Problem 6.1

In three dimensions the solution to the wave equation (6.32) for a point source in space and time (a light flash at $t' = 0$, $\mathbf{x}' = 0$) is a spherical shell disturbance of radius $R = ct$, namely the Green function $G^{(+)}$ (6.44). It may be initially surprising that in one or two dimensions, the disturbance possesses a "wake", even though the source is a "point" in space and time. The solutions for fewer dimensions than three can be found by superposition in the superfluous dimension(s), to eliminate dependence on such variable(s). For example, a flashing line source of uniform amplitude is equivalent to a point source in two dimensions.

(a) Starting with the retarded solution to the three-dimensional wave equation (6.47), show that the source $f(\mathbf{x}', t') = \delta(x')\delta(y')\delta(t')$, equivalent to a $t = 0$ point source at the origin in two spatial dimensions, produces a two-dimensional wave,

$$\Psi(x, y, t) = \frac{2c\,\Theta(ct - \rho)}{\sqrt{c^2t^2 - \rho^2}},$$

where $\rho^2 = x^2 + y^2$ and $\Theta(\xi)$ is the unit step function $[\ \Theta(\xi) = 0\,(1)$ if $\xi < (>)\, 0\ ]$.

(b) Show that a "sheet" source, equivalent to a point pulsed source at the origin in one space dimension, produces a one-dimensional wave proportional to

$$\Phi(x, t) = 2\pi c\,\Theta(ct - |x|).$$

Solution.

(a) Starting with (6.47),

$$\begin{aligned}
\Phi(\mathbf{x}, t) &= \int \frac{[f(\mathbf{x}', t')]_{\text{ret}}}{|\mathbf{x} - \mathbf{x}'|} \mathrm{d}^3x' \\
&= \int \frac{1}{\sqrt{x^2 + y^2 + (z - z')^2}} \delta\left(t - \sqrt{x^2 + y^2 + (z - z')^2}/c\right) \mathrm{d}z' \\
&= \int \frac{1}{\sqrt{\rho^2 + (z - z')^2}} \delta\left(t - \sqrt{\rho^2 + (z - z')^2}/c\right) \mathrm{d}z'
\end{aligned}$$

Recall the property of the Dirac delta function (Chapter 1),

$$\delta\left(f(x)\right) = \sum_i \left|\frac{\mathrm{d}f}{\mathrm{d}x}(x_i)\right|^{-1} \delta(x - x_i),$$

where the x_i's are simple zeroes of $f(x)$. Then

$$\delta\left(f(z')\right) = \delta\left(t - \sqrt{\rho^2 + (z - z')^2}/c\right) = \left|\frac{\mathrm{d}f}{\mathrm{d}z'}(z'_+)\right|^{-1} \delta(z' - z'_+) + \left|\frac{\mathrm{d}f}{\mathrm{d}z'}(z'_-)\right|^{-1} \delta(z' - z'_-),$$

where $z'_\pm = z \pm \sqrt{c^2t^2 - \rho^2}$. Note that $\rho < ct$ for this to be valid, and there are no (real) roots otherwise. This means that $\Phi(\mathbf{x}, t)$ is always zero unless $\rho < ct$. Now,

$$\frac{\mathrm{d}f}{\mathrm{d}z'} = \frac{z - z'}{c\sqrt{\rho^2 + (z - z')^2}}.$$

Therefore,

$$\frac{\mathrm{d}f}{\mathrm{d}z'}(z'_\pm) = \mp\frac{1}{c^2 t}\sqrt{c^2t^2-\rho^2}.$$

Thus,

$$\delta\left(t-\sqrt{\rho^2+(z-z')^2}/c\right) = \frac{c^2 t}{\sqrt{c^2t^2-\rho^2}}\left[\delta(z'-z'_+)+\delta(z'-z'_-)\right].$$

Finally, plugging this back to the integral we obtain

$$\begin{aligned}\Phi(\mathbf{x},t) &= \frac{c^2 t}{\sqrt{c^2t^2-\rho^2}}\left[\frac{1}{\sqrt{\rho^2+(z-z'_+)^2}}+\frac{1}{\sqrt{\rho^2+(z-z'_-)^2}}\right]\\ &= \frac{c^2 t}{\sqrt{c^2t^2-\rho^2}}\left(\frac{2}{ct}\right) = \frac{2c}{\sqrt{c^2t^2-\rho^2}}.\end{aligned}$$

This, combined with the requirement $\rho < ct$, is equivalent to

$$\Phi(\mathbf{x},t) = \Psi(x,y,t) = \Psi(\rho,t) = \frac{2c\,\Theta(ct-\rho)}{\sqrt{c^2t^2-\rho^2}}.$$

(b) A 'sheet' source is given by $f(\mathbf{x}',t') = \delta(x')\delta(t')$. Using similar arguments, we obtain

$$\begin{aligned}\Phi(\mathbf{x},t) &= \int \frac{[f(\mathbf{x}',t')]_{\text{ret}}}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}^3x'\\ &= \int \frac{1}{\sqrt{x^2+(y-y')^2+(z-z')^2}}\delta\left(t-\sqrt{x^2+(y-y')^2+(z-z')^2}/c\right)\mathrm{d}y'\mathrm{d}z'.\end{aligned}$$

By the symmetry of the (infinite) sheet, we can translate our coordinates such that $(x,y,z) = (0,0,z)$ for convenience. Therefore,

$$\Phi(\mathbf{x},t) = \Phi(x,t) = \int \frac{1}{\sqrt{x^2+y'^2+z'^2}}\delta\left(t-\sqrt{x^2+y'^2+z'^2}/c\right)\mathrm{d}y'\mathrm{d}z'.$$

In polar coordinates,

$$\Phi(x,t) = 2\pi\int_0^\infty \frac{1}{\sqrt{x^2+\rho'^2}}\delta\left(t-\sqrt{x^2+\rho'^2}/c\right)\rho'\mathrm{d}\rho'.$$

We again use the property of the Dirac delta function

$$\delta\left(f(\rho')\right) = \delta\left(t-\sqrt{x^2+\rho'^2}/c\right) = \left|\frac{\mathrm{d}f}{\mathrm{d}\rho'}(\rho'_+)\right|^{-1}\delta(\rho'-\rho'_+) + \left|\frac{\mathrm{d}f}{\mathrm{d}\rho'}(\rho'_-)\right|^{-1}\delta(\rho'-\rho'_-),$$

where $\rho'_\pm = \pm\sqrt{c^2t^2-x^2}$, and we require $|x| < ct$. Clearly, ρ'_- is outside the domain of integration. Therefore,

$$\begin{aligned}\Phi(x,t) &= 2\pi\frac{\Theta(ct-|x|)}{\sqrt{x^2+\rho'^2_+}}\left|\frac{\mathrm{d}f}{\mathrm{d}\rho'}(\rho'_+)\right|^{-1}\rho'_+\\ &= \frac{2\pi}{ct}\left|\frac{-\rho'_+}{c\sqrt{x^2+\rho'^2_+}}\right|^{-1}\rho'_+\\ &= 2\pi c\,\Theta(ct-|x|).\end{aligned}$$

□