Problem 3.2

A spherical surface of radius R has charge uniformly distributed over its surface with a density $Q/4\pi R^2$, except for a spherical cap at the north pole, defined by the cone $\theta = \alpha$.

(a) Show that the potential inside the spherical surface can be expressed as

$$\Phi = \frac{Q}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right]\frac{r^l}{R^{l+1}}P_l(\cos\theta),$$

where, for $l = 0$, $P_{l-1}(\cos\alpha) = -1$. What is the potential outside?

(b) Find the magnitude and the direction of the electric field at the origin.

(c) Discuss the limiting forms of the potential (part a) and electric field (part b) as the spherical cap becomes (1) very small, and (2) so large that the area with charge on it becomes a very small cap at the south pole.

Solution.

(a) Since we have azimuthal symmetry, and there are no charges inside the sphere, the potential inside the sphere can be expressed as a sum:

$$\Phi_{\text{in}}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta).$$

Similarly, the potential outside the sphere is

$$\Phi_{\text{out}}(r,\theta) = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta).$$

Now, we have the following boundary conditions:

$$\Phi_{\text{in}}(R,\theta) = \Phi_{\text{out}}(R,\theta),$$
$$\frac{\sigma(\theta)}{\epsilon_0} = E_{r,\text{out}}(R,\theta) - E_{r,\text{in}}(R,\theta).$$

Clearly, from the first boundary condition we require

$$B_l = A_l R^{2l+1}.$$

From the second boundary condition,

$$E_{r,\text{out}}(R,\theta) = -\frac{\partial\Phi_{\text{out}}}{\partial r}(R,\theta) = \sum_{l=0}^{\infty} B_l(l+1)R^{-(l+2)}P_l(\cos\theta),$$

and

$$E_{r,\text{in}}(R,\theta) = -\frac{\partial\Phi_{\text{in}}}{\partial r}(R,\theta) = -\sum_{l=1}^{\infty} A_l l R^{l-1} P_l(\cos\theta).$$

Therefore,

$$\begin{aligned}\frac{\sigma(\theta)}{\epsilon_0} &= \frac{B_0}{R^2} + \sum_{l=1}^{\infty} \left[B_l(l+1)R^{-(l+2)} + A_l l R^{l-1}\right] P_l(\cos\theta)\\ &= \frac{A_0}{R} + \sum_{l=1}^{\infty} (2l+1)\, A_l R^{l-1} P_l(\cos\theta).\end{aligned}$$

For $l \geq 1$,

$$\begin{aligned}A_l &= \frac{1}{2\epsilon_0 R^{l-1}} \int_{-1}^{1} \sigma(\theta) P_l(\cos\theta) \mathrm{d}(\cos\theta)\\ &= \frac{Q}{8\pi\epsilon_0 R^{l+1}} \int_{-1}^{\cos\alpha} P_l(\cos\theta) \mathrm{d}(\cos\theta).\end{aligned}$$

From Equation 3.28,

$$P'_{l+1}(x) - P'_{l-1}(x) = (2l+1)P_l(x).$$

Therefore,

$$A_l = \frac{Q}{8\pi\epsilon_0(2l+1)R^{l+1}} \left[P_{l+1}(x) - P_{l-1}(x)\right]\Bigg|_{-1}^{\cos\alpha}.$$

Now, if $P_{l+1}(x)$ is odd (even) then so is $P_{l-1}(x)$. Thus,

$$A_l = \frac{Q}{8\pi\epsilon_0(2l+1)R^{l+1}} \left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right].$$

Finally, the potential inside the sphere is

$$\Phi_{\text{in}}(r,\theta) = A_0 + \frac{Q}{8\pi\epsilon_0} \sum_{l=1}^{\infty} \frac{1}{2l+1} \left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right] \left(\frac{r^l}{R^{l+1}}\right) P_l(\cos\theta).$$

Similarly, the potential outside the sphere is

$$\Phi_{\text{out}}(r,\theta) = \frac{A_0 R}{r} + \frac{Q}{8\pi\epsilon_0} \sum_{l=1}^{\infty} \frac{1}{2l+1} \left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right] \left(\frac{R^l}{r^{l+1}}\right) P_l(\cos\theta).$$

Since the potential very far away from the sphere is essentially that due to a point charge

$$Q_\alpha = \int R^2 \sigma(\theta) \mathrm{d}\Omega = \frac{Q}{2} \int_{-1}^{\cos\alpha} \mathrm{d}(\cos\theta) = \frac{Q}{2}(1+\cos\alpha).$$

In other words, we must have $A_0 = Q(1+\cos\alpha)/8\pi\epsilon_0 R$.

$$\begin{aligned}\Phi_{\text{in}}(r,\theta) &= \frac{Q}{8\pi\epsilon_0 R}(1+\cos\alpha) + \frac{Q}{8\pi\epsilon_0} \sum_{l=1}^{\infty} \frac{1}{2l+1} \left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right] \left(\frac{r^l}{R^{l+1}}\right) P_l(\cos\theta),\\ \Phi_{\text{out}}(r,\theta) &= \frac{Q}{8\pi\epsilon_0 r}(1+\cos\alpha) + \frac{Q}{8\pi\epsilon_0} \sum_{l=1}^{\infty} \frac{1}{2l+1} \left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right] \left(\frac{R^l}{r^{l+1}}\right) P_l(\cos\theta).\end{aligned}$$

If we define $P_{-1}(\cos\alpha) = -1$, then

$$\Phi_{\text{in}}(r,\theta) = \frac{Q}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\big[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\big]\left(\frac{r^l}{R^{l+1}}\right)P_l(\cos\theta),$$
$$\Phi_{\text{out}}(r,\theta) = \frac{Q}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\big[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\big]\left(\frac{R^l}{r^{l+1}}\right)P_l(\cos\theta).$$

(b) We know that $\mathbf{E} = -\nabla\Phi$. Due to the azimuthal symmetry, $\mathbf{E}$ must point along the $\hat{\mathbf{z}}$ direction. By continuity, we may just take the limit of E as we approach the origin along $\theta = 0$. Therefore,

$$\lim_{r\to 0} E_{r,\text{in}}(r,0) = -A_1 P_1(1) = -\frac{Q}{24\pi\epsilon_0 R^2}\big[P_2(\cos\alpha) - P_0(\cos\alpha)\big],$$
$$\mathbf{E}(0,0) = \frac{Q}{16\pi\epsilon_0 R^2}\left(1-\cos^2\alpha\right)\hat{\mathbf{z}} = \frac{Q\sin^2\alpha}{16\pi\epsilon_0 R^2}\hat{\mathbf{z}}.$$

(c) We first consider the potential. As the spherical cap beomes very small, $\cos\alpha \approx 1 - \alpha^2/2$. Define

$$P(l,\alpha) = \frac{1}{2l+1}\big[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\big].$$

For small enough α, we find that $P(0,\alpha) = 2-\alpha^2/2+\mathcal{O}(\alpha^4)$, and $P(k,\alpha) = -\alpha^2/2+\mathcal{O}(\alpha^4)$ for $k \in \mathbb{Z}_+$. Therefore,

$$\Phi_{\text{in}}(r,\theta) = \frac{Q}{4\pi\epsilon_0 R} - \frac{Q\alpha^2}{16\pi\epsilon_0}\sum_{l=0}^{\infty}\left(\frac{r^l}{R^{l+1}}\right)P_l(\cos\theta),$$
$$\Phi_{\text{out}}(r,\theta) = \frac{Q}{4\pi\epsilon_0 r} - \frac{Q\alpha^2}{16\pi\epsilon_0}\sum_{l=0}^{\infty}\left(\frac{R^l}{r^{l+1}}\right)P_l(\cos\theta).$$

For Φ_{in}, the first term is just a constant which we can ignore, while the second term is the potential due to a point charge. For Φ_{out}, this is just the superposition of the potential due to a full sphere of charge Q and a point charge. For the second case, we let $\beta = \pi - \alpha$ be the small parameter. This time, we expect both Φ_{in} and Φ_{out} to be due to the point charge only. We now consider the electric field at the origin. As the spherical cap becomes very small or very large

$$\mathbf{E}(0,0) \approx \frac{Q}{16\pi\epsilon_0 R^2}\alpha^2\hat{\mathbf{z}} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}\hat{\mathbf{z}},$$

where we have defined

$$q := \frac{Q}{4\pi R^2}\pi(R\alpha)^2.$$

In other words, we can approximate the small spherical cap as a point charge q. As the cap becomes very small, we can think of the electric field at the origin due to the given configuration as the superposition of the field due to the full sphere of charge density $\sigma = Q/4\pi R^2$, which is zero by Gauss's law, plus the field due to the small cap which we can approximate as a small disk with density $-\sigma$ and area $A \approx \pi(R\alpha)^2$. Similarly, as the cap becomes very large, we can think of the charged cap at the south pole as a positive point charge.

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