Recursive Matrix Algorithms: Quick QR decomposition of matrices

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Introduction: QR-decomposition problem

- The problem of the orthogonal decomposition of matrices is still known as the QR-decomposition problem. It is one of the subtasks that are associated with spectral decomposition.
- Given the matrix A, it is required to represent it as a product of two factors,

$$A = QR,$$

  where Q is a unitary matrix (orthogonal in the case of real numbers), R is an upper triangular matrix.
- There are a large number of different approaches [2]-[5] to the problem of computing the orthogonal decomposition, including fast recursive algorithm [6]. However, all these algorithms have cubic complexity.
- In this paper, we consider the algorithm of orthogonal decomposition, **which has the complexity of matrix multiplication. This algorithm was published by A. Schonhage in 1973 year [1].**

Notations

For definiteness, we will consider an algorithm applied to a square matrix A over a field of real numbers. Consider the case of a 2×2 matrix. The desired decomposition $A = QR$ has the form:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix},$$

where $s^2 + c^2 = 1$. If $\gamma = 0$ then we can set $c = 1, \ s = 0$. If $\gamma \neq 0$, then $\quad \Delta = \alpha^2 + \gamma^2 > 0. \quad$ Then we get $c\alpha + s\gamma = a, \quad c\gamma - s\alpha = 0$ and $c = a\alpha/\Delta, \ \ s = a\gamma/\Delta$. Therefore, $1 = s^2 + c^2 = a^2/\Delta$, hence $|a| = \sqrt{\Delta}$.

$$c = \alpha/\sqrt{\Delta}, \qquad s = \gamma/\sqrt{\Delta}.$$

We denote such a matrix Q by $g_{\alpha,\gamma}$.

Sequential QR decomposition: one step

Let the matrix A be given:

- $a_{i,j} = \alpha$, $a_{i+1,j} = \gamma$,
- all the elements to the left of them be zero:
  $\forall (s < j) : (a_{i,s} = 0)$ & $(a_{i+1,s} = 0)$,
- $G_{i,j} = \textbf{diag}(I_{i-1}, g_{\alpha,\gamma}, I_{n-i-1})$.

Then:

- the matrix $G_{i,j}A$ differs from A only in two rows i and $i+1$,
- all the elements to the left of the column j remain zero,
- $a_{i+1,j} = 0$

Algorithm

- (1). First we reset the elements under the diagonal in the left column:

$$A_1 = G_{1,1} G_{2,1} ... G_{n-2,1} G_{n-1,1} A$$

- (2). Then we reset the elements that are under the diagonal in the second column:

$$A_2 = G_{2,2} G_{3,2} ... G_{n-2,2} G_{n-1,2} A_1$$

- (k). Denote $G_{(k)} = G_{k,k} G_{k-1,k} ... G_{n-2,k} G_{n-1,k}$, для $k = 1, 2, .., n-1$. Then, to calculate the elements of the k th column, we need to obtain the product of matrices

$$A_k = G_{(k)} A_{k-1}.$$

- (n-1). At the end of the calculation, the element in the $n-1$ column will be reseted: $A_{n-1} = G_{(n-1)} A_{n-2} = G_{n-1,n-1} A_{n-2}$.

Sequential QR decomposition: complexity

You can find the number of operations. It is necessary to calculate the $(n^2 - n)/2$ turn matrices and for each of them 6 operations must be performed. when calculating A_1, the number of multiplications of the Givens matrices into columns of two elements (4 multiplications and 2 additions) is $(n-1)^2$. When calculating A_2, the number of such multiplications is $(n-2)^2$, and so on. As a result, we get

$$6(n^2 - n)/2 + 6 \sum_{i=1..n-1} i^2 = 3n^2 - 3n + 6(n-1)(2n-1)n/6 \approx 2n^3$$

Here we count the number of all arithmetic operations and the operations of extracting the square root.

QR_G decomposition

Algorithm for M $= \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

- (1). QR_G decomposition of the block C:

$$C = Q_1 C_1, \ \ M_1 = \mathbf{diag}(I, Q_1) M = \begin{pmatrix} A & B \\ C_1 & D_1 \end{pmatrix}.$$

- (2). Cancellation of a parallelogram composed of two triangular blocks: the lower triangular part A^L of the block A and the upper triangular part C_1^U of the block C_1
$\begin{pmatrix} A \\ C_1 \end{pmatrix} = \begin{pmatrix} A_1 \\ 0 \end{pmatrix}, \ M_2 = Q_2 M_1 = \begin{pmatrix} A_1 & B_1 \\ 0 & D_2 \end{pmatrix}.$
- (3). QR_G decomposition of the D_2 block: $D_2 = Q_3 D_3$.
$R = \mathbf{diag}(I, Q_3) M_2 = \begin{pmatrix} A_1 & B_1 \\ 0 & D_3 \end{pmatrix}.$

As a result, we get:

$$M = Q^T R, \quad Q = \mathbf{diag}(I, Q_3) Q_2 \, \mathbf{diag}(I, Q_1).$$

It remains to describe the parallelogram cancellation procedure.

QP-decomposition

We are looking for the factorization of the matrix:

$$M = QP = Q\begin{pmatrix} A^U \\ 0 \end{pmatrix}, \quad M = \begin{pmatrix} A \\ R_1 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \\ f^U & g \\ 0 & h^U \end{pmatrix}$$

$$P1 = Q_{ld}\begin{pmatrix} b \\ f^U \end{pmatrix} = \begin{pmatrix} b^U \\ 0 \end{pmatrix}, P2 = Q_{lu}\begin{pmatrix} a \\ b^U \end{pmatrix} = \begin{pmatrix} a^U \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} d_1 \\ g_1 \end{pmatrix} = Q_{ld}\begin{pmatrix} d_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} c_1 \\ d_2 \end{pmatrix} = Q_{lu}\begin{pmatrix} c \\ d_1 \end{pmatrix},$$

$$P3 = Q_{rd}\begin{pmatrix} g_1 \\ h^U \end{pmatrix} = \begin{pmatrix} g_1^U \\ 0 \end{pmatrix}, P4 = Q_{ru}\begin{pmatrix} d_2 \\ g^U \end{pmatrix} = \begin{pmatrix} d_2^U \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} A_1 \\ 0 \end{pmatrix} = Q\begin{pmatrix} A \\ R_1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} a^U & c_1 \\ 0 & d_2^U \end{pmatrix},$$

$$Q = \mathbf{diag}(I_{n/2}, Q_{ru}, I_{n/2})\,\mathbf{diag}(Q_{lu}, Q_{rd})\,\mathbf{diag}(I_{n/2}, Q_{ld}, I_{n/2})$$

Let multiplication of two matrices of size $n \times n$ needs γn^{β} operations, $n = 2^k$. The number of multiplications of matrix blocks of size $n/2 \times n/2$ is 24. Hence the total number of operations:

$$Cp(2n) = 4Cp(n) + 24M(n/2).$$

$$Cp(2^{k+1}) = 4Cp(2^k) + 24M(2^{k-1}) = 4^k Cp(2^1) + 24\gamma \sum_{i=0}^{k-1} 4^{k-i-1} 2^{i\beta} =$$

$$24\gamma(n^2/4)\frac{2^{k(\beta-2)} - 1}{2^{(\beta-2)} - 1} + 6n^2 = 6\gamma\frac{n^{\beta} - n^2}{2^{\beta} - 4} + 6n^2$$

$$Cp(n) = \frac{6\gamma n^{\beta}}{2^{\beta}(2^{\beta} - 4)} + \frac{3n^2}{2}\left(1 - \frac{\gamma}{2^{\beta} - 4}\right)$$

Complexity of the matrix multiplication is $M(n) = \gamma n^{\beta}$, the complexity of canceling the parallelogram is $Cp(n) = \alpha n^{\beta}$, where α, β, γ are constants, $\alpha = \frac{6\gamma}{2^{\beta}(2^{\beta}-4)}$ and $n = 2^k$ then:

$$C(n) = 2C(n/2) + Cp(n) + 6M(n/2) = 2C(2^{k-1}) + Cp(2^k) + 6M(2^{k-1}) =$$

$$C(2^0)2^k + \sum_{i=0}^{k} 2^{k-i} Cp(2^i) + 6\sum_{i=0}^{k} 2^{k-i} M(2^{i-1}) = \alpha \sum_{i=0}^{k} 2^{k-i} 2^{i\beta} + 6\gamma \sum_{i=0}^{k} 2^{k-}$$

$$(\alpha 2^k + 6\gamma 2^{k-\beta}) \sum_{i=0}^{k} 2^{i(\beta-1)} = (\alpha + 6\gamma 2^{-\beta}) \frac{2^{\beta} n^{\beta} - 2n}{2^{\beta} - 2}$$

$$= \frac{\gamma 6(2^{\beta} - 3)(n^{\beta} - \frac{2n}{2^{\beta}})}{(2^{\beta} - 4)(2^{\beta} - 2)}$$

Conclusion

Thus, presented algorithm has the complexity of matrix multiplication. If we apply the standard matrix multiplication ($2n^3$ operations for the matrix $n \times n$), then we need only $\approx 2.5n^3$ operations.

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