HA 1

Ecm II.

Problem 3.

$$x_1, x_2 \dots x_n \sim iid \ Geom(p) \quad p \in (0,1)$$

$$P(X=k) = (1-p)^{k-1} p, \quad k = 1, 2 \dots$$

(1) $$\mathcal{L} = p(\xi = x_1)\,p(\xi = x_2)\dots p(\xi = x_n) = p(1-p)^{x_1-1}p(1-p)^{x_2-1}\dots =$$

$$= p^n (1-p)^{\sum_{i=1}^n x_i - n}$$

(2) $$\ln\mathcal{L} = n\ln p + \left(\sum_{i=1}^n x_i - n\right)\ln(1-p) \longrightarrow \max_p$$

$$\frac{\partial \ln\mathcal{L}}{\partial p} = \frac{n}{p} + \frac{\sum_{i=1}^n x_i - n}{1-p}(-1) = \frac{n}{p} - \frac{\sum_{i=1}^n x_i - n}{1-p} = 0$$

$$p = \frac{n}{\sum_{i=1}^n x_i}$$

$$\hat{p}_{ML} = \frac{1}{\bar{x}}$$

(3) $$\frac{\partial \ln\mathcal{L}}{\partial p} = \frac{n}{p} - \frac{\sum_{i=1}^n x_i - n}{1-p}$$

$$\frac{\partial^2 \ln\mathcal{L}}{\partial p^2} = -\frac{n}{p^2} - \frac{\sum_{i=1}^n x_i - n}{(1-p)^2}$$

$$I(\hat{p}_{ML}) = -E\left(\frac{\partial^2 \ln\mathcal{L}}{\partial p^2}\right) = \frac{n}{p^2} + \frac{\frac{n}{p} - n}{(1-p)^2} = \frac{n}{p^2} + \frac{n - np}{p(1-p)^2} =$$

$$= \frac{n(1-p)^2 + (n-np)p}{p^2(1-p)^2} = \frac{n(1-p) + np}{p^2(1-p)}$$

$$var(\hat{p}_{ML}) = I^{-1}(\hat{p}_{ML}) = \frac{p_{ML}^2(1-p_{ML})}{n(1-p+p)} = \frac{p_{ML}^2(1-p_{ML})}{n} = \frac{\left(\frac{1}{\bar{x}}\right)^2\left(1-\frac{1}{\bar{x}}\right)}{n}$$

(4) $$var(\hat{p}_{ML}) = var\left(\frac{1}{\bar{x}}\right) = \frac{n^2}{var(\sum_{i=1}^n x_i)} = \frac{n^2}{\sum_{i=1}^n var(x_i)}$$

$$var(x_i) = \left\{\begin{matrix}\text{Geometric} \\ \text{property}\end{matrix}\right\} = \frac{1-p}{p^2}$$

$$\Rightarrow var(\hat{p}_{ML}) = \frac{n^2}{n \cdot \frac{(1-p)}{p^2}} = \frac{np^2}{(1-p)} = \left\{\hat{p}_{ML} = \frac{1}{\bar{x}}\right\} =$$

$$= \frac{n\left(\frac{1}{\bar{x}}\right)^2}{\left(1 - \frac{1}{\bar{x}}\right)}$$