Appendix C

Vector Analysis

C.1 VECTOR IDENTITIES

The following formulas may be shown to hold where Φ is an arbitrary scalar and **A** and **B** are arbitrary vectors.

$$
\begin{aligned}
\nabla \times \nabla\Phi &= 0 \\
\nabla \cdot (\Phi\mathbf{A}) &= \Phi\nabla \cdot (\mathbf{A}) + \mathbf{A} \cdot \nabla\Phi \\
\nabla \times (\Phi\mathbf{A}) &= \nabla\Phi \times \mathbf{A} + \Phi(\nabla \times \mathbf{A}) \\
\nabla \cdot (\nabla \times \mathbf{A}) &= 0 \\
(\mathbf{A} \cdot \nabla)\,\mathbf{A} &= \frac{1}{2}\nabla(\mathbf{A} \cdot \mathbf{A}) - \mathbf{A} \times (\nabla \times \mathbf{A}) \\
\nabla \times (\mathbf{A} \times \mathbf{B}) &= \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla)\,\mathbf{B} + (\mathbf{B} \cdot \nabla)\,\mathbf{A} \\
\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= (\mathbf{A} \cdot \mathbf{C})\,\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\,\mathbf{C}
\end{aligned}
$$

C.2 INTEGRAL THEOREMS

(a) Divergence theorem:

$$\int_A \mathbf{B} \cdot \mathbf{n}dA = \int_V \mathbf{V} \cdot \mathbf{B}dV$$

where V is a volume enclosed by surface A and **n** is a unit normal on A.

(b) Stokes's theorem:

$$\oint \mathbf{B} \cdot d\mathbf{l} = \int_A (\mathbf{V} \times \mathbf{B}) \cdot \mathbf{n}dA$$

where A is a surface bounded by the line traced by the position vector **l** and **n** is a unit normal of A.

C.3 VECTOR OPERATIONS IN VARIOUS COORDINATE SYSTEMS

(a) Cartesian coordinates: (x, y, z)

Coordinate	Symbol	Velocity component	Unit vector
Eastward	x	u	$\mathbf{i}$
Northward	y	v	$\mathbf{j}$
Upward	z	w	$\mathbf{k}$

$$\nabla \Phi = \mathbf{i}\frac{\partial \Phi}{\partial x} + \mathbf{j}\frac{\partial \Phi}{\partial y} + \mathbf{k}\frac{\partial \Phi}{\partial z}$$

$$\nabla \cdot \mathbf{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

$$\mathbf{k} \cdot (\nabla \times \mathbf{V}) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$\nabla_h^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}$$

(b) Cylindrical coordinates: (r, λ, z)

Coordinate	Symbol	Velocity component	Unit vector
Radial	r	u	$\mathbf{i}$
Azimuthal	λ	v	$\mathbf{j}$
Upward	z	w	$\mathbf{k}$

$$\nabla \Phi = \mathbf{i}\frac{\partial \Phi}{\partial r} + \mathbf{j}\frac{1}{r}\frac{\partial \Phi}{\partial \lambda} + \mathbf{k}\frac{\partial \Phi}{\partial z}$$

$$\nabla \cdot \mathbf{V} = \frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \lambda}$$

$$\mathbf{k} \cdot (\nabla \times \mathbf{V}) = \frac{1}{r}\frac{\partial (rv)}{\partial r} - \frac{1}{r}\frac{\partial u}{\partial \lambda}$$

$$\nabla_h^2 \Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \lambda^2}$$

(c) Spherical coordinates: (λ, ϕ, r)

Coordinate	Symbol	Velocity component	Unit vector
Longitude	λ	u	$\mathbf{i}$
Latitude	ϕ	v	$\mathbf{j}$
Height	r	w	$\mathbf{k}$

$$\nabla\Phi = \frac{\mathbf{i}}{r\cos\phi}\frac{\partial\Phi}{\partial\lambda} + \mathbf{j}\frac{1}{r}\frac{\partial\Phi}{\partial\phi} + \mathbf{k}\frac{\partial\Phi}{\partial r}$$

$$\nabla\cdot\mathbf{V} = \frac{1}{r\cos\phi}\left[\frac{\partial u}{\partial\lambda} + \frac{\partial\,(v\cos\phi)}{\partial\phi}\right]$$

$$\mathbf{k}\cdot(\nabla\times\mathbf{V}) = \frac{1}{r\cos\phi}\left[\frac{\partial v}{\partial\lambda} - \frac{\partial\,(u\cos\phi)}{\partial\phi}\right]$$

$$\nabla_h^2\Phi = \frac{1}{r^2\cos^2\phi}\left[\frac{\partial^2\Phi}{\partial\lambda^2} + \cos\phi\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial\Phi}{\partial\phi}\right)\right]$$