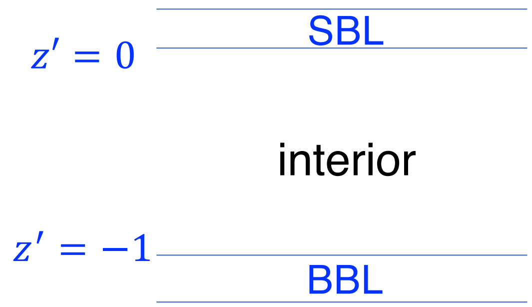
Quasi-geostrophic dynamics for frictional flows

Assumptions: homegeneous, rotational, visicid fluids, flat surface and bottom

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{\partial}{\partial z}(A_V\frac{\partial u}{\partial z})$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \frac{\partial}{\partial z}(A_V\frac{\partial v}{\partial z})$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$



Define the non-dimensional variables:

$$(u, v) = U(u', v') \qquad w = Ww' \qquad (x, y) = L(x', y') \qquad z = Hz' \qquad p = Pp'$$

$$\frac{U}{T}\frac{\partial u'}{\partial t'} + \frac{U^2}{L}u'\frac{\partial u'}{\partial x'} + \frac{U^2}{L}v'\frac{\partial u'}{\partial y'} + \frac{U^2}{L}w'\frac{\partial u'}{\partial z'} - fUv' = -\frac{P}{\rho_0 L}\frac{\partial p'}{\partial x'} + \frac{A_V U}{H^2}\frac{\partial^2 u'}{\partial z'^2}$$

$$\frac{U}{T}\frac{\partial u'}{\partial t'}+\frac{U^2}{L}u'\frac{\partial u'}{\partial x'}+\frac{U^2}{L}v'\frac{\partial u'}{\partial y'}+\frac{U^2}{L}w'\frac{\partial u'}{\partial z'}-fUv'=-\frac{P}{\rho_0 L}\frac{\partial p'}{\partial x'}+\frac{A_V U}{H^2}\frac{\partial^2 u'}{\partial z'^2}$$

Divide the equation by fU, and remove the symbol ':

$$\varepsilon(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z})-v=-\frac{\partial p}{\partial x}+\frac{E_v}{2}\frac{\partial^2 u}{\partial z^2}$$

$$E_v=\frac{2A_V}{fH^2}\sim\varepsilon^2$$

$$\varepsilon\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)+u=-\frac{\partial p}{\partial y}+\frac{E_v}{2}\frac{\partial^2 v}{\partial z^2}$$

Now take the asympotic expansion of the non-dimenional variables:

$$u=u_0+\varepsilon u_1+\varepsilon^2 u_2$$

$$v=v_0+\varepsilon v_1+\varepsilon^2 v_2$$

$$w=w_0+\varepsilon w_1+\varepsilon^2 w_2$$

$$p=p_0+\varepsilon p_1+\varepsilon^2 p_2$$

Substitution into the momentum equation, and to the order of $O(1)$:

$$-v_0 = -\frac{\partial p}{\partial x} \qquad u_0 = -\frac{\partial p}{\partial y}$$

To the order of $O(\varepsilon)$:

$$\varepsilon\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) - v = -\frac{\partial p}{\partial x} + \frac{E_v}{2}\frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial u_0}{\partial t} + u_0\frac{\partial u_0}{\partial x} + v_0\frac{\partial u_0}{\partial y} + w_0\cancel{\frac{\partial u_0}{\partial z}} - v_1 = -\frac{\partial p_1}{\partial x} \qquad (1)$$

$$\frac{\partial v_0}{\partial t} + u_0\frac{\partial v_0}{\partial x} + v_0\frac{\partial v_0}{\partial y} + w_0\cancel{\frac{\partial v_0}{\partial z}} + u_1 = -\frac{\partial p_1}{\partial y} \qquad (2)$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0$$

$z' = 0$ SBL

interior

$z' = -1$ BBL

$\dfrac{\partial(2)}{\partial x} - \dfrac{\partial(1)}{\partial y}$:

$$\frac{d\zeta_0}{dt} = -\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right) = \frac{\partial w_1}{\partial z} = \frac{w_1|_{z'=0} - w_1|_{z'=-1}}{1}$$

$$\frac{d\zeta_0}{dt} = w_1|_{z=0} - w_1|_{z=-1}$$

The Ekman pumping velocity in the bottom boundary layer is:

$$w|_{z'=-1} = \frac{d}{2}\left(\frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y}\right) = \frac{1}{2}\sqrt{\frac{2A_v}{f}}\,\bar{\zeta} = \frac{H}{2}{E_v}^{1/2}\bar{\zeta} \qquad E_v = \frac{2A_V}{fH^2}$$

$$Ww'|_{z'=-1} = \frac{H}{2}{E_v}^{\frac{1}{2}}\frac{U}{L}\left(\frac{\partial v_0}{\partial x'} - \frac{\partial u_0}{\partial y'}\right)$$

$$w'|_{z'=-1} = \frac{1}{2}\cancel{\frac{H}{W}\frac{U}{L}}{E_v}^{\frac{1}{2}}\left(\frac{\partial v_0}{\partial x'} - \frac{\partial u_0}{\partial y'}\right)$$

Remove the symbol ’:

$$w|_{z=-1} = \frac{1}{2}{E_v}^{\frac{1}{2}}\zeta_0 = w_0|_{z=-1} + \varepsilon w_1|_{z=-1}$$

$$w_0|_{z=-1} = 0 \xrightarrow{\frac{\partial w_0}{\partial z} = 0} w_0 = 0$$

$$\Longrightarrow \quad w_1|_{z=-1} = \frac{{E_v}^{\frac{1}{2}}}{2\varepsilon}\zeta_0$$

The Ekman pumping velocity for the surface boundary layer is:

$$w|_{z=0} = \frac{1}{\rho_0 f}\left(\frac{\partial \tau^y}{\partial x} - \frac{\partial \tau^x}{\partial y}\right) = \frac{1}{\rho_0 f}\left(\frac{\partial}{\partial x}\left(\rho_0 A_v \frac{\partial v}{\partial z}\right) - \frac{\partial}{\partial y}\left(\rho_0 A_v \frac{\partial u}{\partial z}\right)\right)$$

$$= \frac{A_v}{f}\left(\frac{\partial}{\partial x}\left(\frac{v_s - \bar{v}}{d}\right) - \frac{\partial}{\partial y}\left(\frac{u_s - \bar{u}}{d}\right)\right)$$

$$= \frac{A_v}{fd}(\zeta_s - \bar{\zeta}) \qquad d = \sqrt{\frac{2A_v}{f}}$$

$$E_v = \frac{2A_V}{fH^2} \qquad = \frac{1}{2}\sqrt{\frac{2A_v}{f}}(\zeta_s - \bar{\zeta}) \quad = \frac{1}{2}E_v^{\frac{1}{2}}H(\zeta_s - \bar{\zeta})$$

$$Ww'|_{z'=0} = \frac{1}{2}E_v^{\frac{1}{2}}H\frac{U}{L}(\zeta_s{}' - \zeta_0) \Longrightarrow w'|_{z'=0} = \frac{1}{2}E_v^{\frac{1}{2}}(\zeta_s{}' - \zeta_0)$$

$$w_0 = 0$$

Remove the symbol ’: $$w|_{z=0} = w_0|_{z=0} + \varepsilon w_1|_{z=0} = \frac{1}{2}E_v^{\frac{1}{2}}(\zeta_s - \zeta_0)$$

$$w_1|_{z=0} = \frac{E_v^{\frac{1}{2}}}{2\varepsilon}(\zeta_s - \zeta_0)$$

$$\frac{d\zeta_0}{dt} = w_1|_{z=0} - w_1|_{z=-1}$$

$$w_1|_{z=0} = \frac{E_v^{\frac{1}{2}}}{2\varepsilon}(\zeta_s - \zeta_0)$$

$$w_1|_{z=-1} = \frac{E_v^{\frac{1}{2}}}{2\varepsilon}\zeta_0$$

$$\frac{d\zeta_0}{dt} = -\frac{E_v^{\frac{1}{2}}}{\varepsilon}\left(\zeta_0 - \frac{1}{2}\zeta_s\right) = -r\left(\zeta_0 - \frac{1}{2}\zeta_s\right)$$

If the surface forcing disappears ($\zeta_s = 0$) : $\frac{d\zeta_0}{dt} = -r\zeta_0$ $\quad \zeta_0 = \zeta_c e^{-rt}$

Circulation:

$$C = \oint \boldsymbol{u_0} d\boldsymbol{r} = \iint \zeta_0 dS$$

geostrophic flow is non-divergent

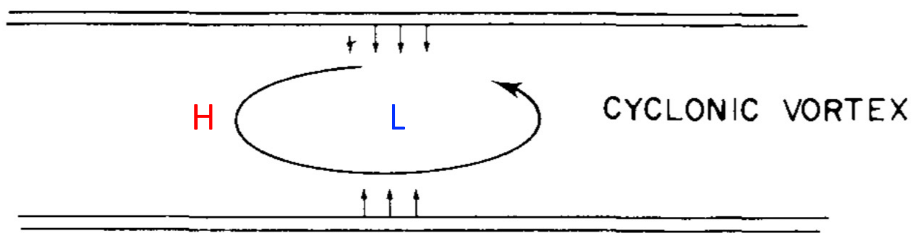
$$\frac{dC}{dt} = \iint \frac{d\zeta_0}{dt} dS + \iint \zeta_0 \cancel{\frac{d}{dt}(dS)} = \iint -r\zeta_0 dS = -rC \Longrightarrow C = C_0 e^{-rt}$$

Spindown time (decay timescale of the flow or vorticity):

$$\zeta_0 = \zeta_c e^{-rt} = \zeta_c e^{-\frac{t}{1/r}}$$

The e-folding decay timescale: $$\tau = \frac{1}{r} = \frac{\varepsilon}{E_v^{\frac{1}{2}}} = \frac{U/fL}{E_v^{\frac{1}{2}}} = \frac{1/fT}{E_v^{\frac{1}{2}}} \sim O(1) \qquad T \sim \frac{1}{f E_v^{\frac{1}{2}}}$$

larger E_v, smaller T, faster disspitation of vorticity and kinetic energy

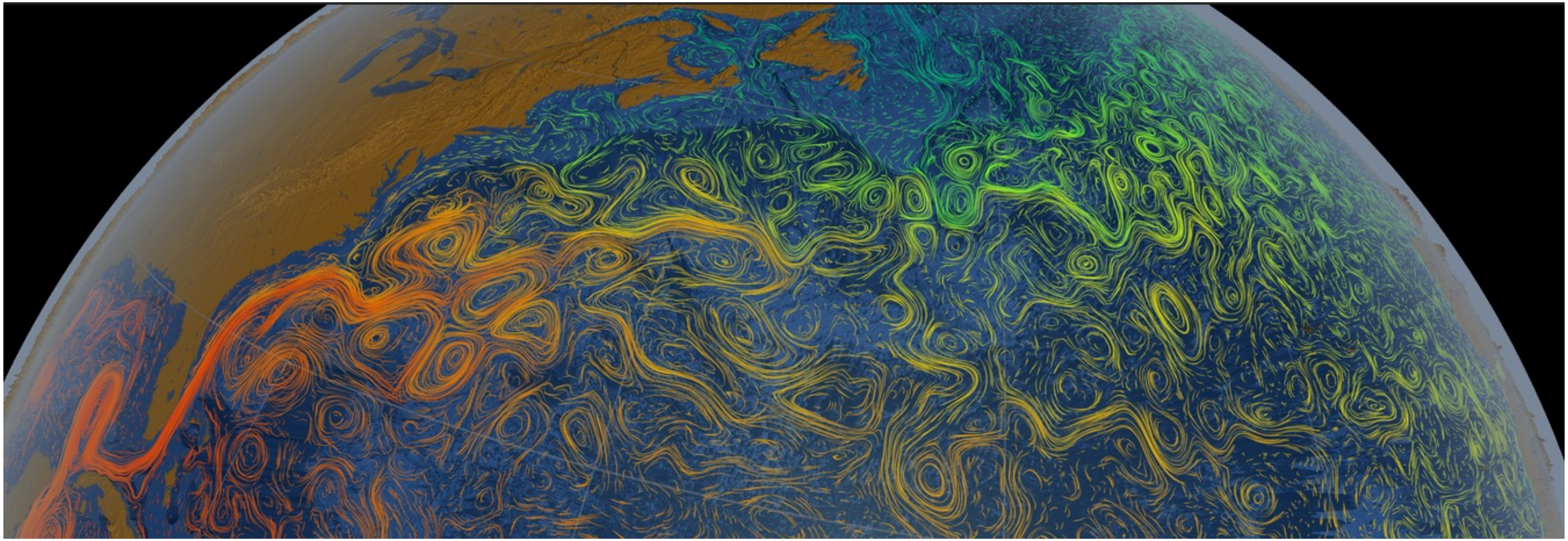


$$w_1|_{z=-1} = \frac{E_v^{\frac{1}{2}}}{2\varepsilon}\zeta_0 > 0 \qquad w_1|_{z=0} = -\frac{E_v^{\frac{1}{2}}}{2\varepsilon}\zeta_0 < 0$$

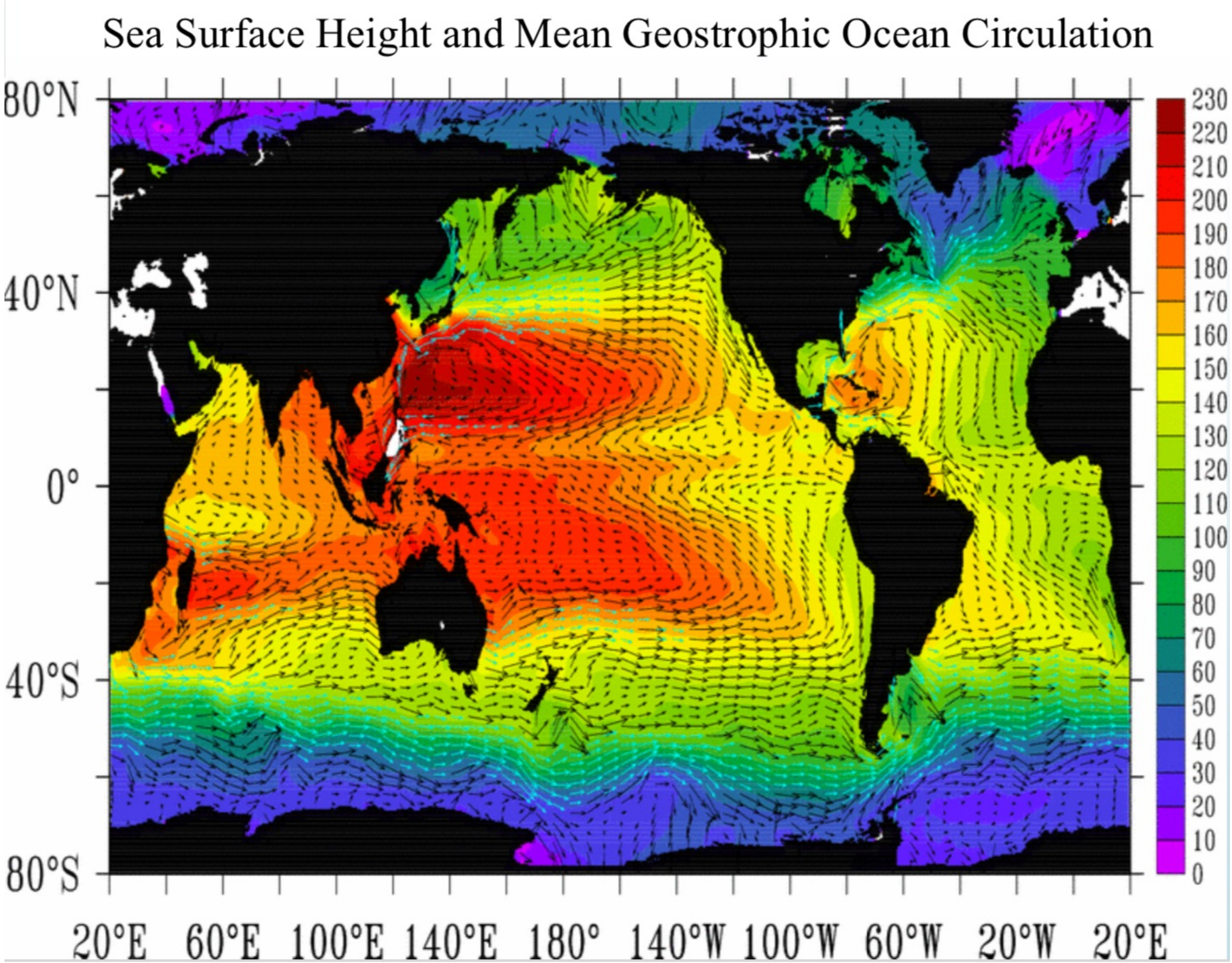
The water column is squeezed ($h \downarrow$), and for PV conservation, $\zeta_0 \downarrow$

The pressure gradient force does negative work, flow deaccelerates

The spindown timescale determines the life time of eddies



Dynamics for the subtropical gyre



For the interior ocean (shallow-water model):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}$$

Take the curl of the equation:

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2} - \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + v\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} - v\frac{\partial^2 u}{\partial y^2} + f\frac{\partial u}{\partial x} + \frac{\partial f}{\partial y}v + f\frac{\partial v}{\partial y} = 0$$

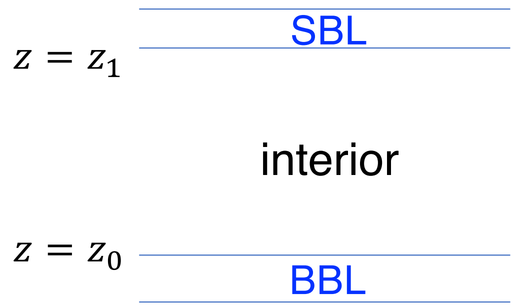
$$\frac{\partial \zeta}{\partial t} + \frac{\partial u}{\partial x}\zeta + u\frac{\partial \zeta}{\partial x} + \frac{\partial v}{\partial y}\zeta + v\frac{\partial \zeta}{\partial y} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta_0 v = 0$$

$$\frac{d\zeta}{dt} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(f + \cancel{\zeta}) + \beta_0 v = 0$$

large scale: $\zeta \ll f$

$$\frac{d\zeta}{dt} + \beta_0 v = f\frac{\partial w}{\partial z}$$

$$\frac{d\zeta}{dt} + \beta_0 v = f\frac{\partial w}{\partial z}$$



Take the vertical integral, and given that $\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0, \frac{\partial \zeta}{\partial z} = 0$

$$\left(\frac{d\zeta}{dt} + \beta_0 v\right)H = f(w|_{z_1} - w|_{z_0})$$

$$= f\left\{\frac{1}{\rho_0}\left[\frac{\partial}{\partial x}\left(\frac{\tau^y}{f}\right) - \frac{\partial}{\partial y}\left(\frac{\tau^x}{f}\right)\right] - \frac{d}{2}\zeta\right\}$$

$$= f\left\{\frac{1}{\rho_0 f}\frac{\partial \tau^y}{\partial x} - \frac{1}{\rho_0 f}\frac{\partial \tau^x}{\partial y} + \frac{\tau^x}{\rho_0 f^2}\beta_0 - \frac{d}{2}\zeta\right\}$$

Define non-dimensional variables:

$$(u, v) = U(u', v') \qquad (x, y) = L(x', y') \qquad t = Tt' \qquad \tau = \tau_0 \tau'$$

$$H\left(\frac{U}{LT}\frac{d\zeta'}{dt\prime} + \beta_0 U v'\right) = \frac{1}{\rho_0}\frac{\tau_0}{L} curl\tau' + \frac{\tau_0}{\rho_0 f}\beta_0 \tau^{x\prime} - \frac{d}{2} f \frac{U}{L}\zeta'$$

$$H\left(\frac{U}{LT}\frac{d\zeta'}{dt}+\beta_0 U v'\right)=\frac{1}{\rho_0}\frac{\tau_0}{L}curl\tau'+\frac{\tau_0}{\rho_0 f}\beta_0\tau^{x'}-\frac{d}{2}f\frac{U}{L}\zeta'$$

$$E_k=\frac{\nu_E}{fH^2}$$

$$H\frac{U^2}{L^2}\left(\frac{d\zeta'}{dt}+\frac{\beta_0 L^2}{U}v'\right)=\frac{\tau_0}{\rho_0 L}\left(curl\tau'+\frac{\beta_0 L}{f}\tau^{x'}\right)-\frac{d}{2}f\frac{U}{L}\zeta'$$

$$d=\sqrt{\frac{\nu_E}{f}}=E_k{}^{1/2}\mathrm{H}$$

<<1

$$\frac{d\zeta'}{dt}+\frac{\beta_0 L^2}{U}v' = \frac{\tau_0 L}{\rho_0 H U^2}\left(curl\tau'+\cancel{\frac{\beta_0 L}{f}}\tau^{x'}\right)-\frac{1}{2}\frac{fL}{U}\frac{d}{H}\zeta'$$

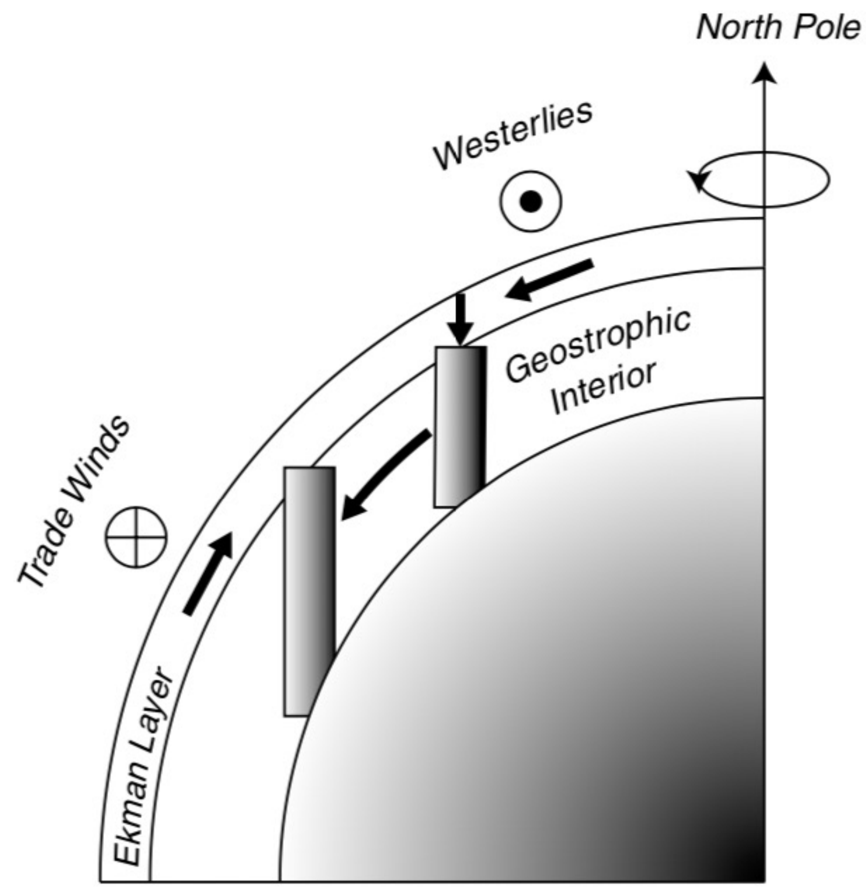
$$\beta \qquad\qquad -\frac{1}{2}\frac{1}{R_o}E_k{}^{1/2}\zeta'$$

Remove the prime $symbol$ $'$

$$\cancel{\frac{d\zeta}{dt}}+\beta v=\frac{\tau_0 L}{\rho_0 H U^2}curl\tau-\cancel{r\zeta}$$

For steady, large-scale flow, and assuming the bottom friction is negligible:

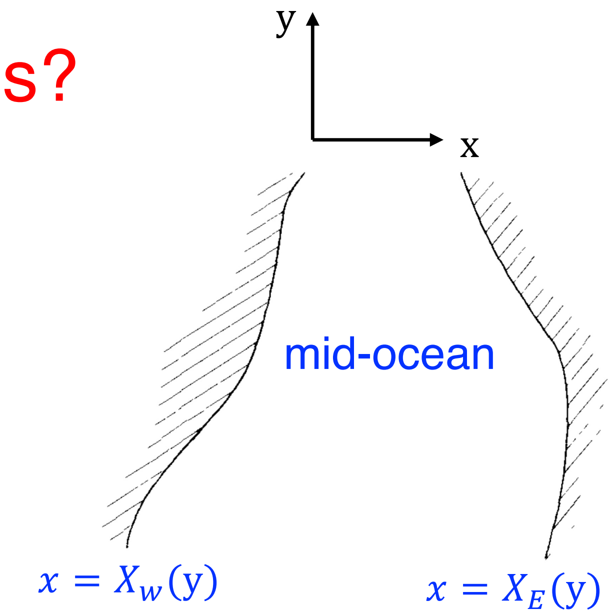
$v = curl\tau$ Sverdrup balance



$$\frac{d}{dt}\left(\frac{f\ +\ \zeta}{h}\right)\ =\ 0$$

Figure 12.7 Ekman pumping that produces a downward velocity at the base of the Ekman layer forces the fluid in the interior of the ocean to move southward. (From Niiler, 1987)

How about at the laternal boundaries?



Laternal boundaries:

West: $x = X_w(y)$ East: $x = X_E(y)$

$$x - X_w(y) = 0 \qquad x - X_E(y) = 0$$

No normal flows:

$$\boldsymbol{u} \cdot \nabla\big(x - X_w(y)\big) = 0 \Longrightarrow u - v\frac{\partial X_w}{\partial y} = 0 \; at \; x = X_w$$

$$\boldsymbol{u} \cdot \nabla\big(x - X_E(y)\big) = 0 \Longrightarrow u - v\frac{\partial X_E}{\partial y} = 0 \; at \; x = X_E$$

For interior geostrophic flow:

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} = -\frac{\partial curl\tau}{\partial y}$$

For a point in the mid-ocean x_0:

$$\int_{x_0}^{x} \frac{\partial u}{\partial x} dx' = -\int_{x_0}^{x} \frac{\partial curl\tau}{\partial y} dx'$$

$$u(x,y) = -\int_{x_0}^{x} \frac{\partial curl\tau}{\partial y} dx' + \mathrm{U}(x_0, \mathrm{y})$$ unknown, and needs to be determined

If the Sverdrup relation is valid at the eastern boundary: $u = v\frac{\partial X_E}{\partial y}$ $at\ x = X_E$

$$-\int_{x_0}^{X_E} \frac{\partial curl\tau}{\partial y} dx' + \mathrm{U}(x_0, \mathrm{y}) \quad = v\frac{\partial X_E}{\partial y} \quad = curl\tau(X_E, y)\frac{\partial X_E}{\partial y}$$

$$\mathrm{U}(x_0, \mathrm{y}) = \int_{x_0}^{X_E} \frac{\partial curl\tau}{\partial y} dx' + curl\tau(X_E, y)\frac{\partial X_E}{\partial y}$$

$$u(x,y) = -\int_{x_0}^{x} \frac{\partial curl\tau}{\partial y} dx' + \int_{x_0}^{X_E} \frac{\partial curl\tau}{\partial y} dx' + curl\tau(X_E, y)\frac{\partial X_E}{\partial y}$$

$$= \int_{x}^{X_E} \frac{\partial curl\tau}{\partial y} dx' + curl\tau(X_E, y)\frac{\partial X_E}{\partial y}$$

$$u(x,y) = \int_{x}^{X_E} \frac{\partial curl\tau}{\partial y} dx' + curl\tau(X_E,y)\frac{\partial X_E}{\partial y}$$

If the Sverdrup relation is also valid at the western boundary: $u = v\frac{\partial X_W}{\partial y}\, at\ x = X_W$

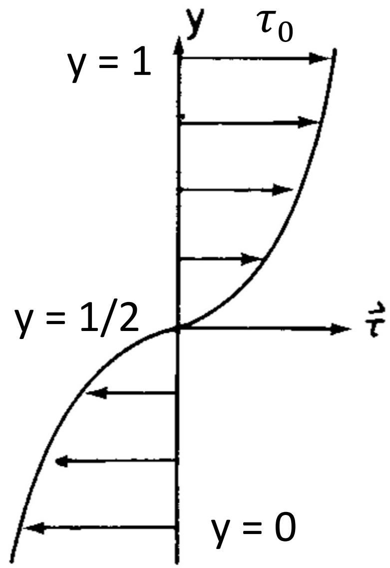
$$u(X_W,y) = \int_{X_W}^{X_E} \frac{\partial curl\tau}{\partial y} dx' + curl\tau(X_E,y)\frac{\partial X_E}{\partial y} \quad = v\frac{\partial X_W}{\partial y} \quad = curl\tau(X_W,y)\frac{\partial X_W}{\partial y}$$

$$\int_{X_W}^{X_E} \frac{\partial curl\tau}{\partial y} dx' + curl\tau(X_E,y)\frac{\partial X_E}{\partial y} - curl\tau(X_W,y)\frac{\partial X_W}{\partial y} = 0$$

$$\frac{\partial}{\partial y}\int_{X_W}^{X_E} curl\tau dx' = 0$$

condition for Sverdrup relation to be valid at both boundaries

$$\frac{\partial}{\partial y}\int_{X_W}^{X_E} curl\tau dx' = 0$$



For the trade wind and westerly wind:

$$\boldsymbol{\tau} = (-\tau_0 cos\pi y, \qquad 0)$$

$0 < y < 1$

$0 < \pi y < \pi$

$$v = curl\tau = -\frac{\partial \tau^x}{\partial y} = -\pi\tau_0 sin\pi y < 0$$

$X_E = \mathrm{C}, X_W = \mathrm{C}$

$$\frac{\partial}{\partial y}\int_{X_W}^{X_E} curl\tau dx' = \frac{\partial}{\partial y}\int_{X_W}^{X_E} -\pi\tau_0 sin\pi y dx' = -\pi^2\tau_0 cos\pi y(X_E - X_W)$$

only =0 at y=1/2

Sverdrup relation cannot hold at both boundaries

If Sverdrup relation holds at the eastern boundary:

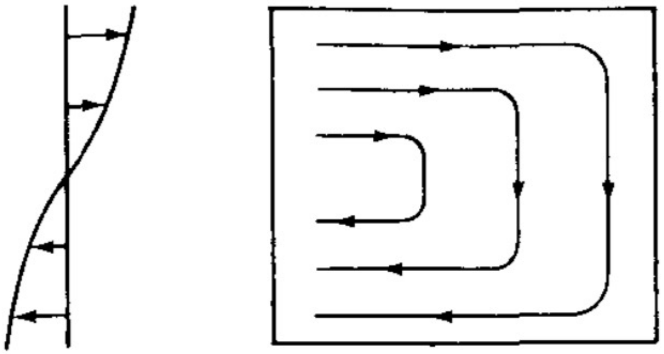
$$u(x,y) = \int_x^{X_E} \frac{\partial curl\tau}{\partial y} dx' + curl\tau(X_E, y)\cancel{\frac{\partial X_E}{\partial y}}$$

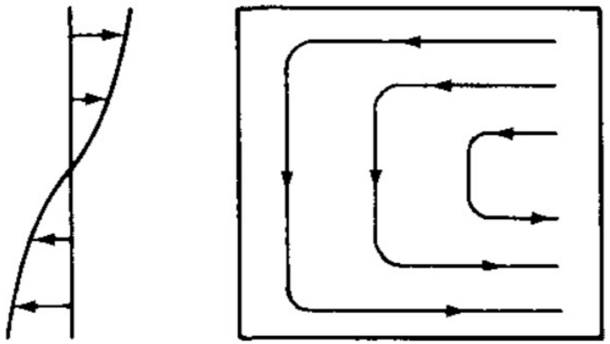
straight coastline $X_E = C$

$$\boxed{curl\tau = -\frac{\partial \tau^x}{\partial y} = -\pi\tau_0 sin\pi y}$$

> 0

$$= \frac{\partial}{\partial y}\int_x^{X_E} curl\tau dx' \;\; = -\pi^2\tau_0 cos\pi y(X_E - x) \;\; \begin{cases} > 0, y > 1/2 \\ < 0, y < 1/2 \end{cases}$$





opposite to the wind direction (X)

If Sverdrup relation holds at the western boundary:

$$u(x,y) = \int_x^{X_W} \frac{\partial curl\tau}{\partial y} dx' + curl\tau(X_W, y)\cancel{\frac{\partial X_W}{\partial y}}$$

straight coastline $X_W = C$

< 0

$$= \frac{\partial}{\partial y}\int_x^{X_W} curl\tau dx' \;\; = -\pi^2\tau_0 cos\pi y(X_W - x) \;\; \begin{cases} < 0, y > 1/2 \\ > 0, y < 1/2 \end{cases}$$

Stommel's model for western boundary intensification — **bottom friction**

$$\cancel{\frac{d\zeta}{dt}} + \beta v = \frac{\tau_0 L}{\rho_0 H U^2} curl\tau - r\zeta$$

For the boundary layers, retain bottom friction, and divide the equation by β:

$$v = curl\tau - \frac{r}{\beta}\zeta$$

For geostrophic flow:

$$\frac{\partial\psi}{\partial x} = curl\tau - \varepsilon_s \nabla^2\psi$$

$$\psi = \underbrace{\psi_I(x,y)}_{\text{interior (mid-ocean) solution}} + \underbrace{\psi_B(x,y)}_{\text{boundary layer correction}}$$

$$\frac{\partial\psi_I}{\partial x}(x,y) = curl\tau$$

$$\varepsilon_s(\nabla^2\psi_I + \nabla^2\psi_B) + \cancel{\frac{\partial\psi_I}{\partial x}} + \frac{\partial\psi_B}{\partial x} = \cancel{curl\tau}$$

West: $\alpha = \dfrac{x-0}{\varepsilon} \sim O(1)$ $\quad \dfrac{\partial}{\partial x} = \dfrac{1}{\varepsilon}\dfrac{\partial}{\partial \alpha}$ $\quad \dfrac{\partial^2}{\partial x^2} = \dfrac{1}{\varepsilon^2}\dfrac{\partial^2}{\partial \alpha^2}$

ε: boundary layer thickness ($\ll 1$)

East: $\alpha = \dfrac{x-1}{\varepsilon} \sim O(1)$

$$\varepsilon_s\left(\cancel{\nabla^2\psi_I} + \frac{1}{\varepsilon^2}\frac{\partial^2\psi_B}{\partial\alpha^2} + \cancel{\frac{\partial^2\psi_B}{\partial y^2}}\right) + \frac{1}{\varepsilon}\frac{\partial\psi_B}{\partial\alpha} = 0$$

Only retain the large terms:

$$\frac{\partial^2\psi_B}{\partial\alpha^2} + \frac{\partial\psi_B}{\partial\alpha} = 0$$

$$\frac{\partial^2\psi_B}{\partial\alpha^2}+\frac{\partial\psi_B}{\partial\alpha}=0$$

$$\psi_B = A(y)e^{\lambda\alpha} \Longrightarrow \lambda^2+\lambda=0 \Longrightarrow \lambda=-1$$

$$\psi_B = A(y)e^{-\alpha}$$

If ψ_B applies to the western boudary: $\alpha = \dfrac{x-0}{\varepsilon} > 0$

ψ_B decays expotentially toward the mid-ocean ✓

If ψ_B applies to the eastern boudary: $\alpha = \dfrac{x-1}{\varepsilon} < 0$

ψ_B grows expotentially toward the mid-ocean ×

The correction applies to the western boundary, and Sverdrup relation holds for the eastern boudary

For the interior and the eastern boundary: $\frac{\partial \psi_I}{\partial x}(x,y) = curl\tau$

$$\int_x^1 \frac{\partial \psi_I}{\partial x} dx' = \int_x^1 curl\tau \, dx'$$

$$\boxed{curl\tau = -\frac{\partial \tau^x}{\partial y} = -\pi\tau_0 sin\pi y}$$

$$\psi_I(1,y) - \psi_I(x,y) = -\pi\tau_0 sin\pi y(1-x)$$

$$\psi_I(x,y) = \pi\tau_0 sin\pi y(1-x)$$

At the western boundary: $\psi_B = A(y)e^{-\alpha}$

$$\psi(0,y) = \psi_I(0,y) + \psi_B(0,y) = \pi\tau_0 \, sin\pi y + A(y) = 0$$

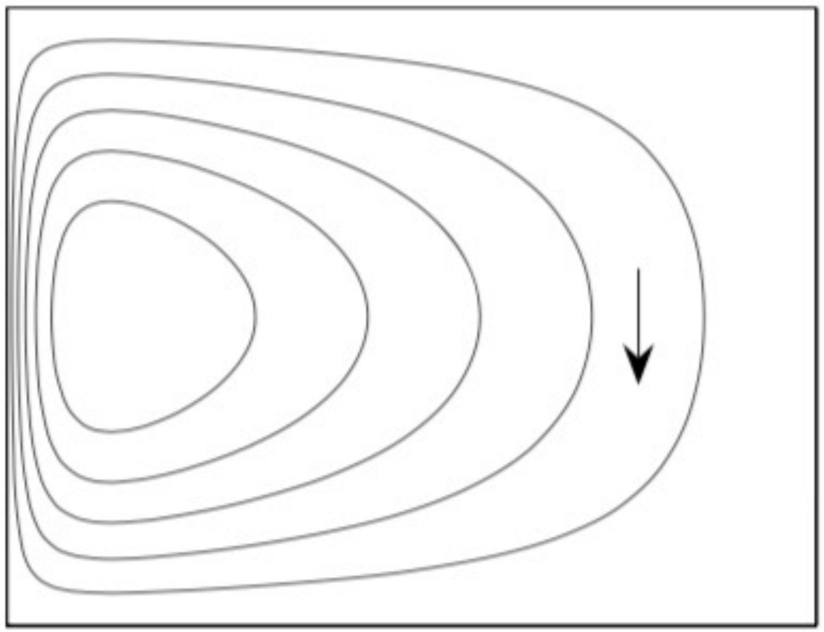
$$A(y) = -\pi\tau_0 sin\pi y$$

$$\boxed{\alpha = \frac{x-0}{\varepsilon}}$$

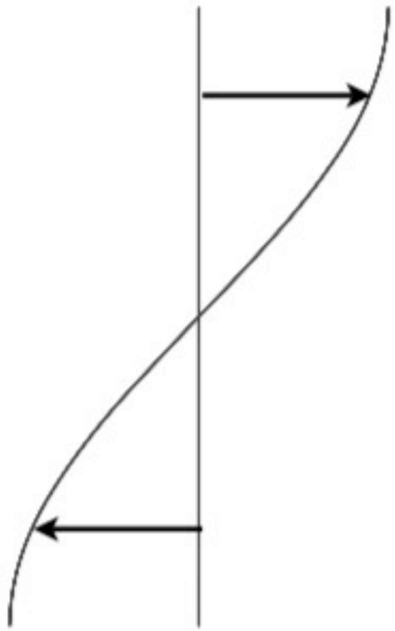
$$\psi(x,y) = \psi_I(x,y) + \psi_B\,(x,y) = \pi\tau_0 sin\pi y(1-x) - \pi\tau_0 \, sin\pi y e^{-x/\varepsilon}$$

$$= (1 - x - e^{-x/\varepsilon})\,\pi\tau_0 \, sin\pi y$$

Streamfunction



Wind stress



Vorticity dynamics

$$\boxed{\frac{d}{dt}\left(\frac{f\ +\ \zeta}{h}\right)\ =\ 0}$$

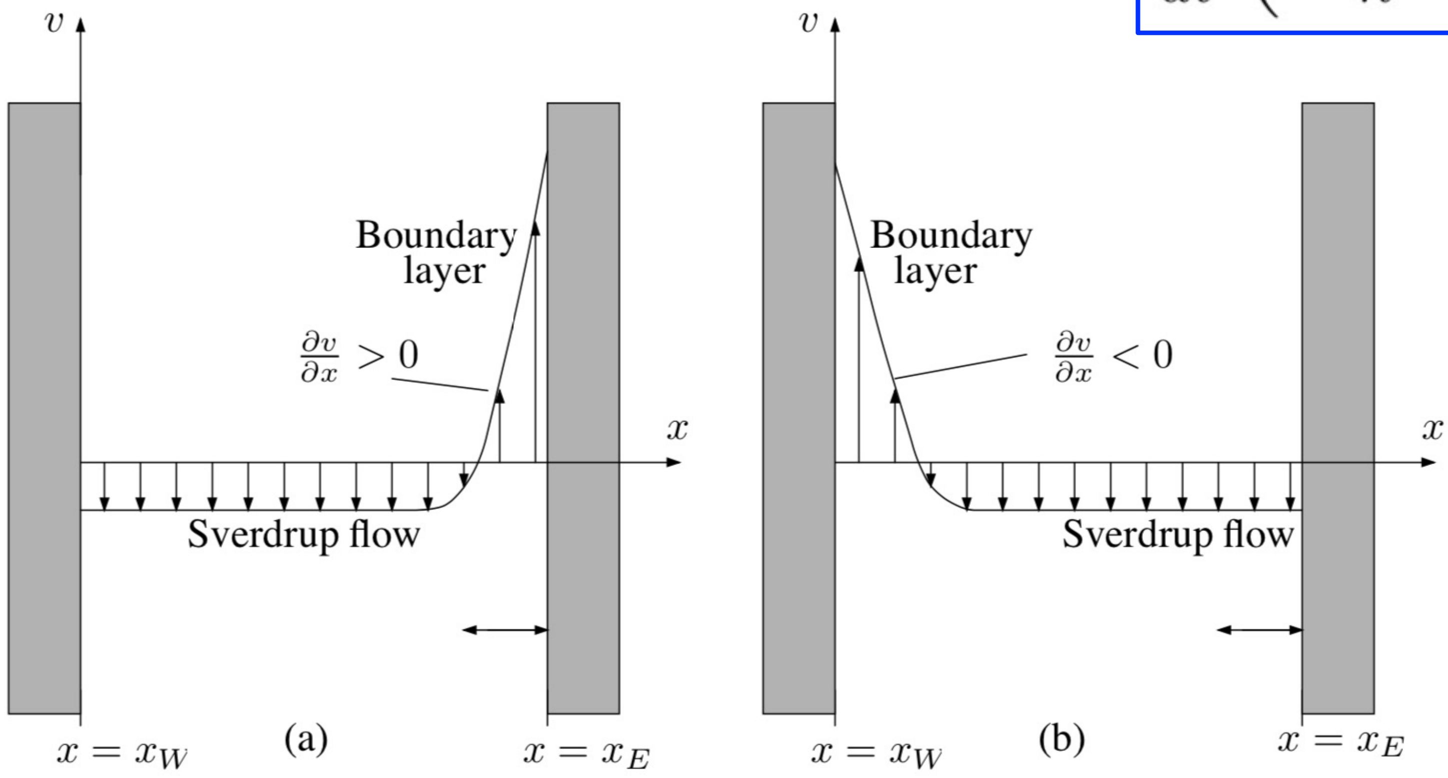


Figure 20-7 The two possible configurations for a northward boundary current to return the southward Sverdrup flow that exists across most of an ocean basin in the mid-latitudes of the Northern Hemisphere: (a) boundary current on the eastern side, (b) boundary current on the western side. The former is to be rejected on dynamic grounds, leaving the latter as the correct configuration.