Chapter 2

Basic Conservation Laws

Atmospheric motions are governed by three fundamental physical principles: conservation of mass, conservation of momentum, and conservation of energy. The mathematical relations that express these laws may be derived by considering the budgets of mass, momentum, and energy for an infinitesimal *control volume* in the fluid. Two types of control volume are commonly used in fluid dynamics. In the *Eulerian* frame of reference, the control volume consists of a parallelepiped of sides δx, δy, and δz, whose position is fixed relative to the coordinate axes. Mass, momentum, and energy budgets will depend on fluxes caused by the flow of fluid through the boundaries of the control volume. (This type of control volume was used in Section 1.2.1.) In the *Lagrangian* frame, however, the control volume consists of an infinitesimal mass of "tagged" fluid particles; thus, the control volume moves about following the motion of the fluid, always containing the same fluid particles.

The Lagrangian frame is particularly useful for deriving conservation laws, since such laws may be stated most simply in terms of a particular mass element of the fluid. The Eulerian system is, however, more convenient for solving most problems because in that system the field variables are related by a set of partial differential equations in which the independent variables are the coordinates x, y, z, and t. In the Lagrangian system, however, it is necessary to follow the time evolution of the fields for various individual fluid parcels. Thus, the independent variables are x_0, y_0, z_0, and t, where x_0, y_0, and z_0 designate the position that a particular parcel passed through at a reference time t_0.

2.1 TOTAL DIFFERENTIATION

The conservation laws to be derived in this chapter contain expressions for the rates of change of density, momentum, and thermodynamic energy following the motion of particular fluid parcels. In order to apply these laws in the Eulerian frame, it is necessary to derive a relationship between the rate of change of a field variable following the motion and its rate of change at a fixed point. The former is called the *substantial*, the *total*, or the *material* derivative (it will be denoted D/Dt). The latter is called the *local* derivative (it is merely the partial derivative with respect to time).

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To derive a relationship between the total derivative and the local derivative, it is convenient to refer to a particular field variable (temperature, for example). For a given air parcel the location (x, y, z) is a function of t so that $x = x(t)$, $y = y(t)$, $z = z(t)$. Following the parcel, T may then be considered as truly a function only of time, and its rate of change is just the total derivative DT/Dt. To relate the total derivative to the local rate of change at a fixed point, we consider the temperature measured on a balloon that moves with the wind. Suppose that this temperature is T_0 at the point x_0, y_0, z_0 and time t_0. If the balloon moves to the point $x_0+\delta x$, $y_0+\delta y$, $z_0+\delta z$ in a time increment δt, then the temperature change recorded on the balloon, δT, can be expressed in a Taylor series expansion as

$$\delta T = \left(\frac{\partial T}{\partial t}\right)\delta t + \left(\frac{\partial T}{\partial x}\right)\delta x + \left(\frac{\partial T}{\partial y}\right)\delta y + \left(\frac{\partial T}{\partial z}\right)\delta z + \text{(higher-order terms)}$$

Dividing through by δt and noting that δT is the change in temperature following the motion so that

$$\frac{DT}{Dt} \equiv \lim_{\delta t \to 0} \frac{\delta T}{\delta t}$$

we find that in the limit $\delta t \to 0$

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \left(\frac{\partial T}{\partial x}\right)\frac{Dx}{Dt} + \left(\frac{\partial T}{\partial y}\right)\frac{Dy}{Dt} + \left(\frac{\partial T}{\partial z}\right)\frac{Dz}{Dt}$$

is the rate of change of T following the motion.

If we now let

$$\frac{Dx}{Dt} \equiv u, \ \frac{Dy}{Dt} \equiv v, \ \frac{Dz}{Dt} \equiv w$$

then u, v, w are the velocity components in the x, y, z directions, respectively, and

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right) \tag{2.1}$$

Using vector notation, this expression may be rewritten as

$$\frac{\partial T}{\partial t} = \frac{DT}{Dt} - \mathbf{U} \cdot \nabla T$$

where $\mathbf{U} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$ is the velocity vector. The term $-\mathbf{U} \cdot \nabla T$ is called the temperature *advection*. It gives the contribution to the local temperature change due to air motion. For example, if the wind is blowing from a cold region toward a warm region, $-\mathbf{U} \cdot \nabla T$ will be negative (cold advection) and the advection term will contribute negatively to the local temperature change. Thus, the local rate of change of temperature equals the rate of change of temperature following the motion (i.e., the heating or cooling of individual air parcels) plus the advective rate of change of temperature.

The relationship between the total derivative and the local derivative given for temperature in (2.1) holds for any of the field variables. Furthermore, the total derivative can be defined following a motion field other than the actual wind field.

Example

We may wish to relate the pressure change measured by a barometer on a moving ship to the local pressure change. The surface pressure decreases by 3 hPa per 180 km in the eastward direction. A ship steaming eastward at 10 km/h measures a pressure fall of 1 hPa per 3 h.

What is the pressure change on an island that the ship is passing? If we take the x axis oriented eastward, then the local rate of change of pressure on the island is

$$\frac{\partial p}{\partial t} = \frac{Dp}{Dt} - u\frac{\partial p}{\partial x}$$

where Dp/Dt is the pressure change observed by the ship and u is the velocity of the ship. Thus,

$$\frac{\partial p}{\partial t} = \frac{-1\text{ hPa}}{3\text{ h}} - \left(10\frac{\text{km}}{\text{h}}\right)\left(\frac{-3\text{ hPa}}{180\text{ km}}\right) = -\frac{1\text{ hPa}}{6\text{ h}}$$

so that the rate of pressure fall on the island is only half the rate measured on the moving ship.

If the total derivative of a field variable is zero, then that variable is a conservative quantity following the motion. The local change is then entirely due to advection. As shown later, field variables that are approximately conserved following the motion play an important role in dynamic meteorology.

2.1.1 Total Differentiation of a Vector in a Rotating System

The conservation law for momentum (Newton's second law of motion) relates the rate of change of the absolute momentum following the motion in an inertial reference frame to the sum of the forces acting on the fluid. For most applications in meteorology it is desirable to refer the motion to a reference frame rotating with Earth. Transformation of the momentum equation to a rotating coordinate system requires a relationship between the total derivative of a vector in an inertial reference frame and the corresponding total derivative in a rotating system.

To derive this relationship, we let $\mathbf{A}$ be an arbitrary vector whose Cartesian components in an inertial frame are given by

$$\mathbf{A} = \mathbf{i}'A'_x + \mathbf{j}'A'_y + \mathbf{k}'A'_z$$

and whose components in a frame rotating with an angular velocity $\boldsymbol{\Omega}$ are

$$\mathbf{A} = \mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z$$

Letting $D_a\mathbf{A}/Dt$ be the total derivative of $\mathbf{A}$ in the inertial frame, we can write

$$\begin{aligned}\frac{D_a\mathbf{A}}{Dt} &= \mathbf{i}'\frac{DA_x'}{Dt} + \mathbf{j}'\frac{DA_y'}{Dt} + \mathbf{k}'\frac{DA_z'}{Dt} \\ &= \mathbf{i}\frac{DA_x}{Dt} + \mathbf{j}\frac{DA_y}{Dt} + \mathbf{k}\frac{DA_z}{Dt} + \frac{D_a\mathbf{i}}{Dt}A_x + \frac{D_a\mathbf{j}}{Dt}A_y + \frac{D_a\mathbf{k}}{Dt}A_z\end{aligned}$$

The first three terms on the preceding line can be combined to give

$$\frac{D\mathbf{A}}{Dt} \equiv \mathbf{i}\frac{DA_x}{Dt} + \mathbf{j}\frac{DA_y}{Dt} + \mathbf{k}\frac{DA_z}{Dt}$$

which is just the total derivative of $\mathbf{A}$ as viewed in the rotating coordinates (i.e., the rate of change in $\mathbf{A}$ following the relative motion).

The last three terms arise because the directions of the unit vectors ($\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$) change their orientation in space as Earth rotates. These terms have a simple form for a rotating coordinate system. For example, consider the eastward directed unit vector:

$$\delta\mathbf{i} = \frac{\partial\mathbf{i}}{\partial\lambda}\delta\lambda + \frac{\partial\mathbf{i}}{\partial\phi}\delta\phi + \frac{\partial\mathbf{i}}{\partial z}\delta z$$

For solid-body rotation $\delta\lambda = \Omega\delta t$, $\delta\phi = 0$, $\delta z = 0$, so that $\delta\mathbf{i}/\delta t = (\partial\mathbf{i}/\partial\lambda)\,(\delta\lambda/\delta t)$ and taking the limit $\delta t \rightarrow 0$,

$$\frac{D_a\mathbf{i}}{Dt} = \Omega\frac{\partial\mathbf{i}}{\partial\lambda}$$

But from Figures 2.1 and 2.2, the longitudinal derivative of $\mathbf{i}$ can be expressed as

$$\frac{\partial\mathbf{i}}{\partial\lambda} = \mathbf{j}\sin\phi - \mathbf{k}\cos\phi$$

However, $\boldsymbol{\Omega} = (0, \Omega\cos\phi, \Omega\sin\phi)$ so that

$$\frac{D_a\mathbf{i}}{Dt} = \Omega(\mathbf{j}\sin\phi - \mathbf{k}\cos\phi) = \boldsymbol{\Omega}\times\mathbf{i}$$

In a similar fashion, it can be shown that $D_a\mathbf{j}/Dt = \boldsymbol{\Omega}\times\mathbf{j}$ and $D_a\mathbf{k}/Dt = \boldsymbol{\Omega}\times\mathbf{k}$. Therefore, the total derivative for a vector in an inertial frame is related to that in a rotating frame by the expression

$$\frac{D_a\mathbf{A}}{Dt} = \frac{D\mathbf{A}}{Dt} + \boldsymbol{\Omega}\times\mathbf{A} \tag{2.2}$$

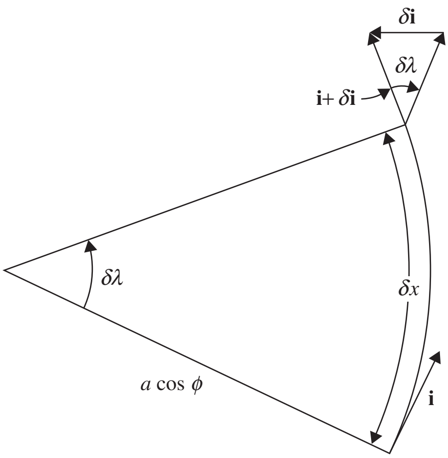

FIGURE 2.1 Longitudinal dependence of the unit vector **i**.

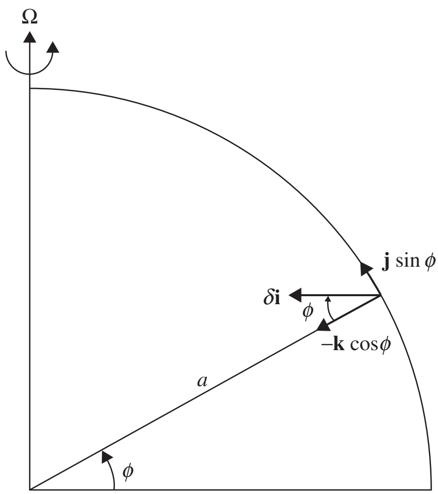

FIGURE 2.2 Resolution of $\delta\mathbf{i}$ in Figure 2.1 into northward and vertical components.

2.2 THE VECTORIAL FORM OF THE MOMENTUM EQUATION IN ROTATING COORDINATES

In an inertial reference frame, Newton's second law of motion may be written symbolically as

$$\frac{D_a \mathbf{U}_a}{Dt} = \sum \mathbf{F} \tag{2.3}$$

The left side represents the rate of change of the absolute velocity $\mathbf{U}_a$, following the motion as viewed in an inertial system. The right side represents the sum of the real forces acting per unit mass. In Section 1.3 we found through simple physical reasoning that when the motion is viewed in a rotating coordinate system, certain additional apparent forces must be included if Newton's second law is to be valid. The same result may be obtained by a formal transformation of coordinates in (2.3).

In order to transform this expression into rotating coordinates, we must first find a relationship between $\mathbf{U}_a$ and the velocity relative to the rotating system, which we will designate $\mathbf{U}$. This relationship is obtained by applying (2.2) to the position vector $\mathbf{r}$ for an air parcel on rotating Earth:

$$\frac{D_a\mathbf{r}}{Dt} = \frac{D\mathbf{r}}{Dt} + \boldsymbol{\Omega} \times \mathbf{r} \tag{2.4}$$

but $D_a\mathbf{r}/Dt \equiv \mathbf{U}_a$ and $D\mathbf{r}/Dt \equiv \mathbf{U}$; therefore, (2.4) may be written as

$$\mathbf{U}_a = \mathbf{U} + \boldsymbol{\Omega} \times \mathbf{r} \tag{2.5}$$

which states simply that the absolute velocity of an object on rotating Earth is equal to its velocity relative to Earth plus the velocity due to the rotation of Earth.

Now we apply (2.2) to the velocity vector $\mathbf{U}_a$ and obtain

$$\frac{D_a\mathbf{U}_a}{Dt} = \frac{D\mathbf{U}_a}{Dt} + \boldsymbol{\Omega} \times \mathbf{U}_a \tag{2.6}$$

Substituting from (2.5) into the right side of (2.6) gives

$$\begin{aligned}\frac{D_a\mathbf{U}_a}{Dt} &= \frac{D}{Dt}(\mathbf{U} + \boldsymbol{\Omega} \times \mathbf{r}) + \boldsymbol{\Omega} \times (\mathbf{U} + \boldsymbol{\Omega} \times \mathbf{r})\\ &= \frac{D\mathbf{U}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{U} - \Omega^2\mathbf{R}\end{aligned} \tag{2.7}$$

where $\boldsymbol{\Omega}$ is assumed to be constant. Here $\mathbf{R}$ is a vector perpendicular to the axis of rotation, with magnitude equal to the distance to the axis of rotation, so that with the aid of a vector identity,

$$\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{R}) = -\Omega^2\mathbf{R}$$

Equation (2.7) states that the acceleration following the motion in an inertial system equals the rate of change in relative velocity following the relative motion in the rotating frame plus the Coriolis acceleration due to relative motion in the rotating frame plus the centripetal acceleration caused by the rotation of the coordinates.

If we assume that the only real forces acting on the atmosphere are the pressure gradient force, gravitation, and friction, we can rewrite Newton's second law (2.3) with the aid of (2.7) as

$$\frac{D\mathbf{U}}{Dt} = -2\boldsymbol{\Omega} \times \mathbf{U} - \frac{\mathbf{1}}{\rho}\nabla p + \mathbf{g} + \mathbf{F}_r \tag{2.8}$$

where $\mathbf{F}_r$ designates the frictional force (see Section 1.2.2), and the centrifugal force has been combined with gravitation in the gravity term $\mathbf{g}$ (see Section 1.3.2). Equation (2.8) is the statement of Newton's second law for motion relative to a rotating coordinate frame. It states that the acceleration following the relative motion in the rotating frame equals the sum of the Coriolis force, the pressure gradient force, effective gravity, and friction. This form of the momentum equation is basic to most work in dynamic meteorology.

2.3 COMPONENT EQUATIONS IN SPHERICAL COORDINATES

For purposes of theoretical analysis and numerical prediction, it is necessary to expand the vectorial momentum equation (2.8) into its scalar components. Since the departure of the shape of Earth from sphericity is entirely negligible for meteorological purposes, it is convenient to expand (2.8) in spherical coordinates so that the (level) surface of Earth corresponds to a coordinate surface. The coordinate axes are then (λ, ϕ, z), where λ is longitude, ϕ is latitude, and z is the vertical distance above the surface of Earth. If the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are now taken to be directed eastward, northward, and upward, respectively, the relative velocity becomes

$$\mathbf{U} \equiv \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$$

where the components u, v, and w are defined as

$$u \equiv r\cos\phi\,\frac{D\lambda}{Dt}, \quad v \equiv r\frac{D\phi}{Dt}, \quad w \equiv \frac{Dz}{Dt} \tag{2.9}$$

Here, r is the distance to the center of Earth, which is related to z by $r = a + z$, where a is the radius of Earth. Traditionally, the variable r in (2.9) is replaced by the constant a. This is a very good approximation, since $z \ll a$ for the regions of the atmosphere with which meteorologists are concerned.

For notational simplicity, it is conventional to define x and y as eastward and northward distance, such that $Dx = a\cos\phi\, D\lambda$ and $Dy = a\,D\phi$. Thus, the horizontal velocity components are $u \equiv Dx/Dt$ and $v \equiv Dy/Dt$ in the eastward and northward directions, respectively. The (x, y, z) coordinate system defined in this way is not, however, a Cartesian coordinate system because the directions of the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ unit vectors are not constant but are functions of position on spherical Earth. This position dependence of the unit vectors must be taken into account when the acceleration vector is expanded into its components on the sphere. Thus, we write

$$\frac{D\mathbf{U}}{Dt} = \mathbf{i}\frac{Du}{Dt} + \mathbf{j}\frac{Dv}{Dt} + \mathbf{k}\frac{Dw}{Dt} + u\frac{D\mathbf{i}}{Dt} + v\frac{D\mathbf{j}}{Dt} + w\frac{D\mathbf{k}}{Dt} \tag{2.10}$$

To obtain the component equations, it is necessary first to evaluate the rates of change of the unit vectors following the motion. We first consider $D\mathbf{i}/Dt$. Expanding the total derivative as in (2.1) and noting that $\mathbf{i}$ is a function only of x (i.e., an eastward-directed vector does not change its orientation if the motion is in the north–south or vertical directions), we get

$$\frac{D\mathbf{i}}{Dt} = u\frac{\partial \mathbf{i}}{\partial x}$$

From Figure 2.1 we see by similarity of triangles,

$$\lim_{\delta x \to 0} \frac{|\delta \mathbf{i}|}{\delta x} = \left|\frac{\partial \mathbf{i}}{\partial x}\right| = \frac{1}{a\cos\phi}$$

and that the vector $\partial\mathbf{i}/\partial x$ is directed toward the axis of rotation. Thus, as is illustrated in Figure 2.2,

$$\frac{\partial \mathbf{i}}{\partial x} = \frac{1}{a\cos\phi}(\mathbf{j}\,\sin\phi - \mathbf{k}\cos\phi)$$

Therefore,

$$\frac{D\mathbf{i}}{Dt} = \frac{u}{a\cos\phi}(\mathbf{j}\,\sin\phi - \mathbf{k}\cos\phi) \tag{2.11}$$

Considering now $D\mathbf{j}/Dt$, we note that $\mathbf{j}$ is a function only of x and y. Thus, with the aid of Figure 2.3 we see that for eastward motion $|\delta\mathbf{j}| = \delta x/(a/\tan\phi)$. Because the vector $\partial\mathbf{j}/\partial x$ is directed in the negative x direction, we then have

$$\frac{\partial \mathbf{j}}{\partial x} = -\frac{\tan\phi}{a}\mathbf{i}$$

From Figure 2.4 it is clear that for northward motion $|\delta\mathbf{j}| = \delta\phi$, but $\delta y = a\delta\phi$, and $\delta\mathbf{j}$ is directed downward so that

$$\frac{\partial \mathbf{j}}{\partial y} = -\frac{\mathbf{k}}{a}$$

Hence,

$$\frac{D\mathbf{j}}{Dt} = -\frac{u\tan\phi}{a}\,\mathbf{i} - \frac{v}{a}\,\mathbf{k} \tag{2.12}$$

Finally, by similar arguments it can be shown that

$$\frac{D\mathbf{k}}{Dt} = \mathbf{i}\frac{u}{a} + \mathbf{j}\frac{v}{a} \tag{2.13}$$

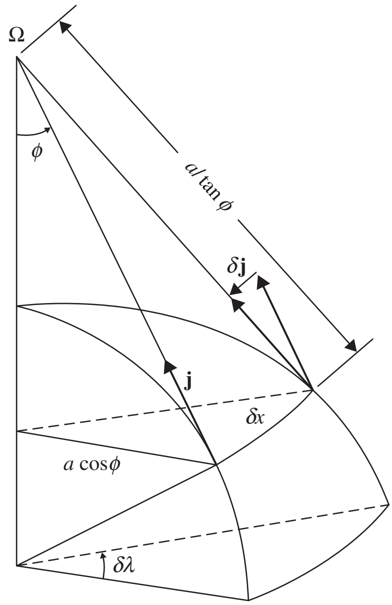

FIGURE 2.3 Dependence of unit vector **j** on longitude.

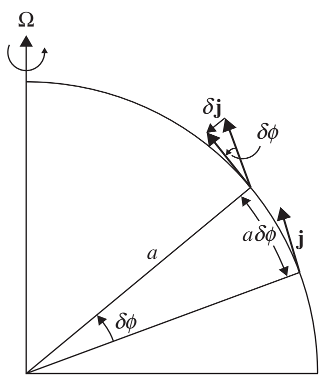

FIGURE 2.4 Dependence of unit vector **j** on latitude.

Substituting (2.11), (2.12), and (2.13) into (2.10) and rearranging the terms, we obtain the spherical polar coordinate expansion of the acceleration following the relative motion

$$\frac{D\mathbf{U}}{Dt} = \left(\frac{Du}{Dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a}\right)\mathbf{i} + \left(\frac{Dv}{Dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a}\right)\mathbf{j} + \left(\frac{Dw}{Dt} - \frac{u^2+v^2}{a}\right)\mathbf{k} \tag{2.14}$$

We next turn to the component expansion of the force terms in (2.8). The Coriolis force is expanded by noting that $\mathbf{\Omega}$ has no component parallel to $\mathbf{i}$ and that its components parallel to $\mathbf{j}$ and $\mathbf{k}$ are $2\Omega\cos\phi$ and $2\Omega\sin\phi$, respectively. Thus, using the definition of the vector cross-product,

$$\begin{aligned} -2\mathbf{\Omega}\times\mathbf{U} &= -2\Omega\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & \cos\phi & \sin\phi\\ u & v & w\end{vmatrix} \\ &= -(2\Omega w\cos\phi - 2\Omega v\sin\phi)\,\mathbf{i} - 2\Omega u\sin\phi\,\mathbf{j} + 2\Omega u\cos\phi\,\mathbf{k} \end{aligned} \tag{2.15}$$

The pressure gradient may be expressed as

$$\nabla p = \mathbf{i}\frac{\partial p}{\partial x} + \mathbf{j}\frac{\partial p}{\partial y} + \mathbf{k}\frac{\partial p}{\partial z} \tag{2.16}$$

and gravity is conveniently represented as

$$\mathbf{g} = -g\,\mathbf{k} \tag{2.17}$$

where g is a positive scalar ($g \cong 9.8$ m s^{-2} at Earth's surface). Finally, recall from Section 1.2.2 that

$$\mathbf{F}_r = \mathbf{i}F_{rx} + \mathbf{j}F_{ry} + \mathbf{k}F_{rz} \tag{2.18}$$

Substituting (2.14) through (2.18) into the equation of motion (2.8) and equating all terms in the $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ directions, respectively, we obtain

$$\frac{Du}{Dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v\sin\phi - 2\Omega w\cos\phi + F_{rx} \tag{2.19}$$

$$\frac{Dv}{Dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u\sin\phi + F_{ry} \tag{2.20}$$

$$\frac{Dw}{Dt} - \frac{u^2+v^2}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + 2\Omega u\cos\phi + F_{rz} \tag{2.21}$$

which are the eastward, northward, and vertical component momentum equations, respectively. The terms proportional to $1/a$ on the left sides in (2.19), (2.20), and (2.21) are called *curvature* terms; they are due to the curvature the Earth.[1] Because they are nonlinear (i.e., they are quadratic in the dependent variables), they are difficult to handle in theoretical analyses. Fortunately, as shown in the next section, the curvature terms are unimportant for midlatitude synoptic-scale motions. However, even when the curvature terms are neglected,

[1] It can be shown that when r is replaced by a, as here (the traditional approximation) the Coriolis terms proportional to $\cos\phi$ in (2.19) and (2.21) must be neglected if the equations are to satisfy angular momentum conservation.

(2.19), (2.20), and (2.21) are still nonlinear partial differential equations, as can be seen by expanding the total derivatives into their local and advective parts:

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

with similar expressions for Dv/Dt and Dw/Dt. In general, the advective acceleration terms are comparable in magnitude to the local acceleration term. The presence of nonlinear advection processes is one reason that dynamic meteorology is an interesting and challenging subject.

2.4 SCALE ANALYSIS OF THE EQUATIONS OF MOTION

Section 1.6 discussed the basic notion of scaling the equations of motion in order to determine whether some terms in the equations are negligible for motions of meteorological concern. Elimination of terms on scaling considerations have the advantage of simplifying the mathematics; as shown in later chapters, the elimination of small terms in some cases has the very important property of completely eliminating or *filtering* an unwanted type of motion. The complete equations of motion—(2.19), (2.20), and (2.21)—describe all types and scales of atmospheric motions. Sound waves, for example, are a perfectly valid class of solutions to these equations. However, sound waves are of negligible importance in dynamical meteorology. Therefore, it will be a distinct advantage if, as turns out to be true, we can neglect the terms that lead to the production of sound waves and filter out this unwanted class of motions.

To simplify (2.19), (2.20), and (2.21) for synoptic-scale motions, we define the following characteristic scales of the field variables based on observed values for midlatitude synoptic systems.

$U \sim 10\,\mathrm{m\,s^{-1}}$	horizontal velocity scale
$W \sim 1\,\mathrm{cm\,s^{-1}}$	vertical velocity scale
$L \sim 10^6\,\mathrm{m}$	length scale [$\sim 1/(2\pi)$ wavelength]
$H \sim 10^4\,\mathrm{m}$	depth scale
$\delta P/\rho \sim 10^3\,\mathrm{m^2\,s^{-2}}$	horizontal pressure fluctuation scale
$L/U \sim 10^5\,\mathrm{s}$	time scale

Horizontal pressure fluctuation δP is normalized by the density ρ in order to produce a scale estimate that is valid at all heights in the troposphere, despite the approximate exponential decrease with height of both δP and ρ. Note that $\delta P/\rho$ has units of a geopotential. Referring back to (1.31), we see that indeed the magnitude of the fluctuation of $\delta P/\rho$ on a surface of constant height must equal the magnitude of the fluctuation of the geopotential on an isobaric surface. The time scale here is an advective one, which is appropriate for pressure systems

TABLE 2.1 Scale Analysis of the Horizontal Momentum Equations

	A	B	C	D	E	F	G
x − Eq.	$\frac{Du}{Dt}$	$-2\Omega v \sin\phi$	$+2\Omega w \cos\phi$	$+\frac{uw}{a}$	$-\frac{uv\tan\phi}{a}$	$= -\frac{1}{\rho}\frac{\partial p}{\partial x}$	$+F_{rx}$
y − Eq.	$\frac{Dv}{Dt}$	$+2\Omega u \sin\phi$		$+\frac{vw}{a}$	$+\frac{u^2\tan\phi}{a}$	$= -\frac{1}{\rho}\frac{\partial p}{\partial y}$	$+F_{ry}$
Scales	U^2/L	$f_0 U$	$f_0 W$	$\frac{UW}{a}$	$\frac{U^2}{a}$	$\frac{\delta P}{\rho L}$	$\frac{\nu U}{H^2}$
(m s^{-2})	10^{-4}	10^{-3}	10^{-6}	10^{-8}	10^{-5}	10^{-3}	10^{-12}

that move at approximately the speed of the horizontal wind, as is observed for synoptic-scale motions. Thus, L/U is the time required to travel a distance L at a speed U, and the substantial differential operator scales as $D/Dt \sim U/L$ for such motions.

It should be pointed out here that the synoptic-scale vertical velocity is not a directly measurable quantity. However, as shown in Chapter 3, the magnitude of w can be deduced from knowledge of the horizontal velocity field.

We can now estimate the magnitude of each term in (2.19) and (2.20) for synoptic-scale motions at a given latitude. It is convenient to consider a disturbance centered at latitude $\phi_0 = 45°$ and introduce the notation

$$f_0 = 2\Omega \sin\phi_0 = 2\Omega \cos\phi_0 \cong 10^{-4}\text{s}^{-1}$$

Table 2.1 shows the characteristic magnitude of each term in (2.19) and (2.20) based on the scaling considerations given before. The molecular friction term is so small that it may be neglected for all motions except the smallest-scale turbulent motions near the ground, where vertical wind shears shear can become very large and the molecular friction term must be retained, as discussed in Chapter 8.

2.4.1 Geostrophic Approximation and Geostrophic Wind

It is apparent from Table 2.1 that for midlatitude synoptic-scale disturbances the Coriolis force (term B) and the pressure gradient force (term F) are in approximate balance. Retaining only these two terms in (2.19) and (2.20) gives as a first approximation the *geostrophic* relationship

$$-fv \approx -\frac{1}{\rho}\frac{\partial p}{\partial x}; \qquad fu \approx -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{2.22}$$

where $f \equiv 2\Omega \sin\phi$ is called the *Coriolis parameter*. The geostrophic balance is a *diagnostic* expression that gives the approximate relationship between the pressure field and horizontal velocity in large-scale extratropical systems. The approximation (2.22) contains no reference to time and therefore cannot be used to predict the evolution of the velocity field. It is for this reason that the geostrophic relationship is called a diagnostic relationship.

By analogy to the geostrophic approximation (2.22), it is possible to define a horizontal velocity field, $\mathbf{V}_g \equiv \mathbf{i}u_g + \mathbf{j}v_g$, called the *geostrophic wind*, that satisfies (2.22) identically. In vectorial form,

$$\mathbf{V}_g \equiv \mathbf{k} \times \frac{1}{\rho f} \nabla p \tag{2.23}$$

Thus, knowledge of the pressure distribution at any time determines the geostrophic wind. It should be kept clearly in mind that (2.23) always defines the geostrophic wind; however, only for large-scale motions away from the equator should the geostrophic wind be used as an approximation to the actual horizontal wind field. For the scales used in Table 2.1, the geostrophic wind approximates the true horizontal velocity to within 10 to 15% in midlatitudes.

2.4.2 Approximate Prognostic Equations: The Rossby Number

To obtain prediction equations, it is necessary to retain the acceleration (term A) in (2.19) and (2.20). The resulting approximate horizontal momentum equations are

$$\frac{Du}{Dt} = fv - \frac{1}{\rho}\frac{\partial p}{\partial x} = f\left(v - v_g\right) = fv_a \tag{2.24}$$

$$\frac{Dv}{Dt} = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y} = -f\left(u - u_g\right) = -fu_a \tag{2.25}$$

where (2.23) is used to rewrite the pressure gradient force in terms of the geostrophic wind. Because the acceleration terms in (2.24) and (2.25) are proportional to the difference between the actual wind and the geostrophic wind, they are about an order of magnitude smaller than the Coriolis force and the pressure gradient force, in agreement with our scale analysis. In the final equalities of (2.24) and (2.25) we define the difference between the actual wind and the geostrophic wind as the *ageostrophic wind.*

The fact that the horizontal flow is in approximate geostrophic balance is helpful for diagnostic analysis. However, it makes actual applications of these equations in weather prognosis difficult because acceleration (which must be measured accurately) is given by the small difference between two large terms. Thus, a small error in measurement of either velocity or pressure gradient will lead to very large errors in estimating the acceleration. This problem of numerical weather prediction is discussed in Chapter 13, and the problem of simplifying the equations for physical understanding based on the smallness of the ageostrophic wind will be taken up in Chapter 6.

A convenient measure of the magnitude of the acceleration compared to the Coriolis force may be obtained by forming the ratio of the characteristic scales for the acceleration and Coriolis force terms: $(U^2/L)/(f_0U)$. This ratio is a nondimensional number called the *Rossby number* after the Swedish meteorologist C. G. Rossby (1898–1957) and is designated by

$$\mathrm{Ro} \equiv U/(f_0L)$$

Thus, the smallness of the Rossby number is a measure of the validity of the geostrophic approximation.

2.4.3 The Hydrostatic Approximation

A similar scale analysis can be applied to the vertical component of the momentum equation (2.21). Because pressure decreases by about an order of magnitude from the ground to the tropopause, the vertical pressure gradient may be scaled by P_0/H, where P_0 is the surface pressure and H is the depth of the troposphere. The terms in (2.21) may then be estimated for synoptic-scale motions and are shown in Table 2.2. As with the horizontal component equations, we consider motions centered at 45° latitude and neglect friction. The scaling indicates that to a high degree of accuracy the pressure field is in *hydrostatic equilibrium*; that is, the pressure at any point is simply equal to the weight of a unit cross-section column of air above that point.

The preceding analysis of the vertical momentum equation is, however, somewhat misleading. It is not sufficient to show merely that the vertical acceleration is small compared to g. Because only that part of the pressure field that varies horizontally is directly coupled to the horizontal velocity field, it is actually necessary to show that the horizontally varying pressure component is itself in hydrostatic equilibrium with the horizontally varying density field. To show this, it is convenient to first define a standard pressure $p_0(z)$, which is the horizontally averaged pressure at each height, and a corresponding standard density $\rho_0(z)$, defined so that $p_0(z)$ and $\rho_0(z)$ are in *exact* hydrostatic balance:

$$\frac{1}{\rho_0}\frac{dp_0}{dz} \equiv -g \tag{2.26}$$

TABLE 2.2 Scale Analysis of the Vertical Momentum Equation

$z-$ Eq.	Dw/Dt	$-2\Omega u\cos\phi$	$-(u^2+v^2)/a$	$= -\rho^{-1}\partial p/\partial z$	$-g$	$+\mathrm{F}_{rz}$
Scales	UW/L	f_0U	U^2/a	$P_0/(\rho H)$	g	νWH^{-2}
m s^{-2}	10^{-7}	10^{-3}	10^{-5}	10	10	10^{-15}

We may then write the total pressure and density fields as

$$\begin{aligned} p(x, y, z, t) &= p_0(z) + p'(x, y, z, t) \\ \rho(x, y, z, t) &= \rho_0(z) + \rho'(x, y, z, t) \end{aligned} \tag{2.27}$$

where p' and ρ' are deviations from the standard values of pressure and density, respectively. For an atmosphere at rest, p' and ρ' would thus be zero. Using the definitions of (2.26) and (2.27) and assuming that ρ'/ρ_0 is much less than unity in magnitude so that $(\rho_0 + \rho')^{-1} \cong \rho_0^{-1}\,(1 - \rho'/\rho_0)$, we find that

$$\begin{aligned} -\frac{1}{\rho}\frac{\partial p}{\partial z} - g &= -\frac{1}{(\rho_0 + \rho')}\frac{\partial}{\partial z}\left(p_0 + p'\right) - g \\ &\approx \frac{1}{\rho_0}\left[\frac{\rho'}{\rho_0}\frac{dp_0}{dz} - \frac{\partial p'}{\partial z}\right] = -\frac{1}{\rho_0}\left[\rho' g + \frac{\partial p'}{\partial z}\right] \end{aligned} \tag{2.28}$$

For synoptic-scale motions, the terms in (2.28) have the magnitudes

$$\frac{1}{\rho_0}\frac{\partial p'}{\partial z} \sim \left[\frac{\delta P}{\rho_0 H}\right] \sim 10^{-1}\,\mathrm{m\,s^{-2}}, \qquad \frac{\rho' g}{\rho_0} \sim 10^{-1}\,\mathrm{m\,s^{-2}}$$

Comparing these with the magnitudes of other terms in the vertical momentum equation (refer to Table 2.2), we see that to a very good approximation the perturbation pressure field is in hydrostatic equilibrium with the perturbation density field so that

$$\frac{\partial p'}{\partial z} + \rho' g = 0 \tag{2.29}$$

Therefore, for synoptic-scale motions, vertical accelerations are negligible and the vertical velocity cannot be determined from the vertical momentum equation. However, we show in Chapter 3 that it is, nevertheless, possible to deduce the vertical motion field indirectly. Moreover, in Chapter 6 we show that the vertical motion field may be estimated, and physically understood, from knowledge of the pressure field alone.

2.5 THE CONTINUITY EQUATION

We turn now to the second of the three fundamental conservation principles: conservation of mass. The mathematical relationship that expresses conservation of mass for a fluid is called the *continuity equation*. This section develops the continuity equation using two alternative methods. The first method is based on an Eulerian control volume, whereas the second is based on a Lagrangian control volume.

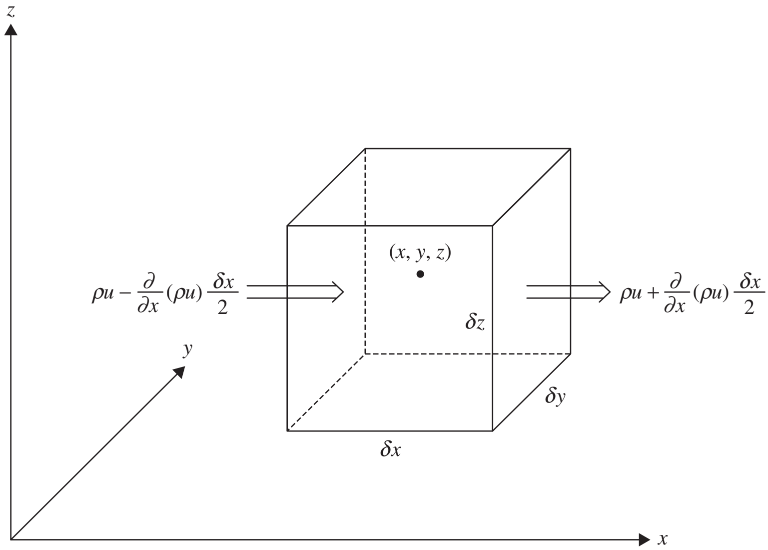

FIGURE 2.5 Mass inflow into a fixed (Eulerian) control volume as a result of motion parallel to the x axis.

2.5.1 A Eulerian Derivation

We consider a volume element $\delta x \delta y \delta z$ that is fixed in a Cartesian coordinate frame as shown in Figure 2.5. For such a fixed control volume, the net rate of mass inflow through the sides must equal the rate of accumulation of mass within the volume. The rate of inflow of mass through the left face per unit area is

$$\left[\rho u - \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}\right]$$

whereas the rate of outflow per unit area through the right face is

$$\left[\rho u + \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}\right]$$

Because the area of each of these faces is $\delta y \delta z$, the net rate of flow into the volume due to the x velocity component is

$$\left[\rho u - \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}\right]\delta y \delta z - \left[\rho u + \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}\right]\delta y \delta z = -\frac{\partial}{\partial x}(\rho u)\delta x \delta y \delta z$$

Similar expressions obviously hold for the y and z directions. Thus, the net rate of mass inflow is

$$-\left[\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)\right]\delta x\delta y\delta z$$

and the mass inflow per unit volume is just $-\nabla \cdot (\rho \mathbf{U})$, which must equal the rate of mass increase per unit volume. Now the increase of mass per unit volume is just the local density change $\partial\rho/\partial t$. Therefore,

$$\frac{\partial \rho}{\partial t}+\nabla \cdot (\rho \mathbf{U})=0 \tag{2.30}$$

Equation (2.30) is the mass divergence form of the continuity equation.

An alternative form of the continuity equation is obtained by applying the vector identity

$$\nabla \cdot (\rho \mathbf{U}) \equiv \rho \nabla \cdot \mathbf{U}+\mathbf{U} \cdot \nabla \rho$$

and the relationship

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t}+\mathbf{U} \cdot \nabla$$

to get

$$\frac{1}{\rho}\frac{D\rho}{Dt}+\nabla \cdot \mathbf{U}=0 \tag{2.31}$$

Equation (2.31) is the velocity divergence form of the continuity equation. It states that the fractional rate of increase of the density *following the motion* of an air parcel is equal to minus the velocity divergence (i.e., convergence). This should be clearly distinguished from (2.30), which states that the *local* rate of change of density is equal to minus the mass divergence.

2.5.2 A Lagrangian Derivation

The physical meaning of divergence can be illustrated by the following alternative derivation of (2.31). Consider a control volume of fixed mass δM that moves with the fluid. Letting $\delta V=\delta x\delta y\delta z$ be the volume, we find that because $\delta M=\rho\delta V=\rho\delta x\delta y\delta z$ is conserved following the motion, we can write

$$\frac{1}{\delta M}\frac{D}{Dt}(\delta M)=\frac{1}{\rho\delta V}\frac{D}{Dt}(\rho\delta V)=\frac{1}{\rho}\frac{D\rho}{Dt}+\frac{1}{\delta V}\frac{D}{Dt}(\delta V)=0 \tag{2.32}$$

but

$$\frac{1}{\delta V}\frac{D}{Dt}(\delta V)=\frac{1}{\delta x}\frac{D}{Dt}(\delta x)+\frac{1}{\delta y}\frac{D}{Dt}(\delta y)+\frac{1}{\delta z}\frac{D}{Dt}(\delta z)$$

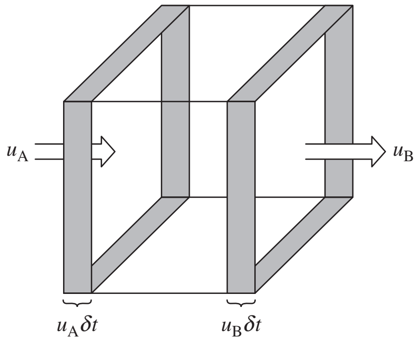

FIGURE 2.6 Change in Lagrangian control volume (shown by shading) as a result of fluid motion parallel to the x axis.

Referring to Figure 2.6, we see that the faces of the control volume in the y, z plane (designated A and B) are advected with the flow in the x direction at speeds $u_A = Dx/Dt$ and $u_B = D(x + \delta x)/Dt$, respectively. Thus, the difference in speeds of the two faces is $\delta u = u_B - u_A = D(x + \delta x)/Dt - Dx/Dt$ or $\delta u = D(\delta x)/Dt$. Similarly, $\delta v = D(\delta y)/Dt$ and $\delta w = D(\delta z)/Dt$. Therefore,

$$\lim_{\delta x, \delta y, \delta z \to 0} \left[\frac{1}{\delta V} \frac{D}{Dt}(\delta V) \right] = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \mathbf{U}$$

so that in the limit $\delta V \to 0$, (2.32) reduces to the continuity equation (2.31); the divergence of the three-dimensional velocity field is equal to the fractional rate of change of the volume of a fluid parcel in the limit $\delta V \to 0$. It is left as a problem for the student to show that the divergence of the *horizontal* velocity field is equal to the fractional rate of change of the horizontal area δA of a fluid parcel in the limit $\delta A \to 0$.

2.5.3 Scale Analysis of the Continuity Equation

Following the technique developed in Section 2.4.3, and again assuming that $|\rho'/\rho_0| \ll 1$, we can approximate the continuity (2.31) as

$$\underset{\text{A}}{\frac{1}{\rho_0}\left(\frac{\partial \rho'}{\partial t} + \mathbf{U} \cdot \nabla \rho'\right)} + \underset{\text{B}}{\frac{w}{\rho_0}\frac{d\rho_0}{dz}} + \underset{\text{C}}{\nabla \cdot \mathbf{U}} \approx 0 \tag{2.33}$$

where ρ' designates the local deviation of density from its horizontally averaged value, $\rho_0(z)$. For synoptic-scale motions, $\rho'/\rho_0 \sim 10^{-2}$ so that, using the characteristic scales given in Section 2.4, we find that term A has magnitude

$$\frac{1}{\rho_0}\left(\frac{\partial \rho'}{\partial t} + \mathbf{U} \cdot \nabla \rho'\right) \sim \frac{\rho'}{\rho_0}\frac{U}{L} \approx 10^{-7}\mathrm{s}^{-1}$$

For motions in which the depth scale H is comparable to the density scale height, $d \ln \rho_0/dz \sim H^{-1}$, so that term B scales as

$$\frac{w}{\rho_0}\frac{d\rho_0}{dz} \backsim \frac{W}{H} \approx 10^{-6}\text{s}^{-1}$$

Expanding term C in Cartesian coordinates, we have

$$\nabla \cdot \mathbf{U} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

For synoptic-scale motions, the terms $\partial u/\partial x$ and $\partial v/\partial y$ tend to be of equal magnitude but opposite sign. Thus, they tend to balance so that

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \backsim 10^{-1}\frac{U}{L} \approx 10^{-6}\,\text{s}^{-1}$$

In addition,

$$\frac{\partial w}{\partial z} \sim \frac{W}{H} \approx 10^{-6}\,\text{s}^{-1}$$

Thus, terms B and C are each an order of magnitude greater than term A, and, to a first approximation, terms B and C balance in the continuity equation. To a good approximation, then

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + w\frac{d}{dz}(\ln \rho_0) = 0$$

or, alternatively, in vector form

$$\nabla \cdot (\rho_0 \mathbf{U}) = 0 \tag{2.34}$$

Thus, for synoptic-scale motions the mass flux computed using the basic state density ρ_0 is nondivergent. This approximation is similar to the idealization of incompressibility, which is often used in fluid mechanics. However, an *incompressible* fluid has a density constant following the motion

$$\frac{D\rho}{Dt} = 0$$

Thus, by (2.31) the velocity divergence vanishes ($\nabla \cdot \mathbf{U} = 0$) in an incompressible fluid, which is not the same as (2.34). Our approximation (2.34) shows that for purely horizontal flow the atmosphere behaves as though it were an incompressible fluid. However, when there is vertical motion, the compressibility associated with the height dependence of ρ_0 must be taken into account.

2.6 THE THERMODYNAMIC ENERGY EQUATION

We now turn to the third fundamental conservation principle: the conservation of energy as applied to a moving fluid element. The first law of thermodynamics is usually derived by considering a system in thermodynamic equilibrium—that is, a system that is initially at rest and, after exchanging heat with its surroundings and doing work on the surroundings, is again at rest. For such a system the first law states that *the change in internal energy of the system is equal to the difference between the heat added to the system and the work done by the system*. A Lagrangian control volume consisting of a specified mass of fluid may be regarded as a thermodynamic system. However, unless the fluid is at rest, it will not be in thermodynamic equilibrium. Nevertheless, the first law of thermodynamics still applies.

To show that this is the case, we note that the total thermodynamic energy of the control volume is considered to consist of the sum of the internal energy (due to the kinetic energy of the individual molecules) and the kinetic energy due to the macroscopic motion of the fluid. The rate of change of this total thermodynamic energy is equal to the rate of diabatic heating plus the rate at which work is done on the fluid parcel by external forces.

If we let e designate the internal energy per unit mass, then the total thermodynamic energy contained in a Lagrangian fluid element of density ρ and volume δV is $\rho\,[e + (1/2)\mathbf{U}\cdot\mathbf{U}]\,\delta V$. The external forces that act on a fluid element may be divided into surface forces, such as pressure and viscosity, and body forces, such as gravity or the Coriolis force. The rate at which work is done on the fluid element by the x component of the pressure force is illustrated in Figure 2.7. Recalling that pressure is a force per unit area and that the rate at which a force does work is given by the dot product of the force and velocity

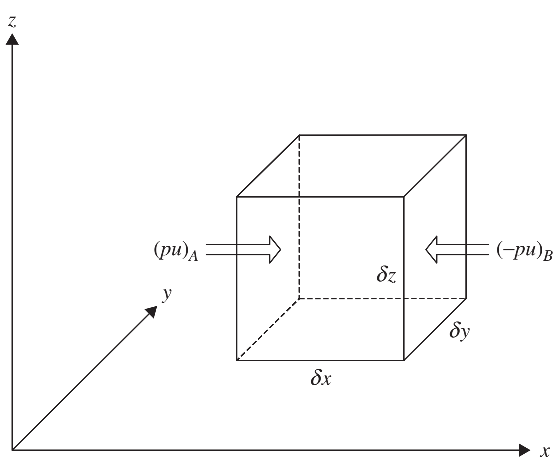

FIGURE 2.7 Rate of working on a fluid element due to the x component of the pressure force.

vectors, we see that the rate at which the surrounding fluid does work on the element due to the pressure force on the two boundary surfaces in the y, z plane is given by

$$(pu)_A \delta y \delta z - (pu)_B \delta y \delta z$$

(The negative sign is needed before the second term because the work done on the fluid element is positive if u is negative across face B.) Now, by expanding in a Taylor series, we can write

$$(pu)_B = (pu)_A + \left[\frac{\partial}{\partial x}(pu)\right]_A \delta x + \cdots$$

Thus, the net rate at which the pressure force does work due to the x component of motion is

$$\left[(pu)_A - (pu)_B\right]\delta y \delta z = -\left[\frac{\partial}{\partial x}(pu)\right]_A \delta V$$

where $\delta V = \delta x \delta y \delta z$.

Similarly, we can show that the net rates at which the pressure force does work due to the y and z components of motion are

$$-\left[\frac{\partial}{\partial y}(pv)\right]\delta V \text{ and } -\left[\frac{\partial}{\partial z}(pw)\right]\delta V$$

respectively. Hence, the total rate at which work is done by the pressure force is simply

$$-\boldsymbol{\nabla} \cdot (p\mathbf{U})\delta V$$

The only body forces of meteorological significance that act on an element of mass in the atmosphere are the Coriolis force and gravity. However, because the Coriolis force, $-2\boldsymbol{\Omega} \times \mathbf{U}$, is perpendicular to the velocity vector, it can do no work. Thus, the rate at which body forces do work on the mass element is just $\rho \mathbf{g} \cdot \mathbf{U}\, \delta V$.

Applying the principle of energy conservation to our Lagrangian control volume (neglecting effects of molecular viscosity), we thus obtain

$$\frac{D}{Dt}\left[\rho\left(e + \frac{1}{2}\mathbf{U}\cdot\mathbf{U}\right)\delta V\right] = -\boldsymbol{\nabla}\cdot(p\mathbf{U})\delta V + \rho\mathbf{g}\cdot\mathbf{U}\delta V + \rho J \delta V \qquad (2.35)$$

Here J is the rate of heating per unit mass due to radiation, conduction, and latent heat release. With the aid of the chain rule of differentiation, we can rewrite (2.35) as

$$\begin{aligned}&\rho\delta V \frac{D}{Dt}\left(e + \frac{1}{2}\mathbf{U}\cdot\mathbf{U}\right) + \left(e + \frac{1}{2}\mathbf{U}\cdot\mathbf{U}\right)\frac{D(\rho\delta V)}{Dt} \\ &= -\mathbf{U}\cdot\boldsymbol{\nabla}p\,\delta V - p\boldsymbol{\nabla}\cdot\mathbf{U}\,\delta V - \rho g w\,\delta V + \rho J\,\delta V\end{aligned} \qquad (2.36)$$

where we have used $\mathbf{g} = -g\mathbf{k}$. Now, from (2.32), the second term on the left in (2.36) vanishes so that

$$\rho\frac{De}{Dt} + \rho\frac{D}{Dt}\left(\frac{1}{2}\mathbf{U}\cdot\mathbf{U}\right) = -\mathbf{U}\cdot\nabla p - p\nabla\cdot\mathbf{U} - \rho g w + \rho J \tag{2.37}$$

This equation can be simplified by noting that if we take the dot product of $\mathbf{U}$ with the momentum equation (2.8), we obtain (neglecting friction)

$$\rho\frac{D}{Dt}\left(\frac{1}{2}\mathbf{U}\cdot\mathbf{U}\right) = -\mathbf{U}\cdot\nabla p - \rho g w \tag{2.38}$$

Subtracting (2.38) from (2.37), we obtain

$$\rho\frac{De}{Dt} = -p\nabla\cdot\mathbf{U} + \rho J \tag{2.39}$$

The terms in (2.37) that were eliminated by subtracting (2.38) represent the balance of mechanical energy due to the motion of the fluid element; the remaining terms represent the thermal energy balance.

Using the definition of geopotential (1.11), we have

$$gw = g\frac{Dz}{Dt} = \frac{D\Phi}{Dt}$$

so that (2.38) can be rewritten as

$$\rho\frac{D}{Dt}\left(\frac{1}{2}\mathbf{U}\cdot\mathbf{U} + \Phi\right) = -\mathbf{U}\cdot\nabla p \tag{2.40}$$

which is referred to as the *mechanical energy equation*. The sum of the kinetic energy plus the gravitational potential energy is called the *mechanical energy*. Thus, (2.40) states that following the motion, the rate of change of mechanical energy per unit volume equals the rate at which work is done by the pressure gradient force.

The thermal energy equation (2.39) can be written in more familiar form by noting from (2.31) that

$$\frac{1}{\rho}\nabla\cdot\mathbf{U} = -\frac{1}{\rho^2}\frac{D\rho}{Dt} = \frac{D\alpha}{Dt}$$

and that for dry air the internal energy per unit mass is given by $e = c_v T$, where $c_v (= 717\ \mathrm{J\ kg^{-1}\ K^{-1}})$ is the specific heat at constant volume. We then obtain

$$c_v\frac{DT}{Dt} + p\frac{D\alpha}{Dt} = J \tag{2.41}$$

which is the usual form of the thermodynamic energy equation. Thus, the first law of thermodynamics indeed is applicable to a fluid in motion. The second term on the left, representing the rate of working by the fluid system (per unit mass), represents a conversion between thermal and mechanical energy. This conversion process enables the solar heat energy to drive the motions of the atmosphere.

2.7 THERMODYNAMICS OF THE DRY ATMOSPHERE

Taking the total derivative of the equation of state (1.25), we obtain

$$p\frac{D\alpha}{Dt} + \alpha\frac{Dp}{Dt} = R\frac{DT}{Dt}$$

Substituting for $pD\alpha/Dt$ in (2.41) and using $c_p = c_v + R$, where $c_p(= 1004 \text{ J kg}^{-1} \text{ K}^{-1})$ is the specific heat at constant pressure, we can rewrite the first law of thermodynamics as

$$c_p\frac{DT}{Dt} - \alpha\frac{Dp}{Dt} = J \tag{2.42}$$

Dividing through by T and again using the equation of state, we obtain the entropy form of the first law of thermodynamics:

$$c_p\frac{D\ln T}{Dt} - R\frac{D\ln p}{Dt} = \frac{J}{T} \equiv \frac{Ds}{Dt} \tag{2.43}$$

Equation (2.43) gives the rate of change of entropy per unit mass following the motion for a thermodynamically *reversible* process. A reversible process is one in which a system changes its thermodynamic state and then returns to the original state without changing its surroundings. For such a process the entropy s defined by (2.43) is a field variable that depends only on the state of the fluid. Thus, Ds is a perfect differential, and Ds/Dt is to be regarded as a total derivative. However, "heat" is not a field variable, so the heating rate J is not a total derivative.[2]

2.7.1 Potential Temperature

For an ideal gas that is undergoing an *adiabatic* process (i.e., a reversible process in which no heat is exchanged with the surroundings), the first law of thermodynamics can be written in differential form as

$$c_pD\ln T - RD\ln p = D\left(c_p\ln T - R\ln p\right) = 0$$

Integrating this expression from a state at pressure p and temperature T to a state in which the pressure is p_s and the temperature is θ, we obtain, after taking the antilogarithm,

$$\theta = T\,(p_s/p)^{R/c_p} \tag{2.44}$$

[2]For a discussion of entropy and its role in the second law of thermodynamics, see Curry and Webster (1999), for example.

This relationship in (2.44) is referred to as Poisson's equation, and the temperature θ defined by (2.44) is called the *potential temperature*. θ is simply the temperature that a parcel of dry air at pressure p and temperature T would have if it were expanded or compressed adiabatically to a standard pressure p_s (usually taken to be 1000 hPa). Thus, every air parcel has a unique value of potential temperature, and this value is conserved for dry adiabatic motion. Because synoptic-scale motions are approximately adiabatic outside regions of active precipitation, θ is a quasi-conserved quantity for such motions.

Taking the logarithm of (2.44) and differentiating, we find that

$$c_p \frac{D \ln \theta}{Dt} = c_p \frac{D \ln T}{Dt} - R \frac{D \ln p}{Dt} \tag{2.45}$$

Comparing (2.43) and (2.45), we obtain

$$c_p \frac{D \ln \theta}{Dt} = \frac{J}{T} = \frac{Ds}{Dt} \tag{2.46}$$

Thus, for reversible processes, changes in fractional potential temperature are indeed proportional to entropy changes. A parcel that conserves entropy following the motion must move along an isentropic (constant θ) surface.

2.7.2 The Adiabatic Lapse Rate

A relationship between the *lapse rate* of temperature (i.e., the rate of *decrease* of temperature with respect to height) and the rate of change of potential temperature with respect to height can be obtained by taking the logarithm of (2.44) and differentiating with respect to height. Using the hydrostatic equation and the ideal gas law to simplify the result gives

$$\frac{T}{\theta} \frac{\partial \theta}{\partial z} = \frac{\partial T}{\partial z} + \frac{g}{c_p} \tag{2.47}$$

For an atmosphere in which the potential temperature is constant with respect to height, the lapse rate is thus

$$-\frac{dT}{dz} = \frac{g}{c_p} \equiv \Gamma_d \tag{2.48}$$

Thus, the dry adiabatic lapse rate is approximately constant throughout the lower atmosphere.

2.7.3 Static Stability

If potential temperature is a function of height, the atmospheric lapse rate, $\Gamma \equiv -\partial \mathrm{T}/\partial z$, will differ from the adiabatic lapse rate and

$$\frac{T}{\theta} \frac{\partial \theta}{\partial z} = \Gamma_d - \Gamma \tag{2.49}$$

If $\Gamma < \Gamma_d$ so that θ increases with height, an air parcel that undergoes an adiabatic displacement from its equilibrium level will be positively buoyant when displaced downward and negatively buoyant when displaced upward, so that it will tend to return to its equilibrium level and the atmosphere is said to be statically stable or *stably stratified*.

Adiabatic oscillations of a fluid parcel about its equilibrium level in a stably stratified atmosphere are referred to as *buoyancy oscillations*. The characteristic frequency of such oscillations can be derived by considering a parcel that is displaced vertically a small distance δz without disturbing its environment. If the environment is in hydrostatic balance, $\rho_0 g = -dp_0/dz$, where p_0 and ρ_0 are the pressure and density of the environment, respectively. The vertical acceleration of the parcel is

$$\frac{Dw}{Dt} = \frac{D^2}{Dt^2}(\delta z) = -g - \frac{1}{\rho}\frac{\partial p}{\partial z} \tag{2.50}$$

where p and ρ are the pressure and density of the parcel, respectively. In the parcel method it is assumed that the pressure of the parcel adjusts instantaneously to the environmental pressure during the displacement: $p = p_0$. This condition must be true if the parcel is to leave the environment undisturbed. Thus, with the aid of the hydrostatic relationship, pressure can be eliminated in (2.50) to give

$$\frac{D^2}{Dt^2}(\delta z) = g\left(\frac{\rho_0 - \rho}{\rho}\right) = g\frac{\theta}{\theta_0} \tag{2.51}$$

where (2.44) and the ideal gas law have been used to express the buoyancy force in terms of potential temperature. Here θ designates the deviation of the potential temperature of the parcel from its basic state (environmental) value $\theta_0(z)$. If the parcel is initially at level $z = 0$ where the potential temperature is $\theta_0(0)$, then for a small displacement δz, we can represent the environmental potential temperature as

$$\theta_0(\delta z) \approx \theta_0(0) + (d\theta_0/dz)\,\delta z$$

If the parcel displacement is adiabatic, the potential temperature of the parcel is conserved. Thus, $\theta(\delta z) = \theta_0(0) - \theta_0(\delta z) = -(d\theta_0/dz)\delta z$, and (2.51) becomes

$$\frac{D^2}{Dt^2}(\delta z) = -N^2\delta z \tag{2.52}$$

where

$$N^2 = g\frac{d\ln\theta_0}{dz}$$

is a measure of the static stability of the environment. Equation (2.52) has a general solution of the form $\delta z = A\exp(iNt)$. Therefore, if $N^2 > 0$, the parcel will oscillate about its initial level with a period $\tau = 2\pi/N$. The corresponding frequency N is the *buoyancy frequency*.[3] For average tropospheric conditions, $N \approx 1.2 \times 10^{-2}\ \mathrm{s}^{-1}$ so that the period of a buoyancy oscillation is about 8 min.

In the case of $N = 0$, examination of (2.52) indicates that no accelerating force will exist and the parcel will be in neutral equilibrium at its new level. However, if $N^2 < 0$ (potential temperature decreasing with height), the displacement will increase exponentially with time. We thus arrive at the familiar gravitational or static stability criteria for dry air:

$d\theta_0/dz > 0$	statically stable
$d\theta_0/dz = 0$	statically neutral
$d\theta_0/dz < 0$	statically unstable

On the synoptic-scale the atmosphere is always stably stratified because any unstable regions that develop are stabilized quickly by convective overturning.

2.7.4 Scale Analysis of the Thermodynamic Energy Equation

If potential temperature is divided into a basic state $\theta_0(z)$ and a deviation $\theta\,(x, y, z, t)$ so that the total potential temperature at any point is given by $\theta_{tot} = \theta_0\,(z) + \theta\,(x, y, z, t)$, the first law of thermodynamics (2.46) can be written approximately for synoptic scaling as

$$\frac{1}{\theta_0}\left(\frac{\partial\theta}{\partial t} + u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y}\right) + w\frac{d\ln\theta_0}{dz} = \frac{J}{c_pT} \tag{2.53}$$

where we have used the facts that for $|\theta/\theta_0| \ll 1$, $|d\theta/dz| \ll d\theta_0/dz$, and

$$\ln\theta_{tot} = \ln\left[\theta_0\left(1 + \theta/\theta_0\right)\right] \approx \ln\theta_0 + \theta/\theta_0$$

Outside regions of active precipitation, diabatic heating is due primarily to net radiative heating. In the troposphere, radiative heating is quite weak so that typically $J/c_p \leq 1^\circ\mathrm{C\ d^{-1}}$ (except near cloud tops, where substantially larger cooling can occur due to thermal emission by the cloud particles). The typical amplitude of horizontal potential temperature fluctuations in a midlatitude synoptic system (above the boundary layer) is $\theta \sim 4^\circ\mathrm{C}$. Thus,

$$\frac{T}{\theta_0}\left(\frac{\partial\theta}{\partial t} + u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y}\right) \sim \frac{\theta U}{L} \sim 4^\circ\mathrm{C\,d^{-1}}$$

The cooling due to vertical advection of the basic state potential temperature (usually called adiabatic cooling) has a typical magnitude of

$$w\left(\frac{T}{\theta_0}\frac{d\theta_0}{dz}\right) = w\left(\Gamma_d - \Gamma\right) \sim 4^\circ\mathrm{C\ d^{-1}}$$

[3] N is often referred to as the Brunt–Väisälä frequency.

where $w \sim 1$ cm s^{-1}, and $\Gamma_d - \Gamma$, the difference between dry adiabatic and actual lapse rates, is $\sim 4°$ C km^{-1}.

Thus, in the absence of strong diabatic heating, the rate of change in the perturbation potential temperature is equal to the adiabatic heating or cooling due to vertical motion in the statically stable basic state, and (2.54) can be approximated as

$$\left(\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y}\right) + w\frac{d\theta_0}{dz} \approx 0 \tag{2.54}$$

Alternatively, if the temperature field is divided into a basic state $T_0(z)$ and a deviation $T(x, y, z, t)$, then, since $\theta/\theta_0 \approx T/T_0$, (2.54) can be expressed to the same order of approximation in terms of temperature as

$$\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) + w(\Gamma_d - \Gamma) \approx 0 \tag{2.55}$$

2.8 THE BOUSSINESQ APPROXIMATION

In some situations, air parcel vertical displacements are relatively small, so the density change following the motion is also small. The *Boussinesq approximation* yields a simplified form of the dynamical equations that are appropriate to this situation. In this approximation, density is replaced by a constant mean value, ρ_0, everywhere except in the buoyancy term in the vertical momentum equation. The horizontal momentum equations (2.24) and (2.25) can then be expressed in Cartesian coordinates as

$$\frac{Du}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + fv + F_{rx} \tag{2.56}$$

and

$$\frac{Dv}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - fu + F_{ry} \tag{2.57}$$

while, with the aid of equations (2.28) and (2.51), the vertical momentum equation becomes

$$\frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + g\frac{\theta}{\theta_0} + F_{rz} \tag{2.58}$$

Here, as in Section 2.7.3 θ designates the perturbation departure of potential temperature from its basic state value $\theta_0(z)$. Thus, the total potential temperature field is given by $\theta_{\text{tot}} = \theta(x, y, z, t) + \theta_0(z)$, and the adiabatic thermodynamic energy equation has a form similar to (2.54),

$$\frac{D\theta}{Dt} = -w\frac{d\theta_0}{dz} \tag{2.59}$$

except, from the full material derivative, we see that the vertical advection of perturbation potential temperature is formally included. Finally, the continuity equation (2.34) under the Boussinesq approximation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.60}$$

2.9 THERMODYNAMICS OF THE MOIST ATMOSPHERE

Although it is convenient as a first approximation to neglect the role of water vapor in atmospheric dynamics, it sometimes plays an important, if not essential, role. Most often it is the change in phase of water, and the accompanying latent heat release, that affects the dynamics, but even just the presence of water vapor can be critical to the buoyancy acceleration. Water vapor is a highly variable constituent in the atmosphere and as such presents a challenge to the ideal gas law, since the gas constant depends on the molecular composition of air. For dry air the ideal gas law is

$$p_d = \rho_d R_d T \tag{2.61}$$

where p_d and ρ_d are the dry air pressure and density, respectively, and R_d is the dry air gas constant. For pure water vapor, the ideal gas law is

$$e = \rho_v R_v T \tag{2.62}$$

where e and ρ_v are the vapor pressure and density, respectively, and R_v is the water vapor gas constant. For moist air (a mixture of dry air and water vapor) we choose

$$p = \rho R_d T_v \tag{2.63}$$

where p and ρ are the pressure and density, respectively, and T_v is the *virtual temperature*, which represents the temperature dry air needs to have the same pressure and density as moist air. This formulation allows use of the dry air gas constant everywhere. By adding (2.61) and (2.62), it can be shown that the virtual temperature is given by

$$T_v = \frac{T}{1 - \frac{e}{p}\left(1 - \frac{R_d}{R_v}\right)} \tag{2.64}$$

We see that $T_v \geq T$, and the difference is typically on the order of a few degrees Celsius or less; nevertheless, even these small changes can make the difference between an air parcel being stable or unstable.

There are many measures of water vapor in the atmosphere, and we summarize here some of the most common. A measure used earlier in the ideal gas law is the vapor pressure that by Dalton's Law is simply the contribution to the total pressure from water vapor. Saturation vapor pressure is determined solely by temperature as given by the Clausius–Clapeyron equation[4]

$$\frac{1}{e_s}\frac{de_s}{dT} = \frac{L}{R_v T^2} \tag{2.65}$$

For water vapor over a plane surface of liquid water (ice), L represents the latent heat of condensation (sublimation). Saturation vapor pressure depends approximately exponentially on temperature and, for temperatures below freezing, is larger above water surfaces than ice. Relative humidity may be defined as the percentage

$$RH = 100 \times \frac{e}{e_s} \tag{2.66}$$

and e_s may apply to either water or ice saturation; if unspecified, the former may be assumed with confidence.

The water vapor mixing ratio is given by the ratio of the mass of water vapor to the mass of dry air in a volume of air, which takes units of either kg/kg or, more commonly, g/kg. From (2.61) and (2.62), the mixing ratio is given by

$$q = 0.622\frac{e}{p - e} \tag{2.67}$$

where the leading coefficient 0.622 reflects the ratio of the molecular weight of water vapor to dry air.

There are several ways to bring moist air to saturation, one of which is by cooling the air at constant pressure without addition or removal of water vapor. Saturation achieved in this manner gives the dew point temperature.

2.9.1 Equivalent Potential Temperature

We previously applied the parcel method to discuss the vertical stability of a dry atmosphere. We found that the stability of a dry parcel with respect to a vertical displacement depends on the lapse rate of potential temperature in the environment such that a parcel displacement is stable, provided that $\partial\theta/\partial z > 0$ (i.e., the actual lapse rate is less than the adiabatic lapse rate). The same condition also applies to parcels in a moist atmosphere when the relative humidity is less than 100%. If, however, a parcel of moist air is forced to rise, it will eventually

[4] For a derivation see, for example, Curry and Webster (1999, p. 108).

become saturated at a level called the lifting condensation level (LCL). A further forced rise will then cause condensation and latent heat release, and the parcel will then cool at the saturated adiabatic lapse rate. If the environmental lapse rate is greater than the saturated adiabatic lapse rate, and the parcel is forced to continue to rise, it will reach a level at which it becomes buoyant relative to its surroundings. It can then freely accelerate upward. The level at which this occurs is called the level of free convection (LFC).

Discussion of parcel dynamics in a moist atmosphere is facilitated by defining a thermodynamic field called the *equivalent potential temperature*. Equivalent potential temperature, designated θ_e, is the potential temperature that a parcel of air would have if all its moisture were condensed and the resultant latent heat used to warm the parcel. The temperature of an air parcel can be brought to its equivalent potential value by raising the parcel from its original level until all the water vapor in the parcel has condensed and fallen out, and then compressing the parcel adiabatically to a pressure of 1000 hPa. Because the condensed water is assumed to fall out, the temperature increase during the compression will be at the dry adiabatic rate and the parcel will arrive back at its original level with a temperature that is higher than its original temperature. Thus, the process is irreversible. Ascent of this type, in which all condensation products are assumed to fall out, is called *pseudoadiabatic* ascent. (It is not a truly adiabatic process because the liquid water that falls out carries a small amount of heat with it.)

A complete derivation of the mathematical expression relating θ_e to the other variables of state is rather involved and will be relegated to Appendix D. For most purposes, however, it is sufficient to use an approximate expression for θ_e that can be immediately derived from the entropy form of the first law of thermodynamics (2.46). If we let q_s denote the mass of water vapor per unit mass of dry air in a saturated parcel (the saturation mixing ratio), then the rate of diabatic heating per unit mass is

$$J = -L_c \frac{Dq_s}{Dt}$$

where L_c is the latent heat of condensation. Therefore, from the first law of thermodynamics

$$c_p \frac{D \ln \theta}{Dt} = -\frac{L_c}{T} \frac{Dq_s}{Dt} \tag{2.68}$$

For a saturated parcel undergoing pseudoadiabatic ascent, the rate of change in q_s following the motion is much larger than the rate of change in T or L_c. Therefore,

$$d \ln \theta \approx -d\left(L_c q_s / c_p T\right) \tag{2.69}$$

Integrating (2.69) from the initial state (θ, q_s, T) to a state where $q_s \approx 0$, we obtain

$$\ln(\theta/\theta_e) \approx -L_c q_s / c_p T$$

where θ_e, the potential temperature in the final state, is approximately the equivalent potential temperature defined earlier. Thus, θ_e, for a saturated parcel is given by

$$\theta_e \approx \theta \exp\left(L_c q_s / c_p T\right) \tag{2.70}$$

The expression in (2.70) may also be used to compute θ_e for an unsaturated parcel provided that the temperature used in the formula is the temperature that the parcel would have if expanded adiabatically to saturation (i.e., T_{LCL}) and the saturation mixing ratio were replaced by the *actual* mixing ratio of the initial state. Thus, equivalent potential temperature is conserved for a parcel during both dry adiabatic and pseudoadiabatic displacements.

An alternative to θ_e, which is sometimes used in studies of convection, is the *moist static energy*, defined as $h \equiv s + L_c q$, where $s \equiv c_p T + gz$ is the *dry static energy*. It can be shown (Problem 2.10) that

$$c_p T d \ln \theta_e \approx dh \tag{2.71}$$

Thus, moist static energy is approximately conserved when θ_e is conserved.

2.9.2 The Pseudoadiabatic Lapse Rate

The first law of thermodynamics (2.68) can be used to derive a formula for the rate of change in temperature with respect to height for a saturated parcel undergoing pseudoadiabatic ascent. Using the definition of θ (2.44), we can rewrite (2.68) for vertical ascent as

$$\frac{d \ln T}{dz} - \frac{R}{c_p} \frac{d \ln p}{dz} = -\frac{L_c}{c_p T} \frac{dq_s}{dz}$$

which, upon noting that $q_s \equiv q_s(T, p)$ and applying the hydrostatic equation and equation of state, can be expressed as

$$\frac{dT}{dz} + \frac{g}{c_p} = -\frac{L_c}{c_p} \left[\left(\frac{\partial q_s}{\partial T} \right)_p \frac{dT}{dz} - \left(\frac{\partial q_s}{\partial p} \right)_T \rho g \right]$$

Thus, as shown in detail in Appendix D, following the ascending saturated parcel

$$\Gamma_s \equiv -\frac{dT}{dz} = \Gamma_d \frac{[1 + L_c q_s / (RT)]}{\left[1 + \varepsilon L_c^2 q_s / \left(c_p R T^2\right)\right]} \tag{2.72}$$

where $\varepsilon = 0.622$ is the ratio of the molecular weight of water to that of dry air, $\Gamma_d \equiv g/c_p$ is the dry adiabatic lapse rate, and Γ_s is the *pseudoadiabatic lapse rate,* which is always less than Γ_d. Observed values of Γ_s range from ~4 K km^{-1} in warm, humid air masses in the lower troposphere to ~6–7 K km^{-1} in the midtroposphere.

2.9.3 Conditional Instability

Section 2.7.3 showed that for dry adiabatic motions the atmosphere is statically stable provided that the lapse rate is less than the dry adiabatic lapse rate (i.e., the potential temperature increases with height). If the lapse rate Γ lies between dry adiabatic and pseudoadiabatic values ($\Gamma_s < \Gamma < \Gamma_d$), the atmosphere is stably stratified with respect to dry adiabatic displacements but unstable with respect to pseudoadiabatic displacements. Such a situation is referred to as *conditional instability* (i.e., the instability is conditional to saturation of the air parcel).

The conditional stability criterion can also be expressed in terms of the gradient of a field variable θ_e^*, defined as the equivalent potential temperature of a hypothetically saturated atmosphere that has the thermal structure of the actual atmosphere.[5] Thus,

$$d \ln \theta_e^* = d \ln \theta + d \left(L_c q_s / c_p T \right) \tag{2.73}$$

where T is the actual temperature, not the temperature after adiabatic expansion to saturation as in (2.70). To derive an expression for conditional instability, we consider the motion of a saturated parcel in an environment in which the potential temperature is θ_0 at the level z_0. At the level $z_0 - \delta z$ the undisturbed environmental air thus has potential temperature

$$\theta_0 - (\partial\theta/\partial z)\,\delta z$$

Suppose that a saturated parcel that has the environmental potential temperature at $z_0 - \delta z$ is raised to the level z_0. When it arrives at z_0, the parcel will have the potential temperature

$$\theta_1 = \left(\theta_0 - \frac{\partial\theta}{\partial z}\delta z \right) + \delta\theta$$

where $\delta\theta$ is the change in parcel potential temperature due to condensation during ascent through vertical distance δz. Assuming a pseudoadiabatic ascent, we see from (2.69) that

$$\frac{\delta\theta}{\theta} \approx -\delta \left(\frac{L_c q_s}{c_p T} \right) \approx -\frac{\partial}{\partial z} \left(\frac{L_c q_s}{c_p T} \right) \delta z$$

[5]Note that θ_e^* is *not* the same as θ_e except in a saturated atmosphere.

so that the buoyancy of the parcel when it arrives at z_0 is proportional to

$$\frac{(\theta_1 - \theta_0)}{\theta_0} \approx -\left[\frac{1}{\theta}\frac{\partial \theta}{\partial z} + \frac{\partial}{\partial z}\left(\frac{L_c q_s}{c_p T}\right)\right]\delta z \approx -\frac{\partial \ln \theta_e^*}{\partial z}\delta z$$

where the last expression arises from the application of (2.73).

The saturated parcel will be warmer than its environment at z_0 provided that $\theta_1 > \theta_0$. Thus, the conditional stability criterion for a saturated parcel is

$$\frac{\partial \theta_e^*}{\partial z}\begin{cases} < 0 & \text{conditionally unstable} \\ = 0 & \text{saturated neutral} \\ > 0 & \text{conditionally stable} \end{cases} \tag{2.74}$$

In Figure 2.8 the vertical profiles of θ, θ_e, and θ_e^* for a typical sounding in the vicinity of an extratropical thunderstorm are shown. It is obvious from

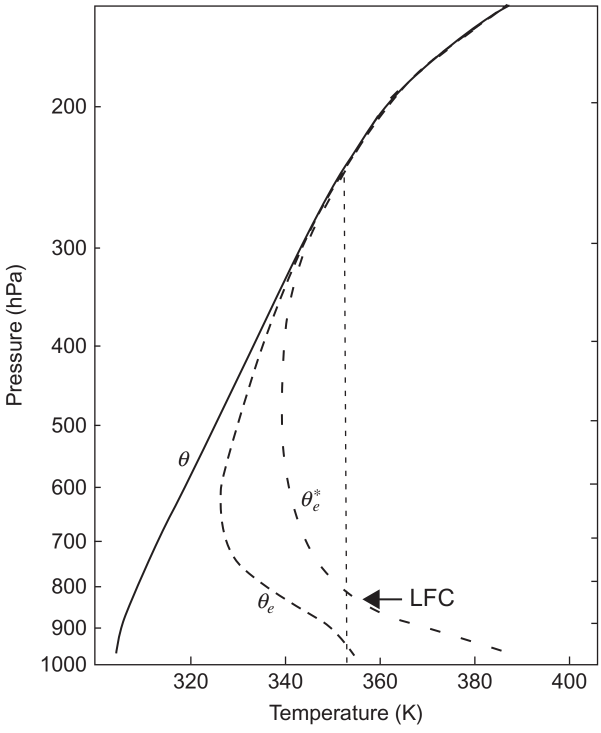

FIGURE 2.8 Schematic sounding for a conditionally unstable environment, characteristic of midwestern North American thunderstorm conditions, showing the vertical profiles of potential temperature θ, equivalent potential temperature θ_e, and equivalent temperature θ_e^* of a hypothetically saturated atmosphere with the same temperature profile. The *dotted line* shows θ_e for a nonentraining parcel raised from the surface. The *arrow* denotes the LFC for the parcel.

this figure that the sounding is conditionally unstable in the lower troposphere. However, this observed profile does not imply that convective overturning will occur spontaneously. The release of conditional instability requires not only that $\partial\theta_e^*/\partial z < 0$, but also parcel saturation at the environmental temperature of the level where the convection begins (i.e., the parcel must reach the LFC). The mean relative humidity in the troposphere is well below 100%, even in the boundary layer. Thus, low-level convergence with resultant forced layer ascent or vigorous vertical turbulent mixing in the boundary layer is required to produce saturation.

The amount of ascent necessary to raise a parcel to its LFC can be estimated simply from Figure 2.8. A parcel rising pseudoadiabatically from a level $z_0 - \delta z$ will conserve the value of θ_e characteristic of the environment at $z_0 - \delta z$. However, the buoyancy of a parcel depends only on the difference in density between the parcel and its environment. Thus, in order to compute the buoyancy of the parcel at z_0, it is not correct simply to compare θ_e of the environment at z_0 to $\theta_e(z_0 - \delta z)$ because if the environment is unsaturated, the difference in θ_e between the parcel and the environment may be due primarily to the difference in mixing ratios, not to any temperature (density) difference. To estimate the buoyancy of the parcel, $\theta_e(z_0 - \delta z)$ should instead be compared to $\theta_e^*(z_0)$, which is the equivalent potential temperature that the environment at z_0 would have if it were isothermally brought to saturation. The parcel will thus become buoyant when raised to the level z_0 if $\theta_e(z_0 - \delta z) > \theta_e^*(z_0)$, for then the parcel temperature will exceed the temperature of the environment at z_0.

From Figure 2.8, we see that θ_e for a parcel raised from about 960 hPa will intersect the θ_e^* curve near 850 hPa, whereas a parcel raised from any level much above 850 hPa will not intersect θ_e^* no matter how far it is forced to ascend. It is for this reason that low-level convergence is usually required to initiate convective overturning over the oceans. Only air near the surface has a sufficiently high value of θ_e to become buoyant when it is forcibly raised. Convection over continental regions, however, can be initiated without significant boundary layer convergence, as strong surface heating can produce positive parcel buoyancy all the way to the surface. Sustained deep convection, however, requires mean low-level moisture convergence.

SUGGESTED REFERENCES

Curry and Webster's *Thermodynamics of Atmospheres and Oceans* contains an excellent treatment of atmospheric thermodynamics.

Pedlosky, *Geophysical Fluid Dynamics*, discusses the equations of motion for a rotating coordinate system and has a thorough discussion of scale analysis at a graduate level.

Salby's *Fundamentals of Atmospheric Physics* contains a thorough development of the basic conservation laws at the graduate level.

PROBLEMS

2.1. A ship is steaming northward at a rate of 10 km h^{-1}.The surface pressure increases toward the northwest at the rate of 5 Pa km^{-1}. What is the pressure tendency recorded at a nearby island station if the pressure aboard the ship decreases at a rate of 100 Pa/3 h?

2.2. The temperature at a point 50 km north of a station is 3°C cooler than at the station. If the wind is blowing from the northeast at 20 m s^{-1} and the air is being heated by radiation at the rate of 1°C h^{-1}, what is the local temperature change at the station?

2.3. Derive the relationship

$$\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = -\Omega^2 \mathbf{R}$$

which was used in Eq. (2.7).

2.4. Derive the expression given in Eq. (2.13) for the rate of change of **k** following the motion.

2.5. Suppose a 1-kg parcel of dry air is rising at a constant vertical velocity. If the parcel is being heated by radiation at the rate of 10^{-1}W kg^{-1}, what must the speed of rise be to maintain the parcel at a constant temperature?

2.6. Derive an expression for the density ρ that results when an air parcel initially at pressure p_s and density ρ_s expands adiabatically to pressure p.

2.7. An air parcel that has a temperature of 20°C at the 1000-hPa level is lifted dry adiabatically. What is its density when it reaches the 500-hPa level?

2.8. Suppose an air parcel starts from rest at the 800-hPa level and rises vertically to 500 hPa while maintaining a constant 1°C temperature excess over the environment. Assuming that the mean temperature of the 800- to 500-hPa layer is 260 K, compute the energy released due to the work of the buoyancy force. Assuming that all the released energy is realized as kinetic energy of the parcel, what will the vertical velocity of the parcel be at 500 hPa?

2.9. Show that for an atmosphere with an adiabatic lapse rate (i.e., constant potential temperature) the geopotential height is given by

$$Z = H_\theta \left[1 - (p/p_0)^{R/c_p}\right]$$

where p_0 is the pressure at $Z = 0$ and $H_\theta \equiv c_p\,\theta/g_0$ is the total geopotential height of the atmosphere.

2.10. The azimuthal velocity component in some hurricanes is observed to have a radial dependence given by $v_\lambda = V_0(r_0/r)^2$ for distances from the center given by $r \geq r_0$. Letting $V_0 = 50\,\mathrm{m\,s^{-1}}$ and $r_0 = 50\,\mathrm{km}$, find the total geopotential difference between the far field ($r \to \infty$) and $r = r_0$, assuming gradient wind balance and $f_0 = 5 \times 10^{-5}\ \mathrm{s^{-1}}$. At what distance from the center does the Coriolis force equal the centrifugal force?

2.11. In the *isentropic* coordinate system (see Section 4.6), potential temperature is used as the vertical coordinate. Because potential temperature in adiabatic flow is conserved following the motion, isentropic coordinates are useful for tracing the actual paths of travel of individual air parcels. Show that the transformation of the horizontal pressure gradient force from z to θ coordinates is given by

$$\frac{1}{\rho}\nabla_z p = \nabla_\theta M$$

where $M \equiv c_p\, T + \Phi$ is the *Montgomery streamfunction*.

2.12. French scientists have developed a high-altitude balloon that remains at constant potential temperature as it circles Earth. Suppose such a balloon is in the lower equatorial stratosphere where the temperature is isothermal at 200 K. If the balloon were displaced vertically from its equilibrium level by a small distance δz, it would tend to oscillate about the equilibrium level. What would be the period of this oscillation?

2.13. Derive the approximate thermodynamic energy equation (2.55) using the scaling arguments of Sections 2.4 and 2.7.

MATLAB Exercises

M2.1. The MATLAB script **standard_T_p.m** defines and plots the temperature and the lapse rate associated with the U.S. Standard Atmosphere as functions of height. Modify this script to compute the pressure and potential temperature, and plot these in the same format used for temperature and lapse rate. [*Hint*: To compute pressure, integrate the hydrostatic equation from the surface upward in increments of δz.] Show that if we define a mean scale height for the layer between z and $z + \delta z$ by letting $H = R\,[T\,(z) + T\,(z + \delta z)]/(2g)$, then $p\,(z + \delta z) = p\,(z)\exp\,[-\delta z/H]$. (Note that as you move upward layer by layer, you must use the local height-dependent value of H in this formula.)

M2.2. The MATLAB script **thermo_profile.m** is a simple script to read in data giving pressure and temperature for a tropical mean sounding. Run this script to plot temperature versus pressure for data in the file **tropical_temp.dat**. Use the hypsometric equation to compute the geopotential height corresponding to each pressure level of the data file. Compute the corresponding potential temperature and plot graphs of the temperature and potential temperature variations with pressure and with geopotential height.