_I = (\delta \boldsymbol{B})_R + (\delta \boldsymbol{B})_{rot}, \tag{2.4}$$

where the terms are, respectively, the change seen in the inertial frame, the change due to the vector itself changing as measured in the rotating frame, and the change due to the rotation. Using (2.2) $(\delta \boldsymbol{B})_{rot} = \boldsymbol{\Omega} \times \boldsymbol{B}\, \delta t$, and so the rates of change of the vector $\boldsymbol{B}$ in the inertial and rotating frames are related by

$$\boxed{\left(\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega} \times \boldsymbol{B}.} \tag{2.5}$$

This relation applies to a vector $\boldsymbol{B}$ that, as measured at any one time, is the same in both inertial and rotating frames.

2.1.2 Velocity and Acceleration in a Rotating Frame

The velocity of a body is not measured to be the same in the inertial and rotating frames, so care must be taken when applying (2.5) to velocity. First apply (2.5) to $\boldsymbol{r}$, the position of a particle to obtain

$$\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega} \times \boldsymbol{r} \tag{2.6}$$

or

$$\boldsymbol{v}_I = \boldsymbol{v}_R + \boldsymbol{\Omega} \times \boldsymbol{r}. \tag{2.7}$$

We refer to $\boldsymbol{v}_R$ and $\boldsymbol{v}_I$ as the relative and inertial velocity, respectively, and (2.7) relates the two. Apply (2.5) again, this time to the velocity $\boldsymbol{v}_R$ to give

$$\left(\frac{\mathrm{d}\boldsymbol{v}_R}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\boldsymbol{v}_R}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega} \times \boldsymbol{v}_R, \tag{2.8}$$

or, using (2.7)

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v}_I - \boldsymbol{\Omega} \times \boldsymbol{r})\right)_I = \left(\frac{\mathrm{d}\boldsymbol{v}_R}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega} \times \boldsymbol{v}_R, \tag{2.9}$$

or

$$\left(\frac{\mathrm{d}\boldsymbol{v}_I}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\boldsymbol{v}_R}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega} \times \boldsymbol{v}_R + \frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t} \times \boldsymbol{r} + \boldsymbol{\Omega} \times \left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_I. \tag{2.10}$$

Then, noting that

$$\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega} \times \boldsymbol{r} = (\boldsymbol{v}_R + \boldsymbol{\Omega} \times \boldsymbol{r}), \tag{2.11}$$

and assuming that the rate of rotation is constant, (2.10) becomes

$$\left(\frac{\mathrm{d}\boldsymbol{v}_R}{\mathrm{d}t}\right)_R = \left(\frac{\mathrm{d}\boldsymbol{v}_I}{\mathrm{d}t}\right)_I - 2\boldsymbol{\Omega} \times \boldsymbol{v}_R - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}). \tag{2.12}$$

This equation may be interpreted as follows. The term on the left-hand side is the rate of change of the relative velocity as measured in the rotating frame. The first term on the right-hand side is the rate of change of the inertial velocity as measured in the inertial frame (the inertial acceleration, which is, by Newton's second law, equal to the force on a fluid parcel divided by its mass). The second and third terms on the right-hand side (including the minus signs) are the *Coriolis force* and the *centrifugal force* per unit mass. Neither of these is a true force — they may be thought of as quasi-forces (i.e., 'as if' forces); that is, when a body is observed from a rotating frame it behaves as if unseen forces are present that affect its motion. If (2.12) is written, as is common, with the terms $+2\boldsymbol{\Omega} \times \boldsymbol{v}_r$ and $+\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$ on the left-hand side then these terms should be referred to as the Coriolis and centrifugal *accelerations.*[2]

Centrifugal force

If $\boldsymbol{r}_\perp$ is the perpendicular distance from the axis of rotation (see Fig. 2.1 and substitute $\boldsymbol{r}$ for $\boldsymbol{C}$), then, because $\boldsymbol{\Omega}$ is perpendicular to $\boldsymbol{r}_\perp$, $\boldsymbol{\Omega} \times \boldsymbol{r} = \boldsymbol{\Omega} \times \boldsymbol{r}_\perp$. Then, using the vector identity $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}_\perp) = (\boldsymbol{\Omega} \cdot \boldsymbol{r}_\perp)\boldsymbol{\Omega} - (\boldsymbol{\Omega} \cdot \boldsymbol{\Omega})\boldsymbol{r}_\perp$ and noting that the first term is zero, we see that the centrifugal force per unit mass is just given by

$$\boldsymbol{F}_{ce} = -\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = \Omega^2 \boldsymbol{r}_\perp. \tag{2.13}$$

This may usefully be written as the gradient of a scalar potential,

$$\boldsymbol{F}_{ce} = -\nabla \Phi_{ce}, \tag{2.14}$$

where $\Phi_{ce} = -(\Omega^2 r_\perp^2)/2 = -(\boldsymbol{\Omega} \times \boldsymbol{r}_\perp)^2/2$.

Coriolis force

The Coriolis force per unit mass is given by

$$\boldsymbol{F}_{Co} = -2\boldsymbol{\Omega} \times \boldsymbol{v}_R. \tag{2.15}$$

It plays a central role in much of geophysical fluid dynamics and will be considered extensively later on. For now, we just note three basic properties:

(i) There is no Coriolis force on bodies that are stationary in the rotating frame.
(ii) The Coriolis force acts to deflect moving bodies at right angles to their direction of travel.
(iii) The Coriolis force does no work on a body because it is perpendicular to the velocity, and so $\boldsymbol{v}_R \cdot (\boldsymbol{\Omega} \times \boldsymbol{v}_R) = 0$.

2.1.3 Momentum Equation in a Rotating Frame

Since (2.12) simply relates the accelerations of a particle in the inertial and rotating frames, then in the rotating frame of reference the momentum equation may be written

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + 2\boldsymbol{\Omega} \times \boldsymbol{v} = -\frac{1}{\rho}\nabla p - \nabla\Phi, \tag{2.16}$$

incorporating the centrifugal term into the potential, Φ. We have dropped the subscript R; henceforth, unless we need to be explicit (as in the next section), all velocities without a subscript will be considered to be relative to the rotating frame.

2.1.4 Mass and Tracer Conservation in a Rotating frame

Let φ be a scalar field that, in the inertial frame, obeys

$$\frac{\mathrm{D}\varphi}{\mathrm{D}t} + \varphi\nabla \cdot \boldsymbol{v}_I = 0. \tag{2.17}$$

Now, observers in both the rotating and inertial frame measure the same value of φ. Further, $\mathrm{D}\varphi/\mathrm{D}t$ is simply the rate of change of φ associated with a material parcel, and therefore is reference frame invariant. Thus, without further ado, we write

$$\left(\frac{\mathrm{D}\varphi}{\mathrm{D}t}\right)_R = \left(\frac{\mathrm{D}\varphi}{\mathrm{D}t}\right)_I, \tag{2.18}$$

where $(\mathrm{D}\varphi/\mathrm{D}t)_R = (\partial\varphi/\partial t)_R + \boldsymbol{v}_R \cdot \nabla\varphi$ and $(\mathrm{D}\varphi/\mathrm{D}t)_I = (\partial\varphi/\partial t)_I + \boldsymbol{v}_I \cdot \nabla\varphi$, and the local temporal derivatives $(\partial\varphi/\partial t)_R$ and $(\partial\varphi/\partial t)_I$ are evaluated at fixed locations in the rotating and inertial frames, respectively.

Further, using (2.7), we have that

$$\nabla \cdot \boldsymbol{v}_I = \nabla \cdot (\boldsymbol{v}_R + \boldsymbol{\Omega} \times \boldsymbol{r}) = \nabla \cdot \boldsymbol{v}_R, \tag{2.19}$$

since $\nabla \cdot (\boldsymbol{\Omega} \times \boldsymbol{r}) = 0$. Thus, using (2.18) and (2.19), (2.17) is equivalent to

$$\frac{\mathrm{D}\varphi}{\mathrm{D}t} + \varphi\nabla \cdot \boldsymbol{v}_R = 0, \tag{2.20}$$

where all observables are measured in the *rotating* frame. Thus, the equation for the evolution of a scalar whose measured value is the same in rotating and inertial frames is unaltered by the presence of rotation. In particular, the mass conservation equation is unaltered by the presence of rotation.

Although we have taken (2.18) as true a priori, the individual components of the material derivative differ in the rotating and inertial frames. In particular

$$\left(\frac{\partial \varphi}{\partial t}\right)_I = \left(\frac{\partial \varphi}{\partial t}\right)_R - (\boldsymbol{\Omega} \times \boldsymbol{r}) \cdot \nabla \varphi, \tag{2.21}$$

because $\boldsymbol{\Omega} \times \boldsymbol{r}$ is the velocity, in the inertial frame, of a uniformly rotating body. Similarly,

$$\boldsymbol{v}_I \cdot \nabla \varphi = (\boldsymbol{v}_R + \boldsymbol{\Omega} \times \boldsymbol{r}) \cdot \nabla \varphi. \tag{2.22}$$

Adding the last two equations reprises and confirms (2.18).

2.2 EQUATIONS OF MOTION IN SPHERICAL COORDINATES

The Earth is very nearly spherical and it might appear obvious that we should cast our equations in spherical coordinates. Although this does turn out to be true, the presence of a centrifugal force causes some complications that we should first discuss. The reader who is willing ab initio to treat the Earth as a perfect sphere and to neglect the horizontal component of the centrifugal force may skip the next section.

2.2.1 ✦ The Centrifugal Force and Spherical Coordinates

The centrifugal force is a potential force, like gravity, and so we may therefore define an 'effective gravity' equal to the sum of the true, or Newtonian, gravity and the centrifugal force. The Newtonian gravitational force is directed approximately toward the centre of the Earth, with small deviations due mainly to the Earth's oblateness. The line of action of the effective gravity will in general differ slightly from this, and therefore have a component in the 'horizontal' plane, that is the plane perpendicular to the radial direction. The magnitude of the centrifugal force is $\Omega^2 r_\perp$, and so the effective gravity is given by

$$\boldsymbol{g} \equiv \boldsymbol{g}_{\mathit{eff}} = \boldsymbol{g}_{grav} + \Omega^2 \boldsymbol{r}_\perp, \tag{2.23}$$

where $\boldsymbol{g}_{grav}$ is the Newtonian gravitational force due to the gravitational attraction of the Earth and $\boldsymbol{r}_\perp$ is normal to the rotation vector (in the direction $\boldsymbol{C}$ in Fig. 2.2), with $r_\perp = r\cos\vartheta$. Both gravity and centrifugal force are potential forces and therefore we may define the *geopotential*, Φ, such that

$$\boldsymbol{g} = -\nabla \Phi. \tag{2.24}$$

Surfaces of constant Φ are not quite spherical because $r_\perp$, and hence the centrifugal force, vary with latitude (Fig. 2.2); this has certain ramifications, as we now discuss.

The components of the centrifugal force parallel and perpendicular to the radial direction are $\Omega^2 r \cos^2\vartheta$ and $\Omega^2 r \cos\vartheta \sin\vartheta$. Newtonian gravity is much larger than either of these, and at the Earth's surface the ratio of centrifugal to gravitational terms is approximately, and no more than,

$$\alpha \approx \frac{\Omega^2 a}{g} \approx \frac{(7.27\times 10^{-5})^2 \times 6.4 \times 10^6}{9.8} \approx 3 \times 10^{-3}. \tag{2.25}$$

(At the equator and pole the horizontal component of the centrifugal force is zero and the effective gravity is aligned with Newtonian gravity.) The angle between $\boldsymbol{g}$ and the line to the centre of the Earth is given by a similar expression and so is also small, typically around 3×10^{-3} radians. However, the horizontal component of the centrifugal force is still large compared to the Coriolis force, the ratio of their magnitudes in mid-latitudes being given by

$$\frac{\text{horizontal centrifugal force}}{\text{Coriolis force}} \approx \frac{\Omega^2 a \cos\vartheta \sin\vartheta}{2\Omega|u|} \approx \frac{\Omega a}{4|u|} \approx 10, \tag{2.26}$$

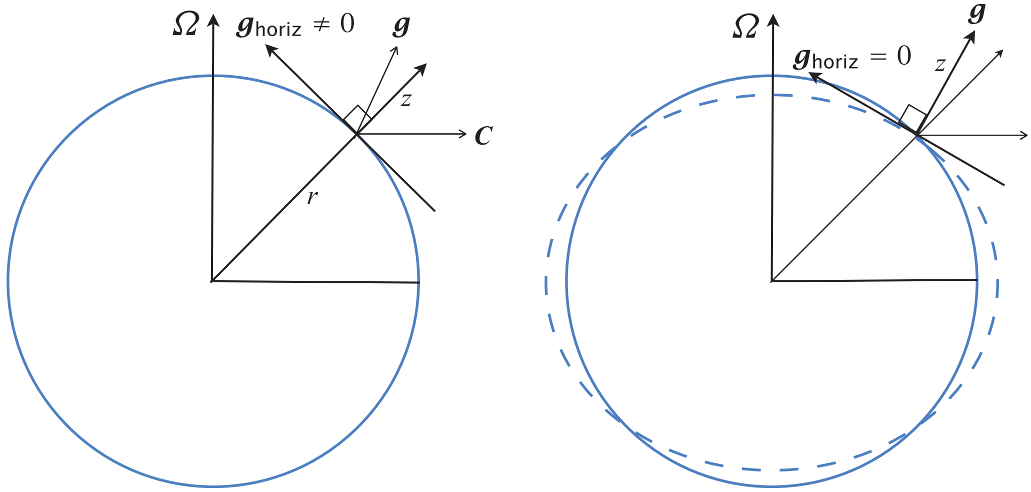

Fig. 2.2 Left: directions of forces and coordinates in true spherical geometry. $\boldsymbol{g}$ is the effective gravity (including the centrifugal force, $\boldsymbol{C}$) and its horizontal component is evidently non-zero. Right: a modified coordinate system, in which the vertical direction is defined by the direction of $\boldsymbol{g}$, and so the horizontal component of $\boldsymbol{g}$ is identically zero. The dashed line schematically indicates a surface of constant geopotential. The differences between the direction of $\boldsymbol{g}$ and the direction of the radial coordinate, and between the sphere and the geopotential surface, are much exaggerated and in reality are similar to the thickness of the lines themselves.

using $u = 10\,\mathrm{m\,s^{-1}}$. The centrifugal term therefore dominates over the Coriolis term, and is largely balanced by a pressure gradient force. Thus, if we adhered to true spherical coordinates, both the horizontal and radial components of the momentum equation would be dominated by a static balance between a pressure gradient and gravity or centrifugal terms. Although in principle there is nothing wrong with writing the equations this way, it obscures the dynamical balances involving the Coriolis force and pressure that determine the large-scale horizontal flow.

A way around this problem is to use the direction of the geopotential force to *define* the vertical direction, and then for all geometric purposes to regard the surfaces of constant Φ as if they were true spheres.[3] The horizontal component of effective gravity is then identically zero, and we have traded a potentially large dynamical error for a very small geometric error. In fact, over time, the Earth has developed an equatorial bulge to compensate for and neutralize the centrifugal force, so that the effective gravity does act in a direction virtually normal to the Earth's surface; that is, the surface of the Earth is an oblate spheroid of nearly constant geopotential. The geopotential Φ is then a function of the vertical coordinate alone, and for many purposes we can just take $\Phi = gz$; that is, the direction normal to geopotential surfaces, the local vertical, is, in this approximation, taken to be the direction of increasing r in spherical coordinates. It is because the oblateness is very small (the polar diameter is about 12 714 km, whereas the equatorial diameter is about 12 756 km) that using spherical coordinates is a very accurate way to map the spheroid. If the angle between effective gravity and a natural direction of the coordinate system were not small then more heroic measures would be called for.

If the solid Earth did not bulge at the equator, the *behaviour* of the atmosphere and ocean would differ significantly from that of the present system. For example, the surface of the ocean is, necessarily, very nearly a geopotential surface; if the solid Earth were exactly spherical then the ocean would perforce become much deeper at low latitudes and the ocean basins would dry out completely at high latitudes. We could still choose to use the spherical coordinate system discussed above to describe the dynamics, but the shape of the surface of the solid Earth would have to be represented by a topography, with the topographic height increasing monotonically polewards nearly everywhere.

2.2.2 Some Identities in Spherical Coordinates

The location of a point is given by the coordinates (λ, ϑ, r) where λ is the angular distance eastwards (i.e., longitude), ϑ is angular distance polewards (i.e., latitude) and r is the radial distance from the centre of the Earth — see Fig. 2.3. (In some other fields of study co-latitude is used as a spherical coordinate.) If a is the radius of the Earth, then we also define $z = r - a$. At a given location we may also define the Cartesian increments $(\delta x, \delta y, \delta z) = (r\cos\vartheta\,\delta\lambda, r\delta\vartheta, \delta r)$.

For a scalar quantity ϕ the material derivative in spherical coordinates is

$$\frac{\mathrm{D}\phi}{\mathrm{D}t} = \frac{\partial\phi}{\partial t} + \frac{u}{r\cos\vartheta}\frac{\partial\phi}{\partial\lambda} + \frac{v}{r}\frac{\partial\phi}{\partial\vartheta} + w\frac{\partial\phi}{\partial r}, \tag{2.27}$$

where the velocity components corresponding to the coordinates (λ, ϑ, r) are

$$(u, v, w) \equiv \left(r\cos\vartheta\frac{\mathrm{D}\lambda}{\mathrm{D}t}, r\frac{\mathrm{D}\vartheta}{\mathrm{D}t}, \frac{\mathrm{D}r}{\mathrm{D}t}\right). \tag{2.28}$$

That is, u is the zonal velocity, v is the meridional velocity and w is the vertical velocity. If we define $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ to be the unit vectors in the direction of increasing (λ, ϑ, r) then

$$\boldsymbol{v} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w. \tag{2.29}$$

Note also that $\mathrm{D}r/\mathrm{D}t = \mathrm{D}z/\mathrm{D}t$.

The divergence of a vector $\boldsymbol{B} = \mathbf{i}\,B^\lambda + \mathbf{j}\,B^\vartheta + \mathbf{k}\,B^r$ is

$$\nabla\cdot\boldsymbol{B} = \frac{1}{\cos\vartheta}\left[\frac{1}{r}\frac{\partial B^\lambda}{\partial\lambda} + \frac{1}{r}\frac{\partial}{\partial\vartheta}(B^\vartheta\cos\vartheta) + \frac{\cos\vartheta}{r^2}\frac{\partial}{\partial r}(r^2B^r)\right]. \tag{2.30}$$

The vector gradient of a scalar is:

$$\nabla\phi = \mathbf{i}\frac{1}{r\cos\vartheta}\frac{\partial\phi}{\partial\lambda} + \mathbf{j}\frac{1}{r}\frac{\partial\phi}{\partial\vartheta} + \mathbf{k}\frac{\partial\phi}{\partial r}. \tag{2.31}$$

The Laplacian of a scalar is:

$$\nabla^2\phi \equiv \nabla\cdot\nabla\phi = \frac{1}{r^2\cos\vartheta}\left[\frac{1}{\cos\vartheta}\frac{\partial^2\phi}{\partial\lambda^2} + \frac{\partial}{\partial\vartheta}\left(\cos\vartheta\frac{\partial\phi}{\partial\vartheta}\right) + \cos\vartheta\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)\right]. \tag{2.32}$$

The curl of a vector is:

$$\operatorname{curl}\boldsymbol{B} = \nabla\times\boldsymbol{B} = \frac{1}{r^2\cos\vartheta}\begin{vmatrix} \mathbf{i}\,r\cos\vartheta & \mathbf{j}\,r & \mathbf{k} \\ \partial/\partial\lambda & \partial/\partial\vartheta & \partial/\partial r \\ B^\lambda r\cos\vartheta & B^\vartheta r & B^r \end{vmatrix}. \tag{2.33}$$

The vector Laplacian $\nabla^2\boldsymbol{B}$ (used for example when calculating viscous terms in the momentum equation) may be obtained from the vector identity:

$$\nabla^2\boldsymbol{B} = \nabla(\nabla\cdot\boldsymbol{B}) - \nabla\times(\nabla\times\boldsymbol{B}). \tag{2.34}$$

Only in Cartesian coordinates does this take the simple form:

$$\nabla^2\boldsymbol{B} = \frac{\partial^2\boldsymbol{B}}{\partial x^2} + \frac{\partial^2\boldsymbol{B}}{\partial y^2} + \frac{\partial^2\boldsymbol{B}}{\partial z^2}. \tag{2.35}$$

The expansion in spherical coordinates is of itself, to most eyes, rather uninformative.

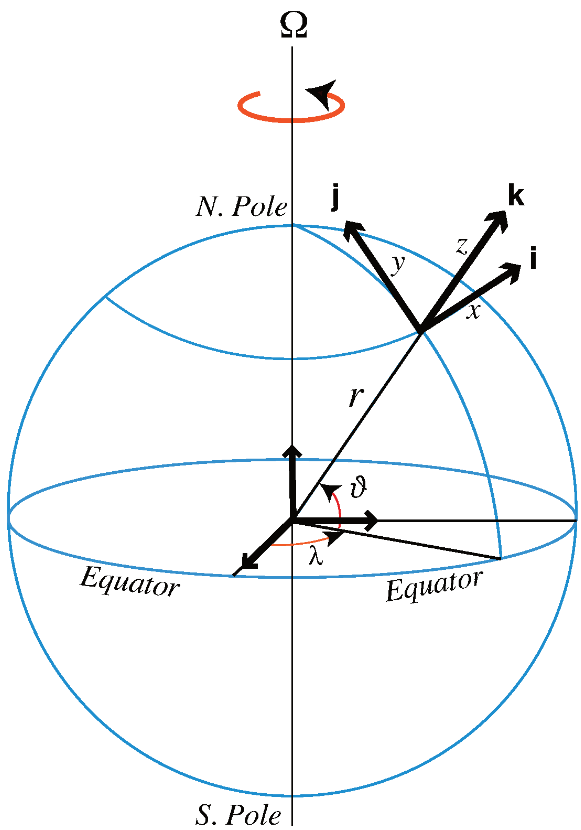

Fig. 2.3 The spherical coordinate system. The orthogonal unit vectors **i**, **j** and **k** point in the direction of increasing longitude λ, latitude ϑ, and altitude z. Locally, one may apply a Cartesian system with variables x, y and z measuring distances along **i**, **j** and **k**.

Rate of change of unit vectors

In spherical coordinates the defining unit vectors are **i**, the unit vector pointing eastwards, parallel to a line of latitude; **j** is the unit vector pointing polewards, parallel to a meridian; and **k**, the unit vector pointing radially outward. The directions of these vectors change with location, and in fact this is the case in nearly all coordinate systems, with the notable exception of the Cartesian one, and thus their material derivative is not zero. One way to evaluate this is to consider geometrically how the coordinate axes change with position. Another way, and the way that we shall proceed, is to first obtain the effective rotation rate $\boldsymbol{\Omega}_{flow}$, relative to the Earth, of a unit vector as it moves with the flow, and then apply (2.3). Specifically, let the fluid velocity be $\boldsymbol{v} = (u, v, w)$. The meridional component, v, produces a displacement $r\delta\vartheta = v\delta t$, and this gives rise to a local effective vector rotation rate around the local zonal axis of $-(v/r)\mathbf{i}$, the minus sign arising because a displacement in the direction of the north pole is produced by negative rotational displacement around the **i** axis. Similarly, the zonal component, u, produces a displacement $\delta\lambda r\cos\vartheta = u\delta t$ and so an effective rotation rate, about the Earth's rotation axis, of $u/(r\cos\vartheta)$. Now, a rotation around the Earth's rotation axis may be written as (see Fig. 2.4)

$$\boldsymbol{\Omega} = \Omega(\mathbf{j}\cos\vartheta + \mathbf{k}\sin\vartheta). \tag{2.36}$$

If the scalar rotation rate is not Ω but is $u/(r\cos\vartheta)$, then the vector rotation rate is

$$\frac{u}{r\cos\vartheta}(\mathbf{j}\cos\vartheta + \mathbf{k}\sin\vartheta) = \mathbf{j}\frac{u}{r} + \mathbf{k}\frac{u\tan\vartheta}{r}. \tag{2.37}$$

Thus, the total rotation rate of a vector that moves with the flow is

$$\boldsymbol{\Omega}_{flow} = -\mathbf{i}\frac{v}{r} + \mathbf{j}\frac{u}{r} + \mathbf{k}\frac{u\tan\vartheta}{r}. \tag{2.38}$$

Applying (2.3) to (2.38), we find

$$\frac{\mathrm{D}\mathbf{i}}{\mathrm{D}t} = \boldsymbol{\Omega}_{flow} \times \mathbf{i} = \frac{u}{r\cos\vartheta}(\mathbf{j}\sin\vartheta - \mathbf{k}\cos\vartheta), \tag{2.39a}$$

$$\frac{\mathrm{D}\mathbf{j}}{\mathrm{D}t} = \boldsymbol{\Omega}_{flow} \times \mathbf{j} = -\mathbf{i}\frac{u}{r}\tan\vartheta - \mathbf{k}\frac{v}{r}, \tag{2.39b}$$

$$\frac{\mathrm{D}\mathbf{k}}{\mathrm{D}t} = \boldsymbol{\Omega}_{flow} \times \mathbf{k} = \mathbf{i}\frac{u}{r} + \mathbf{j}\frac{v}{r}. \tag{2.39c}$$

2.2.3 Equations of Motion

Mass conservation and thermodynamic equation

The mass conservation equation, (1.36a), expanded in spherical co-ordinates, is

$$\frac{\partial\rho}{\partial t} + \frac{u}{r\cos\vartheta}\frac{\partial\rho}{\partial\lambda} + \frac{v}{r}\frac{\partial\rho}{\partial\vartheta} + w\frac{\partial\rho}{\partial r} + \frac{\rho}{r\cos\vartheta}\left[\frac{\partial u}{\partial\lambda} + \frac{\partial}{\partial\vartheta}(v\cos\vartheta) + \frac{1}{r}\frac{\partial}{\partial r}(wr^2\cos\vartheta)\right] = 0. \tag{2.40}$$

Equivalently, using the form (1.36b), this is

$$\frac{\partial\rho}{\partial t} + \frac{1}{r\cos\vartheta}\frac{\partial(u\rho)}{\partial\lambda} + \frac{1}{r\cos\vartheta}\frac{\partial}{\partial\vartheta}(v\rho\cos\vartheta) + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 w\rho) = 0. \tag{2.41}$$

The thermodynamic equation, (1.108), is a tracer advection equation. Thus, using (2.27), its (adiabatic) spherical coordinate form is

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = \frac{\partial\theta}{\partial t} + \frac{u}{r\cos\vartheta}\frac{\partial\theta}{\partial\lambda} + \frac{v}{r}\frac{\partial\theta}{\partial\vartheta} + w\frac{\partial\theta}{\partial r} = 0, \tag{2.42}$$

and similarly for tracers such as water vapour or salt.

Momentum equation

Recall that the inviscid momentum equation is:

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + 2\boldsymbol{\Omega}\times\boldsymbol{v} = -\frac{1}{\rho}\nabla p - \nabla\Phi, \tag{2.43}$$

where Φ is the geopotential. In spherical coordinates the directions of the coordinate axes change with position and so the component expansion of (2.43) is

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = \frac{\mathrm{D}u}{\mathrm{D}t}\mathbf{i} + \frac{\mathrm{D}v}{\mathrm{D}t}\mathbf{j} + \frac{\mathrm{D}w}{\mathrm{D}t}\mathbf{k} + u\frac{\mathrm{D}\mathbf{i}}{\mathrm{D}t} + v\frac{\mathrm{D}\mathbf{j}}{\mathrm{D}t} + w\frac{\mathrm{D}\mathbf{k}}{\mathrm{D}t} \tag{2.44a}$$

$$= \frac{\mathrm{D}u}{\mathrm{D}t}\mathbf{i} + \frac{\mathrm{D}v}{\mathrm{D}t}\mathbf{j} + \frac{\mathrm{D}w}{\mathrm{D}t}\mathbf{k} + \boldsymbol{\Omega}_{flow}\times\boldsymbol{v}, \tag{2.44b}$$

using (2.39). Using either (2.44a) and the expressions for the rates of change of the unit vectors given in (2.39), or (2.44b) and the expression for $\boldsymbol{\Omega}_{flow}$ given in (2.38), (2.44) becomes

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = \mathbf{i}\left(\frac{\mathrm{D}u}{\mathrm{D}t} - \frac{uv\tan\vartheta}{r} + \frac{uw}{r}\right) + \mathbf{j}\left(\frac{\mathrm{D}v}{\mathrm{D}t} + \frac{u^2\tan\vartheta}{r} + \frac{vw}{r}\right) + \mathbf{k}\left(\frac{\mathrm{D}w}{\mathrm{D}t} - \frac{u^2+v^2}{r}\right). \tag{2.45}$$

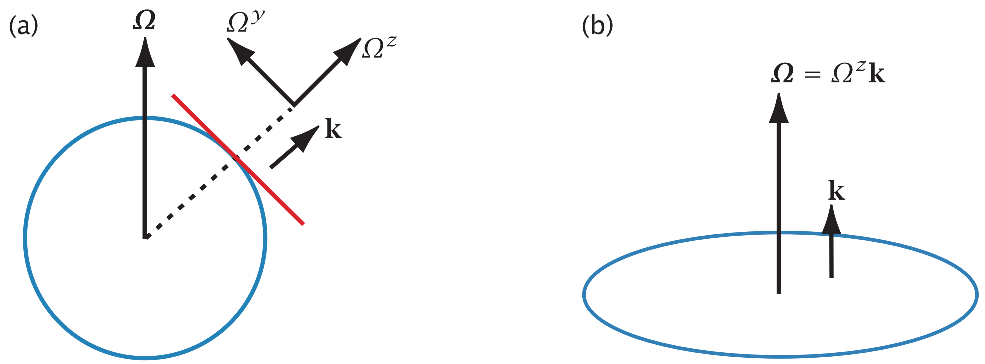

Fig. 2.4 (a) On the sphere the rotation vector $\boldsymbol{\Omega}$ can be decomposed into two components, one in the local vertical and one in the local horizontal, pointing toward the pole. That is, $\boldsymbol{\Omega} = \Omega_y\mathbf{j}+\Omega_z\mathbf{k}$ where $\Omega_y = \Omega\cos\vartheta$ and $\Omega_z = \Omega\sin\vartheta$. In geophysical fluid dynamics, the rotation vector in the local vertical is often the more important component in the horizontal momentum equations. On a rotating disk, (b), the rotation vector $\boldsymbol{\Omega}$ is parallel to the local vertical $\mathbf{k}$.

Using the definition of a vector cross product the Coriolis term is:

$$2\boldsymbol{\Omega}\times\boldsymbol{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2\Omega\cos\vartheta & 2\Omega\sin\vartheta \\ u & v & w \end{vmatrix}$$
$$= \mathbf{i}\,(2\Omega w\cos\vartheta - 2\Omega v\sin\vartheta) + \mathbf{j}\,2\Omega u\sin\vartheta - \mathbf{k}\,2\Omega u\cos\vartheta. \tag{2.46}$$

Using (2.45) and (2.46), and the gradient operator given by (2.31), the momentum equation (2.43) becomes:

$$\frac{\mathrm{D}u}{\mathrm{D}t} - \left(2\Omega + \frac{u}{r\cos\vartheta}\right)(v\sin\vartheta - w\cos\vartheta) = -\frac{1}{\rho r\cos\vartheta}\frac{\partial p}{\partial\lambda}, \tag{2.47a}$$
$$\frac{\mathrm{D}v}{\mathrm{D}t} + \frac{wv}{r} + \left(2\Omega + \frac{u}{r\cos\vartheta}\right)u\sin\vartheta = -\frac{1}{\rho r}\frac{\partial p}{\partial\vartheta}, \tag{2.47b}$$
$$\frac{\mathrm{D}w}{\mathrm{D}t} - \frac{u^2+v^2}{r} - 2\Omega u\cos\vartheta = -\frac{1}{\rho}\frac{\partial p}{\partial r} - g. \tag{2.47c}$$

The terms involving Ω are called Coriolis terms, and the quadratic terms on the left-hand sides involving $1/r$ are often called metric terms.

2.2.4 The Primitive Equations

The so-called *primitive equations* of motion are simplifications of the above equations frequently used in atmospheric and oceanic modelling.[4] Three related approximations are involved:

(i) The hydrostatic approximation. In the vertical momentum equation the gravitational term is assumed to be balanced by the pressure gradient term, so that

$$\frac{\partial p}{\partial z} = -\rho g. \tag{2.48}$$

The advection of vertical velocity, the Coriolis terms, and the metric term $(u^2+v^2)/r$ are all neglected.

(ii) *The shallow-fluid approximation.* We write $r = a + z$ where the constant a is the radius of the Earth and z increases in the radial direction. The coordinate r is then replaced by a except where it is used as the differentiating argument. Thus, for example,

$$\frac{1}{r^2}\frac{\partial(r^2 w)}{\partial r} \to \frac{\partial w}{\partial z}. \tag{2.49}$$

(iii) *The traditional approximation.* Coriolis terms in the horizontal momentum equations involving the vertical velocity, and the still smaller metric terms uw/r and vw/r, are neglected.

The second and third of these approximations should be taken, or not, together, the underlying reason being that they both relate to the presumed small aspect ratio of the motion, so the approximations succeed or fail together. If we make one approximation but not the other then we are being asymptotically inconsistent, and angular momentum and energy conservation are not assured.[5] The hydrostatic approximation also depends on the small aspect ratio of the flow, but in a slightly different way. For large-scale flow in the terrestrial atmosphere and ocean all three approximations are in fact very accurate approximations. We defer a more complete treatment until Section 2.7, in part because a treatment of the hydrostatic approximation is done most easily in the context of the Boussinesq equations, derived in Section 2.4.

Making these approximations, the momentum equations for a shallow layer are

$$\frac{\mathrm{D}u}{\mathrm{D}t} - 2\Omega\sin\vartheta v - \frac{uv}{a}\tan\vartheta = -\frac{1}{a\rho\cos\vartheta}\frac{\partial p}{\partial\lambda}, \tag{2.50a}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + 2\Omega\sin\vartheta u + \frac{u^2\tan\vartheta}{a} = -\frac{1}{\rho a}\frac{\partial p}{\partial\vartheta}, \tag{2.50b}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \tag{2.50c}$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} = \left(\frac{\partial}{\partial t} + \frac{u}{a\cos\vartheta}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\vartheta} + w\frac{\partial}{\partial z}\right). \tag{2.51}$$

We note the ubiquity of the factor $2\Omega\sin\vartheta$, and take the opportunity to define the *Coriolis parameter*, $f \equiv 2\Omega\sin\vartheta$. The associated mass conservation equation for a shallow fluid layer is:

$$\frac{\partial\rho}{\partial t} + \frac{u}{a\cos\vartheta}\frac{\partial\rho}{\partial\lambda} + \frac{v}{a}\frac{\partial\rho}{\partial\vartheta} + w\frac{\partial\rho}{\partial z} + \rho\left[\frac{1}{a\cos\vartheta}\frac{\partial u}{\partial\lambda} + \frac{1}{a\cos\vartheta}\frac{\partial}{\partial\vartheta}(v\cos\vartheta) + \frac{\partial w}{\partial z}\right] = 0, \tag{2.52}$$

or equivalently,

$$\frac{\partial\rho}{\partial t} + \frac{1}{a\cos\vartheta}\frac{\partial(u\rho)}{\partial\lambda} + \frac{1}{a\cos\vartheta}\frac{\partial}{\partial\vartheta}(v\rho\cos\vartheta) + \frac{\partial(w\rho)}{\partial z} = 0. \tag{2.53}$$

2.2.5 Primitive Equations in Vector Form

The primitive equations on a sphere may be written in a compact vector form provided we make a slight reinterpretation of the material derivative of the coordinate axes. Instead of (2.39) we take the material derivative of the unit vectors to be

$$\frac{\mathrm{D}\mathbf{i}}{\mathrm{D}t} = \widetilde{\boldsymbol{\Omega}}_{flow} \times \mathbf{i} = \mathbf{j}\frac{u\tan\vartheta}{a}, \tag{2.54a}$$

$$\frac{\mathrm{D}\mathbf{j}}{\mathrm{D}t} = \widetilde{\boldsymbol{\Omega}}_{flow} \times \mathbf{j} = -\mathbf{i}\frac{u\tan\vartheta}{a}, \tag{2.54b}$$

where $\widetilde{\boldsymbol{\Omega}}_{flow} = \mathbf{k}u\tan\vartheta/a$, which is the vertical component of (2.38) with r replaced by a. Given (2.54), the primitive equations (2.50a) and (2.50b) may be written as

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f}\times\boldsymbol{u} = -\frac{1}{\rho}\nabla_z p, \tag{2.55}$$

where $\boldsymbol{u} = u\mathbf{i} + v\mathbf{j} + 0\,\mathbf{k}$ is the horizontal velocity, $\nabla_z p = [(a\cos\vartheta)^{-1}\partial p/\partial\lambda,\ a^{-1}\partial p/\partial\vartheta]$ is the gradient operator at constant z, and $\boldsymbol{f} = f\mathbf{k} = 2\Omega\sin\vartheta\mathbf{k}$. In (2.55) the material derivative of the horizontal velocity is given by

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \mathbf{i}\frac{\mathrm{D}u}{\mathrm{D}t} + \mathbf{j}\frac{\mathrm{D}v}{\mathrm{D}t} + u\frac{\mathrm{D}\mathbf{i}}{\mathrm{D}t} + v\frac{\mathrm{D}\mathbf{j}}{\mathrm{D}t}. \tag{2.56}$$

The advection of the horizontal wind $\boldsymbol{u}$ is still by the three-dimensional velocity $\boldsymbol{v}$.

The vertical momentum equation is the hydrostatic equation, (2.50c), and the mass conservation equation is

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\boldsymbol{v} = 0 \qquad \text{or} \qquad \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0, \tag{2.57}$$

where $\mathrm{D}/\mathrm{D}t$ is given by (2.51), and the second expression is written out in full in (2.53).

2.2.6 The Vector Invariant Form of the Momentum Equation

The 'vector invariant' form of the momentum equation is so-called because it appears to take the same form in all coordinate systems — there is no advective derivative of the coordinate system to worry about. With the aid of the identity $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\boldsymbol{v}\times\boldsymbol{\omega} + \nabla(\boldsymbol{v}^2/2)$, where $\boldsymbol{\omega} \equiv \nabla\times\boldsymbol{v}$ is the relative vorticity (which we explore at greater length in Chapter 4) the three-dimensional momentum equation, (2.16), may be written:

$$\frac{\partial\boldsymbol{v}}{\partial t} + (2\boldsymbol{\Omega} + \boldsymbol{\omega})\times\boldsymbol{v} = -\frac{1}{\rho}\nabla p - \frac{1}{2}\nabla\boldsymbol{v}^2 + \boldsymbol{g}, \tag{2.58}$$

and this is the vector invariant momentum equation. In spherical coordinates the relative vorticity is given by:

$$\begin{aligned}\boldsymbol{\omega} = \nabla\times\boldsymbol{v} &= \frac{1}{r^2\cos\vartheta}\begin{vmatrix} \mathbf{i}\,r\cos\vartheta & \mathbf{j}\,r & \mathbf{k} \\ \partial/\partial\lambda & \partial/\partial\vartheta & \partial/\partial r \\ ur\cos\vartheta & rv & w \end{vmatrix} \\ &= \mathbf{i}\,\frac{1}{r}\left(\frac{\partial w}{\partial\vartheta} - \frac{\partial(rv)}{\partial r}\right) - \mathbf{j}\frac{1}{r\cos\vartheta}\left(\frac{\partial w}{\partial\lambda} - \frac{\partial}{\partial r}(ur\cos\vartheta)\right) + \mathbf{k}\frac{1}{r\cos\vartheta}\left(\frac{\partial v}{\partial\lambda} - \frac{\partial}{\partial\vartheta}(u\cos\vartheta)\right).\end{aligned} \tag{2.59}$$

We can write the horizontal momentum equations of the primitive equations in a similar way. Making the traditional and shallow fluid approximations, the horizontal components of (2.58) become

$$\frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{f} + \mathbf{k}\zeta)\times\boldsymbol{u} + w\frac{\partial\boldsymbol{u}}{\partial z} = -\frac{1}{\rho}\nabla_z p - \frac{1}{2}\nabla\boldsymbol{u}^2, \tag{2.60}$$

where $\boldsymbol{u} = (u, v, 0)$, $\boldsymbol{f} = \mathbf{k}\,2\Omega\sin\vartheta$ and ∇_z is the horizontal gradient operator (the gradient at a constant value of z). Using (2.59), ζ is given by

$$\zeta = \frac{1}{a\cos\vartheta}\frac{\partial v}{\partial\lambda} - \frac{1}{a\cos\vartheta}\frac{\partial}{\partial\vartheta}(u\cos\vartheta) = \frac{1}{a\cos\vartheta}\frac{\partial v}{\partial\lambda} - \frac{1}{a}\frac{\partial u}{\partial\vartheta} + \frac{u}{a}\tan\vartheta. \tag{2.61}$$

The separate components of the momentum equation are given by:

$$\frac{\partial u}{\partial t} - (f + \zeta)v + w\frac{\partial u}{\partial z} = -\frac{1}{a\cos\vartheta}\left(\frac{1}{\rho}\frac{\partial p}{\partial\lambda} + \frac{1}{2}\frac{\partial\boldsymbol{u}^2}{\partial\lambda}\right), \tag{2.62}$$

and

$$\frac{\partial v}{\partial t} + (f + \zeta)u + w\frac{\partial v}{\partial z} = -\frac{1}{a}\left(\frac{1}{\rho}\frac{\partial p}{\partial \vartheta} + \frac{1}{2}\frac{\partial \boldsymbol{u}^2}{\partial \vartheta}\right). \tag{2.63}$$

2.2.7 Angular Momentum

The zonal momentum equation can be usefully expressed as a statement about axial angular momentum; that is, angular momentum about the rotation axis. The zonal angular momentum per unit mass is the component of angular momentum in the direction of the axis of rotation and it is given by, without making any shallow atmosphere approximation,

$$m = (u + \Omega r \cos\vartheta) r \cos\vartheta. \tag{2.64}$$

The evolution equation for this quantity follows from the zonal momentum equation and has the simple form

$$\frac{\mathrm{D}m}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial \lambda}, \tag{2.65}$$

where the material derivative is

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \frac{u}{r\cos\vartheta}\frac{\partial}{\partial \lambda} + \frac{v}{r}\frac{\partial}{\partial \vartheta} + w\frac{\partial}{\partial r}. \tag{2.66}$$

Using the mass continuity equation, (2.65) can be written as

$$\frac{\mathrm{D}\rho m}{\mathrm{D}t} + \rho m \nabla\cdot\boldsymbol{v} = -\frac{\partial p}{\partial \lambda} \tag{2.67}$$

or

$$\frac{\partial \rho m}{\partial t} + \frac{1}{r\cos\vartheta}\frac{\partial(\rho u m)}{\partial \lambda} + \frac{1}{r\cos\vartheta}\frac{\partial}{\partial \vartheta}(\rho v m\cos\vartheta) + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho m w r^2) = -\frac{\partial p}{\partial \lambda}. \tag{2.68}$$

This is an angular momentum conservation equation.

If the fluid is confined to a shallow layer near the surface of a sphere, then we may replace r, the radial coordinate, by a, the radius of the sphere, in the definition of m, and we define $\widetilde{m} \equiv (u + \Omega a\cos\vartheta)a\cos\vartheta$. Then (2.65) is replaced by

$$\frac{\mathrm{D}\widetilde{m}}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial \lambda}, \tag{2.69}$$

where now

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \frac{u}{a\cos\vartheta}\frac{\partial}{\partial \lambda} + \frac{v}{a}\frac{\partial}{\partial \vartheta} + w\frac{\partial}{\partial z}. \tag{2.70}$$

In the shallow fluid approximation (2.68) becomes

$$\frac{\partial \rho m}{\partial t} + \frac{1}{a\cos\vartheta}\frac{\partial(\rho u m)}{\partial \lambda} + \frac{1}{a\cos\vartheta}\frac{\partial}{\partial \vartheta}(\rho v m\cos\vartheta) + \frac{\partial}{\partial z}(\rho m w) = -\frac{\partial p}{\partial \lambda}, \tag{2.71}$$

which is an angular momentum conservation equation for a shallow atmosphere.

✦ *From angular momentum to the spherical component equations*

An alternative way of deriving the three components of the momentum equation in spherical polar coordinates is to *begin* with (2.65) and the principle of conservation of energy. That is, we take the equations for conservation of angular momentum and energy as true a priori and demand that the forms of the momentum equation be constructed to satisfy these. Expanding the material derivative in (2.65), noting that $\mathrm{D}r/\mathrm{D}t = w$ and $\mathrm{D}\cos\vartheta/\mathrm{D}t = -(v/r)\sin\vartheta$, immediately gives (2.47a). Multiplication by u then yields

$$u\frac{\mathrm{D}u}{\mathrm{D}t} - 2\Omega uv\sin\vartheta + 2\Omega uw\cos\vartheta - \frac{u^2 v\tan\vartheta}{r} + \frac{u^2 w}{r} = -\frac{u}{\rho r\cos\vartheta}\frac{\partial p}{\partial\lambda}. \tag{2.72}$$

Now suppose that the meridional and vertical momentum equations are of the form

$$\frac{\mathrm{D}v}{\mathrm{D}t} + \text{Coriolis and metric terms} = -\frac{1}{\rho r}\frac{\partial p}{\partial\vartheta}, \tag{2.73a}$$

$$\frac{\mathrm{D}w}{\mathrm{D}t} + \text{Coriolis and metric terms} = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{2.73b}$$

but that we do not know what form the Coriolis and metric terms take. To determine that form, construct the kinetic energy equation by multiplying (2.73) by v and w, respectively. Now, the metric terms must vanish when we sum the resulting equations along with (2.72), so that (2.73a) must contain the Coriolis term $2\Omega u\sin\vartheta$ as well as the metric term $u^2\tan\vartheta/r$, and (2.73b) must contain the term $-2\Omega u\cos\phi$ as well as the metric term u^2/r. But if (2.73b) contains the term u^2/r it must also contain the term v^2/r by isotropy, and therefore (2.73a) must also contain the term vw/r. In this way, (2.47) is precisely reproduced, although the sceptic might argue that the uniqueness of the form has not been demonstrated.

A particular advantage of this approach arises in determining the appropriate momentum equations that conserve angular momentum and energy in the shallow-fluid approximation. We begin with (2.69) and expand to obtain (2.50a). Multiplying by u gives

$$u\frac{\mathrm{D}u}{\mathrm{D}t} - 2\Omega uv\sin\vartheta - \frac{u^2 v\tan\vartheta}{a} = -\frac{u}{\rho a\cos\vartheta}\frac{\partial p}{\partial\lambda}. \tag{2.74}$$

To ensure energy conservation, the meridional momentum equation must contain the Coriolis term $2\Omega u\sin\vartheta$ and the metric term $u^2\tan\vartheta/a$, but the vertical momentum equation must have neither of the metric terms appearing in (2.47c). Thus we deduce the following equations:

$$\frac{\mathrm{D}u}{\mathrm{D}t} - \left(2\Omega\sin\vartheta + \frac{u\tan\vartheta}{a}\right)v = -\frac{1}{\rho a\cos\vartheta}\frac{\partial p}{\partial\lambda}, \tag{2.75a}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + \left(2\Omega\sin\vartheta + \frac{u\tan\vartheta}{a}\right)u = -\frac{1}{\rho a}\frac{\partial p}{\partial\vartheta}, \tag{2.75b}$$

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial r} - g. \tag{2.75c}$$

This equation set, when used in conjunction with the thermodynamic and mass continuity equations, conserves appropriate forms of angular momentum and energy. In the hydrostatic approximation the material derivative of w in (2.75c) is *additionally* neglected. Thus, the hydrostatic approximation is mathematically and physically consistent with the shallow-fluid approximation, but it is an additional approximation with slightly different requirements that one may choose, rather than being required, to make. From an asymptotic perspective, the difference lies in the small parameter necessary for either approximation to hold, namely:

$$\text{shallow fluid and traditional approximations:} \qquad \gamma \equiv \frac{H}{a} \ll 1, \tag{2.76a}$$

$$\text{small aspect ratio for hydrostatic approximation:} \qquad \alpha \equiv \frac{H}{L} \ll 1, \tag{2.76b}$$

where L is the horizontal scale of the motion and a is the radius of the Earth. For hemispheric or global scale phenomena $L \sim a$ and the two approximations coincide. (Requirement (2.76b) for the hydrostatic approximation will be derived in Section 2.7.)

2.3 CARTESIAN APPROXIMATIONS: THE TANGENT PLANE

2.3.1 The f-plane

Although the rotation of the Earth is central for many dynamical phenomena, the sphericity of the Earth is not always so. This is especially true for phenomena on a scale somewhat smaller than global where the use of spherical coordinates becomes awkward, and it is more convenient to use a locally Cartesian representation of the equations. Referring to the red line in Fig. 2.4 we will define a plane tangent to the surface of the Earth at a latitude ϑ_0, and then use a Cartesian coordinate system (x, y, z) to describe motion on that plane. For small excursions on the plane, $(x, y, z) \approx (a\lambda \cos\vartheta_0, a(\vartheta - \vartheta_0), z)$. Consistently, the velocity is $\boldsymbol{v} = (u, v, w)$, so that u,v and w are the components of the velocity *in the tangent plane,* in approximately in the east–west, north–south and vertical directions, respectively.

The momentum equations for flow in this plane are then

$$\frac{\partial u}{\partial t} + (\boldsymbol{v}\cdot\nabla)u + 2(\Omega^y w - \Omega^z v) = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{2.77a}$$

$$\frac{\partial v}{\partial t} + (\boldsymbol{v}\cdot\nabla)v + 2(\Omega^z u - \Omega^x w) = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{2.77b}$$

$$\frac{\partial w}{\partial t} + (\boldsymbol{v}\cdot\nabla)w + 2(\Omega^x v - \Omega^y u) = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \tag{2.77c}$$

where the rotation vector $\boldsymbol{\Omega} = \Omega^x\mathbf{i} + \Omega^y\mathbf{j} + \Omega^z\mathbf{k}$ and $\Omega^x = 0$, $\Omega^y = \Omega\cos\vartheta_0$ and $\Omega^z = \Omega\sin\vartheta_0$. If we make the traditional approximation, and so ignore the components of $\boldsymbol{\Omega}$ not in the direction of the local vertical, then the above equations become

$$\frac{\mathrm{D}u}{\mathrm{D}t} - f_0 v = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad \frac{\mathrm{D}v}{\mathrm{D}t} + f_0 u = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \qquad \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \tag{2.78a,b,c}$$

where $f_0 = 2\Omega^z = 2\Omega\sin\vartheta_0$. Defining the horizontal velocity vector $\boldsymbol{u} = (u, v, 0)$, the first two equations may be written as

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f}_0 \times \boldsymbol{u} = -\frac{1}{\rho}\nabla_z p, \tag{2.79}$$

where $\mathrm{D}\boldsymbol{u}/\mathrm{D}t = \partial\boldsymbol{u}/\partial t + \boldsymbol{v}\cdot\nabla\boldsymbol{u}$, $\boldsymbol{f}_0 = 2\Omega\sin\vartheta_0\mathbf{k} = f_0\mathbf{k}$, and $\mathbf{k}$ is the direction perpendicular to the plane. These equations are, evidently, exactly the same as the momentum equations in a system in which the rotation vector is aligned with the local vertical, as illustrated in panel (b) of Fig. 2.4. They will describe flow on the surface of a rotating sphere to a good approximation provided the flow is of limited latitudinal extent so that the effects of sphericity are unimportant; we have made what is known as the *f-plane*. We may in addition make the hydrostatic approximation, in which case (2.78c) becomes the familiar $\partial p/\partial z = -\rho g$.

2.3.2 The Beta-plane Approximation

The magnitude of the vertical component of rotation varies with latitude, and this has important dynamical consequences. We can approximate this effect by allowing the effective rotation vector to vary. Thus, noting that, for small variations in latitude,

$$f = 2\Omega\sin\vartheta \approx 2\Omega\sin\vartheta_0 + 2\Omega(\vartheta - \vartheta_0)\cos\vartheta_0, \tag{2.80}$$

then on the tangent plane we may mimic this by allowing the Coriolis parameter to vary as

$$f = f_0 + \beta y, \tag{2.81}$$

where $f_0 = 2\Omega \sin\vartheta_0$ and $\beta = \partial f/\partial y = (2\Omega\cos\vartheta_0)/a$. This important approximation is known as the *beta-plane*, or *β-plane*, approximation; it captures the the most important *dynamical* effects of sphericity, without the complicating *geometric* effects, which are not essential to describe many phenomena. The momentum equations (2.78) are unaltered except that f_0 is replaced by $f_0 + \beta y$ to represent a varying Coriolis parameter. Thus, sphericity combined with rotation is dynamically equivalent to a *differentially rotating* system. For future reference, we write down the β-plane horizontal momentum equations:

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{u} = -\frac{1}{\rho}\nabla_z p, \tag{2.82}$$

where $\boldsymbol{f} = (f_0 + \beta y)\widehat{\mathbf{k}}$. In component form this equation becomes

$$\frac{\mathrm{D}u}{\mathrm{D}t} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad \frac{\mathrm{D}v}{\mathrm{D}t} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}. \tag{2.83a,b}$$

The mass conservation, thermodynamic and hydrostatic equations in the β-plane approximation are the same as the usual Cartesian, f-plane, forms of those equations.

2.4 EQUATIONS FOR A STRATIFIED OCEAN: THE BOUSSINESQ APPROXIMATION

The density variations in the ocean are quite small compared to the mean density, and we may exploit this to derive somewhat simpler but still quite accurate equations of motion. Let us first examine how much density does vary in the ocean.

2.4.1 Variation of Density in the Ocean

The variations of density in the ocean are due to three effects: the compression of water by pressure (which we denote as $\Delta_p\rho$), the thermal expansion of water if its temperature changes ($\Delta_T\rho$), and the haline contraction if its salinity changes ($\Delta_S\rho$). How big are these? An appropriate equation of state to approximately evaluate these effects is the linear one

$$\rho = \rho_0\left[1 - \beta_T(T - T_0) + \beta_S(S - S_0) + \beta_p p\right], \tag{2.84}$$

where $\beta_T \approx 2\times10^{-4}\,\mathrm{K}^{-1}$, $\beta_S \approx 10^{-3}\,\mathrm{g/kg}^{-1}$ and $\beta_p = 1/(\rho_0 c_s^2) \approx 4.4\times10^{-10}\,\mathrm{Pa}^{-1}$ with $c_s \approx 1500\,\mathrm{m\,s}^{-1}$ (see Table 1.2 on page 33). The three effects may then be evaluated as follows:

Pressure compressibility. We have $\Delta_p\rho \approx \Delta p/c_s^2 \approx \rho_0 gH/c_s^2$ where H is the depth we evaluate the pressure change (quite accurately) using the hydrostatic approximation. Thus,

$$\frac{|\Delta_p\rho|}{\rho_0} \approx \frac{gH}{c_s^2} \sim 4\times10^{-2}, \tag{2.85}$$

with $H = 8\,\mathrm{km}$ and $c_s^2/g \approx 200\,\mathrm{km}$. The latter quantity is the density scale height of the ocean. Thus, the pressure at the bottom of the ocean, enormous as it is, is insufficient to compress the water enough to make a significant change in its density. Changes in density due to dynamical variations of pressure are small if the Mach number is small, and this is also usually the case.

Thermal expansion. We have $\Delta_T\rho \approx -\beta_T\rho_0\Delta T$ and therefore

$$\frac{|\Delta_T\rho|}{\rho_0} \approx \beta_T\Delta T \sim 4\times10^{-3} \tag{2.86}$$

with $\Delta T = 20\,\mathrm{K}$. Evidently we would require temperature differences of order β_T^{-1}, or 5000 K to obtain order one variations in density.

Saline contraction. We have $\Delta_S \rho \approx \beta_S \rho_0 \Delta S$ and therefore

$$\frac{|\Delta_S \rho|}{\rho_0} \approx \beta_S \Delta S \sim 1.5 \times 10^{-3}, \tag{2.87}$$

with $\Delta S = 5 g\,\mathrm{kg}^{-1}$. The fractional change in the density of seawater due to salinity variations is thus also very small.

Evidently, fractional density changes in the ocean are very small due to the above effects.

2.4.2 The Boussinesq Equations

The *Boussinesq equations* are a set of equations that exploit the smallness of density variations in liquids.[6] An asymptotic derivation is given in Appendix A (page 101) but in what follows we are more heuristic. To set notation we write

$$\begin{aligned} \rho &= \rho_0 + \delta\rho(x, y, z, t) && (2.88a)\\ &= \rho_0 + \widehat{\rho}(z) + \rho'(x, y, z, t) && (2.88b)\\ &= \widetilde{\rho}(z) + \rho'(x, y, z, t), && (2.88c) \end{aligned}$$

where ρ_0 is a constant and we assume that

$$|\widehat{\rho}|, |\rho'|, |\delta\rho| \ll \rho_0. \tag{2.89}$$

We need not assume that $|\rho'| \ll |\widehat{\rho}|$, but this is often the case in the ocean. The horizontal gradients (i.e., gradients at constant z, ∇_z) satisfy $\nabla_z p = \nabla_z p' = \nabla_z \delta p$. To obtain the Boussinesq equations we will just use (2.88a), but (2.88c) will be useful for the anelastic equations considered later.

Associated with the reference density is a reference pressure that is defined to be in hydrostatic balance with it. That is,

$$p = p_0(z) + \delta p(x, y, z, t), \tag{2.90}$$

where $|\delta p| \ll p_0$ and

$$\frac{\mathrm{d} p_0}{\mathrm{d} z} \equiv -g\rho_0. \tag{2.91a,b}$$

Momentum equations

Letting $\rho = \rho_0 + \delta\rho$ the momentum equation can be written, without approximation, as

$$(\rho_0 + \delta\rho)\left(\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + 2\boldsymbol{\Omega} \times \boldsymbol{v}\right) = -\nabla\delta p - \frac{\partial p_0}{\partial z}\mathbf{k} - g(\rho_0 + \delta\rho)\mathbf{k}, \tag{2.92}$$

and using (2.91) this becomes, again without approximation,

$$(\rho_0 + \delta\rho)\left(\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + 2\boldsymbol{\Omega} \times \boldsymbol{v}\right) = -\nabla\delta p - g\delta\rho\mathbf{k}. \tag{2.93}$$

If $\delta\rho/\rho_0 \ll 1$ then we may neglect the $\delta\rho$ term on the left-hand side and the above equation becomes

$$\boxed{\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + 2\boldsymbol{\Omega} \times \boldsymbol{v} = -\nabla\phi + b\mathbf{k},} \tag{2.94}$$

where $\phi = \delta p/\rho_0$ and $b = -g\,\delta\rho/\rho_0$ is the *buoyancy*. We should not and do not neglect the term $g\,\delta\rho$, for there is no reason to believe it to be small: $\delta\rho$ may be small, but g is big! Equation (2.94) is the momentum equation in the Boussinesq approximation, and it is common to say that the

Boussinesq approximation ignores all variations of density of a fluid in the momentum equation, except when associated with the gravitational term.

For most large-scale motions in the ocean the *deviation* pressure and density fields are also approximately in hydrostatic balance, and in that case the vertical component of (2.94) becomes

$$\frac{\partial \phi}{\partial z} = b. \tag{2.95}$$

A condition for (2.95) to hold is that vertical accelerations are small *compared to $g\,\delta\rho/\rho_0$, and not compared to the acceleration due to gravity itself.* For more discussion of this point, see Section 2.7.

Mass continuity

The unapproximated mass continuity equation is

$$\frac{\mathrm{D}\delta\rho}{\mathrm{D}t} + (\rho_0 + \delta\rho)\nabla\cdot\boldsymbol{v} = 0. \tag{2.96}$$

Provided that time scales advectively — that is to say that $\mathrm{D}/\mathrm{D}t$ scales in the same way as $\boldsymbol{v}\cdot\nabla$ — then we may approximate this equation by

$$\boxed{\nabla\cdot\boldsymbol{v} = 0,} \tag{2.97}$$

which is the same as that for a constant density fluid. This *absolutely does not* allow one to go back and use (2.96) to say that $D\delta\rho/Dt = 0$; the evolution of density is given by the thermodynamic equation in conjunction with an equation of state, and this should not be confused with the mass conservation equation. Note also that in eliminating the time-derivative of density we eliminate the possibility of sound waves.

Thermodynamic equation and equation of state

The Boussinesq equations are closed by the addition of an equation of state, a thermodynamic equation and, as appropriate, a salinity equation. Neglecting salinity for the moment, a useful starting point is to write the thermodynamic equation, (1.114), as

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} - \frac{1}{c_s^2}\frac{\mathrm{D}p}{\mathrm{D}t} = \frac{\dot{Q}}{(\partial\eta/\partial\rho)_p T} \approx -\dot{Q}\left(\frac{\rho_0\beta_T}{c_p}\right) \tag{2.98}$$

using $(\partial\eta/\partial\rho)_p = (\partial\eta/\partial T)_p(\partial T/\partial\rho)_p \approx -c_p/(T\rho_0\beta_T)$. Given the expansions (2.88a) and (2.90a), (2.98) can be written to a good approximation as

$$\frac{\mathrm{D}\delta\rho}{\mathrm{D}t} - \frac{1}{c_s^2}\frac{\mathrm{D}p_0}{\mathrm{D}t} = -\dot{Q}\left(\frac{\rho_0\beta_T}{c_p}\right), \tag{2.99}$$

or, using (2.91a),

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\delta\rho + \frac{\rho_0 g}{c_s^2}z\right) = -\dot{Q}\left(\frac{\rho_0\beta_T}{c_p}\right). \tag{2.100}$$

The term in brackets on left-hand side is the potential density, as in (1.117). The severest approximation to this is to neglect the second term there, and noting that $b = -g\delta\rho/\rho_0$ we obtain

$$\boxed{\frac{\mathrm{D}b}{\mathrm{D}t} = \dot{b},} \tag{2.101}$$

where $\dot{b} = g\beta_T \dot{Q}/c_p$. The momentum equation (2.94), mass continuity equation (2.97) and thermodynamic equation (2.101) then form a closed set, called the *simple Boussinesq equations*.

In the ocean the compressibility effect can be important and it is convenient to write the thermodynamic equation as

$$\frac{\mathrm{D}b_\sigma}{\mathrm{D}t} = \dot{b}_\sigma, \tag{2.102}$$

where b_σ is the potential buoyancy given by

$$b_\sigma \equiv -g\frac{\delta\rho_\theta}{\rho_0} = -\frac{g}{\rho_0}\left(\delta\rho + \frac{\rho_0 g z}{c_s^2}\right) = b - g\frac{z}{H_\rho}, \tag{2.103}$$

where $H_\rho = c_s^2/g$. Buoyancy itself is obtained from b_σ by the 'equation of state', $b = b_\sigma + gz/H_\rho$.

In many applications we may need to use a still more accurate equation of state. In that case (and see Section 1.7.3) we replace (2.101) by the thermodynamic equations

$$\frac{\mathrm{D}\Theta}{\mathrm{D}t} = \dot{\Theta}, \qquad \frac{\mathrm{D}S}{\mathrm{D}t} = \dot{S}, \tag{2.104a,b}$$

where Θ is an appropriate thermodynamic state variable, such as potential enthalpy or entropy, S is salinity, and an equation of state then gives the buoyancy. The equation of state has the general form $b = b(\Theta, S, p)$, but to be consistent with the level of approximation in the other Boussinesq equations we replace p by the hydrostatic pressure calculated with the reference density, that is by $-\rho_0 g z$, and the equation of state then takes the general form

$$b = b(\Theta, S, z). \tag{2.105}$$

An example of (2.105) is (1.155), taken with the definition of buoyancy $b = -g\delta\rho/\rho_0$. The closed set of equations (2.94), (2.97), (2.104) and (2.105) are sometimes called the general Boussinesq equations, or, in oceanographic contexts, the seawater Boussinesq equations. Using an accurate equation of state and the Boussinesq approximation is the procedure used in many comprehensive ocean general circulation models. The Boussinesq equations, which with the hydrostatic and traditional approximations are often considered to be the oceanic primitive equations, are summarized in the shaded box on the following page.

✦ *Mean stratification and the buoyancy frequency*

The processes that cause density to vary in the vertical often differ from those that cause it to vary in the horizontal. For this reason it is sometimes useful to write $\rho = \rho_0 + \widehat{\rho}(z) + \rho'(x, y, z, t)$ and define $\tilde{b}(z) \equiv -g\widehat{\rho}/\rho_0$ and $b' \equiv -g\rho'/\rho_0$. Using the hydrostatic equation to evaluate pressure, the thermodynamic equation (2.98) becomes, to a good approximation,

$$\frac{\mathrm{D}b'}{\mathrm{D}t} + N^2 w = 0, \tag{2.106}$$

where $\mathrm{D}/\mathrm{D}t$ remains a three-dimensional operator and

$$N^2(z) = \left(\frac{\mathrm{d}\tilde{b}}{\mathrm{d}z} - \frac{g^2}{c_s^2}\right) = \frac{\mathrm{d}\tilde{b}_\sigma}{\mathrm{d}z}, \tag{2.107}$$

where $\tilde{b}_\sigma = \tilde{b} - gz/H_\rho$. The quantity N^2 is a measure of the mean stratification of the fluid, and is equal to the vertical gradient of the mean potential buoyancy. N is known as the buoyancy frequency, something we return to in Section 2.10. Equations (2.106) and (2.107) also hold in the simple Boussinesq equations, but with $c_s^2 = \infty$.

Summary of Boussinesq Equations

The simple Boussinesq equations are, for an inviscid fluid:

momentum equations: $$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{v} = -\nabla\phi + b\mathbf{k}, \tag{B.1}$$

mass conservation: $$\nabla \cdot \boldsymbol{v} = 0, \tag{B.2}$$

buoyancy equation: $$\frac{\mathrm{D}b}{\mathrm{D}t} = \dot{b}. \tag{B.3}$$

A more general form replaces the buoyancy equation by:

thermodynamic equation: $$\frac{\mathrm{D}\Theta}{\mathrm{D}t} = \dot{\Theta}, \tag{B.4}$$

salinity equation: $$\frac{\mathrm{D}S}{\mathrm{D}t} = \dot{S}, \tag{B.5}$$

equation of state: $$b = b(\Theta, S, z). \tag{B.6}$$

An equation of state of the form $b = b(\Theta, S, \phi)$ is not asymptotically correct and good conservation properties are not assured.

2.4.3 Energetics of the Boussinesq System

In a uniform gravitational field but with no other forcing or dissipation, we write the simple Boussinesq equations as

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + 2\boldsymbol{\Omega} \times \boldsymbol{v} = b\mathbf{k} - \nabla\phi, \qquad \nabla \cdot \boldsymbol{v} = 0, \qquad \frac{\mathrm{D}b}{\mathrm{D}t} = 0. \tag{2.108a,b,c}$$

From (2.108a) and (2.108b) the kinetic energy density evolution (cf. Section 1.10) is given by

$$\frac{1}{2}\frac{\mathrm{D}v^2}{\mathrm{D}t} = bw - \nabla \cdot (\phi\boldsymbol{v}), \tag{2.109}$$

where the constant reference density ρ_0 is omitted. Let us now define the potential $\Phi \equiv -z$, so that $\nabla\Phi = -\mathbf{k}$ and

$$\frac{\mathrm{D}\Phi}{\mathrm{D}t} = \nabla \cdot (\boldsymbol{v}\Phi) = -w, \tag{2.110}$$

and using this and (2.108c) gives

$$\frac{\mathrm{D}}{\mathrm{D}t}(b\Phi) = -wb. \tag{2.111}$$

Adding (2.111) to (2.109) and expanding the material derivative gives

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\boldsymbol{v}^2 + b\Phi\right) + \nabla \cdot \left[\boldsymbol{v}\left(\frac{1}{2}\boldsymbol{v}^2 + b\Phi + \phi\right)\right] = 0. \tag{2.112}$$

This constitutes an energy equation for the Boussinesq system, and may be compared to (1.199). The energy density (divided by ρ_0) is just $\boldsymbol{v}^2/2 + b\Phi$. What does the term $b\Phi$ represent? Its integral, multiplied by ρ_0, is the potential energy of the flow minus that of the basic state, or $\int g(\rho - \rho_0)z\,\mathrm{d}z$. If there were a heating term on the right-hand side of (2.108c) this would directly provide a source of potential energy, rather than internal energy as in the compressible system. Because the fluid is incompressible, there is no conversion from kinetic and potential energy into internal energy.

✦ *Energetics with a general equation of state*

Now consider the energetics of the general Boussinesq equations. Suppose first that we allow the equation of state to be a function of pressure; the equations of motion are then (2.108) except that (2.108c) is replaced by

$$\frac{D\Theta}{Dt} = 0, \qquad \frac{DS}{Dt} = 0, \qquad b = b(\Theta, S, \phi). \tag{2.113a,b,c}$$

where Θ is some conservative thermodynamic variable and S is salinity. A little algebraic experimentation will reveal that no energy conservation law of the form (2.112) generally exists for this system! The problem arises because, by requiring the fluid to be incompressible, we eliminate the proper conversion of internal energy to kinetic energy. However, if we use the approximation $b = b(\Theta, S, z)$, the system does conserve an energy, as we now show.[7]

Define the potential, Π, as the integral of b at constant potential temperature and salinity, namely

$$\Pi(\Theta, S, z) \equiv -\int_a^z b\,dz', \tag{2.114}$$

where a is a constant, so that $\partial\Pi/\partial z = -b$. (The quantity Π is related to the dynamic enthalpy of Section 1.7.3.) Taking the material derivative of the left-hand side gives

$$\frac{D\Pi}{Dt} = \left(\frac{\partial\Pi}{\partial\Theta}\right)_{S,z}\frac{D\Theta}{Dt} + \left(\frac{\partial\Pi}{\partial S}\right)_{\Theta,z}\frac{DS}{Dt} + \left(\frac{\partial\Pi}{\partial z}\right)_{\Theta,S}\frac{Dz}{Dt} = -bw, \tag{2.115}$$

using (2.113a,b). Combining (2.115) and (2.109) gives

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\boldsymbol{v}^2 + \Pi\right) + \nabla\cdot\left[\boldsymbol{v}\left(\frac{1}{2}\boldsymbol{v}^2 + \Pi + \phi\right)\right] = 0. \tag{2.116}$$

Thus, energetic consistency is maintained with an arbitrary equation of state, provided that the buoyancy (or density) is taken as a function of z and not pressure — as Appendix A indicates is the proper thing to do.

2.5 EQUATIONS FOR A STRATIFIED ATMOSPHERE: THE ANELASTIC APPROXIMATION

2.5.1 Preliminaries

In the atmosphere the density varies significantly, especially in the vertical. However, deviations of both ρ and p from a statically balanced state are often quite small, and the relative vertical variation of potential temperature is also small. We can usefully exploit these observations to give a somewhat simplified set of equations, useful both for theoretical and numerical analyses because sound waves are eliminated by way of an 'anelastic' approximation.[8] To begin we set

$$\rho = \widetilde{\rho}(z) + \delta\rho(x, y, z, t), \qquad p = \widetilde{p}(z) + \delta p(x, y, z, t), \tag{2.117a,b}$$

where we assume that $|\delta\rho| \ll |\widetilde{\rho}|$ and we define $\widetilde{p}$ such that

$$\frac{\partial\widetilde{p}}{\partial z} \equiv -g\widetilde{\rho}(z). \tag{2.118}$$

The notation is similar to that for the Boussinesq case except that, importantly, the density basic state is now a (given) function of the vertical coordinate. As with the Boussinesq case, the idea is to ignore dynamic variations of density (i.e., of $\delta\rho$) except where associated with gravity. First recall a couple of ideal gas relationships involving potential temperature, θ. If we define $s = \log\theta$ (so that s is entropy divided by c_p) then

$$s = \log\theta = \log T - \frac{R}{c_p}\log p = \frac{1}{\gamma}\log p - \log\rho, \tag{2.119}$$

where $\gamma = c_p/c_v$, implying

$$\delta s = \frac{\delta\theta}{\theta} = \frac{1}{\gamma}\frac{\delta p}{p} - \frac{\delta\rho}{\rho} \approx \frac{1}{\gamma}\frac{\delta p}{\widetilde{p}} - \frac{\delta\rho}{\widetilde{\rho}}. \tag{2.120}$$

Further, if $\tilde{s} \equiv \gamma^{-1}\log\widetilde{p} - \log\widetilde{\rho}$ then

$$\frac{\mathrm{d}\tilde{s}}{\mathrm{d}z} = \frac{1}{\gamma\widetilde{p}}\frac{\mathrm{d}\widetilde{p}}{\mathrm{d}z} - \frac{1}{\widetilde{\rho}}\frac{\mathrm{d}\widetilde{\rho}}{\mathrm{d}z} = -\frac{g\widetilde{\rho}}{\gamma\widetilde{p}} - \frac{1}{\widetilde{\rho}}\frac{\mathrm{d}\widetilde{\rho}}{\mathrm{d}z}. \tag{2.121}$$

In the atmosphere, the left-hand side is, typically, much smaller than either of the two terms on the right-hand side.

2.5.2 The Momentum Equation

The exact inviscid horizontal momentum equation is

$$(\widetilde{\rho} + \delta\rho)\left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f}\times\boldsymbol{u}\right) = -\nabla_z\delta p. \tag{2.122}$$

Neglecting $\delta\rho$ where it appears with $\widetilde{\rho}$ leads to

$$\boxed{\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f}\times\boldsymbol{u} = -\nabla_z\phi,} \tag{2.123}$$

where $\phi = \delta p/\widetilde{\rho}$, and this is similar to the corresponding equation in the Boussinesq approximation.

The vertical component of the inviscid momentum equation is, without approximation,

$$(\widetilde{\rho} + \delta\rho)\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial\widetilde{p}}{\partial z} - \frac{\partial\delta p}{\partial z} - g\widetilde{\rho} - g\delta\rho = -\frac{\partial\delta p}{\partial z} - g\delta\rho, \tag{2.124}$$

using (2.118). Neglecting $\delta\rho$ on the left-hand side we obtain

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\widetilde{\rho}}\frac{\partial\delta p}{\partial z} - g\frac{\delta\rho}{\widetilde{\rho}} = -\frac{\partial}{\partial z}\left(\frac{\delta p}{\widetilde{\rho}}\right) - \frac{\delta p}{\widetilde{\rho}^2}\frac{\partial\widetilde{\rho}}{\partial z} - g\frac{\delta\rho}{\widetilde{\rho}}. \tag{2.125}$$

This is not a useful form for a gaseous atmosphere, since the variation of the mean density cannot be ignored. However, we may eliminate $\delta\rho$ in favour of δs using (2.120) to give

$$\frac{\mathrm{D}w}{\mathrm{D}t} = g\,\delta s - \frac{\partial}{\partial z}\left(\frac{\delta p}{\widetilde{\rho}}\right) - \frac{g}{\gamma}\frac{\delta p}{\widetilde{p}} - \frac{\delta p}{\widetilde{\rho}^2}\frac{\partial\widetilde{\rho}}{\partial z}, \tag{2.126}$$

and using (2.121) gives

$$\frac{\mathrm{D}w}{\mathrm{D}t} = g\,\delta s - \frac{\partial}{\partial z}\left(\frac{\delta p}{\widetilde{\rho}}\right) + \frac{\mathrm{d}\tilde{s}}{\mathrm{d}z}\frac{\delta p}{\widetilde{\rho}}. \tag{2.127}$$

What have these manipulations gained us? Two things:

(i) The gravitational term now involves δs rather than $\delta\rho$ which enables a more direct connection with the thermodynamic equation.

(ii) The potential temperature scale height ($\sim 100\,\mathrm{km}$) in the atmosphere is much larger than the density scale height ($\sim 10\,\mathrm{km}$), and so the last term in (2.127) is small.

The second item thus suggests that we choose our reference state to be one of constant potential temperature. The term $\mathrm{d}\tilde{s}/\mathrm{d}z$ then vanishes and the vertical momentum equation becomes

$$\boxed{\frac{\mathrm{D}w}{\mathrm{D}t} = g\,\delta s - \frac{\partial \phi}{\partial z},} \tag{2.128}$$

where $\delta s = \delta\theta/\theta_0$, where θ_0 is a constant. If we define a buoyancy by $b_a \equiv g\delta s = g\delta\theta/\theta_0$, then (2.123) and (2.128) have the same form as the Boussinesq momentum equations, but with a slightly different definition of buoyancy.

2.5.3 Mass Conservation

Using (2.117a) the mass conservation equation may be written, without approximation, as

$$\frac{\partial \delta\rho}{\partial t} + \nabla \cdot [(\tilde{\rho} + \delta\rho)\boldsymbol{v}] = 0. \tag{2.129}$$

We neglect $\delta\rho$ where it appears with $\tilde{\rho}$ in the divergence term. Further, the local time derivative will be small if time itself is scaled advectively (i.e., $T \sim L/U$ and sound waves do not dominate), giving

$$\boxed{\nabla \cdot \boldsymbol{u} + \frac{1}{\tilde{\rho}}\frac{\partial}{\partial z}(\tilde{\rho} w) = 0.} \tag{2.130}$$

It is here that the eponymous anelastic approximation arises: the elastic compressibility of the fluid is neglected, and this serves to eliminate sound waves. For reference, in spherical coordinates the equation is

$$\frac{1}{a\cos\vartheta}\frac{\partial u}{\partial \lambda} + \frac{1}{a\cos\vartheta}\frac{\partial}{\partial\vartheta}(v\cos\vartheta) + \frac{1}{\tilde{\rho}}\frac{\partial (w\tilde{\rho})}{\partial z} = 0. \tag{2.131}$$

In an ideal gas, the choice of constant potential temperature determines how the reference density $\tilde{\rho}$ varies with height. In some circumstances it is convenient to let $\tilde{\rho}$ be a constant, ρ_0 (effectively choosing a different equation of state), in which case the anelastic equations become identical to the Boussinesq equations, albeit with the buoyancy interpreted in terms of potential temperature in the former and density in the latter.

2.5.4 Thermodynamic Equation

The thermodynamic equation for an ideal gas may be written

$$\frac{\mathrm{D}\ln\theta}{\mathrm{D}t} = \frac{\dot{Q}}{Tc_p}. \tag{2.132}$$

In the anelastic equations, $\theta = \theta_0 + \delta\theta$, where θ_0 is constant, and the thermodynamic equation is

$$\frac{\mathrm{D}\delta s}{\mathrm{D}t} = \frac{\tilde{\theta}}{Tc_p}\dot{Q}. \tag{2.133}$$

Summarizing, the complete set of anelastic equations, with rotation but with no dissipation or diabatic terms, is

$$\boxed{\begin{gathered} \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + 2\boldsymbol{\Omega}\times\boldsymbol{v} = \mathbf{k}b_a - \nabla\phi, \\ \frac{\mathrm{D}b_a}{\mathrm{D}t} = 0, \\ \nabla\cdot(\tilde{\rho}\boldsymbol{v}) = 0, \end{gathered}} \tag{2.134a,b,c}$$

where $b_a = g\delta s = g\delta\theta/\theta_0$. The anelastic equations are sometimes called the 'weak Boussinesq equations', with the original incompressible set then called the 'strong Boussinesq equations'.

The main difference between the anelastic and Boussinesq sets is in the mass continuity equation, and when $\widetilde{\rho} = \rho_0 =$ constant the two equation sets are identical. However, whereas the Boussinesq approximation is a very good one for ocean dynamics, the anelastic approximation is less so for large-scale atmosphere flow: the constancy of the reference potential temperature state is not a particularly good approximation, and the deviations in density from its reference profile are not especially small, leading to inaccuracies in the momentum equation. Nevertheless, the anelastic equations have been used very productively in limited area 'large-eddy simulations' where one does not wish to make the hydrostatic approximation but where sound waves are unimportant.[9] The equations also provide a good jumping-off point for theoretical studies and for the still simpler models of Chapter 5.

2.5.5 ✦ Energetics of the Anelastic Equations

Conservation of energy follows in much the same way as for the Boussinesq equations, except that $\widetilde{\rho}$ enters. Take the dot product of (2.134a) with $\widetilde{\rho}\boldsymbol{v}$ to obtain

$$\widetilde{\rho}\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}\boldsymbol{v}^2\right) = -\nabla\cdot(\phi\widetilde{\rho}\boldsymbol{v}) + b_a\widetilde{\rho}w. \tag{2.135}$$

Now, define a potential $\Phi(z)$ such that $\nabla\Phi = -\mathbf{k}$, and so

$$\widetilde{\rho}\frac{\mathrm{D}\Phi}{\mathrm{D}t} = -w\widetilde{\rho}. \tag{2.136}$$

Combining this with the thermodynamic equation (2.134b) gives

$$\widetilde{\rho}\frac{\mathrm{D}(b_a\Phi)}{\mathrm{D}t} = -wb_a\widetilde{\rho}. \tag{2.137}$$

Adding this to (2.135) gives

$$\widetilde{\rho}\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}\boldsymbol{v}^2 + b_a\Phi\right) = -\nabla\cdot(\phi\widetilde{\rho}\boldsymbol{v}), \tag{2.138}$$

or, expanding the material derivative,

$$\frac{\partial}{\partial t}\left[\widetilde{\rho}\left(\frac{1}{2}\boldsymbol{v}^2 + b_a\Phi\right)\right] + \nabla\cdot\left[\widetilde{\rho}\boldsymbol{v}\left(\frac{1}{2}\boldsymbol{v}^2 + b_a\Phi + \phi\right)\right] = 0. \tag{2.139}$$

This equation has the form

$$\frac{\partial E}{\partial t} + \nabla\cdot\left[\boldsymbol{v}(E + \widetilde{\rho}\phi)\right] = 0, \tag{2.140}$$

where $E = \widetilde{\rho}(\boldsymbol{v}^2/2 + b_a\Phi)$ is the energy density of the flow. This is a consistent energetic equation for the system, and when integrated over a closed domain the total energy is evidently conserved. The total energy density comprises the kinetic energy and a term $\widetilde{\rho}b_a\Phi$, which is analogous to the potential energy of a simple Boussinesq system. However, it is not exactly equal to potential energy because b_a is the buoyancy based on potential temperature, not density; rather, the term combines contributions from both the internal energy and the potential energy into an enthalpy-like quantity.

2.6 PRESSURE AND OTHER VERTICAL COORDINATES

Although using z as a vertical coordinate is a natural choice given our Cartesian worldview, it is not the only option, nor is it always the most useful one. Any variable that has a one-to-one correspondence with z in the vertical, so any variable that varies monotonically with z, could be used; pressure and, more surprisingly, entropy, are common choices. In the atmosphere pressure almost always falls monotonically with height, and using it instead of z provides a useful simplification of the mass conservation and geostrophic relations, as well as a more direct connection with observations, which are often taken at fixed values of pressure. (In the ocean pressure coordinates are essentially almost the same as height coordinates because density is almost constant.) Entropy seems an exotic vertical coordinate, but it is very useful in adiabatic flow and we consider it in Chapter 3.

2.6.1 General Relations

First consider a general vertical coordinate, ξ. Any variable Ψ that is a function of the coordinates (x, y, z, t) may be expressed instead in terms of (x, y, ξ, t) by considering ξ to be a function of the independent variables (x, y, z, t). Derivatives with respect to z and ξ are related by

$$\frac{\partial \Psi}{\partial \xi} = \frac{\partial \Psi}{\partial z}\frac{\partial z}{\partial \xi} \qquad \text{and} \qquad \frac{\partial \Psi}{\partial z} = \frac{\partial \Psi}{\partial \xi}\frac{\partial \xi}{\partial z}. \tag{2.141a,b}$$

Horizontal derivatives in the two coordinate systems are related by the chain rule,

$$\left(\frac{\partial \Psi}{\partial x}\right)_\xi = \left(\frac{\partial \Psi}{\partial x}\right)_z + \left(\frac{\partial z}{\partial x}\right)_\xi \frac{\partial \Psi}{\partial z}, \tag{2.142}$$

and similarly for time.

The material derivative in ξ coordinates may be derived by transforming the original expression in z coordinates using the chain rule, but because (x, y, ξ, t) are independent coordinates, and noting that the 'vertical velocity' in ξ coordinates is just $\dot{\xi}$ (i.e., $\mathrm{D}\xi/\mathrm{D}t$, just as the vertical velocity in z coordinates is $w = \mathrm{D}z/\mathrm{D}t$), we can write down

$$\frac{\mathrm{D}\Psi}{\mathrm{D}t} = \left(\frac{\partial \Psi}{\partial t}\right)_{x,y,\xi} + \boldsymbol{u}\cdot\nabla_\xi \Psi + \dot{\xi}\frac{\partial \Psi}{\partial \xi}, \tag{2.143}$$

where ∇_ξ is the gradient operator at constant ξ. The operator $\mathrm{D}/\mathrm{D}t$ is the same in z or ξ coordinates because it is the total derivative of some property of a fluid parcel, and this is independent of the coordinate system. However, the individual terms within it will differ between coordinate systems.

2.6.2 Pressure Coordinates

In pressure coordinates the analogue of the vertical velocity is $\omega \equiv Dp/Dt$, and the advective derivative itself is given by

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla_p + \omega\frac{\partial}{\partial p}. \tag{2.144}$$

Note, though, that the advective derivative is the same operator as it is in height coordinates, since it is just the total derivative of a given fluid parcel; it is just written with different coordinates.

To obtain an expression for the pressure force, now let $\xi = p$ in (2.142) and apply the relationship to p itself to give

$$0 = \left(\frac{\partial p}{\partial x}\right)_z + \left(\frac{\partial z}{\partial x}\right)_p \frac{\partial p}{\partial z}, \tag{2.145}$$

which, using the hydrostatic relationship, gives

$$\left(\frac{\partial p}{\partial x}\right)_z = \rho\left(\frac{\partial \Phi}{\partial x}\right)_p, \tag{2.146}$$

where $\Phi = gz$ is the *geopotential*. Thus, the horizontal pressure force in the momentum equations is

$$\frac{1}{\rho}\nabla_z p = \nabla_p \Phi, \tag{2.147}$$

where the subscripts on the gradient operator indicate that the horizontal derivatives are taken at constant z or constant p. The horizontal momentum equation thus becomes

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f}\times\boldsymbol{u} = -\nabla_p\Phi, \tag{2.148}$$

where $\mathrm{D}/\mathrm{D}t$ is given by (2.144). The hydrostatic equation in height coordinates is $\partial p/\partial z = -\rho g$ and in pressure coordinates this becomes

$$\frac{\partial \Phi}{\partial p} = -\alpha \qquad \text{or} \qquad \frac{\partial \Phi}{\partial p} = -\frac{p}{RT}. \tag{2.149}$$

The mass continuity equation simplifies attractively in pressure coordinates, if the hydrostatic approximation is used. Recall that the mass conservation equation can be derived from the material form

$$\frac{\mathrm{D}}{\mathrm{D}t}(\rho\,\delta V) = 0, \tag{2.150}$$

where $\delta V = \delta x\,\delta y\,\delta z$ is a volume element. But by the hydrostatic relationship $\rho\delta z = -(1/g)\delta p$ and thus

$$\frac{\mathrm{D}}{\mathrm{D}t}(\delta x\,\delta y\,\delta p) = 0. \tag{2.151}$$

This is completely analogous to the expression for the material conservation of volume in an incompressible fluid, (1.15). Thus, without further ado, we write the mass conservation in pressure coordinates as

$$\nabla_p\cdot\boldsymbol{u} + \frac{\partial\omega}{\partial p} = 0, \tag{2.152}$$

where the horizontal derivative is taken at constant pressure.

The (adiabatic) thermodynamic equation is still $\mathrm{D}\theta/\mathrm{D}t = 0$, and θ may be related to pressure and temperature using its definition and the ideal gas equation to complete the equation set. However, because the hydrostatic equation is written in terms of temperature and not potential temperature it is convenient to write the thermodynamic equation accordingly. To do this we begin with the thermodynamic equation in the form of (1.99b), namely $c_p\mathrm{D}T/\mathrm{D}t - \alpha\,\mathrm{D}p/\mathrm{D}t = 0$. Since $\omega \equiv \mathrm{D}p/\mathrm{D}t$ this equation is simply

$$c_p\frac{\mathrm{D}T}{\mathrm{D}t} - \frac{RT}{p}\omega = 0, \tag{2.153}$$

which is an appropriate thermodynamic equation in pressure coordinates. It is sometimes useful to write this as

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - \omega S_p = 0, \qquad \text{where} \qquad S_p = \frac{\kappa T}{p} - \frac{\partial T}{\partial p} = -\frac{T}{\theta}\frac{\partial\theta}{\partial p}, \tag{2.154a,b}$$

Equations of Motion in Pressure and Log-pressure Coordinates

The adiabatic, inviscid primitive equations in pressure coordinates are:

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{u} = -\nabla_p \Phi, \tag{P.1}$$

$$\frac{\partial \Phi}{\partial p} = \frac{-RT}{p}, \tag{P.2}$$

$$\nabla_p \cdot \boldsymbol{u} + \frac{\partial \omega}{\partial p} = 0, \tag{P.3}$$

$$c_p \frac{\mathrm{D}T}{\mathrm{D}t} - \frac{RT}{p}\omega = 0 \quad \text{or} \quad \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - \omega S_p = 0. \tag{P.4}$$

where $S_p = \kappa T/p - \partial T/\partial p$ and $\kappa = R/c_p$. The above equations are, respectively, the horizontal momentum equation, the hydrostatic equation, the mass continuity equation and the thermodynamic equation. Using hydrostasy and the ideal gas relation, the thermodynamic equation may also be written as

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \omega\frac{\partial s}{\partial p} = 0, \tag{P.5}$$

where $s = T + gz/c_p$ is the dry static energy divided by c_p.

The corresponding equations in log-pressure coordinates are

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{u} = -\nabla_Z \Phi, \tag{P.6}$$

$$\frac{\partial \Phi}{\partial Z} = \frac{RT}{H}, \tag{P.7}$$

$$\nabla_Z \cdot \boldsymbol{u} + \frac{1}{\rho_R}\frac{\partial \rho_R W}{\partial z} = 0, \tag{P.8}$$

$$c_p \frac{\mathrm{D}T}{\mathrm{D}t} + W\frac{RT}{H} = 0 \quad \text{or} \quad \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + WS_Z = 0. \tag{P.9}$$

where $\rho_R = \rho_0 \exp(-Z/H)$ and $S_Z = \kappa T/H + \partial T/\partial Z$. The thermodynamic equation may also be written as

$$\frac{\partial}{\partial t}\frac{\partial \Phi}{\partial Z} + u\frac{\partial}{\partial x}\frac{\partial \Phi}{\partial z} + v\frac{\partial}{\partial y}\frac{\partial \Phi}{\partial z} + WN_*^2 = 0, \tag{P.10}$$

where $N_*^2 = (R/H)S_Z$.

having used the ideal gas equation and the definition of potential temperature, with $\kappa = R/c_p$. Evidently, S_p is an appropriate measure of static stability in pressure coordinates and it is closely related to the buoyancy frequency N, as we see in the next subsection.

The main practical difficulty with the pressure-coordinate equations is the lower boundary condition. Using

$$w \equiv \frac{\mathrm{D}z}{\mathrm{D}t} = \frac{\partial z}{\partial t} + \boldsymbol{u} \cdot \nabla_p z + \omega\frac{\partial z}{\partial p}, \tag{2.155}$$

and (2.149), the boundary condition of $w = 0$ at $z = z_s$ becomes

$$\frac{\partial \Phi}{\partial t} + \boldsymbol{u} \cdot \nabla_p \Phi - \alpha\omega = 0, \tag{2.156}$$

at $p(x, y, z_s, t)$. In theoretical studies, it is common to assume that the lower boundary is in fact a constant pressure surface and simply assume that $\omega = 0$, or that $\omega = -\alpha^{-1}\partial\Phi/\partial t$. For more realistic studies the fact that the level $z = 0$ is not a coordinate surface must be properly considered. For this reason, and especially if the lower boundary is uneven because of the presence of topography, so-called *sigma coordinates* are sometimes used, in which the vertical coordinate is chosen so that the lower boundary is itself a coordinate surface. Sigma coordinates may use height itself as a vertical measure (typical in oceanic applications) or use pressure (typical in atmospheric applications). In the latter case the vertical coordinate is $\sigma = p/p_s$ where $p_s(x, y, t)$ is the surface pressure. The difficulty of applying (2.156) is replaced by a prognostic equation for the surface pressure, derived from the mass conservation equation.

Interestingly, the pressure coordinate equations (collected together in the shaded box on the previous page) are isomorphic to the hydrostatic, salt-free general Boussinesq equations (see the shaded box on page 74) with $z \Leftrightarrow -p$, $w \Leftrightarrow -\omega$, $\phi \Leftrightarrow \Phi$, $b \Leftrightarrow \alpha$, $\Theta \Leftrightarrow \theta$ and an equation of state $b = b(\Theta, z) \Leftrightarrow \alpha = \alpha(\theta, p)$ (and in an ideal gas $\alpha = (\theta R/p_R)(p_R/p)^{1/\gamma}$). The dynamics of one system can often therefore be expected to have an analogue in the other.

2.6.3 Log-pressure Coordinates

A variant of pressure coordinates arises by using *log-pressure* coordinates, in which the vertical coordinate is $Z = -H \ln(p/p_R)$ where p_R is a reference pressure (say 1000 mb) and H is a constant (for example a scale height RT_0/g where T_0 is a constant) so that Z has units of length. (Uppercase letters are conventionally used for some variables in log-pressure coordinates, and these are not to be confused with scaling parameters.) The 'vertical velocity' for the system is now

$$W \equiv \frac{\mathrm{D}Z}{\mathrm{D}t}, \tag{2.157}$$

and the advective derivative is

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla_p + W\frac{\partial}{\partial Z}. \tag{2.158}$$

The horizontal momentum equation is unaltered from (2.148), although we use (2.158) to evaluate the advective derivative. It is straightforward to show that the hydrostatic equation becomes

$$\frac{\partial \Phi}{\partial Z} = \frac{RT}{H}\,. \tag{2.159}$$

The mass continuity equation (2.152) becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial W}{\partial Z} - \frac{W}{H} = 0, \tag{2.160}$$

which may be written as

$$\nabla_Z \cdot \boldsymbol{u} + \frac{1}{\rho_R}\frac{\partial(\rho_R W)}{\partial z} = 0, \tag{2.161}$$

where $\nabla_Z \cdot$ is the divergence at constant Z and $\rho_R = \rho_0 \exp(-Z/H)$, so giving a form similar to the mass conservation equation in the anelastic equations. (The value of the constant ρ_0 may be set to one.)

As with pressure coordinates, it is convenient to write the thermodynamic equation in terms of temperature and not potential temperature, and in an analogous procedure to the one leading to (2.153) we obtain

$$c_p \frac{\mathrm{D}T}{\mathrm{D}t} + W \frac{RT}{H} = 0. \tag{2.162}$$

This equation may be written as

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + WS_Z = 0, \tag{2.163}$$

where

$$S_Z = \frac{\kappa T}{H} + \frac{\partial T}{\partial Z}, \tag{2.164}$$

and we may note that $S_Z = S_p p/H$. Using the hydrostatic equation we may write (2.163) as

$$\frac{\partial}{\partial t}\frac{\partial \Phi}{\partial Z} + u\frac{\partial}{\partial x}\frac{\partial \Phi}{\partial z} + v\frac{\partial}{\partial y}\frac{\partial \Phi}{\partial z} + WN_*^2 = 0, \tag{2.165}$$

where $N_*^2 = (R/H)S_Z$. The quantity N_* is not exactly equal to the square of the buoyancy frequency as normally defined (for an ideal gas $N^2 = (g/\theta)\partial\theta/\partial z$), but the two can be shown to be related by $N_*/N = p/(\rho g H) = RT/gH$, and are equal for an isothermal atmosphere.[10] Integrating the hydrostatic equation between two pressure levels gives, with $\Phi = gz$,

$$z(p_2) - z(p_1) = -\frac{R}{g}\int_{p_1}^{p_2} T \,\mathrm{d}\ln p. \tag{2.166}$$

Thus, the thickness of the layer is proportional to the average temperature of the layer, and at constant temperature the geometric height increases linearly with the logarithm of pressure. At a temperature of 240 K (280 K) the scale height, RT/g, is about 7 km (8.2 km). A useful rule of thumb for Earth's atmosphere (and one that holds at 240 K) is that geometric height increases by about 16 km for each factor of ten decrease in pressure, and pressures of 1000 hPa, 100 hPa, 10 hPa roughly correspond to heights of 0, 16 km 32 km and so on.

2.7 SCALING FOR HYDROSTATIC BALANCE

We first encountered hydrostatic balance in Section 1.3.3; we now look in more detail at the conditions required for it to hold. Along with geostrophic balance, considered in the next section, it is one of the most fundamental balances in geophysical fluid dynamics. The corresponding states, hydrostasy and geostrophy, are not exactly realized, but their approximate satisfaction has profound consequences on the behaviour of the atmosphere and ocean.

2.7.1 Preliminaries

Consider the relative sizes of terms in (2.77c):

$$\frac{W}{T} + \frac{UW}{L} + \frac{W^2}{H} + \Omega U \sim \left|\frac{1}{\rho}\frac{\partial p}{\partial z}\right| + g. \tag{2.167}$$

For most large-scale motion in the atmosphere and ocean the terms on the right-hand side are orders of magnitude larger than those on the left, and therefore must be approximately equal. Explicitly, suppose $W \sim 1\,\mathrm{cm\,s^{-1}}$, $L \sim 10^5\,\mathrm{m}$, $H \sim 10^3\,\mathrm{m}$, $U \sim 10\,\mathrm{m\,s^{-1}}$, $T = L/U$. Then by substituting into (2.167) it seems that the pressure term is the only one which could balance the gravitational term, and we are led to approximate (2.77c) by,

$$\frac{\partial p}{\partial z} = -\rho g. \tag{2.168}$$

This equation, which is a vertical momentum equation, is known as *hydrostatic balance*.

However, (2.168) is not always a useful equation! Let us suppose that the density is a constant, ρ_0 . We can then write the pressure as

$$p(x, y, z, t) = p_0(z) + p'(x, y, z, t), \qquad \text{where} \qquad \frac{\partial p_0}{\partial z} \equiv -\rho_0 g. \tag{2.169}$$

That is, p_0 and ρ_0 are in hydrostatic balance. On the f-plane, the inviscid vertical momentum equation becomes, without approximation,

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z}. \tag{2.170}$$

Thus, *for constant density fluids the gravitational term has no dynamical effect:* there is no buoyancy force, and the pressure term in the horizontal momentum equations can be replaced by p'. Hydrostatic balance, and in particular (2.169), is not a useful vertical momentum equation in this case. If the fluid is stratified, we should therefore subtract off the hydrostatic pressure associated with the mean density before we can determine whether hydrostasy is a useful *dynamical* approximation, accurate enough to determine the horizontal pressure gradients. This is automatic in the Boussinesq equations, where the vertical momentum equation is

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial \phi}{\partial z} + b, \tag{2.171}$$

and the hydrostatic balance of the basic state is already subtracted out. In the more general equation,

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \tag{2.172}$$

we need to compare the advective term on the left-hand side with the pressure variations arising from horizontal flow in order to determine whether hydrostasy is an appropriate vertical momentum equation. Nevertheless, if we only need to determine the pressure for use in an equation of state then we simply need to compare the sizes of the dynamical terms in (2.77c) with g itself, in order to determine whether a hydrostatic approximation will suffice.

2.7.2 Scaling and the Aspect Ratio

In a Boussinesq fluid we write the horizontal and vertical momentum equations as

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{u} = -\nabla_z \phi, \qquad \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial \phi}{\partial z} + b. \tag{2.173a,b}$$

With $\boldsymbol{f} = 0$, (2.173a) implies the scaling

$$\phi \sim U^2. \tag{2.174}$$

If we use mass conservation, $\nabla_z \cdot \boldsymbol{u} + \partial w/\partial z = 0$, to scale vertical velocity then

$$w \sim W = \frac{H}{L}U = \alpha U, \tag{2.175}$$

where $\alpha \equiv H/L$ is the aspect ratio. The advective terms in the vertical momentum equation all scale as

$$\frac{\mathrm{D}w}{\mathrm{D}t} \sim \frac{UW}{L} = \frac{U^2 H}{L^2}. \tag{2.176}$$

Using (2.174) and (2.176) the ratio of the advective term to the pressure gradient term in the vertical momentum equations then scales as

$$\frac{|\mathrm{D}w/\mathrm{D}t|}{|\partial\phi/\partial z|} \sim \frac{U^2 H/L^2}{U^2/H} \sim \left(\frac{H}{L}\right)^2. \tag{2.177}$$

Thus, the condition for hydrostasy, that $|\mathrm{D}w/\mathrm{D}t|/|\partial\phi/\partial z| \ll 1$, is:

$$\boxed{\alpha^2 \equiv \left(\frac{H}{L}\right)^2 \ll 1.} \tag{2.178}$$

The advective term in the vertical momentum may then be neglected. Thus, *hydrostatic balance arises from a small aspect ratio approximation.*

We can obtain the same result more formally by nondimensionalizing the momentum equations. Using uppercase symbols to denote scaling values we write

$$\begin{gathered}(x, y) = L(\widehat{x}, \widehat{y}), \qquad z = H\widehat{z}, \qquad \boldsymbol{u} = U\widehat{\boldsymbol{u}}, \qquad w = W\widehat{w} = \frac{HU}{L}\widehat{w}, \\ t = T; \hat{t} = \frac{L}{U}\hat{t}, \qquad \phi = \Phi\widehat{\phi} = U^2\widehat{\phi}, \qquad b = B\hat{b} = \frac{U^2}{H}\hat{b},\end{gathered} \tag{2.179}$$

where the hatted variables are nondimensional and the scaling for w is suggested by the mass conservation equation, $\nabla_z \cdot \boldsymbol{u} + \partial w/\partial z = 0$. Substituting (2.179) into (2.173) (with $\boldsymbol{f} = 0$) gives us the nondimensional equations

$$\frac{\mathrm{D}\widehat{\boldsymbol{u}}}{\mathrm{D}\hat{t}} = -\nabla\widehat{\phi}, \qquad \alpha^2 \frac{\mathrm{D}\widehat{w}}{\mathrm{D}\hat{t}} = -\frac{\partial\widehat{\phi}}{\partial\widehat{z}} + \hat{b}, \tag{2.180a,b}$$

where $\mathrm{D}/\mathrm{D}\hat{t} = \partial/\partial\hat{t} + \widehat{u}\partial/\partial\widehat{x} + \widehat{v}\partial/\partial\widehat{y} + \widehat{w}\partial/\partial\widehat{z}$ and we use the convention that when ∇ operates on nondimensional quantities the operator itself is nondimensional. From (2.180b) it is clear that hydrostatic balance pertains when $\alpha^2 \ll 1$.

2.7.3 ✦ Effects of Stratification on Hydrostatic Balance

To include the effects of stratification we need to involve the thermodynamic equation, so let us first write down the complete set of non-rotating dimensional equations:

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla_z\phi, \qquad \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial\phi}{\partial z} + b', \tag{2.181a,b}$$

$$\frac{\mathrm{D}b'}{\mathrm{D}t} + wN^2 = 0, \qquad \nabla\cdot\boldsymbol{v} = 0. \tag{2.182a,b}$$

We have written, without approximation, $b = b'(x, y, z, t) + \tilde{b}(z)$, with $N^2 = \mathrm{d}\tilde{b}/\mathrm{d}z$; this separation is useful because the horizontal and vertical buoyancy variations may scale in different ways, and often N^2 may be regarded as given. (We have also redefined ϕ by subtracting off a static component in hydrostatic balance with $\tilde{b}$.) We nondimensionalize (2.182) by first writing

$$\begin{gathered}(x, y) = L(\widehat{x}, \widehat{y}), \qquad z = H\widehat{z}, \qquad \boldsymbol{u} = U\widehat{\boldsymbol{u}}, \quad w = W\widehat{w} = \epsilon\frac{HU}{L}\widehat{w}, \\ t = T\hat{t} = \frac{L}{U}\hat{t}, \qquad \phi = U^2\widehat{\phi}, \qquad b' = \Delta b\hat{b}' = \frac{U^2}{H}\hat{b}', \qquad N^2 = \overline{N}^2\widehat{N}^2,\end{gathered} \tag{2.183}$$

where ϵ is, for the moment, undetermined, $\overline{N}$ is a representative constant value of the buoyancy frequency and Δb scales only the horizontal buoyancy variations. Substituting (2.183) into (2.181) and (2.182) gives

$$\frac{\mathrm{D}\hat{\boldsymbol{u}}}{\mathrm{D}\hat{t}} = -\nabla_z \hat{\phi}, \qquad \epsilon\alpha^2 \frac{\mathrm{D}\widehat{w}}{\mathrm{D}t} = -\frac{\partial\widehat{\phi}}{\partial\widehat{z}} + \hat{b}' \tag{2.184a,b}$$

$$\frac{U^2}{\overline{N}^2 H^2}\frac{\mathrm{D}\hat{b}'}{\mathrm{D}\hat{t}} + \epsilon\widehat{w}\widehat{N}^2 = 0, \qquad \nabla\cdot\hat{\boldsymbol{u}} + \epsilon\frac{\partial\widehat{w}}{\partial\widehat{z}} = 0. \tag{2.185a,b}$$

where now $\mathrm{D}/\mathrm{D}\hat{t} = \partial/\partial\hat{t} + \hat{\boldsymbol{u}}\cdot\nabla_z + \epsilon\widehat{w}\partial/\partial\widehat{z}$. To obtain a non-trivial balance in (2.185a) we choose $\epsilon = U^2/(\overline{N}^2 H^2) \equiv Fr^2$, where Fr is the *Froude number,* a measure of the stratification of the flow. A strong stratification corresponds to a small Froude number. From (2.183), the vertical velocity then scales as

$$W = \frac{Fr^2 UH}{L} \tag{2.186}$$

and if the flow is highly stratified the vertical velocity will be even smaller than a pure aspect ratio scaling might suggest. (There must, therefore, be some cancellation in horizontal divergence in the mass continuity equation; that is, $|\nabla_z\cdot\boldsymbol{u}| \ll U/L$.) With this choice of ϵ the nondimensional Boussinesq equations may be written:

$$\frac{\mathrm{D}\hat{\boldsymbol{u}}}{\mathrm{D}\hat{t}} = -\nabla_z \hat{\phi}, \qquad Fr^2\alpha^2 \frac{\mathrm{D}\widehat{w}}{\mathrm{D}\hat{t}} = -\frac{\partial\widehat{\phi}}{\partial\widehat{z}} + \hat{b}', \tag{2.187a,b}$$

$$\frac{\mathrm{D}\hat{b}'}{\mathrm{D}\hat{t}} + \widehat{w}\widehat{N}^2 = 0, \qquad \nabla\cdot\hat{\boldsymbol{u}} + Fr^2\frac{\partial\widehat{w}}{\partial\widehat{z}} = 0. \tag{2.188a,b}$$

The nondimensional parameters in the system are the aspect ratio and the Froude number (in addition to $\widehat{N}$, but by construction this is just an order one function of z). From (2.187b) the condition for hydrostatic balance to hold is evidently that

$$\boxed{Fr^2\alpha^2 \ll 1,} \tag{2.189}$$

so generalizing the aspect ratio condition (2.178) to a stratified fluid. Because Fr is a measure of stratification, (2.189) formalizes our intuitive expectation that the more stratified a fluid the more vertical motion is suppressed and therefore the more likely hydrostatic balance is to hold. Equation (2.189) is equivalent to

$$\frac{U^2}{\overline{N}^2 H^2}\frac{H^2}{L^2} = \frac{U^2}{L^2\overline{N}^2} \ll 1, \tag{2.190}$$

and in a hydrostatic model the condition is always, by construction, satisfied.

Why bother with any of this scaling? Why not just say that hydrostatic balance holds when $|Dw/Dt| \ll |\partial\phi/\partial z|$? One reason is that we do not have a good idea of the value of w from direct measurements, and it may change significantly in different oceanic and atmospheric parameter regimes. On the other hand the Froude number and the aspect ratio are familiar nondimensional parameters with a wide applicability in other contexts, and which we can control in a laboratory setting or estimate in the ocean or atmosphere. Still, when equations are scaled, ascertaining which parameters are to be regarded as given and which should be derived is often a choice, rather than being set a priori.

2.7.4 Hydrostasy in the Ocean and Atmosphere

Is the hydrostatic approximation in fact a good one in the ocean and atmosphere?

In the ocean

For the large-scale ocean circulation, let $N \sim 10^{-2}\,\mathrm{s}^{-1}$, $U \sim 0.1\,\mathrm{m\,s}^{-1}$ and $H \sim 1\,\mathrm{km}$. Then $Fr = U/(NH) \sim 10^{-2} \ll 1$. Thus, $Fr^2\alpha^2 \ll 1$ even for unit aspect-ratio motion. In fact, for larger scale flow the aspect ratio is also small; for basin-scale flow $L \sim 10^6\,\mathrm{m}$ and $Fr^2\alpha^2 \sim 0.01^2 \times 0.001^2 = 10^{-10}$ and hydrostatic balance is an extremely good approximation.

For intense convection, for example in the Labrador Sea, the hydrostatic approximation may be less appropriate, because the intense descending plumes may have an aspect ratio (H/L) of one or greater and the stratification is very weak. The hydrostatic condition then often becomes the requirement that the Froude number is small. Representative orders of magnitude are $U \sim W \sim 0.1\,\mathrm{m\,s}^{-1}$, $H \sim 1\,\mathrm{km}$ and $N \sim 10^{-3}\,\mathrm{s}^{-1}$ to $10^{-4}\,\mathrm{s}^{-1}$. For these values Fr ranges between 0.1 and 1, and at the upper end of this range hydrostatic balance is violated.

In the atmosphere

Over much of the troposphere $N \sim 10^{-2}\,\mathrm{s}^{-1}$ so that with $U = 10\,\mathrm{m\,s}^{-1}$ and $H = 1\,\mathrm{km}$ we find $Fr \sim 1$. Hydrostasy is then maintained because the aspect ratio H/L is much less than unity. For larger scale synoptic activity a larger vertical scale is appropriate, and with $H = 10\,\mathrm{km}$ both the Froude number and the aspect ratio are much smaller than one; indeed with $L = 1000\,\mathrm{km}$ we find $Fr^2\alpha^2 \sim 0.1^2 \times 0.01^2 = 10^{-6}$ and the flow is hydrostatic to a very good approximation indeed. However, for smaller scale atmospheric motions associated with fronts and, especially, convection, there can be little expectation that hydrostatic balance will be a good approximation.

For large-scale flows in both atmosphere and ocean, the conceptual and practical simplifications afforded by the hydrostatic approximation can hardly be overemphasized.

2.8 GEOSTROPHIC AND THERMAL WIND BALANCE

We now consider the dominant dynamical balance in the horizontal components of the momentum equation. In the horizontal plane (meaning along geopotential surfaces) we find that the Coriolis term is much larger than the advective terms and the dominant balance is between it and the horizontal pressure force. This balance is called *geostrophic balance,* and it occurs when the Rossby number is small, as we now investigate.

2.8.1 The Rossby Number

The *Rossby number* characterizes the importance of rotation in a fluid.[11] It is, essentially, the ratio of the magnitude of the relative acceleration to the Coriolis acceleration, and it is of fundamental importance in geophysical fluid dynamics. It arises from a simple scaling of the horizontal momentum equation, namely

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{u} + \boldsymbol{f}\times\boldsymbol{u} = -\frac{1}{\rho}\nabla_z p, \tag{2.191a}$$

$$\qquad U^2/L \qquad\qquad fU, \tag{2.191b}$$

where U is the approximate magnitude of the horizontal velocity and L is a typical length scale over which that velocity varies. (We assume that $W/H \lesssim U/L$, so that vertical advection does not dominate the advection.) The ratio of the sizes of the advective and Coriolis terms is defined to be the Rossby number,

$$\boxed{Ro \equiv \frac{U}{fL}.} \tag{2.192}$$

If the Rossby number is small then rotation effects are important and, as the values in Table 2.1 indicate, this is the case for large-scale flow in both ocean and atmosphere.

Variable	Scaling symbol	Meaning	Atmos. value	Ocean value
(x, y)	L	Horizontal length scale	10^6 m	10^5 m
t	T	Time scale	1 day (10^5 s)	10 days (10^6 s)
(u, v)	U	Horizontal velocity	10 m s^{-1}	0.1 m s^{-1}
	Ro	Rossby number, U/fL	0.1	0.01

Table 2.1 Scales of large-scale flow in atmosphere and ocean. The choices given are representative of large-scale mid-latitude eddying motion in both systems.

Another intuitive way to think about the Rossby number is in terms of time scales. The Rossby number based on a time scale is

$$Ro_T \equiv \frac{1}{fT}, \tag{2.193}$$

where T is a time scale associated with the dynamics at hand. If the time scale is an advective one, meaning that $T \sim L/U$, then this definition is equivalent to (2.192). Now, $f = 2\Omega \sin \vartheta$, where Ω is the angular velocity of the rotating frame and equal to $2\pi/T_p$ where T_p is the period of rotation (24 hours). Thus,

$$Ro_T = \frac{T_p}{4\pi T \sin \vartheta} = \frac{T_i}{T}, \tag{2.194}$$

where $T_i = 1/f$ is the 'inertial time scale', about three hours in mid-latitudes. Thus, for phenomena with time scales much longer than this, such as the motion of the Gulf Stream or a mid-latitude atmospheric weather system, the effects of the Earth's rotation can be expected to be important, whereas a short-lived phenomenon, such as a cumulus cloud or tornado, may be oblivious to such rotation.

2.8.2 Geostrophic Balance

If the Rossby number is sufficiently small in (2.191a) then the rotation term will dominate the nonlinear advection term, and if the time period of the motion scales advectively then the rotation term also dominates the local time derivative. The only term that can then balance the rotation term is the pressure term, and therefore we must have

$$\boldsymbol{f} \times \boldsymbol{u} \approx -\frac{1}{\rho}\nabla_z p, \tag{2.195}$$

or, in Cartesian component form

$$fu \approx -\frac{1}{\rho}\frac{\partial p}{\partial y}, \qquad fv \approx \frac{1}{\rho}\frac{\partial p}{\partial x}. \tag{2.196}$$

This balance is known as *geostrophic balance,* and its consequences are profound, giving geophysical fluid dynamics a special place in the broader field of fluid dynamics. We *define* the geostrophic velocity by

$$fu_g \equiv -\frac{1}{\rho}\frac{\partial p}{\partial y}, \qquad fv_g \equiv \frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{2.197}$$

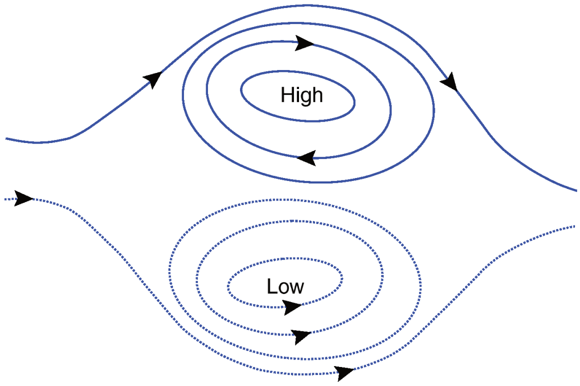

Fig. 2.5 Geostrophic flow with a positive value of the Coriolis parameter f. Flow is parallel to the lines of constant pressure (isobars). Cyclonic flow is anticlockwise around a low pressure region and anticyclonic flow is clockwise around a high. If f were negative, as in the Southern Hemisphere, (anti)cyclonic flow would be (anti)clockwise.

and for low Rossby number flow $u \approx u_g$ and $v \approx v_g$. In spherical coordinates the geostrophic velocity is

$$fu_g = -\frac{1}{\rho a}\frac{\partial p}{\partial \vartheta}, \qquad fv_g = \frac{1}{a\rho\cos\vartheta}\frac{\partial p}{\partial \lambda}, \tag{2.198}$$

where $f = 2\Omega\sin\vartheta$. Geostrophic balance has a number of immediate ramifications:

- Geostrophic flow is parallel to lines of constant pressure (isobars). If $f > 0$ the flow is anticlockwise round a region of low pressure and clockwise around a region of high pressure (see Fig. 2.5).

- If the Coriolis force is constant and if the density does not vary in the horizontal the geostrophic flow is horizontally non-divergent and

$$\nabla_z \cdot \boldsymbol{u}_g = \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = 0. \tag{2.199}$$

  We may define the *geostrophic streamfunction*, ψ, by

$$\psi \equiv \frac{p}{f_0\rho_0}, \qquad \text{whence} \qquad u_g = -\frac{\partial \psi}{\partial y}, \quad v_g = \frac{\partial \psi}{\partial x}. \tag{2.200}$$

  The vertical component of vorticity, ζ, is then given by

$$\zeta = \mathbf{k}\cdot\nabla\times\boldsymbol{v} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \nabla_z^2\psi. \tag{2.201}$$

- If the Coriolis parameter is not constant, then cross-differentiating (2.197) gives, for constant density geostrophic flow,

$$v_g\frac{\partial f}{\partial y} + f\nabla_z\cdot\boldsymbol{u}_g = 0. \tag{2.202}$$

  Using the mass continuity equation, $\nabla_z\cdot\boldsymbol{u}_g = -\partial w/\partial z$, we then obtain

$$\beta v_g = f\frac{\partial w}{\partial z}, \tag{2.203}$$

  where $\beta \equiv \partial f/\partial y = 2\Omega\cos\vartheta/a$. This geostrophic vorticity balance is sometimes known as 'Sverdrup balance', although that expression is better restricted to the case when the vertical velocity comes from a wind stress, as considered in Chapter 19.

2.8.3 Taylor–Proudman Effect

If $\beta = 0$, then (2.203) implies that the vertical velocity is not a function of height. In fact, in that case none of the components of velocity vary with height if density is also constant. To show this, in the limit of zero Rossby number we first write the three-dimensional momentum equation as

$$\boldsymbol{f}_0 \times \boldsymbol{v} = -\nabla\phi - \nabla\chi, \tag{2.204}$$

where $\boldsymbol{f}_0 = 2\boldsymbol{\Omega} = 2\Omega\mathbf{k}$, $\phi = p/\rho_0$, and $\nabla\chi$ represents other potential forces. If $\chi = gz$ then the vertical component of this equation represents hydrostatic balance, and the horizontal components represent geostrophic balance. On taking the curl of this equation, the terms on the right-hand side vanish and the left-hand side becomes

$$(\boldsymbol{f}_0 \cdot \nabla)\boldsymbol{v} - \boldsymbol{f}_0 \nabla \cdot \boldsymbol{v} - (\boldsymbol{v} \cdot \nabla)\boldsymbol{f}_0 + \boldsymbol{v}\nabla \cdot \boldsymbol{f}_0 = 0. \tag{2.205}$$

But $\nabla \cdot \boldsymbol{v} = 0$ by mass conservation, and because $\boldsymbol{f}_0$ is constant both $\nabla \cdot \boldsymbol{f}_0$ and $(\boldsymbol{v} \cdot \nabla)\boldsymbol{f}_0$ vanish. Equation (2.205) thus reduces to

$$(\boldsymbol{f}_0 \cdot \nabla)\boldsymbol{v} = 0, \tag{2.206}$$

which, since $\boldsymbol{f}_0 = f_0\mathbf{k}$, implies $f_0 \partial\boldsymbol{v}/\partial z = 0$, and in particular we have

$$\frac{\partial u}{\partial z} = 0, \quad \frac{\partial v}{\partial z} = 0, \quad \frac{\partial w}{\partial z} = 0. \tag{2.207}$$

All three components of velocity are uniform along the axis of rotation.

A different presentation of this argument proceeds as follows. If the flow is exactly in geostrophic and hydrostatic balance then

$$v = \frac{1}{f_0}\frac{\partial\phi}{\partial x}, \quad u = -\frac{1}{f_0}\frac{\partial\phi}{\partial y}, \qquad \frac{\partial\phi}{\partial z} = -g. \tag{2.208a,b,c}$$

Differentiating (2.208a,b) with respect to z, and using (2.208c) yields

$$\frac{\partial v}{\partial z} = \frac{-1}{f_0}\frac{\partial g}{\partial x} = 0, \qquad \frac{\partial u}{\partial z} = \frac{1}{f_0}\frac{\partial g}{\partial y} = 0. \tag{2.209}$$

Noting that the geostrophic velocities are horizontally non-divergent ($\nabla_z \cdot \boldsymbol{u} = 0$), and using mass continuity then gives $\partial w/\partial z = 0$, as before.

If there is a solid horizontal boundary anywhere in the fluid, for example at the surface, then $w = 0$ at that surface and thus $w = 0$ everywhere. Hence the motion occurs in planes that lie perpendicular to the axis of rotation, and the flow is effectively two dimensional. This result is known as the *Taylor–Proudman effect,* namely that for constant density flow in geostrophic and hydrostatic balance the vertical derivatives of the horizontal and the vertical velocities are zero.[12] At zero Rossby number, if the vertical velocity is zero somewhere in the flow then it is zero everywhere in that vertical column; furthermore, the horizontal flow has no vertical shear, and the fluid moves like a slab. The effects of rotation have provided a *stiffening* of the fluid in the vertical.

In neither the atmosphere nor the ocean do we observe precisely such vertically coherent flow, mainly because of the effects of stratification. However, it is typical of geophysical fluid dynamics that the assumptions underlying a derivation are not fully satisfied, yet there are manifestations of it in real flow. For example, one might have naïvely expected, because $\partial w/\partial z = -\nabla_z \cdot \boldsymbol{u}$, that the scales of the various variables would be related by $W/H \sim U/L$. However, if the flow is rapidly rotating we expect that the horizontal flow will be in near geostrophic balance and therefore nearly divergence free; thus $\nabla_z \cdot \boldsymbol{u} \ll U/L$, and $W \ll HU/L$.

2.8.4 Thermal Wind Balance

Thermal wind balance arises by combining the geostrophic and hydrostatic approximations, and this is most easily done in the context of the anelastic (or Boussinesq) equations, or in pressure coordinates. For the anelastic equations, geostrophic balance may be written

$$-fv_g = -\frac{\partial \phi}{\partial x} = -\frac{1}{a\cos\vartheta}\frac{\partial \phi}{\partial \lambda}, \qquad fu_g = -\frac{\partial \phi}{\partial y} = -\frac{1}{a}\frac{\partial \phi}{\partial \vartheta}. \tag{2.210a,b}$$

Combining these relations with hydrostatic balance, $\partial\phi/\partial z = b$, gives

$$\boxed{\begin{aligned} -f\frac{\partial v_g}{\partial z} &= -\frac{\partial b}{\partial x} = -\frac{1}{a\cos\lambda}\frac{\partial b}{\partial \lambda}, \\ f\frac{\partial u_g}{\partial z} &= -\frac{\partial b}{\partial y} = -\frac{1}{a}\frac{\partial b}{\partial \vartheta}. \end{aligned}} \tag{2.211a,b}$$

These equations represent *thermal wind balance,* and the vertical derivative of the geostrophic wind is the 'thermal wind'. Equation (2.211) may be written in terms of the zonal angular momentum as

$$\frac{\partial m_g}{\partial z} = -\frac{a}{2\Omega\tan\vartheta}\frac{\partial b}{\partial y}, \tag{2.212}$$

where $m_g = (u_g + \Omega a\cos\vartheta)a\cos\vartheta$. Potentially more accurate than geostrophic balance is the so-called gradient wind balance, discussed more in Section 2.9, which retains a centrifugal term in the momentum equation. The meridional momentum equation (2.50b) becomes

$$2u\Omega\sin\vartheta + \frac{u^2}{a}\tan\vartheta \approx -\frac{\partial \phi}{\partial y} = -\frac{1}{a}\frac{\partial \phi}{\partial \vartheta}. \tag{2.213}$$

For large-scale flow this only differs significantly from geostrophic balance very close to the equator. Taking the vertical derivative of (2.213) and using the hydrostatic relation $\partial\phi/\partial z = b$ gives a modified thermal wind relation, and this may be put in the simple form

$$\frac{\partial m^2}{\partial z} = -\frac{a^3\cos^3\vartheta}{\sin\vartheta}\frac{\partial b}{\partial y}, \tag{2.214}$$

where $m = (u + \Omega a\cos\vartheta)a\cos\vartheta$ is the annular angular momentum.

If the density or buoyancy is constant then there is no shear and (2.211) or (2.214) give the Taylor–Proudman result. But suppose that the temperature falls in the poleward direction. Then thermal wind balance implies that the (eastward) wind will increase with height — just as is observed in the atmosphere! In general, a vertical shear of the horizontal wind is associated with a horizontal temperature gradient, and this is one of the most simple and far-reaching effects in geophysical fluid dynamics. The underlying physical mechanism is illustrated in Fig. 2.6.

Geostrophic and thermal wind balance in pressure coordinates

In pressure coordinates geostrophic balance is just

$$\boldsymbol{f} \times \boldsymbol{u}_g = -\nabla_p \Phi, \tag{2.215}$$

where Φ is the geopotential and ∇_p is the gradient operator taken at constant pressure. If f is constant, it follows from (2.215) that the geostrophic wind is non-divergent on pressure surfaces.

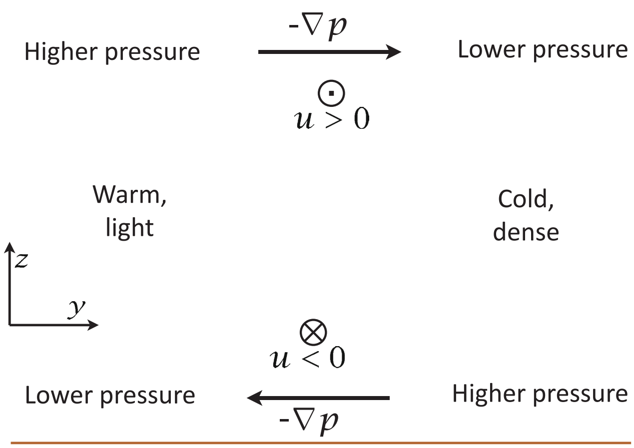

Fig. 2.6 The mechanism of thermal wind. A cold fluid is denser than a warm fluid, so by hydrostasy the vertical pressure gradient is greater where the fluid is cold. Thus, pressure gradients form as shown, where 'higher' and 'lower' mean relative to the average at that height. The horizontal pressure gradients are balanced by the Coriolis force, producing (for $f > 0$) the horizontal winds shown. Only the wind *shear* is given by the thermal wind.

Taking the vertical derivative of (2.215) (that is, its derivative with respect to p) and using the hydrostatic equation, $\partial\Phi/\partial p = -\alpha$, gives the thermal wind equation

$$\boldsymbol{f} \times \frac{\partial \boldsymbol{u}_g}{\partial p} = \nabla_p \alpha = \frac{R}{p}\nabla_p T, \tag{2.216}$$

where the last equality follows using the ideal gas equation and because the horizontal derivative is at constant pressure. In component form this is

$$-f\frac{\partial v_g}{\partial p} = \frac{R}{p}\frac{\partial T}{\partial x}, \qquad f\frac{\partial u_g}{\partial p} = \frac{R}{p}\frac{\partial T}{\partial y}. \tag{2.217}$$

In log-pressure coordinates, with $Z = -H\ln(p/p_R)$, thermal wind is

$$\boldsymbol{f} \times \frac{\partial \boldsymbol{u}_g}{\partial Z} = -\frac{R}{H}\nabla_Z T. \tag{2.218}$$

The effect in all these cases is the same: a horizontal temperature gradient, or a temperature gradient along an isobaric surface, is accompanied by a vertical shear of the horizontal wind.

2.8.5 ✦ Vertical Velocity and Hydrostatic Balance

Scaling for vertical velocity

If the Coriolis parameter is constant then flows that are in geostrophic balance have zero horizontal divergence ($\nabla_x \cdot \boldsymbol{u} = 0$) and zero vertical velocity. We can therefore expect that any flow with small Rossby number will have a correspondingly small vertical velocity. Let us make this statement more precise using the rotating Boussinesq equations, (2.173) with constant Coriolis parameter. Let $\boldsymbol{u} = \boldsymbol{u}_g + \boldsymbol{u}_a$ where the geostrophic flow satisfies $\boldsymbol{f}_0 \times \boldsymbol{u}_g = -\nabla\phi$. The horizontal momentum equation, with corresponding scales for each term, then becomes

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + w\frac{\partial \boldsymbol{u}}{\partial z} + \boldsymbol{f}_0 \times \boldsymbol{u}_a = 0, \tag{2.219}$$

$$\frac{U^2}{L} \qquad \frac{U^2}{L} \qquad \frac{WU}{H} \qquad f_0 U_a. \tag{2.220}$$

This equation suggests a scaling for the ageostrophic flow of

$$U_a = \frac{U}{f_0 L}U = Ro\, U. \tag{2.221}$$

That is, the ageostrophic flow is Rossby number smaller (at least) than the geostrophic flow. To obtain a scaling for the vertical velocity we look to the mass continuity equation written in the form

$$\frac{\partial w}{\partial z} = -\nabla \cdot \boldsymbol{u}_a, \tag{2.222}$$

since only the ageostrophic flow has a divergence. Equations (2.221) and (2.222) suggest the scaling

$$W = Ro\,\frac{HU}{L}. \tag{2.223}$$

That is, the vertical velocity is order Rossby number smaller than an estimate based purely on the mass continuity equation would suggest.

If the Coriolis parameter is not constant then the geostrophic flow itself is divergent and this induces a vertical velocity, as in (2.203). The scaling for vertical velocity is now

$$W = \frac{\beta}{f} HU = Ro_\beta \frac{HU}{L}, \tag{2.224}$$

where $Ro_\beta = \beta L/f$ is the *beta Rossby number*. It is less than one for all flows except those with a truly global scale.

Scaling for hydrostatic balance

Let us nondimensionalize the rotating Boussinesq equations, (2.173), by writing

$$\begin{gathered}(x, y) = L(\hat{x}, \hat{y}), \qquad z = H\hat{z}, \qquad \boldsymbol{u} = U\hat{\boldsymbol{u}}, \qquad t = T\hat{t} = \frac{L}{U}\hat{t}, \qquad \boldsymbol{f} = f_0\widehat{\boldsymbol{f}}, \\ w = \frac{\epsilon HU}{L}\widehat{w}, \qquad \phi = \Phi\widehat{\phi} = f_0 UL\widehat{\phi}, \qquad b = B\hat{b} = \frac{f_0 UL}{H}\hat{b}.\end{gathered} \tag{2.225}$$

These relations are almost the same as (2.179), except for the factor of ϵ in the scaling of w. If the Coriolis parameter is constant or nearly so then, from (2.223), $\epsilon = Ro$, whereas if the Coriolis parameter varies then $\epsilon = Ro_\beta$, as in (2.223). The scaling for ϕ and b' are suggested by geostrophic and thermal wind balance with f_0 a representative value of f. Substituting these values into (2.173) we obtain the scaled momentum equations:

$$Ro\,\frac{\mathrm{D}\hat{\boldsymbol{u}}}{\mathrm{D}\hat{t}} + \widehat{\boldsymbol{f}} \times \hat{\boldsymbol{u}} = -\nabla\widehat{\phi}, \qquad Ro\,\epsilon\alpha^2 \frac{\mathrm{D}\widehat{w}}{\mathrm{D}\hat{t}} = -\frac{\partial\widehat{\phi}}{\partial\hat{z}} - \hat{b}, \tag{2.226a,b}$$

where $\mathrm{D}/\mathrm{D}\hat{t} = \partial/\partial\hat{t} + \hat{\boldsymbol{u}} \cdot \nabla_z + \epsilon\widehat{w}\partial/\partial\hat{z}$. There are two notable aspects to these equations. First and most obviously, when $Ro \ll 1$, (2.226a) reduces to geostrophic balance, $\boldsymbol{f} \times \boldsymbol{u} \approx -\nabla\widehat{\phi}$. Second, the material derivative in (2.226b) is multiplied by three nondimensional parameters, and we can understand the appearance of each as follows:

(i) The aspect ratio dependence (α^2) arises in the same way as for non-rotating flows — that is, because of the presence of w and z in the vertical momentum equation as opposed to (u, v) and (x, y) in the horizontal equations.

(ii) The Rossby number dependence (Ro) arises because in rotating flow the pressure gradient is balanced by the Coriolis force, and the advective terms are Rossby-number smaller.

(iii) The factor ϵ arises because in rotating flow w is smaller than u by ϵ times the aspect ratio. The factor may be the Rossby number itself, or the beta Rossby number.

The factor $Ro\,\epsilon\alpha^2$ is very small for large-scale flow; the reader is invited to calculate representative values. Evidently, a rapidly rotating fluid is more likely to be in hydrostatic balance than a non-rotating fluid, other conditions being equal. The combined effects of rotation and stratification are, not surprisingly, quite subtle and we leave that topic for Chapter 5.

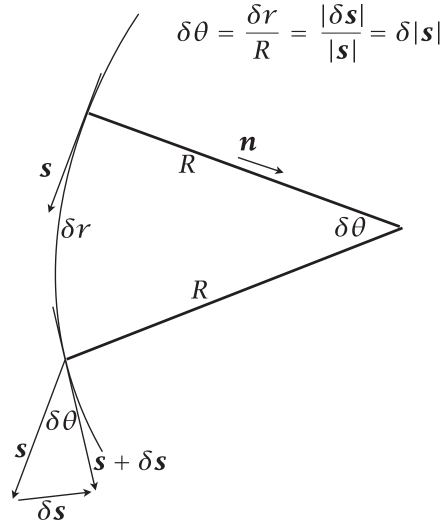

Fig. 2.7 A parcel tracing a curved path with radius of curvature R, moving a small distance δr and through a small angle $\delta\theta$. The parcel may experience centrifugal forces along $\boldsymbol{n}$ as well as Coriolis forces due to Earth's rotation.

2.9 ✦ GRADIENT WIND BALANCE

If a flow follows a curved path then our intuition suggests that it will experience a centrifugal force of some kind in addition to the Coriolis force. We can easily imagine that a parcel then experiences a three-way balance, between Coriolis, centrifugal and pressure forces, and this balance is called *gradient wind balance.* It is a more general, and more accurate, balance than geostrophic balance but it is not always as useful. To illustrate it we will keep matters simple and consider purely horizontal flow of constant density. We first introduce the notion of *natural coordinates* and show how gradient wind balance emerges straightforwardly.[13] We then discuss how gradient wind balance arises in the Eulerian equations of motion in a fixed coordinate system.

2.9.1 Natural Coordinates

For most purposes a coordinate system that is fixed in space, or rotating coincidentally with the Earth, is the most practically useful. However, it is sometimes useful to use a coordinate system that is moving with the local flow. To that end, and restricting our attention to horizontal flow, consider a parcel of fluid moving with velocity $\boldsymbol{u}$. We define a *natural coordinate system* by the set of unit vectors $\boldsymbol{s}$, $\boldsymbol{n}$ and $\mathbf{k}$, where $\boldsymbol{s}$ is the unit vector tangential to the flow, $\boldsymbol{n}$ is the horizontal unit vector normal to the flow and $\mathbf{k}$ is the unit vector in the vertical. Apart from $\mathbf{k}$, all these vectors evolve with the flow. Our goal is to split the horizontal momentum equation into components parallel to and normal to the direction of the local flow and thereby to discern the force balances in either direction. (Readers who trust their intuition may skip ahead to (2.231), which they may find obvious.)

If U is the speed of the parcel then $\boldsymbol{u} = U\boldsymbol{s}$ and so

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = \boldsymbol{s}\frac{\mathrm{d}U}{\mathrm{d}t} + U\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t}. \tag{2.227}$$

Furthermore, if the flow of a parcel follows the curve $r(t)$ — that is, r gives the distance moved by the parcel — then $U = \mathrm{d}r/\mathrm{d}t$. To obtain a useful expression for $\mathrm{d}\boldsymbol{s}/\mathrm{d}t$ we note that, as in Fig. 2.7,

$$\theta \equiv \frac{\delta r}{R} = \frac{|\delta\boldsymbol{s}|}{|\boldsymbol{s}|} = |\delta\boldsymbol{s}|, \tag{2.228}$$

where R is the *radius of curvature* and $|\boldsymbol{s}| = 1$. [The radius of curvature may be evaluated geometrically as follows. Draw a line tangent to the path at some point, and draw a line perpendicular to the tangent through that point. An infinitesimal distance along the curve, construct another perpendicular line in the same manner. The two perpendicular lines meet at the center of curvature, and the distance along one of the perpendicular lines from the center of curvature to the curve itself is the radius of curvature. By convention, the radius of curvature is positive (negative) if the parcel is curving to the left (right).]

The change of $\boldsymbol{s}$ is directed along $\boldsymbol{n}$, and so from (2.228) we infer that

$$\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}r} = \frac{\boldsymbol{n}}{R}. \tag{2.229}$$

Thus, the rate of change of $\boldsymbol{s}$ is given by

$$\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\boldsymbol{n}}{R}U, \tag{2.230}$$

and using this expression in (2.227), the acceleration of the fluid parcel is given by

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = \boldsymbol{s}\frac{\mathrm{d}U}{\mathrm{d}t} + \boldsymbol{n}\frac{U^2}{R}. \tag{2.231}$$

The two terms on the right-hand side may be interpreted as being, respectively, the change in speed of the parcel along its path and the centripetal acceleration owing to the curvature of the path (if the path is a straight line then R is infinite and the centripetal acceleration is zero).

2.9.2 Application to Fluids

Now consider the application of the above to a fluid in a rotating frame of reference. The total time derivatives of (2.231) may be replaced by material derivatives, and additional pressure and Coriolis forces act on the flow. The pressure-gradient force, $-\nabla\phi$, has components along and perpendicular to the flow so that

$$-\nabla\phi = -\left(\frac{\partial\phi}{\partial r}\boldsymbol{s} + \frac{\partial\phi}{\partial n}\boldsymbol{n}\right). \tag{2.232}$$

Given our convention, $\partial\phi/\partial n$ is positive if pressure increases to the left of the trajectory of the flow. The Coriolis force, $-\boldsymbol{f}\times\boldsymbol{u}$, has only a component perpendicular to the flow so that

$$-\boldsymbol{f}\times\boldsymbol{u} = -fU\boldsymbol{n}. \tag{2.233}$$

Because $\boldsymbol{n}$ is directed to the left of the flow, for positive f the Coriolis force is directed to the right of the flow. Using (2.231), (2.232) and (2.233), and neglecting friction, we can write down the components of the momentum equation parallel and perpendicular to the flow, namely

$$\frac{\mathrm{D}U}{\mathrm{D}t} = -\frac{\partial\phi}{\partial r}, \qquad \frac{U^2}{R} + fU = -\frac{\partial\phi}{\partial n}. \tag{2.234a,b}$$

These equations tell us, at least in principle, how a fluid field will evolve from some initial conditions. The radius of curvature can only be determined from the instantaneous velocity field for steady flow: if the flow is unsteady then the radius of curvature for a parcel depends on the pressure field and differs from the radius of curvature of a streamline.

Gradient wind balance, cyclostrophic balance and inertial flow

Gradient flow (or, as it is commonly called, the gradient wind) is the flow that satisfies (2.234b). We may always define a gradient wind at a given point by using this equation, and the only forces that are neglected are frictional ones. Given the pressure field ϕ we may use (2.234b) to calculate the gradient wind using the quadratic formula. Gradient wind balance is thus a better approximation to the real flow than is geostrophic balance (which omits the centrifugal term). The gradient wind will not in general be a steady solution to the equations of motion — the flow will accelerate and the pressure field will change.

If the flow is in a straight line then the radius of curvature is infinite and (2.234b) reduces to $fU = -\partial\phi/\partial n$, which is geostrophic balance. For this to hold exactly the flow need not be steady, just in a straight line. It will be a good approximation to the real flow when $fU \gg U^2/R$ or $U/fR \ll 1$, taking all quantities as positive, which is similar to a condition of low Rossby number. In contrast, cyclostrophic flow occurs when the quantity U/fR is large and the Coriolis force may be neglected, giving $U^2/R = -\partial\phi/\partial n$.

Inertial flow arises when the pressure gradient vanishes and (2.234b) reduces to $U/R + f = 0$. Now, if the pressure gradient vanishes then, from (2.234a) the speed of fluid parcels is constant. Thus, if f is constant the fluid parcels execute circles, known as inertia circles, of radius U/f and period $2\pi/f = 1\,\text{day}/(2\sin\vartheta)$ where ϑ is latitude. Such a motion is evidently not truly inertial, for inertial motion is in a straight line and at a constant speed; other forces — gravitational and centrifugal — must act to keep the flow in the horizontal plane. Inertial motion is rarely a good approximation to flow in the atmosphere or ocean, although occasionally parcels are observed to trace approximations to inertia circles, especially in the ocean.

2.9.3 Gradient Wind Balance in the Two-dimensional Eulerian Equations

Let us now consider gradient wind balance in the Eulerian equations of motion. The basic ideas can be exposed by considering unforced constant-density two dimensional flow in a rotating reference frame, for which the equations of motion are

$$\frac{\mathrm{D}u}{\mathrm{D}t} - fv = -\frac{\partial\phi}{\partial x}, \qquad \frac{\mathrm{D}v}{\mathrm{D}t} + fu = -\frac{\partial\phi}{\partial y}, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{2.235a,b,c}$$

We may introduce a stream function ψ such that $u = -\partial\psi/\partial y$ and $v = \partial\psi/\partial x$. If the flow is geostrophic then $f \times (u, v) = (-\partial\phi/\partial y, \partial\phi/\partial x)$; however, unless the Coriolis parameter f is constant the streamfunction is not proportional to the pressure field. The vorticity (considered further in Chapter 4), which is defined to be the curl of the velocity, is given by

$$\boldsymbol{\omega} \equiv \nabla \times \boldsymbol{u} = \mathbf{k}\zeta, \qquad \text{where} \qquad \zeta = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} \equiv \nabla_z^2\psi. \tag{2.236}$$

Let us form the evolution equations for vorticity and divergence by taking the curl and divergence of (2.235a,b). After a little algebra the vorticity equation is found to be

$$\frac{\partial\zeta}{\partial t} + \boldsymbol{u}\cdot\nabla(\zeta + f) = 0, \qquad \text{or equivalently} \qquad \frac{\partial\zeta}{\partial t} + J(\psi, \zeta + f) = 0, \tag{2.237}$$

where $J(a, b) \equiv (\partial a/\partial x)(\partial b/\partial y) - (\partial a/\partial y)(\partial b/\partial x)$ is the *Jacobian*. If $f = f(y)$ then $J(\psi, f) = \boldsymbol{u}\cdot\nabla f = \beta v$ where $\beta = \partial f/\partial y$. Because the velocity is divergence-free the vorticity equation above is closed; that is, it may be integrated without the need to solve any other evolution equations, and in particular without the need to solve for the pressure field. (In general the vorticity equation alone is not closed.)

If we take the divergence of (2.235a,b) then, using (2.235c) the time derivative disappears and again after a little algebra we obtain the divergence equation,

$$2J(u, v) + \nabla\cdot(f\nabla\psi) = \nabla^2\phi. \tag{2.238}$$

In this context, (2.238) may be regarded as an equation for pressure, ϕ, given the velocity or the streamfunction. Equation (2.238) is commonly referred to as the *gradient wind balance relation*, and it is analogous to (2.234b) in that it generalizes and is more accurate than geostrophic balance. Equation (2.238) is *exact* for two-dimensional, incompressible, inviscid and unforced flow, even when time-dependent. If the Rossby number is small then the second term on the left-hand side of (2.238) dominates the first, and $\nabla \cdot (f \nabla \psi) \approx \nabla^2 \phi$. This is equivalent to geostrophic balance, and if f is constant the streamfunction and pressure field are proportional to each other.

In the more general case the two-dimensional divergence is not zero; that is $\partial_t(\partial_x u + \partial_y v) \neq 0$. However, if the Rossby number is small and if f is nearly constant, then the geostrophic relation implies that the two-dimensional divergence will be small compared to the two-dimensional vorticity. In this case, gradient wind balance will be a good *approximation* to the flow, and a somewhat better approximation, if not more useful, than geostrophic balance alone.

2.10 STATIC INSTABILITY AND THE PARCEL METHOD

In this and the next couple of sections we consider how a fluid might oscillate if it were perturbed away from a resting state. Our focus is on vertical displacements, and the restoring force is gravity. Given that, the simplest way to approach the problem is to consider from first principles the pressure and gravitational forces on a displaced parcel, as in Fig. 2.8. To this end, consider a fluid initially at rest in a constant gravitational field, and therefore in hydrostatic balance. Suppose that a small parcel of the fluid is adiabatically displaced upwards by the small distance δz, without altering the overall pressure field; that is, the fluid parcel instantly assumes the pressure of its environment. If after the displacement the parcel is lighter than its environment, it will accelerate upwards, because the upward pressure gradient force is now greater than the downward gravity force on the parcel; that is, the parcel is *buoyant* (a manifestation of Archimedes' principle) and the fluid is *statically unstable.* If on the other hand the fluid parcel finds itself heavier than its surroundings, the downward gravitational force will be greater than the upward pressure force and the fluid will sink back towards its original position and an oscillatory motion will develop. Such an equilibrium is *statically stable.* Using such simple parcel arguments we will now develop criteria for the stability of the environmental profile.

2.10.1 Stability and the Profile of Potential Density

Consider the case of a stationary fluid whose density varies with altitude. We denote this background state with a tilde, as in $\widetilde{\rho}(z)$. We then displace a fluid parcel adiabatically a small distance from z to δz, as in Fig. 2.8. In such a displacement it is the *potential density* ρ_θ (not the actual density) that is materially conserved, because potential density takes into account the effects of pressure compressibility. Let us also use the pressure at level $z + \delta z$ as the reference level, where potential density equals in situ density.

The parcel at z takes on the potential density of its environment so that $\rho_\theta(z) = \widetilde{\rho}_\theta(z)$ and it preserves this as it rises, so that $\rho_\theta(z+\delta z) = \rho_\theta(z)$. But since $z+\delta z$ is the reference level, the *in situ* density of the displaced parcel, $\rho(z+\delta z)$, is equal to its potential density $\rho_\theta(z+\delta z)$, which is equal to $\widetilde{\rho}_\theta(z)$. Thus, at $z+\delta z$ the environment has in situ density equal to $\widetilde{\rho}_\theta(z+\delta z)$ and the parcel has in situ density equal to $\widetilde{\rho}_\theta(z)$. Putting all this together in a single equation, the difference between the parcel density and the environmental density, $\delta\rho$, is given by

$$\begin{aligned}\delta\rho &= \rho(z+\delta z) - \widetilde{\rho}(z+\delta z) = \rho_\theta(z+\delta z) - \widetilde{\rho}_\theta(z+\delta z) \\ &= \rho_\theta(z) - \widetilde{\rho}_\theta(z+\delta z) = \widetilde{\rho}_\theta(z) - \widetilde{\rho}_\theta(z+\delta z).\end{aligned} \tag{2.239}$$

Thus, for small δz,

$$\delta\rho = -\frac{\partial \widetilde{\rho}_\theta}{\partial z}\delta z, \tag{2.240}$$

where the derivative on the right-hand side is the environmental gradient of potential density. If the right-hand side is positive, the parcel is heavier than its surroundings and the displacement is

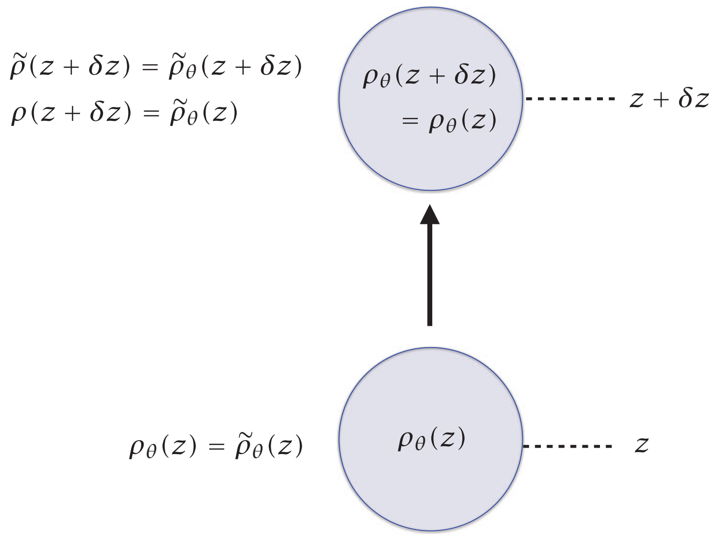

Fig. 2.8 A parcel is adiabatically displaced upward from level z to $z + \delta z$. A tilde denotes the value in the environment, and variables without tildes are those in the parcel.

The parcel preserves its potential density, ρ_θ, which it takes from the environment at level z. If $z+\delta z$ is the reference level, the potential density there is equal to the actual density. The parcel's stability is determined by the difference between its density and the environmental density, as in (2.239). If the difference is positive the displacement is stable, and if negative the displacement is unstable.

stable. That is, *the stability of a parcel of fluid is determined by the gradient of the locally-referenced potential density.*

The conditions for stability are

$$\begin{aligned} \text{Stability}: \quad & \frac{\partial \tilde\rho_\theta}{\partial z} < 0, \\ \text{Instability}: \quad & \frac{\partial \tilde\rho_\theta}{\partial z} > 0. \end{aligned} \tag{2.241a,b}$$

The equation of motion of the fluid parcel is then given by a direct application of Newton's second law, that the mass times the acceleration is given by the force acting on the parcel. The force is the above-derived buoyancy force so we have

$$\frac{\partial^2 \delta z}{\partial t^2} = \frac{g}{\rho}\left(\frac{\partial \tilde\rho_\theta}{\partial z}\right)\delta z = -N^2 \delta z, \tag{2.242}$$

where, noting that $\rho(z) = \tilde\rho_\theta(z)$ to within $O(\delta z)$,

$$N^2 = -\frac{g}{\tilde\rho_\theta}\left(\frac{\partial \tilde\rho_\theta}{\partial z}\right). \tag{2.243}$$

A parcel that is displaced in a stably stratified fluid will thus oscillate at the *buoyancy frequency* N, proportional to the vertical gradient of potential density. (The buoyancy frequency is also known as the Brunt–Väisälä frequency, after its discovers.) The above expression for the buoyancy frequency is a general one, true in both liquids and gases in a constant gravitational field. The quantity $\tilde\rho_\theta$ is the *locally-referenced* potential density of the environment. The reference level turns out not to be important for the atmosphere, but it is for the ocean: parcels at the same level with the same in situ density may have different potential densities if their salinity differs. In contrast, for fresh water in a laboratory setting potential density is virtually equal to in situ density.

2.10.2 A Dry Ideal-gas Atmosphere

Buoyancy frequency

In the atmosphere potential density is related to potential temperature by $\rho_\theta = p_R/(\theta R)$, where p_R is the reference level for potential temperature. Using this expression in (2.243) gives

$$\boxed{N^2 = \frac{g}{\widetilde{\theta}}\left(\frac{\partial \widetilde{\theta}}{\partial z}\right),} \tag{2.244}$$

where $\widetilde{\theta}$ is the environmental potential temperature. The reference value p_R does not appear, and we are free to choose this value arbitrarily — the surface pressure is a common choice. The conditions for stability, (2.241), then correspond to $N^2 > 0$ for stability and $N^2 < 0$ for instability. On average the atmosphere is stable and in the troposphere (the lowest several kilometres of the atmosphere) the average N is about $0.01\,\mathrm{s}^{-1}$, with a corresponding period, $(2\pi/N)$, of about 10 minutes. In the stratosphere (which lies above the troposphere) N is a few times higher than this.

Dry adiabatic lapse rate

The negative of the rate of change of the (real) temperature in the vertical is known as the *temperature lapse rate*, or often just the lapse rate, and denoted $\varGamma$. The lapse rate corresponding to $\partial\theta/\partial z = 0$ is called the *dry adiabatic lapse rate* and denoted $\varGamma_d$. Using $\theta = T(p_0/p)^{R/c_p}$ and $\partial p/\partial z = -\rho g$ we find that the lapse rate and the potential temperature lapse rate are related by

$$\frac{T}{\theta}\frac{\partial\theta}{\partial z} = \frac{\partial T}{\partial z} + \frac{g}{c_p}, \tag{2.245}$$

so that the dry adiabatic lapse rate is given by

$$\varGamma_d = \frac{g}{c_p}, \tag{2.246}$$

as we derived in (1.131). The conditions for static stability corresponding to (2.241) are thus:

$$\boxed{\begin{aligned} &\text{Stability}: & \frac{\partial\widetilde{\theta}}{\partial z} > 0, \quad &\text{or} \quad -\frac{\partial \widetilde{T}}{\partial z} < \varGamma_d. \\ &\text{Instability}: & \frac{\partial\widetilde{\theta}}{\partial z} < 0, \quad &\text{or} \quad -\frac{\partial \widetilde{T}}{\partial z} > \varGamma_d. \end{aligned}} \tag{2.247a,b}$$

The observed lapse rate (look ahead to Fig. 15.25) is often less than $7\,\mathrm{K\,km^{-1}}$ (corresponding to a buoyancy frequency of about $10^{-2}\,\mathrm{s}^{-1}$) whereas a dry adiabatic lapse rate is about $10\,\mathrm{K\,km^{-1}}$. Why the discrepancy? Why is the atmosphere so apparently stable? One reason is that in mid-latitudes heat is transferred upwards by in large-scale weather systems that keep the atmosphere stable even in the absence of convection. A second reason is that the atmosphere contains water vapour and a column of air that contains water may be unstable even if its lapse rate is less than dry adiabatic. If a moist parcel rises then, as it enters a cooler environment, water vapour may condense releasing more heat and leading to more ascent; that is, a moist atmosphere may be unstable when a dry atmosphere is stable. We defer more discussion to Chapters 15 and 18.

2.10.3 A Liquid Ocean

No simple, accurate, analytic expression is available for computing static stability in the ocean. If the ocean had no salt, then the potential density referenced to the surface would generally be a measure of the sign of stability of a fluid column, if not of the buoyancy frequency. However, in the presence of salinity, the surface-referenced potential density is not necessarily even a measure of the sign of stability, because the coefficients of compressibility β_T and β_S vary in different ways with pressure. To see this, suppose two neighbouring fluid elements at the surface have the same potential density, but different salinities and temperatures, and displace them both adiabatically to the deep ocean. Although their potential densities referenced to the surface are still equal, we can say little about their actual densities, and hence their stability relative to each other, without doing a detailed calculation because they will each have been compressed by different amounts. It is the profile of the *locally-referenced* potential density that determines the stability.

A useful expression for stability arises by noting that in an adiabatic displacement

$$\delta\rho_\theta = \delta\rho - \frac{1}{c_s^2}\delta p = 0. \tag{2.248}$$

If the fluid is hydrostatic $\delta p = -\rho g \delta z$, so that if a parcel is displaced adiabatically its density changes according to

$$\left(\frac{\partial \rho}{\partial z}\right)_{\rho_\theta} = -\frac{\rho g}{c_s^2}. \tag{2.249}$$

If a parcel is displaced a distance δz upwards then the density difference between it and its new surroundings is

$$\delta\rho = -\left[\left(\frac{\partial \rho}{\partial z}\right)_{\rho_\theta} - \left(\frac{\partial \widetilde{\rho}}{\partial z}\right)\right]\delta z = \left[\frac{\rho g}{c_s^2} + \left(\frac{\partial \widetilde{\rho}}{\partial z}\right)\right]\delta z, \tag{2.250}$$

where the tilde again denotes the environmental field. It follows that the stratification is given by

$$\boxed{N^2 = -g\left[\frac{g}{c_s^2} + \frac{1}{\widetilde{\rho}}\left(\frac{\partial \widetilde{\rho}}{\partial z}\right)\right].} \tag{2.251}$$

This expression holds for both liquids and gases, and it is proportional to the vertical gradient of potential density. For ideal gases it is the same as (2.244), as a little algebra will show, using $c_s^2 = \gamma p/\rho$. In seawater the expression may be compared to the gradient of (1.144). The factor of g/c_s^2 is small but not negligible; it is a slightly destabilising factor in the sense that a density profile with an in situ density that increases with depth is not necessarily stable. In liquids, a good approximation is to use a reference value ρ_0 for the undifferentiated density in the denominator, whence (2.251) becomes equal to the Boussinesq expression (2.107). On average the ocean is statically stable, with typical values of N in the upper ocean being about 0.01 s^{-1}, falling to 0.001 s^{-1} in the more homogeneous abyssal ocean. These frequencies correspond to periods of about 10 and 100 minutes, respectively.

2.10.4 Gravity Waves and Convection Using the Equations of Motion

The parcel approach to oscillations and stability, while simple and direct, seems divorced from the fluid-dynamical equations of motion. To remedy this, we now use the equations of motion for a stratified Boussinesq fluid to analyze the motion resulting from a small disturbance. Our treatment here is brief and introductory, with a fuller treatment given in Chapter 7.

Consider a Boussinesq fluid, initially at rest, in which the buoyancy varies linearly with height. Linearizing the equations of motion about this basic state gives the linear momentum equations,

$$\frac{\partial u'}{\partial t} = -\frac{\partial \phi'}{\partial x}, \qquad \frac{\partial w'}{\partial t} = -\frac{\partial \phi'}{\partial z} + b', \tag{2.252a,b}$$

the mass continuity and thermodynamic equations,

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0, \qquad \frac{\partial b'}{\partial t} + w' N^2 = 0, \tag{2.253a,b}$$

where $N^2 = \mathrm{d}\tilde{b}/\mathrm{d}z$ is the basic state buoyancy profile, and we assume that the flow is a function only of x and z. A little algebra reduces the above equations to a single one for w',

$$\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\frac{\partial^2}{\partial t^2} + N^2 \frac{\partial^2}{\partial x^2}\right] w' = 0. \tag{2.254}$$

Seeking solutions of the form $w' = \mathrm{Re}\, W \exp[\mathrm{i}(kx + mz - \omega t)]$ yields the dispersion relationship for gravity waves:

$$\boxed{\omega^2 = \frac{k^2 N^2}{k^2 + m^2}.} \tag{2.255}$$

The frequency (look ahead to Fig. 7.2) is always less than N, approaching N for small horizontal scales, $k \gg m$.

Consider two special cases. First, if we neglect pressure perturbations, as in the parcel argument, then the two equations,

$$\frac{\partial w'}{\partial t} = b', \qquad \frac{\partial b'}{\partial t} + w' N^2 = 0, \tag{2.256}$$

form a closed set and give $\omega^2 = N^2$, as in the parcel argument. Second, if we make the hydrostatic approximation and omit $\partial w'/\partial t$ in (2.252b) then the dispersion relation becomes $\omega^2 = k^2 N^2/m^2$. The frequency then grows, artifactually, without bound as the horizontal scale becomes smaller.

If the basic state density increases with height then $N^2 < 0$ and we expect this state to be unstable. Indeed, the disturbance grows exponentially according to $\exp(\sigma t)$ where $\sigma = \mathrm{i}\omega = \pm k\widetilde{N}/(k^2 + m^2)^{1/2}$, and where $\widetilde{N}^2 = -N^2$. We have reproduced the result previously obtained by parcel theory, namely that if the basic state density (or more generally potential density) increases with height the flow is unstable. Most convective activity in the ocean and atmosphere is, in the end, related to an instability of this form.

APPENDIX A: ASYMPTOTIC DERIVATION OF THE BOUSSINESQ EQUATIONS

The Boussinesq equations are those equations that are appropriate when the density variations are very small but gravitational effects are large, and here we provide an asymptotic derivation. Two key results are that the velocity field is divergence-free, and the buoyancy should be taken as a function of z and not p in the equation of state. The first result follows from fact that density variations are presumptively small, and the second follows because the lowest order balance in the vertical momentum equation is $\partial p_0/\partial z = -\rho_0 g$, whence $p_0 = -\rho_0 g z$, and p_0 and not p should be used in the equation of state at lowest order. The following derivation, which assumes familiarity with elementary asymptotics (or which can be taken as a gentle introduction to asymptotics) mainly just formalizes these results.

Let us suppose that the density varies like $\rho(x, y, z, t) = \rho_0 + \delta\rho(x, y, z, t)$, where ρ_0 is a constant and $|\delta\rho| \ll \rho_0$. Specifically, let $\epsilon\rho_0$ be a typical magnitude for $\delta\rho$ where $\epsilon \ll 1$ so that

$$\delta\rho = (\epsilon\rho_0)\delta\widehat{\rho} \qquad \text{and} \qquad \rho = \rho_0(1 + \epsilon\delta\widehat{\rho}) \tag{2.257}$$

where a hat denotes a nondimensional quantity and $\delta\widehat{\rho}$ is an $\mathcal{O}(1)$ quantity.

The dimensional vertical momentum equation, omitting rotation and viscosity for simplicity, is

$$(\rho_0 + \delta\rho)\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial p}{\partial z} - (\rho_0 + \delta\rho)g. \tag{2.258}$$

Now, g is 'big' and variations in density are 'small' and so $g\delta\rho$ is taken to be the same approximate size as the advection term $\rho_0 \mathrm{D}w/\mathrm{D}t$ on the left-hand side. The term $\rho_0 g$ must then be balanced by the pressure gradient. Also, there is no necessary difference between vertical and horizontal scales and velocities. With these points in mind we nondimensionalize with the following scales:

$$(u, v, w) = U(\widehat{u}, \widehat{v}, \widehat{w}), \quad (x, y, z) = L(\widehat{x}, \widehat{y}, \widehat{z}), \quad t = \frac{L}{U}\hat{t}, \quad p = \rho_0 \frac{U^2}{\epsilon}\widehat{p}, \quad g = \frac{U^2}{\epsilon L}\widehat{g}, \tag{2.259}$$

where the hatted quantities are nondimensional and are presumptively $\mathcal{O}(1)$. Equation (2.258) becomes

$$(1 + \epsilon\delta\widehat{\rho})\frac{\mathrm{D}\widehat{w}}{\mathrm{D}\hat{t}} = -\frac{1}{\epsilon}\frac{\partial \widehat{p}}{\partial \widehat{z}} - \frac{1}{\epsilon}(1 + \epsilon\delta\widehat{\rho})\widehat{g}. \tag{2.260}$$

We now take ϵ as an asymptotic ordering parameter and expand the nondimensional fields as series in ϵ. Then, with subscripts denoting the asymptotic order (for nondimensional quantities only), we have

$$\delta\widehat{\rho} = \delta\widehat{\rho}_0 + \epsilon\delta\widehat{\rho}_1 + \epsilon^2\delta\widehat{\rho}_2 \dots, \qquad \widehat{p} = \widehat{p}_0 + \epsilon\widehat{p}_1 + \epsilon^2\widehat{p}_2 \dots, \qquad \widehat{w} = \widehat{w}_0 + \epsilon\widehat{w}_1 + \epsilon^2\widehat{w}_2 \dots, \tag{2.261}$$

and similarly for $\widehat{u}$ and $\widehat{v}$. If we substitute the above series into (2.260) and equate terms with the same power of ϵ, the first two orders are

$$\frac{\partial \widehat{p}_0}{\partial \widehat{z}} = -\widehat{g}, \qquad \frac{\mathrm{D}\widehat{w}_0}{\mathrm{D}\hat{t}} = -\frac{\partial \widehat{p}_1}{\partial \widehat{z}} - \delta\widehat{\rho}_0 g. \tag{2.262a,b}$$

Evidently the leading order pressure, $\widehat{p}_0$, is hydrostatic. We may now revert to dimensional variables and (2.262a) gives $p_0(z) = -\rho_0 g z$, and (2.262b) becomes

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\delta\rho}{\rho_0}g \qquad \text{or} \qquad \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial \phi}{\partial z} + b, \tag{2.263}$$

where p' is the deviation from the hydrostatic pressure, $\phi = p'/\rho_0$ and $b = -g\delta\rho/\rho_0$.

In the horizontal momentum equations only the perturbation pressure p' appears since p_0 is a function of z only. Furthermore, the density must be taken to be the constant ρ_0 (since this is $1/\epsilon$ larger than $\delta\rho$). Then, if $\boldsymbol{u}$ is the horizontal velocity, (u, v), and $\boldsymbol{v}$ is the three-dimensional one, (u, v, w), we obtain

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{u} = -\frac{1}{\rho_0}\nabla_z p' \qquad \text{or} \qquad \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla_z \phi, \tag{2.264}$$

where the gradients (i.e., ∇_z) on the right-hand side are horizontal, at constant z. Equations (2.263) and (2.264) combine to give

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\nabla\phi + b\mathbf{k}, \tag{2.265}$$

where $\mathbf{k}$ is the vertical unit vector and frictional and rotational terms may be added as needed.

The mass continuity equation is

$$\frac{\mathrm{D}}{\mathrm{D}t}(\rho_0 + \delta\rho) + (\rho_0 + \delta\rho)\nabla\cdot\boldsymbol{v} = 0. \tag{2.266}$$

Since $\mathrm{D}\rho_0/\mathrm{D}t = 0$ and ρ_0 is $1/\epsilon$ larger than $\delta\rho$, we see without further ado that the lowest order mass conservation equation is that of an incompressible, divergence-free fluid, namely $\nabla\cdot\boldsymbol{v} = 0$.

Boussinesq thermodynamics

The Boussinesq equations are completed with a thermal equation of state, an equation of evolution for the composition (e.g., S, the salinity), and an equation that provides a thermodynamic state variable Θ (e.g., entropy or internal energy). The thermal equation of state gives the density in terms of pressure, composition and the thermodynamic variable and so is of the form $b = b(p, \Theta, S)$, meaning b is a function of (p, Θ, S). However, to be consistent with the derivation of (2.263) we should use the lowest order variables on the right-hand side of (2.267), and specifically *we should use* p_0 *and not* p *itself*. The equation of state becomes

$$b = b(p_0, \Theta, S) \qquad \text{or} \qquad b = b(z, \Theta, S), \tag{2.267}$$

and an example is (1.155); that equation gives the inverse density, α, in terms of z, potential temperature and salinity, and from which an expression for b immediately follows.

Evolution of the thermodynamic variable is obtained from the first law, and the discussions of Sections 1.6 and 1.7.3 generally apply. The internal energy equation is, with no external source or diffusion,

$$\frac{\mathrm{D}I}{\mathrm{D}t} + p\alpha\nabla\cdot\boldsymbol{v} = 0. \tag{2.268}$$

The lowest order velocities are divergence-free and at lowest order we thus have $\mathrm{D}I/\mathrm{D}t = 0$. (To determine the actual magnitudes of the terms, we can obtain the internal energy from the Gibbs function using $I = g + \eta T - p\alpha = g - T\partial g/\partial T - p\partial g/\partial p$, and with the seawater Gibbs function, (1.146), we obtain $I = c_{p0}T$ + smaller terms. Referring to the values in table 1.2, $c_{p0}\Delta T$ is roughly comparable to $p\alpha$, so that $p\alpha\nabla\cdot\boldsymbol{v}$ is indeed much smaller than $\mathrm{D}I/\mathrm{D}t$.) If we evolved I, we would then need an equation of state of the form $b = b(z, I, S)$ to complete the system.

However, as for the full (non-Boussinesq) system it is often advantageous to evolve potential enthalpy, or potential temperature or entropy, rather than internal energy. We then obtain buoyancy using an accurate equation of state but with the hydrostatic pressure instead of the full pressure. For example, and for idealized or laboratory work with fresh water, a thermodynamic equation and an approximate equation of state might use (1.128) and a simplified form of (1.155) giving

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = 0 \qquad \text{and} \qquad b = g\frac{\delta\alpha}{\alpha_0} = g\left[\frac{gz}{c_s^2} + \beta_T(\theta - \theta_0)\right], \tag{2.269}$$

where θ is potential temperature, θ_0 is a constant reference value and c_s is the speed of sound. The potential temperature is related to the actual temperature by $\theta = T + \beta_T g\theta_0 z/c_p$, but T is not needed to evolve the system. The term in gz/c_s^2 is often small enough to be neglected, in which case the thermodynamic equation becomes an evolution of buoyancy itself, $\mathrm{D}b/\mathrm{D}t = 0$.

Notes

1 The geocentric view was slowly and contentiously being replaced by the Copernican or heliocentric view during Shakespeare's lifetime, with the upheaval of matters thought settled and stable. Galileo, whose telescopes helped confirm the heliocentric view, was born in the same year as Shakespeare, 1564. In the geocentric view Earth's surface is an inertial frame and there is no Coriolis force.

2 The distinction between Coriolis force and acceleration is not always made in the literature, even after noting that the force is considered as a force per unit mass. For a fluid in geostrophic balance, one might either say that there is a balance between the pressure force and the Coriolis force, with no net acceleration, or that the pressure force produces a Coriolis acceleration. The descriptions are equivalent, because of Newton's second law, but should not be conflated.

The Coriolis effect is named after Gaspard Gustave de Coriolis (1792–1843), who discussed the eponymous force in the context of rotating mechanical systems (Coriolis 1832, 1835), but Euler was aware of the effect almost a century before. Persson (1998) provides a historical account.

3 Phillips (1973). A related discussion can be found in Stommel & Moore (1989).

4 Phillips (1966) and White (2002, 2003) form a pleasing set of review articles that synthesize the various forms and approximations of the equations of motion. In the early days of numerical modelling the primitive equations were indeed the most primitive — i.e., the least filtered — equations that could practically be integrated numerically. Associated with increasing computer power there is a tendency for comprehensive numerical models to use non-hydrostatic equations of motion that do not make the shallow-fluid or traditional approximations, and it is conceivable that the meaning of the word 'primitive' may evolve to accommodate them.

5 It is nevertheless possible to derive dynamically consistent equations for a shallow atmosphere that do not make the traditional approximation (Tort & Dubos 2014, Dellar 2011). See White *et al.* (2005) for a related discussion.

6 The Boussinesq approximation is named for Boussinesq (1903), although similar approximations were used earlier by Oberbeck (1879, 1888). Spiegel & Veronis (1960) give a physically based derivation for an ideal gas, and Mihaljan (1962) and Gray & Giorgini (1976) provide more systematic derivations that include the effects of viscosity and diffusion and discussions of the energetics.

7 Young (2010).

8 Various versions of anelastic and pseudo-incompressible equations exist — see Batchelor (1953a), Ogura & Phillips (1962), Gough (1969), Gilman & Glatzmaier (1981), Lipps & Hemler (1982), and Durran (1989), although not all have potential vorticity and energy conservation laws (Bannon 1995, 1996; Scinocca & Shepherd 1992). The system we derive is most similar to that of Ogura & Phillips (1962) and unpublished notes by J. S. A. Green. The connection between the Boussinesq and anelastic equations is discussed by Lilly (1996) and Ingersoll (2005), and the extension to a complex equation of state and inclusion of moisture is discussed by Pauluis (2008).

9 A numerical model that explicitly includes sound waves must take very small timesteps in order to maintain numerical stability, in particular to satisfy the Courant–Friedrichs–Lewy (CFL) criterion. An alternative is to use implicit timestepping that effectively lets the numerics filter the sound waves. If we make the hydrostatic approximation then all sound waves except those that propagate horizontally are eliminated, and there is little numerical need to make the anelastic approximation.

10 Gill (1982) provides a longer discussion. Not all authors differentiate between N and N_*.

11 The Rossby number is named for C.-G. Rossby (see endnote 2 on page 211), but it was also used by Kibel (1940) and is sometimes called the Kibel or Rossby–Kibel number. The notion of geostrophic balance and so, implicitly, that of a small Rossby number, predates both Rossby and Kibel.

12 After Taylor (1921b) and Proudman (1916). The Taylor–Proudman effect is sometimes called the Taylor–Proudman 'theorem', but it is more usefully thought of as a physical effect, with manifestations even when the conditions for its satisfaction are not precisely met. In fact, Hough (1897) seems to have been aware of the effect well before Taylor and Proudman.

13 This discussion owes much to that in Holton (1992). Inertial motion is discussed by Durran (1993).

Jim Holton (1938–2004) made many contributions to atmospheric dynamics over the course of a distinguished career spent almost entirely at the University of Washington in Seattle. In his early career he elucidated, with Richard Lindzen of MIT, the essential mechanism of the quasi-biennial oscillation, or QBO, and he continued to make important contributions to wave–mean-flow interaction, stratosphere-troposphere interaction and stratospheric dynamics more generally throughout his career. He is also known, both to scientists and students, for his popular textbook *An Introduction to Dynamical Meteorology*.