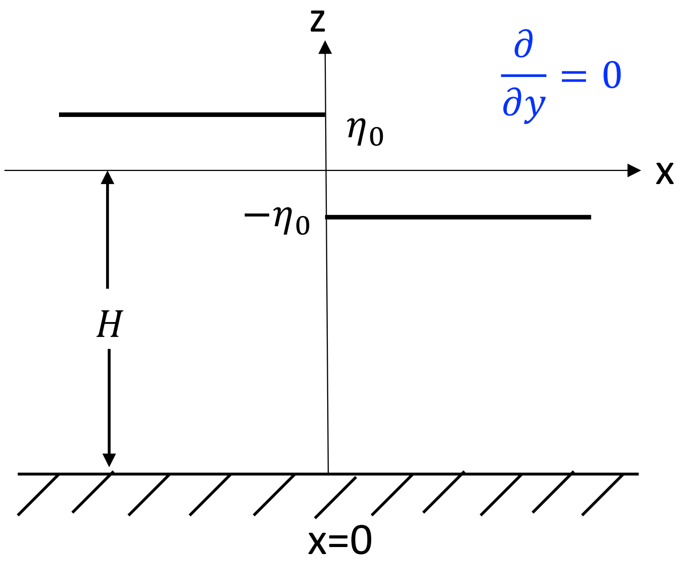
Geostrophic (Rossby) adjustment - barotropic

Initial state (unbalanced):

$$\eta = \begin{cases} \eta_0, & x < 0 \\ -\eta_0, & x > 0 \end{cases}$$

η_0<<H

$$u = v = 0$$



For a steady final state (based on shallow water model):

$$\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}} - fv = -g\frac{\partial \eta}{\partial x}$$

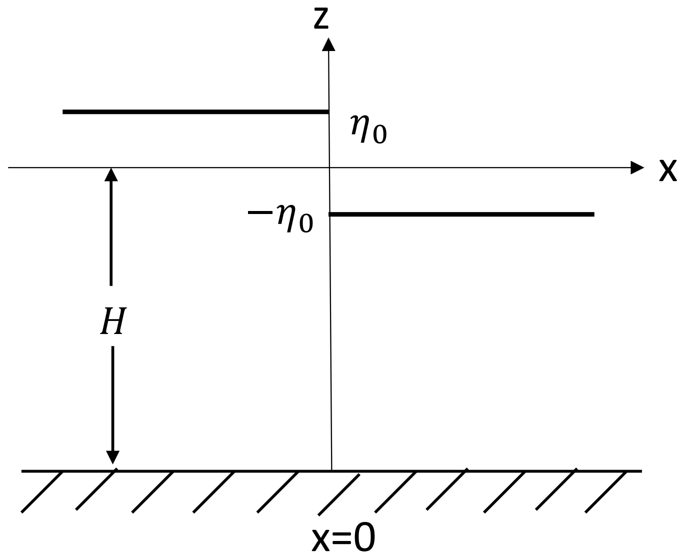
$$\cancel{\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial x}} + \cancel{v\frac{\partial v}{\partial y}} + fu = -g\frac{\partial \eta}{\partial y}$$

geostrophic flow

$$\cancel{\frac{\partial \eta}{\partial t}} + \frac{\partial hu}{\partial x} + \cancel{\frac{\partial hv}{\partial y}} = 0$$

sufficient condition: $u = 0$ everywhere

For potential vorticity conservation:



x < 0:

$$\frac{f}{H+\eta_0} = \frac{f+\frac{\partial v}{\partial x}}{H+\eta} = \frac{f+\frac{g}{f}\frac{\partial^2 \eta}{\partial x^2}}{H+\eta}$$

η_0<<H

$$\cancel{fH} + f\eta = \cancel{fH} + f\eta_0 + \frac{gH}{f}\frac{\partial^2 \eta}{\partial x^2} + \cancel{\frac{g\eta_0}{f}\frac{\partial^2 \eta}{\partial x^2}}$$

$$R^2\,\frac{\partial^2 \eta}{\partial x^2} - \eta = -\eta_0$$

$$R = \frac{\sqrt{gH}}{f}$$

$$\eta = \mathrm{f(x)} + \eta_0$$

$$e^{-\frac{1}{R}x} \rightarrow \infty \ as\ x \rightarrow -\infty$$

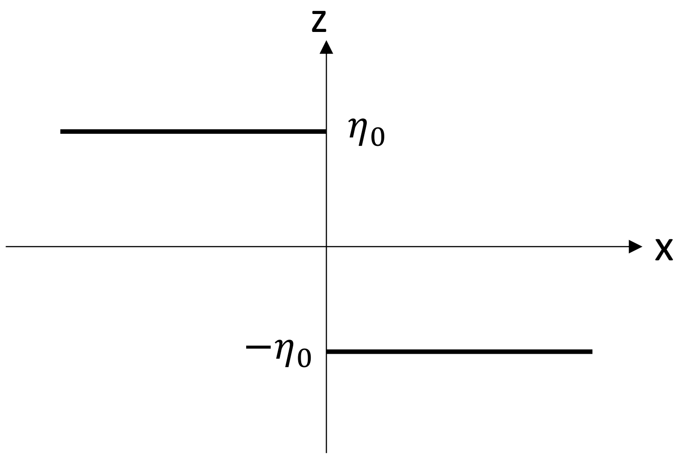
$$\mathrm{f(x)} = \mathrm{C}e^{\lambda x} \Longrightarrow R^2\,\lambda^2 - 1 = 0 \Longrightarrow \lambda = \pm\frac{1}{R} \Longrightarrow \lambda = \frac{1}{R}$$

$$\eta = Ce^{\frac{x}{R}} + \eta_0 \xrightarrow{x\,=\,0,\,\eta\,=\,0,\,\mathrm{C}\,=\,-\eta_0} \eta = -\eta_0 e^{\frac{x}{R}} + \eta_0$$

For PV conservation:

x > 0:

$$\frac{f}{H-\eta_0} = \frac{f+\frac{\partial v}{\partial x}}{H+\eta} = \frac{f+\frac{g}{f}\frac{\partial^2\eta}{\partial x^2}}{H+\eta}$$



$$\cancel{fH} + f\eta = \cancel{fH} - f\eta_0 + \frac{gH}{f}\frac{\partial^2\eta}{\partial x^2} - \cancel{\frac{g\eta_0}{f}\frac{\partial^2\eta}{\partial x^2}}$$

$$R^2\frac{\partial^2\eta}{\partial x^2} - \eta = \eta_0$$

$$\eta = \mathrm{f(x)} - \eta_0$$

$e^{\frac{1}{R}x} \rightarrow \infty \; as \; x \rightarrow \infty$

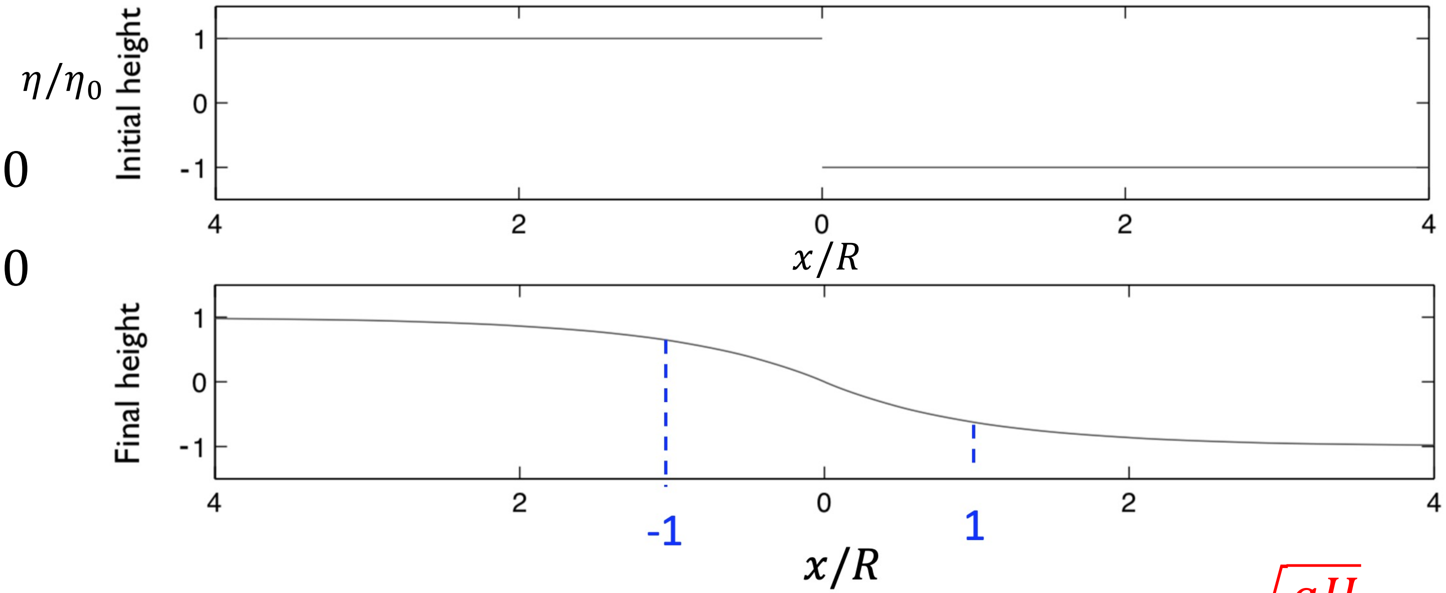
$$\mathrm{f(x)} = \mathrm{C}e^{\lambda x} \Rightarrow R^2\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm\frac{1}{R} \Rightarrow \lambda = -\frac{1}{R}$$

$$\eta = Ce^{-\frac{x}{R}} - \eta_0 \xrightarrow{x=0,\eta=0,\mathrm{C}=\eta_0} \eta = \eta_0 e^{-\frac{x}{R}} - \eta_0$$

$x \to -R, \quad \eta \to \eta_0$

$$\eta = \begin{cases} -\eta_0 e^{\frac{x}{R}} + \eta_0, & x < 0 \\ \eta_0 e^{-\frac{x}{R}} - \eta_0, & x > 0 \end{cases}$$

$x \to R, \quad \eta \to -\eta_0$

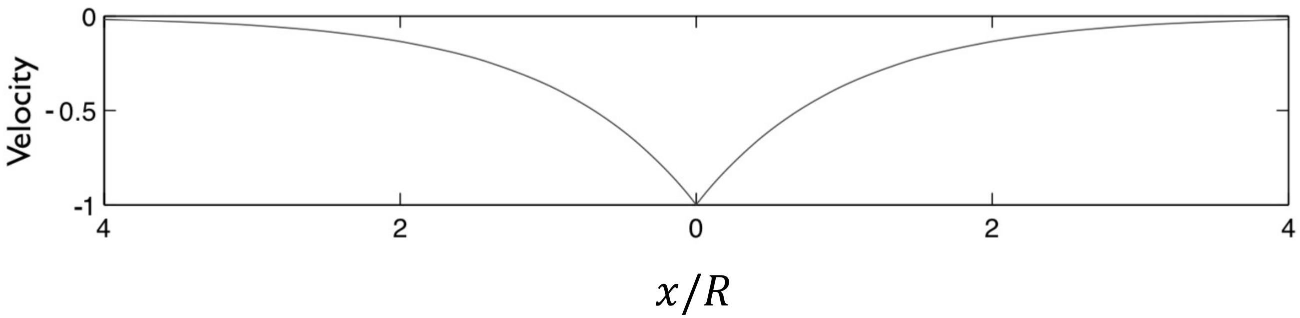


The adjustment spatial scale is the Rossby deformation radius $R = \dfrac{\sqrt{gH}}{f}$

$x \to -R, \quad v \to 0$

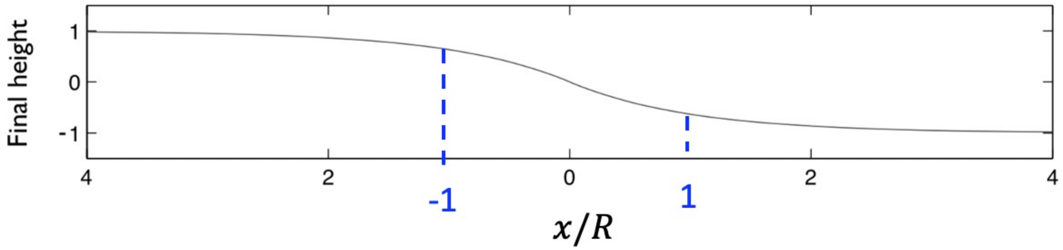
$$v = \frac{g}{f}\frac{\partial \eta}{\partial x} = \begin{cases} -\sqrt{\dfrac{g}{H}}\eta_0 e^{\frac{x}{R}}, & x < 0 \\ -\sqrt{\dfrac{g}{H}}\eta_0 e^{-\frac{x}{R}}, & x > 0 \end{cases}$$

$x \to R, \quad v \to 0$



Energetics of geostrophic adjustment

For unit length in the y-direction:



Potential energy:

$$\int_0^{\eta}\int_0^{1}\int_{x_1}^{x_2}\rho_0 g z\,dx\,dy\,dz = \int_0^{\eta}\int_{x_1}^{x_2}\rho_0 g z\,dx\,dz = \int_{x_1}^{x_2}\int_0^{\eta}\rho_0 g z\,dz\,dx = \frac{1}{2}\rho_0 g\int_{x_1}^{x_2}\eta^2 dx$$

PE change is limited to $(-R\colon R,\ -\eta_0\colon \eta_0)$

$$PE_I = \frac{1}{2}\rho_0 g\int_{-R}^{0}{\eta_0}^2 dx + \frac{1}{2}\rho_0 g\int_0^{R}(-\eta_0)^2 dx = \frac{1}{2}\rho_0 g \mathrm{R}{\eta_0}^2 + \frac{1}{2}\rho_0 g\mathrm{R}{\eta_0}^2 = \rho_0 g\mathrm{R}{\eta_0}^2$$

$$PE_F = \frac{1}{2}\rho_0 g{\eta_0}^2\int_{-R}^{0}(1-e^{\frac{x}{R}})^2 dx + \frac{1}{2}\rho_0 g{\eta_0}^2\int_0^{R}(1-e^{-\frac{x}{R}})^2 dx$$

$$\eta = \begin{cases} -\eta_0 e^{\frac{x}{R}} + \eta_0, & x < 0 \\ \eta_0 e^{-\frac{x}{R}} - \eta_0, & x > 0 \end{cases}$$

$$= \frac{1}{2}\rho_0 g{\eta_0}^2\left(x|_{-R}^{0} - 2Re^{\frac{x}{R}}|_{-R}^{0} + \frac{R}{2}e^{\frac{2x}{R}}|_{-R}^{0} + x|_0^R + 2Re^{-\frac{x}{R}}|_0^R - \frac{R}{2}e^{-\frac{2x}{R}}|_0^R\right)$$

$$= -\frac{1}{2}\rho_0 g R{\eta_0}^2$$

$$\Delta PE = -\frac{3}{2}\rho_0 g R{\eta_0}^2$$

Kinetic energy:

$$KE_I = 0$$

η_0<<H

$$KE_F = \int_{-H}^{0}\int_{-R}^{R}\frac{1}{2}\rho_0 v^2 dxdz$$

$$= \int_{-H}^{0}\int_{-R}^{0}\frac{1}{2}\rho_0{\eta_0}^2\frac{g}{H}e^{\frac{2x}{R}}dxdz + \int_{-H}^{0}\int_{0}^{R}\frac{1}{2}\rho_0{\eta_0}^2\frac{g}{H}e^{-\frac{2x}{R}}dxdz$$

$$= \frac{1}{2}\rho_0{\eta_0}^2 g(\int_{-R}^{0}e^{\frac{2x}{R}}dx + \int_{0}^{R}e^{-\frac{2x}{R}}dx)$$

$$= \frac{1}{2}\rho_0{\eta_0}^2 g(\frac{R}{2}e^{\frac{2x}{R}}|_{-R}^{0} - \frac{R}{2}e^{-\frac{2x}{R}}|_{0}^{R})$$

$$= \frac{1}{2}\rho_0 g R{\eta_0}^2$$

$$\boxed{v = \begin{cases} -\sqrt{\frac{g}{H}}\eta_0 e^{\frac{x}{R}}, & x < 0 \\ -\sqrt{\frac{g}{H}}\eta_0 e^{-\frac{x}{R}}, & x > 0 \end{cases}}$$

$$\Delta PE = -\frac{3}{2}\rho_0 g R{\eta_0}^2$$

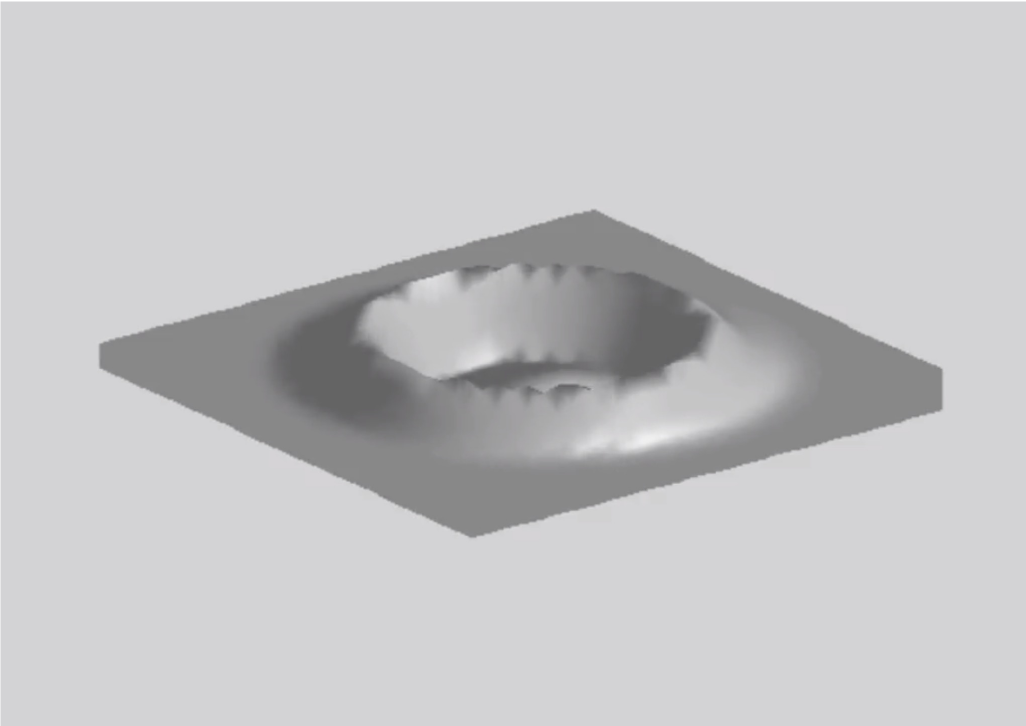
$$\frac{\Delta KE}{|\Delta PE|} = 1/3$$

Only 1/3 of released PE is converted into KE, and the rest of lost PE is radiated away by waves

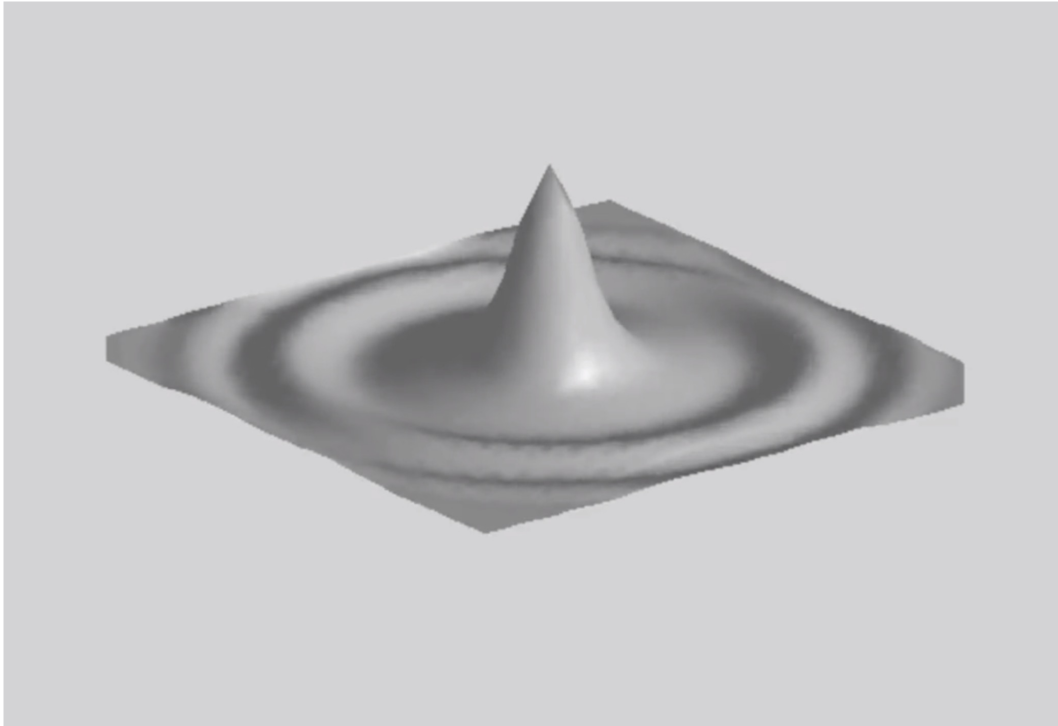
inertial-gravity waves

Adjustment of perturbations

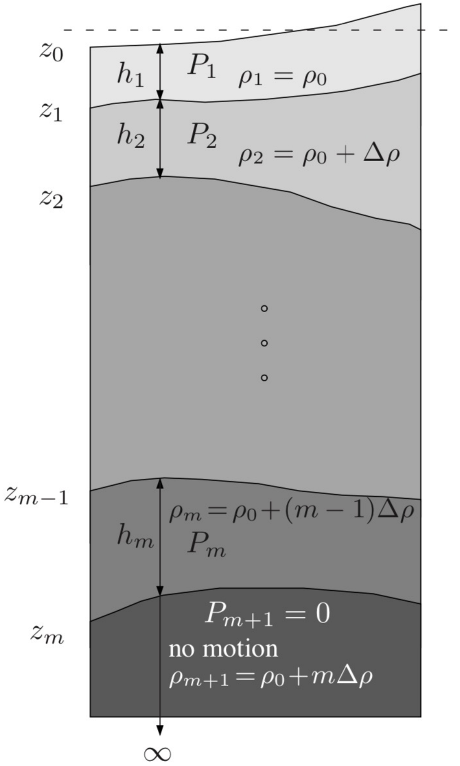
without rotation



with rotation



shallow-water reduced gravity model – one layer



One layer:

$z_1 = -h_1$ $\qquad$ $P_1 = \rho_0 g' h_1$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g'\frac{\partial h}{\partial x} \quad (1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g'\frac{\partial h}{\partial y} \quad (2)$$

$$\frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} + \frac{\partial hv}{\partial y} = 0 \quad \Rightarrow \quad \frac{dh}{dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

$\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(1)$:

$$\frac{d(f+\zeta)}{dt} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(f+\zeta) = 0 \ (1)$$

$$\frac{d}{dt}\left(\frac{f+\zeta}{h}\right) = 0$$

PV conservation

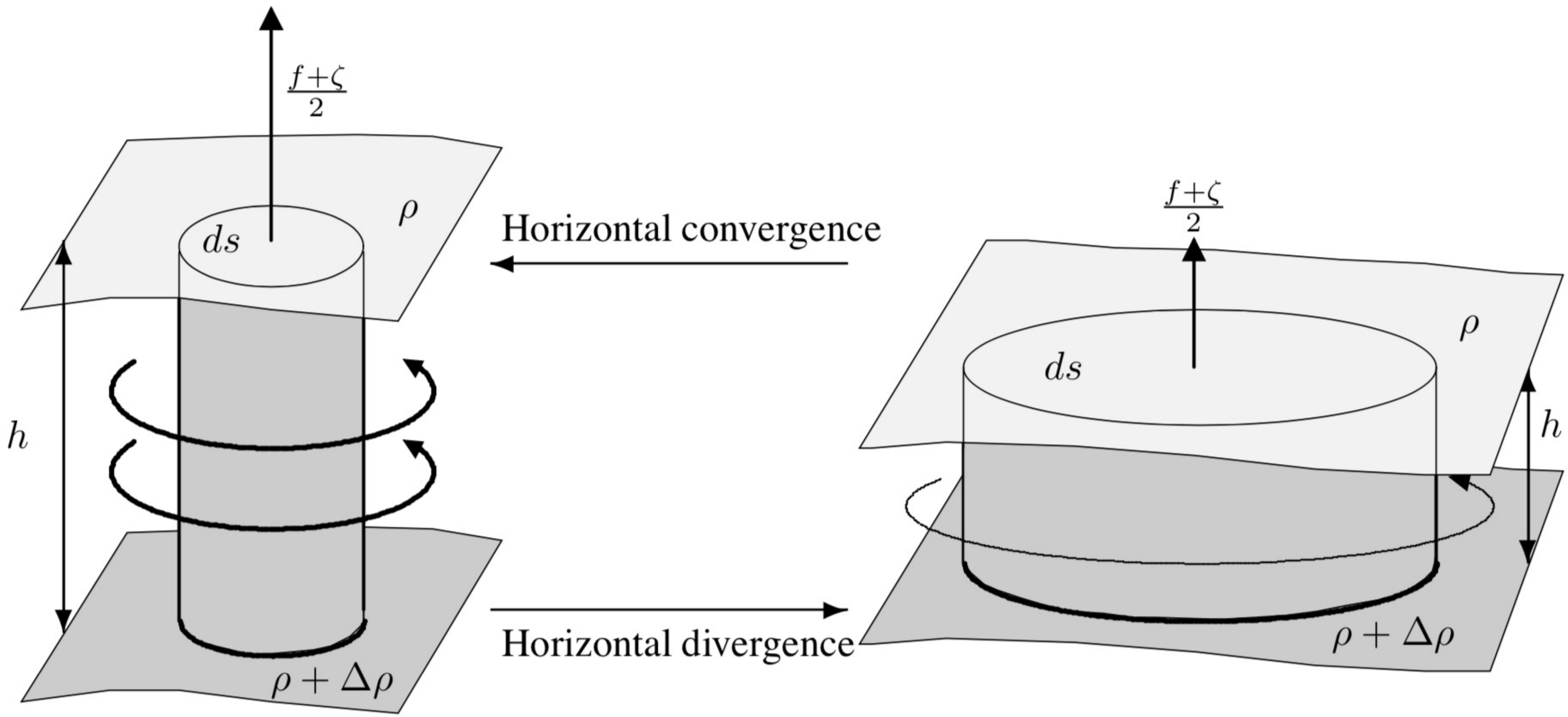
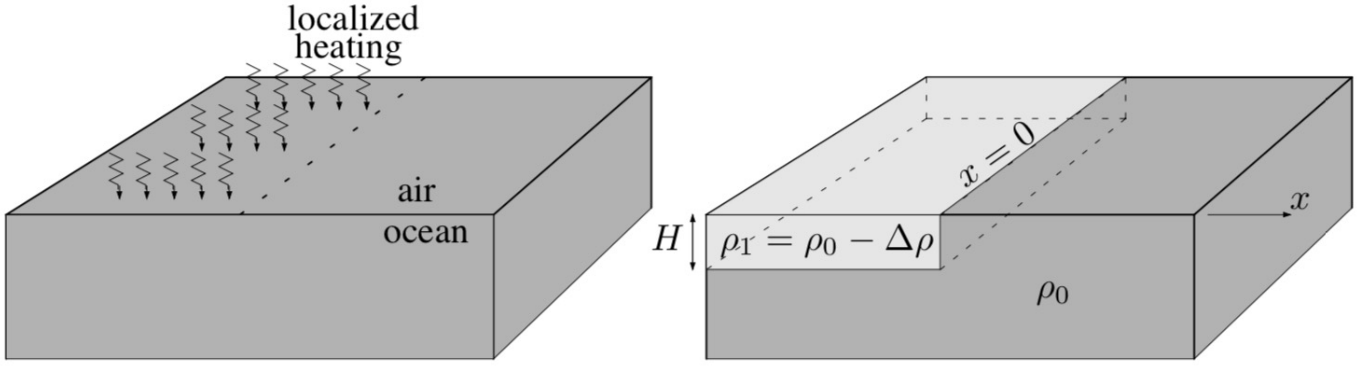


Figure 12-4 Conservation of volume and circulation in a fluid undergoing divergence (squeezing) or convergence (stretching). The products of $h\ ds$ and $(f + \zeta)\ ds$ are conserved during the transformation, implying conservation of $(f + \zeta)/h$, too.

Geostrophic adjustment - baroclinic



(a) Initial state (b) Immediately after heating event

$$\frac{\partial}{\partial y} = 0$$

For the lighter layer:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - fv = -g'\frac{\partial h}{\partial x}$$

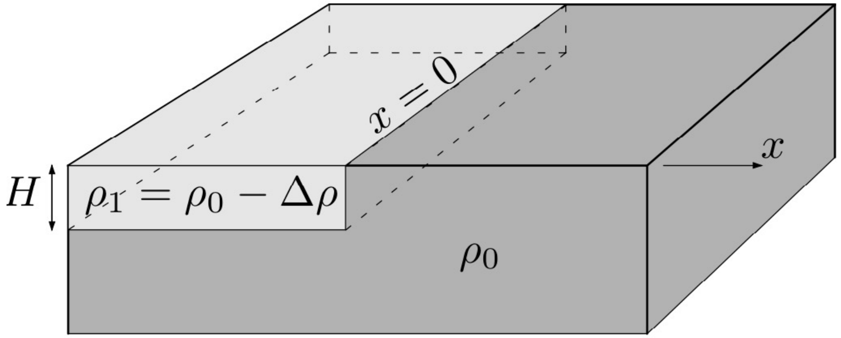
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + fu = 0$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0.$$

Initial state (unbalanced):

$$h = \begin{cases} H, & x < 0 \\ 0, & x > 0 \end{cases}$$

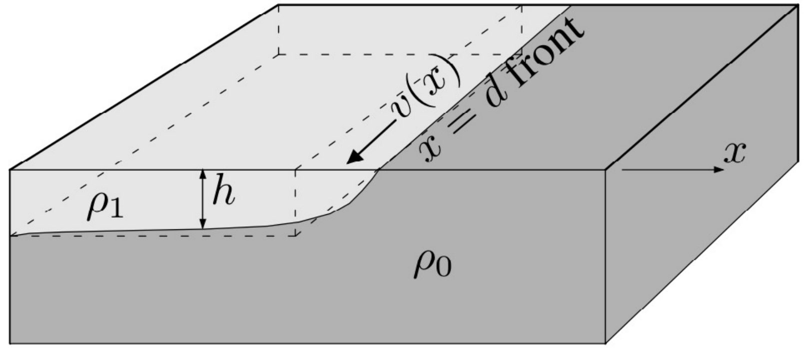
$$u = v = 0$$



Boundary conditions:

$$x \to -\infty, \quad h \to \mathrm{H}, \quad \mathrm{u, v} \to 0$$

$$x \to d, \quad h \to 0$$



(c) After adjustment

Final state (steady): $\dfrac{\partial}{\partial t} = 0$

$$\boxed{\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0} \Longrightarrow \frac{\partial hu}{\partial x} = 0$$

at $x = d$, $h = 0$, $hu = 0$: $hu = 0$ everywhere $\Longrightarrow$ $u = 0$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - fv = -g'\frac{\partial h}{\partial x} \qquad \cancel{\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + fu = 0}$$

$$-fv = -g'\frac{dh}{dx}$$

PV conservation:

$$\frac{f}{H} = \frac{f + \frac{\partial v}{\partial x}}{h} \Rightarrow \frac{f}{H} = \frac{f + \frac{g'}{f}\frac{d^2h}{\partial x^2}}{h}$$

$$fh = fH + \frac{g'H}{f}\frac{d^2h}{\partial x^2}$$

baroclinic deformation radius

$$R = \frac{\sqrt{g'H}}{f}$$

$$R^2\frac{d^2h}{\partial x^2} - h + H = 0$$

$$\mathrm{h} = \mathrm{f(x)} + H \qquad \mathrm{f(x)}:\; R^2\frac{d^2h}{\partial x^2} - h = 0$$

$$x \to -\infty, h \to \mathrm{H}$$

$$\mathrm{f(x)} = Ae^{\lambda x} \Rightarrow R^2\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm\frac{1}{R} \Rightarrow \lambda = \frac{1}{R} \qquad \mathrm{f(x)} = Ae^{x/R}$$

$$f(x) = Ae^{x/R}$$

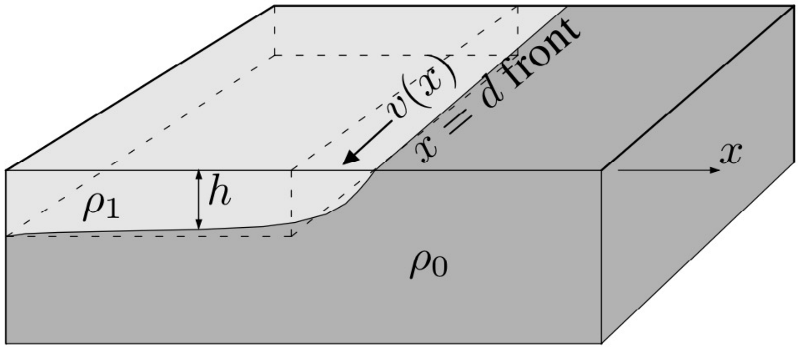
$x \to d, \qquad h \to 0: \qquad f(x) \to -\mathrm{H}$ $\qquad\qquad$ $\mathrm{h} = \mathrm{f(x)} + H$

$$f(x) = Be^{(x-d)/R} \qquad B = -H$$

$$h = H(1 - e^{\frac{x-d}{R}}) \qquad\qquad -fv = -g'\frac{dh}{dx}$$

$$v = -\sqrt{g'H}\, e^{\frac{x-d}{R}}$$

The depleted volume of light water in x < 0 should be equal to the volume of light water in x > 0:



(c) After adjustment

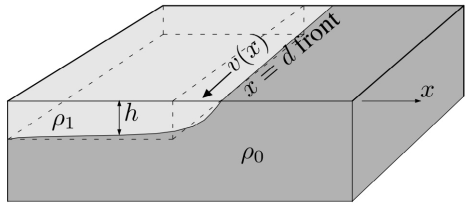
$$\int_{-\infty}^{0}(H - h)dx = \int_0^d h dx$$

$$H\int_{-\infty}^{0} e^{\frac{x-d}{R}} dx = Hd - H\int_0^d e^{\frac{x-d}{R}} dx$$

$$Re^{\frac{x-d}{R}}|_{-\infty}^{0} = d - Re^{\frac{x-d}{R}}|_0^d$$

$d = R$ $\qquad$ adjustment spatial scale is R

Energetics



(c) After adjustment

Initial state: $KE_i = 0$ $\quad PE_i = \frac{1}{2}\rho_0 \int_{-\infty}^{0} g' H^2 dx = \frac{1}{2}\rho_0 g' H^2 x|_{-\infty}^{0}$

Final state: $KE_f = \frac{1}{2}\rho_0 \int_{-\infty}^{R} h v^2 dx = \frac{1}{2}\rho_0 g' H^2 \int_{-\infty}^{R} (1 - e^{\frac{x-R}{R}}) e^{2\frac{x-R}{R}} dx$

$v = -\sqrt{g'H}\, e^{\frac{x-R}{R}}$

$h = H(1 - e^{\frac{x-R}{R}})$

$$= \frac{1}{2}\rho_0 g' H^2 \left(\frac{R}{2} e^{2\frac{x-R}{R}}|_{-\infty}^{R} - \frac{R}{3} e^{3\frac{x-R}{R}}|_{-\infty}^{R}\right) = \frac{1}{12}\rho_0 g' H^2 R \qquad \Delta KE$$

$$PE_f = \frac{1}{2}\rho_0 \int_{-\infty}^{R} g' h^2 dx = \frac{1}{2}\rho_0 g' H^2 \int_{-\infty}^{R} (1 - e^{\frac{x-R}{R}})^2 dx$$

$$= \frac{1}{2}\rho_0 g' H^2 \left(x|_{-\infty}^{R} - 2R e^{\frac{x-R}{R}}|_{-\infty}^{R} + \frac{R}{2} e^{2\frac{x-R}{R}}|_{-\infty}^{R}\right)$$

$$= \frac{1}{2}\rho_0 g' H^2 \left(x|_{-\infty}^{R} - \frac{3}{2} R\right)$$

$$\Delta PE = PE_i - PE_f = \frac{1}{4}\rho_0 g' H^2 R \qquad \frac{\Delta KE}{\Delta PE} = 1/3$$