MTH 451/EGR 551 - Homework 2

1. In this problem we will evaluate the errors in the solution of a linear system of equations by performing the following steps.

   - For $n = 1, 2, \cdots, 15$.
   - Construct the n by n matrix H with elements
   $$h_{ij} = 1/(i + j - 1),\ i, j = 1, 2, \cdots, n.$$
   - Construct the vector $\bar{x} \in \mathcal{R}^n$ with elements $\bar{x}_i = 1 + i - (n + 1)/2,\ i = 1, 2, \cdots, n$.
   - Construct the the vector $b = H\bar{x}$.
   - Solve the linear system $Hx = b$. (You can use the basic LU factorization algorithm developed in class).
   - Compute the absolute error $ea_n = \|\bar{x} - x\|_\infty$, and the relative error $er_n = ea_n/\max(1, \|\bar{x}\|_\infty)$.

   Plot ea_n and er_n versus n, and comment on the accuracy of the numerical solutions.

2. The matrix $A \in \mathcal{R}^{n\times n}$ is said to be tridiagonal if $a_{ij} = 0$, whenever $|i - j| > 1$. Thus, the tridiagonal matrix A has the form

$$A = \begin{bmatrix} a_{11} & a_{12} & & & 0 \\ a_{21} & a_{22} & \ddots & & \vdots \\ & \ddots & \ddots & \ddots & \\ \vdots & & \ddots & \ddots & a_{(n-1)n} \\ 0 & \cdots & & a_{n(n-1)} & a_{nn} \end{bmatrix}.$$

   (a) Develop a algorithm to factor A into a unit diagonal lower triangular matrix L and an upper triangular matrix U. Of course your algorithm should not perform operations related to the zero elements in A.

   (b) Implement your algorithm and use it to solve the linear system,
$$\begin{aligned} 2x_1 - x_2 &= 1, \\ -x_{i-1} + 2x_i - x_{i+1} &= 0,\ i = 2, 3, \cdots, n-1, \\ -x_{n-1} + 2x_n &= 0, \end{aligned}$$
   for the cases where $n = 10$ and $n = 100$. Note that the implementation of your algorithm should only require the diagonal elements, the sub-diagonal elements and the super-diagonal elements of A, i.e., there is no need to store all the zeros.

   (c) How many operations are required to compute the LU factors? How many operations are required to obtain the solution to the linear system once L and U are known?

3. Develop and implement a Cholesky factorization algorithm that produces $A = U^TU$, where A is a symmetric, positive definite matrix, and U is an upper triangular matrix. Use your

implementation to factor the follow matrix

$$A = \begin{bmatrix} 5.76726 & 0.75166 & 0.36371 & 0.68326 & 0.36536 \\ 0.75166 & 5.67889 & 0.23914 & 0.63509 & 0.14912 \\ 0.36371 & 0.23914 & 5.90905 & 0.36502 & 0.82444 \\ 0.68326 & 0.63509 & 0.36502 & 5.51865 & 0.11787 \\ 0.36536 & 0.14912 & 0.82444 & 0.11787 & 5.81971 \end{bmatrix}$$

and solve the linear system $Ax = b$ where $b = [1, 2, 3, 4, 5]^T$

4. Implement the Classical Gram-Schmidt Algorithm (Chapter 2, page 44) and the Modified Gram-Schmidt Agorithm (Chapter 2, page 47) in MATLAB or Octave. Use the implementations to find the QR factors of the following matrices.

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 611 & 196 & -192 & 407 & -8 & -52 & -49 & 29 \\ 196 & 899 & 113 & -192 & -71 & -43 & -8 & -44 \\ -192 & 113 & 899 & 196 & 61 & 49 & 8 & 52 \\ 407 & -192 & 196 & 611 & 8 & 44 & 59 & -23 \\ -8 & -71 & 61 & 8 & 411 & -599 & 208 & 208 \\ -52 & -43 & 49 & 44 & -599 & 411 & 208 & 208 \\ -49 & -8 & 8 & 59 & 208 & 208 & 99 & -911 \\ 29 & -44 & 52 & -23 & 208 & 208 & -911 & 99 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1.000000 & 0.500000 & 0.333333 & 0.250000 & 0.200000 & 0.166667 \\ 0.500000 & 0.333333 & 0.250000 & 0.200000 & 0.166667 & 0.142857 \\ 0.333333 & 0.250000 & 0.200000 & 0.166667 & 0.142857 & 0.125000 \\ 0.250000 & 0.200000 & 0.166667 & 0.142857 & 0.125000 & 0.111111 \\ 0.200000 & 0.166667 & 0.142857 & 0.125000 & 0.111111 & 0.100000 \\ 0.166667 & 0.142857 & 0.125000 & 0.111111 & 0.100000 & 0.090909 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}.$$

In each case determine the error $\|QR - A\|_\infty$ and the error $\|Q^TQ - I\|_\infty$ to compare the performance of the algorithms.