Trigonometry

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March 22, 2022

Contents

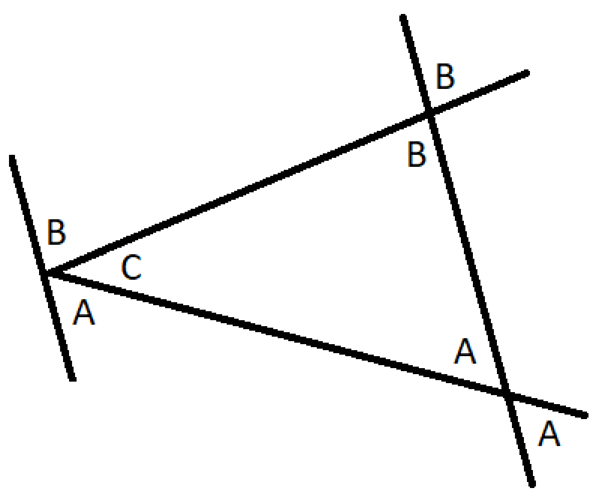
Chapter 1

Trigonometry

Trigonometry is the study of triangles.

1 Triangles

A *triangle* is defined as a planar figure with three *angles* and three *sides*.



The triangle shown has interior angles A, B, C, with three sides AB, BC, CA. A line parallel to AB has been copied and placed to intersect point C, with exterior angles indicated. From this, we deduce that the total angle BCA identifies a straight line. The angle BCA is designated 180°, i.e. 180 *degrees*. Since the angles and side lengths were not individually specified, it follows that *all* triangles have the property that the sum BCA is 180°. Note also that two identical arcs BCA complete a full circle, corresponding to 360°.

1.1 Angle Parameter

An angle A, B, C, etc. is generally represented by the symbol θ (Greek *theta*), a parameter that must be a *dimensionless quantity*. That is, θ must be a pure number such as 3 or -17.5°, but never some number of meters, seconds, or pounds.

1.2 Degrees and Radians

A more natural (dimensionless) unit for measuring angles is the *radian*, defined such that

$$1^\circ = \frac{\pi}{180}\, rad \qquad\qquad 1\, rad = \frac{180^\circ}{\pi}\ .$$

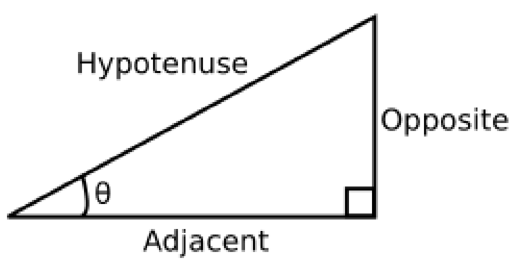
In other words, the 360-degree trip around the circle is equivalent to 2π radians. Note that any angle *outside* of the domain

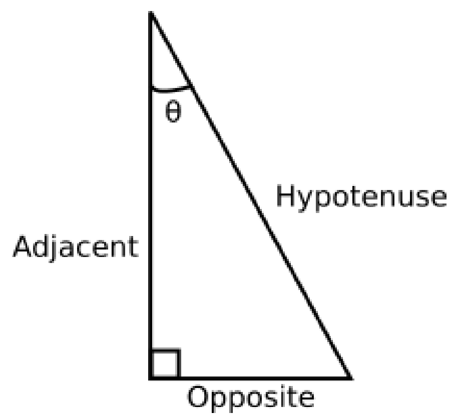
$$0 \le \theta < 2\pi$$

is does the same job as some value *inside* this domain, in the same sense that $370°$ is equivalent to $10°$.

1.3 Right Triangles

A *right triangle* is any triangle that has two sides meeting at $90°$. The side across from (not touching) the 90-degree corner is called the *hypotenuse*. If we label either of the remaining inner angles as θ, then the side touching this is called *adjacent*. The side across from θ is called the *opposite*. A right triangle can be held in any orientation without violating these labels as shown.





Pythagorean Theorem

Right triangles lend themselves to a very useful algebraic identity known as the *Pythagorean theorem*. Using our new terminology for the sides of the triangle, the theorem reads

$$(Opposite)^2 + (Adjacent)^2 = (Hypotenuse)^2 \ .$$

(We have this fact liberally already. The theorem has dozens of elementary proofs.)

2 Sine, Cosine, Tangent

By studying polynomials, we discovered a pair of curves that exhibit periodicity, namely

$$S_x^2 = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \frac{x^8}{315} + \cdots$$
$$C_x^2 = 1 - x^2 + \frac{x^4}{3} - \frac{2x^6}{45} + \frac{x^8}{315} - \cdots .$$

Note that each curve repeats itself on a period of 2π, which is argued by recalling that the unity condition

$$S_x^2 + C_x^2 = 1$$

always holds. This must also mean S_x and S_y correspond to two perpendicular sides of a right triangle with hypotenuse of 1, the family of which traces out a circle whose circumference is 2π.

As a pure switch of notation, let us rewrite the respective S_x and C_x-curves in more conventional terms. In particular, we shall relabel $x \to \theta$, where the Greek 'theta'-symbol simply takes the place of x. Then, we let

$$S_x = \sin(\theta) \qquad C_x = \cos(\theta)$$

such that

$$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots \qquad \cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots .$$

The quantity $\sin(\theta)$ reads 'sine of theta', and $\cos(\theta)$ reads 'cosine of theta'. For a given θ, both $\sin(\theta)$ and $\cos(\theta)$ has its own specific decimal value. Along for the ride comes the tangent curve:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \theta + \frac{1}{3}\theta^3 + \frac{2}{15}\theta^5 + \cdots .$$

2.1 Unit Circle

The set of all right triangles with a hypotenuse of 1 is called the *unit circle* as shown in the figure below. Given a point P on the unit circle, we construct a right triangle whose vertical side is the 'opposite', and whose horizontal side is the 'adjacent'. The diagonal line from the origin to P always has length 1. The angle θ always 'grows' from the line $x = 0$ and sweeps counterclockwise.

By embedding a right triangle in the unit circle, it follows that the sine and cosine are respectively equal to the opposite and adjacent sides of the embedded right triangle for any θ:

$$\sin(\theta) = \text{Opposite} \qquad \cos(\theta) = \text{Adjacent}$$

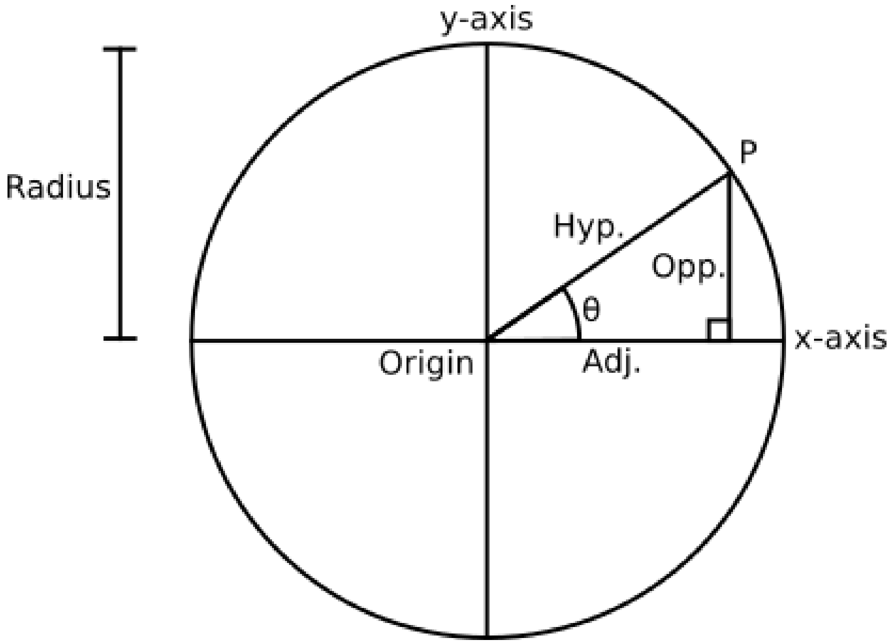
That is, the x-coordinate becomes the 'adjacent' of the triangle, and meanwhile the y-coordinate becomes the 'opposite' side. The ratio of these is the tangent:

$$\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}}$$

2.2 SohCahToa

Any non-unit circle with $R \neq 1$ is also host to the family of embedded right triangles whose hypotenuse is equal to R. In this case, the sine, cosine, and tangent obey the *SohCahToa* mnemonic:

$$\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}} \qquad \cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}} \qquad \tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}}$$



2.3 Phase

Since the θ-variable is only unique on the interval $0 \leq \theta < 2\pi$, and because trigonometric functions repeat themselves, the following relations must hold:

$$\cos(\theta + 2\pi) = \cos\theta \qquad \sin(\theta + 2\pi) = \sin\theta$$

Furthermore, the sine and cosine are the same up to a *phase shift* of $\pi/2$ such that

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta\,.$$

We can also jump across the circle by precisely π radians to flip the sign on either quantity:

$$\cos(\theta + \pi) = -\cos\theta \qquad \sin(\theta + \pi) = -\sin\theta$$

In general, a phase shift is any change in the variable θ, such as $\sin(\theta + \phi)$, or $\cos(\theta + \phi)$, where ϕ is the phase variable. Note also that the tangent obeys identical periodicity and phase relations:

$$\tan(\theta + 2\pi) = \tan\theta \qquad \tan\left(\frac{\pi}{2} - \theta\right) = \tan\theta \qquad \tan(\theta + \pi) = \tan\theta$$

2.4 Trigonometry Tables

Using the polynomial expressions for $\sin(\theta)$ and $\cos(\theta)$, or by reading the plot of either curve, we can generate the so-called *trigonometry tables* below. Note the first and last rows have identical information.

Angle (deg)	Angle (rad)	$\sin(\theta)$	$\cos(\theta)$
0	0	0	1
20	0.349	0.342	0.940
45	$\pi/4 \approx 0.785$	0.707	0.707
70	1.22	0.940	0.342
90	$\pi/2 \approx 1.57$	1	0

Angle (deg)	Angle (rad)	$\sin(\theta)$	$\cos(\theta)$
110	1.92	0.940	-0.342
135	$3\pi/4 \approx 2.36$	0.707	-0.707
160	2.79	0.342	-0.940
180	$\pi \approx 3.14$	0	-1

Angle (deg)	Angle (rad)	$\sin(\theta)$	$\cos(\theta)$
200	3.40	-0.342	-0.940
225	$5\pi/4 \approx 3.93$	-0.707	-0.707
250	4.36	-0.940	-0.342
270	$3\pi/2 \approx 4.71$	-1	0

Angle (deg)	Angle (rad)	$\sin(\theta)$	$\cos(\theta)$
290	5.06	-0.940	0.342
315	$7\pi/4 \approx 5.50$	-0.707	0.707
340	5.93	-0.342	0.940
360	$2\pi \approx 6.28$	0	1

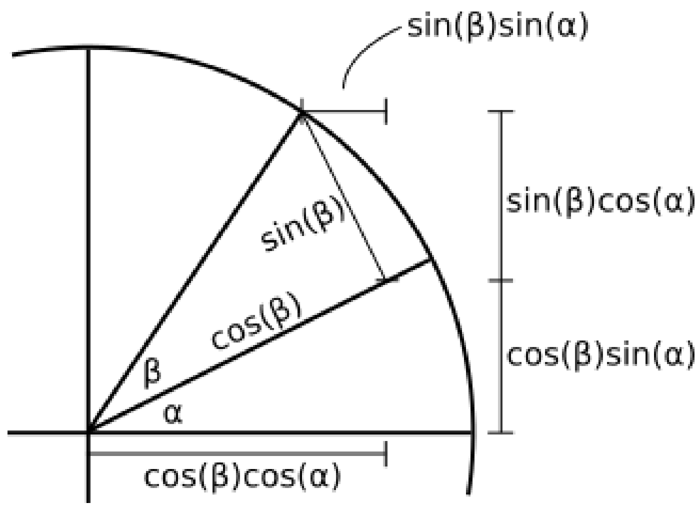
3 Trigonometric Identities

3.1 Angle-Sum Formulas

One can always express an angle θ as the sum of two smaller angles α and β, as sketched out in the figure below. Note first that the line indicated by β projects down onto the line indicated by α, allowing $\cos(\beta)$ and $\sin(\beta)$ to be identified as components of a right triangle. Projecting these onto the x- and y- axes, we discover the two angle-sum formulas:

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$



Moving on to $\tan(\alpha \pm \beta)$, there is no clever argument from geometry to discover the result, so we need brute force (using the two above equations):

$$\tan(\alpha \pm \beta) = \frac{\sin(\alpha \pm \beta)}{\cos(\alpha \pm \beta)} = \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$$

3.2 Reciprocal Identities

The main players of trigonometry are the cosine, sine, tangent

$$\sin(\theta) \qquad \cos(\theta) \qquad \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)},$$

along with their reciprocal counterparts, the *co-secant*, *secant*, and *cotangent*:

$$\csc(\theta) = \frac{1}{\sin(\theta)} \qquad \sec(\theta) = \frac{1}{\cos(\theta)} \qquad \cot(\theta) = \frac{1}{\tan(\theta)}$$

3.3 Fundamental Identities

It's not news that the fundamental identity

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

always holds, which yields two more identities if we divide by $\cos^2(\theta)$ and $\sin^2(\theta)$, respectively:

$$1 + \tan^2(\theta) = \sec^2(\theta)$$

$$\cot^2\theta + 1 = \csc^2(\theta)$$

3.4 Angle-Sum Identities

By considering triangles embedded in the unit circle, we previously deduced the angle-sum formulas:

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\tan(\alpha\pm\beta) = \frac{\sin(\alpha\pm\beta)}{\cos(\alpha\pm\beta)} = \frac{\tan(\alpha)\pm\tan(\beta)}{1\mp\tan(\alpha)\tan(\beta)}$$

Starting with the results (re)iterated here, a myriad of *trigonometric identities* can be derived.

3.5 Product Formulas

Take $\sin(\alpha+\beta)$ and add/subtract to/from $\sin(\alpha-\beta)$ to derive two *product formulas*. Repeat by replacing sin with cos to produce two more.

$$2\sin(\alpha)\cos(\beta) = \sin(\alpha+\beta) + \sin(\alpha-\beta)$$

$$2\cos(\alpha)\sin(\beta) = \sin(\alpha+\beta) - \sin(\alpha-\beta)$$

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha+\beta) + \cos(\alpha-\beta)$$

$$2\sin(\alpha)\sin(\beta) = \cos(\alpha+\beta) - \cos(\alpha-\beta)$$

3.6 Double-Angle Formulas

Let $\alpha=\beta=\theta$ in the product formulas to derive the *double-angle formulas*:

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

$$\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)}$$

3.7 Half-Angle Formulas

The $\cos(2\theta)$ equation can be mixed around to yield the *half-angle formulas*. First replace the $\sin^2(\theta)$ by $1-\cos^2(\theta)$, and then re-factor the θ variable by letting $\theta\to\theta/2$, giving

$$\cos(\theta) = 2\cos^2\left(\frac{\theta}{2}\right) - 1\,.$$

Going from there, the *half-angle formulas* pop out:

$$\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1}{2}+\frac{\cos(\theta)}{2}} \qquad \sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1}{2}-\frac{\cos(\theta)}{2}}$$

In addition, we also have:

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{1+\cos(\theta)} \qquad \cot\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{1-\cos(\theta)}$$

3.8 Superposition Relationships

The *superposition relationships* are useful in applied physics, particularly when summing waveforms. Begin by writing out the product $\sin(\alpha+\beta)\cos(\alpha-\beta)$, and simplify like mad:

$$\begin{aligned}
\sin(\alpha+\beta)\cos(\alpha-\beta) &= (\sin\alpha\cos\beta+\cos\alpha\sin\beta)(\cos\alpha\cos\beta+\sin\alpha\sin\beta) \\
&= (\sin\alpha\cos\alpha)\left(\sin^2\beta+\cos^2\beta\right)+(\sin\beta\cos\beta)\left(\sin^2\alpha+\cos^2\alpha\right) \\
&= \frac{\sin(2\alpha)}{2}+\frac{\sin(2\beta)}{2}
\end{aligned}$$

Note that several trig identities have been used without being mentioned (you should get good at seeing this). Re-factor the α and β variables and arrive at out first result

$$\sin(\alpha)+\sin(\beta)=2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) .$$

One down, three to go. Luckily we can let $\beta\to-\beta$ to get the next equation for free:

$$\sin(\alpha)-\sin(\beta)=2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right) .$$

Proceeding in a similar spirit, start with $\cos(\alpha+\beta)\cos(\alpha-\beta)$ and do it all again:

$$\begin{aligned}
\cos(\alpha+\beta)\cos(\alpha-\beta) &= (\cos\alpha\cos\beta-\sin\alpha\sin\beta)(\cos\alpha\cos\beta+\sin\alpha\sin\beta) \\
&= \cos^2\alpha\cos^2\beta+(1-1)(\sin\alpha\cos\alpha\sin\beta\cos\beta)-\sin^2\alpha\sin^2\beta \\
&= \frac{\cos^2\beta-\sin^2\alpha}{2}+\frac{\cos^2\alpha-\sin^2\beta}{2} \\
&= \frac{\cos(2\alpha)}{2}+\frac{\cos(2\beta)}{2}
\end{aligned}$$

Don't be *too* cavalier about following the algebra here. If the third step didn't seem to follow smoothly from the second one, think harder! As before, re-factor the α and β variables to arrive at our third result

$$\cos(\alpha)+\cos(\beta)=2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) .$$

Lastly, we have to write out $\sin(\alpha+\beta)\sin(\alpha-\beta)$, which goes similarly to the previous calculation:

$$\begin{aligned}
\sin(\alpha+\beta)\sin(\alpha-\beta) &= (\sin\alpha\cos\beta+\cos\alpha\sin\beta)(\sin\alpha\cos\beta-\cos\alpha\sin\beta) \\
&= \sin^2\alpha\cos^2\beta+(1-1)(\sin\alpha\cos\alpha\sin\beta\cos\beta)-\cos^2\alpha\sin^2\beta \\
&= \frac{-\cos^2\alpha+\sin^2\alpha}{2}+\frac{\cos^2\beta-\sin^2\beta}{2} \\
&= -\frac{\cos(2\alpha)}{2}+\frac{\cos(2\beta)}{2}
\end{aligned}$$

Re-factoring α and β, we finally have

$$\cos(\alpha)-\cos(\beta)=-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) .$$

3.9 Inverse Trigonometric Functions

The inverse of the cosine, sine, and tangent are defined as follows:

$$\arccos(\cos(x))=x \qquad \cos(\arccos(x))=x$$

$$\arcsin(\sin(x))=x \qquad \sin(\arcsin(x))=x$$

$$\arctan(\tan(x)) = x \qquad \tan(\arctan(x)) = x$$

The secant, co-secant, and cotangent also have inverse counterparts defined in analogy to those above:

$$\operatorname{arcsec}(\sec(x)) = x \qquad \sec(\operatorname{arcsec}(x)) = x$$

$$\operatorname{arccsc}(\csc(x)) = x \qquad \csc(\operatorname{arccsc}(x)) = x$$

$$\operatorname{arccot}(\cot(x)) = x \qquad \cot(\operatorname{arccot}(x)) = x$$

3.10 Inverse Trigonometric Identities

Not surprisingly, a flurry of identities arise from the inverse trigonometric functions. To get started, take a trivial statement such as

$$\cos(\arccos(x)) = x ,$$

and square each side to write

$$\cos^2(\arccos(x)) = 1 - \sin^2(\arccos(x)) = x^2 ,$$

and separate terms:

$$\sin(\arccos(x)) = \sqrt{1-x^2}$$

Divide through by x to get a similar identity:

$$\tan(\arccos(x)) = \frac{\sqrt{1-x^2}}{x}$$

An alternative derivation entails inspection of a right triangle of Hypotenuse 1, with the adjacent obeying $\cos(\theta) = x$. Naturally we have $\theta = \arccos(x)$, and the Opposite obeys $\sin(\theta) = \sqrt{1-x^2}$.

More identities emerge by tweaking the triangle. For instance, swap the Opposite for the Adjacent to get

$$\sin(\arcsin(x)) = x ,$$

along with

$$\cos(\arcsin(x)) = \sqrt{1-x^2}$$

and

$$\tan(\arcsin(x)) = \frac{x}{\sqrt{1-x^2}} .$$

Instead setting the Hypotenuse to $\sqrt{1+x^2}$ with an Adjacent of 1 and Opposite side x, the appropriate identities emerge:

$$\tan(\arctan(x)) = x$$

$$\sin(\arctan(x)) = \frac{x}{\sqrt{1+x^2}}$$

$$\cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}}$$

Three more triangles are needed to handle inverse identities. For the first of these, let the Hypotenuse be x, the Opposite have length 1, and the Adjacent have length $\sqrt{x^2-1}$, implying $\theta = \arcsin(1/x)$ or $\theta = \operatorname{arccsc}(x)$. Then, we have:

$$\sin(\operatorname{arccsc}(x)) = \frac{1}{x}$$

$$\cos(\operatorname{arccsc}(x)) = \frac{\sqrt{x^2-1}}{x}$$

$$\tan(\operatorname{arccsc}(x)) = \frac{1}{\sqrt{x^2-1}}$$

To proceed to the next triangle, swap the Opposite for the Adjacent to write:

$$\cos(\operatorname{arcsec}(x)) = \frac{1}{x}$$

$$\sin(\operatorname{arcsec}(x)) = \frac{\sqrt{x^2-1}}{x}$$

$$\tan(\operatorname{arcsec}(x)) = \sqrt{x^2-1}$$

Finally, we define the last triangle with Hypotenuse $\sqrt{x^2+1}$, Opposite of length 1, and Adjacent side x. This means $\theta = \operatorname{arccot}(x)$, and we find:

$$\tan(\operatorname{arccot}(x)) = \frac{1}{x}$$

$$\sin(\operatorname{arccot}(x)) = \frac{1}{\sqrt{x^2+1}}$$

$$\cos(\operatorname{arccot}(x)) = \frac{x}{\sqrt{x^2+1}}$$