Chapter 11

Stratification

ABSTRACT

After having studied the effects of rotation in homogeneous fluids, we now turn our attention toward the other distinctive feature of geophysical fluid dynamics, namely, stratification. A basic measure of stratification, the Brunt–Väisälä frequency, is introduced, and the accompanying dimensionless ratio, the Froude number, is defined and given a physical interpretation. The numerical part deals with the handling of unstable stratification in model simulations

11.1 INTRODUCTION

As Chapter 1 stated, problems in geophysical fluid dynamics concern fluid motions with one or both of two attributes, namely, ambient rotation and stratification. In the preceding chapters, attention was devoted exclusively to the effects of rotation, and stratification was avoided by the systematic assumption of a homogeneous fluid. We noted that rotation imparts to the fluid a strong tendency to behave in a columnar fashion—to be vertically rigid.

By contrast, a stratified fluid, consisting of fluid parcels of various densities, will tend under gravity to arrange itself so that the higher densities are found below lower densities. This vertical layering introduces an obvious gradient of properties in the vertical direction, which affects—among other things—the velocity field. Hence, the vertical rigidity induced by the effects of rotation will be attenuated by the presence of stratification. In return, the tendency of denser fluid to lie below lighter fluid imparts a horizontal rigidity to the system.

Because stratification induces a certain degree of decoupling between the various fluid masses (those of different densities), stratified systems typically contain more degrees of freedom than homogeneous systems, and we anticipate that the presence of stratification permits the existence of additional types of motions. When the stratification is mostly vertical (e.g., layers of various densities stacked on top of one another), gravity waves can be sustained internally (Chapter 13). When the stratification also has a horizontal component, additional waves can be permitted. These may lead to motion in equilibrium (Chapter 15), or, if they grow at the expense of the basic potential energy available in the system, may cause instabilities (Chapter 17).

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11.2 STATIC STABILITY

Let us first consider a fluid in static equilibrium. Lack of motion can occur only in the absence of horizontal forces and thus in the presence of horizontal homogeneity. Stratification is then purely vertical (Fig. 11.1).

It is intuitively obvious that if the heavier fluid parcels are found below the lighter fluid parcels, the fluid is stable, whereas if heavier parcels lie above lighter ones, the system is apt to overturn, and the fluid is unstable. Let us now verify this intuition. Take a fluid parcel at a height z above a certain reference level, where the density is $\rho(z)$, and displace it vertically to the higher level $z+h$, where the ambient density is $\rho(z+h)$ (Fig. 11.1). If the fluid is incompressible, our displaced parcel retains its former density despite a slight pressure change, and that new level is subject to a net downward force equal to its own weight minus, by Archimedes' buoyancy principle, the weight of the displaced fluid, thus

$$g[\rho(z)-\rho(z+h)]V,$$

where V is the volume of the parcel. As it is written, this force is positive if it is directed downward. Newton's law (mass times acceleration equals upward force) yields

$$\rho(z)\, V\frac{\mathrm{d}^2h}{\mathrm{d}t^2} = g\,[\rho(z+h) - \rho(z)]\,V. \tag{11.1}$$

Now, geophysical fluids are generally only weakly stratified; the density variations, although sufficient to drive or affect motions, are nonetheless relatively small compared with the average or reference density of the fluid. This remark was the essence of the Boussinesq approximation (Section 3.7). In the present case, this fact allows us to replace $\rho(z)$ on the left-hand side of Eq. (11.1) by the reference density ρ_0 and to use a Taylor expansion to approximate the density difference on the right by

$$\rho(z+h)-\rho(z)\simeq\frac{\mathrm{d}\rho}{\mathrm{d}z}h.$$

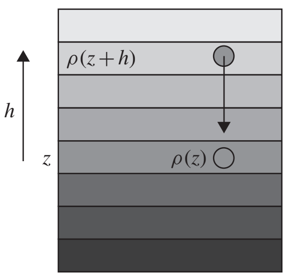

FIGURE 11.1 When an incompressible fluid parcel of density $\rho(z)$ is vertically displaced from level z to level $z+h$ in a stratified environment, a buoyancy force appears because of the density difference $\rho(z)-\rho(z+h)$ between the particle and the ambient fluid.

After a division by V, Eq. (11.1) reduces to

$$\frac{\mathrm{d}^2 h}{\mathrm{d}t^2} - \frac{g}{\rho_0}\frac{\mathrm{d}\rho}{\mathrm{d}z}h = 0, \tag{11.2}$$

which shows that two cases can arise. The coefficient $-(g/\rho_0)\mathrm{d}\rho/\mathrm{d}z$ is either positive or negative. If it is positive ($\mathrm{d}\rho/\mathrm{d}z < 0$, corresponding to a fluid with the greater densities below the lesser densities), we can define the quantity N^2 as

$$N^2 = -\frac{g}{\rho_0}\frac{\mathrm{d}\rho}{\mathrm{d}z}, \tag{11.3}$$

and the solution to the equation has an oscillatory character, with frequency N. Physically, this means that when displaced upward, the parcel is heavier than its surroundings, experiences a downward recalling force, falls down, and, in the process, acquires a vertical velocity; upon reaching its original level, the particle's inertia causes it to go further downward and to become surrounded by heavier fluid. The parcel, now buoyant, is recalled upward, and oscillations persist about the equilibrium level. The quantity N, defined by the square root of Eq. (11.3), provides the frequency of the oscillation and can thus be called the *stratification frequency*. It goes more commonly, however, by the name of Brunt–Väisälä frequency, in recognition of the two scientists who were the first to highlight the importance of this frequency in stratified fluids. (See their biographies at the end of this chapter.)

If the coefficient in Eq. (11.1) is negative (i.e., $\mathrm{d}\rho/\mathrm{d}z > 0$, corresponding to a top-heavy fluid configuration), the solution exhibits exponential growth, a sure sign of instability. The parcel displaced upward is surrounded by heavier fluid, finds itself buoyant, and moves farther and farther away from its initial position. Obviously, small perturbations will ensure not only that the single displaced parcel will depart from its initial position, but that all other fluid parcels will likewise participate in a general overturning of the fluid until it is finally stabilized, with the lighter fluid lying above the heavier fluid. If, however, a permanent destabilization is forced onto the fluid, such as by heating from below or cooling from above, the fluid will remain in constant agitation, a process called *convection*.

11.3 A NOTE ON ATMOSPHERIC STRATIFICATION

In a compressible fluid, such as the air of our planetary atmosphere, density can change in one of two ways: by pressure changes or by internal energy changes. In the first case, a pressure variation resulting in no heat exchange (i.e., an adiabatic compression or expansion) is accompanied by both density and temperature variations: All three quantities increase (or decrease) simultaneously, though not in equal proportions. If the fluid is made of fluid parcels all having the same heat content, the lower parcels, experiencing the weight of those above them, will be more compressed than those in the upper levels, and the system will appear stratified, with the denser and warmer fluid underlying the lighter and colder fluid. But such stratification cannot be dynamically relevant, for if parcels are interchanged adiabatically, they adjust their density and temperature according to the local pressure, and the system is left unchanged.

In contrast, internal energy changes are dynamically important. In the atmosphere, such variations occur because of a heat flux (such as heating in the tropics and cooling at high latitudes, or according to the diurnal cycle) or because of variations in air composition (such as water vapor). Such variations among fluid parcels do remain despite adiabatic compression or expansion and cause density differences that drive motions. It is thus imperative to distinguish, in a compressible fluid, the density variations that are dynamically relevant from those that are not. Such separation of density variations leads to the concept of potential density.

First, we consider a neutral (adiabatic) atmosphere—that is, one consisting of all air parcels having the same internal energy. Further, let us assume that the air, a mixture of various gases, behaves as a single ideal gas. Under these assumptions, we can write the equation of state and the adiabatic conservation law:

$$p = R\rho T, \tag{11.4}$$

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}, \tag{11.5}$$

where p, ρ, and T are, respectively, the pressure, density,[1] and absolute temperature; $R = C_p - C_v$ and $\gamma = C_p/C_v$ are the constants of an ideal gas.[2] Finally, p_0 and ρ_0 are reference pressure and density characterizing the level of internal energy of the fluid; the corresponding reference temperature T_0 is obtained from Eq. (11.4)—that is, $T_0 = p_0/R\rho_0$. Expressing both pressure and density in terms of the temperature, we obtain

$$\frac{p}{p_0} = \left(\frac{T}{T_0}\right)^{\gamma/(\gamma-1)} \tag{11.6a}$$

$$\frac{\rho}{\rho_0} = \left(\frac{T}{T_0}\right)^{1/(\gamma-1)}. \tag{11.6b}$$

Without motion, the atmosphere is in static equilibrium, which requires hydrostatic balance:

$$\frac{dp}{dz} = -\rho g. \tag{11.7}$$

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[1]In contrast with preceding chapters, the variables p and ρ denote here the full pressure and density.

[2]For air, values are $C_p = 1005\ \mathrm{J\,kg^{-1}\,K^{-1}}$, $C_v = 718\ \mathrm{J\,kg^{-1}\,K^{-1}}$, $R = 287\ \mathrm{J\,kg^{-1}\,K^{-1}}$, and $\gamma = 1.40$.

Elimination of p and ρ by the use of Eqs. (11.6a) and (11.6b) yields a single equation for the temperature:

$$\frac{dT}{dz} = -\frac{\gamma - 1}{\gamma}\frac{g}{R} = -\frac{g}{C_p}. \tag{11.8}$$

In the derivation, it was assumed that p_0, ρ_0, and thus T_0 are not dependent on z, in agreement with our premise that the atmosphere is composed of parcels with identical internal energy contents. Equation (11.8) states that the temperature in such an atmosphere must decrease with increasing height at the uniform rate $g/C_p \simeq 10$ K/km. This gradient is called the *adiabatic lapse rate*. Physically, lower parcels are under greater pressure than higher parcels and thus have higher densities and temperatures. This explains why the air temperature is lower on mountain tops than in the valleys below.

It almost goes without saying that the departures from this adiabatic lapse rate—and not the actual temperature gradients—are to be considered in the study of atmospheric motions. We can demonstrate this clearly by redoing here, with a compressible fluid, the analysis of a vertical displacement performed in the previous section with an incompressible fluid. Consider a vertically stratified gas with pressure, density, and temperature, p, ρ, and T, varying with height z but not necessarily according to Eq. (11.8); that is, the heat content in the fluid is not uniform. The fluid is in static equilibrium so that Eq. (11.7) is satisfied. Consider now a parcel at height z; its properties are $p(z)$, $\rho(z)$, and $T(z)$. Imagine then that this fluid parcel is displaced adiabatically upward over a small distance h. According to the hydrostatic equation, this results in a pressure change $\delta p = -\rho g h$, which causes density and temperature changes given by the adiabatic constraints Eqs. (11.5) and (11.6a): $\delta\rho = -\rho g h/\gamma R T$ and $\delta T = -(\gamma - 1)gh/\gamma R$. Thus, the new density is $\rho' = \rho + \delta\rho = \rho - \rho g h/\gamma R T$. But, at that new level, the ambient density is given by the stratification: $\rho(z+h) \simeq \rho(z) + (d\rho/dz)h$. The net force exerted on the parcel is the difference between its own weight and the weight of the displaced fluid at the new location (the buoyancy force), which per volume is

$$\begin{aligned} F &= g\,[\rho_{\text{ambient}} - \rho_{\text{parcel}}] \\ &= g\,[\rho(z+h) - \rho'] \\ &\simeq g\left(\frac{d\rho}{dz} + \frac{\rho g}{\gamma R T}\right)h. \end{aligned}$$

As the ideal gas law ($p = R\rho T$) holds everywhere, the vertical gradients of pressure, density, and temperature are related by

$$\frac{dp}{dz} = RT\frac{d\rho}{dz} + R\rho\frac{dT}{dz}.$$

With the pressure gradient given by the hydrostatic balance (11.7), it follows that the density and temperature gradients are related by

$$\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}z}+\frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}z}+\frac{g}{RT}=0,$$

and the force on the fluid parcel can be expressed in terms of the temperature gradient

$$F\simeq-\frac{\rho g}{T}\left(\frac{\mathrm{d}T}{\mathrm{d}z}+\frac{g}{C_p}\right)h.$$

If

$$N^2=-\frac{g}{\rho}\left(\frac{\mathrm{d}\rho}{\mathrm{d}z}+\frac{\rho g}{\gamma RT}\right) \tag{11.9a}$$

$$=+\frac{g}{T}\left(\frac{\mathrm{d}T}{\mathrm{d}z}+\frac{g}{C_p}\right) \tag{11.9b}$$

is a positive quantity, the force recalls the particle toward its initial level, and the stratification is stable. As we can clearly see, the relevant quantity is not the actual temperature gradient but its departure from the adiabatic gradient $-g/C_p$. As in the previous case of a stably stratified incompressible fluid, the quantity N is the frequency of vertical oscillations. It is called the stratification, or Brunt–Väisälä, frequency.

In order to avoid the systematic subtraction of the adiabatic gradient from the temperature gradient, the concept of potential temperature is introduced. The *potential temperature*, denoted by θ, is defined as the temperature that the parcel would have if it were brought adiabatically to a given reference pressure.[3] From Eq. (11.6a), we have

$$\frac{p}{p_0}=\left(\frac{T}{\theta}\right)^{\gamma/(\gamma-1)}$$

and hence

$$\theta=T\left(\frac{p}{p_0}\right)^{-(\gamma-1)/\gamma}. \tag{11.10}$$

The corresponding density is called the *potential density*, denoted by σ:

$$\sigma=\rho\left(\frac{p}{p_0}\right)^{-1/\gamma}=p_0/\mathrm{R}\,\theta. \tag{11.11}$$

[3] In the atmosphere, this reference pressure is usually taken as the standard sea level pressure of 1013.25 millibars $=1.01325\times10^5$ N/m^2.

The definition of the stratification frequency (11.9b) takes the more compact form:

$$N^2 = -\frac{g}{\sigma}\frac{d\sigma}{dz} = +\frac{g}{\theta}\frac{d\theta}{dz}. \tag{11.12}$$

Comparison with the earlier definition, (11.3), immediately shows that the substitution of potential density for density allows us to treat compressible fluids as incompressible.

During daytime and above land, the lower atmosphere is typically heated from below by the warmer ground and is in a state of turbulent convection. The convective layer not only covers the region where the time-averaged gradient of potential temperature is negative but also penetrates into the region above where it is positive (Fig. 11.2). Consequently, the sign of N^2 at a particular level is not

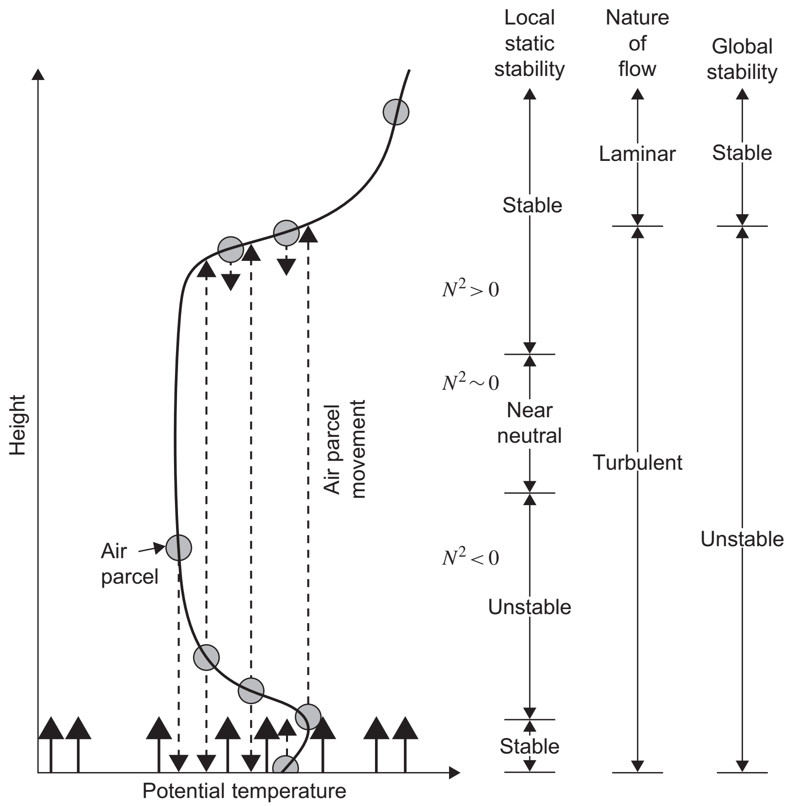

FIGURE 11.2 Typical profile of potential temperature in the lower atmosphere above warm ground. Heating from below destabilizes the air, generating convection and turbulence. Note how the convective layer extends not only over the region of negative N^2 but also slightly beyond, where N^2 is positive. Such a situation shows that a positive value of N^2 may not always be indicative of local stability. Global stability refers then to regions where even a finite amplitude displacement cannot destabilize the fluid parcel. (From *Stull, 1991*)

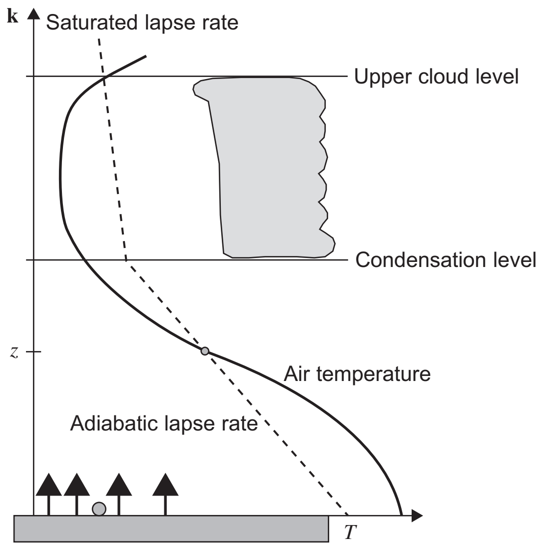

FIGURE 11.3 Fluid parcels located around level z amidst a temperature gradient (curved solid line) locally exceeding the adiabatic lapse rate (dashed line) are in an unstable situation. They move upward and eventually reach their saturation level, condensation takes place, and the lapse rate is decreased. If an inversion is present at higher levels, cloud extension is vertically limited.

unequivocally indicative of stability at that level. For this reason, Stull (1991) advocates the use of a nonlocal criterion to determine static stability. Those considerations apply equally well to the upper ocean under surface cooling.

When the air is moist, the thermodynamics of water vapor affect the situation, and, because the value of C_p for water vapor is higher than that for dry air, the adiabatic lapse rate is reduced. As the temperature of ascending air drops, the relative humidity may reach 100%, in which case condensation occurs and water droplets form a cloud. Condensation liberates latent heat, which reduces the temperature drop if parcels continue to ascend. The lapse is then further reduced to a *saturated adiabatic lapse rate*, as depicted in Fig. 11.3.

11.4 CONVECTIVE ADJUSTMENT

When gravitational instability is present in the ocean or atmosphere, nonhydrostatic movements tend to restore stability through narrow columns of convection, rising plumes and thermals in the atmosphere, and so-called convective chimneys in the ocean (e.g., Marshall & Schott, 1999). These vigorous vertical motions are not resolved by most computer models, and parameterizations called *convection schemes* are introduced to remove the instability and model the mixing associated with convection. Such parameterization can be achieved by additional terms in the governing equations, typically through a much increased eddy viscosity and diffusivity whenever $N^2 \leq 0$ (e.g., Cox, 1984; Marotzke, 1991). Other parameterizations are pieces of computer code of the type (see Fig. 11.4):

```
while there is any denser fluid being on top of lighter fluid
   loop over all layers
      if density of layer above > density of layer below
         mix properties of both layers, with a volume-weighted
            average
      end if
   end loop over all layers
end while
```

Oceanic circulation models (e.g., Bryan, 1969; Cox, 1984) were the first to use this type of parameterization.

The mixing accomplished by such scheme, however, is too strong in practice, because the model mixes fluid properties instantaneously over an entire horizontal grid cell of size $\Delta x \Delta y$, whereas physical convection operates at shorter scales and only partially mixes the physical properties at the spot. Therefore, numerical mixing should preferably be replaced by a mere swapping of fluid masses, under the assumption that convection carries part of the properties without alteration to their new level of equilibrium (e.g., Roussenov, Williams & Roether, 2001). It is clear that some arbitrariness remains and that every application demands its own calibration. Among other things, changing the time step clearly modifies the speed at which mixing takes place.

In atmospheric applications, the situation is more complicated as it may involve condensation, latent heat release, and precipitation during convective movement. Atmospheric convection parameterizations involve delicate adjustments of both temperature and moisture in the vertical (e.g., Betts, 1986; Kuo, 1974).

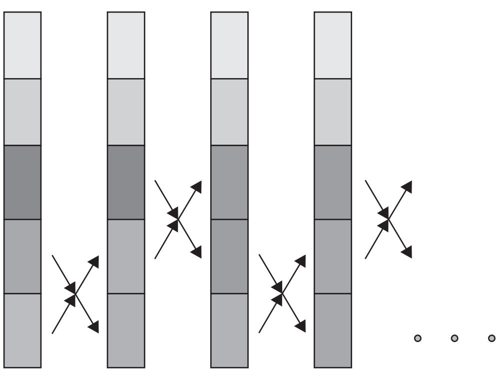

FIGURE 11.4 Illustration of convective adjustment within a fluid heated from below. Grid boxes below heavier neighbors are systematically mixed in pairs until the whole fluid column is rendered stable.

11.5 THE IMPORTANCE OF STRATIFICATION: THE FROUDE NUMBER

It was established in Section 1.5 that rotational effects are dynamically important when the Rossby number is on the order of unity or less. This number compares the distance traveled horizontally by a fluid parcel during one revolution ($\sim U/\Omega$) with the length scale over which the motions take place (L). Rotational effects are important when the former is less than the latter. By analogy, we may ask whether there exists a similar number measuring the importance of stratification. From the remarks in the preceding sections, we can anticipate that the stratification frequency, N, and the height scale, H, of a stratified fluid will play roles similar to those of Ω and L in rotating fluids.

To illustrate how such a dimensionless number can be derived, let us consider a stratified fluid of thickness H and stratification frequency N flowing horizontally at a speed U over an obstacle of length L and height Δz (Fig. 11.5). We can think of a wind in the lower atmosphere blowing over a mountain range. The presence of the obstacle forces some of the fluid to be displaced vertically and, hence, requires some supply of gravitational energy. Stratification will act to restrict or minimize such vertical displacements in some way, forcing the flow to pass around rather than over the obstacle. The greater the restriction, the greater the importance of stratification.

The time passed in the vicinity of the obstacle is approximately the time spent by a fluid parcel to cover the horizontal distance L at the speed U, that is, $T = L/U$. To climb a height of Δz, the fluid needs to acquire a vertical velocity on the order of

$$W = \frac{\Delta z}{T} = \frac{U \Delta z}{L}. \tag{11.13}$$

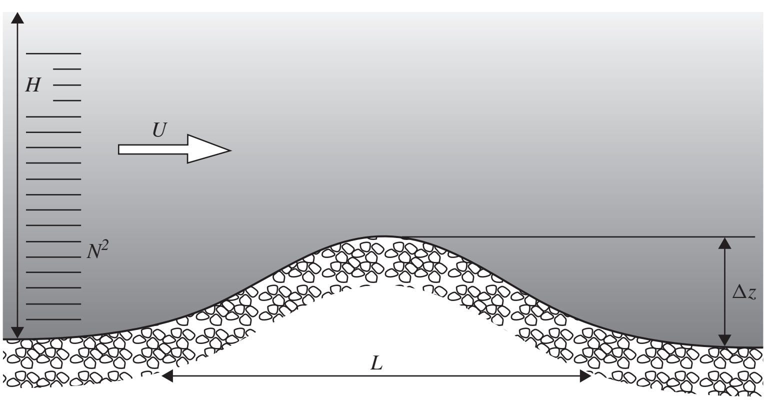

FIGURE 11.5 Situation in which a stratified flow encounters an obstacle, forcing some fluid parcels to move vertically against a buoyancy force.

The vertical displacement is on the order of the height of the obstacle and, in the presence of stratification $\rho(z)$, causes a density perturbation on the order of

$$\Delta\rho = \left|\frac{d\bar{\rho}}{dz}\right| \Delta z$$
$$= \frac{\rho_0 N^2}{g} \Delta z, \tag{11.14}$$

where $\bar{\rho}(z)$ is the fluid's vertical density profile upstream. In turn, this density variation gives rise to a pressure disturbance that scales, via the hydrostatic balance, as

$$\Delta P = gH\Delta\rho$$
$$= \rho_0 N^2 H \Delta z. \tag{11.15}$$

By virtue of the balance of forces in the horizontal, the pressure-gradient force must be accompanied by a change in fluid velocity $[u\partial u/\partial x + v\partial u/\partial y \sim (1/\rho_0)\partial p/\partial x]$:

$$\frac{U^2}{L} = \frac{\Delta P}{\rho_0 L} \quad \Longrightarrow \quad U^2 = N^2 H \Delta z. \tag{11.16}$$

From this last expression, the ratio of vertical convergence, W/H, to horizontal divergence, U/L, is found to be

$$\frac{W/H}{U/L} = \frac{\Delta z}{H} = \frac{U^2}{N^2 H^2}. \tag{11.17}$$

We immediately note that if U is less than the product NH, W/H must be less than U/L, implying that convergence in the vertical cannot fully meet horizontal divergence. Consequently, the fluid is forced to be partially deflected horizontally so that the term $\partial u/\partial x$ can be met by $-\partial v/\partial y$ better than by $-\partial w/\partial z$. The stronger the stratification, the smaller is U compared with NH and, thus, W/H compared with U/L.

From this argument, we conclude that the ratio

$$Fr = \frac{U}{NH}, \tag{11.18}$$

called the *Froude number*, is a measure of the importance of stratification. The rule is as follows: If $Fr \lesssim 1$, stratification effects are important; the smaller Fr, the more important these effects are.

The analogy with the Rossby number of rotating fluids,

$$Ro = \frac{U}{\Omega L}, \tag{11.19}$$

where Ω is the angular rotation rate and L the horizontal scale, is immediate. Both Froude and Rossby numbers are ratios of the horizontal velocity scale by a product of frequency and length scale; for stratified fluids, the relevant frequency and length are naturally the stratification frequency and the height scale, whereas in rotating fluids they are, respectively, the rotation rate and the horizontal length scale.

The analogy can be pursued a little further. Just as the Froude number is a measure of the vertical velocity in a stratified fluid [via Eq. (11.17)], the Rossby number can be shown to be a measure of the vertical velocity in a rotating fluid. We saw (Section 7.2) that strongly rotating fluids (Ro nominally zero) allow no convergence of vertical velocity, even in the presence of topography. This results from the absence[4] of horizontal divergence in geostrophic flows. In reality, the Rossby number cannot be nil, and the flow cannot be purely geostrophic. The nonlinear terms, of relative importance measured by Ro, yield corrective terms to the geostrophic velocities of the same relative importance. Thus, the horizontal divergence, $\partial u/\partial x + \partial v/\partial y$, is not zero but is on the order of RoU/L. Since the divergence is matched by the vertical divergence, $-\partial w/\partial z$, on the order of W/H, we conclude that

$$\frac{W/H}{U/L} = Ro, \tag{11.20}$$

in rotating fluids. Contrasting Eqs. (11.17)–(11.20), we note that, with regard to vertical velocities, the square of the Froude number is the analogue of the Rossby number.

In continuation of the analogy, it is tempting to seek the stratified analogue of the Taylor column in rotating fluids. Recall that Taylor columns occur in rapidly rotating fluids ($Ro = U/\Omega L \ll 1$). Let us then ask what happens when a fluid is very stratified ($Fr = U/NH \ll 1$). By virtue of Eq. (11.17), the vertical displacements are severely restricted ($\Delta z \ll H$), implying that an obstacle causes the fluid at that level to be deflected almost purely horizontally. (In the absence of rotation, there is no tendency toward vertical rigidity, and parcels at levels above the obstacle can flow straight ahead without much disruption.) If the obstacle occupies the entire width of the domain, such a horizontal detour is not allowed, and the fluid at the level of the obstacle is blocked on both upstream and downstream sides. This horizontal blocking in stratified fluids is the analogue of the vertical Taylor columns in rotating fluids. Further analogies between homogeneous rotating fluids and stratified nonrotating fluids have been described by Veronis (1967).

11.6 COMBINATION OF ROTATION AND STRATIFICATION

In the light of the previous remarks, we are now in position to ask what happens when, as in actual geophysical fluids, the effects of rotation and stratification

[4]For the sake of the analogy, we rule out here an possible beta effect.

are simultaneously present. The preceding analysis remains unchanged, except that we now invoke the geostrophic balance [see Eq. (7.4)] in the horizontal momentum equation to obtain the horizontal velocity scale:

$$\Omega U = \frac{\Delta P}{\rho_0 L} \quad \Longrightarrow \quad U = \frac{N^2 H \Delta z}{\Omega L}. \tag{11.21}$$

The ratio of the vertical to horizontal convergence then becomes

$$\begin{aligned}\frac{W/H}{U/L} &= \frac{\Delta z}{H} = \frac{\Omega L U}{N^2 H^2} \\ &= \frac{Fr^2}{Ro}.\end{aligned} \tag{11.22}$$

This is a particular case of great importance. According to our foregoing scaling analysis, the ratio of vertical convergence to horizontal divergence, $(W/H)/(U/L)$, is given by Fr^2, Fr^2/Ro, or Ro, depending on whether vertical motions are controlled by stratification, rotation, or both (Fig. 11.6). Thus, if Fr^2/Ro is less than Ro, stratification restricts vertical motions more than rotation and is the dominant process. The converse is true if Fr^2/Ro is greater than Ro.

Note that Ro is in the denominator of Eq. (11.22), which implies that the influence of rotation is to increase the scale for the vertical velocity when stratification is present. However, since vertical divergence cannot exist without horizontal convergence ($W/H \lesssim U/L$), the following inequality must hold:

$$Fr^2 \lesssim Ro, \tag{11.23}$$

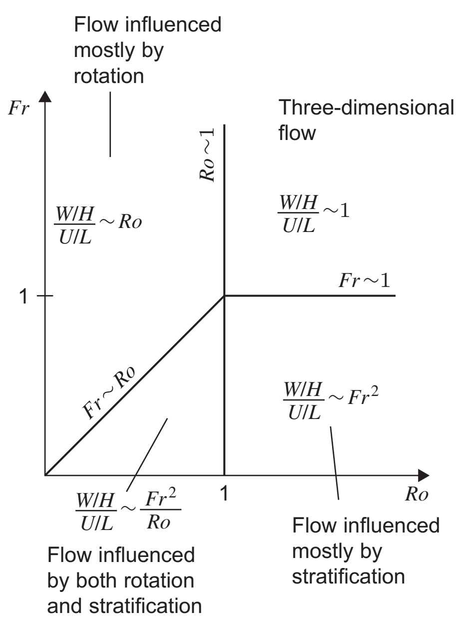

FIGURE 11.6 Recapitulation of the various scalings of the ratio of vertical convergence (divergence), W/H, to horizontal divergence (convergence), U/L, as a function of the Rossby number, $Ro = U/(\Omega L)$, and Froude number, $Fr = U/(NH)$.

that is,

$$\frac{U}{NH} \lesssim \frac{NH}{\Omega L}. \tag{11.24}$$

This sets an upper bound for the magnitude of the flow field in a fluid under given rotation (Ω) and of given stratification (N) in a domain of given dimensions (L, H). If the velocity is imposed externally (e.g., by an upstream condition), the inequality specifies either the horizontal or the vertical length scales of the possible disturbances. Finally, if the system is such that all quantities are externally imposed and that they do not meet Eq. (11.24), then special effects such as Taylor columns or blocking must occur.

Inequality Eq. (11.24) brings a new dimensionless number $NH/\Omega L$, namely, the ratio of the Rossby and Froude numbers. For historical reasons and also because it is more convenient in some dimensional analyses, the square of this quantity is usually defined:

$$Bu = \left(\frac{NH}{\Omega L}\right)^2 = \left(\frac{Ro}{Fr}\right)^2. \tag{11.25}$$

It bears the name of *Burger number*, in honor of Alewyn P. Burger (1927–2003), who contributed to our understanding of geostrophic scales of motions (Burger, 1958). In practice, the Burger number is a useful measure of stratification in the presence of rotation.

In typical geophysical fluids, the height scale is much less than the horizontal length scale ($H \ll L$), but there is also a disparity between the two frequencies Ω and N. Although the rotation rate of the earth corresponds to a period of 24 h, the stratification frequency generally corresponds to much shorter periods, on the order of few to tens of minutes in both the ocean and atmosphere. This implies that generally $\Omega \ll N$ and opens the possibility of a Burger number on the order of unity.

Stratification and rotation influence the flow field to similar degrees if Fr^2/Ro and Ro are on the same order. Such is the case when the Froude number equals the Rossby number and, consequently, the Burger number is unity. The horizontal length scale then assumes a special value:

$$L = \frac{NH}{\Omega}. \tag{11.26}$$

For the values of Ω and N just cited and a height scale H of 100 m in the ocean and 1 km in the atmosphere, this horizontal length scale is on the order of 50 km and 500 km in the ocean and atmosphere, respectively. At this length scale, stratification and rotation go hand in hand. Later on (Chapter 15), it will be shown that the scale defined above is none other than the so-called *internal radius of deformation*.

ANALYTICAL PROBLEMS

11.1. Gulf Stream waters are characterized by surface temperatures around 22°C. At a depth of 800 m below the Gulf Stream, temperature is only 10°C. Using the value $2.1 \times 10^{-4}\,\mathrm{K}^{-1}$ for the coefficient of thermal expansion, calculate the stratification frequency. What is the horizontal length at which both rotation and stratification play comparable roles? Compare this length scale to the width of the Gulf Stream.

11.2. An atmospheric inversion occurs when the temperature increases with altitude, in contrast to the normal situation when the temperature decays with height. This corresponds to a very stable stratification and, hence, to a lack of ventilation (smog, etc.). What is the stratification frequency when the inversion sets in ($\mathrm{d}T/\mathrm{d}z = 0$)? Take $T = 290\,\mathrm{K}$ and $C_p = 1005\,\mathrm{m}^2\,\mathrm{s}^{-2}\,\mathrm{K}^{-1}$).

11.3. A meteorological balloon rises through the lower atmosphere, simultaneously measuring temperature and pressure. The reading, transmitted to the ground station where the temperature and pressure are, respectively, 17°C and 1028 millibars, reveals a gradient $\Delta T/\Delta p$ of 6°C per 100 millibars. Estimate the stratification frequency. If the atmosphere were neutral, what would the reading be?

11.4. Wind blowing from the sea at a speed of 10 m/s encounters Diamond Head, an extinct volcano on the southeastern coast of O'ahu Island in Hawai'i. This volcano is 232 m tall and 20 km wide. Stable air possesses a stratification frequency on the order of $0.02\ \mathrm{s}^{-1}$. How do vertical displacements compare to the height of the volcano? What does this imply about the importance of the stratification? Is the Coriolis force important in this case?

11.5. Redo Problem 11.4 with the same wind speed and stratification but with a mountain range 1000 m high and 500 km wide.

11.6. Vertical soundings of the atmosphere provided the temperature profiles displayed in Fig. 11.7. Analyze the stability of each profile.

NUMERICAL EXERCISES

11.1. Use `medprof.m` to read average Mediterranean temperature and salinity vertical profiles and calculate N^2 for various levels of vertical resolution (averaging data within cells). What do you conclude? (*Hint*: Use `ies80.m` for the state equation.)

11.2. Use the diffusion equation solver of Numerical Exercise 5.4 with a turbulent diffusion coefficient that changes from 10^{-4} to $10^{-2}\ \mathrm{m}^2/\mathrm{s}$ whenever N^2 is negative. Simulate the evolution of a 50-m high water column with

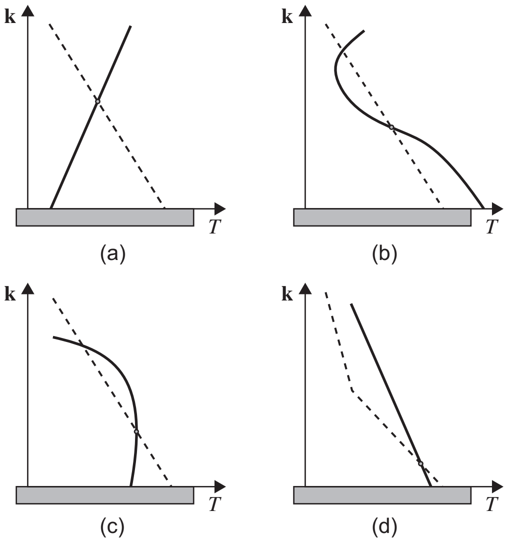

FIGURE 11.7 Various vertical profiles of temperature (solid lines) with the lapse rate (dashed line) corresponding to a particular fluid parcel (dot).

an initially stable vertical temperature gradient of 0.3°C/m subsequently cooled at the surface by a heat loss of 100 W/m^2. Salinity is unchanged. Study the effect of changes in Δz and Δt.

11.3. Implement the algorithm outlined in Section 11.4, to remove any gravitational instability instantaneously. Keep the turbulent diffusivity constant at 10^{-4} m^2/s and simulate the same problem as in Numerical Exercise 11.2.

David Brunt
1886–1965

As a bright young British mathematician, David Brunt began a career in astronomy, analyzing the statistics of celestial variables. Then, turning to meteorology during World War I, he became fascinated with weather forecasting and started to apply his statistical methods to atmospheric observations in the search for primary periodicities. By 1925, he had concluded that weather forecasting by extrapolation of cyclical behavior was not possible and turned his attention to the dynamic approach, which had been initiated in the late nineteenth century by William Ferrel and given new impetus by Vilhelm Bjerknes in recent years.

In 1926, he delivered a lecture at the Royal Meteorological Society on the vertical oscillations of particles in a stratified atmosphere. Lewis F. Richardson then led him to a paper published the preceding year by Finnish scientist Vilho Väisälä, in which the same oscillatory frequency was derived. This quantity is now jointly known as the Brunt–Väisälä frequency.

Continuing his efforts to explain observed phenomena by physical processes, Brunt contributed significantly to the theories of cyclones and anticyclones and of heat transfer in the atmosphere. His studies culminated in a textbook titled *Physical and Dynamical Meteorology* (1934) and confirmed him as a founder of modern meteorology. (*Photo credit: LaFayette, London*)

Vilho Väisälä
1889–1969

Altough he obtained his doctorate in mathematics (at the University of Helsinki, Finland), Vilho Väisälä found the subject rather uninspiring and became interested in meteorology. His positions at various Finnish institutes, including the Ilmala Meteorological Observation Station, required of him to develop instruments for atmospheric observations, which he did with much ingenuity. This eventually led him to establish in 1936 a commercial company for the manufacture of meteorological instrumentation, the Vaisala Company, a company now with branches on five continents and sales across the globe. In addition to his inventions and commercial activities, Väisälä retained an interest in the physics of the atmosphere, publishing over one hundred scientific papers, and mastered nine foreign languages. (*Photo credit: Vaisala Archives, Helsinki*)