Linear Stability Analysis for Planar Interface - Solidification of Dilute Binary Alloy

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1 Directional Solidification

Assume a planar interface moving at a constant velovity 'V'. The steady state Species Transfer Equation,

$$\frac{\partial C_i}{\partial t} = D_{ii}\nabla^2 C_i = 0 \tag{1}$$

Transforming to a moving co-ordinate system,

$$D_{ii}\nabla^2 C_i + V\frac{dC_i}{dz} = 0 \tag{2}$$

Boundary Conditions,

$$C_i = C^l_{i,eq} \qquad \text{at} \quad z = 0 \tag{3}$$

$$VC^l_{i,eq}(1-k_i) = -D_{ii}\left.\frac{dC_i}{dz}\right|_{z=0} = -D_{ii}G_{c,i} \qquad \text{at} \quad z = 0 \tag{4}$$

Solution is given by,

$$C_i = C^l_{i,eq} + \frac{G_{c,i}D_{ii}}{V}\left[1 - \exp\left(\frac{-Vz}{D_{ii}}\right)\right] \tag{5}$$

Adding a linear perturbation

Assume a infinitesimallly small sinosoidal noise on the planar interface normal to 'z'. Now the interface is no more $z = 0$ rather it is at $z = \Phi$ where,

$$z = \Phi = \delta(t)\sin(\omega x) \tag{6}$$

Now the diffusion equation becomes,

$$D_{ii}\nabla^2 \tilde{C}_i + V\frac{\partial \tilde{C}_i}{\partial z} = 0 \tag{7}$$

Boundary Conditions,

$$C_{i,\Phi} = C^l_{i,eq} + b_i \delta \sin(\omega x) \qquad \text{at} \quad z = \Phi \tag{8}$$

$$(V + \dot{\delta}\sin(\omega x))C^l_{i,\Phi}(1-k_i) = -D_{ii}\left.\frac{d\tilde{C}_i}{dz}\right|_{z=\Phi} \qquad \text{at} \quad z = \Phi \tag{9}$$

The interface equilibrium concentration will be no longer constant and will be a function of the curvature at the interface(Gibb's Thompson Condition).

The solution to these equations will be similar to Eq. 5 along with a tern that ensures that the perterbation provided at the interface will die down as $z \to \infty$

$$\tilde{C}_i = C^l_{i,eq} + \frac{G_{c,i}D_{ii}}{V}\left[1 - \exp\left(\frac{-Vz}{D_{ii}}\right)\right] + E_i \sin(\omega x)\exp(-k_{w,i}z) \tag{10}$$

Substituting this in Eq. 7 we get a relation between E_i and $k_{w,i}$,

$$k_{w,i} = \frac{V}{2D_{ii}} + \sqrt{\left(\frac{V}{2D_{ii}}\right)^2 + \omega^2} \tag{11}$$

We can write E_i in terms of b_i, by equating Eq.8 and Eq.10 and using the approximation $\exp(x) = 1 + x$ for small x,

$$E_i = \delta(b_i - G_{c,i}) \tag{12}$$

Now, we solve for the other boundary condition (i.e Stefan condition, Eq. 9). Before this a small simplification using binomial approx. can be performed on Eq. 8,

$$\frac{1}{C^l_{i,\Phi}} = \frac{1}{C^l_{i,eq}(1 + \frac{b_i\delta}{C^l_{i,eq}}\sin(\omega x))} \tag{13}$$

$$\frac{1}{C^l_{i,\Phi}} = \frac{1}{C^l_{i,eq}}\left(1 - \frac{b_i\delta}{C^l_{i,eq}}\sin(\omega x)\right) \tag{14}$$

Finally, on solving Eq. 9, we arrive at,

$$\left.\frac{d\tilde{C}_i}{dz}\right|_{z=\Phi} = G_{c,i}\left(1 - \frac{V\delta\sin(\omega x)}{D_{ii}}\right) - k_{w,i}E_i\sin(\omega x)(1 - \delta\sin(\omega x)) \tag{15}$$

$$\left.\frac{d\tilde{C}_i}{dz}\right|_{z=\Phi} = G_{c,i} - \left(\frac{G_{c,i}V}{D_{ii}} + k_{w,i}(b_i - G_{c,i})\right)\delta\sin(\omega x) \tag{16}$$

$$\begin{aligned} V + \dot{\delta}\sin(\omega x) = & \frac{-D_{ii}}{C^l_{i,eq}(1-k_i)}\left(1 - \frac{b_i\delta}{C^l_{i,eq}}\sin(\omega x)\right) \\ & \left(G_{c,i} - \left(\frac{G_{c,i}V}{D_{ii}} + k_{w,i}(b_i - G_{c,i})\right)\delta\sin(\omega x)\right) \end{aligned} \tag{17}$$

$$\frac{\dot{\delta}}{\delta} = V\tilde{\omega}\left[-\frac{b_i}{G_{c,i}} + \frac{1}{\tilde{\omega}}\left(k_{\omega,i} - \frac{V}{D_{ii}}\right)\right] \tag{18}$$

where $\tilde{w} = k_{\omega,i} - \frac{V}{D_{ii}}(1-k_i)$. Now the only unknown is b_i. But, the term $\frac{\dot{\delta}}{\delta}$ is constant for all the components, so any component $i! = 1$ should be equal to $i = 1$,

$$V\tilde{\omega}\left[-\frac{b_i}{G_{c,i}} + \frac{1}{\tilde{\omega}}\left(k_{\omega,i} - \frac{V}{D_{ii}}\right)\right] = V\tilde{\omega}\left[-\frac{b_1}{G_{c,1}} + \frac{1}{\tilde{\omega}}\left(k_{\omega,1} - \frac{V}{D_{11}}\right)\right] \tag{19}$$

$$b_i = \frac{G_{c,i}}{\tilde{\omega_i}}\left[\frac{b_1}{G_{c,1}}\tilde{\omega_i} + \left(k_{w,i} - \frac{V}{D_{ii}}\right) - \left(k_{w,1} - \frac{V}{D_{11}}\right)\right] \tag{20}$$

Now the only unknown is b_1 and this can be found with the condition of local equilibrium,

$$T|_{z=\phi} - (T_{pure} + \sum_i m_i^l(\tilde{C^l_{i,\phi}})) \quad = \quad -\Gamma\kappa \tag{21}$$

$$T|_{z=0} + G\delta\sin(\omega z) - (T_m + \sum_i m_i^l b_i \delta \sin(\omega x)) \quad = \quad -\Gamma\delta\omega^2\sin(\omega x) \tag{22}$$

Comparing the coefficients of $\delta\sin(\omega x)$ on both sides,

$$G + \Gamma\omega^2 = \sum_i m_i^l b_i \tag{23}$$

Now that all the values are known, we can calculate the critical frequency ω_{crit} for the stability by $\frac{\dot{\delta}}{\delta} = 0$,

$$b_i = G_{c,i}\frac{k_{w,i} - \frac{V}{D_{ii}}}{\tilde{\omega_i}} \tag{24}$$

The critical wavelength $\lambda_{crit} = 2\pi/\omega_{crit}$ and this turns out to be,

$$TBD \tag{25}$$

2 Assumptions

1. The bulk and surface parameters are assumed to be isotropic
2. Analysis carried out when the system is in a steady state in both temperature and composition. (These calculations are valid only when the dendrite shape changes slowly)