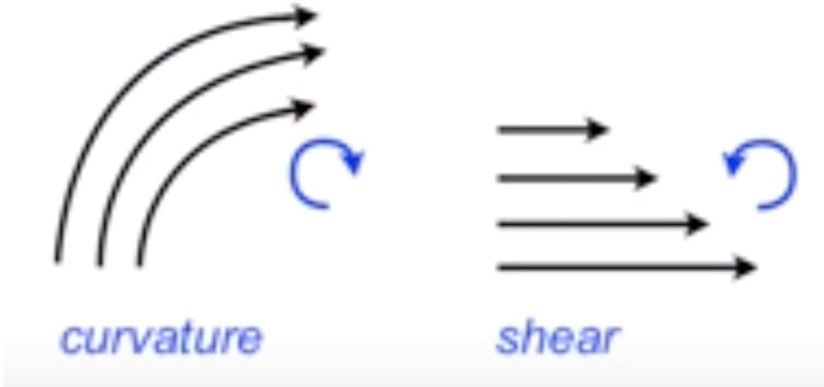
Vorticity



Vorticity: curl of velocity (a measure of spin)

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$$

$$= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\boldsymbol{k}$$

For 2-D flow on the horizontal plane: $\boldsymbol{u} = u\mathbf{i} + v\mathbf{j}$

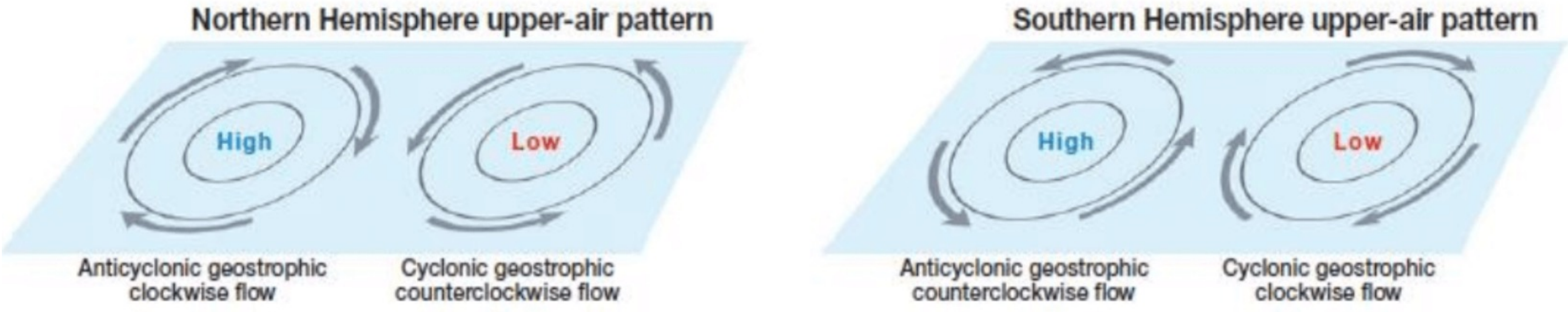
$$\boldsymbol{\omega} = \mathbf{k}\left(\underbrace{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}_{\zeta}\right)$$

Properties:

$$\nabla \cdot \boldsymbol{\omega} = 0$$

$$\nabla \times \nabla f = 0$$

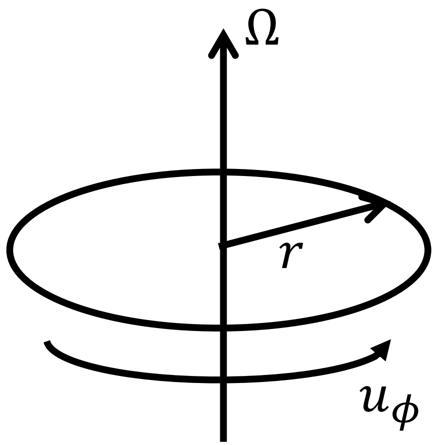
Cyclones and anticyclones



negative ζ positive ζ positive ζ negative ζ

Rigid body motion

$$u_r = 0, \qquad u_\phi = \Omega r, \qquad u_z = 0$$



$$\omega = \frac{1}{r}\begin{vmatrix} \boldsymbol{e_r} & r\boldsymbol{e_\phi} & \boldsymbol{e_z} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ u_r & r u_\phi & u_z \end{vmatrix}$$

$$\omega^z = \frac{1}{r}\frac{\partial}{\partial r}\left(r u_\phi\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r^2 \Omega\right) = 2\Omega$$

The vorticity of a fluid in solid body rotation is twice the angular velocity of the fluid about the axis of rotation, and is pointed in a direction orthogonal to the plane of rotation.

Circulation

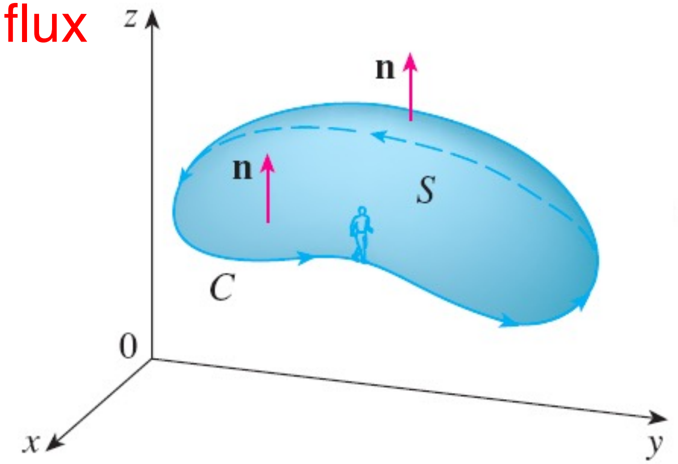
Circulation: the integral of velocity around a closed fluid loop

$$C \equiv \oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{r}$$

Stokes' Theorem:

$$C \equiv \oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{r} = \int_S \underline{\boldsymbol{\omega} \cdot \mathrm{d}\boldsymbol{S}}$$

vortex flux



Vorticity equation – without rotation

The momentum equation:

$$\frac{d\boldsymbol{v}}{dt} = -\frac{1}{\rho}\nabla p - \nabla\Phi + \nu_E \nabla^2 \boldsymbol{v}$$

$$\boxed{(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\boldsymbol{v}\times\boldsymbol{\omega} + \nabla(\boldsymbol{v}^2/2),}$$

$$\frac{\partial\boldsymbol{v}}{\partial t} + \boldsymbol{\omega}\times\boldsymbol{v} = -\frac{1}{\rho}\nabla p - \nabla\Phi - \nabla(\boldsymbol{v}^2/2) + \nu_E\nabla^2\boldsymbol{v}$$

Take the curl of the momentum equation:

$$\boxed{\nabla\times\nabla f = 0}$$

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + \nabla\times(\boldsymbol{\omega}\times\boldsymbol{v}) = \frac{1}{\rho^2}(\nabla\rho\times\nabla p) + \nabla\times\boldsymbol{F}.$$

$$\nabla\times(\boldsymbol{\omega}\times\boldsymbol{v}) = (\boldsymbol{v}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{v} + \boldsymbol{\omega}\nabla\cdot\boldsymbol{v} - \cancel{\boldsymbol{v}\nabla\cdot\boldsymbol{\omega}}.$$

$$\boxed{\nabla\cdot\boldsymbol{\omega} = 0}$$

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{v} - \boldsymbol{\omega}\nabla\cdot\boldsymbol{v} + \frac{1}{\rho^2}(\nabla\rho\times\nabla p) + \nabla\times\boldsymbol{F}.$$

The baroclinic term

$$\frac{1}{\rho^2}\nabla\rho \times \nabla p$$

If the density is only a function of pressure:

$$\rho = \rho(\mathrm{p})$$

Isolines of pressure and density are parallel

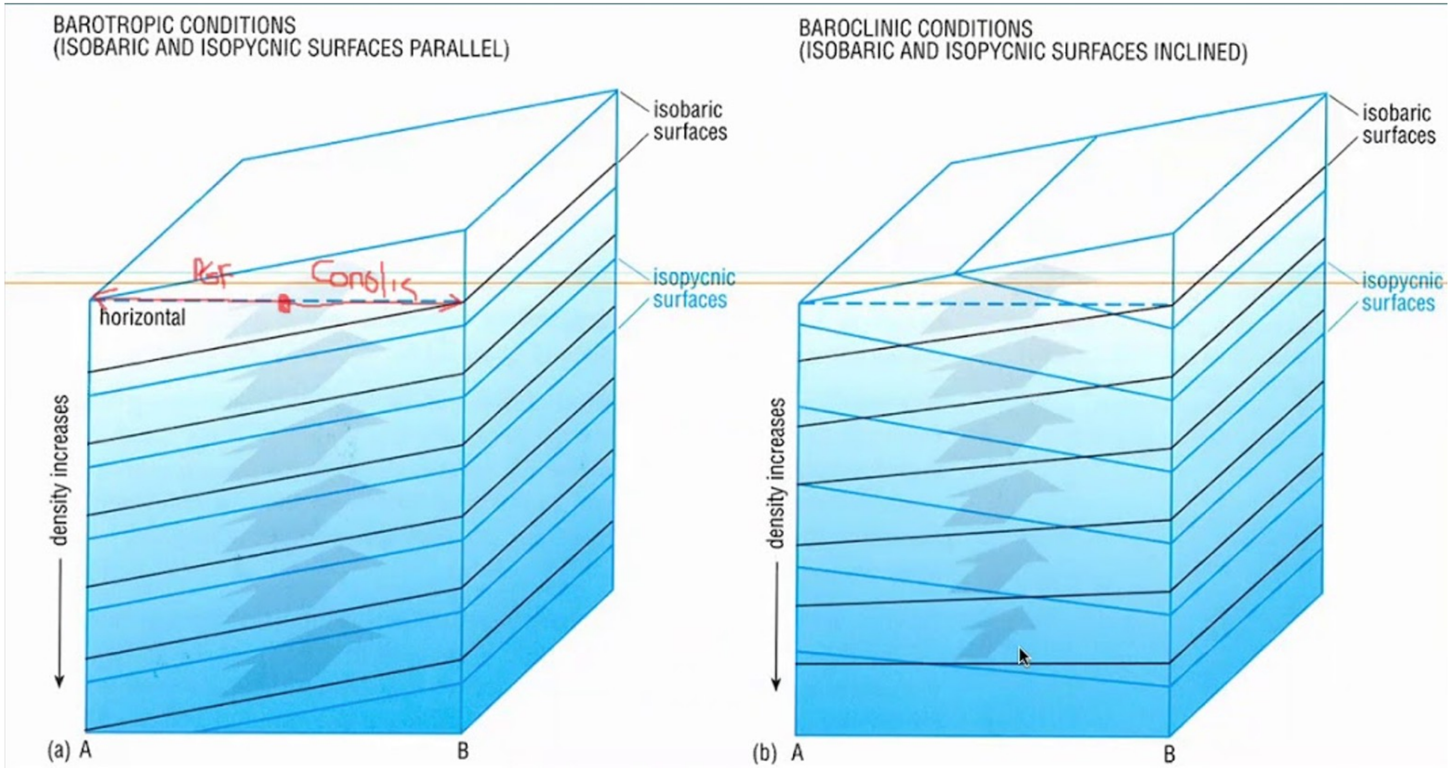
$$\nabla\rho \times \nabla p = 0$$ barotropic fluid

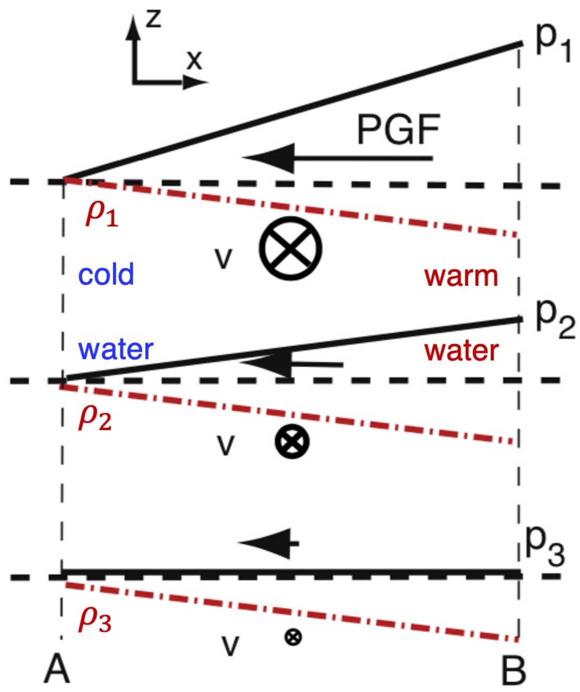
(for constant density?)

Otherwise:

$$\nabla\rho \times \nabla p \neq 0$$ baroclinic fluid

Barotropic and barolinic conditions





$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{v} - \cancel{\boldsymbol{\omega}\nabla \cdot \boldsymbol{v}} + \frac{1}{\rho^2}(\cancel{\nabla\rho \times \nabla p}) + \cancel{\nabla \times \boldsymbol{F}}.$$

For incompressible, barotropic, inviscous flow:

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{v}.$$

For two-dimensional flows $\boldsymbol{u} = u\mathbf{i} + v\mathbf{j}$

$$\boldsymbol{\omega} = \mathbf{k}\underbrace{\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}_{\zeta}$$

If a streamfunction exists:

$$u = -\frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x}$$

$$\frac{\boldsymbol{D\omega}}{Dt} = 0 \qquad \boxed{\frac{d\zeta}{dt} = 0}$$

$$\boxed{\zeta = \nabla^2 \psi}$$

Kelvin's circulation theorem

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p - \nabla \Phi$$ inviscid flow

Rate of change in circulation:

$$\begin{aligned}\frac{\mathrm{D}C}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t}\oint \boldsymbol{v}\cdot \mathrm{d}\boldsymbol{r} &= \oint \left(\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t}\cdot \mathrm{d}\boldsymbol{r} + \boldsymbol{v}\cdot \mathrm{d}\boldsymbol{v}\right) \\ &= \oint \left[\left(-\frac{1}{\rho}\nabla p - \nabla \Phi\right)\cdot \mathrm{d}\boldsymbol{r} + \boldsymbol{v}\cdot \mathrm{d}\boldsymbol{v}\right] \\ &= \oint -\frac{1}{\rho}\nabla p \cdot \mathrm{d}\boldsymbol{r}\end{aligned}$$

$$\boxed{D(\mathrm{d}\boldsymbol{r})/Dt = \mathrm{d}\boldsymbol{v}}$$

$$\oint \frac{1}{\rho}\nabla p \cdot \mathrm{d}\boldsymbol{r} = \int_S \nabla \times \left(\frac{\nabla p}{\rho}\right)\cdot \mathrm{d}\boldsymbol{S} = \int_S \frac{-\nabla\rho \times \nabla p}{\rho^2}\cdot \mathrm{d}\boldsymbol{S}$$

Circulation is conserved

Vortex flux is conserved

For barotropic fluid:

$$\boxed{\frac{\mathrm{D}}{\mathrm{D}t}\oint \boldsymbol{v}\cdot \mathrm{d}\boldsymbol{r} = 0}$$ Stokes' theorem $$\boxed{\frac{\mathrm{D}}{\mathrm{D}t}\int_S \boldsymbol{\omega}\cdot \mathrm{d}\boldsymbol{S} = 0}$$

Circulation in a rotating frame

The absolute velocity in an inertia frame is:

$$\boldsymbol{v}_a = \boldsymbol{v}_r + \boldsymbol{\Omega} \times \boldsymbol{r}$$

Rate of change in circulation from absolute velocity:

$$\frac{\mathrm{D}}{\mathrm{D}t}\oint(\boldsymbol{v}_r + \boldsymbol{\Omega}\times\boldsymbol{r})\cdot \mathrm{d}\boldsymbol{r} = \oint\left[\left(\frac{\mathrm{D}\boldsymbol{v}_r}{\mathrm{D}t} + \boldsymbol{\Omega}\times\boldsymbol{v}_r\right)\cdot \mathrm{d}\boldsymbol{r} + (\cancel{\boldsymbol{v}_r} + \boldsymbol{\Omega}\times\boldsymbol{r})\cdot \mathrm{d}\boldsymbol{v}_r\right]$$

$$\oint(\boldsymbol{\Omega}\times\boldsymbol{r})\cdot \mathrm{d}\boldsymbol{v}_r = \oint\left\{\mathrm{d}[(\boldsymbol{\Omega}\times\boldsymbol{r})\cdot\boldsymbol{v}_r] - (\boldsymbol{\Omega}\times \mathrm{d}\boldsymbol{r})\cdot\boldsymbol{v}_r\right\}$$

$$= \oint\left\{\cancel{\mathrm{d}[(\boldsymbol{\Omega}\times\boldsymbol{r})\cdot\boldsymbol{v}_r]} + (\boldsymbol{\Omega}\times\boldsymbol{v}_r)\cdot \mathrm{d}\boldsymbol{r}\right\}$$

$$\frac{\mathrm{D}}{\mathrm{D}t}\oint(\boldsymbol{v}_r + \boldsymbol{\Omega}\times\boldsymbol{r})\cdot \mathrm{d}\boldsymbol{r} = \oint\left(\frac{\mathrm{D}\boldsymbol{v}_r}{\mathrm{D}t} + 2\boldsymbol{\Omega}\times\boldsymbol{v}_r\right)\cdot \mathrm{d}\boldsymbol{r}$$

$$= -\oint\frac{1}{\rho}\nabla p\cdot \mathrm{d}\boldsymbol{r}$$ = 0 (for barotropic and inviscid fluids)

$$\boxed{\boldsymbol{\omega}_r = \nabla \times \boldsymbol{v}_r}$$

relative vorticity

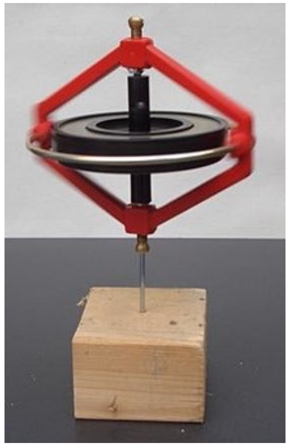
$$\frac{\mathrm{D}}{\mathrm{D}t} \oint (\boldsymbol{v}_r + \boldsymbol{\Omega} \times \boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{r} = 0$$

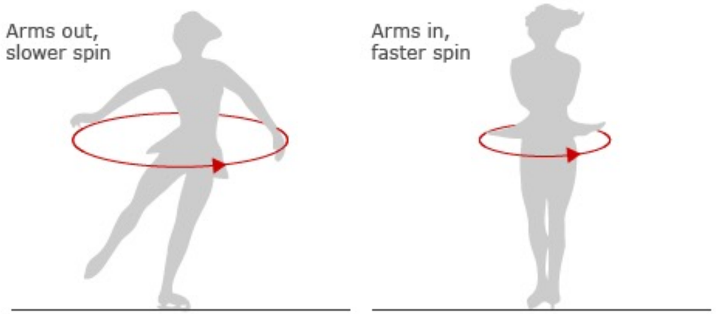
$$\boxed{\nabla \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = 2\boldsymbol{\Omega}}$$

$2\boldsymbol{\Omega}$: planetary (ambient) vorticity

$$\frac{\mathrm{D}}{\mathrm{D}t} \int_S (\underline{\boldsymbol{\omega}_r + 2\boldsymbol{\Omega}}) \cdot \mathrm{d}\boldsymbol{S} = 0$$

$\boldsymbol{\omega}_a$: absolute vorticity





Angular momentum conservation:

$$L = m\omega r^2$$

Vorticity equation in a rotating frame

For inviscid flow:

$$\frac{d\boldsymbol{v}_r}{dt} + 2\boldsymbol{\Omega}\times\boldsymbol{v}_r = -\frac{1}{\rho}\nabla p - \nabla\Phi$$

$$(\boldsymbol{v}_r \cdot \boldsymbol{\nabla})\boldsymbol{v}_r = -\boldsymbol{v}_r\times\boldsymbol{\omega}_r + \nabla(\boldsymbol{v}_r^2/2)$$

$$\frac{\partial \boldsymbol{v}_r}{\partial t} + (2\boldsymbol{\Omega} + \boldsymbol{\omega}_r) \times \boldsymbol{v}_r = -\frac{1}{\rho}\nabla p + \nabla\left(\Phi - \frac{1}{2}\boldsymbol{v}_r^2\right)$$

Take the curl of the equation:

$$\nabla \times [(2\boldsymbol{\Omega} + \boldsymbol{\omega}_r) \times \boldsymbol{v}_r] = (2\boldsymbol{\Omega} + \boldsymbol{\omega}_r)\nabla \cdot \boldsymbol{v}_r + (\boldsymbol{v}_r \cdot \nabla)(2\boldsymbol{\Omega} + \boldsymbol{\omega}_r) - [(2\boldsymbol{\Omega} + \boldsymbol{\omega}_r) \cdot \nabla]\boldsymbol{v}_r$$

$$\frac{D\boldsymbol{\omega}_a}{Dt} = [(2\boldsymbol{\Omega} + \boldsymbol{\omega}_r) \cdot \nabla]\boldsymbol{v}_r - (2\boldsymbol{\Omega} + \boldsymbol{\omega}_r)\nabla \cdot \boldsymbol{v}_r + \frac{1}{\rho^2}(\nabla\rho \times \nabla p)$$

$\boldsymbol{\omega}_a$: absolute vorticity

For incompressible, barotropic fluids: $\boxed{\frac{D\boldsymbol{\omega}_a}{Dt} = (\boldsymbol{\omega}_a \cdot \nabla)\boldsymbol{v}_r}$ $\boxed{\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{v}}$

PV conservation from the circulation theorem

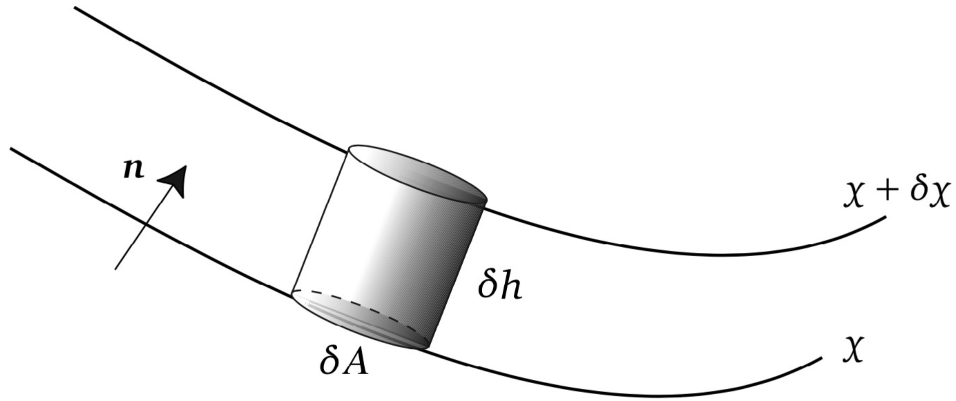
$$\frac{\mathrm{D}}{\mathrm{D}t}[(\boldsymbol{\omega}_a \cdot \boldsymbol{n})\delta A] = 0$$

$$\boldsymbol{\omega}_a \cdot \boldsymbol{n}\,\delta A = \boldsymbol{\omega}_a \cdot \frac{\nabla\chi}{|\nabla\chi|}\frac{\delta V}{\delta h}$$

$$\delta\chi = \delta\boldsymbol{x} \cdot \nabla\chi = \delta h|\nabla\chi|$$

$$\frac{\mathrm{D}}{\mathrm{D}t}\left[\frac{(\boldsymbol{\omega}_a \cdot \nabla\chi)\delta V}{\delta\chi}\right] = 0.$$

$$\frac{\rho\delta V}{\delta\chi}\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{\omega}_a}{\rho} \cdot \nabla\chi\right) = 0$$



χ is any materially conserved tracer: $\mathrm{D}\chi/\mathrm{D}t = 0$

$$\boldsymbol{n} = \nabla\chi/|\nabla\chi|$$

$$\delta V = \delta h\,\delta A$$

$$\boxed{\frac{\mathrm{D}}{\mathrm{D}t}(\widetilde{\boldsymbol{\omega}}_a \cdot \nabla\chi) = 0}$$