MTH 451/EGR 551 - Homework 8
Due Monday April 19, 2021 at 5 p.m.
Please submit all computer codes you developed to solve these problems.

In this assignment you are required to implement the explicit Euler method and the implicit Euler method to solve a system of differential equations. These algorithms are described below.

Algorithm 0.1 Explicit Euler Method

Input: $f(y,t)$, $N > 0$, $h = (t_f - t_i)/N$, t_i, y_i.
Output: $y^{(k)}$, $k = 0, 1, 2, \cdots, N$.

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t^(0) = t_i, y^(0) = y_i
for k = 0, 1, ..., N-1 do
    y^(k+1) = y^(k) + h f(y^(k), t^(k))
    t^(k+1) = t^(k) + h
end for
```

$t^{(0)} = t_i$, $y^{(0)} = y_i$
for $k = 0, 1, \cdots, N-1$ **do**
  $y^{(k+1)} = y^{(k)} + hf(y^{(k)}, t^{(k)})$
  $t^{(k+1)} = t^{(k)} + h$
end for

Algorithm 0.2 Implicit Euler Method

Input: $f(y,t)$, $N > 0$, $h = (t_f - t_i)/N$, t_i, y_i.
Output: $y^{(k)}$, $k = 0, 1, 2, \cdots, N$.

Define $\Psi(y^{(k+1)}) = y^{(k+1)} - y^{(k)} - hf(y^{(k+1)}, t^{(k+1)})$
$t^{(0)} = t_i$, $y^{(0)} = y_i$
for $k = 0, 1, \cdots, N-1$ **do**
  $t^{(k+1)} = t^{(k)} + h$
  Solve $\Psi(y^{(k+1)}) = 0$ via Newton's method to obtain $y^{(k+1)}$
end for

Newton's method

Input: Given $y^{(k+1)}_{(0)}$ and a small number $\epsilon_N > 0$.
Output: $y^{(k+1)}$ that solves $\Psi(y^{(k+1)}) = 0$.

$D\Psi = \frac{\partial}{\partial y^{(k+1)}_{(j)}} \Psi(y^{(k+1)}_{(j)}) = I - h\frac{d}{dy} f(y^{(k+1)}_{(j)}, t^{(k)})$
for $j = 0, 1, \cdots$ **do**
  if $\|\Psi(y^{(k+1)}_{(j)})\| \leq \epsilon_N$ **stop**, $y^{(k+1)} = y^{(k+1)}_{(j)}$
  Solve $D\Psi \Delta y = -\Psi(y^{(k+1)}_{(j)})$
  Set $y^{(k+1)}_{(j+1)} = y^{(k+1)}_{(j)} + \Delta y$
end for

1. Apply the explicit and implicit Euler methods to solve the differential equations

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -1000 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

with initial conditions $y(0) = [1, -2]^T$, in the interval $0 \leq t \leq 5$. Use step sizes (i) $h = 0.1$, (ii) $h = 0.01$ and (iii) $h = 0.001$. Compare the numerical results with the exact solution in each case.

2. Use the explicit and implicit Euler methods to solve the ODE

$$\dot{y}(t) = -50(y(t) - \cos t),\ \ y(0) = 0.$$

Use step sizes (i) $h = 1.974/50$ and (ii) $h = 1.875/50$ for $t = 0$ to $t = 1.5$. Plot $y(t)$ in each case (4 plots).