Monday May 3, 2021 at 5 p.m.

You are not allowed to collaborate with anyone for this exam.

Please submit all computer codes you developed to solve these problem.

1. **[15 points]**
   The loop-closure equations for a four-bar mechanism are as follows.

$$\begin{aligned} f_1 &= r_1 \cos\theta_1 + r_2 \cos\theta_2 - r_3 \cos\theta_3 - r_4 = 0, \\ f_2 &= r_1 \sin\theta_1 + r_2 \sin\theta_2 - r_3 \sin\theta_3 = 0. \end{aligned}$$

   A point P on the coupler has coordinates

$$\begin{aligned} x_P &= r_1 \cos\theta_1 + r_5 \cos(\theta_2 + \Delta), \\ y_P &= r_1 \sin\theta_1 + r_5 \sin(\theta_2 + \Delta). \end{aligned}$$

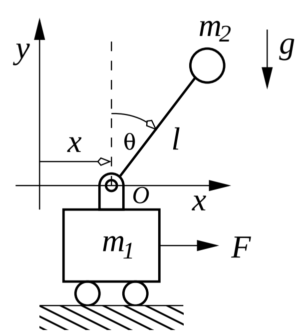
   The model constants are $r_1 = 0.963$, $r_2 = 0.764$, $r_3 = 0.528$, $r_4 = 1.815$, $r_5 = 0.778$, $\Delta = -89.65°$.

   (a) Solve the loop-closure equations for $-40° \le \theta_1 \le 40°$. Use a convergence tolerance $\epsilon_t = 10^{-6}$.

   (b) Plot (x_P, y_P), (θ_1, θ_2) and (θ_1, θ_3).

   (c) Print the values of (x_P, y_P) for $\theta_1 = -40°, -30°, -20°, -10°, 0°, 10°, 20°, 30°, 40°$.

2. **[20 points]**
   The differential equations that determine the behavior of an inverted pendulum system is given as follows.



$$\begin{aligned} (m_1 + m_2)\ddot{x} + m_2 l(\ddot{\theta}\cos\theta - \dot{\theta}^2 \sin\theta) &= F, \\ m_2(l^2\ddot{\theta} + l\ddot{x}\cos\theta) &= m_2 g l \sin\theta. \end{aligned}$$

   The model parameters are $m_1 = 0.5$, $m_2 = 0.1$, $l = 0.25$ and $g = 9.8$.
   Using the `rk34` implementation from Homework 10, or the `rk23` implementation developed in class, or the ROW method develop in problem 3 below;

   (a) Solve the differential equations in the interval $0 \le t \le 3$ with initial conditions $x(0) = 0$, $\dot{x}(0) = 0$, $\theta(0) = 0$, $\dot{\theta}(0) = 0.1$, and $F = 0$. Plot (t, x), $(t, \dot{x})$, (t, θ), $(t, \dot{\theta})$. Print the values of $x, \dot{x}, \theta$ and $\dot{\theta}$ at $t = 3$.

   (b) Solve the differential equations in the interval $0 \le t \le 3$ with initial conditions $x(0) = 0$, $\dot{x}(0) = 0$, $\theta(0) = 0$, $\dot{\theta}(0) = 0.1$, and $F = 133.98x + 115\theta + 52.296\dot{x} + 18.074\dot{\theta}$. Plot (t, x), $(t, \dot{x})$, (t, θ), $(t, \dot{\theta})$. Print the values of $x, \dot{x}, \theta$ and $\dot{\theta}$ at $t = 3$.

3. **[25 points]**
   This problem uses a linearly implicit Runge-Kutta method to solve a system of ordinary differential equations of the form

$$\dot{y} = f(y(t)), \ t \in [t_0, t_f], \quad y(t_0) = y_0,$$

where $y(t) \in \mathcal{R}^n$. Linearly implicit methods are very useful for solving stiff ODEs. These methods are also called Rosenbrock-Wanner methods, or ROW methods.

The following 4-th order ROW method is presented in L. F. Shampine, *Implementation of Rosenbrock Methods*, ACM Transactions on Mathematical Software, Vol. 8, pp. 93-113, 1982.

Input: $f(y)$, y_i, t_i, t_f, h_i, ϵ_t
Output: $y(t)$, $t_i \le t \le t_f$

$h_{\min} = 10^{-9}$, $\beta = 0.9$, $\texttt{fac}_0 = 0.2$, $\texttt{fac}_1 = 5$, $p = 4$
$t_0 = t_i$, $y_0 = y_i$, $h = h_i$, $n = \text{length}(y_i)$
for $j = 0, 1, \cdots$ **do**
  $J = \partial f(y_0)/\partial y$
  $E = I - \frac{1}{2}hJ$
  Solve $Ek_1 = f(y_0)$
  $y_2 = y_0 + hk_1$
  Solve $Ek_2 = f(y_2) - 4k_1$
  $y_3 = y_0 + \frac{24}{25}hk_1 + \frac{3}{25}hk_2$
  Solve $Ek_3 = f(y_3) + \frac{186}{25}k_1 + \frac{6}{5}k_2$
  $y_4 = y_0 + \frac{24}{25}hk_1 + \frac{3}{25}hk_2$
  $Ek_4 = f(y_4) - \frac{56}{125}k_1 - \frac{27}{125}k_2 - \frac{1}{5}k_3$
  $y = y_0 + h\left(\frac{19}{18}k_1 + \frac{1}{4}k_2 + \frac{25}{216}k_3 + \frac{125}{216}k_4\right)$
  $\eta = h(\frac{17}{108}k_1 + \frac{7}{72}k_2 + \frac{125}{216}k_4)$
  $\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{|\eta_i|}{\epsilon_t + \epsilon_t|y_{0i}|}\right)^2}$
  $\hat{h} = h\min(\texttt{fac}_1, \max(\texttt{fac}_0, \beta\sigma^{-1/p}))$
  if $\sigma \le 1$ **then**
    if $t = t_f$, **then** $y(t_f) = y$, **return end if**
    $y_0 = y$
    $t_0 = t$
    if $t + \hat{h} > t_f$, **then** $\hat{h} = t_f - t$ **end if**
  end if
  $h = \hat{h}$
  if $h < h_{\min}$, **then return end if**, stepsize too small
end for
return, too many iterations

Here, $J \in \mathcal{R}^{n\times n}$ is the Jacobian of $f(y)$, and $I \in \mathcal{R}^{n\times n}$ is the identity matrix. Also, note that the stage derivatives, $(k_1, k_2, k_3$, and $k_4)$, are deterimed by solving a linear system of equations.

Implement this ROW method and use it to solve the following problems. Consider the ODEs,

$$\dot{y}_1 = y_2,\ \dot{y}_2 = \mu(1 - y_1^2)y_2 - y_1,\ y_1(0) = 2,\ y_2(0) = 4.$$

(I) Integrate the ODEs using $\mu = 10$, and the final time $t_f = 50$.

(II) Integrate the ODEs using $\mu = 100$, and the final time $t_f = 500$.

(III) Integrate the ODEs using $\mu = 1000$, and the final time $t_f = 5000$.

In each case use a tolerance $\epsilon_t = 10^{-6}$,

(a) Plot $(t, y_1(t))$, $(t, y_2(t))$ and $(y_1(t), y_2(t))$.

(b) Print the values of y_1 and y_2 at the final time.

4. **[40 points]**
In the time interval $t \in [0, 1]$ let $u_1(t) = \sum_{k=1}^{p} z_k t^{k-1}$ and $u_2(t) = \sum_{k=1}^{p} z_{k+p} t^{k-1}$, where $z_1, z_2, \ldots, z_{2p}$ are unknowns.

In fact, we would like to find $z \in \mathcal{R}^{2p}$ that minimizes the cost function

$$J(z) = q_5(1) + \rho(x_2(1)^2 + (y_2(1) - 0.5)^2 + q_2(1)^2 + q_4(1)^2)$$

where

$$\begin{aligned}
\dot{q}_1(t) &= q_2(t) \\
\dot{q}_2(t) &= -cq_2(t)^2 + u_1(t) \\
\dot{q}_3(t) &= q_4(t) \\
\dot{q}_4(t) &= -cq_4(t)^2 + u_2(t) \\
y_2(t) &= l_1 \sin q_1(t) + l_2 \sin q_3(t) \\
s(t) &= \max(y_2(t) - 0.5, 0) \\
\dot{q}_5(t) &= \frac{1}{2}(u_1(t)^2 + u_2(t)^2) + \rho s(t)^2
\end{aligned}$$

Also,

$$x_2(t) = l_1 \cos q_1(t) + l_2 \cos q_3(t),$$

$q_1(0) = 0$, $q_2(0) = 0$, $q_3(0) = 0$, $q_4(0) = 0$ and $q_5(0) = 0$.

Using $l_1 = 0.4$, $l_2 = 0.3$, $c = 10^{-3}$ and $\rho = 10^3$;

(a) Solve the optimization problem for $p = 5$, $p = 10$ and $p = 15$. In each case set the convergence tolerance to $\epsilon_t = 10^{-6}$

(b) Plot $(t, u_1(t))$ and $(t, u_2(t))$ in the interval $0 \le t \le 1$.

(c) Plot $(x_2(t), y_2(t))$ in the interval $0 \le t \le 1$.