Internal waves – without rotation

Assumptions: stratified, non-rotational, inviscid, incompressible, small perturbation

The density in a stratified system with wave perturbation is:

$$\rho = \rho_0 + \bar{\rho}(z) + \rho'(x, y, z, t) \qquad \rho' \ll \bar{\rho}(z) << \rho_0$$

The governing equations are:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{\rho}{\rho_0}g$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial \rho}{\partial t} + w\frac{d\bar{\rho}}{dz} = 0$$

incompressible fluid:

$$\frac{d\rho}{dt} = 0$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = 0$$

$$\frac{\partial \rho}{\partial t} + u\cancel{\frac{\partial \rho'}{\partial x}} + v\cancel{\frac{\partial \rho'}{\partial y}} + w\frac{\partial \bar{\rho}}{\partial z} = 0$$

From the horizontal momentum equations:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -\frac{1}{\rho_0}\nabla_h{}^2 p = -\frac{\partial^2 w}{\partial z \partial t} \qquad (1)$$

From the vertical momentum equation:

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{\rho}{\rho_0} g$$

$$\frac{\partial \rho}{\partial t} + w\frac{d\bar{\rho}}{dz} = 0$$

$$\begin{aligned}\frac{\partial^2 w}{\partial t^2} &= -\frac{1}{\rho_0}\frac{\partial^2 p}{\partial z \partial t} - \frac{g}{\rho_0}\frac{\partial \rho}{\partial t} \\ &= -\frac{1}{\rho_0}\frac{\partial^2 p}{\partial z \partial t} + \frac{g}{\rho_0} w \frac{d\bar{\rho}}{dz} \\ &= -\frac{1}{\rho_0}\frac{\partial^2 p}{\partial z \partial t} - N^2 w \qquad (2)\end{aligned}$$

From (1):

$$-\frac{1}{\rho_0}\frac{\partial^2}{\partial z \partial t}\nabla_h{}^2 p = -\frac{\partial^4 w}{\partial z^2 \partial t^2}$$

From (2):

$$\frac{\partial^2}{\partial t^2}\nabla_h{}^2 w = -\frac{\partial^4 w}{\partial z^2 \partial t^2} - N^2 \nabla_h{}^2 w$$

$$\frac{\partial^2}{\partial t^2}\nabla^2 w + N^2 \nabla_h{}^2 w = 0$$

$$\frac{\partial^2}{\partial t^2}\nabla^2 w + N^2 {\nabla_h}^2 w = 0$$

Assume constant N^2, and apply a wavelike solution:

$$w = w_0 e^{i(kx+ly+mz-\omega t)}$$

$$(-i\omega)^2[(ik)^2 + (il)^2 + (im)^2] + N^2[(ik)^2 + (il)^2] = 0$$

$$\omega^2(k^2 + l^2 + m^2) - N^2(k^2 + l^2) = 0$$

$$\omega^2 = \frac{N^2(k^2 + l^2)}{k^2 + l^2 + m^2} \quad = N^2\frac{{\boldsymbol{K_h}}^2}{\boldsymbol{K}^2}$$

$$= N^2 cos^2\theta$$

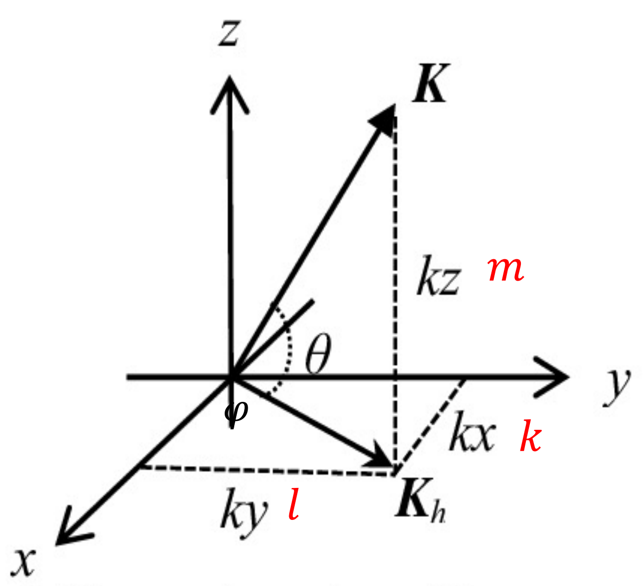


Fig. 1. Wave number vector and its components.

Properties of internal waves:

- the propagation of internal waves are not restricted to the horizontal plane
- $\omega \leq N, \quad \omega = N$ when waves propagate in the horizontal direction

- ω is independent of the wave number magnitude

$(u, v, w) = (u_0, v_0, w_0)e^{i(kx+ly+mz-\omega t)}$, and from the continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$(iku_0 + ilv_0 + imw_0)e^{i(kx+ly+mz-\omega t)} = 0 \qquad \boldsymbol{K} \cdot \boldsymbol{u} = 0$$

- the flow direction is perpendicular to the wave propagation

The group speed:

$$\boxed{\omega^2 = \frac{N^2(k^2 + l^2)}{k^2 + l^2 + m^2}}$$

$$c_{gx} = \frac{\partial \omega}{\partial k} = \frac{N}{K} sin^2\theta cos\varphi$$

$$c_{gy} = \frac{\partial \omega}{\partial l} = \frac{N}{K} sin^2\theta sin\varphi$$

$$c_{gz} = \frac{\partial \omega}{\partial m} = -\frac{N}{K} cos\theta sin\theta$$

$$< 0$$

$$c_z \cdot c_{gz} = \frac{\omega}{m} \cdot \frac{\partial \omega}{\partial m} = \frac{N cos\theta}{K sin\theta} \cdot -\frac{N}{K} cos\theta sin\theta = -\frac{N^2}{K^2} cos^2\theta$$

- vertical phase speed is opposite to the vertical group speed

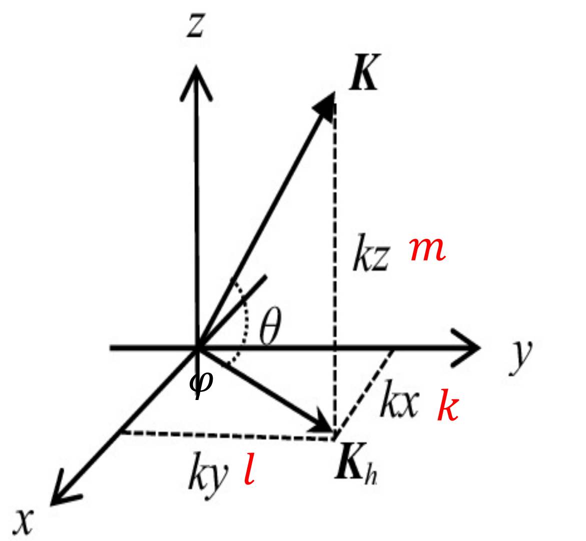


Fig. 1. Wave number vector and its components.

$$\boldsymbol{K} \cdot \boldsymbol{c_g} = k\frac{\partial \omega}{\partial k} + l\frac{\partial \omega}{\partial l} + m\frac{\partial \omega}{\partial m}$$

$$= K cos\theta \cos\varphi \frac{N}{K} sin^2\theta cos\varphi + K cos\theta \sin\varphi \frac{N}{K} sin^2\theta sin\varphi + K sin\theta (-\frac{N}{K} cos\theta sin\theta)$$

$$= N cos\theta sin^2\theta cos^2\varphi + N cos\theta sin^2\theta sin^2\varphi - N cos\theta sin^2\theta$$

$$= 0$$

$$k = K cos\theta \cos\varphi$$

$$l = K cos\theta \sin\varphi$$

$$m = K sin\theta$$

- group speed is perpendicular to the wave propagation direction

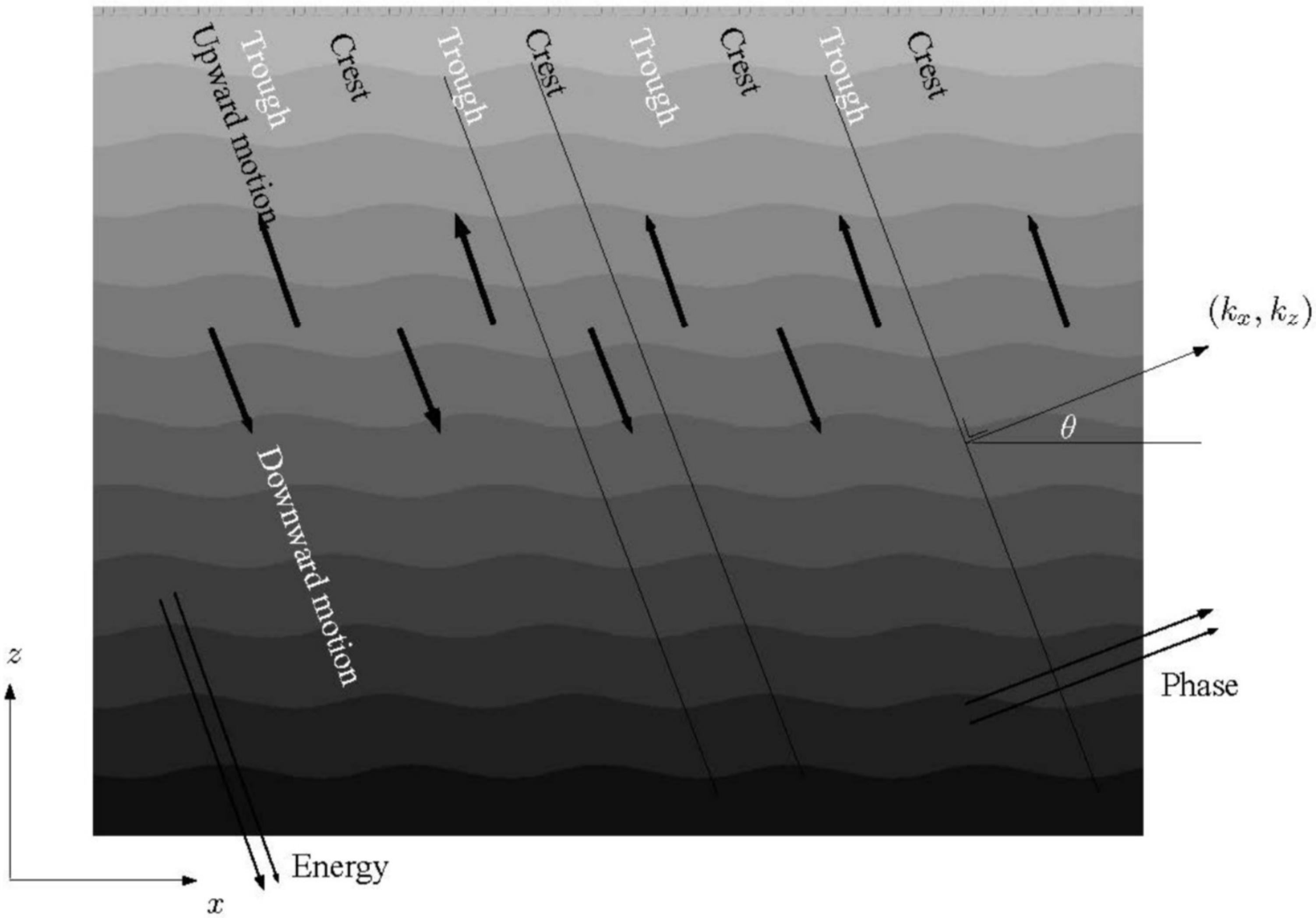


Figure 13-3 Vertical structure of an internal wave.

Internal waves – with rotation

Assumptions: stratified, rotational, inviscid, incompressible, small perturbation

The governing equations are:

$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{\rho}{\rho_0}g$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial \rho}{\partial t} + w\frac{d\bar{\rho}}{dz} = 0$$

From the horizontal momentum equations:

$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \qquad (1)$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \qquad (2)$$

$\frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(2)$:

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - f\zeta = -\frac{1}{\rho_0}{\nabla_h}^2 p$$

$$-\frac{\partial^2 w}{\partial z \partial t} - f\zeta = -\frac{1}{\rho_0}{\nabla_h}^2 p$$

$$-\frac{\partial^2 w}{\partial z \partial t^2} - f\frac{\partial \zeta}{\partial t} = -\frac{1}{\rho_0}\frac{\partial}{\partial t}{\nabla_h}^2 p \qquad (3)$$

$\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(1)$:

$$\frac{\partial \zeta}{\partial t} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

$$\frac{\partial \zeta}{\partial t} = f\frac{\partial w}{\partial z} \qquad (4)$$

Combine (3) and (4), and eliminate ζ:

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial w}{\partial z}\right) + f^2\frac{\partial w}{\partial z} = \frac{1}{\rho_0}\frac{\partial}{\partial t}{\nabla_h}^2 p \qquad (5)$$

From the vertical momentum equation and the density equation:

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{\rho}{\rho_0}g$$

$$\frac{\partial \rho}{\partial t} + w\frac{d\bar{\rho}}{dz} = 0$$

$$\frac{\partial^2 w}{\partial t^2} = -\frac{1}{\rho_0}\frac{\partial^2 p}{\partial z \partial t} - N^2 w \qquad (6)$$

Combine (5) and (6), and eliminate p:

$$\frac{\partial^2}{\partial t^2}\nabla^2 w + f^2\frac{\partial^2 w}{\partial z^2} + N^2{\nabla_h}^2 w = 0$$

$$\frac{\partial^2}{\partial t^2}\nabla^2 w + f^2\frac{\partial^2 w}{\partial z^2} + N^2{\nabla_h}^2 w = 0$$

Assume constant N^2, and apply a wavelike solution $w = w_0 e^{i(kx+ly+mz-\omega t)}$:

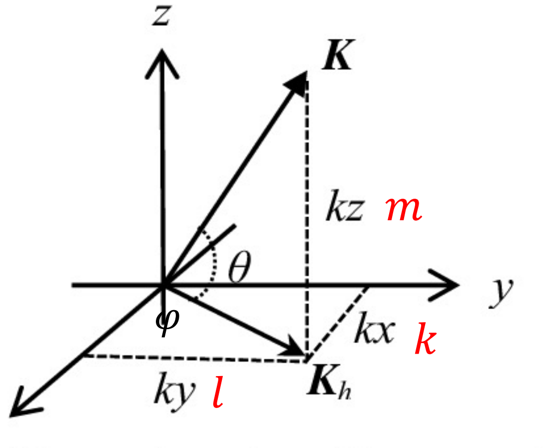


Fig. 1. Wave number vector and its components.

$$(-i\omega)^2[(ik)^2 + (il)^2 + (im)^2] + f^2(im)^2 + N^2[(ik)^2 + (il)^2] = 0$$

$$\omega^2(k^2 + l^2 + m^2) - f^2m^2 - N^2(k^2 + l^2) = 0$$

$$\omega^2 = \frac{N^2(k^2 + l^2) + f^2m^2}{k^2 + l^2 + m^2}$$

$$= \frac{N^2{\boldsymbol{K_h}}^2 + f^2m^2}{\boldsymbol{K}^2}$$

$$= N^2cos^2\theta + f^2sin^2\theta$$

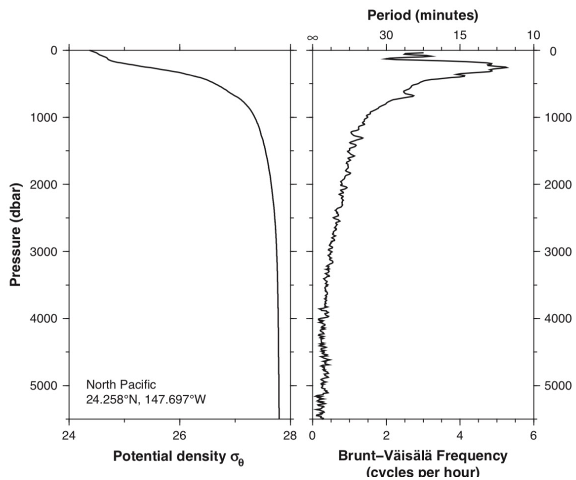
What if $\theta = 0$?

$f^2 < N^2$

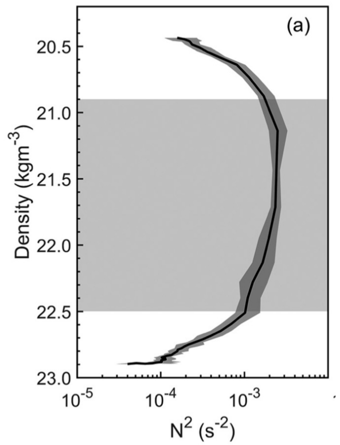
$$\omega^2 - N^2 = (f^2 - N^2)sin^2\theta \le 0$$

$$\omega^2 - f^2 = (N^2 - f^2)cos^2\theta \ge 0$$

$f^2 \le \omega^2 \le N^2$



Yang et al. (2020)



$(u, v, w) = (u_0, v_0, w_0)e^{i(kx+ly+mz-\omega t)}$, and from the continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$(iku_0 + ilv_0 + imw_0)e^{i(kx+ly+mz-\omega t)} = 0 \qquad \boldsymbol{K} \cdot \boldsymbol{u} = 0$$

The flow direction is perpendicular to the wave propagation

Consider $w = w_0 e^{i(kx+mz-\omega t)}$:

$$\omega^2 = \frac{N^2(k^2 + l^2) + f^2 m^2}{k^2 + l^2 + m^2}$$

$$\omega^2 = N^2 \frac{k^2}{k^2 + m^2} + f^2 \frac{m^2}{k^2 + m^2}$$

$$c_{gx} = \frac{\partial \omega}{\partial k} = (N^2 - f^2)\frac{m^2 k}{\omega K^4} \qquad c_{gz} = \frac{\partial \omega}{\partial m} = -(N^2 - f^2)\frac{mk^2}{\omega K^4}$$

$$c_z \cdot c_{gz} = -\frac{\omega}{m} \cdot (N^2 - f^2)\frac{mk^2}{\omega K^4} \quad < 0$$

$$\boldsymbol{K} \cdot \boldsymbol{c_g} = kc_{gx} + mc_{gz} = (N^2 - f^2)\frac{m^2 k^2}{\omega K^4} - (N^2 - f^2)\frac{m^2 k^2}{\omega K^4} \quad = 0$$

The energy propagation direction is perpendicular to the wave propagation direction

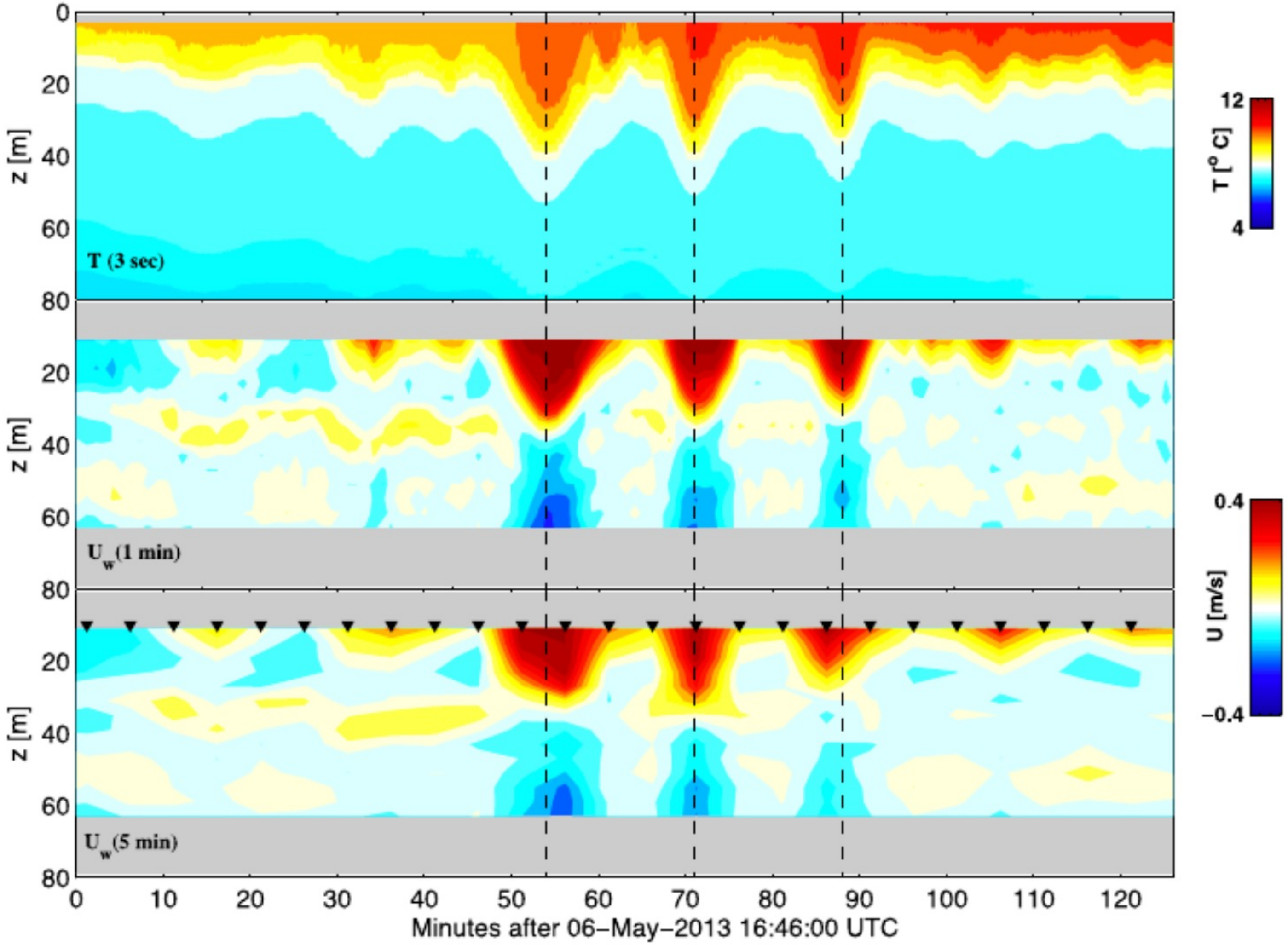
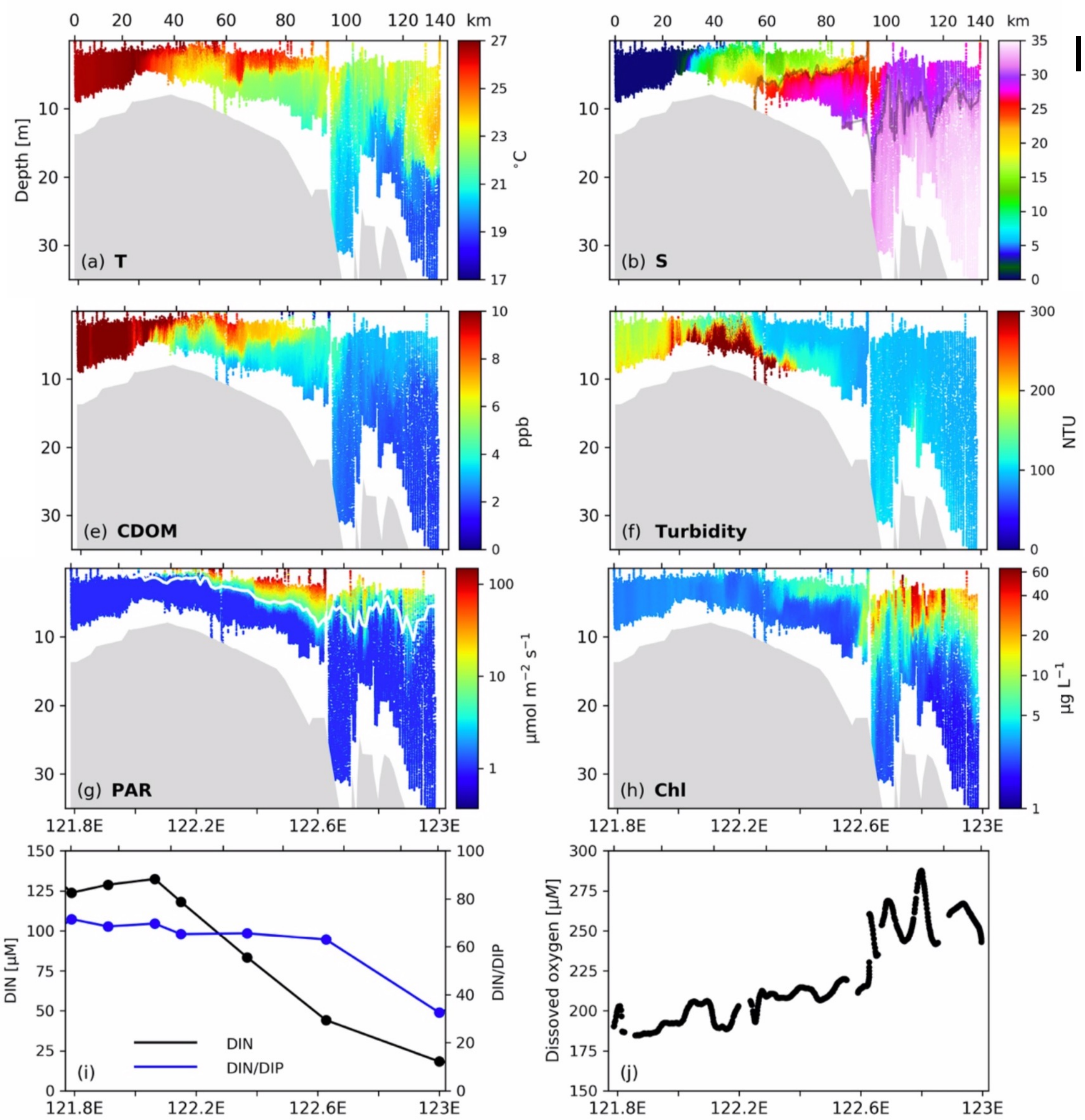


Figure 6. Temperature, 1 min velocity and subsampled 5 min velocity for a wave detected on 6 May 2013. Vertical dashed lines denote detected wave troughs. Black triangles denote the 5 min subsampling time.

Zhang et al. (2014, JGR)



Internal waves generated by plume front

Nash and Moum (2005, Nature)

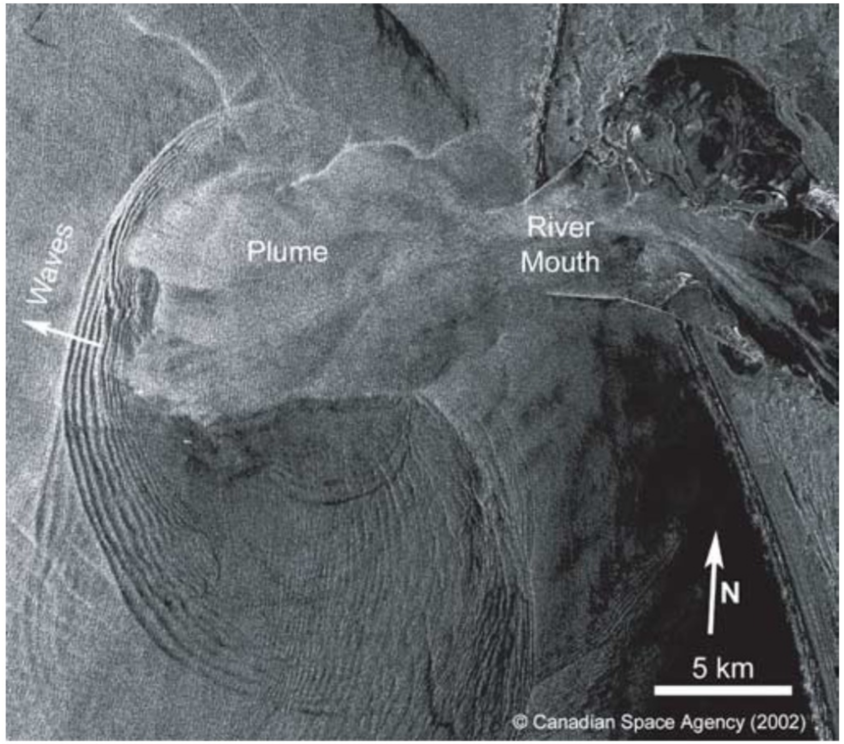


Figure 1 | Synthetic aperture radar (SAR) image of the Columbia River plume on 9 August 2002. Image indicates regions of enhanced surface roughness associated with plume-front and internal wave velocity convergences. Similar features appear in images during all summertime months (April–October; see http://oceanweb.ocean.washington.edu/rise/data.htm for more Columbia River plume images) and from other regions[1,2]. SAR image courtesy of P. Orton, T. Sanders and D. Jay; image was processed at the Alaska Satellite Facility, and is copyright Canadian Space Agency.