Legendre polynomials of odd degree

We start from the equation

$$0 = a_0\alpha(\alpha-1)x^{\alpha-2} + a_1\alpha(\alpha+1)x^{\alpha-1}$$
$$+x^\alpha \sum_{j=0}^{\infty} \left\{a_{j+2}(\alpha+j+1)(\alpha+j+2) - a_j\left[(\alpha+j)(\alpha+j+1) - l(l+1)\right]\right\} x^j.$$

Recall that we set $a_0 \neq 0$. Let $\alpha = 1$. Then $a_1 = 0$. Therefore, the power series solution of the Legendre differential equation is given by (with $j \to 2k$)

$$P(x) = \sum_{k=0}^{\infty} a_{2k}x^{2k+1},$$

where the coefficients satisfy the recursion relation

$$a_{2k+2} = \left[\frac{(2k+1)(2k+2) - l(l+1)}{(2k+2)(2k+3)}\right] a_{2k}.$$

This will terminate when $2k+1 = l$. Clearly, l must be an odd integer. Let $l = 1$. Then $P_1(x) = a_0^{(1)}x$. (Note that this a_0 is not the same as above.) Imposing the normalization condition $P(1) = 1$, we obtain

$$P_1(x) = x.$$

Let $l = 3$. Then $P_3(x) = a_0^{(3)}x + a_2^{(3)}x^3$. From the recursion relation, $P_3(x) = a_0^{(3)}\left(x - \frac{5}{3}x^3\right)$. Normalizing, we get

$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right).$$

Let $l = 5$. Then $P_5(x) = a_0^{(5)}x + a_2^{(5)}x^3 + a_4^{(5)}x^5$. From the recursion relation, we have $P_5(x) = a_0^{(5)}\left(x - \frac{14}{3}x^3 + \frac{21}{5}x^5\right)$. Normalizing, we obtain

$$P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right).$$