Problem 5.4

A magnetic induction $\mathbf{B}$ in a current-free region in a uniform medium is cylindrically symmetric with components $B_z(\rho, z)$ and $B_\rho(\rho, z)$ and with a known $B_z(0, z)$ on the axis of symmetry. The magnitude of the axial field varies slowly in z.

(a) Show that near the axis the axial and radial components of magnetic induction are approximately

$$\begin{aligned} B_z(\rho, z) &\approx B_z(0, z) - \left(\frac{\rho^2}{4}\right)\left[\frac{\partial^2 B_z}{\partial z^2}(0, z)\right] + \dots \\ B_\rho(\rho, z) &\approx -\left(\frac{\rho}{2}\right)\left[\frac{\partial B_z}{\partial z}(0, z)\right] + \left(\frac{\rho^3}{16}\right)\left[\frac{\partial^3 B_z}{\partial z^3}(0, z)\right] + \dots \end{aligned}$$

(b) What are the magnitudes of the neglected terms, or equivalently what is the criterion defining "near" the axis?

Solution. We expand $B_z(\rho, z)$ and $B_\rho(\rho, z)$ as a power series around $\rho = 0$. We have

$$\begin{aligned} B_z(\rho, z) &= \sum_{n=0}^{\infty} \frac{a_n(z)}{n!}\rho^n \\ B_\rho(\rho, z) &= \sum_{n=0}^{\infty} \frac{b_n(z)}{n!}\rho^n. \end{aligned}$$

Now, we impose $\nabla \cdot \mathbf{B} = 0$.

$$\begin{aligned} \nabla \cdot \mathbf{B} &= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho B_\rho) + \frac{\partial}{\partial z}B_z \\ &= \sum_{n=0}^{\infty}(n+1)\frac{b_n(z)}{n!}\rho^{n-1} + \sum_{n=0}^{\infty}\frac{a_n'(z)}{n!}\rho^n \\ &= \frac{b_0(z)}{\rho} + \sum_{n=1}^{\infty}(n+1)\frac{b_n(z)}{n!}\rho^{n-1} + \sum_{n=0}^{\infty}\frac{a_n'(z)}{n!}\rho^n \\ &= \frac{b_0(z)}{\rho} + \sum_{n=0}^{\infty}(n+2)\frac{b_{n+1}(z)}{(n+1)!}\rho^n + \sum_{n=0}^{\infty}\frac{a_n'(z)}{n!}\rho^n \\ &= \frac{b_0(z)}{\rho} + \sum_{n=0}^{\infty}\left[(n+2)\frac{b_{n+1}(z)}{(n+1)!} + \frac{a_n'(z)}{n!}\right]\rho^n. \end{aligned}$$

Since ρ is arbitrary the coefficients must vanish independently. Therefore,

$$b_0(z) = 0 \quad \text{and} \quad b_{n+1} = -\frac{n+1}{n+2}a_n'(z).$$

Since the region is current-free we also have $\nabla \times \mathbf{B} = 0$. Clearly, only the ϕ-component is not trivially zero. We have

$$\begin{aligned}(\nabla \times \mathbf{B})_\phi &= \frac{\partial}{\partial z}B_\rho - \frac{\partial}{\partial \rho}B_z \\ &= \sum_{n=0}^{\infty} \frac{b_n'(z)}{n!}\rho^n - \sum_{n=1}^{\infty} \frac{a_n(z)}{(n-1)!}\rho^{n-1} \\ &= \sum_{n=0}^{\infty} \left[b_n'(z) - a_{n+1}(z)\right] \frac{\rho^n}{n!}.\end{aligned}$$

Therefore,

$$b_n'(z) = a_{n+1}(z).$$

Now,

$$\begin{aligned}b_n'(z) &= -\frac{n}{n+1}a_{n-1}''(z) = a_{n+1}(z) \\ a_{n+1}(z) &= -\frac{n}{n+1}a_{n-1}''(z) \\ a_{n+2}(z) &= -\frac{n+1}{n+2}a_n''(z).\end{aligned}$$

Since $b_0(z) = 0$, the odd $a_n(z)$ vanish. Now, from our recursion relation

$$\begin{aligned}a_2(z) &= -\frac{1}{2}a_0''(z) \\ a_4(z) &= -\frac{3}{4}a_2''(z) = \frac{3}{4} \cdot \frac{1}{2}a_0^{(4)}(z) = \frac{3}{8}a_0^{(4)}(z) \\ a_{2k}(z) &= (-1)^k \frac{(n-1)!!}{n!!}a_0^{(2k)}(z).\end{aligned}$$

Since only the even $a_n(z)$ are nonzero, it follows that only the odd $b_n(z)$ are nonzero. Therefore,

$$b_1 = -\frac{1}{2}a_0'(z), \qquad b_3 = -\frac{3}{4}a_2'(z) = \frac{3}{8}a_0^{(3)}(z), \quad \ldots$$

Plugging this back into the series expansion, we immediately see that $a_0(z) = B(0, z)$. Thus,

$$\begin{aligned}B_z(\rho, z) &\approx a_0(z) + \frac{1}{2}a_2(z)\rho^2 + \ldots \\ &\approx B(0, z) - \left(\frac{\rho^2}{4}\right)\left[\frac{\partial^2 B_z}{\partial z^2}(0, z)\right] + \ldots \\ B_\rho(\rho, z) &\approx b_1(z)\rho + \frac{1}{6}b_3(z)\rho^3 + \ldots \\ &\approx -\left(\frac{\rho}{2}\right)\left[\frac{\partial^2 B_z}{\partial z^2}(0, z)\right] + \left(\frac{\rho^3}{16}\right)\left[\frac{\partial^3 B_z}{\partial z^3}(0, z)\right] + \ldots\end{aligned}$$

We assume that the field does not vary in the higher derivatives of $B_z(0, z)$ near the axis. $\square$