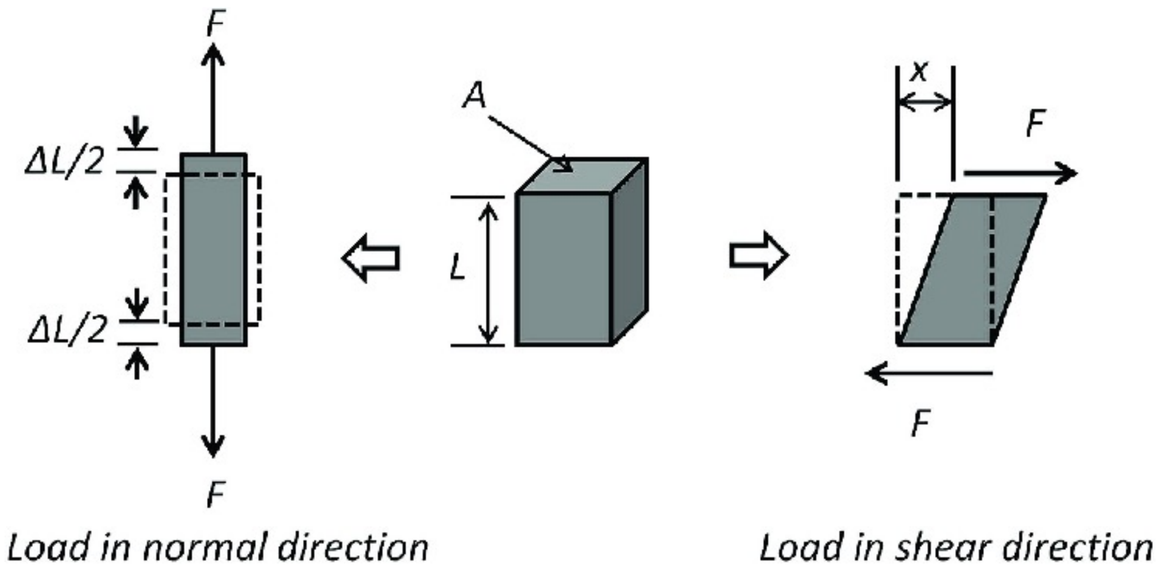
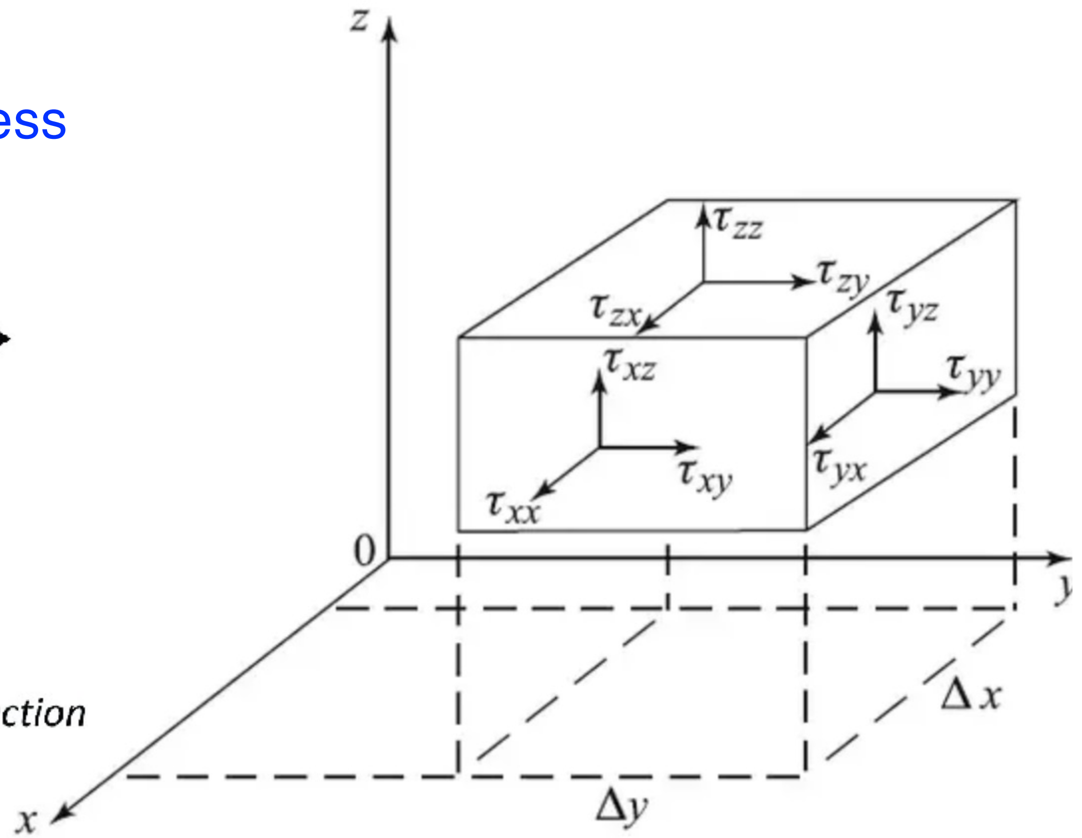
Frictional force term

Stress – second-order tensor (N m^{-2})

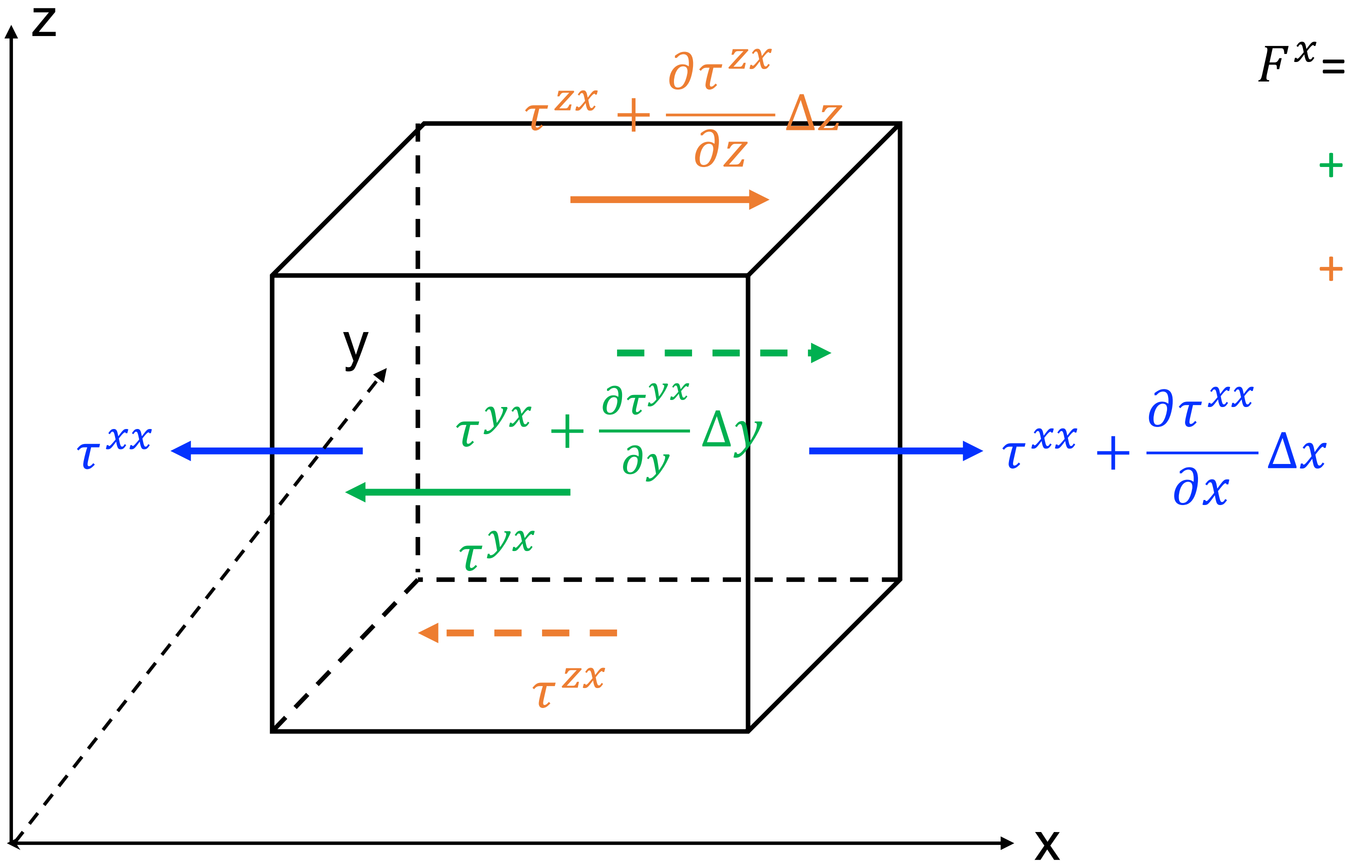
normal stress

shear stress





9 components



$$
\begin{aligned}
F^{x} = & \left(\tau^{xx} + \frac{\partial \tau^{xx}}{\partial x}\Delta x\right)\Delta y \Delta z - \tau^{xx}\Delta y \Delta z \\
& + \left(\tau^{yx} + \frac{\partial \tau^{yx}}{\partial y}\Delta y\right)\Delta x \Delta z - \tau^{yx}\Delta x \Delta z \\
& + \left(\tau^{zx} + \frac{\partial \tau^{zx}}{\partial z}\Delta z\right)\Delta x \Delta y - \tau^{zx}\Delta x \Delta y
\end{aligned}
$$

For per unit volume

$$
F^{x} = \frac{\partial \tau^{xx}}{\partial x} + \frac{\partial \tau^{yx}}{\partial y} + \frac{\partial \tau^{zx}}{\partial z}
$$

For Newtonian fluids, viscous stress is: $\tau^{zx} = \mu \frac{\partial u}{\partial z}$

μ: dynamic viscosity coefficient

Assumption:
μ is constant

$$F^x = \frac{\partial \tau^{xx}}{\partial x} + \frac{\partial \tau^{yx}}{\partial y} + \frac{\partial \tau^{zx}}{\partial z}$$
$$= \frac{1}{\partial x}\left(\mu \frac{\partial u}{\partial x}\right) + \frac{1}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) + \frac{1}{\partial z}\left(\mu \frac{\partial u}{\partial z}\right)$$
$$= \mu \nabla^2 u$$

Laplacian operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$F^y = \frac{\partial \tau^{xy}}{\partial x} + \frac{\partial \tau^{yy}}{\partial y} + \frac{\partial \tau^{zy}}{\partial z}$$
$$= \frac{1}{\partial x}\left(\mu \frac{\partial v}{\partial x}\right) + \frac{1}{\partial y}\left(\mu \frac{\partial v}{\partial y}\right) + \frac{1}{\partial z}\left(\mu \frac{\partial v}{\partial z}\right)$$
$$= \mu \nabla^2 v$$

$$\frac{F^x}{\rho} = \frac{\mu}{\rho} \nabla^2 u$$
$$= \nu \nabla^2 u$$

ν : kinematic viscosity coefficient

$$F^z = \frac{\partial \tau^{xz}}{\partial x} + \frac{\partial \tau^{yz}}{\partial y} + \frac{\partial \tau^{zz}}{\partial z}$$
$$= \frac{1}{\partial x}\left(\mu \frac{\partial w}{\partial x}\right) + \frac{1}{\partial y}\left(\mu \frac{\partial w}{\partial y}\right) + \frac{1}{\partial z}\left(\mu \frac{\partial w}{\partial z}\right)$$
$$= \mu \nabla^2 w$$

	μ ($\mathrm{kg\,m^{-1}\,s^{-1}}$)	ν ($\mathrm{m^2\,s^{-1}}$)
Air	1.8×10^{-5}	1.5×10^{-5}
Water	1.1×10^{-3}	1.1×10^{-6}
Mercury	1.6×10^{-3}	1.2×10^{-7}

The momentum equations

x direction: $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} +$$

$\nu\nabla^2 u$ (for constant ν)

$$\underline{\frac{\partial}{\partial x}\left(\nu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nu\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\nu\frac{\partial u}{\partial z}\right)} + \cdots$$

y direction: $$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} +$$

$\nu\nabla^2 v$

$$\frac{\partial}{\partial x}\left(\nu\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nu\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(\nu\frac{\partial v}{\partial z}\right) + \cdots$$

z direction: $$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} +$$

$\nu\nabla^2 w$

$$\frac{\partial}{\partial x}\left(\nu\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nu\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(\nu\frac{\partial w}{\partial z}\right) + \cdots$$

Expressions for the force terms are given in the inertial frame.

So is the acceleration term.

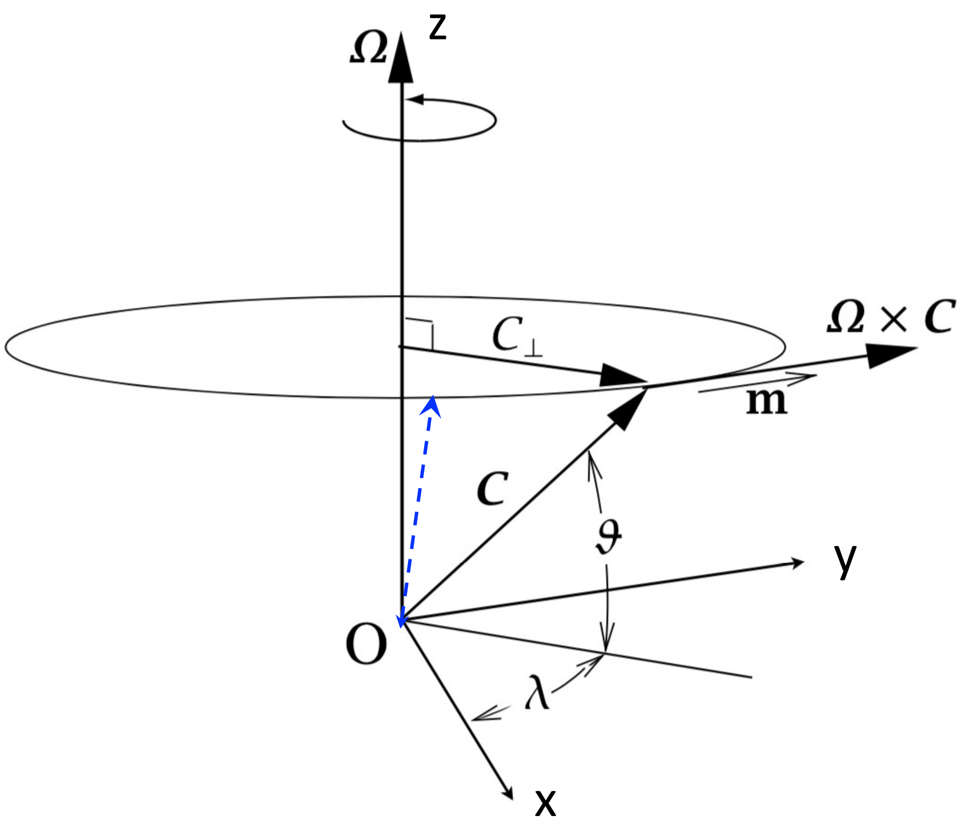


Fig. 2.1 A vector $\boldsymbol{C}$ rotating at an angular velocity $\boldsymbol{\Omega}$. It appears to be a constant vector in the rotating frame, whereas in the inertial frame it evolves according to $(\mathrm{d}\boldsymbol{C}/\mathrm{d}t)_I = \boldsymbol{\Omega}\times\boldsymbol{C}$.

Coriolis Force

The change in C *in* δt with respect to the Intertial frame

$$\delta\boldsymbol{C} = |\boldsymbol{C}|\cos\vartheta\,\delta\lambda\,\boldsymbol{m},$$

$$\delta\lambda = |\boldsymbol{\Omega}|\delta t$$

Let $\hat{\vartheta} = (\pi/2 - \vartheta)$

$$\delta\boldsymbol{C} = |\boldsymbol{C}||\boldsymbol{\Omega}|\sin\hat{\vartheta}\,\boldsymbol{m}\,\delta t = \boldsymbol{\Omega}\times\boldsymbol{C}\,\delta t.$$

$$\left(\frac{\mathrm{d}\boldsymbol{C}}{\mathrm{d}t}\right)_I = \boldsymbol{\Omega}\times\boldsymbol{C}$$

Non-constant vector in the rotating frame $\boldsymbol{B}$

$$\delta \boldsymbol{C} = \boldsymbol{\Omega} \times \boldsymbol{C}\, \delta t$$

$$\boxed{\left(\frac{\mathrm{d}\boldsymbol{C}}{\mathrm{d}t}\right)_I = \boldsymbol{\Omega} \times \boldsymbol{C}}$$

The change of $\boldsymbol{B}$ in the inertial frame:

$$(\delta \boldsymbol{B})_I = (\delta \boldsymbol{B})_R + (\delta \boldsymbol{B})_{rot}$$

With $(\delta \boldsymbol{B})_{rot} = \boldsymbol{\Omega} \times \boldsymbol{B}\, \delta t$

$$\left(\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega} \times \boldsymbol{B}$$

$\boldsymbol{r}$ is a vector from the Earth center pointing to the Earth surface

$$\left(\frac{\mathrm{d}\boldsymbol{v}_I}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\boldsymbol{v}_R}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega} \times \boldsymbol{v}_R + \cancel{\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t} \times \boldsymbol{r}} + \boldsymbol{\Omega} \times \underbrace{\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_I}_{\boldsymbol{v}_I}$$

Subsitute $\boldsymbol{B}$ with $\boldsymbol{v}_R$

$$\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega} \times \boldsymbol{r}$$

$$\boldsymbol{v}_I = \boldsymbol{v}_R + \boldsymbol{\Omega} \times \boldsymbol{r}$$

Centrifugal acceleration

$$\left(\frac{\mathrm{d}\boldsymbol{v}_R}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\boldsymbol{v}_R}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega} \times \boldsymbol{v}_R$$

$$\left(\frac{d\boldsymbol{v}_I}{dt}\right)_I = \left(\frac{d\boldsymbol{v}_R}{dt}\right)_R \underline{+2\boldsymbol{\Omega}\times\boldsymbol{v}_R} + \underline{\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})}$$

Coriolis acceleration

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v}_I - \boldsymbol{\Omega} \times \boldsymbol{r})\right)_I = \left(\frac{\mathrm{d}\boldsymbol{v}_R}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega} \times \boldsymbol{v}_R$$

The Coriolis acceleration

$$2\boldsymbol{\Omega}\times\boldsymbol{v}_R$$

$$\boldsymbol{\Omega} = \Omega\cos\varphi\,\mathbf{j} + \Omega\sin\varphi\,\mathbf{k}$$

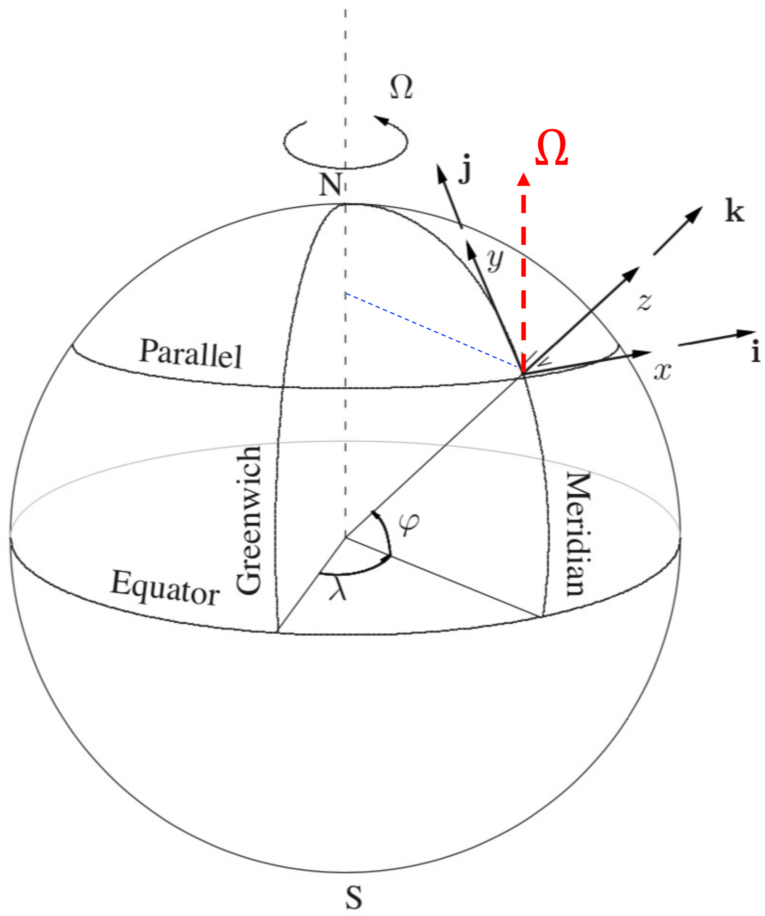
x: $2\Omega\cos\varphi\, w - 2\Omega\sin\varphi\, v$

y: $2\Omega\sin\varphi\, u$

z: $-2\Omega\cos\varphi\, u.$

$$f = 2\Omega\sin\varphi$$
$$f_* = 2\Omega\cos\varphi.$$

f: Coriolis parameter



The momentum equations

$$\left(\frac{d\boldsymbol{v_I}}{dt}\right)_I = \left(\frac{d\boldsymbol{v_R}}{dt}\right)_R + 2\boldsymbol{\Omega}\times\boldsymbol{v}_R + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}) = \text{ force terms}$$

Nonlinear advection term Coriolis term Pressure gradient term

x direction: $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv + f_* w = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u + \cdots$$

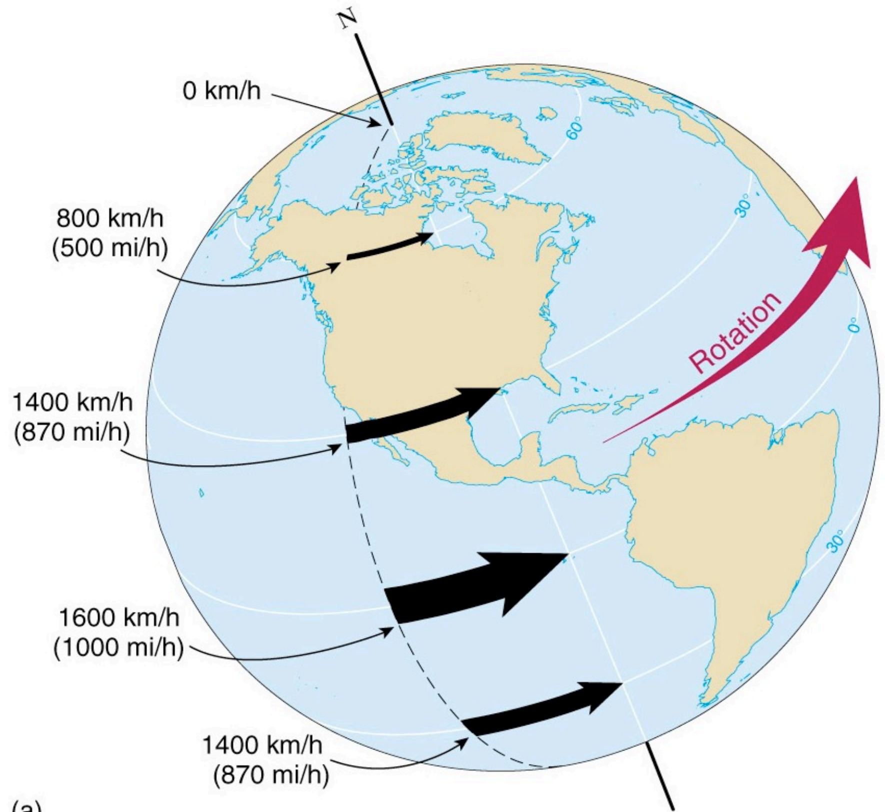
Viscosity term

Local acceleration

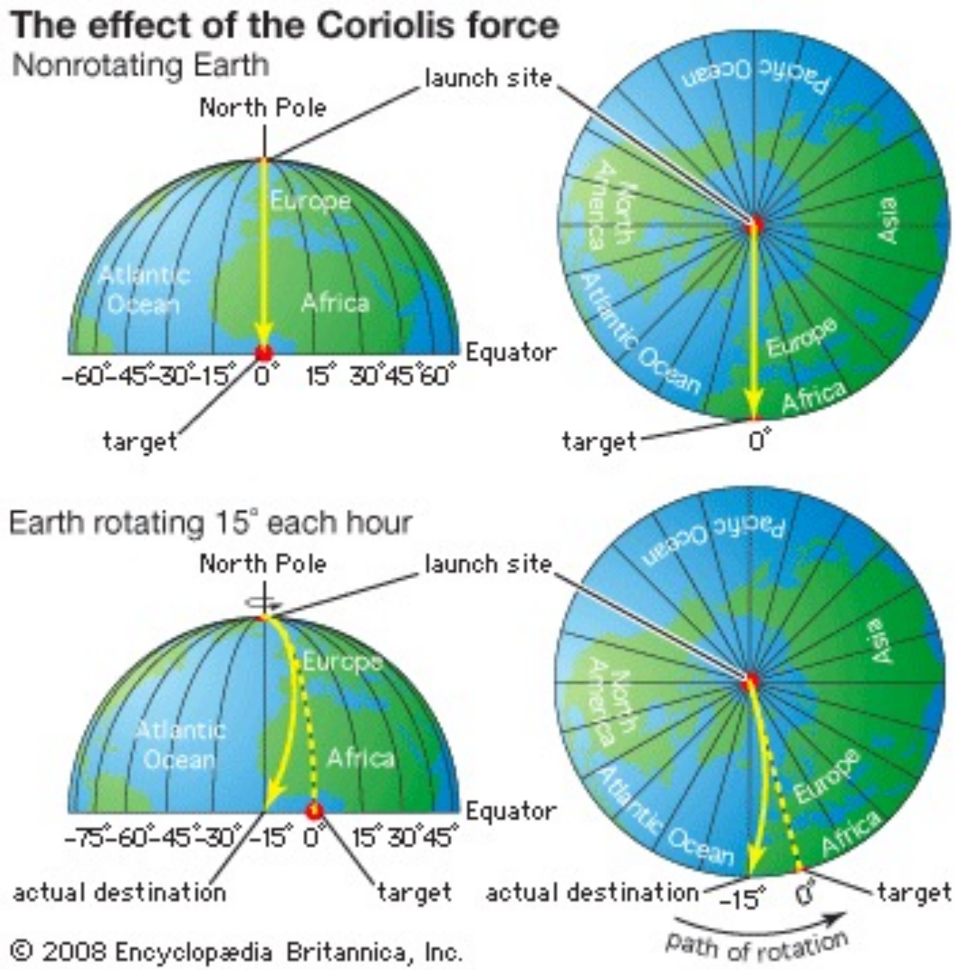
y direction: $$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v + \cdots$$

z direction: $$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} - f_* u = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w + \cdots$$

The Coriolis Force



(a)

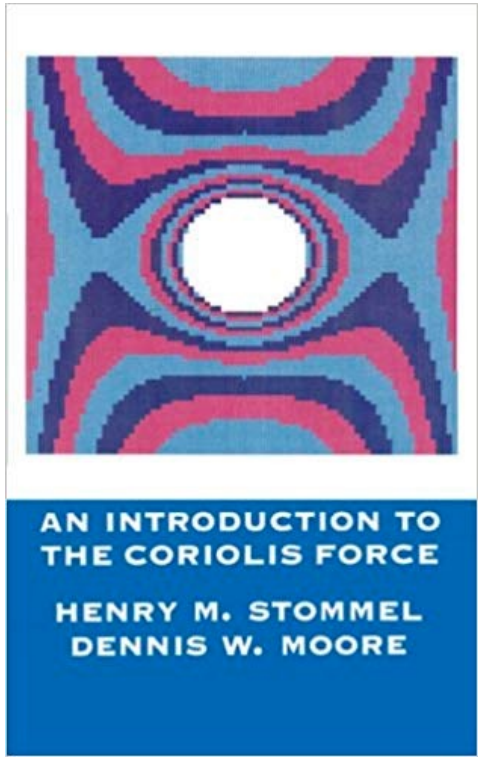


References about Coriolis force

- ***An introduction to the Coriolis Force***, H.M. Stommel and D.W. Moore, 1989

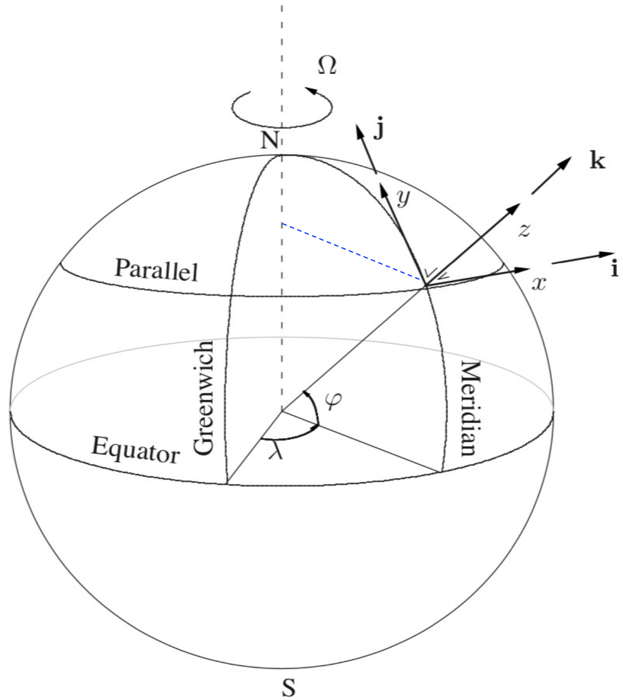
- Persson (2000). ***What is the Coriolis force***? Weather, 55, 165–170.

- Persson (1998). ***How to understand the Coriolis force?*** Bulletin of the American Meteorological Society, 79(7), 1373–1385.



f-plane and β-plane

$$f = 2\Omega\sin\varphi = \underbrace{2\Omega\left(sin\varphi_0\right.}_{f_0} \left. + cos\varphi_0(\varphi-\varphi_0)\right)$$



If $\varphi - \varphi_0$ is small:

f-plane: $f = f_0$ is a constant.

If $\varphi - \varphi_0$ cannot be neglected:

β-plane: $f = f_0 + 2\Omega cos\varphi_0\underbrace{(\varphi-\varphi_0)}_{y/a} = f_0 + \underbrace{\frac{2\Omega\cos\varphi_0}{a}}_{\beta}y$

Centrifugal force term

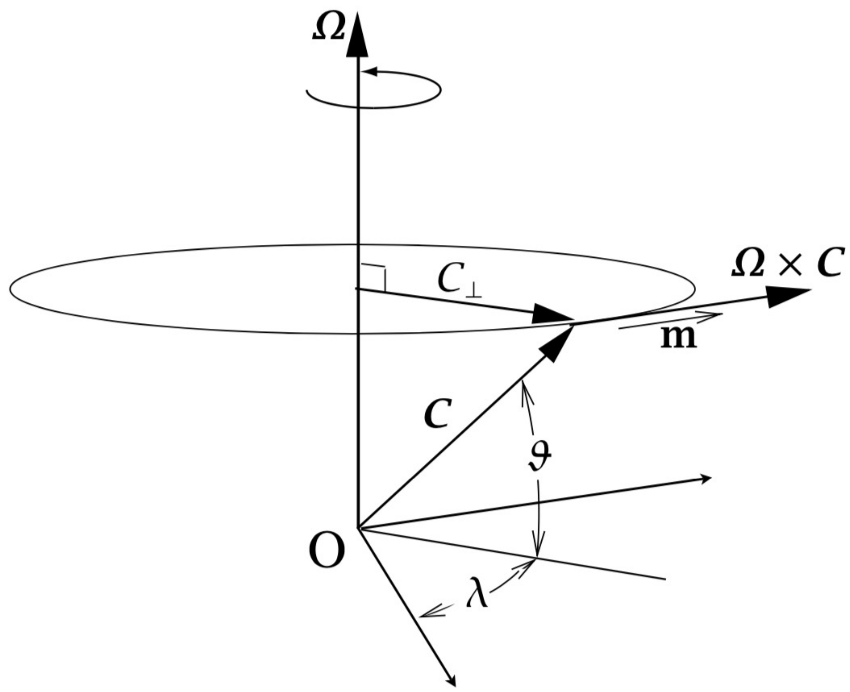
$$-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})$$

Substitute $\boldsymbol{C}$ with $\boldsymbol{r}$:

$$= -\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}_{\perp})$$

$$= -(\boldsymbol{\Omega}\cdot\boldsymbol{r}_{\perp})\boldsymbol{\Omega} + (\boldsymbol{\Omega}\cdot\boldsymbol{\Omega})\boldsymbol{r}_{\perp}$$

$$= \Omega^2\boldsymbol{r}_{\perp}$$



Gravity force term

The Gravity Term in the Momentum Equation The gravitational attraction of two masses M_1 and m is:

$$\mathbf{F}_g = \frac{G\,M_1\,m}{R^2}$$

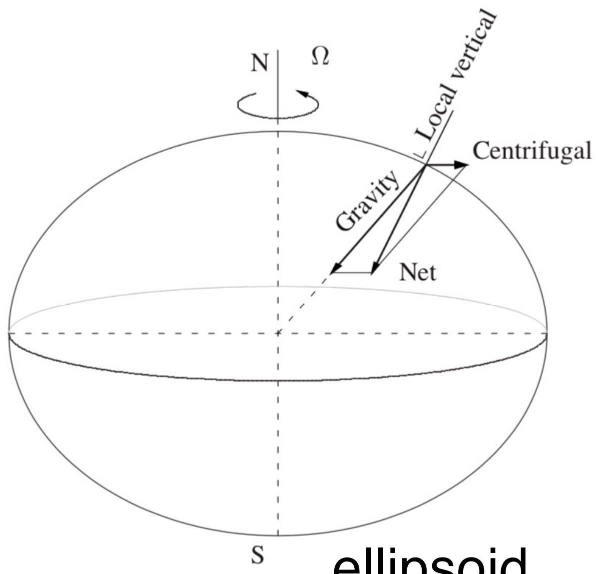
where R is the distance between the masses, and G is the gravitational constant. The vector force $\mathbf{F}_g$ is along the line connecting the two mases.

The force per unit mass due to gravity is:

$$\frac{\mathbf{F}_g}{m} = \mathbf{g}_f = \frac{G\,M_E}{R^2} \tag{7.15}$$

where M_E is the mass of Earth. Adding the centrifugal acceleration to (7.15) gives gravity $\mathbf{g}$ (Figure 7.5):

effective gravity $$\mathbf{g} = \mathbf{g}_f - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{R}) \tag{7.16}$$



ellipsoid

The momentum equations

Nonlinear advection term Coriolis term Pressure gradient term

x direction: $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv + f_* w = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u$$

Viscosity term

Local acceleration

y direction: $$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v$$

z direction: $$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + f_* u = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w - g$$

gravity term

Continuity equation

$$\text{fluid loss} = \int_S \rho \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = \int_V \nabla \cdot (\rho \boldsymbol{v})\, \mathrm{d}V$$

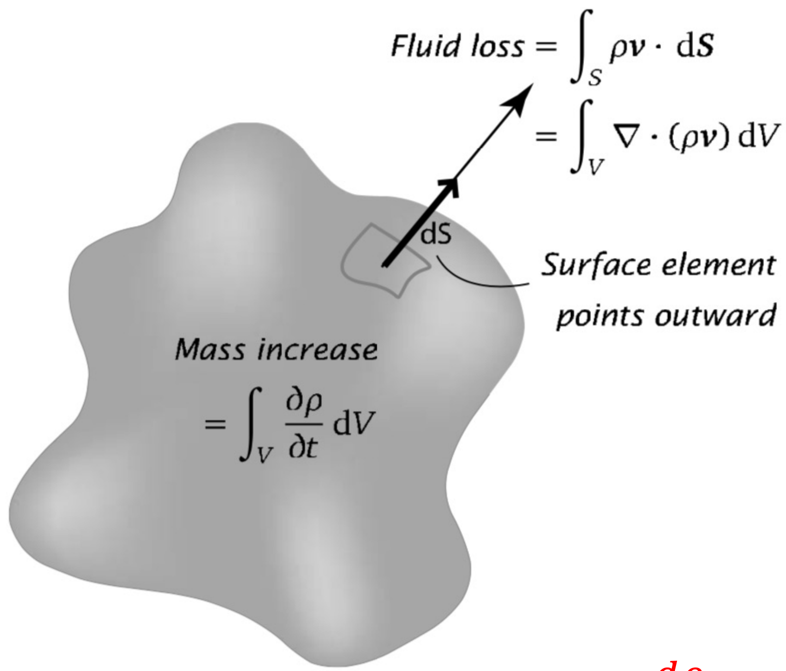
Divergence (Gaussian) theorem

$$\|$$

Vallis (Eq. 1.19)

$$-\frac{dM}{dt} = -\frac{d}{dt}\int_v \rho\, dV = -\int_v \frac{\partial \rho}{\partial t}\, dV$$

$$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v})\right] \mathrm{d}V = 0$$



For any arbitrary volume V,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \quad \text{or} \quad \frac{d\rho}{dt} + \rho \nabla \cdot \boldsymbol{v} = \boldsymbol{0}$$

For incompressible fluids ($\frac{d\rho}{dt} = 0$)

$$\nabla \cdot \boldsymbol{v} = 0$$

The thermodynamic equation

1st Law of Thermodyamics (energy conservation, and for per unit mass):

$$\frac{dI}{dt} = \mathrm{Q} - \mathrm{W}$$

I: internal energy

Q: rate of heat gain

W: rate of work done by pressure force onto surrounding fluids

C_v: specific heat capacity

$$I = C_v T$$

Fourier's Law of heat conduction:

$$Q = \frac{k_T}{\rho} \nabla^2 T$$

Rate of work done by pressure force:

specific volume

$$W = p \frac{d\alpha}{dt} \qquad \boxed{\alpha = 1/\rho}$$

$$C_v \frac{dT}{dt} = \frac{k_T}{\rho} \nabla^2 T - p \frac{d\alpha}{dt}$$

$$\boxed{\alpha = 1/\rho}$$

$$= \frac{k_T}{\rho} \nabla^2 T + \frac{p}{\rho^2} \frac{d\rho}{dt}$$

From the continuity equation:

$$\frac{d\rho}{dt} = -\rho \nabla \cdot \boldsymbol{v}$$

$$C_v \frac{dT}{dt} = \frac{k_T}{\rho} \nabla^2 T - \frac{p}{\rho} \nabla \cdot \boldsymbol{v}$$

For incompressible fluids:

K_T: thermal diffusivity coefficient

$$\frac{dT}{dt} = \frac{k_T}{\rho C_v} \nabla^2 T$$

$$\frac{dS}{dt} = K_S \nabla^2 S$$