MTH 451/EGR 551 - Homework 10
Due Monday May 3, 2021 at 5 p.m.
Please submit all computer codes you developed to solve these problems.

In this assignment you are required to implement the classical 4-th order explicit Runge-Kutta method with an embedded 3-rd order error estimator.

Input: $f(y,t)$, y_i, t_i, t_f, h_i, ϵ_t
Output: $y(t)$, $t_i \le t \le t_f$

$h_{\min} = 10^{-9}$, $\beta = 0.9$, $\texttt{fac}_0 = 0.2$, $\texttt{fac}_1 = 5$, $p = 4$
$t_0 = t_i$, $y_0 = y_i$, $h = h_i$, $n = \text{length}(y_i)$
$k_1 = f(y_0, t_0)$
for $j = 0, 1, \cdots$ **do**
    $y_2 = y_0 + \frac{1}{2}hk_1$
    $\tau_2 = t_0 + \frac{1}{2}h$
    $k_2 = f(y_2, \tau_2)$
    $y_3 = y_0 + \frac{1}{2}hk_2$
    $\tau_3 = t_0 + \frac{1}{2}h$
    $k_3 = f(y_3, \tau_3)$
    $y_4 = y_0 + hk_3$
    $\tau_4 = t_0 + h$
    $k_4 = f(y_4, \tau_4)$
    $y = y_0 + h(\frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4)$
    $t = t_0 + h$
    $k_5 = f(y, t)$
    $\eta = (h/6)(k_4 - k_5)$
    $\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{|\eta_i|}{\epsilon_t + \epsilon_t|y_{0i}|}\right)^2}$
    $\hat{h} = h\min(\texttt{fac}_1, \max(\texttt{fac}_0, \beta\sigma^{-1/p}))$
    if $\sigma \le 1$ **then**
        if $t = t_f$, **then** $y(t_f) = y$, **return end if**
        $y_0 = y$
        $t_0 = t$
        $k_1 = k_5$
        if $t + \hat{h} > t_f$, **then** $\hat{h} = t_f - t$ **end if**
    end if
    $h = \hat{h}$
    if $h < h_{\min}$, **then return end if**, stepsize too small
end for
return, too many iterations

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1. Use your implementation of this algorithm to solve the differential equations.
   (a) $\dot{x} = -0.1x + 1$, $x(0) = 0$, and $0 \le t \le 1$.
   (b) $\dot{x}_1 = x_2$, $\dot{x}_2 = -2\xi\omega_n x_2 - \omega_n^2 x_1$, $x_1(0) = 1$, $x_2(0) = 0$, $0 \le t \le 1$, $\xi = 0.1$, and $\omega_n = 2\pi$.

Use $\epsilon_t = 10^{-6}$. Compare the numerical solutions with the exact solution in each case. Plot the solution.

2. Use your implementation of this algorithm to solve the differential equation

$$\begin{aligned}\dot{y}_1 &= y_2, \\ \dot{y}_2 &= \mu(1 - y_1^2)y_2 - y_1,\end{aligned}$$

with initial conditions $y(0) = [2,\ 4]^T$ and $\epsilon_t = 10^{-6}$. Solve this problem for the case $\mu = 1$ and final time $t = 20$. Plot (i) $(t, y_1(t))$, (ii) $(t, y_2(t))$, and (iii) $(y_1(t), y_2(t))$.