Available online at www.sciencedirect.com
ScienceDirect

International Journal of Rock Mechanics & Mining Sciences 44 (2007) 222–227

International Journal of Rock Mechanics and Mining Sciences

www.elsevier.com/locate/ijrmms

Theory of generalized porothermoelasticity

H.M. Youssef

Faculty of Engineering, Umm Al-Qurah University, P.O. 5555, Makkah, Saudi Arabia

Accepted 4 July 2006
Available online 20 September 2006

Abstract

In this paper, the governing equations, which describe the behavior of thermoelastic porous medium in the context of the theory of generalized thermoelasticity with one relaxation time (Lord-Shulman) has been derived. The energy and the entropy equations have been derived also in general co-ordinates. The uniqueness of the solution for the complete system of the equations of the theorem has been proved.

Keywords: Elasticity; Thermoelasticity; Generalized thermoelasticity; Porous media; Thermo-poroelasticity

1. Introduction

Porous materials make their appearance in a wide variety of settings, natural and artificial, and in diverse technological applications. As a consequence a number of problems arise dealing with, among other issues, statics and strength, fluid flow and heat conduction, and dynamics. In connection with the latter, we note that problems of this kind are encountered in the prediction of the behavior of sound-absorbing materials and in the area of exploration geophysics, the steadily growing literature bearing witness to the importance of the subject [1].

The problem of a fluid-saturated porous material has been studied for many years. A short list of papers pertinent to the present study includes Biot [2–3], Gassmann [4], Biot and Willis [5], Biot [6], Deresiewicz and Skalak [7], Mandl [8], Nur and Byerlee [9], Brown and Korringa [10], Rice and Cleary [11], Burridge and Keller [12], Zimmerman et al. [13–14], Berryman and Milton [15], Thompson and Willis [16], Pride et al. [17], Berryman and Wang [18], Tuncay and Corapcioglu [19], Cheng [20], Charlez and Heugas [21], Abousleiman and Cui [22], Ghassemi and Diek [23] and Tod [24].

The thermo-mechanical coupling in the poroelastic medium turns out to be of much greater complexity than in the classical case of an impermeable elastic solid, since, in addition to thermal and mechanical interaction within each phase, thermal and mechanical coupling occurs between the phases. Thus, a mechanical or thermal change in one phase results in mechanical and thermal changes throughout the aggregate.

Following Biot, we take our physical model to consist of a homogeneous, isotropic, elastic matrix whose interstices are filled with a compressible ideal liquid. Both the solid and liquid form continuous (and interacting) regions and, although viscous stresses in the liquid are neglected, the liquid is assumed capable of exerting a velocity-dependent friction force on the skeleton. The mathematical model consists of two superposed, continuous phases, each separately filling the entire space occupied by the aggregate. Thus, there are two distinct elements at each point of space, each characterized by its own displacement, stress and temperature, and during a thermo-mechanical process they may interact, with a consequent exchange of momentum and energy.

Our development proceeds by obtaining the stress–strain–temperature relations using the theory of the generalized thermoelasticity with one relaxation time, the so-called "Lord-Shulman" model [25]. Here, in addition to the usual isobaric coefficients of thermal expansion of the single-phase materials, two coefficients appear which represent measures of each phase caused by temperature changes in the other phase. As a result of the presence of these coupling coefficients, it follows that the coefficient of

E-mail address: yousefanne@yahoo.com.

1365-1609/$ - see front matter © 2006 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijrmms.2006.07.001

thermal expansion of the dry material differs from that of the saturated one, and the expansion of the liquid in the bulk is not the same as of the liquid phase.

There follow the equations of motion, in Biot's form, including the dynamic and "interface" dissipation proportional to the relative velocity of the two phases. Having in mind, in particular, applications of geophysical interest, we take the coefficient of proportionality in the dissipation term to be independent of frequency, that is, we confine ourselves to low-frequency motions. The last constituent of the theory is the equations of energy flux. Because the two phases will, in general, be at different temperatures at each point of the material, there arise in the energy equations a heat-source term descriptive of heat flux between the phases. We have taken this "interphase heat transfer" to be proportional to the temperature difference between the phases. Finally, we set the uniqueness theorem, and carry out the proof.

2. The basic formulation of the theory

The state of stress at a point of the bulk material is [1,26]

$$\tau_{ij} = (1-\beta)\tau^*_{ij} - \beta p \delta_{ij}, \tag{1}$$

where β is the porosity of the material. Now let [1,26]

$$\sigma_{ij} = (1-\beta)\tau^*_{ij} \quad \text{and} \quad \sigma = -\beta p$$

from which it follows that

$$\tau_{ij} = \sigma_{ij} + \sigma\delta_{ij}. \tag{2}$$

The displacements of the phases are u_i for the skeleton and U_i for the fluid. For the skeleton, we have

$$e_{ij} = \tfrac{1}{2}\left(u_{i,j} + u_{j,i}\right), \quad e_{ii} = e = u_{i,i} \tag{3}$$

and for the fluid, we have

$$\varepsilon = U_{i,i}. \tag{4}$$

Let ρ^{s*} and ρ^{f*} denote the density of the solid and the liquid phases. Then, the density of the aggregate is given by [1,26]

$$\rho = (1-\beta)\rho^{s*} + \beta\rho^{f*} \equiv \rho^s + \rho^f, \tag{5}$$

where

$$\rho^s = (1-\beta)\rho^{s*}, \quad \rho^f = \beta\rho^{f*} \tag{6}$$

represent the densities of the two phases per unit volume of bulk. The mass coefficients are expressed in terms of the densities by means of

$$\rho_{11} = \rho^s - \rho_{12} \quad \text{and} \quad \rho_{22} = \rho^f - \rho_{12}, \tag{7}$$

where ρ_{12} is the dynamic coupling coefficient.

The kinetic energy takes the form

$$KE = \tfrac{1}{2}\left(\rho_{11}\dot{u}_i\dot{u}_i + \rho_{22}\dot{U}_i\dot{U}_i + 2\rho_{12}\dot{u}_i\dot{U}_i\right). \tag{8}$$

By Lagrange's equations, we obtain

$$\rho_{11}\ddot{u}_i + \rho_{12}\ddot{U}_i = \sigma_{ij,j} + \rho^s b^s_i, \tag{9}$$

$$\rho_{12}\ddot{u}_i + \rho_{22}\ddot{U}_i = \sigma_{,i} + \rho^f b^f_i, \tag{10}$$

where b^s_i, b^f_i are the body forces.

The first law of thermodynamics states that [26]

$$\begin{aligned}
&\frac{d}{dt}\int_V \rho\xi \, dV + \frac{d}{dt}\int_V \frac{1}{2}\left(\rho_{11}\dot{u}_i\dot{u}_i + \rho_{22}\dot{U}_i\dot{U}_i + 2\rho_{12}\dot{u}_i\dot{U}_i\right) dV \\
&\quad = \int_V \left(\rho_1 b^s_i\dot{u}_i + \rho_2 b^f_i\dot{U}_i\right) dV - \int_v \left(q^s_{i,i} + q^f_{i,i}\right) dV \\
&\qquad + \int_V \rho\left(h^s + h^f\right) + \int_S \left(\sigma_{ij}\, n_j\, \dot{u}_i + \sigma\, n_i \dot{U}_i\right) dS,
\end{aligned} \tag{11}$$

where ξ is the internal energy per unit mass of the bulk, q^s_i, q^f_i are the heat fluxes, h^s_i, h^f_i are the heat sources in the solid and the liquid, and n_i is the normal to the surface S that bounds the volume bulk V.

Using Gauss's integral theorem with Eqs. (9) and (10), we get

$$\rho\dot{\xi} = \sigma_{ij}\dot{u}_{i,j} + \sigma\dot{\varepsilon} + \rho\left(h^s + h^f\right) - q^s_{i,i} - q^f_{i,i}. \tag{12}$$

Let ϕ be the combination of the internal energy and the entropy (Helmholtz's function):

$$\phi = \xi - T^s\eta^s - T^f\eta^f. \tag{13}$$

Differentiating Eq. (13) with respect to time gives

$$\dot{\phi} = \dot{\xi} - T^s\dot{\eta}^s - T^f\dot{\eta}^f - \dot{T}^s\eta^s - \dot{T}^f\eta^f, \tag{14}$$

where (T^s, T^f) and (η^s, η^f) are the temperatures and the entropy for the solid and the liquid per unit mass of aggregate, respectively.

We will assume that ϕ depends on the variables (e_{ij}, ε, T^s, T^f), i.e.,

$$\phi = \phi\left(e_{ij}, \varepsilon, T^s, T^f\right). \tag{15}$$

Using the chain rule, we obtain

$$\dot{\phi} = \frac{\partial\phi}{\partial e_{ij}}\dot{e}_{ij} + \frac{\partial\phi}{\partial\varepsilon}\dot{\varepsilon} + \frac{\partial\phi}{\partial T_s}\dot{T}^s + \frac{\partial\phi}{\partial T_f}\dot{T}^f. \tag{16}$$

From Eqs. (12), (15) and (16), we obtain

$$\sigma_{ij} = \rho\frac{\partial\phi}{\partial e_{ij}}, \tag{17}$$

$$\sigma = \rho\frac{\partial\phi}{\partial\varepsilon}, \tag{18}$$

$$\eta^s = -\frac{\partial\phi}{\partial T^s}, \tag{19}$$

$$\eta^f = -\frac{\partial\phi}{\partial T^f}, \tag{20}$$

$$q^s_{i,i} + \rho T^s\dot{\eta}^s - \rho h^s + K\left(T^s - T^f\right) = 0,$$

$$q^f_{i,i} + \rho T^f\dot{\eta}^f - \rho h^f - K\left(T^s - T^f\right) = 0$$

and for linearity, we let $T^s \approx T^f \approx T_0$. Hence, we get

$$q^s_{i,i} + \rho T_0\dot{\eta}^s - \rho h^s + K\left(T^s - T^f\right) = 0, \tag{21}$$

$$q^{\mathrm{f}}_{i,i} + \rho T_0 \dot{\eta}^{\mathrm{f}} - \rho h^{\mathrm{f}} - K\left(T^{\mathrm{s}} - T^{\mathrm{f}}\right) = 0, \tag{22}$$

where the term $K(T^{\mathrm{s}} - T^{\mathrm{f}})$ represents the interphase heat transfer resulting from the unequal temperature of the phases at every point of the medium, and K is the interphase thermal conductivity.

To proceed further, we make the following assumptions with regard to the functional dependence of the internal forces and entropy densities:

$$\sigma_{ij} = \sigma_{ij}\left(e_{ij}, \varepsilon, T^{\mathrm{s}}, T^{\mathrm{f}}\right), \tag{23}$$

$$\sigma = \sigma\left(e_{ij}, \varepsilon, T^{\mathrm{s}}, T^{\mathrm{f}}\right), \tag{24}$$

$$\eta^{\mathrm{s}} = \eta^{\mathrm{s}}\left(e_{ij}, \varepsilon, T^{\mathrm{s}}\right), \tag{25}$$

$$\eta^{\mathrm{f}} = \eta^{\mathrm{f}}\left(e_{ij}, \varepsilon, T^{\mathrm{f}}\right). \tag{26}$$

The generalized Fourier's law of heat conduction is [25]

$$q^{\mathrm{s}}_i + \tau^{\mathrm{s}}_0 \dot{q}^{\mathrm{s}}_i = -k^{\mathrm{s}} T^{\mathrm{s}}_{,i}, \tag{27}$$

$$q^{\mathrm{f}}_i + \tau^{\mathrm{f}}_0 \dot{q}^{\mathrm{f}}_i = -k^{\mathrm{f}} T^{\mathrm{f}}_{,i}, \tag{28}$$

where τ^{s}_0 and τ^{f}_0 are the solid and liquid relaxation times, respectively, and k^{s} and k^{f} are the thermal conductivity of the phases, such that

$$k^{\mathrm{s}} = (1-\beta)k^{\mathrm{s}*} \quad \text{and} \quad k^{\mathrm{f}} = \beta k^{\mathrm{f}*},$$

where $k^{\mathrm{s}*}$, $k^{\mathrm{f}*}$are the thermal conductivity of the solid and the liquid, respectively.

The potential function ψ can be expanded as

$$\psi = \rho\phi = \mu e_{ij}e_{ij} + \frac{\lambda}{2}e_{ii}e_{jj} + \frac{1}{2}R\varepsilon^2 + Qe_{ii}\varepsilon + Ce_{ii}\theta^{\mathrm{s}} + D_1 e_{ii}\theta^{\mathrm{f}} + D_2\varepsilon\theta^{\mathrm{f}} + F\varepsilon\theta^{\mathrm{s}} + G\theta^{\mathrm{s}^2} + H\theta^{\mathrm{f}^2} + J\theta^{\mathrm{s}}\theta^{\mathrm{f}} + \ldots, \tag{29}$$

where λ, μ, R and Q are poroelastic moduli, and C, D_1, D_2, F, G, H, and J are additional mixed and thermal coefficients.

In Eq. (29), we have assumed that $\theta^{\mathrm{s}} = T^{\mathrm{s}} - T_0$ and $\theta^{\mathrm{f}} = T^{\mathrm{f}} - T_0$, such that

$$\left|\frac{\theta^{\mathrm{s}}}{T_0}\right| \ll 1 \text{ and } \left|\frac{\theta^{\mathrm{f}}}{T_0}\right| \ll 1,$$

where in the reference state we have $T^{\mathrm{s}} = T^{\mathrm{f}} = T_0$. Hence, we get

$$\sigma_{ij} = 2\mu e_{ij} + \lambda e_{kk}\delta_{ij} + \left(Q\varepsilon + C\theta^{\mathrm{s}} + D_1\theta^{\mathrm{f}}\right)\delta_{ij}, \tag{30}$$

$$\sigma = R\varepsilon + Qe_{kk} + D_2\theta^{\mathrm{f}} + F\theta^{\mathrm{s}}, \tag{31}$$

$$\rho\eta^{\mathrm{s}} = -Ce_{kk} - F\varepsilon - 2G\theta^{\mathrm{s}} - J\theta^{\mathrm{f}}, \tag{32}$$

$$\rho\eta^{\mathrm{f}} = -D_1 e_{kk} - D_2\varepsilon - 2H\theta^{\mathrm{f}} - J\theta^{\mathrm{s}}. \tag{33}$$

Let

$$-C = R_{11} - D_1 = R_{12}, -F = R_{21}, -D_2 = R_{22}$$

in which case we have

$$\rho\eta^{\mathrm{s}} = R_{11}e_{kk} + R_{21}\varepsilon - 2G\theta^{\mathrm{s}} - J\theta^{\mathrm{f}}, \tag{34}$$

$$\rho\eta^{\mathrm{f}} = R_{12}e_{kk} + R_{22}\varepsilon - 2H\theta^{\mathrm{f}} - J\theta^{\mathrm{s}}, \tag{35}$$

$$Qe_{kk} + R\varepsilon = M_2, \tag{36}$$

$$Pe_{kk} + 3Q\varepsilon = M_1, \tag{37}$$

where

$$M_1 = \sigma_{ii} + 3\left(R_{11}\theta^{\mathrm{s}} + R_{12}\theta^{\mathrm{f}}\right) \quad \text{and} \quad M_2 = \sigma + R_{22}\theta^{\mathrm{f}} + R_{21}\theta^{\mathrm{s}}.$$

By solving Eqs. (36) and (37), we get

$$e_{kk} = \frac{1}{\Delta}(RM_1 - 3QM_2), \tag{38}$$

$$\varepsilon = \frac{1}{\Delta}(PM_2 - QM_1), \tag{39}$$

where

$$\Delta = PR - 3Q^2 \neq 0 \quad \text{and} \quad P = 2\mu + 3\lambda.$$

Thus, we can obtain the inverse of Eqs. (30) and (31) as follows:

$$e_{ij} = \frac{\sigma_{ij}}{2\mu} + \left(S_1\sigma_{ii} + S_2\sigma + \alpha^{\mathrm{s}}\theta^{\mathrm{s}} + \alpha^{\mathrm{sf}}\theta^{\mathrm{f}}\right)\delta_{ij}, \tag{40}$$

$$\varepsilon = S_2\sigma_{ii} + S_3\sigma + \alpha^{\mathrm{f}}\theta^{\mathrm{f}} + \alpha^{\mathrm{fs}}\theta^{\mathrm{s}}, \tag{41}$$

where

$$S_1 = \frac{Q^2 - \lambda R}{2\mu\Delta},\ S_2 = -\frac{Q}{\Delta},\ S_3 = \frac{P}{\Delta},$$

$$\alpha^{\mathrm{s}} = \frac{RR_{12} - QR_{21}}{\Delta}, \quad \alpha^{\mathrm{f}} = \frac{RR_{22} - 3QR_{12}}{\Delta},$$
$$\alpha^{\mathrm{sf}} = \frac{RR_{12} - QR_{22}}{\Delta}, \quad \alpha^{\mathrm{fs}} = \frac{RR_{21} - 3QR_{11}}{\Delta},$$

α^{s}, α^{f} are the coefficients of the thermal expansion of the particular phases, and α^{sf}, α^{fs} are the thermoelastic couplings existing between the phases.

Now, we have

$$R_{11} = \alpha^{\mathrm{s}}p + \alpha^{\mathrm{fs}}Q, \quad R_{22} = \alpha^{\mathrm{f}}R + 3\alpha^{\mathrm{sf}}Q,$$

$$R_{12} = \alpha^{\mathrm{f}}Q + \alpha^{\mathrm{sf}}P, \quad R_{21} = 3\alpha^{\mathrm{s}}Q + \alpha^{\mathrm{fs}}R.$$

Next, we return to Eq. (34) for the entropy of the solid phase, and replace the dilatation by their representations (40) and (41). We find, after some manipulations, that

$$\rho\eta^{\mathrm{s}} = \alpha^{\mathrm{s}}\sigma_{ii} + \alpha^{\mathrm{fs}}\sigma + \left(3\alpha^{\mathrm{s}}R_{11} + \alpha^{\mathrm{fs}}R_{21} - 2G\right)\theta^{\mathrm{s}}, \tag{42}$$

where $3\alpha^{\mathrm{s}}R_{12} + \alpha^{\mathrm{fs}}R_{22} - J = 0$ in order to account for the postulated independence of η_{s} of the temperature of the fluid phase.

The specific heat of the solid is given by

$$C^{\mathrm{s}}_E = -T^{\mathrm{s}}\frac{\partial\eta^{\mathrm{s}}}{\partial T^{\mathrm{s}}}. \tag{43}$$

From Eqs. (30), (31) and (42), we find

$$G = -\frac{\rho C^{\mathrm{s}}_E}{2T_0}. \tag{44}$$

We now define the following notation:

$$3\alpha^{s}R_{11}+\alpha^{fs}R_{21}+\frac{\rho C_E^{s}}{T_0}=\frac{\rho C_T^{s}}{T_0} \tag{45}$$

in which case we can write

$$\rho C_T^{s}-\rho C_E^{s}=T_0\left[3(\alpha^{s})^2P+6\alpha^{s}\alpha^{fs}Q+\left(\alpha^{fs}\right)^2R\right]. \tag{46}$$

Finally, we have the entropy equations in the forms

$$\rho\eta^{s}=\alpha^{s}\sigma_{ii}+\alpha^{fs}\sigma+\frac{\rho C_T^{s}}{T_0}\theta^{s}, \tag{47}$$

$$\rho\eta^{f}=\alpha^{sf}\sigma_{ii}+\alpha^{f}\sigma+\frac{\rho C_T^{f}}{T_0}\theta^{f}. \tag{48}$$

The equations of motion take the forms

$$\begin{aligned}&\mu u_{i,jj}+(\lambda+\mu)u_{j,ij}+QU_{i,ii}-R_{11}\theta^{s}_{,i}-R_{12}\theta^{f}_{,i}\\&\quad+\rho^{s}b_i^{s}=\rho_{11}\ddot{u}_i+\rho_{12}\ddot{U}_i,\end{aligned} \tag{49}$$

$$RU_{i,ii}+Qu_{j,ij}-R_{21}\theta^{s}_{,i}-R_{22}\theta^{f}_{,i}+\rho^{f}b_i^{f}=\rho_{12}\ddot{u}_i+\rho_{22}\ddot{U}_i. \tag{50}$$

We get the heat equations from Eqs. (21), (22), (27), (28), (34) and (35) in the forms

$$\begin{aligned}k^{s}\theta^{s}_{,ii}=&\left(\frac{\partial}{\partial t}+\tau_0^{s}\frac{\partial^2}{\partial t^2}\right)\left(F_{11}\theta^{s}+F_{12}\theta^{f}+T_0R_{11}e_{ii}+T_0R_{21}\varepsilon\right)\\&-\rho\left(1+\tau_0^{s}\frac{\partial}{\partial t}\right)h^{s}+k\left(\theta^{s}-\theta^{f}\right),\end{aligned} \tag{51}$$

$$\begin{aligned}k^{f}\theta^{f}_{,ii}=&\left(\frac{\partial}{\partial t}+\tau_0^{f}\frac{\partial^2}{\partial t^2}\right)\left(F_{21}\theta^{s}+F_{22}\theta^{f}+T_0R_{12}e_{ii}+T_0R_{22}\varepsilon\right)\\&-\rho\left(1+\tau_0^{f}\frac{\partial}{\partial t}\right)h^{f}+k\left(\theta^{f}-\theta^{s}\right),\end{aligned} \tag{52}$$

where

$$F_{11}=-2GT_0=\rho C_{E,}^{s}\quad F_{22}=-2HT_0=\rho C_E^{f}$$

$$F_{12}=-\left(3\alpha^{s}R_{12}+\alpha^{fs}R_{22}\right)T_0=-JT_0,$$

$$F_{21}=-\left(3\alpha^{sf}R_{11}+\alpha^{f}R_{21}\right)T_0=-JT_0.$$

The constitutive equations then take the form [27]

$$\sigma_{ij}=2\mu e_{ij}+\lambda e_{kk}\delta_{ij}+\left(Q\varepsilon-R_{11}\theta^{s}-R_{12}\theta^{f}\right)\delta_{ij}, \tag{53}$$

$$\sigma=R\varepsilon+Qe_{kk}-R_{22}\theta^{f}-R_{21}\theta^{s}. \tag{54}$$

The entropy equations can be written as follows:

$$\rho\eta^{s}=R_{11}e_{kk}+R_{21}\varepsilon+\frac{F_{11}}{T_0}\theta^{s}+\frac{F_{12}}{T_0}\theta^{f}, \tag{55}$$

$$\rho\eta^{f}=R_{12}e_{kk}+R_{22}\varepsilon+\frac{F_{21}}{T_0}\theta^{s}+\frac{F_{22}}{T_0}\theta^{f}. \tag{56}$$

3. Uniqueness theorem

Let V be an open regular region of space with boundary S occupied by the reference configuration of a homogeneous isotropic linear thermoelastic solid, where S is assumed closed and bounded [7]. We supplement the equations of generalized porothermoelasticity (9), (10), (49)–(56) by prescribed boundary conditions

$$u_i=\bar{u}_i,\ U_i=\bar{U}_i\quad\text{on}\quad S_1\times[0,\infty), \tag{57}$$

$$p_i=\sigma_{ji}n_j=\bar{p}_i,\ \ell_i=\sigma=\bar{\ell}_i\quad\text{on}\quad S-S_1\times[0,\infty), \tag{58}$$

$$\theta_i^{s}=\bar{\theta}_i^{s},\quad\theta_i^{f}=\bar{\theta}_i^{f}\quad\text{on}\quad S, \tag{59}$$

where $S_1\subset S$. Also, we prescribe the following initial conditions within V at $t=0$:

$$\begin{aligned}&u_i=u_{i0},\ \dot{u}_i=\dot{u}_{i0},\ U_i=U_{i0},\ \dot{U}_i=\dot{U}_{i0},\ \theta^{s}=\theta^{s}_0,\\&\theta^{f}=\theta^{f}_0,\ \dot{\theta}^{s}=\dot{\theta}^{s}_o,\ \dot{\theta}^{f}=\dot{\theta}^{f}_0.\end{aligned} \tag{60}$$

We now state the uniqueness theorem: Given a regular region of space V+S with boundary S, there exists at most one set of single-valued functions $\sigma_{ij}(x_k,t)$, $\sigma(x_k,t)$, $\varepsilon(x_k,t)$ and $\sigma_{ij}(x_k,t)$ of class $C^{(1)}$, and $u_i(x_k,t)$, $U_i(x_k,t)$, $\theta_i^{s}(x_k,t)$ and $\theta_i^{f}(x_k,t)$ of class $C^{(2)}$ in V+S, $t\geqslant 0$, that satisfy Eqs. (9), (10), (49)–(56) and conditions (57)–(60), where all the parameters are all positive.

The proof proceeds as follows. Let there be two sets of functions $\sigma_{ij}^{(I)}$ and $\sigma_{ij}^{(II)}$, $e_{ij}^{(I)}$ and $e_{ij}^{(II)}$, $\sigma^{(I)}$ and $\sigma^{(II)}$, $\varepsilon^{(I)}$ and $\varepsilon^{(II)}$, etc., and let $\sigma_{ij}=\sigma_{ij}^{(I)}-\sigma_{ij}^{(II)}$, etc. By virtue of the linearity of the problem, it is clear that these differences also satisfy the above-mentioned equations (without loss of generality we neglect the terms $K(T^{s}-T^{f})$, the body forces and the heat fluxes) and the homogeneous counterparts of conditions (57)–(60), namely they satisfy the following field equations in $V\times(0,\infty)$ [7]:

$$\rho_{11}\ddot{u}_i+\rho_{12}\ddot{U}_i=\sigma_{ij,j}, \tag{61}$$

$$\rho_{12}\ddot{u}_i+\rho_{22}\ddot{U}_i=\sigma_{,i}, \tag{62}$$

$$\mu u_{i,jj}+(\lambda+\mu)u_{j,ij}+QU_{i,ii}-R_{11}\theta^{s}_{,i}-R_{12}\theta^{f}_{,i}=\rho_{11}\ddot{u}_i+\rho_{12}\ddot{U}_i, \tag{63}$$

$$RU_{i,ii}+Qu_{j,ij}-R_{21}\theta^{s}_{,i}-R_{22}\,\theta^{f}_{,i}=\rho_{12}\ddot{u}_i+\rho_{22}\ddot{U}_i, \tag{64}$$

$$q^{s}_{i,i}=-\rho T_0\dot{\eta}^{s}, \tag{65}$$

$$q^{f}_{i,i}=-\rho T_0\dot{\eta}^{f}, \tag{66}$$

$$\sigma_{ij}=2\mu e_{ij}+\lambda e_{kk}\delta_{ij}+\left(Q\varepsilon-R_{11}\theta^{s}-R_{12}\theta^{f}\right)\delta_{ij}, \tag{67}$$

$$\sigma=R\varepsilon+Qe_{kk}-R_{22}\theta^{f}-R_{21}\theta^{s}, \tag{68}$$

$$q_i^{s}+\tau_0^{s}\dot{q}_i^{s}=-k^{s}\theta^{s}_{,i}, \tag{69}$$

$$q_i^{f}+\tau_0^{f}\dot{q}_i^{f}=-k^{f}\theta^{f}_{,i}, \tag{70}$$

$$\rho\eta^{s} = R_{11}e_{kk} + R_{21}\varepsilon + \frac{F_{11}}{T_0}\theta^{s} + \frac{F_{12}}{T_0}\theta^{f}, \tag{71}$$

$$\rho\eta^{f} = R_{12}e_{kk} + R_{22}\varepsilon + \frac{F_{21}}{T_0}\theta^{s} + \frac{F_{22}}{T_0}\theta^{f}, \tag{72}$$

$$k^{s}\theta^{s}_{,ii} = \left(\frac{\partial}{\partial t} + \tau^{s}_{o}\frac{\partial^2}{\partial t^2}\right)\left(F_{11}\theta^{s} + F_{12}\theta^{f} + T_0R_{11}e_{ii} + T_0R_{21}\varepsilon\right), \tag{73}$$

$$k^{f}\theta^{f}_{,ii} = \left(\frac{\partial}{\partial t} + \tau^{f}_{0}\frac{\partial^2}{\partial t^2}\right)\left(F_{21}\theta^{s} + F_{22}\theta^{f} + T_0R_{12}e_{ii} + T_0R_{22}\varepsilon\right), \tag{74}$$

where

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{75}$$

$$\varepsilon = U_{i,i} \tag{76}$$

together with the following boundary conditions:

$$u_i = 0,\ \ U_i = 0 \quad \text{on} \quad S_1 \times [0,\infty), \tag{77}$$

$$p_i = 0,\ \ \ell_i = 0 \quad \text{on} \quad S - S_1 \times [0,\infty), \tag{78}$$

$$\theta^{s}_{i} = 0,\ \ \theta^{f}_{i} = 0 \quad \text{on} \quad S, \tag{79}$$

where $S_1 \subset S$. Also, we have the following initial conditions for all V, at $t = 0$:

$$u_i = 0,\ \dot{u}_i = 0,\ U_i = 0,\ \dot{U}_i = 0,\ \theta^{s} = 0,\ \theta^{f} = 0,\ \dot{\theta}^{s} = 0,\ \dot{\theta}^{f} = 0. \tag{80}$$

Now, consider the following integral:

$$\int_V \sigma_{ij}\dot{e}_{ij}\,\mathrm{d}V + \int_V \sigma\dot{\varepsilon}\delta_{ij}\,\mathrm{d}V = \int_V \sigma_{ij}\dot{u}_{i,j}\,\mathrm{d}V + \int_V \sigma\dot{U}_{i,i}\delta_{ij}\,\mathrm{d}V. \tag{81}$$

Using integrating by parts, we get

$$\int_V \sigma_{ij}\dot{e}_{ij}\,\mathrm{d}V + \int_V \sigma\dot{\varepsilon}\delta_{ij}\,\mathrm{d}V = -\int_V \sigma_{ij,j}\dot{u}_i\,\mathrm{d}V - \int_V \sigma_{,i}\dot{U}_i\delta_{ij}\,\mathrm{d}V. \tag{82}$$

Hence, we obtain

$$\int_V \sigma_{ij}\dot{e}_{ij}\,\mathrm{d}V + \int_V \sigma\dot{\varepsilon}\delta_{ij}\,\mathrm{d}V + \int_V \sigma_{ij,j}\dot{u}_i\,\mathrm{d}V + \int_V \sigma_{,i}\dot{U}_i\delta_{ij}\,\mathrm{d}V = 0. \tag{83}$$

Upon inserting Eqs. (61) and (62), the latter equation reduces to

$$\int_V \sigma_{ij}\dot{e}_{ij}\,\mathrm{d}V + \int_V \sigma\dot{\varepsilon}\delta_{ij}\,\mathrm{d}V + \int_V \left(\rho_{11}\ddot{u}_i + \rho_{12}\ddot{U}_i\right)\dot{u}_i\,\mathrm{d}V + \int_V \left(\rho_{12}\ddot{u}_i + \rho_{22}\ddot{U}_i\right)\dot{U}_i\delta_{ij}\,\mathrm{d}V = 0. \tag{84}$$

Using Eq. (66), we get

$$\int_V \left(2\mu e_{ij} + \lambda e_{kk}\delta_{ij} + \left(Q\varepsilon - R_{11}\theta^{s} - R_{12}\theta^{f}\right)\delta_{ij}\right)\dot{e}_{ij}\,\mathrm{d}V + \int_V \left(R\varepsilon + Qe_{kk} - R_{22}\theta^{f} - R_{21}\theta^{s}\right)\dot{\varepsilon}\delta_{ij}\,\mathrm{d}V + \int_V \left(\rho_{11}\ddot{u}_i + \rho_{12}\ddot{U}_i\right)\dot{u}_i\,\mathrm{d}V + \int_V \left(\rho_{12}\ddot{u}_i + \rho_{22}\ddot{U}_i\right)\dot{U}_i\delta_{ij}\,\mathrm{d}V = 0, \tag{85}$$

This can be written as follows:

$$\int_V \left[2\mu e_{ij}\dot{e}_{ij} + \lambda e_{kk}\dot{e}_{ij}\delta_{ij} + Q\varepsilon\dot{e}_{ij}\delta_{ij} + R\varepsilon\dot{\varepsilon} + Q\dot{\varepsilon}e_{kk} + \rho_{11}\dot{u}_i\dot{u}_i + \rho_{22}\dot{U}_i\dot{U}_i + \rho_{12}\dot{u}_i\ddot{U}_i + \rho_{12}\ddot{u}_i\dot{U}_i\right]\mathrm{d}V - \int_V \theta^{s}\left(R_{11}\dot{e}_{ij} + R_{21}\dot{\varepsilon}\right) + \theta^{f}\left(R_{12}\dot{e}_{ij} + R_{22}\dot{\varepsilon}\right)\big]\,\mathrm{d}V = 0. \tag{86}$$

Substituting from Eqs. (73) and (74), we get

$$T_0\frac{\mathrm{d}Y}{\mathrm{d}t} + \int_V \Big[k^{s}\theta^{s}_{,i}\theta^{s}_{,i} + k_{f}\theta^{f}_{,i}\theta^{f}_{,i} + \tau^{s}_{o}\Big(F_{11}\theta^{s}\ddot{\theta}^{s} + F_{12}\theta^{s}\ddot{\theta}^{f} + T_0R_{11}\theta^{s}\ddot{e}_{ii} + T_0R_{21}\ddot{\varepsilon}\theta^{s}\Big) + \tau^{f}_{o}\Big(F_{22}\theta^{f}\ddot{\theta}^{f} + F_{21}\theta^{f}\ddot{\theta}^{s} + T_0R_{12}\theta^{f}\ddot{e}_{ii} + T_0R_{22}\ddot{\varepsilon}\theta^{f}\Big)\Big]\,\mathrm{d}V = 0, \tag{87}$$

where

$$Y = \int_V \left[\mu e_{ij}e_{ij} + \frac{\lambda}{2}e_{kk}e_{kk} + Q\varepsilon e_{kk} + \frac{R}{2}\varepsilon^2 + \rho_{11}\dot{u}_i\dot{u}_i + \rho_{22}\dot{U}_i\dot{U}_i + \rho_{12}\dot{u}_i\dot{U}_i + \frac{F_{11}}{2T_0}\theta^{s2} + \frac{F_{22}}{2T_0}\theta^{f2} + \frac{F_{12}}{T_0}\theta^{s}\theta^{f}\right]\mathrm{d}V.$$

From the known inequalities

$$-q^{s}_{i}\theta^{s}_{,i} \geqslant 0, \tag{88}$$

$$-q^{f}_{i}\theta^{f}_{,i} \geqslant 0, \tag{89}$$

Using Eqs. (69)–(72) with the above two inequalities, we get

$$\int_V \Big[k_{s}\theta^{s}_{,i}\theta^{s}_{,i} + k_{f}\theta^{f}_{,i}\theta^{f}_{,i} + \tau^{s}_{o}\Big(F_{11}\theta^{s}\ddot{\theta}^{s} + F_{12}\theta^{s}\ddot{\theta}^{f} + T_0R_{11}\theta^{s}\ddot{e}_{ii} + T_0R_{21}\ddot{\varepsilon}\theta^{s}\Big) + \tau^{f}_{o}\Big(F_{22}\theta^{f}\ddot{\theta}^{f} + F_{21}\theta^{f}\ddot{\theta}^{s} + T_0R_{12}\theta^{f}\ddot{e}_{ii} + T_0R_{22}\ddot{\varepsilon}\theta^{f}\Big)\Big]\,\mathrm{d}V \geqslant 0. \tag{90}$$

Hence, we have

$$T_0\frac{\mathrm{d}Y}{\mathrm{d}t} = T_0\frac{\mathrm{d}}{\mathrm{d}t}\int_V \left[\mu e_{ij}e_{ij} + \frac{\lambda}{2}e_{kk}e_{kk} + Q\varepsilon e_{kk} + \frac{R}{2}\varepsilon^2 + \rho_{11}\dot{u}_i\dot{u}_i + \rho_{22}\dot{U}_i\dot{U}_i + \rho_{12}\dot{u}_i\dot{U}_i + \frac{F_{11}}{2T_0}\theta^{s2} + \frac{F_{22}}{2T_0}\theta^{f2} + \frac{F_{12}}{T_0}\theta^{s}\theta^{f}\right]\mathrm{d}V \leqslant 0. \tag{91}$$

The integral in the left-hand side of Eq. (91) is initially zero, since the difference functions satisfy homogeneous initial conditions. By inequality (91), however, this integral either decreases (or therefore becomes negative) or remains equal to zero. Since the integral is the sum of squares and positive terms, only the latter alternative is possible, i.e.,

$$\int_V \left[\mu e_{ij} e_{ij} + \frac{\lambda}{2} e_{kk} e_{kk} + Q \varepsilon e_{kk} + \frac{R}{2} \varepsilon^2 + \rho_{11} \dot{u}_i \dot{u}_i + \rho_{22} \dot{U}_i \dot{U}_i + \rho_{12} \dot{u}_i \dot{U}_i + \frac{F_{11}}{2T_0} \theta^{s2} + \frac{F_{22}}{2T_0} \theta^{f2} + \frac{F_{12}}{T_0} \theta^s \theta^f \right] \mathrm{d}V = 0. \quad (92)$$

It follows that the difference functions are identically zero throughout the body, for all time. Hence, the two assumed solutions were in fact identical, and therefore only one solution exists. This completes the proof of the theorem.

References

[1] Pecker C, Deresiewiez H. Thermal effects on waves in liquid-filled porous media. Acta Mech 1973;16:45–64.
[2] Biot MA. General theory of three-dimensional consolidation. J Appl Phys 1941;12:155–64.
[3] Biot MA. Theory of propagation of elastic waves in a fluid-saturated porous solid. I. Low-frequency range. J Acoust Soc Am 1956;28:168–78.
[4] Gassmann F. Uber die elastizitat poroser medien. Veirt Naturforsch Gesell Zürich 1951;96:1–23.
[5] Biot MA, Willis DG. The elastic coefficients of the theory of consolidation. J Appl Mech 1957;24:594–601.
[6] Biot MA. Mechanics of deformation and acoustic propagation in porous media. J Appl Phys 1962;33:1482–98.
[7] Deresiewicz H, Skalak R. On uniqueness in dynamic poroelasticity. Bull Seismol Soc Am 1963;53:783–8.
[8] Mandl G. Change in skeletal volume of a fluid-filled porous body under stress. J Mech Phys Solids 1964;12:299–315.
[9] Nur A, Byerlee JD. An exact effective stress law for elastic deformation of rock with fluids. J Geophys Res 1971;76:6414–9.
[10] Brown RJS, Korringa J. On the dependence of the elastic properties of a porous rock on the compressibility of the pore fluid. Geophysics 1975;40:608–16.
[11] Rice JR, Cleary MP. Some basic stress diffusion solutions for fluid-saturated elastic porous media with compressible constituents. Rev Geophys Space Phys 1976;14:227–41.
[12] Burridge R, Keller JB. Poroelasticity equations derived from microstructure. J Acoust Soc Am 1981;70:1140–6.
[13] Zimmerman RW, Somerton WH, King MS. Compressibility of porous rocks. J Geophys Res 1986;91:12765–77.
[14] Zimmerman RW, Myer LR, Cook NGW. Grain and void compression in fractured and porous rock. Int J Rock Mech Min Sci Geomech Abstr 1994;31:179–84.
[15] Berryman JG, Milton GW. Exact results for generalized Gassmann's equations in composite porous media with two constituents. Geophysics 1991;56:1950–60.
[16] Thompson M, Willis JR. A reformulation of the equations of anisotropic poroelasticity. J Appl Mech 1991;58:612–6.
[17] Pride SR, Gangi AF, Morgan FD. Deriving the equations of motion for porous isotropic media. J Acoust Soc Am 1992;92:3278–90.
[18] Berryman JG, Wang HF. The elastic coefficients of double-porosity models for fluid transport in jointed rock. J Geophys Res 1995;100:24611–27.
[19] Tuncay K, Corapcioglu MY. Effective stress principle for saturated fractured porous media. Water Resour Res 1995;31:3103–6.
[20] Cheng AHD. Integral equation for poroelasticity in frequency domain with BEM solution. J Eng Mech 1991;117(5):1136–57.
[21] Charlez PA, Heugas O. Measurement of thermoporoelastic properties of rocks: theory and applications. In: Hudson JA, editor, Proceedings of the Eurock '92, 1992. p. 42–6.
[22] Abousleiman Y, Cui L. Poroelastic solutions in transversely isotropic media for wellbore and cylinder. Int J Solids Struct 1998;35:4905–6.
[23] Ghassemi A, Diek A. Porothermoelasticity for swelling shales. J Petrol Sci Eng 2002;34:123–35.
[24] Tod SR. An anisotropic fractured poroelastic effective medium theory. Geophys J Int 2003;155:1006–20.
[25] Lord H, Shulman Y. A generalized dynamical theory of thermoelasticity. J Mech Phys Solids 1967;15:299–309.
[26] Coussy O. Poromechanics. New York: Wiley; 2004.
[27] Norris A. On the correspondence between poroelasticity and thermoelasticity. J Appl Phys 1992;71:1138–41.