The action is:

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} (\nabla^\mu a)(\nabla_\mu a) - V\left(\frac{a}{f}\right) \right] \quad \text{with } V(x) = \frac{\Lambda^4}{c_z}\left(1 - \sqrt{1 - 4c_z \sin^2 \frac{x}{2}}\right)$$

Metric: $d^2s = (1+2\phi)\,dt^2 - (1-2\phi)\,dr^2 - r^2 d\Omega^2$

$$\Rightarrow g_{\mu\nu} = \begin{pmatrix} 1+2\phi & & & \\ & -(1-2\phi) & & \\ & & -r^2 & \\ & & & -r^2\sin^2\theta \end{pmatrix}$$

Christoffel symbols (from Mathematica's notebook "christoffel.nb") of the form $\Gamma^\mu_{\nu\nu}$:

$$\Gamma^r_{rr} = \frac{\phi'}{-1+2\phi} \simeq -\phi'(1+2\phi)$$

$$\Gamma^r_{\theta\theta} = \frac{r}{-1+2\phi} \simeq -r(1+2\phi)$$

$$\Gamma^r_{\varphi\varphi} = \frac{r\sin^2\theta}{-1+2\phi} \simeq -r\sin^2\theta\,(1+2\phi)$$

$$\Gamma^r_{tt} = \frac{\phi'}{1-2\phi} \simeq \phi'(1+2\phi)$$

$$\Gamma^\theta_{\varphi\varphi} = -\cos\theta\sin\theta$$

$$\Gamma^t_{rr} = \frac{-\dot\phi}{1+2\phi} \simeq -\dot\phi(1-2\phi)$$

$$\Gamma^t_{tt} = \frac{\dot\phi}{1+2\phi} \simeq \dot\phi(1-2\phi)$$

Now, define $\alpha \equiv \frac{a}{f_a}$ and make $r \to \tilde r = mr$ & $t \to \tilde t = mt$:

$$\mathcal{L} = \int \frac{d^4\tilde x}{m^4} \sqrt{-g} \left[ \overset{\Lambda^4}{\frac{f_a^2 m^2}{2}} (\tilde\nabla^\mu\alpha)(\tilde\nabla_\mu\alpha) - V(\alpha) \right]$$

drop the ~

$\tilde V(\alpha) \equiv \dfrac{V(\alpha)}{\Lambda^4}$

$$\Rightarrow \mathcal{L} = \int \frac{d^4x}{m^4} \sqrt{-g}\, \Lambda^4 \left[ \frac{1}{2} (\nabla^\mu\alpha)(\nabla_\mu\alpha) - \tilde V(\alpha) \right]$$

Equation of motion for scalar in curved spacetime:

$$\Box\alpha = -\frac{\partial\tilde{V}}{\partial\alpha}, \quad \text{with } \Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$$

We have

$$\Box\alpha = g^{\mu\nu}\nabla_\mu\partial_\nu\alpha = g^{\mu\nu}\left(\partial_\mu\partial_\nu\alpha - \Gamma^\beta_{\mu\nu}\partial_\beta\alpha\right)$$

$$= g^{tt}\left(\ddot{\alpha} - \Gamma^\beta_{tt}\partial_\beta\alpha\right) + g^{rr}\left(\alpha'' - \Gamma^\beta_{rr}\partial_\beta\alpha\right) - \underbrace{\left(g^{\theta\theta}\Gamma^\beta_{\theta\theta} + g^{\varphi\varphi}\Gamma^\beta_{\varphi\varphi}\right)\partial_\beta\alpha}_{\text{assuming spherically symmetrical } \alpha\equiv\alpha(t,r)}$$

$$\cong (1-2\phi)\left[\ddot{\alpha} - (1-2\phi)\dot{\phi}\dot{\alpha} - (1+2\phi)\phi'\alpha'\right]$$

$$-(1+2\phi)\left[\alpha'' + (1-2\phi)\dot{\phi}\dot{\alpha} + (1+2\phi)\phi'\alpha'\right]$$

$$-\frac{1}{\cancel{r^2}_{r}}\cdot\cancel{r}(1+2\phi)\alpha' - \frac{\cancel{\csc}^2\theta}{\cancel{r^2}_{r}}\,\cancel{r}\,\cancel{\sin}^2\theta(1+2\phi)\alpha'$$

$$= (1-2\phi)\ddot{\alpha} - (1+2\phi)\alpha'' - 2\dot{\phi}\dot{\alpha} - 2\phi'\alpha' - (1+2\phi)\frac{2\alpha'}{r} + \mathcal{O}(\phi^2)$$

E.o.m becomes:

$$(1-2\phi)\ddot{\alpha} - (1+2\phi)\alpha'' - (1+2\phi)\frac{2\alpha'}{r} - 2\dot{\phi}\dot{\alpha} - 2\phi'\alpha' = -\frac{\partial\tilde{V}}{\partial\alpha}$$

$$\Rightarrow \ddot{\alpha} \cong (1+4\phi)\left(\alpha'' + \frac{2\alpha'}{r}\right) + 2\dot{\phi}\dot{\alpha} + 2\phi'\alpha' - (1+2\phi)\frac{\partial\tilde{V}}{\partial\alpha}$$

$\hookrightarrow$ Wrong!

The action is:

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} (\nabla^\mu a)(\nabla_\mu a) - V\left(\frac{a}{f}\right) \right] \quad \text{with } V(x) = \frac{\Lambda^4}{c_z} \left( 1 - \sqrt{1 - 4 c_z \sin^2 \frac{x}{2}} \right)$$

Metric: $d^2 s = (1+2\phi)\, dt^2 - (1-2\phi)\left(dr^2 + r^2 d\Omega^2\right)$ ← not in the paper

$$\Rightarrow g_{\mu\nu} = \begin{pmatrix} 1+2\phi & & & \\ & -(1-2\phi) & & \\ & & -r^2(1-2\phi) & \\ & & & -r^2 \sin^2\theta (1-2\phi) \end{pmatrix}$$

Christoffel symbols (from Mathematica's notebook "christoffel.nb") of the form $\Gamma^\mu_{\nu\nu}$, with $\mu = t, r$:

$$\Gamma^r_{rr} = \frac{\phi'}{-1+2\phi} \cong -\phi'(1+2\phi)$$

$$\Gamma^r_{\theta\theta} = -\frac{r(-1+2\phi + r\phi')}{-1+2\phi} \cong r(-1+r\phi')$$

$$\Gamma^r_{\varphi\varphi} = -\frac{r\sin^2\theta(-1+2\phi+r\phi')}{-1+2\phi} \cong r\sin^2\theta(-1+r\phi')$$

$$\Gamma^r_{tt} = \frac{\phi'}{1-2\phi} \cong \phi'(1+2\phi)$$

$$\Gamma^t_{rr} = -\frac{\dot\phi}{1+2\phi} \cong -\dot\phi(1-2\phi)$$

$$\Gamma^t_{\theta\theta} = -\frac{r^2\dot\phi}{1+2\phi} \cong -r^2\dot\phi(1-2\phi)$$

$$\Gamma^t_{\varphi\varphi} = -\frac{r^2\sin^2\theta\,\dot\phi}{1+2\phi} \cong -r^2\sin^2\theta\,\dot\phi(1-2\phi)$$

$$\Gamma^t_{tt} = \frac{\dot\phi}{1+2\phi} \cong \dot\phi(1-2\phi)$$

Equation of motion for scalar in curved spacetime:

$$\Box\alpha = -\frac{\partial\tilde{V}}{\partial\alpha}, \quad \text{with } \Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$$

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$$= g^{tt}\left(\ddot{\alpha} - \Gamma^\beta_{tt}\partial_\beta\alpha\right) + g^{rr}\left(\alpha'' - \Gamma^\beta_{rr}\partial_\beta\alpha\right) - \underbrace{\left(g^{\theta\theta}\Gamma^\beta_{\theta\theta} + g^{\varphi\varphi}\Gamma^\beta_{\varphi\varphi}\right)\partial_\beta\alpha}_{}$$

assuming spherically symmetrical $\alpha \equiv \alpha(t,r)$

$$\tilde{=} (1-2\phi)\left[\ddot{\alpha} - \dot{\phi}(1-2\phi)\dot{\alpha} - \phi'(1+2\phi)\alpha'\right]$$

$$- (1+2\phi)\left[\alpha'' + \dot{\phi}(1-2\phi)\dot{\alpha} + \phi'(1+2\phi)\alpha'\right]$$

$$+ \frac{(1+2\phi)}{r^2}\left[-r^2\dot{\phi}(1-2\phi)\dot{\alpha} + r(-1+r\phi')\alpha'\right]$$

$$+ \frac{(1+2\phi)\csc^2\theta}{r^2}\left[-r^2\sin^2\theta\,\dot{\phi}(1-2\phi)\dot{\alpha} + r\sin^2\theta(-1+r\phi')\alpha'\right]$$

$$\tilde{=} (1-2\phi)\ddot{\alpha} - \dot{\phi}\dot{\alpha} - \cancel{\phi'\alpha'} - (1+2\phi)\alpha'' - \dot{\phi}\dot{\alpha} - \cancel{\phi'\alpha'}$$

$$- \dot{\phi}\dot{\alpha} - (1+2\phi)\frac{\alpha'}{r} + \cancel{\phi'\alpha'} - \dot{\phi}\dot{\alpha} - (1+2\phi)\frac{\alpha'}{r} + \cancel{\phi'\alpha'}$$

E.o.m becomes:

$$(1-2\phi)\ddot{\alpha} - (1+2\phi)\alpha'' - (1+2\phi)\frac{2\alpha'}{r} - 4\dot{\phi}\dot{\alpha} = -\frac{\partial\tilde{V}}{\partial\alpha}$$

$$\Rightarrow \ddot{\alpha} \tilde{=} (1+4\phi)\left(\alpha'' + \frac{2\alpha'}{r}\right) + 4\dot{\phi}\dot{\alpha} - (1+2\phi)\frac{\partial\tilde{V}}{\partial\alpha}$$

✓ correct!