Shallow water model

Assumptions:

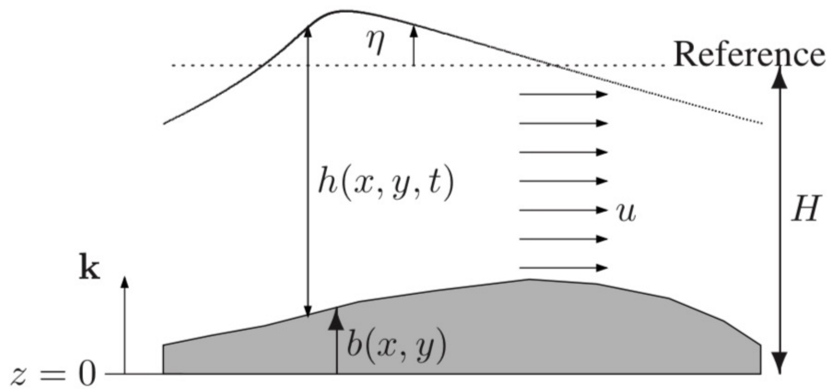
thin layer ($H \ll L$)

inviscid

motions initially independent of z

$$\frac{\partial}{\partial z} = 0 \text{ always}$$

homogeneous (constant density)



$$-g\frac{\partial \eta}{\partial x}$$

$$\text{x:}\quad \frac{\partial \mathrm{u}}{\partial \mathrm{t}} + \mathrm{u}\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \mathrm{v}\frac{\partial \mathrm{u}}{\partial \mathrm{y}} + \cancel{\mathrm{w}\frac{\partial \mathrm{u}}{\partial \mathrm{z}}} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$\text{y:}\quad \frac{\partial \mathrm{v}}{\partial \mathrm{t}} + \mathrm{u}\frac{\partial \mathrm{v}}{\partial \mathrm{x}} + \mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{y}} + \cancel{\mathrm{w}\frac{\partial \mathrm{v}}{\partial \mathrm{z}}} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

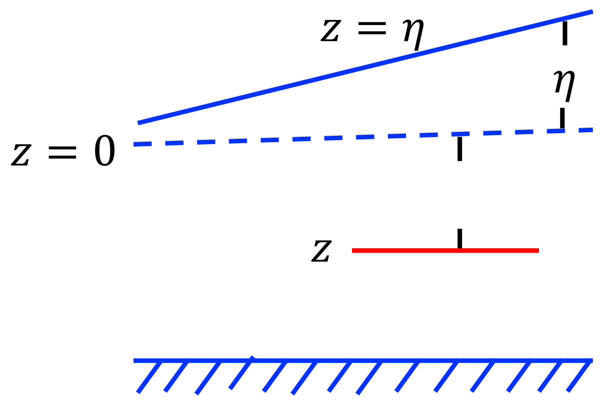
$$-g\frac{\partial \eta}{\partial y}$$

Hydrostatic balance $\frac{\partial p}{\partial z} + \rho g = 0$

If $\rho = \rho_0$ everywhere, vertically integrating the equation from z to the surface η :

$$\int_z^\eta \frac{\partial p}{\partial z} dz + \rho_0 g(\eta - z) = 0$$

$$p|_z = \underset{P_o}{p|_\eta} + \rho_0 g(\eta - z)$$



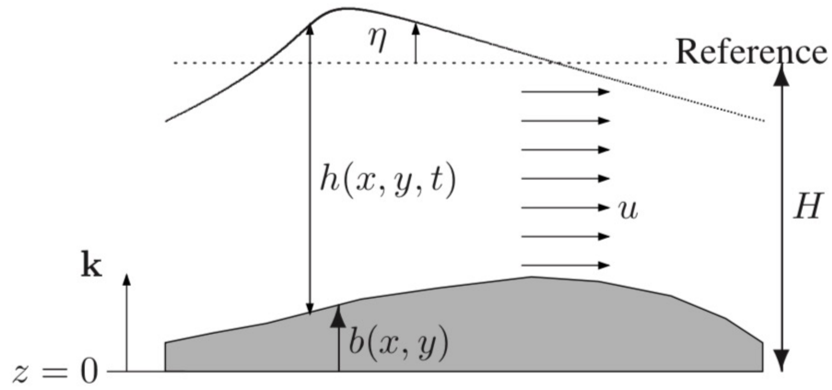
The the horizontal momentum equations become:

$$-\frac{1}{\rho_0}\frac{\partial}{\partial x}(P_o + \rho_0 g\eta - \rho_0 g\mathrm{z}) = -\mathrm{g}\frac{\partial \eta}{\partial x}$$

$$-\frac{1}{\rho_0}\frac{\partial}{\partial y}(P_o + \rho_0 g\eta - \rho_0 g\mathrm{z}) = -\mathrm{g}\frac{\partial \eta}{\partial y}$$

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$



Integrate vertically from b to η:

$$\int_{z=b}^{z=b+h} \frac{\partial w}{\partial z} dz = -\int_{z=b}^{z=b+h} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz$$

$w|_{b+h} - w|_b$

$$\frac{\partial \eta}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} = -h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$\boxed{\frac{\partial \eta}{\partial t} + \frac{\partial hu}{\partial x} + \frac{\partial hv}{\partial y} = 0}$$

Boudary conditions

$$w|_{b+h} = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}$$

$$w|_b = u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y}$$

$$\boxed{\eta = h + b - H}$$

Shallow water waves

Linear wave dynamics

Assumption: $R_O \ll 1$

c: wave speed

$$R_{OT} = \frac{\frac{U}{T}}{fU} = \frac{1}{fT} \sim \frac{1}{fL/c} = \frac{c}{fL} \, (c \gg U) \sim 1$$

The horizontal momentum equations are:

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$

The continuity equation is:

$$\frac{\partial \eta}{\partial t} + \frac{\partial hu}{\partial x} + \frac{\partial hv}{\partial y} = 0$$

$$\frac{\partial \eta}{\partial t} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} = 0$$

For flat bottom, $\eta = h - H$:

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \eta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} = 0$$

$$\boxed{\Delta H \frac{c}{L}} \qquad \frac{\Delta H}{T} \qquad H\frac{U}{L} \qquad \cancel{\Delta H \frac{U}{L}} \qquad \cancel{U\frac{\Delta H}{L}}$$

Then the continuity equation is reduced to:

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

Inertia-gravity waves (Poincaré waves)

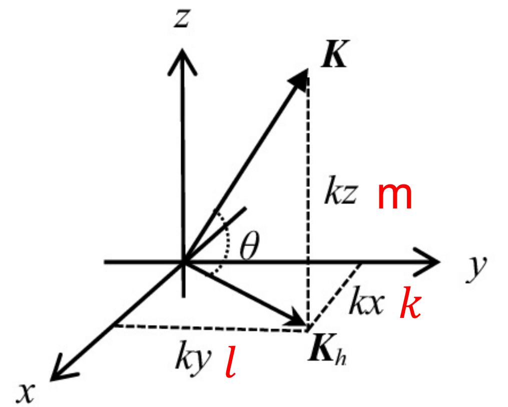


Fig. 1. Wave number vector and its components.

Assumption: flat bottom

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

Give a wave solution:

$$u = Ue^{i(kx+ly-\omega t)}$$

$$v = Ve^{i(kx+ly-\omega t)}$$

$$\eta = Ae^{i(kx+ly-\omega t)}$$

$$-i\omega U - fV = -gikA$$

$$-i\omega V + fU = -gilA$$

$$-i\omega A + ikHU + ilHV = 0$$

$$\begin{pmatrix} -i\omega & -f & gik \\ f & -i\omega & gil \\ ikH & ilH & -i\omega \end{pmatrix} \begin{bmatrix} U \\ V \\ A \end{bmatrix} = 0$$

This homogeneous equation has non-trivial solutions only if the determinant of the coefficient matrix vanishes:

$$\omega[\omega^2 - f^2 - gH(k^2 + l^2)] = 0$$ dispersion relation

$$\boxed{R_d = \frac{\sqrt{gH}}{f}}$$

Rossby deformation radius (barotropic)

1. $\omega = 0, \ \frac{\partial}{\partial t} = 0,$ geostrophic flow

2. $\omega = \sqrt{f^2 + gH\underbrace{(k^2 + l^2)}_{K^2}}$

   a. rotation is weak, $f^2 \ll gHK^2, \ \ \lambda \ll R_d$ (short-wave limit)

   $$\omega = \sqrt{gH}K, \ \ c = \sqrt{gH},$$ gravity waves

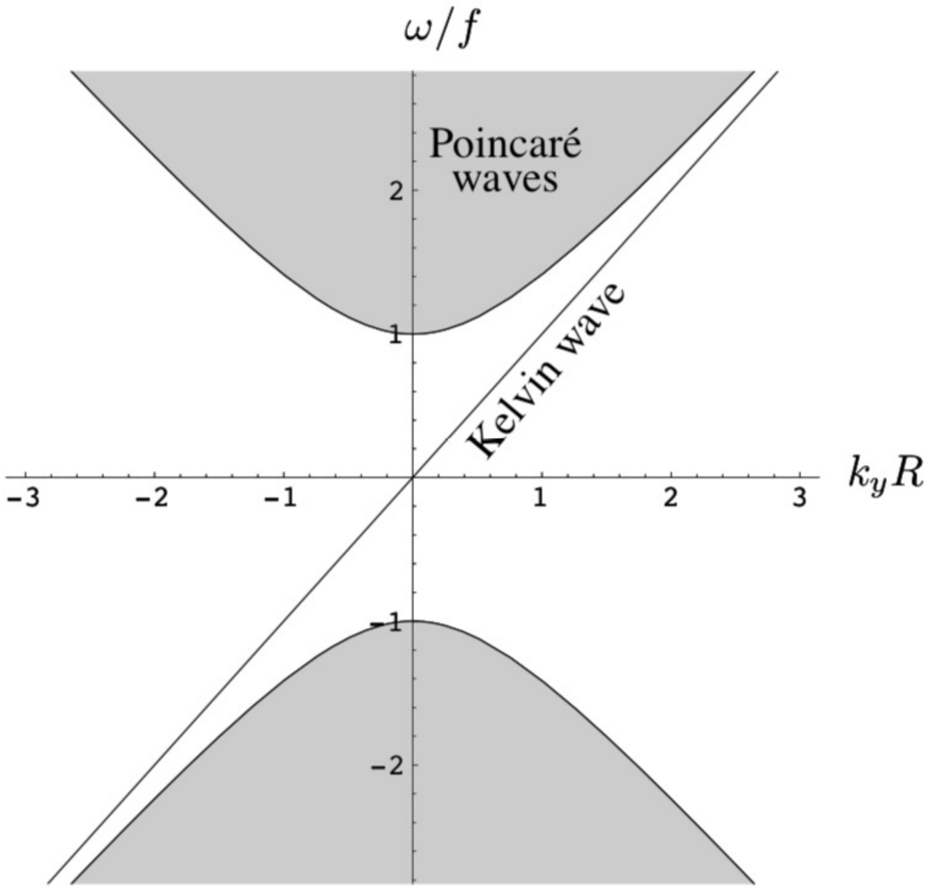
   b. rotation is important, $f^2 \gg gHK^2, \ \ \lambda \gg R_d$ (long-wave limit)

   $K\ (k, l)\ is\ small$

   pressure gradient term is negliable, equations reduced to inertial-motion

   $$\omega \sim f,$$ inertial oscillations

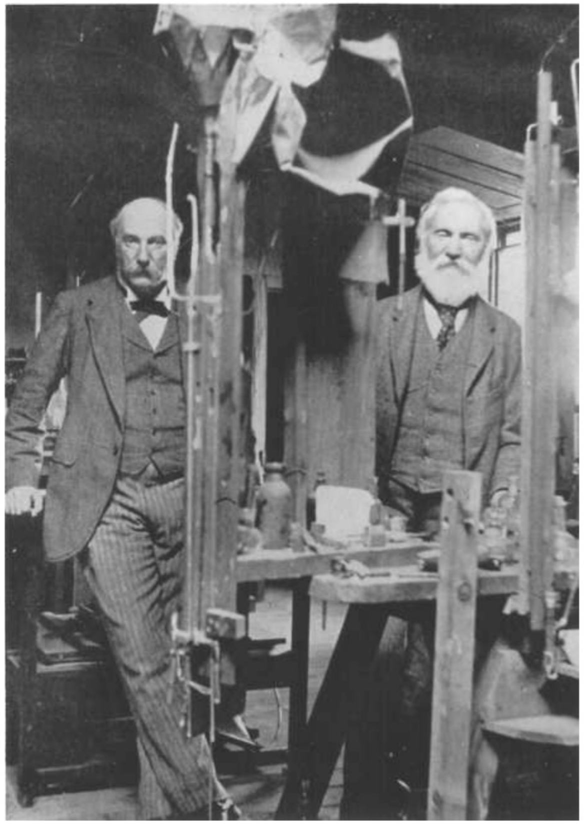
Dispersion relation diagram



$$\omega = \sqrt{f^2 + gH(k^2 + l^2)}$$

Figure 9-3 Recapitulation of the dispersion relation of Kelvin and Poincaré waves on the f-plane and on a flat bottom. While Poincaré waves (gray shades) can travel in all directions and occupy therefore a continuous spectrum in terms of k_y, the Kelvin wave (diagonal line) propagates only along a boundary.

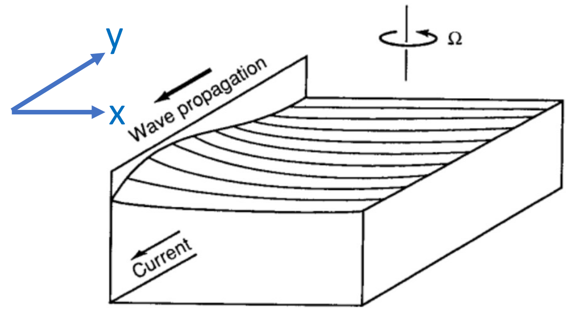
The Kelvin wave



William Thomson, Lord Kelvin
1824 – 1907
(Standing at right, in laboratory of Lord Rayleigh, left)

Named professor of natural philosophy at the University of Glasgow, Scotland, at age 22, William Thomson became quickly regarded as the leading inventor and scientist of his time. In 1892, he was named Baron Kelvin of Largs for his technological and theoretical contributions leading to the successful laying of a transatlantic cable. A friend of James P. Joule, he helped establish a firm theory of thermodynamics and first defined the absolute scale of temperature. He also made major contributions to the study of heat engines. With Hermann von Helmholtz, he estimated the ages of the earth and sun, and ventured into fluid mechanics. His theory of the so-called Kelvin wave was published in 1879 (under the name William Thomson). His more than 300 original papers left hardly any aspect of science untouched. He is quoted as saying that he could understand nothing of which he could not make a model. (*Photo by A.G. Webster*)

The Kelvin wave



Assumptions: flat bottom

one side boundary (y-axis)

velocity normal to the boundary is zero everywhere (u=0)

The momentum equations:

geostrophic flow $$\cancel{\frac{\partial u}{\partial t}} - fv = -g\frac{\partial \eta}{\partial x} \quad (1)$$

$$\frac{\partial v}{\partial t} + \cancel{fu} = -g\frac{\partial \eta}{\partial y} \quad (2) \Rightarrow \frac{\partial^2 v}{\partial t^2} = -g\frac{\partial}{\partial y}\left(\frac{\partial \eta}{\partial t}\right) = \overset{c^2}{\underline{gH}}\frac{\partial^2 v}{\partial y^2}$$

The continuity equation:

$$v = F_1(x)e^{i(ly-\omega t)} + F_2(x)e^{i(ly+\omega t)}$$

$$\frac{\partial \eta}{\partial t} + H\left(\cancel{\frac{\partial u}{\partial x}} + \frac{\partial v}{\partial y}\right) = 0 \quad (3)$$

$$v = F_1(x)e^{i(ly-\omega t)} + F_2(x)e^{i(ly+\omega t)}$$

$$\frac{\partial v}{\partial t} = -g\frac{\partial \eta}{\partial y} \qquad (2)$$

$$\eta = -\frac{1}{g}\int \frac{\partial v}{\partial t}dy = -\frac{1}{g}\int [-i\omega F_1(x)e^{i(ly-\omega t)} + i\omega F_2(x)e^{i(ly+\omega t)}]dy$$

$$= \frac{\omega}{g}\int [F_1(x)e^{i(ly-\omega t)} - F_2(x)e^{i(ly+\omega t)}]d(iy)$$

$$= \frac{\omega}{gl}\int [F_1(x)e^{i(ly-\omega t)} - F_2(x)e^{i(ly+\omega t)}]d(ily)$$

$$= \frac{\omega}{gl}[F_1(x)e^{i(ly-\omega t)} - F_2(x)e^{i(ly+\omega t)}]$$

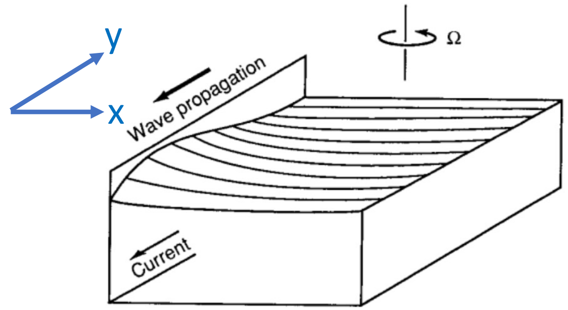
$$c = \frac{\omega}{l} = \sqrt{gH}$$

$$\sqrt{\frac{H}{g}}$$

$$v = F_1(x)e^{i(ly-\omega t)} + F_2(x)e^{i(ly+\omega t)}$$

$$\eta = \sqrt{\frac{H}{g}}\left[F_1(x)e^{i(ly-\omega t)} - F_2(x)e^{i(ly+\omega t)}\right]$$

$$fv = g\frac{\partial \eta}{\partial x} \qquad (1)$$



toward the open ocean $(x \to \infty), v \to \infty$

$$\frac{\partial F_1(x)}{\partial x} - \frac{f}{\sqrt{gH}}F_1(x) = 0 \Longrightarrow \cancel{F_1(x) = C_1 e^{x/R_d}}$$

$$\boxed{R_d = \frac{\sqrt{gH}}{f}}$$

Rossby deformation radius (barotropic)

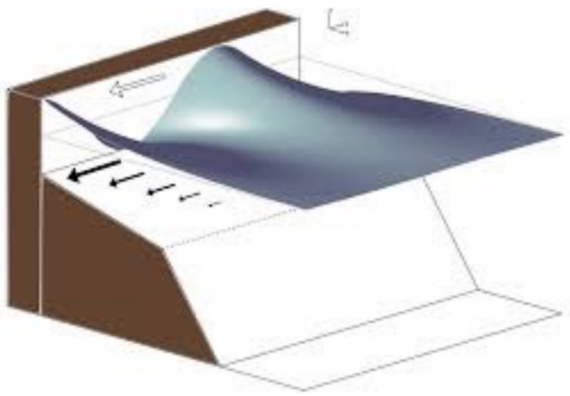
$$\frac{\partial F_2(x)}{\partial x} + \frac{f}{\sqrt{gH}}F_2(x) = 0 \Longrightarrow F_2(x) = C_2 e^{-x/R_d}$$

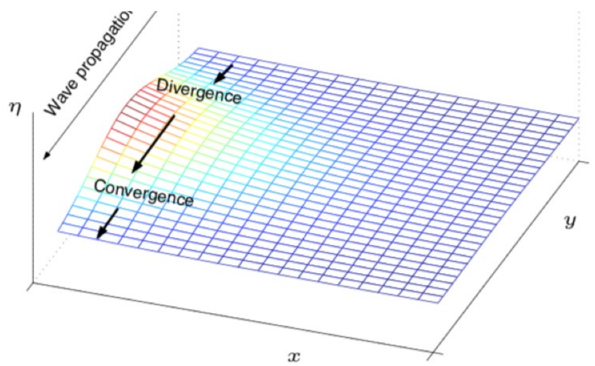
$$v = Ae^{-x/R_d}e^{i(ly+\omega t)}$$

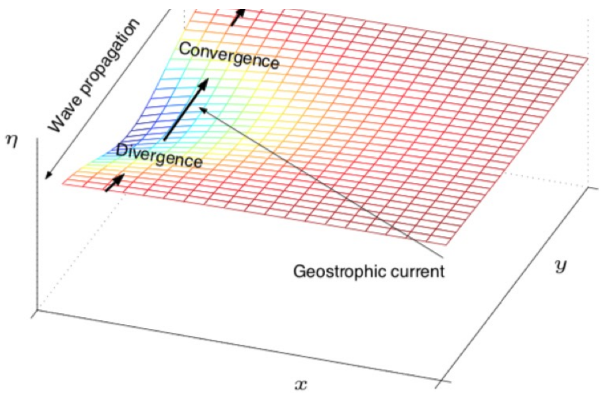
$$\eta = -A\sqrt{\frac{H}{g}}e^{-x/R_d}e^{i(ly+\omega t)}$$

waves propagate toward the +y direction (boundary on the right) in the northern hemisphere

How about in the southern hemisphere?







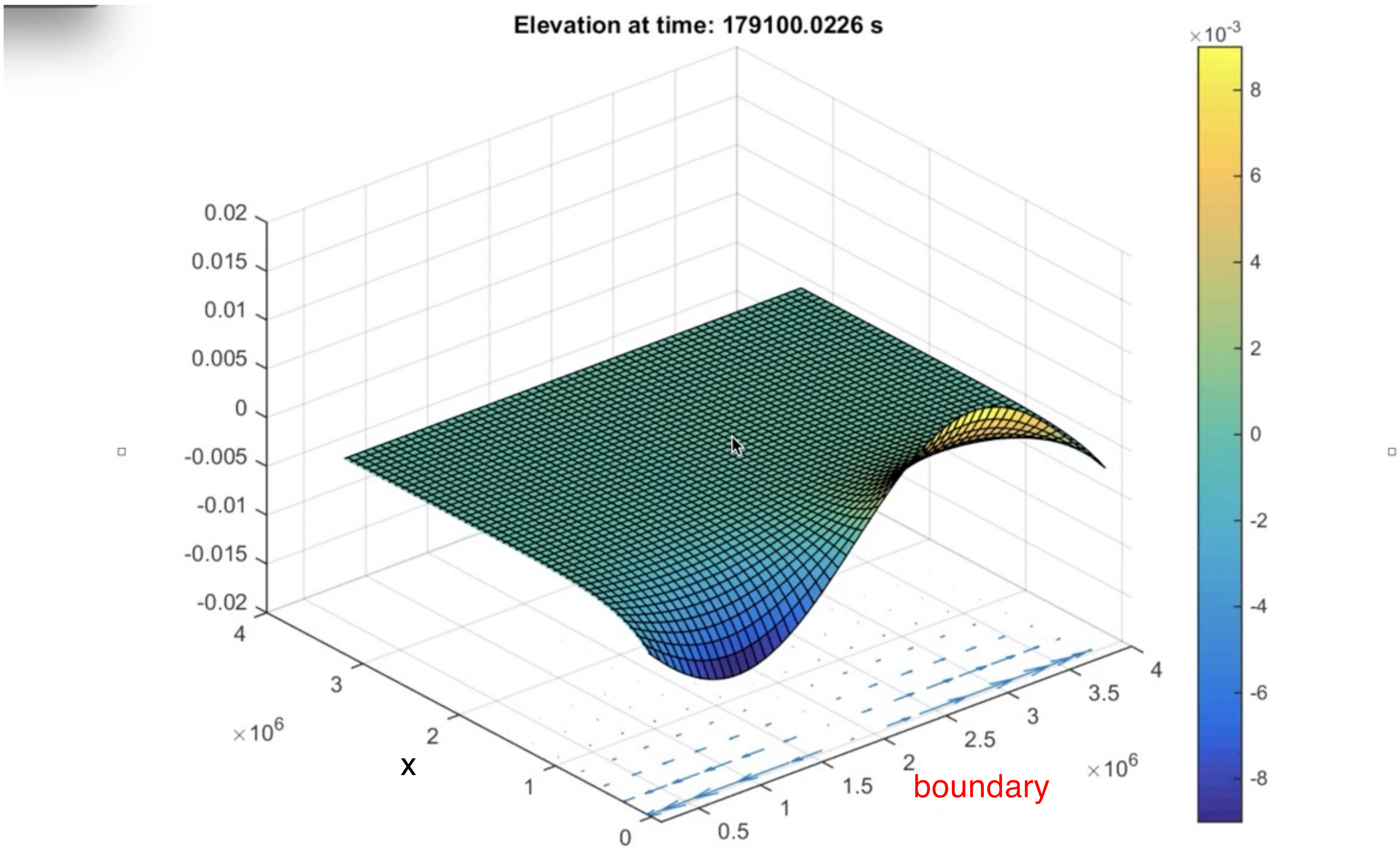
$$u = 0$$

$$v = Ae^{-x/R_d}e^{i(ly+\omega t)}$$

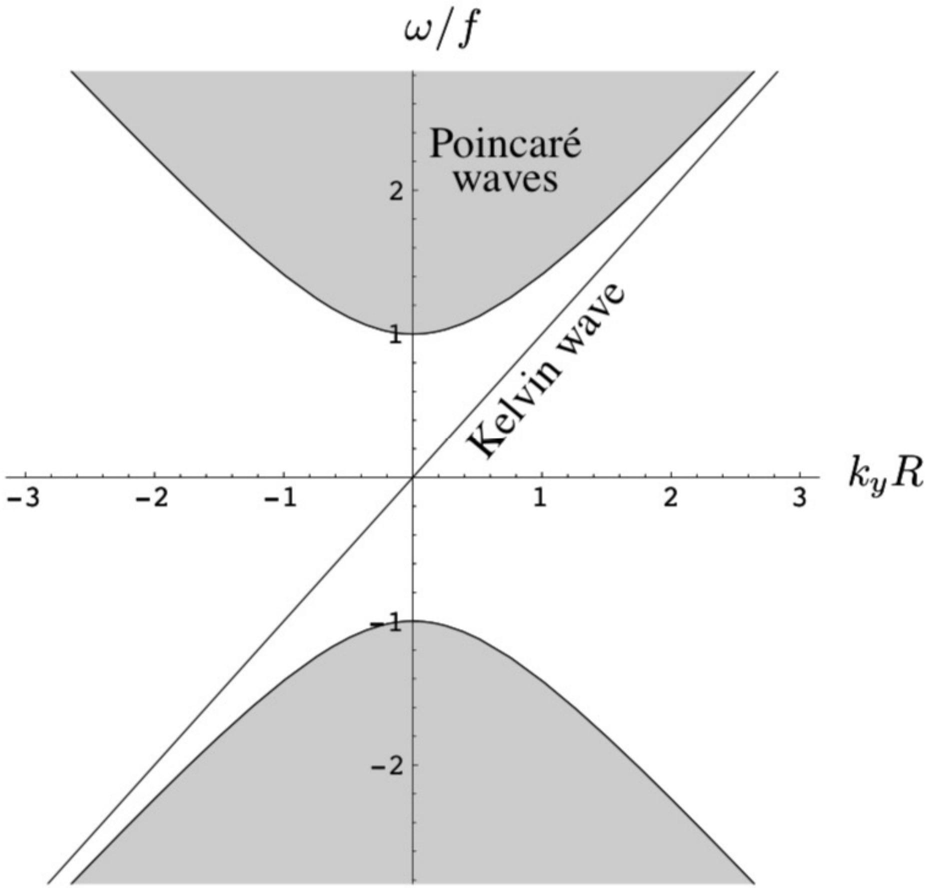
$$\eta = -A\sqrt{\frac{H}{g}}e^{-x/R_d}e^{i(ly+\omega t)}$$

- Kelvin waves propagate with the boundary on the right (left) in the Northern (Southern) Hemisphere
- The wave speed is gravity wave speed ($c = \sqrt{gH}$)
- Velocity perpendicular to the boundary is nail; along-boundary flow is geostrophic
- Surface elevation and along-boundary velocity decay from the boundary to the interior ocean, and the decay scale is R_d (trapped wave)

An upwelling wave ($\eta > 0$) has currents flowing in the direction of wave propagation ($v < 0$)


Elevation at time: 179100.0226 s
0.02
0.015
0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
4
3
2
1
0
×10^6
x
0.5
1
1.5
2
2.5
3
3.5
4
×10^6
boundary
×10^-3
8
6
4
2
0
-2
-4
-6
-8

Dispersion relation diagram



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