Distinguishing attributes of geophysical fluids

1. Rotation

Rotation rate of the Earth:

$$\Omega = \frac{2\pi \text{ radians}}{\text{time of one revolution}} \qquad \boxed{7.2921 \times 10^{-5} \text{ s}^{-1}}$$

If fluid motions evolve on a **time scale comparable to or longer than the time of one rotation**, then the fluid can feel the effect of ambient rotation:

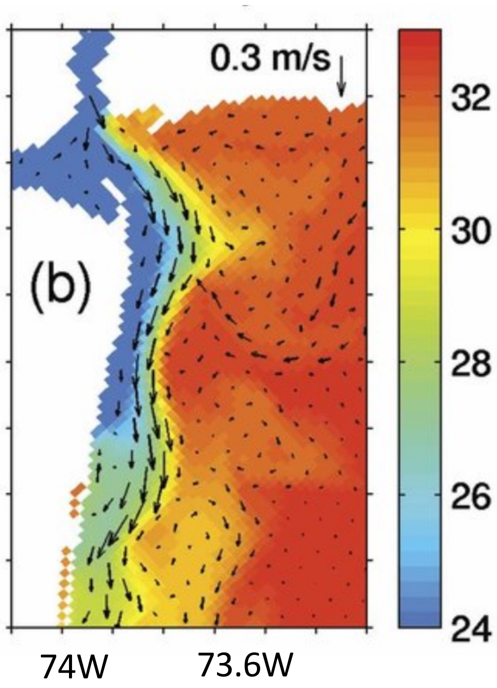
$$\omega = \frac{\text{time of one revolution}}{\text{motion time scale}} = \frac{2\pi/\Omega}{T} = \frac{2\pi}{\Omega T} = \frac{2\pi U}{\Omega L}$$

Rossby number

$\omega \leq 1$, rotation effect is important

Exercise

Choi and Wilkin (2007)



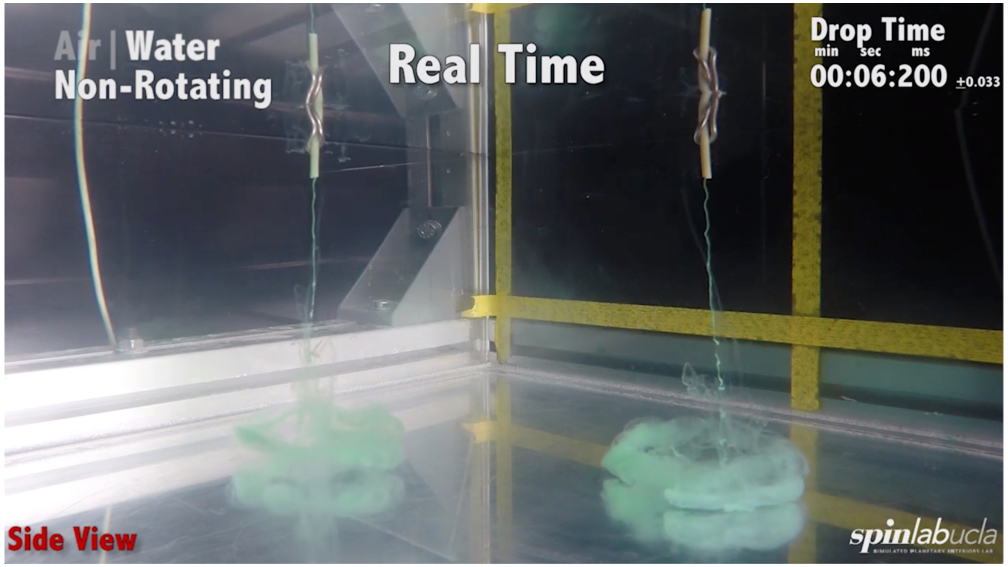
Case 1

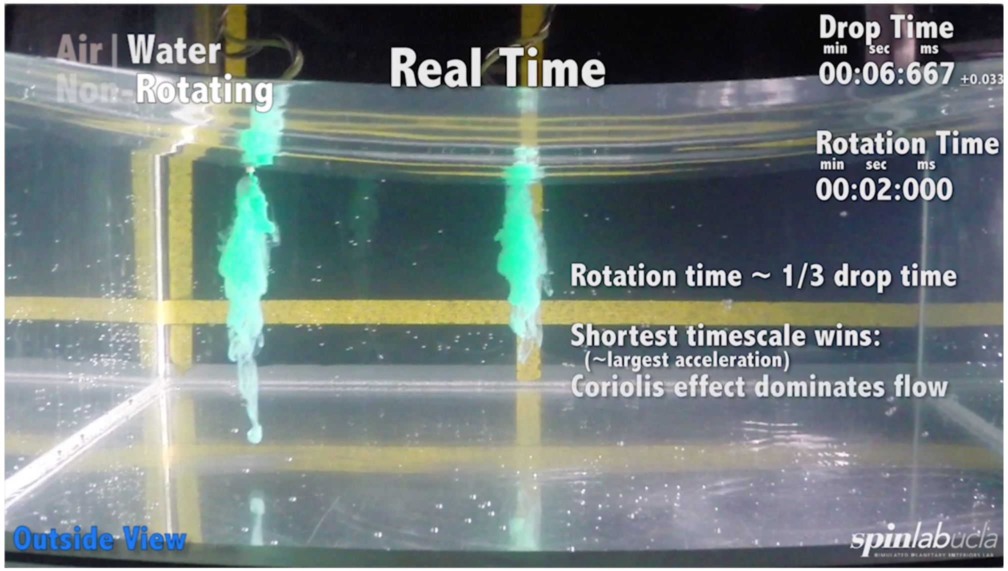
An coastal current with a width of 20 km and speed of 20 cm s^{-1}

Case 2

A tornado at mid-latitude with a scale of 300 m and speed of 30 m s^{-1}

The effect of rotation on fluid motions – imparting rigidity



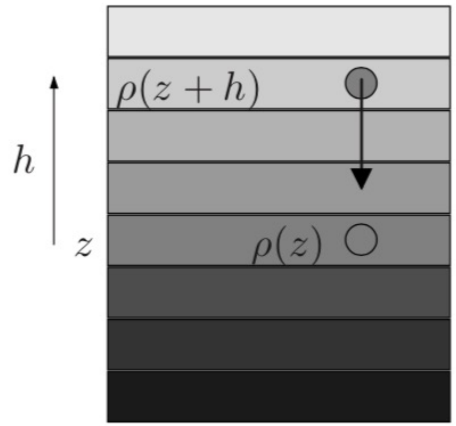


2. Stratification: breaking rigidity

Displace a water parcel from z to z + h:

gravitational force: $\rho(z)gV$

buoyancy force: $\rho(z+h)gV$



By Newton's Second Law:

$$[\rho(z+h)-\rho(z)]\, gV = \rho(z)V\frac{d^2h}{dt^2}$$

$$\frac{d\rho}{dz}hgV = \rho(z)\mathrm{V}\frac{d^2h}{dt^2}$$

$$\frac{d^2h}{dt^2} - \overset{N^2}{\underline{\frac{g}{\rho(z)}\frac{d\rho}{dz}}}h = 0$$

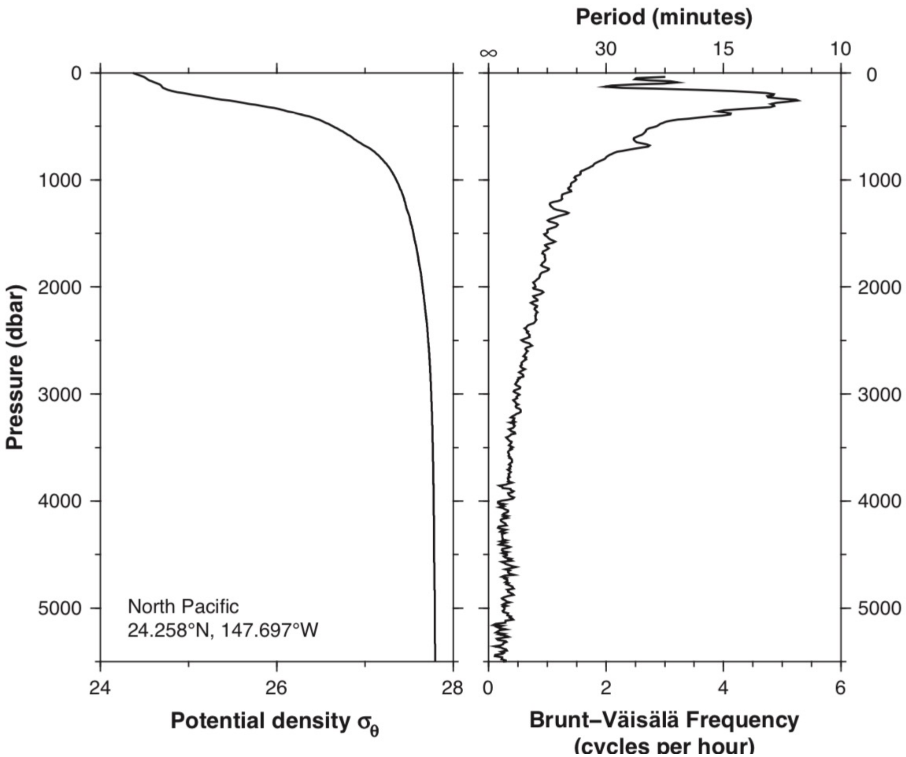
$$h = AsinNt$$

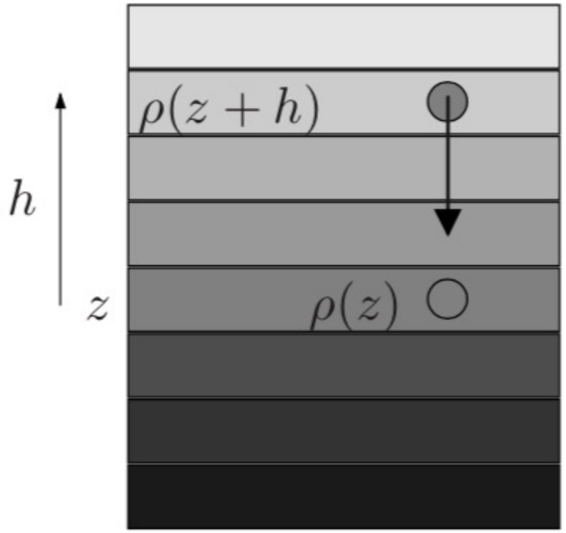
N is the vertical vibration frequency of the parcel

$$N^2 = -\frac{g}{\rho(z)}\frac{d\rho}{dz}$$

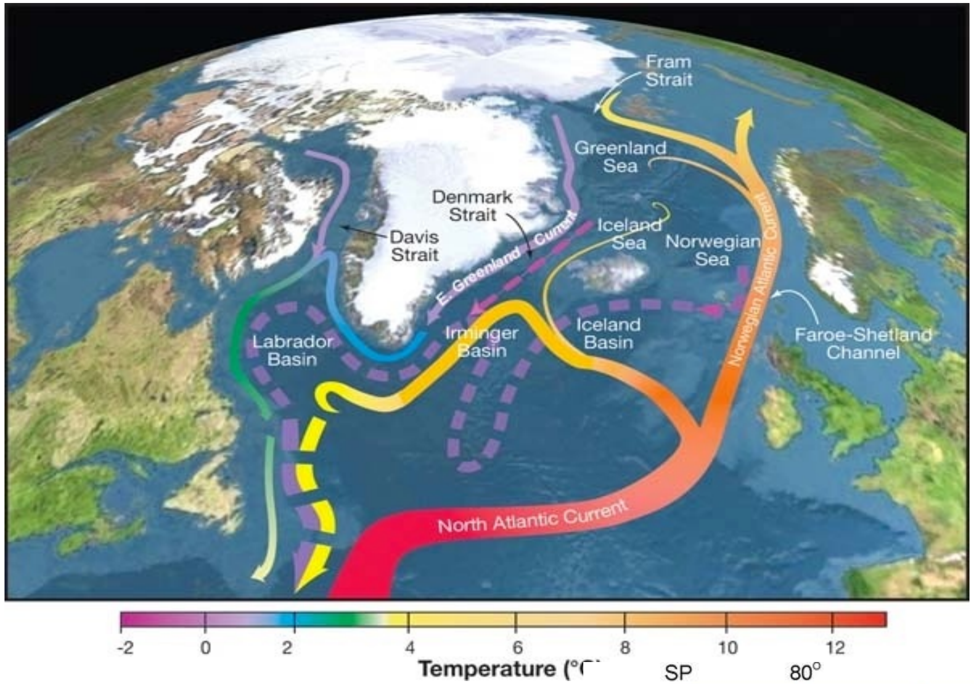
$N^2 > 0, \quad \frac{d\rho}{dz} < 0,$ stable water column

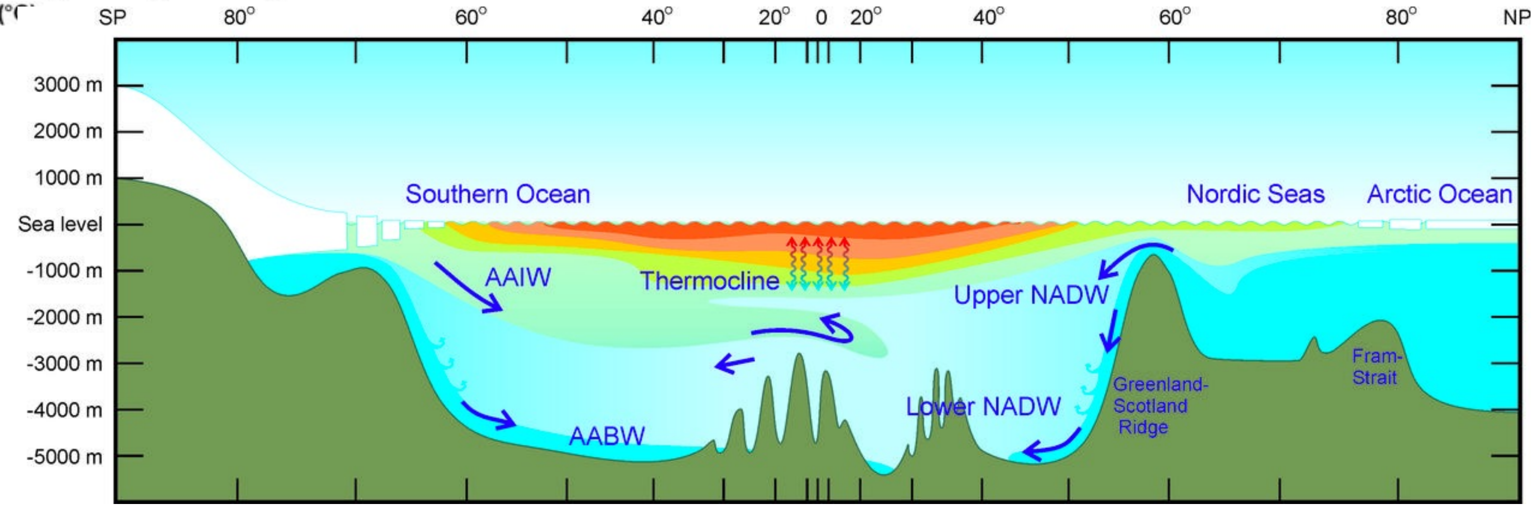
$N^2 < 0, \quad \frac{d\rho}{dz} > 0,$ unstable water column (convection)



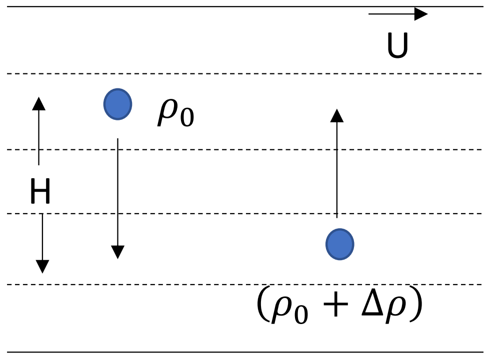


Deep convection in the North Atlantic ocean – ocean to atmosphere heat loss





The importance of stratification: the Froude number



For per unit volume,

Potential energy change:

$$\Delta PE = (\rho_0 + \Delta\rho)gH - \rho_0 gH = \Delta\rho gH$$

Kinetic energy:

$$KE = \frac{1}{2}\rho_0 U^2 + \frac{1}{2}(\rho_0 + \Delta\rho)\,U^2 \approx \rho_0 U^2$$

$$\sigma = \frac{KE}{\Delta PE} = \frac{\rho_0 U^2}{\Delta\rho gH} \sim \frac{U^2}{N^2 H^2}$$

Froude number: $Fr = \frac{U}{NH}$

$\sigma > 1$, PE change consumes a small portion of the KE of the system, so it takes little cost to break stratification, stratification is unimportant

$\sigma \leq 1$, PE change consumes all KE of the system, or KE is not sufficient to supply ΔPE , stratification cannot be broken and is important

Syllabus

1. Primitive governing equations
2. Equation approximations and simplications
3. Inertial motions
4. Geostrophic balance and thermal wind relation
5. Ekman layer dynamics
6. Gravity waves
7. Shallow-water models (barotropic, multi-layer model, reduced gravity model)
8. Shallow water waves (inertial-gravity waves, Kelvin waves)
9. Vorticity dynamics
10. Geostrophic adjustment
11. Quasi-geostrophic motions (Rossby waves)
12. Coastal dynamics

$$\underbrace{\frac{d\boldsymbol{v}}{dt}}_{\text{acceleration}} + \overbrace{f\boldsymbol{k}\times\boldsymbol{v}}^{\text{Coriolis force}} = \underbrace{-\frac{1}{\rho_0}\nabla p}_{\text{pressure gradient}} + \overbrace{\frac{\rho}{\rho_0}\boldsymbol{g}}^{\text{gravity}} + \underbrace{\nu_E\nabla^2\boldsymbol{v}}_{\text{friction}}$$

Part I. Primitive governing equations

Key points:

1. The **principles** for deriving the different governing equations (momentum, continuity, density, tracer)

2. The expressions of the **governing equations** and the **physical interpretations** of each term

Momentum equation

For an inertial frame, Newton's second Law:

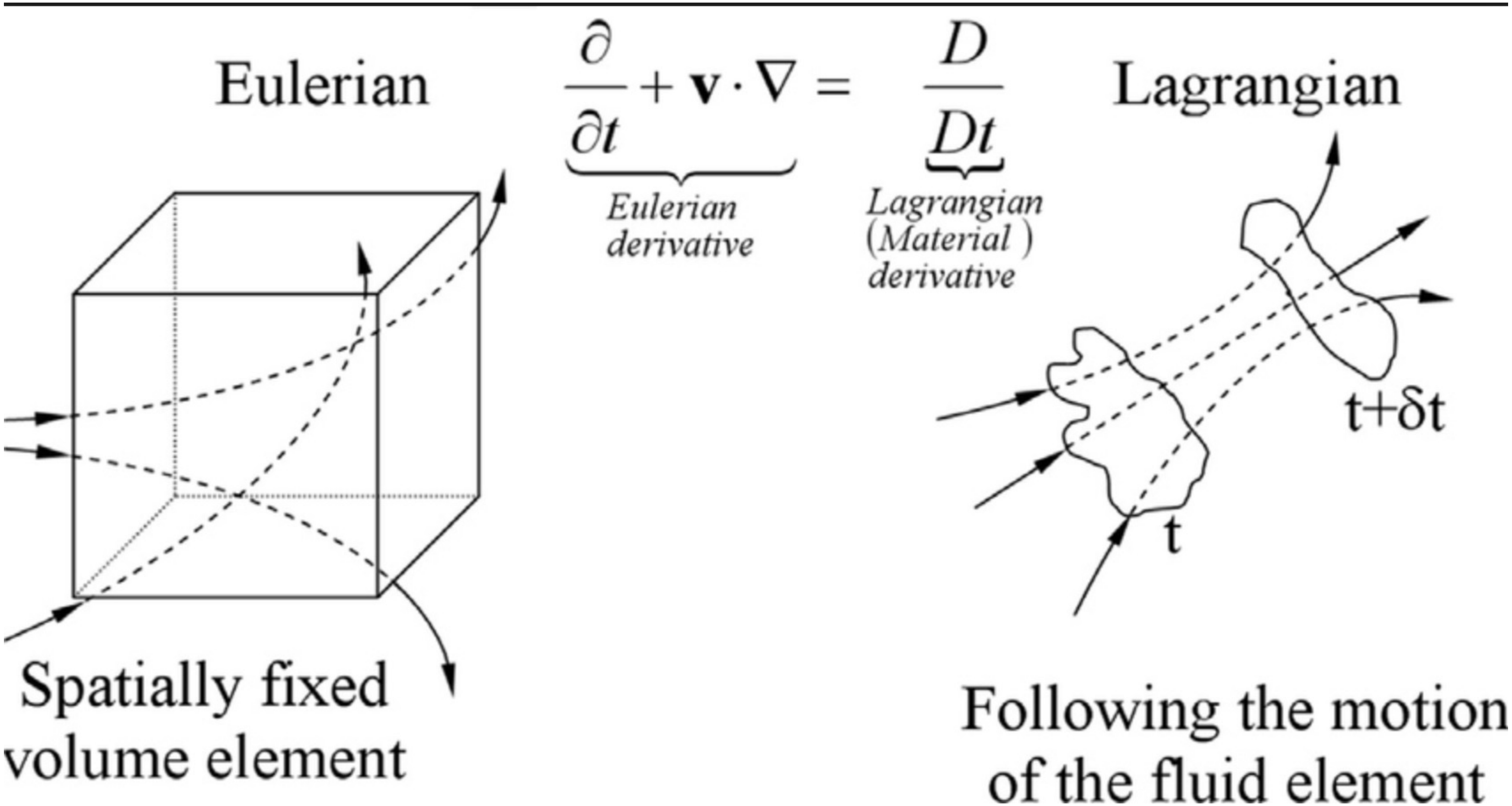
$$m\mathbf{a} = \mathbf{F}$$

For unit volume:

$$\rho \, d\mathbf{u}/dt = \mathbf{F}$$

$$d\mathbf{u}/dt = \mathbf{F}/\rho$$

Eulerian and Lagrangian methods



For a scalar $\varphi(x, y, z, t)$:

$$\delta\varphi = \frac{\partial\varphi}{\partial t}\delta t + \frac{\partial\varphi}{\partial x}\delta x + \frac{\partial\varphi}{\partial y}\delta y + \frac{\partial\varphi}{\partial z}\delta z$$

$$\frac{\delta\varphi}{\delta t} = \frac{\partial\varphi}{\partial t} + u\frac{\partial\varphi}{\partial x} + v\frac{\partial\varphi}{\partial y} + w\frac{\partial\varphi}{\partial z}$$

$$\frac{d\varphi}{dt} = \frac{\partial\varphi}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\varphi$$

$$(\boldsymbol{u}\cdot\boldsymbol{\nabla})\varphi$$

For a vector $\boldsymbol{u}(x, y, z, t)$:

$$\frac{d\boldsymbol{u}}{dt} = \frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}$$

$\boldsymbol{u} = (u, v, w)$

$$\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} \longrightarrow$$ **non-linear advection term**

local acceleration term

$$\text{x direction: } \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$\text{y direction: } \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$\text{z direction: } \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

Pressure gradient force

Derivation of Pressure Term Consider the forces acting on the sides of a small cube of fluid (Figure 7.4). The net force δF_x in the x direction is

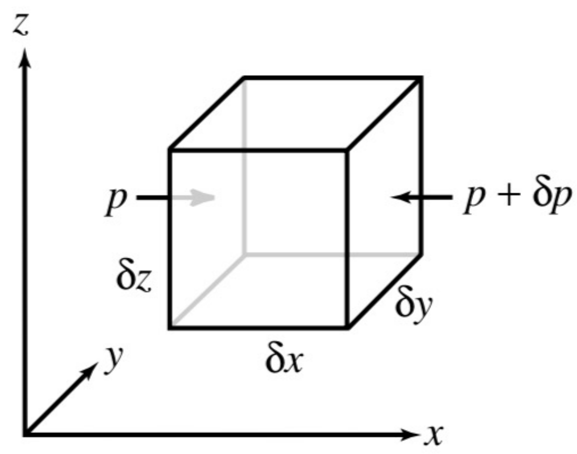
$$\delta F_x = p\,\delta y\,\delta z - (p + \delta p)\,\delta y\,\delta z$$
$$\delta F_x = -\delta p\,\delta y\,\delta z$$

But

$$\delta p = \frac{\partial p}{\partial x}\,\delta x$$

and therefore

$$\delta F_x = -\frac{\partial p}{\partial x}\,\delta x\,\delta y\,\delta z$$
$$\delta F_x = -\frac{\partial p}{\partial x}\,\delta V$$



$$f(x) = f(x_0) + \left.\frac{df(x)}{dx}\right|_{x=x_0}(x - x_0) + \left.\frac{d^2 f(x)}{2!\,dx^2}\right|_{x=x_0}(x - x_0)^2 + \ldots$$

Dividing by the mass of the fluid δm in the box, the acceleration of the fluid in the x direction is:

$$a_x = \frac{\delta F_x}{\delta m} = -\frac{\partial p}{\partial x}\frac{\delta V}{\delta m}$$

$$\boxed{a_x = -\frac{1}{\rho}\frac{\partial p}{\partial x}} \qquad (7.13)$$

The momentum equations

x direction: $$\frac{\partial \mathrm{u}}{\partial \mathrm{t}}+\mathrm{u}\frac{\partial \mathrm{u}}{\partial \mathrm{x}}+\mathrm{v}\frac{\partial \mathrm{u}}{\partial \mathrm{y}}+\mathrm{w}\frac{\partial \mathrm{u}}{\partial \mathrm{z}}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\cdots$$

y direction: $$\frac{\partial \mathrm{v}}{\partial \mathrm{t}}+\mathrm{u}\frac{\partial \mathrm{v}}{\partial \mathrm{x}}+\mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{y}}+\mathrm{w}\frac{\partial \mathrm{v}}{\partial \mathrm{z}}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\cdots$$

z direction: $$\frac{\partial \mathrm{w}}{\partial \mathrm{t}}+\mathrm{u}\frac{\partial \mathrm{w}}{\partial \mathrm{x}}+\mathrm{v}\frac{\partial \mathrm{w}}{\partial \mathrm{y}}+\mathrm{w}\frac{\partial \mathrm{w}}{\partial \mathrm{z}}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\cdots$$