The momentum equation

x direction: $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv + \cancel{f_* w} =$$

$$\frac{U}{T} \qquad \frac{U^2}{L} \qquad \frac{U^2}{L} \qquad W\frac{U}{H} \qquad fU \qquad \cancel{fW}$$

$W \ll U$

$$-\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(A_H\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_H\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_V\frac{\partial u}{\partial z}\right)$$

$$\frac{P}{\rho_0 L} \qquad A_H\frac{U}{L^2} \qquad A_H\frac{U}{L^2} \qquad A_V\frac{U}{H^2}$$

Non-dimensional numbers

Rossby number (R_o) = scale of nonlinear advection term / scale of Coriolis term

$$= \frac{U^2}{L} / fU = U/fL$$

measure of the role of Earth's rotation in motions

$R_o \ll 1$: rotation is important for the motion

Ekman number (E_k) = scale of vertical viscosity term / scale of Coriolis term

$$= A_V \frac{U}{H^2} / fU = \frac{A_V}{fH^2}$$

measure of the importance of frictional force

$E_k \ll 1$: friction can be neglected

Reynolds number (Re) = scale of the nonlinear term / scale of viscous term

$$= \frac{U^2}{L} / \nu \frac{U}{L^2} = \frac{UL}{\nu}$$

small Re: viscous flow

large Re: inviscous, turbulent flow

The momentum equation

From the x-direction equation:

$$\frac{P}{\rho_0 L} \sim fU \sim \frac{U^2}{L} \sim A_H \frac{U}{L^2} \sim A_V \frac{U}{H^2}$$

As $H \ll L$, $W \ll U$

z direction:

$$\cancel{\frac{\partial w}{\partial t}} + \cancel{u\frac{\partial w}{\partial x}} + \cancel{v\frac{\partial w}{\partial y}} + \cancel{w\frac{\partial w}{\partial z}} + \cancel{f_* u} =$$

$$\cancel{\frac{W}{T}} \quad \cancel{\frac{UW}{L}} \quad \cancel{\frac{UW}{L}} \quad \cancel{W\frac{W}{H}} \quad \cancel{fU}$$

$$-\frac{1}{\rho_0}\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left(\cancel{A_H \frac{\partial w}{\partial x}}\right) + \frac{\partial}{\partial y}\left(\cancel{A_H \frac{\partial w}{\partial y}}\right) + \frac{\partial}{\partial z}\left(\cancel{A_V \frac{\partial w}{\partial z}}\right) - \frac{\rho}{\rho_0} g$$

$$\frac{P}{\rho_0 H} \quad \cancel{A_H \frac{W}{L^2}} \quad \cancel{A_H \frac{W}{L^2}} \quad \cancel{A_V \frac{W}{H^2}} \quad \frac{\Delta\rho}{\rho_0} g$$

$$\frac{P}{\rho_0 H} \gg fU$$

$$\frac{P}{\rho_0 H} \gg \frac{UW}{L}$$

$$\frac{P}{\rho_0 H} \gg A_H \frac{W}{L^2}$$

$$\frac{P}{\rho_0 H} \gg A_V \frac{W}{H^2}$$

$$\frac{\partial p}{\partial z} + \rho g = 0$$ Hydrostatic equation (balance)

Hydrostatic balance: $$\frac{\partial p}{\partial z} + \rho g = 0$$

$$p = \tilde{p}(z) + p'(x, y, z, t) \qquad \tilde{p}(z) = P_0 - \rho_0 g z$$

$$\frac{\partial \tilde{p}}{\partial z} + \rho_0 g = 0 \qquad \frac{\partial p'}{\partial z} + \Delta\rho g = 0$$

$$\frac{P}{H} \quad g\Delta\rho$$

$$\frac{P}{\rho_0 L} \sim \mathrm{fU} \qquad \frac{gH\Delta\rho}{P} = \frac{gH\Delta\rho}{\rho_0 f L U} = \frac{U}{fL} \cdot \frac{gH\Delta\rho}{\rho_0 U^2}$$

$$= R_o \frac{N^2}{(U/H)^2}$$ Richardson number

The momentum equations

$$\frac{\partial p'}{\partial x}$$

x direction: $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(A_H\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_H\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_V\frac{\partial u}{\partial z}\right)$$

y direction: $$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(A_H\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_H\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_V\frac{\partial v}{\partial z}\right)$$

$$\frac{\partial p'}{\partial y}$$

z direction: $$\frac{\partial p}{\partial z} + \rho g = 0$$ Hydrostatic equation

$$\frac{\partial p'}{\partial z} + \Delta\rho g = 0$$

Summary of the governing equations

Momentum equation: $$\frac{d\boldsymbol{v}}{dt} + f\boldsymbol{k}\times\boldsymbol{v} = -\frac{1}{\rho}\nabla p + \nu \nabla^2\boldsymbol{v} + \overset{-\nabla\Phi}{\boldsymbol{g}}$$

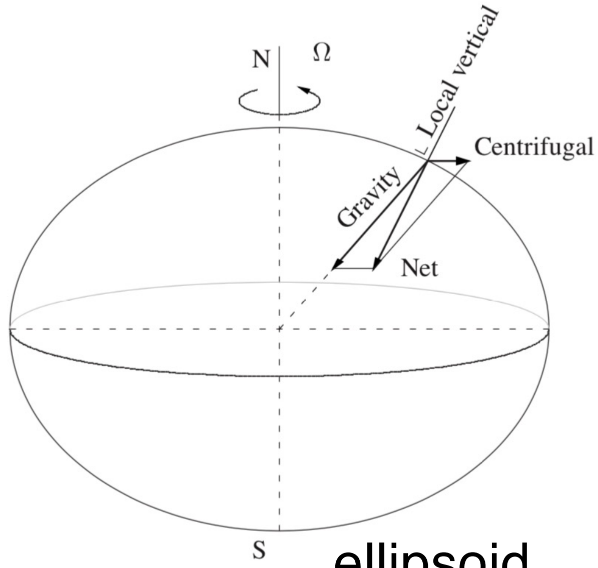
Continuity equation: $$\frac{d\rho}{dt} + \rho\nabla\cdot\boldsymbol{v} = 0$$

Tracer equations: $$\frac{dT}{dt} = K\nabla^2 T$$

$$\frac{dS}{dt} = K\nabla^2 S$$

Equation of state:

$$\rho = \rho(\mathrm{T}, \mathrm{S}, \mathrm{P})$$



ellipsoid

Energy equation – Mechanical energy

$\Phi = gz$

For constant density:

$$f\boldsymbol{k}\times\boldsymbol{v} + \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\nabla(\phi + \Phi) + \nu\nabla^2\boldsymbol{v},$$

$\boxed{\phi = p/\rho_0}$ $\boxed{\boldsymbol{g} = -\nabla\Phi}$

$$(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\boldsymbol{v}\times\boldsymbol{\omega} + \nabla(\boldsymbol{v}^2/2),$$

$\boxed{\boldsymbol{\omega} \equiv \nabla\times\boldsymbol{v}}$

Ingore the viscous term:

$$f\boldsymbol{k}\times\boldsymbol{v} + \frac{\partial\boldsymbol{v}}{\partial t} + \boldsymbol{\omega}\times\boldsymbol{v} = -\nabla B,$$

$\boxed{B = (\phi + \Phi + \boldsymbol{v}^2/2)}$

Bernoulli function

Multiply the equation by $\rho_0\boldsymbol{v}$:

$$f\rho_0\boldsymbol{v}\cdot(\cancel{\boldsymbol{k}\times\boldsymbol{v}}) + \frac{1}{2}\frac{\partial\rho_0\boldsymbol{v}^2}{\partial t} + \rho_0\boldsymbol{v}\cdot(\cancel{\boldsymbol{\omega}\times\boldsymbol{v}}) = -\rho_0\boldsymbol{v}\cdot\nabla B$$

For incompressibe fluids:

$\boxed{K = \rho_0\boldsymbol{v}^2/2}$

$$\frac{\partial K}{\partial t} + \nabla\cdot(\rho_0\boldsymbol{v}B) = 0.$$

Mechanical energy is conserved unless pressure does work

E: mechanical energy

$$\frac{\partial}{\partial t}\left[\underline{\rho_0\left(\frac{1}{2}\boldsymbol{v}^2 + \Phi\right)}\right] + \nabla\cdot\left[\rho_0\boldsymbol{v}\left(\frac{1}{2}\boldsymbol{v}^2 + \Phi + \phi\right)\right] = 0.$$

$$\boxed{\frac{\partial E}{\partial t} + \nabla\cdot[\boldsymbol{v}(E + p)] = 0.}$$

$$\frac{\partial K}{\partial t} + \nabla \cdot (\rho_0 \boldsymbol{v} B) = 0.$$

For a volume through which there is no mass flux in and out of surface (like bounded by rigid walls):

$$\frac{\mathrm{d}\widehat{K}}{\mathrm{d}t} \equiv \frac{\mathrm{d}}{\mathrm{d}t} \int_V K \,\mathrm{d}V = -\int_V \nabla \cdot (\rho_0 \boldsymbol{v} B) \,\mathrm{d}V = -\int_S \rho_0 B \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = 0,$$

The kinetic energy is conserved

The potential energy:

$$\widehat{P} = \int_V \rho_0 g z \,\mathrm{d}V,$$

There is no exchange between kinetic energy and potential energy

The Bernoulli function

$$f\boldsymbol{k}\times\boldsymbol{v} + \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\omega}\times\boldsymbol{v} = -\nabla B,$$

$$\boxed{B = (\phi + \Phi + \boldsymbol{v}^2/2)}$$

For steady flow, multiply the equation by $\boldsymbol{v}$:

$$\boldsymbol{v}\cdot\nabla B = 0$$

$$\mathrm{D}B/\mathrm{D}t = 0$$

The Bernoulli function is constant along a streamline

Energy equation – Total energy

For fluids with density variation:

$$\rho f \boldsymbol{k} \times \boldsymbol{v} + \rho \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\nabla p - \rho \nabla \Phi$$

Multiply the equation by $\boldsymbol{v}$:

$$\frac{1}{2}\rho \frac{\mathrm{D}\boldsymbol{v}^2}{\mathrm{D}t} = -\boldsymbol{v} \cdot \nabla p - \rho \boldsymbol{v} \cdot \nabla \Phi = -\nabla \cdot (p\boldsymbol{v}) + p\nabla \cdot \boldsymbol{v} - \rho \boldsymbol{v} \cdot \nabla \Phi$$

From the thermodynamic equation, considering adiabatic flow:

$$\rho \frac{\mathrm{D}I}{\mathrm{D}t} = -p\nabla \cdot \boldsymbol{v}.$$

$$C_v \frac{dT}{dt} = \frac{k_T}{\rho} \nabla^2 T - \frac{p}{\rho} \nabla \cdot \boldsymbol{v}$$

$$\rho \frac{\mathrm{D}\Phi}{\mathrm{D}t} = \rho \boldsymbol{v} \cdot \nabla \Phi.$$

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}\boldsymbol{v}^2 + I + \Phi\right)\right] + \nabla \cdot \left[\rho \boldsymbol{v}\left(\frac{1}{2}\boldsymbol{v}^2 + I + \Phi + p/\rho\right)\right] = 0$$

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}\boldsymbol{v}^2 + I + \Phi\right) = -\nabla \cdot (p\boldsymbol{v}),$$

$$\boxed{\frac{\partial E}{\partial t} + \nabla \cdot [\boldsymbol{v}(E + p)] = 0}$$

Total energy is conserved unless pressure does work

Initial and boundary conditions

Initial condition: $\frac{\partial}{\partial t}$

Boundary condition: $\frac{\partial}{\partial x}, \frac{\partial^2}{\partial x^2}$

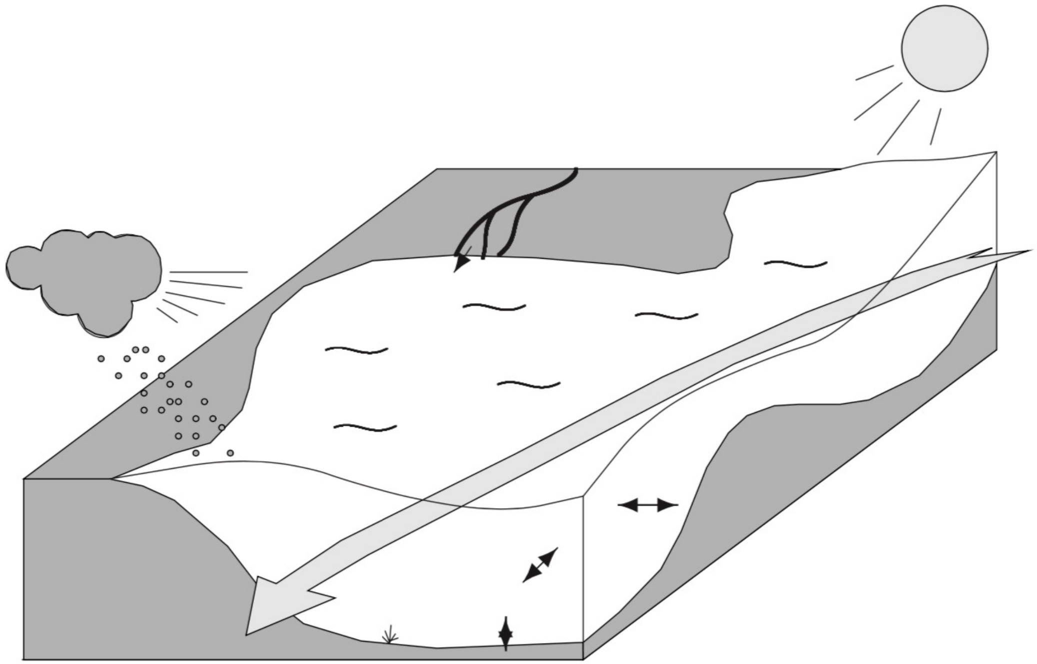


Figure 4-1 Schematic representation of possible exchanges between a coastal system under investigation and the surrounding environment. Boundary conditions must specify the influence of this outside world on the evolution within the domain. Exchanges may take place at the air-sea interface, in bottom layers, along coasts and/or at any other boundary of the domain.

Kinematic boundary conditions — velocity

Principle: flow cannot penetrate a solid wall

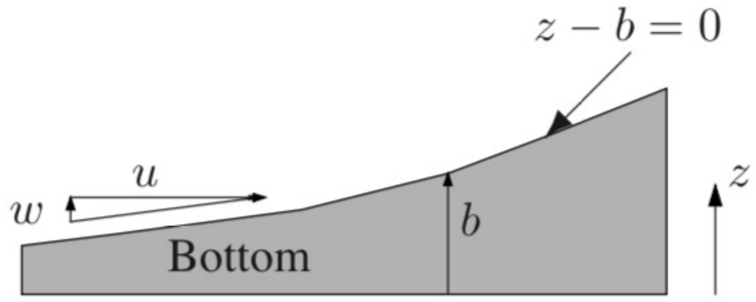
Bottom boudary: $\underbrace{}^{G}z - b(x, y) = 0$

$$\nabla G = \left(-\frac{\partial b}{\partial x}, -\frac{\partial b}{\partial y}, 1\right)$$

Considering impermeability:

$$\boldsymbol{u} \cdot \nabla G = 0$$

$$\boxed{w = u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y}}$$



For flat bottom:

$$w = 0$$

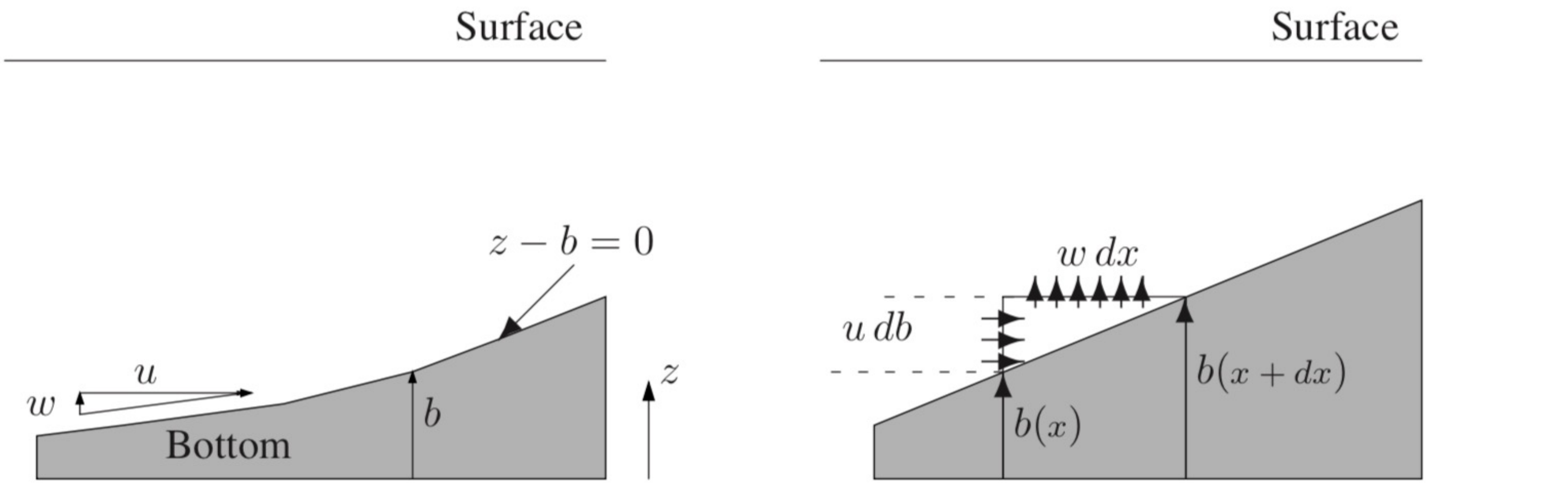


Figure 4-2 Notation and two physical interpretations of the bottom boundary condition illustrated here in a (x, z) plane for a topography independent of y. The impermeability of the bottom imposes that the velocity be tangent to the bottom defined by $z - b = 0$. In terms of the fluid budget, which can be extended to a finite volume approach, expressing that the horizontal inflow matches the vertical outflow requires $u\,(b(x+dx) - b(x)) = w\,dx$, which for $dx \to 0$ leads to (4.28). Note that the velocity ratio w/u is equal to the topographic slope db/dx, which scales like the ratio of vertical to horizontal length scales, *i.e.*, the aspect ratio.

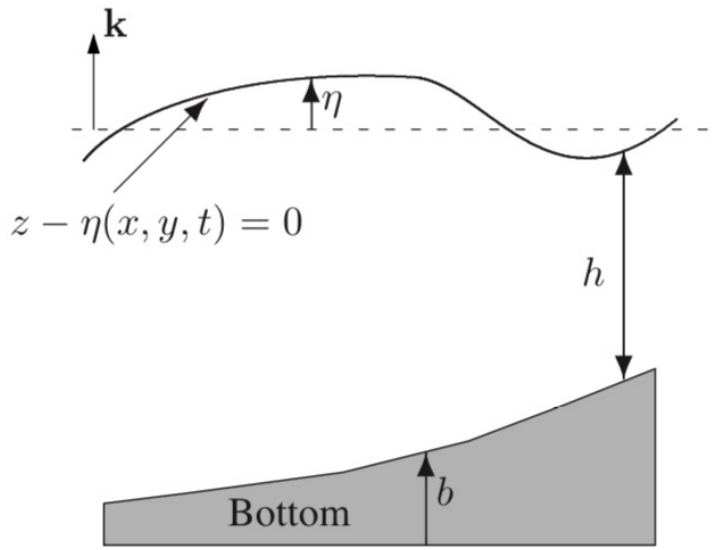
Surface boudary: the boundary is moving with the fluid (free surface)

$$z - \eta(x, y, t) = 0$$

Without precipication and evaporation, it can be considered as a material surface:

$$\frac{d}{dt}(z - \eta) = 0$$

$$\boxed{w = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}}$$



Rigid-lid approximation:

$$\eta = C$$

$$w = 0$$

Tangential velocity for fixed boundaries

No-slip boundary condition:

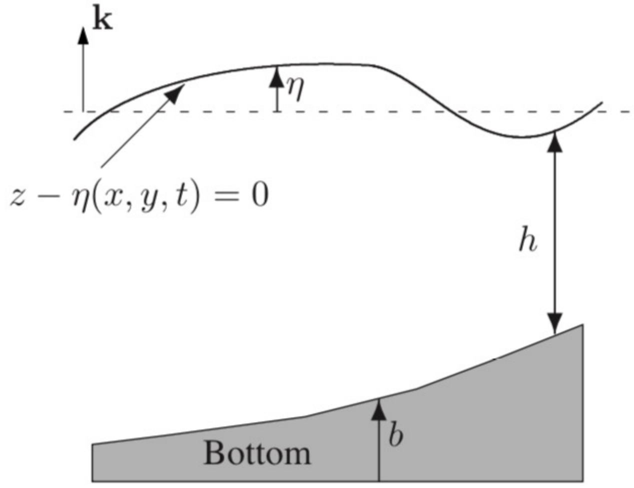
$$\boldsymbol{u} \cdot \boldsymbol{s} = \boldsymbol{0}$$

Free-slip boundary condition (for very thin boundary layer):

$$\boldsymbol{u} \cdot \boldsymbol{s} \neq \boldsymbol{0}$$

$\boldsymbol{s}$: unit vector in the tangential direction of the boundary

Dynamic boundary conditions — force



Pressure:

$$p_{\text{atm}} = p_{\text{sea}} \quad \text{at air-sea interface.}$$

$$p_{\text{sea}}(z = 0) = p_{\text{atm at sea level}} + \rho_0 g \eta$$

Surface stress:

Stress must be cotinuous along moving boudaries (sea surface)

$$\rho_0 \nu_E \left(\frac{\partial u}{\partial z} \right) \bigg|_{\text{at surface}} = \tau^x, \quad \rho_0 \nu_E \left(\frac{\partial v}{\partial z} \right) \bigg|_{\text{at surface}} = \tau^y$$

The stress must be equal to the wind stress that is parameterized by:

$$U_{10} = \sqrt{u_{10}^2 + v_{10}^2} \qquad \tau^x = C_d \, \rho_{\text{air}} \, U_{10} u_{10}, \quad \tau^y = C_d \, \rho_{\text{air}} \, U_{10} v_{10}$$

Bottom stress:

linear parameterization: $\boldsymbol{\tau} = r_1 \boldsymbol{u}$

quadratic parameterization: $\boldsymbol{\tau} = r_2 |\boldsymbol{u}|\boldsymbol{u}$

logrithmic parameterization: $\boldsymbol{\tau} = \frac{K^2}{ln^2(z/z_0)} |\boldsymbol{u}|\boldsymbol{u}$

K: von Kármán's constant
z_0: roughness length

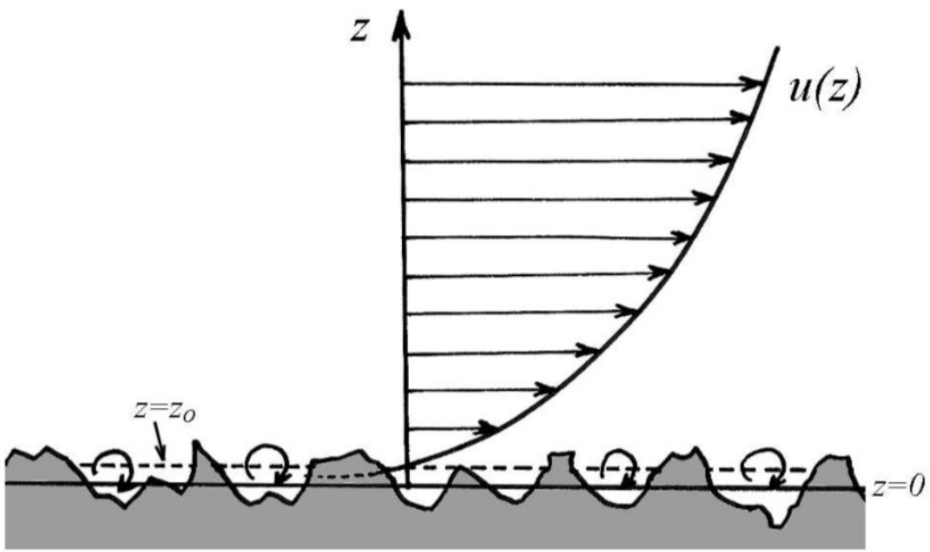


Figure 8-2 Velocity profile in the vicinity of a rough wall. The roughness heigh z_0 is smaller than the averaged height of the surface asperities. So, the velocity u falls to zero somewhere within the asperities, where local flow degenerates into small vortices between the peaks, and the negative values predicted by the logarithmic profile are not physically realized.

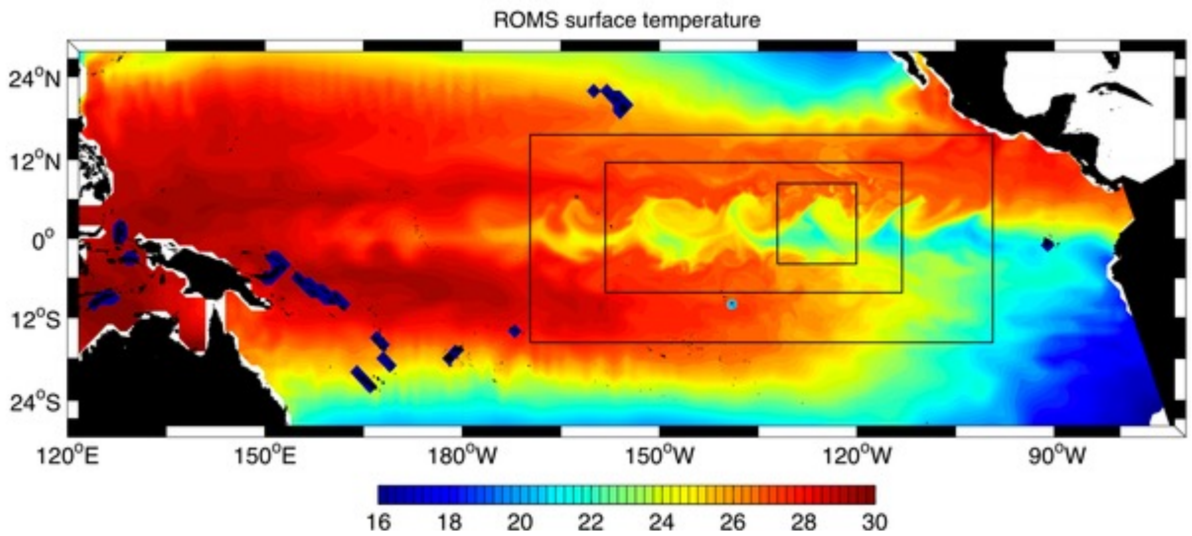
Open boundary conditions

Dirichlet condition: Prescribing the value of the variable at the boundary ($\phi = \phi_0$)

Newman condition: setting the gradient to impose the diffusive flux of a quantity ($\kappa\frac{\partial\phi}{\partial n}$)

Cauchy condition: Prescribing a total, advective plus diffusive, flux ($u\phi - \kappa\frac{\partial\phi}{\partial x}$)

Data source for open boundary conditions**: observations, modelling results**



nested models