## Potential of a square sheet

(a) Obtain the scalar potential at a distance $d$ from the center of and perpendicular to a square sheet with a constant dipole moment density $D$.

(b) How does the potential behave for arbitrarily large $d$?

*Solution.*

(a) Suppose the square sheet lies on the $x$-$y$ plane and is centered at the origin, such the observation point $P$ lies along the $z$-axis. We use (1.26):

$$\Phi(\mathbf{x}) = -\frac{1}{4\pi\epsilon_0}\int_S D(\mathbf{x}')\mathrm{d}\Omega,$$

where $\mathrm{d}\Omega$ is the solid angle subtended by the square sheet $S$ from the observation point $P$. Without loss of generality, we let $P$ be a distance $d$ above the center, $\mathcal{O}$, of the square sheet. In our convention, the unit normal to $S$ is $-\hat{\mathbf{z}}$. Also, let $2a$ be the side length of the square. Then, since the dipole moment density is constant,

$$\begin{aligned}
\Phi(\mathbf{x}) &= -\frac{D}{4\pi\epsilon_0}\int_S \mathrm{d}\Omega = -\frac{D}{4\pi\epsilon_0}\int_S \frac{\cos\theta}{|\mathbf{x}-\mathbf{x}'|^2}\mathrm{d}a' \\
\Phi(P) &= -\frac{D}{4\pi\epsilon_0}\int_{-a}^{a}\int_{-a}^{a}\frac{\cos\theta}{x^2+y^2+d^2}\mathrm{d}x\mathrm{d}y \\
&= -\frac{D}{4\pi\epsilon_0}\int_{-a}^{a}\int_{-a}^{a}\frac{d}{(x^2+y^2+d^2)^{3/2}}\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Note that $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{x}{\sqrt{1+x^2}} = \dfrac{1}{(1+x^2)^{3/2}}$. Therefore, if we let $u = \dfrac{x}{\sqrt{y^2+d^2}}$,

$$\begin{aligned}
\Phi(P) &= -\frac{Dd}{4\pi\epsilon_0}\int_{-a}^{a}\frac{1}{y^2+d^2}\left(\frac{x}{\sqrt{x^2+y^2+d^2}}\right)\Bigg|_{x=-a}^{x=a}\mathrm{d}y \\
&= -\frac{Dd}{4\pi\epsilon_0}\int_{-a}^{a}\frac{1}{y^2+d^2}\frac{2a}{\sqrt{a^2+y^2+d^2}}\mathrm{d}y.
\end{aligned}$$

Using Mathematica,

$$\begin{aligned}
\Phi(P) &= -\frac{Dd}{4\pi\epsilon_0}\left(\frac{2}{d}\arctan\left[\frac{ay}{d\sqrt{a^2+y^2+d^2}}\right]\right)\Bigg|_{y=-a}^{y=a} \\
&= -\frac{D}{\pi\epsilon_0}\arctan\left[\frac{a^2}{d\sqrt{2a^2+d^2}}\right],
\end{aligned}$$

where we have used the fact that arctan is an odd function.

(b) We may further write

$$\Phi(P) = -\frac{D}{\pi\epsilon_0}\arctan\left[\frac{a^2}{d^2}\left(1+2\frac{a^2}{d^2}\right)^{-1/2}\right].$$

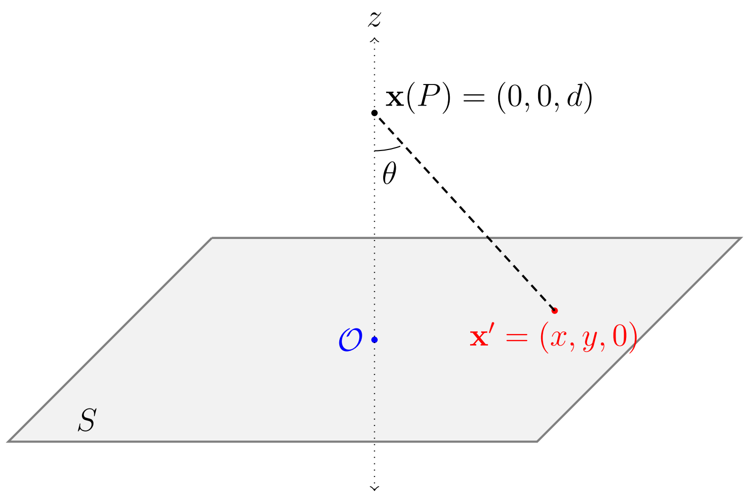


Figure 1: A square sheet with constant dipole moment density $D$.

For arbitrarily large $d$, we may expand this as a power series in $a/d$.

$$\Phi(P) = -\frac{D}{\pi\epsilon_0}\left[\frac{a^2}{d^2} - \frac{a^4}{d^4} + \dots\right].$$

If we only keep the leading order term, and use the fact that the area of the square sheet is $A = 4a^2$, we obtain

$$\Phi(P) \approx -\frac{1}{4\pi\epsilon_0}\frac{DA}{d^2}.$$

This is the potential due to a dipole $\mathbf{p} = DA(-\hat{\mathbf{z}})$.

$\square$