Equiangular lines in $\mathbb{R}^{17}$ and the characteristic polynomial of a Seidel matrix

Gary Greaves

Division of Mathematical Sciences,
Nanyang Technological University

12th July 2019

Equiangular line systems

- Let $\mathcal{L}$ be a system of n lines spanned by $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^d$ with $\langle \mathbf{v}_i, \mathbf{v}_i \rangle = 1$.

- $\mathcal{L}$ is **equiangular** if $|\langle \mathbf{v}_i, \mathbf{v}_j \rangle| = \alpha$; ("*common angle* α").

- **Problem:** given d, what is the largest possible size $N(d)$ of an Equiangular Line System (ELS) in $\mathbb{R}^d$?

Estimates for $N(d)$

- Real ETFs give $N(d) = \Omega(d\sqrt{d})$.

- MUB construction gives $N(d) = \Theta(d^2)$.

Upper bounds

- Gerzon (1976): $N(d) \leqslant d(d+1)/2$;

- $N_\alpha(d) :=$ largest cardinality ELS with common angle α.

- For $\alpha^2 \leqslant 1/(d+2)$: $\qquad N_\alpha(d) \leqslant d(1-\alpha^2)/(1-\alpha^2 d)$.

- Barg and Yu (2014), Okuda and Yu (2016), King and Tang (2016), Glazyrin and Yu (2018), De Laat et al. (2018): SDP upper bounds;

- Bukh (2016), Jiang and Polyanski (2017), Balla, Dräxler, Keevash, Sudakov (2018): for fixed α, $N_\alpha(d) = O(d)$.

Bounds for small dimensions

- GG, Koolen, Munemasa, Szöllősi (2016): $N(14) \leqslant 29$ and $N(16) \leqslant 41$;
- GG and Yatsyna (2019): $N(17) \leqslant 49$;
- Szöllősi (2017): $N(18) \geqslant 54$;
- GG (2018): $N(18) \leqslant 60$;
- Lin and Yu (2019+): $N(18) \geqslant 56$;
- Azarija and Marc (2018): $N(19) \leqslant 75$ and $N(20) \leqslant 95$;
- GG, Syatriadi, and Yatsyna (2020+): $N(19) \leqslant 74$ and $N(20) \leqslant 94$;

Below is a table with bounds for $N(d)$ for $d \leqslant 20$.

d	2	3	4	5	6	7 – 13	14	15	16	17	18	19	20
$N(d)$	3	6	6	10	16	28	28 29	36	40 41	48 49	56 60	72 74	90 94

Seidel matrices

Equiangular lines $l_1, \ldots, l_n$ common angle $\alpha > 0$

$\Updownarrow$

Unit spanning vectors $\mathbf{v}_i \ : \ l_i = \langle \mathbf{v}_i \rangle$ $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \pm\alpha$

$\Updownarrow$

Gram matrix $M = (\langle \mathbf{v}_i, \mathbf{v}_j \rangle)_{ij}$ $\begin{pmatrix} 1 & \pm\alpha & \pm\alpha \\ \pm\alpha & 1 & \pm\alpha \\ \pm\alpha & \pm\alpha & 1 \end{pmatrix}$

$\Updownarrow$

Seidel matrix $S = \dfrac{(M - I)}{\alpha}$ $\begin{pmatrix} 0 & \pm1 & \pm1 \\ \pm1 & 0 & \pm1 \\ \pm1 & \pm1 & 0 \end{pmatrix}$

Multiplicity of the smallest eigenvalue

Unit vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ in $\mathbb{R}^d$

n vectors

$$\Updownarrow$$

$$B = \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \\ | & | & & | \end{pmatrix}$$

rank $= d$

$$\Updownarrow$$

Gram matrix $M = B^\top B$

smallest eigenvalue $[0]^{n-d}$

$$\Updownarrow$$

Seidel matrix $S = \dfrac{(M - I)}{\alpha}$

smallest eigenvalue $\left[\dfrac{-1}{\alpha}\right]^{n-d}$

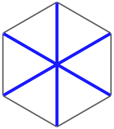
Theorem (Relative bound)

Let $\mathcal{L}$ be an equiangular line system of n lines in $\mathbb{R}^d$ whose Seidel matrix has smallest eigenvalue λ_0 and suppose $\lambda_0^2 \geqslant d+2$.

$$n \leqslant \frac{d(\lambda_0^2 - 1)}{\lambda_0^2 - d}.$$

Equality implies that S has 2 distinct eigenvalues.

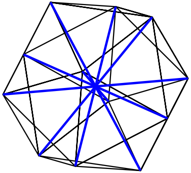
$\mathbb{R}^2$:



$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

Spectrum: $\{[-2]^1, [1]^2\}$;

$\mathbb{R}^3$:



$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 & -1 & -1 \\ 1 & -1 & 1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & 1 & 0 \end{pmatrix}$$

Spectrum: $\{[-\sqrt{5}]^3, [\sqrt{5}]^3\}$.

Relative bound in low dimensions

GG, Koolen, Munemasa, Szöllősi (2016): "Spectrum is determined for systems close to the relative bound"

d	λ_0	$\frac{d(\lambda_0^2-1)}{\lambda_0^2-d}$	$\left\lfloor \frac{d(\lambda_0^2-1)}{\lambda_0^2-d} \right\rfloor$	Spectrum
14	-5	≈ 30	30	$\{[-5]^{16}, [5]^9, [7]^5\}$
15	-5	36	36	$\{[-5]^{21}, [7]^{15}\}$
16	-5	≈ 42	42	$\{[-5]^{26}, [7]^7, [9]^9\}$
17	-5	51	51	$\{[-5]^{34}, [10]^{17}\}$
18	-5	≈ 61	61	$\{[-5]^{43}, [11]^9, [12]^1, [13]^8\}$
19	-5	76	76	$\{[-5]^{57}, [15]^{19}\}$
20	-5	96	96	$\{[-5]^{76}, [19]^{20}\}$

Relative bound in low dimensions

GG, Koolen, Munemasa, Szöllősi (2016): "Spectrum is determined for systems close to the relative bound"

d	λ_0	$\frac{d(\lambda_0^2-1)}{\lambda_0^2-d}$	$\left\lfloor \frac{d(\lambda_0^2-1)}{\lambda_0^2-d} \right\rfloor$	Spectrum
14	-5	≈ 30	30	$\{[-5]^{16}, [5]^9, [7]^5\}$
15	-5	36	36	$\{[-5]^{21}, [7]^{15}\}$
16	-5	≈ 42	42	$\{[-5]^{26}, [7]^7, [9]^9\}$
17	-5	51	51	$\{[-5]^{34}, [10]^{17}\}$
18	-5	≈ 61	61	$\{[-5]^{43}, [11]^9, [12]^1, [13]^8\}$
19	-5	76	76	$\{[-5]^{57}, [15]^{19}\}$
20	-5	96	96	$\{[-5]^{76}, [19]^{20}\}$

Relative bound in low dimensions

GG, Koolen, Munemasa, Szöllősi (2016): "Spectrum is determined for systems close to the relative bound"

d	λ_0	$\frac{d(\lambda_0^2-1)}{\lambda_0^2-d}$	$\left\lfloor \frac{d(\lambda_0^2-1)}{\lambda_0^2-d} \right\rfloor$	Spectrum
14	-5	≈ 30	30	$\{[-5]^{16}, [5]^9, [7]^5\}$
15	-5	36	36	$\{[-5]^{21}, [7]^{15}\}$
16	-5	≈ 42	42	$\{[-5]^{26}, [7]^7, [9]^9\}$
17	-5	51	51	$\{[-5]^{34}, [10]^{17}\}$
18	-5	≈ 61	61	$\{[-5]^{43}, [11]^9, [12]^1, [13]^8\}$
19	-5	76	76	$\{[-5]^{57}, [15]^{19}\}$
20	-5	96	96	$\{[-5]^{76}, [19]^{20}\}$

Note: even eigenvalues *cannot* have multiplicity greater than 1.

Relative bound in low dimensions

GG, Koolen, Munemasa, Szöllősi (2016): "Spectrum is determined for systems close to the relative bound"

d	λ_0	$\frac{d(\lambda_0^2-1)}{\lambda_0^2-d}$	$\left\lfloor \frac{d(\lambda_0^2-1)}{\lambda_0^2-d} \right\rfloor$	Spectrum
14	-5	≈ 30	30	$\{[-5]^{16}, [5]^9, [7]^5\}$
15	-5	36	36	$\{[-5]^{21}, [7]^{15}\}$
16	-5	≈ 42	42	$\{[-5]^{26}, [7]^7, [9]^9\}$
17	-5	51	51	$\{[-5]^{34}, [10]^{17}\}$
18	-5	≈ 61	61	$\{[-5]^{43}, [11]^9, [12]^1, [13]^8\}$
19	-5	76	76	$\{[-5]^{57}, [15]^{19}\}$
20	-5	96	96	$\{[-5]^{76}, [19]^{20}\}$

Note: even eigenvalues *cannot* have multiplicity greater than 1.

Equiangular lines in $\mathbb{R}^{14}$

- Suppose there are $n > 2 \cdot 14$ equiangular lines in $\mathbb{R}^{14}$.
- Lemmens and Seidel (1973): $\implies \lambda_0 = -5$.
- Relative bound: $n \leqslant 30.54 \cdots \notin \mathbb{N}$.
- Suppose we have 30 lines in $\mathbb{R}^{14}$, with corresponding Seidel matrix S having eigenvalues

$$-5 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{14}.$$

It follows that

$$14 = \sum_{i=1}^{14} (\lambda_i - 6)^2$$

Equiangular lines in $\mathbb{R}^{14}$

- Suppose there are $n > 2 \cdot 14$ equiangular lines in $\mathbb{R}^{14}$.
- Lemmens and Seidel (1973): $\implies \lambda_0 = -5$.
- Relative bound: $n \leqslant 30.54 \cdots \notin \mathbb{N}$.
- Suppose we have 30 lines in $\mathbb{R}^{14}$, with corresponding Seidel matrix S having eigenvalues
$$-5 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{14}.$$
It follows that
$$14 = \sum_{i=1}^{14} (\lambda_i - 6)^2 \geqslant 14 \sqrt[14]{\prod (\lambda_i - 6)^2} \geqslant 14$$

Equiangular lines in $\mathbb{R}^{14}$

- Suppose there are $n > 2 \cdot 14$ equiangular lines in $\mathbb{R}^{14}$.
- Lemmens and Seidel (1973): $\implies \lambda_0 = -5$.
- Relative bound: $n \leqslant 30.54 \cdots \notin \mathbb{N}$.
- Suppose we have 30 lines in $\mathbb{R}^{14}$, with corresponding Seidel matrix S having eigenvalues
$$-5 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{14}.$$
It follows that
$$14 = \sum_{i=1}^{14} (\lambda_i - 6)^2 \geqslant 14 \sqrt[14]{\prod (\lambda_i - 6)^2} \geqslant 14$$
$$\implies \lambda_i \in \{5, 7\}.$$

Equiangular lines in $\mathbb{R}^{17}$

- Suppose there are $n > 2 \cdot 17$ equiangular lines in $\mathbb{R}^{17}$.
- Lemmens and Seidel (1973): $\implies \lambda_0 = -5$.
- Relative bound: $n \leqslant 51$ (but equality is not possible).
- Suppose we have 50 lines in $\mathbb{R}^{17}$, with corresponding Seidel matrix S having eigenvalues
$$-5 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{17}.$$
It follows that
$$25 = \sum_{i=1}^{17} (\lambda_i - 10)^2.$$

Equiangular lines in $\mathbb{R}^{17}$

- Suppose there are $n > 2 \cdot 17$ equiangular lines in $\mathbb{R}^{17}$.
- Lemmens and Seidel (1973): $\implies \lambda_0 = -5$.
- Relative bound: $n \leqslant 51$ (but equality is not possible).
- Suppose we have 50 lines in $\mathbb{R}^{17}$, with corresponding Seidel matrix S having eigenvalues

$$-5 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{17}.$$

It follows that

$$25 = \sum_{i=1}^{17} (\lambda_i - 10)^2.$$

Note: $(\lambda_i - 10)^2$ are +ve algebraic integers with sum 25.

Characteristic polynomial modulo 2^k

Let S be a Seidel matrix of order n.

- GG and Yatsyna (2019):
  - for n even, there are $\leqslant 2^{\binom{k-2}{2}}$ congruence classes for $\chi_S(x)$ modulo $2^k\mathbb{Z}[x]$.
  - for n odd, there are $\leqslant 2 \cdot 2^{\binom{k-2}{2}}$ congruence classes for $\chi_S(x)$ modulo $2^k\mathbb{Z}[x]$.
- Conjecture: These upper bounds are sharp for large n.

Characteristic polynomial modulo 2^k

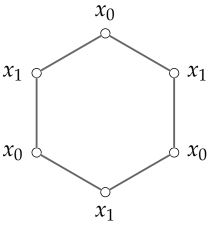
Let S be a Seidel matrix of order n.

- GG and Yatsyna (2019):
  - for n even, there are $\leqslant 2^{\binom{k-2}{2}}$ congruence classes for $\chi_S(x)$ modulo $2^k\mathbb{Z}[x]$.
  - for n odd, there are $\leqslant 2 \cdot 2^{\binom{k-2}{2}}$ congruence classes for $\chi_S(x)$ modulo $2^k\mathbb{Z}[x]$.
- Conjecture: These upper bounds are sharp for large n.

$$\det(S+I) = \left| \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \end{pmatrix} \right| = \left| \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & -2 & -2 & 0 & 0 \end{pmatrix} \right|$$

$$\implies 2^{n-1} \text{ divides } \det(S+I).$$

A key lemma



- Let Γ be a graph.
- $D_N = \langle r, s \mid r^N, s^2, (rs)^2 \rangle$ acts on the set of closed N-walks.
- $\mathrm{fix}_\Gamma(g)$ denotes the set of closed N-walks fixed by $g \in D_N$.

Burnside: $|D_N|$ divides $\displaystyle\sum_{g \in D_N} |\mathrm{fix}_\Gamma(g)|$.

Key Lemma (GG and Yatsyna 2019)

Let A be a graph-adjacency matrix. For $l \geqslant 2$, we have

$$\sum_{d \mid 2l} \varphi(2l/d)\,\mathrm{tr}(A^d) + l\mathbf{1}^\top A^l \mathbf{1} \equiv 0 \pmod{4l}.$$

The candidate characteristic polynomials

Theorem (GG and Yatsyna 2019)

Let S be a Seidel matrix corresponding to 50 equiangular lines in $\mathbb{R}^{17}$. Then

$$\chi_S(x) = (x+5)^{33}(x-9)^{10}(x-11)^5(x^2-20x+95),$$
$$\chi_S(x) = (x+5)^{33}(x-7)(x-9)^9(x-11)^7, \text{ or}$$
$$\chi_S(x) = (x+5)^{33}(x-9)^{12}(x-11)^4(x-13).$$

However, there does not exist a Seidel matrix having any of these characteristic polynomials.

I.e., $\nexists$ a system of 50 equiangular lines in $\mathbb{R}^{17}$.

Idea of the Seidel matrix nonexistence proof

- Let S be an $n \times n$ Seidel matrix with spectrum $\{[\lambda_1]^{e_1}, \dots, [\lambda_m]^{e_m}\}$.
- Let $S[r]$ denote the principal submatrix of S obtained by deleting the r-th row and column of S.

Proposition (Graph angle theory)

$$\chi_{S[j]}(x) = \chi_S(x) \sum_{i=1}^{m} \frac{\alpha_{ij}^2}{x - \lambda_i}, \qquad \forall j \in \{1, \dots, n\};$$

$$e_i = \sum_{j=1}^{n} \alpha_{ij}^2, \qquad \forall i \in \{1, \dots, m\}.$$

Idea of the Seidel matrix nonexistence proof

Lemma

There does not exist a Seidel matrix with characteristic polynomial

$$f(x) = (x+5)^{33}(x-9)^{12}(x-11)^4(x-13).$$

Proof.

- Suppose S is a Seidel matrix with $\chi_S(x) = f(x)$;
- $\chi_{S[j]}(x) = (x+5)^{32}(x-9)^{11}(x-11)^3(x^3-28x^2+243x-r)$;
- By interlacing, $r \in \{616, \dots, 624\}$;
- Using modular restriction, we find $r = 616$;
- Then $(\alpha_{1j}^2, \alpha_{2j}^2, \alpha_{3j}^2, \alpha_{4j}^2) = (83/126, 2/7, 0, 1/18)$;
- But $50 \cdot 83/126 = 32.93\cdots \neq 33$.

$\square$

Thanks for listening!

arXiv:1806.08323