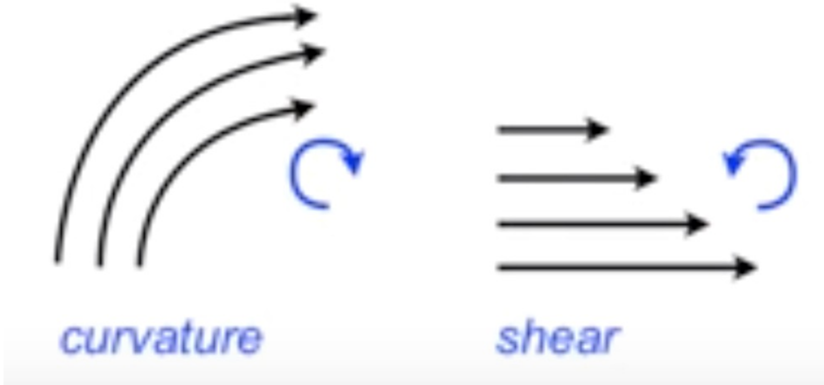
Vorticity



Vorticity: curl of velocity (a measure of spin)

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$$

$$= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\boldsymbol{k}$$

For 2-D flow on the horizontal plane: $\boldsymbol{u} = u\mathbf{i} + v\mathbf{j}$

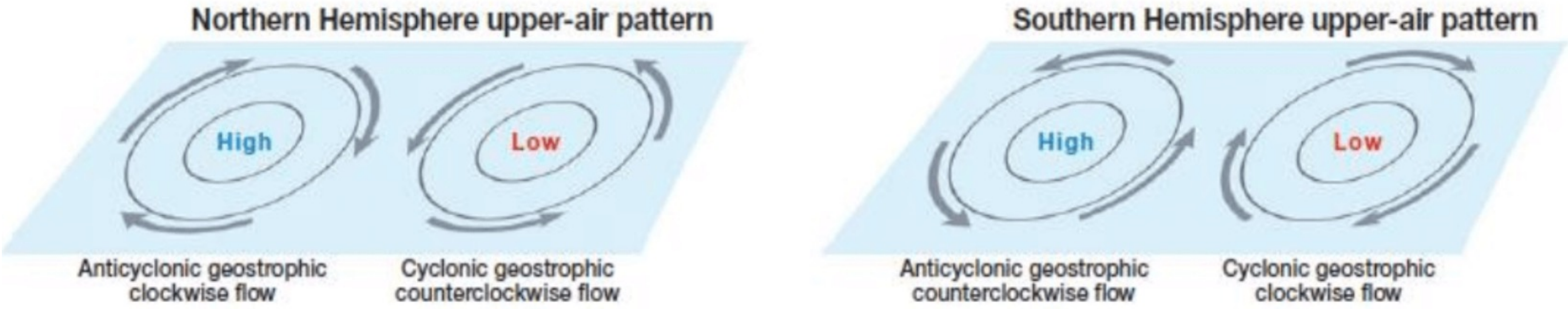
$$\boldsymbol{\omega} = \mathbf{k}\left(\underbrace{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}_{\zeta}\right)$$

Properties:

$$\nabla \cdot \boldsymbol{\omega} = 0$$

$$\nabla \times \nabla f = 0$$

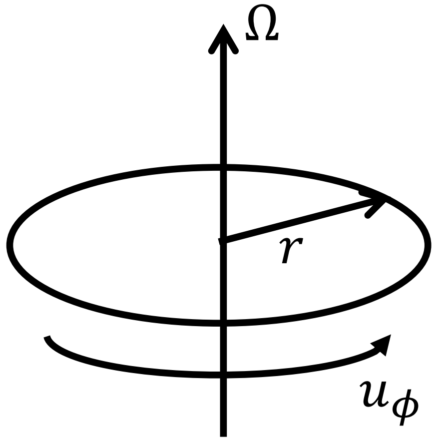
Cyclones and anticyclones



negative ζ | positive ζ | positive ζ | negative ζ

Rigid body motion

$$u_r = 0, \qquad u_\phi = \Omega r, \qquad u_z = 0$$



$$\omega = \frac{1}{r}\begin{vmatrix} \boldsymbol{e_r} & r\boldsymbol{e_\phi} & \boldsymbol{e_z} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ u_r & r u_\phi & u_z \end{vmatrix}$$

$$\omega^z = \frac{1}{r}\frac{\partial}{\partial r}\left(r u_\phi\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r^2\Omega\right) = 2\Omega$$

The vorticity of a fluid in solid body rotation is twice the angular velocity of the fluid about the axis of rotation, and is pointed in a direction orthogonal to the plane of rotation.

Circulation

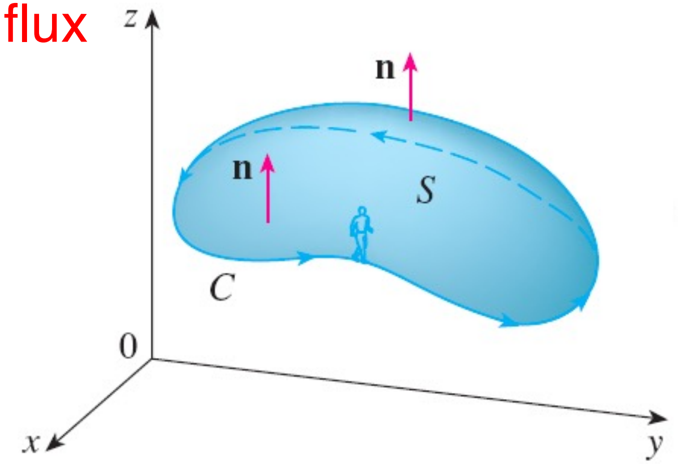
Circulation: the integral of velocity around a closed fluid loop

$$C \equiv \oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{r}$$

Stokes' Theorem:

$$C \equiv \oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{r} = \int_S \underline{\boldsymbol{\omega} \cdot \mathrm{d}\boldsymbol{S}}$$

vortex flux



Vorticity equation – without rotation

The momentum equation:

$$\frac{d\boldsymbol{v}}{dt} = -\frac{1}{\rho}\nabla p - \nabla\Phi + \nu_E \nabla^2 \boldsymbol{v}$$

$$\boxed{(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\boldsymbol{v}\times\boldsymbol{\omega} + \nabla(\boldsymbol{v}^2/2),}$$

$$\frac{\partial\boldsymbol{v}}{\partial t} + \boldsymbol{\omega}\times\boldsymbol{v} = -\frac{1}{\rho}\nabla p - \nabla\Phi - \nabla(\boldsymbol{v}^2/2) + \nu_E \nabla^2 \boldsymbol{v}$$

Take the curl of the momentum equation:

$$\boxed{\nabla\times\nabla f = 0}$$

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + \nabla\times(\boldsymbol{\omega}\times\boldsymbol{v}) = \frac{1}{\rho^2}(\nabla\rho\times\nabla p) + \nabla\times\boldsymbol{F}.$$

$$\nabla\times(\boldsymbol{\omega}\times\boldsymbol{v}) = (\boldsymbol{v}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{v} + \boldsymbol{\omega}\nabla\cdot\boldsymbol{v} - \cancel{\boldsymbol{v}\nabla\cdot\boldsymbol{\omega}}.$$

$$\boxed{\nabla\cdot\boldsymbol{\omega} = 0}$$

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{v} - \boldsymbol{\omega}\nabla\cdot\boldsymbol{v} + \frac{1}{\rho^2}(\nabla\rho\times\nabla p) + \nabla\times\boldsymbol{F}.$$

The baroclinic term

$$\frac{1}{\rho^2}\nabla\rho \times \nabla p$$

If the density is only a function of pressure:

$$\rho = \rho(\mathrm{p})$$

Isolines of pressure and density are parallel

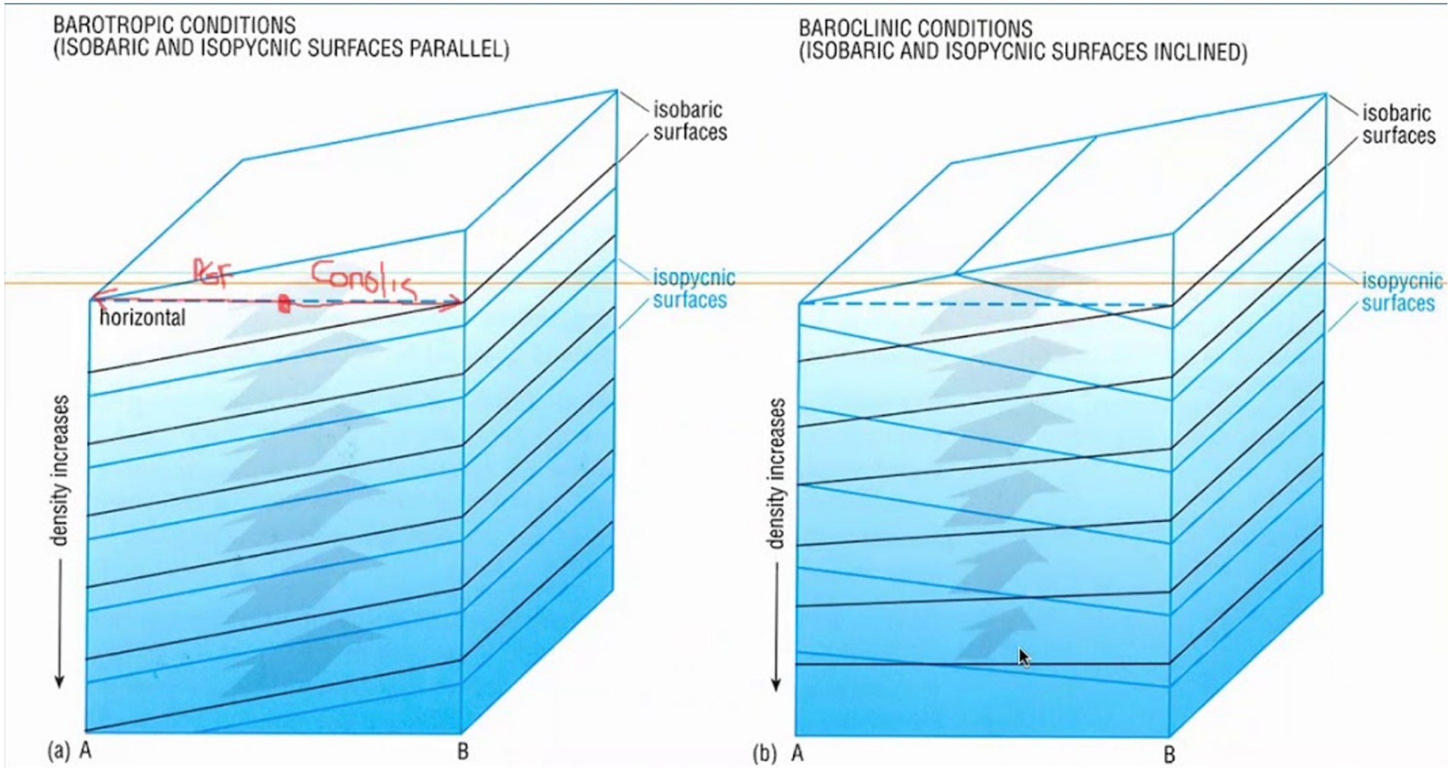
$$\nabla\rho \times \nabla p = 0$$ barotropic fluid

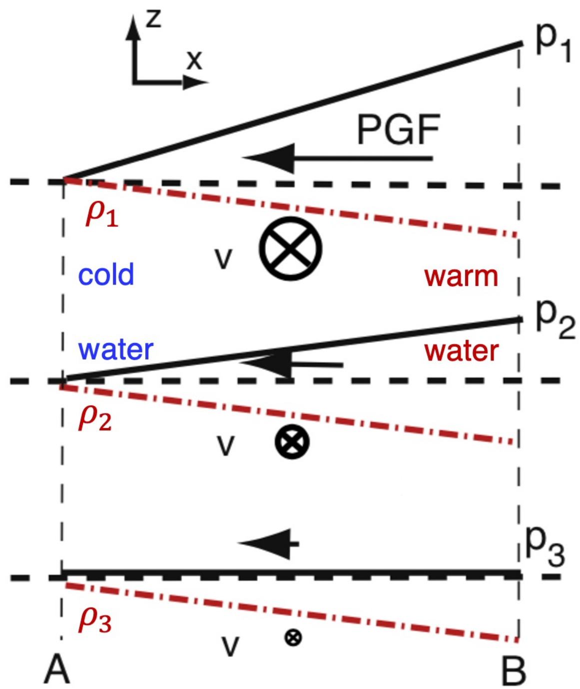
(for constant density?)

Otherwise:

$$\nabla\rho \times \nabla p \neq 0$$ baroclinic fluid

Barotropic and barolinic conditions





$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{v} - \cancel{\boldsymbol{\omega}\nabla\cdot\boldsymbol{v}} + \cancel{\frac{1}{\rho^2}(\nabla\rho\times\nabla p)} + \cancel{\nabla\times\boldsymbol{F}}.$$

For incompressible, barotropic, inviscous flow:

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{v}.$$

For two-dimensional flows $\boldsymbol{u} = u\mathbf{i} + v\mathbf{j}$

$$\boldsymbol{\omega} = \mathbf{k}\underbrace{\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}_{\zeta}$$

If a streamfunction exists:

$$u = -\frac{\partial\psi}{\partial y}, \qquad v = \frac{\partial\psi}{\partial x}$$

$$\frac{\boldsymbol{D\omega}}{Dt} = 0 \qquad \boxed{\frac{d\zeta}{dt} = 0}$$

$$\boxed{\zeta = \nabla^2\psi}$$

Kelvin's circulation theorem

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p - \nabla \Phi$$ inviscid flow

Rate of change in circulation:

$$\begin{aligned}\frac{\mathrm{D}C}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t}\oint \boldsymbol{v}\cdot \mathrm{d}\boldsymbol{r} &= \oint \left(\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t}\cdot \mathrm{d}\boldsymbol{r} + \boldsymbol{v}\cdot \mathrm{d}\boldsymbol{v}\right)\\ &= \oint \left[\left(-\frac{1}{\rho}\nabla p - \nabla\Phi\right)\cdot \mathrm{d}\boldsymbol{r} + \boldsymbol{v}\cdot \mathrm{d}\boldsymbol{v}\right] \\ &= \oint -\frac{1}{\rho}\nabla p \cdot \mathrm{d}\boldsymbol{r}\end{aligned}$$

$$\boxed{D(\mathrm{d}\boldsymbol{r})/Dt = \mathrm{d}\boldsymbol{v}}$$

$$\oint \frac{1}{\rho}\nabla p\cdot \mathrm{d}\boldsymbol{r} = \int_S \nabla\times\left(\frac{\nabla p}{\rho}\right)\cdot \mathrm{d}\boldsymbol{S} = \int_S \frac{-\nabla\rho\times\nabla p}{\rho^2}\cdot \mathrm{d}\boldsymbol{S}$$

For barotropic fluid:

Circulation is conserved

$$\boxed{\frac{\mathrm{D}}{\mathrm{D}t}\oint \boldsymbol{v}\cdot \mathrm{d}\boldsymbol{r} = 0}$$

Stokes' theorem

Vortex flux is conserved

$$\boxed{\frac{\mathrm{D}}{\mathrm{D}t}\int_S \boldsymbol{\omega}\cdot \mathrm{d}\boldsymbol{S} = 0}$$

Circulation in a rotating frame

The absolute velocity in an inertia frame is:

$$\boldsymbol{v}_a = \boldsymbol{v}_r + \boldsymbol{\Omega} \times \boldsymbol{r}$$

Rate of change in circulation from absolute velocity:

$$\frac{\mathrm{D}}{\mathrm{D}t} \oint (\boldsymbol{v}_r + \boldsymbol{\Omega} \times \boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{r} = \oint \left[\left(\frac{\mathrm{D}\boldsymbol{v}_r}{\mathrm{D}t} + \boldsymbol{\Omega} \times \boldsymbol{v}_r \right) \cdot \mathrm{d}\boldsymbol{r} + (\cancel{\boldsymbol{v}_r} + \boldsymbol{\Omega} \times \boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{v}_r \right]$$

$$\oint (\boldsymbol{\Omega} \times \boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{v}_r = \oint \left\{ \mathrm{d}[(\boldsymbol{\Omega} \times \boldsymbol{r}) \cdot \boldsymbol{v}_r] - (\boldsymbol{\Omega} \times \mathrm{d}\boldsymbol{r}) \cdot \boldsymbol{v}_r \right\}$$

$$= \oint \left\{ \cancel{\mathrm{d}[(\boldsymbol{\Omega} \times \boldsymbol{r}) \cdot \boldsymbol{v}_r]} + (\boldsymbol{\Omega} \times \boldsymbol{v}_r) \cdot \mathrm{d}\boldsymbol{r} \right\}$$

$$\frac{\mathrm{D}}{\mathrm{D}t} \oint (\boldsymbol{v}_r + \boldsymbol{\Omega} \times \boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{r} = \oint \left(\frac{\mathrm{D}\boldsymbol{v}_r}{\mathrm{D}t} + 2\boldsymbol{\Omega} \times \boldsymbol{v}_r \right) \cdot \mathrm{d}\boldsymbol{r}$$

$$= - \oint \frac{1}{\rho} \nabla p \cdot \mathrm{d}\boldsymbol{r}$$ = 0 (for barotropic and inviscid fluids)

$$\boxed{\boldsymbol{\omega}_r = \nabla \times \boldsymbol{v}_r}$$

relative vorticity

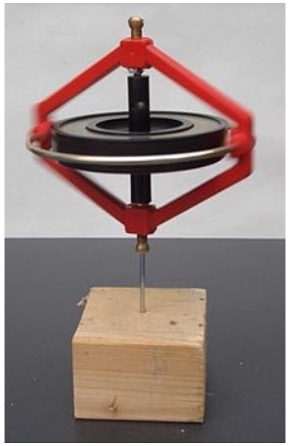
$$\frac{\mathrm{D}}{\mathrm{D}t} \oint (\boldsymbol{v}_r + \boldsymbol{\Omega} \times \boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{r} = 0$$

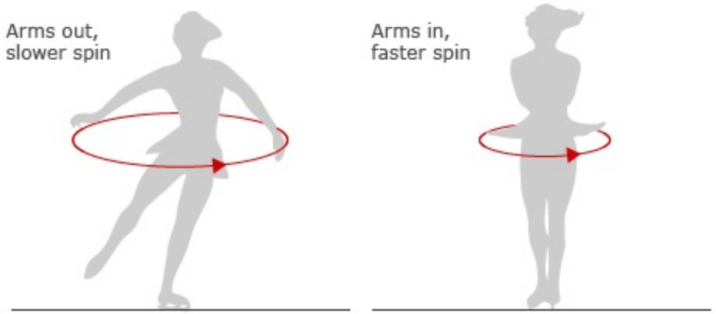
$$\boxed{\nabla \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = 2\boldsymbol{\Omega}}$$

$2\boldsymbol{\Omega}$: planetary (ambient) vorticity

$$\frac{\mathrm{D}}{\mathrm{D}t} \int_S \underbrace{(\boldsymbol{\omega}_r + 2\boldsymbol{\Omega})}_{\boldsymbol{\omega}_a} \cdot \mathrm{d}\boldsymbol{S} = 0$$

$\boldsymbol{\omega}_a$: absolute vorticity





Angular momentum conservation:

$$L = m\omega r^2$$

Vorticity equation in a rotating frame

For inviscid flow:

$$\frac{d\boldsymbol{v}_r}{dt} + 2\boldsymbol{\Omega}\times\boldsymbol{v}_r = -\frac{1}{\rho}\nabla p - \nabla\Phi$$

$$(\boldsymbol{v}_r \cdot \boldsymbol{\nabla})\boldsymbol{v}_r = -\boldsymbol{v}_r\times\boldsymbol{\omega}_r + \nabla(\boldsymbol{v}_r^2/2)$$

$$\frac{\partial\boldsymbol{v}_r}{\partial t} + (2\boldsymbol{\Omega} + \boldsymbol{\omega}_r)\times\boldsymbol{v}_r = -\frac{1}{\rho}\nabla p + \nabla\left(\Phi - \frac{1}{2}\boldsymbol{v}_r^2\right)$$

Take the curl of the equation:

$$\nabla\times[(2\boldsymbol{\Omega}+\boldsymbol{\omega}_r)\times\boldsymbol{v}_r] = (2\boldsymbol{\Omega}+\boldsymbol{\omega}_r)\nabla\cdot\boldsymbol{v}_r + (\boldsymbol{v}_r\cdot\nabla)(2\boldsymbol{\Omega}+\boldsymbol{\omega}_r) - [(2\boldsymbol{\Omega}+\boldsymbol{\omega}_r)\cdot\nabla]\boldsymbol{v}_r$$

$$\frac{D\boldsymbol{\omega}_a}{Dt} = [(2\boldsymbol{\Omega}+\boldsymbol{\omega}_r)\cdot\nabla]\boldsymbol{v}_r - (2\boldsymbol{\Omega}+\boldsymbol{\omega}_r)\nabla\cdot\boldsymbol{v}_r + \frac{1}{\rho^2}(\nabla\rho\times\nabla p)$$

$\boldsymbol{\omega}_a$: absolute vorticity

For incompressible, barotropic fluids: $\boxed{\frac{D\boldsymbol{\omega}_a}{Dt} = (\boldsymbol{\omega}_a\cdot\nabla)\boldsymbol{v}_r}$ $\boxed{\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{v}}$

Potential vorticity conservation from the circulation theorem

$$\frac{\mathrm{D}}{\mathrm{D}t}\left[(\boldsymbol{\omega}_a \cdot \boldsymbol{n})\delta A\right] = 0$$

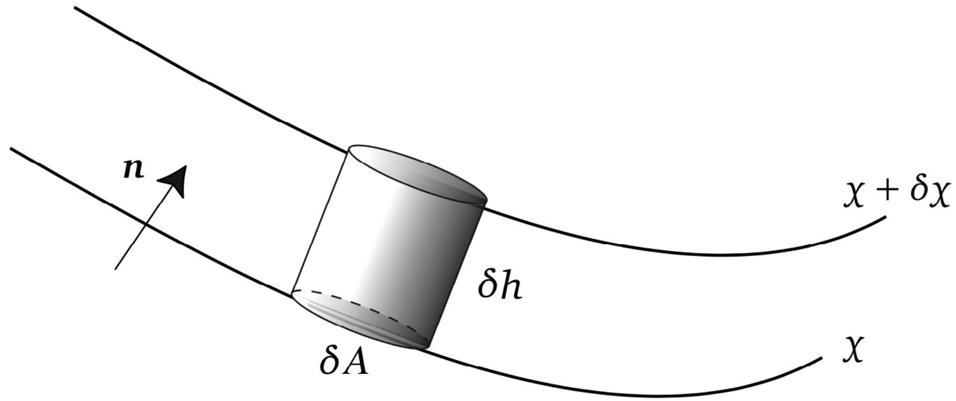
$$\boldsymbol{\omega}_a \cdot \boldsymbol{n}\,\delta A = \boldsymbol{\omega}_a \cdot \frac{\nabla\chi}{|\nabla\chi|}\frac{\delta V}{\delta h}$$

$$\delta\chi = \delta\boldsymbol{x}\cdot\nabla\chi = \delta h|\nabla\chi|$$

$$\frac{\mathrm{D}}{\mathrm{D}t}\left[\frac{(\boldsymbol{\omega}_a\cdot\nabla\chi)\delta V}{\delta\chi}\right] = 0.$$

$$\frac{\rho\delta V}{\delta\chi}\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{\omega}_a}{\rho}\cdot\nabla\chi\right) = 0$$

$$\boxed{\frac{\mathrm{D}}{\mathrm{D}t}\left(\widetilde{\boldsymbol{\omega}}_a\cdot\nabla\chi\right) = 0}$$



χ is any materially conserved tracer: $\mathrm{D}\chi/\mathrm{D}t = 0$

$$\boldsymbol{n} = \nabla\chi/|\nabla\chi|$$

$$\delta V = \delta h\,\delta A$$

Potential vorticity conservation for the shallow water model

The horizontal momentum equations (**homogeneous, invisicd,** $\frac{\partial}{\partial z} = 0$):

$$\frac{\partial \mathrm{u}}{\partial \mathrm{t}} + \mathrm{u}\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \mathrm{v}\frac{\partial \mathrm{u}}{\partial \mathrm{y}} - fv = -g\frac{\partial \eta}{\partial x} \quad (1)$$

$$\frac{\partial \mathrm{v}}{\partial \mathrm{t}} + \mathrm{u}\frac{\partial \mathrm{v}}{\partial \mathrm{x}} + \mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{y}} + fu = -g\frac{\partial \eta}{\partial y} \quad (2)$$

Taking the curl of the momentum equations by doing $\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(1)$:

$$\frac{\partial}{\partial \mathrm{t}}\left(\frac{\partial \mathrm{v}}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{\partial u}{\partial x}\frac{\partial \mathrm{v}}{\partial x} + u\frac{\partial^2 v}{\partial x^2} - \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + v\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} - v\frac{\partial^2 u}{\partial y^2} + f\frac{\partial u}{\partial x} + \frac{\partial f}{\partial y}v + f\frac{\partial v}{\partial y} = 0$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial u}{\partial x}\zeta + u\frac{\partial \zeta}{\partial x} + \frac{\partial v}{\partial y}\zeta + v\frac{\partial \zeta}{\partial y} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v = 0$$

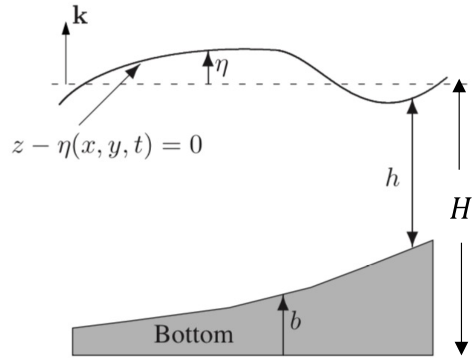
$$\frac{d\zeta}{dt} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(f + \zeta) + \beta v = 0$$

$$\boxed{\frac{df}{dt} = \frac{\partial f}{dt} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} = \beta v}$$

$$\frac{d(f + \zeta)}{dt} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(f + \zeta) = 0$$

$$\frac{d(f+\zeta)}{dt} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(f+\zeta) = 0 \ (1)$$

The continuity equation:



$$\frac{\partial \eta}{\partial t} + \frac{\partial hu}{\partial x} + \frac{\partial hv}{\partial y} = 0$$

$$\frac{dh}{dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \ \ (2)$$

$$\boxed{\eta = h + b - H}$$

Combining (1) and (2):

$$\frac{d(f+\zeta)}{dt} - \frac{f+\zeta}{h}\frac{dh}{dt} = 0$$

f: planetary vorticity

ζ: relative vorticity

$f+\zeta$: absolute vorticity

$$\boxed{\frac{d}{dt}\left(\frac{f+\zeta}{h}\right) = 0}$$

potential vorticity

$$\frac{\zeta}{f} \sim \frac{U/L}{f} \sim \frac{U}{fL}$$ Rossby number

Horizontal divergence/convergence in PV conservation

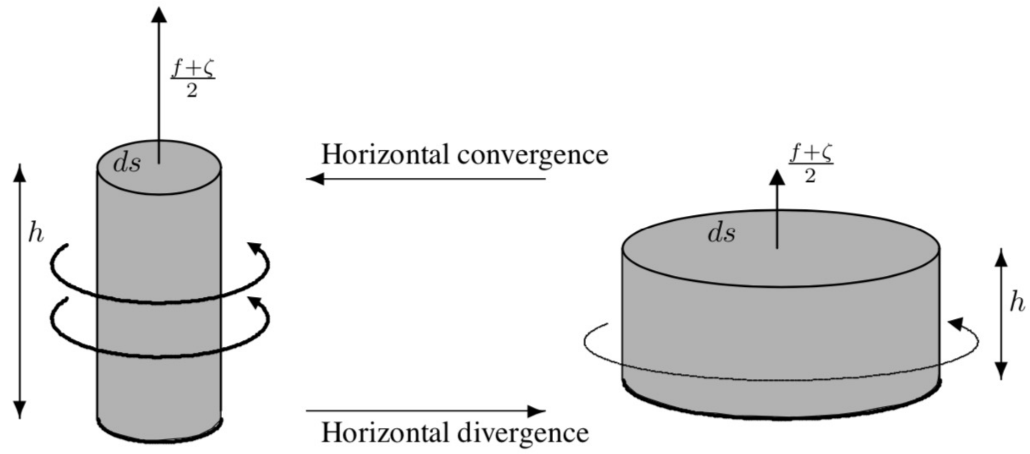
For volume conservation of a fluid column with a cross-section of ds and thickness of h :

$$\frac{d}{dt}(hds) = 0$$

$$\frac{dh}{dt}ds + h\frac{d}{dt}(ds) = 0$$

$$\frac{dh}{dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \quad (2)$$

$$\frac{d}{dt}(ds) = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)ds$$



divergent flow, $\frac{d}{dt}(ds) > 0$, h decreases

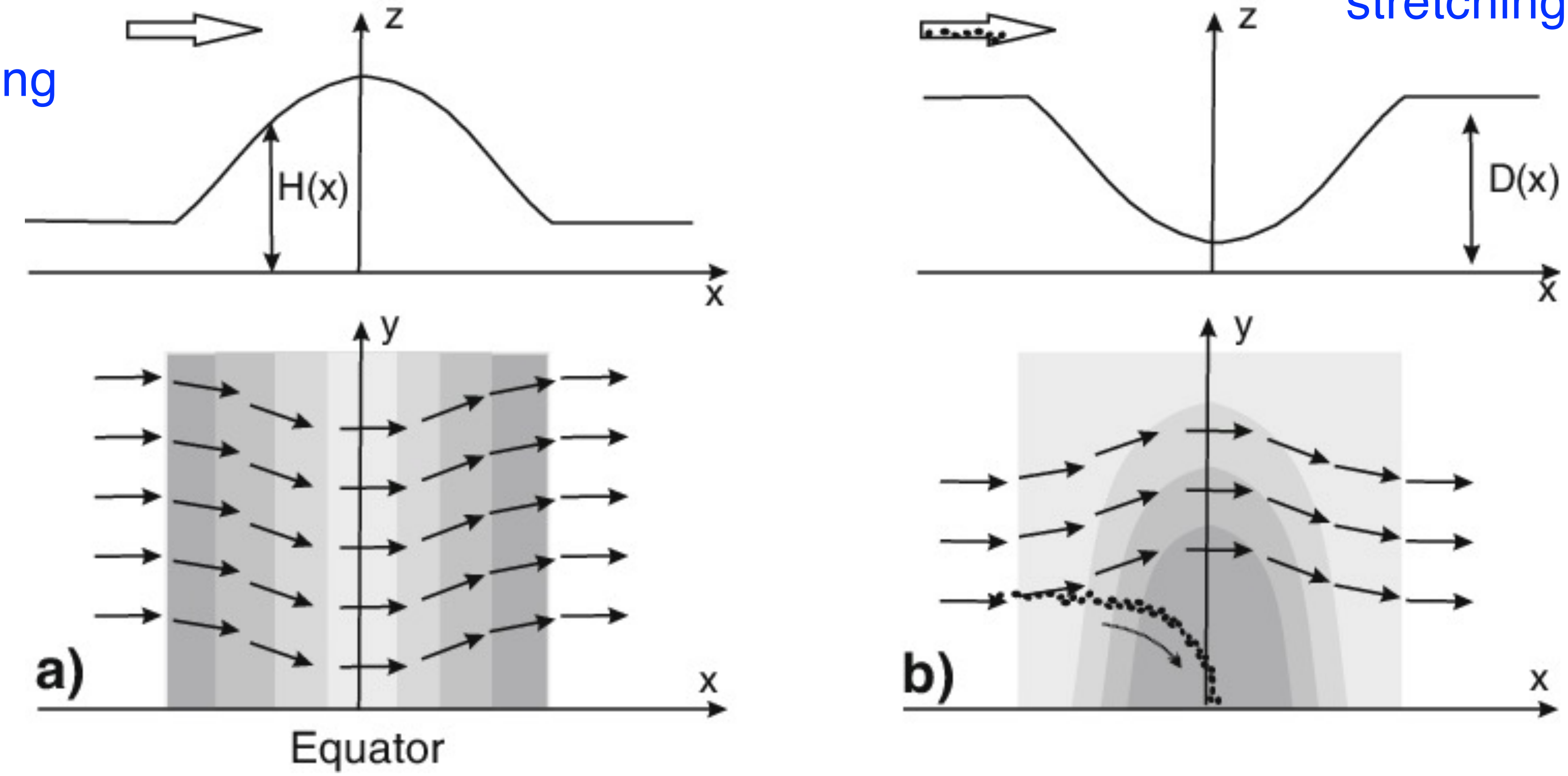
convergent flow, $\frac{d}{dt}(ds) < 0$, h increases

$$\frac{d}{dt}\left(\frac{f\ +\ \zeta}{h}\right)\ =\ 0$$

For large scale flow, $\zeta \ll f$

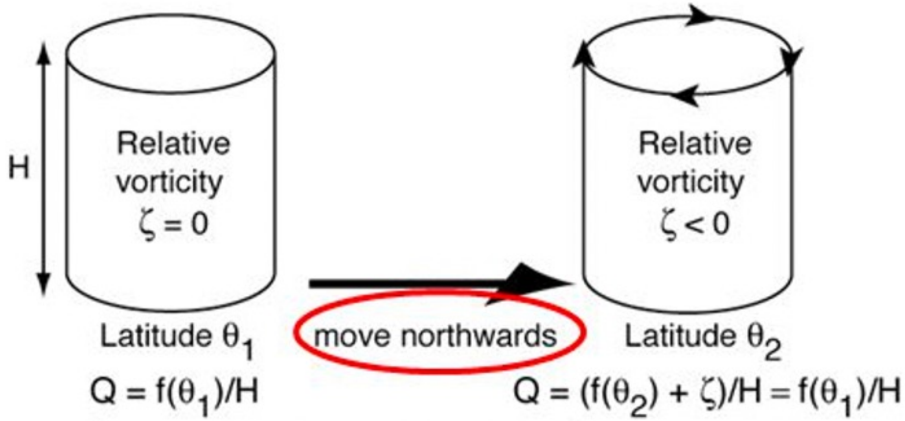
squeezing

stretching



$$\frac{d}{dt}\left(\frac{f + \zeta}{h}\right) = 0$$

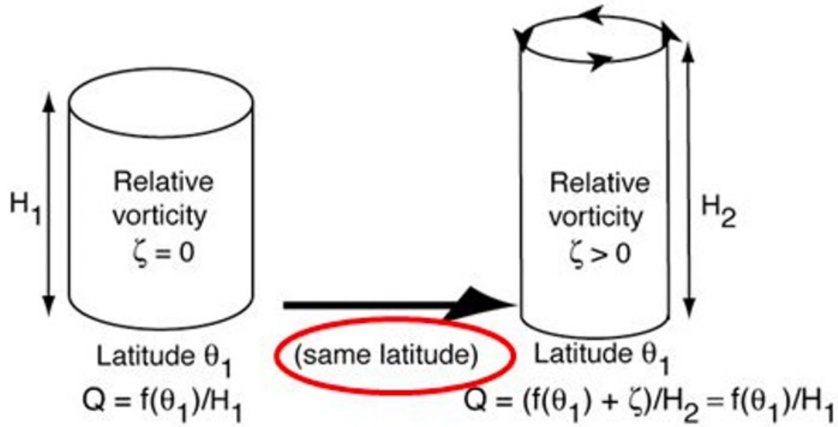
h is constant



Conservation of potential vorticity Q in the absence of stretching (northern hemisphere): balance of planetary vorticity and relative vorticity

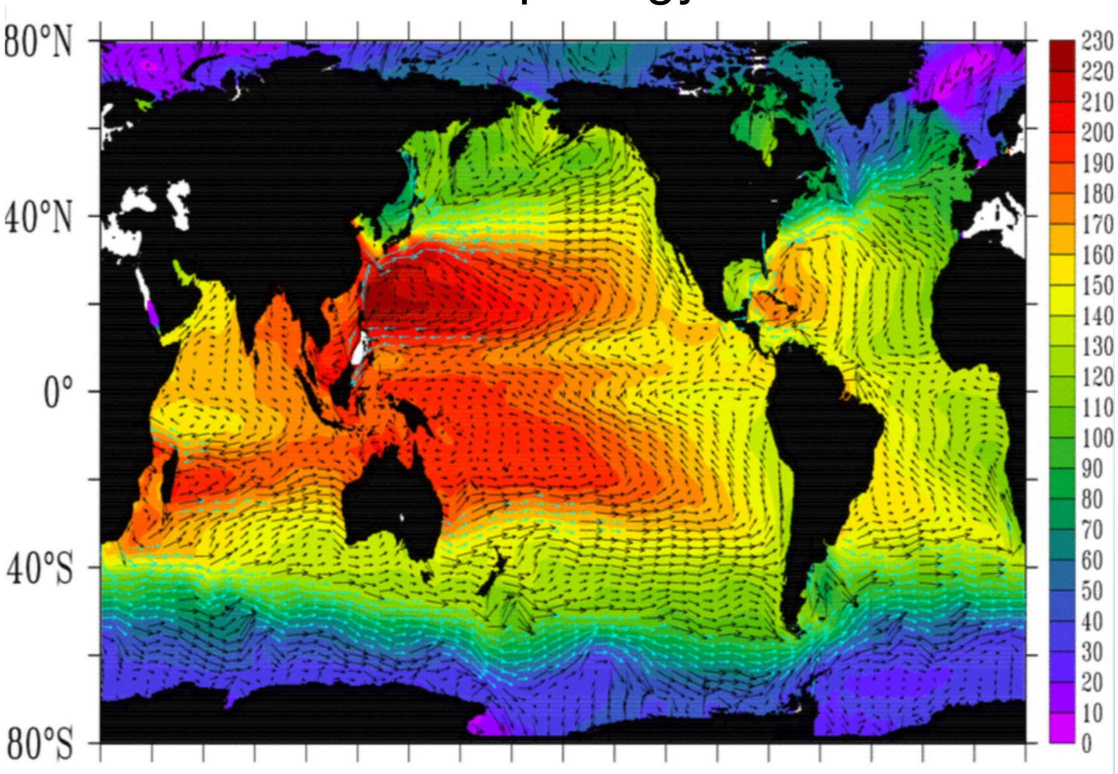
f is constant

eddies generated by topography

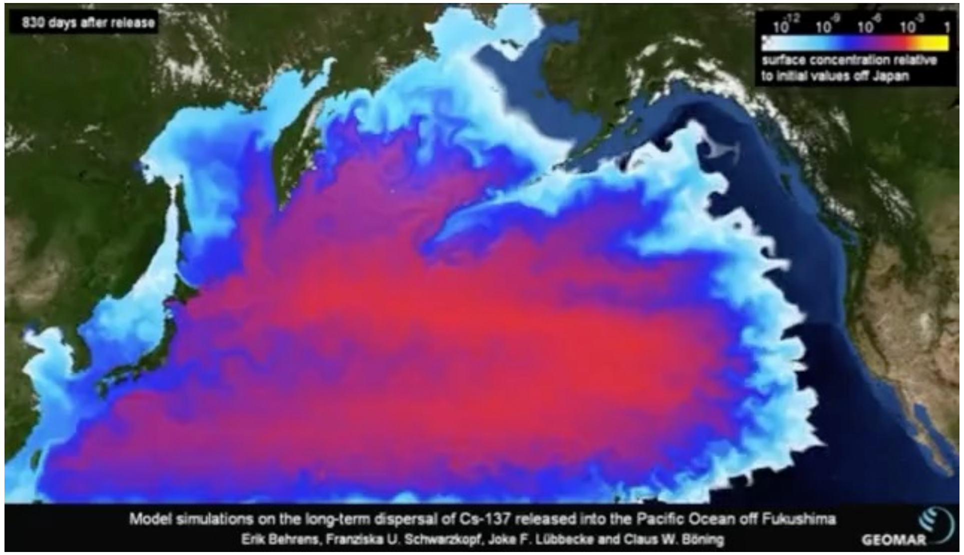


Conservation of potential vorticity Q in the absence of planetary vorticity change (northern hemisphere): balance of relative vorticity and stretching

Subtropical gyre

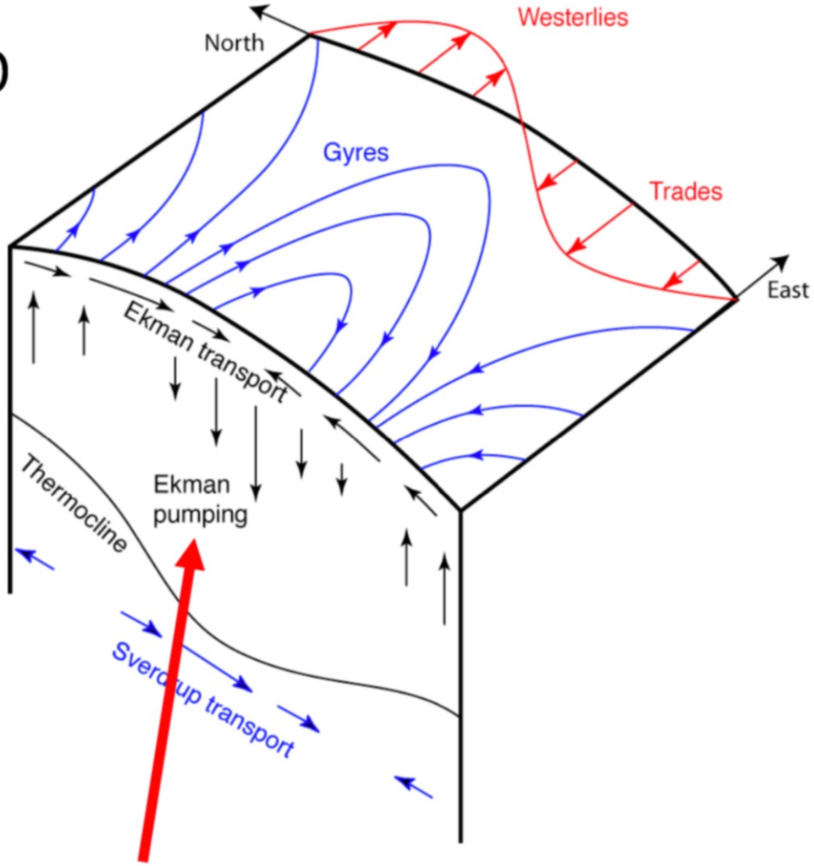


Dispersion of nuclear water released off Fukushima



Why do the interior currents in the subtropical gyre flow towards the equator?

Sverdrup



$$\frac{d}{dt}\left(\frac{f + \zeta}{h}\right) = 0$$

- Ekman pumping provides the squashing or stretching.
- The water columns must respond. They do this by changing latitude.
- (They do not spin up in place for the large-scale circulation.)

Squashing -> equatorward movement   Stretching -> poleward

TRUE in both Northern and Southern Hemisphere

DPO Fig. 7.13

Excercise

As shown in Figure 1, a vertically uniform but laterally sheared coastal current must climb a bottom escarpment. Assuming that the jet velocity still vanishes offshore, determine the velocity profile and the width of the jet downstream of the escarpment using h_1 =200m, h_2 =160m, U_1 =0.5m/s (maximum velocity in the area with depth h_1), L_1 =10km and f =10^{-4} s^{-1}. (that is, you should obtain U_2 and L_2, and plot the velocity profile). What would happen if the downstream depth were only 100m?

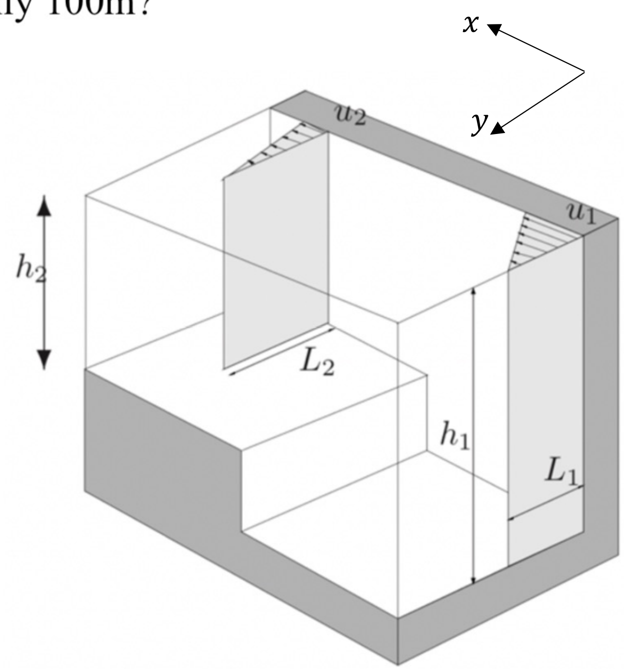


Figure 1: A sheared coastal jet negotiating a bottom escarpment.

Homework 3

1. (50 pts) Assume that the atmospheric Ekman layer over the earth's surface at latitude 45 °N can be modelled with a turbulent kinematic viscosity $\nu = 10\ m^2/s$. If the geostrophic velocity above the layer is only in the x-direction and uniformly 10 m/s, what is the magnitude and direction of the Ekman transport? Is there any vertical velocity? (Hint: the dynamics of the atmosphere bottom Ekman layer is similar to that of the ocean bottom Ekman layer)

$f = 2\omega \sin\varphi \approx 10^{-4} s^{-1}$, $d = \sqrt{\frac{2\nu}{f}} \approx 440\ m$

Bottom Ekman Transport:

$$U = -\frac{d}{2}(\bar{u} + \bar{v}) \approx -2.2\times10^{3}\ m^2/s$$

$$V = \frac{d}{2}(\bar{u} - \bar{v}) \approx 2.2\times10^{3}\ m^2/s$$

magnitude $\sim 10^3\ m^2/s$, direction $135°$ (Northwest)

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = \int_0^d \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz = -\frac{d}{2}\left(\frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y}\right) = 0$$

Continuity equation:

$$\frac{\partial w}{\partial z} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0, \quad w|_{z=d} - w|_{z=0} = 0$$

$w|_{z=d} = w|_{z=0} = 0$ no vertical velocity

2. (50 pts) Internal waves are generated along the coast of Norway by the M_2 surface tide that has a period of 12.42 h. If the buoyancy frequency N is 2×10^{-3} s^{-1}, at which possible angles can the energy propagate with respect to the horizontal plane if the Earth's rotation is not considered?