*Shootings of water thread down the slope*
*of the huge green stone —*
*The white eddy-rose that blossom'd up*
*against the stream in the scollop,*
*by fits and starts, obstinate in resurrection —*
*It* is the life *that we live.*

Samuel Taylor Coleridge, *The Eddy-Rose*, adapted from notebook, 1799.

CHAPTER 11

# Basics of Incompressible Turbulence

TURBULENCE IS HIGH REYNOLDS NUMBER FLUID FLOW, dominated by nonlinearity, containing both spatial and temporal disorder. No definition is perfect, and it is hard to disentangle a definition from a property, but this statement captures the essential aspects. A turbulent flow has eddies with a spectrum of sizes between some upper and lower bounds, the former often determined by the forcing scale or the domain scale, and the latter usually by viscosity. The individual eddies come and go, and are inherently unpredictable. Rather like life, turbulent flows are endlessly fascinating and not a little frustrating.[1]

The circulation of the atmosphere and ocean is, *inter alia,* the motion of a forced-dissipative fluid subject to various constraints such as rotation and stratification. The larger scales are orders of magnitude larger than the dissipation scale (the scale at which molecular viscosity becomes important) and at many if not all scales the motion is highly nonlinear and quite unpredictable. Thus, we can justifiably say that the atmosphere and ocean are turbulent fluids. We are not simply talking about the small-scale flows traditionally regarded as turbulent; rather, our main focus will be the large-scale flows associated with baroclinic instability and greatly influenced by rotation and stratification, a kind of turbulence known as *geostrophic turbulence*. However, before discussing turbulence in the atmosphere and ocean, in this chapter we consider from a fairly elementary standpoint the basic theory of two- and three-dimensional turbulence, and in particular the theory of inertial ranges. We do not provide a comprehensive discussion of turbulence; rather, we provide an introduction to those aspects of most interest or relevance to the dynamical oceanographer or meteorologist. In the next chapter we consider the effects of rotation and stratification, and after that we look at turbulent diffusion.

## 11.1 THE FUNDAMENTAL PROBLEM OF TURBULENCE

Turbulence is a difficult subject because it is nonlinear, and because, and relatedly, there are interactions between scales of motion. Let us first see what difficulties these bring, beginning with the closure problem itself.

### 11.1.1 The Closure Problem

Although in a turbulent flow it may be virtually impossible to predict the detailed motion of each eddy, the statistical properties — time averages for example — are not necessarily changing and we might like to predict such averages. Effectively, we accept that we cannot predict the weather, but we can try to predict the climate. Even though we know the equations that determine the system, this task proves to be very difficult because the equations are nonlinear, and we come up against the *closure problem*. To see what this is, let us decompose the velocity field into mean and fluctuating components,

$$\boldsymbol{v} = \overline{\boldsymbol{v}} + \boldsymbol{v}'. \tag{11.1}$$

Here $\overline{\boldsymbol{v}}$ is the mean velocity field, and $\boldsymbol{v}'$ is the deviation from that mean. The mean might be a time average, in which case $\overline{\boldsymbol{v}}$ is a function only of space and not of time, or it might be a time mean over a finite period (e.g., a season if we are dealing with the weather), or it might be some form of ensemble mean. The average of the deviation is, by definition, zero; that is $\overline{\boldsymbol{v}'} = 0$. The idea is to substitute (11.1) into the momentum equation and try to obtain a closed equation for the mean quantity $\overline{\boldsymbol{v}}$. Rather than dealing with the full Navier–Stokes equations, let us carry out this program for a model nonlinear system that obeys

$$\frac{\mathrm{d}u}{\mathrm{d}t} + uu + ru = 0, \tag{11.2}$$

where $r$ is a constant. The average of this equation is:

$$\frac{\mathrm{d}\overline{u}}{\mathrm{d}t} + \overline{uu} + r\overline{u} = 0. \tag{11.3}$$

The value of the term $\overline{uu}$ (i.e., $\overline{u^2}$) is not deducible simply by knowing $\overline{u}$, since it involves correlations between eddy quantities, namely $\overline{u'u'}$. That is, $\overline{uu} = \overline{u}\,\overline{u} + \overline{u'u'} \neq \overline{u}\,\overline{u}$. We can go to the next order to try (vainly!) to obtain an equation for $\overline{u}\,\overline{u}$. First multiply (11.2) by $u$ to obtain an equation for $u^2$, and then average it to yield

$$\frac{1}{2}\frac{\mathrm{d}\overline{u^2}}{\mathrm{d}t} + \overline{uuu} + r\overline{u^2} = 0. \tag{11.4}$$

This equation contains the undetermined cubic term $\overline{uuu}$. An equation determining this would contain a quartic term, and so on in an unclosed hierarchy. Many methods of closing the hierarchy make assumptions about the relationship of $(n+1)$th order terms to $n$th order terms, for example by supposing that

$$\overline{uuuu} = \alpha\overline{uu}\,\overline{uu} + \beta\overline{uuu}, \tag{11.5}$$

where $\alpha$ and $\beta$ are some parameters, and closures set in physical space or in spectral space (i.e., acting on the Fourier transformed variables) have both been proposed. If we know that the variables are distributed normally then such closures can sometimes be made exact, but this is not generally the case in turbulence and all closures that have been proposed so far are, at best, approximations.

This same closure problem arises in the Navier–Stokes equations. If density is constant (say $\rho = 1$) the $x$-momentum equation for an averaged flow is

$$\frac{\partial\overline{u}}{\partial t} + (\overline{\boldsymbol{v}}\cdot\nabla)\overline{u} = -\frac{\partial\overline{p}}{\partial x} - \nabla\cdot\overline{\boldsymbol{v}'u'}. \tag{11.6}$$

Written out in full in Cartesian coordinates, the last term is

$$\nabla\cdot\overline{\boldsymbol{v}'u'} = \frac{\partial}{\partial x}\overline{u'u'} + \frac{\partial}{\partial y}\overline{u'v'} + \frac{\partial}{\partial z}\overline{u'w'}. \tag{11.7}$$

These terms, and the similar ones in the $y$- and $z$-momentum equations, represent the effects of eddies on the mean flow and are known as *Reynolds stress* terms. The 'closure problem' of turbulence may be thought of as finding a representation of such Reynolds stress terms in terms of mean flow quantities. Nobody has been able to close the system, in any useful way, without introducing physical assumptions not directly deducible from the equations of motion themselves. Indeed, not only has the problem not been solved, it is not clear that in general a useful closed-form solution actually exists.

### 11.1.2 Triad Interactions in Turbulence

The nonlinear term in the equations of motion not only leads to difficulties in closing the equations, but it leads to interactions among different length scales, and in this section we write the equations of motion in a form that makes this explicit. For algebraic simplicity we will restrict our attention to two-dimensional flows, but very similar considerations also apply in three dimensions, and the details of the algebra following are not of themselves important to subsequent sections.[2]

The equation of motion for an incompressible fluid in two dimensions — see for example (4.67) or (5.119) — may be written as

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) = F + \nu \nabla^2 \zeta, \qquad \zeta = \nabla^2 \psi. \tag{11.8}$$

We include a forcing and viscous term but no Coriolis term. Let us suppose that the fluid is contained in a square, doubly-periodic domain of side $L$, and let us expand the streamfunction and vorticity in Fourier series so that, with a tilde denoting a Fourier coefficient,

$$\psi(x, y, t) = \sum_{\boldsymbol{k}} \widetilde{\psi}(\boldsymbol{k}, t) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}, \qquad \zeta(x, y, t) = \sum_{\boldsymbol{k}} \tilde{\zeta}(\boldsymbol{k}, t) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{11.9}$$

where $\boldsymbol{k} = \mathbf{i}k^x + \mathbf{j}k^y$, $\tilde{\zeta} = -k^2 \widetilde{\psi}$ where $k^2 = k^{x2} + k^{y2}$ and, to ensure that $\psi$ is real, $\widetilde{\psi}(k^x, k^y, t) = \widetilde{\psi}^*(-k^x, -k^y, t)$, where $^*$ denotes the complex conjugate, and this property is known as conjugate symmetry. The summations are over all positive and negative $x$- and $y$-wavenumbers, and $\widetilde{\psi}(\boldsymbol{k}, t)$ is shorthand for $\widetilde{\psi}(k^x, k^y, t)$. Substituting (11.9) in (11.8) gives, with (for the moment) $F$ and $\nu$ both zero,

$$\begin{aligned}\frac{\partial}{\partial t} \sum_{\boldsymbol{k}} \tilde{\zeta}(\boldsymbol{k}, t) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} = &\sum_{\boldsymbol{p}} p^x \widetilde{\psi}(\boldsymbol{p}, t) \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \times \sum_{\boldsymbol{q}} q^y \tilde{\zeta}(\boldsymbol{q}, t) \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}} \\ &- \sum_{\boldsymbol{p}} p^y \widetilde{\psi}(\boldsymbol{p}, t) \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \times \sum_{\boldsymbol{q}} q^x \tilde{\zeta}(\boldsymbol{q}, t) \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}},\end{aligned} \tag{11.10}$$

where $\boldsymbol{p}$ and $\boldsymbol{q}$ are, like $\boldsymbol{k}$, horizontal wave vectors. We may obtain an evolution equation for the wavevector $\boldsymbol{k}$ by multiplying (11.10) by $\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})$ and integrating over the domain, and using the fact that the Fourier modes are orthogonal; that is

$$\int \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}} \,\mathrm{d}A = L^2 \delta(\boldsymbol{p} + \boldsymbol{q}), \tag{11.11}$$

where $\delta(\boldsymbol{p} + \boldsymbol{q})$ equals unity if $\boldsymbol{p} = -\boldsymbol{q}$ and is zero otherwise. Using this, (11.10) becomes, restoring the forcing and dissipation terms,

$$\frac{\partial}{\partial t} \widetilde{\psi}(\boldsymbol{k}, t) = \sum_{\boldsymbol{p},\boldsymbol{q}} A(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q}) \widetilde{\psi}(\boldsymbol{p}, t) \widetilde{\psi}(\boldsymbol{q}, t) + \widetilde{F}(\boldsymbol{k}) - \nu k^2 \widetilde{\psi}(\mathbf{k}, t), \tag{11.12}$$

where $A(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q}) = (q^2/k^2)(p^x q^y - p^y q^x)\delta(\boldsymbol{p} + \boldsymbol{q} - \mathbf{k})$ is an 'interaction coefficient', and the summation is over all $\boldsymbol{p}$ and $\boldsymbol{q}$; however, only those wavevector triads with $\boldsymbol{p} + \boldsymbol{q} = \mathbf{k}$ make a non-zero contribution, because of the presence of the delta function.

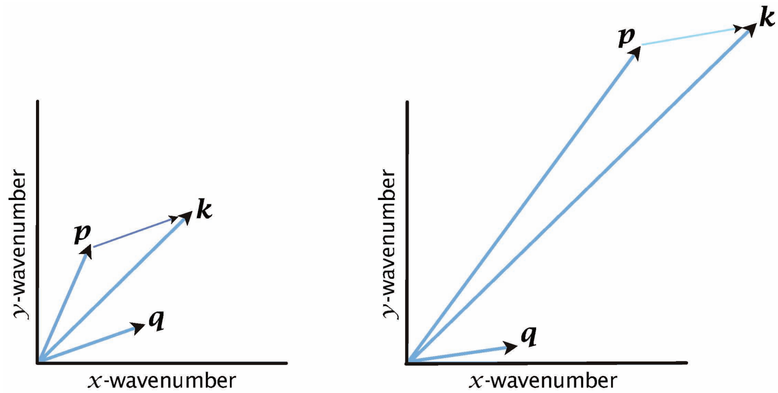


**Fig. 11.1** Two interacting triads, each with $\boldsymbol{k} = \boldsymbol{p} + \boldsymbol{q}$. On the left, a local triad with $k \sim p \sim q$. On the right, a non-locpwal triad with $k \sim p \gg q$.

Consider, then, a fluid in which just two Fourier modes are initially excited, with wavevectors $\boldsymbol{p}$ and $\boldsymbol{q}$, along with their conjugate-symmetric partners at $-\boldsymbol{p}$ and $-\boldsymbol{q}$. These modes interact, obeying (11.12), to generate third and fourth wavenumbers, $\boldsymbol{k} = \boldsymbol{p} + \boldsymbol{q}$ and $\boldsymbol{m} = \boldsymbol{p} - \boldsymbol{q}$ (along with their conjugate-symmetric partners). These four wavenumbers can interact among themselves to generate several additional wavenumbers, $\boldsymbol{k} + \boldsymbol{p}$, $\boldsymbol{k} + \boldsymbol{m}$ and so on, so potentially filling out the entire spectrum of wavenumbers. The individual interactions are called *triad interactions,* and it is by way of such interactions that energy is transferred between scales in turbulent flows, in both two and three dimensions. The dissipation term does not lead to interactions between modes with different wavevectors; rather, it acts like a drag on each Fourier mode, with a coefficient that increases with wavenumber and therefore that preferentially affects small scales.

The selection rule for triad interactions — that $\boldsymbol{k} = \boldsymbol{p} + \boldsymbol{q}$ — does not restrict the scales of these interacting wavevectors, and the types of triad interactions fall between two extremes:

*(i)* local interactions, in which $k \sim p \sim q$;

*(ii)* non-local interactions, in which $k \sim p \gg q$.

These two kinds of triads are schematically illustrated in Fig. 11.1. Without a detailed analysis of the solutions of the equations of motion — an analysis that is in general impossible for fully-developed turbulence — we cannot say with certainty whether one kind of triad interaction dominates. The theory of Kolmogorov considered below, and its two-dimensional analogue, assume that it is the *local* triads that are most important in transferring energy; this is a reasonable assumption because, from the perspective of a small eddy, large eddies appear as a nearly-uniform flow and so translate the small eddies around without distortion and thus without transferring energy between scales.

## 11.2 THE KOLMOGOROV THEORY

The foundation of many theories of turbulence is the spectral theory of Kolmogorov.[3] This theory does not close the equations as explicit a manner as (11.5), but it does provide a prediction for the energy spectrum of a turbulent flow (i.e., how much energy is present at a particular spatial scale) and it does this by suggesting a relationship between the energy spectrum (a second-order quantity in velocity) and the spectral energy flux (a third-order quantity).

### 11.2.1 The Physical Picture

Consider high Reynolds number ($Re$) incompressible flow that is being maintained by some external force. Then the evolution of a system that has $\rho = 1$ is governed by

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\nabla p + \boldsymbol{F} + \nu \nabla^2 \boldsymbol{v} \qquad (11.13)$$

and

$$\nabla \cdot \boldsymbol{v} = 0. \tag{11.14}$$

Here, $\boldsymbol{F}$ is some force we apply to maintain fluid motion — for example, we stir the fluid with a spoon. (One might argue that such stirring is not a force, like gravity, but a continuous changing of the boundary conditions. Having noted this, we treat it as a force.) A simple scale analysis of these equations seems to indicate that the ratio of the size of the inertial terms on the left-hand side to the viscous term is the Reynolds number $VL/\nu$, where $V$ and $L$ are velocity and length scales. To be explicit let us consider the ocean, and take $V = 0.1\,\mathrm{m\,s^{-1}}$, $L = 1000\,\mathrm{km}$ and $\nu = 10^{-6}\,\mathrm{m^2\,s^{-1}}$. Then $Re = VL/\nu \approx 10^{11}$, and it seems that we can neglect the viscous term on the right-hand side of (11.13). But this can lead to a paradox, as if the fluid is being forced this forcing is likely to put energy into the fluid. To see this, we obtain the energy budget for (11.13) by multiplying by $\boldsymbol{v}$ and integrating over a domain. If there is no flow into or out of our domain, the inertial terms in the momentum equation conserve energy and, recalling Section 1.10, the energy equation is

$$\frac{\mathrm{d}\widehat{E}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int \frac{1}{2}\boldsymbol{v}^2\,\mathrm{d}V = \int \left(\boldsymbol{F}\cdot\boldsymbol{v} + \nu\boldsymbol{v}\cdot\nabla^2\boldsymbol{v}\right)\,\mathrm{d}V = \int \left(\boldsymbol{F}\cdot\boldsymbol{v} - \nu\boldsymbol{\omega}^2\right)\,\mathrm{d}V, \tag{11.15}$$

where $\widehat{E}$ is the total energy. If we neglect the viscous term we are led to an inconsistency, since the forcing term is a source of energy ($\overline{\boldsymbol{F}\cdot\boldsymbol{v}} > 0$), because a force will normally, on average, produce a velocity that is correlated with the force itself. Without viscosity, energy keeps on increasing.

What is amiss? It is true that for motion with a 1000 km length scale and a velocity of a few centimetres per second we can neglect viscosity when considering the balance of forces in the momentum equation. But this does not mean that there is no motion at much smaller length scales — indeed we seem to be led to the inescapable conclusion that there must be some motion at smaller scales in order to remove energy. Scale analysis of the momentum equation suggests that viscous terms will be comparable with the inertial terms at a scale $L_\nu$ where the Reynolds number based on that scale is of order unity, giving

$$L_\nu \sim \frac{\nu}{V}. \tag{11.16}$$

This is a very small scale for geophysical flows, of order millimetres or less. Where and how are such small scales generated? Boundaries are one important region; if there is high Reynolds number flow above a solid boundary, for example the wind above the ground, then viscosity *must* become important in bringing the velocity to zero in order that it can satisfy the no-slip condition at the surface, as illustrated in Fig. 5.4.

Motion on very small scales may also be generated in the fluid interior. How might this happen? Suppose the forcing acts only at large scales, and its direct action is to set up some correspondingly large-scale flow, composed of eddies and shear flows and such-like. Then typically there will be an instability in the flow, and a smaller eddy will grow: initially, the large-scale flow may be treated as an unchanging shear flow, and the disturbance while small will obey linear equations of motion similar to those applicable in an idealized Kelvin–Helmholtz instability. This instability clearly must draw from the large scale quasi-stationary flow, and it will eventually saturate at some finite amplitude. Although it has grown in intensity, it is still typically smaller than the large scale flow that fostered it (remember how the growth rate of the shear instability gets larger as the wavelength of the perturbation decreased). As it reaches finite amplitude, the perturbation itself may become unstable, and smaller eddies will feed off its energy and grow, and so on.[4] Vortex stretching plays an important role in all this, stretching line elements and creating eddies, and energy, at small scales. The general picture that emerges is of a large-scale flow that is unstable to eddies somewhat smaller in scale. These eddies grow, and develop still smaller eddies, and energy is transferred to smaller and smaller scales in a cascade-like process, sketched in Fig. 11.2. Finally, eddies are generated that are sufficiently small that they feel the effects of viscosity, and energy is drained away. Thus, there is a flux of kinetic energy from the large to the small scales, where it is dissipated into heat.

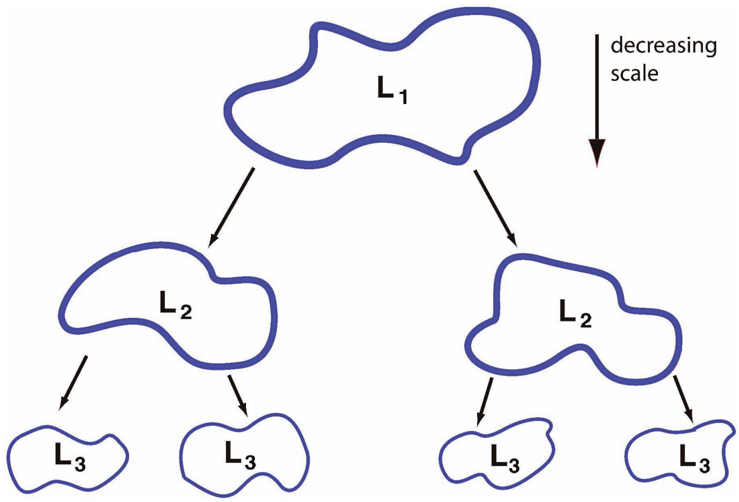


**Fig. 11.2** The passage of energy to smaller scales: eddies at large scale break up into ones at smaller scale, thereby transferring energy to smaller scales. (The eddies in reality are embedded within each other.)

If the passage occurs between eddies of similar sizes (i.e., if it is spectrally local) the transfer is said to be a cascade.

### 11.2.2 Inertial-range Theory

Given the above picture it becomes possible to predict what the energy spectrum is. Let us suppose that the flow is statistically isotropic (i.e., the same in all directions) and homogeneous (i.e., the same everywhere; all isotropic flows are homogeneous, but not vice versa). Homogeneity precludes the presence of solid boundaries but can be achieved in a periodic domain, and the finite domain puts an upper limit, sometimes called the outer scale, on the size of eddies.

If we decompose the velocity field into Fourier components, then in a finite domain we may write

$$u(x,y,z,t) = \sum_{\boldsymbol{k}} \widetilde{u}(\boldsymbol{k},t) \mathrm{e}^{\mathrm{i}(k^x x + k^y y + k^z z)}, \tag{11.17}$$

where $\widetilde{u}$ is the Fourier transformed field of $u$, with similar identities for $v$ and $w$, and $\boldsymbol{k} = (k^x, k^y, k^z)$. The sum is a triple sum over all wavenumbers $(k^x, k^y, k^z)$, and in a finite domain these wavenumbers are quantized. Finally, to ensure that $u$ is real we require that $\widetilde{u}(-\boldsymbol{k}) = \widetilde{u}^*(\boldsymbol{k})$, where the asterisk denotes the complex conjugate. Using Parseval's theorem (and assuming density is unity, as we shall throughout this chapter) the energy in the fluid is given by

$$\frac{1}{V}\int_V E \,\mathrm{d}V = \frac{1}{2V}\int_V \left(u^2 + v^2 + w^2\right) \mathrm{d}V = \frac{1}{2}\sum_{\boldsymbol{k}} \left(|\widetilde{u}|^2 + |\widetilde{v}|^2 + |\widetilde{w}|^2\right) \equiv \sum_{\boldsymbol{k}} \mathcal{E}_{\boldsymbol{k}}, \tag{11.18}$$

where $E$ is the energy density per unit mass, $V$ is the volume of the domain, and the last equality serves to define the discrete energy spectrum $\mathcal{E}_{\boldsymbol{k}}$. We will now assume that the turbulence is isotropic, and that the domain is sufficiently large that the sums in the above equations may be replaced by integrals. We may then write

$$\overline{E} = \frac{1}{V}\widehat{E} = \frac{1}{2V}\int_V \boldsymbol{v}^2 \,\mathrm{d}V = \int \mathcal{E}(k)\,\mathrm{d}k, \tag{11.19}$$

where $\overline{E}$ is the average energy, $\widehat{E}$ is the total energy and $\mathcal{E}(k)$ is the energy spectral density, or the energy spectrum, so that $\mathcal{E}(k)\delta k$ is the energy in the small wavenumber interval $\delta k$. Because of the assumed isotropy, the energy is a function only of the scalar wavenumber $k$, where $k^2 = k^{x2} + k^{y2} + k^{z2}$. The units of $\mathcal{E}(k)$ are $L^3/T^2$ and the units of $\overline{E}$ are $L^2/T^2$.

We now suppose that the fluid is stirred at large scales and, via the nonlinear terms in the momentum equation, that this energy is transferred to small scales where it is dissipated by viscosity. The key assumption is to suppose that, if the forcing scale is sufficiently larger than the dissipation

### Dimensions and the Kolmogorov Spectrum

| **Quantity** | **Dimension** |
|---|---|
| Wavenumber, $k$ | 1/L |
| Energy per unit mass, $E$ | $U^2 = L^2/T^2$ |
| Energy spectrum, $\mathcal{E}(k)$ | $EL = L^3/T^2$ |
| Energy flux, $\varepsilon$ | $E/T = L^2/T^3$ |

If $\mathcal{E} = f(\varepsilon, k)$ then the only dimensionally consistent relation for the energy spectrum is

$$\mathcal{E} = \mathcal{K}\varepsilon^{2/3}k^{-5/3},$$

where $\mathcal{K}$ is a dimensionless constant.

scale, there exists a range of scales that is intermediate between the large scale and the dissipation scale and where neither forcing nor dissipation are explicitly important to the dynamics. This assumption, known as the *locality hypothesis,* depends on the nonlinear transfer of energy being sufficiently local (in spectral space). This intermediate range is known as the *inertial range,* because the inertial terms and not forcing or dissipation must dominate in the momentum balance. If the rate of energy input per unit volume by stirring is equal to $\varepsilon$, then if we are in a steady state there must be a flux of energy from large to small scales that is also equal to $\varepsilon$, and an energy dissipation rate, also $\varepsilon$.

Now, we have no general theory for the energy spectrum of a turbulent fluid, but we might suppose it takes the general form

$$\mathcal{E}(k) = g(\varepsilon, k, k_0, k_\nu), \tag{11.20}$$

where the right-hand side denotes a function of the spectral energy flux or cascade rate $\varepsilon$, the wavenumber $k$, the forcing wavenumber $k_0$ and the wavenumber at which dissipation acts, $k_\nu$ (and $k_\nu \sim L_\nu^{-1}$). The function $g$ will of course depend on the particular nature of the forcing. Now, the locality hypothesis essentially says that at some scale within the inertial range the flux of energy to smaller scales depends only on processes occurring at or near that scale. That is to say, the energy flux is only a function of $\mathcal{E}$ and $k$, or equivalently that the energy spectrum can be a function *only* of the energy flux $\varepsilon$ and the wavenumber itself. From a physical point of view, as energy cascades to smaller scales the details of the forcing are forgotten but the effects of viscosity are not yet apparent, and the energy spectrum takes the form,

$$\mathcal{E}(k) = g(\varepsilon, k). \tag{11.21}$$

The function $g$ is assumed to be *universal,* the same for every turbulent flow.

Let us now use dimensional analysis to give us the form of the function $g(\varepsilon, k)$ (see the shaded box above). In (11.21), the left-hand side has dimensions $L^3/T^2$; the factor $T^{-2}$ can only be balanced by $\varepsilon^{2/3}$ because $k$ has no time dependence; that is, (11.21), and its dimensions, must take the form

$$\mathcal{E}(k) = \varepsilon^{2/3} g(k), \tag{11.22a}$$

$$\frac{L^3}{T^2} \sim \frac{L^{4/3}}{T^2} g(k), \tag{11.22b}$$

where $g(k)$ is some function. Evidently $g(k)$ must have dimensions $L^{5/3}$, and the functional rela-

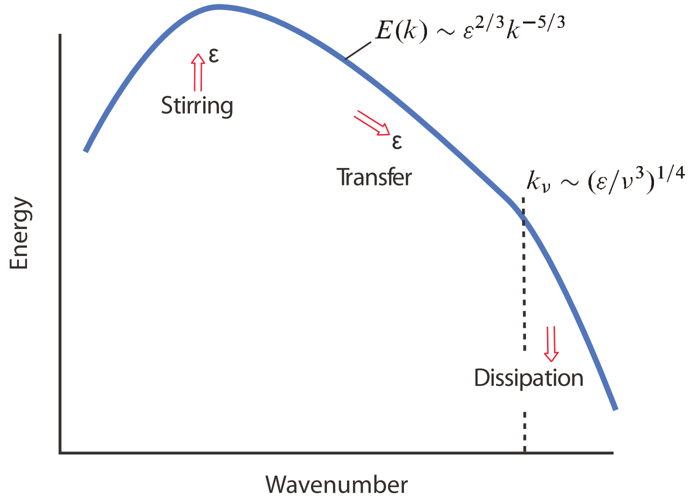


**Fig. 11.3** The energy spectrum in three-dimensional turbulence, in the theory of Kolmogorov. Energy is supplied at some rate $\varepsilon$; it is cascaded to small scales, where it is ultimately dissipated by viscosity. There is no systematic energy transfer to scales larger than the forcing scale, so here the energy falls off.

tionship we must have, if the physical assumptions are right, is

$$\boxed{\mathcal{E}(k) = \mathcal{K}\varepsilon^{2/3}k^{-5/3}.} \tag{11.23}$$

This is the famous 'Kolmogorov -5/3 spectrum', enshrined as one of the cornerstones of turbulence theory. It is sketched in Fig. 11.3, and some experimental results are shown in Fig. 11.4. The parameter $\mathcal{K}$ is a dimensionless constant, undetermined by this theory; it is known as Kolmogorov's constant and experimentally its value is found to be about 1.5.[5]

An equivalent, and revealing, way to derive this result is to first define an eddy turnover time $\tau_k$, which is the time taken for a parcel with velocity $v_k$ to move a distance $1/k$, $v_k$ being the velocity associated with the (inverse) scale $k$. On dimensional considerations $v_k = [\mathcal{E}(k)k]^{1/2}$ so that

$$\tau_k = \left[k^3\mathcal{E}(k)\right]^{-1/2}. \tag{11.24}$$

Kolmogorov's assumptions are then equivalent to setting

$$\varepsilon \sim \frac{v_k^2}{\tau_k} = \frac{k\mathcal{E}(k)}{\tau_k}. \tag{11.25}$$

If we demand that $\varepsilon$ be a constant then (11.24) and (11.25) yield (11.23).

### *The viscous scale and energy dissipation*

At some small length scale we should expect viscosity to become important and the scaling theory we have just set up will fail. What is that scale? In the inertial range friction is unimportant because the time scales on which it acts are too long for it be important and dynamical effects dominate. In the momentum equation the viscous term is $\nu\nabla^2 u$ so that a viscous or dissipation time scale at a scale $k^{-1}$, $\tau_k^\nu$, is

$$\tau_k^\nu \sim \frac{1}{k^2\nu}, \tag{11.26}$$

so that the viscous time scale decreases with scale. The eddy turnover time, $\tau_k$ — that is, the inertial time scale — in the Kolmogorov spectrum is

$$\tau_k = \varepsilon^{-1/3}k^{-2/3}. \tag{11.27}$$

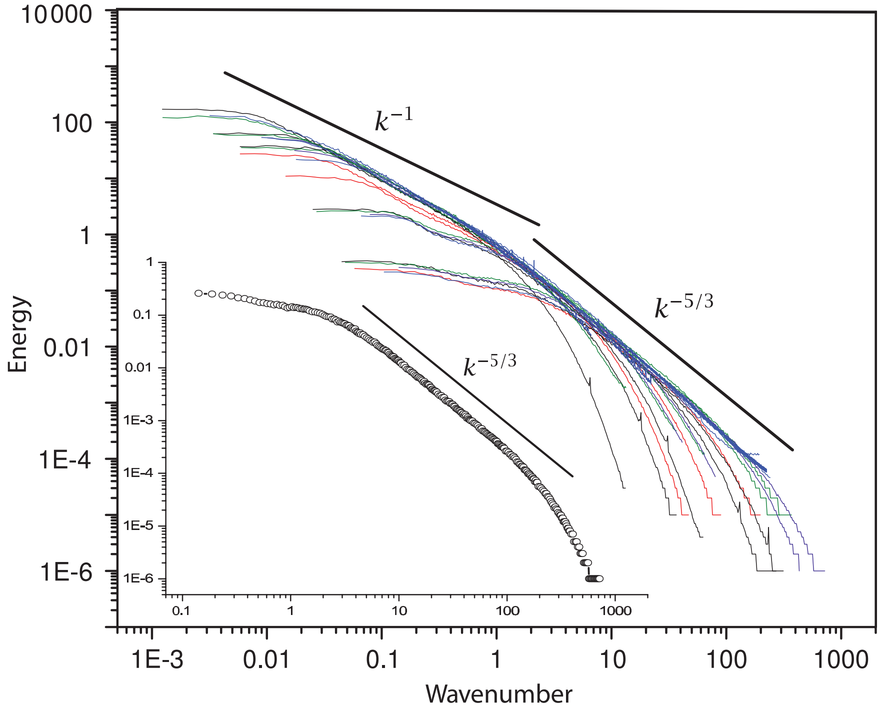


**Fig. 11.4** The energy spectrum of 3D turbulence measured in some experiments at the Princeton Superpipe facility.[6] The outer plot shows the spectra from a large number of experiments at different Reynolds numbers up to $10^6$, with the magnitude of their spectra appropriately rescaled. Smaller scales show a good $-5/3$ spectrum, whereas at larger scales the eddies feel the effects of the pipe wall and the spectra are a little shallower. The inner plot shows the spectrum in the centre of the pipe in a single experiment at $Re \approx 10^6$.

The wavenumber at which dissipation becomes important is then given by equating the above two time scales, yielding the dissipation wavenumber, $k_\nu$ and the associated length scale, $L_\nu$,

$$k_\nu \sim \left(\frac{\varepsilon}{\nu^3}\right)^{1/4}, \qquad L_\nu \sim \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}. \tag{11.28a,b}$$

$L_\nu$ is called the *Kolmogorov scale*. It is the *only* quantity which can be created from the quantities $\nu$ and $\varepsilon$ that has the dimensions of length. (It is the same as the scale given by (11.16) provided that in that expression $V$ is the velocity magnitude at the Kolmogorov scale.) Thus, for $L \gg L_\nu$, $\tau_k \ll \tau_k^\nu$ and inertial effects dominate. For $L \ll L_\nu$, $\tau_k^\nu \ll \tau_k$ and frictional effects dominate. In fact for length scales smaller than the dissipation scale, (11.27) is inaccurate; the energy spectrum falls off more rapidly than $k^{-5/3}$ and the inertial time scale falls off less rapidly than (11.27) implies, and dissipation dominates even more.

Given the dissipation scale, let us estimate the average energy dissipation rate, $\mathrm{d}/\mathrm{d}T\overline{E}$. This is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{E} = \frac{1}{V}\int \nu \boldsymbol{v}\cdot\nabla^2\boldsymbol{v}\,\mathrm{d}V. \tag{11.29}$$

The length at which dissipation acts is the Kolmogorov scale and, noting that $v_k^2 \sim \varepsilon^{2/3}k^{-2/3}$ and using (11.28a), the average energy dissipation rate scales as

$$\mathrm{d}/\mathrm{d}t\,\overline{E} \sim \nu k_\nu^2 v_{k_\nu}^2 \sim \nu k_\nu^2 \frac{\varepsilon^{2/3}}{k_\nu^{2/3}} \sim \varepsilon. \tag{11.30}$$

That is, the energy dissipation rate is equal to the energy cascade rate. On the one hand this seems sensible, but on the other hand it is *independent of the viscosity.* In particular, in the limit of viscosity tending to zero the energy dissipation remains finite! Surely the energy dissipation rate must go to zero if viscosity goes to zero? To see that this is not the case, consider that energy is input at some large scales, and the magnitude of the stirring largely determines the energy input and cascade rate. The scale at which viscous effects then become important is determined by the viscous scale, $L_\nu$, given by (11.28b). *As viscosity tends to zero $L_\nu$ becomes smaller in just such a way as to preserve the constancy of the energy dissipation.* This is one of the most important results in three-dimensional turbulence. Now, we established in Section 1.10 that the Euler equations (i.e., the fluid equations with the viscous term omitted from the outset) do conserve energy. This means that the Euler equations are a *singular limit* of the Navier–Stokes equations: the behaviour of the Navier–Stokes equations as viscosity tends to zero is different from the behaviour resulting from simply omitting the viscous term from the equations ab initio.

How big is $L_\nu$ in the atmosphere? A crude estimate, perhaps wrong by an order of magnitude, comes from noting that $\varepsilon$ has units of $U^3/L$, and that at length scales of the order of 100 m in the atmospheric boundary layer (where there might be a three-dimensional energy cascade to small scales) velocity fluctuations are of the order of $1\,\mathrm{cm\,s^{-1}}$, giving $\varepsilon \approx 10^{-8}\,\mathrm{m^2\,s^{-3}}$. Using (11.28b) we then find the dissipation scale to be of the order of a millimetre or so. In the ocean the dissipation scale is also of the order of millimetres. Various inertial range properties, in both three and two dimensions, are summarized in the shaded box on the facing page.

## *Degrees of freedom*

How many degrees of freedom does a turbulent fluid like the atmosphere potentially have? We might estimate this number, $N$ say, by the expression

$$N \sim \left(\frac{L}{L_\nu}\right)^3, \tag{11.31}$$

where $L$ is the length scale of the energy-containing eddies at the large scale. If we take $L = 1000$ km and $L_\nu = 1$ mm this gives about $10^{27}$! On a rather more general basis, we can obtain an expression for $N$ using (11.28b), to give

$$N \sim L^3 \left(\frac{\varepsilon}{\nu^3}\right)^{3/4}, \tag{11.32}$$

or, using $\varepsilon \sim U^3/L$,

$$N \sim \left(\frac{UL}{\nu}\right)^{9/4} = Re^{9/4}, \tag{11.33}$$

where $Re$ is the Reynolds number based on the large-scale flow. For typical large-scale atmospheric flows with $U \sim 10\,\mathrm{m\,s^{-1}}$, $L \sim 10^6$ m and $\nu = 10^5\,\mathrm{m^2\,s^{-1}}$, $Re \sim 10^{12}$ and again $N \sim 10^{27}$. Obviously, this number is very approximate, but nevertheless the number of potential degrees of freedom in the atmosphere is truly enormous, greater than Avogadro's number. Thus trying to model the turbulent atmosphere explicitly is akin to trying to model the gas in a room by following the motion of each individual molecule, and it seems unnecessary. How *should* we model it? That, in a nutshell, is the (unsolved) problem of turbulence.

### Inertial Range Properties in 3D and 2D Turbulence

For reference, a few inertial range properties are listed below, omitting non-dimensional constants.

| | 3D energy range | 2D enstrophy range | |
|---|---|---|---|
| Energy spectrum | $\varepsilon^{2/3}k^{-5/3}$ | $\eta^{2/3}k^{-3}$ | (T.1) |
| Turnover time | $\varepsilon^{-1/3}k^{-2/3}$ | $\eta^{-1/3}$ | (T.2) |
| Viscous scale, $L_\nu$ | $(\nu^3/\varepsilon)^{1/4}$ | $(\nu^3/\eta)^{1/6}$ | (T.3) |
| Passive tracer spectrum | $\chi\varepsilon^{-1/3}k^{-5/3}$ | $\chi\eta^{-1/3}k^{-1}$ | (T.4) |

In these expressions:

$$\nu = \text{viscosity},\ \ k = \text{wavenumber},\ \ \varepsilon = \text{energy cascade rate},$$
$$\eta = \text{enstrophy cascade rate},\ \ \chi = \text{tracer variance cascade rate}.$$

### 11.2.3 A Final Note on our Assumptions

The assumptions of homogeneity and isotropy that are made in the Kolmogorov theory are ansatzes, in that we make them because we want to have a tractable model of turbulence (and certainly we can conceive of an experiment in which turbulence is for most practical purposes homogeneous and isotropic). The essential *physical* assumptions are: (i) that there exists an inertial range in which the energy flux is constant; and (ii) that the energy is cascaded from large to small scales in a series of small steps, as the energy spectra will then be determined by spectrally local quantities. The second assumption is the locality assumption and without it we could have, instead of (11.23),

$$\mathcal{E}(k) = C\varepsilon^{2/3}k^{-5/3}g(k/k_0)h(k/k_\nu), \tag{11.34}$$

where $g$ and $h$ are unknown functions. We essentially postulate that there exists a range of intermediate wavenumbers over which $g(k/k_0) = h(k/k_\nu) = 1$.

The first, and less obvious, assumption might be called the *non-intermittency* assumption, and it demands that rare events (in time or space) with large amplitudes do not dominate the energy flux or the dissipation rate. If they were to do so, then the flux would fluctuate strongly, the turbulent statistics would not be completely characterized by $\varepsilon$ and Kolmogorov's theory would not be exactly right. Note that in the theory $\varepsilon$ is the mean energy cascade rate, and $\varepsilon^{2/3}$ is the two-thirds power of the mean, which is not equal to the mean of the two-thirds power. In fact, in high Reynolds turbulence the $-5/3$ spectra is often observed to a fairly high degree of accuracy (e.g., as in Fig. 11.4), although the higher-order statistics (e.g., higher-order structure functions) predicted by the theory are often found to be in error, and it is generally believed that Kolmogorov's theory is not exact.[7]

## 11.3 TWO-DIMENSIONAL TURBULENCE

Two-dimensional turbulence behaves in a profoundly different way from three-dimensional turbulence, largely because of the presence of another quadratic invariant, the enstrophy (defined below;

see also Section 5.6.3). In two dimensions, the vorticity equation for incompressible flow is:

$$\frac{\partial \zeta}{\partial t} + \boldsymbol{u} \cdot \nabla \zeta = F + \nu \nabla^2 \zeta, \tag{11.35}$$

where $\boldsymbol{u} = u\boldsymbol{i} + v\boldsymbol{j}$ and $\zeta = \boldsymbol{k} \cdot \nabla \times \boldsymbol{u}$ and $F$ is a stirring term. In terms of a streamfunction, $u = -\partial\psi/\partial y$, $v = \partial\psi/\partial x$, and $\zeta = \nabla^2\psi$, and (11.35) may be written as

$$\frac{\partial \nabla^2 \psi}{\partial t} + J(\psi, \nabla^2\psi) = F + \nu \nabla^4 \psi. \tag{11.36}$$

We obtain an energy equation by multiplying by $-\psi$ and integrating over the domain, and an enstrophy equation by multiplying by $\zeta$ and integrating. When $F = \nu = 0$ we find

$$\widehat{E} = \frac{1}{2}\int_A (u^2 + v^2)\,\mathrm{d}A = \frac{1}{2}\int_A (\nabla\psi)^2\,\mathrm{d}A, \qquad \frac{\mathrm{d}\widehat{E}}{\mathrm{d}t} = 0, \tag{11.37a}$$

$$\widehat{Z} = \frac{1}{2}\int_A \zeta^2\,\mathrm{d}A = \frac{1}{2}\int_A (\nabla^2\psi)^2\,\mathrm{d}A, \qquad \frac{\mathrm{d}\widehat{Z}}{\mathrm{d}t} = 0, \tag{11.37b}$$

where the integral is over a finite area with either no-normal flow or periodic boundary conditions. The quantity $\widehat{E}$ is the energy, and $\widehat{Z}$ is known as the *enstrophy.* The enstrophy invariant arises because the vortex stretching term, so important in three-dimensional turbulence, vanishes identically in two dimensions. In fact, because vorticity is conserved on parcels it is clear that the integral of *any* function of vorticity is zero when integrated over $A$; that is, from (11.35)

$$\frac{\mathrm{D}g(\zeta)}{\mathrm{D}t} = 0 \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_A g(\zeta)\,\mathrm{d}A = 0, \tag{11.38}$$

where $g(\zeta)$ is an arbitrary function. Of this infinity of conservation properties, enstrophy conservation (with $g(\zeta) = \zeta^2$) in particular has been found to have enormous consequences to the flow of energy between scales, as we will soon discover.[8]

### 11.3.1 Energy and Enstrophy Transfer

In three-dimensional turbulence we posited that energy is cascaded to small scales via vortex stretching. In two dimensions that mechanism is absent, and there is reason to expect energy to be transferred to *larger* scales. This counter-intuitive behaviour arises from the twin integral constraints of energy and enstrophy conservation, and the following three arguments illustrate why this should be so.

#### *I. Vorticity elongation*

Consider a band or a patch of vorticity, as in Fig. 11.5, in a nearly inviscid fluid. The vorticity of each element of fluid is conserved as the fluid moves. Now, we should expect that the quasi-random motion of the fluid will act to elongate the band but, as its area must be preserved, the band narrows and so vorticity gradients will increase. This is equivalent to the enstrophy moving to smaller scales. Now, the energy in the fluid is

$$\widehat{E} = -\frac{1}{2}\int \psi\zeta\,\mathrm{d}A, \tag{11.39}$$

where the streamfunction is obtained by solving the Poisson equation $\nabla^2\psi = \zeta$. If the vorticity is locally elongated primarily only in one direction (as it must be to preserve area), the integration involved in solving the Poisson equation will lead to the scale of the streamfunction becoming larger in the direction of stretching, but virtually no smaller in the perpendicular direction. Because stretching occurs, on average, in all directions, the overall scale of the streamfunction will increase in all directions, and the cascade of enstrophy to small scales will be accompanied by a transfer of energy to large scales.

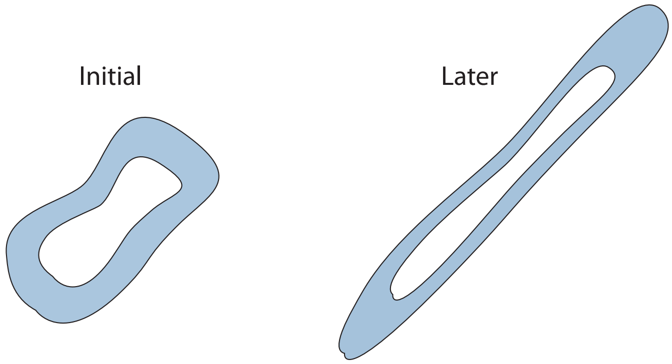


**Fig. 11.5** In two-dimensional incompressible flow, a band of fluid is elongated, but its area is preserved. Elongation is followed by folding and more elongation, producing filaments as in Fig. 11.8. As vorticity is tied to fluid parcels, the values of the vorticity in the shaded area (and in the hole) are maintained; thus, vorticity gradients increase and the enstrophy is thereby, on average, moved to smaller scales.

*II. An energy-enstrophy conservation argument*

A moment's thought will reveal that the distribution of energy and enstrophy in wavenumber space are respectively analogous to the distribution of mass and moment of inertia of a lever, with wavenumber playing the role of distance from the fulcrum. Any re-arrangement of mass such that its distribution also becomes wider must be such that the centre of mass moves toward the fulcrum. Thus, analogously, any rearrangement of a flow that preserves both energy and enstrophy, and that causes the distribution to spread out in wavenumber space, will tend to move energy to small wavenumbers and enstrophy to large. To prove this we begin with expressions for the average energy and enstrophy:

$$\overline{E} = \int \mathcal{E}(k)\,\mathrm{d}k, \qquad \overline{Z} = \int \mathcal{Z}(k)\,\mathrm{d}k = \int k^2\mathcal{E}(k)\,\mathrm{d}k, \tag{11.40}$$

where $\mathcal{E}(k)$ and $\mathcal{Z}(k)$ are the energy and enstrophy spectra. A wavenumber characterizing the spectral location of the energy is the centroid,

$$k_e = \frac{\int k\mathcal{E}(k)\,\mathrm{d}k}{\int \mathcal{E}(k)\,\mathrm{d}k}, \tag{11.41}$$

and, for simplicity, we normalize units so that the denominator is unity. The spreading out of the energy distribution is formalized by setting

$$I \equiv \int (k-k_e)^2\mathcal{E}(k)\,\mathrm{d}k, \qquad \frac{\mathrm{d}I}{\mathrm{d}t} > 0. \tag{11.42}$$

Here, $I$ measures the width of the energy distribution, and this is assumed to increase. Expanding out the integral gives

$$\begin{aligned} I &= \int k^2\mathcal{E}(k)\,\mathrm{d}k - 2k_e\int k\mathcal{E}(k)\,\mathrm{d}k + k_e^2\int \mathcal{E}(k)\,\mathrm{d}k \\ &= \int k^2\mathcal{E}(k)\,\mathrm{d}k - k_e^2\int \mathcal{E}(k)\,\mathrm{d}k, \end{aligned} \tag{11.43}$$

where the last equation follows because $k_e = \int k\mathcal{E}(k)\,\mathrm{d}k$ is, from (11.41), the energy-weighted centroid. Because both energy and enstrophy are conserved, (11.43) gives

$$\frac{\mathrm{d}k_e^2}{\mathrm{d}t} = -\frac{1}{\overline{E}}\frac{\mathrm{d}I}{\mathrm{d}t} < 0. \tag{11.44}$$

Thus, the centroid of the distribution moves to smaller wavenumbers and to larger scales (see Fig. 11.6).

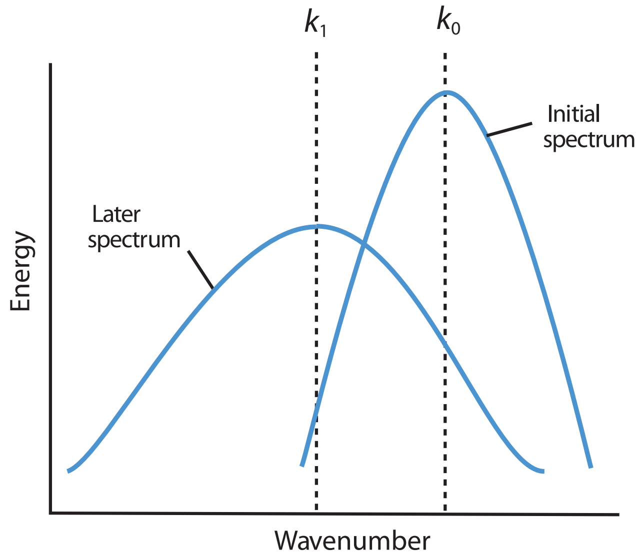


**Fig. 11.6** In two-dimensional flow, the centroid of the energy spectrum will move to large scales (smaller wavenumber) provided that the width of the distribution increases — as can be expected in a nonlinear, eddying flow.

An appropriately defined measure of the centre of the enstrophy distribution, on the other hand, moves to higher wavenumbers. The demonstration follows easily if we work with the inverse wavenumber, which is a direct measure of length. Let $q = 1/k$ and assume that the enstrophy distribution spreads out by nonlinear interactions, so that, analogously to (11.42),

$$J \equiv \int (q - q_e)^2 \mathcal{Y}(q)\, \mathrm{d}q, \qquad \frac{\mathrm{d}J}{\mathrm{d}t} > 0, \tag{11.45}$$

where $\mathcal{Y}(q)$ is such that the enstrophy is $\int \mathcal{Y}(q)\mathrm{d}q$ and

$$q_e = \frac{\int q\mathcal{Y}(q)\, \mathrm{d}q}{\int \mathcal{Y}(q)\, \mathrm{d}q}. \tag{11.46}$$

Expanding the integrand in (11.45) and using (11.46) gives

$$J = \int q^2 \mathcal{Y}(q)\, \mathrm{d}q - q_e^2 \int \mathcal{Y}(q)\, \mathrm{d}q. \tag{11.47}$$

But $\int q^2 \mathcal{Y}(q)\, \mathrm{d}q$ is conserved, because this is the energy. Thus,

$$\frac{\mathrm{d}J}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t} q_e^2 \int \mathcal{Y}(q)\, \mathrm{d}q, \tag{11.48}$$

whence

$$\frac{\mathrm{d}q_e^2}{\mathrm{d}t} = -\frac{1}{\overline{\overline{Z}}} \frac{\mathrm{d}J}{\mathrm{d}t} < 0. \tag{11.49}$$

Thus, the length scale characterizing the enstrophy distribution gets smaller, and the corresponding wavenumber gets larger.

### *III. A similarity argument*

Consider an initial value problem, in which a fluid with some initial distribution of energy is allowed to freely evolve, unencumbered by boundaries. We note two aspects of the problem:

*(i)* there is no externally imposed length scale (because of the way the problem is posed);

*(ii)* the energy is conserved (this being an assumption).

It is the second condition that limits the argument to two dimensions, for in three dimensions energy is quickly cascaded to small scales and dissipated, but let us here posit that this does not occur. These two assumptions are then sufficient to infer the general direction of transfer of energy, using a rather general similarity argument. To begin, write the energy per unit mass of the fluid as

$$\overline{E} = U^2 = \int \mathcal{E}(k,t)\,\mathrm{d}k, \tag{11.50}$$

where $\mathcal{E}(k,t)$ is the energy spectrum and $U$ is proportional to the square root of the total energy and has units of velocity. On dimensional considerations we could write

$$\mathcal{E}(k,t) = U^2 L \widehat{\mathcal{E}}(\widehat{k},\hat{t}), \tag{11.51}$$

where $\widehat{\mathcal{E}}$, and its arguments, are nondimensional quantities, and $L$ is some length scale. However, if, over time, the initial conditions are forgotten then there is no length scale in the problem and the only parameters available to determine the energy spectrum are energy, time, and wavenumber, that is $U$, $t$ and $k$. A little thought reveals that the most general form for the energy spectrum is then

$$\mathcal{E}(k,t) = U^3 t \widehat{\mathcal{E}} = U^3 t\, g(Ukt), \tag{11.52}$$

where $g$ is an arbitrary function of its arguments. The argument of $g$ is the only nondimensional grouping of $U$, $t$ and $k$, and $U^3 t$ provides the proper dimensions for $\mathcal{E}$. Conservation of energy now implies that the integral

$$I = \int_0^\infty t\, g(Ukt)\,\mathrm{d}k \tag{11.53}$$

is *not* a function of time. Defining $\vartheta = Ukt$, this requirement is met if

$$\int_0^\infty g(\vartheta)\,\mathrm{d}\vartheta = \text{constant}. \tag{11.54}$$

Now, the spectrum is a function of $k$ only through the combination $\vartheta = Ukt$. Thus, as time proceeds features in the spectrum move to smaller $k$. Suppose, for example, that the energy is initially peaked at some wavenumber $k_p$; the product $tk_p$ is preserved, so $k_p$ must diminish with time and the energy must move to larger scales. Similarly, the energy weighted mean wavenumber, $k_e$, moves to smaller wavenumbers, or larger scales. To see this explicitly, we have

$$k_e = \frac{\int k\mathcal{E}\,\mathrm{d}k}{\int \mathcal{E}\,\mathrm{d}k} = \frac{\int k\mathcal{E}\,\mathrm{d}k}{U^2} = \int kUt\, g(Ukt)\,\mathrm{d}k = \int \frac{\vartheta g(\vartheta)}{Ut}\,\mathrm{d}\vartheta = \frac{C}{Ut}, \tag{11.55}$$

where all the integrals are over the interval $(0,\infty)$ and $C = \int \vartheta g(\vartheta)\,\mathrm{d}\vartheta$ is a constant. Thus, the wavenumber centroid of the energy distribution decreases with time, and the characteristic scale of the flow, $1/k_e$, increases with time. Interestingly, the enstrophy does not explicitly enter this argument, and in general it is not conserved; rather, it is the requirement that energy be conserved that limits the argument to two dimensions; if we accept ab initio that energy *is* conserved, it must be transferred to larger scales.[9]

### 11.3.2 Inertial Ranges in Two-dimensional Turbulence

If, unlike the case in three dimensions, energy is transferred to larger scales in inviscid, nonlinear, two-dimensional flow then we might expect the inertial ranges of two-dimensional turbulence to be quite different from their three-dimensional counterparts. But before looking in detail at the inertial ranges themselves, let us establish a couple of general properties of forced-dissipative flow in two dimensions.

### *Some properties of forced-dissipative flow*

We will first show that, unlike the case in three dimensions, energy dissipation goes to zero as the Reynolds number rises. In the absence of forcing terms, the total dissipation of energy is, from (11.35)),

$$\frac{\mathrm{d}\widehat{E}}{\mathrm{d}t} = -\nu \int \zeta^2 \,\mathrm{d}A. \tag{11.56}$$

Energy dissipation can only remain finite as $\nu \to 0$ if vorticity becomes infinite. However, this cannot happen because vorticity is conserved on parcels except for the action of viscosity, meaning that $\mathrm{D}\zeta/\mathrm{D}t = \nu\nabla^2\zeta$. However, the viscous term can only *reduce* the value of vorticity on a parcel, and so vorticity can never become infinite if it is not so initially, and therefore using (11.56) energy dissipation goes to zero with $\nu$. (In three dimensions vorticity becomes infinite as viscosity goes to zero because of the effect of vortex stretching.) This conservation of energy is related to the fact that energy is trapped at large scales, even in forced-dissipative flow. On the other hand, enstrophy is transferred to small scales and therefore we expect it to be dissipated at large wavenumbers, even as the Reynolds number becomes very large.

We can show that energy is trapped at large scales in forced-dissipative two-dimensional flow (in a sense that will be made explicit) by the following argument.[10] Suppose that the forcing of the fluid is confined to a particular scale, characterized by the wavenumber $k_f$, and that dissipation is effected by a linear drag and a small viscosity. The equation of motion is

$$\frac{\partial\zeta}{\partial t} + J(\psi,\zeta) = F - r\zeta + \nu\nabla^2\zeta, \tag{11.57}$$

where $F$ is the stirring and $r$ and $\nu$ are positive constants. This leads to the following energy and enstrophy equations:

$$\frac{\mathrm{d}\widehat{E}}{\mathrm{d}t} = -2r\widehat{E} - \int \psi F \,\mathrm{d}A - \int \nu\zeta^2 \,\mathrm{d}A \approx -2r\widehat{E} - \int \psi F \,\mathrm{d}A, \tag{11.58a}$$

$$\frac{\mathrm{d}\widehat{Z}}{\mathrm{d}t} = -2r\widehat{Z} + \int \zeta F \,\mathrm{d}A - D_Z \approx -2r\widehat{Z} - k_f^2 \int \psi F \,\mathrm{d}A - D_Z, \tag{11.58b}$$

where $D_Z = \int \nu(\nabla\zeta)^2 \,\mathrm{d}A$ is the enstrophy dissipation, which is positive. To obtain the right-most expressions, in (11.58a) we assume there is no dissipation of energy by the viscous term, and in (11.58b) we assume that the forcing is confined to wavenumbers near $k_f$. Consider a statistically steady state and also write $\widehat{E} = \int \mathcal{E}(k)\,\mathrm{d}k$ and $\widehat{Z} = \int k^2\mathcal{E}(k)\,\mathrm{d}k$, where the integrations are over all wavenumbers. If we then eliminate the integral involving $\psi F$ between (11.58a) and (11.58b) we obtain

$$\int k^2\mathcal{E}(k)\,\mathrm{d}k + \frac{D_Z}{2r} = \int k_f^2\mathcal{E}(k)\,\mathrm{d}k, \tag{11.59}$$

Now, from the obvious inequality $\int (k-k_e)^2\mathcal{E}(k)\,\mathrm{d}k \geq 0$, where $k_e$ is the energy centroid defined in (11.41), we obtain

$$\int \left(k^2 - k_e^2\right)\mathcal{E}(k)\,\mathrm{d}k \geq 0. \tag{11.60}$$

Combining (11.59) and (11.60) gives

$$\int \left(k_f^2 - k_e^2\right)\mathcal{E}(k)\,\mathrm{d}k \geq \frac{D_Z}{2r} > 0. \tag{11.61}$$

Thus, in a statistically steady state, $k_f$ is larger than $k_e$, and the energy containing scale, as characterized by $k_e^{-1}$, must be larger than the forcing scale $k_f^{-1}$. This demonstration (like argument II in Section 11.3.1) relies both on the conservation of energy and enstrophy by the nonlinear terms and on the particular relationship between the energy and enstrophy spectra.

This result, as well as the arguments of Section 11.3.1, suggest that in a forced-dissipative two-dimensional fluid, energy is transferred to larger scales and enstrophy is transferred to small scales. To obtain a statistically steady state some friction, such as the Rayleigh drag of (11.57), is necessary to remove energy at large scales, and enstrophy must be removed at small scales, but if the forcing scale is sufficiently well separated in spectral space from such frictional effects then two inertial ranges may form — an *energy inertial range* carrying energy to larger scales, and an *enstrophy inertial range* carrying enstrophy to small scales (Fig. 11.7). These ranges are analogous to the three-dimensional inertial range of Section 11.2, and similar conditions must apply if the ranges are to be truly inertial — in particular we must assume spectral locality of the energy or enstrophy transfer. Given that, we can then calculate their properties, as follows.

### *The enstrophy inertial range*

In the enstrophy inertial range the enstrophy cascade rate $\eta$, equal to the rate at which enstrophy is supplied by stirring, is assumed constant. By analogy with (11.25) we may assume that this rate is given by

$$\eta \sim \frac{k^3 \mathcal{E}(k)}{\tau_k}. \tag{11.62}$$

With $\tau_k$ (still) given by (11.24) we obtain

$$\boxed{\mathcal{E}(k) = \mathcal{K}_\eta \eta^{2/3} k^{-3},} \tag{11.63}$$

where $\mathcal{K}_\eta$ is, we presume, a universal constant, analogous to the Kolmogorov constant of (11.23). This, and various other properties in two- and three-dimensional turbulence, are summarized in the shaded box on page 423.

The velocity and time at a particular wavenumber then scale as

$$v_k \sim \eta^{1/3} k^{-1}, \qquad t_k \sim l_k/v_k \sim 1/(k v_k) \sim \eta^{-1/3}. \tag{11.64a,b}$$

We may also obtain (11.64) by substituting (11.63) into (11.24). Thus, *the eddy turnover time in the enstrophy range of two-dimensional turbulence is length-scale invariant.* The appropriate viscous scale is given by equating the inertial and viscous terms in (11.35). Using (11.64a) we obtain, analogously to (11.28a), the viscous wavenumber

$$k_\nu \sim \left(\frac{\eta^{1/3}}{\nu}\right)^{1/2}. \tag{11.65}$$

The enstrophy dissipation, analogously to (11.30) goes to a finite limit given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{Z} = \nu \int_A \zeta \nabla^2 \zeta \, \mathrm{d}A \sim \nu k_\nu^4 v_{k_\nu}^2 \sim \eta, \tag{11.66}$$

using (11.64a) and (11.65). Thus, the enstrophy dissipation in two-dimensional turbulence is (at least according to this theory) independent of the viscosity.

### *Energy inertial range*

The energy inertial range of two-dimensional turbulence is quite similar to that of three-dimensional turbulence, except in one major respect: the energy flows from smaller to larger scales! Because the atmosphere and ocean behave in some ways as two-dimensional fluids, this has profound consequences on their behaviour, and is something we return to in the next chapter. The upscale

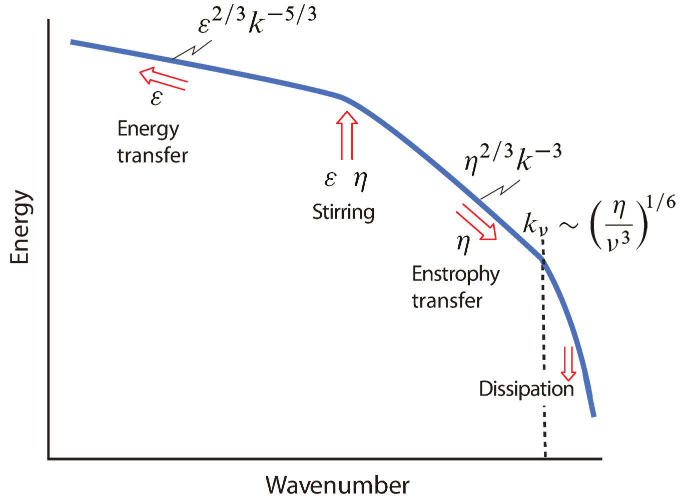


**Fig. 11.7** The energy spectrum of two-dimensional turbulence. (Compare with Fig. 11.3.) Energy supplied at some rate $\varepsilon$ is transferred to large scales, whereas enstrophy supplied at some rate $\eta$ is transferred to small scales, where it may be dissipated by viscosity. If the forcing is localized at a scale $k_f^{-1}$ then $\eta \approx k_f^2 \varepsilon$.

energy flow is known as the *inverse cascade,* and the associated energy spectrum is, as in the three-dimensional case,

$$\mathcal{E}(k) = \mathcal{K}_\varepsilon \varepsilon^{2/3} k^{-5/3}, \tag{11.67}$$

where $\mathcal{K}_\varepsilon$ is a nondimensional constant — sometimes called the Kolmogorov–Kraichnan constant, and not necessarily equal to $\mathcal{K}$ in (11.23) — and $\varepsilon$ is the rate of energy transfer to larger scales. Of course we now need a mechanism to remove energy at large scales, otherwise it will pile up at the scale of the domain and a statistical steady state will not be achieved. Introducing a linear drag, $-r\zeta$, into the vorticity equation, as in (11.57), is one means of removing energy, and such a term may be physically justified by appeal to Ekman layer theory (Section 5.7). Although such a term appears to be scale invariant, its effects will be felt primarily at large scales because at smaller scales the time scale of the turbulence is much shorter than that of the friction, and we may estimate the scale at which the drag becomes important by equating the two time scales. The turbulent timescale is given by (11.27), and equating this to the frictional time scale $r^{-1}$ gives $r^{-1} = \varepsilon^{-1/3} k_r^{-2/3}$, or

$$k_r = \left(\frac{r^3}{\varepsilon}\right)^{1/2} \qquad \text{or} \qquad L_r = \left(\frac{\varepsilon}{r^3}\right)^{1/2}, \tag{11.68}$$

where $k_r$ is the frictional wavenumber, and frictional effects are important at scales *larger* than the frictional scale $L_r$.

### 11.3.3 † More Phenomenology

The phenomenology of two-dimensional turbulence is not quite as settled as the above arguments imply. Note, for example, that time scale (11.64b) is independent of length scale, whereas in three-dimensional turbulence the time scale decreases with length scale, which seems more physical and more conducive to spectrally local interactions. A useful heuristic measure of this locality is given by estimating the contributions to the straining rate, $S(k)$, from motions at all scales larger than $k^{-1}$. The strain rate scales like the shear, so that an estimate of the total strain rate is given by

$$S(k) = \left[\int_{k_0}^{k} \mathcal{E}(p) p^2 \, \mathrm{d}p\right]^{1/2}, \tag{11.69}$$

where $k_0$ is the wavenumber of the largest scale present. The contributions to the integrand from a given wavenumber octave are given by

$$\int_p^{2p} \mathcal{E}(p')p'^3 \, \mathrm{d}\log p' \sim \mathcal{E}(p)p^3. \tag{11.70}$$

In three dimensions, use of the −5/3 spectrum indicates that the contributions from each octave below a given wavenumber $k$ increase with wavenumber, being a maximum close to $k$, and this is a posteriori consistent with the locality hypothesis. However, in two-dimensional turbulence with a −3 spectrum each octave makes the same contribution. That is to say, the contributions to the strain rate at a given wavenumber, as defined by (11.69), are not spectrally local. This does not prove that the enstrophy transfer is spectrally non-local, but nor does it build confidence in the theory.

Dimensionally the strain rate is the inverse of a time, and if this is a spectrally non-local quantity then, instead of (11.24), we might use the inverse of the strain rate as an eddy turnover time giving

$$\tau_k = \left[ \int_{k_0}^{k} p^2 \mathcal{E}(p) \, \mathrm{d}p \right]^{-1/2}. \tag{11.71}$$

This has the advantage over (11.24) in that it is a non-increasing function of wavenumber, whereas if the spectrum is steeper than $k^{-3}$, (11.24) implies a time scale increasing with wavenumber. Using this in (11.62) gives a prediction for the enstrophy inertial range, namely

$$\mathcal{E}(k) = \mathcal{K}_\eta \eta^{2/3} \left[\log(k/k_0)\right]^{-1/3} k^{-3}, \tag{11.72}$$

which is similar to (11.63) except for a logarithmic correction. This expression is, of course, spectrally non-local, in contradiction to our original assumption: it has arisen by noting the spectral locality inherent in (11.69), and proposing a reasonable, although ad hoc, solution.

The discussion above suggests that the phenomenology of the forward enstrophy cascade is on the verge of being internally inconsistent, and that the $k^{-3}$ spectral slope might be the shallowest limit that is likely to be actually achieved in nature or in any particular computer simulation or laboratory experiment rather than a robust, universal slope. To see this, suppose the detailed fluid dynamics strives in some way to produce a slope shallower than $k^{-3}$; then, using (11.70), the strain is local and the shallow slope is forbidden by the Kolmogorovian scaling results. However, if the dynamics organizes itself into structures with a slope steeper than $k^{-3}$ the strain is quite non-local. The fundamental assumption of Kolmogorov scaling is not satisfied, and there is no internal inconsistency. The theory then simply does not apply, and a slope steeper than $k^{-3}$ is not theoretically inconsistent.

There are two other potential issues with the theory of two-dimensional turbulence. One is that enstrophy is only one of an infinity of invariants of inviscid two-dimensional flow, and the theory takes no account of the presence of others. The second is that, as in three-dimensional turbulence, if there is strong intermittency the flow cannot be fully characterized by single enstrophy and energy cascade rates. In two-dimensional turbulence the main form of intermittency may be the formation of coherent vortices, discussed more below. In spite of these problems, the notions of a forward transfer of enstrophy and an inverse transfer of energy are quite robust, and have considerable numerical support. Indeed, the realization that in two-dimensional turbulence energy is transferred to larger scales was arguably one of the most important developments in fluid mechanics in the second half of the twentieth century, with important ramifications in rotating and stratified *three-dimensional* fluids.[11]

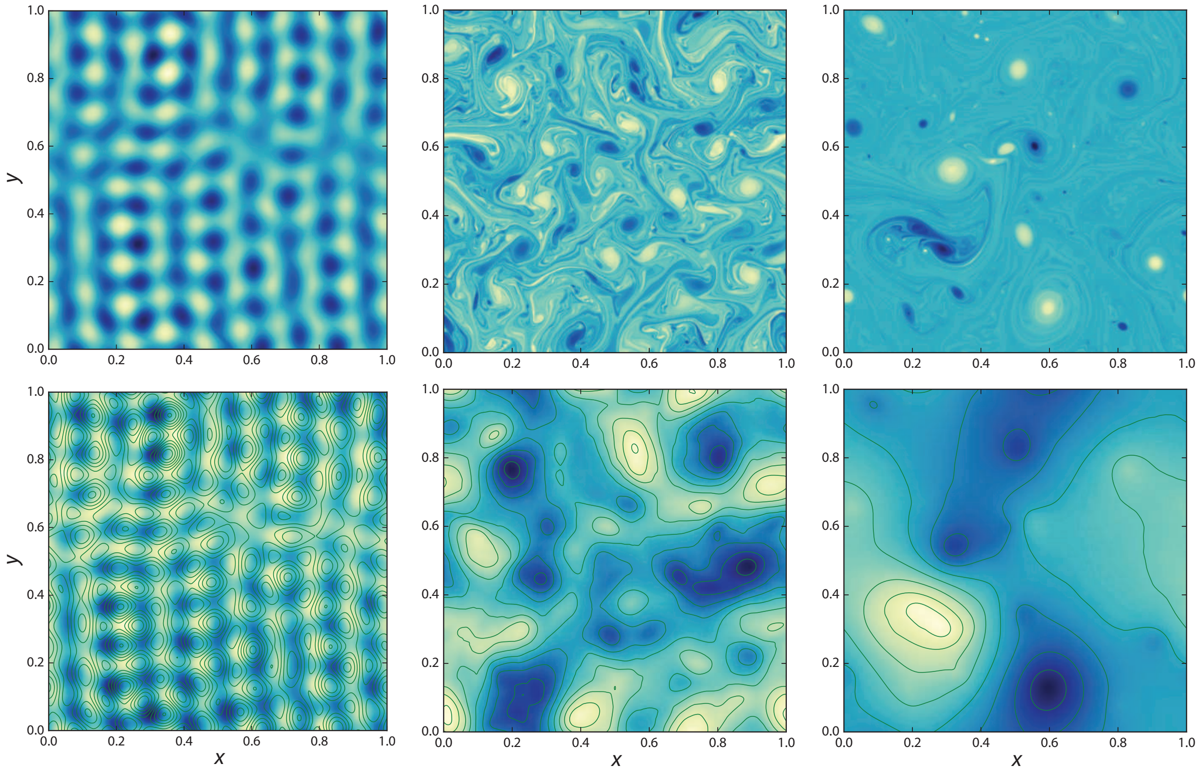


**Fig. 11.8** Nearly-free evolution of vorticity (top) and streamfunction (bottom) in a doubly-periodic domain obeying the two-dimensional vorticity equation with no forcing but with a weak viscous term, in a numerical simulation with $512^2$ equivalent grid points. Time proceeds from left to right. The initial conditions have just a few non-zero Fourier modes (around wavenumber 9) with randomly generated phases. Kelvin–Helmholtz instability leads to vortex formation and roll-up (as in Fig. 9.6), and like-signed vortices merge (with an example in the top right panel) ultimately leading to a state of just two oppositely-signed vortices. Between the vortices, enstrophy cascades to smaller scales. The scale of the streamfunction grows larger, reflecting the transfer of energy to larger scales.

### 11.3.4 Numerical Illustrations

Numerical simulations nicely illustrate both the classical phenomenology and its shortcomings. In the simulations shown in Fig. 11.8 the vorticity field is initialized quasi-randomly, with little structure in the initial field, and with only a few non-zero Fourier components; the flow then freely evolves, save for the effects of a weak viscosity.

Vortices soon form, and between them enstrophy is cascaded to small scales where it is dissipated, producing a flat and nearly featureless landscape. The energy cascade to larger scales is reflected in the streamfunction field (bottom row of Fig. 11.8), the length scale of which slowly grows larger with time. The vortices themselves form through a roll-up mechanism, similar to that illustrated in Fig. 9.6, and their presence provides problems to the phenomenology. Because circular vortices are nearly exact, stable solutions of the inviscid equations they can 'store' enstrophy, disrupting the relationship between enstrophy flux and enstrophy itself that is assumed in the Kolmogorov–Kraichnan phenomenology. When vortices merge, the enstrophy dissipation rate increases rapidly for a short period of time, so providing a source of intermittency.

Nevertheless, some forced-dissipative numerical simulations suggest that the presence of vortices may be confined to scales close to that of the forcing, and if the resolution is sufficiently high

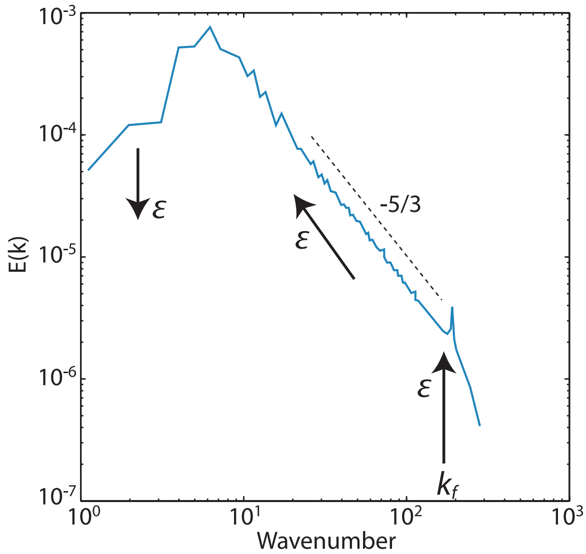


**Fig. 11.9** The energy spectrum in a numerical simulation of forced-dissipative two-dimensional turbulence. The fluid is stirred at wavenumber $k_f$ and dissipated at large scales with a linear drag, and there is a $k^{-5/3}$ spectrum at intermediate scales. The arrows indicate the direction of the energy flux, $\varepsilon$.[12]

then the −5/3 inverse cascade and −3 forward enstrophy cascade appear. Certainly, if the forcing is spectrally localized, then a well-defined −5/3 spectrum robustly forms, as illustrated in Fig. 11.9. The forward $k^{-3}$ spectrum is typically more delicate, being influenced by the presence of coherent vortices, but it does arise in some numerical simulations when the resolution is sufficiently high.[13] The atmosphere itself is not observed to have an inverse −5/3 spectrum at large scales; indeed there is no well-defined inverse energy cascade in the sense described above. The atmosphere does have an approximate −3 cascade (see Fig. 12.9 in the next chapter), but whether we should attribute this to a classical forward enstrophy cascade is not settled.

## 11.4 PREDICTABILITY OF TURBULENCE

*Small differences in the initial conditions may produce very great ones in the final phenomenon... Prediction becomes impossible... A tenth of a degree more or less at any given point, and the cyclone will burst here and not there, and extend its ravages over districts it might otherwise have spared.*
Henri Poincaré, *Science and Method*, 1908.

Forecasting the weather is hard. That this is so stems from the fact that the atmosphere is chaotic, and chaotic systems are unpredictable virtually by definition. However, the atmosphere (and turbulence in general) is certainly not a *low-dimensional* chaotic system (meaning a system with only a few degrees of freedom), and the connection between atmospheric unpredictability and the so-called 'sensitive dependence on initial conditions' of low-dimensional systems is not as straightforward as it might seem. In this section we expand on and clarify these issues, beginning with an informal discussion of a few aspects of low-dimensional dynamical systems.

### 11.4.1 Low-dimensional Chaos and Unpredictability

Chaos, or temporal disorder leading to effective indeterminism, is a ubiquitous property of nonlinear dynamical systems. Much of this was known to Poincaré, but in its modern reincarnation it stems in part from the 'Lorenz equations'.[14] These are a set of three coupled nonlinear ordinary differential equations, originally derived by way of a rather ad hoc truncation of the fluid equations governing a two-dimensional convective system: the streamfunction of a convective role is written as $\psi(x, z, t) = X(t) \sin kx \sin \pi z$, and the temperature perturbation as $\theta(x, z, t) =$

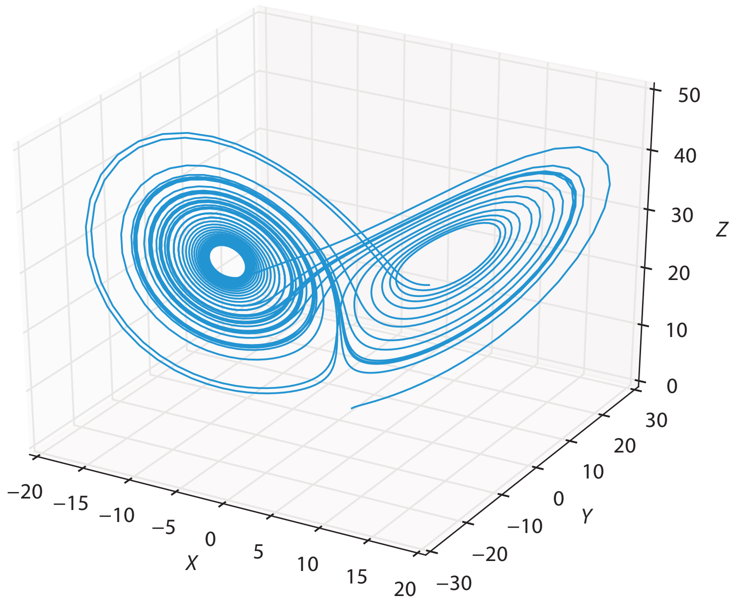


**Fig. 11.10** A solution of the Lorenz equations, with $\sigma = 10$, $r = 28$ and $b = 8/3$. The plot shows a trajectory of a solution in phase space, with initial conditions $X = Y = Z = 1$.

$Y(t)\cos kx\ \sin \pi z + Z(t)\sin 2\pi z$, where $x$ and $z$ are the horizontal and vertical coordinates in physical space, $k$ is a wavenumber, and $X, Y, Z$ are amplitudes (not coordinates). Thus, $X$ represents the rotational speed of the convection roll, $Y$ the temperature difference horizontally across the roll, and $Z$ the deviation temperature from a background vertical stratification. The resulting equations are, in notation standard for them,

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \sigma(Y - X), \qquad \frac{\mathrm{d}Y}{\mathrm{d}t} = rX - Y - XZ, \qquad \frac{\mathrm{d}Z}{\mathrm{d}t} = XY - bZ, \tag{11.73a,b,c}$$

where the parameters are: $\sigma$, the Prandtl number; $r$, proportional to the Rayleigh number; and $b$, a wavenumber dependent dissipation coefficient. The behaviour of the system varies with the parameters, and a well studied set uses $\sigma = 10$, $r = 28$ and $b = 8/3$. A typical solution of the system, a 'flow', is given in Fig. 11.10, and evidently the behaviour is quite complex. It is aperiodic, and the frequency spectrum (not shown) is quite broad. Now, suppose that at any given instant the flow is perturbed slightly. Or to put it another way, suppose that we are trying to predict the future behaviour of the system by integrating the equations of motion but that we have inaccurate knowledge about the system at some particular time. We find that the evolution of the original flow and that of the perturbed flow diverge from each other, and after a little while the two systems are completely different (Fig. 11.11). Because we can never expect to have completely accurate information about the state of the system, the system is thus unpredictable; that is, the details of the evolution depend sensitively on the initial conditions, and small errors in the initial conditions grow to finite amplitude. Let us make two other points:

*(i)* The time taken for the trajectories to diverge depends on the magnitude of the initial perturbation. Small perturbations grow exponentially at first, and at any given point in the trajectory, the smaller the perturbation the longer the predictability.

*(ii)* Once the perturbation has reached finite amplitude, the predictability time — the time for the error to become as large as the solution itself — will typically be of the order of the characteristic advective time of the system, the time for a convective roll to overturn.

Deterministic unpredictability is in fact a common feature of nonlinear dynamical systems, and may be taken as an informal definition of a chaotic system.

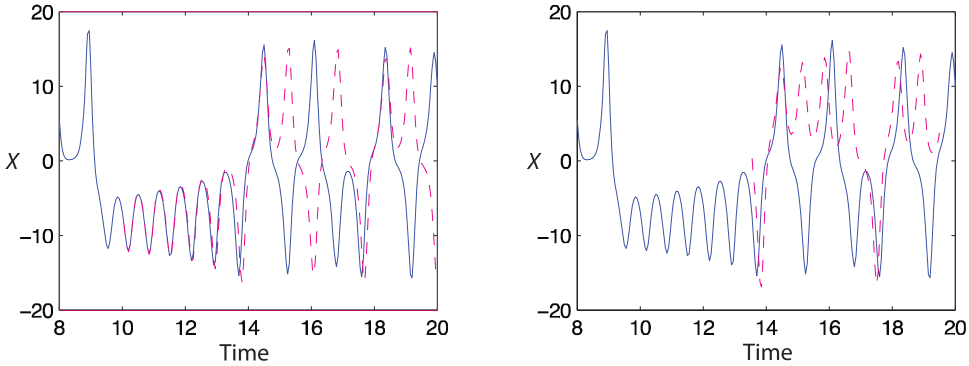


**Fig. 11.11** Examples of the evolution of the variable $X$ in the Lorenz model subject to a small perturbation at time 10 (left panel) and time 13.5 (right panel). The original and perturbed systems are the solid and dashed lines, respectively.

### 11.4.2 ✦ Predictability of a Turbulent Flow

A turbulent flow will be unpredictable if turbulence is chaotic. Because it is our common experience that turbulence is both spatially and temporally disordered, it seems, perhaps with the benefit of hindsight, that turbulence must be chaotic and unpredictable. However, this has not always been evident, and it relies on the non-trivial recognition that chaotic systems exist.[15]

Presuming that a turbulent flow *is* unpredictable, can we estimate its predictability time, namely the time taken for an initially small error to completely contaminate the system? Let us first note that turbulent flows in general contain multiple scales of motion, and let us suppose that the initial error is confined to small scales. The predictability time of the system may be taken as the time taken for the error to contaminate all scales of motion. There are two possible routes that the error may take in affecting the larger scales. In the first we suppose, following classical turbulence phenomenology, that errors on a small scale will mostly contaminate the motion on the next larger scale (in a logarithmic sense), and that this contamination occurs on the local eddy turnover time. Eddies on this larger scale then grow and affect the next larger scale, and the error field is so cascaded upscale via local triad interactions finally reaching the largest scales of the fluid. This mechanism does not rely on there being an inverse cascade of energy — it is only the error, or the contamination, that is cascaded upscale. In the second route we suppose that errors occurring on the small scale immediately contaminate the largest scales, with an initial error equal to the amplitude of the small scale, and that the large-scale error then grows exponentially.

#### *I. Error growth via a local cascade*

Let us suppose that the error is initially confined to some small scale characterized by the (inverse of) the wavenumber $k_1$, as determined by the resolution of our observing network. For modes at that scale the error may be considered finite rather than infinitesimal, and it will saturate and contaminate the next largest scale in a time scale comparable to the eddy turnover time at that scale. Thus, in general, errors initially confined to a scale $k$ will contaminate the scale $2k$ after a time $\tau_k$, with $\tau_k$ given by (11.24). The total time taken for errors to propagate from the small scale $k_1$ to the largest scale $k_0$ is then given by

$$T = \int_{k_0}^{k_1} \tau_k \, \mathrm{d}(\ln k) = \int_{k_0}^{k_1} [k^3 \mathcal{E}(k)]^{-1/2} \, \mathrm{d}(\ln k), \tag{11.74}$$

treating the wavenumber spectrum as continuous. The logarithmic integral arises because the cascade proceeds logarithmically — error cascades from $k$ to $2k$ in a time $\tau_k$. For an energy spectrum of the form $E = Ak^{-n}$ this becomes

$$T = \frac{2}{A^{1/2}(n-3)}\left[k^{(n-3)/2}\right]_{k_0}^{k_1}, \tag{11.75}$$

for $n \neq 3$, and $T = A^{-1/2}\ln(k_1/k_0)$ for $n = 3$. If in two-dimensional turbulence we have $n = 3$ and $A = \eta^{2/3}$, and if in three-dimensional turbulence we have $n = 5/3$ and $A = \varepsilon^{2/3}$, then the respective predictability times are given by

$$T_{2d} \sim \eta^{-1/3}\ln(k_1/k_0), \qquad T_{3d} \sim \varepsilon^{-1/3}k_0^{-2/3}. \tag{11.76a,b}$$

As $k_1 \to \infty$, that is as the initial error is confined to smaller and smaller scales, predictability time grows larger for two-dimensional turbulence (and for $n \geq 3$ in general), but remains finite for three-dimensional turbulence.

### *II. Error growth via a direct interaction*

Let us now assume that the small scale error directly affects the large scales, where the error then grows exponentially until it saturates. That is, if $\varphi$ is a measure of the amplitude of the large-scale error then

$$\varphi \sim \varphi_0 \exp(\sigma t), \tag{11.77}$$

where $\sigma$ is the inverse of the eddy turnover time at the large scale, and $\varphi_0$ is the amplitude of its initial error and this, we assume, is equal to the amplitude of the motion at the poorly-observed small scales at wavenumber $k_1$. In the enstrophy cascade of two-dimensional turbulence the eddy turnover time is given by $\tau_k \sim \eta^{-1/3}$ and $A \sim \eta^{2/3}$, and so we take

$$\sigma = \eta^{1/3} = A^{1/2}, \qquad \varphi_0 = Ak_1^{-n}, \tag{11.78}$$

where $n = 3$. The time, $T'_{2d}$, needed for the error to saturate the large scales, $k_0$, is then approximately given by the solution of

$$Ak_0^{-n} = Ak_1^{-n}\exp\left(A^{1/2}T'_{2d}\right), \tag{11.79}$$

giving

$$T'_{2d} \sim \eta^{-1/3}\ln(k_1/k_0). \tag{11.80}$$

In three-dimensional turbulence the eddy turnover time is given by $\tau_k \sim \varepsilon^{-1/3}k^{-2/3}$ and $A \sim \varepsilon^{2/3}$, and so we take

$$\sigma = k_0^{2/3}A^{1/2}, \qquad \varphi_0 = Ak_1^{-n}, \tag{11.81}$$

where $n = 5/3$. The time, $T'_{3d}$, needed for an error to saturate the large scale is then approximately given by the solution of

$$Ak_0^{-n} = Ak_1^{-n}\exp\left(k_0^{2/3}A^{1/2}T'_{3d}\right), \tag{11.82}$$

giving

$$T'_{3d} \sim k_0^{-2/3}\varepsilon^{-1/3}\ln(k_1/k_0). \tag{11.83}$$

The estimates (11.80) and (11.83) are to be compared with (11.76). For two-dimensional turbulence, the estimates are equal (reflecting the scale independence of the eddy turnover time), whereas for three-dimensional turbulence the estimate from (11.76) is much shorter than that from (11.83) if $k_1 \gg k_0$, meaning that the local cascade mechanism of error growth will dominate.

### 11.4.3 Implications and Weather Predictability

In two-dimensional flow the predictability time, (11.76a), increases without bound as the scale of the initial error decreases. This is consistent with what has been rigorously proven about the two-dimensional Navier–Stokes equations, namely that provided the initial conditions are sufficiently smooth, the solutions have a continuous dependence on the initial conditions, and a change in solution at some later time may be bounded by reducing the magnitude of the change in the initial conditions. This does not mean that two-dimensional flow is in practice necessarily predictable: a small error or small amount of noise in the system will still render a flow truly unpredictable sometime in the future, but we can put off that time indefinitely if we know the initial conditions well enough.

In three dimensions, with a spectrum of $k^{-5/3}$, the predictability-time estimate from (11.76b) is not dependent on the scale of the initial error. Thus, even if the initial error is confined to smaller and smaller scales, the predictability time is bounded. The time it takes for such errors to spread to the largest scales is simply a few large eddy turnover times, essentially because the eddy turnover times of the small scales are so small. For such a fluid, there is no unique error doubling time, because the error growth rate is a function of scale.

In the troposphere the large-scale flow behaves more like a two-dimensional fluid than a three-dimensional fluid and from scales from a few hundred to a few thousand kilometres it has, roughly, a $k^{-3}$ spectrum (look ahead to Fig. 12.9). If this spectrum extended indefinitely to small scales the predictability time would be correspondingly large, but at scales smaller than about 100 km or so, the atmosphere starts to behave more three dimensionally and the predictability time cannot be significantly extended by making observations at still finer scales. That is, the effective limit to predictability is governed by the horizontal scale at which the atmosphere turns three dimensional. (Hypothetically, we might be able to increase the predictability time if we could observe scales well into the viscous regime where the spectrum steepens again, but this is not a practicable proposition.) Putting in the numbers gives a predictability limit of about 12 days (but there is at least a factor of two uncertainty in such a calculation), and small perturbations that are impossible to observe will change the course of the large-scale weather systems on this time scale. The 'butterfly effect' has its origins in this argument: a butterfly flapping its wings is, so it goes, able to change the course of the weather a week or so later.[16]

As regards our attempts to predict weather, as the atmosphere becomes observed more and more accurately, the initial error will become concentrated at smaller and smaller scales, eventually reaching the scale at which the atmosphere ceases to behave as a quasi-two-dimensional fluid and where its spectrum flattens. The initial error growth rate will then increase, indicating the unavoidable onset of diminishing returns in adding resolution to our observing systems and models. Unlike the situation in a low-order chaotic system, the growth rate of errors in a turbulent flow is not, in general, exponential (except for a pure −3 spectrum) even for a small initial error. This is because the initial error will never be properly infinitesimal, in that a given error will nearly always project on to some scale at finite amplitude.[17]

## 11.5 ✦ SPECTRA OF PASSIVE TRACERS

In fluid dynamics we are often concerned with the transport of tracers by turbulent flows. An *active tracer* is one that affects the flow itself (potential vorticity and salt are examples), whereas a *passive tracer* does not affect the flow field (a neutrally buoyant dye, for example), and its dynamics are often simpler. In some circumstances an active tracer can be considered to be approximately passive; the atmospheric temperature field at very large scales turns out to be one example, and its transport is obviously a key determinant of our climate. Let us therefore spend some time discussing the dynamics of a passive tracer; in this section we will consider their spectra, and in Chapter 13 we will consider their diffusive transport.

We consider a tracer that obeys

$$\frac{D\varphi}{Dt} = F_{[\varphi]} + \kappa \nabla^2 \varphi, \tag{11.84}$$

where $F_{[\varphi]}$ is the stirring of the dye and $\kappa$ is its diffusivity, and $\kappa$ in general differs from the kinematic molecular viscosity $\nu$. If $\varphi$ is temperature, the ratio of viscosity to diffusivity is called the *Prandtl number* and denoted $\sigma$, so that $\sigma \equiv \nu/\kappa$. If $\varphi$ is a passive tracer, the ratio is sometimes called the *Schmidt number,* but we shall call it the Prandtl number in all cases. We assume that the tracer variance is created at some well-defined scale $k_0$, and that $\kappa$ is sufficiently small that dissipation only occurs at very small scales. (Dissipation only reduces the tracer *variance,* not the amount of tracer itself.) The turbulent flow will generically tend to stretch patches of dye into elongated filaments, in much the same way as vorticity in two-dimensional turbulence is filamented — Fig. 11.5 applies just as well to a passive tracer in either two or three dimensions as it does to vorticity in two dimensions. Thus we expect a transfer of tracer variance from large to small scales. If the dye is stirred at a rate $\chi$ then, by analogy with our treatment of the cascade of energy, we posit that

$$\mathcal{K}_\chi \chi \propto \frac{\mathcal{P}(k)k}{\tau_k}, \tag{11.85}$$

where $\mathcal{P}(k)$ is the spectrum of the tracer, $k$ is the wavenumber, $\tau_k$ is an eddy time scale and $\mathcal{K}_\chi$ is a constant, not necessarily the same constant in all cases. (In the rest of the section, Kolmogorov-like constants will be denoted by $\mathcal{K}$, differentiated by miscellaneous superscripts or subscripts.) We will also assume that $\tau_k$ is given by

$$\tau_k = [k^3 \mathcal{E}(k)]^{-1/2}. \tag{11.86}$$

Suppose that the turbulent spectrum is given by $\mathcal{E}(k) = Ak^{-n}$. Using (11.86), (11.85) becomes

$$\mathcal{K}_\chi \chi = \frac{\mathcal{P}(k)k}{[Ak^{3-n}]^{-1/2}}, \tag{11.87}$$

and

$$\boxed{\mathcal{P}(k) = \mathcal{K}_\chi A^{-1/2} \chi k^{(n-5)/2}.} \tag{11.88}$$

We see that the steeper the energy spectrum the shallower the tracer spectrum. If the energy spectrum is steeper than −3 then (11.86) may not be a good estimate of the eddy turnover time, and we use instead

$$\tau_k = \left[ \int_{k_0}^{k} p^2 \mathcal{E}(p) \, dp \right]^{-1/2}, \tag{11.89}$$

where $k_0$ is the low-wavenumber limit of the spectrum. If the energy spectrum is shallower than −3, then the integrand is dominated by the contributions from high wavenumbers and (11.89) effectively reduces to (11.86). If the energy spectrum is steeper than −3, then the integrand is dominated by contributions from low wavenumbers. For $k \gg k_0$ we can approximate the integral by $[k_0^3 E(k_0)]^{-1/2}$, that is the eddy-turnover time at large scales, $\tau_{k_0}$, given by (11.86). The tracer spectrum then becomes

$$\boxed{\mathcal{P}(k) = \mathcal{K}'_\chi \chi \tau_{k_0} k^{-1},} \tag{11.90}$$

where $\mathcal{K}'_\chi$ is a constant. In all these cases the tracer cascade is to smaller scales even if, as may happen in two-dimensional turbulence, energy is cascading to larger scales.

The scale at which diffusion becomes important is given by equating the turbulent time scale $\tau_k$ to the diffusive time scale $(\kappa k^2)^{-1}$. This is independent of the flux of tracer, $\chi$, essentially because the equation for the tracer is linear. Determination of expressions for these scales in two and three dimensions is left as a problem for the reader.

### 11.5.1 Examples of Tracer Spectra

*Energy inertial range flow in three dimensions*

Consider a range of wavenumbers over which neither viscosity nor diffusivity directly influence the turbulent motion and the tracer. Then, in (11.88), $A = \mathcal{K}\varepsilon^{2/3}$ where $\varepsilon$ is the rate of energy transfer to small scales, $\mathcal{K}$ is the Kolmogorov constant, and $n = 5/3$. The tracer spectrum becomes[18]

$$\mathcal{P}(k) = \mathcal{K}_\chi^{3d}\varepsilon^{-1/3}\chi k^{-5/3}, \tag{11.91}$$

where $\mathcal{K}_\chi^{3d}$ is a (putatively universal) constant. It is interesting that the $-5/3$ exponent appears in both the energy spectrum and the passive tracer spectrum. Using (11.86), this is the only spectral slope for which this occurs. Experiments show that this range does, at least approximately, exist with a value of $\mathcal{K}_\chi^{3d}$ of about 0.5–0.6 in three dimensions.

*Inverse energy-cascade range in two-dimensional turbulence*

Suppose that the energy injection occurs at a smaller scale than the tracer injection, so that there exists a range of wavenumbers over which energy is cascading to larger scales while tracer variance is simultaneously cascading to smaller scales. The tracer spectrum is then

$$\mathcal{P}(k) = \mathcal{K}_\chi^{2d}\varepsilon^{-1/3}\chi k^{-5/3}, \tag{11.92}$$

the same as (11.91), although $\varepsilon$ is now the energy cascade rate to larger scales and the constant $\mathcal{K}_\chi^{2d}$ does not necessarily equal $\mathcal{K}_\chi^{3d}$.

*Enstrophy inertial range in two-dimensional turbulence*

In the forward enstrophy inertial range the eddy time scale is $\tau_k = \eta^{-1/3}$ (assuming of course that the classical phenomenology holds). Directly from (11.85) the corresponding tracer spectrum is then

$$\mathcal{P}(k) = \mathcal{K}_\chi^{2d^*}\eta^{-1/3}\chi k^{-1}. \tag{11.93}$$

The passive tracer spectrum now has the same slope as the spectrum of vorticity variance (i.e., the enstrophy spectrum), which is perhaps reassuring since the tracer and vorticity obey similar equations in two dimensions.

*The viscous-advective range of large Prandtl number flow*

If the Prandtl number $\sigma = \nu/\kappa \gg 1$ (and in seawater $\sigma \approx 7$) then there may exist a range of wavenumbers in which viscosity is important, but not tracer diffusion. The energy spectrum is then very steep, and (11.90) will apply. The straining then comes from wavenumbers near the viscous scale, so that for three dimensional flow the appropriate $k_0$ to use in (11.90) is the viscous wavenumber, and $k_0 = k_\nu = (\varepsilon/\nu^3)^{1/4}$. The dynamical time scale at this wavenumber is given by

$$\tau_{k_\nu} = \left(\frac{\nu}{\varepsilon}\right)^{1/2}, \tag{11.94}$$

and using this and (11.90) the tracer spectrum in this viscous-advective range becomes

$$\mathcal{P}(k) = \mathcal{K}_B'\left(\frac{\nu}{\varepsilon}\right)^{1/2}\chi k^{-1}. \tag{11.95}$$

This spectral form applies for $k_\nu < k < k_\kappa$, where $k_\kappa$ is the wavenumber at which diffusion becomes important, found by equating the eddy turnover time given by (11.94) with the diffusive time scale $(\kappa k^2)^{-1}$. This gives

$$k_\kappa = \left(\frac{\varepsilon}{\nu\kappa^2}\right)^{1/4}, \tag{11.96}$$

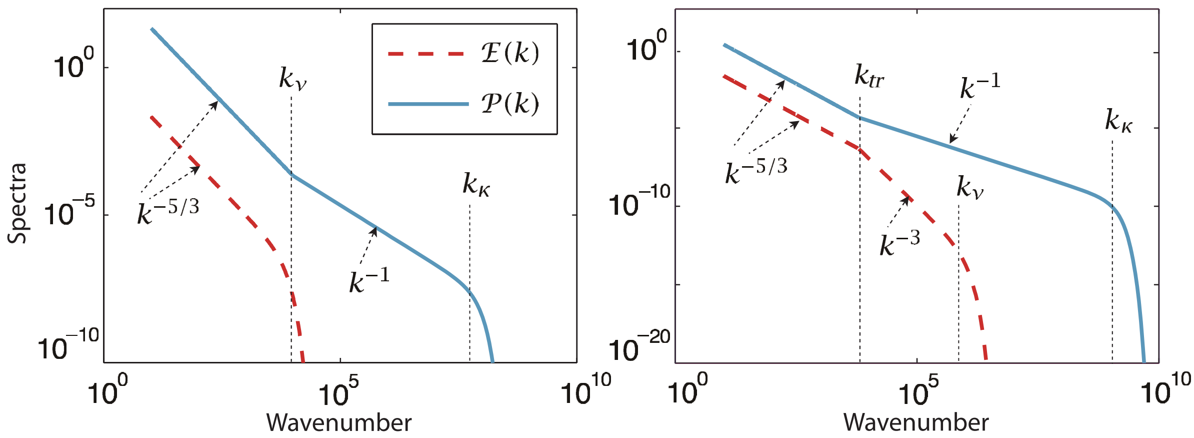


**Fig. 11.12** The energy spectra, $\mathcal{E}(k)$ and passive tracer spectra $\mathcal{P}(k)$ in large Prandtl number three-dimensional (left) and two-dimensional (right) turbulence. In three dimensions $\mathcal{P}(k)$ is given by (11.91) for $k < k_\nu$ and by (11.100) for $k > k_\nu$. In two dimensions, if $k_{tr}$ marks the transition between a $k^{-5/3}$ inverse energy cascade and a $k^{-1}$ forward enstrophy cascade, then $\mathcal{P}(k)$ is given by (11.92) for $k < k_{tr}$ and by (11.101) for $k > k_{tr}$. In both two and three dimensions the tracer spectra fall off rapidly for $k > k_\kappa$.

and $k_\kappa$ is known as the Batchelor wavenumber (and its inverse is the Batchelor scale). Beyond $k_\kappa$, the diffusive flux is not constant and the tracer spectrum can be expected to decay as wavenumber increases. A heuristic way to calculate the spectrum in the diffusive range is to first note that the flux of the tracer is not constant but diminishes according to

$$\frac{\mathrm{d}\chi'(k)}{\mathrm{d}k} = -2\kappa k^2 \mathcal{P}(k), \tag{11.97}$$

where $\chi'$ is the wavenumber-dependent rate of tracer transfer. Let us still assume that $\chi'$ and $\mathcal{P}(k)$ are related by an equation of the form (11.85), where now $\tau_k$ is a constant, given by (11.94). Thus,

$$\mathcal{K}_B \chi' = \frac{\mathcal{P}(k)k}{\tau_{k_\kappa}} = \frac{\mathcal{P}(k)k}{(\nu/\varepsilon)^{1/2}}, \tag{11.98}$$

where $\mathcal{K}_B$ is a constant. Using (11.97) and (11.98) we obtain

$$\frac{\mathrm{d}\chi'}{\mathrm{d}k} = -2\mathcal{K}_B \kappa k \left(\frac{\nu}{\varepsilon}\right)^{1/2} \chi'. \tag{11.99}$$

Solving this, using $\chi' = \chi$ (where $\chi$ is a constant) for small $k$, gives

$$\mathcal{P}(k) = \mathcal{K}_B \left(\frac{\nu}{\varepsilon}\right)^{1/2} \chi k^{-1} \exp\left[-\mathcal{K}_B (k/k_\kappa)^2\right]. \tag{11.100}$$

This reduces to (11.95) if $k \ll k_\kappa$, and is known as the Batchelor spectrum.[19] Its high-wavenumber part $k > k_\kappa$ is known as the viscous-diffusive subrange. The spectrum, and its two-dimensional analogue, are illustrated in Fig. 11.12.

In two dimensions the viscous-advective range occurs for wavenumbers greater than $k_\nu = (\eta/\nu^3)^{1/6}$. The appropriate time scale within this subrange is $\eta^{-1/3}$, which gives a spectrum with precisely the same form as (11.93). At sufficiently high wavenumbers tracer diffusion becomes important, with the diffusive scale now given by equating the eddy turnover time $\eta^{-1/3}$ with the viscous time scale $(\kappa k^2)^{-1}$. This gives the diffusive wavenumber, analogous to (11.96), of $k_\kappa = (\eta/\kappa^3)^{1/6}$. Using (11.99) and the procedure above we then obtain an expression for the spectrum in the region $k > k_\nu$, that is a two-dimensional analogue of (11.100), namely

$$\mathcal{P}(k) = \mathcal{K}_B' \eta^{-1/3} \chi k^{-1} \exp\left[-\mathcal{K}_B'(k/k_\kappa)^2\right]. \tag{11.101}$$

For $k \ll k_\kappa$ this reduces to (11.93), possibly with a different value of the Kolmogorov-like constant.

### † *The inertial-diffusive range of small Prandtl number flow*

For small Prandtl number ($\nu/\kappa \ll 1$) the energy inertial range may coexist with a range over which tracer variance is being dissipated, giving us the so-called inertial-diffusive range. The tracer will begin to be dissipated at a wavenumber obtained by equating a dynamical eddy turnover time with a diffusive time, and this gives a diffusive wavenumber

$$k_\kappa' = \begin{cases} (\varepsilon/\kappa^3)^{1/4} & \text{in three dimensions,} \\ (\eta/\kappa^3)^{1/6} & \text{in two dimensions.} \end{cases} \tag{11.102}$$

Beyond the diffusive wavenumber the flux of the tracer is no longer constant but diminishes according to (11.97).

Given a non-constant flux and an eddy-turnover time that varies with wavenumber there is no self-evidently correct way to proceed. One way is to assume that $\chi$ and $\mathcal{P}(k)$ are related by $\mathcal{K}_\chi'' \chi = \mathcal{P}(k)k/\tau_k$ (as in (11.98), but with a potentially different proportionality constant) and with $\tau_k$ given by (11.86); that is, $\tau_k = \varepsilon^{-1/3}k^{-2/3}$ in three-dimensional turbulence. Using this in (11.97) leads to

$$\mathcal{P}(k) = \mathcal{K}_\chi'' \chi \varepsilon^{-1/3} k^{-5/3} \exp[-(\mathcal{K}_\chi'' 3/2)(k/k_\kappa')^{4/3}], \tag{11.103}$$

where $\chi$ is the tracer flux at the beginning of the tracer dissipation range. (A similar expression emerges in two-dimensional turbulence.) However, given such a steep spectrum an argument based on spectral locality is likely to be suspect. Another argument posits a particular relationship between the tracer spectrum and energy spectrum in the inertial-diffusive range, and this leads to

$$\mathcal{P}(k) = \frac{\mathcal{K}_B''}{3} \chi_0 \varepsilon^{2/3} \kappa^{-3} k^{-17/3} = \mathcal{K}_B'' \chi_0 \varepsilon^{-1/3} k^{-5/3} g(k/k_\kappa), \tag{11.104}$$

where $g(\alpha) = \alpha^{-4}/3$ and $\mathcal{K}_B''$ is a constant.[20]

## *Notes*

1 Horace Lamb has been quoted as saying that when he died and went to Heaven he hoped for enlightenment on two things, quantum electrodynamics and turbulence, although he was only optimistic about the former. Werner Heisenberg expressed a similar sentiment. But Heaven may be the wrong place to seek such enlightenment, for turbulence is the invention of the Devil, put on Earth to torment us.

2 The algebra of the three-dimensional case is more complicated because of the pressure term and because the momentum equation is a vector equation. Nevertheless, in incompressible flow we can take the divergence of the momentum equation to obtain an elliptic equation for pressure of the form $\nabla^2 p = Q(\boldsymbol{v})$, where the right-hand side is a quadratic function of velocity and its derivatives, Fourier transform this and then proceed much as in the two-dimensional case.

3 A. N. Kolmogorov (1903–1987) was a Russian mathematician and theoretical physicist, who made seminal contributions to turbulence (with famous papers in 1941 and in 1962), to probability and statistics, and to classical mechanics (e.g., the Kolmogorov–Arnold–Moser theorem). Yaglom (1994) provides more details on both the man and his scientific contributions.

4 The process has been encapsulated in the following ditty by Lewis Fry (L.F.) Richardson (1881–1953), his own summary of Richardson (1920):

*Big whorls have little whorls, that feed on their velocity;*
*And little whorls have lesser whorls, and so on to viscosity.*

The verse follows a well-known one by the mathematician Augustus De Morgan (in *A Budget of Paradoxes* 1872), *'Great fleas have little fleas upon their backs to bite 'em...'*, which in turn is a parody on a poem by Jonathan Swift. Richardson himself was a British scientist best known as the person who (following earlier work by Cleveland Abbe in 1901 and Vilhelm Bjerknes in 1904) envisioned weather forecasting in its current form — that is, numerical weather prediction. However, as described in his 1922 book, instead of an electronic computer performing the calculations, he imagined, perhaps fancifully, a hall full of people performing calculations in unison all directed by a conductor at the front. His first numerical forecast, calculated by hand, was wildly inaccurate because he failed to initialize his atmosphere properly and because his timestep was too long and did not satisfy the CFL condition, and unrealistic gravity waves dominated the solution. However, it was a prescient and important effort. He also worked on turbulence, and seems to have envisioned the turbulent cascade prior to Kolmogorov (to wit the verse above), and the 'Richardson number', a measure of fluid stratification, is named after him. He also made contributions to the theory of war and was known as a pacifist — he was a conscientious objector and drove ambulances in the first World War, and resigned from the UK Meteorological Office because it became part of the Air Ministry.

5 Kolmogorov (1941) obtained the result in a slightly different way, using distances in real space rather than wavenumber and deriving the equivalent result for the longitudinal structure function, $D(r) \equiv \langle [\boldsymbol{u}_l(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}_l(\boldsymbol{x})]^2 \rangle \sim r^{2/3}$. It was Obukhov (1941) who gave an argument in spectral space and first wrote down that $E(k) \sim k^{-5/3}$. Kolmogorov's argument is regarded as more general and hence the $5/3$ spectrum is usually named for him, but sometimes it is called the 'Kolmogorov–Obukhov' 5/3 spectrum.

6 Results kindly provided by R. Zhao.

7 The first observations confirming the -5/3 predictions were from a tidal channel (Grant *et al.* 1962). These results were initially presented at a turbulence conference in Marseille in 1961, ironically at the same time as Kolmogorov presented a modification of his original theory that incorporated a local mean dissipation rate, to try to take intermittency into account, recognizing that his first theory was incomplete (Kolmogorov 1962). It is said to have been L. D. Landau who pointed out the consequences of intermittency to Kolmogorov, soon after the K41 theory first appeared.

8 Two early papers on two-dimensional turbulence are those of Lee (1951) and Fjørtoft (1953), the former noting the incompatibility of the material conservation of vorticity with Kolmogorov's energy inertial range, and the latter recognizing the two-dimensional nature of large-scale atmospheric motion. Batchelor (1953b) noted the tendency of energy to concentrate in small wavenumbers as a consequence of energy and enstrophy conservation. The theory was developed by Kraichnan (1967), who predicted the spectral slopes of the two-dimensional cascades, Leith (1968) and Batchelor (1969). Lilly (1969) performed some early numerical integrations. For reviews see Kraichnan & Montgomery (1980), Danilov & Gurarie (2001) and Boffetta & Ecke (2012).

9 This similarity argument is due to Batchelor (1969), and its validity was explored by Bartello & Warn (1996) using numerical simulations of decaying two-dimensional turbulence. They found that the similarity hypothesis is not quantitatively accurate, and in particular that higher-order moments of the vorticity do not obey the predictions of the theory. This failure may be ascribed to the fact that in two-dimensional flow vorticity is conserved on parcels, and in the presence of coherent vortices this is an effective constraint that is not included in the theory. In flow with a finite deformation radius coherent vortices are found to be less important and an analogous similarity hypothesis appears to work better (Iwayama *et al.* 2002).

10 Arbic *et al.* (2007), Colin de Verdière (1980). See also Scott (2001).

11 For example, Peltier & Stuhne (2002) and Smith & Waleffe (1999).

12 Adapted from Maltrud & Vallis (1991).

13 For numerical simulations illustrating these and other properties of two-dimensional turbulence see, among others, McWilliams (1984), Maltrud & Vallis (1991), Oetzel & Vallis (1997), Lindborg & Alvelius (2000) and Smith *et al.* (2002). Also look at Jupiter through a telescope! A statistical-mechanical argument that in two-dimensional turbulence there is tendency for a small number of vortices to form was given by Onsager (1949).

14 The Lorenz equations were written down by Lorenz (1963), based on some earlier work of Saltzman (1962), and inspired a veritable industry of study. The field of chaos, or more generally nonlinear dynamics, has grown enormously since then, prompted also by work in mathematics occurring at about the same time, and its development is sometimes regarded as one of the true revolutions of science in the twentieth century. Aubin & Dahan Dalmedico (2002) write a history. The correspondence of the Lorenz equations to a real fluid system is tenuous, but the importance of the properties they demonstrate transcends this; we regard the equations simply as an example of a chaotic system with some fluid relevance. The equations and variations about them have reappeared in studies of, among other things, lasers, dynamos, chemical reactions, mechanical waterwheels and El Niño.

15 That a turbulent flow is, *inter alia*, chaotic and unpredictable follows from the work of Lorenz (1963), Ruelle & Takens (1971) and others who showed that fluid turbulence was generically a consequence of a small number of bifurcations as some controlling parameter (such as the Reynolds number) is changed. Prior to this, turbulence was sometimes thought, following Landau (1944), to be a large collection of periodic motions with incommensurate frequencies that would have complex and non-repeating but presumably predictable behaviour. Notwithstanding the Landau picture, it seems to have been known, well before the development of nonlinear dynamical systems theory in the 1960s and 1970s, that the weather was inherently unpredictable. Poincaré was perhaps the first to properly understand this at the turn of the twentieth century, although in the review of a book by the thermodynamicist P. Duhem, W. S. Franklin wrote in 1898 that 'An infinitesimal cause produces a finite effect. Long range detailed weather prediction is therefore impossible... the flight of a grasshopper in Montana may turn a storm aside from Philadelphia to New York!'

Weather forecasters themselves long seem to have intuited that the atmosphere was intrinsically, and not just practically, difficult to forecast. In the 1941 novel *Storm* by G. R. Stewart (1895–1980, a professor of English at UC Berkeley) we find a forecaster recalling his old professor's saying that 'A Chinaman sneezing in Shen-si may set men to shoveling snow in New York City'. (*Storm* is also notable because it used female names for intense storms, a practice that became common among forecasters in World War II and that was used by the US Weather Service from 1953–1978, after which gender equity obtained.) In a more academic setting, the predictability problem is mentioned in the book by Godske *et al.* (1957), much of which was written in the 1930s and 1940s, and Thompson (1957) and Novikov (1959) studied the unpredictability of atmospheric flows from the perspective of turbulence, evidently unaware of, or at least uninfluenced by, either Poincaré or Landau. Phillip Thompson himself was in the Joint Numerical Weather Prediction Unit of the US government in the 1950s, whose task was to numerically produce weather forecasts, and this practical experience undoubtedly confronted and guided his theoretical thinking. These various strands came together and were clarified by the dynamical systems viewpoint coupled with the view of the atmosphere as a geostrophically turbulent fluid, and this led to the viewpoint we describe here, and to estimates of the limit to predictability of the atmospheric weather of about two weeks, although at any given time the predictability may certainly be shorter or longer than that.

16 The more technical phrase 'sensitive dependence on initial conditions' is a paraphrase of one of Poincaré, but the catchier one 'butterfly effect' is more recent. It seems to have first appeared in a meteorological context in Smagorinsky (1969), where we find 'Would the flutter of a butterfly's wings ultimately amplify to the point where the numerical simulation departed from reality…? If not the flutter of the butterfly's wings, the disturbance might be the result of instrumental errors in the initial conditions…' The phrase became more well known following a lecture by Lorenz to the American Association for the Advancement of Science (AAAS) in 1972 entitled 'Predictability: does the flap of a butterfly's wings in Brazil set off a tornado in Texas?' The shape of the Lorenz attractor in phase space also resembles a butterfly (Fig. 11.10), and the two (not unrelated) phenomena are sometimes conflated. The death of a butterfly also changes the course of history in a science fiction story by Ray Bradbury, *A Sound of Thunder* (1952). See also endnote 15 above and the brief history

by Hilborn (2004).

17 A related point is that the predictability of a turbulent system is not well characterized by its spectrum of Lyapunov exponents: in a turbulent system in three dimensions the largest Lyapunov exponent is likely to be associated with very small scales of motion, and the error growth associated with this effectively saturates at small scales.

18 First derived by Obukhov (1949). See also Corrsin (1951).

19 Batchelor (1959) also suggests that the constant $\mathcal{K}_B$ in (11.100) should have the value 2. There is some observational support for the $k^{-1}$ viscous-advective range in the temperature spectra of the ocean, one of the first measurements being that of Grant *et al.* (1968). Aside from their intrinsic interest, the viscous and diffusive scales are used in microstructure theory and measurements that lead to estimates of the ocean's energy dissipation rate (Gregg 1998, Stips 2005).

20 Batchelor (1959). There is some numerical support for the $-17/3$ spectrum using a large-eddy simulation (LES) model (Chasnov 1991). See also O'Gorman & Pullin (2005).