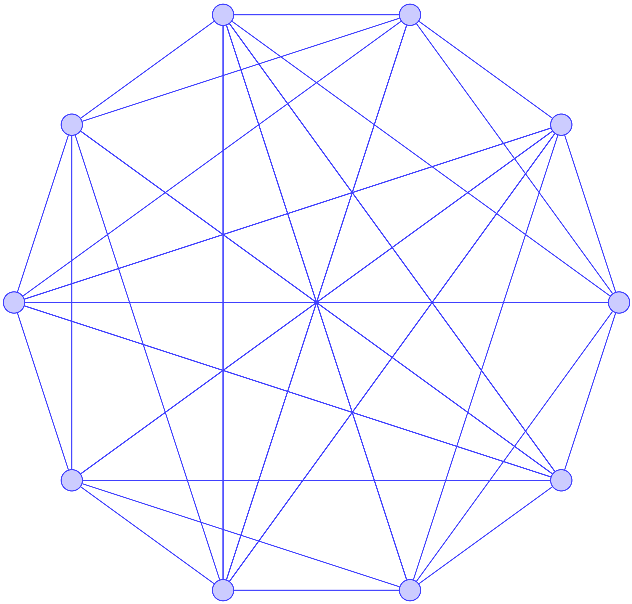
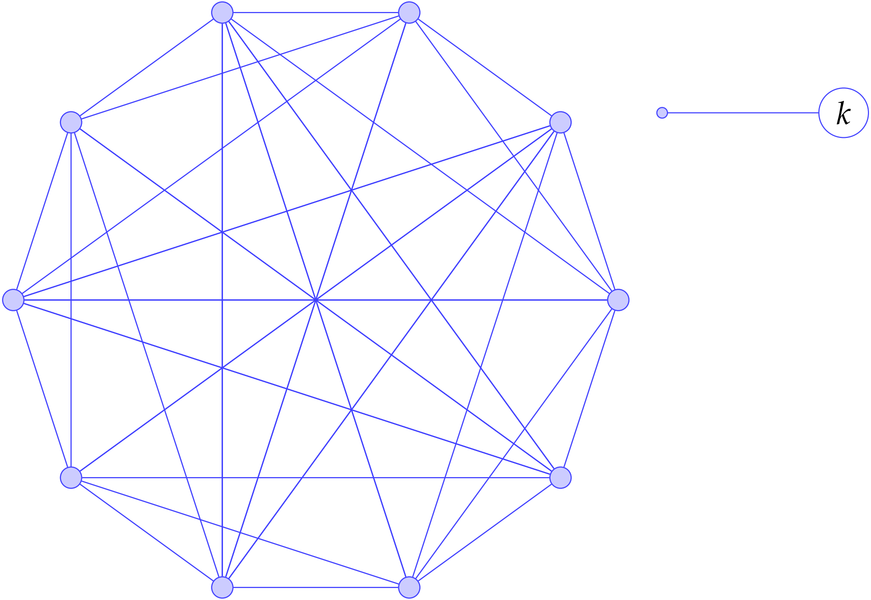
Edge-regular graphs and regular cliques

Gary Greaves

NTU, Singapore

7th August 2017





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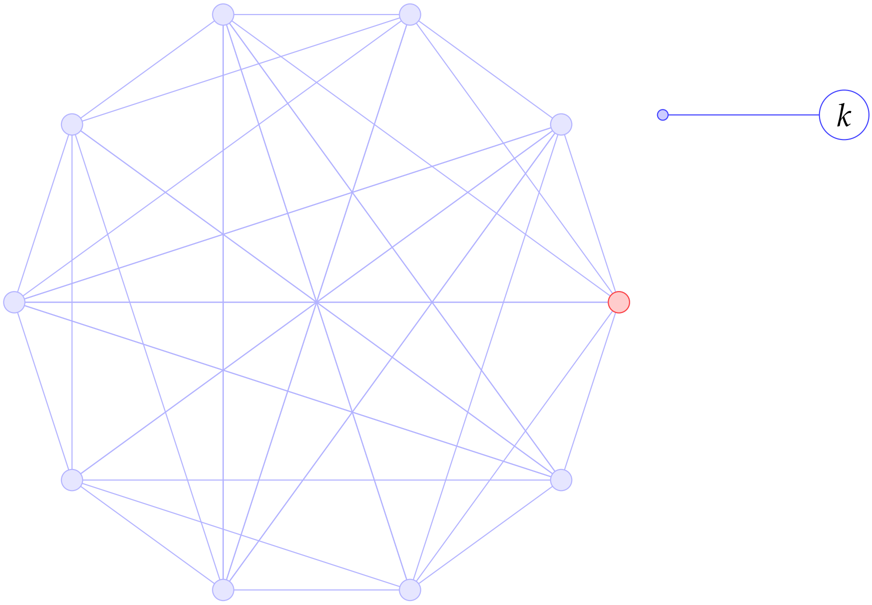
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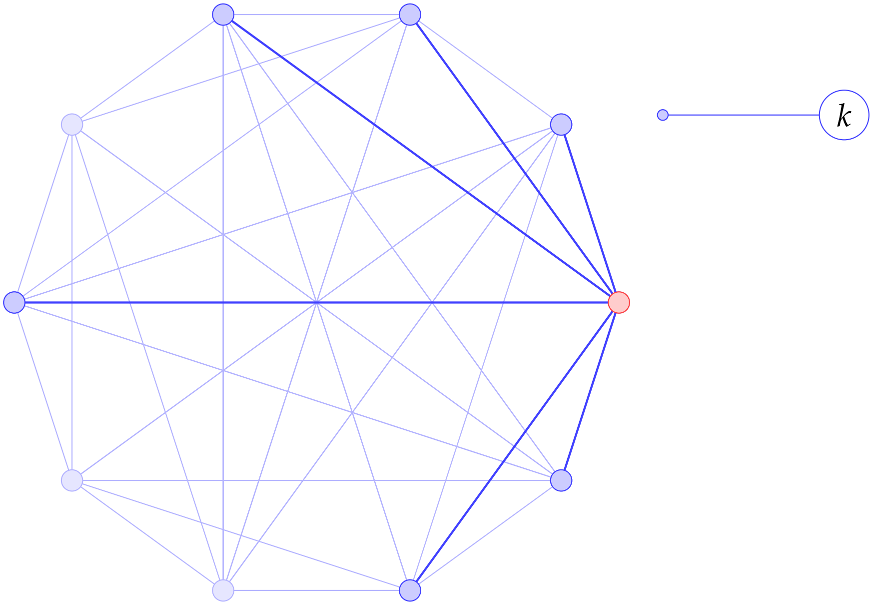
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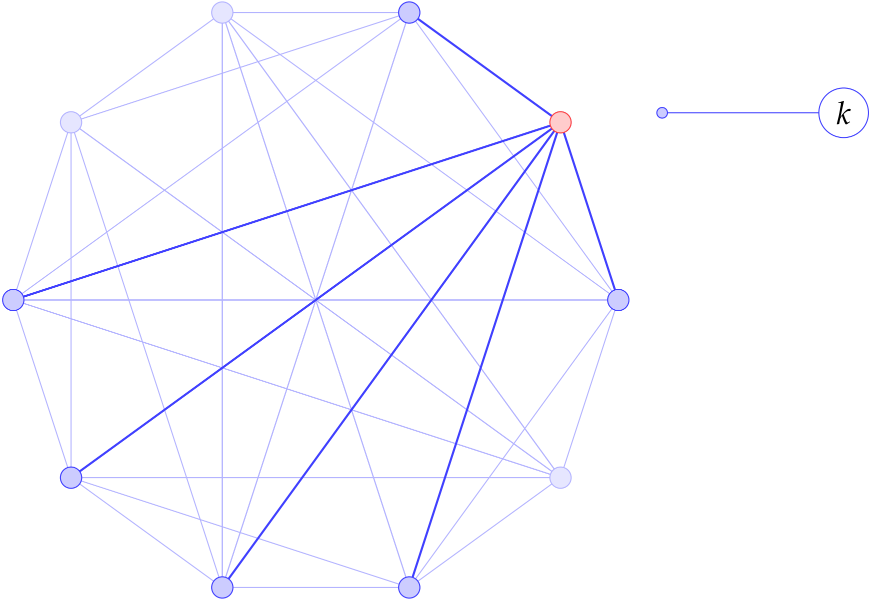
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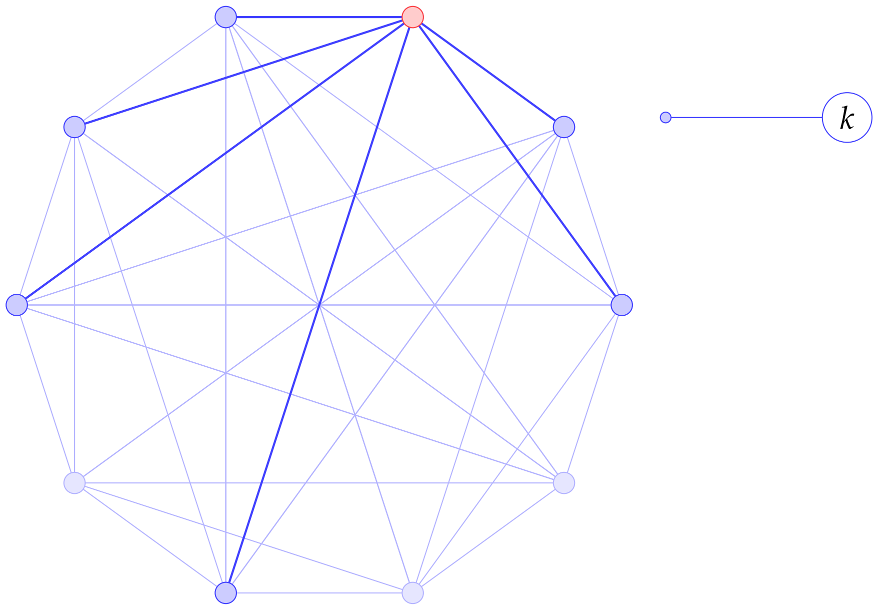
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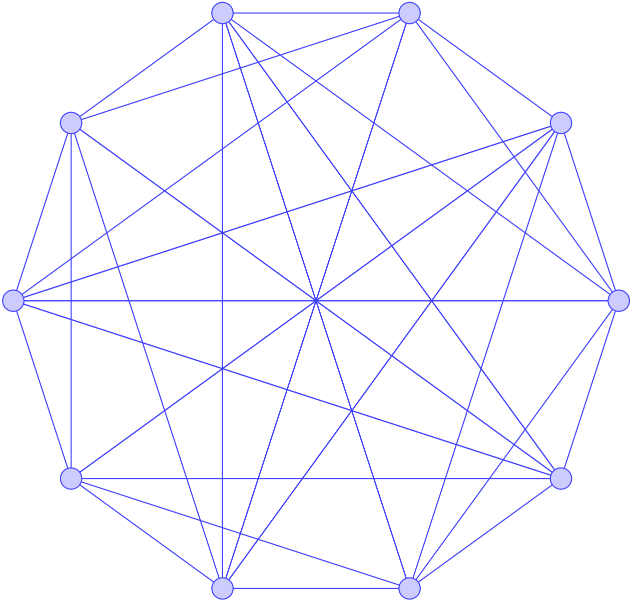

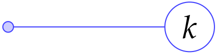

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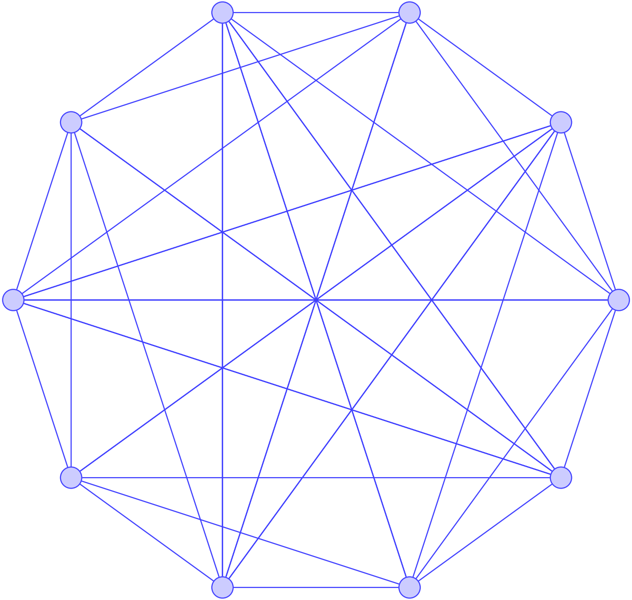

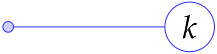

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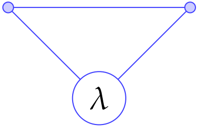

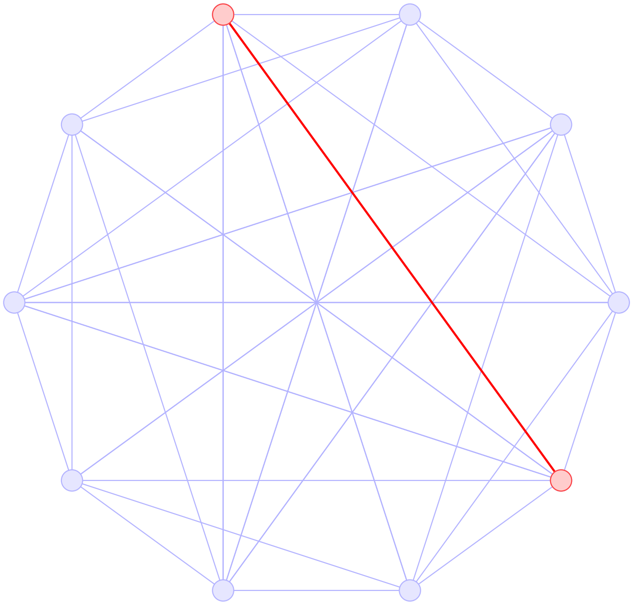

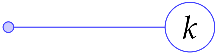

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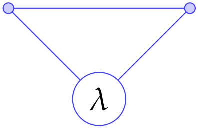

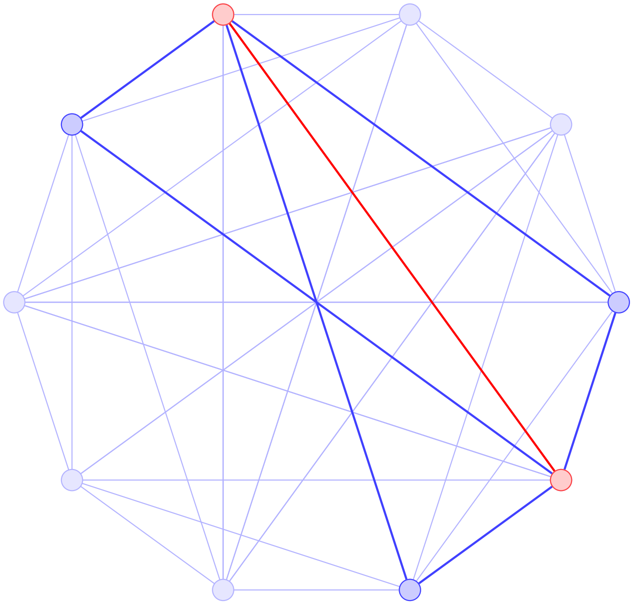

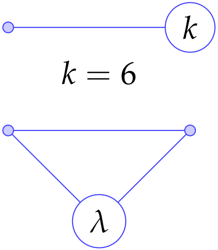
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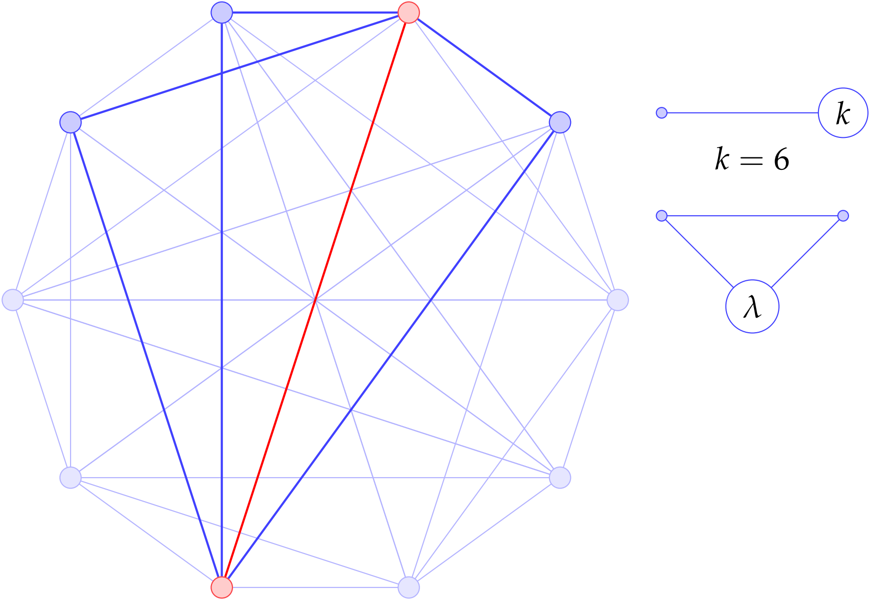
λ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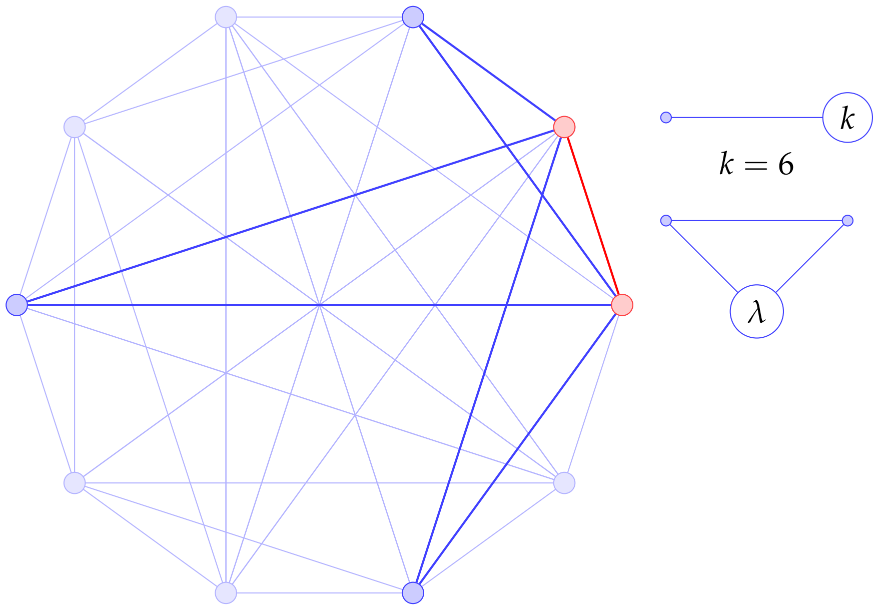

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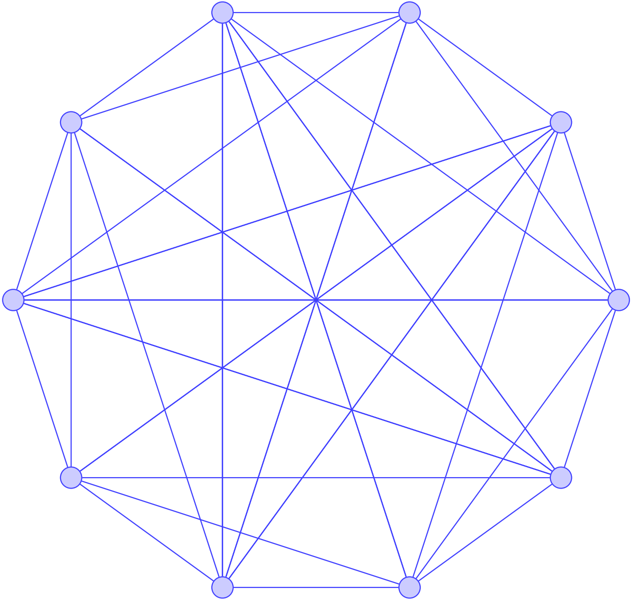

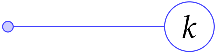

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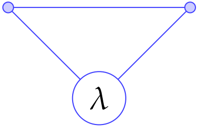
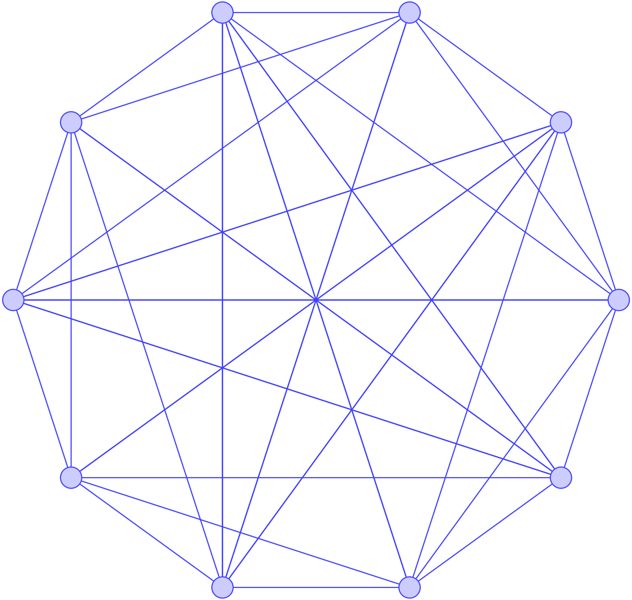
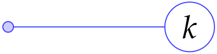


$\lambda = 3$





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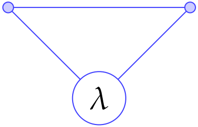

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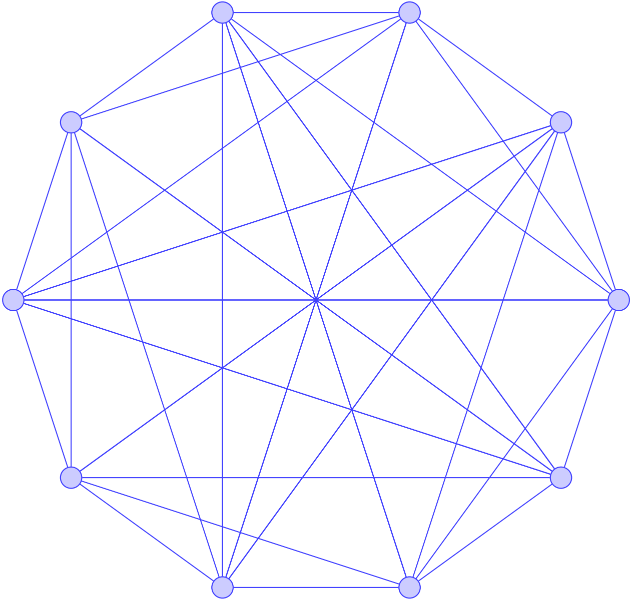
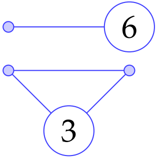
edge-regular

$\mathrm{erg}(10, 6, 3)$





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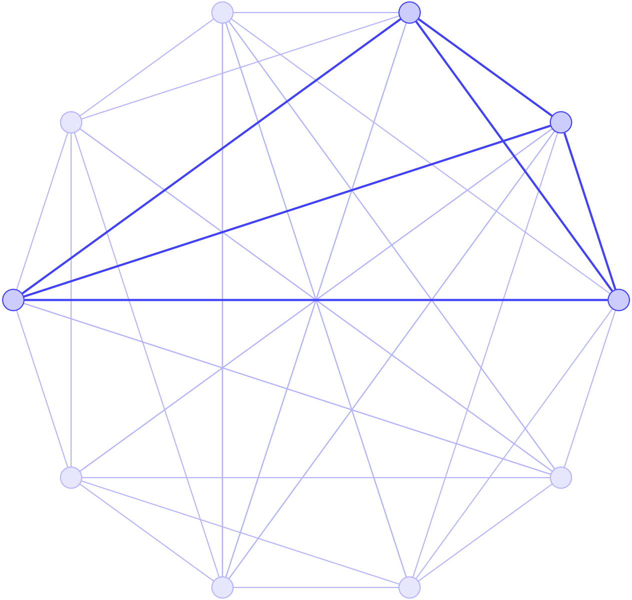

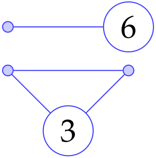

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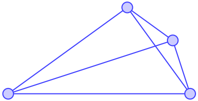
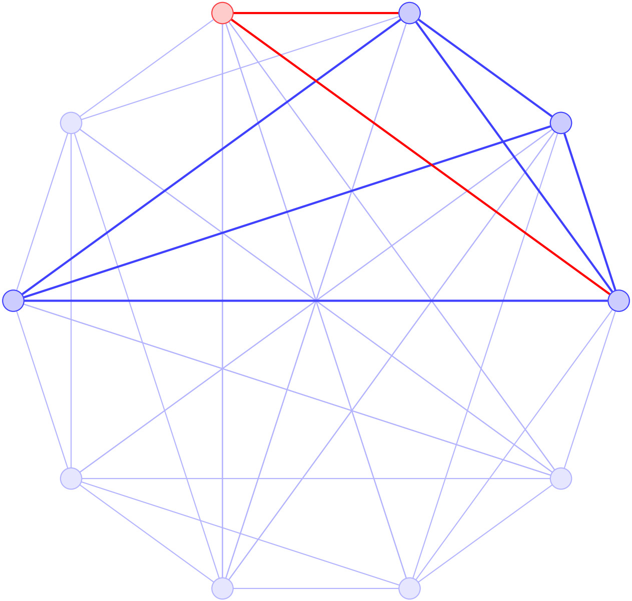
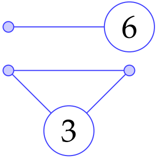


clique

of order 4





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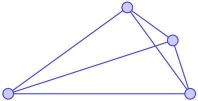

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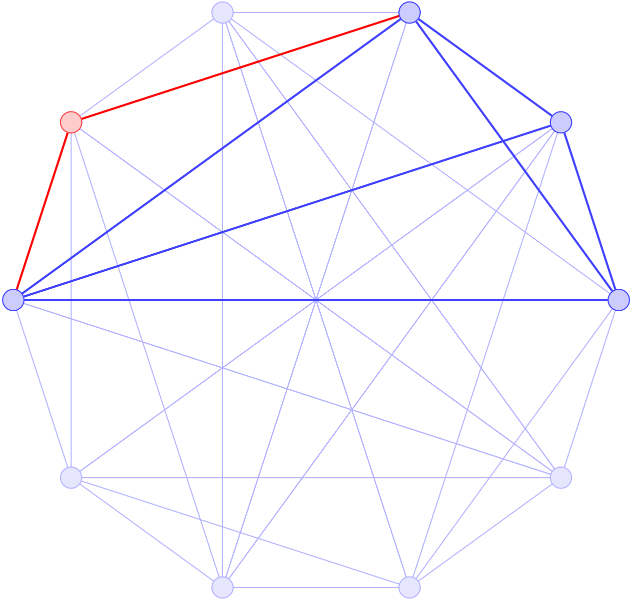

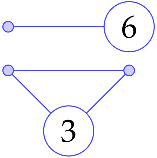

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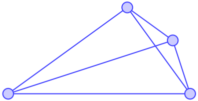

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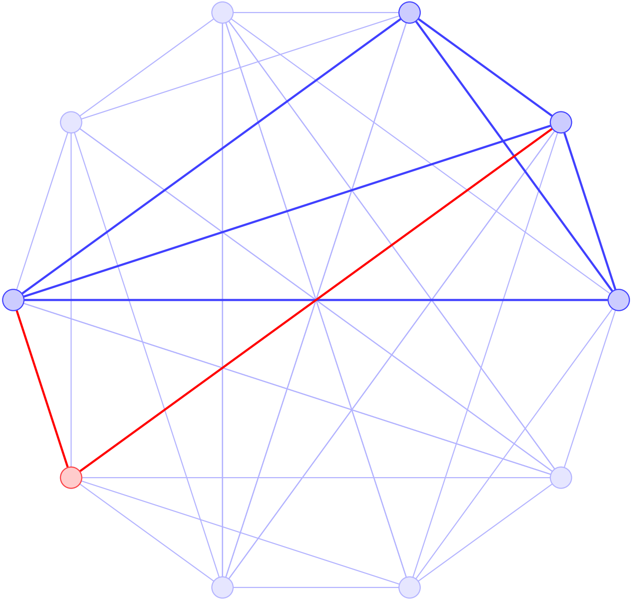

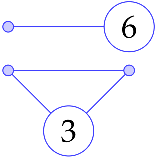

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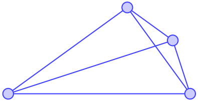

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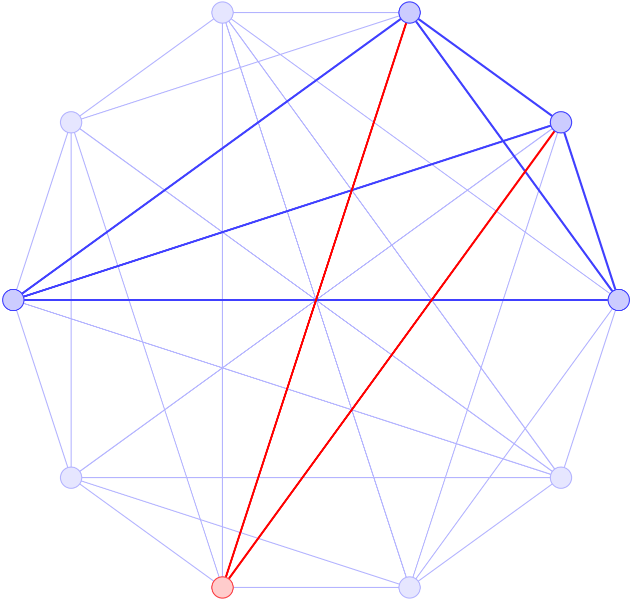

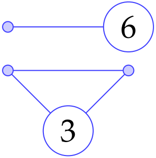

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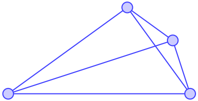

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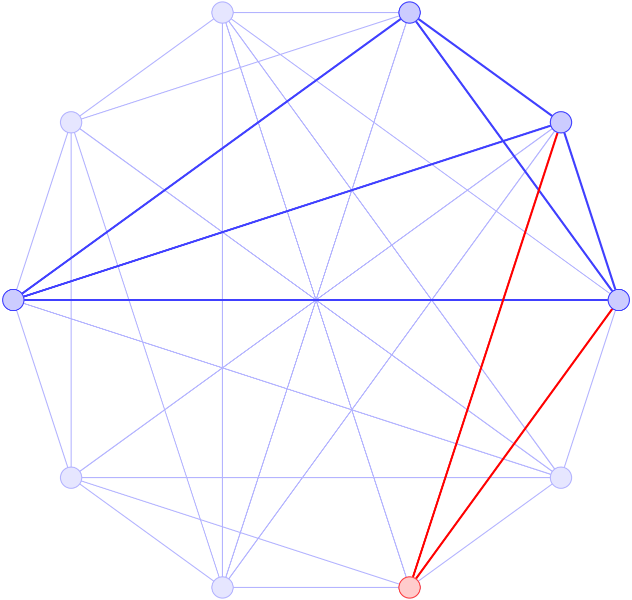

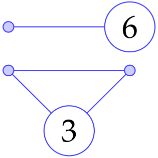

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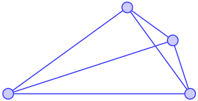

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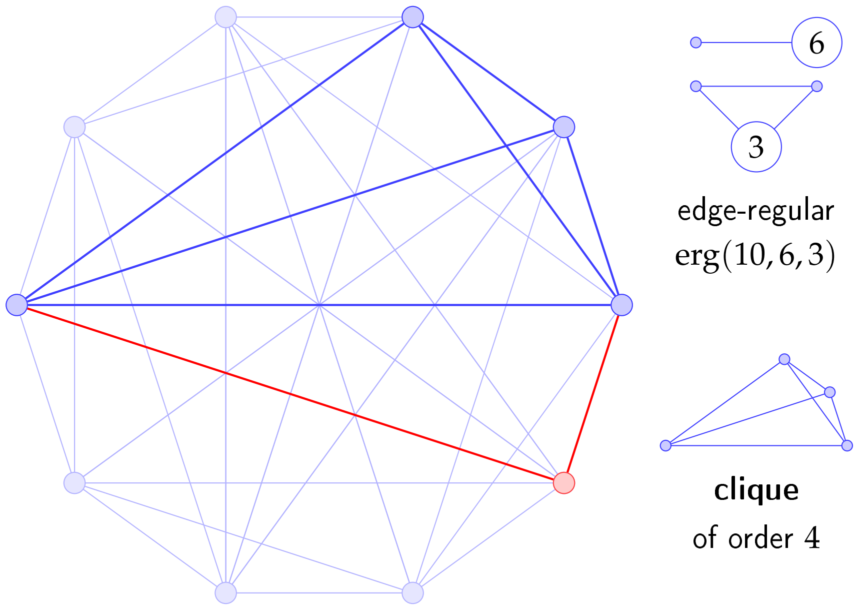
6
3
edge-regular
erg(10, 6, 3)
clique
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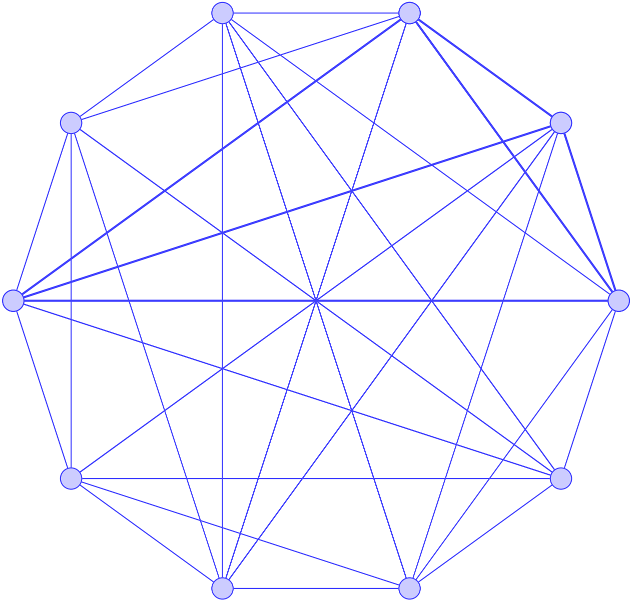

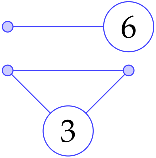

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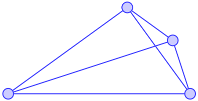

2-**regular** clique
of order 4

Theorem (Neumaier 1981)

Let Γ be edge-regular with a regular clique.
Suppose Γ is vertex-transitive and edge-transitive.
Then Γ is strongly regular.

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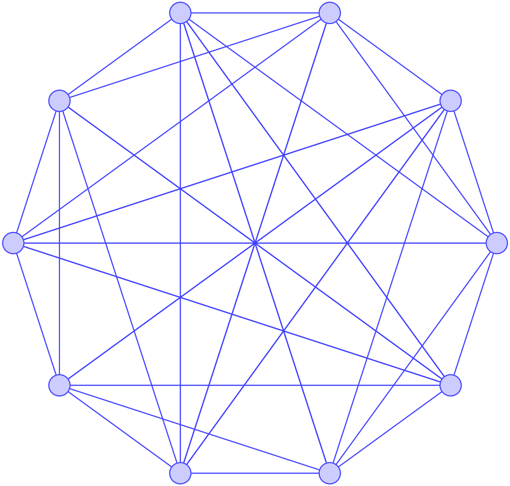

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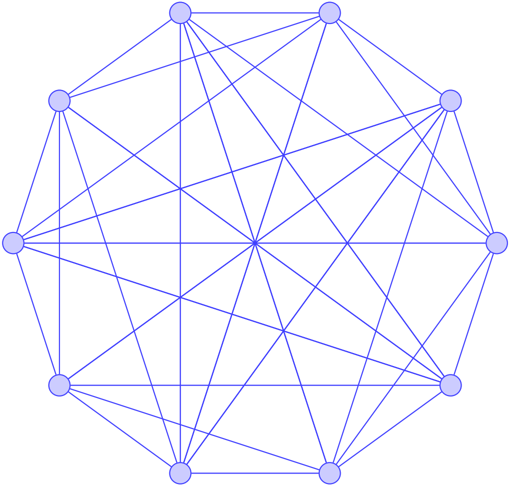

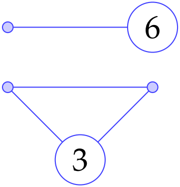

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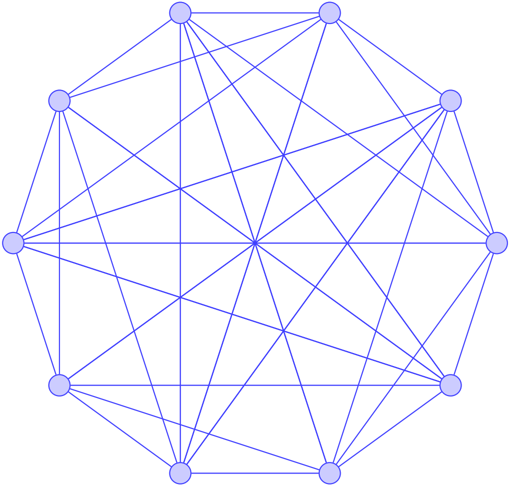

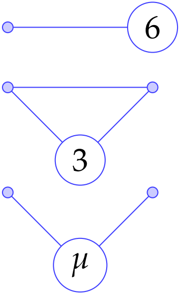

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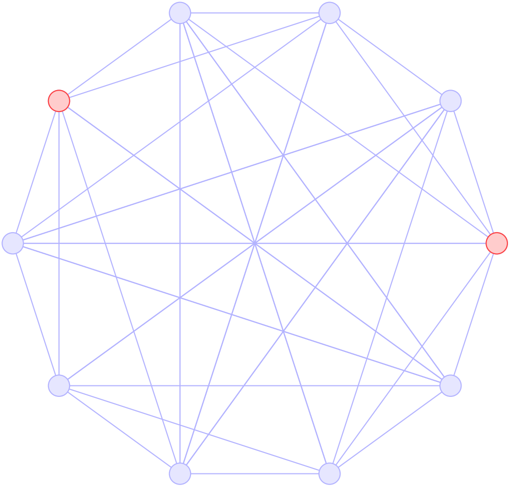

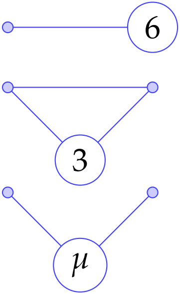

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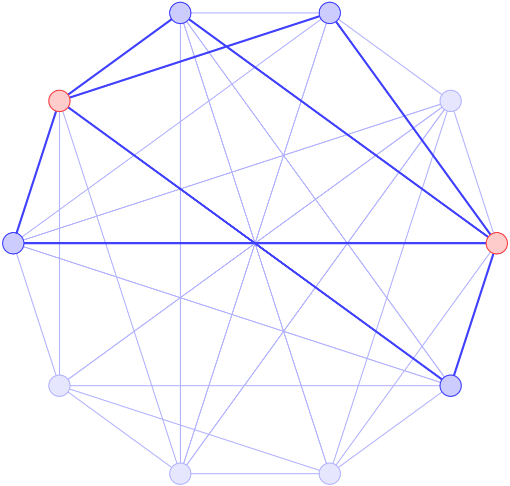

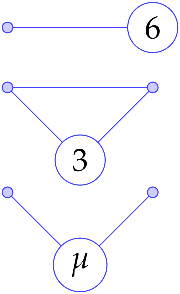

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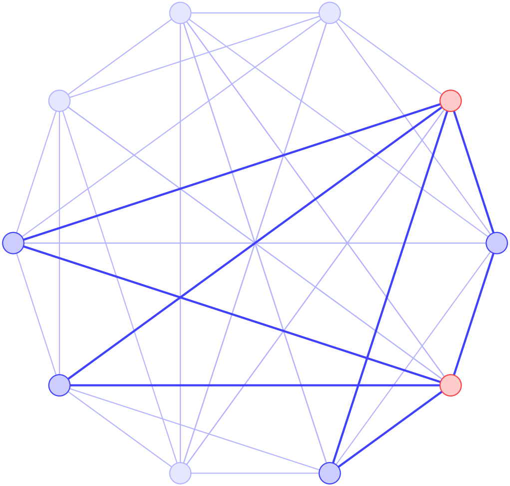

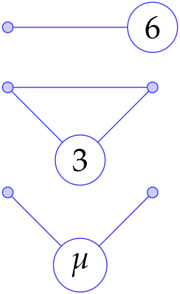

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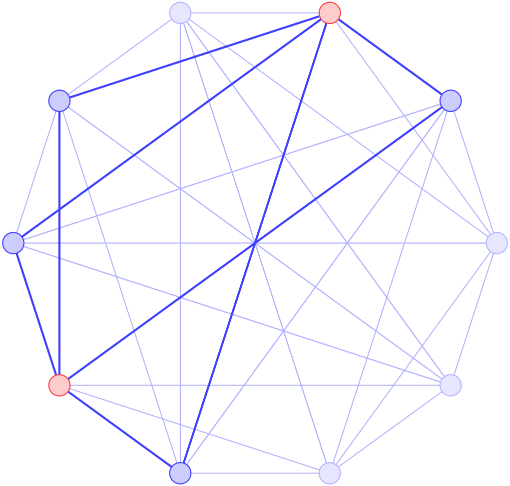

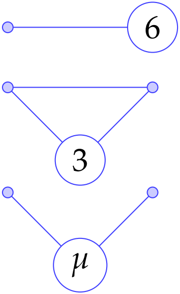

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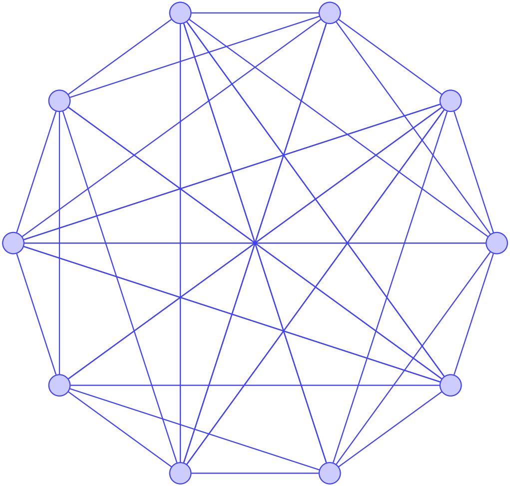

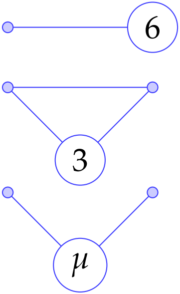

$\mu = 4$

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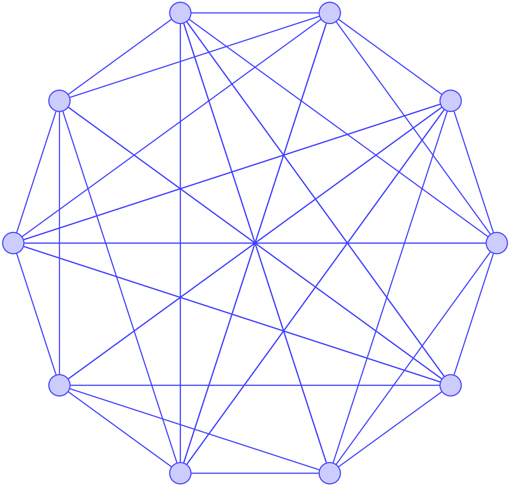

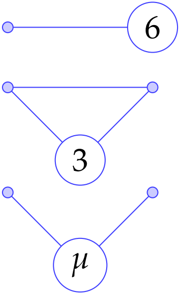

$\mu = 4$

strongly regular
$\mathrm{srg}(10, 6, 3, 4)$

Question (Neumaier 1981)

Is every edge-regular graph with a regular clique strongly regular?

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Answer (GG and Koolen 2017+)

No.

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Answer (GG and Koolen 2017+)

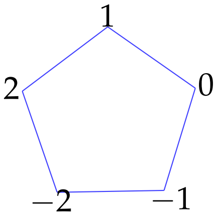
No. There exist infinitely many non-strongly-regular, edge-regular vertex-transitive graphs with regular cliques.

Cayley graphs

- Let G be an (additive) group and $S \subseteq G$ a (symmetric) generating subset, i.e., $s \in S \implies -s \in S$ and $G = \langle S \rangle$.
- The **Cayley graph** $\mathrm{Cay}(G, S)$ has vertex set G and edge set
$$\{\{g, g+s\} : g \in G \text{ and } s \in S\}.$$

Example

$\Gamma = \mathrm{Cay}(\mathbb{Z}_5, S)$ Generating set $S = \{-1, 1\}$



A construction

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$

Generating set S

$$\begin{array}{ll} (01,0) & (01,\pm 1) \\ (10,0) & (10,\pm 2) \\ (11,0) & (11,\pm 3) \end{array}$$

A construction

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$
- Γ is **edge-regular** $(28, 9, 2)$:

Generating set S

$$\begin{array}{ll} (01,0) & (01,\pm 1) \\ (10,0) & (10,\pm 2) \\ (11,0) & (11,\pm 3) \end{array}$$

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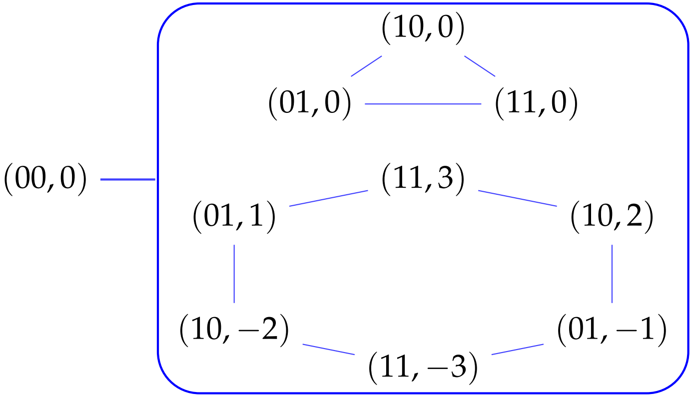

A construction

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$
- Γ is edge-regular $(28, 9, 2)$;

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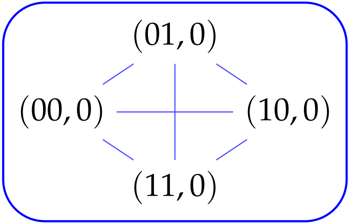

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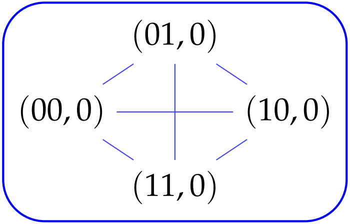

$(a, b) \quad b \neq 0$

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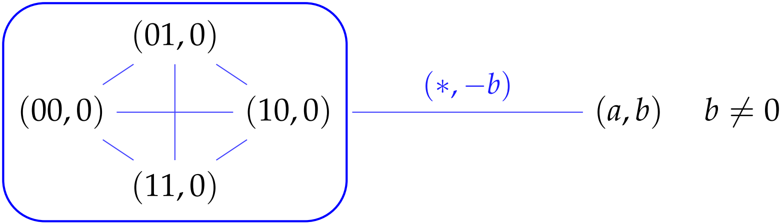

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- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$
- Γ is edge-regular $(28, 9, 2)$;
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- Γ is **not strongly regular**:

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	$(11, 0)$
$(01, 0)$	$(01, 1)$
$(11, 3)$	$(10, 0)$
$(10, -2)$	$(01, -1)$
$(11, -3)$	$(10, 2)$

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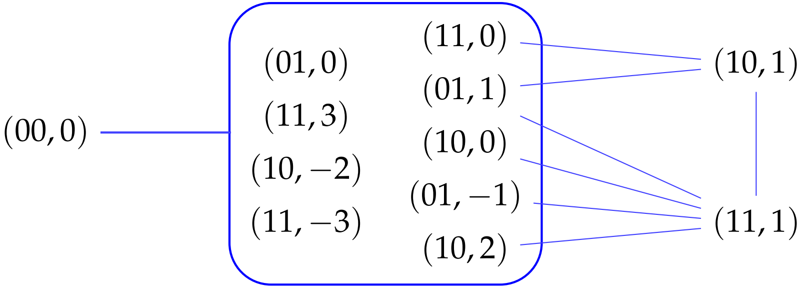

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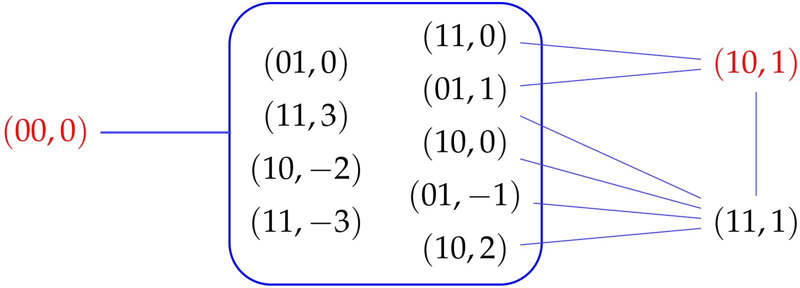

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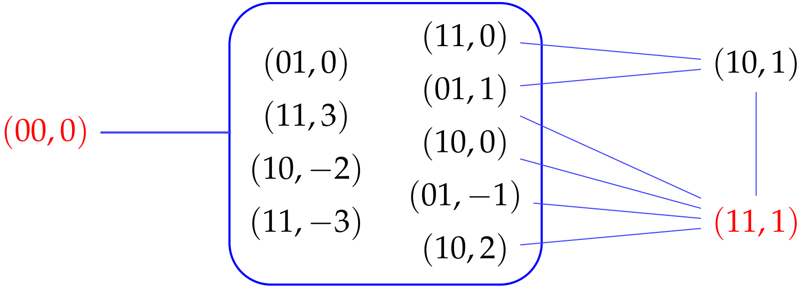

Some infinite families

- Generalise: $\mathbb{Z}_2^2 \oplus \mathbb{Z}_7$ to $\mathbb{Z}_{(c+1)/2} \oplus \mathbb{Z}_2^2 \oplus \mathbb{F}_q$.
- Works for $q \equiv 1 \pmod 6$ such that the 3rd cyclotomic number $c = c_q^3(1,2)$ is odd.
- Then there exists an $\mathrm{erg}(2(c+1)q, 2c+q, 2c)$ having a 1-regular clique of order $2c+2$.
- Take $p \equiv 1 \pmod 3$ a prime s.t. $2 \not\equiv x^3 \pmod p$. Then there exist a such that $c_{p^a}^3(1,2)$ is odd.

An example in the wild

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MSC 05C

КЭЛИ-ДЕЗА ГРАФЫ, ИМЕЮЩИЕ МЕНЕЕ 60 ВЕРШИН

С.В. ГОРЯИНОВ, Л.В. ШАЛАГИНОВ

Abstract. Deza graph, which is the Cayley graph is called the Cayley-Deza graph. The paper describes all non-isomorphic Cayley-Deza graphs of diameter 2 having less than 60 vertices.

Keywords: Deza graph, Cayley graph, graph isomorphism, automorphism group.

1. Введение

В этой статье мы начинаем изучение графов Деза, которые являются графами Кэли. Графы Деза принято рассматривать как обобщение сильно регулярных графов. В ряде исследований было выяснено, что графы Деза наследуют некоторые свойства сильно регулярных графов. Например, в [1] показано, что вершинная связность графа Деза, полученного из сильно регулярного графа с помощью инволюции, совпадает с валетностью.

$\mathrm{erg}(24, 8, 2)$ with a 1-regular clique

Open problems

- Smallest non-strongly-regular, edge-regular graph with regular clique
- All known examples have 1-regular cliques
- Find general construction that includes $\mathrm{erg}(24, 8, 2)$

Willem Haemers,
Felix Lazebnik,
and Andrew Woldar;

Congratulations!