Problem 4.8

A very long, right circular, cylindrical shell of dielectric constant ϵ/ϵ_0 and inner and outer radii a and b, respectively, is placed in a previously uniform electric field E_0 with its axis perpendicular to the field. The medium inside and outside the cylinder has a dielectric constant of unity.

(a) Determine the potential and electric field in the three regions, neglecting end effects.

(b) Sketch the lines of force for a typical case of $b \simeq 2a$.

(c) Discuss the limiting forms of your solution appropriate for a solid dielectric cylinder in a uniform field, and a cylindrical cavity in a uniform dielectric.

Solution.

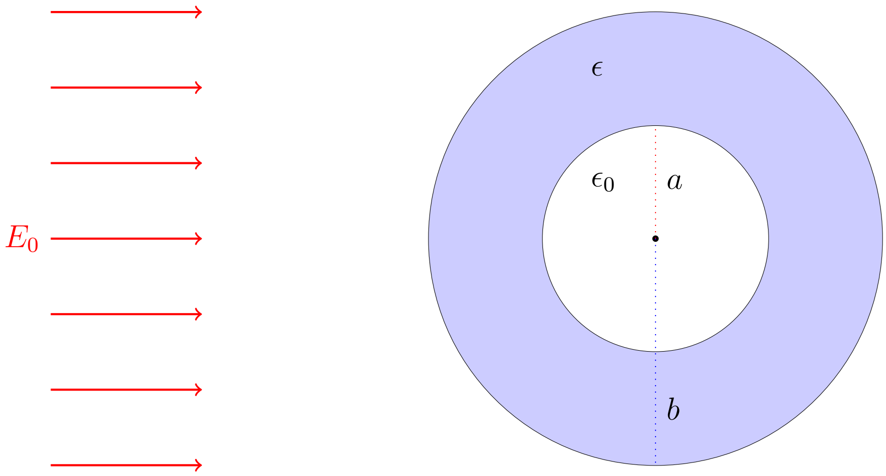

Figure 1: Cross section of the cylinder.

(a) Note that there is no z dependence so our problem is reduced to a two-dimensional problem. Recall from Section 2.11 that the general solution is given by (Equation 2.71)

$$\Phi(\rho,\phi) = a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi + \alpha_n) + \sum_{n=1}^{\infty} b_n \rho^{-n} \sin(n\phi + \beta_n).$$

Since the system is symmetric under $\phi \to -\phi$, this becomes

$$\Phi(\rho,\phi) = a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty} a_n \rho^n \cos(n\phi) + \sum_{n=1}^{\infty} b_n \rho^{-n} \cos(n\phi).$$

We consider three regions: I. $\rho < a$, II. $a < \rho < b$, and III. $\rho > b$. Note that in the first region the potential must be finite at $\rho = 0$, therefore

$$\Phi_I(\rho,\phi) = A_0 + \sum_{n=1}^{\infty} A_n \rho^n \cos(n\phi).$$

We write the potential in region II and III as

$$\Phi_{II}(\rho,\phi) = B_0 + C_0 \ln\rho + \sum_{n=1}^{\infty} B_n \rho^n \cos(n\phi) + \sum_{n=1}^{\infty} C_n \rho^{-n} \cos(n\phi),$$
$$\Phi_{III}(\rho,\phi) = U_0 + V_0 \ln\rho + \sum_{n=1}^{\infty} U_n \rho^n \cos(n\phi) + \sum_{n=1}^{\infty} V_n \rho^{-n} \cos(n\phi).$$

Now, very far away the electric field must be $E_0\hat{\mathbf{x}}$. Imposing this condition, we have

$$\begin{aligned} E_0 = -\frac{\partial \Phi_{III}}{\partial x}\bigg|_{\rho\to\infty} &= -\hat{\mathbf{x}}\cdot\nabla\Phi_{III}\bigg|_{\rho\to\infty} = \left[-\cos\phi\frac{\partial\Phi_{III}}{\partial\rho} + \frac{1}{\rho}\sin\phi\frac{\partial\Phi_{III}}{\partial\phi}\right]\bigg|_{\rho\to\infty} \\ &= -\cos\phi\left(\frac{V_0}{\rho} + \sum_{n=1}^{\infty} nU_n\rho^{n-1}\cos(n\phi)\right) - \frac{\sin\phi}{\rho}\left(\sum_{n=1}^{\infty} nU_n\rho^n \sin(n\phi)\right). \end{aligned}$$

Clearly, for this equality to hold the RHS must have no ρ dependence, therefore we set the other terms to zero such that the equality becomes

$$E_0 = -U_1\left(\cos^2\phi + \sin^2\phi\right) = -U_1.$$

The potential very far away becomes

$$\Phi_{III}(\rho,\phi) = U_0 - E_0\rho\cos\phi + \sum_{n=1}^{\infty} V_n\rho^{-n}\cos(n\phi).$$

Now, we impose the boundary conditions on $\rho = b$, since there are no charges on the surface

$$\begin{aligned} \epsilon_0 \frac{\partial\Phi_{III}}{\partial\rho}\bigg|_{\rho=b} &= \epsilon\frac{\partial\Phi_{II}}{\partial\rho}\bigg|_{\rho=b}, \\ \frac{\partial\Phi_{III}}{\partial\phi}\bigg|_{\rho=b} &= \frac{\partial\Phi_{III}}{\partial\phi}\bigg|_{\rho=b}; \end{aligned}$$

where

$$\begin{aligned} \epsilon_0\frac{\partial\Phi_{III}}{\partial\rho}\bigg|_{\rho=b} &= -\epsilon_0 E_0\cos\phi - \epsilon_0\sum_{n=1}^{\infty} nV_n b^{-(n+1)}\cos(n\phi), \\ \epsilon\frac{\partial\Phi_{II}}{\partial\rho}\bigg|_{\rho=b} &= \frac{\epsilon C_0}{b} + \epsilon\sum_{n=1}^{\infty} nB_n b^{n-1}\cos(n\phi) - \epsilon\sum_{n=1}^{\infty} nC_n b^{-(n+1)}\cos(n\phi), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial\Phi_{III}}{\partial\phi}\bigg|_{\rho=b} &= E_0 b\sin\phi - \sum_{n=1}^{\infty} nV_n b^{-n}\sin(n\phi), \\ \frac{\partial\Phi_{II}}{\partial\phi}\bigg|_{\rho=b} &= -\sum_{n=1}^{\infty} nB_n b^n \sin(n\phi) - \sum_{n=1}^{\infty} nC_n b^{-n}\sin(n\phi). \end{aligned}$$

Similarly, we can impose the boundary conditions on $\rho = a$, again there are no charges on the surface

$$
\begin{aligned}
\epsilon_0 \left.\frac{\partial \Phi_I}{\partial \rho}\right|_{\rho=a} &= \epsilon \left.\frac{\partial \Phi_{II}}{\partial \rho}\right|_{\rho=a}, \\
\left.\frac{\partial \Phi_I}{\partial \phi}\right|_{\rho=a} &= \left.\frac{\partial \Phi_{III}}{\partial \phi}\right|_{\rho=a};
\end{aligned}
$$

where

$$
\begin{aligned}
\epsilon_0 \left.\frac{\partial \Phi_I}{\partial \rho}\right|_{\rho=a} &= \epsilon_0 \sum_{n=1}^{\infty} n A_n a^{n-1} \cos(n\phi), \\
\epsilon \left.\frac{\partial \Phi_{II}}{\partial \rho}\right|_{\rho=a} &= \frac{\epsilon C_0}{a} + \epsilon \sum_{n=1}^{\infty} n B_n a^{n-1} \cos(n\phi) - \epsilon \sum_{n=1}^{\infty} n C_n a^{-(n+1)} \cos(n\phi),
\end{aligned}
$$

and

$$
\begin{aligned}
\left.\frac{\partial \Phi_I}{\partial \phi}\right|_{\rho=a} &= -\sum_{n=1}^{\infty} n A_n a^n \sin(n\phi), \\
\left.\frac{\partial \Phi_{II}}{\partial \phi}\right|_{\rho=a} &= -\sum_{n=1}^{\infty} n B_n a^n \sin(n\phi) - \sum_{n=1}^{\infty} n C_n a^{-n} \sin(n\phi).
\end{aligned}
$$

Using the orthogonality of trigonometric functions, from the previous equations we get

$$
\begin{aligned}
A_1 &= B_1 + C_1 a^{-2}, \\
\epsilon_0 A_1 &= \epsilon B_1 - \epsilon C_1 a^{-2}, \\
V_1 - E_0 b^2 &= B_1 b^2 + C_1, \\
\epsilon_0 E_0 b^2 + \epsilon_0 V_1 &= \epsilon C_1 - \epsilon B_1 b^2,
\end{aligned}
$$

for $n = 1$; while for $n > 1$ we have

$$
\begin{aligned}
A_n &= B_n + C_n a^{-2n}, \\
\epsilon_0 A_n &= \epsilon B_n - \epsilon C_n a^{-2n}, \\
V_n &= B_n b^{2n} + C_n, \\
\epsilon_0 V_n &= \epsilon C_n - \epsilon B_n b^{2n}.
\end{aligned}
$$

It is also clear that $C_0 = 0$. We find that these equations can be solved only for the $n = 1$ case. We obtain the following coefficients:

$$
\begin{aligned}
A_1 &= -\frac{4 b^2 \epsilon_0 \epsilon E_0}{b^2(\epsilon + \epsilon_0)^2 - a^2(\epsilon - \epsilon_0)^2}, \\
B_1 &= -\frac{2 b^2 \epsilon_0 (\epsilon + \epsilon_0) E_0}{b^2(\epsilon + \epsilon_0)^2 - a^2(\epsilon - \epsilon_0)^2}, \\
C_1 &= -\frac{2 a^2 b^2 \epsilon_0 (\epsilon - \epsilon_0) E_0}{b^2(\epsilon + \epsilon_0)^2 - a^2(\epsilon - \epsilon_0)^2}, \\
V_1 &= \frac{b^2 (b^2 - a^2)(\epsilon^2 - \epsilon_0^2) E_0}{b^2(\epsilon + \epsilon_0)^2 - a^2(\epsilon - \epsilon_0)^2}.
\end{aligned}
$$

Without loss of generality, we may choose $A_0 = B_0 = U_0 = 0$. Thus, the potential is

$$\begin{aligned}
\Phi_I(\rho,\phi) &= -\frac{4b^2\epsilon_0\epsilon E_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\rho\cos\phi,\\
\Phi_{II}(\rho,\phi) &= -\frac{2b^2\epsilon_0(\epsilon+\epsilon_0)E_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\rho\cos\phi - \frac{2a^2b^2\epsilon_0(\epsilon-\epsilon_0)E_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\frac{\cos\phi}{\rho},\\
\Phi_{III}(\rho,\phi) &= -E_0\rho\cos\phi + \frac{b^2(b^2-a^2)(\epsilon^2-\epsilon_0^2)E_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\frac{\cos\phi}{\rho}.
\end{aligned}$$

The electric field is

$$\begin{aligned}
\mathbf{E}_I = -\nabla\Phi_I &= \frac{4b^2\epsilon_0\epsilon E_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\left(\cos\phi\hat{\boldsymbol{\rho}} - \sin\phi\hat{\boldsymbol{\phi}}\right)\\
&= \frac{4b^2\epsilon_0\epsilon E_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\hat{\mathbf{x}},\\
\mathbf{E}_{II} = -\nabla\Phi_{II} &= \left(\frac{2b^2\epsilon_0(\epsilon+\epsilon_0)E_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2} - \frac{2a^2b^2\epsilon_0(\epsilon-\epsilon_0)E_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\frac{1}{\rho^2}\right)\cos\phi\hat{\boldsymbol{\rho}}\\
&\quad - \left(\frac{2b^2\epsilon_0(\epsilon+\epsilon_0)E_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2} + \frac{2a^2b^2\epsilon_0(\epsilon-\epsilon_0)E_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\frac{1}{\rho^2}\right)\sin\phi\hat{\boldsymbol{\phi}},\\
&= \frac{2b^2\epsilon_0(\epsilon+\epsilon_0)E_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\hat{\mathbf{x}} - \frac{2a^2b^2\epsilon_0(\epsilon-\epsilon_0)E_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\frac{1}{\rho^2}\left(\cos\phi\hat{\boldsymbol{\rho}} + \sin\phi\hat{\boldsymbol{\phi}}\right)\\
\mathbf{E}_{III} = -\nabla\Phi_{III} &= \left(E_0 + \frac{b^2(b^2-a^2)(\epsilon^2-\epsilon_0^2)E_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\frac{1}{\rho^2}\right)\cos\phi\hat{\boldsymbol{\rho}}\\
&\quad - \left(E_0 - \frac{b^2(b^2-a^2)(\epsilon^2-\epsilon_0^2)E_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\frac{1}{\rho^2}\right)\sin\phi\hat{\boldsymbol{\phi}}\\
&= E_0\hat{\mathbf{x}} + \frac{b^2(b^2-a^2)(\epsilon^2-\epsilon_0^2)E_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\frac{1}{\rho^2}\left(\cos\phi\hat{\boldsymbol{\rho}} + \sin\phi\hat{\boldsymbol{\phi}}\right).
\end{aligned}$$

(b) For the case $b = 2a$, $\epsilon = 1.5\epsilon_0$, we provide a sketch of the electric field using Mathematica (see Figure 2). The field lines can be easily traced out by following the arrows.

(c) We now discuss the limiting forms of the solution. For a solid dielectric cylinder in a uniform field, we take the limit $a \to 0$. The potential outside the cylinder is now

$$\Phi_{\text{out}} = \lim_{a\to 0}\Phi_{III} = -E_0\rho\cos\phi + \frac{b^2(\epsilon-\epsilon_0)E_0}{(\epsilon+\epsilon_0)}\frac{\cos\phi}{\rho};$$

while the potential inside the cylinder becomes

$$\Phi_{\text{in}} = \lim_{a\to 0}\Phi_{II}(\rho,\phi) = -\frac{2\epsilon_0 E_0}{(\epsilon+\epsilon_0)}\rho\cos\phi.$$

For a cylindrical cavity in a uniform dielectric, we take the limit $b \to \infty$. The potential outside the cavity is now

$$\Phi_{\text{out}} = \lim_{b\to\infty}\Phi_{II} = -\frac{2\epsilon_0 E_0}{(\epsilon+\epsilon_0)}\rho\cos\phi - \frac{2a^2\epsilon_0(\epsilon-\epsilon_0)E_0}{(\epsilon+\epsilon_0)^2}\frac{\cos\phi}{\rho};$$

while the potential inside the cylinder becomes

$$\Phi_{\text{in}} = \lim_{b\to\infty}\Phi_I(\rho,\phi) = -\frac{4\epsilon_0\epsilon E_0}{(\epsilon+\epsilon_0)^2}\rho\cos\phi.$$

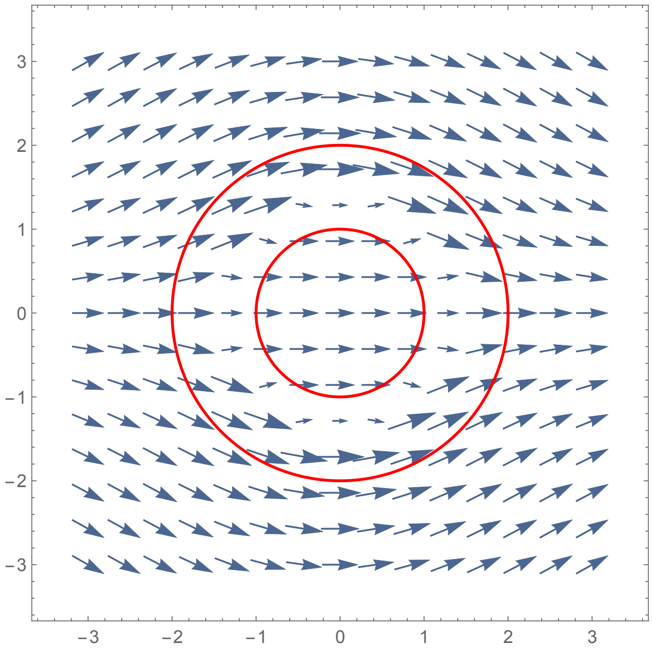

Figure 2: A sketch of the electric field (where a=1).

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