Problem 1.5

The time-averaged potential of a neutral hydrogen atom is given by

$$\Phi = \frac{q}{4\pi\epsilon_0}\frac{e^{-\alpha r}}{r}\left(1+\frac{\alpha r}{2}\right),$$

where q is the magnitude of the electronic charge, and $\alpha^{-1} = a_0/2$, a_0 being the Bohr radius. Find the distribution of charge (both continuous and discrete) that will give this potential and interpret your result physically.

Solution. We can rewrite the potential as

$$\Phi = \frac{q}{4\pi\epsilon_0}e^{-\alpha r}\left(\frac{1}{r}+\frac{\alpha}{2}\right).$$

Note the product rule for the Laplace operator: $\nabla^2(fg) = g\nabla^2 f + f\nabla^2 g + 2\nabla f\cdot\nabla g$. Therefore,

$$\begin{aligned}
\nabla^2\Phi &= \frac{q}{4\pi\epsilon_0}\left[\left(\frac{1}{r}+\frac{\alpha}{2}\right)\nabla^2 e^{-\alpha r} + e^{-\alpha r}\nabla^2\left(\frac{1}{r}+\frac{\alpha}{2}\right) + 2\nabla e^{-\alpha r}\cdot\nabla\left(\frac{1}{r}+\frac{\alpha}{2}\right)\right] \\
&= \frac{q}{4\pi\epsilon_0}\left[\left(\frac{1}{r}+\frac{\alpha}{2}\right)\nabla^2 e^{-\alpha r} + e^{-\alpha r}\nabla^2\frac{1}{r} + 2\nabla e^{-\alpha r}\cdot\nabla\frac{1}{r}\right].
\end{aligned}$$

Since $\Phi = \Phi(r)$, $\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\Phi\right)$. We also know that $\nabla^2\frac{1}{r} = -4\pi\delta^3(r)$.

$$\begin{aligned}
\nabla^2\Phi &= \frac{q}{4\pi\epsilon_0}e^{-\alpha r}\left[\left(\frac{\alpha^3}{2}-\frac{2\alpha}{r^2}\right) - 4\pi\delta^3(r) + \frac{2\alpha}{r^2}\right] \\
&= \frac{q}{4\pi\epsilon_0}e^{-\alpha r}\left(\frac{\alpha^3}{2} - 4\pi\delta^3(r)\right) \\
&= \frac{q\alpha^3}{8\pi\epsilon_0}e^{-\alpha r} - \frac{q}{\epsilon_0}\delta^3(r).
\end{aligned}$$

Since $\nabla^2\Phi = -\rho/\epsilon_0$,

$$\rho(r) = q\delta^3(r) - \frac{q\alpha^3}{8\pi}e^{-\alpha r}.$$

We can physically interpret this charge distribution as a point charge (discrete) surrounded by a ‘cloud’ of charges (continuous) whose density decreases exponentially in r.

□