Equiangular lines

Mini-course

Gary Greaves

Graph theory, algebraic combinatorics and mathematical physics,
Centre de recherches mathématiques
Université de Montréal
Canada

August 2022

Contents

Chapter 1

Introduction

In this first chapter, we introduce the basic theory for equiangular lines. There has been much work in this area, and we do not attempt to give a comprehensive account of it. For a nice overview of the history of equiangular lines, we refer to the appendix of Kao-Yu [24].

1.1 Equiangular lines

Definition 1.1. A set of lines $l_1, l_2, \ldots, l_n$ through the origin of $\mathbb{R}^d$ is called **equiangular** if the angle between any two lines is constant. More precisely, let each l_i be spanned by a unit vector $\mathbf{u}_i$, then X is equiangular if there exists an $\alpha \in \mathbb{R}$ such that $|\langle \mathbf{u}_i, \mathbf{u}_j \rangle| = \alpha$ for all $i \neq j$. Here, we are using the usual Euclidean inner product: $\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{x}^\intercal \mathbf{y}$. We call α the **common angle** of the equiangular line system X.

Now for some examples.

Example 1.2. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be an orthonormal basis for $\mathbb{R}^n$. Then $\{[\mathbf{v}_1], \ldots, [\mathbf{v}_n]\}$ is an equiangular line system with angle $\alpha = 0$.

Example 1.3. The lines through the antipodes of a regular hexagon centred at the origin of $\mathbb{R}^2$ form a set of 3 equiangular lines in $\mathbb{R}^2$ with angle $\alpha = 1/2$. Alternatively, the set $\{[(1, 0)], [(1/2, \sqrt{3}/2)], [(1/2, -\sqrt{3}/2)]\}$ is an equiangular line system with angle $\alpha = 1/2$.

Example 1.4. The lines through the antipodes of a regular icosahedron centred at the origin of $\mathbb{R}^3$ form a set of 6 equiangular lines in $\mathbb{R}^3$ with angle $\alpha = 1/\sqrt{5}$. Alternatively, take all cyclic permutations of the vector

$$\mathbf{v} = \frac{1}{\sqrt{\varphi + 2}}(\varphi, \pm 1, 0),$$

where $\varphi = (1 + \sqrt{5})/2$. Let $\mathbf{u}_1, \ldots, \mathbf{u}_6$ be the resulting 6 unit vectors. Then the set $\{[\mathbf{u}_1], \ldots, [\mathbf{u}_6]\}$ is an equiangular line system with angle $\alpha = 1/\sqrt{5}$.

Example 1.5. Take all permutations of the entries of the vector

$$\mathbf{v} = \frac{1}{\sqrt{24}}(3, 3, -1, -1, -1, -1, -1, -1).$$

Let $\mathbf{u}_1, \ldots, \mathbf{u}_{28}$ be the resulting 28 unit vectors. Then the set $\{[\mathbf{u}_1], \ldots, [\mathbf{u}_{28}]\}$ is an equiangular line system with angle $\alpha = 1/3$. Each unit spanning vector $\mathbf{u}_i$ is orthogonal to the all-ones vector. Therefore, the lines lie in a 7-dimensional subspace of $\mathbb{R}^8$. Hence, there are 28 equiangular lines in $\mathbb{R}^7$.

Main problem. Given d, what is the largest cardinality $N(d)$ of a set of equiangular lines in $\mathbb{R}^d$?

1.1.1 Aim of this mini-course

Our aim is to introduce the tools required to find the values of $N(d)$ for $d \leqslant 23$. By the end of the course, one should be aware of the main ideas used to establish the following table (Table 1.1) of values for $N(d)$.

d	2	3	4	5	6	7–14	15	16	17	18	19	20	21	22	23
$N(d)$	3	6	6	10	16	28	36	40	48	57–59	72–74	90–94	126	176	276

Table 1.1: Bounds for the sequence $N(d)$ for $2 \leqslant d \leqslant 23$. A single number is given in the cases where the exact number is known. The latest developments can be found in [17].

The main tool at our disposal is the theory of *Seidel matrices*, which we introduce next.

1.1.2 Seidel matrices

Start with a set of equiangular lines $X = \{[\mathbf{u}_1], \dots, [\mathbf{u}_n]\}$ with angle $\alpha > 0$. Note that it is reasonable to assume that $\alpha > 0$, since the case when $\alpha = 0$ corresponds to orthonormal bases. Form the Gram matrix $G(X) = (\langle \mathbf{u}_i, \mathbf{u}_j \rangle)_{i,j}$. Then the diagonal entries of G are equal to 1 and the off-diagonal entries are equal to $\pm\alpha$. The **Seidel matrix** for X is given by $S(X) := (G(X) - I)/\alpha$.

Observe that the smallest eigenvalue of the Seidel matrix $S(X)$ is $-1/\alpha$. Assume that the vectors $\mathbf{u}_1, \dots, \mathbf{u}_n$ span $\mathbb{R}^d$. Then the multiplicity of $-1/\alpha$ is $n - d$.

This process can be reversed. Indeed, let S be an $n \times n$ Seidel matrix, that is, a symmetric $\{0, 1\}$-matrix with zeros on the diagonal. Let λ_0 be the smallest eigenvalue of S with multiplicity $n - d$. Then $G = S/\lambda_0 + I$ is a positive semidefinite matrix with rank d. Hence, G is the Gram matrix for a set of n equiangular lines in $\mathbb{R}^d$.

1.1.3 Eigenvalues of Seidel matrices

Let M be a real symmetric matrix with r distinct eigenvalues $\theta_1 < \dots < \theta_r$ such that θ_i has multiplicity m_i. We write the spectrum of M as $\{[\theta_1]^{m_1}, \dots, [\theta_r]^{m_r}\}$. Less accurately, we may write that a real symmetric matrix has spectrum $\{\{[\theta_1]^{m_1}, \dots, [\theta_r]^{m_r}\}\}$ implying that θ_i may be equal to θ_j for some i and j. In this instance, the multiplicity of an eigenvalue θ of M is equal to the sum of the m_i for which $\theta_i = \theta$.

Let S be a Seidel matrix of order n. Then S is a real symmetric matrix with diagonal entries 0. Moreover, the diagonal entries of S^2 are all equal to $n - 1$. Putting the above facts about Seidel matrices together, we obtain the following equations for the traces of S and S^2:

Proposition 1.6 (Basic properties of Seidel matrices)**.** *Let S be a Seidel matrix of order n.*

$$\operatorname{tr} S = 0; \tag{1.1}$$
$$\operatorname{tr} S^2 = n(n-1). \tag{1.2}$$

1.2 Absolute and relative bounds

In this section, we prove two classical upper bounds for $N(d)$. Furthermore, we consider the effects of equality being attained in each case.

Proposition 1.7. *The space of degree-k homogeneous polynomials with n variables has dimension $\binom{k+n-1}{n-1}$.*

Proof. Suppose our variables are $x_1, \dots, x_n$. Then the standard basis consists of all monomials $x_1^{e_1} x_2^{e_2} \dots x_n^{e_n}$, where the exponents e_i are non-negative integers satisfying $e_1 + e_2 + \dots + e_n = k$. ∎

The next lemma is a technical result that we will use in the proofs of the absolute and relative bounds below. We will use this lemma to show that when the relative bound or the absolute bound is attained, then the Seidel matrix of the resulting equiangular line system has just two distinct eigenvalues.

Lemma 1.8. *Let X be a set of n equiangular lines in $\mathbb{R}^d$ with common angle α. Then*

$$n(1 - d\alpha^2) \leqslant d(1 - \alpha^2). \tag{1.3}$$

Furthermore, in the case of equality, the Seidel matrix $S(X)$ has spectrum

$$\left\{ \left[-\sqrt{\frac{d(n-1)}{n-d}} \right]^{n-d}, \left[\sqrt{\frac{(n-d)(n-1)}{d}} \right]^{d} \right\}.$$

Proof. Let $S = S(X)$ be the Seidel matrix for X having smallest eigenvalue $\lambda_0 = -1/\alpha$ with multiplicity $n - d$. Denote by $\lambda_1, \ldots, \lambda_d$ the eigenvalues of S not equal to λ_0. By the Cauchy-Schwarz inequality, we have

$$\left(\sum_{i=1}^{d} \lambda_i \right)^2 \leqslant d \sum_{i=1}^{d} \lambda_i^2. \tag{1.4}$$

Using Proposition 1.6, this inequality becomes $\lambda_0^2(n-d)^2 \leqslant d(n(n-1) - \lambda_0^2(n-d))$, which simplifies to

$$n(\lambda_0^2 - d) \leqslant d(\lambda_0^2 - 1). \tag{1.5}$$

In the case of equality, we have $\alpha = \sqrt{\frac{n-d}{d(n-1)}}$ and equality in the Cauchy-Schwarz inequality (1.4), which implies that $\lambda_1 = \lambda_2 = \cdots = \lambda_d$. Since $\operatorname{tr} S = 0$, we find that S has spectrum

$$\left\{ \left[\frac{-1}{\alpha} \right]^{n-d}, \left[\frac{n-d}{d\alpha} \right]^{d} \right\}.$$

□

The so-called relative bound [28] follows as a corollary.

Theorem 1.9 (Relative bound)**.** *Let X be a set of n equiangular lines in $\mathbb{R}^d$ with common angle α satisfying $d < 1/\alpha^2$. Then $n \leqslant d(1-\alpha^2)/(1-d\alpha^2)$. Furthermore, in the case of equality, the Seidel matrix $S(X)$ has spectrum*

$$\left\{ \left[-\sqrt{\frac{d(n-1)}{n-d}} \right]^{n-d}, \left[\sqrt{\frac{(n-d)(n-1)}{d}} \right]^{d} \right\}.$$

The absolute bound first appeared in Lemmens and Seidel's seminal paper [26] where it was attributed to Gerzon. We present a proof in the style of Koornwinder [25].

Theorem 1.10 (Absolute bound)**.** $N(d) \leqslant \binom{d+1}{2}$. *Furthermore, if there exists a set X of equiangular lines in $\mathbb{R}^d$ with* $\operatorname{card} X = \binom{d+1}{2}$ *then the Seidel matrix $S(X)$ has spectrum*

$$\left\{ \left[-\sqrt{d+2} \right]^{d(d-1)/2}, \left[\frac{d-1}{2}\sqrt{d+2} \right]^{d} \right\}.$$

Proof. Let α be the angle for X. Write $X = \{[\mathbf{u}_1], \dots, [\mathbf{u}_n]\}$ and, for each $i \in \{1, \dots, n\}$, define the polynomial

$$f_i(\mathbf{x}) := \langle \mathbf{u}_i, \mathbf{x} \rangle^2 - \alpha^2 \langle \mathbf{x}, \mathbf{x} \rangle.$$

Note that each f_i belongs to the space of degree-2 homogeneous polynomials with d variables, which has dimension $\binom{d+1}{2}$ by Proposition 1.7. Therefore, it suffices to show that the polynomials $f_1, \dots, f_n$ are linearly independent. To do this, we will use the fact that, for $i \neq j$:

$$f_i(\mathbf{u}_j) = 0;$$
$$f_i(\mathbf{u}_i) = 1 - \alpha^2.$$

Suppose that

$$c_1 f_1(\mathbf{x}) + \dots + c_n f_n(\mathbf{x}) = 0 \tag{1.6}$$

Substituting $\mathbf{u}_i$ into (1.6) yields $c_i(1-\alpha^2) = 0$, giving $c_i = 0$. Therefore the f_i are linearly independent and hence $n \leqslant \binom{d+1}{2}$, as required.

In the case of equality: $n = \binom{d+1}{2}$, the set $\{f_1, \dots, f_n\}$ is a basis for the space of degree-2 homogeneous polynomials with d variables. Therefore, there exist coefficients $c_1, \dots, c_n$ such that

$$\langle \mathbf{x}, \mathbf{x} \rangle = \sum_{i=1}^{n} c_i f_i(\mathbf{x}). \tag{1.7}$$

Substituting $\mathbf{u}_i$ for $\mathbf{x}$ in (1.7) yields $c_i = 1/(1-\alpha^2)$. Furthermore,

$$(1 + \alpha^2(n-1)) \langle \mathbf{x}, \mathbf{x} \rangle = \sum_{i=1}^{n} \langle \mathbf{u}_i, \mathbf{x} \rangle^2. \tag{1.8}$$

Denote by $\mathbf{e}_i$ the standard basis vector with ith entry equal to 1 and the rest equal to 0. Substituting $\mathbf{e}_j$ for $\mathbf{x}$ in (1.8) yields

$$1 + \alpha^2(n-1) = \sum_{i=1}^{n} \langle \mathbf{u}_i, \mathbf{e}_j \rangle^2. \tag{1.9}$$

Since each $\mathbf{u}_i$ is a unit vector, summing (1.9) over j gives

$$d(1 + \alpha^2(n-1)) = \sum_{j=1}^{d} \sum_{i=1}^{n} \langle \mathbf{u}_i, \mathbf{e}_j \rangle^2 = n.$$

Thus $\alpha^2 = \frac{n-d}{d(n-1)}$. This corresponds to equality in (1.3). By Lemma 1.8, since $n = \binom{d+1}{2}$, we find that the Seidel matrix $S(X)$ has spectrum

$$\left\{ \left[-\sqrt{d+2} \right]^{d(d-1)/2}, \left[\frac{d-1}{2} \sqrt{d+2} \right]^{d} \right\}. \qquad \square$$

Now we consider when the absolute bound (Theorem 1.10) can be attained. Examples 1.3, 1.4, and 1.5, show that the absolute bound can be attained when $d = 2$, $d = 3$, and $d = 7$. However, these small examples account for almost all the known cases where this bound can be attained. Since a Seidel matrix is a $\{0, \pm 1\}$-matrix, all of its eigenvalues must be algebraic integers (zeros of a monic integer polynomial) and its spectrum must be invariant under Galois conjugation. Therefore, by Theorem 1.10, if there exists an equiangular line system attaining the absolute bound in $\mathbb{R}^d$ and $d \neq d(d-1)/2$ then $\sqrt{d+2}$ must be a (rational) integer. Since, $d = d(d-1)/2$ implies $d = 3$, we find that $d+2$ must be a square when $d \neq 3$. We will see later (Lemma 2.2) that Seidel matrices cannot have even eigenvalues with multiplicity more than 1. Therefore, we have the following corollary.

Corollary 1.11 (Necessary condition for absolute equality)**.** *Suppose $N(d) = \binom{d+1}{2}$. Then $d = 2$, $d = 3$, or $d + 2$ is the square of an odd integer.*

Corollary 1.11 shows us that it is relatively rare that the absolute bound can be attained. We have seen that the absolute bound can be attained when $d = 2$, $d = 3$, and $d = 7$. It can also be attained when $d = 23$. These four values of d are the only known instances where the absolute value can be attained - it is an open question if there are any more d for which the absolute value can be attained. Bannai-Munemasa-Venkov [1] and Nebe-Venkov [30] showed that there are infinitely many d such that $d+2$ is the square of an odd integer for which the absolute bound cannot be attained.

1.3 Constructions

We have seen upper bounds for $N(d)$. In this section, we use constructions to produce lower bounds for $N(d)$.

1.3.1 De Caen's infinite quadratic construction

The first construction is an infinite construction due to De Caen [3]. Together with the absolute bound, this shows that $N(d)$ is quadratic in d, i.e., $N(d) = \Theta(d^2)$.

Theorem 1.12. *For each positive integer t and $d = 3 \cdot 2^{2t-1} - 1$ there exists an equiangular set of $\frac{2}{9}(d+1)^2$ lines in $\mathbb{R}^d$.*

Note that the Seidel matrices corresponding to De Caen's constructions have precisely four distinct eigenvalues.

Using De Caen's construction, Greaves et al. [14] obtained the general lower bound

$$N(d) \geqslant \frac{32d^2 + 328d + 296}{1089}.$$

Corollary 1.13. *For $d \geqslant 58$, we have $N(d) > 2d$.*

1.3.2 Inside the Witt design

The **Witt design** on 23 points [12, Page 241] is a 4-$(23, 7, 1)$ design. We will not worry about the definition of designs here. Instead, we give a construction of the Witt design. Start with the polynomial

$$p(x) = x^{11} + x^9 + x^7 + x^6 + x^5 + x + 1 \in \mathbb{F}_2[x].$$

Let $P = \{p(x)^i \mod x^{23} - 1\}$ and let $B \subset P$ be the subset of polynomials having precisely 7 nonzero coefficients. E.g., $x^{22} + x^{21} + x^{18} + x^{12} + x^8 + x^4 + x \equiv p(x)^{39} \mod x^{23} - 1$ is in B. The cardinality of B is 253 and the elements of B correspond to the blocks of the Witt design. Let N be the $\mathbb{Z}$-matrix whose columns are the coefficient vectors of the polynomials in B. Then N is a 23×253 matrix. Now define the Seidel matrix $\mathcal{W}$ as

$$\mathcal{W} := \begin{bmatrix} J - I & J - 2N \\ J - 2N^\intercal & N^\intercal N - 5I - 2J \end{bmatrix},$$

where J and I are the all-ones and identity matrices of the appropriate size. We refer to Godsil and Royle [12, Page 260] to check that $\mathcal{W}$ is, in fact, a Seidel matrix that has spectrum

$$\left\{ [-5]^{253}, [55]^{23} \right\}.$$

Therefore $\mathcal{W}$ corresponds to an equiangular set of cardinality 276 in $\mathbb{R}^{23}$.

Seidel matrices for large equiangular sets in lower dimensions can be found as principal submatrices of $\mathcal{W}$. For example:

- 176 equiangular lines in $\mathbb{R}^{22}$ [26];
- 126 equiangular lines in $\mathbb{R}^{21}$ [26];
- 90 equiangular lines in $\mathbb{R}^{20}$ [35];
- 72 equiangular lines in $\mathbb{R}^{19}$ [34];
- 56 equiangular lines in $\mathbb{R}^{18}$ [27];
- 48 equiangular lines in $\mathbb{R}^{17}$ [14, Example 5.18].

Potentially not in the Witt design. Although we do not know if a system of 57 equiangular lines in $\mathbb{R}^{18}$ can be obtained as a principal submatrix of $\mathcal{W}$, we can nonetheless construct 57 equiangular lines in $\mathbb{R}^{18}$. We give two constructions taken from [16].

$$\setcounter{MaxMatrixCols}{57}\begin{bmatrix}
2&2&0&0&+&0&0&0&0&0&0&0&0&0&0&0&0&-&0&-&0&-&0&0&+&0&0&+&+&+&+&+&+&+&+&+&-&+&0&0&+&+&-&0&0&0&0&+&-&-&0&0&0&-&0&0&+\\
0&0&2&2&0&0&0&0&0&0&0&0&0&-&0&0&-&+&0&0&+&0&0&-&-&-&-&0&0&-&-&+&0&+&0&0&0&0&+&-&+&-&-&+&0&+&-&0&0&0&0&-&-&0&0&-&0\\
0&-&+&0&2&2&-&0&0&0&0&0&-&0&0&0&+&0&0&0&-&0&+&0&+&-&-&0&0&+&-&0&0&0&0&+&0&-&-&+&+&-&0&-&+&+&+&0&+&0&0&-&+&0&0&0&0\\
0&-&+&-&0&0&2&2&+&-&-&0&-&0&0&0&0&0&0&+&+&-&0&0&-&0&-&0&0&0&+&0&+&0&+&-&-&0&-&0&0&0&-&0&+&0&0&+&0&+&0&0&+&0&0&0&0\\
+&0&0&0&0&0&0&0&2&2&0&0&0&0&+&-&0&0&0&0&0&+&+&+&-&+&-&-&-&0&+&0&-&0&0&-&0&+&-&0&+&0&0&+&-&0&0&+&+&0&0&-&0&+&0&0&+\\
-&0&0&0&+&-&0&0&+&-&2&+&0&0&+&-&0&0&-&0&0&+&+&-&-&-&0&0&0&0&0&0&+&+&+&+&+&+&+&+&-&-&0&0&0&0&0&0&0&0&0&0&0&0&+&+&+\\
0&-&0&-&0&0&-&-&0&0&0&2&0&0&0&+&-&0&+&-&+&0&+&-&0&0&-&0&0&0&0&+&-&-&0&0&0&+&+&+&0&+&-&-&0&0&0&0&0&+&+&+&+&+&-&-&+\\
0&0&+&0&0&0&0&0&0&0&-&-&2&+&0&0&-&0&+&+&-&0&0&0&-&+&0&-&0&+&0&+&0&0&-&+&+&0&+&-&0&+&0&-&+&0&0&0&+&0&+&-&0&-&0&+&+\\
0&0&-&0&0&+&0&0&-&+&+&0&0&2&0&0&0&0&0&0&0&+&0&+&-&0&0&+&0&-&-&+&+&+&-&0&-&+&0&-&0&0&0&-&0&+&-&0&-&+&-&-&+&+&-&+&0\\
0&-&0&+&+&0&+&-&0&0&0&0&0&+&2&+&0&0&+&0&0&+&-&0&0&-&0&-&+&0&0&-&0&+&-&-&+&0&0&0&0&+&-&0&0&-&+&+&-&0&-&0&0&0&+&0&0\\
+&0&+&-&-&+&+&0&0&0&0&-&-&0&-&2&0&0&0&-&+&0&+&+&0&-&+&-&-&+&0&-&-&+&0&+&0&+&+&0&-&0&0&0&-&+&+&0&0&+&0&0&-&0&0&+&0\\
-&+&0&+&0&0&-&+&0&0&0&0&0&-&-&0&2&-&+&+&0&+&0&+&0&0&-&-&+&+&0&0&0&0&0&-&0&0&+&0&0&0&0&-&-&+&-&+&-&+&+&+&0&0&+&0&0\\
0&0&0&0&0&0&0&0&0&0&0&+&-&-&0&-&0&2&0&-&0&0&+&0&0&0&+&0&+&+&0&0&-&0&-&-&-&-&+&-&+&+&-&0&0&0&0&-&0&0&-&0&+&-&+&+&0\\
-&+&0&0&-&0&0&+&-&+&+&0&0&0&+&0&0&+&-&-&-&-&-&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&+&+&+&+&+&+&+&+&+&+&+&+&+&+&+&+&+\\
0&0&0&0&0&+&-&+&0&0&0&-&0&0&0&0&0&-&+&-&+&0&0&-&-&+&0&+&+&+&+&-&0&+&-&0&+&-&0&0&0&-&0&+&-&0&0&0&+&-&-&+&+&+&-&0&+\\
0&0&0&0&0&+&0&0&-&+&0&0&+&0&0&-&+&0&+&0&+&+&+&-&0&-&-&0&+&-&+&0&0&-&+&0&-&0&0&-&-&0&+&0&+&-&+&-&0&+&-&0&0&-&0&+&+\\
+&0&+&0&+&0&0&0&0&0&+&0&-&+&-&0&0&0&+&+&0&0&0&0&0&+&+&+&-&0&-&+&-&0&+&0&+&-&0&-&0&0&+&+&0&-&+&+&-&+&0&+&0&0&+&-&+\\
0&0&0&-&0&+&0&+&+&-&-&+&0&0&0&0&-&-&-&0&+&+&-&+&+&0&0&+&+&0&-&-&-&+&0&0&0&0&0&-&+&0&+&0&+&-&-&-&0&0&+&0&-&+&+&0&+
\end{bmatrix}$$

Figure 1.1: The matrix F_1 consisting of 57 (column) vectors in $\mathbb{Z}^{18}$. The entries $\pm$ should be replaced with ± 1 entries in the obvious way.

The Seidel matrix $S_1 = F_1^\intercal F_1/2 - 5I_{57}$ corresponding to F_1 has characteristic polynomial $\mathrm{Char}_{S_1}(x) = (x+5)^{39}(x-4)(x-7)(x-9)^2(x-11)^9(x-13)^4(x-15)$. Thus the 57 vectors in Figure 1.1 span 57 equiangular lines in $\mathbb{R}^{18}$.

$$\setcounter{MaxMatrixCols}{57}\begin{bmatrix}
2&2&2&0&0&-&+&-&+&+&0&0&0&-&+&0&0&0&0&-&0&0&0&0&0&+&-&-&-&+&+&+&-&+&+&-&+&+&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&-&0&2&2&+&0&-&0&0&0&0&0&0&-&0&0&+&0&0&0&0&+&-&+&+&-&+&-&0&0&0&0&0&0&0&0&0&+&0&-&-&-&+&0&0&-&0&0&0&0&0&+&-&+&0&0\\
0&0&0&-&+&2&2&0&+&0&0&0&0&0&0&0&0&+&0&0&0&0&0&0&0&-&+&0&0&0&0&0&0&+&-&0&0&0&0&-&+&+&-&+&0&-&-&+&+&-&-&-&0&-&-&+&+\\
0&0&+&0&0&-&0&2&2&0&0&0&0&+&0&+&-&0&0&+&0&0&0&-&-&+&+&+&+&+&-&0&-&0&0&+&0&0&0&0&0&0&0&0&-&+&-&0&0&+&0&-&+&-&+&-&+\\
0&0&0&+&-&0&0&0&+&2&2&0&0&0&0&0&0&+&0&0&0&0&0&0&0&-&+&0&0&0&0&0&0&-&+&0&0&0&0&-&-&+&-&+&0&+&+&-&-&-&-&+&0&-&-&-&-\\
0&0&0&0&0&0&0&+&0&0&+&2&2&-&+&0&0&-&0&-&0&0&0&0&0&0&0&+&-&+&-&-&+&-&-&+&+&+&0&-&0&-&-&+&0&0&0&0&0&+&+&0&0&+&-&0&0\\
0&0&0&0&0&0&0&0&0&+&0&0&0&2&2&0&0&0&0&0&-&-&0&0&0&-&0&0&-&0&0&+&+&-&-&0&-&+&+&0&+&-&+&-&+&0&-&+&+&0&-&0&+&0&0&-&-\\
0&0&0&0&0&0&0&0&0&+&0&0&0&0&0&2&2&0&0&0&-&+&0&0&0&-&0&0&-&0&0&-&-&+&+&0&-&-&+&0&-&-&+&+&+&0&-&+&-&0&+&0&-&0&0&-&+\\
0&-&+&0&0&0&0&0&0&0&-&0&0&0&0&-&+&2&2&0&0&0&-&0&0&0&0&+&+&0&0&+&0&-&+&0&+&+&-&-&+&0&0&0&+&-&0&0&0&0&+&+&0&+&+&-&+\\
+&0&0&0&0&+&-&0&0&0&+&-&+&0&0&0&0&-&+&2&2&0&0&+&-&+&0&+&0&0&0&0&+&+&+&0&0&0&+&0&0&0&0&0&+&+&+&+&+&-&+&-&0&0&0&0&0\\
0&-&+&+&-&0&0&+&-&0&0&0&0&0&0&0&0&0&0&-&2&-&-&0&0&-&-&0&0&+&+&-&0&0&0&-&0&-&-&+&+&0&0&+&-&0&-&+&0&0&0&0&-&-&+&0&-\\
-&+&0&0&0&0&-&0&0&0&0&-&+&+&0&0&0&+&-&-&0&2&-&0&0&0&+&+&0&-&+&-&0&0&0&-&+&+&0&+&+&0&0&0&+&+&0&-&-&0&0&-&+&0&0&0&0\\
0&+&0&+&-&+&0&0&-&0&+&-&+&-&0&0&0&0&+&0&0&-&2&0&+&0&+&0&0&+&-&0&-&-&0&-&0&-&0&0&+&0&0&-&+&-&0&0&-&+&-&-&+&0&+&0&+\\
+&0&-&0&0&0&+&-&0&0&0&+&-&0&-&+&-&0&+&0&0&0&0&2&0&0&+&+&0&0&0&-&-&0&0&0&-&+&-&+&0&-&-&-&0&0&0&-&+&+&0&0&-&0&0&-&-\\
-&+&0&0&0&0&+&0&0&0&0&0&0&0&-&+&-&0&0&0&0&0&0&+&2&0&0&+&0&+&+&+&+&0&+&+&0&0&+&+&+&+&+&+&0&+&0&0&0&+&0&+&0&+&0&+&+\\
-&0&0&+&-&0&0&-&0&+&-&-&+&0&0&-&+&0&+&-&0&+&0&-&-&+&0&0&+&+&-&0&-&0&0&+&-&0&+&+&0&-&-&0&-&0&0&0&+&0&-&0&-&+&-&+&0\\
0&0&-&0&0&0&-&0&+&-&+&+&-&0&+&0&0&0&-&0&0&0&-&0&+&0&-&+&+&+&-&0&0&0&+&-&+&-&+&+&0&0&0&0&0&-&-&-&+&-&-&-&0&+&0&0&0\\
-&0&+&+&-&+&0&0&0&-&0&0&0&-&0&-&+&0&0&0&0&-&-&+&0&0&0&0&-&-&-&-&0&+&0&-&-&0&0&0&0&+&+&0&-&+&-&-&0&+&0&+&+&0&-&-&0
\end{bmatrix}$$

Figure 1.2: The matrix F_2 consisting of 57 (column) vectors in $\mathbb{Z}^{18}$. The entries $\pm$ should be replaced with ± 1 entries in the obvious way.

The Seidel matrix $S_2 = F_2^\intercal F_2/2 - 5I_{57}$ corresponding to F_2 in Figure 1.2 has characteristic polynomial $\mathrm{Char}_{S_2}(x) = (x+5)^{39}(x-7)^4(x-9)(x-11)^5(x-13)^6(x^2-25x+152)$.

1.3.3 From strongly regular graphs

Large sets of equiangular lines often correspond to strongly regular graphs. This material is well-known and somewhat standard, so we will not go far into it and merely reference Brouwer and Haemers' book [2].

Lemma 1.14. *Let Γ be a (simple) connected n-vertex, k-regular graph whose adjacency matrix A has eigenvalues*

$$k = \lambda_1 > \lambda_2 \geqslant \cdots \geqslant \lambda_n.$$

Then the Seidel matrix $J - I - 2A$ has eigenvalues $n - 1 - 2k$ and $-1 - 2\lambda_i$ for $i \in \{2, \ldots, n\}$.

Here we note a few more equiangular line sets that correspond to strongly regular graphs.

n	d	SRG	SRG spectrum	Seidel spectrum
276	23	$(276, 140, 58, 84)$	$\{[140]^1, [2]^{252}, [-28]^{23}\}$	$\{[-5]^{253}, [55]^{23}\}$
176	22	$(176, 70, 18, 34)$	$\{[70]^1, [2]^{154}, [-18]^{21}\}$	$\{[-5]^{155}, [35]^{22}\}$
126	21	$(126, 50, 13, 24)$	$\{[50]^1, [2]^{105}, [-13]^{20}\}$	$\{[-5]^{105}, [25]^{21}\}$
40	16	$(40, 12, 2, 4)$	$\{[12]^1, [2]^{24}, [-4]^{15}\}$	$\{[-5]^{24}, [7]^{15}, [15]^1\}$
36	15	$(36, 20, 10, 12)$	$\{[20]^1, [2]^{20}, [-4]^{15}\}$	$\{[-5]^{21}, [7]^{15}\}$
28	7	$(28, 15, 6, 10)$	$\{[15]^1, [1]^{20}, [-5]^7\}$	$\{[-3]^{21}, [9]^7\}$
16	6	$(16, 5, 0, 2)$	$\{[5]^1, [1]^{10}, [-3]^5\}$	$\{[-3]^{10}, [5]^6\}$
10	5	$(10, 3, 0, 1)$	$\{[3]^1, [1]^5, [-2]^4\}$	$\{[-3]^5, [3]^5\}$

Table 1.2: Some equiangular line systems that correspond to strongly regular graphs.

Using the constructions in the section, we can further strengthen Corollary 1.13:

Corollary 1.15. *For all $d \geqslant 6$ such that $d \neq 14$, we have $N(d) > 2d$.*

1.4 Proving maximality

We can prove the maximality of a given construction using either the absolute bound or the relative bound. In order to use the relative bound, we need to pin down the angle that gives rise to the largest possible equiangular line system in a given dimension.

The following result from [26] is attributed to Neumann.

Theorem 1.16 (Neumann). *Assume there exists n equiangular lines in $\mathbb{R}^d$ with $n > 2d$ and common angle α. Then $1/\alpha$ is an odd integer.*

Theorem 1.16 greatly restricts the possibility for which common angle α corresponds to the maximum value of $N(d)$. Accordingly, we introduce the function $N_{1/\alpha}(d)$, which is defined to be the largest n such that there exists n equiangular lines in $\mathbb{R}^d$ having common angle α. We can now write $N(d)$ as the maximum of $N_{1/\alpha}(d)$ over all α.

Theorem 1.17. *[26, Theorem 3.7] Suppose $d < (2m + 1)^2$ for some $m \in \mathbb{N}$. Then*

$$N(d) \leqslant \max(N_3(d), N_5(d), \ldots, N_{2m+1}(d)).$$

Combining Corollary 1.15 with Theorem 1.16 tells us that it is prudent to study the behaviour of $N_k(d)$ when k is an odd integer. The next two results help us find which values of the common angle α to consider in the search for $N(d)$ for a given dimension d.

Theorem 1.18 (Lemmens-Seidel [26]). *$N_3(d) = 28$ for $d \in \{7, \ldots, 15\}$ and $N_3(d) = 2(d - 1)$ for $d \geqslant 15$.*

Theorem 1.19 (Cao-Koolen-Lin-Yu [4])**.** *$N_5(d) = 276$ for $d \in \{23, \ldots, 185\}$ and $N_5(d) = \lfloor \frac{3}{2}(d-1) \rfloor$ for $d \geqslant 185$.*

We note in passing that the asymptotics for $N_a(d)$ have recently been determined in the celebrated work of Jiang et al. [23]:

$$N_{2m+1}(d) = \left\lfloor \frac{m+1}{m}(d-1) \right\rfloor \quad \text{for all positive integers } m \text{ and sufficiently large } d.$$

Now we can consider the maximality of the constructions of the previous section.

- $N(2) = 3$: absolute bound.
- $N(3) = 6$: absolute bound.
- $N(4) = 6$: exercise.
- $N(5) = 10$. Suppose for a contradiction that there exist 11 equiangular lines in $\mathbb{R}^5$. Then, by Theorem 1.16, $1/\alpha$ is an odd integer. Since $d = 5 < (2m+1)^2$ for all $m \in \mathbb{N}$, Theorem 1.9 gives $N_{2m+1}(5) \leqslant \frac{5 \cdot m(m+1)}{m^2+m-1} = 5 + 5/(m^2+m-1) < 11$, which contradicts Theorem 1.17.
- $N(6) = 16$. Suppose for a contradiction that there exist 17 equiangular lines in $\mathbb{R}^6$. Then, by Theorem 1.16, $1/\alpha$ is an odd integer. Since $d = 6 < (2m+1)^2$ for all $m \in \mathbb{N}$, Theorem 1.9 gives $N_{2m+1}(6) \leqslant \frac{6 \cdot 4m(m+1)}{4m^2+4m-5} = 6 + 30/(4m^2+4m-5) < 17$, which contradicts Theorem 1.17.
- $N(7) = 28$: absolute bound.
- $N(8) = \cdots = N(13) = 28$. For each $d \in \{8, \ldots, 13\}$, suppose for a contradiction that there exist 29 equiangular lines in $\mathbb{R}^d$. Then, by Theorem 1.16, $1/\alpha = 2m+1$ for some $m \in \mathbb{N}$. By Theorem 1.18, we have $m \geqslant 2$. Since $d < (2m+1)^2$ for all $m \geqslant 2$, Theorem 1.9 gives $N_{2m+1}(d) \leqslant \frac{d \cdot 4m(m+1)}{4m^2+4m+1-d}$. It is straightforward to check that, for $d \in \{8, \ldots, 13\}$ and $m \geqslant 2$, we have $\frac{d \cdot 4m(m+1)}{4m^2+4m+1-d} \leqslant 26$, which contradicts Theorem 1.17.

Using the theory developed so far, we can further establish some values of $N(d)$. We leave these as exercises. However, for some values of d, it takes more work to find upper bounds for $N(d)$ that match the lower bounds we find from constructions. In the sequel, we will develop more restrictions for equiangular line systems via the study of Seidel matrices. In particular, we will develop the theory required to prove that $N(14) = 28$, $N(16) = 40$, and $N(17) = 48$.

1.5 Exercises

1. Prove Lemma 1.14.
2. Prove Theorem 1.16.
3. Prove Theorem 1.17.
4. Show that $N(4) = 6$.
5. Show that $N(14) \leqslant 30$.
6. Show that $N(15) = 36$.

7. Show that $N(16) \leqslant 42$.
8. Show that $N(17) \leqslant 50$.
9. Show that $N(18) \leqslant 61$.
10. Show that $N(19) \leqslant 76$.
11. Show that $N(20) \leqslant 96$.
12. Show that $N(21) = 126$.
13. Show that $N(22) = 176$.
14. Show that $N(23) = 276$.

Chapter 2

Characteristic polynomials of Seidel matrices

In this chapter, we introduce various spectral restrictions for Seidel matrices. We largely follow [18] and [15].

The symbols I, J, and O will (respectively) always denote the identity matrix, the all-ones matrix, and the all-zeros matrix; the order of each matrix should be clear from the context in which it is used, however, the order will sometimes be indicated by a subscript. We use $\mathbf{1}$ to denote the all-ones (column) vector.

2.1 Basic properties of Seidel matrices

Recall that a **Seidel matrix** is a symmetric $\{0, \pm 1\}$-matrix S with zero diagonal and all off-diagonal entries nonzero.

2.1.1 Seidel matrices modulo 2

Let $\mathcal{M}_n$ denote the ring of integer matrices of order n. Let S be a Seidel matrix. Since we can write $S = J - I - 2A$ where A is a graph adjacency matrix, the next lemma follows immediately.

Lemma 2.1. *Let S be a Seidel matrix of order n and $k \in \mathbb{N}$. Then modulo $2\mathcal{M}_n$ we have*

$$S^k \equiv \begin{cases} J - I, & \text{if } k \text{ is odd;} \\ nJ - I, & \text{if } k \text{ is even.} \end{cases}$$

Next, we have the following lemma about matrices congruent to $J - I$ modulo $2\mathcal{M}_n$. We denote the characteristic polynomial of a matrix M by $\mathrm{Char}_M(x) := \det(xI - M)$.

Lemma 2.2. *Let M be an $n \times n$ matrix congruent to $J - I$ modulo $2\mathcal{M}_n$. Then modulo $2\mathbb{Z}[x]$ we have*

$$\mathrm{Char}_S(x) \equiv \begin{cases} (x+1)^n & \text{if } n \text{ is even,} \\ x(x+1)^{n-1} & \text{if } n \text{ is odd.} \end{cases}$$

It follows from this lemma that, if its order is even, then a Seidel matrix cannot have any even integer eigenvalues. Furthermore, a Seidel matrix of odd order must have a simple eigenvalue. Indeed, we record this consequence as a corollary.

Corollary 2.3. *Let S be a Seidel matrix of odd order. Then S has an eigenvalue of multiplicity* 1.

2.1.2 A relation for characteristic polynomials

In this section, we establish a relation between the characteristic polynomial of a Seidel matrix S and the characteristic polynomial of a graph in the switching class of S. If Γ is a graph with adjacency matrix A, then its Seidel matrix has the form $S = J - I - 2A$. The characteristic polynomial $\mathrm{Char}_S(x)$ of S can be written as $\mathrm{Char}_S(x) = \mathrm{Char}_{J-2A}(x+1)$. With this in mind, we instead consider the relation between $\mathrm{Char}_A(x)$ and $\mathrm{Char}_{J-2A}(x)$.

Lemma 2.4. *Let A be a matrix of order n. Write $\mathrm{Char}_{J-2A}(x) = \sum_{i=0}^n a_i x^{n-i}$ and $\mathrm{Char}_A(x) = \sum_{i=0}^n b_i x^{n-i}$. Then*

$$a_r = (-2)^r \left(b_r + \frac{1}{2} \sum_{i=1}^{r} b_{r-i}\, \mathbf{1}^\top A^{i-1} \mathbf{1} \right).$$

Proof. By the matrix determinant lemma,

$$\mathrm{Char}_{J-2A}(x) = \mathrm{Char}_{-2A}(x) - \mathbf{1}^\top \operatorname{adj}(xI + 2A)\mathbf{1}.$$

Write $\mathrm{Char}_{-2A}(x) = \sum_{i=0}^n c_i x^{n-i}$. The adjugate matrix can be written [9, p. 38] as

$$\operatorname{adj}(xI + 2A) = \sum_{i=0}^{n-1} (-2A)^{n-1-i} \sum_{j=0}^{i} x^{i-j} c_j.$$

Note that we have $c_i = (-2)^i b_i$ for all $i \in \{0, \dots, n\}$. The result then follows by equating coefficients. □

The next result is a standard result from linear algebra, which can be proved by iteratively handshaking.

Lemma 2.5. *Let A be a symmetric integer matrix whose diagonal entries are all even. Then $\mathbf{1}^\top A^i \mathbf{1}$ is even for all integers $i \geqslant 1$.*

Now we record a couple of corollaries to Lemma 2.4. First, a surprisingly strong restriction on $\mathrm{Char}_{J-2A}(x)$ where A is the adjacency matrix of a graph of order n even.

Corollary 2.6. *Let A be the adjacency matrix of a graph of order n even and write $\mathrm{Char}_{J-2A}(x) = \sum_{i=0}^n a_i x^{n-i}$. Then 2^r divides a_r for all $r \in \{0, \dots, n\}$.*

Proof. By Lemma 2.4, it suffices to show that $\mathbf{1}^\top A^{i-1} \mathbf{1}$ is even for all $i \geqslant 1$. By Lemma 2.5, for all $i \geqslant 2$, we have that $\mathbf{1}^\top A^{i-1} \mathbf{1}$ is even, and, for $i = 1$, we have $\mathbf{1}^\top A^{i-1} \mathbf{1} = n$, which is also even. □

Denote by $\mathscr{S}_n$ the set of all Seidel matrices of order n. Given a positive integer e, define the set

$$\mathscr{P}_{n,e} \coloneqq \{\mathrm{Char}_S(x) \mod 2^e \mathbb{Z}[x] \mid S \in \mathscr{S}_n\}.$$

Corollary 2.7. *Let n be an even integer and e be a positive integer. Then the cardinality of $\mathscr{P}_{n,e}$ is at most $2^{\binom{e-2}{2}}$.*

Proof. Let A be an adjacency matrix of a graph of order n. It is clear that the trace of $J - 2A$ equals n and the trace of $(J - 2A)^2$ equals n^2. Write $\mathrm{Char}_{J-2A}(x) = \sum_{i=0}^n a_i x^{n-i}$. Obviously $a_0 = 1$. Furthermore, using Newton's identities, we see that $a_1 = -n$ and $a_2 = 0$. Then use Corollary 2.6. □

Clearly, if n is small compared to e then the cardinality of $\mathscr{P}_{n,e}$ will be strictly less than $2^{\binom{e-2}{2}}$. Indeed, for $n = 2$ the cardinality of $\mathscr{P}_{n,e}$ is 1 for all e. However, it is straightforward to check that the bound in Corollary 2.7 is sharp for small ($e \leqslant 5$) values of e and large enough even n.

Conjecture 2.8. *For all integers $e \geqslant 2$, there exists $N \in \mathbb{N}$ such that $|\mathscr{P}_{n,e}| = 2^{\binom{e-2}{2}}$ for all even $n > N$.*

We will need a corollary of the following well-known result from linear algebra.

Lemma 2.9. *Let A be a symmetric integer matrix of order n odd whose diagonal entries are all zero. Then $\det A$ is even.*

Corollary 2.10. *Let A be a symmetric integer matrix of order n whose diagonal entries are all zero and write $\mathrm{Char}_A(x) = \sum_{i=0}^{n} b_i x^{n-i}$. Then b_r is even for all odd r.*

We now establish a result similar to Corollary 2.6.

Lemma 2.11. *Let A be an adjacency matrix of a graph of order n and write $\mathrm{Char}_{J-2A}(x) = \sum_{i=0}^{n} a_i x^{n-i}$. Then 2^r divides a_r for all r even.*

Proof. Let $\mathrm{Char}_A(x) = \sum_{i=0}^{n} b_i x^{n-i}$. Using Lemma 2.4, it suffices to show that $\sum_{i=1}^{r} b_{r-i}\, \mathbf{1}^\top A^{i-1} \mathbf{1}$ is even. By Lemma 2.5, for all $i \geqslant 2$, we have that $\mathbf{1}^\top A^{i-1} \mathbf{1}$ is even. And, by Corollary 2.10, the coefficient b_{r-1} is even for all r even. $\square$

Lemma 2.11 is not quite strong enough to establish the upper bound that we want to prove (Theorem 2.15 below). The extra strength required comes from a delightful congruence (Lemma 2.13) due to Harary and Schwenk.

2.2 Counting walks with Harary and Schwenk

In this section, we apply Burnside's lemma to find a congruence modulo $2N$ for a weighted sum of traces of powers of a graph-adjacency matrix. This is a crucial ingredient for the proof of Theorem 2.15, below.

Let Γ be a graph and let $\mathbf{x}$ be a closed walk of length N in Γ; we write $\mathbf{x} = x_0 x_1 \dots x_{N-1}$ where x_i is adjacent to x_{i+1} for each $i \in \{0, \dots, N-1\}$ with indices reduced modulo N. There is a natural correspondence between the vertices of the closed walk $\mathbf{x}$ and the vertices of a regular N-gon. Hence, under this correspondence, we consider the dihedral group D_N of order $2N$ acting on the set of closed N-walks of Γ. Let $N \geqslant 3$ and write $D_N = \langle r, s \mid r^N, s^2, (rs)^2 \rangle$. For $g \in D_N$, we denote by $\mathrm{fix}_\Gamma(g)$ the set of closed N-walks of Γ fixed by g.

Lemma 2.12. *Let Γ be a simple graph with adjacency matrix A and let $N \geqslant 3$. Then*

(i) $|\mathrm{fix}_\Gamma(r^k)| = \mathrm{tr}\big(A^{\gcd(k,N)}\big)$, *for all $k \in \mathbb{Z}$;*

(ii) $|\mathrm{fix}_\Gamma(r^{2k}s)| = 0$, *for all $k \in \mathbb{Z}$;*

(iii) $|\mathrm{fix}_\Gamma(r^{2k+1}s)| = \begin{cases} \mathbf{1}^\top A^{N/2} \mathbf{1}, & \text{if } N \text{ is even} \\ 0, & \text{if } N \text{ is odd} \end{cases}$, *for all $k \in \mathbb{Z}$.*

Proof. Let $\mathbf{x} = x_0 x_1 \dots x_{N-1}$ be a closed N-walk that is fixed by some element $g \in D_N$. Observe that, if c is a cycle of the group element g, then for each i and j in c, we have $x_i = x_j$.

First, suppose that $g = r^k$ for some $k \in \mathbb{Z}$. Then g has order $m = N/\gcd(k, N)$. Therefore, g consists of N/m cycles each of length m. It follows that, for all $i \in \{0, \dots, N/m-1\}$, we have $x_{im} x_{im+1} \dots x_{im+m-1} = x_0 x_1 \dots x_{m-1}$. Hence $\mathrm{fix}_\Gamma(g)$ consists of closed N/m-walks repeated m times. Since $\mathrm{tr}\big(A^k\big)$ is equal to the number of closed walks of length k, we have established (i).

Now suppose N is odd and $g = r^k s$ for some $k \in \mathbb{Z}$. In this case, g consists of $\lfloor N/2 \rfloor$ cycles of length 2. It follows that two adjacent vertices of $\mathbf{x}$ must be equal, but there are no such closed walks since Γ has no loops. Whence we have (ii) and (iii) for N odd. It remains to assume that N is even.

Suppose that $g = r^{2k}s$ for some $k \in \mathbb{Z}$. In this case, g consists of $N/2$ cycles of length 2. Then two adjacent vertices of $\mathbf{x}$ must be equal, but there are no such closed walks since Γ has no loops. This yields (ii).

Finally, suppose that $g = r^{2k+1}s$ for some $k \in \mathbb{Z}$. In this case, g consists of $N/2 - 1$ cycles of length 2. Without loss of generality, we can assume that x_0 and $x_{N/2}$ are fixed by g. Then, for each $i \in \{1, \dots, N/2-1\}$, we must have $x_i = x_{N-i}$. Hence $\operatorname{fix}_\Gamma(g)$ consists of closed N-walks made up of an $N/2$-walk together with its inverse. □

For a positive integer a, we use $\varphi(a)$ to denote Euler's totient function of a. The following two lemmas were first discovered by Harary and Schwenk [21] in 1979.

Lemma 2.13. *Let Γ be a graph with adjacency matrix A and let $N \geqslant 4$ be an even integer. Then*

$$\sum_{d \mid N} \varphi\left(\frac{N}{d}\right) \operatorname{tr}\left(A^d\right) + \frac{N}{2}\mathbf{1}^\top A^{N/2}\mathbf{1} \equiv 0 \mod 2N.$$

Proof. The Dihedral group D_N (of order $2N$) acts on closed walks of length N. By Burnside's lemma, the number of orbits of closed walks of length N is equal to

$$\frac{1}{2N} \sum_{g \in D_N} |\operatorname{fix}_\Gamma(g)|.$$

Then, by Lemma 2.12, we have

$$\sum_{g \in D_N} |\operatorname{fix}_\Gamma(g)| = \sum_{d \mid N} \varphi\left(\frac{N}{d}\right) \operatorname{tr}\left(A^d\right) + \frac{N}{2}\mathbf{1}^\top A^{N/2}\mathbf{1}.$$ □

Note that we also have a similar congruence when N is odd, which also follows from Lemma 2.12.

Lemma 2.14. *Let Γ be a graph with adjacency matrix A and let $N \geqslant 3$ be an odd integer. Then*

$$\sum_{d \mid N} \varphi(N/d) \operatorname{tr}\left(A^d\right) \equiv 0 \mod 2N.$$

Theorem 2.15 ([18, Corollary 3.13])**.** *Let n be an odd integer and e be a positive integer. Then the cardinality of $\mathscr{P}_{n,e}$ is at most $2^{\binom{e-2}{2}+1}$.*

Conjecture 2.16. *For all integers $e \geqslant 3$, there exists $N \in \mathbb{N}$ such that $|\mathscr{P}_{n,e}| = 2^{\binom{e-2}{2}+1}$ for all odd $n > N$.*

2.3 Type-2 polynomials

It was shown in Section 2.1.2 that the coefficients of characteristic polynomials of Seidel matrices satisfy certain modular constraints. In particular, combining the results of Section 2.1.2 leads to the following theorem.

Theorem 2.17. *Let S be a Seidel matrix of order n and write $\operatorname{Char}_{S+I}(x) = \sum_{i=0}^n a_i x^{n-i}$. Then $a_0 = 1$, $a_1 = -n$, and $a_2 = 0$. Furthermore, if n is even then 2^i divides a_i for all $i \in \{0, \dots, n\}$. Otherwise, 2^{i-1} divides a_i for all odd $i \in \{0, \dots, n\}$ and 2^i divides a_i for all even $i \in \{0, \dots, n\}$.*

Motivated by the above theorem, we consider polynomials whose coefficients satisfy related modular conditions.

Definition 2.18. Let $p(x) = \sum_{i=0}^n a_i x^{n-i}$ be a monic polynomial in $\mathbb{Z}[x]$. We say p is **type 2** if 2^i divides a_i for all $i \geqslant 0$ and **weakly type 2** if 2^{i-1} divides a_i for all $i \geqslant 1$.

The next result follows from Lemma 2.4.

Lemma 2.19. *Let S be a Seidel matrix of order n and κ be an odd integer. Then $\mathrm{Char}_{S-\kappa I}(x)$ is weakly type 2. Furthermore, if n is even then $\mathrm{Char}_{S-\kappa I}(x)$ is type 2.*

Note the following equivalent definition of (weakly)-type-2 polynomials. A monic integer polynomial $p(x)$ is type 2 if and only if $p(2x)/2^{\deg p} \in \mathbb{Z}[x]$ and is weakly type 2 if and only if $p(2x)/2^{\deg p-1} \in \mathbb{Z}[x]$.

Recall that the **content** $c(p)$ of a polynomial $p \in \mathbb{Z}[x]$ is the greatest common divisor of its coefficients. For $p \in \mathbb{Q}[x]$, the content $c(p)$ is defined to be $c(vp)/v$ where $v \in \mathbb{N}$ satisfying $vp \in \mathbb{Z}[x]$. The following lemma deals with the factorisation of type-2 and weakly-type-2 polynomials.

Lemma 2.20. *Let $p \in \mathbb{Z}[x]$ be a monic polynomial. Suppose $p = qr$ where $q, r \in \mathbb{Z}[x]$. Then*

(a) p is type 2 if and only if q and r are both type 2;

(b) p is weakly type 2 if and only if q and r are both weakly type 2 and at least one of them is type 2.

Proof. Since p is monic, both q and r are also monic. Observe that

$$\frac{p(2x)}{2^{\deg p}} = \frac{q(2x)}{2^{\deg q}} \cdot \frac{r(2x)}{2^{\deg r}}$$

and both $q(2x)/2^{\deg q}$ and $r(2x)/2^{\deg r}$ are monic polynomials in $\mathbb{Q}[x]$. It follows that there exist positive integers u and v such that $u \cdot q(2x)/2^{\deg q}$ and $v \cdot r(2x)/2^{\deg r}$ are both in $\mathbb{Z}[x]$ each with content equal to 1. Since the content is multiplicative, we obtain

$$\frac{uv}{2} \cdot c\left(\frac{p(2x)}{2^{\deg p-1}}\right) = c\left(uv \cdot \frac{p(2x)}{2^{\deg p}}\right) = c\left(u\frac{q(2x)}{2^{\deg q}}\right) \cdot c\left(v\frac{r(2x)}{2^{\deg r}}\right) = 1. \tag{2.1}$$

If q and r are both type 2 then

$$\frac{p(2x)}{2^{\deg p}} = \frac{q(2x)}{2^{\deg q}} \cdot \frac{r(2x)}{2^{\deg r}} \in \mathbb{Z}[x].$$

Hence p is type 2. Conversely, suppose p is type 2. Then $p(2x)/2^{\deg p}$ is a monic polynomial in $\mathbb{Z}[x]$, which implies $c\left(p(2x)/2^{\deg p}\right) = 1$. Consequently, we must have that $p(2x)/2^{\deg p-1} \in \mathbb{Z}[x]$ and $c\left(p(2x)/2^{\deg p-1}\right) = 2$. By (2.1), we have $uv = 1$ and hence $u = v = 1$. Therefore, q and r are both type 2.

If q and r are both weakly type 2 and at least one of them is type 2, then

$$\frac{p(2x)}{2^{\deg p-1}} = \frac{q(2x)}{2^{\deg q-1}} \cdot \frac{r(2x)}{2^{\deg r}} = \frac{q(2x)}{2^{\deg q}} \cdot \frac{r(2x)}{2^{\deg r-1}} \in \mathbb{Z}[x].$$

Hence, p is weakly type 2. Conversely, suppose p is weakly type 2. Then the polynomial $p(2x)/2^{\deg p-1}$ has integer coefficients with leading coefficient 2, which implies $c\left(p(2x)/2^{\deg p-1}\right)$ is equal to 1 or 2. If $c\left(p(2x)/2^{\deg p-1}\right) = 2$, then $p(2x)/2^{\deg p} \in \mathbb{Z}[x]$, i.e., p is type 2 and hence q and r are both type 2, as above. Otherwise, we must have $c\left(p(2x)/2^{\deg p-1}\right) = 1$. Then $uv = 2$ by (2.1). Hence $\{u, v\} = \{1, 2\}$, which implies that both q and r are weakly type 2 and one of them is type 2. $\square$

2.4 Getting close to the relative bound

The next lemma can be thought of as an extension of the relative bound.

Lemma 2.21. *Let d be a positive integer and let S be a Seidel matrix of order n with smallest eigenvalue $\lambda_0 \in \mathbb{Z}$ of multiplicity at least $n-d > 1$. Let κ be a closest odd integer to $(d-n)\lambda_0/d$. Define*

$$\theta := \min\left\{\eta \in \mathbb{N} \mid \eta 4^{(\eta-\gamma(n))/\eta} > n(n-1) - \lambda_0^2(n-d) + 2\kappa\lambda_0(n-d) + d\kappa^2\right\},$$

where $\gamma(n) = 1$ if n is odd and $\gamma(n) = 0$, otherwise. If $\theta \leqslant d$ then

$$\mathrm{Char}_S(x) = (x-\lambda_0)^{n-d}(x-\kappa)^{d+1-\theta}\phi(x),$$

for some monic integer polynomial $\phi(x)$ of degree $\theta - 1$.

Proof. By Lemma 2.2, since its multiplicity is greater than 1, the eigenvalue λ_0 must be odd. Let $\lambda_1, \dots, \lambda_d$ be the d other eigenvalues of S. Since $\operatorname{tr} S = 0$ and $\operatorname{tr} S^2 = n(n-1)$, we have

$$\sum_{i=1}^{d} \lambda_i = -\lambda_0(n-d) \qquad \text{and} \qquad \sum_{i=1}^{d} \lambda_i^2 = n(n-1) - \lambda_0^2(n-d).$$

Combining the above yields

$$\sum_{i=1}^{d} (\lambda_i - \kappa)^2 = n(n-1) - \lambda_0^2(n-d) + 2\kappa\lambda_0(n-d) + d\kappa^2. \tag{2.2}$$

The minimum value of the polynomial $n(n-1) - \lambda_0^2(n-d) + 2x\lambda_0(n-d) + dx^2$ is attained when $x = (d-n)\lambda_0/d$. Hence, the minimum value of $n(n-1) - \lambda_0^2(n-d) + 2x\lambda_0(n-d) + dx^2$ for x an odd integer is attained when $x = \kappa$.

Let $\mathcal{R} = n(n-1) - \lambda_0^2(n-d) + 2\kappa\lambda_0(n-d) + d\kappa^2$ and let $\eta = d$. From (2.2), we have $\sum_{i=1}^{\eta}(\lambda_i - \kappa)^2 = \mathcal{R}$. Suppose that η satisfies $\eta 4^{(\eta-\gamma(n))/\eta} > \mathcal{R}$. By Lemma 2.19, the characteristic polynomial

$$\mathrm{Char}_{S-\kappa I}(x) = \mathrm{Char}_S(x+\kappa) = x^{d-\eta}(x-\lambda_0+\kappa)^{n-d}\prod_{i=1}^{\eta}(x-\lambda_i+\kappa)$$

is weakly type 2 and is type 2 if n is even. By Lemma 2.20, the polynomial $\prod_{i=1}^{\eta}(x-\lambda_i+\kappa)$ is also weakly type 2, or type 2 if n is even. In particular, the constant term $\prod_{i=1}^{\eta}(\lambda_i - \kappa)$ is divisible by $2^{\eta-\gamma(n)}$. Thus, we write $\prod_{i=1}^{\eta}(\lambda_i - \kappa) = 2^{\eta-\gamma(n)} \cdot k_\eta$ where $k_\eta \in \mathbb{Z}$. Using the inequality of arithmetic and geometric means, we obtain

$$\eta^\eta 4^{\eta-\gamma(n)} > \mathcal{R}^\eta = \left(\sum_{i=1}^{\eta}(\lambda_i-\kappa)^2\right)^\eta \geqslant \eta^\eta \prod_{i=1}^{\eta}(\lambda_i-\kappa)^2 = \eta^\eta 4^{\eta-\gamma(n)} \cdot k_\eta^2.$$

This implies that $k_\eta = 0$ and, without loss of generality, we can assume $\lambda_\eta = \lambda_d = \kappa$. Then

$$\mathrm{Char}_{S-\kappa I}(x) = x^{d+1-\eta}(x-\lambda_0+\kappa)^{n-d}\prod_{i=1}^{\eta-1}(x-\lambda_i+\kappa).$$

Furthermore, we have $\sum_{i=1}^{\eta-1}(\lambda_i-\kappa)^2 = \mathcal{R}$ and, by Lemma 2.20, the polynomial $\prod_{i=1}^{\eta-1}(x-\lambda_i+\kappa)$ is weakly type 2, or type 2 if n is even.

Suppose $\theta \leqslant d$. Note that for $\eta > 0$, the function $\Psi(\eta) = \eta 4^{(\eta-\gamma(n))/\eta}$ is increasing. Hence, for each integer η where $\theta \leqslant \eta \leqslant d$, we have $\eta 4^{(\eta-\gamma(n))/\eta} > \mathcal{R}$. Inductively repeat the steps above from $\eta = d$ to $\eta = \theta$. We obtain $\lambda_\theta = \cdots = \lambda_d = \kappa$ and thus

$$\mathrm{Char}_S(x) = (x-\lambda_0)^{n-d}(x-\kappa)^{d+1-\theta}\prod_{i=1}^{\theta-1}(x-\lambda_i),$$

as required. ∎

Remark 2.22. It is interesting to note that the extremal case of Lemma 2.21 (when $\theta = 1$) characterises Seidel matrices having precisely two distinct eigenvalues. Such Seidel matrices correspond to regular two-graphs [35].

Example 2.23. Set $d = 14$ and suppose there exists a Seidel matrix S of order 29 with smallest eigenvalue $\lambda_0 = -5$. We check the assumptions of Lemma 2.21. We have $\kappa = 5$ and

$$n(n-1) - \lambda_0^2(n-d) + 2\kappa\lambda_0(n-d) + d\kappa^2 = 37.$$

Thus, $\theta = 11$. Indeed, $11 \cdot 4^{10/11} = 38.79\cdots > 37$, and $10 \cdot 4^{9/10} = 34.82\cdots < 37$. Hence, by Lemma 2.21, we must have

$$\mathrm{Char}_S(x) = (x+5)^{15}(x-5)^4\phi(x),$$

for some monic integer polynomial $\phi(x)$ of degree 10.

Corollary 2.24. *Let S be a Seidel matrix.*

- *If S corresponds to 29 equiangular lines in $\mathbb{R}^{14}$ then*

$$\mathrm{Char}_S(x) = (x+5)^{15}(x-5)^4\phi(x),$$

for some monic polynomial ϕ of degree 10 in $\mathbb{Z}[x]$.

- *If S corresponds to 41 equiangular lines in $\mathbb{R}^{16}$ then*

$$\mathrm{Char}_S(x) = (x+5)^{25}(x-7)^3\phi(x),$$

for some monic polynomial ϕ of degree 13 in $\mathbb{Z}[x]$.

- *If S is a Seidel matrix for 49 equiangular lines in $\mathbb{R}^{17}$ then*

$$\mathrm{Char}_S(x) = (x+5)^{32}(x-9)^4\phi(x),$$

for some monic polynomial ϕ of degree 13 in $\mathbb{Z}[x]$.

The next step is to consider the possibilities for the polynomial $\phi(x)$ in each case of Corollary 2.24.

2.5 Enumerating candidate polynomials

2.5.1 Candidate characteristic polynomials

Now we can bring together different aspects of this chapter to enumerate all candidate characteristic polynomials that can potentially correspond to an equiangular line system in $\mathbb{R}^d$ larger than one of the constructions. We consider $(d,n) \in \{(14,29),(16,41),(17,49)\}$. Let S be the Seidel matrix corresponding to n equiangular in $\mathbb{R}^d$. Since, in each case n is odd, by Lemma 2.19, the characteristic polynomial $\mathrm{Char}_S(x-1)$ is weakly type 2. Furthermore, by Lemma 2.20, each polynomial $\phi(x-1)$, with $\phi(x)$ from Lemma 2.24, is also weakly type 2. Write $\phi(x) = \sum_{t=0}^{\theta-1} b_t x^{\theta-1-t}$. Suppose

$$\mathrm{Char}_S(x) = (x-\lambda_0)^{n-d}(x-\kappa)^{d+1-\theta}\phi(x),$$

with κ and θ as in Lemma 2.21. It is obvious that $b_0 = 1$. And we can find b_1 and b_2, using a basic fact about Seidel matrices: the traces of S and S^2 are expressed in terms of n as $\operatorname{tr} S = 0$ and $\operatorname{tr} S^2 = n(n-1)$. Hence, using Newton's identities, we have

$$b_1 = \lambda_0(n-d) + \kappa(d+1-\theta) \quad \text{and} \quad b_2 = \left(b_1^2 + \lambda_0^2(n-d) + \kappa^2(d+1-\theta) - n(n-1)\right)/2.$$

It is also obvious that $\phi(x)$ is totally real, i.e., all its zeros are real. One last condition is that the characteristic polynomial $\mathrm{Char}_S(x)$ must belong to a congruence class of $\mathscr{P}_{n,7}$. (Here we choose $e = 7$ since it is small enough that we can enumerate all the congruence classes of $\mathscr{P}_{n,7}$ simply by randomly generating Seidel matrices of order n, computing their characteristic polynomials modulo 2^7 until we obtain 2^{11} polynomials, which is the upper bound by Theorem 2.15.)

In summary, we need to find all totally-real, integer polynomials $\phi(x)$ with the following properties:

(i) $b_0 = 1$, $b_1 = \lambda_0(n-d) + \kappa(d+1-\theta)$, and

$$b_2 = \left(b_1^2 + \lambda_0^2(n-d) + \kappa^2(d+1-\theta) - n(n-1)\right)/2,$$

(ii) $\phi(x-1)$ is weakly type 2.

(iii) $(x-\lambda_0)^{n-d}(x-\kappa)^{d+1-\theta}\phi(x)$ belongs to a congruence class in $\mathscr{P}_{n,7}$.

2.5.2 Polynomial enumeration algorithm.

We use a method due to Robinson to find all possibilities for $\phi(x)$ for Lemma 2.21. This method has been detailed by Smyth [32] and McKee and Smyth [29]. We use the fact that $\phi(x)$ is totally-real, its top three coefficients are fixed, and $\phi(x-1)$ is weakly type 2.

First, we state a result about the interlacing of the zeros of a totally real polynomial and its derivative. This result is a straightforward consequence of Rolle's theorem.

Proposition 2.25. *Let $d \geqslant 2$ and let $p(x)$ be a degree-d polynomial having zeros $\alpha_1 \leqslant \cdots \leqslant \alpha_d$. Denote by $\beta_1 \leqslant \cdots \leqslant \beta_{d-1}$ the zeros of its derivative $p'(x)$. Then*

$$\alpha_1 \leqslant \beta_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_{d-1} \leqslant \beta_{d-1} \leqslant \alpha_d.$$

Fix the top three coefficients 1, c_1, and c_2. Let $\mathfrak{P} \subset \mathbb{Z}[x]$ be the set of monic totally-real polynomials having top three coefficients equal to 1, c_1, and c_2, i.e., of the form

$$x^d + c_1x^{d-1} + c^2x^{d-2} + \sum_{i=3}^{d} a_ix^{d-i},$$

for some coefficients a_i. We would like to find all polynomials in $\mathfrak{P}$. Suppose $p(x) \in \mathfrak{P}$. Then

$$p(x) = x^d + c_1x^{d-1} + c_2x^{d-2} + \cdots + a_{d-1}x + a_d,$$

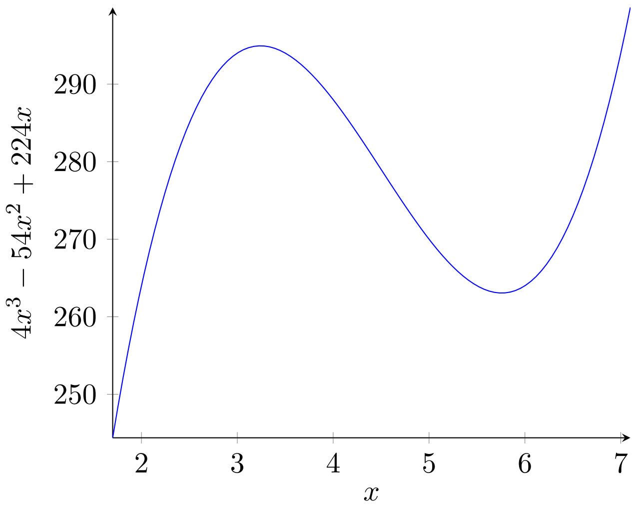
for some integers $a_3, \dots, a_d$. For $r = d, d-1, \dots, 1$, define

$$p_r(x) = \frac{r!}{d!}\frac{\mathrm{d}^{d-r}}{\mathrm{d}x^{d-r}}p(x) = x^r + c_1\frac{r}{d}x^{r-1} + c_2\frac{r(r-1)}{d(d-1)}x^{r-2} + \cdots + a_r\frac{r!(d-r)!}{d!}.$$

Then, by Proposition 2.25, we have that p_r is totally-real for each r. Given a candidate for $p_i(x)$, that is, given values for $a_3, \dots, a_i$, we seek the (possibly empty) range of values for a_{i+1} such that $p_{i+1}(x)$ is totally-real. To find this range of values for a_{i+1}, we set $f(x) = p_{i+1}(x) - a_{i+1}\frac{(i+1)!(d-i-1)!}{d!}$. Let l and u be the minimum (resp. maximum) over the set of local maxima (resp. minima) of $f(x)$. Then $a_{i+1}\frac{(i+1)!(d-i-1)!}{d!}$ must belong to the interval $[-l, -u]$. Hence $\frac{-l\cdot d!}{(i+1)!(d-i-1)!} \leqslant a_{i+1} \leqslant \frac{-u\cdot d!}{(i+1)!(d-i-1)!}$

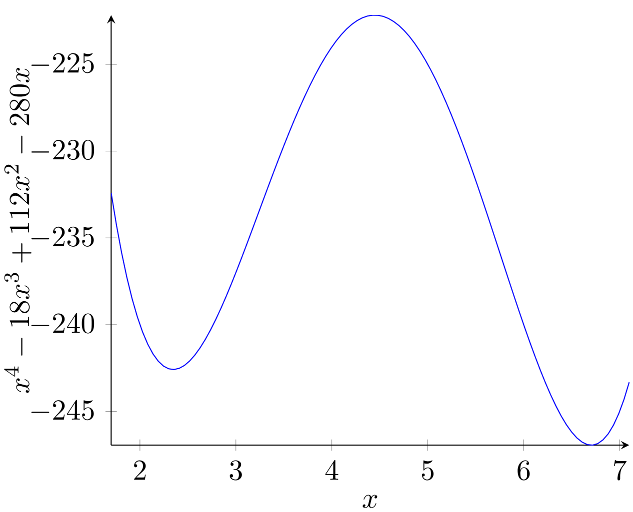
In each step of the above algorithm, a range of values is found for the constant term of a polynomial. Our modification is to apply divisibility "checks" to reduce the number of possible values for the constant term at each iteration. To illustrate how the algorithm works, we provide a toy example below.

Example 2.26. Suppose we want to find all polynomials $f(x) = x^4 - 18x^3 + 112x^2 + a_3x + a_4$ such that all roots of f are real and f is type 2. Since f is totally real, the derivative $f'(x) = 4x^3 - 54x^2 + 224x + a_3$ must also be totally-real.



Hence, $a_3 \in \{-294, \ldots, -264\}$. Now, we use the fact that f is type 2, that is, 8 divides a_3 and 16 divides a_4. Since 8 divides a_3, there are only four possibilities for a_3, which are -288, -280, -272, and -264. For each a_3, we can find the range of possible values for a_4 that ensures that f is totally real.

- When $a_3 = -288$ we must have $a_4 \in \{256, \ldots, 262\}$.
- When $a_3 = -280$ we must have $a_4 \in \{223, \ldots, 242\}$.



- When $a_3 = -272$ we must have $a_4 \in \{185, \ldots, 194\}$.
- When $a_3 = -264$ we must have $a_4 \in \{144\}$.

Now we impose the condition that 16 divides a_4. In total, we obtain 5 possible polynomials for f:

$$\begin{aligned}
&x^4 - 18x^3 + 112x^2 - 288x + 256, \quad x^4 - 18x^3 + 112x^2 - 280x + 224,\\
&x^4 - 18x^3 + 112x^2 - 280x + 240, \quad x^4 - 18x^3 + 112x^2 - 272x + 192,\\
&x^4 - 18x^3 + 112x^2 - 264x + 144.
\end{aligned}$$

2.5.3 Enumeration of candidate characteristic polynomials

Proposition 2.27. *Let S be a Seidel matrix corresponding to n equiangular lines in $\mathbb{R}^d$. Then $\mathrm{Char}_S(x) \in P_{n,d}$ where*

1. *$P_{29,14}$ consists of the elements of*

$$E_{29,14} = \left\{ \begin{array}{l} (x+5)^{15}(x-5)^{10}(x-7)^2(x^2-11x+16), \\ (x+5)^{15}(x-3)(x-5)^9(x-7)^2(x^2-13x+32), \\ (x+5)^{15}(x-5)^{10}(x-7)(x^3-18x^2+93x-128), \\ (x+5)^{15}(x-3)(x-5)^{11}(x^2-17x+68), \\ (x+5)^{15}(x-3)^2(x-5)^8(x-7)^2(x^2-15x+52), \\ (x+5)^{15}(x-3)(x-4)(x-5)^{10}(x-9)^2 \end{array} \right\}$$

   together with the 25 polynomials listed in [15, Table 4].

2. *$P_{41,16}$ consists of the elements of*

$$E_{41,16} = \left\{ \begin{array}{l} (x+5)^{25}(x-7)^9(x-9)^4(x-11)(x^2-15x+48), \\ (x+5)^{25}(x-3)(x-7)^6(x-8)(x-9)^8 \end{array} \right\}$$

   together with the 20 polynomials listed in [15, Table 6].

3. *$P_{49,17}$ consists of the elements of*

$$E_{49,17} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C},$$

   where

$$\mathcal{A} = \{(x+5)^{32}(x-9)^{16}(x-16)\}$$
$$\mathcal{B} = \{(x+5)^{32}(x-8)(x-9)^8(x^2-20x+95)^4, (x+5)^{32}(x-8)(x-9)^{12}(x^2-22x+113)^2\}$$
$$\mathcal{C} = \left\{ \begin{array}{l} (x+5)^{32}(x-7)(x-9)^{14}(x-12)(x-15), \\ (x+5)^{32}(x-7)(x-8)(x-9)^{12}(x-11)^2(x-15), \\ (x+5)^{32}(x-9)^{13}(x-11)^2(x^2-21x+92), \\ (x+5)^{32}(x-7)^2(x-8)(x-9)^{10}(x-11)^2(x-13)^2, \\ (x+5)^{32}(x-9)^{13}(x-13)^2(x^2-17x+64), \\ (x+5)^{32}(x-9)^{12}(x-11)^3(x^2-19x+72), \\ (x+5)^{32}(x-7)(x-9)^{10}(x-11)^4(x^2-19x+76), \\ (x+5)^{32}(x-4)(x-9)^{10}(x-11)^6 \end{array} \right\}$$

   together with the 164 polynomials listed in [16, Table 2], the 11 polynomials listed in [16, Table 4], and the 8 polynomials listed in [16, Table 6].

2.6 Exercises

1. Prove Lemma 2.1.
2. Prove Lemma 2.2.
3. Let S be a Seidel matrix of odd order. Prove that S has an eigenvalue of multiplicity 1.
4. Prove that $\sqrt{2}$ cannot be an eigenvalue of any Seidel matrix.
5. Prove Lemma 2.5.
6. Prove Lemma 2.9.

Chapter 3

Nonexistence of Seidel matrices having a prescribed characteristic polynomial

In this chapter, we describe a procedure for showing the nonexistence of a Seidel matrix having characteristic polynomial $p(x)$, where $p(x)$ is some fixed polynomial. We largely follow [15] and [16].

3.1 Principal submatrices

Our main approach for showing that a Seidel matrix S having a certain spectrum does not exist is to consider the principal submatrices of S and their characteristic polynomials.

Let M be a real symmetric matrix of order n. We write $\Lambda(M)$ for the set of distinct eigenvalues of M and define the polynomial

$$\operatorname{Min}_M(x) := \prod_{\lambda \in \Lambda(M)} (x - \lambda),$$

which is the minimal polynomial of M. Define

$$\operatorname{Quo}_M(x) := \operatorname{Char}_M(x) / \operatorname{Min}_M(x),$$

and denote by $M[i]$ the principal submatrix of M obtained by deleting its ith row and column.

Cauchy's interlacing theorem, below, provides bounds for the eigenvalues of principal submatrices of M.

Theorem 3.1 ([5, 10, 22]). *Let M be a real symmetric matrix having eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$ and suppose $M[i]$, for some $i \in \{1, \dots, n\}$, has eigenvalues $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_{n-1}$. Then*

$$\lambda_1 \leqslant \mu_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{n-1} \leqslant \mu_{n-1} \leqslant \lambda_n.$$

Given $e \in \mathbb{N}$ and polynomials $f(x) = \prod_{i=0}^{e}(x - \lambda_i)$ and $g(x) = \prod_{i=1}^{e}(x - \mu_i)$ such that $\lambda_0 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_e$, and $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_e$, we say that g **interlaces** f if $\lambda_0 \leqslant \mu_1 \leqslant \lambda_1 \leqslant \cdots \leqslant \mu_e \leqslant \lambda_e$. Note that, if a polynomial $\mathfrak{f}(x)$ interlaces $\operatorname{Char}_M(x)$ then we can write $\mathfrak{f}(x) = \operatorname{Quo}_M(x) f(x)$, where $f(x)$ is a monic integer polynomial that interlaces $\operatorname{Min}_M(x)$.

The next result is a condition on the sum of the characteristic polynomials of principal submatrices of a matrix.

Theorem 3.2 ([36, Page 116]). *Let M be a real matrix of order n. Then*

$$\sum_{i=1}^{n} \operatorname{Char}_{M[i]}(x) = \frac{\mathrm{d}}{\mathrm{d}x} \operatorname{Char}_M(x). \tag{3.1}$$

3.2 Interlacing characteristic polynomials and interlacing configurations

Definition 3.3. Let $p(x) \in \mathbb{Z}[x]$ be a monic polynomial of degree n and write

$$p(x) = \sum_{t=0}^{n} a_t x^{n-t},$$

where $a_0 = 1$, $a_1 = 0$, and $a_2 = -\binom{n}{2}$. An **interlacing characteristic polynomial** for $p(x)$ is defined to be a totally-real, integer polynomial $\mathfrak{f}(x) = \sum_{t=0}^{n-1} b_t x^{n-1-t}$ such that

(i) $b_0 = 1$, $b_1 = 0$, $b_2 = -\binom{n-1}{2}$,

(ii) $\mathfrak{f}(x)$ interlaces $p(x)$,

(iii) $\mathfrak{f}(x-1)$ is weakly type 2 and is type 2 if $n-1$ is even,

(iv) $\mathfrak{f}(x)$ is in a congruence class of $\mathscr{P}_{n-1,7}$, if $n-1$ is odd.

Denote by $\operatorname{Deck}(p)$ the set of all interlacing characteristic polynomials for $p(x)$.

Lemma 3.4. *Suppose $p(x)$ is the characteristic polynomial of a Seidel matrix S. Then there exist nonnegative integers $n_{\mathfrak{f}}$ for each $\mathfrak{f}(x) \in \operatorname{Deck}(p)$ such that*

$$\sum_{\mathfrak{f}(x) \in \operatorname{Deck}(p)} n_{\mathfrak{f}} \cdot \mathfrak{f}(x) = p'(x). \tag{3.2}$$

The (row) vector $\mathbf{n}$ indexed by $\operatorname{Deck}(p)$ whose $\mathfrak{f}(x)$-entry is $n_{\mathfrak{f}}$, for each $\mathfrak{f}(x) \in \operatorname{Deck}(p)$ is called an **interlacing configuration for** $p(x)$.

Example 3.5. Set $p(x) = x^7 - 21x^5 + 26x^4 + 71x^3 - 132x^2 + 45x + 10$.

$$\operatorname{Deck}(p) = \left\{ \begin{array}{l} \mathfrak{f}_1(x) = x^6 - 15x^4 + 8x^3 + 51x^2 - 40x - 5 \\ \mathfrak{f}_2(x) = x^6 - 15x^4 + 16x^3 + 27x^2 - 48x + 19 \\ \mathfrak{f}_3(x) = x^6 - 15x^4 + 16x^3 + 43x^2 - 80x + 35 \\ \mathfrak{f}_4(x) = x^6 - 15x^4 + 24x^3 + 3x^2 - 24x + 11 \end{array} \right\}$$

There is just one interlacing configuration of $p(x)$: $(3, 2, 0, 2)$. One can check that $3\mathfrak{f}_1(x) + 2\mathfrak{f}_2(x) + 2\mathfrak{f}_4(x) = \frac{\mathrm{d}}{\mathrm{d}x} p(x)$. Furthermore, we can find a Seidel matrix S such that $\operatorname{Char}_S(x) = p(x)$:

$$S = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 0 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 0 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 0 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 0 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 1 & 1 & -1 & 0 \end{bmatrix}.$$

Since the interlacing configuration of $p(x)$ is $(3, 2, 0, 2)$, among the characteristic polynomials of the order-6 principal submatrices of S, 3 are equal to $\mathfrak{f}_1$, 2 are equal to $\mathfrak{f}_2$, and 2 are equal to $\mathfrak{f}_4$. Indeed, we have $\operatorname{Char}_{S[1]}(x) = \mathfrak{f}_4(x)$, $\operatorname{Char}_{S[2]}(x) = \mathfrak{f}_4(x)$, $\operatorname{Char}_{S[3]}(x) = \mathfrak{f}_1(x)$, $\operatorname{Char}_{S[4]}(x) = \mathfrak{f}_2(x)$, $\operatorname{Char}_{S[5]}(x) = \mathfrak{f}_1(x)$, $\operatorname{Char}_{S[6]}(x) = \mathfrak{f}_1(x)$, and $\operatorname{Char}_{S[7]}(x) = \mathfrak{f}_2(x)$.

3.2.1 Constructing $\mathrm{Deck}(p)$ and enumerating interlacing configurations

This section deals with the practical issues of constructing $\mathrm{Deck}(p)$ and enumerating interlacing configurations.

We write $\Lambda(p)$ for the set of distinct zeros of the polynomial $p(x)$ and define the polynomial

$$\mathrm{Min}_p(x) := \prod_{\lambda \in \Lambda(p)} (x - \lambda).$$

By Definition 3.3, if $\mathfrak{f}(x) \in \mathrm{Deck}(p)$ then $\mathfrak{f}(x)$ interlaces $p(x)$ and the top three coefficients of $\mathfrak{f}(x)$ are fixed (see (i) of Definition 3.3). Consequently, we have the following lemma.

Lemma 3.6. *Let $p(x)$ be a polynomial in $\mathbb{Z}[x]$. Suppose $\mathrm{Min}_p(x) = \sum_{i=0}^{e} a_i x^{e-i}$. Then, for all $\mathfrak{f}(x) \in \mathrm{Deck}(p)$,*

$$\mathfrak{f}(x) = \frac{p(x)}{\mathrm{Min}_p(x)} \sum_{i=0}^{e-1} b_i x^{e-1-i},$$

where $b_0 = 1$, $b_1 = a_1$, $b_2 = a_2 + n - 1$, and $b_i \in \mathbb{Z}$ for $i \in \{3, \dots, e-1\}$.

We write $\mathbf{x} \geqslant \mathbf{0}$ to indicate that all entries of the vector $\mathbf{x}$ are nonnegative. The **coefficient vector** of a polynomial $f(x) = \sum_{t=0}^{n-1} c_t x^{n-1-t}$ of degree $n-1$ is defined to be the (row) vector $(c_0, c_1, \dots, c_{n-1})$. Given a set $\mathfrak{P}$ of polynomials each of degree $n-1$, the **coefficient matrix** $C(\mathfrak{P})$ is defined as the $|\mathfrak{P}| \times n$ matrix whose rows are the coefficient vectors for each polynomial in $\mathfrak{P}$. If $\mathfrak{P}$ is a singleton, i.e., $\mathfrak{P} = \{\mathfrak{f}(x)\}$ then we merely write $C(\mathfrak{P})$ as $C(\mathfrak{f})$. Note that (3.2) can be written as a vector equation as

$$\mathbf{n} C(\mathrm{Deck}(p)) = C(p'). \tag{3.3}$$

Therefore, to find all interlacing configurations for $p(x)$, we need to find all vectors $\mathbf{n} \geqslant \mathbf{0}$ satisfying $\mathbf{n} C(\mathrm{Deck}(p)) = C(p')$.

Example 3.7. Let $p(x) = (x-7)^6(x-3)^6(x+5)^{13}(x^2-5x-2)$. We will construct $\mathrm{Deck}(p)$. First,

$$\mathrm{Min}_p(x) = (x-7)(x-3)(x+5)(x^2-5x-2) = x^5 - 10x^4 - 6x^3 + 260x^2 - 467x - 210.$$

By Lemma 3.6, each interlacing characteristic polynomial $\mathfrak{f}(x) \in \mathrm{Deck}(p)$ has the form

$$\mathfrak{f}(x) = (x-7)^5(x-3)^5(x+5)^{12}(x^4 - 10x^3 + 20x^2 + b_3 x + b_4).$$

Write $q(x) = x^4 - 10x^3 + 20x^2 + b_3 x + b_4$. By Lemma 2.20, the polynomial $q(x-1) = x^4 - 14x^3 + 56x^2 + c_3 x + c_4$ must be type 2. Furthermore, $q(x)$ must also be totally-real. Now we can apply the polynomial enumeration algorithm from Section 2.5.2 to obtain all possibilities for $q(x-1)$ and hence for $q(x)$. There are 24 possibilities for $q(x)$, which we list in Table 3.1. We can further reduce this list using interlacing. Since $\mathfrak{f}(x)$ interlaces $p(x)$, the factor $q(x)$ must interlace $\mathrm{Min}_p(x)$. This leaves us with 18 possibilities for $q(x)$. Hence, we obtain $\mathrm{Deck}(p)$, which consists of 18 polynomials.

There are 335 interlacing configurations for $p(x)$. One of these is

$$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 24, 0, 0, 0, 0, 3, 0, 0),$$

which corresponds to $\mathrm{Deck}(p) = \{\mathfrak{f}_1, \dots, \mathfrak{f}_{18}\}$ given in Table 3.1. The matrix S below is a Seidel matrix that corresponds to this interlacing configuration.

	Deck(p)	$q(x)$	interlaces $\mathrm{Min}_p(x)$?
1		$x^4 - 10x^3 + 20x^2 - 14x + 3$	No
2		$x^4 - 10x^3 + 20x^2 - 6x - 5$	No
3		$x^4 - 10x^3 + 20x^2 + 2x - 13$	No
4	$\mathfrak{f}_1$	$x^4 - 10x^3 + 20x^2 + 10x - 21$	Yes
5	$\mathfrak{f}_2$	$x^4 - 10x^3 + 20x^2 + 10x - 5$	Yes
6	$\mathfrak{f}_3$	$x^4 - 10x^3 + 20x^2 + 18x - 45$	Yes
7	$\mathfrak{f}_4$	$x^4 - 10x^3 + 20x^2 + 18x - 29$	Yes
8	$\mathfrak{f}_5$	$x^4 - 10x^3 + 20x^2 + 18x - 13$	Yes
9	$\mathfrak{f}_6$	$x^4 - 10x^3 + 20x^2 + 18x + 3$	Yes
10	$\mathfrak{f}_7$	$x^4 - 10x^3 + 20x^2 + 26x - 69$	Yes
11	$\mathfrak{f}_8$	$x^4 - 10x^3 + 20x^2 + 26x - 53$	Yes
12	$\mathfrak{f}_9$	$x^4 - 10x^3 + 20x^2 + 26x - 37$	Yes
13	$\mathfrak{f}_{10}$	$x^4 - 10x^3 + 20x^2 + 26x - 21$	Yes
14	$\mathfrak{f}_{11}$	$x^4 - 10x^3 + 20x^2 + 26x - 5$	Yes
15	$\mathfrak{f}_{12}$	$x^4 - 10x^3 + 20x^2 + 34x - 93$	Yes
16	$\mathfrak{f}_{13}$	$x^4 - 10x^3 + 20x^2 + 34x - 77$	Yes
17	$\mathfrak{f}_{14}$	$x^4 - 10x^3 + 20x^2 + 34x - 61$	Yes
18	$\mathfrak{f}_{15}$	$x^4 - 10x^3 + 20x^2 + 34x - 45$	Yes
19	$\mathfrak{f}_{16}$	$x^4 - 10x^3 + 20x^2 + 42x - 117$	Yes
20	$\mathfrak{f}_{17}$	$x^4 - 10x^3 + 20x^2 + 42x - 101$	Yes
21		$x^4 - 10x^3 + 20x^2 + 42x - 85$	No
22	$\mathfrak{f}_{18}$	$x^4 - 10x^3 + 20x^2 + 50x - 141$	Yes
23		$x^4 - 10x^3 + 20x^2 + 50x - 125$	No
24		$x^4 - 10x^3 + 20x^2 + 50x - 165$	No

Table 3.1: The 24 possibilities for $\xi(x)$.

$$S = \begin{bmatrix}
0 & + \\
+ & 0 & + & + & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & + & + & + & + & + & + & + \\
+ & + & 0 & + & + & + & + & + & + & + & + & + & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\
+ & + & + & 0 & - & - & - & - & - & - & - & - & + & + & + & + & + & + & + & + & - & - & - & - & - & - & - \\
+ & - & + & - & 0 & + & + & + & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - \\
+ & - & + & - & + & 0 & + & + & + & - & - & + & - & + & + & - & - & + & + & - & + & - & - & + & - & + & + \\
+ & - & + & - & + & + & 0 & + & - & + & + & - & - & + & + & - & + & - & - & + & - & + & + & - & - & + & + \\
+ & - & + & - & + & + & + & 0 & - & + & + & - & + & - & - & + & - & + & + & - & - & + & + & - & + & - & - \\
+ & - & + & - & + & + & - & - & 0 & + & + & + & + & - & + & - & + & - & + & - & + & - & - & + & + & - & + \\
+ & - & + & - & - & - & + & + & + & 0 & + & + & + & - & + & - & - & + & - & + & - & + & + & - & - & + & - \\
+ & - & + & - & - & - & + & + & + & + & 0 & + & - & + & - & + & + & - & + & - & - & + & + & - & + & - & + \\
+ & - & + & - & + & + & - & - & + & + & + & 0 & - & + & - & + & - & + & - & + & + & - & - & + & - & + & - \\
+ & - & - & + & + & - & - & + & + & + & - & - & 0 & + & + & + & + & + & - & - & + & - & + & - & + & - & - \\
+ & - & - & + & - & + & + & - & - & - & + & + & + & 0 & + & + & + & + & - & - & + & - & + & - & - & + & + \\
+ & - & - & + & - & + & + & - & + & + & - & - & + & + & 0 & + & - & - & + & + & - & + & - & + & - & + & + \\
+ & - & - & + & + & - & - & + & - & - & + & + & + & + & + & 0 & - & - & + & + & - & + & - & + & + & - & - \\
+ & - & - & + & + & - & + & - & + & - & + & - & + & + & - & - & 0 & + & + & + & + & - & + & - & + & - & + \\
+ & - & - & + & - & + & - & + & - & + & - & + & + & + & - & - & + & 0 & + & + & + & - & + & - & - & + & - \\
+ & - & - & + & - & + & - & + & + & - & + & - & - & - & + & + & + & + & 0 & + & - & + & - & + & + & - & + \\
+ & - & - & + & + & - & + & - & - & + & - & + & - & - & + & + & + & + & + & 0 & - & + & - & + & - & + & - \\
+ & + & - & - & + & + & - & - & + & - & - & + & + & + & - & - & + & + & - & - & 0 & + & + & + & + & + & - \\
+ & + & - & - & - & - & + & + & - & + & + & - & - & - & + & + & - & - & + & + & + & 0 & + & + & + & + & - \\
+ & + & - & - & - & - & + & + & - & + & + & - & + & + & - & - & + & + & - & - & + & + & 0 & + & - & - & + \\
+ & + & - & - & + & + & - & - & + & - & - & + & - & - & + & + & - & - & + & + & + & + & + & 0 & - & - & + \\
+ & + & - & - & + & - & - & + & + & - & + & - & + & - & - & + & + & - & + & - & + & + & - & - & 0 & + & + \\
+ & + & - & - & - & + & + & - & - & + & - & + & - & + & + & - & - & + & - & + & + & + & - & - & + & 0 & + \\
+ & + & - & - & - & + & + & - & + & - & + & - & - & + & + & - & + & - & + & - & - & - & + & + & + & + & 0
\end{bmatrix}$$

3.2.2 Certificates of infeasibility

If there exists a Seidel matrix whose characteristic polynomial is equal to $p(x)$ then there must exist an interlacing configuration for $p(x)$, i.e., a solution $\mathbf{n}$ to (3.3) where $\mathbf{n} \geqslant \mathbf{0}$. In the other direction, if there does not exist a nonnegative solution to (3.3) then there does not exist a Seidel matrix having $p(x)$ as its characteristic polynomial. Farkas' Lemma allows a convenient way to verify that there is no such solution by providing a solution to a dual linear system.

Theorem 3.8 (Farkas' Lemma [8])**.** *Let A be a real $n \times m$ matrix and let $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{x} \in \mathbb{R}^n$. Then the linear system*

$$\mathbf{x}^\top A = \mathbf{b}^\top,\ \mathbf{x} \geqslant \mathbf{0} \tag{3.4}$$

has no solution if and only if the linear system

$$A\mathbf{y} \geqslant \mathbf{0},\ \mathbf{b}^\top \mathbf{y} < 0 \tag{3.5}$$

has a solution, where $\mathbf{y} \in \mathbb{R}^m$.

Proof. To prove that at most one of the systems can have a solution is easy. Indeed, suppose that both (3.4) and (3.5) hold for some $\mathbf{x}$ and $\mathbf{y}$. Then

$$\begin{aligned} \mathbf{x}^\intercal A\mathbf{y} &= (\mathbf{x}^\intercal A)\mathbf{y} = \mathbf{b}^\intercal \mathbf{y} < 0 \\ &= \mathbf{x}^\intercal (A\mathbf{y}) \geqslant 0, \end{aligned}$$

which is a contradiction. The rest of the proof is left as an exercise. □

In the following corollary, note that $p(x)$ divides $\operatorname{Min}_p(x)p'(x)$ and $\operatorname{Min}_p(x)\mathfrak{f}(x)$ for each $\mathfrak{f}(x) \in \operatorname{Deck}(p)$.

Corollary 3.9. *Let $p(x)$ be a polynomial where $\operatorname{Min}_p(x)$ has degree e. Suppose there exists a vector $\mathbf{c} \in \mathbb{R}^e$ such that*

$$C\left(\left\{\frac{\operatorname{Min}_p(x)\mathfrak{f}(x)}{p(x)}\ :\ \mathfrak{f}(x) \in \operatorname{Deck}(p)\right\}\right)\mathbf{c} \geqslant \mathbf{0} \quad \textit{and} \quad C\left(\left\{\frac{\operatorname{Min}_p(x)p'(x)}{p(x)}\right\}\right)\mathbf{c} < 0.$$

Then $p(x)$ is not the characteristic polynomial of any Seidel matrix.

Proof. By Lemma 3.4, Lemma 3.6, and Farkas' Lemma (Theorem 3.8), it follows that $p(x)$ does not have an interlacing configuration. $\square$

We call the vector $\mathbf{c}$ from Corollary 3.9 a **certificate of infeasibility** for $p(x)$.

Example 3.10. We will show that the polynomial $p(x) = (x-11)(x-4)(x-5)^{12}(x+5)^{15}$ has a certificate of infeasibility. We have $\mathrm{Min}_p(x) = (x-11)(x-4)(x-5)(x+5)$ and

$$\mathrm{Deck}(p) = \begin{Bmatrix} (x-5)^{11}(x+5)^{14}(x^3-15x^2+47x-9), & (x-5)^{11}(x+5)^{14}(x^3-15x^2+47x-1), \\ (x-5)^{11}(x+5)^{14}(x^3-15x^2+47x+7), & (x-5)^{12}(x+5)^{14}(x^2-10x-3) \end{Bmatrix}.$$

The coefficient matrix $A = C\left(\left\{\frac{\mathrm{Min}_p(x)\mathfrak{f}(x)}{p(x)} \ : \ \mathfrak{f}(x) \in \mathrm{Deck}(p)\right\}\right)$ is equal to

$$A = \begin{bmatrix} 1 & -15 & 47 & -9 \\ 1 & -15 & 47 & -1 \\ 1 & -15 & 47 & 7 \\ 1 & -15 & 47 & 15 \end{bmatrix}.$$

And

$$C\left(\left\{\frac{\mathrm{Min}_p(x)p'(x)}{p(x)}\right\}\right) = (29, -453, 1363, -285).$$

We claim that $\mathbf{c} = (423, 0, 0, 44)^\intercal$ is a certificate of infeasibility for $p(x)$. Indeed, one can check that

$$A\mathbf{c} = (27, 379, 731, 1083)^\intercal \geqslant \mathbf{0} \quad \text{and} \quad (29, -453, 1363, -285)\mathbf{c} = -273 < 0.$$

Since $p(x)$ has a certificate of infeasibility, it cannot have an interlacing configuration. Therefore, there does not exist a Seidel matrix whose characteristic polynomial is equal to $p(x)$. Note that such a Seidel matrix would correspond to a system of 29 equiangular lines in $\mathbb{R}^{14}$.

Certificates of infeasibility provide us with a convenient way to show that there cannot exist a Seidel matrix having a given characteristic polynomial. However, not all polynomials have certificates of infeasibility. This motivates the next section.

3.2.3 Warranted polynomials

Definition 3.11. Let $p(x)$ be a totally-real polynomial in $\mathbb{Z}[x]$. Suppose that $p(x)$ has at least one interlacing configuration and $\mathrm{Min}_p(x)$ has degree e. We say that $\mathfrak{f}(x) \in \mathrm{Deck}(p)$ is $p(x)$-**warranted** if the $\mathfrak{f}(x)$-entry of every interlacing configuration for $p(x)$ is positive. Equivalently, by Corollary 3.9, the interlacing characteristic polynomial $\mathfrak{f}(x)$ is $p(x)$-warranted if there exists $\mathbf{c} \in \mathbb{R}^e$ such that the $\mathfrak{h}(x)$-entry of

$$C\left(\left\{\frac{\mathrm{Min}_p(x)\mathfrak{g}(x)}{p(x)} \ : \ \mathfrak{g}(x) \in \mathrm{Deck}(p)\right\}\right)\mathbf{c}$$

is negative for $\mathfrak{h}(x) = \mathfrak{f}(x)$, nonnegative for $\mathfrak{h}(x) \in \mathrm{Deck}(p)\backslash\{\mathfrak{f}(x)\}$, and

$$C\left(\left\{\frac{\mathrm{Min}_p(x)p'(x)}{p(x)}\right\}\right)\mathbf{c} < 0.$$

The vector $\mathbf{c}$ is called the **certificate of warranty** for $\mathfrak{f}(x)$.

Lemma 3.12. *Let S be a Seidel matrix of order n. Suppose that $\mathfrak{f}(x) \in \mathrm{Deck}(\mathrm{Char}_S)$ is $\mathrm{Char}_S(x)$-warranted. Then there exists $i \in \{1, \dots, n\}$ such that $\mathrm{Char}_{S[i]}(x) = \mathfrak{f}(x)$.*

Example 3.13. Let $p(x) = (x-7)^2(x-3)^2(x-5)^8(x+5)^{15}(x^2-15x+52)$. We claim that

$$\mathfrak{f}(x) = (x-7)^2(x-3)^2(x-5)^7(x+5)^{15}(x^3-15x^2+55x-17) \in \mathrm{Deck}(p)$$

is $p(x)$-warranted. Note that $\dfrac{\mathrm{Min}_p(x)\mathfrak{f}(x)}{p(x)} = x^5 - 25x^4 + 226x^3 - 882x^2 + 1325x - 357$. Set

$$A = C\left(\left\{\frac{\mathrm{Min}_p(x)\mathfrak{h}(x)}{p(x)} \;:\; \mathfrak{h}(x) \in \mathrm{Deck}(p)\right\}\right),$$

then

$$A = \begin{bmatrix}
1 & -25 & 226 & -890 & 1405 & -525 \\
1 & -25 & 226 & -882 & 1277 & -85 \\
1 & -25 & 226 & -882 & 1293 & -165 \\
1 & -25 & 226 & -882 & 1309 & -245 \\
1 & -25 & 226 & -882 & 1325 & -357 \\
1 & -25 & 226 & -874 & 1165 & 275 \\
1 & -25 & 226 & -874 & 1181 & 195 \\
1 & -25 & 226 & -874 & 1197 & 115 \\
1 & -25 & 226 & -874 & 1213 & 35 \\
1 & -25 & 226 & -866 & 1085 & 475 \\
1 & -25 & 226 & -866 & 1101 & 395 \\
1 & -25 & 226 & -866 & 1117 & 315 \\
1 & -25 & 226 & -858 & 1005 & 675 \\
1 & -25 & 226 & -858 & 1021 & 595 \\
1 & -25 & 226 & -850 & 925 & 875 \\
1 & -25 & 226 & -842 & 829 & 1155
\end{bmatrix}$$

And

$$C\left(\left\{\frac{\mathrm{Min}_p(x)p'(x)}{p(x)}\right\}\right) = (29, -725, 6554, -25490, 36841, -4345).$$

The vector $\mathbf{c} = (0,0,0,708,507,154)^\intercal$ is a certificate of warranty for $\mathfrak{f}(x)$. Indeed,

$$A\mathbf{c} = (1365, 9893, 5685, 1477, -7659, 14213, 10005, 5797, 1589, 10117, 5909, 1701, 6021, 1813, 1925, 2037)^\intercal,$$

which is nonnegative, except for the entry corresponding to $\mathfrak{f}(x)$, which is negative and

$$(29, -725, 6554, -25490, 36841, -4345)\mathbf{c} = -37663 < 0.$$

By Lemma 3.12, if there exists a Seidel matrix S such that $\mathrm{Char}_S(x) = p(x)$ then there must exist $i \in \{1, \dots, 29\}$ such that $\mathrm{Char}_{S[i]}(x) = \mathfrak{f}(x)$.

3.3 Compatibility of polynomials

Let $p(x)$ be a totally-real polynomial. Once we have found a warranted interlacing characteristic polynomial $\mathfrak{f}(x) \in \mathrm{Deck}(p)$, we can reduce to a subset of $\mathrm{Deck}(p)$ that consists of those polynomials that are compatible with $\mathfrak{f}(x)$. In this section, we introduce the notion of compatibility for interlacing characteristic polynomials.

3.3.1 Angle vectors

Let M be a real symmetric matrix of order n. For each $\lambda \in \Lambda(M)$, denote by $\mathcal{E}(\lambda)$ the eigenspace of λ and let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be the standard basis of $\mathbb{R}^n$. Denote by P_λ the orthogonal projection of $\mathbb{R}^n$ onto $\mathcal{E}(\lambda)$. For a vector $\mathbf{v}$, we write $\mathbf{v}(i)$ to denote its ith entry.

Theorem 3.14 (Spectral Decomposition Theorem)**.** *Let M be a real symmetric matrix. Then*

$$M = \sum_{\lambda \in \Lambda(M)} \lambda P_\lambda.$$

Denote by $\boldsymbol{\alpha}_\lambda$ the **angle vector** for $\lambda \in \Lambda(M)$, that is, for each $i \in \{1, \dots, n\}$,

$$\boldsymbol{\alpha}_\lambda(i) = ||P_\lambda \mathbf{e}_i||.$$

Proposition 3.15 (See [6, (4.2.8)] or [11])**.** *Let M be a real symmetric matrix of order n. Then, for each $i \in \{1, \dots, n\}$, we have*

$$\mathrm{Char}_{M[i]}(x) = \mathrm{Char}_M(x) \sum_{\lambda \in \Lambda(M)} \frac{\boldsymbol{\alpha}_\lambda^2(i)}{x - \lambda}.$$

Given a polynomial $p(x) \in \mathbb{Q}[x]$, we denote its derivative by $p'(x)$. In the next result, we give a convenient expression for the entries of an angle vector.

Lemma 3.16. *Let M be a real symmetric matrix of order n. Let $i \in \{1, \dots, n\}$ and suppose that $\mathrm{Char}_{M[i]}(x) = \mathrm{Quo}_M(x) \cdot f_i(x)$ for some polynomial $f_i(x) \in \mathbb{Z}[x]$. Then, for each $\lambda \in \Lambda(M)$, we have*

$$\boldsymbol{\alpha}_\lambda(i) = \sqrt{\frac{f_i(\lambda)}{\mathrm{Min}_M'(\lambda)}}.$$

Proof. By Proposition 3.15,

$$\mathrm{Char}_{M[i]}(x) = \mathrm{Char}_M(x) \sum_{\lambda \in \Lambda(M)} \frac{\boldsymbol{\alpha}_\lambda^2(i)}{x - \lambda}.$$

Dividing both sides by $\mathrm{Quo}_M(x)$, we obtain

$$f_i(x) = \mathrm{Min}_M(x) \sum_{\lambda \in \Lambda(M)} \frac{\boldsymbol{\alpha}_\lambda^2(i)}{x - \lambda}.$$

Thus, for each $\lambda \in \Lambda(M)$, we have

$$f_i(\lambda) = \boldsymbol{\alpha}_\lambda^2(i) \prod_{\mu \in \Lambda(M) \backslash \{\lambda\}} (\lambda - \mu).$$

On the other hand, we also have

$$\mathrm{Min}_M'(x) = \sum_{\lambda \in \Lambda(M)} \prod_{\mu \in \Lambda(M) \backslash \{\lambda\}} (x - \mu).$$

Therefore, $\boldsymbol{\alpha}_\lambda^2(i) = f_i(\lambda) / \mathrm{Min}_M'(\lambda)$. The statement of the lemma follows since, by definition, $\boldsymbol{\alpha}_\lambda(i)$ is nonnegative. □

The next result generalises the fact that a unit eigenvector of a simple eigenvalue λ can be expressed in terms of the angle vector of λ. See [7] for a survey on a related result.

Lemma 3.17. *Let M be a real symmetric matrix of order n and let λ be an eigenvalue of M of multiplicity e. Let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_e$ be an orthonormal basis for the eigenspace $\mathcal{E}(\lambda)$. For all $i \in \{1, \dots, n\}$, we have that*

$$\boldsymbol{\alpha}_\lambda^2(i) = \sum_{k=1}^{e} \mathbf{u}_k^2(i).$$

Proof. Firstly, we can write

$$P_\lambda = \mathbf{u}_1\mathbf{u}_1^\intercal + \cdots + \mathbf{u}_e\mathbf{u}_e^\intercal.$$

Hence, for each $i \in \{1, \dots, n\}$, we have

$$P_\lambda \mathbf{e}_i = \mathbf{u}_1(i)\mathbf{u}_1 + \cdots + \mathbf{u}_e(i)\mathbf{u}_e.$$

Therefore,

$$\boldsymbol{\alpha}_\lambda^2(i) = \|P_\lambda \mathbf{e}_i\|^2 = \langle P_\lambda \mathbf{e}_i, P_\lambda \mathbf{e}_i \rangle = \sum_{k=1}^{e}\sum_{l=1}^{e} \mathbf{u}_k(i)\mathbf{u}_l(i)\langle \mathbf{u}_k, \mathbf{u}_l\rangle = \sum_{k=1}^{e} \mathbf{u}_k^2(i)$$

since $\mathbf{u}_1, \dots, \mathbf{u}_e$ are orthonormal. □

Next, we introduce the notion of *compatibility* for polynomials.

3.3.2 Seidel-compatible polynomials

In this section, we introduce the notions of angles and Seidel-compatibility for polynomials.

Let $\mathfrak{f}(x) = \dfrac{p(x)}{\mathrm{Min}_p(x)} f(x) \in \mathrm{Deck}(p)$. For each $\lambda \in \Lambda(p)$, define the **angle** $\alpha_\lambda(\mathfrak{f})$ of $\mathfrak{f}(x)$ with respect to λ as

$$\alpha_\lambda(\mathfrak{f}) := \sqrt{\frac{f(\lambda)}{\mathrm{Min}_p'(\lambda)}}.$$

Now we can introduce the notion of *compatibility* for polynomials with respect to $p(x)$. Let $\Sigma(p)$ be the set of simple zeros of $p(x)$, define $\mathrm{Sim}_p(x)$ as

$$\mathrm{Sim}_p(x) := \prod_{\lambda \in \Sigma(p)} (x - \lambda),$$

and, given a factor $\xi(x)$ of $\mathrm{Sim}_p(x)$, define

$$\mathrm{Q}_{p,\xi}(x) := \mathrm{Min}_p(x)/\xi(x).$$

Definition 3.18. Let $p(x)$ be a totally-real monic integer polynomial and let $\xi(x) \in \mathbb{Z}[x]$ be a factor of $\mathrm{Sim}_p(x)$ with zero-set $\Lambda(\xi)$. Let $\mathfrak{f}(x)$ and $\mathfrak{g}(x)$ be distinct polynomials in $\mathrm{Deck}(p)$. We say that $\mathfrak{f}(x)$ and $\mathfrak{g}(x)$ are $\xi(x)$-**Seidel-compatible** if there exists $\boldsymbol{\delta} \in \{\pm 1\}^{\Lambda(\xi)}$ such that

$$\sum_{\lambda \in \Lambda(\xi)} \mathrm{Q}_{p,\xi}(\lambda)\boldsymbol{\delta}(\lambda)\alpha_\lambda(\mathfrak{f})\alpha_\lambda(\mathfrak{g}) \equiv R(p, \xi) \pmod 2, \tag{3.6}$$

where

$$R(p,\xi) := \begin{cases} \mathrm{Q}_{p,\xi}(1) + \mathrm{Q}_{p,\xi}(0), & \text{if } \deg p \text{ is odd;} \\ \left(\mathrm{Q}_{p,\xi}(1) - \mathrm{Q}_{p,\xi}(-1)\right)/2, & \text{if } \deg p \text{ is even.} \end{cases}$$

If $\mathfrak{f}(x)$ and $\mathfrak{g}(x)$ are $\xi(x)$-Seidel-compatible with respect to $p(x)$ for every factor $\xi(x) \in \mathbb{Z}[x]$ of $\mathrm{Sim}_p(x)$ then we say that $\mathfrak{f}(x)$ and $\mathfrak{g}(x)$ are **Seidel-compatible** with respect to S.

Any polynomial $\mathfrak{f}(x) \in \mathrm{Deck}(p)$ is considered to be Seidel-compatible with itself.

Lemma 3.19. *Let S be a Seidel matrix of order n. Suppose that $\mathfrak{f}(x) \in \mathrm{Deck}(\mathrm{Char}_S)$ is $\mathrm{Char}_S(x)$-warranted. Then for all $j \in \{1,\dots,n\}$, the polynomials $\mathrm{Char}_{S[j]}(x)$ and $\mathfrak{f}(x)$ are Seidel-compatible with respect to $\mathrm{Char}_S(x)$.*

Proof. Set $p(x) = \mathrm{Char}_S(x)$. Since $\mathfrak{f}(x) \in \mathrm{Deck}(\mathrm{Char}_S)$ is $\mathrm{Char}_S(x)$-warranted, then, by Lemma 3.12, there exists $i \in \{1,\dots,n\}$ such that $\mathrm{Char}_{S[i]}(x) = \mathfrak{f}(x)$. Let $j \in \{1,\dots,n\}$ and let $\mathfrak{g}(x) = \mathrm{Char}_{S[j]}(x)$. If $\mathfrak{f}(x) = \mathfrak{g}(x)$ then we are done. Otherwise, we have $i \neq j$. Let $\xi(x) \in \mathbb{Z}[x]$ be a factor of $\mathrm{Sim}_p(x)$. For each $\lambda \in \Lambda(\xi)$, denote by $\mathbf{u}_\lambda$ a unit eigenvector for λ. By the Spectral Decomposition Theorem, we have

$$\mathrm{Q}_{p,\xi}(S)_{i,j} = \sum_{\lambda\in\Lambda(\xi)} \mathrm{Q}_{p,\xi}(\lambda)\mathbf{u}_\lambda(i)\mathbf{u}_\lambda(j) = \sum_{\lambda\in\Lambda(\xi)} \mathrm{Q}_{p,\xi}(\lambda)\boldsymbol{\delta}(\lambda)\alpha_\lambda(\mathfrak{f})\alpha_\lambda(\mathfrak{g})$$

where $\boldsymbol{\delta} \in \{\pm1\}^{\Lambda(\xi)}$ and $\mathbf{u}_\lambda(i)^2 = \alpha_\lambda(\mathfrak{f})^2$, $\mathbf{u}_\lambda(j)^2 = \alpha_\lambda(\mathfrak{g})^2$ by Lemma 3.16 and Lemma 3.17. Clearly, $\mathrm{Q}_{p,\xi}(S)_{i,j}$ is an off-diagonal entry of $\mathrm{Q}_{p,\xi}(S)$, which is an integer matrix. Furthermore, the parities of the off-diagonal entries of $\mathrm{Q}_{p,\xi}(S)$ can be determined by using Lemma 2.1. It follows that $\mathrm{Q}_{p,\xi}(S)_{i,j}$ also satisfies (3.6) and hence, $\mathfrak{g}(x)$ and $\mathfrak{f}(x)$ are Seidel-compatible with respect to $\mathrm{Char}_S(x)$. $\square$

Example 3.20. Let $p(x) = (x+5)^{25}(x-3)(x-7)^6(x-8)(x-9)^8$. The coefficient matrix

$$A = C\left(\left\{\frac{\mathrm{Min}_p(x)\mathfrak{f}(x)}{p(x)} \ : \ \mathfrak{f}(x) \in \mathrm{Deck}(p)\right\}\right)$$

is equal to

$$A = \begin{bmatrix} 1 & -22 & 168 & -522 & 567 \\ 1 & -22 & 168 & -514 & 511 \\ 1 & -22 & 168 & -514 & 527 \\ 1 & -22 & 168 & -506 & 455 \\ 1 & -22 & 168 & -498 & 399 \end{bmatrix}.$$

The polynomials $\mathfrak{f}_1(x)$ and $\mathfrak{f}_2(x)$ given by

$$\mathfrak{f}_1(x) = (x+5)^{24}(x-7)^5(x-9)^7(x^2-10x+17)(x^2-12x+31),$$
$$\mathfrak{f}_2(x) = (x+5)^{24}(x-7)^6(x-9)^7(x^3-15x^2+63x-57)$$

are each $p(x)$-warranted, with certificates of warranty given by $(-133896, 0, 0, -304, -43)$ and $(-6402648, 0, 0, -14059, -1562)$, respectively.

Next, to check compatibility, we compute the angles $\alpha_\lambda(\mathfrak{f})$ for $\lambda \in \Sigma(p) = \{3, 8\}$ and $\mathfrak{f} = \mathfrak{f}_1, \mathfrak{f}_2$. Here we have

$$\mathrm{Min}_p(x) = (x+5)(x-3)(x-7)(x-8)(x-9) \quad \text{and} \quad \mathrm{Min}_p'(x) = 5x^4 - 88x^3 + 384x^2 + 476x - 3873.$$

$$f_1(x) = (x^2-10x+17)(x^2-12x+31) \quad \text{and} \quad f_2(x) = (x-7)(x^3-15x^2+63x-57).$$

Hence,

$$\begin{bmatrix} \alpha_3(\mathfrak{f}_1)^2 & \alpha_8(\mathfrak{f}_1)^2 \\ \alpha_3(\mathfrak{f}_2)^2 & \alpha_8(\mathfrak{f}_2)^2 \end{bmatrix} = \begin{bmatrix} 1/60 & 1/65 \\ 1/10 & 1/65 \end{bmatrix}.$$

Then

$$\sum_{\lambda \in \Lambda(\xi)} \mathrm{Q}_{p,\xi}(\lambda)\boldsymbol{\delta}(\lambda)\alpha_\lambda(\mathfrak{f}_1)\alpha_\lambda(\mathfrak{f}_2) = \pm\frac{192}{\sqrt{600}} \pm \frac{1}{5},$$

which cannot be an integer.

By Lemma 3.19, there does not exist a Seidel matrix with characteristic polynomial $p(x)$. Note that such a Seidel matrix would correspond to a system of 41 equiangular lines in $\mathbb{R}^{16}$.

3.3.3 Computational shortcuts

In the section, we develop some results that help us check compatibility for polynomials. Empirically, we find that checking compatibility for polynomials can be quite computationally expensive, since it requires arithmetic in potentially high-degree number fields (as high as degree 120 over $\mathbb{Q}$ for some polynomials corresponding to 49 equiangular lines in $\mathbb{R}^{17}$). Next we establish some tools that will enable us to more efficiently check compatibility (without having to compute the angles for the interlacing characteristic polynomials).

Let $p(x)$ be a monic, totally-real, integer polynomial and let $\xi(x) \in \mathbb{Z}[x]$ be an irreducible factor of $\mathrm{Sim}_p(x)$. Suppose $\mathfrak{f}(x) = \mathrm{Quo}_p(x)f(x)$ and $\mathfrak{g}(x) = \mathrm{Quo}_p(x)g(x)$ are monic integer polynomials that interlace $p(x)$. Note that, for each $\lambda \in \Lambda(\xi)$, we can write

$$\mathrm{Q}_{p,\xi}(\lambda)\boldsymbol{\delta}(\lambda)\alpha_\lambda(\mathfrak{f})\alpha_\lambda(\mathfrak{g}) = \boldsymbol{\delta}(\lambda)\,\mathrm{Q}_{p,\xi}(\lambda)\frac{\sqrt{fg(\lambda)}}{|\,\mathrm{Min}_p'(\lambda)|} = \frac{\boldsymbol{\delta}^*(\lambda)\sqrt{fg(\lambda)}}{\xi'(\lambda)} \tag{3.7}$$

where $\boldsymbol{\delta}^*(\lambda)/\,\mathrm{Min}_p'(\lambda) = \boldsymbol{\delta}(\lambda)/|\,\mathrm{Min}_p'(\lambda)|$. Since $f(\lambda)/\,\mathrm{Min}_p'(\lambda) \geqslant 0$ and $g(\lambda)/\,\mathrm{Min}_p'(\lambda) \geqslant 0$, we have $fg(\lambda) = f(\lambda)g(\lambda) \geqslant 0$. Thus, the square root of $fg(\lambda)$ is a real number.

Lemma 3.21. *Let $p(x)$ be a monic, totally-real, integer polynomial, and let $\xi(x) \in \mathbb{Z}[x]$ be an irreducible factor of $\mathrm{Sim}_p(x)$. Let $\mathfrak{f}(x) = \mathrm{Quo}_p(x)f(x)$ and $\mathfrak{g}(x) = \mathrm{Quo}_p(x)g(x)$ be distinct monic integer polynomials that interlace $p(x)$. Suppose that there exists a polynomial $h(x) \in \mathbb{Q}[x]$ such that $h^2(\lambda) = fg(\lambda)$ for all $\lambda \in \Lambda(\xi)$. Then there exists $\boldsymbol{\delta} \in \{\pm 1\}^{\Lambda(\xi)}$ such that*

$$\sum_{\lambda \in \Lambda(\xi)} \mathrm{Q}_{p,\xi}(\lambda)\boldsymbol{\delta}(\lambda)\alpha_\lambda(\mathfrak{f})\alpha_\lambda(\mathfrak{g}) \in \mathbb{Q}.$$

Proof. Let $\pi(x) \in \mathbb{Q}[x]$ be the unique polynomial of degree at most $|\Lambda(\xi)| - 1$ such that $\pi(x) \equiv h(x) \mod \xi(x)$. For each $\lambda \in \Lambda(\xi)$, we have that $\pi(\lambda) = h(\lambda) = \boldsymbol{\delta}^*(\lambda)\sqrt{fg(\lambda)}$ where $\boldsymbol{\delta}^* \in \{\pm 1\}^{\Lambda(\xi)}$. Consider the $|\Lambda(\xi)|$ distinct interpolation points $(\lambda, \pi(\lambda))$ for $\lambda \in \Lambda(\xi)$. Then we can write $\pi(x)$ as the unique interpolation polynomial in Lagrange form

$$\pi(x) = \sum_{\lambda \in \Lambda(\xi)} \pi(\lambda)L_\lambda(x),$$

where, for each $\lambda \in \Lambda(\xi)$, the polynomial $L_\lambda(x)$ is the Lagrange polynomial

$$L_\lambda(x) = \prod_{\mu \in \Lambda(\xi)\backslash\{\lambda\}} \frac{x-\mu}{\lambda-\mu}.$$

Let ω be the coefficient of $x^{|\Lambda(\xi)|-1}$ in $\pi(x)$. Using (3.7), observe that

$$\omega = \sum_{\lambda \in \Lambda(\xi)} \frac{\pi(\lambda)}{\xi'(\lambda)} = \sum_{\lambda \in \Lambda(\xi)} \frac{\boldsymbol{\delta}^*(\lambda)\sqrt{fg(\lambda)}}{\xi'(\lambda)} = \sum_{\lambda \in \Lambda(\xi)} \mathrm{Q}_{p,\xi}(\lambda)\boldsymbol{\delta}(\lambda)\alpha_\lambda(\mathfrak{f})\alpha_\lambda(\mathfrak{g})$$

where $\boldsymbol{\delta}^*(\lambda)/\,\mathrm{Min}_p'(\lambda) = \boldsymbol{\delta}(\lambda)/|\,\mathrm{Min}_p'(\lambda)|$ for all $\lambda \in \Lambda(\xi)$. The statement of the lemma follows since $\omega \in \mathbb{Q}$. □

Let K be the splitting field of $\xi(x)$ and let $\mathrm{Gal}(K/\mathbb{Q})$ be the Galois group of K over $\mathbb{Q}$. The Galois group $\mathrm{Gal}(K/\mathbb{Q})$ acts transitively on $\Lambda(\xi)$, the set of zeros of $\xi(x)$. Thus for all $\lambda, \mu \in \Lambda(\xi)$, there exists $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ such that $\sigma(\lambda) = \mu$. We will use this fact in the proof of Proposition 3.22 and Lemma 3.24.

Proposition 3.22. *Let $p(x)$ be a monic, totally-real, integer polynomial, and let $\xi(x) \in \mathbb{Z}[x]$ be an irreducible factor of $\mathrm{Sim}_p(x)$. Let $\mathfrak{f}(x) = \mathrm{Quo}_p(x)f(x)$ and $\mathfrak{g}(x) = \mathrm{Quo}_p(x)g(x)$ be distinct monic integer polynomials that interlace $p(x)$. Let $\rho(x)$ be the minimal polynomial of $fg(\lambda)$ over $\mathbb{Q}$ for some $\lambda \in \Lambda(\xi)$ and suppose that $\rho(x^2)$ is reducible over $\mathbb{Q}$. Then there exists $\boldsymbol{\delta} \in \{\pm 1\}^{\Lambda(\xi)}$ such that*

$$\sum_{\lambda \in \Lambda(\xi)} \mathrm{Q}_{p,\xi}(\lambda)\boldsymbol{\delta}(\lambda)\alpha_\lambda(\mathfrak{f})\alpha_\lambda(\mathfrak{g}) \in \mathbb{Q}.$$

Proof. Let e be the degree of $\mathbb{Q}(\sqrt{fg(\lambda)})$ over $\mathbb{Q}$. Note that the minimal polynomial of $\sqrt{fg(\lambda)}$ divides $\rho(x^2)$, which has degree $2\deg\rho$. This implies that e divides $2\deg\rho$ and moreover, $e < 2\deg\rho$ since $\rho(x^2)$ is reducible over $\mathbb{Q}$. On the other hand, $e \geqslant \deg\rho$ since $\mathbb{Q}(fg(\lambda)) \subseteq \mathbb{Q}(\sqrt{fg(\lambda)})$. It follows that $e = \deg\rho$ and thus $\mathbb{Q}(\sqrt{fg(\lambda)}) = \mathbb{Q}(fg(\lambda)) \subseteq \mathbb{Q}(\lambda)$. Hence, there exists a polynomial $h(x) \in \mathbb{Q}[x]$ such that $\sqrt{fg(\lambda)} = h(\lambda)$, which implies that $fg(\lambda) = h^2(\lambda)$. Let $\mu \in \Lambda(\xi)$ and let K be the splitting field of $\xi(x)$. There exists $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ such that $\sigma(\lambda) = \mu$. Hence, we obtain

$$\sigma\left(fg\left(\lambda\right)\right) = \sigma\left(h^2(\lambda)\right) \implies fg\left(\mu\right) = h^2(\mu).$$

Therefore, we have $h^2(\lambda) = fg(\lambda)$ for all $\lambda \in \Lambda(\xi)$ and, by Lemma 3.21, the conclusion follows. □

We can also use the above to check Seidel-compatibility. We can first construct the polynomial $h(x)$ from the proof of Proposition 3.22 and the polynomial $\pi(x)$ in the proof of Lemma 3.21. Using (3.7), we see that ω from the proof of Lemma 3.21 is the same as the left hand side of (3.6). Thus, to check Seidel-compatibility, we can check if ω has the same parity as $R(p,\xi)$.

Example 3.23. Set $p(x) = (x+5)^{32}(x-9)^{13}(x-11)^2(x^2-21x+92)$. Then we have

$$\begin{aligned}
\mathrm{Min}_p(x) &= (x+5)(x-9)(x-11)(x^2-21x+92),\\
\mathrm{Quo}_p(x) &= (x+5)^{31}(x-9)^{12}(x-11),\\
\mathrm{Sim}_p(x) &= x^2-21x+92.
\end{aligned}$$

Let $f(x) = x^4 - 36x^3 + 454x^2 - 2356x + 4241$ and $g(x) = x^4 - 36x^3 + 454x^2 - 2348x + 4169$. Then $\mathfrak{f}(x) = \mathrm{Quo}_p(x)f(x)$ and $\mathfrak{g}(x) = \mathrm{Quo}_p(x)g(x)$ both interlace $p(x)$. Since $\mathrm{Sim}_p(x)$ is irreducible, we take $\xi(x) = \mathrm{Sim}_p(x)$. Let $\lambda = (21-\sqrt{73})/2$ be a zero of $\xi(x)$. Then the minimal polynomial of $fg(\lambda)$ is $\rho(x) = x^2 - 10521x + 5308416$ and $\rho(x^2) = (x^2-123x+2304)(x^2+123x+2304)$. By Proposition 3.22, $\mathfrak{f}(x)$ and $\mathfrak{g}(x)$ are compatible. For $h(x) = 9x-33$ we have $fg(\lambda) = h^2(\lambda)$. It follows that $\pi(x) = h(x) = 9x-33$ and $\omega = 9$. Finally, since $\mathrm{Q}_{p,\xi}(x) = x^3 - 15x^2 - x + 495$, we have $R(p,\xi) \coloneqq \mathrm{Q}_{p,\xi}(1) + \mathrm{Q}_{p,\xi}(0) = 975$. Thus, $\mathfrak{f}(x)$ and $\mathfrak{g}(x)$ are Seidel-compatible with respect to $p(x)$.

Next we develop tools to show that two polynomials are not Seidel-compatible.

Lemma 3.24. *Let $p(x)$ be a monic, totally-real, integer polynomial, and let $\xi(x) \in \mathbb{Z}[x]$ be an irreducible factor of $\mathrm{Sim}_p(x)$ having splitting field K. Let $\mathfrak{f}(x) = \mathrm{Quo}_p(x)f(x)$ and $\mathfrak{g}(x) = \mathrm{Quo}_p(x)g(x)$ be distinct monic integer polynomials that interlace $p(x)$. Let $\rho(x)$ be the minimal polynomial of $fg(\lambda)$ over $\mathbb{Q}$ for some $\lambda \in \Lambda(\xi)$. Suppose that $\deg\rho = |\Lambda(\xi)|$ and $\rho(x^2)$ is irreducible over $\mathbb{Q}$. Further, suppose $\sqrt{fg(\lambda)} \notin K$. Then, for each $\boldsymbol{\delta} \in \{\pm 1\}^{\Lambda(\xi)}$,*

$$\sum_{\lambda \in \Lambda(\xi)} \mathrm{Q}_{p,\xi}(\lambda)\boldsymbol{\delta}(\lambda)\alpha_\lambda(\mathfrak{f})\alpha_\lambda(\mathfrak{g}) \in \mathbb{Q} \implies \sum_{\lambda \in \Lambda(\xi)} \mathrm{Q}_{p,\xi}(\lambda)\boldsymbol{\delta}(\lambda)\alpha_\lambda(\mathfrak{f})\alpha_\lambda(\mathfrak{g}) = 0.$$

Proof. Let $\boldsymbol{\delta} \in \{\pm 1\}^{\Lambda(\xi)}$ such that

$$\sum_{\lambda \in \Lambda(\xi)} \mathrm{Q}_{p,\xi}(\lambda)\boldsymbol{\delta}(\lambda)\alpha_\lambda(\mathfrak{f})\alpha_\lambda(\mathfrak{g}) \in \mathbb{Q}.$$

For any $\lambda, \mu \in \Lambda(\xi)$ there exists $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ such that $\sigma(\lambda) = \mu$. Since $\rho\left(fg(\lambda)\right) = 0$, we obtain $\rho\left(fg(\mu)\right) = 0$. Since $\rho(x)$ is irreducible over $\mathbb{Q}$, we conclude that ρ is the minimal polynomial of $fg(\lambda)$ over $\mathbb{Q}$ for all $\lambda \in \Lambda(\xi)$.

Next, for any $\lambda \in \Lambda(\xi)$ we have $\mathbb{Q}(fg(\lambda)) \subseteq \mathbb{Q}(\lambda)$. Since the degree of $\rho(x)$ is equal to $|\Lambda(\xi)|$ and $\rho(x)$ is the minimal polynomial of $fg(\lambda)$, we have $\mathbb{Q}(fg(\lambda)) = \mathbb{Q}(\lambda)$. Now suppose, for a contradiction, that there exist distinct $\mu, \nu \in \Lambda(\xi)$ such that $fg(\mu) = fg(\nu)$. We can write $\mu = \pi(fg(\mu))$ for some $\pi(x) \in \mathbb{Q}[x]$, since $\mathbb{Q}(fg(\mu)) = \mathbb{Q}(\mu)$. Take $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ such that $\sigma(\mu) = \nu$, which yields us $\nu = \pi(fg(\nu))$. However, this leads us to $\mu = \pi(fg(\mu)) = \pi(fg(\nu)) = \nu$, which is a contradiction. Hence

$$\rho(x) = \prod_{\lambda \in \Lambda(\xi)} \left(x - fg(\lambda)\right).$$

Moreover, K is the splitting field of $\rho(x)$.

Let L be the splitting field of $\rho(x^2)$ over $\mathbb{Q}$, which contains K. We have a tower of fields $\mathbb{Q} \subseteq K \subseteq L$ where L is Galois over $\mathbb{Q}$ so L is Galois over K. Hence, if $\sigma \in \mathrm{Gal}(L/K)$ then $\sigma\left(\sqrt{fg(\lambda)}\right)$ is equal to either $\sqrt{fg(\lambda)}$ or $-\sqrt{fg(\lambda)}$ for each $\lambda \in \Lambda(\xi)$. Since $K/\mathbb{Q}$ is a normal extension, if $\sqrt{fg(\lambda)}$ does not belong to K for some $\lambda \in \Lambda(\xi)$ then $\sqrt{fg(\lambda)}$ does not belong to K for all $\lambda \in \Lambda(\xi)$.

For a fixed $\lambda \in \Lambda(\xi)$, there exists $\sigma \in \mathrm{Gal}(L/K)$ such that $\sigma\left(\sqrt{fg(\lambda)}\right) = -\sqrt{fg(\lambda)}$. Otherwise, if $\sqrt{fg(\lambda)}$ is fixed by all elements of $\mathrm{Gal}(L/K)$ then $\sqrt{fg(\lambda)} \in K$, which is a contradiction. Using (3.7), we write

$$\omega = \sum_{\lambda \in \Lambda(\xi)} \omega_\lambda \text{ where } \omega_\lambda = \frac{\boldsymbol{\delta}^*(\lambda)\sqrt{fg(\lambda)}}{\xi'(\lambda)}$$

and $\boldsymbol{\delta}^*(\lambda)/\operatorname{Min}_p'(\lambda) = \boldsymbol{\delta}(\lambda)/|\operatorname{Min}_p'(\lambda)|$ for all $\lambda \in \Lambda(\xi)$. Take any $\sigma \in \mathrm{Gal}(L/K)$ such that $\sigma(\omega_\mu) = -\omega_\mu$ for some $\mu \in \Lambda(\xi)$. Then we can partition the set $\Lambda(\xi)$ into two sets I_+ and I_- such that $\sigma(\omega_\lambda) = \omega_\lambda$ for all $\lambda \in I_+$ and $\sigma(\omega_\mu) = -\omega_\mu$ for all $\mu \in I_-$. This implies that $\sum_{\mu \in I_-} \omega_\mu = 0$ and $\sum_{\lambda \in I_+} \omega_\lambda = \omega$ since $\sigma(\omega) = \omega$. We then apply the same procedure on I_+, where we take any $\sigma \in \mathrm{Gal}(L/K)$ such that $\sigma(\omega_\mu) = -\omega_\mu$ for some $\mu \in I_+$. Since $|\Lambda(\xi)|$ is finite, after finitely many steps on partitioning I_+, we arrive at the last subset $\mathcal{I}$ where the only possible partition is $\mathcal{I}_+ = \emptyset$ and $\mathcal{I}_- = \mathcal{I}$. Therefore, we conclude that $\omega = \sum_{\lambda \in \Lambda(\xi)} \omega_\lambda = 0$. □

Now we have the following corollary of Lemma 3.24.

Corollary 3.25. *Let $p(x)$ be a monic, totally-real, integer polynomial, and let $\xi(x)$ be an irreducible factor of $\operatorname{Sim}_p(x)$. Let $\mathfrak{f}(x) = \operatorname{Quo}_p(x)f(x)$ and $\mathfrak{g}(x) = \operatorname{Quo}_p(x)g(x)$ be distinct monic integer polynomials that interlace $p(x)$. Let $\rho(x)$ be the minimal polynomial of $fg(\lambda)$ over $\mathbb{Q}$ for some $\lambda \in \Lambda(\xi)$. Suppose that $\deg \rho = |\Lambda(\xi)|$ and $\rho(x^2)$ is irreducible over $\mathbb{Q}$. Let G and H be the Galois groups of $\rho(x)$ and $\rho(x^2)$ over $\mathbb{Q}$, respectively, and suppose that $|G| < |H|$. If $R(p, \xi)$ is odd then $\mathfrak{f}(x)$ and $\mathfrak{g}(x)$ are not Seidel-compatible.*

Example 3.26. Set $p(x) = (x+5)^{32}(x-9)^{14}(x-11)(x^2-23x+116)$. Then we have

$$\begin{aligned}
\operatorname{Min}_p(x) &= (x+5)(x-9)(x-11)(x^2-23x+116),\\
\operatorname{Quo}_p(x) &= (x+5)^{31}(x-9)^{13},\\
\operatorname{Sim}_p(x) &= (x-11)(x^2-23x+116).
\end{aligned}$$

Let $f(x) = x^4 - 38x^3 + 508x^2 - 2810x + 5363$ and $g(x) = x^4 - 38x^3 + 508x^2 - 2802x + 5291$. Then $\mathfrak{f}(x) = \mathrm{Quo}_p(x)f(x)$ and $\mathfrak{g}(x) = \mathrm{Quo}_p(x)g(x)$ both interlace $p(x)$. Let $\xi(x) = x^2 - 23x + 116$ be an irreducible factor of $\mathrm{Sim}_p(x)$ and $\lambda = (23 - \sqrt{65})/2$ be a zero of $\xi(x)$. Then the minimal polynomial of $fg(\lambda)$ is $\rho(x) = x^2 - 11105x + 1433600$ and $\rho(x^2)$ is irreducible over $\mathbb{Q}$. Furthermore, the Galois groups of $\rho(x)$ and $\rho(x^2)$ are $G = S_2$ and $H = D_4$, respectively. Hence $2 = |G| < |H| = 8$. Finally, since $\mathrm{Q}_{p,\xi}(x) = x^3 - 15x^2 - x + 495$, we have $R(p, \xi) := \mathrm{Q}_{p,\xi}(1) + \mathrm{Q}_{p,\xi}(0) = 975$. Therefore, by Corollary 3.25, the polynomials $\mathfrak{f}(x)$ and $\mathfrak{g}(x)$ are not Seidel-compatible.

By Lemma 3.19, there does not exist a Seidel matrix with characteristic polynomial $p(x)$. Note that such a Seidel matrix would correspond to a system of 49 equiangular lines in $\mathbb{R}^{17}$.

In Corollary 3.25, the condition that the cardinality of the Galois group of $\rho(x)$ over $\mathbb{Q}$ is strictly smaller than that of $\rho(x^2)$ implies, in the notation of Lemma 3.24, that $\sqrt{fg(\lambda)}$ is not in K. Indeed, let $K \subseteq L$ be splitting fields of $\rho(x)$ and $\rho(x^2)$ over $\mathbb{Q}$, respectively. If $K = L$ then $G \cong H$. Therefore, if $|G| < |H|$ then we conclude that K is a proper subfield of L and it follows that $\sqrt{fg(\lambda)}$ is not in K. Otherwise, if $\sqrt{fg(\lambda)} \in K$ then $\sqrt{fg(\lambda)} \in K$ for all $\lambda \in \Lambda(\xi)$, which will imply that $K = L$.

Note that we cannot take the condition $|G| < |H|$ for granted, since for example, we have the polynomials $\rho_1(x) = x^3 - 11x^2 + 27x - 13$, $\rho_2(x) = x^4 - 14x^3 + 34x^2 - 14x + 1$, and $\rho_3(x) = x^4 - 14x^3 + 45x^4 - 29x + 4$. These polynomials have the property that $\rho_i(x^2)$ is irreducible and the Galois group of $\rho_i(x)$ is isomorphic to that of $\rho_i(x^2)$ for each $i \in \{1, 2, 3\}$. Furthermore, the polynomials $\rho_1(x^2)$, $\rho_2(x^2)$, and $\rho_3(x^2)$ have the Galois groups S_3, D_4, and S_4, respectively.

3.4 Exercises

1. Use Jacobi's formula to prove Theorem 3.2.
2. Prove Theorem 3.8.
3. Verify the enumeration of $\mathrm{Deck}(p)$ in Example 3.10.
4. Verify that $\mathfrak{f}_1(x)$ and $\mathfrak{f}_2(x)$ are warranted in Example 3.20.

Bibliography

[1] E. Bannai, A. Munemasa, and B. Venkov, The nonexistence of certain tight spherical designs. Algebra i Analiz, 16(4):1–23, 2004.

[2] A.E. Brouwer and W. H. Haemers, Spectra of Graphs, Springer, New York, (2012).

[3] D. De Caen, Large equiangular sets of lines in Euclidean space, Electron. J. Comb., 7, #55, 3 pp. (electronic) (2000).

[4] M.-Y. Cao, J. H. Koolen, Y.-C. Lin, and W.-H. Yu, *The Lemmens-Seidel conjecture and forbidden subgraphs*, J. Combin. Theory Ser. A **185**, 105538 (2022).

[5] A.-L. Cauchy, *Sur l'équation à l'aide de laquelle on détermine les inégalités séculaires des mouvements des planètes.* In Œuvres complètes, IIième Série, Gauthier-Villars, Paris, 1829.

[6] D. Cvetković, P. Rowlinson, and S. Simić, *Eigenspaces of graphs*, Encyclopedia of Mathematics and its Applications, vol. 66, Cambridge University Press, Cambridge, 1997.

[7] P.B. Denton, S.J. Parke, T. Tao, and X. Zhang, Eigenvectors from Eigenvalues: a survey of a basic identity in linear algebra, Bull. Am. Math. Soc. 59, 31–58 (2021).

[8] J. Farkas, *Theorie der einfachen Ungleichungen*, Journal für die reine und angewandte Mathematik **124**, 1–27 (1902).

[9] R.A. Frazer, W.J. Duncan, and A.R. Collar, *Elementary Matrices and Some Applications to Dynamics and Differential Equations*, Cambridge University Press, Cambridge, England, (1955).

[10] S. Fisk, *A very short proof of Cauchy's interlace theorem for eigenvalues of Hermitian matrices.* Amer. Math. Monthly **112**(2), 118–118 (2005).

[11] C.D. Godsil and B.D. McKay, *Spectral conditions for the reconstructibility of a graph*, J. Combin. Theory Ser. B, **30**(3), 285–289 (1981).

[12] C.D. Godsil and G. Royle, Algebraic Graph Theory. Graduate Texts in Mathematics. New York: Springer, 2000.

[13] G.R.W. Greaves, *Equiangular line systems and switching classes containing regular graphs*, Linear Algebra Appl. **536**, 31–51 (2018).

[14] G. Greaves, J. Koolen, A. Munemasa and F. Szöllősi, *Equiangular lines in Euclidean spaces*, J. Combin. Theory Ser. A **138**, 208–235 (2016).

[15] G.R.W. Greaves, J. Syatriadi, and P. Yatsyna, *Equiangular lines in low dimensional Euclidean spaces*, Combinatorica **41**(6), 839–872 (2021).

[16] G.R.W. Greaves, J. Syatriadi, and P. Yatsyna, *Equiangular lines in Euclidean spaces: dimensions 17 and 18*, arXiv:2104.04330.

[17] G.R.W. Greaves and J. Syatriadi, *Real equiangular lines in dimension 18 and the De Caen-Jacobi identity for complementary subgraphs*, arXiv:2206.04267.

[18] G.R.W. Greaves and P. Yatsyna, *On equiangular lines in 17 dimensions and the characteristic polynomial of a Seidel matrix.* Math. Comp. **88**(320), 3041–3061 (2019).

[19] J. Haantjes, Equilateral point-sets in elliptic two-and three-dimensional spaces. Nieuw Arch. Wiskunde (2), 22:355–362, (1948).

[20] J. Hage, T. Harju, and E. Welzl, *Euler graphs, triangle-free graphs and bipartite graphs in switching classes*, Fund. Inform. **58**, 23–37 (2003).

[21] F. Harary and A.J. Schwenk, *The spectral approach to determining the number of walks in a graph*, Pacific J. Math. **80**(2), 443–449 (1979).

[22] S.-G. Hwang, *Cauchy's interlace theorem for eigenvalues of Hermitian matrices.* Amer. Math. Monthly **111**(2), 157–159 (2004).

[23] Z. Jiang, J. Tidor, Y. Yao, S. Zhang, and Y. Zhao, Equiangular lines with a fixed angle. Annals of Mathematics, 194(3):729–743, 2021.

[24] W.-J. Kao and W.-H. Yu, *Four-point semidefinite bound for equiangular lines*, arXiv:2203.05828 (2022).

[25] Koornwinder, T. H., A note on the absolute bound for systems of lines. Indagationes Mathematicae (Proceedings), 79, 152–153 (1976).

[26] P.W.H. Lemmens and J.J. Seidel, *Equiangular lines*, J. Algebra **24**(3), 494–512 (1973).

[27] Y.-C. Lin and W.-H. Yu, *Saturated configuration and new large construction of equiangular lines*, Linear Algebra Appl. **588**, 272–281 (2020).

[28] J.H. van Lint and J.J. Seidel, *Equilateral point sets in elliptic geometry*, Indag. Math. **28**(3), 335–348 (1966).

[29] J.F. McKee and C.J. Smyth, *Salem numbers of trace -2 and traces of totally positive algebraic integers.* Algorithmic number theory, 327–337, Lecture Notes in Comput. Sci., 3076, Springer, Berlin, 2004.

[30] G. Nebe and B. Venkov, On tight spherical designs. Algebra i Analiz, 24(3):163–171, 2012.

[31] J.J. Seidel, *Graphs and two-graphs*, in: Proc. 5th Southeastern Conf. on Combinatorics, Graph Theory, and Computing, Utilitas Mathematica Publishing Inc., Winnipeg, Canada, 1974.

[32] C.J. Smyth, *Totally positive algebraic integers of small trace*, Ann. Inst. Fourier, Grenoble, **33**(3), 1–28 (1984).

[33] E. Spence, The strongly regular (40,12,2,4) graphs, Electron. J. Comb., 7, #22, 4 pp. (2000).

[34] F. Szöllősi, *A remark on a construction of D. S. Asche*, Discrete Comput. Geom. **61**(1), 120–122 (2019).

[35] D.E. Taylor, *Regular 2-graphs*, Proc. London Math. Soc. **35**, 257–274 (1977).

[36] R.C. Thompson, *Principal submatrices V: Some results concerning principal submatrices of arbitrary matrices*, J. Res. Natl. Bur. Standards Sect. B **72**(2), 115–125 (1968).

[37] E. Witt, *Die 5-fach transitiven Gruppen von Mathieu*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg, vol. 12, Springer, 256–264 (1937).