The waves broke and spread their waters swiftly over the shore. One after another they massed themselves and fell; the spray tossed itself back with the energy of their fall. The waves were steeped deep-blue save for a pattern of diamond-pointed light on their backs which rippled as the backs of great horses ripple with muscles as they move. The waves fell; withdrew and fell again.

Virginia Woolf, *The Waves*, 1931.

CHAPTER 6

Wave Fundamentals

WAVES ARE EVERYWHERE: on the sea-shore, on piano wires, in football stadiums, and filling the space between the distant stars and Earth. This chapter provides an introduction to their properties, paying particular attention to a wave that is especially important to the large scale flow in both ocean and atmosphere — the Rossby wave. We start with an elementary introduction to wave kinematics, discussing such basic concepts as phase speed and group velocity. Then, beginning with Section 6.4, we discuss the dynamics of Rossby waves, and this part may be considered to be the natural follow-on from the geostrophic theory of the previous chapter. Finally, in Section 6.7, we return to group velocity in a more general way and illustrate the results using Poincaré waves, with more applications to gravity and Rossby waves in later chapters.

The reason for such an ordering of topics is that wave kinematics without a dynamical example is jejune and dry, yet understanding wave dynamics of any sort is hardly possible without appreciating at least some of its formal structure, and readers should flip pages back and forth through the chapter as needed. Those readers who wish to cut to the chase may skip the first few sections and begin at Section 6.4, referring back as needed. (Many of the key elementary results are summarized in the shaded box on page 218.) Other readers may wish to skip the sections on Rossby waves altogether and, after absorbing the sections on the wave theory move on to Chapter 7 on gravity waves, returning to Rossby waves (or not) later on. The Rossby wave and gravity wave discussions are largely independent of each other, although they both require that the reader is familiar with such ideas as group velocity and phase speed. Rossby waves and gravity waves can co-exist and close to the equator the two kinds of waves become more intertwined; we deal with the ensuing waves in Chapter 8. We also extend our discussion of Rossby waves in a global atmospheric context in Chapter 16.

6.1 FUNDAMENTALS AND FORMALITIES

6.1.1 Definitions and Kinematics

A wave may be more easily recognized than defined. Loosely speaking, a wave is a propagating disturbance that has a characteristic relationship between its frequency and size, and a linear wave may be defined as a disturbance that satisfies a *dispersion relation*. (Nonlinear waves exist, but the curious reader must look elsewhere to learn about them.[1]) In order to see what all this means, and what a dispersion relation is, suppose that a disturbance, $\psi(\boldsymbol{x}, t)$ (where ψ might be velocity, streamfunction, pressure, etc.), satisfies the equation

$$L(\psi) = 0, \tag{6.1}$$

where L is a linear operator, typically a polynomial in time and space derivatives; one example is $L(\psi) = \partial\nabla^2\psi/\partial t + \beta\partial\psi/\partial x$. If (6.1) has constant coefficients (if β is constant in this example) then harmonic solutions may often be found that are a superposition of *plane waves,* each of which satisfy

$$\psi = \mathrm{Re}\,\widetilde{\psi}\mathrm{e}^{\mathrm{i}\theta(\boldsymbol{x},t)} = \mathrm{Re}\,\widetilde{\psi}\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}, \tag{6.2}$$

where $\widetilde{\psi}$ is a complex constant, θ is the phase, ω is the wave frequency and $\boldsymbol{k}$ is the vector wavenumber (k, l, m) (also written as (k^x, k^y, k^z) or, in subscript notation, k_i). The prefix Re denotes the real part of the expression, but we will drop it if there is no ambiguity.

Earlier, we said that waves are characterized by having a particular relationship between the frequency and wavevector known as the *dispersion relation*. This is an equation of the form

$$\omega = \Omega(\boldsymbol{k}), \tag{6.3}$$

where $\Omega(\boldsymbol{k})$, or $\Omega(k_i)$, and meaning $\Omega(k, l, m)$, is some function determined by the form of L in (6.1) and which thus depends on the particular type of wave — the function is different for sound waves, light waves and the Rossby waves and gravity waves we will encounter in this book (peek ahead to (6.60) and (7.56), and there is more discussion in Section 6.1.3). Unless it is necessary to explicitly distinguish the function Ω from the frequency ω, we will often write $\omega = \omega(\boldsymbol{k})$.

If the medium in which the waves are propagating is inhomogeneous then (6.1) will probably not have constant coefficients (for example, β may vary with y). Nevertheless, if the medium is varying sufficiently slowly, wave solutions may often still be found with the general form

$$\psi(\boldsymbol{x}, t) = a(\boldsymbol{x}, t)\mathrm{e}^{\mathrm{i}\theta(\boldsymbol{x},t)}, \tag{6.4}$$

where the amplitude $a(\boldsymbol{x}, t)$ varies slowly compared to the phase, θ. The frequency and wavenumber are then defined by

$$\boldsymbol{k} \equiv \nabla\theta, \qquad \omega \equiv -\frac{\partial\theta}{\partial t}. \tag{6.5}$$

The example of (6.2) is clearly just a special case of this. Equation (6.5) implies the formal relation between $\boldsymbol{k}$ and ω:

$$\frac{\partial\boldsymbol{k}}{\partial t} + \nabla\omega = 0. \tag{6.6}$$

The WKB method, described in Appendix A at the end of this chapter, is one way of finding such solutions.

6.1.2 Wave Propagation and Phase Speed

A common property of waves is that they propagate through space with some velocity, which in special cases might be zero. Waves in fluids may carry energy and momentum but not normally, at least to a first approximation, fluid parcels themselves. Further, it turns out that the speed at which properties like energy are transported (the group speed) may be different from the speed at which the wave crests themselves move (the phase speed). Let's try to understand this statement, beginning with the phase speed. A summary of key results is given on page 218.

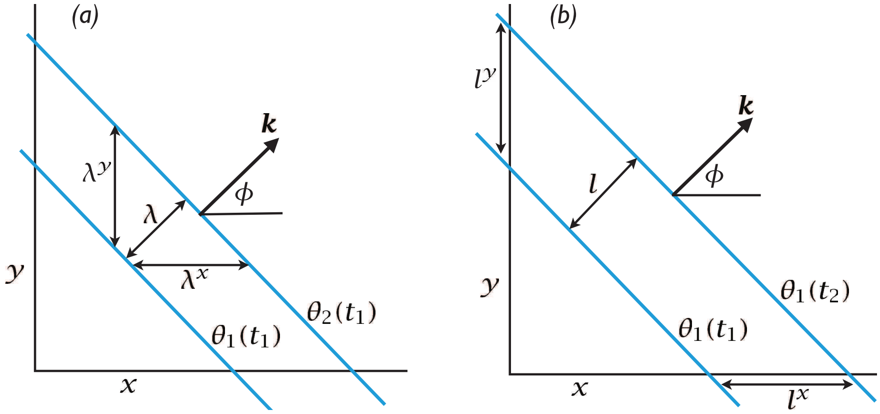

Fig. 6.1 The propagation of a two-dimensional wave. (a) Two lines of constant phase (e.g., two wavecrests) at a time t_1. The wave is propagating in the direction $\boldsymbol{k}$ with wavelength λ. (b) The same line of constant phase at two successive times. The phase speed is the speed of advancement of the wavecrest in the direction of travel, and so $c_p = l/(t_2 - t_1)$. The phase speed in the x-direction is the speed of propagation of the wavecrest along the x-axis, and $c_p^x = l^x/(t_2 - t_1) = c_p/\cos\phi$.

Phase speed

Consider the propagation of monochromatic plane waves, for that is all that is needed to introduce the phase speed. Given (6.2) a wave will propagate in the direction of $\boldsymbol{k}$ (Fig. 6.1). At a given instant and location we can align our coordinate axis along this direction, and we write $\boldsymbol{k}\cdot\boldsymbol{x} = Kx^*$, where x^* increases in the direction of $\boldsymbol{k}$ and $K^2 = |\boldsymbol{k}|^2$ is the magnitude of the wavenumber. With this, we can write (6.2) as

$$\psi = \operatorname{Re}\widetilde{\psi}\mathrm{e}^{\mathrm{i}(Kx^* - \omega t)} = \operatorname{Re}\widetilde{\psi}\mathrm{e}^{\mathrm{i}K(x^* - ct)}, \tag{6.7}$$

where $c = \omega/K$. From this equation it is evident that the phase of the wave propagates at the speed c in the direction of $\boldsymbol{k}$, and we define the *phase speed* by

$$c_p \equiv \frac{\omega}{K}. \tag{6.8}$$

The wavelength of the wave, λ, is the distance between two wavecrests — that is, the distance between two locations along the line of travel whose phase differs by 2π — and evidently this is given by

$$\lambda = \frac{2\pi}{K}. \tag{6.9}$$

In (for simplicity) a two-dimensional wave, and referring to Fig. 6.1, the wavelength and wave vectors in the x- and y-directions are given by,

$$\lambda^x = \frac{\lambda}{\cos\phi}, \quad \lambda^y = \frac{\lambda}{\sin\phi}, \qquad k^x = K\cos\phi, \quad k^y = K\sin\phi. \tag{6.10}$$

In general, lines of constant phase intersect both the coordinate axes and propagate along them. The speed of propagation along these axes is given by

$$c_p^x = c_p\frac{l^x}{l} = \frac{c_p}{\cos\phi} = c_p\frac{K}{k^x} = \frac{\omega}{k^x}, \qquad c_p^y = c_p\frac{l^y}{l} = \frac{c_p}{\sin\phi} = c_p\frac{K}{k^y} = \frac{\omega}{k^y}, \tag{6.11}$$

using (6.8) and (6.10), and again referring to Fig. 6.1 for notation. The speed of phase propagation along any one of the axes is in general *larger* than the phase speed in the primary direction of

Wave Fundamentals

- A wave is a propagating disturbance that has a characteristic relationship between its frequency and size, known as the dispersion relation. Waves typically arise as solutions to a linear problem of the form
$$L(\psi) = 0, \tag{WF.1}$$
where L is, commonly, a linear operator in space and time. Two examples are
$$\frac{\partial^2\psi}{\partial t^2} - c^2\nabla^2\psi = 0 \qquad \text{and} \qquad \frac{\partial}{\partial t}\nabla^2\psi + \beta\frac{\partial\psi}{\partial x} = 0. \tag{WF.2}$$
The first example is so common in all areas of physics it is sometimes called 'the' wave equation. The second example gives rise to Rossby waves.
- Solutions to the governing equation are often sought in the form of plane waves that have the form
$$\psi = \mathrm{Re}\, A\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}, \tag{WF.3}$$
where A is the wave amplitude, $\boldsymbol{k} = (k, l, m)$ is the wavevector, and ω is the frequency.
- The dispersion relation connects the frequency and wavevector through an equation of the form $\omega = \Omega(\boldsymbol{k})$ where Ω is some function. The relation is normally derived by substituting a trial solution like (WF.3) into the governing equation (WF.1). For the examples of (WF.2) we obtain $\omega = c^2K^2$ and $\omega = -\beta k/K^2$ where $K^2 = k^2 + l^2 + m^2$ or, in two dimensions, $K^2 = k^2 + l^2$.
- The phase speed is the speed at which the wave crests move. In the direction of propagation and in the x, y and z directions the phase speeds are given by, respectively,
$$c_p = \frac{\omega}{K}, \qquad c_p^x = \frac{\omega}{k}, \quad c_p^y = \frac{\omega}{l}, \quad c_p^z = \frac{\omega}{m}, \tag{WF.4}$$
where $K = 2\pi/\lambda$ and λ is the wavelength. The wave crests have both a speed (c_p) and a direction of propagation (the direction of $\boldsymbol{k}$), like a vector, but the components defined in (WF.4) are not the components of that vector.
- The group velocity is the velocity at which a wave packet or wave group moves. It is a vector and is given by
$$\boldsymbol{c}_g = \frac{\partial\omega}{\partial\boldsymbol{k}} \quad \text{with components} \quad c_g^x = \frac{\partial\omega}{\partial k}, \quad c_g^y = \frac{\partial\omega}{\partial l}, \quad c_g^z = \frac{\partial\omega}{\partial m}. \tag{WF.5}$$
Many physical quantities of interest are transported at the group velocity.
- If the coefficients of the wave equation are not constant (for example if the medium is inhomogeneous) then, if the coefficients are only slowly varying, approximate solutions may sometimes be found in the form
$$\psi = \mathrm{Re}\, A(\boldsymbol{x}, t)\mathrm{e}^{\mathrm{i}\theta(\boldsymbol{x},t)}, \tag{WF.6}$$
where the amplitude A is also slowly varying and the local wavenumber and frequency are related to the phase, θ, by $\boldsymbol{k} = \nabla\theta$ and $\omega = -\partial\theta/\partial t$. The dispersion relation is then a *local* one of the form $\omega = \Omega(\boldsymbol{k}; x, t)$.

the wave. The phase speeds are clearly *not* components of a vector: for example, $c_p^x \neq c_p \cos\phi$. Analogously, the wavevector $\boldsymbol{k}$ is a true vector, whereas the wavelength λ is not.

To summarize, the phase speed and its components are given by

$$c_p = \frac{\omega}{K}, \qquad c_p^x = \frac{\omega}{k^x}, \quad c_p^y = \frac{\omega}{k^y}. \tag{6.12}$$

Phase velocity

Although it is not particularly useful, there is a way of defining a phase speed so that it is a true vector, and which might then be called phase velocity. We define the phase velocity to be the velocity that has the magnitude of the phase speed and the direction in which wave crests are propagating; that is

$$\boldsymbol{c}_p \equiv \frac{\omega}{K}\frac{\boldsymbol{k}}{|K|} = c_p \frac{\boldsymbol{k}}{|K|}, \tag{6.13}$$

where $\boldsymbol{k}/|K|$ is the unit vector in the direction of wave-crest propagation. The components of the phase velocity in the x- and y-directions are then given by

$$c_p^x = c_p \cos\phi, \qquad c_p^y = c_p \sin\phi. \tag{6.14}$$

Defined this way, the quantity given by (6.13) is a vector velocity. However, the components in the x- and y-directions are manifestly not the speed at which wave crests propagate in those directions, and it is a misnomer to call these quantities phase speeds. Still, it can be helpful to ascribe a direction to the phase speed and the quantity given by (6.13) may then be useful.

6.1.3 The Dispersion Relation

The above description is kinematic, in that it applies to almost any disturbance that has a wavevector and a frequency. The particular *dynamics* of a wave are determined by the relationship between the wavevector and the frequency; that is, by the *dispersion relation*. Once the dispersion relation is known a great many of the properties of the wave follow in a more-or-less straightforward manner. Picking up from (6.3), the dispersion relation is a functional relationship between the frequency and the wavevector of the general form

$$\omega = \Omega(\boldsymbol{k}). \tag{6.15}$$

Perhaps the simplest example of a linear operator that gives rise to waves is the one-dimensional equation

$$\frac{\partial \psi}{\partial t} + c\frac{\partial \psi}{\partial x} = 0. \tag{6.16}$$

Substituting a trial solution of the form $\psi = \mathrm{Re}\, A e^{\mathrm{i}(kx-\omega t)}$ we obtain $(-\mathrm{i}\omega + c\mathrm{i}k)A = 0$, giving the dispersion relation

$$\omega = ck. \tag{6.17}$$

The phase speed of this wave is $c_p = \omega/k = c$. A few other examples of governing equations, dispersion relations and phase speeds are:

$$\frac{\partial \psi}{\partial t} + \boldsymbol{c}\cdot\nabla\psi = 0, \qquad \omega = \boldsymbol{c}\cdot\boldsymbol{k}, \qquad c_p = |\boldsymbol{c}|\cos\theta, \qquad c_p^x = \frac{\boldsymbol{c}\cdot\boldsymbol{k}}{k}, \qquad c_p^y = \frac{\boldsymbol{c}\cdot\boldsymbol{k}}{l}, \tag{6.18a}$$

$$\frac{\partial^2 \psi}{\partial t^2} - c^2\nabla^2\psi = 0, \qquad \omega^2 = c^2K^2, \qquad c_p = \pm c, \qquad c_p^x = \pm\frac{cK}{k}, \qquad c_p^y = \pm\frac{cK}{l}, \tag{6.18b}$$

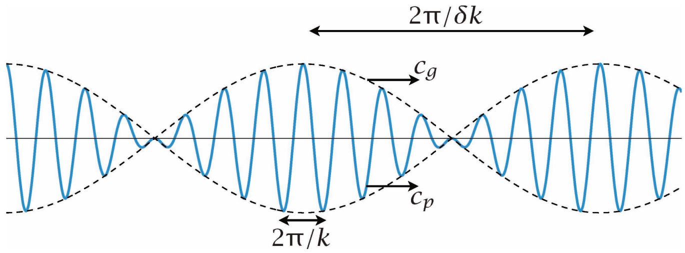

Fig. 6.2 Superposition of two sinusoidal waves with wavenumbers k and $k + \delta k$, producing a wave (solid line) that is modulated by a slowly varying wave envelope or packet (dashed).

The envelope moves at the group velocity, $c_g = \partial\omega/\partial k$, and the phase moves at the group speed, $c_p = \omega/k$.

$$\frac{\partial}{\partial t}\nabla^2\psi + \beta\frac{\partial\psi}{\partial x} = 0, \qquad \omega = \frac{-\beta k}{K^2}, \qquad c_p = \frac{\omega}{K}, \qquad c_p^x = -\frac{\beta}{K^2}, \qquad c_p^y = -\frac{\beta k/l}{K^2}, \tag{6.18c}$$

where $K^2 = k^2 + l^2$ and θ is the angle between $\boldsymbol{c}$ and $\boldsymbol{k}$, and the examples are all two-dimensional, with variation in x and y only.

A wave is said to be *nondispersive* if the phase speed is independent of the wavelength. This condition is satisfied for the simple example (6.16) but is manifestly not satisfied for (6.18c), and these waves (Rossby waves, in fact) are *dispersive*. Waves of different wavelengths then travel at different speeds so that a group of waves will spread out — disperse — even if the medium is homogeneous. When a wave is dispersive there is another characteristic speed at which the waves propagate, the group velocity, and we come to this shortly.

Most media are inhomogeneous, but if the medium varies sufficiently slowly in space and time — and in particular if the variations are slow compared to the wavelength and period — we may still have a *local* dispersion relation between frequency and wavevector,

$$\omega = \Omega(\boldsymbol{k}; \boldsymbol{x}, t), \tag{6.19}$$

where x and t are slowly varying parameters. We'll resume our discussion of this topic in Section 6.3, but before that we must introduce the group velocity.

6.2 GROUP VELOCITY

Information and energy do not, in general, propagate at the phase speed. Rather, most quantities of interest propagate at the *group velocity*, a quantity of enormous importance in wave theory.[2] Roughly speaking, group velocity is the velocity at which a packet or a group of waves will travel, whereas the individual wave crests travel at the phase speed. To introduce the idea we will consider the superposition of plane waves, noting that a truly monochromatic plane wave already fills all space uniformly so that there can be no propagation of energy from place to place.

6.2.1 Superposition of Two Waves

Consider the linear superposition of two waves. Limiting attention to the one-dimensional case for simplicity, consider a disturbance represented by

$$\psi = \mathrm{Re}\,\widetilde{\psi}(\mathrm{e}^{\mathrm{i}(k_1 x - \omega_1 t)} + \mathrm{e}^{\mathrm{i}(k_2 x - \omega_2 t)}). \tag{6.20}$$

Let us further suppose that the two waves have similar wavenumbers and frequency, and, in particular, that $k_1 = k + \Delta k$ and $k_2 = k - \Delta k$, and $\omega_1 = \omega + \Delta\omega$ and $\omega_2 = \omega - \Delta\omega$. With this, (6.20) becomes

$$\begin{aligned}\psi &= \mathrm{Re}\,\widetilde{\psi}\mathrm{e}^{\mathrm{i}(kx-\omega t)}[\mathrm{e}^{\mathrm{i}(\Delta k x - \Delta\omega t)} + \mathrm{e}^{-\mathrm{i}(\Delta k x - \Delta\omega t)}] \\ &= 2\,\mathrm{Re}\,\widetilde{\psi}\mathrm{e}^{\mathrm{i}(kx-\omega t)}\cos(\Delta k\, x - \Delta\omega\, t).\end{aligned} \tag{6.21}$$

The resulting disturbance, illustrated in Fig. 6.2 has two aspects: a rapidly varying component, with wavenumber k and frequency ω, and a more slowly varying envelope, with wavenumber Δk and frequency $\Delta\omega$. The envelope modulates the fast oscillation, and moves with velocity $\Delta\omega/\Delta k$; in the limit $\Delta k \to 0$ and $\Delta\omega \to 0$ this is the *group velocity*, $c_g = \partial\omega/\partial k$. Group velocity is equal to the phase speed, ω/k, only when the frequency is a linear function of wavenumber. The energy in the disturbance must move at the group velocity — note that the node of the envelope moves at the speed of the envelope and no energy can cross the node. These concepts generalize to more than one dimension, and if the wavenumber is the three-dimensional vector $\boldsymbol{k} = (k, l, m)$ then the three-dimensional envelope propagates at the group velocity given by

$$\boldsymbol{c}_g = \frac{\partial\omega}{\partial\boldsymbol{k}} \equiv \left(\frac{\partial\omega}{\partial k}, \frac{\partial\omega}{\partial l}, \frac{\partial\omega}{\partial m}\right). \tag{6.22}$$

The group velocity is also written as $\boldsymbol{c}_g = \nabla_{\boldsymbol{k}}\omega$ or, in subscript notation, $c_{gi} = \partial\Omega/\partial k_i$, with the subscript i denoting the component of a vector.

6.2.2 Superposition of Many Waves

Now consider a slight extension of the above arguments to the case in which many waves are excited. The essential assumption of this derivation is that the wavenumber distribution is sufficiently narrow so that the dispersion relation can be approximated as

$$\omega(k) \approx \omega(k_0) + \left.\frac{\mathrm{d}\omega}{\mathrm{d}k}\right|_{k_0}(k - k_0). \tag{6.23}$$

We treat only the one-dimensional case but the three-dimensional generalization is possible.

A superposition of plane waves, each satisfying some dispersion relation, can be represented by the Fourier integral

$$\psi(x,t) = \int_{-\infty}^{\infty} \widetilde{A}(k)\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,\mathrm{d}k. \tag{6.24a}$$

The function $\widetilde{A}(k)$ is given by the initial conditions:

$$\widetilde{A}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(x,0)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x. \tag{6.24b}$$

Note that if the waves are dispersionless and $\omega = ck$ where c is a constant, then

$$\psi(x,t) = \int_{-\infty}^{+\infty} \widetilde{A}(k)\mathrm{e}^{\mathrm{i}k(x-ct)}\,\mathrm{d}k = \psi(x - ct, 0), \tag{6.25}$$

by comparison with (6.24a) at $t = 0$. That is, the initial condition simply translates at a speed c, with no change in structure.

Now consider the case for which the disturbance is a *wave packet* — essentially a nearly plane wave or superposition of waves that is confined to a finite region of space. We will consider a case with the initial condition

$$\psi(x,0) = a(x)\mathrm{e}^{\mathrm{i}k_0 x}, \tag{6.26}$$

where $a(x,t)$, rather like the envelope in Fig. 6.3, modulates the amplitude of the wave on a scale much longer than that of the wavelength $2\pi/k_0$, and more slowly than the wave period. That is,

$$\frac{1}{a}\frac{\partial a}{\partial x} \ll k_0, \qquad \frac{1}{a}\frac{\partial a}{\partial t} \ll k_0 c, \tag{6.27a,b}$$

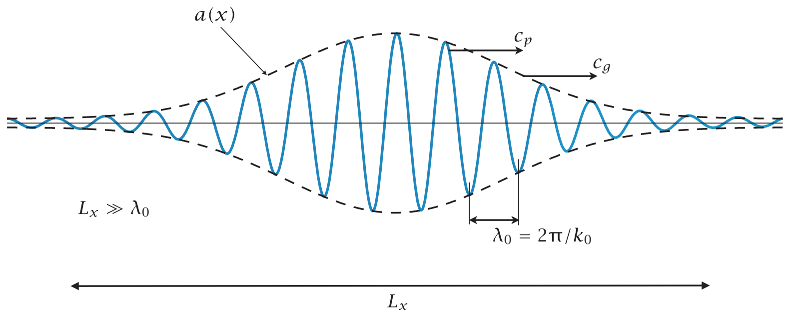

Fig. 6.3 A wave packet. The envelope, $a(x)$, has a scale, L_x, much larger than the wavelength, λ_0, of the wave embedded inside. The envelope moves at the group velocity, c_g, and the phase of the waves at the phase speed, c_p.

and the disturbance is essentially a slowly modulated plane wave. We suppose that $a(x, 0)$ is peaked around some value x_0 and is very small if $|x-x_0| \gg k_0^{-1}$; that is, $a(x, 0)$ is small if we are sufficiently many wavelengths of the plane wave away from the peak, as is the case in Fig. 6.3. We would like to know how such a packet evolves.

We can express the envelope as a Fourier integral by first writing the initial conditions as a Fourier integral,

$$\psi(x,0) = \int_{-\infty}^{\infty} \widetilde{A}(k) \mathrm{e}^{\mathrm{i}kx} \, \mathrm{d}k \quad \text{where} \quad \widetilde{A}(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \psi(x,0) \mathrm{e}^{-ikx} \, \mathrm{d}x, \tag{6.28a,b}$$

so that, using (6.26),

$$\widetilde{A}(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} a(x,0) \mathrm{e}^{\mathrm{i}(k_0-k)x} \, \mathrm{d}x \quad \text{and} \quad a(x) = \int_{-\infty}^{\infty} \widetilde{A}(k) \mathrm{e}^{\mathrm{i}(k-k_0)x} \, \mathrm{d}k. \tag{6.29a,b}$$

We still haven't made much progress beyond (6.24). To do so, we note first that $a(x)$ is confined in space, so that to a good approximation the limits of the integral in (6.29a) can be made finite, $\pm L$ say, provided $L \gg k_0^{-1}$. We then note that when $(k_0 - k)$ is large the integrand in (6.29a) oscillates rapidly; successive intervals in x therefore cancel each other and make a small net contribution to the integral. Thus, the integral is dominated by values of k near k_0, and $\widetilde{A}(k)$ is peaked near k_0. (The finite spatial extent of $a(x, 0)$ is needed for this argument.)

We can now evaluate how the wave packet evolves. Beginning with (6.24a) we have

$$\psi(x,t) = \int_{-\infty}^{\infty} \widetilde{A}(k) \exp\{\mathrm{i}(kx - \omega(k)t)\} \, \mathrm{d}k \tag{6.30a}$$

$$\approx \int_{-\infty}^{\infty} \widetilde{A}(k) \exp\left\{ \mathrm{i}[k_0 x - \omega(k_0)t] + \mathrm{i}(k-k_0)x - \mathrm{i}(k-k_0) \left.\frac{\partial \omega}{\partial k}\right|_{k=k_0} t \right\} \mathrm{d}k, \tag{6.30b}$$

having expanded $\omega(k)$ in a Taylor series about k_0, using (6.23). We thus have

$$\begin{aligned} \psi(x,t) &= \exp\{\mathrm{i}[k_0 x - \omega(k_0)t]\} \int \int_{-\infty}^{\infty} \widetilde{A}(k) \exp\left\{ \mathrm{i}(k-k_0) \left[x - \left.\frac{\partial \omega}{\partial k}\right|_{k=k_0} t \right] \right\} \mathrm{d}k \\ &= \exp\{\mathrm{i}[k_0 x - \omega(k_0)t]\}\, a(x - c_g t), \end{aligned} \tag{6.31}$$

using (6.29b) and where $c_g = \partial\omega/\partial k$ evaluated at $k = k_0$. That is to say, the envelope $a(x, t)$ moves at the group velocity, keeping its initial shape.

The group velocity has a meaning beyond that implied by the derivation above: it turns out to be a quite general property of waves that energy (and certain other quadratic properties) propagate at the group velocity. This is to be expected, at least in the presence of coherent wave packets, because there is no energy outside the wave envelope so the energy must propagate with the envelope.

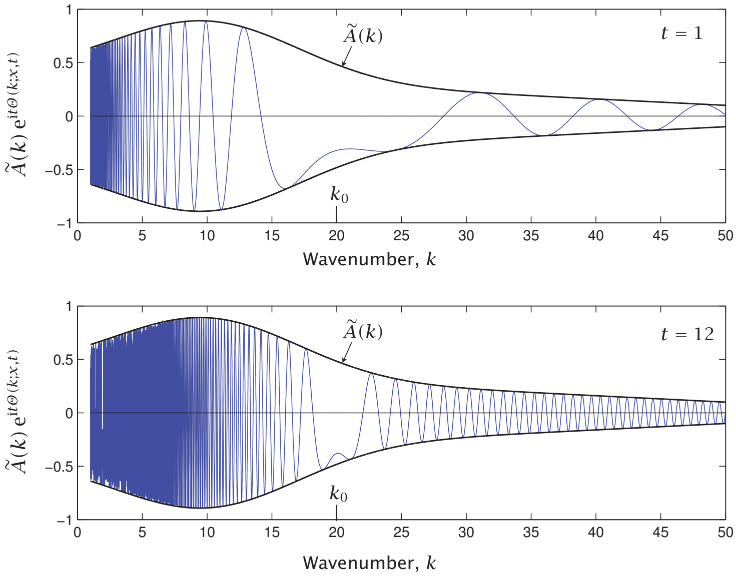

Fig. 6.4 The integrand of (6.32), namely the function that when integrated over wavenumber gives the wave amplitude at a particular x and t.

The example shown is for a Rossby wave with $\omega = -\beta/k$, with $\beta = 400$ and $x/t = 1$, and hence $k_0 = 20$, for two times $t = 1$ and $t = 12$. (The amplitude of the envelope, $\tilde{A}(k)$ is arbitrary.) At the later time the oscillations are much more rapid in k, so that the contribution is more peaked from wavenumbers near to k_0.

6.2.3 ✦ The Method of Stationary Phase

We will now relax the assumption that wavenumbers are confined to a narrow band but (since there is no free lunch) we confine ourselves to seeking solutions at large t; that is, we will be seeking a description of waves far from their source. Consider a disturbance of the general form

$$\psi(x,t) = \int_{-\infty}^{\infty} \tilde{A}(k)\mathrm{e}^{\mathrm{i}[kx-\omega(k)t]}\,\mathrm{d}k = \int_{-\infty}^{\infty} \tilde{A}(k)\mathrm{e}^{\mathrm{i}\Theta(k;x,t)t}\,\mathrm{d}k, \tag{6.32}$$

where $\Theta(k;x,t) \equiv kx/t - \omega(k)$. (Here we regard Θ as a function of k with parameters x and t; we will sometimes just write $\Theta(k)$ with $\Theta'(k) = \partial\Theta/\partial k$.) Now, a standard result in mathematics (known as the 'Riemann–Lebesgue lemma') states that

$$I = \lim_{t\to\infty} \int_{-\infty}^{\infty} f(k)\mathrm{e}^{\mathrm{i}kt}\,\mathrm{d}k = 0, \tag{6.33}$$

provided that $f(k)$ is integrable and $\int_{-\infty}^{\infty} f(k)\,\mathrm{d}k$ is finite. Intuitively, as t increases the oscillations in the integral increase and become much faster than any variation in $f(k)$; successive oscillations thus cancel and the integral becomes very small.

Looking at (6.32), with $\tilde{A}$ playing the role of $f(k)$, the integral will be small if Θ is everywhere varying with k. However, if there is a region where Θ does not vary with k — that is, if there is a region where the phase is stationary and $\partial\Theta/\partial k = 0$ — then there will be a contribution to the integral from that region. Thus, for large t, an observer will predominantly see waves for which $\Theta'(k) = 0$ and so, using the definition of Θ, for which

$$\frac{x}{t} = \frac{\partial\omega}{\partial k}. \tag{6.34}$$

In other words, at some space-time location (x,t) the waves that dominate are those whose group velocity $\partial\omega/\partial k$ is x/t. An example is plotted in Fig. 6.4 with a dispersion relation $\omega = -\beta/k$; the wavenumber that dominates, k_0 say, is thus given by solving $\beta/k_0^2 = x/t$, which for $x/t = 1$ and $\beta = 400$ gives $k_0 = 20$.

We may actually approximately calculate the contribution to $\psi(x,t)$ from waves moving with the group velocity. Let us expand $\Theta(k)$ around the point, k_0, where $\Theta'(k_0) = 0$. We obtain

$$\psi(x,t) = \int_{-\infty}^{\infty} \widetilde{A}(k) \exp\left\{\mathrm{i}t\left[\Theta(k_0) + (k-k_0)\Theta'(k_0) + \tfrac{1}{2}(k-k_0)^2\Theta''(k_0)\ldots\right]\right\} \mathrm{d}k \tag{6.35}$$

The higher order terms are small because $k - k_0$ is presumed small (for if it is large the integral vanishes), and the term involving $\Theta'(k_0)$ is zero. The integral becomes

$$\psi(x,t) = \widetilde{A}(k_0)\mathrm{e}^{\mathrm{i}\Theta(k_0)} \int_{-\infty}^{\infty} \exp\left\{\mathrm{i}t\tfrac{1}{2}(k-k_0)^2\Theta''(k_0)\right\} \mathrm{d}k. \tag{6.36}$$

We thus have to evaluate a Gaussian, and because $\int_{-\infty}^{\infty} \mathrm{e}^{-cx^2}\,\mathrm{d}x = \sqrt{\pi/c}$ we obtain

$$\psi(x,t) \approx \widetilde{A}(k_0)\mathrm{e}^{\mathrm{i}\Theta(k_0)t}\left[\frac{-2\pi}{(\mathrm{i}t\theta''(k_0))}\right]^{1/2} = \widetilde{A}(k_0)\left[\frac{2\mathrm{i}\pi}{(t\theta''(k_0))}\right]^{1/2} \mathrm{e}^{\mathrm{i}(k_0x-\omega(k_0)t)}. \tag{6.37}$$

The solution is therefore a plane wave, with wavenumber k_0 and frequency $\omega(k_0)$, slowly modulated by an envelope determined by the form of $\Theta(k_0; x,t)$, where k_0 is the wavenumber such that $x/t = c_g = \partial\omega/\partial k|_{k=k_0}$.

6.3 RAY THEORY

Most waves propagate in a medium that is inhomogeneous; for example, in the Earth's atmosphere and ocean the stratification varies with altitude and the Coriolis parameter varies with latitude. In these cases it can be hard to obtain the solution of a wave problem by Fourier methods, even approximately. Nonetheless, the idea of signals propagating at the group velocity is a robust one, and we can often obtain some of the information we want — and in particular the trajectory of a wave — using a recipe known as *ray theory*.[3]

6.3.1 Introduction

In an inhomogeneous medium let us suppose that the solution to a particular wave problem is

$$\psi(\boldsymbol{x},t) = a(\boldsymbol{x},t)\mathrm{e}^{\mathrm{i}\theta(\boldsymbol{x},t)}, \tag{6.38}$$

where a is the wave amplitude and θ the phase, and a varies slowly in a sense we will make more precise shortly. The local wavenumber and frequency are defined by,

$$k_i \equiv \frac{\partial\theta}{\partial x_i}, \qquad \omega \equiv -\frac{\partial\theta}{\partial t}, \tag{6.39}$$

where the first expression is equivalent to $\boldsymbol{k} \equiv \nabla\theta$ and so $\nabla\times\boldsymbol{k} = 0$. We suppose that the amplitude a varies slowly over a wavelength and a period; that is $|\Delta a|/|a|$ is small over the length $1/k$ and the period $1/\omega$ or

$$\frac{|\partial a/\partial x|}{a} \ll |k|, \qquad \frac{|\partial a/\partial t|}{a} \ll \omega, \tag{6.40}$$

and similarly in the other directions. We will assume that the wavenumber and frequency as defined by (6.39) are the same as those that would arise if the medium were homogeneous and a were a constant. Thus, we may obtain a local dispersion relation from the governing equation by keeping the spatially (and possibly temporally) varying parameters fixed and obtain

$$\omega = \Omega(k_i; x_i, t), \tag{6.41}$$

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and then allow x_i and t to vary, albeit slowly.

Let us now consider how the wavevector and frequency might change with position and time. It follows from their definitions above that the wavenumber and frequency are related by

$$\frac{\partial k_i}{\partial t} + \frac{\partial \omega}{\partial x_i} = 0, \tag{6.42}$$

where we use a subscript notation for vectors and repeated indices are summed. Using (6.42) and (6.41) gives

$$\frac{\partial k_i}{\partial t} + \frac{\partial \Omega}{\partial x_i} + \frac{\partial \Omega}{\partial k_j}\frac{\partial k_j}{\partial x_i} = 0 \qquad \text{or} \qquad \frac{\partial k_i}{\partial t} + \frac{\partial \Omega}{\partial x_i} + \frac{\partial \Omega}{\partial k_j}\frac{\partial k_i}{\partial x_j} = 0, \tag{6.43a,b}$$

where to get (6.43b) we use $\partial k_j/\partial x_i = \partial k_i/\partial x_j$, true because $\boldsymbol{k}$ has no curl. Equation (6.43b) may be written as

$$\frac{\partial \boldsymbol{k}}{\partial t} + \boldsymbol{c}_g \cdot \nabla \boldsymbol{k} = -\nabla \Omega, \tag{6.44}$$

where

$$\boldsymbol{c}_g = \frac{\partial \Omega}{\partial \boldsymbol{k}} = \left(\frac{\partial \Omega}{\partial k}, \frac{\partial \Omega}{\partial l}, \frac{\partial \Omega}{\partial m} \right) \tag{6.45}$$

is, once more, the group velocity. The left-hand side of (6.44) is similar to an advective derivative, but uses the group velocity. Evidently, if the dispersion relation for frequency is not an explicit function of space then *the wavevector is propagated at the group velocity.*

The frequency is, in general, a function of space, wavenumber and time, and from the dispersion relation, (6.41), its variation is governed by

$$\frac{\partial \omega}{\partial t} = \frac{\partial \Omega}{\partial t} + \frac{\partial \Omega}{\partial k_i}\frac{\partial k_i}{\partial t} = \frac{\partial \Omega}{\partial t} - \frac{\partial \Omega}{\partial k_i}\frac{\partial \omega}{\partial x_i}, \tag{6.46}$$

using (6.42). Using the definition of group velocity, we may write (6.46) as

$$\frac{\partial \omega}{\partial t} + \boldsymbol{c}_g \cdot \nabla \omega = \frac{\partial \Omega}{\partial t}. \tag{6.47}$$

As with (6.44) the left-hand side is like an advective derivative, but uses a group velocity. Thus, if the dispersion relation is not a function of time, the frequency also propagates at the group velocity.

Motivated by (6.44) and (6.47) we define a *ray* as the trajectory traced by the group velocity, and we see that if the function Ω is not an explicit function of space or time, then *both the wavevector and the frequency are constant along a ray.*

6.3.2 Ray Theory in Practice

What use is ray theory? The idea is that we may use (6.44) and (6.47) to track a group of waves from one location to another without solving the full wave equations of motion. We can then sometimes solve interesting problems using ordinary differential equations (ODEs) rather than partial differential equations (PDEs).

Suppose that the initial conditions consist of a group of waves at a position x_0, for which the amplitude and wavenumber vary only slowly with position. We also suppose that we know the dispersion relation for the waves at hand; that is, we know the functional form of $\Omega(k; x, t)$. Now, the total derivative following the group velocity is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \boldsymbol{c}_g \cdot \nabla, \tag{6.48}$$

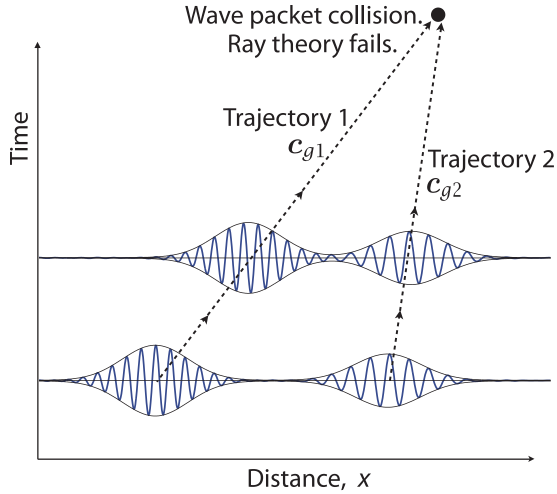

Fig. 6.5 Idealised trajectory of two wavepackets, each with a different wavelength and moving with a different group velocity, as might be calculated using ray theory. If the wave packets collide ray theory must fail. Ray theory gives only the trajectory of the wave packet, not the detailed structure of the waves within a packet.

so that (6.44) and (6.47) may be written as

$$\frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t} = -\nabla\Omega, \qquad \frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{\partial\Omega}{\partial t}. \tag{6.49a,b}$$

These are ordinary differential equations for wavevector and frequency, solvable provided we know the right-hand sides — that is, provided we know the space and time location at which the dispersion relation [i.e., $\Omega(k; x, t)$] is to be evaluated. But the location *is* known because it is moving with the group velocity and so

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{c}_g\,, \tag{6.49c}$$

where $\boldsymbol{c}_g = \partial\Omega/\partial\boldsymbol{k}|_{\boldsymbol{x},t}$ (i.e., $c_{gi} = \partial\Omega/\partial k_i|_{\boldsymbol{x},t}$). The set (6.49a) is a triplet of ordinary differential equations for the wavevector, frequency and position of a wave group. The equations may be solved, albeit sometimes numerically, to give the trajectory of a wave packet or collection of wave packets as schematically illustrated in Fig. 6.5. Of course, if the medium or the wavepacket amplitude is not slowly varying ray theory will fail, and this will perforce happen if two wave packets collide.

The evolution of the amplitude of the wave packet is not given by ray theory. However, the evolution of a related quantity — the wave activity — may be calculated if the group velocity is known. In Section 6.7 we will find that the wave activity, $\mathcal{A}$, satisfies $\partial\mathcal{A}/\partial t + \nabla\cdot(\boldsymbol{c}_g\mathcal{A}) = 0$; that is, the flux of wave activity is along a ray. Another way to calculate the evolution of a wave and its amplitude in a varying medium is to use 'WKB theory' — see Appendix A. Before all that we shift gears and turn our attention to a specific wave, the Rossby wave, but the reader whose interest is more in the general properties of waves may skip forward to Section 6.7.

6.4 ROSSBY WAVES

Rossby waves are perhaps the most important large-scale wave in the atmosphere and ocean, although gravity waves rival them in some contexts.[4] They are most easily described using the quasi-geostrophic equations, as follows.

6.4.1 The Linear Equation of Motion

The relevant equation of motion is the inviscid, adiabatic potential vorticity equation in the quasi-geostrophic system, as discussed in Chapter 5, namely

$$\frac{\partial q}{\partial t} + \boldsymbol{u}\cdot\nabla q = 0, \tag{6.50}$$

where $q(x, y, z, t)$ is the potential vorticity and $\boldsymbol{u}(x, y, z, t)$ is the horizontal velocity. The velocity is related to a streamfunction by $u = -\partial\psi/\partial y$, $v = \partial\psi/\partial x$ and the potential vorticity is some function of the streamfunction, which might differ from system to system. Two examples, one applying to a continuously stratified system and the second to a single layer system, are

$$q = f + \zeta + \frac{\partial}{\partial z}\left(S(z)\frac{\partial\psi}{\partial z}\right), \qquad q = \zeta + f - k_d^2\psi, \tag{6.51a,b}$$

where $S(z) = f_0^2/N^2$, $\zeta = \nabla^2\psi$ is the relative vorticity and $k_d = 1/L_d$ is the inverse radius of deformation for a shallow water system. (Note that definitions of k_d and L_d can vary, typically by factors of 2, π, etc.) Boundary conditions may be needed to form a complete system.

We now *linearize* (6.50); that is, we suppose that the flow consists of a time-independent component (the 'basic state') plus a perturbation, with the perturbation being small compared with the mean flow. The basic state must satisfy the time-independent equation of motion, and it is common and useful to linearize about a zonal flow, $\overline{u}(y, z)$. The basic state is then purely a function of y and so we write

$$q = \overline{q}(y, z) + q'(x, y, t), \qquad \psi = \overline{\psi}(y, z) + \psi'(x, y, z, t) \tag{6.52}$$

with a similar notation for the other variables. Note that $\overline{u} = -\partial\overline{\psi}/\partial y$ and $\overline{v} = 0$. Substituting into (6.50) gives, without approximation,

$$\frac{\partial q'}{\partial t} + \overline{\boldsymbol{u}} \cdot \nabla\overline{q} + \overline{\boldsymbol{u}} \cdot \nabla q' + \boldsymbol{u}' \cdot \nabla\overline{q} + \boldsymbol{u}' \cdot \nabla q' = 0. \tag{6.53}$$

The primed quantities are presumptively small so we neglect terms involving their products. Further, we are assuming that we are linearizing about a state that is a solution of the equations of motion, so that $\overline{\boldsymbol{u}} \cdot \nabla\overline{q} = 0$. Finally, since $\overline{v} = 0$ and $\partial\overline{q}/\partial x = 0$ we obtain

$$\boxed{\frac{\partial q'}{\partial t} + \overline{u}\frac{\partial q'}{\partial x} + v'\frac{\partial\overline{q}}{\partial y} = 0.} \tag{6.54}$$

This equation or one very similar appears very commonly in studies of Rossby waves. To proceed, let us consider the simple example of waves in a single layer.

6.4.2 Waves in a Single Layer

Consider a system obeying (6.50) and (6.51b). The equation could be written in spherical coordinates with $f = 2\Omega\sin\vartheta$, but the dynamics are more easily illustrated on a Cartesian β-plane for which $f = f_0 + \beta y$, and since f_0 is a constant it does not appear in our subsequent derivations.

Infinite deformation radius

If the scale of motion is much less than the deformation scale then we make the approximation that $k_d = 0$ and the equation of motion may be written as

$$\frac{\partial\zeta}{\partial t} + \boldsymbol{u} \cdot \nabla\zeta + \beta v = 0. \tag{6.55}$$

We linearize about a constant zonal flow, $\overline{u}$, by writing

$$\psi = \overline{\psi}(y) + \psi'(x, y, t), \tag{6.56}$$

where $\overline{\psi} = -\overline{u}y$. Substituting(6.56) into (6.55) and neglecting the nonlinear terms involving products of ψ' gives

$$\frac{\partial}{\partial t}\nabla^2\psi' + \overline{u}\frac{\partial\nabla^2\psi'}{\partial x} + \beta\frac{\partial\psi'}{\partial x} = 0. \tag{6.57}$$

This equation is just a single-layer version of (6.54), with $\partial\overline{q}/\partial y = \beta$, $q' = \nabla^2\psi'$ and $v' = \partial\psi'/\partial x$.

The coefficients in (6.57) are not functions of y or z; this is not a requirement for wave motion to exist but it does enable solutions to be found more easily. Let us seek solutions in the form of a plane wave, namely

$$\psi' = \operatorname{Re}\widetilde{\psi}\mathrm{e}^{\mathrm{i}(kx+ly-\omega t)}, \tag{6.58}$$

where $\widetilde{\psi}$ is a complex constant. Solutions of this form are valid in a domain with doubly-periodic boundary conditions; solutions in a channel can be obtained using a meridional variation of $\sin ly$, with no essential changes to the dynamics. The amplitude of the oscillation is given by $\widetilde{\psi}$ and the phase by $kx + ly - \omega t$, where k and l are the x- and y-wavenumbers and ω is the frequency of the oscillation.

Substituting (6.58) into (6.57) yields

$$[(-\omega + \overline{u}k)(-K^2) + \beta k]\widetilde{\psi} = 0, \tag{6.59}$$

where $K^2 = k^2 + l^2$. For non-trivial solutions the above equation implies

$$\boxed{\omega = \overline{u}k - \frac{\beta k}{K^2}}\,, \tag{6.60}$$

and this is the *dispersion relation* for barotropic Rossby waves. Evidently the velocity U Doppler shifts the frequency by the amount Uk. The components of the phase speed and group velocity are given by, respectively,

$$c_p^x \equiv \frac{\omega}{k} = \overline{u} - \frac{\beta}{K^2}, \qquad c_p^y \equiv \frac{\omega}{l} = \overline{u}\frac{k}{l} - \frac{\beta k}{K^2 l}, \tag{6.61a,b}$$

and

$$c_g^x \equiv \frac{\partial\omega}{\partial k} = \overline{u} + \frac{\beta(k^2 - l^2)}{(k^2 + l^2)^2}, \qquad c_g^y \equiv \frac{\partial\omega}{\partial l} = \frac{2\beta kl}{(k^2 + l^2)^2}. \tag{6.62a,b}$$

The phase speed in the absence of a mean flow is *westward,* with waves of longer wavelengths travelling more quickly, and the eastward current speed required to hold the waves of a particular wavenumber stationary (i.e., $c_p^x = 0$) is $U = \beta/K^2$. The background flow $\overline{u}$ evidently just provides a uniform shift to the phase speed, and (in this case) can be transformed away by a change of coordinate. The x-component of the group velocity may also be written as the sum of the phase speed plus a positive quantity, namely

$$c_g^x = c_p^x + \frac{2\beta k^2}{(k^2 + l^2)^2}. \tag{6.63}$$

This means that the zonal group velocity for Rossby wave packets moves eastward faster than its zonal phase speed. A stationary wave ($c_p^x = 0$) can only propagate eastward, and this has implications for the 'downstream development' of Rossby wave packets.[5]

Finite deformation radius

For a finite deformation radius the basic state $\Psi = -\overline{u}y$ is still a solution of the original equations of motion, but the potential vorticity corresponding to this state is $q = \overline{u}yk_d^2 + \beta y$ and its gradient is $\nabla q = (\beta + \overline{u}k_d^2)\mathbf{j}$. The linearized equation of motion is thus

$$\left(\frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x}\right)(\nabla^2\psi' - \psi' k_d^2) + (\beta + \overline{u}k_d^2)\frac{\partial \psi'}{\partial x} = 0. \tag{6.64}$$

Substituting $\psi' = \widetilde{\psi}e^{i(kx+ly-\omega t)}$ we obtain the dispersion relation,

$$\boxed{\omega = \frac{k(\overline{u}K^2 - \beta)}{K^2 + k_d^2} = \overline{u}k - k\frac{\beta + \overline{u}k_d^2}{K^2 + k_d^2}\,.} \tag{6.65}$$

The corresponding components of phase speed and group velocity are

$$c_p^x = \overline{u} - \frac{\beta + \overline{u}k_d^2}{K^2 + k_d^2} = \frac{\overline{u}K^2 - \beta}{K^2 + k_d^2}, \qquad c_p^y = \overline{u}\frac{k}{l} - \frac{k}{l}\left(\frac{\overline{u}K^2 - \beta}{K^2 + k_d^2}\right), \tag{6.66a,b}$$

and

$$c_g^x = \overline{u} + \frac{(\beta + \overline{u}k_d^2)(k^2 - l^2 - k_d^2)}{(k^2 + l^2 + k_d^2)^2}\,, \qquad c_g^y = \frac{2kl(\beta + \overline{u}k_d^2)}{(k^2 + l^2 + k_d^2)^2}\,. \tag{6.67a,b}$$

The uniform velocity field now no longer provides just a simple Doppler shift of the frequency, nor a uniform addition to the phase speed. From (6.66a) the waves are stationary when $K^2 = \beta/\overline{u} \equiv K_s^2$; that is, the current speed required to hold waves of a particular wavenumber stationary is $\overline{u} = \beta/K^2$. However, this is *not* simply the magnitude of the phase speed of waves of that wavenumber in the absence of a current — this is given by

$$c_p^x = \frac{-\beta}{K_s^2 + k_d^2} = \frac{-\overline{u}}{1 + k_d^2/K_s^2} \neq -\overline{u}. \tag{6.68}$$

Why is there a difference? It is because the current does not just provide a uniform translation, but, if k_d is non-zero, it also modifies the basic potential vorticity gradient. The basic state height field η_0 is sloping; that is $\eta_0 = -(f_0/g)\overline{u}y$, and the ambient potential vorticity field increases with y and $q = (\beta + Uk_d^2)y$. Thus, the basic state defines a preferred frame of reference, and the problem is not Galilean invariant.[6]

We also note that, from (6.67a), the group velocity is negative (westward) if the x-wavenumber is sufficiently small compared to the y-wavenumber or the deformation wavenumber. That is, said a little loosely, *long waves move information westward and short waves move information eastward*, and this is a common property of Rossby waves. The x-component of the phase speed, on the other hand, is always westward relative to the mean flow.

6.4.3 The Mechanism of Rossby Waves

The fundamental mechanism underlying Rossby waves may be understood as follows. Consider a material line of stationary fluid parcels along a line of constant latitude, and suppose that some disturbance causes their displacement to the line marked $\eta(t = 0)$ in Fig. 6.6. In the displacement, the potential vorticity of the fluid parcels is conserved, and in the simplest case of barotropic flow on the β-plane the potential vorticity is the absolute vorticity, $\beta y + \zeta$. Thus, in either hemisphere, a northward displacement leads to the production of negative relative vorticity and a southward displacement leads to the production of positive relative vorticity. The relative vorticity gives rise

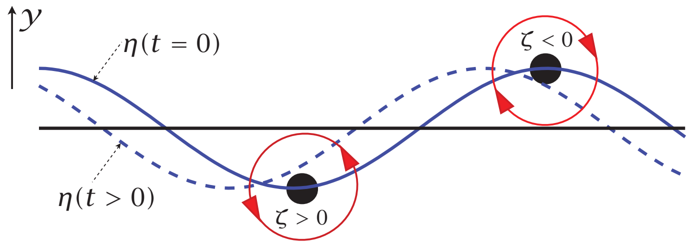

Fig. 6.6 A two-dimensional (x–y) Rossby wave. An initial disturbance displaces a material line at constant latitude (the straight horizontal line) to the solid line marked $\eta(t = 0)$. Conservation of potential vorticity, $\beta y + \zeta$, leads to the production of relative vorticity, ζ, as shown. The associated velocity field (arrows on the circles) then advects the fluid parcels, and the material line evolves into the dashed line with the phase propagating westward.

to a velocity field which, in turn, advects the parcels in the material line in the manner shown, and the wave propagates westwards.

In more complicated situations, such as flow in two layers, considered below, or in a continuously stratified fluid, the mechanism is essentially the same. A displaced fluid parcel carries with it its potential vorticity and, in the presence of a potential vorticity gradient in the basic state, a potential vorticity anomaly is produced. The potential vorticity anomaly produces a velocity field (an example of potential vorticity inversion) which further displaces the fluid parcels, leading to the formation of a Rossby wave. The vital ingredient is a basic state potential vorticity gradient, such as that provided by the change of the Coriolis parameter with latitude.

6.4.4 Rossby Waves in Two Layers

Now consider the dynamics of the two-layer model, linearized about a state of rest. The two, coupled, linear equations describing the motion in each layer are

$$\frac{\partial}{\partial t}\left[\nabla^2\psi_1' + F_1(\psi_2' - \psi_1')\right] + \beta\frac{\partial \psi_1'}{\partial x} = 0, \tag{6.69a}$$

$$\frac{\partial}{\partial t}\left[\nabla^2\psi_2' + F_2(\psi_1' - \psi_2')\right] + \beta\frac{\partial \psi_2'}{\partial x} = 0, \tag{6.69b}$$

where $F_1 = f_0^2/g'H_1$ and $F_2 = f_0^2/g'H_2$. By inspection (6.69) may be transformed into two uncoupled equations: the first is obtained by multiplying (6.69a) by F_2 and (6.69b) by F_1 and adding, and the second is the difference of (6.69a) and (6.69b). Then, defining

$$\overline{\psi} = \frac{F_1\psi_2' + F_2\psi_1'}{F_1 + F_2}, \qquad \tau = \frac{1}{2}(\psi_1' - \psi_2'), \tag{6.70a,b}$$

(think 'τ for temperature'), (6.69) become

$$\frac{\partial}{\partial t}\nabla^2\overline{\psi} + \beta\frac{\partial \overline{\psi}}{\partial x} = 0, \tag{6.71a}$$

$$\frac{\partial}{\partial t}\left[(\nabla^2 - k_d^2)\tau\right] + \beta\frac{\partial \tau}{\partial x} = 0, \tag{6.71b}$$

where now $k_d = (F_1 + F_2)^{1/2}$. The internal radius of deformation for this problem is the inverse of this, namely

$$L_d = k_d^{-1} = \frac{1}{f_0}\left(\frac{g'H_1H_2}{H_1 + H_2}\right)^{1/2}. \tag{6.72}$$

The variables $\overline{\psi}$ and τ are the *normal modes* for the two-layer model, as they oscillate independently of each other. (For the continuous equations the analogous modes are the eigenfunctions of $\partial_z[(f_0^2/N^2)\partial_z\phi] = \lambda^2\phi$.) The equation for $\overline{\psi}$, *the barotropic mode,* is identical to that of the single-layer, rigid-lid model, namely (6.57) with $U = 0$, and its dispersion relation is just

$$\omega = -\frac{\beta k}{K^2}. \tag{6.73}$$

The barotropic mode corresponds to synchronous, depth-independent, motion in the two layers, with no undulations in the dividing interface.

The displacement of the interface is given by $2f_0\tau/g'$ and so is proportional to the amplitude of τ, the *baroclinic mode*. The dispersion relation for the baroclinic mode is

$$\omega = -\frac{\beta k}{K^2 + k_d^2}. \tag{6.74}$$

The mass transport associated with this mode is identically zero, since from (6.70) we have

$$\psi_1 = \overline{\psi} + \frac{2F_1\tau}{F_1 + F_2}, \qquad \psi_2 = \overline{\psi} - \frac{2F_2\tau}{F_1 + F_2}, \tag{6.75a,b}$$

and this implies

$$H_1\psi_1 + H_2\psi_2 = (H_1 + H_2)\overline{\psi}. \tag{6.76}$$

The left-hand side is proportional to the total mass transport, which is evidently associated with the barotropic mode.

The dispersion relation and associated group and phase velocities are plotted in Fig. 6.7. The x-component of the phase speed, ω/k, is negative (westwards) for both baroclinic and barotropic Rossby waves. The group velocity of the barotropic waves is always positive (eastwards), but the group velocity of long baroclinic waves may be negative (westwards). For very short waves, $k^2 \gg k_d^2$, the baroclinic and barotropic velocities coincide and their phase and group velocities are equal and opposite. With a deformation radius of 50 km, typical for the mid-latitude ocean, a nondimensional frequency of unity in the figure corresponds to a dimensional frequency of $5\times10^{-7}\,\mathrm{s}^{-1}$ or a period of about 100 days. In an atmosphere with a deformation radius of 1000 km a nondimensional frequency of unity corresponds to $1\times10^{-5}\,\mathrm{s}^{-1}$ or a period of about 7 days. nondimensional velocities of unity correspond to respective dimensional velocities of about $0.25\,\mathrm{m\,s^{-1}}$ (ocean) and $10\,\mathrm{m\,s^{-1}}$ (atmosphere).

The deformation radius only affects the baroclinic mode. For scales much smaller than the deformation radius, $K^2 \gg k_d^2$, we see from (6.71b) that the baroclinic mode obeys the same equation as the barotropic mode so that

$$\frac{\partial}{\partial t}\nabla^2\tau + \beta\frac{\partial \tau}{\partial x} = 0. \tag{6.77}$$

Using this and (6.71a) implies that

$$\frac{\partial}{\partial t}\nabla^2\psi_i + \beta\frac{\partial \psi_i}{\partial x} = 0, \qquad i = 1, 2. \tag{6.78}$$

That is to say, the two layers themselves are uncoupled from each other. At the other extreme, for very long baroclinic waves the relative vorticity is unimportant.

6.5 ROSSBY WAVES IN STRATIFIED QUASI-GEOSTROPHIC FLOW

6.5.1 Preliminaries

Let us now consider the dynamics of linear waves in stratified quasi-geostrophic flow on a β-plane, with a resting basic state. (In Chapter 16 we explore the role of Rossby waves in a more realistic

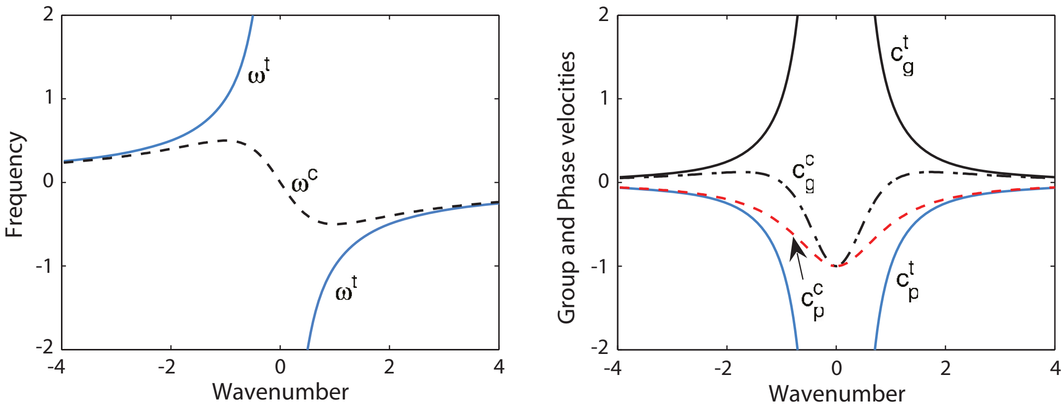

Fig. 6.7 Left: the dispersion relation for barotropic (ω^t, solid line) and baroclinic (ω^c, dashed line) Rossby waves in the two-layer model, calculated using (6.73) and (6.74) with $k^y = 0$, plotted for both positive and negative zonal wavenumbers and frequencies. The wavenumber is nondimensionalized by k_d, and the frequency is nondimensionalized by β/k_d. Right: the corresponding zonal group and phase velocities, $c_g = \partial\omega/\partial k^x$ and $c_p = \omega/k^x$, with superscript 't' or 'c' for the barotropic or baroclinic mode, respectively. The velocities are nondimensionalized by β/k_d^2.

setting.) The interior flow is governed by the potential vorticity equation, (5.118), and linearizing this about a state of rest gives

$$\frac{\partial}{\partial t}\left[\nabla^2\psi' + \frac{1}{\widetilde{\rho}(z)}\frac{\partial}{\partial z}\left(\widetilde{\rho}(z)F(z)\frac{\partial\psi'}{\partial z}\right)\right] + \beta\frac{\partial\psi'}{\partial x} = 0, \tag{6.79}$$

where $\widetilde{\rho}$ is the density profile of the basic state and $F(z) = f_0^2/N^2$. (F is the square of the inverse Prandtl ratio, N/f_0.) In the Boussinesq approximation $\widetilde{\rho} = \rho_0$, i.e., a constant. The vertical boundary conditions are determined by the thermodynamic equation, (5.120). If the boundaries are flat, rigid, slippery surfaces then $w = 0$ at the boundaries and if there is no surface buoyancy gradient the linearized thermodynamic equation is

$$\frac{\partial}{\partial t}\left(\frac{\partial\psi'}{\partial z}\right) = 0. \tag{6.80}$$

We apply this at the ground and, with somewhat less justification, at the tropopause: we assume the higher static stability of the stratosphere inhibits vertical motion. If the ground is not flat or if friction provides a vertical velocity by way of an Ekman layer, the boundary condition must be modified, but we will stay with the simplest case and apply (6.80) at $z = 0$ and $z = H$.

6.5.2 Wave Motion

As in the single-layer case, we seek solutions of the form

$$\psi' = \mathrm{Re}\,\widetilde{\psi}(z)\mathrm{e}^{\mathrm{i}(kx+ly-\omega t)}, \tag{6.81}$$

where $\widetilde{\psi}(z)$ will determine the vertical structure of the waves. The case of a sphere is more complicated but introduces no truly new physical phenomena.

Substituting (6.81) into (6.79) gives

$$\omega\left[-K^2\widetilde{\psi}(z) + \frac{1}{\widetilde{\rho}}\frac{\mathrm{d}}{\mathrm{d}z}\left(\widetilde{\rho}F(z)\frac{\mathrm{d}\widetilde{\psi}}{\mathrm{d}z}\right)\right] - \beta k\widetilde{\psi}(z) = 0. \tag{6.82}$$

Now, let us suppose that $\widetilde{\psi}$ satisfies

$$\frac{1}{\widetilde{\rho}}\frac{\mathrm{d}}{\mathrm{d}z}\left(\widetilde{\rho}F(z)\frac{\mathrm{d}\widetilde{\psi}}{\mathrm{d}z}\right) = -\Gamma\widetilde{\psi}, \tag{6.83}$$

where Γ is some constant (in fact it is an eigenvalue, as discussed below). Equation (6.82) becomes

$$-\omega\left[K^2+\Gamma\right]\widetilde{\psi} - \beta k\widetilde{\psi} = 0, \tag{6.84}$$

and the dispersion relation follows, namely

$$\boxed{\omega = -\frac{\beta k}{K^2+\Gamma}\,.} \tag{6.85}$$

The value of Γ is obtained by solving the eigenvalue problem, (6.83), for the vertical structure; the boundary conditions, derived from (6.80), are $\partial\widetilde{\psi}/\partial z = 0$ at $z = 0$ and $z = H$. The resulting eigenvalues, Γ are proportional to the inverse of the squares of the deformation radii for the problem and the eigenfunctions are the vertical structure functions.

A simple example

Consider the case in which $F(z)$ and $\widetilde{\rho}$ are constant, and in which the domain is confined between two rigid surfaces at $z = 0$ and $z = H$. Then the eigenvalue problem for the vertical structure is

$$F\frac{\mathrm{d}^2\widetilde{\psi}}{\mathrm{d}z^2} = -\Gamma\widetilde{\psi}, \tag{6.86a}$$

with boundary conditions of

$$\frac{\mathrm{d}\widetilde{\psi}}{\mathrm{d}z} = 0, \qquad \text{at } z = 0, H. \tag{6.86b}$$

There is a sequence of solutions to this, namely

$$\widetilde{\psi}_n(z) = \cos(n\pi z/H), \qquad n = 1, 2\ldots, \tag{6.87}$$

with corresponding eigenvalues

$$\Gamma_n = n^2\frac{F\pi^2}{H^2} = (n\pi)^2\left(\frac{f_0}{NH}\right)^2, \qquad n = 1, 2\ldots. \tag{6.88}$$

Equation (6.88) may be used to define the deformation radii for this problem, namely

$$L_n \equiv \frac{1}{\sqrt{\Gamma_n}} = \frac{NH}{n\pi f_0}. \tag{6.89}$$

The first deformation radius is the same as the expression obtained by dimensional analysis, namely NH/f_0, (or NH/f if f varies) except for a factor of π. (Definitions of the deformation radii both with and without the factor of π are common in the literature, and neither is obviously more correct. In the latter case, the first deformation radius in a problem with uniform stratification is given by NH/f_0, equal to $\pi/\sqrt{\Gamma_1}$.) In addition to these baroclinic modes, the case with $n = 0$, that is with $\widetilde{\psi} = 1$, is also a solution of (6.86) for any $F(z)$.

Using (6.85) and (6.88) the dispersion relation becomes

$$\boxed{\omega = -\frac{\beta k}{K^2 + (n\pi)^2(f_0/NH)^2}\,, \qquad n = 0, 1, 2\ldots\,,} \tag{6.90}$$

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and the horizontal wavenumbers k and l are also quantized in a finite domain. The dynamics of the barotropic mode are independent of height and independent of the stratification of the basic state, and so these Rossby waves are *identical* with the Rossby waves in a homogeneous fluid contained between two flat rigid surfaces. The structure of the baroclinic modes, which in general depends on the structure of the stratification, becomes increasingly complex as the vertical wavenumber n increases. This increasing complexity naturally leads to a certain delicacy, making it rare that they can be unambiguously identified in nature. The eigenproblem for a realistic atmospheric profile is further complicated because of the lack of a rigid lid at the top of the atmosphere.[7]

6.6 ENERGY PROPAGATION AND REFLECTION OF ROSSBY WAVES

We now consider how energy is fluxed in Rossby waves. To keep matters algebraically simple we consider waves in a single layer and without a mean flow, but we allow for a finite radius of deformation. To remind ourselves, the dynamics are governed by the evolution of potential vorticity and the linearized evolution equation is

$$\frac{\partial}{\partial t}\left(\nabla^2 - k_d^2\right)\psi + \beta\frac{\partial \psi}{\partial x} = 0. \tag{6.91}$$

The dispersion relation follows in the usual way and is

$$\omega = \frac{-k\beta}{K^2 + k_d^2}, \tag{6.92}$$

which is a simplification of (6.65), and the group velocities are

$$c_g^x = \frac{\beta(k^2 - l^2 - k_d^2)}{(K^2 + k_d^2)^2}, \qquad c_g^y = \frac{2\beta k l}{(K^2 + k_d^2)^2}, \tag{6.93a,b}$$

which are simplifications of (6.67), and as usual $K^2 = k^2 + l^2$.

To obtain an energy equation multiply (6.91) by $-\psi$ to obtain, after a couple of lines of algebra,

$$\frac{1}{2}\frac{\partial}{\partial t}\left((\nabla\psi)^2 + k_d^2\psi^2\right) - \nabla\cdot\left(\psi\nabla\frac{\partial\psi}{\partial t} + \mathbf{i}\frac{\beta}{2}\psi^2\right) = 0, \tag{6.94}$$

where $\mathbf{i}$ is the unit vector in the x-direction. The first group of terms are the energy itself, or more strictly the energy density. (An energy density is an energy per unit mass or per unit volume, depending on the context.) The term $(\nabla\psi)^2/2 = (u^2 + v^2)/2$ is the kinetic energy and $k_d^2\psi^2/2$ is the potential energy, proportional to the displacement of the free surface, squared. The second term is the energy flux, so that we may write

$$\frac{\partial E}{\partial t} + \nabla\cdot\boldsymbol{F} = 0, \tag{6.95}$$

where $E = (\nabla\psi)^2/2 + k_d^2\psi^2$ and $\boldsymbol{F} = -\left(\psi\nabla\partial\psi/\partial t + \mathbf{i}\beta\psi^2\right)$. We haven't yet used the fact that the disturbance has a dispersion relation, and if we do so we may expect, following the derivations of Section 6.2, that the energy moves at the group velocity. Let us now demonstrate this explicitly.

We assume a solution of the form

$$\psi = A(x)\cos(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t) = A(x)\cos(kx + ly - \omega t) \tag{6.96}$$

where $A(x)$ is assumed to vary slowly compared to the nearly plane wave. (Note that $\boldsymbol{k}$ is the wave vector, to be distinguished from $\mathbf{k}$, the unit vector in the z-direction.) The kinetic energy in a wave is given by

$$\mathrm{KE} = \frac{A^2}{2}\left(\psi_x^2 + \psi_y^2\right), \tag{6.97}$$

Essentials of Rossby Waves

- Rossby waves owe their existence to a gradient of potential vorticity in the fluid. If a fluid parcel is displaced, it conserves its potential vorticity and so its relative vorticity will in general change. The relative vorticity creates a velocity field that displaces neighbouring parcels, whose relative vorticity changes and so on.

- A common source of a potential vorticity gradient is differential rotation, or the β-effect and the associated Rossby waves are called *planetary waves*. In the presence of non-zero β the ambient potential vorticity increases northward and the phase of the Rossby waves propagates westward. In general, Rossby waves propagate pseudo-westwards, meaning to the left of the direction of increasing potential vorticity.

- A common equation of motion for Rossby waves is

$$\frac{\partial q'}{\partial t} + \overline{u}\frac{\partial q'}{\partial x} + v'\frac{\partial \overline{q}}{\partial y} = 0, \tag{RW.1}$$

  with an overbar denoting the basic state and a prime a perturbation. In the case of a single layer of fluid with no mean flow this equation becomes

$$\frac{\partial}{\partial t}(\nabla^2 + k_d^2)\psi' + \beta\frac{\partial \psi'}{\partial x} = 0, \tag{RW.2}$$

  with dispersion relation

$$\omega = \frac{-\beta k}{k^2 + l^2 + k_d^2}. \tag{RW.3}$$

- In the absence of a mean flow (i.e., $\overline{u} = 0$), the phase speed in the zonal direction ($c_p^x = \omega/k$) is always negative, or westward, and is larger for large waves. For (RW.3) the components of the group velocity are given by

$$c_g^x = \frac{\beta(k^2 - l^2 - k_d^2)}{(k^2 + l^2 + k_d^2)^2}, \qquad c_g^y = \frac{2\beta kl}{(k^2 + l^2 + k_d^2)^2}. \tag{RW.4}$$

  The group velocity is westward if the zonal wavenumber is sufficiently small, and eastward if the zonal wavenumber is sufficiently large.

- Rossby waves exist in stratified fluids, and have a similar dispersion relation to (RW.3) with an appropriate vertical wavenumber appearing in place of the inverse deformation radius, k_d.

- The reflection of such Rossby waves at a wall is specular, meaning that the group velocity of the reflected wave makes the same angle with the wall as the group velocity of the incident wave. The energy flux of the reflected wave is equal and opposite to that of the incoming wave in the direction normal to the wall.

so that, averaged over a wave period,

$$\overline{\mathrm{KE}} = \frac{A^2}{2}(k^2 + l^2)\frac{\omega}{2\pi}\int_0^{2\pi/\omega} \sin^2(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)\,\mathrm{d}t. \tag{6.98}$$

The time-averaging produces a factor of one half, and applying a similar procedure to the potential energy we obtain

$$\overline{\mathrm{KE}} = \frac{A^2}{4}(k^2 + l^2), \qquad \overline{\mathrm{PE}} = \frac{A^2}{4}k_d^2, \tag{6.99}$$

so that the average total energy is

$$\overline{E} = \frac{A^2}{4}(K^2 + k_d^2), \tag{6.100}$$

where $K^2 = k^2 + l^2$.

The flux, $\boldsymbol{F}$, is given by

$$\boldsymbol{F} = -\left(\psi\nabla\frac{\partial\psi}{\partial t} + \mathbf{i}\frac{\beta}{2}\psi^2\right) = -A^2\cos^2(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)\left(\boldsymbol{k}\omega - \mathbf{i}\frac{\beta}{2}\right), \tag{6.101}$$

so that evidently the energy flux has a component in the direction of the wavevector, $\boldsymbol{k}$, and a component in the x-direction. Averaging over a wave period straightforwardly gives us additional factors of one half:

$$\overline{\boldsymbol{F}} = -\frac{A^2}{2}\left(\boldsymbol{k}\omega + \mathbf{i}\frac{\beta}{2}\right). \tag{6.102}$$

We now use the dispersion relation $\omega = -\beta k/(K^2 + k_d^2)$ to eliminate the frequency, giving

$$\overline{\boldsymbol{F}} = \frac{A^2\beta}{2}\left(\boldsymbol{k}\frac{k}{K^2 + k_d^2} - \mathbf{i}\frac{1}{2}\right), \tag{6.103}$$

and writing this in component form we obtain

$$\overline{\boldsymbol{F}} = \frac{A^2\beta}{4}\left[\mathbf{i}\left(\frac{k^2 - l^2 - k_d^2}{K^2 + k_d^2}\right) + \mathbf{j}\left(\frac{2kl}{K^2 + k_d^2}\right)\right]. \tag{6.104}$$

Comparison of (6.104) with (6.93) and (6.100) reveals that

$$\overline{\boldsymbol{F}} = \mathbf{c}_g\overline{E} \tag{6.105}$$

so that the energy propagation equation (6.95), when averaged over a wave, becomes

$$\boxed{\frac{\partial\overline{E}}{\partial t} + \nabla\cdot\mathbf{c}_g\overline{E} = 0.} \tag{6.106}$$

It is interesting that the variation of A plays no role in the above manipulations, so that the derivation appears to go through if the amplitude $A(\boldsymbol{x}, t)$ is in fact a constant and the wave is a single plane wave. This seems hard to reconcile with our previous discussion, in which we noted that the group velocity was the velocity of a wave *packet* involving a superposition of plane waves. Indeed, the derivative of the frequency with respect to wavenumber means little if there is only one wavenumber. In fact there is nothing wrong with the above derivation if A is a constant and only a single plane wave is present. The resolution of the paradox arises by noting that a plane wave fills all of space and time; in this case there is no convergence of the energy flux and the energy propagation equation is trivially true.

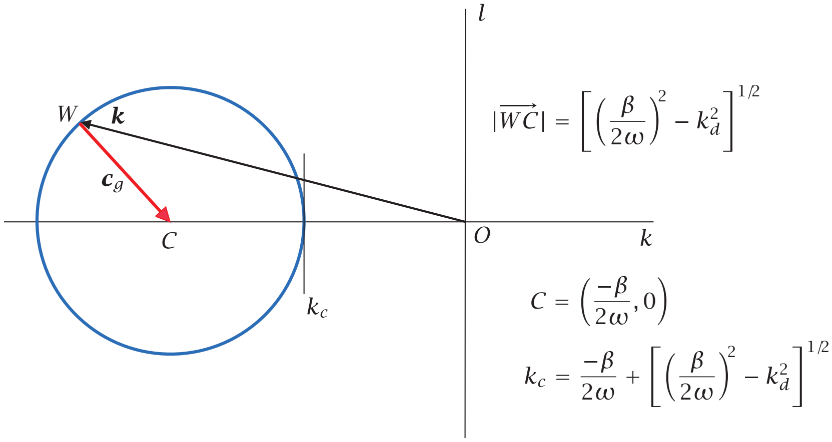

Fig. 6.8 The energy propagation diagram for Rossby waves. The wavevectors of a given frequency all lie in a circle of radius $[(\beta/2\omega)^2 - k_d^2]^{1/2}$, centred at the point C. The closest distance of the circle to the origin is k_c, and if the deformation radius is infinite then $k_c = 0$ and the circle touches the origin. For a given wavenumber $\boldsymbol{k}$, the group velocity is along the line directed from W to C.

6.6.1 ✦ Rossby Wave Reflection

We now consider how Rossby waves might be reflected from a solid boundary. The topic has an obvious oceanographic relevance, for the reflection of Rossby waves turns out to be one way of interpreting why intense oceanic boundary currents form on the western sides of ocean basins, not the east. Rossby waves also reflect off the western boundary in equatorial regions during the El Niño phenomenon. There is also an atmospheric relevance, for meridionally propagating Rossby waves may effectively be reflected as they approach a 'turning latitude' where the meridional wavenumber goes to zero, as considered in Chapter 16. As a preliminary we give a useful graphic interpretation of Rossby wave propagation.[8]

The energy propagation diagram

The dispersion relation for Rossby waves, $\omega = -\beta k/(k^2 + l^2 + k_d^2)$, may be rewritten as

$$(k + \beta/2\omega)^2 + l^2 = (\beta/2\omega)^2 - k_d^2. \tag{6.107}$$

For constant ω this equation is the parametric representation of a circle, meaning that the wavevector (k, l) must lie on a circle centred at the point $(-\beta/2\omega, 0)$ and with radius $[(\beta/2\omega)^2 - k_d^2]^{1/2}$, as illustrated in Fig. 6.8. If k_d is zero the circle touches the origin, and if it is non-zero the distance of the closest point to the circle, k_c say, is given by $k_c = -\beta/2\omega + [(\beta/2\omega)^2 - k_d^2]^{1/2}$. For low frequencies, specifically if $\omega \ll \beta/2k$, then $k_c \approx -\omega k_d^2/\beta$. The radius of the circle is a positive real number only when $\omega < \beta/2k_d$. This is the maximum frequency possible, and it occurs when $l = 0$ and $k = k_d$ and when $c_g^x = c_g^y = 0$.

The group velocity, and hence the energy flux, can be visualized graphically from Fig. 6.8. By direct manipulation of the expressions for group velocity and frequency (Equations (RW.3) and (RW.4) on page 235) we find that

$$c_g^x = \frac{-2\omega}{K^2 + k_d^2}\left(k + \frac{\beta}{2\omega}\right), \qquad c_g^y = \frac{-2\omega}{K^2 + k_d^2}\, l. \tag{6.108a,b}$$

(To check this, it is easiest to begin with the right-hand sides and use the dispersion relation for ω.) Now, since the centre of the circle of wavevectors is at the position $(-\beta/2\omega, 0)$, and referring to

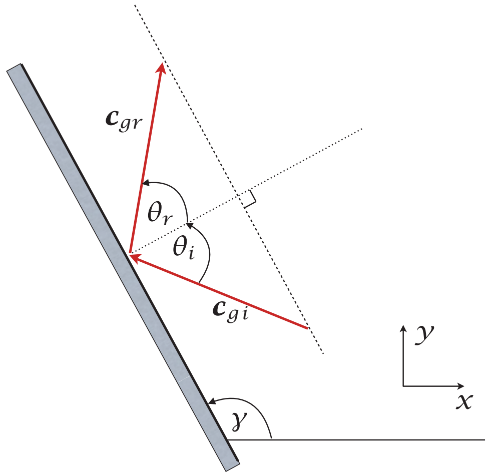

Fig. 6.9 The reflection of a Rossby wave at a western wall, in physical space. A Rossby wave with a westward group velocity impinges at an angle θ_i to a wall, inducing a reflected wave moving eastward at an angle θ_r. The reflection is specular, with $\theta_r = \theta_i$, and energy conserving, with $|\boldsymbol{c}_{gr}| = |\boldsymbol{c}_{gi}|$ — see text and Fig. 6.10.

Fig. 6.8, we have

$$\boldsymbol{c}_g = \frac{2\omega}{K^2 + k_d^2}\boldsymbol{R} \tag{6.109}$$

where $\boldsymbol{R} = \overrightarrow{WC}$ is the vector directed from W to C, that is from the end of the wavevector itself to the centre of the circle around which all the wavevectors lie.

Equation (6.109) and Fig. 6.8 allow for a useful visualization of the energy and phase. The phase propagates in the direction of the wave vector, and for Rossby waves this is always westward. The group velocity is in the direction of the wave vector to the centre of the circle, and this can be either eastward (if $k^2 > l^2 + k_d^2$) or westward ($k^2 < l^2 + k_d^2$). Interestingly, the velocity vector is normal to the wave vector. To see this, consider a purely westward propagating wave for which $l = 0$. Then $v = \partial\psi/\partial x = \mathrm{i}k\widetilde{\psi}$ and $u = -\partial\psi/\partial y = -\mathrm{i}l\widetilde{\psi} = 0$. We now see how some of these properties can help us understand the reflection of Rossby waves.

Reflection at a wall

Consider Rossby waves incident on a wall making an angle γ with the x-axis, and suppose that somehow these waves are reflected back into the fluid interior. This is a reasonable expectation, for the wall cannot normally simply absorb all the wave energy, and if reflection does occur it will have the following two properties:

(i) The incident and reflected wave will have the same wavenumber component along the wall.

(ii) The incident and reflected wave will have the same frequency.

To understand these properties, first consider the case in which the wall is oriented meridionally along the y-axis with $\gamma = 90°$. For our immediate concerns there is loss of generality in this choice, because we may simply choose coordinates so that y is parallel to the wall and the β-effect, which differentiates x from y, does not enter the initial argument. The incident and reflected waves are

$$\psi_i(x, y, t) = A_i \exp\left[\mathrm{i}\left(k_i x + l_i y - \omega_i t\right)\right], \qquad \psi_r(x, y, t) = A_r \exp\left[\mathrm{i}\left(k_r x + l_r y - \omega_r t\right)\right], \tag{6.110}$$

with subscripts i and r denoting incident and reflected. At the wall, which we take to be at $x = 0$, the normal velocity $u = -\partial\psi/\partial y$ must be zero, so that

$$A_i l_i \exp\left[\mathrm{i}\left(l_i y - \omega_i t\right)\right] + A_r l_r \exp\left[\mathrm{i}\left(l_r y - \omega_r t\right)\right] = 0. \tag{6.111}$$

For this equation to hold for all y and all time then we must have

$$l_r = l_i, \qquad \omega_r = \omega_i. \tag{6.112}$$

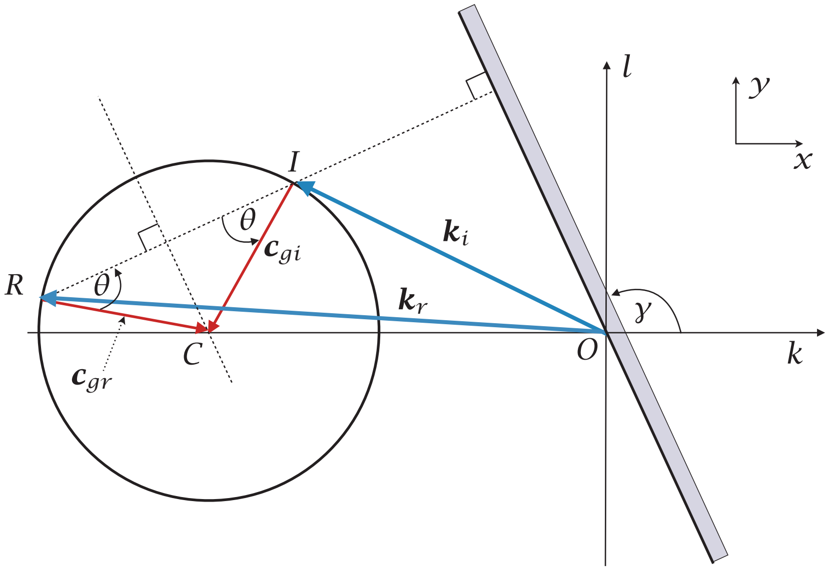

Fig. 6.10 Graphical representation of the reflection of a Rossby wave at a western wall, in spectral space. The incident wave has wavevector $\boldsymbol{k}_i$, ending at point I. Construct the wavevector circle of constant frequency through point I with radius $(\beta/2\omega)^2 - k_d^2)^{1/2}$ and centre $C = (-\beta/2\omega, 0)$; the group velocity vector then lies along $\overrightarrow{IC}$ and is directed westward. The reflected wave has a wavevector $\boldsymbol{k}_r$ such that its projection on the wall is equal to that of $\boldsymbol{k}_i$, and this fixes the point R. The group velocity of the reflected wave then lies along $\overrightarrow{RC}$, and it can be seen that $\boldsymbol{c}_{gr}$ makes the same angle to the wall as does $\boldsymbol{c}_{gi}$, except that it is directed eastward. The reflection is therefore specular and is such that the energy flux directed away from the wall is equal to the energy flux directed toward the wall.

This result is independent of the detailed dynamics of the waves, requiring only that the velocity is determined from a streamfunction. If we consider Rossby-wave dynamics specifically, the x- and y-coordinates are not arbitrary and the y-axis cannot be taken to be aligned with the wall; however, the underlying result still holds, meaning that the *projection* of the incident wavevector, $\boldsymbol{k}_i$ on the wall must equal the *projection* of the reflected wavevector, $\boldsymbol{k}_r$. The magnitude of the wavevector (the wavenumber) is not in general conserved by reflection. Finally, given these results and using (6.111) we see that the incident and reflected amplitudes are related by

$$A_r = -A_i. \tag{6.113}$$

Now let's delve a little deeper into the problem.

Generally, when we consider a wave to be incident on a wall, we are supposing that the *group velocity* is directed toward the wall. Suppose that a wave of given frequency, ω, and wavevector, $\boldsymbol{k}_i$, and with westward group velocity is incident on a predominantly western wall, as in Fig. 6.9. (Similar reasoning, *mutatis mutandis,* can be applied to a wave incident on an eastern wall.) Let us suppose that incident wave, $\boldsymbol{k}_i$ lies at the point I on the wavenumber circle, and the group velocity is found by drawing a line from I to the centre of the circle, C (so $\boldsymbol{c}_{gi} \propto \overrightarrow{IC}$), and in this case the vector is directed westward.

The projection of the $\boldsymbol{k}_i$ must be equal to the projection of the reflected wave vector, $\boldsymbol{k}_r$, and both wavevectors must lie in the same wavenumber circle, centred at $-\beta/2\omega$, because the frequencies of the two waves are the same. We may then graphically determine the wavevector of the reflected wave using the construction of Fig. 6.10. (This is a figure in spectral space and the position of the wall does *not* imply that it is an eastern boundary.) Given the wavevector, the group velocity of the reflected wave follows by drawing a line from the wavevector to the centre of the circle (the line

$\overrightarrow{RC}$). We see from the figure that the reflected group velocity is directed eastward and that it forms the same angle to the wall as does the incident wave; that is, the reflection is *specular.* Since the amplitudes of the incoming and reflected wave are the same, the components of the energy flux perpendicular to the wall are equal and opposite. Furthermore, we can see from the figure that the wavenumber of the reflected wave has a larger magnitude than that of the incident wave. For waves reflecting off an eastern boundary, the reverse is true. Put simply, at a western boundary incident long waves are reflected as short waves, whereas at an eastern boundary incident short waves are reflected as long waves.

Quantitatively solving for the wavenumbers of the reflected wave is a little tedious in the case when the wall is at angle, but easy enough if the wall is along the y-axis. We know the frequency, ω, and the y-wavenumber, l, so that the x-wavenumber may be deduced from the dispersion relation

$$\omega = \frac{-\beta k_i}{k_i^2 + l^2 + k_d^2} = \frac{-\beta k_r}{k_r^2 + l^2 + k_d^2}. \tag{6.114}$$

We obtain

$$k_i = \frac{-\beta}{2\omega} + \sqrt{\left(\frac{\beta}{2\omega}\right)^2 - (l^2 + k_d^2)}, \qquad k_r = \frac{-\beta}{2\omega} - \sqrt{\left(\frac{\beta}{2\omega}\right)^2 - (l^2 + k_d^2)}. \tag{6.115a,b}$$

The signs of the square-root terms are chosen for reflection at a western boundary, for which, as we noted, the reflected wave has a larger (absolute) wavenumber than the incident wave. For reflection at an eastern boundary we reverse the signs.

Oceanographic relevance

The behaviour of Rossby waves at lateral boundaries is not surprisingly of some oceanographic importance. Suppose that Rossby waves are generated in the middle of the ocean, for example by the wind or by some fluid dynamical instability in the ocean. Shorter waves will tend to propagate eastward, and be reflected back at the eastern boundary as long waves, and long waves will tend to propagate westward, being reflected back as short waves.

Reflection at the western boundary is complicated by friction and by other waves. In mid-latitudes the reflection at a western boundary generates Rossby waves that have a short *zonal* length scale (the meridional scale of the reflected wave is the same as the incident wave if the wall is meridional), which means that their *meridional* velocity is large. Now, if the zonal wavenumber is much larger than both the meridional wavenumber l and the inverse deformation radius k_d then, using either (6.62) or (6.67) the group velocity in the x-direction is given by $c_g^x = \overline{u} + \beta/k^2$, where $\overline{u}$ is the zonal mean flow. If the mean flow is westward, so that $\overline{u}$ is negative, then very short waves will be unable to escape from the boundary; specifically, if $k > \sqrt{-\beta/U}$ then the waves will be trapped in a western boundary layer. Even with no mean flow, the short zonal length scale means that frictional effects will be large.

In some circumstances Rossby waves incident on a boundary also have the option of generating coastal Kelvin waves. Also, at a western boundary at the equator Rossby waves may also reflecting back as equatorial Kelvin waves, which we introduce in the next chapter. This effect may be particularly important in the dynamics of El Niño, as we discuss in Section 22.7.

6.7 ✦ GROUP VELOCITY, REVISITED

We now return to a more general discussion of group velocity. Our goal is to show that the group velocity arises in fairly general ways, not just from methods stemming from Fourier analysis or from ray theory. We first give a simple and direct derivation of group velocity that is valid in the simple but important special case of a homogeneous medium.Then, in Section 6.7.2, we give a rather general derivation of the *group velocity property,* namely that conserved quantities that are quadratic in the wave amplitude — that is, *wave activities* — are transported at the group velocity.

6.7.1 Group Velocity in Homogeneous Media

Consider waves propagating in a medium in which the wave equation has the form

$$L(\psi) = \Lambda\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right]\psi(x,t) = 0, \tag{6.116}$$

where Λ is a polynomial operator in the space and time derivatives, with constant coefficients, and its arguments are in square brackets. For simplicity we restrict attention to waves in one dimension, and a simple example is $\Lambda = \partial(\partial_{xx})/\partial t + \beta\partial/\partial x$ so that $L(\psi) = \partial(\partial_{xx}\psi)/\partial t + \beta\partial\psi/\partial x$. We will seek a solution of the form

$$\psi(x,t) = A(x,t)\mathrm{e}^{\mathrm{i}\theta(x,t)}, \tag{6.117}$$

where θ is the phase of the disturbance and $A(x,t)$ is the slowly varying amplitude, so that the solution has the form of a wave packet. The phase is such that $k = \partial\theta/\partial x$ and $\omega = -\partial\theta/\partial t$, and the slowly varying nature of the envelope $A(x,t)$ is formalized by demanding that

$$\frac{1}{A}\frac{\partial A}{\partial x} \ll k, \qquad \frac{1}{A}\frac{\partial A}{\partial t} \ll \omega. \tag{6.118}$$

The space and time derivatives of ψ are then given by

$$\frac{\partial\psi}{\partial x} = \left(\frac{\partial A}{\partial x} + \mathrm{i}A\frac{\partial\theta}{\partial x}\right)\mathrm{e}^{\mathrm{i}\theta} = \left(\frac{\partial A}{\partial x} + \mathrm{i}Ak\right)\mathrm{e}^{\mathrm{i}\theta}, \tag{6.119a}$$

$$\frac{\partial\psi}{\partial t} = \left(\frac{\partial A}{\partial t} + \mathrm{i}A\frac{\partial\theta}{\partial t}\right)\mathrm{e}^{\mathrm{i}\theta} = \left(\frac{\partial A}{\partial t} - \mathrm{i}A\omega\right)\mathrm{e}^{\mathrm{i}\theta}, \tag{6.119b}$$

so that the wave equation becomes

$$\Lambda\psi = \Lambda\left[\frac{\partial}{\partial t} - \mathrm{i}\omega, \frac{\partial}{\partial x} + \mathrm{i}k\right]A = 0. \tag{6.120}$$

Noting that the space and time derivative of A are small compared to k and ω we expand the polynomial in a Taylor series about (ω, k) to obtain

$$\Lambda[-\mathrm{i}\omega, \mathrm{i}k]A + \frac{\partial\Lambda}{\partial(-\mathrm{i}\omega)}\frac{\partial A}{\partial t} + \frac{\partial\Lambda}{\partial(\mathrm{i}k)}\frac{\partial A}{\partial x} \approx 0. \tag{6.121}$$

Now, the dispersion relation for plane waves is $\Lambda[-\mathrm{i}\omega, \mathrm{i}k] = 0$. Taking this to be satisfied the first term in (6.121) vanishes giving

$$\frac{\partial A}{\partial t} - \frac{\partial\Lambda/\partial k}{\partial\Lambda/\partial\omega}\frac{\partial A}{\partial x} = \frac{\partial A}{\partial t} + \frac{\partial\omega}{\partial k}\frac{\partial A}{\partial x} = 0, \tag{6.122}$$

having used $(\partial\Lambda/\partial k)/(\partial\Lambda/\partial\omega) = -(\partial\omega/\partial k)_\Lambda$. Then, since $c_g \equiv \partial\omega/\partial k$, we have

$$\boxed{\frac{\partial A}{\partial t} + c_g\frac{\partial A}{\partial x} = 0,} \tag{6.123}$$

meaning that *the envelope moves at the group velocity.*

6.7.2 ✦ The Group Velocity Property

In Section 6.6 we found that, when averaged over the phase, the energy of a Rossby wave, $\overline{E}$, obeys

$$\frac{\partial \overline{E}}{\partial t} + \nabla \cdot c_g \overline{E} = 0. \tag{6.124}$$

This equation tells us that the energy flux is equal to $c_g \overline{E}$, and it turns out to be a very general feature of waves. It is called the *group velocity property,* and it is not restricted to Rossby waves, or energy, or homogeneous media; it holds for almost any conserved quantity that is quadratic in the wave amplitude, and we now demonstrate this in a more general way.[9] A quantity that is quadratic and conserved is known as a *wave activity.* (The corresponding local quantity, such as the wave activity per unit volume, is strictly called the wave activity *density,* but also often just wave activity.) The group velocity property is useful because if we can determine $\boldsymbol{c}_g$ then we know straight away how wave activities propagate. Energy itself is sometimes a wave activity but often is not: in a growing baroclinic wave energy is drawn from the background state and is not conserved. However, we will see in Chapter 10 that even in a growing baroclinic disturbance it is possible to define a conserved wave activity.

The formal procedure

The derivation, which is rather formal, will hold for waves and wave activities that satisfy the following three assumptions:

(i) The wave activity, A, and flux, $\boldsymbol{F}$, obey the general conservation relation

$$\frac{\partial A}{\partial t} + \nabla \cdot \boldsymbol{F} = 0. \tag{6.125}$$

(ii) Both the wave activity and the flux are quadratic functions of the wave amplitude.

(iii) The waves themselves are of the general form

$$\psi = \widetilde{\psi} e^{i\theta(\boldsymbol{x},t)} + \text{c.c.}, \qquad \theta = \boldsymbol{k} \cdot \boldsymbol{x} - \omega t, \qquad \omega = \omega(\boldsymbol{k}), \tag{6.126a,b,c}$$

where (6.126c) is the dispersion relation, and ψ is any wave field. We will carry out the derivation in the case in which $\widetilde{\psi}$ is treated as a constant in space, but the treatment applies when the amplitude varies slowly over a wavelength (that is, when the wave is 'WKB-able').

The wave activity and flux in (6.125) are not unique, as (6.125) is unaffected by the following transformation,

$$A \to A + \nabla \cdot \boldsymbol{C}, \qquad \boldsymbol{F} \to \boldsymbol{F} - \frac{\partial \boldsymbol{C}}{\partial t}, \tag{6.127}$$

where $\boldsymbol{C}$ is any vector, as well as the addition of any non-divergent vector to $\boldsymbol{F}$ or a constant to A. The conditions above, in particular that the wave activity and flux are quadratic functions, remove the ambiguity and provide a flux that is related to the activity by the group velocity; that is, by $\boldsymbol{F} = \boldsymbol{c}_g A$, as we see below.

To proceed, from assumption *(ii)*, the wave activity must have the general form

$$A = b + a e^{2i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} + a^* e^{-2i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}, \tag{6.128a}$$

where the asterisk, *, denotes complex conjugacy, and b is a real constant and a is a complex constant. For example, suppose that $A = \psi^2$ and $\psi = c e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} + c^* e^{-i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}$, then we find that (6.128a) is satisfied with $a = c^2$ and $b = 2cc^*$. Similarly, the flux has the general form

$$\boldsymbol{F} = \boldsymbol{g} + \boldsymbol{f} e^{2i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} + \boldsymbol{f}^* e^{-2i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}. \tag{6.128b}$$

where $\boldsymbol{g}$ is a real constant vector (not gravity) and $\boldsymbol{f}$ is a complex constant vector. The mean activity and mean flux are obtained by averaging over a cycle; the oscillating terms vanish on integration and therefore the wave activity and flux are given by

$$\overline{A} = b, \qquad \overline{\boldsymbol{F}} = \boldsymbol{g}, \tag{6.129}$$

where the overbar denotes the mean.

Now consider a wave with a slightly different phase, $\theta + \mathrm{i}\,\delta\theta$, where $\delta\theta$ is small compared with θ. Thus, we formally replace $\boldsymbol{k}$ by $\boldsymbol{k} + \mathrm{i}\,\boldsymbol{\delta k}$ and ω by $\omega + \mathrm{i}\,\delta\omega$ where, to satisfy the dispersion relation, we have

$$\omega + \mathrm{i}\,\delta\omega = \omega(\boldsymbol{k} + \mathrm{i}\,\boldsymbol{\delta k}) \approx \omega(\boldsymbol{k}) + \mathrm{i}\,\boldsymbol{\delta k} \cdot \frac{\partial \omega}{\partial \boldsymbol{k}}, \tag{6.130}$$

and therefore

$$\delta\omega = \boldsymbol{\delta k} \cdot \frac{\partial \omega}{\partial \boldsymbol{k}} = \boldsymbol{\delta k} \cdot \boldsymbol{c}_g, \tag{6.131}$$

where $\boldsymbol{c}_g \equiv \partial\omega/\partial\boldsymbol{k}$ is the group velocity.

The new wave has the general form

$$\psi' = (\widetilde{\psi} + \delta\widetilde{\psi})\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}\mathrm{e}^{-\boldsymbol{\delta k}\cdot\boldsymbol{x} + \delta\omega t} + \text{c.c.}, \tag{6.132}$$

and, analogously to (6.128), the associated wave activity and flux have the forms:

$$A' = \left[b + \delta b + (a + \delta a)\mathrm{e}^{2\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} + (a^* + \delta a^*)\mathrm{e}^{-2\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}\right]\mathrm{e}^{-2\boldsymbol{\delta k}\cdot\boldsymbol{x} + 2\delta\omega t} \tag{6.133a}$$

$$\boldsymbol{F}' = \left[\boldsymbol{g} + \delta\boldsymbol{g} + (\boldsymbol{f} + \delta\boldsymbol{f})\mathrm{e}^{2\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} + (\boldsymbol{f}^* + \delta\boldsymbol{f}^*)\mathrm{e}^{-2\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}\right]\mathrm{e}^{-2\boldsymbol{\delta k}\cdot\boldsymbol{x} + 2\delta\omega t}, \tag{6.133b}$$

where the δ quantities are small. If we now demand that A' and $\boldsymbol{F}'$ satisfy assumption (i), then substituting (6.133) into (6.125) gives, on averaging over the phase of a wave and after a little algebra,

$$(\boldsymbol{g} + \delta\boldsymbol{g}) \cdot \boldsymbol{\delta k} = (b + \delta b)\delta\omega, \tag{6.134}$$

and therefore at first order in δ quantities, $\boldsymbol{g} \cdot \boldsymbol{\delta k} = b\delta\omega$. Using (6.131) and (6.129) we obtain

$$\boldsymbol{c}_g = \frac{\boldsymbol{g}}{b} = \frac{\overline{\boldsymbol{F}}}{\overline{A}} \qquad \text{or} \qquad \overline{\boldsymbol{F}} = \boldsymbol{c}_g\overline{A}. \tag{6.135}$$

Using this the conservation law, (6.125), becomes

$$\frac{\partial \overline{A}}{\partial t} + \nabla \cdot (\boldsymbol{c}_g\overline{A}) = 0. \tag{6.136}$$

Thus, for waves satisfying our three assumptions, the flux velocity — that is, the propagation velocity of the wave activity — is equal to the group velocity. Henceforth, we will normally denote $\overline{A}$ by $\mathcal{A}$, and $\overline{\boldsymbol{F}}$ by $\boldsymbol{\mathcal{F}}$.

A remark: In this section and in Section 6.1 we have seen various derivations of the group velocity, some more kinematic, some more physical, some more general. Why are there different ways to derive it? Which is the 'real' derivation? Perhaps the answer is that quantity $\partial\omega/\partial k$ is a fundamental velocity for a wave. If, for example, the energy propagates at this speed then the wave envelope must move at this speed, and vice versa. The karma of mathematics means that the derivations, even if stylistically very different, must give the same answer.

6.8 ENERGY PROPAGATION OF POINCARÉ WAVES

In the final section of this chapter we discuss the energetics of Poincaré waves (first encountered in Section 3.8.2) and show explicitly that the energy propagation occurs at the group velocity. We begin with the one-dimensional problem as this shows the essential aspects and the algebra is a little simpler.

6.8.1 Energetics in One Dimension

The one-dimensional (i.e., no variations in the y-direction), inviscid linear shallow-water equations on the f-plane, linearized about a state of rest, are

$$\frac{\partial u}{\partial t} - f_0 v = -g\frac{\partial h}{\partial x}, \qquad \frac{\partial v}{\partial t} + f_0 u = 0, \qquad \frac{\partial \eta}{\partial t} = -H\frac{\partial u}{\partial x}. \tag{6.137a,b,c}$$

To obtain the dispersion relation we differentiate the first equation with respect to t and substitute from the second and third to obtain

$$\frac{\partial^2 u}{\partial t^2} - Hg\frac{\partial u}{\partial x} + f_0^2 u = 0, \tag{6.138}$$

whence, assuming solutions of the form $u = \mathrm{Re}\,\tilde{u}e^{i(kx-\omega t)}$, we obtain the dispersion relation,

$$\omega^2 = f_0^2 + Hgk^2, \tag{6.139}$$

similar to (3.121) on page 125. Differentiating with respect to k gives $2\omega\,\partial\omega/\partial k = 2kHg$ or

$$c_g = \frac{Hg}{c_p}, \tag{6.140}$$

where $c_g = \partial\omega/\partial k$ and $c_p = \omega/k$. Using (6.139) and (6.140) the ratio of the group and phase velocities is found to be

$$\frac{c_g}{c_p} = \frac{L_d^2 k^2}{1 + L_d^2 k^2}, \tag{6.141}$$

where $L_d = \sqrt{gH}/f$ is the deformation radius. This ratio is always less than unity, tending to zero in the long-wave limit ($kL_d \ll 1$) and to unity for short waves ($kL_d \gg 1$).

The energy equations are obtained by multiplying the three equations of (6.137) by u, v and η respectively, and adding, to give

$$\frac{\partial E}{\partial t} + \frac{\partial F}{\partial x} = 0, \tag{6.142a}$$

where

$$E = \frac{1}{2}(Hu^2 + Hv^2 + g\eta^2), \qquad F = gHu\eta, \tag{6.142b}$$

are the energy density and the energy flux, respectively. Here in the linear approximation the energy is transported only by the pressure term, whereas in the full nonlinear equations there is also an advective transport.

The group velocity property for Poincaré waves

To specialize to the case of propagating *waves* we need to average over a wavelength and use the phase relationships between u, v and η implied by the equations of motion. Writing $u = \mathrm{Re}\,\tilde{u}e^{i(kx-\omega t)}$, and similarly for v and η, we have

$$\tilde{v} = -\mathrm{i}f\frac{\tilde{\eta}}{Hk}, \qquad \tilde{u} = \omega\frac{\tilde{\eta}}{Hk}. \tag{6.143a,b}$$

The kinetic energy, averaged over a wavelength, is then

$$\mathrm{KE} = \frac{1}{2}H(\overline{u^2} + \overline{v^2}) = \frac{1}{4}(\omega^2 + f^2)\frac{\tilde{\eta}^2}{Hk^2} = \frac{1}{4}\frac{\omega^2 + f^2}{\omega^2 - f^2}g\tilde{\eta}^2 \tag{6.144}$$

using (6.143) and the dispersion relation, with the extra factor of one half arising from the averaging over a wavelength. Similarly, the potential energy of the wave is

$$\mathrm{PE} = \frac{1}{2}g\overline{\eta^2} = \frac{1}{4}g\tilde{\eta}^2. \tag{6.145}$$

Thus, the ratio of kinetic to potential energy is just

$$\frac{\mathrm{KE}}{\mathrm{PE}} = \frac{\omega^2 + f^2}{\omega^2 - f^2} = 1 + \frac{2}{k^2L_d^2}, \tag{6.146}$$

using the dispersion relation, and where $L_d = \sqrt{gH/f}$ is the deformation radius. Thus, the kinetic energy is always *greater* than the potential energy (there is no equipartition in this problem), with the ratio approaching unity for small scales (large k).

The total energy (kinetic plus potential) is then

$$\mathrm{KE} + \mathrm{PE} = \frac{1}{4}\left(\frac{\omega^2 + f^2}{\omega^2 - f^2} + 1\right)g\tilde{\eta}^2 = \frac{1}{2}\frac{\omega^2}{k^2H}\tilde{\eta}^2 = \frac{1}{2}\frac{c_p^2}{H}\tilde{\eta}^2, \tag{6.147}$$

again using the dispersion relation. The energy flux, F, averaged over a wavelength, is

$$F = gH\overline{u\eta} = \frac{1}{2}\frac{g\omega}{k}\tilde{\eta}^2 = \frac{1}{2}gc_p\tilde{\eta}^2. \tag{6.148}$$

From (6.147) an (6.148) the flux and the energy are evidently related by

$$\boxed{F = \frac{Hg}{c_p}E = c_gE,} \tag{6.149}$$

using (6.140). That is, the energy flux is equal to the group velocity times the energy itself. Note that in this problem there is no flux in the y direction, because v and η are exactly out of phase from (6.143a).

6.8.2 ✦ Energetics in Two Dimensions

The derivations of the preceding section carry through, *mutatis mutandis,* in the full two-dimensional case. We will give only the key results and allow the reader to fill in the algebra.

As derived in Section 3.8.2 the dispersion relation is

$$\omega^2 = f_0^2 + gH(k^2 + l^2). \tag{6.150}$$

The relation between the components of the group velocity and the phase speed is very similar to the one-dimensional case, and in particular we have

$$c_g^x = \frac{\partial\omega}{\partial k} = gH\frac{k}{\omega} = \frac{gH}{c_p^x}, \qquad c_g^y = \frac{\partial\omega}{\partial l} = gH\frac{l}{\omega} = \frac{gH}{c_p^y}. \tag{6.151}$$

The magnitude of the group velocity is $c_g \equiv |\boldsymbol{c}_g| = (c_g^{x2} + c_g^{y2})^{1/2}$. The magnitude of the phase speed, in the direction of travel of the wave crests, is $c_p = \omega/(k^2 + l^2)^{1/2}$ (note that in general this is *smaller* than the phase speed in either the x or y directions, ω/k or ω/l). Thus, we have

$$c_g^2 = (gH)^2 \frac{k^2 + l^2}{\omega^2} = \frac{(gH)^2}{c_p^2}, \qquad \boldsymbol{c}_g = \left(\frac{gH}{c_p K}\right)\boldsymbol{k}, \tag{6.152}$$

which is analogous to (6.140). The ratio of the magnitudes of the group and phase velocities is, analogously to (6.141),

$$\frac{c_g}{c_p} = \frac{gH}{c_p^2} = \frac{L_d^2 K^2}{1 + L_d^2 K^2}, \tag{6.153}$$

where $K^2 = k^2 + l^2$. As in the one-dimensional case the group velocity is large for short waves, in which rotation plays no role, and small for long waves.

The energy equation is found to be

$$\frac{\partial E}{\partial t} + \nabla \cdot \boldsymbol{F} = 0, \tag{6.154a}$$

with

$$E = \frac{1}{2}(Hu^2 + Hv^2 + g\eta^2), \qquad F = gH(u\mathbf{i} + v\mathbf{j})\eta. \tag{6.154b}$$

From the equations of motion the phase relations between the fields are

$$\widetilde{v} = \frac{\omega l - \mathrm{i}kf}{HK^2}\widetilde{\eta}, \qquad \widetilde{u} = \frac{\omega k - \mathrm{i}lf}{HK^2}\widetilde{\eta}, \tag{6.155}$$

so that the kinetic energy is given by, similarly to (6.144),

$$\mathrm{KE} = \frac{1}{2}H(\overline{u^2} + \overline{v^2}) = \frac{1}{4}(\omega^2 + f^2)\frac{\widetilde{\eta}^2}{Hk^2} = \frac{1}{4}\frac{\omega^2 + f^2}{\omega^2 - f^2}g\widetilde{\eta}^2, \tag{6.156}$$

and the potential energy by

$$\mathrm{PE} = \frac{1}{2}g\overline{\eta^2} = \frac{1}{4}g\widetilde{\eta}^2. \tag{6.157}$$

The ratio of the kinetic and potential energies is given by

$$\frac{\mathrm{KE}}{\mathrm{PE}} = \frac{\omega^2 + f^2}{\omega^2 - f^2} = 1 + \frac{2}{K^2 L_d^2}. \tag{6.158}$$

The total (kinetic plus potential) energy is given by

$$E = \mathrm{KE} + \mathrm{PE} = \frac{1}{4}\left(\frac{\omega^2 + f^2}{\omega^2 - f^2} + 1\right)g\widetilde{\eta}^2 = \frac{1}{2}\frac{\omega^2}{K^2 H}\widetilde{\eta}^2 = \frac{1}{2}\frac{c_p^2}{H}\widetilde{\eta}^2, \tag{6.159}$$

The energy flux, $\boldsymbol{F}$, averaged over a wavelength, is

$$\boldsymbol{F} = gH\overline{\boldsymbol{u}\eta} = \frac{1}{2}\frac{g\omega}{k^2 + l^2}\widetilde{\eta}^2\boldsymbol{k} = \frac{1}{2}\frac{g\omega}{K^2}\widetilde{\eta}^2\boldsymbol{k}, \tag{6.160}$$

using (6.155) and where $\boldsymbol{k} = k\mathbf{i} + l\mathbf{j}$ is the wavevector of the wave.

From (6.159) and (6.160), and using (6.152), the flux and the energy are related by

$$\boxed{\boldsymbol{F} = \boldsymbol{c}_g E.} \tag{6.161}$$

That is, the energy flux is equal to the group velocity times the energy itself.

APPENDIX A: THE WKB APPROXIMATION FOR LINEAR WAVES

The WKB method (after Wentzel, Kramers and Brillouin, the last people to discover the technique[10]) is a way of finding approximate solutions to certain linear differential equations in which the term with the highest derivative is multiplied by a small parameter. The theory for such equations is quite extensive but our interests are modest, being mainly in dispersive waves, and WKB theory can be used to find approximate solutions in cases in which the coefficients of the wave equation vary slowly in space or time. Consider an equation of the form

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}z^2} + m^2(z)\xi = 0. \tag{6.162}$$

Such an equation commonly arises in wave problems. If m^2 is positive the equation has wavelike solutions, and if m is constant the solution has the harmonic form

$$\xi = \mathrm{Re}\, A_0 \mathrm{e}^{\mathrm{i}mz}, \tag{6.163}$$

where A_0 is a complex constant. If m varies only slowly with z — meaning that the variations in m only occur on a scale much longer than $1/m$ — one might reasonably expect that the harmonic solution above would provide a decent first approximation; that is, we expect the solution to locally look like a plane wave with local wavenumber $m(z)$. However, we might also expect that the solution would not be *exactly* of the form $\exp(\mathrm{i}m(z)z)$, because the phase of ξ is $\theta(z) = mz$, so that $\mathrm{d}\theta/\mathrm{d}z = m + z\mathrm{d}m/\mathrm{d}z \neq m$. Thus, in (6.163) m is not the wavenumber unless m is constant.

The condition that variations in m, or in the wavelength $\lambda \sim m^{-1}$, occur only slowly may be variously expressed as

$$\lambda\left|\frac{\partial\lambda}{\partial z}\right| \ll \lambda \qquad \text{or} \qquad \left|\frac{\partial m^{-1}}{\partial z}\right| \ll 1 \qquad \text{or} \qquad \left|\frac{\partial m}{\partial z}\right| \ll m^2. \tag{6.164a,b,c}$$

This condition will generally be satisfied if variations in the background state, or in the medium, occur on a scale much longer than the wavelength. Let us first find a solution by way of a perturbation expansion.

A.1 Solution by Perturbation Expansion

To explicitly recognize the rapid oscillations of the wave compared to its slow variations in amplitude we rescale the coordinate z with a small parameter ϵ. Thus, we let $\widehat{z} = \epsilon z$ where $\epsilon \ll 1$ (ϵ may be similar to the nondimensional parameter $|\mathrm{d}\lambda/\mathrm{d}z|$) and the new variable $\widehat{z}$ then varies by $\mathcal{O}(1)$ over the scale on which m varies. Equation (6.162) becomes

$$\epsilon^2\frac{\mathrm{d}^2\xi}{\mathrm{d}\widehat{z}^2} + m^2(\widehat{z})\xi = 0, \tag{6.165}$$

and we may now suppose that all variables are $\mathcal{O}(1)$. If m were constant the solution would be of the form $\xi = A\exp(m\widehat{z}/\epsilon)$ and this suggests that we look for a solution to (6.165) of the form

$$\xi(z) = \mathrm{e}^{g(\widehat{z})/\epsilon}, \tag{6.166}$$

where $g(\widehat{z})$ is some as yet unknown function. We then have, with primes denoting derivatives,

$$\xi' = \frac{1}{\epsilon}g'\,\mathrm{e}^{g/\epsilon}, \qquad \xi'' = \left(\frac{1}{\epsilon^2}g'^2 + \frac{1}{\epsilon}g''\right)\mathrm{e}^{g/\epsilon}. \tag{6.167a,b}$$

Using these expressions in (6.165) yields

$$\epsilon g'' + g'^2 + m^2 = 0, \tag{6.168}$$

and if we let $g = \int h \, \mathrm{d}\hat{z}$ we obtain

$$\epsilon \frac{\mathrm{d}h}{\mathrm{d}\hat{z}} + h^2 + m^2 = 0. \tag{6.169}$$

To obtain a solution of this equation we expand h in powers of the small parameter ϵ,

$$h(\hat{z};\epsilon) = h_0(\hat{z}) + \epsilon h_1(\hat{z}) + \epsilon^2 h_2(\hat{z}) + \cdots. \tag{6.170}$$

Substituting this in (6.169) and setting successive powers of ϵ to zero gives, at first and second order,

$$h_0^2 + m^2 = 0, \qquad 2h_0 h_1 + \frac{\mathrm{d}h_0}{\mathrm{d}\hat{z}} = 0. \tag{6.171a,b}$$

The solutions of these equations are

$$h_0 = \pm \mathrm{i} m, \qquad h_1(\hat{z}) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\hat{z}} \ln \frac{m(\hat{z})}{m_0}, \tag{6.172a,b}$$

where m_0 is a constant. Now, ignoring higher-order terms, (6.166) may be written in terms of h_0 and h_1 as

$$\xi(\hat{z}) = \exp\left(\int h_0 \, \mathrm{d}\hat{z}/\epsilon\right) \exp\left(\int h_1 \, \mathrm{d}\hat{z}\right), \tag{6.173}$$

and, using (6.172) and with z in place of $\hat{z}$, we obtain

$$\boxed{\xi(z) = A_0 m^{-1/2} \exp\left(\pm \mathrm{i} \int m \, \mathrm{d}z\right),} \tag{6.174}$$

where A_0 is a constant, and this is the WKB solution to (6.162). Explicitly, the solution is

$$\xi(z) = A_0 m^{-1/2} \exp\left(\mathrm{i} \int m \, \mathrm{d}z\right) + A_0^* m^{-1/2} \exp\left(-\mathrm{i} \int m \, \mathrm{d}z\right), \tag{6.175}$$

or, in terms of real quantities,

$$\xi(z) = B_0 m^{-1/2} \cos\left(\int m \, \mathrm{d}z\right) + C_0 m^{-1/2} \sin\left(\int m \, \mathrm{d}z\right), \tag{6.176}$$

where B_0, and C_0 are real constants.

A property of (6.174) is that the derivative of the phase is just m; that is, m is indeed the local wavenumber. A crucial aspect of the derivation is that m varies slowly, so that there is a small parameter, ϵ, in the problem. Having said this, WKB theory can often provide qualitative guidance even when there is little scale separation between the variation of the background state and the wavelength. Asymptotics often works when it does not have to.

A.2 Alternate Derivation

A quick and informative, but less systematic, way to obtain the same result is to seek solutions of (6.162) in the form

$$\xi = A(z) \mathrm{e}^{\mathrm{i}\theta(z)}, \tag{6.177}$$

where $A(z)$ and $\theta(z)$ are both presumptively real. Using (6.177) in (6.162) yields

$$\mathrm{i}\left[2\frac{\mathrm{d}A}{\mathrm{d}z}\frac{\mathrm{d}\theta}{\mathrm{d}z} + A\frac{\mathrm{d}^2\theta}{\mathrm{d}z^2}\right] + \left[A\left(\frac{\mathrm{d}\theta}{\mathrm{d}z}\right)^2 - \frac{\mathrm{d}^2 A}{\mathrm{d}z^2} - m^2 A\right] = 0. \tag{6.178}$$

The terms in square brackets must each be zero. The WKB approximation is to assume that the amplitude varies sufficiently slowly that $|A^{-1}\mathrm{d}^2A/\mathrm{d}z^2| \ll m^2$, and hence that the term involving $\mathrm{d}^2A/\mathrm{d}z^2$ may be neglected. The real and imaginary parts of (6.178) become

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}z}\right)^2 = m^2, \qquad 2\frac{\mathrm{d}A}{\mathrm{d}z}\frac{\mathrm{d}\theta}{\mathrm{d}z} + A\frac{\mathrm{d}^2\theta}{\mathrm{d}z^2} = 0. \tag{6.179a,b}$$

These two equations are very similar to (6.171). The solution of the first one is

$$\theta = \pm \int m \, \mathrm{d}z, \tag{6.180}$$

and substituting this into (6.179b) gives

$$2\frac{\mathrm{d}A}{\mathrm{d}z}m + A\frac{\mathrm{d}m}{\mathrm{d}z} = 0, \qquad \text{with solution} \qquad A = A_0 m^{-1/2}. \tag{6.181a,b}$$

Using (6.180) and (6.181b) in (6.177) recovers (6.174). Using (6.179a) and the real part of (6.178) we see that the condition for the validity of the approximation is that

$$\left|A^{-1}\frac{\mathrm{d}^2A}{\mathrm{d}z^2}\right| \ll m^2, \quad \text{which using (6.181b) is} \quad \left|\frac{1}{m^{-1/2}}\frac{\mathrm{d}^2m^{-1/2}}{\mathrm{d}z^2}\right| \ll m^2. \tag{6.182a,b}$$

Equation (6.164) expresses a similar condition to (6.182b).

Notes

1 For example, *Linear and Nonlinear Waves* by G. B. Whitham, *Nonlinear Dispersive Waves* by M. J. Ablowitz, or Boyd (1980).

2 Group velocity seems to have been first articulated in about 1841 by the Irish mathematician and physicist William Rowan Hamilton (1806–1865), who is also remembered for his formulation of 'Hamiltonian mechanics'. Hamilton may have been motivated by optics, and it was George Stokes, Osborne Reynolds and John Strutt (better known as Lord Rayleigh) who further developed and generalized the idea in a more hydrodynamic context.

3 More detailed treatments of ray theory and related matters are given by Whitham (1974), Lighthill (1978) and LeBlond & Mysak (1980).

4 What are now called Rossby waves were probably first discovered in a theoretical context by Hough (1897, 1898). He considered the linear shallow water equations on a sphere (i.e., Laplace's tidal equations) expanding the solution in powers of the sine of latitude, and obtained two classes of waves: long, rotationally modified, gravity waves and a balanced wave dependent on variations in Coriolis parameter. However, his work was mainly aimed at understanding ocean tides and it was not until the topic was revisited by Rossby (1939) that the meteorological relevance was appreciated. Rossby used the beta-plane approximation in Cartesian co-ordinates, and the simplicity of the presentation along with the meteorological context led to the work attracting significant notice.

5 Chang & Orlanski (1994), Orlanski & Sheldon (1995). Edmund Chang made number of very helpful remarks to me on waves more generally. I would also like to thank Emma Howard for comments on this chapter and Chapter 18.

6 The non-Doppler effect also even in models in height coordinates. See White (1977).

7 See Chapman & Lindzen (1970).

8 As in Longuet-Higgins (1964).

9 The form of this derivation was originally given by Hayes (1977) in the context of wave energy. Vanneste & Shepherd (1998) provide further discussion, in particular of the uniqueness or otherwise of the wave activity density and flux.

10 A description of the WKB method, also called the JWKB method and sometimes the geometrical optics approximation, can be found in many books on perturbation methods, for example Simmonds & Mann (1998), Holmes (2013) and Bender & Orszag (1978). Developments in perturbation theory and multiple scale analysis have generalized the method to the extent that 'WKB' does not appear as a separate topic, or in the index, in the well-known book by Kevorkian & Cole (2011). Wentzel, Kramers and Brillouin separately (or at least in different articles) presented the technique in 1926 as a way to find approximate solutions of the Schrödinger equation. Harold Jeffreys, a mathematical geophysicist, had proposed a similar technique in 1924, and Rayleigh in 1912 had already addressed some aspects of the theory. A mathematical treatment of the topic was in fact given by Joseph Liouville and George Green in 1837, with even earlier relevant work by Francesco Carlini, an Italian astronomer and director of an observatory in Milan, in 1817. The story thus affirms the hypothesis that methods are named after the last people to discover them...

Further Reading

Waves

Bühler, O. 2009. *Waves and Mean Flows.*
A modern, advanced, discussion of waves, mean flows and their interaction, including the transformed Eulerian mean, the generalized Lagrangian mean, and more.

LeBlond, P. H. & Mysak, L. A. 1980. *Waves in the Ocean.*
A comprehensive review of the dynamics of many kinds of waves in the ocean.

Pedlosky, J., 2003. *Waves in the Ocean and Atmosphere: Introduction to Wave Dynamics.*
A compact, informal introduction to the main waves to be found in the atmosphere and ocean, in the style of lecture notes.

Instabilities

The following two books cover most forms of hydrodynamic instability.

Chandrasekhar, S., 1961. *Hydrodynamic and Hydromagnetic Stability*.
This book has become an enduring classic, but has no discussion of baroclinic instability at all.

Drazin P. & Reid, W. H., 1981. *Hydrodynamic Stability*.
A standard text on hydrodynamic instability theory. It has straightforward and extensive discussions of most of the standard cases, but alas only a brief treatment of baroclinic instability.

Turbulence

There are numerous books on turbulence, but one place to start is

Tennekes, H. & Lumley, J., 1972. *A First Course in Turbulence.*
The book remains a classic introduction to the subject.

Another book that has stood the test of time is

Monin, A. S. & Yaglom, A. M., 1971. *Statistical Fluid Mechanics.*
The two volumes are encyclopædic in content and contain a wealth of information, especially on turbulent diffusion. They are not nearly as daunting as they seem.

Two more modern general introductions and references are

Davidson, P. D., 2015. *Turbulence: An Introduction for Scientists and Engineers.*

Pope, S. B., 2000. *Turbulent Flows.*
Both of these are written at about the graduate student level.

Two books containing review and synthesis articles covering a range of topics related to jets and turbulence on both large and small scales, in atmospheres and oceans, are

Galperin, B. & Read, P. L. Eds., 2017. *Zonal Jets: Phenomenology, Genesis, Physics.*

Baumert, H. Z., Simpson, J. & Sündermann, J. Eds., 2005. *Marine Turbulence: Theories, Observations and Models.*