Appendix A

Elements of Fluid Mechanics

ABSTRACT

Basic principles of fluid mechanics are recalled and summarized. It is shown how budgets can be established on infinitesimal volumes. The distinction is made between Eulerian and Lagrangian approaches to fluid dynamics. For reference, a few equations and operators are expressed in cylindrical and spherical coordinates. Finally, the link between vorticity and rotation is outlined.

A.1 BUDGETS

Most physical principles of fluid mechanics can be cast as budgets of one quantity or another, with the simplest budget being the one for mass conservation. We begin here with the one-dimensional (1D) version, from which the 3D generalization is immediate.

For a 1D budget, we consider a very short (infinitesimal) segment of fluid, of length dx, along the x-axis of the system (Fig. A.1), for which we state that the mass within this segment at one moment, say time $t+\mathrm{d}t$, is the mass that was there at a previous moment, say time t, augmented by the amount of inflow on the left, say at $x-\mathrm{d}x$, minus the amount of outflow from the right, say x, during the elapsed time dt:

$$\underbrace{\rho(x,t+\mathrm{d}t)\mathrm{d}x}_{\text{mass at time } t+\mathrm{d}t} = \underbrace{\rho(x,t)\mathrm{d}x}_{\text{mass at time } t} + \underbrace{u(x-\mathrm{d}x,t)\mathrm{d}t\,\rho(x-\mathrm{d}x,t)}_{\text{mass entering}} - \underbrace{u(x,t)\mathrm{d}t\,\rho(x,t)}_{\text{mass exiting}}. \tag{A.1}$$

After division by the time interval dt and space interval dx, this budget can be recast as

$$\frac{\rho(x,t+\mathrm{d}t)-\rho(x,t)}{\mathrm{d}t} + \frac{\rho(x,t)u(x,t)-\rho(x-\mathrm{d}x,t)u(x-\mathrm{d}x,t)}{\mathrm{d}x} = 0. \tag{A.2}$$

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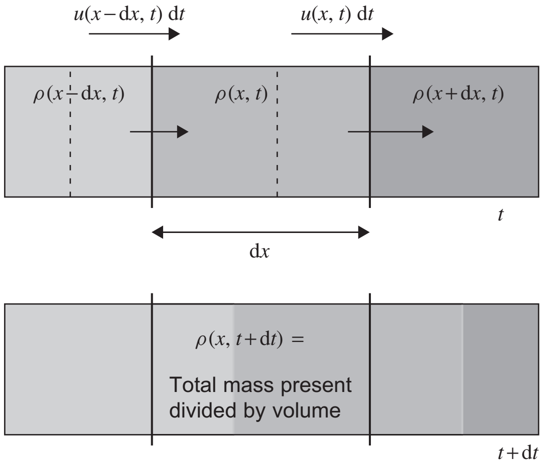

FIGURE A.1 One-dimensional mass conservation.

In the limit of vanishing dt and dx, the differences become derivatives, and the one-dimensional mass conservation equation is obtained:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0. \tag{A.3}$$

Note that we assumed such an infinitesimal limit exists, meaning that "infinitesimal" is extremely small compared with the scales of macroscopic properties, yet large compared with the size of the molecules constituting the fluid for which the budget is established. This is the essence of continuum mechanics.

Similarly, for a three-dimensional domain (Fig. A.2), the budget calculation yields:

$$\begin{aligned}
\rho(x,y,z,t+\mathrm{d}t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \rho(x,y,z,t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&+ u(x-\mathrm{d}x,y,z,t)\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}z\,\rho(x-\mathrm{d}x,y,z,t) - u(x,y,z,t)\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}z\,\rho(x,y,z,t) \\
&+ v(x,y-\mathrm{d}y,z,t)\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}z\,\rho(x,y-\mathrm{d}y,z,t) - v(x,y,z,t)\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}z\,\rho(x,y,z,t) \\
&+ w(x,y,z-\mathrm{d}z,t)\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\rho(x,y,z-\mathrm{d}z,t) - w(x,y,z,t)\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\rho(x,y,z,t).
\end{aligned}$$

Division by the infinitesimal volume dxdydz and time interval dt provides in the continuous limit:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0. \tag{A.4}$$

This is the mass conservation equation, also called the *continuity equation*.

Newton's second law of physics stating that mass times acceleration is equal to the sum of forces can likewise be cast as a budget, this time with

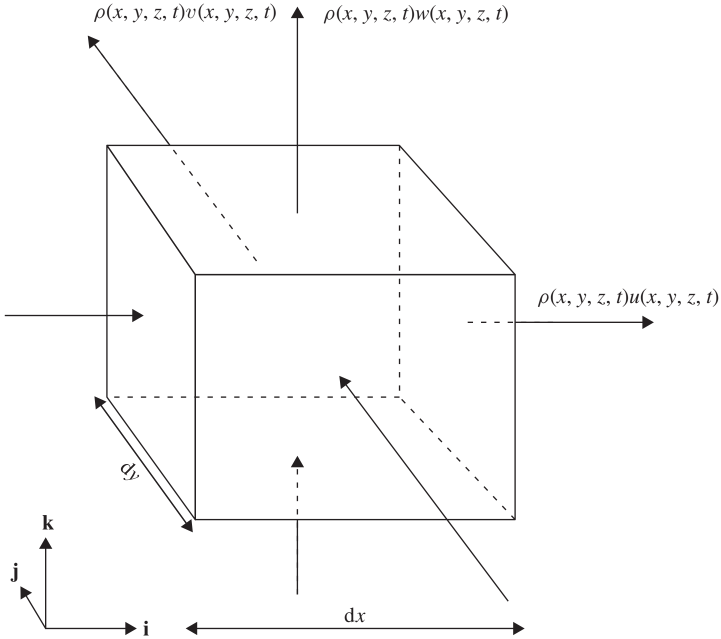

FIGURE A.2 Infinitesimal volume with mass inflow and outflow across boundaries for the three-dimensional mass budget.

momentum (mass times velocity, here per unit volume) being the quantity for which the budget is written and with forces (per unit volume) acting as sources. For clarity, the budget is established here in the two-dimensional case (Fig. A.3).

A suitable departure point is the mass budget equation (A.1) in which we replace density by the product of density with velocity. The sources are forces per volume. We also progress from one to two dimensions. Thus, we write

$$\begin{aligned}\rho u|_{\text{at } x,y,t+\mathrm{d}t}\,\mathrm{d}x\,\mathrm{d}y = {} & \rho u|_{\text{at } x,y,t}\,\mathrm{d}x\,\mathrm{d}y \\ & + \rho u\,u|_{\text{at } x-\mathrm{d}x,y}\,\mathrm{d}y\,\mathrm{d}t - \rho u\,u|_{\text{at } x,y}\,\mathrm{d}y\,\mathrm{d}t \\ & + \rho u\,v|_{\text{at } x,y-\mathrm{d}y}\,\mathrm{d}x\,\mathrm{d}t - \rho u\,v|_{\text{at } x,y}\,\mathrm{d}x\,\mathrm{d}t \\ & + \text{Sum of forces in the } x\text{-direction}\end{aligned} \tag{A.5a}$$

$$\begin{aligned}\rho v|_{\text{at } x,y,t+\mathrm{d}t}\,\mathrm{d}x\,\mathrm{d}y = {} & \rho v|_{\text{at } x,y,t}\,\mathrm{d}x\,\mathrm{d}y \\ & + \rho v\,u|_{\text{at } x-\mathrm{d}x,y}\,\mathrm{d}y\,\mathrm{d}t - \rho v\,u|_{\text{at } x,y}\,\mathrm{d}y\,\mathrm{d}t \\ & + \rho v\,v|_{\text{at } x,y-\mathrm{d}y}\,\mathrm{d}x\,\mathrm{d}t - \rho v\,v|_{\text{at } x,y}\,\mathrm{d}x\,\mathrm{d}t \\ & + \text{Sum of forces in the } y\text{-direction.}\end{aligned} \tag{A.5b}$$

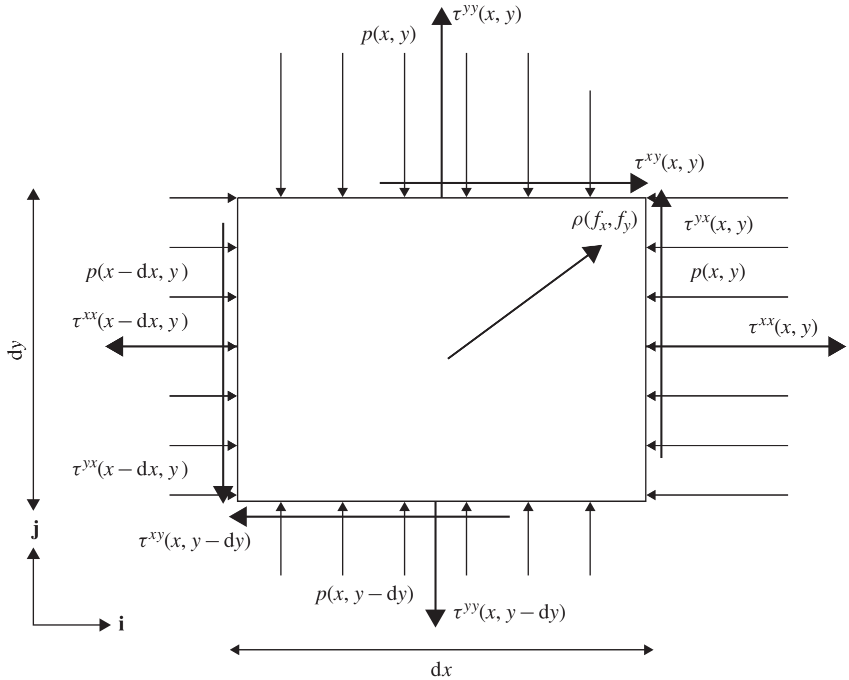

FIGURE A.3 Two-dimensional element subject to the typical forces encountered in a fluid flow: pressure p (a normal force per unit area), shear stress τ (a tangential force per unit area), and an internal (body) force $\rho(f_x, f_y)$, which is usually the gravitational force.

The forces applied to the fluid element are as follows:

$$\begin{aligned}\text{Sum of forces in the } x\text{-direction} = {} & p|_{\text{at } x-\mathrm{d}x,y}\,\mathrm{d}y - p|_{\text{at } x,y}\,\mathrm{d}y \\ & - \tau^{xx}|_{\text{at } x-\mathrm{d}x,y}\,\mathrm{d}y + \tau^{xx}|_{\text{at } x,y}\,\mathrm{d}y \\ & - \tau^{xy}|_{\text{at } x,y-\mathrm{d}y}\,\mathrm{d}x + \tau^{xy}|_{\text{at } x,y}\,\mathrm{d}x \\ & + \rho f_x\,\mathrm{d}x\,\mathrm{d}y \end{aligned} \tag{A.6a}$$

$$\begin{aligned}\text{Sum of forces in the } y\text{-direction} = {} & p|_{\text{at } x,y-\mathrm{d}y}\,\mathrm{d}x - p|_{\text{at } x,y}\,\mathrm{d}x \\ & - \tau^{yx}|_{\text{at } x-\mathrm{d}x,y}\,\mathrm{d}y + \tau^{yx}|_{\text{at } x,y}\,\mathrm{d}y \\ & - \tau^{yy}|_{\text{at } x,y-\mathrm{d}y}\,\mathrm{d}x + \tau^{yy}|_{\text{at } x,y}\,\mathrm{d}x \\ & + \rho f_y\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{A.6b}$$

Here, the force (f_x, f_y) is the body force per unit mass, so that the product $\rho(f_x, f_y)$ is the body force per unit volume. Note that the stresses τ depend on the nature of the fluid and its flow, and that the tangential stresses τ^{xy} and τ^{yx} act in different directions (Fig. A.3) but must have the same strength. This equality

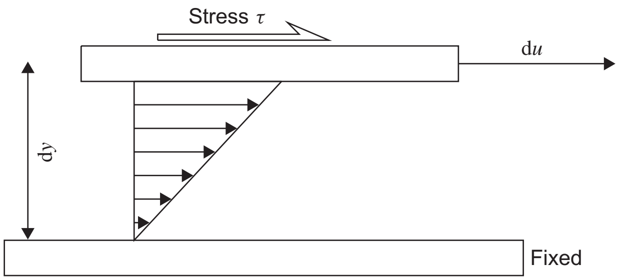

FIGURE A.4 Creeping flow with stress proportional to shear: $\tau \propto \mathrm{d}u/\mathrm{d}y$.

of stresses, $\tau^{xy}(x,y) = \tau^{yx}(x,y)$, proceeds from the fact that, if this were not the case, the infinitesimal element would be subjected to an uncompensated torque.

A so-called constitutive equation must relate the stress components to the fluid flow, usually its velocity shear (see Fig. A.4).

With these forces and dividing by $\mathrm{d}x\,\mathrm{d}y$, we obtain the momentum budget in the x-direction.

$$\begin{aligned}\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u u) + \frac{\partial}{\partial y}(\rho u v) + \\ = \rho f_x - \frac{\partial p}{\partial x} + \frac{\partial \tau^{xx}}{\partial x} + \frac{\partial \tau^{xy}}{\partial y},\end{aligned} \tag{A.7}$$

and similarly in the y-direction. Generalization to three dimensions is straightforward and leads to Eq. (3.2) with Eq. (3.3).

Note that in solid mechanics, Newton's second law is presented by following a given mass along its path (called the Lagrangian approach) rather than by performing a budget over a fixed part of space (called the Eulerian approach). Because the physical law is the same, we should be able to reach the same governing equations by either approach. To show that this is possible, we express the Eulerian derivative of a field $F(x, y, z, t)$, which may be any property of the fluid or flow field, as

$$\frac{\partial F}{\partial t} = \text{derivative of } F \text{ with respect to } t, \text{ at fixed } x, y, z. \tag{A.8}$$

In other words, this is the time change of F as perceived by an observer at a fixed location. In contrast, the Lagrangian approach considers the change moving with a fluid parcel, the position of which changes over time, $(x, y, z) = [x(t), y(t), z(t)]$. This time dependence of the coordinates describes the trajectory of the fluid parcel. The time change of F, taking the displacement over time into account, is the *total* time derivative of F:

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \text{derivative of } F(x(t), y(t), z(t), t) \text{ with respect to } t. \tag{A.9}$$

This change of F for a fluid parcel is obtained by the chain rule of derivatives:

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial F}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{\partial F}{\partial t}. \tag{A.10}$$

Because $[x(t), y(t), z(t)]$ is the trajectory of the fluid parcel, the change in position over time $\mathrm{d}x/\mathrm{d}t$ is nothing else than the parcel velocity u, and similarly $\mathrm{d}y/\mathrm{d}t = v$ and $\mathrm{d}z/\mathrm{d}t = w$, so that we can express the Lagrangian derivative $\mathrm{d}F/\mathrm{d}t$, also called the *material derivative*, as

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z}. \tag{A.11}$$

This relates the Lagrangian derivative to the Eulerian derivative, permitting a switch from one approach to the other. The difference between the two expressions, that is, the sum of terms with velocity components, is the *advection* contribution.

The passage from Eulerian to Lagrangian formulation also permits a manipulation of the mass-conservation equation (A.4) by using the material derivative (A.11):

$$\frac{1}{\mathfrak{v}}\frac{\mathrm{d}\mathfrak{v}}{\mathrm{d}t} = -\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}, \tag{A.12}$$

with $\mathfrak{v} = 1/\rho$ being the volume per unit mass. The expression $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z$ is the divergence of the flow field. It is positive when the flow diverges and negative when it converges. It follows that a fluid volume is dilated (shrinking) and density drops (increases) when the flow diverges (converges).

In two dimensions, we can proceed one step further by relating the divergence to the area $\mathcal{S}$ containing the fluid element:

$$\frac{1}{\mathcal{S}}\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}t} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}. \tag{A.13}$$

This last expression becomes useful in the study of vorticity. It is left as an exercise to the reader to formulate the momentum equation in a Lagrangian way and to interpret the resulting equation.

A.2 EQUATIONS IN CYLINDRICAL COORDINATES

The preceding equations assumed a rectangular (Cartesian) system of coordinates, but in geophysical fluid dynamics, we occasionally encounter circular structures, such as vortices, for which the use of cylindrical coordinates is more convenient. The three coordinates of space are then the radial distance r, the azimuthal angle θ (in radians), and the vertical coordinate z.

In cylindrical coordinates, the material derivative becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z}. \tag{A.14}$$

In this notation, u is the radial velocity, v the azimuthal velocity (positive for a parcel turning in the trigonometric sense, increasing θ), and w the vertical velocity.

Mass conservation and horizontal components of the momentum equations are as follows:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r u) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \tag{A.15a}$$

$$\rho\left(\frac{\mathrm{d}u}{\mathrm{d}t} - \frac{v^2}{r} - f v + f_* w\right) = -\frac{\partial p}{\partial r} + F_r \tag{A.15b}$$

$$\rho\left(\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{uv}{r} + fu\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + F_\theta \tag{A.15c}$$

$$\rho\left(\frac{\mathrm{d}w}{\mathrm{d}t} - f_* u\right) = -\frac{\partial p}{\partial z} - \rho g + F_z \tag{A.15d}$$

where F_r, F_θ, and F_z are the stress terms. The Laplacian of a scalar field ψ reads

$$\nabla^2 \psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2}. \tag{A.16}$$

Polar coordinates are cylindrical coordinates in two dimensions, with the z dependence dropped.

A.3 EQUATIONS IN SPHERICAL COORDINATES

When the dimension of the domain is comparable to the earth's radius, and especially when the entire globe is the domain, spherical coordinates are preferred. The three coordinates of space are then the radial distance r from the center of the earth (which is often cropped to z along the local vertical and measured form the mean sea level), longitude λ, and latitude[1] φ (both expressed in radians rather than degrees). The material derivative becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \frac{u}{r\cos\varphi}\frac{\partial}{\partial \lambda} + \frac{v}{r}\frac{\partial}{\partial \varphi} + w\frac{\partial}{\partial r}. \tag{A.17}$$

[1]Contrary to classical spherical coordinates, we do not use the polar angle but latitude.

Equations (3.1) through (3.3) become:

$$\frac{\partial}{\partial t}(\rho\cos\varphi)+\frac{\partial}{\partial\lambda}\left(\frac{\rho u}{r}\right)+\frac{\partial}{\partial\varphi}\left(\frac{\rho v\cos\varphi}{r}\right)+\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\rho w\cos\varphi\right)=0 \quad \text{(A.18a)}$$

$$\rho\left(\frac{\mathrm{d}u}{\mathrm{d}t}-\frac{uv\tan\varphi}{r}+\frac{uw}{r}-fv+f_*w\right)=-\frac{1}{r\cos\varphi}\frac{\partial p}{\partial\lambda}+F_\lambda \quad \text{(A.18b)}$$

$$\rho\left(\frac{\mathrm{d}v}{\mathrm{d}t}+\frac{u^2\tan\varphi}{r}+\frac{vw}{r}+fu\right)=-\frac{1}{r}\frac{\partial p}{\partial\varphi}+F_\varphi \quad \text{(A.18c)}$$

$$\rho\left(\frac{\mathrm{d}w}{\mathrm{d}t}-\frac{u^2+v^2}{r}-f_*u\right)=-\frac{\partial p}{\partial r}-\rho g+F_r, \quad \text{(A.18d)}$$

in which $f=2\Omega\sin\varphi$ and $f_*=2\Omega\cos\varphi$. The components F_λ, F_φ, and F_r of the frictional force have complicated expressions and need not be reproduced here. For a detailed development of these equations, the reader is referred to Chapter 4 of the book by Gill (1982). The Laplacian of a scalar field ψ reads

$$\nabla^2\psi=\frac{1}{r^2\cos\varphi^2}\frac{\partial^2\psi}{\partial\lambda^2}+\frac{1}{r^2\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial\psi}{\partial\varphi}\right)+\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right). \quad \text{(A.19)}$$

It is worth noting that since the radius of the earth is much longer than the thickness of either atmosphere or ocean, some vertical derivatives may be approximated as

$$\frac{1}{r^2}\frac{\partial(r^2a)}{\partial r}\simeq\frac{\partial a}{\partial z} \quad \text{(A.20a)}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial a}{\partial r}\right)\simeq\frac{\partial^2 a}{\partial z^2}. \quad \text{(A.20b)}$$

A.4 VORTICITY AND ROTATION

Vorticity, as it name indicates, quantifies the rotation rate of a fluid parcel. Because rotation is also defined by an axis around which the spin occurs, vorticity ought to be a vector. For simplicity, however, we start by considering the case of a flow in the horizontal plane, so that rotation takes place around the vertical axis, and the vorticity vector is directed along this axis. Only its intensity matters, which is defined as

$$\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}. \quad \text{(A.21)}$$

First let us consider a flow in solid-body rotation around the origin of the axes (left part of Fig. A.5). The flow field is then ($u=-\Omega y$, $v=+\Omega x$), and the vorticity defined by Eq. (A.21) is $\zeta=2\Omega$, twice the rotation rate of the flow, and except for the factor 2, this seems intuitive.

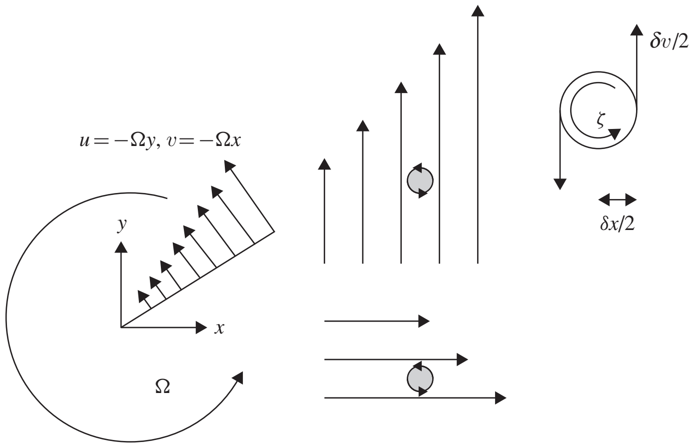

FIGURE A.5 Vorticity and rotation.

Not only flows with curved trajectories have vorticity; rectilinear shear flows, too, possess vorticity, as depicted in the middle of Fig. A.5. Take for example the flow $v(x)$ in which fluid parcels located at different x positions travel at different velocities in the y-direction, some overtaking others in a slipping movement. A stick placed across the flow would see one tip proceeding faster than the other and would effectively be rotated by the flow. This rotation is expressed mathematically by the vorticity:

$$\zeta = \frac{\mathrm{d}v}{\mathrm{d}x}. \tag{A.22}$$

The sign of vorticity is such that it is positive for rotation in the trigonometric (counterclockwise) sense seen downward along the vertical axis.

In three-dimensions, vorticity is the curl of the vector velocity, and its three components are as follows:

$$\zeta_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \tag{A.23}$$

$$\zeta_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \tag{A.24}$$

$$\zeta_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{A.25}$$

ANALYTICAL PROBLEMS

A.1. Verify that the velocity components in cylindrical coordinates are

$$u = \frac{\mathrm{d}r}{\mathrm{d}t}, \quad v = r\frac{\mathrm{d}\theta}{\mathrm{d}t}, \quad w = \frac{\mathrm{d}z}{\mathrm{d}t}. \tag{A.26}$$

Can you interpret these formulas? (*Hint*: Apply the definition of the material derivative.)

A.2. Determine the vorticity of an eddy in which the velocity field is

$$u = -\frac{\partial \psi}{\partial y}, \quad v = +\frac{\partial \psi}{\partial x} \tag{A.27}$$

with the streamfunction ψ given as

$$\psi = \omega L^2 \exp\left(-\frac{x^2+y^2}{L^2}\right). \tag{A.28}$$

In particular, calculate the value at the origin and at $x = 3L, y = 0$.

A.3. Assume a two-dimensional flow, for which, in cylindrical coordinates, the radial velocity component is zero, whereas the azimuthal component is only depending on r:

$$v = v(r). \tag{A.29}$$

Calculate the circulation[2] around a circle of radius R, centered at the origin. Relate the result to the vorticity distribution within the surface delimited by the circle. (*Hint*: Show that vorticity is $\zeta_z = (1/r)(\mathrm{d}/\mathrm{d}r)(rv)$.)

A.4. Knowing that the divergence of a flow is the relative change of density over time, can you derive the expression of the divergence operator in cylindrical and spherical coordinates? (*Hint*: Look at the mass-conservation equation.)

NUMERICAL EXERCISE

A.1. Plot the velocity and vorticity fields of Analytical Problem A.2.

[2] Circulation is tangential velocity integrated along the chosen path, here simply the product of the azimuthal velocity by the circumference of the circle.