MTH 451/EGR 551 - Homework 9
Due Monday April 26, 2021 at 5 p.m.
Please submit all computer codes you developed to solve these problems.

In this assignment you are required to implement the classical 4-order explicit Runge-Kutta method, which can be described as follows.

Input: $f(y, t)$, $N > 0$, $h = (t_f - t_i)/N$, t_i, y_i
Output: $y^{(k)}$, $k = 0, 1, 2, \cdots, N$

$t^{(0)} = t_i$, $y^{(0)} = y_i$
for $k = 0, 1, \cdots, N-1$ **do**
  $k_1 = f(y^{(k)}, t^{(k)})$
  $y_2 = y^{(k)} + \frac{1}{2}hk_1$
  $\tau_2 = t^{(k)} + \frac{1}{2}h$
  $k_2 = f(y_2, \tau_2)$
  $y_3 = y^{(k)} + \frac{1}{2}hk_2$
  $\tau_3 = t^{(k)} + \frac{1}{2}h$
  $k_3 = f(y_3, \tau_3)$
  $y_4 = y^{(k)} + hk_3$
  $\tau_4 = t^{(k)} + h$
  $k_4 = f(y_4, \tau_4)$
  $y^{(k+1)} = y^{(k)} + h(\frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4)$
  $t^{(k+1)} = t^{(k)} + h$
end for

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1. Apply the explicit Euler method, the implicit Euler method and the classical 4-th order Runge-Kutta method to the following problems.

   (a) $\dot{x} = -0.1x + 1$, $x(0) = 0$, and $0 \le t \le 1$.

   (b) $\dot{x}_1 = x_2$, $\dot{x}_2 = -2\xi\omega_n x_2 - \omega_n^2 x_1$, $x_1(0) = 1$, $x_2(0) = 0$, $0 \le t \le 1$, $\xi = 0.1$, and $\omega_n = 2\pi$.

   For each of the numerical methods use the following step sizes; $h = 10^{-1},\ 10^{-2},\ 10^{-3},\ 10^{-4}$. Compare the numerical solutions with the exact solution in each case. Plot the solution.

   For part (b) the exact solution for $x_1(t)$ is

   $$x_1(t) = x_1(0)e^{-\xi\omega_n t}\left(\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right) + \frac{x_2(0)}{\omega_d}e^{-\xi\omega_n t}\sin\omega_d t,$$

   where $\omega_d = \omega_n\sqrt{1-\xi^2}$. Also, $x_2(t) = \dot{x}_1(t) = \frac{dx_1}{dt}$.

2. Use the classical 4-th order Runge-Kutta method to solve the differential equation

   $$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= \mu(1 - y_1^2)y_2 - y_1, \end{aligned}$$

   with initial conditions $y(0) = [2,\ 4]^T$. Solve this problem for the case $\mu = 1$ and final time $t = 20$. Plot (i) $(t, y_1(t))$, (ii) $(t, y_2(t))$, and (iii) $(y_1(t), y_2(t))$.