Problem 3.17

The Dirichlet Green function for the unbounded space between the planes at $z = 0$ and $z = L$ allows discussion of a point charge or a distribution of charge between parallel conducting planes held at zero potential.

(a) Using cylindrical coordinates show that one form of the Green function is

$$G(\mathbf{x}, \mathbf{x}') = \frac{4}{L} \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \sin\left(\frac{n\pi z}{L}\right) \sin\left(\frac{n\pi z'}{L}\right) I_m\left(\frac{n\pi}{L}\rho_<\right) K_m\left(\frac{n\pi}{L}\rho_>\right).$$

(b) Show that an alternative form of the Green function is

$$G(\mathbf{x}, \mathbf{x}') = 2 \sum_{m=-\infty}^{\infty} \int_0^\infty \mathrm{d}k\, e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') \frac{\sinh(kz_<) \sinh[k(L - z_>)]}{\sinh(kL)}.$$

Solution.

(a) We start with the Green function expansion (Equation 3.140)

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{2\pi^2} \sum_{m=-\infty}^{\infty} \int_0^\infty \mathrm{d}k\; e^{im(\phi-\phi')} \cos[k(z - z')] g_m(k, \rho, \rho'),$$

with

$$g_m(k, \rho, \rho') = \psi_1(k\rho_<)\psi_2(k\rho_>),$$

where $\psi_1(k\rho) = AI_m(k\rho)$ and $\psi_2(k\rho) = K_m(k\rho)$, since $g_m(k, \rho, \rho')$ must be finite at $\rho = 0$ and vanish as $\rho \to \infty$. Following the arguments in Section 3.11, we obtain $A = 4\pi$. Using the trigonometric identity $\cos(x - y) = \cos x \cos y + \sin x \sin y$, we can rewrite the Green function as

$$G(\mathbf{x}, \mathbf{x}') = \frac{2}{\pi} \sum_{m=-\infty}^{\infty} \int_0^\infty \mathrm{d}k\; e^{im(\phi-\phi')} [\cos(kz) \cos(kz') + \sin(kz) \sin(kz')] I_m(k\rho_<) K_m(k\rho_>).$$

Now, we require the Green function to vanish at $z = 0$ and $z = L$. Therefore, the $\cos(kz)\cos(kz')$ terms must vanish; and $k = n\pi/L$, where $n \in \mathbb{Z}$. Thus, if we make the replacement $\displaystyle\int_0^\infty \mathrm{d}k \to \frac{2\pi}{L} \sum_{n=1}^{\infty}$ (the factor of 2 accounts for negative n),

$$G(\mathbf{x}, \mathbf{x}') = \frac{4}{L} \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \sin\left(\frac{n\pi z}{L}\right) \sin\left(\frac{n\pi z'}{L}\right) I_m\left(\frac{n\pi}{L}\rho_<\right) K_m\left(\frac{n\pi}{L}\rho_>\right).$$

(b) Another way of expressing the Green function is by expanding using undetermined functions of z instead of ρ; i.e., we use the generalization of Equation 3.108,

$$\frac{1}{\rho}\delta(\rho - \rho') = \int_0^\infty k J_m(k\rho) J_m(k\rho') \mathrm{d}k.$$

Therefore,

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_0^\infty \mathrm{d}k\, k e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') g_m(k, z, z'),$$

with

$$g_m(k, z, z') = \psi_1(kz_<)\psi_2(kz_>).$$

Clearly, if we substitute this in Equation 3.138:

$$\nabla_x^2 G(\mathbf{x}, \mathbf{x}') = -\frac{4\pi}{\rho}\delta(\rho - \rho')\delta(\phi - \phi')\delta(z - z'),$$

we obtain the following differential equation

$$\frac{\mathrm{d}^2 g_m}{\mathrm{d}z^2} - k^2 g_m = -4\pi\delta(z - z').$$

For $z \neq z'$, the solutions are just the hyperbolic functions. Now, we apply the boundary conditions for ψ_1 $[z < z']$ and ψ_2 $[z > z']$. We want ψ_1 to vanish at $z = 0$, therefore $\psi_1(kz) = A\sinh(kz)$. We also want $\psi_2(kz) = A'\sinh(kz) + B'\cosh(kz)$ to vanish at $z = L$, and this is satisfied by choosing $A' = -\cosh(kL)$ and $B' = \sinh(kL)$; such that $\psi_2(kz) = \sinh(kL)\cosh(kz) - \cosh(kL)\sinh(kz) = \sinh[k(L - z)]$. Therefore,

$$g_m(k, z, z') = A\sinh(kz_<)\sinh[k(L - z_>)].$$

The normalization constant A is determined from the discontinuity in the slope. Using similar arguments as Equation 3.144,

$$\begin{aligned}
\left.\frac{\mathrm{d}g_m}{\mathrm{d}z}\right|_+ - \left.\frac{\mathrm{d}g_m}{\mathrm{d}z}\right|_- &= -kA\sinh(kz')\cosh[k(L - z')] - kA\cosh(kz')\sinh[k(L - z')] \\
-4\pi &= -kA\sinh(kL) \\
A &= \frac{4\pi}{k\sinh(kL)}.
\end{aligned}$$

Thus,

$$G(\mathbf{x}, \mathbf{x}') = 2\sum_{m=-\infty}^{\infty} \int_0^\infty \mathrm{d}k\, e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') \frac{\sinh(kz_<)\sinh[k(L - z_>)]}{\sinh(kL)}.$$

$\square$