Section 5.12: Magnetic shielding

Consider a spherical shell of inner (outer) radius a (b), made of material of permeability μ, and placed in a formerly uniform constant magnetic induction $\mathbf{B}_0$. We wish to find the fields $\mathbf{B}$ and $\mathbf{H}$ everywhere in space. Since there are no free currents present, the magnetic field is derivable from a scalar potential; $\mathbf{H} = -\nabla\Phi_M$. Since $\mathbf{B} = \mu\mathbf{H}$, then $\nabla \cdot \mathbf{H} = 0$ for all regions and the potential satisfies the Laplace equation everywhere. Since the problem involves a spherical shell we adopt spherical coordinates. Moreover, we let $\mathbf{B_0} = B_0\hat{\mathbf{z}}$. Since there is an azimuthal symmetry, the potential in each region can be expressed as

$$\Phi_M = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta).$$

We denote the regions: I. ($0 < r < a$), II. ($a < r < b$), and III. ($r > b$). Now, since the magnetic induction is $\mathbf{B}_0 = \mu_0\mathbf{H}_0$ very far away, then $\Phi_M^{(\mathrm{III})} \simeq -H_0 z = -H_0 r\cos\theta$ as r grows very large. Therefore,

$$\Phi_M^{(\mathrm{III})} = -H_0 r\cos\theta + \sum_{l=0}^{\infty} \frac{\alpha_l}{r^{l+1}} P_l(\cos\theta).$$

Also, the potential must be finite at $r = 0$. Thus, the potential in the remaining regions can be expressed as

$$\Phi_M^{(\mathrm{II})} = \sum_{l=0}^{\infty} \left(\beta_l r^l + \frac{\gamma_l}{r^{l+1}} \right) P_l(\cos\theta),$$
$$\Phi_M^{(\mathrm{I})} = \sum_{l=0}^{\infty} \delta_l r^l P_l(\cos\theta).$$

We now apply the boundary conditions. From Equations 5.88 and 5.89, we see that the normal component of $\mathbf{B}$ and the tangential component $\mathbf{H}$ must be continuous. Therefore, we get the following conditions

$$\left.\frac{\partial\Phi_M^{(\mathrm{III})}}{\partial\theta}\right|_{r=b} = \left.\frac{\partial\Phi_M^{(\mathrm{II})}}{\partial\theta}\right|_{r=b}, \qquad \left.\frac{\partial\Phi_M^{(\mathrm{II})}}{\partial\theta}\right|_{r=a} = \left.\frac{\partial\Phi_M^{(\mathrm{I})}}{\partial\theta}\right|_{r=a},$$
$$\mu_0\left.\frac{\partial\Phi_M^{(\mathrm{III})}}{\partial r}\right|_{r=b} = \mu\left.\frac{\partial\Phi_M^{(\mathrm{II})}}{\partial r}\right|_{r=b}, \qquad \mu\left.\frac{\partial\Phi_M^{(\mathrm{II})}}{\partial r}\right|_{r=a} = \mu_0\left.\frac{\partial\Phi_M^{(\mathrm{I})}}{\partial r}\right|_{r=a}.$$

Let $P_l^{(')}(\cos\theta) = \partial_\theta\left[P_l(\cos\theta)\right]$. Now, we list the terms for each equation:

$$\left.\frac{\partial\Phi_M^{(\mathrm{III})}}{\partial\theta}\right|_{r=b} = H_0 b\sin\theta + \sum_{l=1}^{\infty} \frac{\alpha_l}{b^{l+1}} P_l^{(')}(\cos\theta),$$
$$\left.\frac{\partial\Phi_M^{(\mathrm{II})}}{\partial\theta}\right|_{r=b} = \sum_{l=1}^{\infty} \left(\beta_l b^l + \frac{\gamma_l}{b^{l+1}} \right) P_l^{(')}(\cos\theta);$$

$$\left.\frac{\partial\Phi_M^{(\mathrm{II})}}{\partial\theta}\right|_{r=a} = \sum_{l=1}^{\infty} \left(\beta_l a^l + \frac{\gamma_l}{a^{l+1}} \right) P_l^{(')}(\cos\theta),$$
$$\left.\frac{\partial\Phi_M^{(\mathrm{I})}}{\partial\theta}\right|_{r=a} = \sum_{l=1}^{\infty} \delta_l a^l P_l^{(')}(\cos\theta);$$

$$\mu_0 \frac{\partial \Phi_M^{(\mathrm{III})}}{\partial r}\bigg|_{r=b} = -\mu_0 H_0 \cos\theta - \mu_0 \sum_{l=0}^{\infty} (l+1) \frac{\alpha_l}{b^{l+2}} P_l(\cos\theta),$$

$$\mu \frac{\partial \Phi_M^{(\mathrm{II})}}{\partial r}\bigg|_{r=b} = \mu \sum_{l=1}^{\infty} l\beta_l b^{l-1} P_l(\cos\theta) - \mu \sum_{l=0}^{\infty} (l+1) \frac{\gamma_l}{b^{l+2}} P_l(\cos\theta);$$

$$\mu \frac{\partial \Phi_M^{(\mathrm{II})}}{\partial r}\bigg|_{r=a} = \mu \sum_{l=1}^{\infty} l\beta_l a^{l-1} P_l(\cos\theta) - \mu \sum_{l=0}^{\infty} (l+1) \frac{\gamma_l}{a^{l+2}} P_l(\cos\theta),$$

$$\mu_0 \frac{\partial \Phi_M^{(\mathrm{I})}}{\partial r}\bigg|_{r=a} = \mu_0 \sum_{l=1}^{\infty} l\delta_l a^{l-1} P_l(\cos\theta).$$

Note that the first term has a factor of $\sin\theta$, therefore if we wish to find coefficients that do not depend on θ only the $P_1^{(\prime)}(\cos\theta) = -\sin\theta$ terms can be nonzero. Similarly, the fifth term has a factor of $\cos\theta$, therefore only the $P_1(\cos\theta)$ terms can be nonzero. We have the following matrix equation

$$\begin{pmatrix} -b^{-2} & b & b^{-2} & 0 \\ 0 & a & a^{-2} & -a \\ -2\mu_0 b^{-3} & -\mu & 2\mu b^{-3} & 0 \\ 0 & \mu & -2\mu a^{-3} & -\mu_0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \\ \delta_1 \end{pmatrix} = \begin{pmatrix} -bH_0 \\ 0 \\ \mu_0 H_0 \\ 0 \end{pmatrix}.$$

Inverting the matrix equation, we get

$$\alpha_1 = \left[\frac{(\mu'-1)(2\mu'+1)}{(2\mu'+1)(\mu'+2) - 2(\mu'-1)^2 a^3/b^3}\right](b^3 - a^3)H_0,$$

$$\beta_1 = -\left[\frac{3(2\mu'+1)}{(2\mu'+1)(\mu'+2) - 2(\mu'-1)^2 a^3/b^3}\right]H_0,$$

$$\gamma_1 = -\left[\frac{3(\mu'-1)a^3}{(2\mu'+1)(\mu'+2) - 2(\mu'-1)^2 a^3/b^3}\right]H_0,$$

$$\delta_1 = -\left[\frac{9\mu'}{(2\mu'+1)(\mu'+2) - 2(\mu'-1)^2 a^3/b^3}\right]H_0,$$

where $\mu' = \mu/\mu_0$.