Problem 3.3

A thin, flat, conducting, circular disc of radius R is located in the x-y plane with its center at the origin, and is maintained at a fixed potential V. With the information that the charge density on a disc at fixed potential is proportional to $(R^2 - \rho^2)^{-1/2}$, where ρ is the distance out from the center of the disc,

(a) show that for $r > R$ the potential is

$$\Phi(r, \theta, \phi) = \frac{2V}{\pi} \frac{R}{r} \sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1} \left(\frac{R}{r}\right)^{2l} P_{2l}(\cos\theta).$$

(b) find the potential for $r < R$.

(c) What is the capacitance of the disc?

Solution.

(a) We first only consider the region $r > R$. By azimuthal symmetry and the requirement that the potential vanish at infinity, we have the following series ansatz:

$$\Phi(r, \theta, \phi) = \Phi(r, \theta) = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta).$$

Note the symmetry $\theta \to \pi - \theta$; therefore, the odd Legendre polynomials must vanish.

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} B_{2l} r^{-(2l+1)} P_{2l}(\cos\theta).$$

Suppose the surface-charge density is given by

$$\sigma(\rho) = \frac{\alpha}{\sqrt{R^2 - \rho^2}},$$

where α is a constant to be determined. From Coulomb's law, along the axis of symmetry, and using the expansion for $1/|\mathbf{x} - \mathbf{x}'|$ (Equation 3.38), we have

$$\begin{aligned}
\Phi(z) &= \frac{1}{4\pi\epsilon_0} \int_D \frac{\sigma(\rho')}{|\mathbf{x} - \mathbf{x}'|} \mathrm{d}S \\
&= \frac{\alpha}{4\pi\epsilon_0} \int_0^{2\pi} \mathrm{d}\phi' \int_0^R \mathrm{d}\rho' \sum_{l=0}^{\infty} \frac{\rho'}{\sqrt{R^2 - \rho'^2}} \frac{\rho'^l}{z^{l+1}} P_l(0) \\
&= \frac{\alpha}{2\epsilon_0} \sum_{l=0}^{\infty} \frac{1}{z^{2l+1}} P_{2l}(0) \int_0^R \frac{\rho'^{2l+1}}{\sqrt{R^2 - \rho'^2}} \mathrm{d}\rho' \\
&= \frac{\alpha}{2\epsilon_0} \sum_{l=0}^{\infty} \frac{1}{z^{2l}} \frac{(-1)^l}{2l+1} R^{2l+1} \\
&= \frac{\alpha}{2\epsilon_0} \sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1} \left(\frac{R}{z}\right)^{2l+1} \\
\Phi(r, \theta) &= \frac{\alpha}{2\epsilon_0} \sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1} \left(\frac{R}{r}\right)^{2l+1} P_{2l}(\cos\theta).
\end{aligned}$$

We have used the fact that $P_l(0)$ is nonzero only for even l. Now, we impose the boundary condition $\Phi(R, \pi/2) = V$; i.e.

$$V = \frac{\alpha}{2\epsilon_0} \sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1} P_{2l}(0) = \frac{\alpha}{2\epsilon_0} \left(\frac{\pi}{2}\right) = \alpha \frac{\pi}{4\epsilon_0}.$$

Thus,

$$\Phi(r, \theta, \phi) = \frac{2V}{\pi} \frac{R}{r} \sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1} \left(\frac{R}{r}\right)^{2l} P_{2l}(\cos\theta).$$

(b) For $r < R$, we first consider the region above the disk, since the potential in the region below the disk is determined by symmetry. This time, however, the domain is $0 < \theta < \pi/2 = \beta$. Therefore, we have the series ansatz

$$\Phi(r, \theta) = \sum_{\nu=0}^{\infty} A_\nu r^\nu P_\nu(\cos\theta).$$

But, we know that $P_\nu(\cos\beta) = 0$ for odd ν. Therefore,

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} A_{2l+1} r^{2l+1} P_{2l+1}(\cos\theta).$$

Now, we can no longer expand $1/|\mathbf{x} - \mathbf{x}'|$, since it is not always the case that $z > \rho$. Therefore, we shall use the traditional approach. Again, we use Coulomb's law to find the potential along the symmetry axis

$$\begin{aligned}
\Phi(z) &= \frac{1}{4\pi\epsilon_0} \int_D \frac{\sigma(\rho')}{|\mathbf{x} - \mathbf{x}'|} \mathrm{d}S \\
&= \frac{1}{2\epsilon_0} \int_0^R \mathrm{d}\rho' \rho' \frac{\alpha}{\sqrt{R^2 - \rho'^2}} \frac{1}{\sqrt{z^2 + \rho'^2}} \\
&= \frac{\alpha}{2\epsilon_0} \left(\frac{\pi}{2} - \arctan\left(\frac{z}{R}\right)\right) \\
&= \frac{2V}{\pi} \left(\frac{\pi}{2} - \arctan\left(\frac{z}{R}\right)\right) \\
&= V - \frac{2V}{\pi} \arctan\left(\frac{z}{R}\right) \\
&= V - \frac{2V}{\pi} \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l+1)!} \left(\frac{z}{R}\right)^{2l+1} \\
\Phi(r, \theta) &= V - \frac{2V}{\pi} \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l+1)!} \left(\frac{r}{R}\right)^{2l+1} P_{2l+1}(\cos\theta)
\end{aligned}$$

Note that this solution has the expected form (plus some constant). It is also clear that it satisfies the boundary condition $\Phi(0) = V$. Now, we require the potential to be continuous along the boundary $r = R$; it can be seen that $\Phi_{\text{in}}(R, 0) = \Phi_{\text{out}}(R, 0) = \text{V}/2$. It remains to be checked whether the potential remains continuous along different elevations.

Since we require the potential above and below the disk to be symmetric, the potential below the disk is simply

$$\Phi(r,\theta) = V + \frac{2V}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{(2l+1)!}\left(\frac{r}{R}\right)^{2l+1} P_{2l+1}(\cos\theta).$$

The change in sign arises from the fact that the second term is an odd function of θ. It may seem unintuitive that our solution contains odd Legendre polynomials when it should be (plane) symmetric. However, in the region of validity for each solution, the domain of the Legendre polynomials are only half intervals; any interchange $\theta \to \pi - \theta$ will take us outside the original domain. Thus, it makes no sense to impose symmetry conditions on the Legendre polynomials. The symmetry condition must only be imposed after we obtain the solution for one region.

(c) The total charge in the disk is simply

$$Q = \int_D \sigma \mathrm{d}S = 2\pi\alpha \int_0^R \mathrm{d}\rho' \frac{\rho'}{\sqrt{R^2 - \rho'2}} = 2\pi R\alpha = 8\epsilon_0 RV.$$

Therefore, the capacitance is $C = Q/V = 8\epsilon_0 R$.

$\square$