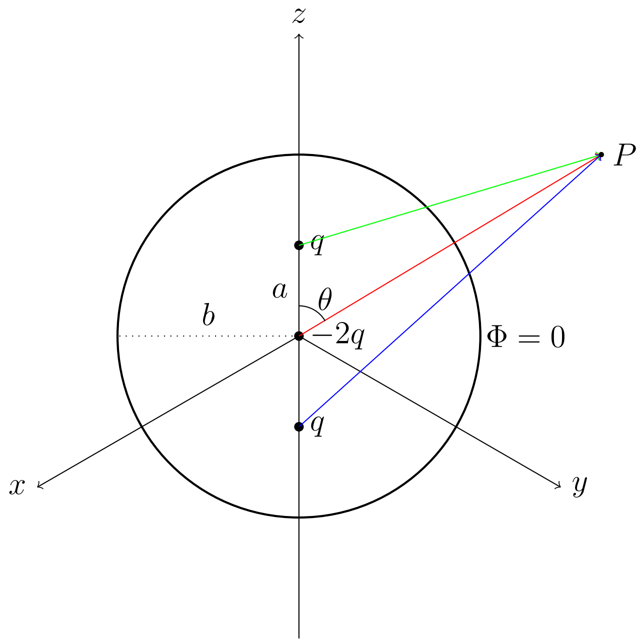
## Problem 3.7

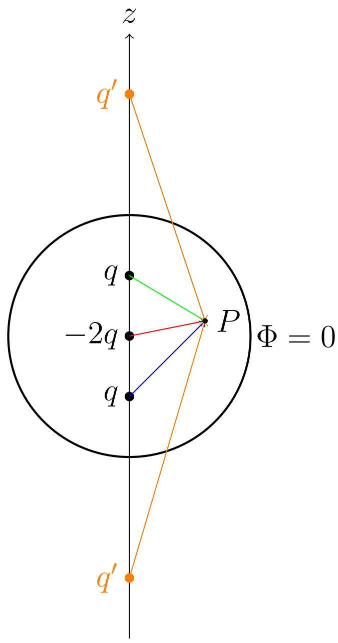
Three point charges $(q, -2q, q)$ are located in a straight line with separation $a$ and with the middle charge $(-2q)$ at the origin of a grounded conducting spherical shell of radius $b$, as indicated in the sketch.

(a) Write down the potential of the three charges in the absence of the grounded conducting sphere. Find the limiting form of the potential as $a \to 0$, but the product $qa^2 = Q$ remains finite. Write this latter answer in spherical coordinates.

(b) The presence of the grounded sphere of radius $b$ alters the potential for $r < b$. The added potential can be viewed as caused by the surface-charge density induced on the inner surface at $r = b$ or by image charges located at $r > b$. Use linear superposition to satisfy the boundary conditions and find the potential everywhere inside the sphere for $r < a$ and $r > a$. Show that in the limit $a \to 0$,

$$\Phi(r, \theta, \phi) \to \frac{Q}{2\pi\epsilon_0 r^3}\left(1 - \frac{r^5}{b^5}\right) P_2(\cos\theta).$$




*Solution.*

(a) Again, the system has azimuthal symmetry, such that $\Phi(r, \theta, \phi) \to \Phi(r, \theta)$. Let $r$ denote the distance from the origin $(-2q)$ to the observation point $P$. Similarly, let $r_\pm$ be the distance from the upper (lower) charge $q$ to $P$. By Coulomb's law,

$$\begin{aligned}\Phi(r, \theta) &= \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_+} + \frac{q}{r_-} - \frac{2q}{r}\right)\\ &= \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{r^2 + a^2 - 2ar\cos\theta}} + \frac{q}{\sqrt{r^2 + a^2 + 2ar\cos\theta}} - \frac{2q}{r}\right).\end{aligned}$$

Assuming $r > a$,

$$\begin{aligned}
\Phi(r,\theta) &= \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r\sqrt{1+(a/r)^2-2(a/r)\cos\theta}} + \frac{q}{r\sqrt{1+(a/r)^2+2(a/r)\cos\theta}} - \frac{2q}{r}\right) \\
&= \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r}\sum_{l=0}^{\infty} P_l(\cos\theta)\left(\frac{a}{r}\right)^l + \frac{q}{r}\sum_{l=0}^{\infty} P_l(-\cos\theta)\left(\frac{a}{r}\right)^l - \frac{2q}{r}\right) \\
&= \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r}\sum_{l=0}^{\infty} P_l(\cos\theta)\left(\frac{a}{r}\right)^l + \frac{q}{r}\sum_{l=0}^{\infty} (-1)^l P_l(\cos\theta)\left(\frac{a}{r}\right)^l - \frac{2q}{r}\right) \\
&= \frac{1}{4\pi\epsilon_0}\left(\frac{2q}{r}\sum_{l=0}^{\infty} P_{2l}(\cos\theta)\left(\frac{a}{r}\right)^{2l} - \frac{2q}{r}\right) \\
&= \frac{1}{4\pi\epsilon_0}\left(\frac{2q}{r}\sum_{l=1}^{\infty} P_{2l}(\cos\theta)\left(\frac{a}{r}\right)^{2l}\right) \\
&= \frac{q}{2\pi\epsilon_0 r}\sum_{l=1}^{\infty} P_{2l}(\cos\theta)\left(\frac{a}{r}\right)^{2l}.
\end{aligned}$$

In the limiting case, $r \gg a$ and

$$\begin{aligned}
\Phi(r,\theta) &= \frac{q}{2\pi\epsilon_0 r}\left[P_2(\cos\theta)\frac{a^2}{r^2} + \mathcal{O}(a^4/r^4)\right] \\
&\approx \frac{qa^2}{2\pi\epsilon_0 r^3}P_2(\cos\theta) \\
&\to \frac{Q}{2\pi\epsilon_0 r^3}P_2(\cos\theta).
\end{aligned}$$

In fact, this is the limiting form of the potential due to a linear quadrupole far away.

(b) Using our results from Problem 2.2, a charge $q$ a distance $a$ ($< b$) from the center of a grounded sphere will produce an image charge $q' = -qb/a$ at a distance $b^2/a$. Clearly, the upper (lower) charge $q$ will produce a corresponding image charge above (below). While the charge $-2q$ will not produce an image charge, it would add a constant potential. Therefore, the potential inside the sphere is

$$\begin{aligned}
\Phi(r,\theta) &= V_0 + \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_+} + \frac{q'}{r'_+} + \frac{q}{r_-} + \frac{q'}{r'_-} - \frac{2q}{r}\right) \\
&= V_0 + \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{r^2+a^2-2ar\cos\theta}} - \frac{b}{a}\frac{q}{\sqrt{r^2+(b^2/a)^2-2(b^2/a)r\cos\theta}}\right. \\
&\qquad \left. + \frac{q}{\sqrt{r^2+a^2+2ar\cos\theta}} - \frac{b}{a}\frac{q}{\sqrt{r^2+(b^2/a)^2+2(b^2/a)r\cos\theta}} - \frac{2q}{r}\right) \\
&= V_0 + \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{r^2+a^2-2ar\cos\theta}} - \frac{1}{b}\frac{q}{\sqrt{(ra/b^2)^2+1-2(ra/b^2)\cos\theta}}\right. \\
&\qquad \left. + \frac{q}{\sqrt{r^2+a^2+2ar\cos\theta}} - \frac{1}{b}\frac{q}{\sqrt{(ra/b^2)^2+1+2(ra/b^2)\cos\theta}} - \frac{2q}{r}\right).
\end{aligned}$$

Suppose $r > a$. Then

$$\begin{aligned}\Phi(r,\theta) &= V_0 + \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r}\sum_{l=0}^{\infty} P_l(\cos\theta)\left(\frac{a}{r}\right)^l - \frac{q}{b}\sum_{l=0}^{\infty} P_l(\cos\theta)\left(\frac{ra}{b^2}\right)^l\right. \\ &\qquad \left. + \frac{q}{r}\sum_{l=0}^{\infty} P_l(-\cos\theta)\left(\frac{a}{r}\right)^l - \frac{q}{b}\sum_{l=0}^{\infty} P_l(-\cos\theta)\left(\frac{ra}{b^2}\right)^l - \frac{2q}{r}\right) \\ &= V_0 + \frac{1}{2\pi\epsilon_0}\left(\frac{q}{r}\sum_{l=0}^{\infty} P_{2l}(\cos\theta)\left(\frac{a}{r}\right)^{2l} - \frac{q}{b}\sum_{l=0}^{\infty} P_{2l}(\cos\theta)\left(\frac{ra}{b^2}\right)^{2l} - \frac{q}{r}\right).\end{aligned}$$

Imposing the boundary condition $\Phi(b) = 0$ gives us $V_0 = q/2\pi\epsilon_0 b$, as expected. Therefore, if $r > a$ then

$$\Phi(r,\theta) = \frac{q}{2\pi\epsilon_0 b} + \frac{1}{2\pi\epsilon_0}\left(\frac{q}{r}\sum_{l=0}^{\infty} P_{2l}(\cos\theta)\left(\frac{a}{r}\right)^{2l} - \frac{q}{b}\sum_{l=0}^{\infty} P_{2l}(\cos\theta)\left(\frac{ra}{b^2}\right)^{2l} - \frac{q}{r}\right).$$

Similarly, if $r < a$ then

$$\Phi(r,\theta) = \frac{q}{2\pi\epsilon_0 b} + \frac{1}{2\pi\epsilon_0}\left(\frac{q}{a}\sum_{l=0}^{\infty} P_{2l}(\cos\theta)\left(\frac{r}{a}\right)^{2l} - \frac{q}{b}\sum_{l=0}^{\infty} P_{2l}(\cos\theta)\left(\frac{ra}{b^2}\right)^{2l} - \frac{q}{r}\right)$$

In the limiting form $a \to 0$, we have $r \sim b \gg a$. Therefore

$$\begin{aligned}\Phi(r,\theta) &\approx \frac{q}{2\pi\epsilon_0 b} + \frac{1}{2\pi\epsilon_0}\left[\frac{q}{r}\left(1 + \frac{a^2}{r^2}P_2(\cos\theta)\right) - \frac{q}{b}\left(1 + \frac{r^2a^2}{b^4}P_2(\cos\theta)\right) - \frac{q}{r}\right] \\ &\approx \frac{1}{2\pi\epsilon_0}\frac{qa^2}{r^3}\left(1 - \frac{r^5}{b^5}\right)P_2(\cos\theta) \\ &\to \frac{Q}{2\pi\epsilon_0 r^3}\left(1 - \frac{r^5}{b^5}\right)P_2(\cos\theta).\end{aligned}$$

□