Problem 3.10

For the cylinder in Problem 3.9 the cylindrical surface is made of two equal half-cylinders, one at potential V and the other at potential $-V$, so that

$$V(\phi, z) = \begin{cases} V & \text{for } -\pi/2 < \phi < \pi/2 \\ -V & \text{for } \pi/2 < \phi < 3\pi/2 \end{cases}.$$

(a) Find the potential inside the cylinder.

(b) Assuming $L \gg b$, consider the potential at $z = L/2$ as a function of ρ and ϕ and compare it with two-dimensional Problem 2.13.

Solution.

(a) We are to find the potential inside a cylinder of radius b and length L. The potential is zero at $z = 0$ and $z = L$. The basic solution takes the form

$$\Phi(\rho, \phi, z) = R(\rho)Q(\phi)Z(z).$$

The potential must be single valued when the full azimuth is allowed. Therefore,

$$Q(\phi) = A \sin(m\phi) + B \cos(m\phi),$$

where m is an integer. Also, the potential must vanish when $z = 0$ or $z = L$. Therefore, we must replace the separation constant k^2 with $-k^2$, such that

$$Z(z) = \sin(kz) = \sin(n\pi z/L),$$

where n is an integer. Clearly, the radial solution will be

$$R(\rho) = CI_m(k\rho) + DK_m(k\rho) = CI_m(n\pi\rho/L),$$

since we require the potential to be finite at $\rho = 0$. Thus, the general solution takes the form

$$\Phi(\rho, \phi, z) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} I_m(n\pi\rho/L)\sin(n\pi z/L)\left(A_{mn}\sin(m\phi) + B_{mn}\cos(m\phi)\right).$$

Now,

$$\Phi(b, \phi, z) = V(\phi, z) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} I_m(n\pi b/L)\sin(n\pi z/L)\left(A_{mn}\sin(m\phi) + B_{mn}\cos(m\phi)\right).$$

Using the orthogonality relations,

$$\int_0^L \mathrm{d}z \sin(n\pi z/L) \int_0^{2\pi} \mathrm{d}\phi \sin(m\phi)V(\phi, z) = I_m(n\pi b/L) \cdot \frac{L}{2} \cdot \pi \cdot A_{mn},$$
$$\int_0^L \mathrm{d}z \sin(n\pi z/L) \int_0^{2\pi} \mathrm{d}\phi \cos(m\phi)V(\phi, z) = I_m(n\pi b/L) \cdot \frac{L}{2} \cdot \pi \cdot B_{mn}.$$

Rewriting, we have

$$\begin{aligned}
A_{mn} &= \frac{1}{I_m(n\pi b/L)}\frac{2}{\pi L}\frac{L}{n\pi}\left(1-(-1)^n\right)\int_0^{2\pi} \mathrm{d}\phi \sin{(m\phi)} V(\phi, z), \\
B_{mn} &= \frac{1}{I_m(n\pi b/L)}\frac{2}{\pi L}\frac{L}{n\pi}\left(1-(-1)^n\right)\int_0^{2\pi} \mathrm{d}\phi \cos{(m\phi)} V(\phi, z).
\end{aligned}$$

Evaluating, we find that $A_{mn} = 0$. Also, it is clear that B_{mn} is zero for even n. We find that B_{mn} is only nonzero if both m and n are odd; i.e.

$$\begin{aligned}
B_{2p+1,2q+1} &= \frac{1}{I_{2p+1}((2q+1)\pi b/L)}\frac{4}{(2q+1)\pi^2}\cdot 4V\frac{(-1)^p}{2p+1} \\
&= \frac{(-1)^p}{I_{2p+1}((2q+1)\pi b/L)}\frac{16V}{(2p+1)(2q+1)\pi^2}.
\end{aligned}$$

Thus, the potential is given by

$$\Phi(\rho,\phi,z) = \frac{16V}{\pi^2}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^p\frac{\sin{((2q+1)\pi z/L)}}{(2p+1)(2q+1)}\frac{I_{2p+1}((2q+1)\pi\rho/L)}{I_{2p+1}((2q+1)\pi b/L)}\cos{((2p+1)\phi)}.$$

(b) Suppose $L \gg b$ and let $z = L/2$. Then, using Equation 3.102,

$$\begin{aligned}
\Phi(\rho,\phi,z) &= \frac{16V}{\pi^2}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(-1)^{p+q}}{(2p+1)(2q+1)}\left(\frac{\rho}{b}\right)^{2p+1}\cos{((2p+1)\phi)} \\
&= \frac{16V}{\pi^2}\sum_{p=0}^{\infty}\frac{(-1)^p}{(2p+1)}\left(\frac{\rho}{b}\right)^{2p+1}\cos{((2p+1)\phi)}\sum_{q=0}^{\infty}\frac{(-1)^q}{2q+1} \\
&= \frac{16V}{\pi^2}\sum_{p=0}^{\infty}\frac{(-1)^p}{(2p+1)}\left(\frac{\rho}{b}\right)^{2p+1}\cos{((2p+1)\phi)}\cdot\frac{\pi}{4} \\
&= \frac{4V}{\pi}\sum_{p=0}^{\infty}\frac{(-1)^p}{(2p+1)}\left(\frac{\rho}{b}\right)^{2p+1}\cos{((2p+1)\phi)} \\
&= \frac{4V}{\pi}\sum_{p=0}^{\infty}\frac{(-1)^p}{(2p+1)}\left(\frac{\rho}{b}\right)^{2p+1}\Re(e^{i(2p+1)\phi}) \\
&= \frac{4V}{\pi}\arctan\left(\frac{\rho}{b}e^{i\phi}\right) \\
&= \frac{2V}{\pi}\arg\left(\frac{i-e^{i\phi}\rho/b}{i+e^{i\phi}\rho/b}\right) \\
&= \frac{2V}{\pi}\arctan\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right),
\end{aligned}$$

consistent with the solution for Problem 2.13.

$\square$

Problem 3.12

An infinite, thin, plane sheet of conducting material has a circular hole of radius a cut in it. A thin, flat disc of the same material and slightly smaller radius lies in the plane, filling the hole, but separated from the sheet by a very narrow insulating ring. The disc is maintained at a fixed potential V, while the infinite sheet is kept at zero potential.

(a) Using appropriate cylindrical coordinates, find an integral expression involving Bessel functions for the potential at any point above the plane.

(b) Show that the potential a perpendicular distance z above the center of the disc is

$$\Phi_0(z) = V\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right).$$

(c) Show that the potential a perpendicular distance z above the edge of the disc is

$$\Phi_a(z) = \frac{V}{2}\left[1 - \frac{\kappa z}{\pi a}K(\kappa)\right],$$

where $\kappa = 2a/(z^2+4a^2)^{1/2}$, and $K(\kappa)$ is the complete elliptic integral of the first kind.

Solution.

(a) By planar symmetry, we only need to determine the potential in the upper half-space. Moreover, the system possesses azimuthal symmetry, and the potential must vanish as z tends to infinity. Using Equation 3.106, we have the following general solution

$$\Phi(\rho, z) = \int_0^\infty \frac{1}{2}B_0(k)e^{-kz}J_0(k\rho)\mathrm{d}k.$$

We now apply the boundary condition

$$\Phi(\rho, 0) = V(\rho) = \frac{1}{2}\int_0^\infty B_0(k)J_0(k\rho)\mathrm{d}k.$$

Using Equation 3.109, we have

$$B_0(k) = 2k\int_0^\infty \rho V(\rho)J_0(k\rho)\mathrm{d}\rho = 2kV\int_0^a \rho J_0(k\rho)\mathrm{d}\rho.$$

Thus,

$$\Phi(\rho, z) = V\int_0^\infty \left(\int_0^a \rho' J_0(k\rho')\mathrm{d}\rho'\right) ke^{-kz}J_0(k\rho)\mathrm{d}k.$$

(b) Note that $J_0(0) = 1$.

$$\begin{aligned}
\Phi_0(z) = \Phi(0, z) &= V\int_0^\infty \left(\frac{a}{k}J_1(ka)\right) ke^{-kz}(1)\mathrm{d}k \\
&= Va\int_0^\infty J_1(ka)e^{-kz}\mathrm{d}k \\
&= V\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right).
\end{aligned}$$

(c) Similarly,

$$\Phi_a(z) = \Phi(a, z) = Va \int_0^\infty J_1(ka)e^{-kz} J_0(ka)\mathrm{d}k.$$

We use the result in p.117 Section 2.6 of Okui, *Complete Elliptic Integrals Resulting from Infinite Integrals of Bessel Functions. II* (1975),

$$\begin{aligned}\Phi_a(z) &= Va \int_0^\infty J_1(ka)e^{-kz} J_0(ka)\mathrm{d}k \\ &= Va\left(\frac{1}{2a} - \frac{z\kappa}{2\pi a^2}K(\kappa)\right) \\ &= \frac{V}{2}\left[1 - \frac{\kappa z}{\pi a}K(\kappa)\right],\end{aligned}$$

where $\kappa = 2a/(z^2 + 4a^2)^{1/2}$, and $K(\kappa)$ is the complete elliptic integral of the first kind.

□