Below, a myriad, myriad waves hastening, lifting up their necks,
Tending in ceaseless flow toward the track of the ship,
Waves of the ocean bubbling and gurgling, blithely prying,
Waves, undulating waves, liquid, uneven, emulous waves,
Toward that whirling current, laughing and buoyant.

Walt Whitman, *After the Sea-Ship*, in *Leaves of Grass*, 1881.

CHAPTER 10

Waves, Mean-Flows, and their Interaction

WAVE–MEAN-FLOW INTERACTION is concerned with how some mean-flow, perhaps a time or zonal average, interacts with a wave-like departure from that mean, and this chapter provides an elementary introduction to this topic. It is 'elementary' because our derivations and discussion are obtained by straightforward manipulations of the equations of motion in the simplest case that illustrates the relevant principle. It is implicit in what we do that it is a sensible thing to decompose the fields into a mean plus some departure, and one case when this is so is when the departure is of small amplitude. Departures from the mean — generically called *eddies* — are in reality not always small; for example, in the mid-latitude troposphere the eddies are often of similar amplitude to the mean-flow, and Chapters 12 and 13 explore this from the standpoint of turbulence. However, in this chapter we will assume that eddies are indeed of small amplitude, and, in particular, that eddy–mean-flow interaction is larger than eddy–eddy interaction.

A *wave* is an eddy that satisfies, at least approximately, a dispersion relation. It is the presence of such a dispersion relation that enables a number of results to be obtained that would otherwise be out of our reach. It is implicit in defining waves this way that they are generally of small amplitude, for it is this that allows the equations of motion to be sensibly linearized and a dispersion relation to be obtained (although some waves have finite amplitude and still satisfy a dispersion relation), and the interaction with the mean-flow calculated. The qualitative nature of such interaction can then provide insights into the finite-amplitude problem, and one goal of wave–mean-flow theory is to provide a way of qualitatively understanding more realistic situations, and to suggest diagnostics that might be used to analyze both observations and numerical solutions of the nonlinear problem. In this chapter we will largely concern ourselves with a *zonal* mean, since this is the simplest and often most useful case because of the presence of simple boundary conditions. We will also be mainly concerned with quasi-geostrophic dynamics on a β-plane (and hence Rossby waves), using Boussinesq dynamics, since here the concepts are most clearly illustrated. Thus, in this chapter the reader will find an introduction to such matters as the 'transformed Eulerian mean', the 'Eliassen–Palm flux' and the 'non-acceleration result', and later in the chapter we look at how some related ideas can be used to prove stability of a flow without invoking normal modes. Impatient readers who are anxious for real examples may wish to first look at Chapters 15 and 17 and then come back to this chapter as needed.

10.1 QUASI-GEOSTROPHIC WAVE–MEAN-FLOW INTERACTION

10.1.1 Preliminaries

To fix our dynamical system and notation, we write down the Boussinesq quasi-geostrophic potential vorticity equation

$$\frac{\partial q}{\partial t} + J(\psi, q) = D, \tag{10.1}$$

where D represents any non-conservative terms and the potential vorticity in a Boussinesq system is

$$q = \beta y + \zeta + \frac{\partial}{\partial z}\left(\frac{f_0}{N^2} b\right), \tag{10.2}$$

where ζ is the relative vorticity and b is the buoyancy perturbation from the background state characterized by N^2. (In an ideal gas $q = \beta y + \zeta + (f_0/\rho_R)\partial_z(\rho_R b/N^2)$, where ρ_R is a specified density profile, and most of our derivations can be extended to that case.) We will refer to lines of constant b as isentropes. In terms of the streamfunction, the variables are

$$\zeta = \nabla^2\psi, \qquad b = f_0\frac{\partial \psi}{\partial z}, \qquad q = \beta y + \left[\nabla^2 + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\right)\right]\psi. \tag{10.3}$$

where $\nabla^2 \equiv (\partial_x^2 + \partial_y^2)$. The potential vorticity equation holds in the fluid interior; the boundary conditions on (10.3) are provided by the thermodynamic equation

$$\frac{\partial b}{\partial t} + J(\psi, b) + wN^2 = H, \tag{10.4}$$

where H represents heating terms. The vertical velocity at the boundary, w, is zero in the absence of topography and Ekman friction, and if H is also zero the boundary condition is just

$$\frac{\partial b}{\partial t} + J(\psi, b) = 0. \tag{10.5}$$

Equations (10.1) and (10.5) are the evolution equations for the system and if both D and H are zero they conserve both the total energy, $\widehat{E}$ and the total enstrophy, $\widehat{Z}$:

$$\begin{aligned} \frac{\mathrm{d}\widehat{E}}{\mathrm{d}t} &= 0, \qquad \widehat{E} = \frac{1}{2}\int_V (\nabla\psi)^2 + \frac{f_0^2}{N^2}\left(\frac{\partial\psi}{\partial z}\right)^2 \mathrm{d}V, \\ \frac{\mathrm{d}\widehat{Z}}{\mathrm{d}t} &= 0, \qquad \widehat{Z} = \frac{1}{2}\int_V q^2\,\mathrm{d}V, \end{aligned} \tag{10.6}$$

where V is a volume bounded by surfaces at which the normal velocity is zero, or that has periodic boundary conditions. The enstrophy is also conserved layerwise; that is, the horizontal integral of q^2 is conserved at every level.

10.1.2 Potential Vorticity Flux in the Linear Equations

Let us decompose the fields into a mean (to be denoted with an overbar) plus a perturbation (denoted with a prime), and let us suppose the perturbation fields are of small amplitude. (In linear problems, such as those considered in Chapter 9, we decomposed the flow into a 'basic state' plus a perturbation, with the basic state fixed in time. Our approach here is similar, but soon we will allow the mean state to evolve.) The linearized quasi-geostrophic potential vorticity equation is then

$$\frac{\partial q'}{\partial t} + \overline{u}\frac{\partial q'}{\partial x} + u'\frac{\partial \overline{q}}{\partial x} + \overline{v}\frac{\partial q'}{\partial y} + v'\frac{\partial \overline{q}}{\partial y} = D', \tag{10.7}$$

where D' represents eddy forcing and dissipation and, in terms of streamfunction,

$$\left(u'(x,y,z,t), v'(x,y,z,t)\right) = \left(-\frac{\partial \psi'}{\partial y}, \frac{\partial \psi'}{\partial x}\right), \tag{10.8a}$$

$$q'(x,y,z,t) = \nabla^2\psi' + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi'}{\partial z}\right). \tag{10.8b}$$

If the mean is a zonal mean then $\partial\overline{q}/\partial x = 0$ and $\overline{v} = 0$ (because v is purely geostrophic) and (10.7) simplifies to

$$\frac{\partial q'}{\partial t} + \overline{u}\frac{\partial q'}{\partial x} + v'\frac{\partial \overline{q}}{\partial y} = D', \tag{10.9}$$

where

$$\overline{q} = \beta y - \frac{\partial \overline{u}}{\partial y} + \frac{\partial}{\partial z}\left(\frac{f_0}{N^2}\overline{b}\right), \quad \text{and} \quad \frac{\partial \overline{q}}{\partial y} = \beta - \frac{\partial^2 \overline{u}}{\partial y^2} - \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \overline{u}}{\partial z}\right). \tag{10.10a,b}$$

using thermal wind, $f_0\partial\overline{u}/\partial z = -\partial b/\partial y$.

Multiplying by q' and zonally averaging gives the enstrophy equation:

$$\frac{1}{2}\frac{\partial}{\partial t}\overline{q'^2} = -\overline{v'q'}\frac{\partial \overline{q}}{\partial y} + \overline{D'q'}. \tag{10.11}$$

The quantity $\overline{v'q'}$ is the meridional flux of potential vorticity; this is downgradient (by definition) when the first term on the right-hand side is positive (i.e., $\overline{v'q'}\partial\overline{q}/\partial y < 0$), and it then acts to increase the variance of the perturbation. (This occurs, for example, when the flux is diffusive so that $\overline{v'q'} = -\kappa\partial\overline{q}/\partial y$, where κ may vary but is everywhere positive.) This argument may be inverted: for inviscid flow ($D = 0$), if the waves are growing, as for example in the canonical models of baroclinic instability discussed in Chapter 9, then *the potential vorticity flux is downgradient*.

If the second term on the right-hand side of (10.11) is negative, as it will be if D' is a dissipative process (e.g., if $D' = A\nabla^2 q'$ or if $D' = -rq'$, where A and r are positive) then a statistical balance can be achieved between enstrophy production via downgradient transport, and dissipation. If the waves are steady (by which we mean statistically steady, neither growing nor decaying in amplitude) and conservative (i.e., $D' = 0$) then we must have

$$\overline{v'q'} = 0. \tag{10.12}$$

Similar results follow for the buoyancy at the boundary; we start by linearizing the thermodynamic equation (10.5) to give

$$\frac{\partial b'}{\partial t} + \overline{u}\frac{\partial b'}{\partial x} + v'\frac{\partial \overline{b}}{\partial y} = H', \tag{10.13}$$

where H' is a diabatic source term. Multiplying (10.13) by b' and averaging gives

$$\frac{1}{2}\frac{\partial}{\partial t}\overline{b'^2} = -\overline{v'b'}\frac{\partial \overline{b}}{\partial y} + \overline{H'b'}. \tag{10.14}$$

Thus growing adiabatic waves have a downgradient flux of buoyancy at the boundary. In the Eady problem there is no interior gradient of basic-state potential vorticity and all the terms in (10.11) are zero, but the perturbation grows at the boundary. If the waves are steady and adiabatic then, analogously to (10.12),

$$\overline{v'b'} = 0. \tag{10.15}$$

The boundary conditions and fluxes may be absorbed into the interior definition of potential vorticity and its fluxes by way of the delta-function boundary layer construction, described in Section 5.4.3. In models with discrete vertical layers or a finite number of levels it is common practice to absorb the boundary conditions into the definition of potential vorticity at top and bottom.

10.1.3 Wave–Mean-Flow Interaction

In linear problems we usually suppose that the mean-flow is fixed and that the zonal mean terms, $\overline{u}$ and $\overline{q}$ in (10.9), are functions only of y and z. However, in reality we might expect that the mean-flow would change because of momentum and heat flux convergences arising from the eddy–eddy interactions. To calculate these changes we begin with the potential vorticity equation (10.1) and, in the usual way, express the variables as a zonal mean plus an eddy term and obtain

$$\frac{\partial \overline{q}}{\partial t} + \nabla \cdot (\overline{\boldsymbol{u}}\,\overline{q}) + \nabla \cdot (\overline{\boldsymbol{u}' q'}) = \overline{D}. \tag{10.16}$$

Now, since the mean-flow is a zonal mean, and $\overline{v} = 0$, the first term is zero and the mean-flow evolves according to

$$\frac{\partial \overline{q}}{\partial t} + \frac{\partial}{\partial y}\overline{v' q'} = \overline{D}. \tag{10.17}$$

Similarly, at the boundary the mean buoyancy evolution equation is

$$\frac{\partial \overline{b}}{\partial t} + \frac{\partial}{\partial y}\overline{v' b'} = \overline{H}. \tag{10.18}$$

To obtain $\overline{u}$ from $\overline{q}$ and $\overline{b}$ we use thermal wind balance to define a streamfunction Ψ. That is, since

$$f_0 \frac{\partial \overline{u}}{\partial z} = -\frac{\partial \overline{b}}{\partial y}, \qquad \text{then} \qquad \left(\overline{u}, \frac{1}{f_0}\overline{b}\right) = \left(-\frac{\partial \Psi}{\partial y}, \frac{\partial \Psi}{\partial z}\right) \tag{10.19a,b}$$

whence, using (10.10a), the potential vorticity is

$$\overline{q}(y,z,t) - \beta y = \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \Psi}{\partial z}\right) + \frac{\partial^2 \Psi}{\partial y^2}. \tag{10.20}$$

If $\overline{q}$ is known in the interior from (10.18), and $\overline{b}$ (i.e., $f_0 \partial\Psi/\partial z$) is known at the boundaries, then $\overline{u}$ and $\overline{b}$ in the interior may be obtained using (10.20) and (10.19b). The equations are also summarized in the grey box on page 390.

To close the system we suppose that the eddy terms themselves evolve according to (10.9) and (10.13). If in those equations we were to include the eddy–eddy interaction terms we would simply recover the full system, so in neglecting those terms we have constructed an eddy–mean-flow system, commonly called a *wave–mean-flow* system because by eliminating the nonlinear terms in the perturbation equation the eddies will often be wavelike. Non quasi-geostrophic wave–mean-flow systems may be constructed in a similar fashion: for example, we could construct a system using the primitive equations with separate equations for eddy and zonal-mean temperature and velocity fields, and an example involving gravity waves is given in Chapter 17.

It is important to realise that such systems do differ from linear ones. In constructing linear systems we posit that the eddy terms are small compared to the mean-flow and thus neglect the eddy–eddy interaction terms and keep the mean-flow fixed. In a wave–mean-flow problem we similarly suppose the eddy terms are small, and we neglect eddy–eddy interaction terms where they produce another eddy, because the terms involving the mean-flow are larger. However, in the mean-flow equation, (10.16), there are no mean-flow terms that are larger, so we keep the eddy–eddy terms and allow the mean-flow to evolve. Such a justification is hardly a rigorous one, since if the eddy terms are small then the effects on the mean-flow will be small, and so one might suppose that the mean-flow should be held fixed. The wave–mean-flow equations really can only be justified on a case-by-case basis with a detailed examination of the size of the terms and the rate at which they evolve, and that is the subject of weakly nonlinear theory. Another justification for wave–mean-flow problems is that they lead to insight into the behaviour of the full system.

We now consider some more properties of the waves themselves — how they propagate and what they conserve — beginning with a discussion of the potential vorticity flux and its relative, the Eliassen–Palm flux

10.2 THE ELIASSEN–PALM FLUX

The eddy flux of potential vorticity may be expressed in terms of vorticity and buoyancy fluxes as

$$v'q' = v'\zeta' + f_0 v' \frac{\partial}{\partial z}\left(\frac{b'}{N^2}\right). \tag{10.21}$$

The second term on the right-hand side can be written as

$$\begin{aligned} f_0 v' \frac{\partial}{\partial z}\left(\frac{b'}{N^2}\right) &= f_0 \frac{\partial}{\partial z}\left(\frac{v'b'}{N^2}\right) - f_0 \frac{\partial v'}{\partial z}\frac{b'}{N^2} \\ &= f_0 \frac{\partial}{\partial z}\left(\frac{v'b'}{N^2}\right) - f_0 \frac{\partial}{\partial x}\left(\frac{\partial \psi'}{\partial z}\right)\frac{b'}{N^2} \\ &= f_0 \frac{\partial}{\partial z}\left(\frac{v'b'}{N^2}\right) - \frac{f_0^2}{2N^2}\frac{\partial}{\partial x}\left(\frac{\partial \psi'}{\partial z}\right)^2, \end{aligned} \tag{10.22}$$

using $b' = f_0 \partial\psi'/\partial z$.

Similarly, the flux of relative vorticity can be written

$$v'\zeta' = -\frac{\partial}{\partial y}(u'v') + \frac{1}{2}\frac{\partial}{\partial x}(v'^2 - u'^2), \tag{10.23}$$

Using (10.22) and (10.23), (10.21) becomes

$$v'q' = -\frac{\partial}{\partial y}(u'v') + \frac{\partial}{\partial z}\left(\frac{f_0}{N^2}v'b'\right) + \frac{1}{2}\frac{\partial}{\partial x}\left((v'^2 - u'^2) - \frac{b'^2}{N^2}\right). \tag{10.24}$$

Thus the meridional potential vorticity flux, in the quasi-geostrophic approximation, can be written as the divergence of a vector: $v'q' = \nabla \cdot \boldsymbol{\mathcal{E}}$ where

$$\boldsymbol{\mathcal{E}} \equiv \frac{1}{2}\left((v'^2 - u'^2) - \frac{b'^2}{N^2}\right)\mathbf{i} - (u'v')\,\mathbf{j} + \left(\frac{f_0}{N^2}v'b'\right)\mathbf{k}. \tag{10.25}$$

A particularly useful form of this arises after zonally averaging, for then (10.24) becomes

$$\overline{v'q'} = -\frac{\partial}{\partial y}\overline{u'v'} + \frac{\partial}{\partial z}\left(\frac{f_0}{N^2}\overline{v'b'}\right). \tag{10.26}$$

The vector defined by

$$\boldsymbol{\mathcal{F}} \equiv -\overline{u'v'}\,\mathbf{j} + \frac{f_0}{N^2}\overline{v'b'}\,\mathbf{k} \tag{10.27}$$

is called the (quasi-geostrophic) *Eliassen–Palm (EP) flux*,[1] and its divergence, given by (10.26), gives the poleward flux of potential vorticity:

$$\overline{v'q'} = \nabla_x \cdot \boldsymbol{\mathcal{F}}, \tag{10.28}$$

where $\nabla_x \cdot \equiv (\partial/\partial y, \partial/\partial z)\cdot$ is the divergence in the meridional plane. Unless the meaning is unclear, the subscript x on the meridional divergence will be dropped.

10.2.1 The Eliassen–Palm Relation

On dividing by $\partial\overline{q}/\partial y$ and using (10.28), the enstrophy equation (10.11) becomes

$$\boxed{\frac{\partial \mathcal{A}}{\partial t} + \nabla\cdot\boldsymbol{\mathcal{F}} = \mathcal{D},} \tag{10.29a}$$

where

$$\mathcal{A} = \frac{\overline{q'^2}}{2\partial\overline{q}/\partial y}, \qquad \mathcal{D} = \frac{\overline{D'q'}}{\partial\overline{q}/\partial y}, \tag{10.29b}$$

and $\boldsymbol{\mathcal{F}}$ is given by (10.27). Equation (10.29a) is known as the *Eliassen–Palm relation,* and it is a conservation law (when $\mathcal{D} = 0$) for the *wave activity density* $\mathcal{A}$. (We also encountered wave activities in Section 6.7.2). The conservation law is exact (in the linear approximation) if the mean-flow is constant in time; it will be a good approximation if $\partial\overline{q}/\partial y$ varies slowly compared to the variation of $\overline{q'^2}$. In this instance $\mathcal{A}$ is the *pseudomomentum density*, $\mathcal{P}$, but other kinds of wave activity density exist — the pseudoenergy density for example, which we will encounter later.

If we integrate (10.29a) over a meridional area A bounded by walls where the eddy activity vanishes, and if $\mathcal{D} = 0$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_A \mathcal{A}\,\mathrm{d}A = 0. \tag{10.30}$$

The integral is a wave activity — a quantity that is quadratic in the amplitude of the perturbation and that is conserved in the absence of forcing and dissipation. If there is no ambiguity we will drop the word density and also refer to $\mathcal{A}$ and $\mathcal{P}$ as wave activities. ('Wave action' is related to wave activity, but specifically means energy divided by the frequency; it is also conserved in many problems.) Note that neither the perturbation energy nor the perturbation enstrophy are wave activities of the linearized equations, because there can be an exchange of energy or enstrophy between mean and perturbation — indeed, this is how a perturbation grows in baroclinic or barotropic instability! This is already evident from (10.11), or in general take (10.7) with $D' = 0$ and multiply by q' to give the enstrophy equation,

$$\frac{1}{2}\frac{\partial q'^2}{\partial t} + \frac{1}{2}\overline{\boldsymbol{u}}\cdot\nabla q'^2 + \boldsymbol{u}'q'\cdot\nabla\overline{q} = 0, \tag{10.31}$$

where here the overbar is an average (although it need not be a zonal average). Integrating this over a volume V gives

$$\frac{\mathrm{d}\hat{Z}'}{\mathrm{d}t} \equiv \frac{\mathrm{d}}{\mathrm{d}t}\int_V \frac{1}{2}q'^2\,\mathrm{d}V = -\int_V \boldsymbol{u}'q'\cdot\nabla\overline{q}\,\mathrm{d}V. \tag{10.32}$$

The right-hand side does not, in general, vanish and so $\hat{Z}'$ is not in general conserved.

10.2.2 The Group Velocity Property for Rossby Waves

The vector $\boldsymbol{\mathcal{F}}$ describes how the wave activity propagates. We noted in Chapter 6 that in the case in which the disturbance is composed of plane or almost plane waves that satisfy a dispersion relation, then $\boldsymbol{\mathcal{F}} = \boldsymbol{c}_g\mathcal{A}$, where $\boldsymbol{c}_g$ is the group velocity and (10.29a) becomes

$$\frac{\partial \mathcal{A}}{\partial t} + \nabla\cdot(\mathcal{A}\boldsymbol{c}_g) = 0. \tag{10.33}$$

This is a useful property, because if we can diagnose $\boldsymbol{c}_g$ from observations we can use (10.29a) to determine how wave activity density propagates. Let us demonstrate this explicitly for the pseudomomentum in Rossby waves.

The Boussinesq quasi-geostrophic equation on the β-plane, linearized around a uniform zonal flow and with constant static stability, is

$$\frac{\partial q'}{\partial t} + \overline{u}\frac{\partial q'}{\partial x} + v'\frac{\partial \overline{q}}{\partial y} = 0, \tag{10.34}$$

where $q' = [\nabla^2 + (f_0^2/N^2)\partial^2/\partial z^2]\psi'$ and, if $\overline{u}$ is constant, $\partial\overline{q}/\partial y = \beta$. Thus we have

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)\left[\nabla^2\psi' + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi'}{\partial z}\right)\right] + \beta\frac{\partial \psi'}{\partial x} = 0. \tag{10.35}$$

Seeking solutions of the form

$$\psi' = \mathrm{Re}\,\widetilde{\psi}\mathrm{e}^{\mathrm{i}(kx+ly+mz-\omega t)}, \tag{10.36}$$

we find the dispersion relation,

$$\omega = \overline{u}k - \frac{\beta k}{\kappa^2}, \tag{10.37}$$

where $\kappa^2 = (k^2 + l^2 + m^2 f_0^2/N^2)$, and the group velocity components:

$$c_g^y = \frac{2\beta kl}{\kappa^4}, \qquad c_g^z = \frac{2\beta km f_0^2/N^2}{\kappa^4}. \tag{10.38}$$

Also, if $u' = \mathrm{Re}\,\widetilde{u}\exp[\mathrm{i}(kx + ly + mz - \omega t)]$, and similarly for the other fields, then

$$\begin{aligned} \widetilde{u} &= -\mathrm{Re}\ \mathrm{i}l\widetilde{\psi}, \qquad & \widetilde{v} &= \mathrm{Re}\ \mathrm{i}k\widetilde{\psi}, \\ \widetilde{b} &= \mathrm{Re}\,\mathrm{i}mf_0\widetilde{\psi}, \qquad & \widetilde{q} &= -\mathrm{Re}\,\kappa^2\widetilde{\psi}. \end{aligned} \tag{10.39}$$

The wave activity density is then

$$\mathcal{A} = \frac{1}{2}\frac{\overline{q'^2}}{\beta} = \frac{\kappa^4}{4\beta}|\widetilde{\psi}^2|, \tag{10.40}$$

where the additional factor of 2 in the denominator arises from the averaging. Using (10.39) the EP flux, (10.27), is

$$\mathcal{F}^y = -\overline{u'v'} = \frac{1}{2}kl|\widetilde{\psi}^2|, \qquad \mathcal{F}^z = \frac{f_0}{N^2}\overline{v'b'} = \frac{f_0^2}{2N^2}km|\widetilde{\psi}^2|. \tag{10.41}$$

Using (10.38), (10.40) and (10.41) we obtain

$$\boxed{\boldsymbol{\mathcal{F}} = (\mathcal{F}^y, \mathcal{F}^z) = \boldsymbol{c}_g\mathcal{A}.} \tag{10.42}$$

If the properties of the medium are slowly varying, so that a (spatially varying) group velocity can still be defined, then this is a useful expression to estimate how the wave activity propagates in the atmosphere and in numerical simulations.

10.2.3 ✦ The Orthogonality of Modes

It is a direct consequence of the conservation of wave activity that disturbance modes are orthogonal in the 'wave activity norm', defined later on, and thus are a useful measure of the amplitude of a particular mode.[2] To explore this, we start with the linearized potential vorticity equation,

$$\frac{\partial q'}{\partial t} + \overline{u}\frac{\partial q'}{\partial x} + v'\frac{\partial \overline{q}}{\partial y} = 0. \tag{10.43}$$

Let us formally seek solutions of the form $\psi' = \mathrm{Re}\,\Psi \exp(\mathrm{i}kx)$ where Ψ is the sum of *modes,*

$$\Psi = \sum_n \widetilde{\psi}_n(y,z)\mathrm{e}^{-\mathrm{i}kc_n t}, \tag{10.44}$$

where n is an identifier of the modes. The modes satisfy

$$(\overline{u}\Delta_k^2 + \overline{q}_y)\widetilde{\psi}_n = c_n\Delta_k^2\widetilde{\psi}_n, \tag{10.45}$$

where

$$\Delta_k^2 = \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\right) - k^2. \tag{10.46}$$

The upper and lower boundary conditions (at $z = 0, -H$) are given by the thermodynamic equation

$$\frac{\partial b'}{\partial t} + \overline{u}\frac{\partial b'}{\partial x} + v'\frac{\partial \overline{b}}{\partial y} = 0, \tag{10.47}$$

and if we simplify further by supposing $\partial\overline{u}/\partial z = 0$ then the boundary condition becomes

$$\frac{\partial \psi_z'}{\partial t} + \overline{u}\frac{\partial \psi_z'}{\partial x} = 0. \tag{10.48}$$

There are no meridional buoyancy fluxes at the boundary. If N^2 is a constant (a simplifying but not essential assumption) then we can let $\widetilde{\psi}_n(y,z) = \psi_n(y)\cos pz$, with $p = j\pi/H$ where j is an integer and the mode n now labels only the meridional modes. The corresponding potential vorticity modes are given by

$$q_n = \Delta_{k,m}^2\psi_n, \qquad \Delta_{k,m}^2 = \frac{\partial^2}{\partial y^2} - \frac{f_0^2}{N^2}m^2 - k^2, \tag{10.49}$$

and the boundary conditions are then built in to any solution we construct from (10.45) and (10.49).[3] We may then consider a single zonal and a single vertical wavenumber. (If there is no horizontal variation of the shear, the meridional modes are harmonic functions, for example $\psi_n \propto \sin(n\pi y/L)$ for a channel of width L.)

For a given basic state we may imagine solving (10.45), numerically or analytically, and determining the modes. However, these modes are not orthogonal in the sense of either energy or enstrophy. That is, denoting the inner product by

$$\langle a, b\rangle \equiv \frac{1}{2L}\int_L ab\,\mathrm{d}y, \tag{10.50}$$

then, in general,

$$I_E = \langle \psi_n, q_m\rangle \neq 0, \qquad I_Z = \langle q_n, q_m\rangle \neq 0, \tag{10.51a,b}$$

for $n \neq m$, where $q_n = \Delta_{k,p}^2\psi_n$. Perturbation energy and enstrophy are thus not wave activities of the linearized equations, and it is not meaningful to talk about the energy or enstrophy of a particular mode. However, by the same token we may expect orthogonality in the wave activity norm. To prove this and understand what it means, suppose that at $t = 0$ the disturbance consists of two modes, n and m, so that at a later time $q = (q_n\mathrm{e}^{-\mathrm{i}kc_n t} + q_m\mathrm{e}^{-\mathrm{i}kc_m t} + \text{c.c.})$, where $c_m \neq c_n$ and we assume that both are real. The wave activity is

$$P \equiv \int \mathcal{A}\,\mathrm{d}y\,\mathrm{d}z = \left\langle q_n, q_m^*/\overline{q}_y\right\rangle \mathrm{e}^{-\mathrm{i}k(c_n - c_m)t} + \left\langle q_m, q_m^*/\overline{q}_y\right\rangle + \left\langle q_n, q_n^*/\overline{q}_y\right\rangle + \text{c.c.} \tag{10.52}$$

The second and third terms on the right-hand side are the wave activities of each mode, and these are constants (to see this, consider the case when the disturbance is just a single mode). Now, because $\mathrm{d}P/\mathrm{d}t = 0$ the first term must vanish if $c_n \neq c_m$, implying the modes are orthogonal and, in particular,

$$\mathrm{Re} \int \frac{1}{\overline{q}_y} q_n q_m^* \, \mathrm{d}y = 0, \tag{10.53}$$

for $n \neq m$. The inner product weighted by $1/\overline{q}_y$ defines the wave activity norm. Orthogonality is a useful result, for it means that the wave activity is a proper measure of the amplitude of a given mode unlike, for example, energy. The conservation of wave activity will lead to a particularly straightforward derivation of the necessary conditions for stability, given in Section 10.6.

10.3 THE TRANSFORMED EULERIAN MEAN

The so-called *transformed Eulerian mean,* or TEM, is a transformation of the equations of motion that provides a useful framework for discussing eddy effects under a wide range of conditions.[4] It is useful because, as we shall see, it is equivalent to a very natural form of averaging the equations that serves to eliminate eddy fluxes in the thermodynamic equation and collect them together, in a simple form, in the momentum equation, and in so doing it highlights the role of potential vorticity fluxes. The TEM also provides a natural separation between diabatic and adiabatic effects or between advective and diffusive fluxes and, in the case in which the flow is adiabatic, a pleasing simplification of the equations. In later chapters we will use the TEM to better understand the mid-latitude troposphere and the dynamics of the Antarctic Circumpolar Current, and as a framework for the parameterization of eddy fluxes. Of course, there being no free lunch, the TEM brings with it its own difficulties, and in particular the implementation of boundary conditions can cause difficulties, especially in the actual numerical integration of the equations.

10.3.1 Quasi-Geostrophic Form

For simplicity we will use the Boussinesq equations on the beta-plane. The zonally-averaged Eulerian mean equations for the zonally-averaged zonal velocity and buoyancy may then be written as (see Section 2.2.6)

$$\frac{\partial \overline{u}}{\partial t} - (f + \overline{\zeta})\overline{v} + \overline{w}\frac{\partial \overline{u}}{\partial z} = -\frac{\partial}{\partial y}\overline{u'v'} - \frac{\partial}{\partial z}\overline{u'w'} + \overline{F}, \tag{10.54a}$$

$$\frac{\partial \overline{b}}{\partial t} + \overline{v}\frac{\partial \overline{b}}{\partial y} + \overline{w}\frac{\partial \overline{b}}{\partial z} = -\frac{\partial}{\partial y}\overline{v'b'} - \frac{\partial}{\partial z}\overline{w'b'} + \overline{S}, \tag{10.54b}$$

where $\overline{F}$ and $\overline{S}$ represent frictional and heating terms, respectively, and the meridional velocity, $\overline{v}$, is purely ageostrophic. Using quasi-geostrophic scaling we neglect the vertical eddy flux divergences and all ageostrophic velocities except when multiplied by f_0 or N^2. The above equations then become

$$\frac{\partial \overline{u}}{\partial t} = f_0\overline{v} - \frac{\partial}{\partial y}\overline{u'v'} + \overline{F}, \tag{10.55a}$$

$$\frac{\partial \overline{b}}{\partial t} = -N^2\overline{w} - \frac{\partial}{\partial y}\overline{v'b'} + \overline{S}. \tag{10.55b}$$

These two equations are connected by the thermal wind relation,

$$f_0\frac{\partial \overline{u}}{\partial z} = -\frac{\partial \overline{b}}{\partial y}, \tag{10.56}$$

which is a combination of the geostrophic v-momentum equation ($f_0\overline{u} = -\partial\overline{\phi}/\partial y$) and hydrostasy ($\partial\overline{\phi}/\partial z = \overline{b}$). One less than ideal aspect of (10.55) is that in the extratropics the dominant balance is usually between the first two terms on the right-hand sides of each equation, even in time-dependent cases. Thus, the Coriolis force closely balances the divergence of the eddy momentum fluxes, and the advection of the mean stratification (N^2w, or 'adiabatic cooling') often balances the divergence of eddy heat flux, with heating being a small residual. This may lead to an underestimation of the importance of diabatic heating, as this is ultimately responsible for the mean meridional circulation. Furthermore, the link between $\overline{u}$ and $\overline{b}$ via thermal wind dynamically couples buoyancy and momentum, and obscures the understanding of how the eddy fluxes influence these fields — is it through the eddy heat fluxes or momentum fluxes, or some combination?

To address this issue we combine the terms N^2w and the eddy flux in (10.55b) into a single total or *residual* (so recognizing the cancellation between the mean and eddy terms) heat transport term that in a steady state is balanced by the diabatic term $\overline{S}$. To do this, we first note that because $\overline{v}$ and $\overline{w}$ are related by mass conservation we can define a mean meridional streamfunction ψ_m such that

$$(\overline{v}, \overline{w}) = \left(-\frac{\partial\psi_m}{\partial z}, \frac{\partial\psi_m}{\partial y}\right). \tag{10.57}$$

The velocities then satisfy $\partial\overline{v}/\partial y + \partial\overline{w}/\partial z = 0$ automatically. If we define a *residual streamfunction* by

$$\psi^* \equiv \psi_m + \frac{1}{N^2}\overline{v'b'}, \tag{10.58a}$$

the components of the *residual mean meridional circulation* are then given by

$$(\overline{v}^*, \overline{w}^*) = \left(-\frac{\partial\psi^*}{\partial z}, \frac{\partial\psi^*}{\partial y}\right), \tag{10.58b}$$

and

$$\overline{v}^* = \overline{v} - \frac{\partial}{\partial z}\left(\frac{1}{N^2}\overline{v'b'}\right), \qquad \overline{w}^* = \overline{w} + \frac{\partial}{\partial y}\left(\frac{1}{N^2}\overline{v'b'}\right). \tag{10.59}$$

Note that by construction, the residual overturning circulation satisfies

$$\frac{\partial\overline{v}^*}{\partial y} + \frac{\partial\overline{w}^*}{\partial z} = 0. \tag{10.60}$$

Substituting (10.59) into (10.55a) and (10.55b) the zonal momentum and buoyancy equations then take the simple forms

$$\boxed{\begin{aligned} \frac{\partial\overline{u}}{\partial t} &= f_0\overline{v}^* + \overline{v'q'} + \overline{F}, \\ \frac{\partial\overline{b}}{\partial t} &= -N^2\overline{w}^* + \overline{S}, \end{aligned}} \tag{10.61a,b}$$

which are known as the (quasi-geostrophic) *transformed Eulerian mean equations,* or TEM equations. The potential vorticity flux, $\overline{v'q'}$, is given in terms of the heat and vorticity fluxes by (10.26), and is equal to the divergence of the Eliassen–Palm flux as in (10.28).

The TEM equations make it apparent that we may consider the potential vorticity fluxes, rather than the separate contributions of the vorticity and heat fluxes, to force the circulation. If we know the potential vorticity flux as well as $\overline{F}$ and $\overline{S}$, then (10.60) and (10.61), along with thermal wind balance

$$f_0\frac{\partial\overline{u}}{\partial z} = -\frac{\partial\overline{b}}{\partial y}, \tag{10.62}$$

form a complete set. The meridional overturning circulation is obtained by eliminating time derivatives from (10.61) using (10.62), giving

$$f_0^2 \frac{\partial^2 \psi^*}{\partial z^2} + N^2 \frac{\partial^2 \psi^*}{\partial y^2} = f_0 \frac{\partial}{\partial z} \overline{v' q'} + f_0 \frac{\partial \overline{F}}{\partial z} + \frac{\partial \overline{S}}{\partial y}. \tag{10.63}$$

Thus, the residual or net overturning circulation is driven by the (vertical derivative of the) potential vorticity fluxes and the diabatic terms — 'driven' in the sense that if we know those terms we can calculate the overturning circulation, although of course the fluxes themselves depend on the circulation. Note that this equation applies at every instant, even if the equations are not in a steady state.

Use of the equations in TEM form is particularly useful when the eddy potential vorticity flux arises from wave activity, for example from Rossby waves. The potential vorticity flux is the convergence of the EP flux $\boldsymbol{\mathcal{F}}$, as in (10.28), and if the eddies satisfy a dispersion relation the components of the EP flux are equal to the group velocity multiplied by the wave activity density $\mathcal{A}$, as in (10.42). Thus, knowing the group velocity tells us a great deal about how momentum is transported by waves. We'll use the TEM to deduce the mean-flow acceleration in Sections 10.4, 10.5 and, in particular, in Section 17.3.

Connection to potential vorticity and wave–mean-flow interaction

If we cross-differentiate (10.61) then, after using the residual mass continuity equation (10.60), we recover the zonally-averaged potential vorticity equation, namely

$$\frac{\partial \overline{q}}{\partial t} = -\frac{\partial}{\partial y} \overline{v' q'} - \frac{\partial \overline{F}}{\partial y}, \qquad \text{where} \qquad \overline{q}(y, t) = \frac{\partial}{\partial z}\left(\frac{f_0}{N^2}\overline{b}\right) - \frac{\partial \overline{u}}{\partial y}, \tag{10.64a,b}$$

which is essentially the same as (10.18) and (10.20), noting that we may add βy to the definition of zonally-averaged potential vorticity with no effect.

The corresponding equation for the evolution of eddy potential vorticity is, in its inviscid form,

$$\left(\frac{\partial}{\partial t} + \overline{u}(y, t)\frac{\partial}{\partial x}\right) q' + v' \frac{\partial \overline{q}}{\partial y} = 0, \tag{10.65}$$

as in (10.7). Equations (10.64) and (10.65) are a closed set of quasi-linear equations, and we have recovered the wave–mean-flow system described in Section 10.1.3.

10.3.2 The TEM in Isentropic Coordinates

The residual circulation has an illuminating interpretation if we think of the fluid as comprising multiple layers of shallow water, or equivalently if we cast the problem in isentropic coordinates (Section 3.10). Using the notation of a shallow water system, the momentum and mass conservation equation can be written as

$$\frac{\partial u}{\partial t} + \boldsymbol{u} \cdot \nabla u - f v = F, \qquad \frac{\partial h}{\partial t} + \nabla \cdot (h\boldsymbol{u}) = S. \tag{10.66a,b}$$

The quantity h is the thickness — the separation between two isentropic surfaces — and S is a thickness source term. (The field h plays the same role as σ in Section 3.10.) With quasi-geostrophic scaling, so that variations in Coriolis parameter and layer thickness are small, zonally averaging in a conventional way gives

$$\frac{\partial \overline{u}}{\partial t} - f_0 \overline{v} = \overline{v' \zeta'} + \overline{F}, \qquad \frac{\partial \overline{h}}{\partial t} + H \frac{\partial \overline{v}}{\partial y} = -\frac{\partial}{\partial y} \overline{v' h'} + \overline{S}. \tag{10.67a,b}$$

Quasi-Geostrophic Wave–Mean-Flow Interaction

The inviscid and unforced Boussinesq quasi-geostrophic set of wave–mean-flow equations is

$$\frac{\partial q'}{\partial t} + \overline{u}\frac{\partial q'}{\partial x} + v'\frac{\partial \overline{q}}{\partial y} = 0, \tag{WMF.1a}$$

$$\frac{\partial \overline{q}}{\partial t} + \frac{\partial}{\partial y}\overline{v'q'} = 0, \tag{WMF.1b}$$

along with similar equations as needed for buoyancy at the boundary (see main text). The eddy terms are

$$q' = \left[\nabla^2 + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\right)\right]\psi', \qquad (u', v') = \left(-\frac{\partial \psi'}{\partial y}, \frac{\partial \psi'}{\partial x}\right). \tag{WMF.2a,b}$$

The mean-flow terms are

$$\overline{q}(y,t) = \beta y - \frac{\partial \overline{u}}{\partial y} + \frac{\partial}{\partial z}\left(\frac{f_0}{N^2}\overline{b}\right), \tag{WMF.3}$$

and

$$\frac{\partial \overline{q}}{\partial y} = \beta - \frac{\partial^2 \overline{u}}{\partial y^2} - \frac{\partial}{\partial z}\left(\frac{f_0}{N^2}\frac{\partial \overline{b}}{\partial y}\right) = \beta - \frac{\partial^2 \overline{u}}{\partial y^2} - \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \overline{u}}{\partial z}\right), \tag{WMF.4}$$

using thermal wind. To solve for the mean-flow we may define a streamfunction Ψ such that

$$\left(\overline{u}, \frac{1}{f_0}\overline{b}\right) = \left(-\frac{\partial \Psi}{\partial y}, \frac{\partial \Psi}{\partial z}\right), \tag{WMF.5}$$

whence

$$\overline{q}(y,t) - \beta y = \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \Psi}{\partial z}\right) + \frac{\partial^2 \Psi}{\partial y^2}. \tag{WMF.6}$$

Given $\overline{q}$ from (WMF.1b) we solve (WMF.6) to give $\overline{u}$ and $\overline{b}$. Equivalently, we may derive a single equation for the zonal wind by differentiating (WMF.1b) with respect to y and, using (WMF.4), we obtain

$$\left[\frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\right)\right]\frac{\partial \overline{u}}{\partial t} = \frac{\partial^2}{\partial y^2}\overline{v'q'}. \tag{WMF.7}$$

The evolution of the mean-flow may also usefully be written in TEM form as

$$\frac{\partial \overline{u}}{\partial t} - f_0\overline{v}^* + \overline{v'q'} = 0, \tag{WMF.8a}$$

$$\frac{\partial \overline{b}}{\partial t} + N^2\overline{w}^* = 0, \tag{WMF.8b}$$

where $\overline{v}^*$ and $\overline{w}^*$ are found by solving the elliptic equation (10.63), and the value of $\partial\overline{q}/\partial y$, for use in (WMF.1a), is obtained using (WMF.4).

The overbars in these equations denote averages taken along isentropes — i.e., they are averages for a given layer — but are otherwise conventional, and the meridional velocity is purely ageostrophic. By analogy with (10.59), we define the residual circulation by

$$\overline{v}^* \equiv \overline{v} + \frac{1}{H}\overline{v'h'}, \tag{10.68}$$

where H is the mean thickness of the layer. Using (10.68) in (10.67) gives

$$\frac{\partial \overline{u}}{\partial t} - f_0\overline{v}^* = \overline{v'q'} + \overline{F}, \qquad \frac{\partial \overline{h}}{\partial t} + H\frac{\partial \overline{v}^*}{\partial y} = \overline{S}, \tag{10.69a,b}$$

where

$$\overline{v'q'} = \overline{v'\zeta'} - \frac{f_0}{H}\overline{v'h'}, \tag{10.70}$$

is the meridional potential vorticity flux in a shallow water system. From (10.68) we see that the residual velocity is a measure of the *total meridional thickness flux*, eddy plus mean, in an isentropic layer. This is often a more useful quantity than the Eulerian velocity $\overline{v}$ because it is generally the former, not the latter, that is constrained by the external forcing. What we have done, of course, is to effectively use a thickness-weighted mean in (10.66b); to see this, define the thickness-weighted mean by

$$\overline{v}_* \equiv \frac{\overline{hv}}{\overline{h}}. \tag{10.71}$$

(We use $\overline{v}_*$ to denote a thickness- or mass-weighted mean, and $\overline{v}^*$ to denote a residual velocity; the quantities are closely related, as we will see.) From (10.71) we have

$$\overline{v}_* = \overline{v} + \frac{1}{\overline{h}}\overline{v'h'}, \tag{10.72}$$

then the zonal average of (10.66b) is just

$$\frac{\partial \overline{h}}{\partial t} + \frac{\partial}{\partial y}(\overline{h}\overline{v}_*) = \overline{S}, \tag{10.73}$$

which is the same as (10.69b) if we take $H = \overline{h}$. Similarly, if we use the thickness weighted velocity (10.72) in the momentum equation (10.67a) we obtain (10.69a).

Evidently, if the mass-weighted meridional velocity is used in the momentum and thickness equations then the eddy mass flux does not enter the equations explicitly: the only eddy flux in (10.69) is that of potential vorticity. That is, in isentropic coordinates the equations in TEM form are equivalent to the equations that arise from a particular form of averaging — thickness weighted averaging — rather than the conventional Eulerian averaging. A similar correspondence occurs in height coordinates, as we now see.

10.3.3 Connection between the Residual and Thickness-weighted Circulation

It is evident from the above arguments that, in a shallow water system or in isentropic coordinates, the residual velocity is a measure of the total (i.e., mean plus eddy) thickness transport. In height coordinates, the definition of residual velocity, (10.58), does not lend itself so easily to such an interpretation. However, the residual velocity in height coordinates is, in fact, also a measure of the total thickness transport, or equivalently of the mass transport between two isentropic surfaces, as we now discover. Specifically, we show that averaging the total transport in isentropic layers is equivalent to the mass transport evaluated by the TEM formalism in height coordinates, and

Aspects of the TEM Formulation

Properties and features

- The residual mean circulation is equivalent to the total mass-weighted (eddy plus Eulerian mean) circulation, and it is this circulation that is driven by the diabatic forcing.
- There are no explicit eddy fluxes in the buoyancy budget; the only eddy term is the flux of potential vorticity, and this is the divergence of the Eliassen–Palm flux; that is $\overline{v'q'} = \nabla_x \cdot \boldsymbol{\mathcal{F}}$.
- The residual circulation, $\overline{v}^*$, becomes part of the solution, just as $\overline{v}$ is part of the solution in an Eulerian mean formulation.

But note

- The TEM formulation does not solve the parameterization problem, and eddy fluxes are still present in the equations.
- The theory and practice are well developed for a zonal average, but less so for three-dimensional, non-zonal flow. This is because the geometry enforces simple boundary conditions in the zonal mean case.[6]
- The boundary conditions on the residual circulation are neither necessarily simple nor easily determined; for example, at a horizontal boundary $\overline{w}^*$ is not zero if there are horizontal buoyancy fluxes.

Examples of the use of the TEM and its relatives in the general circulation of the atmosphere and ocean arise in Sections 15.2, 15.4, 17.3, 17.7 and 21.7.

specifically that the thickness-weighted mean, $\overline{v}_*$, is equivalent to the residual velocity, $\overline{v}^*$, in height coordinates. Our demonstration is for a Boussinesq system, but the extension to a compressible gas is reasonably straightforward.[5]

Consider two isentropic surfaces, η_1 and η_2 with mean positions $\overline{\eta}_1$ and $\overline{\eta}_2$, as in Fig. 10.1. (We use z to denote the vertical coordinate, and η to denote the location of isentropic surfaces.) The meridional transport between these surfaces is given by

$$T = \int_{\eta_2}^{\eta_1} v \,\mathrm{d}z. \tag{10.74}$$

If the velocity does not vary with height within the layer (and in the limit of layer thickness going to zero this is the case) then $T = vh$ where $h = \eta_1 - \eta_2$ is the thickness of the isentropic layer. The zonally-averaged transport is then given by

$$\overline{T} = \frac{1}{L}\int_L T\,\mathrm{d}x = \frac{1}{L}\int_L \left(\int_{\eta_2}^{\eta_1} v\,\mathrm{d}z\right)\mathrm{d}x = \overline{\int_{\eta_2}^{\eta_1} v\,\mathrm{d}z} = \overline{vh} = \overline{v}\overline{h} + \overline{v'h'}, \tag{10.75}$$

with obvious notation, and with an overbar denoting a zonal average. Letting the distance between isentropes shrink to zero this result allows us to write

$$\overline{v}_* \equiv \frac{\overline{v\sigma}^b}{\overline{\sigma}} = \overline{v}^b + \frac{\overline{v'\sigma'}^b}{\overline{\sigma}}, \tag{10.76}$$

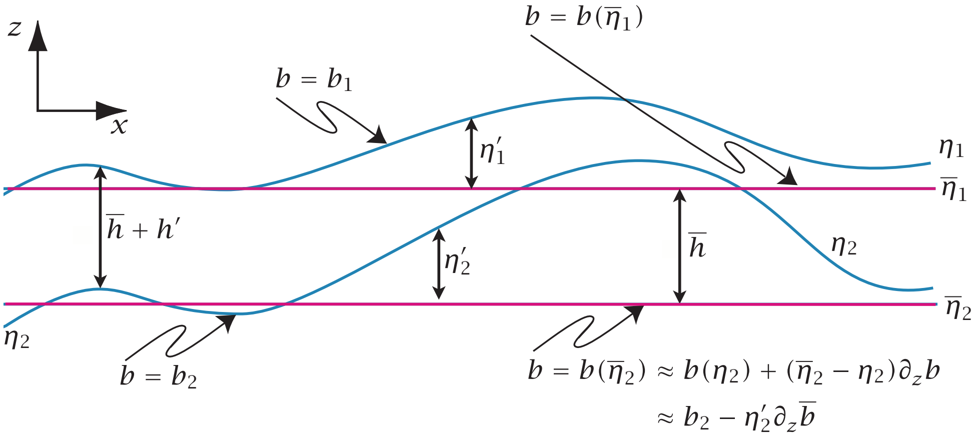

Fig. 10.1 Two isentropic surfaces, η_1 and η_2, and their mean positions, $\overline{\eta}_1$ and $\overline{\eta}_2$. The departure of an isentrope from its mean position is proportional to the temperature perturbation at the mean position of the isentrope, and the variations in thickness (h') of the isentropic layer are proportional to the vertical derivative of this.

where $\overline{(\cdot)}^b$ denotes an average along an isentrope and $\overline{\sigma} = \overline{\partial z/\partial b}$ is the thickness density, a measure of the thickness between two isentropes. Equation (10.76) is analogous to (10.72), for a continuously stratified system. The averaged quantity $\overline{v}_*$ is not proportional to the average of the velocity at constant height, or even to the average along an isentrope; rather, it is the *thickness-weighted* zonal average of the velocity *between* two isentropic surfaces, Δb apart, of mean separation proportional to $\overline{\sigma}\Delta b$. Our goal is to express this transport in terms of Eulerian-averaged quantities, at a constant height z.

Let us first connect an average along an isentrope of some variable χ to its average at constant height by writing, for small isentropic displacements,

$$\overline{\chi}^b = \overline{\chi(z+\eta')}^z \approx \overline{\chi(z) + \eta'\partial\chi/\partial z}^z, \tag{10.77}$$

where the superscript explicitly denotes how the zonal average is taken, and η' is the displacement of the isentrope from its mean position. This can be expressed in terms of the temperature perturbation at the location of the mean isentrope by Taylor-expanding b around its value on that mean isentrope. That is,

$$b(\eta) = b(\overline{\eta}) + \left(\frac{\partial b}{\partial z}\right)_{z=\overline{\eta}} (\eta - \overline{\eta}) + \dots, \tag{10.78}$$

where $\overline{\eta} = \overline{\eta}(z)$, giving

$$\eta' \approx \frac{-b'}{\partial_z b(\overline{\eta})} \approx -\frac{b'}{\partial_z \overline{b}^z}, \tag{10.79}$$

where $\eta' = \eta - \overline{\eta}$ and $b' = b(\overline{\eta}) - b(\eta)$. Using (10.79) in (10.77) (and omitting the superscript z on $\partial_z \overline{b}$) we obtain, with $\chi = v$,

$$\overline{v}^b = \overline{v}^z - \frac{\overline{b'\partial_z v'}^z}{\partial_z \overline{b}}. \tag{10.80}$$

Note that if v is in thermal wind balance with b then the second term vanishes identically, but we will not invoke this.

We now transform the second term on the right-hand side of (10.76) to an average at constant z. The variations in thickness of an isothermal layer are given by

$$\sigma' \approx \overline{\sigma}\frac{\partial \eta'}{\partial z} = -\overline{\sigma}\frac{\partial}{\partial z}\left(\frac{b'}{\partial_z \overline{b}}\right), \tag{10.81}$$

using (10.79). Thus, neglecting terms that are third-order in amplitude,

$$\overline{v'\sigma'}^b = -\overline{\sigma}\overline{v'\frac{\partial}{\partial z}\left(\frac{b'}{\partial_z \overline{b}}\right)}^z. \tag{10.82}$$

Using both (10.80) and (10.82), (10.76) becomes

$$\overline{v}_* = \overline{v}^z - \frac{\overline{b'\partial_z v'}^z}{\partial_z \overline{b}} - \overline{v'\frac{\partial}{\partial z}\left(\frac{b'}{\partial_z \overline{b}}\right)}^z = \overline{v}^z - \frac{\partial}{\partial z}\overline{\left(\frac{v'b'}{\partial_z \overline{b}}\right)}^z. \tag{10.83}$$

The right-hand side of the last equation is the TEM form of the residual velocity; thus, we have shown that

$$\overline{v}_* \equiv \frac{\overline{v\sigma}}{\overline{\sigma}} = \overline{v}^b + \frac{\overline{v'\sigma'}^b}{\overline{\sigma}} \approx \overline{v}^z - \frac{\partial}{\partial z}\left(\frac{\overline{v'b'}^z}{\partial_z \overline{b}}\right) \equiv \overline{v}^*. \tag{10.84}$$

We see the equivalence of the thickness-weighted mean velocity on the left-hand side and the residual velocity on the right-hand side. In the quasi-geostrophic limit $N^2 = \partial_z \overline{b}$ and $\overline{\sigma}$ is a reference thickness.

10.4 THE NON-ACCELERATION RESULT

We now consider further the interpretation and application of the potential vorticity flux and its relatives, using a quasi-geostrophic framework. We first derive an important result in wave–mean-flow dynamics, the non-acceleration condition.[7] This result shows that under certain conditions, to be made precise below, waves have no net effect on the mean-flow, an important and somewhat counter-intuitive result.

10.4.1 A Derivation from the Potential Vorticity Equation

Consider how the potential vorticity fluxes affect the mean fields. The unforced and inviscid zonally-averaged potential vorticity equation is

$$\frac{\partial \overline{q}}{\partial t} + \frac{\partial \overline{v'q'}}{\partial y} = 0. \tag{10.85}$$

Now, in quasi-geostrophic theory the geostrophically balanced velocity and buoyancy can be determined from the potential vorticity via an elliptic equation, and in particular

$$\overline{q} - \beta y = \frac{\partial^2 \overline{\psi}}{\partial y^2} + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \overline{\psi}}{\partial z}\right), \tag{10.86}$$

where $\overline{\psi}$ is such that $(\overline{u}, \overline{b}/f_0) = (-\partial\overline{\psi}/\partial y, \partial\overline{\psi}/\partial z)$. Differentiating (10.85) with respect to y we obtain

$$\left[\frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\right)\right]\frac{\partial \overline{u}}{\partial t} = (\nabla \cdot \boldsymbol{\mathcal{F}})_{yy}, \tag{10.87}$$

where $\nabla\cdot\boldsymbol{\mathcal{F}} = \overline{v'q'}$ is the divergence of the EP flux (in the y–z plane, i.e., $\nabla_x\cdot\boldsymbol{\mathcal{F}}$). This is determined using the wave activity equation for pseudomomentum which, reprising (10.29a), is

$$\frac{\partial \mathcal{P}}{\partial t} + \nabla\cdot\boldsymbol{\mathcal{F}} = \mathcal{D}, \tag{10.88}$$

now using $\mathcal{P}$ for the wave activity since we are specifically talking about pseudomomentum. If the waves are statistically steady (i.e., $\partial\mathcal{P}/\partial t = 0$) and have no dissipation ($\mathcal{D} = 0$) then evidently $\nabla\cdot\boldsymbol{\mathcal{F}} = 0$. If there is no acceleration at the boundaries then the solution of (10.87) is

$$\frac{\partial \overline{u}}{\partial t} = 0. \tag{10.89}$$

This is a *non-acceleration result*. That is to say, under certain conditions the tendency of the mean fields, and in particular of the zonally-averaged zonal flow, are independent of the waves. To be explicit, those conditions are:

(i) The waves are steady (so that, using the wave activity equation $\mathcal{P}$ does not vary).

(ii) The waves are conservative; that is, $\mathcal{D} = 0$ in (10.29a). Given this and item *(i)*, the Eliassen–Palm relation implies that $\nabla\cdot\boldsymbol{\mathcal{F}} = 0$ and the potential vorticity flux is zero.

(iii) The waves are of small amplitude (all of our analysis has neglected terms that are cubic in perturbation amplitude).

(iv) The waves do not affect the boundary conditions (so there are no boundary contributions to the acceleration).

Given the way we have derived it, the result does not seem too surprising; however, it can be powerful and counter-intuitive, for it means that steady waves (i.e., those whose amplitude does not vary) do not affect the zonal flow. However, they *do* affect the Eulerian meridional overturning circulation, and the relative vorticity flux may also be non-zero. In fact, the non-acceleration theorem is telling us that the changes in the vorticity flux are exactly compensated for by changes in the meridional circulation, and there is no net effect on the zonally-averaged zonal flow. It is *irreversibility*, often manifested by the breaking of waves, that leads to permanent changes in the mean-flow.

The derivation of this result by way of the momentum equation, which one might expect to be more natural, is rather awkward because one must consider momentum and buoyancy fluxes separately. Furthermore, the zonally-averaged meridional circulation comes into play: for example, the meridional velocity, $\overline{v}$, is small because it is purely ageostrophic, but it is not zero and we cannot neglect it because it is multiplied by the Coriolis parameter, which is large. Thus, the eddy vorticity fluxes can affect both the meridional circulation and the acceleration of the zonal mean-flow, and it might seem impossible to disentangle the two effects without completely solving the equations of motion. Nevertheless, we *can* proceed by way of the momentum and buoyancy equations if we use the transformed Eulerian mean and this provides a useful alternate derivation, as follows.

10.4.2 Using TEM to Give the Non-Acceleration Result

We may use the TEM formalism to obtain the non-acceleration result. The explanation is largely equivalent to that given above, but the explication may be useful.

A two-dimensional case

Consider two-dimensional incompressible flow on the β-plane, for which there is no buoyancy flux. The linearized vorticity equation is

$$\frac{\partial \zeta'}{\partial t} + \overline{u}\frac{\partial \zeta'}{\partial x} + v'\frac{\partial \overline{\zeta}}{\partial y} = D', \tag{10.90}$$

from which we derive, analogously to (10.29a), the Eliassen–Palm relation

$$\frac{\partial \mathcal{P}}{\partial t} + \frac{\partial \mathcal{F}}{\partial y} = \mathcal{D}, \tag{10.91}$$

where $\mathcal{F} = -\overline{u'v'}$, $\mathcal{D}$ represents non-conservative forces, and

$$\mathcal{P} = \frac{\overline{\zeta'^2}}{2\partial_y\overline{\zeta}} = \frac{1}{2}\overline{\eta'^2}\frac{\partial\overline{\zeta}}{\partial y}. \tag{10.92}$$

The quantity $\eta' \equiv -\zeta'/\partial_y\overline{\zeta}$ is proportional to the meridional particle displacement in a disturbance. Now consider the x-momentum equation

$$\frac{\partial u}{\partial t} = -\frac{\partial u^2}{\partial x} - \frac{\partial uv}{\partial y} - \frac{\partial \phi}{\partial x} + fv. \tag{10.93}$$

Zonally averaging, noting that $\overline{v} = 0$, gives

$$\frac{\partial \overline{u}}{\partial t} = -\frac{\partial \overline{uv}}{\partial y} = \overline{v'\zeta'} = \frac{\partial \mathcal{F}}{\partial y}. \tag{10.94}$$

Finally, combining (10.91) and (10.94) gives

$$\frac{\partial}{\partial t}(\overline{u} + \mathcal{P}) = \mathcal{D}. \tag{10.95}$$

In the absence of non-conservative terms (i.e., if $\mathcal{D} = 0$) the quantity $\overline{u} + \mathcal{P}$ is constant.[8] Further, if the waves are steady and conservative then $\mathcal{P}$ is constant and, therefore, so is $\overline{u}$. This is the non-acceleration result.

The stratified case

In the stratified case we can use the TEM form of the momentum equation to derive a similar result. The unforced zonally-averaged zonal momentum equation can be written as

$$\frac{\partial \overline{u}}{\partial t} - f_0\overline{v}^* = \nabla\cdot\boldsymbol{\mathcal{F}}, \tag{10.96}$$

and using the Eliassen–Palm relation, (10.29a), this may be written as

$$\frac{\partial}{\partial t}(\overline{u} + \mathcal{P}) - f_0\overline{v}^* = \mathcal{D}, \tag{10.97}$$

and so again $\mathcal{P}$ is related to the momentum of the flow. If, furthermore, the waves are steady ($\partial\mathcal{P}/\partial t = 0$) and conservative ($\mathcal{D} = 0$), then $\partial\overline{u}/\partial t - f_0\overline{v}^* = 0$. However, under these same conditions the residual circulation will also be zero. This is because the residual meridional circulation $(\overline{v}^*, \overline{w}^*)$ arises via the necessity to keep the temperature and velocity fields in thermal wind balance, and is thus determined by an elliptic equation, namely (10.63). If the waves are steady and adiabatic then, since $\overline{v'q'} = 0$, the right-hand side of the equation is zero and it becomes

$$f_0^2\frac{\partial^2\psi^*}{\partial z^2} + N^2\frac{\partial^2\psi^*}{\partial y^2} = 0. \tag{10.98}$$

If $\psi^* = 0$ at the boundaries, then the unique solution of this is $\psi^* = 0$ everywhere. At the meridional boundaries we may certainly suppose that ψ^* vanishes if these are quiescent latitudes, and at the horizontal boundaries the buoyancy flux will vanish if the waves there are steady, because from (10.14) we have

$$\overline{v'b'}\frac{\partial \overline{b}}{\partial y} = -\frac{1}{2}\frac{\partial}{\partial t}\overline{b'^2} = 0. \tag{10.99}$$

Under these circumstances, then, the residual meridional circulation vanishes in the interior and, from (10.96), the mean-flow is steady, thus reprising the non-acceleration result.

Compare (10.96) with the momentum equation in conventional Eulerian form, namely

$$\frac{\partial \overline{u}}{\partial t} - f_0\overline{v} = \overline{v'\zeta'}. \tag{10.100}$$

There is no reason that the vorticity flux should vanish when waves are present, even if they are steady. However, such a flux is (under non-acceleration conditions) precisely compensated by the meridional circulation $f_0\overline{v}$, something that is hard to infer or intuit directly from (10.100); even when non-acceleration conditions do not apply there will be a significant cancellation between the Coriolis and eddy terms. The difficulty boils down to the fact that, in contrast to $\overline{v'q'}$, $\overline{v'\zeta'}$ is not the flux of a wave activity.

Unlike the proof of the non-acceleration result given in Section 10.4.1, the above argument does not use the invertibility property of potential vorticity directly, suggesting an extension to the primitive equations, and the reader may pursue that elsewhere.[9] Various results regarding the TEM and non-acceleration are summarized in the shaded box on the following page.

10.4.3 The EP Flux and Form Drag

It may seem a little magical that the zonal flow is driven by the Eliassen–Palm flux via (10.96). The poleward vorticity flux is clearly related to the momentum flux convergence, but why should a poleward buoyancy flux affect the momentum? The TEM form of the momentum equation may be written as

$$\frac{\partial \overline{u}}{\partial t} = \frac{\partial}{\partial z}\left(\frac{f_0}{N^2}\overline{v'b'}\right) + F_m, \tag{10.101}$$

where $F_m = \overline{v'\zeta'} + f_0\overline{v}^*$ represents forces from the momentum flux and Coriolis force. The first term on the right-hand side certainly does not look like a force; however, it turns out to be directly proportional to the *form drag* between isentropic layers. Recall from Section 3.6 that the form drag, τ_d, at an interface between two layers of shallow water is

$$\tau_d = -\overline{\eta'\frac{\partial p'}{\partial x}}, \tag{10.102}$$

where η is the interfacial displacement. But from (10.79) $\eta' = -b'/N^2$ and with this and geostrophic balance we have

$$\tau_d = \frac{\rho_0 f_0}{N^2}\overline{v'b'}. \tag{10.103}$$

Thus, the vertical component of the EP flux (i.e., the meridional buoyancy flux) is in fact a real stress acting on a fluid layer and equal to the momentum flux caused by the wavy interface. The net momentum convergence into an infinitesimal layer of mean thickness $\overline{h}$ is then (cf. (3.81)),

$$F_d = \overline{h}\frac{\partial \tau_d}{\partial z} = \overline{h}\rho_0 f_0\frac{\partial}{\partial z}\left(\frac{\overline{v'b'}}{N^2}\right), \tag{10.104}$$

and a layer of mean thickness $\overline{h}$ is accelerated according to

$$\frac{\partial \overline{u}}{\partial t} = f_0\frac{\partial}{\partial z}\left(\frac{\overline{v'b'}}{\partial_z \overline{b}}\right) + F_m. \tag{10.105}$$

TEM, Residual Velocities and Non-Acceleration

For a Boussinesq quasi-geostrophic system, the TEM form of the unforced momentum equation and the thermodynamic equation are:

$$\frac{\partial \overline{u}}{\partial t} - f_0 \overline{v}^* = \nabla \cdot \boldsymbol{\mathcal{F}}, \qquad \frac{\partial \overline{b}}{\partial t} + \overline{w}^* N^2 = \overline{S}, \tag{T.1}$$

where $N^2 = \partial \overline{b}_0 / \partial z$, $\overline{S}$ represents diabatic effects, $\boldsymbol{\mathcal{F}}$ is the Eliassen–Palm (EP) flux and its divergence is the potential vorticity flux; thus, $\nabla \cdot \boldsymbol{\mathcal{F}} = \nabla_x \cdot \boldsymbol{\mathcal{F}} = \overline{v'q'}$. The residual velocities are

$$\overline{v}^* = \overline{v} - \frac{\partial}{\partial z}\left(\frac{1}{N^2}\overline{v'b'}\right), \qquad \overline{w}^* = \overline{w} + \frac{\partial}{\partial y}\left(\frac{1}{N^2}\overline{v'b'}\right). \tag{T.2}$$

Spherical coordinate and ideal gas versions of these take a similar form. We may define a meridional overturning streamfunction such that $(\overline{v}^*, \overline{w}^*) = (-\partial\psi^*/\partial z, \partial\psi^*/\partial y)$, and using thermal wind to eliminate time-derivatives in (T.1) we obtain

$$f_0^2 \frac{\partial^2 \psi^*}{\partial z^2} + N^2 \frac{\partial^2 \psi^*}{\partial y^2} = f_0 \frac{\partial}{\partial z}\overline{v'q'} + \frac{\partial \overline{S}}{\partial y}. \tag{T.3}$$

The manipulations (given in the main text) that lead to the above equations may seem formal, in that they simply transform the momentum and thermodynamic equations from one form to another. However, the resulting equations have two potential advantages over the untransfomed ones:

(i) The residual meridional velocity is approximately equal to the average thickness-weighted velocity between two neighbouring isentropic surfaces, and so is a measure of the total (Eulerian mean plus eddy) meridional transport of thickness or buoyancy.

(ii) The EP flux is directly related to certain conservation properties of waves. The divergence of the EP flux is the meridional flux of potential vorticity:

$$\boldsymbol{\mathcal{F}} = -(\overline{u'v'})\,\mathbf{j} + \left(\frac{f_0}{N^2}\overline{v'b'}\right)\mathbf{k}, \qquad \nabla \cdot \boldsymbol{\mathcal{F}} = \overline{v'q'}. \tag{T.4}$$

Furthermore, the EP flux satisfies, to second order in wave amplitude,

$$\frac{\partial \mathcal{P}}{\partial t} + \nabla \cdot \boldsymbol{\mathcal{F}} = \mathcal{D}, \qquad \text{where} \quad \mathcal{P} = \frac{\overline{q'^2}}{2\partial \overline{q}/\partial y}, \quad \mathcal{D} = \frac{\overline{D'q'}}{\partial \overline{q}/\partial y}. \tag{T.5}$$

The quantity $\mathcal{P}$ is a *wave activity density,* specifically the *pseudomomentum,* and $\mathcal{D}$ is its dissipation. For nearly plane waves, $\mathcal{P}$ and $\boldsymbol{\mathcal{F}}$ are connected by the *group velocity property,*

$$\boldsymbol{\mathcal{F}} = (\mathcal{F}^y, \mathcal{F}^z) = \boldsymbol{c}_g \mathcal{P}, \tag{T.6}$$

where $\boldsymbol{c}_g$ is the group velocity of the waves. If the waves are steady $(\partial\mathcal{P}/\partial t = 0)$ and dissipationless $(\mathcal{D} = 0)$ then $\nabla \cdot \boldsymbol{\mathcal{F}} = 0$ and using (T.1) and (T.3) there is no wave-induced acceleration of the mean-flow; this is the 'non-acceleration' result. Commonly there is enstrophy dissipation, or wave-breaking, and $\nabla \cdot \boldsymbol{\mathcal{F}} < 0$; such *wave drag* leads to flow deceleration and/or a poleward residual meridional velocity.

The appearance of the buoyancy flux is really a consequence of the way we have chosen to average the equations: obtaining (10.105) involved averaging the forces over an isentropic layer, and given this it can only be the residual circulation that contributes to the Coriolis force. One might say that the vertical component of the EP flux is a force in drag, masquerading as a buoyancy flux.

10.5 ✦ INFLUENCE OF EDDIES ON THE MEAN-FLOW IN THE EADY PROBLEM

We now consider the eddy fluxes in the Eady problem, and, in particular, how these might feed back on to the mean-flow. Because of the simplicity of the setting the problem can be fully solved in both the Eulerian or residual frameworks and it is therefore a very instructive, albeit algebraically complex, example.[10]

10.5.1 Formulation

Let us first distinguish between the basic flow, the zonal mean fields, and the perturbation. The basic flow is the flow around which the equations of motion are linearized; this flow is unstable, and the perturbations, assumed to be small, grow exponentially with time. Because the perturbations are formally always small they do not affect the basic flow, but they do produce changes in the zonal mean velocity and buoyancy fields. In Eulerian form this is represented by,

$$\frac{\partial \overline{u}}{\partial t} = f_0 \overline{v} - \frac{\partial \overline{u'v'}}{\partial y}, \qquad \frac{\partial \overline{b}}{\partial t} = -N^2 \overline{w} - \frac{\partial \overline{b'v'}}{\partial y}, \tag{10.106}$$

and the TEM version of these equations is

$$\frac{\partial \overline{u}}{\partial t} = f_0 \overline{v}^* + \overline{v'q'}, \qquad \frac{\partial \overline{b}}{\partial t} = -N^2 \overline{w}^*, \tag{10.107}$$

where in the Eady problem $\partial_y(\overline{u'v'})$ and $\overline{v'q'}$ are both zero. We can calculate the perturbation quantities from the solution to the Eady problem (e.g., calculate $\overline{v'b'}$) and thus infer the structure of the mean-flow tendencies $\partial\overline{u}/\partial t$ and $\partial\overline{b}/\partial t$ and the meridional circulation, $(\overline{v}, \overline{w})$ or $(\overline{v}^*, \overline{w}^*)$. All of these fields are perturbation quantities and all are exponentially growing, and so in reality they will eventually have a finite effect on the pre-existing zonal flow, but in the Eady problem, or any similar linear problem, such rectification is assumed to be small and is neglected.

Using the thermal wind relation, $f_0 \partial_z \overline{u} = -\partial_y \overline{b}$ to eliminate time derivatives in (10.106) gives an equation for the meridional streamfunction ψ_E, namely,

$$\frac{L^2}{L_d^2}\frac{\partial^2 \psi_E}{\partial z^2} + \frac{\partial^2 \psi_E}{\partial y^2} = -\frac{1}{N^2}\frac{\partial^2 \overline{b'v'}}{\partial y^2}, \tag{10.108}$$

where $(\overline{v}, \overline{w}) = (-\partial\psi_E/\partial z, \partial\psi_E/\partial y)$ and we have nondimensionalized z with D and y with L. The boundary conditions are that $\psi_E = 0$ at $y = 0, L$ and $z = 0, D$. Similarly, and analogously to (10.63), we obtain an equation for the residual streamfunction, ψ^*, namely

$$\frac{L^2}{L_d^2}\frac{\partial^2 \psi^*}{\partial z^2} + \frac{\partial^2 \psi^*}{\partial y^2} = 0, \tag{10.109}$$

where now the boundary conditions are that $N^2 \overline{w}^* = \partial \overline{v'b'}/\partial y$ at the upper and lower boundaries, and $\overline{v} = 0$ at the lateral boundaries. In terms of the residual streamfunction this is

$$\psi^* = \frac{1}{N^2}\overline{v'b'}, \text{ at } z = 0,\ 1, \qquad \psi^* = 0, \text{ at } y = 0,\ 1. \tag{10.110}$$

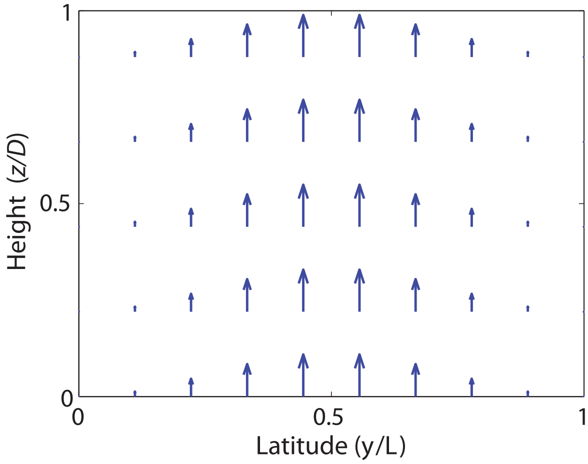

Fig. 10.2 The Eliassen–Palm vector in the Eady problem. It is directed purely vertically, .

The residual and overturning circulations are related by (10.58a), and (10.108) and (10.109) are, at one level, simply different representations of the same problem, connected by a simple mathematical transformation. However, the residual streamfunction better represents the total transport of the fluid. Equation (10.109) is particularly simple, because of the absence of potential vorticity fluxes in the interior, and it is apparent that the residual circulation is driven by boundary sources. We care only about the spatial structure of the right-hand sides of (10.108) and of the boundary conditions of (10.110). The former is given by

$$-\frac{\partial^2 \overline{b'v'}}{\partial y^2} \propto -\frac{\partial^2}{\partial y^2} \sin^2 ly = -2l^2 \cos 2ly. \tag{10.111}$$

The eddy heat fluxes in the Eady problem are independent of height, as may be calculated explicitly from the solutions of Chapter 9. In fact, the result follows without detailed calculation, by first noting that the eddy potential vorticity flux is zero because the basic state has zero QG potential vorticity and therefore none may be generated. Further, because the basic state does not vary in y there can be no momentum flux convergence in the y-direction, and so the momentum flux itself is zero if it is zero on the boundary. Thus [using for example (10.27) and (10.28)] the eddy heat flux is independent of height and the EP vectors are directed purely vertically (Fig. 10.2).

The boundary conditions for the residual circulation are

$$\psi^*(y,0) = \psi^*(y,1) \propto \sin^2 ly. \tag{10.112}$$

10.5.2 ✦ Solution

The solutions to (10.108) and (10.109) may be obtained either analytically or numerically. In a domain $0 < y < 1$ and $0 < z < 1$ the residual streamfunction for $l = \pi$ is given by:

$$\begin{aligned}\psi^* &= \sum_{n=1}^{\infty} A_n \sin[(2n-1)ly] \frac{\cosh[L_d \pi(2n-1)(z-0.5)/L]}{\cosh[L_d\pi(2n-1)/2L]},\\ A_n &= \frac{2}{\pi(2n-1)} - \frac{1}{\pi(2n-1)-2l} - \frac{1}{\pi(2n-1)+2l}.\end{aligned} \tag{10.113}$$

The solution is obtained by first projecting the boundary conditions (proportional to $\sin^2 ly$, or $(1-\cos 2ly)/2$) on to the eigenfunctions of the horizontal part of the Laplacian (i.e., sine functions), and this gives the coefficients of A_n. The vertical structure is then obtained by solving $(L/L_d)^2\partial_z^2\psi^* =$

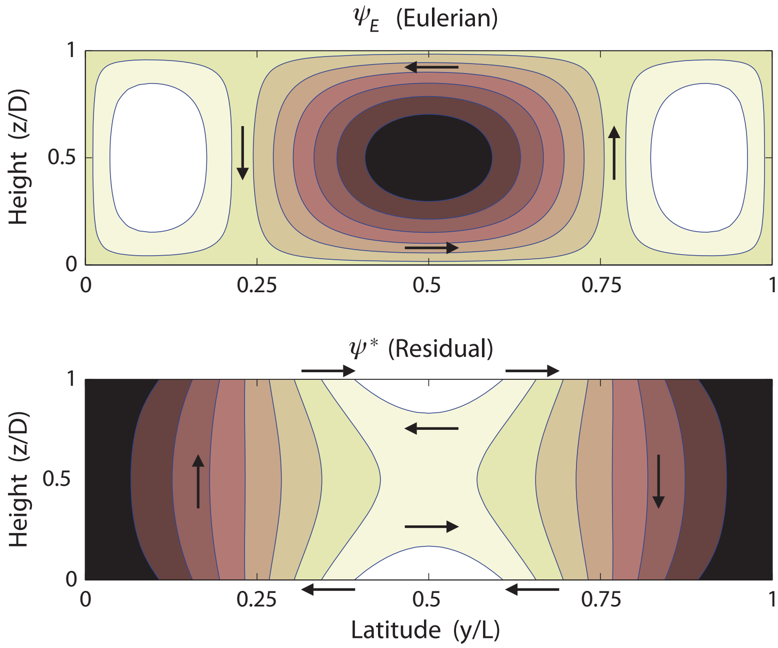

Fig. 10.3 The Eulerian streamfunction (top) and the residual streamfunction for the Eady problem, calculated using (10.108) and (10.109), with $L^2/L_d^2 = 9$.

$-\partial_y^2\psi^*$, which gives the cosh functions. The series converges very quickly, and the first term in the series captures the dominant structure of the solution, essentially because, for $l = \pi$, $\sin ly$ is not unlike $\sin^2 ly$ on the interval $[0, 1]$.

The Eulerian circulation is obtained from the residual circulation using (10.58a), and so by the addition of a field independent of z and proportional to $\sin^2 ly$. The resulting structure is dominated by this and the first term of (10.113) (proportional to $\sin ly$) and, noting that the circulation is symmetric about $z = 0.5$, we obtain a circulation dominated by a single cell, with equatorward motion aloft and poleward motion near the surface (Fig. 10.3). The heat flux convergence in high latitudes is leading to mean rising motion, with the precise shape of the streamfunction determined by the boundary conditions. Although this is true, the heat flux arises *because* of the motion of fluid parcels, so it may be a little misleading to infer, as one might from the Eulerian streamfunction, that the heat flux *causes* the individual parcels to rise or sink in this fashion. The residual streamfunction is a better indicator of the total mass transport and, perhaps as one might intuitively expect, these show parcels rising in the low latitudes and sinking in high latitudes, providing a tendency to flatten the isopycnals and to reduce the meridional temperature gradient.

The residual circulation also shows fluid entering or leaving the domain at the boundary — what does this represent? Suppose that instead of solving the continuous problem we had posed the problem in a finite number of layers (and we explicitly consider the two-layer problem below). As the number of layers increases the solution to the linear baroclinic instability problem approaches that of the Eady problem (e.g., Fig. 9.13); however, as we saw in Section 10.3, the residual circulation is closed in the layered model, and the sum over all the layers of the meridional transport vanishes. Now, in the layered model the vertical boundary conditions are built in to the representation by way of a redefinition of the potential vorticity of the top and bottom layers, so that, in the layered version of the Eady problem there appears to be a potential vorticity gradient in these two layers, instead of a buoyancy gradient at the boundary. The residual circulation is then closed by a return flow that occurs only in the top and bottom layers, and as the number of layers increases this flow is confined to a thinner and thinner layer, and to a delta-function in the continuous limit. To indicate this we have placed arrows just above and below the domain in Fig. 10.3. (This equivalence between boundary conditions and delta-function sources is the same

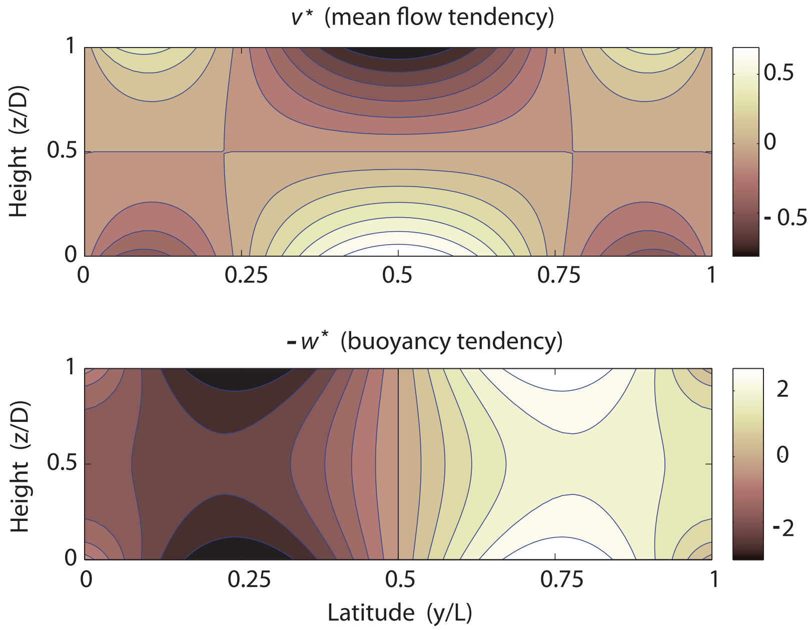

Fig. 10.4 The tendency of the zonal mean-flow $(\partial\overline{u}/\partial t)$ and the buoyancy $(\partial\overline{b}/\partial t)$ for the Eady problem. Lighter (darker) shading means a positive (negative) tendency, but the units themselves are arbitrary.

as that giving rise to the delta-function boundary layer of Section 5.4.3.)

The effect on the mean-flow is inferred directly from the residual circulation: the mean-flow acceleration is proportional to $\overline{v}^*$ and the buoyancy tendency is proportional to $-\overline{w}^*$, and these are plotted in Figs. 10.4 and 10.5. Because there is no momentum flux convergence in the problem the zonal flow tendency is entirely baroclinic — its vertical integral is zero — and over most of the domain is such as to reduce the mean shear. Consistently (using thermal wind) the buoyancy tendency is such as to reduce the meridional temperature gradient; that is, the instabilities act to transport heat polewards and so reduce the instability of the mean-flow.

10.5.3 The Two-level Problem

The residual circulation and mean-flow tendencies can also be calculated for the two-level (Phillips) problem, with the β-effect. The potential vorticity fluxes in each layer are non-zero and the mean-flow equations are, for $i = 1, 2$,

$$\frac{\partial \overline{u}_i}{\partial t} = f_0 \overline{v}_i^* + \overline{v_i' q_i'}, \qquad \frac{\partial \overline{b}}{\partial t} = -N^2 \overline{w}^*. \tag{10.114}$$

The vertical velocity and buoyancy are evaluated at mid-depth, and the thermal wind equation is $\overline{u}_1 - \overline{u}_2 = -(H/2)\partial_y \overline{b}$ where H is the total depth of the fluid and, by mass conservation, $\overline{v}_1^* = -\overline{v}_2^*$. If we define a residual streamfunction ψ^* such that

$$\overline{v}_1^* = -\overline{v}_2^* = \psi^*, \qquad \overline{w}^* = \frac{\partial \psi^*}{\partial y}, \tag{10.115}$$

then eliminating time derivatives in (10.114) gives an equation for the residual streamfunction,

$$\frac{\partial^2 \psi^*}{\partial y^2} - \frac{k_d^2}{2}\psi^* = \frac{2 f_0 L^2}{N^2 H}(\overline{v_1' q_1'} - \overline{v_2' q_2'}), \tag{10.116}$$

where $k_d^2/2 = [2 f_0/(NH)]^2$, and we have nondimensionalized vertical scales by D and horizontal scales by L. As in the Eady problem, it is only the spatial structures of the terms on the right-hand

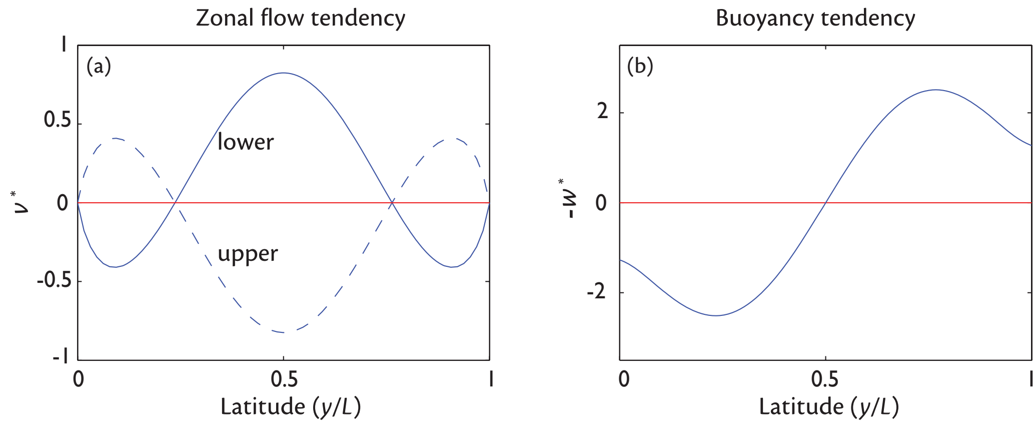

Fig. 10.5 (a) The tendency of the zonal mean-flow ($\partial\overline{u}/\partial t$) just below the upper lid (dashed) and just above the surface (solid) in the Eady problem. The vertically integrated tendency is zero. (b) The vertically averaged buoyancy tendency.

side that are relevant, and these may be calculated from the solutions to the two-level instability problem. The main difference from the Eady problem is that the interior potential vorticity fluxes are non-zero, even in the case with $\beta = 0$: effectively, the boundary fluxes of the Eady problem are absorbed into the potential vorticity fluxes of the two layers. Solving for the residual circulation and interpreting the mean-flow tendencies is left as an exercise for the reader.

10.6 ✦ NECESSARY CONDITIONS FOR INSTABILITY

As we noted in Chapter 9, necessary conditions for instability, or sufficient conditions for stability, can be very useful because when satisfied they obviate the need to perform a detailed calculation. In the remainder of this chapter we use the conservation of wave activities — pseudomomentum and pseudoenergy — to derive such conditions. In sections 9.3 and 9.4.3 we derived such conditions assuming the instability to be of normal-mode form. Here we give derivations that are both more general and, in some ways, simpler; they utilize the fact that the potential vorticity flux may be written as a divergence of a vector and therefore vanishes when integrated over a domain, aside from possible boundary contributions.

10.6.1 Stability Conditions from Pseudomomentum Conservation

Consider the perturbation enstrophy equation,

$$\frac{1}{2}\frac{\partial}{\partial t}\overline{q'^2} = -\frac{\partial\overline{q}}{\partial y}\nabla_x \cdot \boldsymbol{\mathcal{F}}, \tag{10.117}$$

where $\boldsymbol{\mathcal{F}}$ is the Eliassen–Palm flux given by (10.27), the overbar is a zonal mean and the divergence is in the y–z plane. Dividing by $\partial\overline{q}/\partial y$ and integrating over a domain A which is such that the Eliassen–Palm flux vanishes at the boundaries gives the pseudomomentum conservation law,

$$\int_A \frac{\partial}{\partial t}\left(\frac{\overline{q'^2}}{\partial_y\overline{q}}\right) \mathrm{d}y\,\mathrm{d}z = 0. \tag{10.118}$$

Equation (10.118) implies that, in the *norm* $\left[q'^2/\partial_y\overline{q}\right]$, the perturbation cannot grow unless $\partial\overline{q}/\partial y$ changes sign somewhere in the domain, or at the boundaries. This result does not depend upon the instability being of normal-mode form. The simplest result of all occurs in a barotropic problem with no vertical variation. Then $\partial\overline{q}/\partial y = \partial/\partial\overline{\zeta}_a y = \beta - \partial^2\overline{u}/\partial y^2$, and demanding that this must change sign for an instability reprises the inflection point (Rayleigh–Kuo) condition. In the more general case, if $\partial\overline{q}/\partial y$ changes sign along a vertical line then the instability is called a baroclinic instability, and if it changes sign along a horizontal line the instability is barotropic — these may be taken as the definitions of those terms. A mixed instability has a change of sign along both horizontal and vertical lines.

10.6.2 Inclusion of Boundary Terms

Suppose now that the flow is contained between two flat boundaries, at $z = 0$ and $z = H$. The relevant equations of motion are the potential vorticity evolution in the interior, supplemented by the thermodynamic equation at the boundary. For unforced and inviscid flow these give (cf. (10.11) and (10.14)),

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\frac{\overline{q'^2}}{\partial_y\overline{q}}\right) = -\overline{v'q'}, \qquad 0 < z < H, \tag{10.119}$$

and

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\frac{\overline{b'^2}}{\partial_y\overline{b}}\right) = -\overline{v'b'}, \qquad z = 0,\ H. \tag{10.120}$$

The poleward flux of potential vorticity is

$$\overline{v'q'} = -\frac{\partial}{\partial y}\overline{u'v'} + \frac{\partial}{\partial z}\left(\frac{f_0}{N^2}\overline{v'b'}\right), \tag{10.121}$$

and integrating this expression with respect to both y and z gives

$$\int_A \overline{v'q'}\,\mathrm{d}y\,\mathrm{d}z = \left[\frac{f_0}{N^2}\overline{v'b'}\right]_0^H, \tag{10.122}$$

assuming that the meridional boundaries are at quiescent latitudes. Integrating (10.119) over y and z, and using (10.122) gives

$$\frac{\partial}{\partial t}\iint \frac{1}{2}\frac{\overline{q'^2}}{\partial_y\overline{q}}\,\mathrm{d}y\,\mathrm{d}z = -\left[\frac{f_0}{N^2}\overline{v'b'}\right]_0^H. \tag{10.123}$$

Using (10.120) to eliminate $\overline{v'b'}$ finally gives

$$\frac{\partial}{\partial t}\left\{\iint \frac{1}{2}\frac{\overline{q'^2}}{\partial_y\overline{q}}\,\mathrm{d}y\,\mathrm{d}z - \int\left[\frac{1}{2}\frac{f_0}{N^2}\frac{\overline{b'^2}}{\partial_y\overline{b}}\right]_0^H \mathrm{d}y\right\} = 0. \tag{10.124}$$

If this expression is positive or negative definite the perturbation cannot grow and therefore the basic state is stable. Stability thus depends on the meridional gradient of potential vorticity in the interior, and the meridional gradient of buoyancy at the boundary. If $\partial\overline{q}/\partial y$ changes sign in the interior, or $\partial\overline{b}/\partial y$ changes sign at the boundary, we have the potential for instability. If these are both one signed, then various possibilities exist, and using the thermal wind relation ($f_0\partial\overline{u}/\partial z = -\partial\overline{b}/\partial y$) we obtain the following:

I. *A stable case:*

$$\frac{\partial\overline{q}}{\partial y} > 0 \ \text{ and } \ \left.\frac{\partial u}{\partial z}\right|_{z=0} < 0 \ \text{ and } \ \left.\frac{\partial u}{\partial z}\right|_{z=H} > 0 \implies \text{stability.} \tag{10.125}$$

Stability also ensues if all inequalities are switched.

II. *Instability via interior–surface interactions:*

$$\frac{\partial \overline{q}}{\partial y} > 0 \ \text{ and } \ \left.\frac{\partial u}{\partial z}\right|_{z=0} > 0 \ \text{ or } \ \left.\frac{\partial u}{\partial z}\right|_{z=H} < 0 \implies \text{potential instability.} \tag{10.126}$$

The condition $\partial q/\partial y > 0$ and $(\partial u/\partial z)_{z=0} > 0$ is the most common criterion for instability that is met in the atmosphere. In the troposphere we can sometimes ignore contributions of the buoyancy fluxes at the tropopause ($z = H$), and stability is then determined by the interior potential vorticity gradient and the surface buoyancy gradient. Similarly, in the ocean contributions from the ocean floor are normally very small.

III. *Instability via edge wave interaction:*

$$\left.\frac{\partial u}{\partial z}\right|_{z=0} > 0 \ \text{ and } \ \left.\frac{\partial u}{\partial z}\right|_{z=H} > 0 \implies \text{potential instability.} \tag{10.127}$$

(And similarly, with both inequalities switched.) Such an instability may occur where the troposphere acts like a lid, as for example in the Eady problem. If $\partial \overline{q}/\partial y = 0$ and there is no lid at $z = H$ (e.g., the Eady problem with no lid) then the instability disappears.

One consequence of the upper boundary condition is that it provides a condition on the depth of the disturbance. In the Eady problem the evolution of the system is determined by temperature evolution at the surface,

$$\frac{\mathrm{D}b}{\mathrm{D}t} = 0 \qquad \text{at} \quad z = 0,\ H, \tag{10.128}$$

(where $b = f_0 \partial\psi/\partial z$) and zero potential vorticity in the interior, which implies that

$$\nabla^2\psi + k_d^2 H^2 \frac{\partial^2 \psi}{\partial z^2} = 0, \qquad 0 < z < H, \tag{10.129}$$

where $k_d = f_0/(HN)$. Assuming a solution of the form $b \sim \sin kx$ then the Poisson equation (10.129) becomes

$$H^2 k_d^2 \frac{\partial^2 \psi}{\partial z^2} = k^2 \psi, \tag{10.130}$$

with solutions $\psi = A\exp(-\alpha z) + B\exp(\alpha z)$, where $\alpha^2 = k^2 N^2/f_0^2$. The scale height of the disturbance is thus

$$h \sim \frac{f_0 L}{2\pi N}. \tag{10.131}$$

where $L \sim 2\pi/k$ is the horizontal scale of the disturbance. If the upper boundary is higher than this, it cannot interact strongly with the surface, because the disturbances at either boundary decay before reaching the other. Put another way, if the structure of the disturbance is such that it is shallower than H, the presence of the upper boundary is not felt. In the Eady problem, we know that the upper boundary must be important, because it is only by its presence that the flow can be unstable. Thus, all unstable modes in the Eady problem must be 'deep' in this sense, which can be verified by direct calculation. This condition gives rise to a physical interpretation of the high-wavenumber cut-off: if L is too small, the modes are too shallow to span the full depth of the fluid, and from (10.131) the condition for stability is thus

$$L < L_c = 2\pi\frac{NH}{f_0} \qquad \text{or} \qquad K > K_c = \frac{f_0}{NH} = L_d^{-1}, \tag{10.132}$$

where L_c and K_c are the critical length scales and wavenumbers. Wavenumbers larger than the reciprocal of the deformation radius are stable in the Eady problem. If β is non-zero, this condition does not apply, because the necessary condition for instability can be satisfied by a combination of a surface temperature gradient and an interior gradient of potential vorticity provided by β, as in condition (*II.*) in Section 10.6.2. Thus, we may expect that, if $\beta \neq 0$, higher wavenumbers ($k > k_d$) may be unstable but if so they will be shallow, and this may be confirmed by explicit calculation (see Figs. 9.12 and 9.19). In the two-level model shallow modes are, by construction, not allowed so that high wavenumbers will be stable, with or without beta.

10.7 ✦ NECESSARY CONDITIONS FOR INSTABILITY: USE OF PSEUDOENERGY

In this section we derive another necessary condition for instability, sometimes called an 'Arnold condition', that is based on the conservation properties of energy and enstrophy. Such conditions can be derived more generally by variational methods, and these lead to somewhat stronger results (in particular, nonlinear results that do not require the perturbation to be small) but our derivations will be elementary and direct.[11]

10.7.1 Two-dimensional Flow

First consider inviscid, incompressible two-dimensional flow governed by the equation of motion

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0, \tag{10.133}$$

where $q = \zeta + f = \nabla^2\psi + f$ is the absolute vorticity and ψ is the streamfunction. In a steady state, the streamfunction and the potential vorticity are functions of each other so that

$$q = Q(\Psi) \qquad \text{and} \qquad \psi = \Psi(Q), \tag{10.134}$$

where Q is a differentiable but otherwise arbitrary function of its argument, and Ψ its functional inverse. Equation (10.133) is then

$$\frac{\partial q}{\partial t} = -\frac{\mathrm{d}Q}{\mathrm{d}\psi} J(\Psi, \Psi) = 0, \tag{10.135}$$

and all steady solutions are of the form (10.134). We shall prove that if $\mathrm{d}\Psi/\mathrm{d}Q > 0$ then the flow is stable, in a sense to be made explicit below. Consider the evolution of perturbations about such a steady state, so that

$$q = Q + q', \qquad \psi = \Psi + \psi', \tag{10.136}$$

and we suppose that the perturbation vanishes at the domain boundary or that the boundary conditions are periodic. The potential vorticity perturbation satisfies, in the linear approximation,

$$\frac{\partial q'}{\partial t} + J(\psi', Q) + J(\Psi, q') = 0. \tag{10.137}$$

Now, because potential vorticity is conserved on parcels, any function of potential vorticity is also materially conserved, and in particular

$$\frac{\mathrm{D}\Psi(q)}{\mathrm{D}t} = \frac{\partial \Psi}{\partial t} + J(\psi, \Psi) = 0. \tag{10.138}$$

Linearizing this using (10.136) gives

$$\frac{\mathrm{d}\Psi}{\mathrm{d}Q}\frac{\partial q'}{\partial t} + J(\psi', \Psi) + J\left(\Psi, \frac{\mathrm{d}\Psi}{\mathrm{d}Q} q'\right) = 0. \tag{10.139}$$

We now form an energy equation from (10.137) by multiplying by $-\psi'$ and integrating over the domain. Integrating the first term by parts we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{1}{2}(\nabla\psi')^2 \,\mathrm{d}A = \int \psi' J(\Psi, q')\,\mathrm{d}A. \tag{10.140}$$

Similarly, from (10.139) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{1}{2}\frac{\mathrm{d}\Psi}{\mathrm{d}Q}q'^2 \,\mathrm{d}A = -\int \left[q' J(\psi', \Psi) + q' J\left(\Psi, \frac{\mathrm{d}\Psi}{\mathrm{d}Q}q'\right)\right] \mathrm{d}A. \tag{10.141}$$

The second term in square brackets vanishes. This follows using the property of Jacobians, obtained by integrating by parts, that

$$\langle aJ(b,c)\rangle = \langle bJ(c,a)\rangle = \langle cJ(a,b)\rangle = -\langle cJ(b,a)\rangle\,, \tag{10.142}$$

where the angle brackets denote horizontal integration. Using this we have

$$\begin{aligned}\left\langle q' J\left(\Psi, \frac{\mathrm{d}\Psi}{\mathrm{d}Q}q'\right)\right\rangle = -\left\langle \frac{\mathrm{d}\Psi}{\mathrm{d}Q}q' J(\Psi, q')\right\rangle &= -\frac{1}{2}\left\langle \frac{\mathrm{d}\Psi}{\mathrm{d}Q}J(\Psi, q'^2)\right\rangle \\ &= -\frac{1}{2}\left\langle q'^2 J\left(\frac{\mathrm{d}\Psi}{\mathrm{d}Q}, \Psi\right)\right\rangle = 0.\end{aligned} \tag{10.143}$$

Adding (10.140) and (10.141) the remaining nonlinear terms cancel and we obtain the conservation law,

$$\frac{\mathrm{d}\widehat{H}}{\mathrm{d}t} = 0, \qquad \text{where} \qquad \widehat{H} = \frac{1}{2}\int\left[(\nabla\psi')^2 + \frac{\mathrm{d}\Psi}{\mathrm{d}Q}q'^2\right]\mathrm{d}A. \tag{10.144}$$

The quantity $\widehat{H}$ is known as the *pseudoenergy* of the disturbance and because it is a conserved quantity, quadratic in the wave amplitude, it is (like pseudomomentum) a wave activity. Its conservation holds whether the disturbance is growing, decaying or neutral.

If $\mathrm{d}\Psi/\mathrm{d}Q$ is positive everywhere the pseudoenergy is a positive-definite quantity, and the growth of the disturbance is then largely prevented and the basic state is said to be *stable in the sense of Liapunov*. This means that the magnitude of the perturbation, as measured by some norm, is bounded by its initial magnitude. In the case here we define the norm

$$||\psi||^2 \equiv \int\left[(\nabla\psi)^2 + \frac{\mathrm{d}\Psi}{\mathrm{d}Q}(\nabla^2\psi)^2\right]\mathrm{d}A, \tag{10.145}$$

so that

$$||\psi'(t)||^2 = ||\psi'(0)||^2. \tag{10.146}$$

If $\mathrm{d}\Psi/\mathrm{d}Q > 0$ then, although the energy of the disturbance can grow, its final amplitude is bounded by the initial value of the pseudoenergy, because if perturbation energy is to grow perturbation enstrophy must shrink but it cannot shrink past zero. Normal-mode instability, in which modes grow exponentially, is completely precluded.

If the pseudoenergy is *negative definite* then stability is also assured, but this is a less common situation for it demands that $\mathrm{d}\Psi/\mathrm{d}Q$ be sufficiently negative so that the (negative of the) enstrophy contribution is always larger than the energy contribution, and this can usually only be satisfied in a sufficiently small domain. To see this, suppose that $q' = \nabla^2\psi'$, and that in the domain under consideration the Laplacian operator has eigenvalues $-k^2$, where

$$\nabla^2\psi' = -k^2\psi' \tag{10.147}$$

and the smallest eigenvalue, by magnitude, is k_0^2. Then, using Poincaré's inequality,

$$\int (\nabla^2\psi')^2 \,\mathrm{d}A \geq k_0^2 \int (\nabla\psi')^2 \,\mathrm{d}A, \tag{10.148}$$

a sufficient condition to make $\widehat{H}$ negative definite is that

$$\frac{\mathrm{d}\Psi}{\mathrm{d}Q} < -\frac{1}{k_0^2}. \tag{10.149}$$

As the domain gets bigger, k_0 diminishes and this condition becomes harder to satisfy.[12]

Parallel shear flow and Fjørtoft's condition

Consider the stability of a zonal flow (i.e., a flow in the x-direction), that varies only with y. The flow stability condition is then

$$\frac{\mathrm{d}\Psi}{\mathrm{d}Q} = \frac{\mathrm{d}\Psi/\mathrm{d}y}{\mathrm{d}Q/\mathrm{d}y} = -\frac{U - U_s}{\beta - U_{yy}} > 0, \tag{10.150}$$

where U_s is a constant, representing an arbitrary, constant, zonal flow. The last equality follows because the problem is Galilean invariant, and we are therefore at liberty to choose U_s arbitrarily. To connect this with Fjørtoft's condition (Chapter 9) multiply the top and bottom by $(\beta - U_{yy})$, whence we see that a sufficient condition for stability is that $(U - U_s)(\beta - U_{yy})$ is everywhere negative. The derivation here, unlike our earlier one in Section 9.3.2, makes it clear that the condition does not apply only to normal-mode instabilities.

10.7.2 ✦ Stratified Quasi-Geostrophic Flow

The extension of the pseudoenergy arguments to quasi-geostrophic flow is mostly straightforward, but with a complication from the vertical boundary conditions at the surface and at an upper boundary, and the trusting reader may wish to skip straight to the results, (10.155)–(10.157).[13] For definiteness, we consider Boussinesq, β-plane quasi-geostrophic flow confined between flat rigid surfaces at $z = 0$ and $z = H$. The interior flow is governed by the familiar potential vorticity equation $\mathrm{D}q/\mathrm{D}t = 0$ and the buoyancy equation $\mathrm{D}b/\mathrm{D}t = 0$ at the two boundaries, where

$$q = \nabla^2\psi + \beta y + \frac{\partial}{\partial z}\left(S(z)\frac{\partial\psi}{\partial z}\right), \qquad b = f_0\frac{\partial\psi}{\partial z}, \tag{10.151}$$

and $S(z) = f_0^2/N^2$ is positive. The basic state $(\psi = \Psi, q = Q, b = B_1, B_2)$ satisfies

$$\begin{gathered} \psi = \Psi(Q), \quad 0 < z < H, \\ \psi = \Psi_1(B_1), \quad z = 0 \qquad \text{and} \qquad \psi = \Psi_2(B_2), \quad z = H. \end{gathered} \tag{10.152}$$

Analogously to the barotropic case, we obtain the equations of motion for the interior perturbation

$$\frac{\partial q'}{\partial t} + J(\psi', Q) + J(\Psi, q') = 0, \tag{10.153a}$$

$$\frac{\mathrm{d}\Psi}{\mathrm{d}Q}\frac{\partial q'}{\partial t} + J(\psi', \Psi) + J\left(\Psi, \frac{\mathrm{d}\Psi}{\mathrm{d}Q}q'\right) = 0, \tag{10.153b}$$

and at the two boundaries

$$\frac{\partial b'}{\partial t} + J(\psi', B_i) + J(\Psi_i, b') = 0, \tag{10.154a}$$

$$\frac{d\Psi_i}{dB_i}\frac{\partial b'}{\partial t} + J(\psi', \Psi_i) + J\left(\Psi_i, \frac{d\Psi_i}{dB_i}b'\right) = 0, \tag{10.154b}$$

for $i = 1, 2$. (By $d\Psi_i/dB_i$ we mean the derivative of Ψ_i with respect to its argument, evaluated at B_i.) From these equations, we form the pseudoenergy by multiplying (10.153a) by $-\psi'$, (10.153b) by q', and (10.154a) by ψ', (10.154b) by b'. After some manipulation we obtain the pseudoenergy conservation law:

$$\frac{d\widehat{H}}{dt} = 0, \qquad \text{where} \qquad \widehat{H} = \mathcal{E} + \mathcal{Z} + \mathcal{B}_1 + \mathcal{B}_2\,, \tag{10.155}$$

and

$$\begin{aligned} \mathcal{E} &= \frac{1}{2}\left\{(\nabla\psi')^2 + S\left(\frac{\partial\psi'}{\partial z}\right)^2\right\}, & \mathcal{Z} &= \frac{1}{2}\left\{\frac{d\Psi}{dQ}q'^2\right\}, \\ \mathcal{B}_1 &= \frac{1}{2}\left\langle\frac{S(0)}{f_0}\frac{d\Psi_1}{dB_1}b'(0)^2\right\rangle, & \mathcal{B}_2 &= -\frac{1}{2}\left\langle\frac{S(H)}{f_0}\frac{d\Psi_2}{dB_2}b'(H)^2\right\rangle. \end{aligned} \tag{10.156}$$

where the curly brackets denote a three-dimensional integration over the fluid interior, and the angle brackets denote a horizontal integration over the boundary surfaces at 0 and H. The pseudoenergy $\widehat{H}$ is positive-definite, and therefore stability is assured in that norm, if all of the following conditions are satisfied:

$$\frac{d\Psi}{dQ} > 0, \qquad \frac{1}{f_0}\frac{d\Psi_1}{dB_1} > 0, \qquad \frac{1}{f_0}\frac{d\Psi_2}{dB_2} < 0. \tag{10.157}$$

If the flow is compressible, the potential vorticity is $q = \nabla^2\psi + \beta y + \rho_R^{-1}\partial_z(\rho_R S\partial_z\psi)$, where $\rho_R = \rho_R(z)$, but the final stability conditions are unaltered. If the upper boundary is then removed to infinity where $\rho_R(z) = 0$ then only the lower boundary condition contributes to (10.157). In the layered form of the quasi-geostrophic equations the vertical boundary conditions are built in to the definitions of potential vorticity in the top and bottom layers. In this case, a sufficient condition for stability is that $d\Psi/dQ > 0$ in each layer. Indeed, an alternative derivation of (10.155)–(10.157) would be to incorporate the boundary conditions on buoyancy into the definition of potential vorticity by the delta-function construction of Section 5.4.3.

Zonal shear flow

Consider now zonally uniform zonal flows, such as might give rise to baroclinic instability in a channel. The fields are then functions of y and z only, and the sufficient conditions for stability are:

$$\begin{aligned} \frac{d\Psi}{dQ} &= \frac{\partial\Psi/\partial y}{\partial Q/\partial y} = -\frac{U}{dQ/dy} > 0, \\ \frac{d\Psi_1}{dB_1} &= \frac{d\Psi_1/dy}{dB_1/dy} = \frac{U(0)}{dU(0)/dz} > 0, \\ \frac{d\Psi_2}{dB_2} &= \frac{d\Psi_2/dy}{dB_2/dy} = \frac{U(H)}{dU(H)/dz} < 0, \end{aligned} \tag{10.158}$$

using the thermal wind relation, and setting $f_0 = 1$ (its value is irrelevant). These results generalize Fjørtoft's condition to the stratified case,[14] and as in that case we are at liberty to add a uniform zonal flow to all the velocities.

10.7.3 ✦ Applications to Baroclinic Instability

We may use the stability conditions derived above to provide a few more results about baroclinic instability, including an alternative derivation of the minimum shear criterion in two-layer flow, and a derivation of the high-wavenumber cut-off to instability. In what follows we do not derive any new criteria; rather, the derivations make it apparent that the criteria are not restricted to perturbations of normal-mode form.

Minimum shear in two-layer flow

We consider two layers of equal depth, on a flat-bottomed β-plane with basic state

$$\Psi_1 = -U_1 y, \qquad \Psi_2 = -U_2 y \tag{10.159a}$$

$$Q_1 = \beta y - \frac{k_d^2}{2}(U_2 - U_1)y, \qquad Q_2 = \beta y - \frac{k_d^2}{2}(U_1 - U_2)y. \tag{10.159b}$$

This state is characterized by $Q_i = \gamma_i \Psi_i$ where

$$\gamma_1 = -\frac{(\beta + k_d^2 \widehat{U})}{(\overline{U} + \widehat{U})}, \qquad \gamma_2 = -\frac{(\beta - k_d^2 \widehat{U})}{(\overline{U} - \widehat{U})}, \tag{10.160}$$

with $\overline{U} = (U_1 + U_2)/2$ and $\widehat{U} = (U_1 - U_2)/2$. The barotropic flow does not affect the stability properties, so without loss of generality we may choose $\overline{U} < -\widehat{U}$, and this makes $\gamma_1 > 0$. Then γ_2 is also positive if $\beta > k_d^2 \widehat{U}/2$. Thus, a sufficient condition for stability is that

$$\widehat{U} < \frac{\beta}{k_d^2}, \tag{10.161}$$

as obtained in Chapter 9. However, we now see that the stability condition does not apply only to normal-mode instabilities.[15]

Use of pseudomomentum conservation provides an alternative derivation of the same result. The flow will also be stable if in both layers $\partial Q/\partial y > 0$, for then the conserved pseudomomentum will be positive definite. If $U_1 > U_2$ then, from (10.159) $\mathrm{d}Q_1/\mathrm{d}y > 0$. The flow will be stable if $\mathrm{d}Q_2/\mathrm{d}y > 0$, and this gives

$$\widehat{U} = \frac{1}{2}(U_1 - U_2) < \frac{\beta}{k_d^2}, \tag{10.162}$$

as in (10.161).

The high-wavenumber cut-off in two-layer baroclinic instability

We can use a pseudoenergy argument to show that there is a high-wavenumber cut-off to two-layer baroclinic instability, with the basic state (10.159). The conserved pseudoenergy analogous to (10.155) and (10.156) is readily found to be

$$\widehat{H} = \left\langle (\nabla\psi_1')^2 + (\nabla\psi_2')^2 + \frac{1}{2}k_d^2(\psi_1' - \psi_2')^2 + \frac{q_1'^2}{\gamma_1} + \frac{q_2'^2}{\gamma_2} \right\rangle = 0. \tag{10.163}$$

Let us choose (without loss of generality) the barotropic flow to be $\overline{U} = \beta/k_d^2$. We then have $\gamma_1 = \gamma_2 = -1/k_d^2$, and the pseudoenergy is then just the actual energy minus k_d^{-2} times the total enstrophy. If we define $\psi = (\psi_1' + \psi_2')/2$ and $\tau = (\psi_1' - \psi_2')/2$ then, using (12.41a) and (12.44), (10.163) may be expressed as

$$\widehat{H} = \left\langle (\nabla\psi)^2 + (\nabla\tau)^2 + k_d^2\tau^2 - k_d^{-2}\left\{(\nabla^2\psi)^2 + \left[(\nabla^2 - k_d^2)\tau\right]^2\right\}\right\rangle. \tag{10.164}$$

Now, let us express the fields as Fourier sums,

$$(\tau, \psi) = \sum_{k,l} (\widetilde{\tau}_{k,l}, \widetilde{\psi}_{k,l}) \mathrm{e}^{\mathrm{i}(kx+ly)}. \tag{10.165}$$

(This expression assumes a doubly-periodic domain; essentially the same end-result is obtained in a channel.) The pseudoenergy may then be written as

$$\widehat{H} = \sum_{k,l} \left[K^2 \widetilde{\psi}_{k,l}^2 (k_d^2 - K^2) + K'^2 \widetilde{\tau}_{k,l}^2 (k_d^2 - K'^2) \right], \tag{10.166}$$

where $K^2 = k^2 + l^2$ and $K'^2 = K^2 + k_d^2$. If the deformation radius is sufficiently large (or the domain sufficiently small) that $K^2 > k_d^2$, then the pseudoenergy is *negative-definite*, so the flow is stable, no matter what the shear may be. Such a situation might arise on a planet whose circumference was less than the deformation radius, or in a small ocean basin. In the linear problem, in which perturbation modes do not interact, horizontal wavenumbers with $k^2 > k_d^2$ are stable and there is thus a high-wavenumber cut-off to instability, as was found in Chapter 9 by direct calculation.

Notes

1 After Eliassen & Palm (1961).

2 Andrews & McIntyre (1976), Ripa (1981) and Held (1985).

3 These restrictions on the basic state are not necessary to prove orthogonality, but they make the algebra simpler. Also, we pay no attention here to the nature of the eigenvalues of (10.45), which, in general, consist of both a discrete and a continuous spectrum. See Farrell (1984) and McIntyre & Shepherd (1987).

4 The TEM was introduced by Andrews & McIntyre (1976, 1978) and Boyd (1976). A precursor is the paper of Riehl & Fultz (1957), who noted the shortcomings of zonal averaging in uncovering the meaning of indirect cells in laboratory experiments, and by extension the atmosphere.

5 The main result of this subsection was originally obtained by McIntosh & McDougall (1996). I thank A. Plumb for a discussion about the derivation given here. It is in fact possible to write exact TEM-like equations wholly in terms of the thickness-weighted averaged quantities and without taking a zonal average. The literature is extensive and at times hard to follow, but the reader may usefully look to de Szoeke & Bennett (1993) and Young (2012), who show that thickness-weighted averaged equations may be derived that are identical to the unweighted equations except for the appearance, in the horizontal momentum equations, of an eddy forcing by the divergence of three-dimensional Eliassen–Palm vectors. The divergence of these EP vectors is related to the eddy flux of the full potential vorticity, in an analogous manner to (but more general than) the quasi-geostrophic result.

6 This problem can be worked around in some cases (Plumb 1990, Greatbatch 1998).

7 Non-acceleration arguments have a long history, with contributions from Charney & Drazin (1961), Eliassen & Palm (1961), Holton (1974) and, in particular, Boyd (1976) and Andrews & McIntyre (1978). Dunkerton (1980) reviews and provides examples. Non-acceleration is now so prevalent in the literature that it could be written nonacceleration.

8 Conservation laws of this ilk, their connection to the underlying symmetries of the basic state and (relatedly) their finite-amplitude extension, are discussed by McIntyre & Shepherd (1987) and Shepherd (1990). Conservation of momentum is related to the translational invariance of the medium whereas conservation of $\mathcal{P}$ is related to the translational invariance of the basic state, and hence the appellation 'pseudomomentum'.

9 See Andrews & McIntyre (1978) and Young (2012).

10 Steve Garner and Raffaele Ferrari both provided very helpful input to this section. Shepherd (1983) considers the two-layer problem.

11 The original papers are Arnold (1965, 1966), with many results being developed by Holm *et al.* (1985).

12 The stability criterion is sometimes referred to as 'Arnold's second condition'. More discussion is given in Holm *et al.* (1985) and McIntyre & Shepherd (1987).

13 Blumen (1968), but the method we use is more direct.

14 Pedlosky (1964) derived these conditions by a normal-mode approach.

15 Pierini & Vulpiani (1981) and Vallis (1985) further consider the finite-amplitude case.