Chi-square test

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May 6, 2021

1 Definition [1, 2]

The Pearson's χ^2-test is a test used for hypothesis concerning probabilities of multinomial random variables. It tests a null hypothesis stating that the frequency distribution of certain events observed in a sample is consistent with a particular theoretical distribution.

The test statistic is of following kind

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i} = N \sum_{i=1}^{n} \frac{(O_i/N - p_i)^2}{p_i} \tag{1}$$

where

- χ^2 = Pearson's cumulative test statistic, which asymptotically approaches a χ^2 distribution.
- O_i = number of observation of type i
- N = total number of observations
- $E_i = Np_i$ = the expected (theoretical) count of type i
- n = the number of cells in the table $(rows \cdot columns)$

If the test is fasle, then the appropriate test statistic has approximately a noncentral χ^2 distribution with the same degrees of freedom df and a noncentrality parameter λ, which depends on alternative considered.

2 Test types [1]

Pearson's chi-squared test is used to assess three types of comparison: goodness of fit, homogeneity, and independence.

- A test of **goodness of fit** establishes whether an observed frequency distribution differs from a theoretical distribution.
  χ^2 test statistic with degrees of freedom $df = n - 1$
- A test of **homogeneity** compares the distribution of counts for two or more groups using the same categorical variable.
  χ^2 test statistic with degrees of freedom $df = (rows - 1) \cdot (columns - 1)$
- A test of **independence** assesses whether observations consisting of measures on two variables, expressed in a contingency table, are independent of each other.
  χ^2 test statistic with degrees of freedom $df = (rows - 1) \cdot (columns - 1)$

2.1 Test of goodness of fit

Let $X = (X_1, ..., X_k)$ be a multinomial ranom variable with parameters $n, p_1, ..., p_k$. Suppose we wish to test

$$H_0 : \quad p_i = p_{i,e} \qquad i = 1, 2, ..., k \tag{2}$$

against

$$H_a : \text{not all p's are as given by } H_0 \tag{3}$$

where the $p_{i,e}$ are given expected numbers. The value of the chi-square test-statistic is

$$\chi^2_{H_0} = \sum_{i=1}^{k} \frac{(x_i - np_{i,e})^2}{np_{i,e}} = \sum_{i=1}^{k} \frac{(p_i - p_{i,e})^2}{p_{i,e}} \tag{4}$$

The chi-square test reject H_0 if

$$\chi^2_{H_0} > \chi^2_{k-1,1-\alpha}, \tag{5}$$

where α is the significance level, and $\chi^2_{k-1,1-\alpha}$ is the quantile of order $1-\alpha$ of the χ^2 distribution with $k-1$ degrees of freedom. The p-value of the test is

$$p - value = P\left(X < \chi^2_{H_0}\right) \tag{6}$$

To evaluate the power of the test let's precisely define the alternative H_a as follow.

$$H_a: \quad p_i = p_{i,a} \qquad i = 1, 2, ..., k \tag{7}$$

Thus the power o the test is

$$Power = P\left(X_a > \chi^2_{k-1,1-\alpha}\right) \tag{8}$$

where X_a is a ranom variable that follows the noncentral χ^2 distribution with the noncentrality parameter

$$\lambda = n \sum_{i=1}^{k} \frac{(p_{i,a} - p_{i,e})^2}{p_{i,a}} \tag{9}$$

2.2 Test of independence

Let $X = (X_{ij}) \in \mathbb{R}^{r \times c}$ be a multinomial random variable with parameters n, p_{ij} where $i = 1, 2, ..., r,\ j = 1, 2, ..., c$ and $\sum_{i=1}^{r}\sum_{j=1}^{c} p_{ij} = 1$. Suppopse we wish to test independence

$$H_0: \quad p_{ij} = p_{ij,e} = p_{i\cdot,e}\,p_{\cdot j,e} \qquad i = 1, 2, ..., r \quad j = 1, 2, ..., c \tag{10}$$

against

$$H_a: \text{not all the equatons given under } H_0 \text{ are satisfied} \tag{11}$$

where $p_{i\cdot} = \sum_{j=1}^{c} p_{ij}$ and $p_{\cdot j} = \sum_{i=1}^{r} p_{ij}$. The value of the chi-square test-statistic is

$$\chi^2_{H_0} = \sum_{i=1}^{r}\sum_{j=1}^{c} \frac{(x_{ij} - x_{i\cdot}x_{\cdot j}/n)^2}{x_{i\cdot}x_{\cdot j}/n} \tag{12}$$

where $x_{i\cdot} = \sum_{j=1}^{c} x_{ij}$ and $x_{\cdot j} = \sum_{i=1}^{r} x_{ij}$. Using parameters p_* we observe that

$$\chi^2_{H_0} = \frac{1}{n} \sum_{i=1}^{r}\sum_{j=1}^{c} \frac{\left(\frac{x_{ij}}{n} - \frac{x_{i\cdot}}{n}\frac{x_{\cdot j}}{n}\right)^2}{\frac{x_{i\cdot}}{n}\frac{x_{\cdot j}}{n}} \tag{13}$$

$$\chi^2_{H_0} = \frac{1}{n} \sum_{i=1}^{r}\sum_{j=1}^{c} \frac{(p_{ij} - p_{i\cdot}p_{\cdot j})^2}{p_{i\cdot}p_{\cdot j}} = \frac{1}{n} \sum_{i=1}^{r}\sum_{j=1}^{c} \frac{(p_{ij} - p_{ij,e})^2}{p_{ij,e}} \tag{14}$$

The chi-square test reject H_0 if

$$\chi^2_{H_0} > \chi^2_{(r-1)\cdot(c-1),1-\alpha}, \tag{15}$$

To evaluate the power of the test let's precisely define the alternative H_a as follow.

$$H_a: \quad p_{ij} = p_{i\cdot}p_{\cdot j} + \frac{c_{ij}}{\sqrt{n}}, \qquad i = 1,2,...,r \quad j = 1,2,...,c, \quad where \quad \sum_{i=1}^{r}\sum_{j=1}^{c} c_{ij} = 0, \tag{16}$$

Thus the power o the test is

$$Power = P\left(X_a > \chi^2_{(r-1)\cdot(c-1),1-\alpha}\right) \tag{17}$$

where X_a is a ranom variable that follows the noncentral χ^2 distribution with the noncentrality parameter

$$\lambda = \sum_{i=1}^{r}\sum_{j=1}^{c} \frac{c_{ij}^2}{p_{i\cdot}p_{\cdot j}} - \sum_{i=1}^{r} \frac{c_{i\cdot}^2}{p_{i\cdot}} - \sum_{j=1}^{c} \frac{c_{\cdot j}^2}{p_{\cdot j}}, \tag{18}$$

where $c_{i\cdot} = \sum_{j=1}^{c} c_{ij}$ and $c_{\cdot j} = \sum_{i=1}^{r} c_{ij}$.

If $\Delta_{ij} = c_{ij}/\sqrt{n}$, then

$$\lambda = \frac{1}{n}\left[\sum_{i=1}^{r}\sum_{j=1}^{c} \frac{\Delta_{ij}^2}{p_{i\cdot}p_{\cdot j}} - \sum_{i=1}^{r} \frac{\Delta_{i\cdot}^2}{p_{i\cdot}} - \sum_{j=1}^{c} \frac{\Delta_{\cdot j}^2}{p_{\cdot j}}\right], \tag{19}$$

2.3 Test of homogenity

Let $X_i = (X_{ij}) \in \mathbb{R}^c$ be a multinomial random variable with parameters n_i, p_{ij} for $i = 1,2,...,r$ and $\sum_{j=1}^{c} p_{ij} = 1$. Suppopse we wish to test homogenity

$$H_0: \quad p_{1j} = p_{2j} = \cdots = p_{rj} = p_{j,e} \quad j = 1,2,...,c \tag{20}$$

against

$$H_a: \text{not all the equatons given under } H_0 \text{ are satisfied} \tag{21}$$

The value of the chi-square test-statistic is

$$\chi^2_{H_0} = \sum_{i=1}^{r}\sum_{j=1}^{c} \frac{(x_{ij} - x_{i\cdot}x_{\cdot j}/n)^2}{x_{i\cdot}x_{\cdot j}/n} \tag{22}$$

where $x_{i\cdot} = \sum_{j=1}^{c} x_{ij} = n_i$ and $n = \sum_{i=1}^{r} n_i$. The chi-square test reject H_0 if

$$\chi^2_{H_0} > \chi^2_{(r-1)\cdot(c-1),1-\alpha}, \tag{23}$$

To evaluate the power of the test let's precisely define the alternative H_a as follow.

$$H_a: p_{ij} = p_j + \frac{c_{ij}}{\sqrt{n}}, \qquad j = 1,2,...,c \quad where \quad \sum_{j=1}^{c} c_{ij} = 0, \tag{24}$$

Thus the power o the test is

$$Power = P\left(X_a > \chi^2_{(r-1)\cdot(c-1),1-\alpha}\right) \tag{25}$$

where X_a is a ranom variable that follows the noncentral χ^2 distribution with the noncentrality parameter

$$\lambda = \sum_{j=1}^{c} \frac{1}{p_j}\left[\sum_{i=1}^{r} c_{ij}^2 \frac{n_i}{n} - \left(\sum_{i=1}^{r} c_{ij}\frac{n_i}{n}\right)^2\right], \tag{26}$$

If $\Delta_{ij} = c_{ij}/\sqrt{n}$, then

$$\lambda = \frac{1}{n} \sum_{j=1}^{c} \frac{1}{p_j} \left[\sum_{i=1}^{r} \Delta_{ij}^2 \frac{n_i}{n} - \left(\sum_{i=1}^{r} \Delta_{ij} \frac{n_i}{n} \right)^2 \right] \tag{27}$$

3 Sample size

References

[1] Chi-squared-test `https://en.wikipedia.org/wiki/Pearson%27s_chi-squared_test`.

[2] Guenther, W. (1977). *Power and Sample Size for Approximate Chi-Square Tests.* The American Statistician, 31(2), 83-85.