Strange as it may sound, the power of mathematics rests on its evasion of all unnecessary thought and on its wonderful saving of mental operations.
Ernst Mach (1838–1916), quoted in Bell (1937).

CHAPTER 4

Vorticity and Potential Vorticity

VORTICITY AND POTENTIAL VORTICITY both play a central role in geophysical fluid dynamics, especially in the dynamics of the large scale circulation. In this chapter we define and discuss these quantities and deduce some of their dynamical properties and effects. Along the way we will come across *Kelvin's circulation theorem,* one of the most fundamental conservation laws in all of fluid mechanics, to which the conservation of potential vorticity is intimately tied.

4.1 VORTICITY AND CIRCULATION

4.1.1 Preliminaries

Vorticity, $\boldsymbol{\omega}$, is defined to be the curl of velocity and so is given by

$$\boldsymbol{\omega} \equiv \nabla \times \boldsymbol{v}. \tag{4.1}$$

Circulation, C, is defined to be the integral of velocity around a closed fluid loop and so is given by

$$C \equiv \oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{r} = \int_S \boldsymbol{\omega} \cdot \mathrm{d}\boldsymbol{S}, \tag{4.2}$$

where the second expression uses Stokes' theorem and S is any surface bounded by the loop. The circulation around the path is equal to the integral of the normal component of vorticity over *any* surface bounded by that path. The circulation is not a field like vorticity and velocity; rather, we think of the circulation around a particular material line of finite length, and so its value generally depends on the path chosen. If δS is an infinitesimal surface element whose normal points in the direction of the unit vector $\hat{\boldsymbol{n}}$, then

$$\hat{\boldsymbol{n}} \cdot (\nabla \times \boldsymbol{v}) = \frac{1}{\delta S} \oint_{\delta r} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{r}, \tag{4.3}$$

where the line integral is around the infinitesimal area. Thus at a point the component of vorticity in the direction of $\boldsymbol{n}$ is proportional to the circulation around the surrounding infinitesimal fluid element, divided by the elemental area bounded by the path of the integral. A heuristic test for the presence of vorticity is to imagine a small paddle wheel in the flow; the paddle wheel acts as a 'circulation-meter', and rotates if the vorticity is non-zero. Vorticity might seem to be similar

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to angular momentum, in that it is a measure of spin. However, unlike angular momentum, *the value of vorticity at a point does not depend on the particular choice of an axis of rotation;* indeed, the definition of vorticity makes no reference at all to an axis of rotation or to a coordinate system. Rather, vorticity is a measure of the *local* spin of a fluid element.

4.1.2 Simple Axisymmetric Examples

Consider axisymmetric motion in two dimensions, so that the flow is confined to a plane. We use cylindrical coordinates (r, ϕ, z), where z is the direction perpendicular to the plane, with velocity components (u^r, u^ϕ, u^z). For axisymmetric flow $u^z = u^r = 0$ but $u^\phi \neq 0$. The following two examples are quite instructive. (A third example that the reader may wish to consider is solid body rotation on a sphere, which has a vorticity gradient in latitude.)

Rigid body motion

For a body in rigid body rotation, the velocity distribution is given by

$$u^\phi = \Omega r, \tag{4.4}$$

where Ω is the angular velocity of the fluid and r is the distance from the axis of rotation. Associated with this rotation is a vorticity given by

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v} = \omega^z \mathbf{k}, \tag{4.5}$$

where

$$\omega^z = \frac{1}{r}\frac{\partial}{\partial r}(r u^\phi) = \frac{1}{r}\frac{\partial}{\partial r}(r^2 \Omega) = 2\Omega. \tag{4.6}$$

The vorticity of a fluid in solid body rotation is thus twice the angular velocity of the fluid about the axis of rotation, and is pointed in a direction orthogonal to the plane of rotation.

The '$v r$' vortex

This vortex is so-called because the tangential velocity (historically denoted by 'v' in this context) is such that the product vr is constant. In our notation we would have

$$u^\phi = \frac{K}{r}, \tag{4.7}$$

where K is a constant determining the vortex strength. Evaluating the z-component of vorticity gives

$$\omega^z = \frac{1}{r}\frac{\partial}{\partial r}(r u^\phi) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{K}{r}\right) = 0, \tag{4.8}$$

except where $r = 0$, at which the expression is singular and the vorticity is infinite. Our paddle wheel rotates when placed at the vortex center, but, less obviously, does not if placed elsewhere.

The circulation around a circle that encloses the origin is given by

$$C = \oint \frac{K}{r} r \, \mathrm{d}\phi = 2\pi K. \tag{4.9}$$

This does not depend on the radius, and so it is true even as the radius tends to zero. Since the vorticity is the circulation divided by the area, the vorticity at the origin must be infinite. Consider now an integration path that does *not* enclose the origin, for example the contour A–B–C–D–A in Fig. 4.1. Over the segments A–B and C–D the velocity is orthogonal to the contour, and so the contribution is zero. Over B–C and D–A we have

$$C_{BC} = \frac{K}{r_2}\phi r_2 = K\phi, \qquad C_{DA} = -\frac{K}{r_1}\phi r_1 = -K\phi. \tag{4.10}$$

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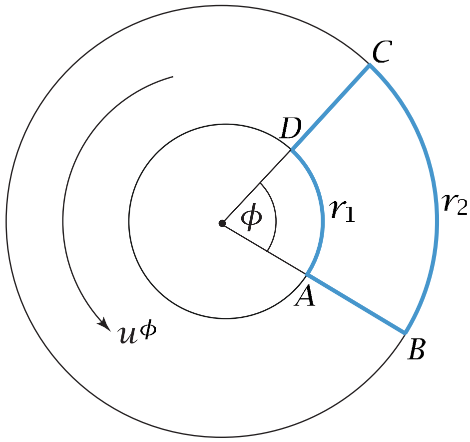

Fig. 4.1 Evaluation of circulation in the axisymmetric vr vortex. The circulation around the path A–B–C–D is zero. This result does not depend on the radii r_1 or r_2 or the angle ϕ, and the circulation around any infinitesimal path not enclosing the origin is zero. Thus the vorticity is zero everywhere except at the origin.

Adding these two expressions we see that the net circulation around the contour C_{ABCDA} is zero. If we shrink the integration path to an infinitesimal size then, within the path, by Stokes' theorem, the vorticity is zero. We can of course place the path anywhere we wish, except surrounding the origin, and obtain this result. Thus the vorticity is everywhere zero, except at the origin.

4.2 THE VORTICITY EQUATION

Using the vector identity $\boldsymbol{v} \times (\nabla \times \boldsymbol{v}) = \nabla(\boldsymbol{v} \cdot \boldsymbol{v})/2 - (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$, we write the momentum equation as

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{v} = -\frac{1}{\rho}\nabla p - \frac{1}{2}\nabla \boldsymbol{v}^2 + \boldsymbol{F}, \tag{4.11}$$

where $\boldsymbol{F}$ represents viscous and body forces. Taking the curl of (4.11) gives the vorticity equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \boldsymbol{v}) = \frac{1}{\rho^2}(\nabla \rho \times \nabla p) + \nabla \times \boldsymbol{F}. \tag{4.12}$$

Now, the vector identity

$$\nabla \times (\boldsymbol{a} \times \boldsymbol{b}) = (\boldsymbol{b} \cdot \nabla)\boldsymbol{a} - (\boldsymbol{a} \cdot \nabla)\boldsymbol{b} + \boldsymbol{a}\nabla \cdot \boldsymbol{b} - \boldsymbol{b}\nabla \cdot \boldsymbol{a}, \tag{4.13}$$

implies that the second term on the left-hand side of (4.12) may be written as

$$\nabla \times (\boldsymbol{\omega} \times \boldsymbol{v}) = (\boldsymbol{v} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{v} + \boldsymbol{\omega}\nabla \cdot \boldsymbol{v} - \boldsymbol{v}\nabla \cdot \boldsymbol{\omega}. \tag{4.14}$$

Because vorticity is the curl of velocity its divergence vanishes, and so (4.12) becomes

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{v} - \boldsymbol{\omega}\nabla \cdot \boldsymbol{v} + \frac{1}{\rho^2}(\nabla \rho \times \nabla p) + \nabla \times \boldsymbol{F}. \tag{4.15}$$

The divergence term may be eliminated with the aid of the mass-conservation equation to give

$$\boxed{\frac{\mathrm{D}\widetilde{\boldsymbol{\omega}}}{\mathrm{D}t} = (\widetilde{\boldsymbol{\omega}} \cdot \nabla)\boldsymbol{v} + \frac{1}{\rho^3}(\nabla \rho \times \nabla p) + \frac{1}{\rho}\nabla \times \boldsymbol{F},} \tag{4.16}$$

where $\widetilde{\boldsymbol{\omega}} \equiv \boldsymbol{\omega}/\rho$. We will set $\boldsymbol{F} = 0$ in most of what follows.

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The third term on the right-hand side of (4.15), as well as the second term on the right-hand side of (4.16), is variously called the *baroclinic* term, the *non-homentropic* term, or the *solenoidal* term. (A solenoidal vector has no divergence, hence the name.) The solenoidal vector, $\boldsymbol{S}_o$, is defined by

$$\boldsymbol{S}_o \equiv \frac{1}{\rho^2}\nabla\rho \times \nabla p = -\nabla\alpha \times \nabla p. \tag{4.17}$$

A solenoid is a tube directed perpendicular to both $\nabla\alpha$ and ∇p, with elements of length proportional to $\nabla p \times \nabla\alpha$. If the isolines of p and α are parallel to each other, then solenoids do not exist. This occurs when the density is a function only of pressure, for then

$$\nabla\rho \times \nabla p = \nabla\rho \times \nabla\rho \,\frac{\mathrm{d}p}{\mathrm{d}\rho} = 0. \tag{4.18}$$

The solenoidal vector may also be written

$$\boldsymbol{S}_o = -\nabla\eta \times \nabla T. \tag{4.19}$$

This follows most easily by first writing the momentum equation in the form $\partial\boldsymbol{v}/\partial t + \boldsymbol{\omega}\times\boldsymbol{v} = T\nabla\eta - \nabla B$, and taking its curl. Evidently the solenoidal term vanishes if: (i) isolines of pressure and density are parallel; (ii) isolines of temperature and entropy are parallel; (iii) density, entropy, temperature or pressure are constant. A *barotropic* fluid has by definition $\rho = \rho(p)$ and therefore no solenoids. A *baroclinic* fluid is one for which ∇p is not parallel to $\nabla\rho$. From (4.16) we see that the baroclinic term must be balanced by terms involving velocity or its tendency and therefore, in general, *a baroclinic fluid is a moving fluid,* even in the presence of viscosity.

For a barotropic fluid the vorticity equation takes the simple form,

$$\frac{\mathrm{D}\widetilde{\boldsymbol{\omega}}}{\mathrm{D}t} = (\widetilde{\boldsymbol{\omega}}\cdot\nabla)\boldsymbol{v}. \tag{4.20}$$

If the fluid is also incompressible, meaning that $\nabla\cdot\boldsymbol{v} = 0$, then we have the even simpler form,

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{v}. \tag{4.21}$$

When expanded into components, the terms on the right-hand side of (4.20) or (4.21) can be divided into 'stretching' and 'tipping' (or 'tilting') terms, and we return to that in Section 4.3.1.

An integral conservation property

Consider a single Cartesian component in (4.15). Then, using superscripts to denote components,

$$\begin{aligned}\frac{\partial\omega^x}{\partial t} &= -\boldsymbol{v}\cdot\nabla\omega^x - \omega^x\nabla\cdot\boldsymbol{v} + (\boldsymbol{\omega}\cdot\nabla)v^x + S_o^x \\ &= -\nabla\cdot(\boldsymbol{v}\omega^x) + \nabla\cdot(\boldsymbol{\omega}v^x) + S_o^x,\end{aligned} \tag{4.22}$$

where S_o^x is the (x-component of the) solenoidal term. Equation (4.22) may be written as

$$\frac{\partial\omega^x}{\partial t} + \nabla\cdot(\boldsymbol{v}\omega^x - \boldsymbol{\omega}v^x) = S_o^x, \tag{4.23}$$

and this implies the Cartesian tensor form of the vorticity equation, namely

$$\frac{\partial\omega_i}{\partial t} + \frac{\partial}{\partial x_j}(v_j\omega_i - v_i\omega_j) = S_{oi}, \tag{4.24}$$

with summation over repeated indices. The tendency of the components of vorticity is thus given by the solenoidal term plus the divergence of a vector field, and if the solenoidal term vanishes the volume integrated vorticity can only be altered by boundary effects. However, in both the atmosphere and the ocean the solenoidal term *is* important, although we will see in Section 4.5 that a useful conservation law for a scalar quantity can still be obtained.

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4.2.1 Two-dimensional Flow

In two-dimensional flow the fluid is confined to a surface, and independent of the dimension normal to that surface. In the simplest case in Cartesian geometry the flow is on a flat plane, and the velocity normal to the plane and the rate of change of any quantity normal to that plane are zero. Let the normal direction be the z-direction; the fluid velocity in the plane, $\boldsymbol{u}$, is $\boldsymbol{u} = u\mathbf{i} + v\mathbf{j}$, and the velocity normal to the plane, w, is zero. Only one component of vorticity is non-zero and this is

$$\boldsymbol{\omega} = \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right). \tag{4.25}$$

That is, in two-dimensional flow the vorticity is perpendicular to the velocity. We let $\zeta \equiv \omega^z = \boldsymbol{\omega}\cdot\mathbf{k}$. Both the stretching and tilting terms vanish in two-dimensional flow, and the two-dimensional vorticity equation becomes, for incompressible flow,

$$\frac{\mathrm{D}\zeta}{\mathrm{D}t} = 0, \tag{4.26}$$

where $\mathrm{D}\zeta/\mathrm{D}t = \partial\zeta/\partial t + \boldsymbol{u}\cdot\nabla\zeta$. That is, in two-dimensional flow vorticity is conserved following the fluid elements; each material parcel of fluid keeps its value of vorticity even as it is being advected around. Furthermore, specification of the vorticity completely determines the flow field. To see this, we use the incompressibility condition to define a streamfunction ψ such that

$$u = -\frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x}, \qquad \zeta = \nabla^2\psi. \tag{4.27a,b,c}$$

Given the vorticity, the Poisson equation (4.27c) can be solved for the streamfunction and the velocity fields obtained through (4.27a,b), and this process is called 'inverting the vorticity'.

Numerical integration of (4.26) is then a process of timestepping plus inversion. The vorticity equation may then be written as an advection equation for vorticity,

$$\frac{\partial \zeta}{\partial t} + \boldsymbol{u}\cdot\nabla\zeta = 0, \tag{4.28}$$

in conjunction with (4.27). The vorticity is stepped forward one timestep using a finite-difference representation of (4.28), and the vorticity inverted to obtain a velocity using (4.27).

Two-dimensional flow is not restricted to a Cartesian plane — it exists on the surface of a sphere for example. In that case the velocity normal to the spherical surface (the 'vertical velocity') vanishes, and the equations are naturally expressed in spherical coordinates. Nevertheless, vorticity (absolute vorticity if the sphere is rotating) is still conserved on parcels as they move over the spherical surface.

4.3 VORTICITY AND CIRCULATION THEOREMS

4.3.1 The 'Frozen-in' Property of Vorticity

Let us first consider some simple topological properties of the vorticity field and its evolution. We define a *vortex line* to be a line drawn through the fluid which is everywhere in the direction of the local vorticity. This definition is analogous to that of a streamline, which is everywhere in the direction of the local velocity. A *vortex tube* is formed by the collection of vortex lines passing through a closed curve (Fig. 4.2). A *material line* is just a line that connects material fluid elements. Suppose we draw a vortex line through the fluid; such a line obviously connects fluid elements and therefore defines a coincident material line. As the fluid moves the material line deforms, and the vortex line also evolves in a manner determined by the equations of motion. A remarkable

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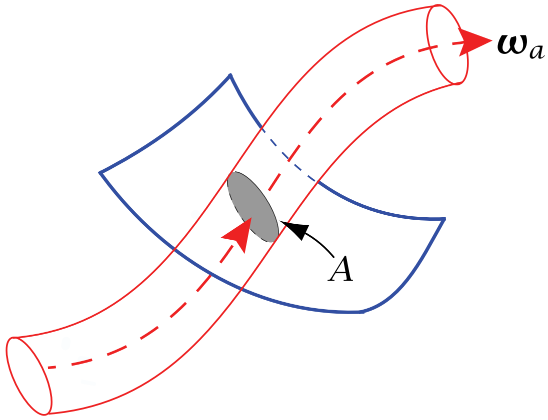

Fig. 4.2 A vortex tube passing through a material sheet. The circulation is the integral of the velocity around the boundary of A, and is equal to the integral of the normal component of vorticity over A.

property of vorticity is that, for an unforced and inviscid barotropic fluid, the flow evolution is such that a vortex line remains coincident with the material line that it was initially associated with. Put another way, a vortex line always contains the same material elements — the vorticity is 'frozen' or 'glued' to the material fluid.[1]

To prove this we consider how an infinitesimal material line element $\delta\boldsymbol{l}$ evolves, $\delta\boldsymbol{l}$ being the infinitesimal material element connecting $\boldsymbol{l}$ with $\boldsymbol{l}+\delta\boldsymbol{l}$. The rate of change of $\delta\boldsymbol{l}$ following the flow is given by

$$\frac{\mathrm{D}\delta\boldsymbol{l}}{\mathrm{D}t} = \frac{1}{\delta t}\left[\delta\boldsymbol{l}(t+\delta t) - \delta\boldsymbol{l}(t)\right], \tag{4.29}$$

which follows from the definition of the material derivative in the limit $\delta t \to 0$. From the Taylor expansion of $\delta\boldsymbol{l}(t)$ and the definition of velocity it is also apparent that

$$\delta\boldsymbol{l}(t+\delta t) = \boldsymbol{l}(t) + \delta\boldsymbol{l}(t) + (\boldsymbol{v}+\delta\boldsymbol{v})\delta t - (\boldsymbol{l}(t) + \boldsymbol{v}\delta t) = \delta\boldsymbol{l} + \delta\boldsymbol{v}\,\delta t, \tag{4.30}$$

as illustrated in Fig. 4.3. Substituting (4.30) into (4.29) gives $\mathrm{D}\delta\boldsymbol{l}/\mathrm{D}t = \delta\boldsymbol{v}$, as expected, and because $\delta\boldsymbol{v} = (\delta\boldsymbol{l}\cdot\nabla)\boldsymbol{v}$ we obtain

$$\frac{\mathrm{D}\delta\boldsymbol{l}}{\mathrm{D}t} = (\delta\boldsymbol{l}\cdot\nabla)\boldsymbol{v}. \tag{4.31}$$

Comparing this with (4.16), we see that vorticity evolves in the same way as a line element in an unforced barotropic fluid. To see what this means, at some initial time we can define an infinitesimal material line element parallel to the vorticity at that location, that is,

$$\delta\boldsymbol{l}(\boldsymbol{x}, t=0) = A\boldsymbol{\omega}(\boldsymbol{x}, t=0), \tag{4.32}$$

where A is a constant. Then, for all subsequent times the magnitude of the vorticity of that fluid element, even as it moves to a new location $\boldsymbol{x}'$, remains proportional to the length of the fluid element at that point and is oriented in the same way; that is $\boldsymbol{\omega}(\boldsymbol{x}', t) = A^{-1}\delta\boldsymbol{l}(\boldsymbol{x}', t)$.

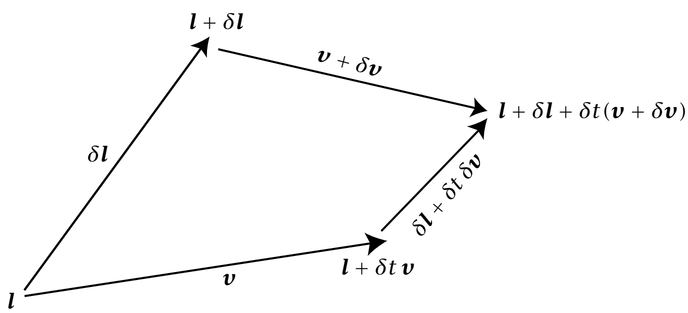

Fig. 4.3 Evolution of an infinitesimal material line $\delta\boldsymbol{l}$ from time t to time $t+\delta t$. It can be seen from the diagram that $\mathrm{D}\delta\boldsymbol{l}/\mathrm{D}t = \delta\boldsymbol{v}$.

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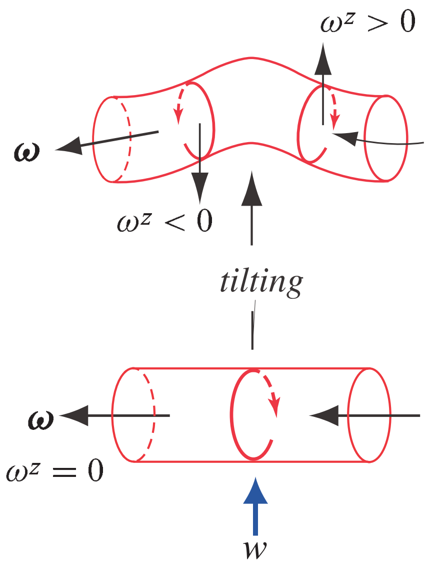

Fig. 4.4 The tilting of vorticity. Suppose that the vorticity, $\boldsymbol{\omega}$ is initially directed horizontally, as in the lower figure, so that ω^z, its vertical component, is zero. The material lines, and therefore also the vortex lines, are tilted by the positive vertical velocity w thus creating a non-zero vertically oriented vorticity. This mechanism is important in creating vertical vorticity in the atmospheric boundary layer, and is connected to the β-effect in large-scale flow.

To verify this result in a different way note that a vortex line element is determined by the condition $\delta \boldsymbol{l} = A\boldsymbol{\omega}$ which implies $\boldsymbol{\omega} \times \delta\boldsymbol{l} = 0$. Now, for any line element we have that

$$\frac{\mathrm{D}}{\mathrm{D}t}(\boldsymbol{\omega} \times \delta\boldsymbol{l}) = \frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} \times \delta\boldsymbol{l} - \frac{\mathrm{D}\delta\boldsymbol{l}}{\mathrm{D}t} \times \boldsymbol{\omega}. \tag{4.33}$$

We also have that

$$\frac{\mathrm{D}\delta\boldsymbol{l}}{\mathrm{D}t} = \delta\boldsymbol{v} = (\delta\boldsymbol{l} \cdot \nabla)\boldsymbol{v} \quad \text{and} \quad \frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{v}. \tag{4.34}$$

If the line element is initially a vortex line element then, at $t = 0$, $\delta\boldsymbol{l} = A\boldsymbol{\omega}$ and, using (4.34), the right-hand side of (4.33) vanishes. Thus, the *tendency* of $\boldsymbol{\omega}\times\delta\boldsymbol{l}$ is zero, and the vortex line continues to be a material line.

Stretching and tilting

The terms on the right-hand side of (4.20) or (4.21) may be interpreted in terms of 'stretching' and 'tipping' (or 'tilting'). Consider a single Cartesian component of (4.21),

$$\frac{\mathrm{D}\omega^x}{\mathrm{D}t} = \omega^x \frac{\partial u}{\partial x} + \omega^y \frac{\partial u}{\partial y} + \omega^z \frac{\partial u}{\partial z}. \tag{4.35}$$

The second and third terms on the right-hand side are the tilting or tipping terms because they involve changes in the orientation of the vorticity vector. They tell us that vorticity in the x-direction may be generated from vorticity in the y- and z-directions if the advection acts to tilt the material lines. Because vorticity is tied to these lines, vorticity oriented in one direction becomes oriented in another, as in Fig. 4.4.

The first term on the right-hand side of (4.35) is the stretching term, and it acts to intensify the x-component of vorticity if the velocity is increasing in the x-direction — that is, if the material lines are being stretched (Fig. 4.5). The effect arises because a vortex line is tied to a material line, and therefore vorticity is amplified in proportion to the stretching of the material line aligned with it. This effect is important in tornadoes, to give one example. If the fluid is incompressible, stretching of a fluid mass in one direction must be accompanied by convergence in another, and this leads to the conservation of circulation, as we now discuss.

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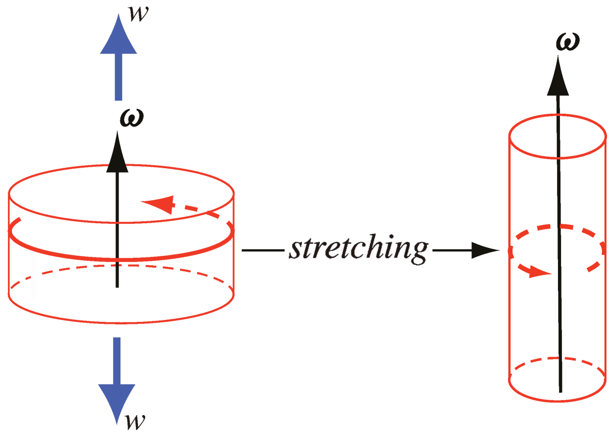

Fig. 4.5 A vertical velocity, w, stretches the cylinder. Vorticity is tied to material lines and so is amplified in the direction of the stretching. However, because the volume of fluid is conserved, the end surfaces shrink, the material lines through the cylinder ends converge and the integral of vorticity over a material surface (the circulation) remains constant.

4.3.2 Kelvin's Circulation Theorem

Kelvin's circulation theorem states that under certain circumstances the circulation around a material fluid parcel is conserved; that is, the circulation is conserved 'moving with the flow'.[2] The primary restrictions are that body forces are conservative (i.e., they are representable as potential forces, and therefore that the flow be inviscid), and that the fluid is barotropic with $\rho = \rho(p)$. Of these, the latter is more restrictive for geophysical fluids. The circulation in the theorem is defined with respect to an inertial frame of reference; specifically, the velocity in (4.39) is the velocity relative to an inertial frame. To prove the theorem, we begin with the inviscid momentum equation,

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p - \nabla \Phi, \tag{4.36}$$

where $\nabla\Phi$ represents the conservative body forces on the system. Applying the material derivative to the circulation, (4.2), gives

$$\begin{aligned}\frac{\mathrm{D}C}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t}\oint \boldsymbol{v}\cdot \mathrm{d}\boldsymbol{r} &= \oint \left(\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t}\cdot \mathrm{d}\boldsymbol{r} + \boldsymbol{v}\cdot \mathrm{d}\boldsymbol{v}\right)\\ &= \oint \left[\left(-\frac{1}{\rho}\nabla p - \nabla\Phi\right)\cdot \mathrm{d}\boldsymbol{r} + \boldsymbol{v}\cdot \mathrm{d}\boldsymbol{v}\right]\\ &= \oint -\frac{1}{\rho}\nabla p\cdot \mathrm{d}\boldsymbol{r},\end{aligned} \tag{4.37}$$

using (4.36) and $D(\mathrm{d}\boldsymbol{r})/Dt = \mathrm{d}\boldsymbol{v}$, where $\mathrm{d}\boldsymbol{r}$ is the line element and with the line integration being over a closed, material, circuit. The second and third terms on the second line vanish separately, because they are exact differentials integrated around a closed loop. The term on the last line vanishes if the density is constant or, more generally, if the density is a function of pressure alone, in which case ∇p is parallel to $\nabla\rho$. To see this, note that

$$\oint \frac{1}{\rho}\nabla p\cdot \mathrm{d}\boldsymbol{r} = \int_S \nabla\times\left(\frac{\nabla p}{\rho}\right)\cdot \mathrm{d}\boldsymbol{S} = \int_S \frac{-\nabla\rho\times\nabla p}{\rho^2}\cdot \mathrm{d}\boldsymbol{S}, \tag{4.38}$$

using Stokes' theorem where S is any surface bounded by the path of the line integral. The integral evidently vanishes identically if p is a function of ρ alone. The right-most expression above is the integral of the solenoidal vector, and if it is zero (4.37) becomes

$$\boxed{\frac{\mathrm{D}}{\mathrm{D}t}\oint \boldsymbol{v}\cdot \mathrm{d}\boldsymbol{r} = 0.} \tag{4.39}$$

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This is Kelvin's circulation theorem. In words, *the circulation around a material loop is invariant for a barotropic fluid that is subject only to conservative forces.* Using Stokes' theorem, the circulation theorem may also be written as

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_S \boldsymbol{\omega}\cdot \mathrm{d}\boldsymbol{S} = 0. \tag{4.40}$$

That is, the area integral of the normal component of vorticity across any material surface is constant, under the same conditions. This form is both natural and useful, and it arises because of the way vorticity is tied to material fluid elements. Kelvin's circulation theorem is the one conservation law that is unique to fluids. Unlike, say, the conservation of energy, it has no analogue in solid body mechanics. Potential vorticity conservation, which we come to later on, is an extension of circulation conservation.

Stretching and circulation

Let us informally consider how vortex stretching and mass conservation work together to give the circulation theorem. Let the fluid be incompressible so that the volume of a fluid mass is constant, and consider a surface normal to a vortex tube, as in Fig. 4.5. Let the volume of a small material box around the surface be δV, the length of the material lines be δl and the surface area be δA. Then

$$\delta V = \delta l\,\delta A. \tag{4.41}$$

Because of the frozen-in property, the vorticity passing through the surface is proportional to the length of the material lines. That is, $\omega \propto \delta l$, and

$$\delta V \propto \omega\,\delta A. \tag{4.42}$$

The right-hand side is just the circulation around the surface. Now, if the corresponding material tube is stretched δl increases, but the volume, δV, remains constant by mass conservation. Thus, the circulation given by the right-hand side of (4.42) also remains constant. In other words, because of the frozen-in property vorticity is amplified by the stretching, but the vortex lines get closer together in such a way that the product $\omega\,\delta A$ remains constant and circulation is conserved.

4.3.3 Baroclinic Flow and the Solenoidal Term

In baroclinic flow, the circulation is not generally conserved, and from (4.37) we have

$$\frac{\mathrm{D}C}{\mathrm{D}t} = -\oint \frac{\nabla p}{\rho}\cdot \mathrm{d}\boldsymbol{r} = -\oint \frac{\mathrm{d}p}{\rho}, \tag{4.43}$$

and this is called the baroclinic circulation theorem.[3] Recalling the fundamental thermodynamic relation $T\,\mathrm{d}\eta = \mathrm{d}I + p\,\mathrm{d}\alpha$, where $\alpha = \rho^{-1}$, we have

$$\alpha\,\mathrm{d}p = \mathrm{d}(p\alpha) - T\,\mathrm{d}\eta + \mathrm{d}I, \tag{4.44}$$

and the first and last terms on the right-hand side will vanish upon integration around a circuit. The solenoidal term on the right-hand side of (4.43) may therefore be written as

$$S_o \equiv -\oint \alpha\,\mathrm{d}p = \oint T\,\mathrm{d}\eta = -\oint \eta\,\mathrm{d}T = -R\oint T\,\mathrm{d}\log p, \tag{4.45}$$

where the last equality holds only for an ideal gas. Using Stokes' theorem, S_o can also be written as

$$S_o = -\int_S \nabla\alpha\times\nabla p\cdot \mathrm{d}\boldsymbol{S} = -\int_S \left(\frac{\partial\alpha}{\partial T}\right)_p \nabla T\times\nabla p\cdot \mathrm{d}\boldsymbol{S} = \int_S \nabla T\times\nabla\eta\cdot \mathrm{d}\boldsymbol{S}. \tag{4.46}$$

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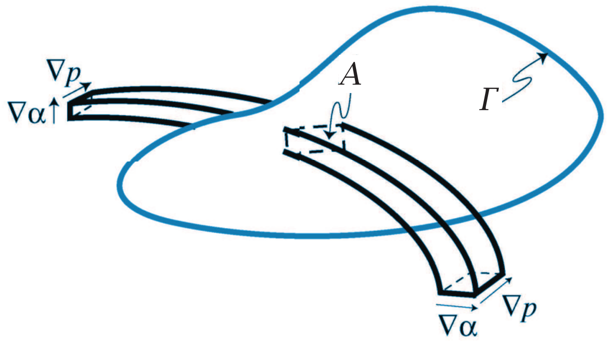

Fig. 4.6 Solenoids and the circulation theorem. Solenoids are tubes perpendicular to both $\nabla\alpha$ and ∇p, and they have a non-zero cross-sectional area if isolines of α and p do not coincide. The rate of change of circulation over a material surface is given by the sum of all the solenoidal areas crossing the surface. If $\nabla\alpha \times \nabla p = 0$ there are no solenoids.

The rate of change of the circulation across a surface depends on the existence of this solenoidal term (Fig. 4.6). However, even if the solenoidal vector is in non-zero, circulation *is* conserved if the material path is in a surface of constant entropy, η, *and if* $\mathrm{D}\eta/\mathrm{D}t = 0$. The solenoidal term then vanishes and, because $\mathrm{D}\eta/\mathrm{D}t = 0$, entropy remains constant on that same material loop as it evolves. This result gives rise to the conservation of potential vorticity, discussed in Section 4.5.

4.3.4 Circulation in a Rotating Frame

The absolute and relative velocities are related by $\boldsymbol{v}_a = \boldsymbol{v}_r + \boldsymbol{\Omega} \times \boldsymbol{r}$, so that in a rotating frame the rate of change of circulation is given by

$$\frac{\mathrm{D}}{\mathrm{D}t} \oint (\boldsymbol{v}_r + \boldsymbol{\Omega} \times \boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{r} = \oint \left[\left(\frac{\mathrm{D}\boldsymbol{v}_r}{\mathrm{D}t} + \boldsymbol{\Omega} \times \boldsymbol{v}_r \right) \cdot \mathrm{d}\boldsymbol{r} + (\boldsymbol{v}_r + \boldsymbol{\Omega} \times \boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{v}_r \right]. \tag{4.47}$$

But $\oint \boldsymbol{v}_r \cdot \mathrm{d}\boldsymbol{v}_r = 0$ and, integrating by parts,

$$\begin{aligned} \oint (\boldsymbol{\Omega} \times \boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{v}_r &= \oint \left\{ \mathrm{d}[(\boldsymbol{\Omega} \times \boldsymbol{r}) \cdot \boldsymbol{v}_r] - (\boldsymbol{\Omega} \times \mathrm{d}\boldsymbol{r}) \cdot \boldsymbol{v}_r \right\} \\ &= \oint \left\{ \mathrm{d}[(\boldsymbol{\Omega} \times \boldsymbol{r}) \cdot \boldsymbol{v}_r] + (\boldsymbol{\Omega} \times \boldsymbol{v}_r) \cdot \mathrm{d}\boldsymbol{r} \right\}. \end{aligned} \tag{4.48}$$

The first term on the right-hand side is zero and so (4.47) becomes

$$\frac{\mathrm{D}}{\mathrm{D}t} \oint (\boldsymbol{v}_r + \boldsymbol{\Omega} \times \boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{r} = \oint \left(\frac{\mathrm{D}\boldsymbol{v}_r}{\mathrm{D}t} + 2\boldsymbol{\Omega} \times \boldsymbol{v}_r \right) \cdot \mathrm{d}\boldsymbol{r} = -\oint \frac{\mathrm{d}p}{\rho}, \tag{4.49}$$

where the second equality uses the momentum equation. The last term vanishes if the fluid is barotropic, and if so the circulation theorem is, unsurprisingly,

$$\frac{\mathrm{D}}{\mathrm{D}t} \oint (\boldsymbol{v}_r + \boldsymbol{\Omega} \times \boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{r} = 0, \qquad \text{or} \qquad \frac{\mathrm{D}}{\mathrm{D}t} \int_S (\boldsymbol{\omega}_r + 2\boldsymbol{\Omega}) \cdot \mathrm{d}\boldsymbol{S} = 0, \tag{4.50a,b}$$

where the second equation uses Stokes' theorem and we have used $\nabla \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = 2\boldsymbol{\Omega}$, and where $\boldsymbol{\omega}_r = \nabla \times \boldsymbol{v}_r$ is the *relative vorticity*.[4]

4.3.5 The Circulation Theorem for Hydrostatic Flow

Kelvin's circulation theorem holds for hydrostatic flow, with a slightly different form. For simplicity we restrict attention to the f-plane, and start with the hydrostatic momentum equations,

$$\frac{\mathrm{D}\boldsymbol{u}_r}{\mathrm{D}t} + 2\boldsymbol{\Omega} \times \boldsymbol{u}_r = -\frac{1}{\rho} \nabla_z p, \qquad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - \nabla\Phi, \tag{4.51a,b}$$

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where $\Phi = gz$ is the gravitational potential and $\boldsymbol{\Omega} = \Omega\mathbf{k}$. The advecting field is three-dimensional, and in particular we still have $\mathrm{D}\delta\boldsymbol{r}/\mathrm{D}t = \delta\boldsymbol{v} = (\delta\boldsymbol{r}\cdot\nabla)\boldsymbol{v}$. Thus, using (4.51) we have

$$
\begin{aligned}
\frac{\mathrm{D}}{\mathrm{D}t}\oint(\boldsymbol{u}_r + \boldsymbol{\Omega}\times\boldsymbol{r})\cdot\mathrm{d}\boldsymbol{r} &= \oint\left[\left(\frac{\mathrm{D}\boldsymbol{u}_r}{\mathrm{D}t} + \boldsymbol{\Omega}\times\boldsymbol{v}_r\right)\cdot\mathrm{d}\boldsymbol{r} + (\boldsymbol{u}_r + \boldsymbol{\Omega}\times\boldsymbol{r})\cdot\mathrm{d}\boldsymbol{v}_r\right] \\
&= \oint\left(\frac{\mathrm{D}\boldsymbol{u}_r}{\mathrm{D}t} + 2\boldsymbol{\Omega}\times\boldsymbol{u}_r\right)\cdot\mathrm{d}\boldsymbol{r} \\
&= \oint\left(-\frac{1}{\rho}\nabla p - \nabla\Phi\right)\cdot\mathrm{d}\boldsymbol{r},
\end{aligned}
\tag{4.52}
$$

as with (4.49), having used $\boldsymbol{\Omega}\times\boldsymbol{v}_r = \boldsymbol{\Omega}\times\boldsymbol{u}_r$, and where the gradient operator ∇ is three-dimensional. The last term on the right-hand side vanishes because it is the integral of the gradient of a potential around a closed path. The first term vanishes if the fluid is barotropic, so that the circulation theorem is

$$
\frac{\mathrm{D}}{\mathrm{D}t}\oint(\boldsymbol{u}_r + \boldsymbol{\Omega}\times\boldsymbol{r})\cdot\mathrm{d}\boldsymbol{r} = 0. \tag{4.53}
$$

Using Stokes' theorem we have the equivalent form

$$
\frac{\mathrm{D}}{\mathrm{D}t}\int_S(\boldsymbol{\omega}_{hy} + 2\boldsymbol{\Omega})\cdot\,\mathrm{d}\boldsymbol{S} = 0, \tag{4.54}
$$

where the subscript 'hy' denotes hydrostatic and, in Cartesian coordinates,

$$
\boldsymbol{\omega}_{hy} = \nabla\times\boldsymbol{u}_r = -\mathbf{i}\frac{\partial v_r}{\partial z} + \mathbf{j}\frac{\partial u_r}{\partial z} + \mathbf{k}\left(\frac{\partial v_r}{\partial x} - \frac{\partial u_r}{\partial y}\right). \tag{4.55}
$$

4.4 VORTICITY EQUATION IN A ROTATING FRAME

Perhaps the easiest way to derive the vorticity equation appropriate for a rotating reference frame is to begin with the momentum equation in the form

$$
\frac{\partial\boldsymbol{v}_r}{\partial t} + (2\boldsymbol{\Omega} + \boldsymbol{\omega}_r)\times\boldsymbol{v}_r = -\frac{1}{\rho}\nabla p - \nabla\left(\Phi + \frac{1}{2}\boldsymbol{v}_r^2\right), \tag{4.56}
$$

where the potential Φ contains the gravitational and centrifugal forces. Take the curl of this and use the identity (4.13), which here implies

$$
\nabla\times[(2\boldsymbol{\Omega} + \boldsymbol{\omega}_r)\times\boldsymbol{v}_r] = (2\boldsymbol{\Omega} + \boldsymbol{\omega}_r)\nabla\cdot\boldsymbol{v}_r + (\boldsymbol{v}_r\cdot\nabla)(2\boldsymbol{\Omega} + \boldsymbol{\omega}_r) - [(2\boldsymbol{\Omega} + \boldsymbol{\omega}_r)\cdot\nabla]\boldsymbol{v}_r, \tag{4.57}
$$

(noting that $\nabla\cdot(2\boldsymbol{\Omega} + \boldsymbol{\omega}) = 0$), to give the vorticity equation

$$
\frac{\mathrm{D}\boldsymbol{\omega}_r}{\mathrm{D}t} = [(2\boldsymbol{\Omega} + \boldsymbol{\omega}_r)\cdot\nabla]\,\boldsymbol{v} - (2\boldsymbol{\Omega} + \boldsymbol{\omega}_r)\nabla\cdot\boldsymbol{v}_r + \frac{1}{\rho^2}(\nabla\rho\times\nabla p). \tag{4.58}
$$

If the rotation rate, $\boldsymbol{\Omega}$, is a constant then $\mathrm{D}\boldsymbol{\omega}_r/\mathrm{D}t = \mathrm{D}\boldsymbol{\omega}_a/\mathrm{D}t$ where $\boldsymbol{\omega}_a = 2\boldsymbol{\Omega} + \boldsymbol{\omega}_r$ is the absolute vorticity. The only difference between the vorticity equation in the rotating and inertial frames of reference is in the presence of the solid-body vorticity $2\boldsymbol{\Omega}$ on the right-hand side. The second term on the right-hand side may be folded into the material derivative using mass continuity, and after a little manipulation (4.58) becomes

$$
\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{\omega}_a}{\rho}\right) = \frac{1}{\rho}(2\boldsymbol{\Omega} + \boldsymbol{\omega}_r)\cdot\nabla\boldsymbol{v}_r + \frac{1}{\rho^3}(\nabla\rho\times\nabla p). \tag{4.59}
$$

However, note that it is the absolute vorticity, $\boldsymbol{\omega}_a$, that now appears on the left-hand side. If ρ is constant, $\boldsymbol{\omega}_a$ may be replaced by $\boldsymbol{\omega}_r$.

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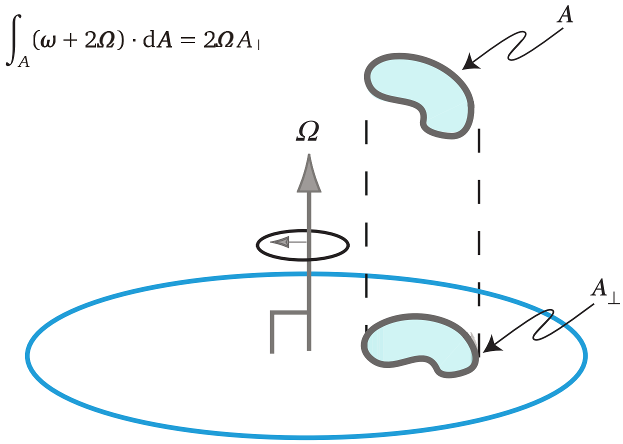

Fig. 4.7 The projection of a material circuit on to the equatorial plane. If a fluid element moves poleward, keeping its orientation to the local vertical fixed (i.e., it stays horizontal), then the area of its projection on to the equatorial plane increases. If its total (absolute) circulation is to be maintained, then the vertical component of its relative vorticity must diminish; that is, $\int_A (\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \mathrm{d}\boldsymbol{A} = \int_A (\zeta + f)\,\mathrm{d}A =$ constant. Thus, the β term in $\mathrm{D}(\zeta + f)/\mathrm{D}t = \mathrm{D}\zeta/\mathrm{D}t + \beta v = 0$ arises from the *tilting* of a parcel relative to the axis of rotation as it moves meridionally.

4.4.1 The Circulation Theorem and the Beta Effect

What are the implications of the circulation theorem on a rotating, spherical planet? Let us define relative circulation over some material loop as

$$C_r \equiv \oint \boldsymbol{v}_r \cdot \mathrm{d}\boldsymbol{r}. \tag{4.60}$$

Because $\boldsymbol{v}_r = \boldsymbol{v}_a - \boldsymbol{\Omega} \times \boldsymbol{r}$ (where $\boldsymbol{r}$ is the distance from the axis of rotation), we use Stokes' theorem to give

$$C_r = C_a - \int 2\boldsymbol{\Omega} \cdot \mathrm{d}\boldsymbol{S} = C_a - 2\Omega A_\perp, \tag{4.61}$$

where C_a is the total or absolute circulation and $A_\perp$ is the area enclosed by the projection of the material circuit on to the plane normal to the rotation vector; that is, on to the equatorial plane (Fig. 4.7). If the solenoidal term is zero, then the circulation theorem, (4.50), may be written as

$$\frac{\mathrm{D}}{\mathrm{D}t}(C_r + 2\Omega A_\perp) = 0. \tag{4.62}$$

Thus, the relative circulation around a circuit changes if the orientation of the plane changes; that is, if the area of its projection on to the equatorial plane changes. In large scale dynamics the most common cause of this is when a fluid parcel changes its latitude. For example, consider the flow of a two-dimensional, infinitesimal, horizontal (i.e., tangent to the the radial vector), constant-density fluid parcel at a latitude ϑ with area A, so that the projection of its area on to the equatorial plane is $A_\perp = A \sin\vartheta$ and $C_r = \zeta_r A$. If the fluid surface moves, but remains horizontal, its area is preserved (because it is incompressible) and directly from (4.62) its relative vorticity changes as

$$\frac{\mathrm{D}\zeta_r}{\mathrm{D}t} = -\frac{2\Omega}{A}\frac{\mathrm{D}A_\perp}{\mathrm{D}t} = -2\Omega\frac{\mathrm{D}}{\mathrm{D}t}\sin\vartheta = -v_r\frac{2\Omega\cos\vartheta}{a} = -\beta v_r, \tag{4.63}$$

where

$$\beta \equiv \frac{\mathrm{d}f}{\mathrm{d}y} = \frac{2\Omega}{a}\cos\vartheta. \tag{4.64}$$

The means by which the vertical component of the relative vorticity of a parcel changes by virtue of its latitudinal displacement is known as the *beta effect*, or the *β-effect*. It is a manifestation of the tilting term in the vorticity equation, and it is often the most important means by which relative vorticity does change in large-scale flow. The β-effect arises in the full vorticity equation, as we now see.

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4.4.2 The Vertical Component of the Vorticity Equation

In large-scale dynamics, the most important, although not the largest, component of the vorticity is often the vertical one, because this contains much of the information about the horizontal flow. We can obtain an explicit expression for its evolution by taking the vertical component of (4.58), although care must be taken because the unit vectors ($\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$) are functions of position (see Section 2.2).

An alternative derivation begins with the horizontal momentum equations,

$$\frac{\partial u}{\partial t} - v(\zeta + f) + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{1}{2}\frac{\partial}{\partial x}(u^2 + v^2) + F^x \tag{4.65a}$$

$$\frac{\partial v}{\partial t} + u(\zeta + f) + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \frac{1}{2}\frac{\partial}{\partial y}(u^2 + v^2) + F^y, \tag{4.65b}$$

where in this section we again drop the subscript r on variables measured in the rotating frame. Cross-differentiating gives, after a little algebra,

$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{D}t}(\zeta + f) = &-(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial u}{\partial z}\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\frac{\partial w}{\partial x}\right)\\ &+ \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right) + \left(\frac{\partial F^y}{\partial x} - \frac{\partial F^x}{\partial y}\right).\end{aligned} \tag{4.66}$$

We interpret the various terms as follows:

$\mathrm{D}\zeta/\mathrm{D}t = \partial\zeta/\partial t + \boldsymbol{v}\cdot\nabla\zeta$. The material derivative of the vertical component of the vorticity.

$\mathrm{D}f/\mathrm{D}t = v\partial f/\partial y = v\beta$. The β-effect. The vorticity is affected by the meridional motion of the fluid, so that, apart from the terms on the right-hand side, $(\zeta + f)$ is conserved on parcels. Because the Coriolis parameter changes with latitude this is like saying that the system has differential rotation. This effect is precisely that due to the change in orientation of fluid surfaces with latitude, as discussed in Section 4.4.1 and illustrated Fig. 4.7.

$-(\zeta + f)(\partial u/\partial x + \partial v/\partial y)$. The divergence term, which gives rise to vortex stretching. In an incompressible fluid this may be written $(\zeta + f)\partial w/\partial z$, so that vorticity is amplified if the vertical velocity increases with height, so stretching the material lines and the vorticity.

$(\partial u/\partial z)(\partial w/\partial y) - (\partial v/\partial z)(\partial w/\partial x)$. The tilting term, whereby a vertical component of vorticity may be generated by a vertical velocity acting on a horizontal vorticity. See Fig. 4.4.

$\rho^{-2}\left[(\partial\rho/\partial x)(\partial p/\partial y) - (\partial\rho/\partial y)(\partial p/\partial x)\right] = \rho^{-2}J(\rho, p)$. The solenoidal term, also called the non-homentropic or baroclinic term, arising when isosurfaces of pressure and density are not parallel.

$(\partial F^y/\partial x - \partial F^x/\partial y)$. The forcing and friction term. If the only contribution is from molecular viscosity then this term is $\nu\nabla^2\zeta$.

Two-dimensional and shallow water vorticity equations

In an inviscid two-dimensional incompressible flow, all of the terms on the right-hand side of (4.66) vanish and we have the simple equation

$$\frac{\mathrm{D}(\zeta + f)}{\mathrm{D}t} = 0, \tag{4.67}$$

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implying that the absolute vorticity, $\zeta_a \equiv \zeta + f$, is materially conserved. If f is a constant, then (4.67) reduces to (4.28), and background rotation plays no role. If f varies linearly with y, so that $f = f_0 + \beta y$, then (4.67) becomes

$$\frac{\partial \zeta}{\partial t} + \boldsymbol{u} \cdot \nabla \zeta + \beta v = 0, \tag{4.68}$$

which is known as the two-dimensional β-plane vorticity equation.

For inviscid shallow water flow, we can show that (see Chapter 3)

$$\frac{\mathrm{D}(\zeta + f)}{\mathrm{D}t} = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right). \tag{4.69}$$

In this equation the vanishing of the tilting term is perhaps the only aspect which is not immediately apparent, but this succumbs to a little thought.

4.5 POTENTIAL VORTICITY CONSERVATION

Too much of a good thing is wonderful.
Mae West (1892–1990).

Although Kelvin's circulation theorem is a general statement about vorticity conservation, in its original form it is not always a practically useful statement for two reasons. First, it is not a statement about a *field*, such as vorticity itself. Second, it is not satisfied for baroclinic flow, such as is found in the atmosphere and ocean. (Non-conservative forces such as viscosity also lead to circulation non-conservation, but this applies to virtually all conservation laws and does not diminish them.) It turns out that it is possible to derive a beautiful conservation law that overcomes both of these failings and one that, furthermore, is extraordinarily useful in geophysical fluid dynamics. This is the conservation of *potential vorticity* (PV) introduced first by Rossby and then in a more general form by Ertel.[5] The idea is that we can use a scalar field that is being advected by the flow to keep track of, or to take care of, the evolution of fluid elements. For a baroclinic fluid this scalar field must be chosen in a special way (it must be a function of the density and pressure alone), but there is no restriction to a barotropic fluid. Then using the scalar evolution equation in conjunction with the vorticity equation gives us a scalar conservation equation. In the next few subsections we derive the equation for potential vorticity conservation in a number of superficially different ways — different explications but the same explanation.[6]

4.5.1 PV Conservation from the Circulation Theorem

Barotropic fluids

Let us begin with the simple case of a barotropic fluid. For an infinitesimal volume we write Kelvin's theorem as

$$\frac{\mathrm{D}}{\mathrm{D}t}\left[(\boldsymbol{\omega}_a \cdot \boldsymbol{n})\delta A\right] = 0, \tag{4.70}$$

where $\boldsymbol{n}$ is a unit vector normal to an infinitesimal surface δA. Now consider a volume bounded by two isosurfaces of values χ and $\chi + \delta\chi$, where χ is any materially conserved tracer, thus satisfying $\mathrm{D}\chi/\mathrm{D}t = 0$, so that δA initially lies in an isosurface of χ (see Fig. 4.8). Since $\boldsymbol{n} = \nabla\chi/|\nabla\chi|$ and the infinitesimal volume $\delta V = \delta h\, \delta A$, where δh is the separation between the two surfaces, we have

$$\boldsymbol{\omega}_a \cdot \boldsymbol{n}\,\delta A = \boldsymbol{\omega}_a \cdot \frac{\nabla\chi}{|\nabla\chi|}\frac{\delta V}{\delta h}. \tag{4.71}$$

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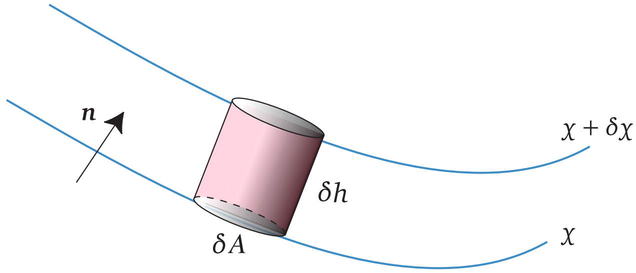

Fig. 4.8 An infinitesimal fluid element, bounded by two isosurfaces of the conserved tracer χ. As $D\chi/Dt = 0$, then $D\delta\chi/Dt = 0$.

Now, the value of δh may be obtained from

$$\delta\chi = \delta \boldsymbol{x} \cdot \nabla\chi = \delta h |\nabla\chi|, \tag{4.72}$$

and using this in (4.70) we obtain

$$\frac{D}{Dt}\left[\frac{(\boldsymbol{\omega}_a \cdot \nabla\chi)\delta V}{\delta\chi}\right] = 0. \tag{4.73}$$

Since χ is conserved on material elements then so is $\delta\chi$ and it may be taken out of the differentiation. The mass of the volume element $\rho\,\delta V$ is also conserved, so that (4.73) becomes

$$\frac{\rho\delta V}{\delta\chi}\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}_a}{\rho}\cdot\nabla\chi\right) = 0 \tag{4.74}$$

or

$$\frac{D}{Dt}\left(\widetilde{\boldsymbol{\omega}}_a \cdot \nabla\chi\right) = 0, \tag{4.75}$$

where $\widetilde{\boldsymbol{\omega}}_a = \boldsymbol{\omega}_a/\rho$. Equation (4.75) is a statement of potential vorticity conservation for a barotropic fluid. The field χ may be chosen arbitrarily, provided that it is materially conserved.

The general case

For a baroclinic fluid the above derivation fails simply because the statement of the conservation of circulation, (4.70) is not, in general, true: there are solenoidal terms on the right-hand side and from (4.43) and (4.45) we have

$$\frac{D}{Dt}\left[(\boldsymbol{\omega}_a \cdot \boldsymbol{n})\delta A\right] = \boldsymbol{S}_o \cdot \boldsymbol{n}\,\delta A, \qquad \boldsymbol{S}_o = -\nabla\alpha \times \nabla p = -\nabla\eta \times \nabla T. \tag{4.76a,b}$$

However, the right-hand side of (4.76a) may be annihilated by choosing the circuit around which we evaluate the circulation to be such that the solenoidal term is identically zero. Given the form of $\boldsymbol{S}_o$, this occurs if the values of any of p, ρ, η, T are constant on that circuit; that is, if $\chi = p, \rho, \eta$ or T. But the derivation also demands that χ be a materially conserved quantity, which usually restricts the choice of χ to be η (or potential temperature), or to be ρ itself if the thermodynamic equation is $D\rho/Dt = 0$. Thus, the conservation of potential vorticity for inviscid, adiabatic flow is

$$\frac{D}{Dt}\left(\widetilde{\boldsymbol{\omega}}_a \cdot \nabla\theta\right) = 0, \tag{4.77}$$

where $D\theta/Dt = 0$. For diabatic flow source terms appear on the right-hand side, and we derive these later on. A summary of this derivation is provided by Fig. 4.9.

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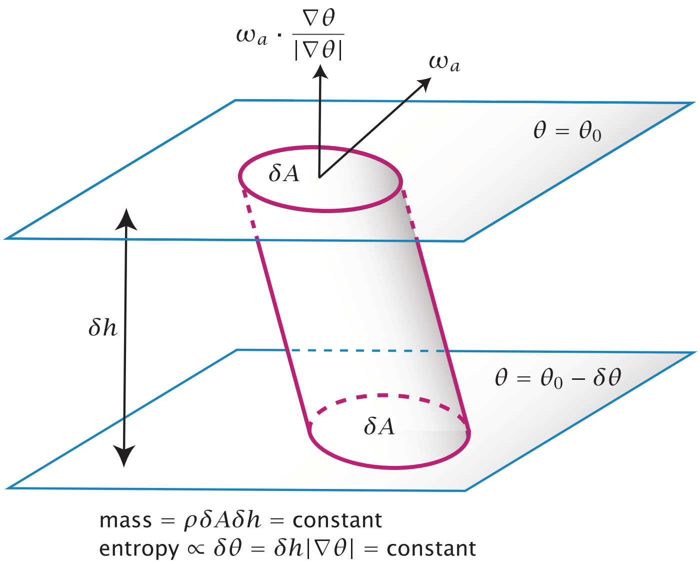

Fig. 4.9 Geometry of potential vorticity conservation. The circulation equation is $\mathrm{D}[(\boldsymbol{\omega}_a \cdot \boldsymbol{n})\delta A]/\mathrm{D}t = \boldsymbol{S}_o \cdot \boldsymbol{n}\delta A$, where $\boldsymbol{S}_o \propto \nabla\theta \times \nabla T$. We choose $\boldsymbol{n} = \nabla\theta/|\nabla\theta|$, where θ is materially conserved, to annihilate the solenoidal term on the right-hand side, and we note that $\delta A = \delta V/\delta h$, where δV is the volume of the cylinder, and the height of the column is $\delta h = \delta\theta/|\nabla\theta|$. The circulation is $C \equiv \boldsymbol{\omega}_a \cdot \boldsymbol{n}\delta A = \boldsymbol{\omega}_a \cdot (\nabla\theta/|\nabla\theta|)(\delta V/\delta h) = [\rho^{-1}\boldsymbol{\omega}_a \cdot \nabla\theta](\delta M/\delta\theta)$, where $\delta M = \rho\delta V$ is the mass of the cylinder. As δM and $\delta\theta$ are materially conserved, so is the potential vorticity $\rho^{-1}\boldsymbol{\omega}_a \cdot \nabla\theta$.

4.5.2 PV Conservation from the Frozen-in Property

In this section we show that potential vorticity conservation is a consequence of the frozen-in property of vorticity. This is not surprising, because the circulation theorem itself has a similar origin. Thus, this derivation is not independent of the derivation in the previous section, just a re-expression of it. We first consider the case in which the solenoidal term vanishes from the outset.

Barotropic fluids

If χ is a materially conserved tracer then the difference in χ between two infinitesimally close fluid elements is also conserved and

$$\frac{\mathrm{D}}{\mathrm{D}t}(\chi_1 - \chi_2) = \frac{\mathrm{D}\delta\chi}{\mathrm{D}t} = 0. \tag{4.78}$$

But $\delta\chi = \nabla\chi \cdot \delta\boldsymbol{l}$, where $\delta\boldsymbol{l}$ is the infinitesimal vector connecting the two fluid elements. Thus

$$\frac{\mathrm{D}}{\mathrm{D}t}(\nabla\chi \cdot \delta\boldsymbol{l}) = 0. \tag{4.79}$$

However, as the line element and the vorticity (divided by density) obey the same equation, we can replace the line element by vorticity (divided by density) in (4.79) to obtain again

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\nabla\chi \cdot \boldsymbol{\omega}_a}{\rho}\right) = 0. \tag{4.80}$$

That is, the potential vorticity, $Q = (\tilde{\boldsymbol{\omega}}_a \cdot \nabla\chi)$ is a material invariant, where χ is any scalar quantity that satisfies $\mathrm{D}\chi/\mathrm{D}t = 0$.

Baroclinic fluids

In baroclinic fluids we cannot casually substitute the vorticity for that of a line element in (4.79) because of the presence of the solenoidal term, and in any case a little more detail would not be amiss. From (4.79) we obtain

$$\delta\boldsymbol{l} \cdot \frac{\mathrm{D}\nabla\chi}{\mathrm{D}t} + \nabla\chi \cdot \frac{\mathrm{D}\delta\boldsymbol{l}}{\mathrm{D}t} = 0, \tag{4.81}$$

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or, using (4.31),

$$\delta \boldsymbol{l} \cdot \frac{\mathrm{D}\nabla\chi}{\mathrm{D}t} + \nabla\chi \cdot [(\delta\boldsymbol{l} \cdot \nabla)\boldsymbol{v}] = 0. \tag{4.82}$$

Now, let us choose $\delta\boldsymbol{l}$ to correspond to a vortex line, so that at the initial time $\delta\boldsymbol{l} = \epsilon\widetilde{\boldsymbol{\omega}}_a$. (Note that in this case the association of $\delta\boldsymbol{l}$ with a vortex line can only be made instantaneously, and we cannot set $\mathrm{D}\delta\boldsymbol{l}/\mathrm{D}t \propto \mathrm{D}\boldsymbol{\omega}_a/\mathrm{D}t$.) Then,

$$\widetilde{\boldsymbol{\omega}}_a \cdot \frac{\mathrm{D}\nabla\chi}{\mathrm{D}t} + \nabla\chi \cdot [(\widetilde{\boldsymbol{\omega}}_a \cdot \nabla)\boldsymbol{v}] = 0, \tag{4.83}$$

or, using the vorticity equation (4.16),

$$\widetilde{\boldsymbol{\omega}}_a \cdot \frac{\mathrm{D}\nabla\chi}{\mathrm{D}t} + \nabla\chi \cdot \left(\frac{\mathrm{D}\widetilde{\boldsymbol{\omega}}_a}{\mathrm{D}t} - \frac{1}{\rho^3}\nabla\rho \times \nabla p\right) = 0. \tag{4.84}$$

This may be written as

$$\frac{\mathrm{D}}{\mathrm{D}t}\widetilde{\boldsymbol{\omega}}_a \cdot \nabla\chi = \frac{1}{\rho^3}\nabla\chi \cdot (\nabla\rho \times \nabla p). \tag{4.85}$$

The term on the right-hand side is, in general, non-zero for an arbitrary choice of scalar, but it will evidently vanish if ∇p, $\nabla\rho$ and $\nabla\chi$ are coplanar. If χ is any function of p and ρ this will be satisfied, but χ must also be a materially conserved scalar. If, as for an ideal gas, $\rho = \rho(\eta, p)$ (or $\eta = \eta(p, \rho)$) where η is the entropy (which is materially conserved), and if χ is a function of entropy η alone, then χ satisfies both conditions. Explicitly, the solenoidal term vanishes because

$$\nabla\chi \cdot (\nabla\rho \times \nabla p) = \frac{\mathrm{d}\chi}{\mathrm{d}\eta}\nabla\eta \cdot \left[\left(\frac{\partial\rho}{\partial p}\nabla p + \frac{\partial\rho}{\partial\eta}\nabla\eta\right) \times \nabla p\right] = 0. \tag{4.86}$$

Thus, provided χ satisfies the two conditions

$$\frac{\mathrm{D}\chi}{\mathrm{D}t} = 0 \quad \text{and} \quad \chi = \chi(p, \rho), \tag{4.87}$$

then (4.85) becomes

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{\omega}_a \cdot \nabla\chi}{\rho}\right) = 0. \tag{4.88}$$

The natural choice for χ is potential temperature, whence

$$\boxed{\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{\omega}_a \cdot \nabla\theta}{\rho}\right) = 0.} \tag{4.89}$$

The presence of a density term in the denominator is not necessary for incompressible flows (i.e., if $\nabla \cdot \boldsymbol{v} = 0$).

4.5.3 PV Conservation: an Algebraic Derivation

Finally, we give an algebraic derivation of potential vorticity conservation. We will take the opportunity to include frictional and, diabatic processes, although these may also be included in the derivations above.[7] We begin with the frictional vorticity equation in the form

$$\frac{\mathrm{D}\widetilde{\boldsymbol{\omega}}_a}{\mathrm{D}t} = (\widetilde{\boldsymbol{\omega}}_a \cdot \nabla)\boldsymbol{v} + \frac{1}{\rho^3}(\nabla\rho \times \nabla p) + \frac{1}{\rho}(\nabla \times \boldsymbol{F}), \tag{4.90}$$

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where $\boldsymbol{F}$ represents any non-conservative force term on the right-hand side of the momentum equation (i.e., $\mathrm{D}\boldsymbol{v}/\mathrm{D}t = -\rho^{-1}\nabla p + \boldsymbol{F}$). We have also the equation for our materially conserved scalar χ,

$$\frac{\mathrm{D}\chi}{\mathrm{D}t} = \dot{\chi}, \tag{4.91}$$

where $\dot{\chi}$ represents any sources and sinks of χ. Now

$$(\widetilde{\boldsymbol{\omega}}_a \cdot \nabla)\frac{\mathrm{D}\chi}{\mathrm{D}t} = \widetilde{\boldsymbol{\omega}}_a \cdot \frac{\mathrm{D}\nabla\chi}{\mathrm{D}t} + [(\widetilde{\boldsymbol{\omega}}_a \cdot \nabla)\boldsymbol{v}] \cdot \nabla\chi, \tag{4.92}$$

which may be obtained just by expanding the left-hand side. Thus, using (4.91),

$$\widetilde{\boldsymbol{\omega}}_a \cdot \frac{\mathrm{D}\nabla\chi}{\mathrm{D}t} = (\widetilde{\boldsymbol{\omega}}_a \cdot \nabla)\,\dot{\chi} - [(\widetilde{\boldsymbol{\omega}}_a \cdot \nabla)\boldsymbol{v}] \cdot \nabla\chi. \tag{4.93}$$

Now take the dot product of (4.90) with $\nabla\chi$:

$$\nabla\chi \cdot \frac{\mathrm{D}\widetilde{\boldsymbol{\omega}}_a}{\mathrm{D}t} = \nabla\chi \cdot [(\widetilde{\boldsymbol{\omega}}_a \cdot \nabla)\boldsymbol{v}] + \nabla\chi \cdot \left[\frac{1}{\rho^3}(\nabla\rho \times \nabla p)\right] + \nabla\chi \cdot \left[\frac{1}{\rho}(\nabla \times \boldsymbol{F})\right]. \tag{4.94}$$

The sum of the last two equations yields

$$\frac{\mathrm{D}}{\mathrm{D}t}(\widetilde{\boldsymbol{\omega}}_a \cdot \nabla\chi) = \widetilde{\boldsymbol{\omega}}_a \cdot \nabla\dot{\chi} + \nabla\chi \cdot \left[\frac{1}{\rho^3}(\nabla\rho \times \nabla p)\right] + \frac{\nabla\chi}{\rho} \cdot (\nabla \times \boldsymbol{F}). \tag{4.95}$$

This equation reprises (4.85), but with the addition of frictional and diabatic terms. As before, the solenoidal term is annihilated if we choose $\chi = \theta(p,\rho)$, so giving the evolution equation for potential vorticity in the presence of forcing and diabatic terms, namely

$$\boxed{\frac{\mathrm{D}}{\mathrm{D}t}(\widetilde{\boldsymbol{\omega}}_a \cdot \nabla\theta) = \widetilde{\boldsymbol{\omega}}_a \cdot \nabla\dot{\theta} + \frac{\nabla\theta}{\rho} \cdot (\nabla \times \boldsymbol{F}).} \tag{4.96}$$

4.5.4 Effects of Salinity and Moisture

For seawater the equation of state may be written as

$$\theta = \theta(\rho, p, S), \tag{4.97}$$

where θ is the potential temperature and S is the salinity. In the absence of diabatic terms and saline diffusion the potential temperature is a materially conserved quantity. However, because of the presence of salinity, potential temperature cannot be used to annihilate the solenoidal term; that is

$$\nabla\theta \cdot (\nabla\rho \times \nabla p) = \left(\frac{\partial\theta}{\partial S}\right)_{p,\rho} \nabla S \cdot (\nabla\rho \times \nabla p) \neq 0. \tag{4.98}$$

Strictly speaking then, *there is no potential vorticity conservation principle for seawater.* However, such a blunt statement overemphasizes the non-conservation of potential vorticity because the saline effect is small. In fact, we can derive an approximate potential vorticity conservation law, as follows.[8]

Suppose that we use potential density to try to annihilate the solenoidal term. Potential density is adiabatically conserved but, like θ, it is a function of salinity so that

$$\nabla\rho_\theta \cdot (\nabla\rho \times \nabla p) = \left(\frac{\partial\rho_\theta}{\partial S}\right)_{p,\rho} \nabla S \cdot (\nabla\rho \times \nabla p) \neq 0. \tag{4.99}$$

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Now, potential density may be written as function of salinity and potential temperature (or entropy) with no pressure dependence and therefore we can rewrite the above expression as

$$(\nabla\rho \times \nabla p)\cdot\nabla\rho_\theta = (\nabla\rho_\theta \times \nabla\rho)\cdot\nabla p \tag{4.100a}$$

$$= \left[\left(\frac{\partial\rho_\theta}{\partial S}\nabla S + \frac{\partial\rho_\theta}{\partial\theta}\nabla\theta\right)\times\left(\frac{\partial\rho}{\partial S}\nabla S + \frac{\partial\rho}{\partial\theta}\nabla\theta + \frac{\partial\rho}{\partial p}\nabla p\right)\right]\cdot\nabla p \tag{4.100b}$$

$$= \left[\frac{\partial\rho_\theta}{\partial S}\nabla S \times \frac{\partial\rho}{\partial\theta}\nabla\theta + \frac{\partial\rho_\theta}{\partial\theta}\nabla\theta \times \frac{\partial\rho}{\partial S}\nabla S\right]\cdot\nabla p \tag{4.100c}$$

$$= \left[\frac{\partial\rho_\theta}{\partial S}\frac{\partial\rho}{\partial\theta} - \frac{\partial\rho_\theta}{\partial\theta}\frac{\partial\rho}{\partial S}\right](\nabla S \times \nabla\theta)\cdot\nabla p, \tag{4.100d}$$

If the term in square brackets in (4.100d) is zero then potential vorticity is conserved, and this is the case if the density and potential density are related by

$$\rho(S,\theta,p) = \rho_\theta(S,\theta) + F(p), \tag{4.101}$$

where F is some function of p; the result also follows directly using (4.101) and (4.100a). Equation (4.101) does not exactly hold because the compressibility of seawater is not in fact just a function of pressure (this is the thermobaric effect). However, as can be seen from (1.156b), the equation holds to a good approximation, a result related to the fact that the speed of sound in seawater is nearly constant. Thus, to this approximation, potential vorticity is adiabatically conserved in seawater if potential density is used as the scalar variable. The derivation does not care whether density itself is a function of salinity; rather, it asks that the difference between density and potential density is a function only of pressure.

Similarly, in a moist atmosphere there is, strictly, no conservation of a conventional potential vorticity because potential temperature is a function of density, pressure and water vapour, although the moisture dependence is usually weak. Condensational heating also provides a diabatic source term that provides a source of potential vorticity. These effects may accounted for in part by using a virtual potential temperature in the definition of potential vorticity, and including the contribution of liquid water to the entropy.[9] In any case, and as with seawater, the compositional effects are fairly small, especially in mid-latitudes, and the dynamics of potential vorticity conservation play a central role in the large-scale dynamics of both atmosphere and ocean.

4.5.5 Effects of Rotation, and Summary Remarks

In a rotating frame the potential vorticity conservation equation is obtained simply by replacing $\boldsymbol{\omega}_a$ by $\boldsymbol{\omega} + 2\boldsymbol{\Omega}$, where $\boldsymbol{\Omega}$ is the rotation rate of the rotating frame. The operator $\mathrm{D}/\mathrm{D}t$ is reference-frame invariant, and so may be evaluated using the usual formulae with velocities measured in the rotating frame.

We have generally referred to the quantity $\boldsymbol{\omega}_a \cdot \nabla\theta/\rho$ as the potential vorticity; however, this form (often referred to as the Ertel or Rossby–Ertel potential vorticity) is not unique. If θ is a materially conserved variable, then so is $g(\theta)$ where g is any function, so that $\boldsymbol{\omega}_a \cdot \nabla g(\theta)/\rho$ is also a potential vorticity. In the atmosphere θ itself is in fact commonly used, whereas in the ocean potential density is the more appropriate scalar, with $f\partial\rho_\theta/\partial z$ being a common approximation for low Rossby number flows.

The conservation of potential vorticity has profound consequences in fluid dynamics, especially in a rotating, stratified fluid. The non-conservative terms are often small, and large-scale flow in both the ocean and the atmosphere is well characterized by conservation of potential vorticity. Such conservation is a very powerful constraint on the flow, and indeed it turns out that potential vorticity is usually a more useful quantity for baroclinic, or non-homentropic, fluids than for barotropic fluids, because the required use of a special conserved scalar imparts additional information; in barotropic fluids potential vorticity has little more power than vorticity itself.

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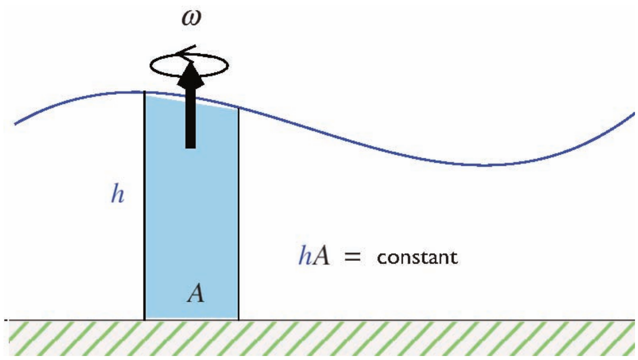

Fig. 4.10 The volume of a column of fluid, hA, is conserved. Furthermore, the vorticity is tied to material lines so that ζA is also a material invariant, where $\zeta = \boldsymbol{\omega} \cdot \mathbf{k}$ is the vertical component of the vorticity. From this, ζ/h must be materially conserved, or $\mathrm{D}(\zeta/h)/\mathrm{D}t = 0$, which is the conservation of potential vorticity in a shallow water system. With rotation this generalizes to $\mathrm{D}[(\zeta + f)/h]/\mathrm{D}t = 0$.

4.6 ✦ POTENTIAL VORTICITY IN THE SHALLOW WATER SYSTEM

In Chapter 3 we derived potential vorticity conservation by direct manipulation of the shallow water equations. In this short section we show that shallow water potential vorticity is also derivable from the conservation of circulation. Specifically, we will begin with the three-dimensional form of Kelvin's theorem, and then make the small aspect ratio assumption (which is the key assumption underlying shallow water dynamics), and thereby recover shallow water potential vorticity conservation (see also Fig. 4.10).

We begin with

$$\frac{\mathrm{D}}{\mathrm{D}t}(\boldsymbol{\omega}_3 \cdot \delta \boldsymbol{S}) = 0, \tag{4.102}$$

where $\boldsymbol{\omega}_3$ is the curl of the three-dimensional velocity and $\delta \boldsymbol{S} = \boldsymbol{n}\, \delta S$ is an arbitrary infinitesimal vector surface element, with $\boldsymbol{n}$ being a unit vector pointing in the direction normal to the surface. If we separate the vorticity and the surface element into vertical and horizontal components we can write (4.102) as

$$\frac{\mathrm{D}}{\mathrm{D}t}\left[(\zeta + f)\delta A + \boldsymbol{\omega}_h \cdot \delta \boldsymbol{S}_h\right] = 0, \tag{4.103}$$

where $\boldsymbol{\omega}_h$ and $\delta \boldsymbol{S}_h$ are the horizontally directed components of the vorticity and the surface element, and $\delta A = \mathbf{k}\, \delta \boldsymbol{S}$ is the area of a horizontal cross-section of a fluid column. In Cartesian form the horizontal component of the vorticity is

$$\boldsymbol{\omega}_h = \mathbf{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) - \mathbf{j}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) = \mathbf{i}\frac{\partial w}{\partial y} - \mathbf{j}\frac{\partial w}{\partial x}, \tag{4.104}$$

where vertical derivatives of the horizontal velocity are zero by virtue of the nature of the shallow water system. Now, the vertical velocity in the shallow water system is smaller than the horizontal velocity by the order of the aspect ratio — the ratio of the fluid depth to the horizontal scale of the motion. Furthermore, the size of the horizontally directed surface element is also smaller than the vertically-directed component by the aspect ratio; that is,

$$|\boldsymbol{\omega}_h| \sim \alpha|\zeta| \qquad \text{and} \qquad |\delta \boldsymbol{S}_h| \sim \alpha|\delta A|, \tag{4.105}$$

where $\alpha = H/L$ is the aspect ratio. Thus $\boldsymbol{\omega}_h \cdot \delta \boldsymbol{S}_h$ is smaller than the term $\zeta \delta A$ by the aspect number squared, and in the small aspect ratio approximation should be neglected. Kelvin's circulation theorem, (4.103), becomes

$$\frac{\mathrm{D}}{\mathrm{D}t}[(\zeta + f)\delta A] = 0 \qquad \text{or} \qquad \frac{\mathrm{D}}{\mathrm{D}t}\left[\frac{(\zeta + f)}{h} h\, \delta A\right] = 0, \tag{4.106a,b}$$

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where h is the depth of the fluid column. But $h\,\delta A$ is the volume of the fluid column, and this is constant. Thus, (4.106b) gives, as in (3.96),

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\zeta + f}{h}\right) = 0, \tag{4.107}$$

where, because horizontal velocities are independent of the vertical coordinate, the advection is purely horizontal.

4.7 POTENTIAL VORTICITY IN APPROXIMATE, STRATIFIED MODELS

If approximate models of stratified flow (Boussinesq, hydrostatic and so on) are to be useful then they should conserve an appropriate form of potential vorticity, and we consider a few such cases.

4.7.1 The Boussinesq Equations

A Boussinesq fluid is incompressible; that is, the volume of a fluid element is conserved and the flow is divergence-free, with $\nabla\cdot\boldsymbol{v} = 0$. The equation for vorticity itself is then isomorphic to that for a line element. However, the Boussinesq equations are not barotropic — $\nabla\rho$ is not parallel to ∇p — and although the pressure gradient term $\nabla\phi$ disappears on taking its curl (or equivalently disappears on integration around a closed path) the buoyancy term $\mathbf{k}b$ does not, and it is this term that prevents Kelvin's circulation theorem from holding. Specifically, the evolution of circulation in the Boussinesq equations obeys

$$\frac{\mathrm{D}}{\mathrm{D}t}[(\boldsymbol{\omega}_a\cdot\boldsymbol{n})\delta A] = (\nabla\times b\mathbf{k})\cdot\boldsymbol{n}\,\delta A, \tag{4.108}$$

where here, as in (4.70), $\boldsymbol{n}$ is a unit vector orthogonal to an infinitesimal surface element of area δA. The right-hand side is annihilated if we choose $\boldsymbol{n}$ to be parallel to ∇b, because $\nabla b\cdot\nabla\times(b\mathbf{k}) = 0$. In the simple Boussinesq equations the thermodynamic equation is

$$\frac{\mathrm{D}b}{\mathrm{D}t} = 0, \tag{4.109}$$

and potential vorticity conservation is therefore (with $\boldsymbol{\omega}_a = \boldsymbol{\omega} + 2\boldsymbol{\Omega}$)

$$\frac{\mathrm{D}Q}{\mathrm{D}t} = 0, \qquad Q = (\boldsymbol{\omega} + 2\boldsymbol{\Omega})\cdot\nabla b. \tag{4.110a,b}$$

Expanding (4.110b) in Cartesian coordinates with $2\boldsymbol{\Omega} = f\mathbf{k}$ we obtain:

$$Q = (v_x - u_y)b_z + (w_y - v_z)b_x + (u_z - w_x)b_y + f b_z. \tag{4.111}$$

In the general Boussinesq equations b itself is not materially conserved. We cannot expect to obtain a conservation law if salinity is present, but if the equation of state and the thermodynamic equation are:

$$b = b(\theta, z), \qquad \frac{\mathrm{D}\theta}{\mathrm{D}t} = 0, \tag{4.112}$$

then potential vorticity conservation follows, because taking $\boldsymbol{n}$ to be parallel to $\nabla\theta$ will cause the right-hand side of (4.108) to vanish; that is,

$$\nabla\theta\cdot\nabla\times(b\mathbf{k}) = \left(\frac{\partial\theta}{\partial z}\nabla z + \frac{\partial\theta}{\partial b}\nabla b\right)\cdot\nabla\times(b\mathbf{k}) = 0. \tag{4.113}$$

The materially conserved potential vorticity in the Boussinesq approximation, Q_B, is thus

$$\boxed{Q_B = \boldsymbol{\omega}_a\cdot\nabla\theta.} \tag{4.114}$$

Note that if the equation of state is $b = b(\theta,\phi)$, where ϕ is the pressure, then potential vorticity is not conserved because then, in general, $\nabla\phi\cdot\nabla\times(b\mathbf{k}) \neq 0$.

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4.7.2 The Hydrostatic Equations

Making the hydrostatic approximation has no effect on whether or not the circulation theorem is satisfied. Thus, in a baroclinic hydrostatic fluid we have

$$\frac{\mathrm{D}}{\mathrm{D}t}\int(\boldsymbol{\omega}_{hy}+2\boldsymbol{\Omega})\cdot\,\mathrm{d}\boldsymbol{S}=-\int\nabla\alpha\times\nabla p\cdot\,\mathrm{d}\boldsymbol{S}, \tag{4.115}$$

where, from (4.55) $\boldsymbol{\omega}_{hy} = \nabla\times\boldsymbol{u} = -\mathbf{i}v_z + \mathbf{j}u_z + \mathbf{k}(v_x - u_y)$, but the gradient operator and material derivative are fully three-dimensional. Derivation of potential vorticity conservation then proceeds, as in Section 4.5.1, by choosing the circuit over which the circulation is calculated to be such that the right-hand side vanishes; that is, to be such that the solenoidal term is annihilated. Precisely as before, this occurs if the circuit is barotropic, and without further ado we write

$$\frac{\mathrm{D}Q_{hy}}{\mathrm{D}t}=\frac{\mathrm{D}}{\mathrm{D}t}\left[\frac{(\boldsymbol{\omega}_{hy}+2\boldsymbol{\Omega})\cdot\nabla\theta}{\rho}\right]=0. \tag{4.116}$$

Expanding the expression for Q_{hy} in Cartesian coordinates gives

$$\boxed{Q_{hy}=\frac{1}{\rho}\left[(v_x-u_y)\theta_z-v_z\theta_x+u_z\theta_y+2\Omega\theta_z\right].} \tag{4.117}$$

In spherical coordinates the hydrostatic approximation is usually accompanied by the traditional approximation and the expanded expression for a conserved potential vorticity is more complicated. It can still be derived from Kelvin's theorem, but this is left as an exercise for the reader.

4.7.3 Potential Vorticity on Isentropic Surfaces

If we begin with the primitive equations in isentropic coordinates then potential vorticity conservation follows quite simply. Cross-differentiating the horizontal momentum equations (3.178) gives the vorticity equation

$$\frac{\mathrm{D}}{\mathrm{D}t}(\zeta+f)+(\zeta+f)\nabla_\theta\cdot\boldsymbol{u}=0, \tag{4.118}$$

where $\mathrm{D}/\mathrm{D}t = \partial/\partial t + \boldsymbol{u}\cdot\nabla_\theta$. The thermodynamic equation is

$$\frac{\mathrm{D}\sigma}{\mathrm{D}t}+\sigma\nabla\cdot\boldsymbol{u}=0, \tag{4.119}$$

where $\sigma = \partial z/\partial b$ (Boussinesq) or $\partial p/\partial\theta$ (ideal gas) is the thickness of an isopycnal layer. Eliminating the divergence between (4.118) and (4.119) gives

$$\boxed{\frac{\mathrm{D}Q_{IS}}{\mathrm{D}t}=0,\quad\text{where}\quad Q_{IS}=\left(\frac{\zeta+f}{\sigma}\right).} \tag{4.120}$$

The derivation, and the result, are precisely the same as with the shallow water equations (Sections 3.7.1 and 4.6).

A connection between isentropic and height coordinates

The hydrostatic potential vorticity written in height coordinates may be transformed into a form that reveals its intimate connection with isentropic surfaces. Let us make the Boussinesq approximation for which the hydrostatic potential vorticity is, with no rotation,

$$Q_{hy}=(v_x-u_y)b_z-v_zb_x+u_zb_y, \tag{4.121}$$

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where b is the buoyancy. We can write this as

$$Q_{hy} = b_z \left[\left(v_x - v_z \frac{b_x}{b_z} \right) - \left(u_y - u_z \frac{b_y}{b_z} \right) \right]. \tag{4.122}$$

But the terms in the inner brackets are just the horizontal velocity derivatives at constant b. To see this, note that

$$\left(\frac{\partial v}{\partial x} \right)_b = \left(\frac{\partial v}{\partial x} \right)_z + \frac{\partial v}{\partial z} \left(\frac{\partial z}{\partial x} \right)_b = \left(\frac{\partial v}{\partial x} \right)_z - \frac{\partial v}{\partial z} \left(\frac{\partial b}{\partial x} \right)_z \bigg/ \frac{\partial b}{\partial z}, \tag{4.123}$$

with a similar expression for $(\partial u/\partial y)_b$. (These relationships follow from standard rules of partial differentiation. Derivatives with respect to z are taken at constant x and y.) Thus, we obtain

$$Q_{hy} = \frac{\partial b}{\partial z} \left[\left(\frac{\partial v}{\partial x} \right)_b - \left(\frac{\partial u}{\partial y} \right)_b \right] = \frac{\partial b}{\partial z} \zeta_b. \tag{4.124}$$

Thus, potential vorticity is simply the horizontal vorticity evaluated on a surface of constant buoyancy, multiplied by the vertical derivative of buoyancy. An analogous derivation, with a similar result, proceeds for the ideal gas equations, with potential temperature replacing buoyancy.

4.8 ✦ THE IMPERMEABILITY OF ISENTROPES TO POTENTIAL VORTICITY

An interesting property of isentropic surfaces is that they are 'impermeable' to potential vorticity, meaning that the mass integral of potential vorticity ($\int Q\rho \, \mathrm{d}V$) over a volume bounded by an isentropic surface remains constant, even in the presence of diabatic sources, provided the surfaces do not intersect a non-isentropic surface such as the ground.[10] This may seem surprising, especially because unlike most conservation laws the result does not require adiabatic flow, and for that reason it leads to interesting interpretations of a number of phenomena. However, impermeability is a consequence of the definition of potential vorticity rather than the equations of motion, and in that sense it is a kinematic and not dynamical property.

To derive the result we define $s \equiv \rho Q = \nabla \cdot (\theta \boldsymbol{\omega}_a)$, where $\boldsymbol{\omega}_a$ is the absolute vorticity, and integrate over some volume V to give

$$I = \int_V s \, \mathrm{d}V = \int_V \nabla \cdot (\theta \boldsymbol{\omega}_a) \, \mathrm{d}V = \int_S \theta \boldsymbol{\omega}_a \cdot \mathrm{d}\boldsymbol{S}, \tag{4.125}$$

using the divergence theorem, where S is the surface surrounding the volume V. If this is an isentropic surface then we have

$$I = \theta \int_S \boldsymbol{\omega}_a \cdot \mathrm{d}\boldsymbol{S} = \theta \int_V \nabla \cdot \boldsymbol{\omega}_a \, \mathrm{d}V = 0, \tag{4.126}$$

again using the divergence theorem. That is, over a volume wholly enclosed by a single isentropic surface the integral of s vanishes. If the volume is bounded by more than one isentropic surface none of which intersect the surface, for example by concentric spheres of different radii as in Fig. 4.11(a), the result still holds. The quantity s is called 'potential vorticity concentration', or 'PV concentration'. The integral of s over a volume is akin to the total amount of a conserved material property, such as salt content, and so may be called 'PV substance'. That is, the PV concentration is the amount of potential vorticity substance per unit volume and

$$\text{PV substance} = \int s \, \mathrm{d}V = \int \rho Q \, \mathrm{d}V. \tag{4.127}$$

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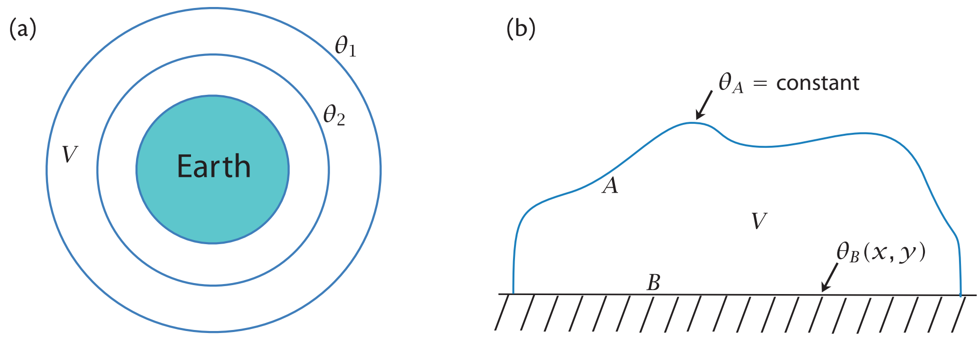

Fig. 4.11 (a) Two isentropic surfaces that do not intersect the ground. The integral of PV concentration over the volume between them, V, is zero, even if there is heating and the contours move. (b) An isentropic surface, A, intersects the ground, B, thus enclosing a volume V. The rate of change of PV concentration over the volume is given by an integral over B.

Suppose now that the fluid volume is enclosed by an isentrope that intersects the ground, as in Fig. 4.11(b). Let A denote the isentropic surface, B denote the ground, θ_A the constant value of θ on the isentrope, and $\theta_B(x, y, t)$ the non-constant value of θ on the ground. The integral of s over the volume is then

$$\begin{aligned} I = \int_V \nabla \cdot (\theta \boldsymbol{\omega}_a) \, \mathrm{d}V &= \theta_A \int_A \boldsymbol{\omega}_a \cdot \mathrm{d}\boldsymbol{S} + \int_B \theta_B \boldsymbol{\omega}_a \cdot \mathrm{d}\boldsymbol{S} \\ &= \theta_A \int_{A+B} \boldsymbol{\omega}_a \cdot \mathrm{d}\boldsymbol{S} + \int_B (\theta_B - \theta_A) \boldsymbol{\omega}_a \cdot \mathrm{d}\boldsymbol{S} \\ &= \int_B (\theta_B - \theta_A) \boldsymbol{\omega}_a \cdot \mathrm{d}\boldsymbol{S}. \end{aligned} \tag{4.128}$$

The first term on the second line vanishes after using the divergence theorem. Thus, the value of I, and hence its rate of change, is a function *only of an integral over the surface B,* and the PV flux there must be calculated using the full equations of motion. However, we do not need to be concerned with any flux of PV concentration through the isentropic surface; put another way, the PV substance in a volume can change only when isentropes enclosing the volume intersect a boundary such as the Earth's surface.

4.8.1 Interpretation and Application

Motion of the isentropic surface

How can the above results hold in the presence of heating? The isentropic surfaces must move in such a way that the total amount of PV concentration contained between them nevertheless stays fixed, and we now demonstrate this explicitly. The potential vorticity equation may be written

$$\frac{\partial Q}{\partial t} + \boldsymbol{v} \cdot \nabla Q = S_Q, \tag{4.129}$$

where, from (4.96), $S_Q = (\boldsymbol{\omega}_a/\rho) \cdot \nabla\dot{\theta} + \nabla\theta \cdot (\nabla \times \boldsymbol{F})/\rho$. Using mass continuity this may be written as

$$\frac{\partial s}{\partial t} + \nabla \cdot \boldsymbol{J} = 0, \tag{4.130}$$

where $\boldsymbol{J} \equiv \rho \boldsymbol{v} Q + \boldsymbol{N}$ and $\nabla \cdot \boldsymbol{N} = -\rho S_Q$. Written in this way, the quantity $\boldsymbol{J}/s$ is a notional velocity, $\boldsymbol{v}_Q$ say, and s satisfies

$$\frac{\partial s}{\partial t} + \nabla \cdot (\boldsymbol{v}_Q s) = 0. \tag{4.131}$$

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That is, s evolves as if it were being fluxed by the velocity $\boldsymbol{v}_Q$. The concentration of a chemical tracer χ (i.e., χ is the amount of tracer per unit volume) obeys a similar equation, to wit

$$\frac{\partial \chi}{\partial t} + \nabla \cdot (\boldsymbol{v}\chi) = 0. \tag{4.132}$$

However, whereas (4.132) implies that $\mathrm{D}(\chi/\rho)/\mathrm{D}t = 0$, (4.131) does not imply that $\partial Q/\partial t + \boldsymbol{v}_Q \cdot \nabla Q = 0$ because $\partial \rho/\partial t + \nabla \cdot (\rho \boldsymbol{v}_Q) \neq 0$.

Now, the impermeability result tells us that there can be no notional velocity across an isentropic surface. How can this be satisfied by the equations of motion? We write the right-hand side of (4.129) as

$$\rho S_Q = \nabla \cdot (\dot{\theta}\boldsymbol{\omega}_a + \theta \nabla \times \boldsymbol{F}) = \nabla \cdot (\dot{\theta}\boldsymbol{\omega}_a + \boldsymbol{F} \times \nabla\theta). \tag{4.133}$$

Thus, $\boldsymbol{N} = -\dot{\theta}\boldsymbol{\omega}_a - \boldsymbol{F} \times \nabla\theta$ and we may write the $\boldsymbol{J}$ vector as

$$\boldsymbol{J} = \rho \boldsymbol{v} Q - \dot{\theta}\boldsymbol{\omega}_a - \boldsymbol{F} \times \nabla\theta = \rho Q(\boldsymbol{v}_\perp + \boldsymbol{v}_\parallel) - \dot{\theta}\boldsymbol{\omega}_\parallel - \boldsymbol{F} \times \nabla\theta, \tag{4.134}$$

where, making use of the thermodynamic equation,

$$\boldsymbol{v}_\parallel = \boldsymbol{v} - \frac{\boldsymbol{v} \cdot \nabla\theta}{|\nabla\theta|^2}\nabla\theta, \qquad \boldsymbol{v}_\perp = -\frac{\partial\theta/\partial t}{|\nabla\theta|^2}\nabla\theta, \tag{4.135a}$$

$$\boldsymbol{\omega}_\parallel = \boldsymbol{\omega}_a - \frac{\boldsymbol{\omega}_a \cdot \nabla\theta}{|\nabla\theta|^2}\nabla\theta = \boldsymbol{\omega}_a - \frac{Q\rho}{|\nabla\theta|^2}\nabla\theta. \tag{4.135b}$$

The subscripts '$\perp$' and '$\parallel$' denote components perpendicular and parallel to the local isentropic surface, and $\boldsymbol{v}_\perp$ is the velocity of the isentropic surface normal to itself. Equation (4.134) may be verified by using (4.135) and $\mathrm{D}\theta/\mathrm{D}t = \dot{\theta}$.

The 'parallel' terms in (4.135) are all vectors parallel to the local isentropic surface, and therefore do not lead to any flux of PV concentration across that surface. Furthermore, the term $\rho Q \boldsymbol{v}_\perp$ is ρQ multiplied by the normal velocity of the surface. That is to say, the notional velocity associated with the flux normal to the isentropic surface is equal to the normal velocity of the isentropic surface itself, and so it too provides no flux of PV concentration across that surface (even though there may well be a mass flux across the surface). Put simply, the isentropic surface always moves in such a way as to ensure that there is no flux of PV concentration across it. In our proof of the impermeability result in the previous subsection we used the fact that the potential vorticity multiplied by the density is the divergence of a vector. In the demonstration above we used the fact that the terms *forcing* potential vorticity are the divergence of a vector.

† *Dynamical choices of PV flux and a connection to Bernoulli's theorem*

If we add a non-divergent vector to the flux, $\boldsymbol{J}$, then it has no effect on the evolution of s. This gauge invariance means that the notional velocity, $v_Q = \boldsymbol{J}/(\rho Q)$ is similarly non-unique, although it does not mean that there are not dynamical choices for it that are more appropriate in given circumstances. To explore this, let us obtain a general expression for $\boldsymbol{J}$ by starting with the definition of s, so that

$$\begin{aligned}\frac{\partial s}{\partial t} &= \nabla\theta \cdot \frac{\partial \boldsymbol{\omega}_a}{\partial t} + \boldsymbol{\omega}_a \cdot \nabla\frac{\partial\theta}{\partial t} \\ &= \nabla\theta \cdot \nabla \times \frac{\partial \boldsymbol{v}}{\partial t} + \nabla \cdot \left(\boldsymbol{\omega}_a \frac{\partial\theta}{\partial t}\right) = -\nabla \cdot \boldsymbol{J}',\end{aligned} \tag{4.136}$$

where

$$\boldsymbol{J}' = \nabla\theta \times \frac{\partial \boldsymbol{v}}{\partial t} - \frac{\partial\theta}{\partial t}\boldsymbol{\omega}_a + \nabla\phi \times \nabla\chi. \tag{4.137}$$

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The last term in this expression is an arbitrary divergence-free vector. If we choose $\phi = \theta$ and $\chi = B$, where B is the Bernoulli function given by $B = I + \boldsymbol{v}^2/2 + p/\rho$ where I is the internal energy per unit mass, then

$$\boldsymbol{J}' = \nabla\theta \times \left(\nabla B + \frac{\partial \boldsymbol{v}}{\partial t}\right) - \boldsymbol{\omega}_a(\dot{\theta} - \boldsymbol{v}\cdot\nabla\theta), \tag{4.138}$$

having used the thermodynamic equation $\mathrm{D}\theta/\mathrm{D}t = \dot{\theta}$. Now, the momentum equation may be written, without approximation, in the form

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\boldsymbol{\omega}_a \times \boldsymbol{v} + T\nabla\eta + \boldsymbol{F} - \nabla B, \tag{4.139}$$

where η is the specific entropy ($\mathrm{d}\eta = c_p \,\mathrm{d}\ln\theta$). Using (4.138) and (4.139) gives

$$\boldsymbol{J}' = \rho Q\boldsymbol{v} - \dot{\theta}\boldsymbol{\omega}_a + \nabla\theta \times \boldsymbol{F}, \tag{4.140}$$

which is the same as (4.134). Furthermore, using (4.137) for steady flow,

$$\boldsymbol{J} = \nabla\theta \times \nabla B. \tag{4.141}$$

That is, the flux of potential vorticity (in this gauge) is aligned with the intersection of θ- and B-surfaces. For steady *inviscid and adiabatic* flow the Bernoulli function is constant along streamlines; that is, surfaces of constant Bernoulli function are aligned with streamlines, and, because θ is materially conserved, streamlines are formed at intersecting θ- and B-surfaces, as in (1.204). In the presence of forcing, this property is replaced by (4.141), and the flux of PV concentration is along such intersections.

This choice of gauge leading to (4.140) is physical in that it reduces to the true advective flux $\boldsymbol{v}\rho Q$ for unforced, adiabatic flow, but it is not a unique choice, nor is it mandated by the dynamics. Choosing $\chi = 0$ leads to the flux

$$\boldsymbol{J}_1 = \rho Q\boldsymbol{v} - \dot{\theta}\boldsymbol{\omega}_a + \nabla\theta \times (\boldsymbol{F} - \nabla B), \tag{4.142}$$

and using (4.137) this vanishes for steady flow, which is a potentially useful property.

Summary remarks

The impermeability result is kinematic, but can provide an interesting point of view and useful diagnostic tool.[11] We make the following summary remarks:

- There can be no net transport of potential vorticity across an isentropic surface, and the total amount of potential vorticity in a volume wholly enclosed by isentropic surfaces is zero. Thus, and with hindsight trivially, the amount of potential vorticity contained between two isentropes isolated from the Earth's surface in the Northern Hemisphere is the negative of the corresponding amount in the Southern Hemisphere.
- Potential vorticity flux lines (i.e., lines everywhere parallel to $\boldsymbol{J}$) can either close in on themselves or begin and end at boundaries (e.g., the ground or the ocean surface). However, $\boldsymbol{J}$ may change its character. Thus, for example, at the base of the oceanic mixed layer $\boldsymbol{J}$ may change from being a diabatic flux above to an adiabatic advective flux below. There may be a similar change in character at the atmospheric tropopause.
- The flux vector $\boldsymbol{J}$ is defined only to within the curl of a vector. Thus the vector $\boldsymbol{J}' = \boldsymbol{J} + \nabla\times\boldsymbol{A}$, where $\boldsymbol{A}$ is arbitrary, is as valid as is $\boldsymbol{J}$ in the above derivations.

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Notes

1 The frozen-in property — that vortex lines are material lines — was derived by Helmholtz (1858) and is sometimes called Helmholtz's theorem.

2 The theorem originates with William Thomson (1824–1907), who became Lord Kelvin in 1892. The circulation theorem was published in Thomson (1869) and is a conservation law that is unique to a fluid: unlike, for example, the conservation of energy, it has no analogue in solid-body mechanics. Thomson was born in Belfast but spent most of his life in Scotland, becoming a professor at the University of Glasgow in 1846 (at the age of 22!) and staying there for 53 years. A prolific and creative scientist, he made a lasting impact on both fluid dynamics and thermodynamics — among other achievements he proposed an absolute temperature scale and a formulation of the second law of thermodynamics. Later in life he turned to engineering and was one of the proponents of a telegraph cable under the Atlantic. To his credit he also had some grand failures — his estimates of the age of Earth and how long oxygen would last in the atmosphere were both wrong by orders of magnitude.

3 Silberstein (1896) proved that 'the necessary and sufficient condition for the generation of vortical flow...influenced only by conservative forces ...is that the surface of constant pressure and surface of constant density...intersect', as we derived in Section 4.2, and this leads to (4.43). Bjerknes (1898a,b) explicitly put this into the form of a circulation theorem and applied it to problems of meteorological and oceanographic importance (see Thorpe *et al.* 2003), and the theorem is sometimes called the Bjerknes theorem or the Bjerknes–Silberstein theorem.

Vilhelm Bjerknes (1862–1951) was a physicist and hydrodynamicist who in 1917 moved to the University of Bergen as founding head of the Bergen Geophysical Institute. Here he did what was probably his most influential work in meteorology, setting up and contributing to the 'Bergen School of Meteorology'. Among other things he and his colleagues were among the first to consider, as a practical proposition, the use of numerical methods — initial data in conjunction with the fluid equations of motion — to forecast the state of the atmosphere, based on earlier work describing how that task might be done (Abbe 1901, Bjerknes 1904). Inaccurate initial velocity fields compounded with the shear complexity of the effort ultimately defeated them, but the effort was continued (also unsuccessfully) by L. F. Richardson (Richardson 1922), before J. Charney, R. Fjørtoft and J. Von Neumann eventually made what may be regarded as the first successful numerical forecast (Charney *et al.* 1950). Their success can be attributed to the used of a simplified, filtered, set of equations and the use of an electronic computer.

4 The result (4.50) was given by Poincaré (1893) although it is sometimes attributed to Bjerknes (1902).

5 The first derivation of the PV conservation law was given for the hydrostatic shallow water equations by Rossby (1936), with a generalization to the stratified case, via the use of isentropic coordinates, in Rossby (1938) and Rossby (1940). In the 1936 paper Rossby noted — his Eq. (75) — that a fluid column satisfies $f + \zeta = cD$, where c is a constant and D is the thickness of a fluid column; equivalently, $(f + \zeta)/D$ is a material invariant. The expression 'potential vorticity' was introduced in Rossby (1940), as follows: *'This quantity, which may be called the potential vorticity, represents the vorticity the air column would have it it were brought, isopycnally or isentropically, to a standard latitude (f_0) and stretched or shrunk vertically to a standard depth D_0 or weight Δ_0.'* (Rossby's italics.) That is,

$$\text{potential vorticity} = \zeta_0 = \left(\frac{\zeta + f}{D}\right) D_0 - f_0, \tag{4.143}$$

which follows from his Eq. (11), and this is the sense he uses it in that paper. However, potential vorticity has come to mean the quantity $(\zeta + f)/D$, which of course does not have the dimensions of vorticity. We use it in this latter, now conventional, sense throughout this book. Ironically, quasi-geostrophic potential vorticity as usually defined does have the dimensions of vorticity.

The expression for potential vorticity in a non-hydrostatic, continuously stratified fluid was given by Ertel (1942a), and its relationship to circulation was given by Ertel (1942b). It is now commonly known as the *Ertel potential vorticity*, or the *Rossby–Ertel* potential vorticity. Interestingly, in Rossby

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(1940) we find the Fermat-like comment 'It is possible to derive corresponding results for an atmosphere in which the potential temperature varies continuously with elevation.... The generalized treatment will be presented in another place.' Given his prior (1938) derivation of the stratified quantity in isentropic coordinates, he must not have regarded his own derivations as very general. Opinions differ as to whether Rossby's and Ertel's derivations were independent, and Cressman (1996) remarks that the origin of the concept of potential vorticity is a 'delicate one that has aroused some passion in private correspondences'. In fact, Ertel visited MIT in autumn 1937 and presumably talked to Rossby and became aware of his work. It seems almost certain that Ertel knew of Rossby's shallow water and isentropic theorems, but it is also clear that Ertel subsequently provided a significant generalization, most likely independently. Rossby and Ertel apparently remained on good terms, but further collaboration was stymied by World War II. They later published a pair of short joint papers, one in German and the other in English, describing related conservation theorems (Ertel & Rossby 1949a,b). English translations of a number of Ertel's papers are to be found in Schubert *et al.* (2004). I thank Roger Samelson for enlightening me about the history of Rossby and Ertel.

6 Native French speakers may be confused by the difference. In English, an explication is a particular way of performing an analysis or presenting an explanation, so there can be different explications of the same mechanism.

7 Truesdell (1951, 1954) and Obukhov (1962) were early explorers of the consequences of heating and friction on potential vorticity. The work of F. P. Bretherton, R. E. Dickinson and J. S. A. Green (e.g., Bretherton 1966a, Dickinson 1969, Green 1970) helped bring potential vorticity ideas further into the mainstream of GFD.

8 Many thanks to Stephen Griffies for pointing out this argument, and for many other comments and discussions on matters related to this book. A study of saline effects on potential vorticity is to be found in Straub (1999).

9 Schubert *et al.* (2001) provide more discussion of these matters. They derive a 'moist PV' that is an extension of the dry Ertel PV to moist atmospheres and that has invertibility and impermeability properties.

10 Haynes & McIntyre (1987, 1990). See also Danielsen (1990), Schär (1993), who obtained the result (4.141), Bretherton & Schär (1993) and Davies-Jones (2003).

11 See, for example, McIntyre & Norton (1990) and Marshall & Nurser (1992). The latter use $\boldsymbol{J}$ vectors to study the creation and transport of potential vorticity in the oceanic thermocline.

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