Quasi-geostrophic dynamics for stratified fluids

$$\rho \;=\; \bar{\rho}(z) \;+\; \rho'(x,y,z,t) \quad \text{with} \quad |\rho'| \;\ll\; |\bar{\rho}|$$

$$p = \bar{p}(z) + p'(x,y,z,t)$$

Governing equations:

$$\begin{aligned}
\overset{\text{S}}{\frac{du}{dt}} - \overset{\text{L}}{f_0 v} - \overset{\text{S}}{\beta_0 y v} \;&=\; \overset{\text{L}}{-\frac{1}{\rho_0}\frac{\partial p'}{\partial x}} \\
\frac{dv}{dt} + f_0 u + \beta_0 y u \;&=\; -\frac{1}{\rho_0}\frac{\partial p'}{\partial y} \\
0 \;&=\; -\frac{\partial p'}{\partial z} - \rho' g \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \;&=\; 0 \\
\frac{\partial \rho'}{\partial t} + u\frac{\partial \rho'}{\partial x} + v\frac{\partial \rho'}{\partial y} + w\frac{d\bar{\rho}}{dz} \;&=\; 0
\end{aligned}$$

Balance of large terms:

$$\begin{aligned}
-\, f_0 v \;&=\; -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} \\
+\, f_0 u \;&=\; -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}
\end{aligned}$$

$$\boxed{\begin{aligned}
u_g \;&=\; -\frac{1}{f_0\rho_0}\frac{\partial p'}{\partial y} \\
v_g \;&=\; +\frac{1}{f_0\rho_0}\frac{\partial p'}{\partial x}
\end{aligned}}$$

Plug u_g and v_g into the small terms of the momentum equations, and neglect the vertical advection term:

$$\frac{du}{dt} - f_0 v - \beta_0 y v = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$
$$\frac{dv}{dt} + f_0 u + \beta_0 y u = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}$$

$$J(a,b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial b}{\partial x}\frac{\partial a}{\partial y}$$

$$\boxed{\begin{aligned} u_g &= -\frac{1}{f_0\rho_0}\frac{\partial p'}{\partial y} \\ v_g &= +\frac{1}{f_0\rho_0}\frac{\partial p'}{\partial x} \end{aligned}}$$

$$-\frac{1}{\rho_0 f_0}\frac{\partial^2 p'}{\partial y\partial t} - \frac{1}{\rho_0^2 f_0^2} J\left(p', \frac{\partial p'}{\partial y}\right) - f_0 v - \frac{\beta_0}{\rho_0 f_0} y \frac{\partial p'}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$
$$+\frac{1}{\rho_0 f_0}\frac{\partial^2 p'}{\partial x\partial t} + \frac{1}{\rho_0^2 f_0^2} J\left(p', \frac{\partial p'}{\partial x}\right) + f_0 u - \frac{\beta_0}{\rho_0 f_0} y \frac{\partial p'}{\partial y} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}$$

$$\begin{aligned} u = u_g + u_a &= -\frac{1}{\rho_0 f_0}\frac{\partial p'}{\partial y} - \frac{1}{\rho_0 f_0^2}\frac{\partial^2 p'}{\partial t\partial x} \\ &\quad - \frac{1}{\rho_0^2 f_0^3} J\left(p', \frac{\partial p'}{\partial x}\right) + \frac{\beta_0}{\rho_0 f_0^2} y \frac{\partial p'}{\partial y} \\ v = v_g + v_a &= +\frac{1}{\rho_0 f_0}\frac{\partial p'}{\partial x} - \frac{1}{\rho_0 f_0^2}\frac{\partial^2 p'}{\partial t\partial y} \\ &\quad - \frac{1}{\rho_0^2 f_0^3} J\left(p', \frac{\partial p'}{\partial y}\right) - \frac{\beta_0}{\rho_0 f_0^2} y \frac{\partial p'}{\partial x} \end{aligned}$$

Substitution into the continuity equation:

$$\frac{\partial w}{\partial z} = \frac{1}{\rho_0 f_0^2}\left[\frac{\partial}{\partial t}\nabla^2 p' + \frac{1}{\rho_0 f_0} J(p', \nabla^2 p') + \beta_0 \frac{\partial p'}{\partial x}\right]$$

The density equation: $$\frac{\partial \rho'}{\partial t} + u\frac{\partial \rho'}{\partial x} + v\frac{\partial \rho'}{\partial y} + w\frac{\partial \bar{\rho}}{\partial z} = 0$$

$$\boxed{\begin{aligned} u_g &= -\frac{1}{f_0\rho_0}\frac{\partial p'}{\partial y} \\ v_g &= +\frac{1}{f_0\rho_0}\frac{\partial p'}{\partial x} \end{aligned}}$$

Plug u_g and v_g into the density equation:

$$\frac{\partial \rho'}{\partial t} + \frac{1}{\rho_0 f_0} J(p', \rho') - \frac{\rho_0 N^2}{g} w = 0$$

Divide the equation by $\frac{N^2}{g}$, and take the z-derivative:

$$\boxed{0 = -\frac{\partial p'}{\partial z} - \rho' g}$$

$$\frac{\partial}{\partial t \partial z}\left(\frac{g}{N^2}\rho'\right) + \frac{1}{\rho_0 f_0}\left[\frac{\partial p'}{\partial x}\frac{\partial}{\partial z}\left(\frac{g}{N^2}\frac{\partial \rho'}{\partial y}\right) - \frac{\partial p'}{\partial y}\frac{\partial}{\partial z}\left(\frac{g}{N^2}\frac{\partial \rho'}{\partial x}\right)\right] - \rho_0\frac{\partial w}{\partial z} = 0$$

$$-\frac{\partial}{\partial t \partial z}\left(\frac{1}{N^2}\frac{\partial p'}{\partial z}\right) + \frac{1}{\rho_0 f_0}\left\{\frac{\partial p'}{\partial x}\frac{\partial}{\partial y}\left[\frac{\partial}{\partial z}\left(-\frac{1}{N^2}\frac{\partial p'}{\partial z}\right)\right] - \frac{\partial p'}{\partial y}\frac{\partial}{\partial x}\left[\frac{\partial}{\partial z}\left(-\frac{1}{N^2}\frac{\partial p'}{\partial z}\right)\right]\right\} - \rho_0\frac{\partial w}{\partial z} = 0$$

$$-\frac{\partial}{\partial t}\left[\frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial p'}{\partial z}\right)\right] - \frac{1}{\rho_0 f_0} J\left(p', \frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial p'}{\partial z}\right)\right) - \rho_0\frac{\partial w}{\partial z} = 0$$

$$\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial}{\partial t}\left[\frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial p'}{\partial z}\right)\right] - \frac{1}{{\rho_0}^2 f_0} J\left(p', \frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial p'}{\partial z}\right)\right)$$

$$\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial}{\partial t}\left[\frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial p'}{\partial z}\right)\right] - \frac{1}{\rho_0{}^2 f_0} J\left(p', \frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial p'}{\partial z}\right)\right)$$

From the derivations based on the momentum equations:

$$\frac{\partial w}{\partial z} = \frac{1}{\rho_0 f_0{}^2}\left[\frac{\partial}{\partial t}\nabla^2 p' + \frac{1}{\rho_0 f_0} J(p', \nabla^2 p') + \beta_0 \frac{\partial p'}{\partial x}\right]$$

$$\frac{\partial}{\partial t}\left[\nabla^2 p' + \frac{\partial}{\partial z}\left(\frac{f_0{}^2}{N^2}\frac{\partial p'}{\partial z}\right)\right] + \frac{1}{\rho_0 f_0} J\left(p', \nabla^2 p' + \frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial p'}{\partial z}\right)\right) + \beta_0 \frac{\partial p'}{\partial x} = 0$$

The geostrophic flows have streamfunction ψ:

$$u_g = -\frac{1}{\rho_0 f_0}\frac{\partial p'}{\partial y} = -\frac{\partial \psi}{\partial y}$$

$$v_g = \frac{1}{\rho_0 f_0}\frac{\partial p'}{\partial x} = \frac{\partial \psi}{\partial x} \quad \Longrightarrow \quad p' = \rho_0 f_0 \psi$$

$+\beta_0 y$ $\qquad\qquad\qquad$ $J(\psi, \beta_0 y)$

$$\rho_0 f_0 \frac{\partial}{\partial t}\left[\nabla^2 \psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right)\right] + \rho_0 f_0 J\left(\psi, \nabla^2 \psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right)\right) + \rho_0 f_0 \underline{\beta_0 \frac{\partial \psi}{\partial x}} = 0$$

$$\frac{\partial}{\partial t}\left[\nabla^2\psi+\frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right)+\beta_0 y\right]+J\left(\psi,\nabla^2\psi+\frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right)\right)+J(\psi,\beta_0 y)=0$$

ζ planetary vorticity

$$q=\nabla^2\psi+\frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right)+\beta_0 y$$

potential vorticity

$$\frac{\partial q}{\partial t}+J(\psi,q)=0$$

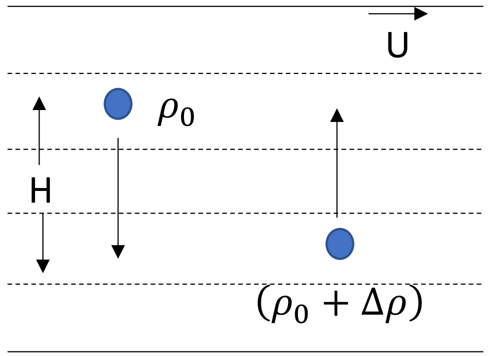
$$\frac{\partial q}{\partial t}+\frac{\partial\psi}{\partial x}\frac{\partial q}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial q}{\partial x}=0$$

$$\frac{\partial q}{\partial t}+u_g\frac{\partial q}{\partial x}+v_g\frac{\partial q}{\partial y}=0$$

$$\frac{dq}{dt}=0$$

potential vorticity conservation

The importance of stratification: the Froude number



For per unit volume,

Potential energy change:

$$\Delta PE = (\rho_0 + \Delta\rho)gH - \rho_0 gH = \Delta\rho gH$$

Kinetic energy:

$$KE = \frac{1}{2}\rho_0 U^2 + \frac{1}{2}(\rho_0 + \Delta\rho)\,U^2 \approx \rho_0 U^2$$

$$\sigma = \frac{KE}{\Delta PE} = \frac{\rho_0 U^2}{\Delta\rho gH} \sim \frac{U^2}{N^2 H^2}$$

Froude number: $Fr = \frac{U}{NH}$

$\sigma > 1$, PE change consumes a small portion of the KE of the system, so it takes little cost to break stratification, stratification is unimportant

$\sigma \leq 1$, PE change consumes all KE of the system, or KE is not sufficient to supply ΔPE , stratification cannot be broken and is important relative vorticity planetary vorticity

$$q = \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right) + \beta_0 y$$

$$-\frac{g}{\rho_0 f_0}\frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\rho'\right)$$ vertical stretching

$$0 = -\frac{\partial p'}{\partial z} - \rho' g \qquad p' = \rho_0 f_0 \psi \qquad \frac{\partial\psi}{\partial z} = -\frac{g}{\rho_0 f_0}\rho'$$

relative vorticity: $\frac{U}{L}$ vertical stretching: $\frac{f_0^2 UL}{N^2H^2}$

$$\frac{\frac{U}{L}}{\frac{f_0^2 UL}{N^2H^2}} = \frac{N^2H^2}{f_0^2L^2} = \frac{\frac{U^2}{f_0^2L^2}}{\frac{U^2}{N^2H^2}} = \left(\frac{R_0}{Fr}\right)^2$$

Bu: Burger number

$Bu < 1,$ rotation is more important, vertical stretching dominates the PV

$Bu > 1,$ stratification is more important, relative vorticity dominates the PV

Burger number:

$$Bu = \left(\frac{R_0}{Fr}\right)^2 = \frac{U^2}{f^2L^2} \Big/ \frac{U^2}{N^2H^2} = \frac{N^2H^2}{f^2L^2} = \frac{g'H}{f^2L^2} = \frac{R^2}{L^2}$$

a measure of relative importance of rotation and stratification

$$\boxed{N^2 = -\frac{g}{\rho_0}\frac{d\rho}{dz} \simeq \frac{g}{\rho_0}\frac{\Delta\rho}{H} = \frac{g'}{H}}$$

$L < R$, $Bu > 1$, $Fr < R_0$, motion is more affected by stratification

$L > R$, $Bu < 1$, $R_0 < Fr$, motion is more affected by rotation

$$Fr^2 = \frac{U^2}{N^2H^2} = \frac{U^2}{g'H} \qquad Fr = \frac{U}{\sqrt{g'H}}$$

$\sqrt{g'H}$: internal gravity wave speed

The final solutions:

$$
\begin{aligned}
u_g &= -\frac{\partial \psi}{\partial y} \\
v_g &= +\frac{\partial \psi}{\partial x} \\
u_a &= -\frac{1}{f_0}\frac{\partial^2 \psi}{\partial t \partial x} - \frac{1}{f_0} J\left(\psi, \frac{\partial \psi}{\partial x}\right) + \frac{\beta_0}{f_0}\, y\, \frac{\partial \psi}{\partial y} \\
v_a &= -\frac{1}{f_0}\frac{\partial^2 \psi}{\partial t \partial y} - \frac{1}{f_0} J\left(\psi, \frac{\partial \psi}{\partial y}\right) - \frac{\beta_0}{f_0}\, y\, \frac{\partial \psi}{\partial x} \\
w &= -\frac{f_0}{N^2}\left[\frac{\partial^2 \psi}{\partial t \partial z} + J\left(\psi, \frac{\partial \psi}{\partial z}\right)\right] \\
p' &= \rho_0 f_0\, \psi \\
\rho' &= -\frac{\rho_0 f_0}{g}\frac{\partial \psi}{\partial z}\, .
\end{aligned}
$$

Energetics of quasi-geostrophic dynamics in stratified fluids

$$\frac{dq}{dt} = 0$$

$$\frac{d}{dt}(\nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right) + \beta_0 y) = 0$$

For f-plane:

$$\frac{d}{dt}\nabla^2\psi + \frac{d}{dt}\left[\frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right)\right] = 0$$

Multiply the equation by ψ and do a volume integration:

$$\frac{d}{dt}\iiint \frac{1}{2}\rho_0|\nabla\psi|^2 dxdydz + \frac{d}{dt}\iiint \frac{1}{2}\rho_0\frac{f_0^2}{N^2}\left(\frac{\partial\psi}{\partial z}\right)^2 dxdydz = 0$$

KE PE

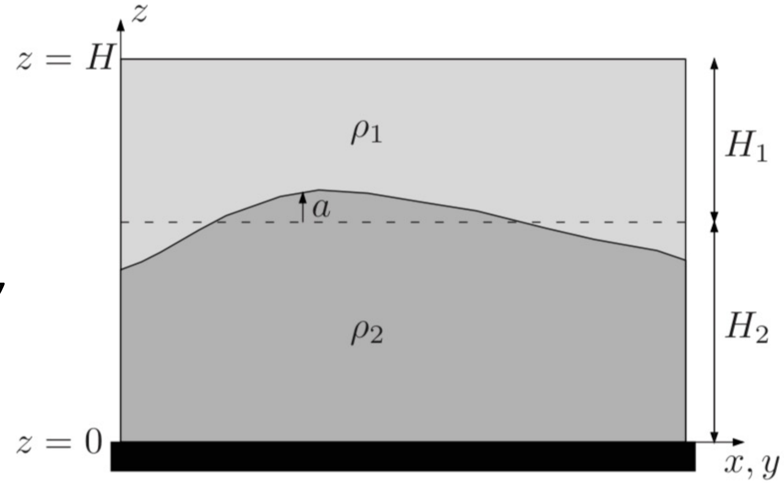
Available potential energy

$$PE(a) = \iint \left[\int_0^{H_2+a} \rho_2 gz dz + \int_{H_2+a}^{H} \rho_1 gz dz\right] dxdy$$

$$= \iint \left[\frac{1}{2}(\rho_1 + \Delta\rho) g (H_2 + a)^2 + \frac{1}{2}\rho_1 g [H^2 - (H_2 + a)^2]\right] dxdy$$

$$= \iint \frac{1}{2}\Delta\rho g H_2^2 \, dxdy + \iint \frac{1}{2}\rho_1 g H^2 \, dxdy$$

$$+ \iint \Delta\rho g H_2 \overset{=0}{a} \, dxdy + \iint \frac{1}{2}\Delta\rho g a^2 \, dxdy$$

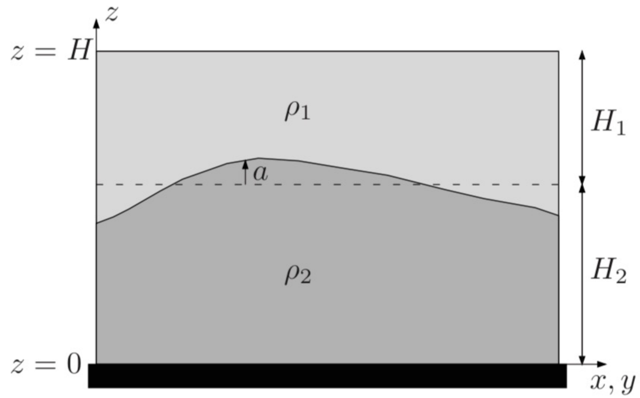


$$PE(a) - \mathrm{PE}(a=0) = \iint \frac{1}{2}\Delta\rho g a^2 \, dxdy = \iint \frac{1}{2}\rho_0 H N^2 a^2 \, dxdy$$

$$N^2 = -\frac{g}{\rho_0}\frac{d\bar{\rho}}{dz} = \frac{g\Delta\rho}{\rho_0 H}$$

$$= \iiint \frac{1}{2}\rho_0 N^2 a^2 dxdydz$$

$$\rho' = \bar{\rho}(z-a) - \bar{\rho}(z) = -a\frac{d\bar{\rho}}{dz} = \frac{\rho_0 N^2}{g}\,a$$



$$PE(a) - \mathrm{PE}(a=0) = \iiint \frac{1}{2}\rho_0 N^2 a^2 dxdydz$$

$$= \iiint \frac{1}{2}\rho_0 N^2 \left(\frac{\rho' g}{\rho_0 N^2}\right)^2 dxdydz$$

$$= \iiint \frac{1}{2}\frac{1}{\rho_0 N^2}\left(-\rho_0 f_0 \frac{\partial\psi}{\partial z}\right)^2 dxdydz$$

$$\boxed{\rho' = -\frac{\rho_0 f_0}{g}\frac{\partial\psi}{\partial z}}$$

$$= \iiint \frac{1}{2}\rho_0 \frac{f_0^2}{N^2}\left(\frac{\partial\psi}{\partial z}\right)^2 dxdydz$$

available potential energy (APE)

APE: difference between existing PE and PE if the stratified system is not perturbed

$$\frac{d}{dt}\iiint \frac{1}{2}\rho_0 |\nabla\psi|^2 dxdydz + \frac{d}{dt}\iiint \frac{1}{2}\rho_0 \frac{f_0^2}{N^2}\left(\frac{\partial\psi}{\partial z}\right)^2 dxdydz = 0$$

Planetary waves in stratified fluids

For quasi-geostrophic model: $$\frac{\partial q}{\partial t} + J(\psi, q) = 0$$

$$q = \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right) + \beta_0 y$$

Linearization of the PV conservation equation, and take constant N^2:

$$\frac{\partial}{\partial t}\left[\nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right) + \beta_0 y\right] + J\cancel{\left(\psi, \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right)\right)} + J(\psi, \beta_0 y) = 0$$

$$\frac{\partial}{\partial t}\left(\nabla^2\psi + \frac{f_0^2}{N^2}\frac{\partial^2\psi}{\partial z^2}\right) + \beta_0\frac{\partial\psi}{\partial x} = 0$$

Assume the motion is bounded by a flat bottom and free surface, apply a wave solution of ψ as:

$$\varphi(x, y, z, t) = \phi(\mathrm{z})e^{i(kx+ly-\omega t)}$$

$$\frac{d^2\phi}{dz^2} - \frac{N^2}{f_0^2}\left(k^2 + l^2 + \frac{\beta_0 k}{\omega}\right)\phi = 0$$

Boundary conditions:

$z = 0$:

$$w = -\frac{f_0}{N^2}\left[\frac{\partial^2\psi}{\partial t\partial z} + \cancel{J\left(\psi, \frac{\partial\psi}{\partial z}\right)}\right] = 0$$

$$\frac{d\phi}{dz} = 0$$

$z = h$:

$$p = \bar{p}(z) + p'(x,y,z,t)$$

$$p' = \rho_0 g\eta$$

$$\boxed{p' = \rho_0 f_0 \psi}$$

$$w = \frac{\partial\eta}{\partial t} + \cancel{u\frac{\partial\eta}{\partial x}} + \cancel{v\frac{\partial\eta}{\partial y}} = \frac{1}{\rho_0 g}\frac{\partial p'}{\partial t} = \frac{f_0}{g}\frac{\partial\psi}{\partial t} = -\frac{f_0}{N^2}\frac{\partial^2\psi}{\partial t\partial z}$$

$$\frac{\partial^2\psi}{\partial t\partial z} + \frac{N^2}{g}\frac{\partial\psi}{\partial t} = 0$$

$$\frac{d\phi}{dz} + \frac{N^2}{g}\phi = 0$$

The solution for ϕ that satisfies both boundary conditions is:

$$\phi(z) = Acosmz$$

Substitution into the governing equation: $$\frac{d^2\phi}{dz^2} - \frac{N^2}{f_0^2}\left(k^2 + l^2 + \frac{\beta_0 k}{\omega}\right)\phi = 0$$

$$\omega = -\frac{\beta_0 k}{k^2 + l^2 + m^2 f_0^2/N^2}$$

barotropic Rossby wave

$$\boxed{\omega = -\beta_0 R^2 \frac{k}{1 + R^2(k^2 + l^2)}}$$

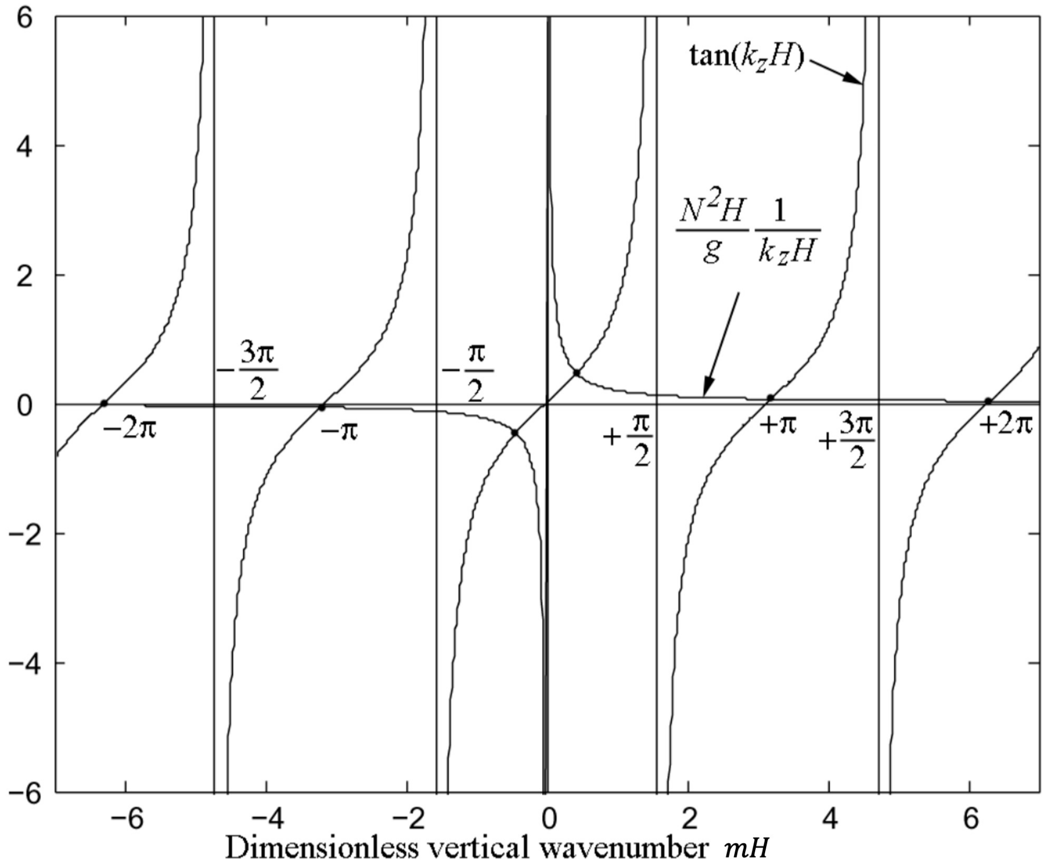
Substitution into the boundary condition at $z = H$: $$\frac{d\phi}{dz} + \frac{N^2}{g}\phi = 0$$

$$-AmsinmH + \frac{N^2}{g}AcosmH = 0$$

$$tanmH = \frac{N^2}{gm} = \frac{N^2 H}{g}\frac{1}{mH}$$

$$tanmH = \frac{N^2 H}{g} \frac{1}{mH}$$

$$\boxed{\omega = -\frac{\beta_0 k}{k^2 + l^2 + m^2 f_0^2 / N^2}}$$



$$\frac{N^2 H}{g} \sim \frac{g}{\rho_0} \frac{\Delta\rho}{H} \frac{H}{g} \sim \frac{\Delta\rho}{\rho_0}$$

For the first solution: mH is small

$$mH = \frac{N^2 H}{g} \frac{1}{mH}$$

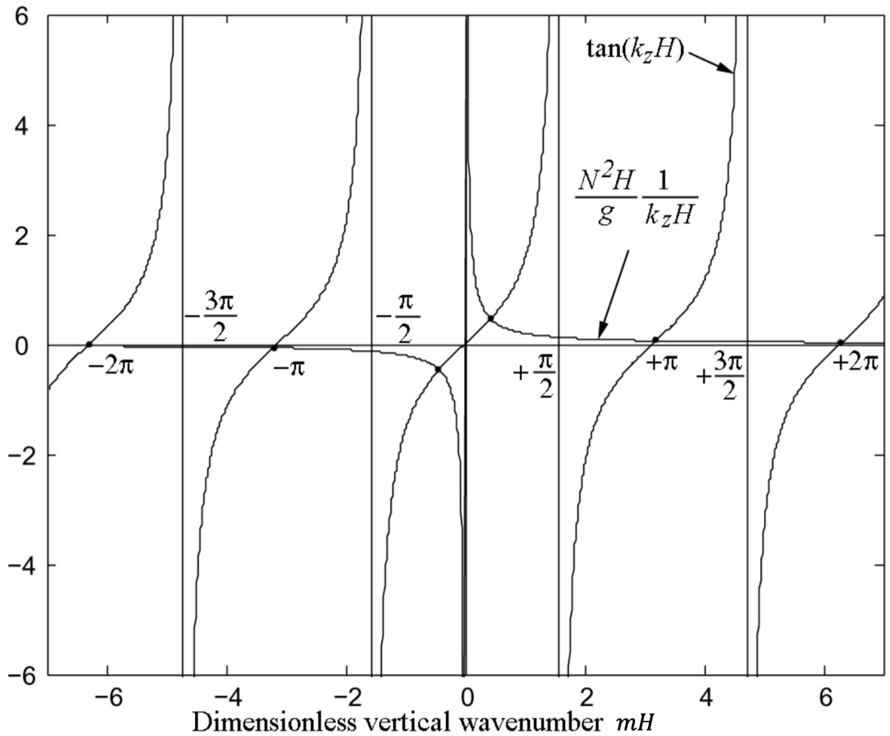
$$m = \frac{N}{\sqrt{gH}}$$

$$\omega = -\frac{\beta_0 k}{k^2 + l^2 + f_0^2 / gH}$$

$$= -\beta_0 R^2 \frac{k}{1 + R^2 (k^2 + l^2)}$$

barotropic Rossby wave

$$\boxed{\omega = -\frac{\beta_0 k}{k^2 + l^2 + m^2 f_0^2 / N^2}}$$



For larger solutions:

$$tan mH \sim 0$$

$$m_j = j\frac{\pi}{H}, j = 1, 2, 3, \ldots$$

baroclinic modes

$$\omega_j = -\frac{\beta_0 k}{k^2 + l^2 + (j\pi f_0 / NH)^2}$$

$$c_x < 0$$

westward propagation

$$R_j = \frac{1}{j}\frac{NH}{\pi f_0}$$

The maximum wave speed in the x direction is when

$$k^2 + l^2 \to 0$$

long waves

$$|c_x|_{max} = \beta_0 R_j^2$$