Problem 1.14

Consider the electrostatic Green functions of Section 1.10 for Dirichlet and Neumann boundary conditions on the surface S bounding the volume V. Apply Green's theorem (1.35) with integration variable $\mathbf{y}$ and $\phi = G(\mathbf{x}, \mathbf{y})$, $\psi = G(\mathbf{x}', \mathbf{y})$, with $\nabla_y^2 G(\mathbf{z}, \mathbf{y}) = -4\pi\delta(\mathbf{y} - \mathbf{z})$. Find an expression for the difference $[G(\mathbf{x}, \mathbf{x}') - G(\mathbf{x}', \mathbf{x})]$ in terms of an integral over the boundary surface S.

(a) For Dirichlet boundary conditions on the potential and the associated boundary condition on the Green function, show that $G_D(\mathbf{x}, \mathbf{x}')$ must be symmetric in $\mathbf{x}$ and $\mathbf{x}'$.

(b) For Neumann boundary conditions, use the boundary condition (1.45) for $G_N(\mathbf{x}, \mathbf{x}')$ to show that $G_N(\mathbf{x}, \mathbf{x}')$ is not symmetric in general, but that $G_N(\mathbf{x}, \mathbf{x}') - F(\mathbf{x})$ is symmetric in $\mathbf{x}$ and $\mathbf{x}'$, where

$$F(\mathbf{x}) = \frac{1}{S} \oint_S G_N(\mathbf{x}, \mathbf{y}) \mathrm{d}a_y.$$

(c) Show that the addition of $F(\mathbf{x})$ to the Green function does not affect the potential $\Phi(\mathbf{x})$.

Solution. We start with Green's theorem (1.35)

$$\int_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) \mathrm{d}^3x = \oint_S \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right] \mathrm{d}a.$$

Making the substitutions $\phi = G(\mathbf{x}, \mathbf{y})$, $\psi = G(\mathbf{x}', \mathbf{y})$, and using $\mathbf{y}$ as the integration variable,

$$\int_V \left[G(\mathbf{x}, \mathbf{y})\nabla_y^2 G(\mathbf{x}', \mathbf{y}) - G(\mathbf{x}', \mathbf{y})\nabla_y^2 G(\mathbf{x}, \mathbf{y})\right] \mathrm{d}^3y = \oint_S \left[G(\mathbf{x}, \mathbf{y})\frac{\partial}{\partial n_y} G(\mathbf{x}', \mathbf{y}) - G(\mathbf{x}', \mathbf{y})\frac{\partial}{\partial n_y} G(\mathbf{x}, \mathbf{y})\right] \mathrm{d}a_y.$$

Using the fact that $\nabla_y^2 G(\mathbf{z}, \mathbf{y}) = -4\pi\delta(\mathbf{y} - \mathbf{z})$, and assuming $\mathbf{x}, \mathbf{x}' \in V$,

$$\text{LHS} : -4\pi \int_V \left[G(\mathbf{x}, \mathbf{y})\delta(\mathbf{x}' - \mathbf{y}) - G(\mathbf{x}', \mathbf{y})\delta(\mathbf{x} - \mathbf{y})\right] \mathrm{d}^3y = -4\pi \left(G(\mathbf{x}, \mathbf{x}') - G(\mathbf{x}', \mathbf{x})\right).$$

Therefore,

$$\left[G(\mathbf{x}, \mathbf{x}') - G(\mathbf{x}', \mathbf{x})\right] = -\frac{1}{4\pi} \oint_S \left[G(\mathbf{x}, \mathbf{y})\frac{\partial}{\partial n_y} G(\mathbf{x}', \mathbf{y}) - G(\mathbf{x}', \mathbf{y})\frac{\partial}{\partial n_y} G(\mathbf{x}, \mathbf{y})\right] \mathrm{d}a_y.$$

(a) For Dirichlet boundary conditions: $G_D(\mathbf{z}, \mathbf{y}) = 0$ for $\mathbf{y} \in S$. Clearly, RHS $= 0$. Thus,

$$G_D(\mathbf{x}, \mathbf{x}') = G_D(\mathbf{x}', \mathbf{x}).$$

(b) For Neumann boundary conditions, we use (1.45): for $\mathbf{y} \in S$,

$$\frac{\partial}{\partial n_y} G_N(\mathbf{z}, \mathbf{y}) = -\frac{4\pi}{S}.$$

Now,

$$\text{RHS} : \frac{1}{S} \oint_S \left[G_N(\mathbf{x}, \mathbf{y}) - G_N(\mathbf{x}', \mathbf{y})\right] \mathrm{d}a_y = F(\mathbf{x}) - F(\mathbf{x}').$$

Thus,

$$G_N(\mathbf{x}, \mathbf{x}') - G_N(\mathbf{x}', \mathbf{x}) = F(\mathbf{x}) - F(\mathbf{x}') \quad \Longleftrightarrow \quad G_N(\mathbf{x}, \mathbf{x}') - F(\mathbf{x}) = G_N(\mathbf{x}', \mathbf{x}) - F(\mathbf{x}').$$

(c) We start again with Green's theorem (1.35), with $\phi = \Phi(\mathbf{x}')$, $\psi = G(\mathbf{x},\mathbf{x}') + F(\mathbf{x})$, and integrating over x'. Since $F(\mathbf{x})$ does not depend on $\mathbf{x}'$, $\nabla'^2 F(\mathbf{x}) = 0$. Therefore,

$$\begin{aligned}
\text{LHS} :& \int_V \left[\Phi(\mathbf{x}')\nabla'^2 G(\mathbf{x},\mathbf{x}') - \nabla'^2\Phi(\mathbf{x}')\left(G(\mathbf{x},\mathbf{x}') + F(\mathbf{x})\right)\right] \mathrm{d}^3x' \\
=& -4\pi \int_V \Phi(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')\mathrm{d}^3x' + \int_V \left(-\nabla'^2\Phi(\mathbf{x}')\right) G(\mathbf{x},\mathbf{x}')\mathrm{d}^3x' + F(\mathbf{x})\int_V \left(-\nabla'^2\Phi(\mathbf{x}')\right)\mathrm{d}^3x' \\
=& -4\pi\Phi(\mathbf{x}) + \frac{1}{\epsilon_0}\int_V \rho(\mathbf{x})G(\mathbf{x},\mathbf{x}')\mathrm{d}^3x' + \frac{F(\mathbf{x})}{\epsilon_0}\int_V \rho(\mathbf{x}')\mathrm{d}^3x'.
\end{aligned}$$

Also note that $\dfrac{\partial}{\partial n'}F(\mathbf{x}) = 0$. Now, using Neumann boundary conditions,

$$\begin{aligned}
\text{RHS} :& \oint_S \left[\Phi(\mathbf{x}')\frac{\partial}{\partial n'}G_N(\mathbf{x},\mathbf{x}') - \frac{\partial}{\partial n'}\Phi(\mathbf{x}')\left(G_N(\mathbf{x},\mathbf{x}') + F(\mathbf{x})\right)\right]\mathrm{d}a' \\
&= -\frac{4\pi}{S}\oint_S \Phi(\mathbf{x}')\mathrm{d}a' - \oint_S \frac{\partial}{\partial n'}\Phi(\mathbf{x}')\, G_N(\mathbf{x},\mathbf{x}')\mathrm{d}a' - \oint_S \frac{\partial}{\partial n'}\Phi(\mathbf{x}')F(\mathbf{x})\mathrm{d}a' \\
&= -4\pi\langle\Phi\rangle_S - \oint_S \frac{\partial}{\partial n'}\Phi(\mathbf{x}')\, G_N(\mathbf{x},\mathbf{x}')\mathrm{d}a' + F(\mathbf{x})\oint_S \left(-\nabla'\Phi(\mathbf{x}')\right)\cdot\hat{\mathbf{n}}'\mathrm{d}a' \\
&= -4\pi\langle\Phi\rangle_S - \oint_S \frac{\partial}{\partial n'}\Phi(\mathbf{x}')\, G_N(\mathbf{x},\mathbf{x}')\mathrm{d}a' + F(\mathbf{x})\int_V \left(-\nabla'^2\Phi(\mathbf{x}')\right)\mathrm{d}^3x' \\
&= -4\pi\langle\Phi\rangle_S - \oint_S \frac{\partial}{\partial n'}\Phi(\mathbf{x}')\, G_N(\mathbf{x},\mathbf{x}')\mathrm{d}a' + \frac{F(\mathbf{x})}{\epsilon_0}\int_V \rho(\mathbf{x}')\mathrm{d}^3x'.
\end{aligned}$$

Thus,

$$\Phi(\mathbf{x}) = \langle\Phi\rangle_S + \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x})G_N(\mathbf{x},\mathbf{x}')\mathrm{d}^3x' + \frac{1}{4\pi}\oint_S \frac{\partial}{\partial n'}\Phi(\mathbf{x}')\, G_N(\mathbf{x},\mathbf{x}')\mathrm{d}a',$$

which is exactly (1.46), since the contributions of $F(\mathbf{x})$ cancel out.

$\square$