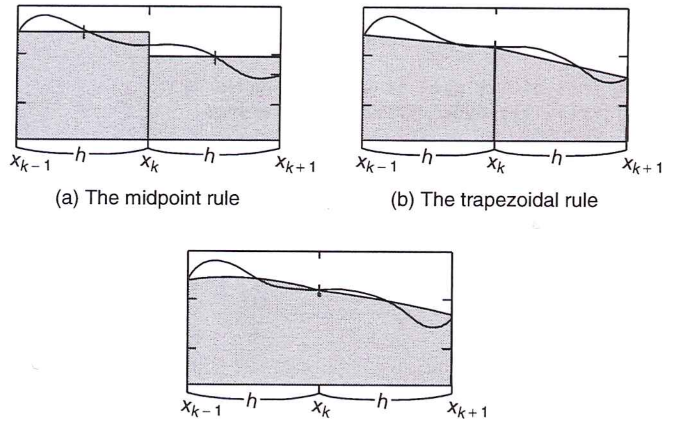

(a) The midpoint rule

(b) The trapezoidal rule

(c) Simpson's rule

Figure 5.3 Various methods of numerical integration.

These three integration rules are based on approximating the target function (integrand) to the zeroth-, first- and second-degree polynomial, respectively. Since the first two integrations are obvious, we are going to derive just Simpson's rule (5.5.4). For simplicity, we shift the graph of $f(x)$ by $-x_k$ along the x axis, or, equivalently, make the variable substitution $t = x - x_k$ so that the abscissas of the three points on the curve of $f(x)$ change from $x = \{x_k - h, x_k, x_k + h\}$ to $t = \{-h, 0, +h\}$. Then, in order to find the coefficients of the second-degree polynomial

$$p_2(t) = c_1 t^2 + c_2 t + c_3 \tag{5.5.5}$$

matching the points $(-h, f_{k-1})$, $(0, f_k)$, $(+h, f_{k+1})$, we should solve the following set of equations:

$$\begin{aligned} p_2(-h) &= c_1(-h)^2 + c_2(-h) + c_3 = f_{k-1} \\ p_2(0) &= c_1 0^2 + c_2 0 + c_3 \qquad\;\; = f_k \\ p_2(+h) &= c_1(+h)^2 + c_2(+h) + c_3 = f_{k+1} \end{aligned}$$

to determine the coefficients c_1, c_2, and c_3 as

$$c_3 = f_k, \qquad c_2 = \frac{f_{k+1} - f_{k-1}}{2h}, \qquad c_1 = \frac{1}{h^2}\left(\frac{f_{k+1} + f_{k-1}}{2} - f_k\right)$$

Integrating the second-degree polynomial (5.5.5) with these coefficients from $t = -h$ to $t = h$ yields

$$\int_{-h}^{h} p_2(t)\,dt = \frac{1}{3}c_1 t^3 + \frac{1}{2}c_2 t^2 + c_3 t\Big|_{-h}^{h} = \frac{2}{3}c_1 h^3 + 2c_3 h$$

$$= \frac{2h}{3}\left(\frac{f_{k+1} + f_{k-1}}{2} - f_k + 3f_k\right) = \frac{h}{3}(f_{k-1} + 4f_k + f_{k+1})$$

This is the Simpson integration formula (5.5.4).

Now, as a preliminary work toward diagnosing the errors of the above integration formulas, we take the Taylor series expansion of the integral function

$$g(x) = \int_{x_k}^{x} f(t)\,dt \quad \text{with } g'(x) = f(x),\ g^{(2)}(x) = f'(x),\ g^{(3)}(x) = f^{(2)}(x) \tag{5.5.6}$$

about the lower bound x_k of the integration interval to write

$$g(x) = g(x_k) + g'(x_k)(x - x_k) + \frac{1}{2}g^{(2)}(x_k)(x - x_k)^2 + \frac{1}{3!}g^{(3)}(x_k)(x - x_k)^3 + \cdots$$

Substituting Eq. (5.5.6) together with $x = x_{k+1}$ and $x_{k+1} - x_k = h$ into this yields

$$\int_{x_k}^{x_{k+1}} f(x)\,dx = 0 + hf(x_k) + \frac{h^2}{2}f'(x_k) + \frac{h^3}{3!}f^{(2)}(x_k) + \frac{h^4}{4!}f^{(3)}(x_k) + \frac{h^5}{5!}f^{(4)}(x_k) + \cdots \tag{5.5.7}$$

First, for the error analysis of the midpoint rule, we substitute x_{k-1} and $-h = x_{k-1} - x_k$ in place of x_{k+1} and h in this equation to write

$$\int_{x_k}^{x_{k-1}} f(x)\,dx = 0 - hf(x_k) + \frac{h^2}{2}f'(x_k) - \frac{h^3}{3!}f^{(2)}(x_k) + \frac{h^4}{4!}f^{(3)}(x_k) - \frac{h^5}{5!}f^{(4)}(x_k) + \cdots$$

and subtract this equation from Eq. (5.5.7) to write

$$\int_{x_k}^{x_{k+1}} f(x)\,dx - \int_{x_k}^{x_{k-1}} f(x)\,dx = \int_{x_k}^{x_{k+1}} f(x)\,dx + \int_{x_{k-1}}^{x_k} f(x)\,dx$$

$$= \int_{x_{k-1}}^{x_{k+1}} f(x)\,dx = 2hf(x_k) + \frac{2h^3}{3!}f^{(2)}(x_k) + \frac{2h^5}{5!}f^{(4)}(x_k) + \cdots \tag{5.5.8}$$

Substituting x_k and $x_{mk} = (x_k + x_{k+1})/2$ in place of x_{k-1} and x_k in this equation and noting that $x_{k+1} - x_{mk} = x_{mk} - x_k = h/2$, we obtain

$$\int_{x_k}^{x_{k+1}} f(x)\,dx = hf(x_{mk}) + \frac{h^3}{3 \times 2^3}f^{(2)}(x_{mk}) + \frac{h^5}{5 \times 4 \times 3 \times 2^5}f^{(4)}(x_{mk}) + \cdots$$

$$\int_{x_k}^{x_{k+1}} f(x)\,dx - hf(x_{mk}) = \frac{h^3}{24}f^{(2)}(x_{mk}) + \frac{h^5}{1920}f^{(4)}(x_{mk}) + \cdots = O(h^3) \tag{5.5.9}$$