Chapter 5

Atmospheric Oscillations

Linear Perturbation Theory

If one is interested in producing an accurate forecast of the circulation at some future time, a detailed numerical model based on the primitive equations and including processes such as latent heating, radiative transfer, and boundary layer drag should produce the best results. However, the inherent complexity of such a model generally precludes any simple interpretation of the physical processes that produce the predicted circulation. If we wish to gain physical insight into the fundamental nature of atmospheric motions, it is helpful to employ simplified models in which certain processes are omitted and compare the results with those of more complete models. It is difficult to gain an appreciation for the processes that produce the wave-like character observed in many meteorological disturbances through the study of numerical integrations alone. Therefore, analytical treatment of idealized representations of the atmosphere are valuable and are the main topic of this chapter.

First, we discuss the *perturbation method*, a simple technique that is useful for qualitative analysis of atmospheric waves. We then use this method to examine several types of waves in the atmosphere. In Chapters 6 and 7, the perturbation method is used to develop the quasi-geostrophic equations and to study the development of *synoptic-wave* disturbances, respectively.

5.1 THE PERTURBATION METHOD

In the perturbation method, all field variables are divided into two parts: a *basic state* portion, which is usually assumed to be independent of time and longitude, and a perturbation portion, which is the local deviation of the field from the basic state. Thus, for example, if $\overline{u}$ designates a time and longitude-averaged zonal velocity, and u' is the deviation from that average, then the complete zonal velocity field is $u(x, t) = \overline{u} + u'(x, t)$. In that case, for example, the inertial acceleration $u\partial u/\partial x$ can be written

$$u\frac{\partial u}{\partial x} = \left(\overline{u} + u'\right)\frac{\partial}{\partial x}\left(\overline{u} + u'\right) = \overline{u}\frac{\partial u'}{\partial x} + u'\frac{\partial u'}{\partial x}$$

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The basic assumptions of perturbation theory are that the basic state variables must themselves satisfy the governing equations when the perturbations are set to zero, and the perturbation fields must be small enough so that all terms in the governing equations that involve products of the perturbations can be neglected. Terms that depend linearly on perturbation fields drop by half when the perturbation amplitude is cut in half, but terms that are quadratic in the perturbation field become smaller by a factor of four. Therefore, the nonlinear terms become smaller faster than the linear terms and can be neglected in the small-amplitude limit. If $|u'/\overline{u}| \ll 1$, then

$$\left|\overline{u}\partial u'/\partial x\right| \gg \left|u'\partial u'/\partial x\right|$$

If terms that are products of the perturbation variables are neglected, the nonlinear governing equations are reduced to linear differential equations in the perturbation variables in which the basic state variables are specified coefficients. These equations can then be solved by standard methods to determine the character and structure of the perturbations in terms of the known basic state. For equations with constant coefficients, the solutions are sinusoidal or exponential in character. Solution of perturbation equations then determines such characteristics as the propagation speed, vertical structure, and conditions for growth or decay of the waves. The perturbation technique is especially useful in studying the stability of a given basic state flow with respect to small superposed perturbations. This application is the subject of Chapter 7.

5.2 PROPERTIES OF WAVES

Wave motions are oscillations in field variables (e.g., velocity and pressure) that propagate in space and time. In this chapter we are concerned with linear sinusoidal wave motions. Many of the mechanical properties of such waves are also features of a familiar system, the linear harmonic oscillator. An important property of the harmonic oscillator is that the period, or time required to execute a single oscillation, is independent of the amplitude of the oscillation. For most natural vibratory systems, this condition holds only for oscillations of sufficiently small amplitude. The classical example of such a system is the simple pendulum (Figure 5.1) consisting of a mass M suspended by a massless string of length l, free to perform small oscillations about the equilibrium position $\theta = 0$. The component of the gravity force parallel to the direction of motion is $-Mg\sin\theta$. Thus, the equation of motion for the mass M is

$$Ml\frac{d^2\theta}{dt^2} = -Mg\sin\theta$$

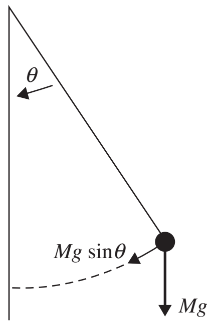

FIGURE 5.1 A simple pendulum.

Now for small displacements, $\sin\theta \approx \theta$ so that the governing equation becomes

$$\frac{d^2\theta}{dt^2} + \nu^2\theta = 0 \tag{5.1}$$

where $\nu^2 \equiv g/l$. The harmonic oscillator equation (5.1) has the general solution

$$\theta = \theta_1 \cos \nu t + \theta_2 \sin \nu t = \theta_0 \cos(\nu t - \alpha)$$

where $\theta_1, \theta_2, \theta_0$, and α are constants determined by the initial conditions (see Problem 5.1) and ν is the frequency of oscillation. The complete solution can thus be expressed in terms of an amplitude θ_0 and a phase $\phi(t) = \nu t - \alpha$. The phase varies linearly in time by a factor of 2π radians per wave period.

Propagating waves can also be characterized by their amplitudes and phases. In a propagating wave, however, phase depends not only on time but on one or more space variables as well. Thus, for a one-dimensional wave propagating in the x direction, $\phi(x, t) = kx - \nu t - \alpha$. Here the *wave number*, k, is defined as 2π divided by the wavelength. For propagating waves the phase is constant for an observer moving at the *phase speed* $c \equiv \nu/k$. This may be verified by observing that if phase is to remain constant following the motion,

$$\frac{D\phi}{Dt} = \frac{D}{Dt}(kx - \nu t - \alpha) = k\frac{Dx}{Dt} - \nu = 0$$

Thus, $Dx/Dt = c = \nu/k$ for phase to be constant. For $\nu > 0$ and $k > 0$ we have $c > 0$. In that case, if $\alpha = 0$, $\phi = k(x - ct)$, so x must increase with increasing t for ϕ to remain constant. Phase then propagates in the positive direction, as illustrated for a sinusoidal wave in Figure 5.2.

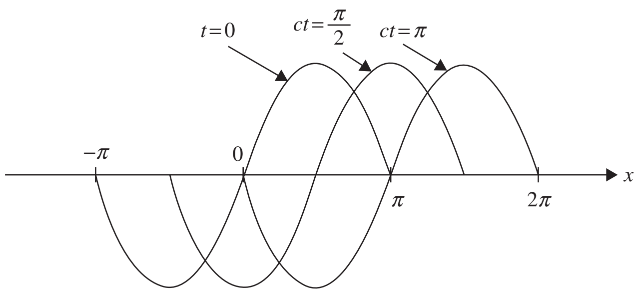

FIGURE 5.2 A sinusoidal wave traveling in the positive x direction at speed c. (Wave number is assumed to be unity.)

5.2.1 Fourier Series

The representation of a perturbation as a simple sinusoidal wave might seem an oversimplification, since disturbances in the atmosphere are never purely sinusoidal. It can be shown, however, that any reasonably well-behaved function of longitude can be represented in terms of a zonal mean plus a *Fourier* series of sinusoidal components:

$$f(x) = \sum_{s=1}^{\infty} (A_s \sin k_s x + B_s \cos k_s x) \tag{5.2}$$

where $k_s = 2\pi s/L$ is the *zonal* wave number (units m^{-1}), L is the distance around a latitude circle, and s, the *planetary* wave number, is an integer designating the number of waves around a latitude circle. The coefficients A_s are calculated by multiplying both sides of (5.2) by $\sin(2\pi nx/L)$, where n is an integer, and integrating around a latitude circle. Applying the orthogonality relationships

$$\int_0^L \sin\frac{2\pi sx}{L} \sin\frac{2\pi nx}{L} dx = \begin{Bmatrix} 0, & s \neq n \\ L/2, & s = n \end{Bmatrix}$$

we obtain

$$A_s = \frac{2}{L}\int_0^L f(x) \sin\frac{2\pi sx}{L} dx$$

In a similar fashion, multiplying both sides in (5.2) by $\cos(2\pi nx/L)$ and integrating gives

$$B_s = \frac{2}{L}\int_0^L f(x) \cos\frac{2\pi sx}{L} dx$$

A_s and B_s are called the *Fourier coefficients*, and

$$f_s\,(x) = A_s \sin k_s x + B_s \cos k_s x \tag{5.3}$$

is called the sth *Fourier component* or sth harmonic of the function $f(x)$. If the Fourier coefficients are computed for, say, the longitudinal dependence of the (observed) geopotential perturbation, the largest-amplitude Fourier components will be those for which s is close to the observed number of troughs or ridges around a latitude circle. When only qualitative information is desired, it is usually sufficient to limit the analysis to a single typical Fourier component and assume that the behavior of the actual field will be similar to that of the component. The expression for a Fourier component may be written more compactly using complex exponential notation. According to the Euler formula,

$$\exp\,(i\phi) = \cos\phi + i \sin\phi$$

where $i \equiv (-1)^{1/2}$ is the imaginary unit. Thus, we can write

$$\begin{aligned} f_s\,(x) &= \mathrm{Re}[C_s \exp\,(ik_s x)] \\ &= \mathrm{Re}[C_s \cos k_s x + iC_s \sin k_s x] \end{aligned} \tag{5.4}$$

where Re[ ] denotes "real part of" and C_s is a complex coefficient. Comparing (5.3) and (5.4), we see that the two representations of $f_s(x)$ are identical, provided that

$$B_s = \mathrm{Re}[C_s] \quad \text{and} \quad A_s = -\mathrm{Im}[C_s]$$

where Im[ ] stands for "imaginary part of." This exponential notation will generally be used for applications of the perturbation theory that follows and also in Chapter 7.

5.2.2 Dispersion and Group Velocity

A fundamental property of linear oscillators is that the frequency of oscillation ν depends only on the physical characteristics of the oscillator, not on the motion itself. For propagating waves, however, ν generally depends on the wave number of the perturbation as well as the physical properties of the medium. Thus, because $c = \nu/k$, the phase speed also depends on the wave number except in the special case where $\nu \propto k$. For waves in which the phase speed varies with k, the various sinusoidal components of a disturbance originating at a given location are at a later time found in different places—that is, they are dispersed. Such waves are referred to as *dispersive*, and the formula that relates ν and k is called a *dispersion relationship*. Some types of waves, such as acoustic waves, have phase speeds that are independent of the wave number. In such *nondispersive* waves a spatially localized disturbance consisting of a number of Fourier wave components (a *wave group*) will preserve its shape as it propagates in space at the phase speed of the wave.

For dispersive waves, however, the shape of a wave group will not remain constant as the group propagates. Because the individual Fourier components of a wave group may either reinforce or cancel each other, depending on the relative phases of the components, the energy of the group will be concentrated in limited regions as illustrated in Figure 5.3. Furthermore, the group generally broadens in the course of time—that is, the energy is *dispersed*.

When waves are dispersive, the speed of the wave group is generally different from the average phase speed of the individual Fourier components. Hence, as shown in Figure 5.4, individual wave components may move either more rapidly or more slowly than the wave group as the group propagates along. Surface waves in deep water (such as a ship wake) are characterized by dispersion in which individual wave crests move twice as fast as the wave group. In synoptic-scale atmospheric disturbances, however, the group velocity exceeds the phase velocity. The resulting downstream development of new disturbances will be discussed later.

An expression for *group velocity*, which is the velocity at which the observable disturbance (and thus the energy) propagates, can be derived as follows: We consider the superposition of two horizontally propagating waves

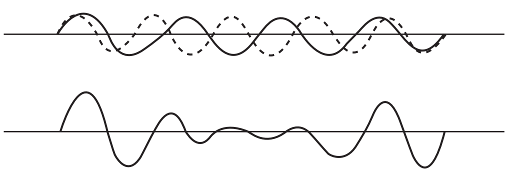

FIGURE 5.3 Wave groups formed from two sinusoidal components of slightly different wavelengths. For nondispersive waves, the pattern in the lower part of the diagram propagates without change of shape. For dispersive waves, the shape of the pattern changes in time.

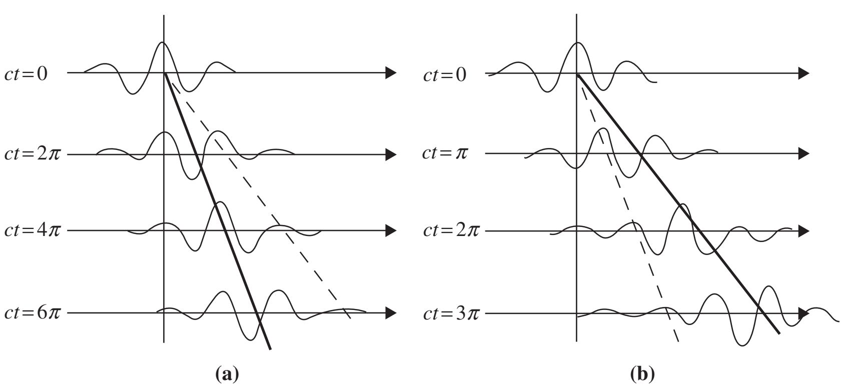

FIGURE 5.4 Schematic showing propagation of wave groups: (a) group velocity less than phase speed and (b) group velocity greater than phase speed. *Heavy lines* show group velocity, and *light lines* show phase speed.

of equal amplitude but slightly different wavelengths, with wave numbers and frequencies differing by $2\delta k$ and $2\delta\nu$, respectively. The total disturbance is thus

$$\Psi(x,t) = \exp\{i[(k+\delta k)x - (\nu+\delta\nu)t]\} + \exp\{i[(k-\delta k)x - (\nu-\delta\nu)t]\}$$

where for brevity the Re[ ] notation in (5.4) is omitted and it is understood that only the real part of the right side has physical meaning. Rearranging terms and applying the Euler formula gives

$$\begin{aligned}\Psi &= \left[e^{i(\delta kx-\delta\nu t)} + e^{-i(\delta kx-\delta\nu t)}\right] e^{i(kx-\nu t)} \\ &= 2\cos(\delta kx - \delta\nu t)\, e^{i(kx-\nu t)}\end{aligned} \tag{5.5}$$

The disturbance (5.5) is the product of a high-frequency *carrier wave* of wavelength $2\pi/k$ whose phase speed, ν/k, is the average for the two Fourier components, and a low-frequency *envelope* of wavelength $2\pi/\delta k$ that travels at the speed $\delta\nu/\delta k$. Thus, in the limit as $\delta k \rightarrow 0$, the horizontal velocity of the envelope, or *group velocity*, is just

$$c_{gx} = \partial\nu/\partial k$$

The wave energy thus propagates at the group velocity. This result applies generally to arbitrary wave envelopes provided that the wavelength of the wave group, $2\pi/\delta k$, is large compared to the wavelength of the dominant component, $2\pi/k$.

5.2.3 Wave Properties in Two and Three Dimensions

Developing a complete understanding of wave properties in two and three dimensions is sufficiently important to the remainder of this chapter and aspects of later chapters that we devote this section to a vector-oriented description of wave properties. We discuss two dimensions for ease of presentation, but the notation is sufficiently general that properties for three dimensions should be clear.

A two-dimensional plane wave in a scalar field, f, may be expressed as

$$f(x,y,t) = \mathrm{Re}\left\{A\, e^{i(kx+ly-\nu t)}\right\} = \mathrm{Re}\left\{A\, e^{i\phi}\right\} \tag{5.6}$$

Independent variables (x, y) and t represent space and time, respectively. k and l are the x and y wave numbers (m^{-1}), and ν is the frequency (s^{-1}). The wave is determined by *amplitude*, A (units of f), and *phase angle*, ϕ. Note that ϕ is a linear function of the independent variables, and it proves useful to consider the space and time properties of ϕ separately.

At any instant in time, $\phi = kx + ly + C$, C constant, so ϕ is uniform along lines of constant $kx + ly$. This implies $d\phi = \frac{\partial\phi}{\partial x}\delta x + \frac{\partial\phi}{\partial y}\delta y = 0$ for ϕ constant.

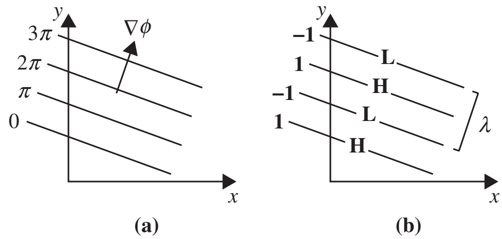

FIGURE 5.5 Two-dimensional plane wave at a fixed time: (a) phase angle, ϕ, and (b) phase, $e^{i\phi}$. Wavelength is denoted λ. Note that if $\nu > 0$, the wave travels in the direction of the wave vector, $\nabla\phi$.

Therefore, the slope of these lines is $\frac{\delta y}{\delta x}\Big|_\phi = -k/l$. Constant values of

$$e^{i\phi} = e^{i(\phi+2\pi n)} \tag{5.7}$$

where n is an integer, define lines of constant *phase*, such as lows and highs (Figure 5.5). The *wave vector* is defined by

$$\mathbf{K} = \nabla\phi \tag{5.8}$$

$\mathcal{K} = |\mathbf{K}|$ is the *total wave number*, and therefore $\lambda = \frac{2\pi}{\mathcal{K}}$ is the *wavelength*—that is, the distance between lines of constant phase.

At any fixed point in space, $\phi = C - \nu t$, C constant, so ϕ is a linear function of time. Therefore, frequency is defined as

$$\nu = -\frac{\partial\phi}{\partial t} \tag{5.9}$$

which is the rate at which lines of constant phase pass a fixed point in space. The wave *period* is $\frac{2\pi}{|\nu|}$, which is the length of time between points of constant phase (units: seconds). The wave *phase speed* (m s^{-1}) is determined by how fast lines of constant phase move along the wave vector

$$c = \frac{\nu}{\mathcal{K}} = -\frac{1}{|\nabla\phi|}\frac{\partial\phi}{\partial t} \tag{5.10}$$

This definition represents a generalization of phase speed from the one-dimensional example given in the previous section. In particular, for two and three dimensions, the phase speed is *not* the same as $c_x = \frac{\nu}{k}$ and $c_y = \frac{\nu}{l}$, which define the rate at which phase lines travel along the x and y coordinate axes, respectively. Furthermore, c, c_x, and c_y do not satisfy rules of vector addition, so $c^2 \neq c_x^2 + c_y^2$.

Note that, in general, ϕ and A may be complex numbers. If ϕ has an imaginary part, $\phi = \phi_r + i\phi_i$, then $e^{i\phi} = e^{i(\phi_r+i\phi_i)} = e^{i\phi_r}e^{-\phi_i} \equiv A^*e^{i\phi_r}$. ϕ_r is the wave phase angle as interpreted in the preceding, and $A^* = Ae^{-\phi_i}$ is a modified amplitude that depends on time and/or space. For example, if the frequency, ν, contributes the imaginary part, then the wave has *time-dependent* amplitude that grows or decays with time. Such waves are referred to as *unstable* to distinguish them from the *neutral* waves with fixed A. In Chapter 7 we seek waves that grow in time, and therefore will study conditions that contribute to imaginary frequency.

We return now to group velocity, which provides the speed and direction in which an observer must travel in order to follow waves of a certain wavelength and frequency. Rather than a pure plane wave, which exists over all space, we take the phase angle, ϕ in (5.6), to be a slowly varying function of space and time due to variable ν and $\mathbf{K}$. Equations (5.8) and (5.9) define the wave vector and frequency respectively, even when these quantities vary in space and time. Together, (5.8) and (5.9) imply

$$\frac{\partial \mathbf{K}}{\partial t} + \nabla \nu = 0 \tag{5.11}$$

Frequency is defined by a dispersion relationship, so that ν is a function of $\mathbf{K}$. As such, $\nabla \nu$ may be evaluated using the chain rule

$$\nabla \nu = (\nabla_k \nu \cdot \nabla)\mathbf{K} \tag{5.12}$$

where $\nabla_k \nu = \left(\frac{\partial \nu}{\partial k}, \frac{\partial \nu}{\partial l}\right)$ is the group velocity, $\mathbf{C}_g$. Therefore, (5.11) can be written as

$$\frac{\partial \mathbf{K}}{\partial t} + (\mathbf{C}_g \cdot \nabla)\mathbf{K} = 0 \tag{5.13}$$

Consequently, in a frame of reference moving with the group velocity, the wave vector is conserved; that is, we follow a group of waves with fixed wavelength and frequency.

5.2.4 A Wave Solution Strategy

Approximations to the full equations governing atmospheric dynamics will be solved for wave motions many times. Even though aspects of each individual case are different, a guide to the general approach to solving these problems is as follows:

1. **Choose a basic state** The basic state is a simplified representation of the atmosphere. Typically we wish to eliminate all complications except those essential to the motions of interest. For example, in this chapter we will often make use of basic states that are at rest and in hydrostatic balance. The basic state is like a musical instrument, and we wish to know what tones naturally emerge.
2. **Linearize the governing equations** Given a basic state, the governing equations can be linearized using the perturbation method of Section 5.1

3. **Assume wave solutions of the form in equation (5.6)** If the coefficients of the linearized equations do not depend on an independent variable, then the functions that depend on the independent variable may be represented as waves, $e^{i\phi}$, as in equation (5.6), provided that the independent variable is periodic. If, for example, the domain of interest is not periodic in the y direction, or the basic state varies in y, then we cannot assume wave solutions in that direction. Instead, a general functional form must be assumed in that direction, such as $F(y)e^{i(kx-\nu t)}$, which typically leads to an ordinary differential equation that must be solved for the structure function $F(y)$.
4. **Solve for the dispersion and polarization relationships** Substituting the wave form into the linearized governing equations typically results in a simpler closed set of algebraic equations or, in cases having nonconstant coefficients, with a set of ordinary differential equations. Normally the wave number is taken as the independent variable, and the frequency is determined as a function of the wave number; this gives the dispersion relationship. The wave amplitude is unconstrained, but the relationships between variables, the polarization relations, must be determined. Variables are *in phase* if they share the same form and sign (e.g., sin(x)) and *out of phase* if they have the same form and opposite sign (e.g., sin(x) and −sin(x)). If the variables are 90° out of phase, they are *in quadrature* (e.g., sin(x) and cos(x)).

5.3 SIMPLE WAVE TYPES

Waves in fluids result from the action of restoring forces on fluid parcels that have been displaced from their equilibrium positions. The restoring forces may be due to compressibility, gravity, rotation, or electromagnetic effects. This section considers the two simplest examples of linear waves in fluids: acoustic waves and shallow water gravity waves.

5.3.1 Acoustic or Sound Waves

Sound waves, or acoustic waves, are *longitudinal waves*. That is, they are waves in which the particle oscillations are parallel to the direction of propagation. Sound is propagated by the alternating adiabatic compression and expansion of the medium. As an example, Figure 5.6 shows a schematic section along a tube that has a diaphragm at its left end. If the diaphragm is set into vibration, the air adjacent to it will be alternately compressed and expanded as the diaphragm moves inward and outward. The resulting oscillating pressure gradient force will be balanced by an oscillating acceleration of the air in the adjoining region, which will cause an oscillating pressure oscillation further into the tube, and so on. The result of this continual adiabatic increase and decrease of pressure through alternating compression and rarefaction is, as shown in Figure 5.6, a sinusoidal pattern of pressure and velocity perturbations that propagates to the right down the tube. Individual air parcels do not, however, have a net rightward

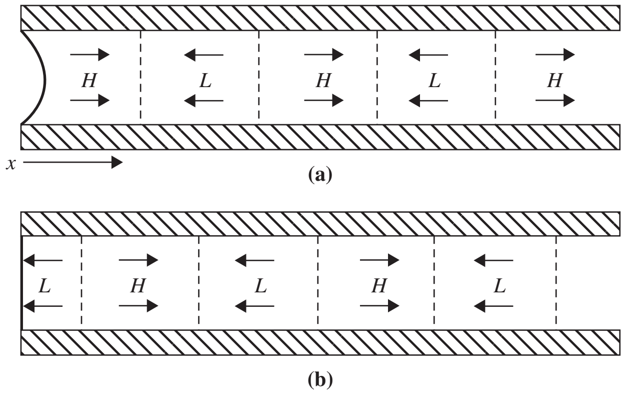

FIGURE 5.6 Schematic diagram illustrating the propagation of a sound wave in a tube with a flexible diaphragm at the left end. Labels H and L designate centers of high- and low-perturbation pressure. *Arrows* show velocity perturbations. The situation one-quarter period (b) later than in (a) for propagation in the positive x direction.

motion; they only oscillate back and forth, while the pressure pattern moves rightward at the speed of sound.

To introduce the perturbation method, we consider the problem illustrated by Figure 5.6—that is, one-dimensional sound waves propagating in a straight pipe parallel to the x axis. To exclude the possibility of *transverse* oscillations (i.e., oscillations in which the particle motion is at right angles to the direction of phase propagation), we assume at the outset that $v = w = 0$. In addition, we eliminate all dependence on y and z by assuming that $u = u(x,t)$. With these restrictions, the momentum equation, continuity equation, and thermodynamic energy equation for adiabatic motion are, respectively,

$$\frac{Du}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \tag{5.14}$$

$$\frac{D\rho}{Dt} + \rho\frac{\partial u}{\partial x} = 0 \tag{5.15}$$

$$\frac{D\ln\theta}{Dt} = 0 \tag{5.16}$$

where for this case $D/Dt = \partial/\partial t + u\partial/\partial x$. Recalling from (2.44) and the ideal gas law that potential temperature may be expressed as

$$\theta = (p/\rho R)\,(p_s/p)^{R/c_p}$$

where $p_s = 1000$ hPa, we may eliminate θ in (5.16) to give

$$\frac{1}{\gamma}\frac{D\ln p}{Dt} - \frac{D\ln\rho}{Dt} = 0 \tag{5.17}$$

where $\gamma = c_p/c_v$. Eliminating ρ between (5.15) and (5.17) gives

$$\frac{1}{\gamma}\frac{D\ln p}{Dt} + \frac{\partial u}{\partial x} = 0 \tag{5.18}$$

The dependent variables are now divided into constant basic state portions (denoted by overbars) and perturbation portions (denoted by primes):

$$\begin{aligned} u(x,t) &= \overline{u} + u'(x,t) \\ p(x,t) &= \overline{p} + p'(x,t) \\ \rho(x,t) &= \overline{\rho} + \rho'(x,t) \end{aligned} \tag{5.19}$$

Substituting (5.19) into (5.14) and (5.18) we obtain

$$\frac{\partial}{\partial t}\left(\overline{u}+u'\right) + \left(\overline{u}+u'\right)\frac{\partial}{\partial x}\left(\overline{u}+u'\right) + \frac{1}{(\overline{\rho}+\rho')}\frac{\partial}{\partial x}\left(\overline{p}+p'\right) = 0$$

$$\frac{\partial}{\partial t}\left(\overline{p}+p'\right) + \left(\overline{u}+u'\right)\frac{\partial}{\partial x}\left(\overline{p}+p'\right) + \gamma\left(\overline{p}+p'\right)\frac{\partial}{\partial x}\left(\overline{u}+u'\right) = 0$$

We next observe that provided $\left|\rho'/\overline{\rho}\right| \ll 1$, we can use the binomial expansion to approximate the density term as

$$\frac{1}{(\overline{\rho}+\rho')} = \frac{1}{\overline{\rho}}\left(1+\frac{\rho'}{\overline{\rho}}\right)^{-1} \approx \frac{1}{\overline{\rho}}\left(1-\frac{\rho'}{\overline{\rho}}\right)$$

Neglecting products of the perturbation quantities and noting that the basic state fields are constants, we obtain the linear perturbation equations[1]

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)u' + \frac{1}{\overline{\rho}}\frac{\partial p'}{\partial x} = 0 \tag{5.20}$$

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)p' + \gamma\overline{p}\frac{\partial u'}{\partial x} = 0 \tag{5.21}$$

Eliminating u' by operating on (5.21) with $(\partial/\partial t + \overline{u}\partial/\partial x)$ and substituting from (5.20), we get[2]

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)^2 p' - \frac{\gamma\overline{p}}{\overline{\rho}}\frac{\partial^2 p'}{\partial x^2} = 0 \tag{5.22}$$

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[1] It is not necessary that the perturbation velocity be small compared to the mean velocity for linearization to be valid. It is only required that quadratic terms in the perturbation variables be small compared to the dominant linear terms in (5.20) and (5.21).

[2] Note that the squared differential operator in the first term expands in the usual way as

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)^2 = \frac{\partial^2}{\partial t^2} + 2\overline{u}\frac{\partial^2}{\partial t\partial x} + \overline{u}^2\frac{\partial^2}{\partial x^2}$$

which is a form of the standard *wave equation* familiar from electromagnetic theory. A simple solution representing a plane sinusoidal wave propagating in x is

$$p' = A \exp\left[ik\left(x - ct\right)\right] \tag{5.23}$$

where for brevity we omit the Re{ } notation, but it is to be understood that only the real part of (5.23) has physical significance. Substituting the assumed solution (5.23) into (5.22), we find that the phase speed c must satisfy

$$\left(-ikc + ik\overline{u}\right)^2 - \left(\gamma\overline{p}/\overline{\rho}\right)\left(ik\right)^2 = 0$$

where we have canceled out the factor $A \exp\left[ik\left(x - ct\right)\right]$, which is common to both terms. Solving for c gives

$$c = \overline{u} \pm \left(\gamma\overline{p}/\overline{\rho}\right)^{1/2} = \overline{u} \pm \left(\gamma R\overline{T}\right)^{1/2} \tag{5.24}$$

Therefore, (5.23) is a solution of (5.22), provided that the phase speed satisfies (5.24). According to (5.24), the speed of wave propagation relative to the zonal current is $c - \overline{u} = \pm c_s$, where $c_s \equiv \left(\gamma R\overline{T}\right)^{1/2}$ is called the *adiabatic speed of sound*.

The mean zonal velocity here plays only a role of *Doppler shifting* the sound wave so that the frequency relative to the ground corresponding to a given wave number k is

$$\nu = kc = k\left(\overline{u} \pm c_s\right)$$

Thus, in the presence of a wind, the frequency as heard by a fixed observer depends on the location of the observer relative to the source. If $\overline{u} > 0$, the frequency of a stationary source will appear to be higher for an observer to the east (downstream) of the source ($c = \overline{u} + c_s$) than for an observer to the west (upstream) of the source $c = \overline{u} - c_s$.

5.3.2 Shallow Water Waves

The second example of pure wave motion concerns the horizontally propagating oscillations known as shallow water waves. This analysis provides a clear introduction to the primary wave motions in the extratropical latitudes: Rossby waves and gravity waves.

We linearize the shallow water equations (derived in Section 4.5) for a resting basic state having water depth $\overline{h}$ and fixed ambient rotation given by a constant Coriolis parameter f. The linearized momentum (4.33) and mass continuity (4.37) equations are

$$\begin{aligned}\frac{\partial u'}{\partial t} &= -g\frac{\partial h'}{\partial x} + fv' \\ \frac{\partial v'}{\partial t} &= -g\frac{\partial h'}{\partial y} - fu' \\ \frac{\partial h'}{\partial t} &= -\overline{h}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right)\end{aligned} \tag{5.25}$$

Primes denote perturbation values—that is, departures from the state of rest. (5.25) represent a set of three coupled first-order partial differential equations in the unknowns, u', v', h'. One approach to solving these equations is to form and solve a single third-order partial differential equation. This is accomplished by taking $\partial/\partial t$ of the third equation in (5.25), which gives

$$\frac{\partial^2 h'}{\partial t^2} = -\overline{h}\left(\frac{\partial^2 u'}{\partial t \partial x} + \frac{\partial^2 v'}{\partial t \partial y}\right) \tag{5.26}$$

The terms on the right side of (5.26) may be replaced using the first two equations of (5.25):

$$\frac{\partial^2 h'}{\partial t^2} = \overline{h}\left(g\nabla_h^2 h' - f\zeta'\right) \tag{5.27}$$

Again take $\partial/\partial t$, which gives the third-order equation

$$\frac{\partial^3 h'}{\partial t^3} + \left(f^2 - g\overline{h}\nabla_h^2\right)\frac{\partial h'}{\partial t} = 0 \tag{5.28}$$

where $\partial\zeta'/\partial t$ is replaced using the linearized version of (4.38) (noting that f is taken constant here):

$$\frac{\partial \zeta'}{\partial t} = -f\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) \tag{5.29}$$

Assuming that the lateral boundaries are periodic and, since the coefficients f and $g\overline{h}$ are constant, we may assume wave solutions of the form

$$h' = \mathrm{Re}\left\{Ae^{i(kx+ly-\nu t)}\right\} \tag{5.30}$$

Using (5.30) in (5.28) gives a cubic polynomial for the frequency

$$\nu^3 - \nu\left[f^2 + g\overline{h}\left(k^2 + l^2\right)\right] = 0 \tag{5.31}$$

This is the dispersion relationship for shallow water waves. Clearly $\nu = 0$ is a solution, and if $\nu \neq 0$, then

$$\nu^2 = f^2 + g\overline{h}\left(k^2 + l^2\right) \tag{5.32}$$

For readers familiar with linear algebra, we note an alternative solution method. Taking solutions of the form (5.30) for h', u', and v', and substituting directly into (5.25), converts the set of partial differential equations to algebraic equations that may be written as

$$\mathbf{Ax} = 0 \tag{5.33}$$

where $\mathbf{x}$ is the column vector of the unknowns $[u'v'h']^{\mathrm{T}}$, and

$$\mathbf{A} = \begin{bmatrix} -i\nu & -f & ikg \\ f & -i\nu & ilg \\ \overline{h}k & -\overline{h}l & -\nu \end{bmatrix} \tag{5.34}$$

A nontrivial solution to (5.33) is obtained only if $\mathbf{A}$ is not invertible. This is enforced by setting the determinant of $\mathbf{A}$ to zero, which gives (5.31).

Turning now to the structure of the waves, consider first the case where $\nu = 0$. Appealing to (5.25), we see that since the waves are stationary, the left sides are zero. Clearly, these waves depend on rotation, since when $f = 0$, the wave amplitude, A, must also be zero. For non zero f, (5.25) implies

$$\begin{aligned} v' &= \frac{g}{f}\frac{\partial h}{\partial x} \\ u' &= -\frac{g}{f}\frac{\partial h}{\partial y} \end{aligned} \tag{5.35}$$

The stationary waves are in a state of geostrophic balance and are an example of **Rossby waves** in the limit of constant Coriolis parameter. The case of propagating Rossby waves, with nonconstant Coriolis parameter, will be explored in Section 5.7.

Solutions for $\nu \neq 0$ are called **inertia–gravity waves**, since particle oscillations depend on both gravitational and inertial forces. The inertial aspect will be discussed in Section 5.7 so that we may focus here on the gravitational aspect. In the limiting case $f = 0$, we have simply **gravity waves**, and from (5.32) the wave speed becomes

$$c = \sqrt{g\overline{h}} \tag{5.36}$$

Since the waves all move at the same speed, they are nondispersive. For a fluid $\overline{h} = 4$ km deep, the shallow water gravity wave speed is approximately 200 m s^{-1}. Thus, long waves on the ocean surface travel very rapidly. It should be emphasized again that this theory applies only to waves of wavelength

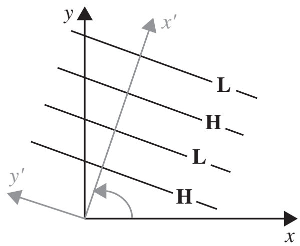

FIGURE 5.7 Coordinate rotation to align the y axis with wave fronts so that $l = 0$.

much greater than $\bar{h}$. Such long waves may be produced by very large-scale disturbances such as earthquakes.[3]

In order to further expose wave structure and the physics of motion, we rotate coordinates such that the y direction is oriented along the phase lines (Figure 5.7). Since this yields the same interpretation as $l = 0$, we omit the prime notation on x and y. In this case, the general expressions for flow normal to and along phase lines, u' and v', respectively, are

$$\begin{aligned} u' &= \frac{\nu k g}{\nu^2 - f^2} h' \\ v' &= \frac{-ikgf}{\nu^2 - f^2} h' \end{aligned} \tag{5.37}$$

From (4.34) and the third equation in (5.25),

$$\frac{\partial w'}{\partial z} = -\frac{\partial u'}{\partial x} = \frac{1}{\bar{h}} \frac{\partial h'}{\partial t} \tag{5.38}$$

Integrating in z gives the w' field

$$w'(z) = \frac{1}{\bar{h}} \frac{\partial h'}{\partial t} \int_0^z dz' = \frac{-ikcz}{\bar{h}} h' \tag{5.39}$$

Taking the real part of w' gives

$$\mathrm{Re}\{w'\} = -\frac{kcz}{\bar{h}} \mathrm{Re}\{ih'\} = \frac{kczA}{\bar{h}} \sin(\phi) \tag{5.40}$$

This shows that w' is in *quadrature* with h', so that w' will lead (lag) h' by 90° if the waves move to the right (left).

Setting $\nu = c = 0$ in (5.37) and (5.39) recovers the geostrophic flow of stationary Rossby waves: $u' = 0$ (no flow normal to phase lines), $v' = -ikg/f$,

[3] Long waves excited by underwater earthquakes or volcanic eruptions are called *tsunamis.*

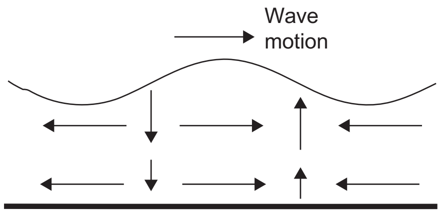

FIGURE 5.8 Depiction of shallow water gravity wave height and velocity fields for the case where the coordinates have been rotated as shown in Figure 5.7.

and $w = 0$. For the inertia gravity waves ($\nu \neq 0$), notice that the flow along phase lines is zero in the absence of ambient rotation. In that case, the physics of wave propagation is particularly clear. From (5.32) we see that $\nu^2 - f^2$ is strictly positive, and in the rotated coordinates, $u' = c/\bar{h}$. This shows that the u' field is either *in phase*, or *out of phase*, with the h' field depending on the sign of c. Waves that move to the right ($c > 0$) are shown in Figure 5.8. Convergence and divergence of the u field raise and lower the height of the fluid, respectively, moving the wave to the right in Figure 5.8. As expected from mass conservation, the w field exhibits rising motion near convergence, which increases linearly with height from zero at the bottom boundary. Changing the sign of c has no effect on h', but reverses the u and w fields so that the wave moves to the left.

Finally, consider the vorticity and potential vorticity (PV) of the waves in the rotated coordinates. Since there is no variation in the y direction, the relative vorticity is simply

$$\zeta' = \frac{\partial v'}{\partial x} = \frac{k^2 g f}{\nu^2 - f^2}\, h' \tag{5.41}$$

A linearized version of the shallow water potential vorticity equation (4.39) is obtained from (5.29) and the third equation in (5.25), which yields

$$\frac{\partial}{\partial t}\left[\zeta' - \frac{f}{\bar{h}}\, h'\right] = 0 \tag{5.42}$$

Linearization has removed all advection terms from the PV conservation equation, so that (5.42) states that the linearized PV

$$Q' = \zeta' - \frac{f}{\bar{h}} h' \tag{5.43}$$

is conserved *locally*, at every point in space. This is an extremely powerful constraint on the linear dynamics, which we will exploit in Section 5.6 when solving initial-vale problems for (5.25).

For shallow water Rossby waves, vorticity in the rotated coordinates becomes

$$\zeta'_{RW} = -\frac{k^2 g}{f}\, h' \tag{5.44}$$

which is out of phase with the height field and larger for shorter waves. Rossby wave PV is given by ζ reduced by $\frac{f}{\bar{h}}h'$, which has the same sign as ζ except for unrealistically long waves.

For shallow water inertia–gravity waves, vorticity in the rotated coordinates becomes

$$\zeta'_{GW} = \frac{f}{\bar{h}} h' \tag{5.45}$$

which is in phase with the height field, and nonzero provided there exists ambient rotation. From (5.45) and (5.43), it is clear that inertia–gravity wave PV is identically zero.

A summary of the essential differences between Rossby and inertia–gravity waves for rotating shallow water follows:

- Rossby waves are stationary for f constant, whereas inertia–gravity waves move very fast.
- Inertia–gravity waves have small relative vorticity but are strongly divergent, whereas Rossby waves have geostrophic winds with zero divergence and large relative vorticity.
- Inertia–gravity waves have zero potential vorticity, whereas Rossby waves generally have nonzero potential vorticity.

These properties generalize to waves in a stratified atmosphere, considered in the next two sections. They provide a useful framework for eliminating inertia–gravity waves from the full nonlinear equations, yielding the quasi-geostrophic equations, which we use to understand the dynamics of extratropical weather systems in Chapters 6 and 7.

5.4 INTERNAL GRAVITY (BUOYANCY) WAVES

The nature of gravity wave propagation in the stratified atmosphere is considered now in the absence of ambient rotation. Atmospheric gravity waves can only exist when the atmosphere is stably stratified so that a fluid parcel displaced vertically will undergo buoyancy oscillations (see Section 2.7.3). Because the buoyancy force is the restoring force responsible for gravity waves, the term *buoyancy wave* is actually more appropriate as a name for these waves. However, in this text we will generally use the traditional name *gravity wave*.

In a fluid, such as the ocean, which is bounded both above and below, gravity waves propagate primarily in the horizontal plane, since vertically traveling waves are reflected from the boundaries to form standing waves. However, in a fluid that has no upper boundary, such as the atmosphere, gravity waves may propagate vertically as well as horizontally. In vertically propagating waves the phase is a function of height. Such waves are referred to as *internal* waves. Although internal gravity waves are not generally of great importance for synoptic-scale weather forecasting (and indeed are nonexistent in the filtered quasi-geostrophic models), they can be important in mesoscale motions. For example, they are responsible for the occurrence of mountain *lee waves*. They also are believed to be an important mechanism for transporting energy and momentum into the middle atmosphere, and are often associated with the formation of clear air turbulence (CAT).

5.4.1 Pure Internal Gravity Waves

For simplicity we neglect the Coriolis force and limit our discussion to two-dimensional internal gravity waves propagating in the x, z plane. An expression for the frequency of such waves can be obtained by modifying the parcel theory developed in Section 2.7.3.

Internal gravity waves are transverse waves in which the parcel oscillations are parallel to the phase lines as indicated in Figure 5.9. A parcel displaced a distance δs along a line tilted at an angle α to the vertical as shown in Figure 5.8 will undergo a vertical displacement $\delta z = \delta s \cos\alpha$. For such a parcel the *vertical* buoyancy force per unit mass is just $-N^2\delta z$, as was shown in (2.52). Thus, the component of the buoyancy force parallel to the tilted path along which the parcel oscillates is just

$$-N^2\delta z\cos\alpha = -N^2\left(\delta s\cos\alpha\right)\cos\alpha = -\left(N\cos\alpha\right)^2\delta s$$

The momentum equation for the parcel oscillation is then

$$\frac{d^2\left(\delta s\right)}{dt^2} = -\left(N\cos\alpha\right)^2\delta s \tag{5.46}$$

which has the general solution $\delta s = \exp\left[\pm i\left(N\cos\alpha\right)t\right]$. Thus, the parcels execute a simple harmonic oscillation at the frequency $\nu = N\cos\alpha$. This frequency depends only on the static stability (measured by the buoyancy frequency N) and the angle of the phase lines to the vertical.

The preceding heuristic derivation can be verified by considering the linearized equations for two-dimensional internal gravity waves. For simplicity, we

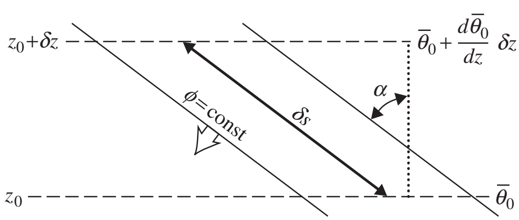

FIGURE 5.9 Parcel oscillation path (*heavy arrow*) for pure gravity waves with phase lines tilted at an angle α to the vertical.

employ the Boussinesq approximation of Section 2.8, in which density is treated as a constant except where it is coupled with gravity in the buoyancy term of the vertical momentum equation. Thus, in this approximation the atmosphere is considered to be incompressible, and local density variations are assumed to be small perturbations of the constant basic state density field. Because the vertical variation of the basic state density is neglected except where coupled with gravity, the Boussinesq approximation is only valid for motions in which the vertical scale is less than the atmospheric scale height $H(\approx 8\,\mathrm{km})$.

Neglecting the effects of rotation, the basic equations for two-dimensional motion of an incompressible atmosphere may be written as

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0 \tag{5.47}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}+\frac{1}{\rho}\frac{\partial p}{\partial z}+g=0 \tag{5.48}$$

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0 \tag{5.49}$$

$$\frac{\partial \theta}{\partial t}+u\frac{\partial \theta}{\partial x}+w\frac{\partial \theta}{\partial z}=0 \tag{5.50}$$

where the potential temperature θ is related to pressure and density by

$$\theta=\frac{p}{\rho R}\left(\frac{p_s}{p}\right)^{\kappa}$$

which, after taking logarithms on both sides, yields

$$\ln\theta=\gamma^{-1}\ln p-\ln\rho+\text{constant} \tag{5.51}$$

We now linearize (5.47) through (5.51) by letting

$$\begin{aligned} \rho &= \rho_0+\rho' & u &= \overline{u}+u' \\ p &= \overline{p}(z)+p' & w &= w' \\ \theta &= \overline{\theta}(z)+\theta' \end{aligned} \tag{5.52}$$

where the basic state zonal flow $\overline{u}$ and the density ρ_0 are both assumed to be constant. The basic state pressure field must satisfy the hydrostatic equation

$$d\overline{p}/dz=-\rho_0 g \tag{5.53}$$

while the basic state potential temperature must satisfy (5.51) so that

$$\ln\overline{\theta}=\gamma^{-1}\ln\overline{p}-\ln\rho_0+\text{constant} \tag{5.54}$$

The linearized equations are obtained by substituting from (5.52) into (5.47) through (5.51) and neglecting all terms that are products of the perturbation variables. Thus, for example, the last two terms in (5.48) are approximated as

$$\begin{aligned}\frac{1}{\rho}\frac{\partial p}{\partial z}+g &= \frac{1}{\rho_0+\rho'}\left(\frac{d\overline{p}}{dz}+\frac{\partial p'}{\partial z}\right)+g \\ &\approx \frac{1}{\rho_0}\frac{d\overline{p}}{dz}\left(1-\frac{\rho'}{\rho_0}\right)+\frac{1}{\rho_0}\frac{\partial p'}{\partial z}+g=\frac{1}{\rho_0}\frac{\partial p'}{\partial z}+\frac{\rho'}{\rho_0}g\end{aligned} \tag{5.55}$$

where (5.53) has been used to eliminate $\overline{p}$. The perturbation form of (5.51) is obtained by noting that

$$\begin{aligned}\ln\left[\overline{\theta}\left(1+\frac{\theta'}{\overline{\theta}}\right)\right] &= \gamma^{-1}\ln\left[\overline{p}\left(1+\frac{p'}{\overline{p}}\right)\right] \\ &\quad -\ln\left[\rho_0\left(1+\frac{\rho'}{\rho_0}\right)\right]+\text{const.}\end{aligned} \tag{5.56}$$

Now, recalling that $\ln(ab)=\ln(a)+\ln(b)$ and that $\ln(1+\varepsilon)\approx\varepsilon$ for any $\varepsilon\ll 1$, we find with the aid of (5.54) that (5.56) may be approximated by

$$\frac{\theta'}{\overline{\theta}}\approx\frac{1}{\gamma}\frac{p'}{\overline{p}}-\frac{\rho'}{\rho_0}$$

Solving for ρ' yields

$$\rho'\approx-\rho_0\frac{\theta'}{\overline{\theta}}+\frac{p'}{c_s^2} \tag{5.57}$$

where $c_s^2\equiv\overline{p}\gamma/\rho_0$ is the square of the speed of sound. For buoyancy wave motions $\left|\rho_0\theta'/\overline{\theta}\right|\gg\left|p'/c_s^2\right|$; that is, density fluctuations due to pressure changes are small compared with those due to temperature changes. Therefore, to a first approximation,

$$\theta'/\overline{\theta}=-\rho'/\rho_0 \tag{5.58}$$

Using (5.55) and (5.58), the linearized version of the equation set (5.47) through (5.50), we can write as

$$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)u'+\frac{1}{\rho_0}\frac{\partial p'}{\partial x}=0 \tag{5.59}$$

$$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)w'+\frac{1}{\rho_0}\frac{\partial p'}{\partial z}-\frac{\theta'}{\overline{\theta}}g=0 \tag{5.60}$$

$$\frac{\partial u'}{\partial x}+\frac{\partial w'}{\partial z}=0 \tag{5.61}$$

$$\left(\frac{\partial}{\partial t}+\overline{u}\frac{\partial}{\partial x}\right)\theta'+w'\frac{d\overline{\theta}}{dz}=0 \tag{5.62}$$

Subtracting $\partial(5.59)/\partial z$ from $\partial(5.60)/\partial x$, we can eliminate p' to obtain

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\left(\frac{\partial w'}{\partial x}-\frac{\partial u'}{\partial z}\right)-\frac{g}{\bar{\theta}}\frac{\partial \theta'}{\partial x}=0 \tag{5.63}$$

which is just the y component of the vorticity equation. With the aid of (5.61) and (5.62), u' and θ' can be eliminated from (5.63) to yield a single equation for w':

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)^2\left(\frac{\partial^2 w'}{\partial x^2}+\frac{\partial^2 w'}{\partial z^2}\right)+N^2\frac{\partial^2 w'}{\partial x^2}=0 \tag{5.64}$$

where $N^2 \equiv g\,d\ln\bar{\theta}/dz$ is the square of the buoyancy frequency, which is assumed to be constant.[4]

Equation (5.64) has harmonic wave solutions of the form

$$w'=\mathrm{Re}\left[\hat{w}\exp(i\phi)\right]=w_r\cos\phi-w_i\sin\phi \tag{5.65}$$

where $\hat{w}=w_r+iw_i$ is a complex amplitude with real part w_r and imaginary part w_i, and $\phi=kx+mz-\nu t$ is the phase, which is assumed to depend linearly on z as well as on x and t. Here the horizontal wave number k is real because the solution is always sinusoidal in x. The vertical wave number $m=m_r+im_i$ may, however, be complex, in which case m_r describes sinusoidal variation in z and m_i describes exponential decay or growth in z, depending on whether m_i is positive or negative. When m is real, the total wave number may be regarded as a vector $\boldsymbol{\kappa}\equiv(k,m)$, directed perpendicular to lines of constant phase, and in the direction of phase increase, whose components, $k=2\pi/L_x$ and $m=2\pi/L_z$, are inversely proportional to the horizontal and vertical wavelengths, respectively. Substitution of the assumed solution into (5.64) yields the dispersion relationship

$$(\nu-\bar{u}k)^2\left(k^2+m^2\right)-N^2k^2=0$$

so that

$$\hat{\nu}\equiv\nu-\bar{u}k=\pm Nk\Big/\left(k^2+m^2\right)^{1/2}=\pm Nk/\left|\boldsymbol{\kappa}\right| \tag{5.66}$$

where $\hat{\nu}$, the *intrinsic frequency*, is the frequency relative to the mean wind. Here, the plus sign is to be taken for eastward phase propagation and the minus sign for westward phase propagation, relative to the mean wind.

[4]Strictly speaking, N^2 cannot be exactly constant if ρ_0 is constant. However, for shallow disturbances the variation of N^2 with height is unimportant.

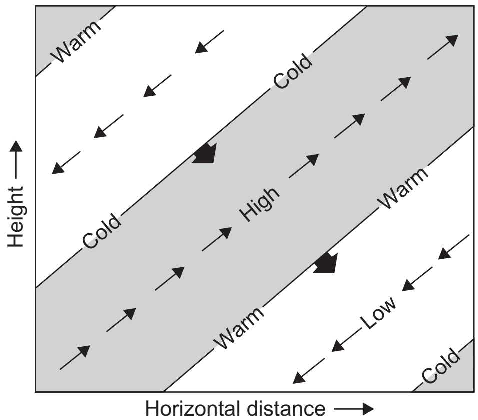

FIGURE 5.10 Idealized cross-section showing phases of pressure, temperature, and velocity perturbations for an internal gravity wave. *Thin arrows* indicate the perturbation velocity field; *blunt solid arrows*, the phase velocity. *Shading* shows regions of upward motion.

If we let $k > 0$ and $m < 0$, then lines of constant phase tilt eastward with increasing height as shown in Figure 5.10 (i.e., for $\phi = kx + mz$ to remain constant as x increases, z must also increase when $k > 0$ and $m < 0$). The choice of the positive root in (5.66) then corresponds to eastward and downward phase propagation relative to the mean flow, with horizontal and vertical phase speeds (relative to the mean flow) given by $c_x = \hat{\nu}/k$ and $c_z = \hat{\nu}/m$, respectively.[5] The components of the group velocity, $c_{gx}$and c_{gz}, however, are given by

$$c_{gx} = \frac{\partial \nu}{\partial k} = \overline{u} \pm \frac{Nm^2}{\left(k^2 + m^2\right)^{3/2}} \tag{5.67}$$

$$c_{gz} = \frac{\partial \nu}{\partial m} = \pm \frac{(-Nkm)}{\left(k^2 + m^2\right)^{3/2}} \tag{5.68}$$

where the upper or lower signs are chosen in the same way as in (5.66). Thus, the vertical component of group velocity has a sign opposite to that of the vertical phase speed relative to the mean flow (downward phase propagation implies upward energy propagation). Furthermore, it is easily shown from (5.45) that the group velocity vector is parallel to lines of constant phase. Internal gravity waves thus have the remarkable property that group velocity is perpendicular

[5]Note that phase speed is not a vector. The phase speed in the direction perpendicular to constant phase lines (i.e., the blunt arrows in Figure 5.10) is given by $\nu/(k^2 + m^2)^{1/2}$, which is not equal to $\left(c_x^2 + c_z^2\right)^{1/2}$.

to the direction of phase propagation. Because energy propagates at the group velocity, this implies that energy propagates parallel to the wave crests and troughs, rather than perpendicular to them as in acoustic waves or shallow water gravity waves. In the atmosphere, internal gravity waves generated in the troposphere by cumulus convection, by flow over topography, and by other processes may propagate upward many scale heights into the middle atmosphere, even though individual fluid parcel oscillations may be confined to vertical distances much less than a kilometer.

Referring again to Figure 5.10, it is evident that the angle of the phase lines to the local vertical is given by

$$\cos\alpha = L_z/(L_x^2 + L_z^2)^{1/2} = \pm k/(k^2 + m^2)^{1/2} = \pm k/\left|\boldsymbol{\kappa}\right|$$

Thus, $\hat{\nu} = \pm N\cos\alpha$ (i.e., gravity wave frequencies must be less than the buoyancy frequency) in agreement with the heuristic parcel oscillation model (5.46). The tilt of phase lines for internal gravity waves depends only on the ratio of the intrinsic wave frequency to the buoyancy frequency and is independent of wavelength.

5.5 LINEAR WAVES OF A ROTATING STRATIFIED ATMOSPHERE

Rotation is now considered by extending the analysis of linear waves in a stratified atmosphere. As in the shallow water analysis of Section 5.3.2, rotation permits the existence of stationary Rossby waves. Gravity waves with horizontal scales greater than a few hundred kilometers and periods greater than a few hours are hydrostatic, but they are influenced by the Coriolis effect and are characterized by parcel oscillations that are elliptical rather than straight lines as in the pure gravity wave case. This *elliptical polarization* can be understood qualitatively by observing that the Coriolis effect resists horizontal parcel displacements in a rotating fluid, but in a manner somewhat different from that in which the buoyancy force resists vertical parcel displacements in a statically stable atmosphere. In the latter case the resistive force is opposite to the direction of parcel displacement, whereas in the former it is at right angles to the horizontal parcel velocity.

5.5.1 Pure Inertial Oscillations

Section 3.2.3 showed that a parcel put into horizontal motion in a resting atmosphere with a constant Coriolis parameter executes a circular trajectory in an anticyclonic sense. A generalization of this type of inertial motion to the case with a geostrophic mean zonal flow can be derived using a parcel argument similar to that used for the buoyancy oscillation in Section 2.7.3.

If the basic state flow is assumed to be a zonally directed geostrophic wind u_g, and it is assumed that the parcel displacement does not perturb the pressure field, the approximate equations of motion become

$$\frac{Du}{Dt} = fv = f\frac{Dy}{Dt} \tag{5.69}$$

$$\frac{Dv}{Dt} = f\left(u_g - u\right) \tag{5.70}$$

We consider a parcel that is moving with the geostrophic basic state motion at a position $y = y_0$. If the parcel is displaced across stream by a distance δy, we can obtain its new zonal velocity from the integrated form of (5.49):

$$u\left(y_0 + \delta y\right) = u_g\left(y_0\right) + f\delta y \tag{5.71}$$

The geostrophic wind at $y_0 + \delta y$ can be approximated as

$$u_g\left(y_0 + \delta y\right) = u_g\left(y_0\right) + \frac{\partial u_g}{\partial y}\delta y \tag{5.72}$$

Using (5.71) and (5.72) to evaluate (5.50) at $y_0 + \delta y$ yields

$$\frac{Dv}{Dt} = \frac{D^2\delta y}{Dt^2} = -f\left(f - \frac{\partial u_g}{\partial y}\right)\delta y = -f\frac{\partial M}{\partial y}\delta y \tag{5.73}$$

where we have defined the *absolute momentum*, $M \equiv fy - u_g$.

This equation is mathematically of the same form as (2.52), the equation for the motion of a vertically displaced particle in a stratified atmosphere. Depending on the sign of the coefficient on the right side in (5.73), the parcel will either be forced to return to its original position or will accelerate further from that position. This coefficient thus determines the condition for *inertial instability*:

$$f\frac{\partial M}{\partial y} = f\left(f - \frac{\partial u_g}{\partial y}\right)\begin{cases} > 0 & \text{stable} \\ = 0 & \text{neutral} \\ < 0 & \text{unstable}\end{cases} \tag{5.74}$$

Viewed in an inertial reference frame, instability results from an imbalance between the pressure gradient and inertial forces for a parcel displaced radially in an axisymmetric vortex. In the Northern Hemisphere, where f is positive, the flow is inertially stable provided that the absolute vorticity of the basic flow, $\partial M/\partial y$, is positive. In the Southern Hemisphere, however, inertial stability requires that the absolute vorticity be negative. Observations show that for extratropical synoptic-scale systems the flow is always inertially stable, although near neutrality often occurs on the anticyclonic shear side of upper-level jet streaks. The occurrence of inertial instability over a large area would immediately trigger inertially unstable motions, which would mix the fluid laterally just as convection mixes it vertically, and reduce the shear until the absolute vorticity times f was again positive. (This explains why anticyclonic shears cannot become arbitrarily large.)

5.5.2 Rossby and Inertia–Gravity Waves

When the flow is both inertially and gravitationally stable, parcel displacements are resisted by both rotation and buoyancy. The resulting oscillations are called *inertia–gravity waves*. The dispersion relation for such waves can be analyzed using a variant of the parcel method applied in Section 5.4 We consider parcel oscillations along a slantwise path in the (y, z) plane as shown in Figure 5.11. For a vertical displacement δz, the buoyancy force component parallel to the slope of the parcel oscillation is $-N^2 \delta z \cos\alpha$, and for a meridional displacement δy the Coriolis (inertial) force component parallel to the slope of the parcel path is $-f^2\, \delta y \sin\alpha$, where we have assumed that the geostrophic basic flow is constant in latitude. Thus, the harmonic oscillator equation for the parcel (5.46) is modified to the form

$$\frac{D^2 \delta s}{Dt^2} = -(f \sin\alpha)^2\, \delta s - (N \cos\alpha)^2\, \delta s \tag{5.75}$$

where δs is again the perturbation parcel displacement.

The frequency now satisfies the dispersion relationship

$$\nu^2 = N^2 \cos^2\alpha + f^2 \sin^2\alpha \tag{5.76}$$

Since in general $N^2 > f^2$, (5.76) indicates that inertia–gravity wave frequencies must lie in the range $f \leq |\nu| \leq N$. The frequency approaches N as the trajectory slope approaches the vertical, and approaches f as the trajectory slope approaches the horizontal. For typical midlatitude tropospheric conditions, inertia–gravity wave periods are in the approximate range of 12 min to 15 h. Rotational effects become important, however, only when the second term

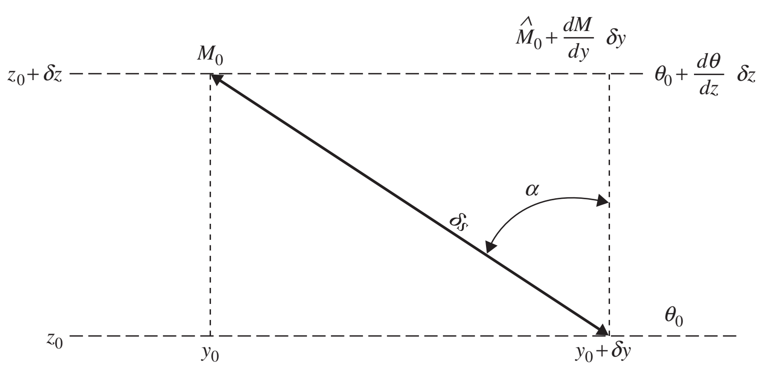

FIGURE 5.11 Parcel oscillation path in the meridional plane for an inertia–gravity wave. (See text for definition of symbols.)

on the right in (5.76) is similar in magnitude to the first term. This requires that $\tan^2\alpha \sim N^2/f^2 = 10^4$, in which case it is clear from (5.76) that $\nu \ll N$. Thus, only low-frequency gravity waves are modified significantly by Earth's rotation, and these have very small parcel trajectory slopes.

The heuristic parcel derivation can again be verified using the linearized dynamical equations. In this case, however, it is necessary to include rotation. The small parcel trajectory slopes of the relatively long period waves that are altered significantly by rotation imply that the horizontal scales are much greater than the vertical scales for these waves. Therefore, we may assume that the motions are in hydrostatic balance. If in addition we assume a motionless basic state, the linearized equations (5.59) through (5.62) are replaced by the set

$$\frac{\partial u'}{\partial t} - fv' + \frac{1}{\rho_0}\frac{\partial p'}{\partial x} = 0 \tag{5.77}$$

$$\frac{\partial v'}{\partial t} + fu' + \frac{1}{\rho_0}\frac{\partial p'}{\partial y} = 0 \tag{5.78}$$

$$\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\theta'}{\overline{\theta}}g = 0 \tag{5.79}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \tag{5.80}$$

$$\frac{\partial \theta'}{\partial t} + w'\frac{d\overline{\theta}}{dz} = 0 \tag{5.81}$$

The hydrostatic relationship in (5.79) may be used to eliminate θ' in (5.81) to yield

$$\frac{\partial}{\partial t}\left(\frac{1}{\rho_0}\frac{\partial p'}{\partial z}\right) + N^2 w' = 0 \tag{5.82}$$

Letting

$$u' = \mathrm{Re}\left[\hat{u}\exp i\left(kx + ly + mz - \nu t\right)\right]$$
$$v' = \mathrm{Re}\left[\hat{v}\exp i\left(kx + ly + mz - \nu t\right)\right]$$
$$w' = \mathrm{Re}\left[\hat{w}\exp i\left(kx + ly + mz - \nu t\right)\right]$$
$$p'/\rho_0 = \mathrm{Re}\left[\hat{p}\exp i\left(kx + ly + mz - \nu t\right)\right]$$

and substituting into (5.77), (5.78), and (5.82), we obtain

$$\hat{u} = \left(\nu^2 - f^2\right)^{-1} (\nu k + ilf)\,\hat{p} \tag{5.83}$$

$$\hat{v} = \left(\nu^2 - f^2\right)^{-1} (\nu l - ikf)\,\hat{p} \tag{5.84}$$

$$\hat{w} = -\left(\nu m/N^2\right)\hat{p} \tag{5.85}$$

which with the aid of (5.80) yields the dispersion relation

$$m^2\nu^3 - \left[N^2(k^2 + l^2) + f^2 m^2\right]\nu = 0 \tag{5.86}$$

As in the shallow water dispersion relationship (5.31), there exists a zero root corresponding to stationary Rossby waves. Time independence implies geostrophic flow from (5.77) and (5.78), and from (5.81) that $w' = 0$.

For $\nu \neq 0$, we have the classic dispersion relationship for hydrostatic inertia–gravity waves,

$$\nu^2 = f^2 + N^2\left(k^2 + l^2\right)m^{-2} \tag{5.87}$$

Because hydrostatic waves must have $(k^2 + l^2)/m^2 \ll 1$, (5.87) indicates that for vertical propagation to be possible (m real) the frequency must satisfy the inequality $|f| < |\nu| \ll N$. Equation (5.87) is just the limit of (5.76) when we let

$$\sin^2\alpha \to 1, \cos^2\alpha = \left(k^2 + l^2\right)/m^2$$

which is consistent with the hydrostatic approximation. Note also that (5.87) may be approximated exactly by the shallow water dispersion relationship (5.32) if one uses the "equivalent depth" of $\overline{h} = \frac{N^2}{gm^2}$ for the water depth.

If axes are chosen to make $l = 0$ (Figure 5.7), it may be shown (see Problem 5.14) that the ratio of the vertical to the horizontal components of group velocity is given by

$$\left|c_{gz}/c_{gx}\right| = |k/m| = \left(\nu^2 - f^2\right)^{1/2}/N \tag{5.88}$$

Thus, for fixed v, inertia–gravity waves propagate more closely to the horizontal than pure internal gravity waves. However, as in the latter case the group velocity vector is again parallel to lines of constant phase.

Eliminating $\hat{p}$ between (5.83) and (5.84) for the case $l = 0$ yields the relationship $\hat{v} = -if\hat{u}/\nu$, from which it is easily verified that if $\hat{u}$ is real, the perturbation horizontal motions satisfy the relations

$$u' = \hat{u}\cos(kx + mz - \nu t), \quad v' = \hat{u}(f/\nu)\sin(kx + mz - \nu t) \tag{5.89}$$

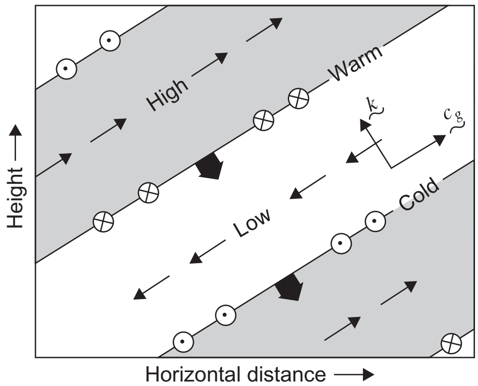

FIGURE 5.12 Vertical section in a plane containing the wave vector $\boldsymbol{k}$ showing the phase relationships among velocity, geopotential, and temperature fluctuations in an upward propagating inertia–gravity wave with $m < 0$, $\nu > 0$, and $f > 0$ (Northern Hemisphere). *Thin sloping lines* denote the surfaces of constant phase (perpendicular to the wave vector), and *thick arrows* show the direction of phase propagation. *Thin arrows* show the perturbation zonal and vertical velocity fields. Meridional wind perturbations are shown by *arrows* pointed into the page (northward) and out of the page (southward). Note that the perturbation wind vector turns clockwise (anticyclonically) with height. (*After Andrews et al., 1987.*)

so that the horizontal velocity vector rotates anticyclonically (that is, clockwise in the Northern Hemisphere) with time. As a result, parcels follow elliptical trajectories in a plane orthogonal to the wave number vector. Equations (5.89) also show that the horizontal velocity vector turns anticyclonically with height for waves with upward energy propagation (e.g., waves with $m < 0$ and $\nu < 0$). These characteristics are illustrated by the vertical cross-section shown in Figure 5.12. The anticyclonic turning of the horizontal wind with height and time is a primary method for identifying inertia–gravity oscillations in meteorological data.

A linearized vorticity equation is derived from $\partial/\partial x$ (5.78) $-\partial/\partial y$ (5.77), which gives

$$\frac{\partial \zeta'}{\partial t} = -f\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = f\frac{\partial w'}{\partial z} \tag{5.90}$$

From (5.81) $\partial w'/\partial z$ may be eliminated from (5.90) to give the linearized potential vorticity equation

$$\frac{\partial \Pi'}{\partial t} = 0 \tag{5.91}$$

This states that the linearized PV

$$\Pi' = \zeta' + f\frac{\partial}{\partial z}\left(\frac{\theta'}{d\overline{\theta}/dz}\right) \tag{5.92}$$

is conserved at every point in space, as was the case for linear shallow water waves. Polarization relationships (5.83) and (5.84) for the $l = 0$ case can be used to show that for Rossby waves

$$\zeta' = -\frac{k^2}{f}\hat{p} \tag{5.93}$$

and for inertia–gravity waves

$$\zeta' = \frac{fm^2}{N^2}\hat{p} \tag{5.94}$$

This result may be used to show (see Problem 5.16) that if the buoyancy frequency, N, is constant, inertia-gravity waves have zero linearized potential vorticity (5.92). Therefore, as in linearized shallow water, the linearized potential vorticity equation (5.91) effectively filters gravity waves from the dynamics. We will exploit this fact in the next section, where we show that geostrophic balance emerges in the long-time limit for arbitrary initial states. This result also motivates the derivation of the quasi-geostrophic equations in Chapter 6, which essentially generalizes (5.91) to include nonlinear advection of linearized PV (5.92).

5.6 ADJUSTMENT TO GEOSTROPHIC BALANCE

This section investigates the process by which geostrophic balance is achieved from an unbalanced initial condition—that is, the *adjustment* process. For simplicity we utilize the linearized shallow water equations of Section 5.3.2; similar considerations apply to a continuously stratified atmosphere. The problem addressed here is the solution of the full linearized equations (5.25) in the limit of large time, starting from an arbitrary initial condition.

The key to solving this problem comes from (5.42), the linearized shallow water potential vorticity equation,

$$\frac{\partial Q'}{\partial t} = 0 \tag{5.95}$$

where Q' designates the perturbation potential vorticity. This implies the conservation relationship

$$Q'(x, y, t) = \zeta'/f - h'/\overline{h} = \text{Const.} \tag{5.96}$$

Thus, if we know the distribution of Q' at the initial time, we know Q' for all time

$$Q'(x, y, t) = Q'(x, y, 0)$$

and the final adjusted state can be determined without solving the time-dependent problem. Note that, in general, u', v', and h' evolve in time, but only in such a way that Q' is constant at every point in space.

This problem was first solved by Rossby in the 1930s and is often referred to as the Rossby adjustment problem. As a simplified, albeit somewhat unrealistic, example of the adjustment process, we consider an idealized shallow water system on a rotating plane with initial conditions

$$u', v' = 0; \quad h' = -h_0 \mathrm{sgn}\,(x) \tag{5.97}$$

where $\mathrm{sgn}(x) = 1$ for $x > 0$ and $\mathrm{sgn}(x) = -1$ for $x < 0$. This corresponds to an initial step function in h' at $x = 0$, with the fluid motionless. Thus, from (5.96)

$$\left(\zeta'/f\right) - \left(h'/\bar{h}\right) = \left(h_0/\bar{h}\right) \mathrm{sgn}\,(x) \tag{5.98}$$

Using (5.98) to eliminate ζ' in (5.27) yields

$$\frac{\partial^2 h'}{\partial t^2} - c^2 \left(\frac{\partial^2 h'}{\partial x^2} + \frac{\partial^2 h'}{\partial y^2}\right) + f^2 h' = -f^2 h_0 \mathrm{sgn}\,(x) \tag{5.99}$$

where $c^2 = g\bar{h}$.

Because initially h' is independent of y, it will remain so for all time. Thus, in the final steady state (5.99) becomes

$$-c^2 \frac{d^2 h'}{dx^2} + f^2 h' = -f^2 h_0 \,\mathrm{sgn}\,(x) \tag{5.100}$$

which has the solution

$$\frac{h'}{h_0} = \begin{Bmatrix} -1 + \exp\left(-x/\lambda_R\right) & \text{for } x > 0 \\ +1 - \exp\left(+x/\lambda_R\right) & \text{for } x < 0 \end{Bmatrix} \tag{5.101}$$

where $\lambda_R \equiv f_0^{-1}\sqrt{g\bar{h}}$ is the Rossby *radius of deformation*. Hence, the radius of deformation may be interpreted as the horizontal length scale over which the height field adjusts during the approach to geostrophic equilibrium. For $|x| \gg \lambda_R$ the original h' remains unchanged. Substituting from (5.101) into (5.25) shows that the steady velocity field is geostrophic and nondivergent:

$$u' = 0, \quad \text{and} \quad v' = \frac{g}{f}\frac{\partial h'}{\partial x} = -\frac{g h_0}{f \lambda_R} \exp\left(-\left|x\right|/\lambda_R\right) \tag{5.102}$$

The steady-state solution (5.102) is shown in Figure 5.13.

Note that (5.102) could not be derived merely by setting $\partial/\partial t = 0$ in (5.25). That would yield geostrophic balance, and *any* distribution of h' would satisfy

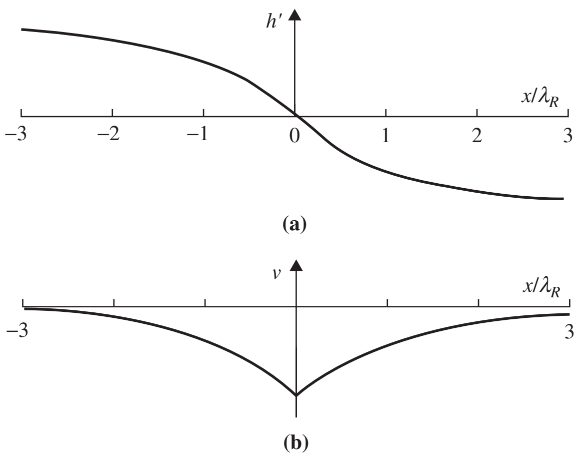

FIGURE 5.13 The geostrophic equilibrium solution corresponding to adjustment from the initial state defined in (5.97). (a) Final surface elevation profiles and (b) the geostrophic velocity profile in the final state. (*After Gill, 1982.*)

the equations:

$$fu' = -g\frac{\partial h'}{\partial y}, \quad fv' = g\frac{\partial h'}{\partial x}, \quad \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0$$

Only by combining (5.25) to obtain the potential vorticity equation, and requiring the flow to satisfy potential vorticity conservation at all intermediate times, can the degeneracy of the geostrophic final state be eliminated. In other words, although any height field can satisfy the steady-state versions of (5.25), there is only one field that is consistent with a given initial state; this field can be found readily because it can be computed from the distribution of potential vorticity, which is conserved.

Although the final state can be computed without solving the time-dependent equation, if the evolution of the adjustment process is required, it is necessary to solve (5.99) subject to the initial conditions (5.97), which is beyond the scope of this discussion. We can, however, compute the amount of energy that is dispersed by gravity waves during the adjustment process. This only requires computing the energy change between initial and final states.

The potential energy per unit horizontal area is given by

$$\int_0^{h'} \rho g z dz = \rho g h'^2/2$$

Thus, the potential energy released per unit length in y during adjustment is

$$\int_{-\infty}^{+\infty} \frac{\rho g h_0^2}{2} dx - \int_{-\infty}^{+\infty} \frac{\rho g h'^2}{2} dx$$
$$= 2 \int_{0}^{+\infty} \frac{\rho g h_0^2}{2} \left[1 - \left(1 - e^{-x/\lambda_R} \right)^2 \right] dx \tag{5.103}$$
$$= \frac{3}{2} \rho g h_0^2 \lambda_R$$

In the nonrotating case ($\lambda_R \to \infty$) all potential energy available initially is released (converted to kinetic energy) so that there is an infinite energy release. (Energy is radiated away in the form of gravity waves, leaving a flat free surface extending to $|x| \to \infty$ as $t \to \infty$.)

In the rotating case only the finite amount given in (5.103) is converted to kinetic energy, and only a portion of this kinetic energy is radiated away. The rest remains in the steady geostrophic circulation. The kinetic energy in the steady-state per unit length is

$$2 \int_{0}^{+\infty} \rho \bar{h} \frac{v'^2}{2} dx = \rho \bar{h} \left(\frac{g h_0}{f \lambda_R} \right)^2 \int_{0}^{+\infty} e^{-2x/\lambda_R} dx = \frac{1}{2} \rho g h_0^2 \lambda_R \tag{5.104}$$

Thus, in the rotating case a finite amount of potential energy is released, but only one-third of the potential energy released goes into the steady geostrophic flow. The remaining two-thirds is radiated away in the form of inertia–gravity waves. This simple analysis illustrates the following points

- It is difficult to extract the potential energy of a rotating fluid. Although there is an infinite reservoir of potential energy in this example (because h' is finite as $|x| \to \infty$), only a finite amount is converted before geostrophic balance is achieved.
- Conservation of potential vorticity allows one to determine the steady-state geostrophically adjusted velocity and height fields without carrying out a time integration.
- The length scale for the steady solution is the Rossby radius λ_R.

5.7 ROSSBY WAVES

The wave type that is of most importance for large-scale meteorological processes is the *Rossby wave*, or *planetary wave*. In an inviscid barotropic fluid of constant depth (where the divergence of the horizontal velocity must vanish), the Rossby wave is an absolute vorticity-conserving motion that owes its existence to the variation of the Coriolis parameter with latitude. More generally, in a baroclinic atmosphere, the Rossby wave is a potential vorticity-conserving motion that owes its existence to the gradient of potential vorticity.

The variation of the Coriolis parameter with latitude can be approximated by expanding the latitudinal dependence of f in a Taylor series about a reference latitude ϕ_0 and retaining only the first two terms to yield

$$f = f_0 + \beta y \tag{5.105}$$

where $\beta \equiv (df/dy)_{\phi_0} = 2\Omega\cos\phi_0/a$ and $y = 0$ at ϕ_0. This approximation is usually referred to as the *midlatitude β-plane* approximation. Rossby wave propagation can be understood in a qualitative fashion by considering a closed chain of fluid parcels initially aligned along a circle of latitude. Recall that the absolute vorticity η is given by $\eta = \zeta + f$, where ζ is the relative vorticity and f is the Coriolis parameter. Assume that $\zeta = 0$ at time t_0. Now suppose that at t_1, δy is the meridional displacement of a fluid parcel from the original latitude. Then at t_1 we have

$$(\zeta + f)_{t_1} = f_{t_0}$$

or

$$\zeta_{t_1} = f_{t_0} - f_{t_1} = -\beta\delta y \tag{5.106}$$

where $\beta \equiv df/dy$ is the planetary vorticity gradient at the original latitude.

From (5.106) it is evident that if the chain of parcels is subject to a sinusoidal meridional displacement under absolute vorticity conservation, the resulting perturbation vorticity will be positive for a southward displacement and negative for a northward displacement.

This perturbation vorticity field will induce a meridional velocity field, which advects the chain of fluid parcels southward west of the vorticity maximum and northward west of the vorticity minimum, as indicated in Figure 5.14.

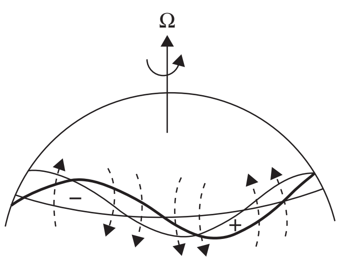

FIGURE 5.14 Perturbation vorticity field and induced velocity field (*dashed arrows*) for a meridionally displaced chain of fluid parcels. The *heavy wavy line* shows original perturbation position; the *light line* shows westward displacement of the pattern due to advection by the induced velocity.

Thus, the fluid parcels oscillate back and forth about their equilibrium latitude, and the pattern of vorticity maxima and minima propagates to the west. This westward propagating vorticity field constitutes a Rossby wave. Just as a positive vertical gradient of potential temperature resists vertical fluid displacements and provides the restoring force for gravity waves, the meridional gradient of absolute vorticity resists meridional displacements and provides the restoring mechanism for Rossby waves.

The speed of westward propagation, c, can be computed for this simple example by letting $\delta y = a \sin [k (x - ct)]$, where a is the maximum northward displacement. Then $v = D (\delta y)/Dt = -kca \cos [k (x - ct)]$, and

$$\zeta = \partial v/\partial x = k^2 ca \sin [k (x - ct)]$$

Substitution for δy and ζ in (5.106) then yields

$$k^2 ca \sin [k (x - ct)] = -\beta a \sin [k (x - ct)]$$

or

$$c = -\beta/k^2 \tag{5.107}$$

Thus, the phase speed is westward relative to the mean flow and is inversely proportional to the square of the zonal wave number.

5.7.1 Free Barotropic Rossby Waves

The dispersion relationship for barotropic Rossby waves may be derived formally by finding wave-type solutions of the linearized barotropic vorticity equation. The barotropic vorticity equation (4.41) states that the vertical component of absolute vorticity is conserved following the horizontal motion. For a midlatitude β-plane this equation has the form

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\zeta + \beta v = 0 \tag{5.108}$$

We now assume that the motion consists of a constant basic state zonal velocity plus a small horizontal perturbation:

$$u = \overline{u} + u', \quad v = v', \quad \zeta = \partial v'/\partial x - \partial u'/\partial y = \zeta'$$

We define a perturbation streamfunction ψ' according to

$$u' = -\partial\psi'/\partial y, \quad v' = \partial\psi'/\partial x$$

from which $\zeta' = \nabla^2\psi'$. The perturbation form of (5.108) is then

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\nabla^2\psi' + \beta\frac{\partial\psi'}{\partial x} = 0 \tag{5.109}$$

where as usual we have neglected terms involving the products of perturbation quantities. We seek a solution of the form

$$\psi' = \mathrm{Re}\,[\Psi \exp(i\phi)]$$

where $\phi = kx + ly - \nu t$. Here k and l are wave numbers in the zonal and meridional directions, respectively. Substituting for ψ' in (5.109) gives

$$(-\nu + k\overline{u})\left(-k^2 - l^2\right) + k\beta = 0$$

which may immediately be solved for ν:

$$\nu = \overline{u}k - \beta k/K^2 \tag{5.110}$$

where $K^2 \equiv k^2 + l^2$ is the total horizontal wave number squared.

Recalling that $c = \nu/k$, we find that the zonal phase speed relative to the mean wind is

$$c - \overline{u} = -\beta/K^2 \tag{5.111}$$

which reduces to (5.109) when the mean wind vanishes and $l \to 0$. Thus, the Rossby wave zonal phase propagation is always *westward* relative to the mean zonal flow. Furthermore, the Rossby wave phase speed depends inversely on the square of the horizontal wave number. Therefore, Rossby waves are dispersive waves whose phase speeds increase rapidly with increasing wavelength.

For a typical midlatitude synoptic-scale disturbance, with similar meridional and zonal scales ($l \approx k$) and zonal wavelength of order 6000 km, the Rossby wave speed relative to the zonal flow calculated from (5.111) is approximately -8 m s^{-1}. Because the mean zonal wind is generally westerly and greater than 8 m s^{-1}, synoptic-scale Rossby waves usually move eastward, but at a phase speed relative to the ground that is somewhat less than the mean zonal wind speed. For longer wavelengths the westward Rossby wave phase speed may be large enough to balance the eastward advection by the mean zonal wind so that the resulting disturbance is stationary relative to Earth's surface. From (5.111) it is clear that the free Rossby wave solution becomes stationary when

$$K^2 = \beta/\overline{u} \equiv K_s^2 \tag{5.112}$$

The significance of this condition is discussed in the next subsection.

Unlike the phase speed, which is always westward relative to the mean flow, the zonal group velocity for a Rossby wave may be either eastward or westward relative to the mean flow, depending on the ratio of the zonal and meridional wave numbers (see Problem 5.20). Stationary Rossby modes (i.e., modes with $c = 0$) have zonal group velocities that are eastward relative to the ground. Synoptic-scale Rossby waves also tend to have zonal group velocities that are eastward relative to the ground. For synoptic waves, advection by the mean zonal wind is generally larger than the Rossby phase speed so that the phase speed is also eastward relative to the ground, but is slower than the zonal group velocity. As shown earlier in Figure 5.4b, this implies that new disturbances tend to develop downstream of existing disturbances, which is an important consideration for forecasting.

It is possible to carry out a less restrictive analysis of free planetary waves using the perturbation form of the full primitive equations. In that case the structure of the free modes depends critically on the boundary conditions at the surface and the upper boundary. The results of such an analysis are mathematically complicated, but qualitatively yield waves with horizontal dispersion properties similar to those in the shallow water model. It turns out that the free oscillations allowed in a hydrostatic gravitationally stable atmosphere consist of eastward- and westward-moving gravity waves that are slightly modified by Earth's rotation and westward-moving Rossby waves that are slightly modified by gravitational stability. These free oscillations are the normal modes of oscillation of the atmosphere. As such, they are continually excited by the various forces acting on the atmosphere. Planetary scale-free oscillations, although they can be detected by careful observational studies, generally have rather weak amplitudes. Presumably this is because the forcing is quite weak at the large phase speeds characteristic of most such waves. An exception is the 16-day period zonal wave number 1 normal mode, which can be quite strong in the winter stratosphere.

5.7.2 Forced Topographic Rossby Waves

Although free propagating Rossby modes are only rather weakly excited in the atmosphere, forced stationary Rossby modes are of primary importance for understanding the planetary-scale circulation pattern. Such modes may be forced by longitudinally dependent diabatic heating patterns or by flow over topography. Of particular importance for the Northern Hemisphere extratropical circulation are stationary Rossby modes forced by flow over the Rockies and the Himalayas. It is just the topographic Rossby wave that was described qualitatively in the discussion of streamline deflections in potential vorticity-conserving flows crossing mountain ranges in Section 4.3.

As the simplest possible dynamical model of topographic Rossby waves, we use a modified version of the the shallow potential vorticity equation (4.39). We assume that the upper boundary is at a fixed height H, and the lower boundary is at the variable height $h_T(x, y)$ where $|h_T| \ll H$. We also approximate ζ by the geostrophic value, ζ_g, and assume that $|\zeta_g| \ll f_0$. We can then approximate (4.39) by

$$H\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\boldsymbol{\nabla}\right)(\zeta_g + f) = -f_0\frac{Dh_T}{Dt} \tag{5.113}$$

Linearizing and applying the midlatitude β-plane approximation yields

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\zeta_g' + \beta v_g' = -\frac{f_0}{H}\bar{u}\frac{\partial h_T}{\partial x} \tag{5.114}$$

We now examine solutions of (5.114) for the special case of a sinusoidal lower boundary. We specify the topography to have the form

$$h_T(x, y) = \text{Re}\,[h_0 \exp(ikx)] \cos ly \tag{5.115}$$

and represent the geostrophic wind and vorticity by the perturbation streamfunction

$$\psi(x, y) = \text{Re}\,[\psi_0 \exp(ikx)] \cos ly \tag{5.116}$$

Then (5.114) has a steady-state solution with complex amplitude given by

$$\psi_0 = f_0 h_0 / \left[H\left(K^2 - K_s^2\right)\right] \tag{5.117}$$

The streamfunction is either exactly in phase (ridges over the mountains) or exactly out of phase (troughs over the mountains), with the topography depending on the sign of $K^2 - K_s^2$. For long waves ($K < K_s$) the topographic vorticity source in (5.114) is balanced primarily by the meridional advection of planetary vorticity (the β effect). For short waves ($K > K_s$) the source is balanced primarily by the zonal advection of relative vorticity.

The topographic wave solution (5.117) has the unrealistic characteristic that when the wave number exactly equals the critical wave number K_s, the amplitude goes to infinity. From (5.112) it is clear that this singularity occurs at the zonal wind speed for which the free Rossby mode becomes stationary. Thus, it may be thought of as a resonant response of the barotropic system.

Charney and Eliassen (1949) used the topographic Rossby wave model to explain the winter mean longitudinal distribution of 500-hPa heights in Northern Hemisphere midlatitudes. They removed the resonant singularity by including boundary layer drag in the form of Ekman pumping, which for the barotropic vorticity equation is simply a linear damping of the relative vorticity (see Section 8.3.4). The vorticity equation thus takes the form

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\zeta_g' + \beta v_g' + r\zeta_g' = -\frac{f_0}{H}\bar{u}\frac{\partial h_T}{\partial x} \tag{5.118}$$

where $r \equiv \tau_e^{-1}$ is the inverse of the spin-down time defined in (8.37).

For steady flow, (5.118) has a solution with complex amplitude

$$\psi_0 = f_0 h_0 / \left[H\left(K^2 - K_s^2 - i\varepsilon\right)\right] \tag{5.119}$$

where $\varepsilon \equiv rK^2\,(k\bar{u})^{-1}$. Thus, boundary layer drag shifts the response's phase and removes the singularity at resonance. However, the amplitude is still a maximum for $K = K_s$ and the trough in the streamfunction occurs one-quarter cycle east of the mountain crest, in approximate agreement with observations.

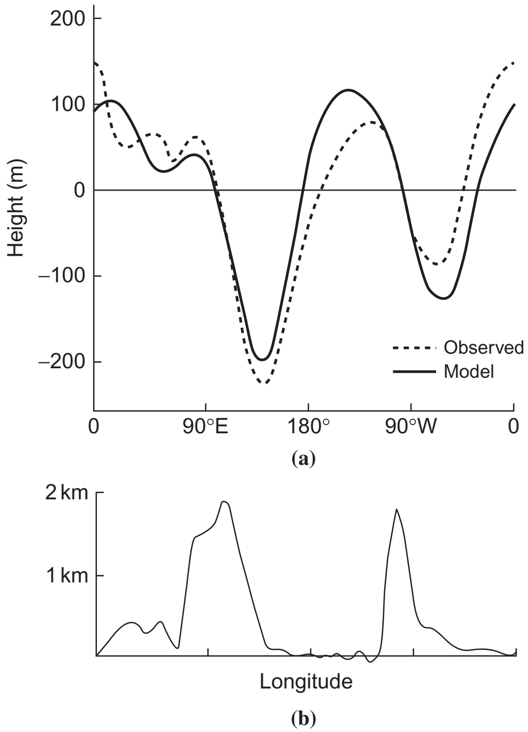

FIGURE 5.15 (a) Longitudinal variation of the disturbance in geopotential height ($\equiv f_0\Psi/g$) in the Charney–Eliassen model for the parameters given in the text (*solid line*) compared with the observed 500-hPa height perturbations at 45°N in January (*dashed line*). (b) Smoothed profile of topography at 45°N used in the computation. (*After Held, 1983.*)

By use of a Fourier expansion, (5.118) can be solved for realistic distributions of topography. The results for an x-dependence of h_T given by a smoothed version of Earth's topography at 45°N, a meridional wave number corresponding to a latitudinal half-wavelength of 35°, $\tau_e = 5$ days, $\bar{u} = 17$ m s^{-1}, $f_0 = 10^{-4}$ s^{-1}, and $H = 8$ km are shown in Figure 5.15. Despite its simplicity, the Charney–Eliassen model does a remarkable job of reproducing the observed 500-hPa stationary wave pattern in Northern Hemisphere midlatitudes.

SUGGESTED REFERENCES

Chapman and Lindzen, *Atmospheric Tides*: *Thermal and Gravitational*, is the classic reference on both observational and theoretical aspects of tides, a class of atmospheric motions for which the linear perturbations method has proved to be particularly successful.

Gill, *Atmosphere-Ocean Dynamics*, has a very complete treatment of gravity, inertia–gravity, and Rossby waves, with particular emphasis on observed oscillations in the oceans.

Hildebrand, *Advanced Calculus for Applications*, is one of many standard textbooks that discuss the mathematical techniques used in this chapter, including the representation of functions in Fourier series and the general properties of the wave equation.

Nappo, *An Introduction to Atmospheric Gravity Waves*, is an excellent introduction to the theory and observation of gravity waves in the atmosphere.

Scorer, *Natural Aerodynamics*, contains an excellent qualitative discussion on many aspects of waves generated by barriers such as lee waves.

PROBLEMS

5.1. Show that the Fourier component $F(x) = \mathrm{Re}\,[C \exp(imx)]$ can be written as

$$F(x) = |C| \cos m(x + x_0)$$

where $x_0 = m^{-1} \sin^{-1}(C_i/|C|)$ and C_i stands for the imaginary part of C.

5.2. In the study of atmospheric wave motions, it is often necessary to consider the possibility of amplifying or decaying waves. In such a case we might assume that a solution has the form

$$\psi = A \cos(kx - \nu t - kx_0) \exp(\alpha t)$$

where A is the initial amplitude, α the amplification factor, and x_0 the initial phase. Show that this expression can be written more concisely as

$$\psi = \mathrm{Re}\left[Be^{ik(x-ct)}\right]$$

where both B and c are complex constants. Determine the real and imaginary parts of B and c in terms of A, α, k, ν, and x_0.

5.3. Several of the wave types discussed in this chapter are governed by equations that are generalizations of the wave equation

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{\partial^2 \psi}{\partial x^2}$$

This equation can be shown to have solutions corresponding to waves of arbitrary profile moving at the speed c in both positive and negative x directions. We consider an arbitrary initial profile of the field ψ; $\psi = f(x)$ at $t = 0$. If the profile is translated in the positive x direction at speed c without change of shape, then $\psi = f(x')$, where x' is a coordinate moving at speed c so that $x = x' + ct$. Thus, in terms of the fixed coordinate x we can write $\psi = f(x - ct)$, corresponding to a profile that moves in the positive x direction at speed c without change of shape. Verify that $\psi = f(x - ct)$ is a solution for any arbitrary continuous profile $f(x - ct)$. [*Hint*: Let $x - ct = x'$ and differentiate f using the chain rule.]

5.4. Assuming that the pressure perturbation for a one-dimensional acoustic wave is given by (5.23), find the corresponding solutions of the zonal wind and density perturbations. Express the amplitude and phase for u' and ρ' in terms of the amplitude and phase of p'.

5.5. Show that for isothermal motion ($DT/Dt = 0$) the acoustic wave speed is given by $(gH)^{1/2}$, where $H = RT/g$ is the scale height.

5.6. In Section 5.3.1 the linearized equations for acoustic waves were developed for the special situation of one-dimensional propagation in a horizontal tube. Although this situation does not appear to be directly applicable to the atmosphere, there is a special atmospheric mode, the *Lamb wave*, which is a horizontally propagating acoustic mode with no vertical velocity perturbation ($w' = 0$). Such oscillations have been observed following violent explosions such as volcanic eruptions and atmospheric nuclear tests. Using (5.20) and (5.21), plus the linearized forms of the hydrostatic equation, and the continuity equation (5.15), derive the height dependence of the perturbation fields for the Lamb mode in an isothermal basic state atmosphere, assuming that the pressure perturbation at the lower boundary ($z = 0$) has the form (5.23). Determine the vertically integrated kinetic energy density per unit horizontal area for this mode.

5.7. If the surface height perturbation in a shallow water gravity wave is given by

$$h' = \mathrm{Re}\left[Ae^{ik(x-ct)}\right]$$

find the corresponding velocity perturbation $u'(x,t)$. Sketch the phase relationship between h' and u' for an eastward propagating wave.

5.8. Assuming that the vertical velocity perturbation for a two-dimensional internal gravity wave is given by (5.65), obtain the corresponding solution for the u', p', and θ' fields. Use these results to verify the approximation

$$\left|\rho_0\theta'/\overline{\theta}\right| \gg \left|p'/c_s^2\right|$$

which was used in (5.58).

5.9. For the situation in the preceding problem, express the vertical flux of horizontal momentum, $\rho_0\overline{u'w'}$, in terms of the amplitude A of the vertical velocity perturbation. Hence, show that the momentum flux is positive for waves in which phase speed propagates eastward and downward.

5.10. Show that if (5.60) is replaced by the hydrostatic equation (i.e., the terms in w' are neglected), the resulting frequency equation for internal gravity waves is just the asymptotic limit of (5.66) for waves in which $|k| \ll |m|$.

5.11. **(a)** Show that the intrinsic group velocity vector in two-dimensional internal gravity waves is parallel to lines of constant phase.

(b) Show that in the long-wave limit ($|k| \ll |m|$) the magnitude of the zonal component of the group velocity equals the magnitude of the zonal phase speed so that energy propagates one wavelength per wave period.

5.12. Determine the perturbation horizontal and vertical velocity fields for stationary gravity waves forced by flow over sinusoidally varying topography given the following conditions: The height of the ground is $h = h_0 \cos kx$, where $h_0 = 50$ m is a constant; $N = 2 \times 10^{-2}\ \mathrm{s}^{-1}$; $\overline{u} = 5\ \mathrm{m\,s}^{-1}$; and $k = 3 \times 10^{-3}\ \mathrm{m}^{-1}$. *Hint*: For small-amplitude topography ($h_0 k \ll 1$) we can approximate the lower boundary condition by

$$w' = Dh/Dt = \overline{u}\partial h/\partial x \quad \text{at z} = 0$$

5.13. Verify the group velocity relationship for inertia–gravity waves that are given in (5.88).

5.14. Show that when $\overline{u} = 0$ the wave number vector $\boldsymbol{\kappa}$ for an internal inertia–gravity wave is perpendicular to the group velocity vector.

5.15. Derive an expression for the group velocity of a barotropic Rossby wave with dispersion relation (5.110). Show that for stationary waves the group velocity always has an eastward zonal component relative to Earth. Hence, Rossby wave energy propagation must be downstream of topographic sources.

5.16. Prove that the internal gravity waves of Section 5.5.2 have zero linearized potential vorticity in the case of constant buoyancy frequency, N.

MATLAB Exercises

M5.1. **(a)** The MATLAB script **phase_demo.m** shows that the Fourier series $F(x) = A\sin(kx) + B\cos(kx)$ is equivalent to the form $F(x) = \mathrm{Re}\, C\exp(ikx)$ where A and B are real coefficients and C is a complex coefficient. Modify the MATLAB script to confirm that the expression $F(x) = |C|\cos k(x + x_0) = |C|\cos(kx + \alpha)$ represents the same Fourier series where $kx_0 \equiv \alpha = \sin^{-1}(C_i/|C|)$, C_i stands for the imaginary part of C, α is the "phase" defined in the MATLAB script. Plot your results as the third subplot in the script.

(b) By running the script for several input phase angles (e.g., 0, 30, 60, and 90°), determine the relationship between α and the location of the maximum of $F(x)$.

M5.2. In this problem you will examine the formation of a "wave envelope" for a combination of dispersive waves. The example is that of a deep water wave in which the group velocity is half of the phase velocity. The MATLAB script **grp_vel_3.m** has code to show the wave height field at four different times for a group composed of various numbers of waves with differing wave numbers and frequencies. Study the code and determine the period and wavelength of the carrier wave. Then run the script several times, varying the number of wave modes from 4 to 32. Determine the half-width of the envelope at time $t = 0$ (top line on the graph) as a function of the number of modes in the group. The half-width is here defined as two times the distance from the point of maximum amplitude ($x = 0$) to the point along the envelope where the amplitude is half the maximum. You can estimate this from the graph using the **ginput** command to determine the distance. Use MATLAB to plot a curve of the half-width versus number of wave modes.

M5.3. The script **geost_adjust_1.m** together with the function **yprim_adj_1.m** illustrates one-dimensional geostrophic adjustment of the velocity field in a barotropic model for a sinusoidally varying initial height field. The equations are a simplification of (5.25) in the text for the case of no y dependence. Initially $u' = v' = 0,\ \phi' \equiv gh' = 9.8\cos(kx)$. The final balanced wind in this case will have only a meridional component. Run the program **geost_adjust_1.m** for the cases of latitude 30 and 60°, choosing a time value of at least 10 days. For each of these cases choose values of wavelength of 2000, 4000, 6000, and 8000 km (a total of eight runs). Construct a table showing the initial and final values of the fields u', v', ϕ' and the ratio of the final energy to the initial energy. You can read the values from the MATLAB graphs by using the **ginput** command or, for greater accuracy, add lines in the MATLAB code to print out the values needed. Modify the MATLAB script to determine the partition of final state energy per unit mass between the kinetic energy ($v'^2/2$) and the potential energy $\phi'^2/(2gH)$. Compute the ratio of final kinetic energy to potential energy for each of your eight cases and show this in a table.

M5.4. The script **geost_adjust_2.m** together with the function **yprim_adj_2.m** extends Problem M5.3 by using Fourier expansion to examine the geostrophic adjustment for an isolated initial height disturbance of the form $h_0(x) = -h_m/\left[1 + (x/L)^2\right]$. The version given here uses 64 Fourier modes and employs a fast Fourier transform algorithm (FFT). (There are 128 modes in the FFT, but only one-half of them provide real information.) In this case you may run the model for only 5 days of integration time (it requires a lot of computation compared to the previous case). Choose an initial zonal scale of the disturbance of 500 km and run the model for latitudes (in degrees) of 15, 30, 45, 60, 75, and 90 (six runs). Study the animations for each case. Note that the zonal flow is entirely in a gravity wave mode that propagates away from the initial disturbance. The meridional flow has a propagating gravity wave component, but also a geostrophic part (cyclonic flow). Use the **ginput** command to estimate the zonal scale of the final geostrophic flow (v component) by measuring the distance from the negative velocity maximum just west of the center to the positive maximum just east of the center. Plot a curve showing the zonal scale as a function of latitude. Compare this scale with the Rossby radius of deformation defined in the text. (Note that $gH = 400\ \mathrm{m^2\ s^{-2}}$ in this example.)

M5.5. This problem examines the variation of phase velocity for Rossby waves as the zonal wavelength is varied. Run the MATLAB program **rossby_1.m** with zonal wavelengths specified as 5000, 10,000 and 20,000 km. For each of these cases try different values of the mean zonal wind until you find the mean wind for which the Rossby wave is approximately stationary.

M5.6. The MATLAB script **rossby_2.m** shows an animation of the Rossby waves generated by a vorticity disturbance initially localized in the center of the domain, with mean wind zero. Run the script and note that the waves excited have westward phase speeds, but that disturbances develop on the eastward side of the original disturbance. By following the development of these disturbances, make a crude estimate of the characteristic wavelength and the group velocity for the disturbances appearing to the east of the original disturbance at time $t = 7.5$ days. (Wavelength can be estimated by using **ginput** to measure the distance between adjacent troughs.) Compare your estimate with the group velocity formula derived from (5.108). Can you think of a reason why your estimate for group velocity may differ from that given in the formula?

M5.7. The MATLAB script **rossby_3.m** gives the surface height and meridional velocity disturbances for topographic Rossby waves generated by flow over an isolated ridge. The program uses a Fourier series approach to the solution. Ekman damping with a 2-day damping time is included to minimize the effect of waves propagating into the mountain from upstream (but this cannot be entirely avoided). Run this program for input zonal mean winds from 10 to

100 m/s at 10-m/s intervals. For each run use **ginput** to estimate the scale of the leeside trough by measuring the zonal distance between the minimum and maximum in the meridional velocity. (This should be equal approximately to one-half the wavelength of the dominant disturbance.) Compare your results in each case with the zonal wavelength for resonance given by solving (5.112) to determine the resonant $L_x = 2\pi/k$, where $K^2 = k^2 + l^2$ and $l = \pi/8 \times 10^6$ in units of 1/m. (Do not expect exact agreement because the actual disturbance corresponds to the sum over many separate zonal wavelengths.)