

Chi-square test

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1 Definition [1, 2]

The Pearson's χ^2 -test is a test used for hypothesis concerning probabilities of multinomial random variables. It tests a null hypothesis stating that the frequency distribution of certain events observed in a sample is consistent with a particular theoretical distribution.

The test statistic is of following kind

$$\chi^2 = \sum_{i=1}^n \frac{(O_i - E_i)^2}{E_i} = N \sum_{i=1}^n \frac{(O_i/N - p_i)^2}{p_i} \quad (1)$$

where

- χ^2 = Pearson's cumulative test statistic, which asymptotically approaches a χ^2 distribution.
- O_i = number of observation of type i
- N = total number of observations
- $E_i = Np_i$ = the expected (theoretical) count of type i
- n = the number of cells in the table (*rows* · *columns*)

If the test is false, then the appropriate test statistic has approximately a noncentral χ^2 distribution with the same degrees of freedom df and a noncentrality parameter λ , which depends on alternative considered.

2 Test types [1]

Pearson's chi-squared test is used to assess three types of comparison: goodness of fit, homogeneity, and independence.

- A test of **goodness of fit** establishes whether an observed frequency distribution differs from a theoretical distribution.
 χ^2 test statistic with degrees of freedom $df = n - 1$
- A test of **homogeneity** compares the distribution of counts for two or more groups using the same categorical variable.
 χ^2 test statistic with degrees of freedom $df = (rows - 1) \cdot (columns - 1)$
- A test of **independence** assesses whether observations consisting of measures on two variables, expressed in a contingency table, are independent of each other.
 χ^2 test statistic with degrees of freedom $df = (rows - 1) \cdot (columns - 1)$

2.1 Test of goodness of fit

Let $X = (X_1, \dots, X_k)$ be a multinomial random variable with parameters n, p_1, \dots, p_k . Suppose we wish to test

$$H_0 : p_i = p'_i \quad i = 1, 2, \dots, k \quad (2)$$

against

$$H_a : \text{not all } p\text{'s are as given by } H_0 \quad (3)$$

where the p'_i are given expected numbers. The value of the chi-square test-statistic is

$$\chi_{H_0}^2 = \sum_{i=1}^k \frac{(x_i - np'_i)^2}{np'_i} = \sum_{i=1}^k \frac{(\hat{p}_i - p'_i)^2}{p'_i} \quad (4)$$

The chi-square test reject H_0 if

$$\chi_{H_0}^2 > \chi_{k-1, 1-\alpha}^2, \quad (5)$$

where α is the significance level, and $\chi_{k-1, 1-\alpha}^2$ is the quantile of order $1 - \alpha$ of the χ^2 distribution with $k - 1$ degrees of freedom. The p-value of the test is

$$p - value = P(X < \chi_{H_0}^2) \quad (6)$$

To evaluate the power of the test let's precisely define the alternative H_a as follow.

$$H_a : \quad p_i = p''_i \quad i = 1, 2, \dots, k \quad (7)$$

Thus the power of the test is

$$Power = P(X_a > \chi_{k-1, 1-\alpha}^2) \quad (8)$$

where X_a is a random variable that follows the noncentral χ^2 distribution with the noncentrality parameter

$$\lambda = n \sum_{i=1}^k \frac{(p''_i - p'_i)^2}{p'_i} \quad (9)$$

2.2 Test of independence

Let $X = (X_{ij}) \in \mathbb{R}^{r \times c}$ be a multinomial random variable with parameters n, p_{ij} where $i = 1, 2, \dots, r$, $j = 1, 2, \dots, c$ and $\sum_{i=1}^r \sum_{j=1}^c p_{ij} = 1$. Suppose we wish to test independence

$$H_0 : \quad p_{ij} = p_{i \cdot} p_{\cdot j} \quad i = 1, 2, \dots, r \quad j = 1, 2, \dots, c \quad (10)$$

against

$$H_a : \text{not all the equations given under } H_0 \text{ are satisfied} \quad (11)$$

where $p_{i \cdot} = \sum_{j=1}^c p_{ij}$ and $p_{\cdot j} = \sum_{i=1}^r p_{ij}$. The value of the chi-square test-statistic is

$$\chi_{H_0}^2 = \sum_{i=1}^r \sum_{j=1}^c \frac{(x_{ij} - x_{i \cdot} x_{\cdot j} / n)^2}{x_{i \cdot} x_{\cdot j} / n} \quad (12)$$

where $x_{i \cdot} = \sum_{j=1}^c x_{ij}$ and $x_{\cdot j} = \sum_{i=1}^r x_{ij}$. Using parameters p_* we observe that

$$\chi_{H_0}^2 = \frac{1}{n} \sum_{i=1}^r \sum_{j=1}^c \frac{\left(\frac{x_{ij}}{n} - \frac{x_{i \cdot}}{n} \frac{x_{\cdot j}}{n} \right)^2}{\frac{x_{i \cdot}}{n} \frac{x_{\cdot j}}{n}} \quad (13)$$

$$\chi_{H_0}^2 = \frac{1}{n} \sum_{i=1}^r \sum_{j=1}^c \frac{(p_{ij} - p_{i \cdot} p_{\cdot j})^2}{p_{i \cdot} p_{\cdot j}} = \frac{1}{n} \sum_{i=1}^r \sum_{j=1}^c \frac{(p_{ij} - p_{ij,e})^2}{p_{ij,e}} \quad (14)$$

The chi-square test reject H_0 if

$$\chi_{H_0}^2 > \chi_{(r-1) \cdot (c-1), 1-\alpha}^2, \quad (15)$$

To evaluate the power of the test let's precisely define the alternative H_a as follow.

$$H_a : p_{ij} = p_{i \cdot} p_{\cdot j} + \frac{c_{ij}}{\sqrt{n}}, \quad i = 1, 2, \dots, r \quad j = 1, 2, \dots, c, \quad \text{where} \quad \sum_{i=1}^r \sum_{j=1}^c c_{ij} = 0, \quad (16)$$

Thus the power of the test is

$$\text{Power} = P\left(X_a > \chi_{(r-1) \cdot (c-1), 1-\alpha}^2\right) \quad (17)$$

where X_a is a random variable that follows the noncentral χ^2 distribution with the noncentrality parameter

$$\lambda = \sum_{i=1}^r \sum_{j=1}^c \frac{c_{ij}^2}{p_{i \cdot} p_{\cdot j}} - \sum_{i=1}^r \frac{c_{i \cdot}^2}{p_{i \cdot}} - \sum_{j=1}^c \frac{c_{\cdot j}^2}{p_{\cdot j}}, \quad (18)$$

where $c_{i \cdot} = \sum_{j=1}^c c_{ij}$ and $c_{\cdot j} = \sum_{i=1}^r c_{ij}$.

If $\Delta_{ij} = c_{ij}/\sqrt{n}$, then

$$\lambda = \frac{1}{n} \left[\sum_{i=1}^r \sum_{j=1}^c \frac{\Delta_{ij}^2}{p_{i \cdot} p_{\cdot j}} - \sum_{i=1}^r \frac{\Delta_{i \cdot}^2}{p_{i \cdot}} - \sum_{j=1}^c \frac{\Delta_{\cdot j}^2}{p_{\cdot j}} \right], \quad (19)$$

2.3 Test of homogeneity

Let $X_i = (X_{ij}) \in \mathbb{R}^c$ be a multinomial random variable with parameters n_i, p_{ij} for $i = 1, 2, \dots, r$ and $\sum_{j=1}^c p_{ij} = 1$. Suppose we wish to test homogeneity

$$H_0 : p_{1j} = p_{2j} = \dots = p_{rj} = p_j \quad j = 1, 2, \dots, c \quad (20)$$

against

$$H_a : \text{not all the equations given under } H_0 \text{ are satisfied} \quad (21)$$

The value of the chi-square test-statistic is

$$\chi_{H_0}^2 = \sum_{i=1}^r \sum_{j=1}^c \frac{(x_{ij} - x_{i \cdot} x_{\cdot j} / n)^2}{x_{i \cdot} x_{\cdot j} / n} \quad (22)$$

where $x_{i \cdot} = \sum_{j=1}^c x_{ij} = n_i$ and $n = \sum_{i=1}^r n_i$. The chi-square test rejects H_0 if

$$\chi_{H_0}^2 > \chi_{(r-1) \cdot (c-1), 1-\alpha}^2, \quad (23)$$

To evaluate the power of the test let's precisely define the alternative H_a as follow.

$$H_a : p_{ij} = p_j + \frac{c_{ij}}{\sqrt{n}}, \quad j = 1, 2, \dots, c \quad \text{where} \quad \sum_{j=1}^c c_{ij} = 0, \quad (24)$$

Thus the power of the test is

$$\text{Power} = P\left(X_a > \chi_{(r-1) \cdot (c-1), 1-\alpha}^2\right) \quad (25)$$

where X_a is a random variable that follows the noncentral χ^2 distribution with the noncentrality parameter

$$\lambda = \sum_{j=1}^c \frac{1}{p_j} \left[\sum_{i=1}^r c_{ij}^2 \frac{n_i}{n} - \left(\sum_{i=1}^r c_{ij} \frac{n_i}{n} \right)^2 \right], \quad (26)$$

If $\Delta_{ij} = c_{ij}/\sqrt{n}$, then

$$\lambda = \frac{1}{n} \sum_{j=1}^c \frac{1}{p_j} \left[\sum_{i=1}^r \Delta_{ij}^2 \frac{n_i}{n} - \left(\sum_{i=1}^r \Delta_{ij} \frac{n_i}{n} \right)^2 \right] \quad (27)$$

3 Sample size

References

- [1] Chi-squared-test https://en.wikipedia.org/wiki/Pearson%27s_chi-squared_test.
- [2] Guenther, W. (1977). *Power and Sample Size for Approximate Chi-Square Tests*. The American Statistician, 31(2), 83-85.