Problem 5.8

A localized cylindrically symmetric current distribution is such that the current flows only in the azimuthal direction; the current density is a function only of r and θ (or ρ and z): $\mathbf{J} = \hat{\boldsymbol{\phi}} J(r,\theta)$. The distribution is "hollow" in the sense that there is a current-free region near the origin, as well as outside.

(a) Show that the magnetic field can be derived from the azimuthal component of the vector potential, with a multipole expansion

$$A_\phi(r,\theta) = -\frac{\mu_0}{4\pi}\sum_L m_L r^L P_L^1(\cos\theta)$$

in the interior and

$$A_\phi(r,\theta) = -\frac{\mu_0}{4\pi}\sum_L \mu_L r^{-L-1} P_L^1(\cos\theta)$$

outsie the current distribution.

(b) Show that the internal and external multipole moments are

$$m_L = -\frac{1}{L(L+1)}\int \mathrm{d}^3x\, r^{-L-1} P_L^1(\cos\theta) J(r,\theta)$$

and

$$\mu_L = -\frac{1}{L(L+1)}\int \mathrm{d}^3x\, r^{L} P_L^1(\cos\theta) J(r,\theta)$$

Solution. The vector potential is given by Equation 5.32

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}^3x'.$$

Now,

$$\begin{aligned}
\mathbf{A}(\mathbf{x}) &= \frac{\mu_0}{4\pi}\int \frac{J(r',\theta')}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}^3x'\hat{\boldsymbol{\phi}} \\
A_\phi(r,\theta) &= \frac{\mu_0}{4\pi}\int \frac{J(r',\theta')}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}^3x'.
\end{aligned}$$

Expanding $1/|\mathbf{x}-\mathbf{x}'|$ in terms of spherical harmonics (Equation 3.70) and noting that the current flows only in the azimuthal direction, we get

$$\begin{aligned}
A_\phi(r,\theta) &= \frac{\mu_0}{4\pi}\int J(r',\theta')\, 4\pi \sum_L \frac{1}{2L+1} Y_{L1}(\cos\theta) Y_{L1}^*(\cos\theta')\frac{r_<^L}{r_>^{L+1}}\mathrm{d}^3x' \\
&= \frac{\mu_0}{4\pi}\int J(r',\theta')\sum_L \frac{(L-1)!}{(L+1)!} P_L^1(\cos\theta) P_L^1(\cos\theta')\frac{r_<^L}{r_>^{L+1}}\mathrm{d}^3x' \\
&= \frac{\mu_0}{4\pi}\sum_L \frac{P_L^1(\cos\theta)}{L(L+1)}\int J(r',\theta')\frac{r_<^L}{r_>^{L+1}} P_L^1(\cos\theta')\mathrm{d}^3x'.
\end{aligned}$$

In the interior $r_< = r$ and $r_> = r'$. Therefore,

$$\begin{aligned} A_\phi(r,\theta) &= \frac{\mu_0}{4\pi}\sum_L \frac{P_L^1(\cos\theta)}{L(L+1)}\int J(r',\theta')\frac{r^L}{r'^{L+1}}P_L^1(\cos\theta')\mathrm{d}^3x' \\ &= \frac{\mu_0}{4\pi}\sum_L r^L P_L^1(\cos\theta)\frac{1}{L(L+1)}\int J(r',\theta')r'^{-L-1}P_L^1(\cos\theta')\mathrm{d}^3x' \\ &= -\frac{\mu_0}{4\pi}\sum_L m_L r^L P_L^1(\cos\theta). \end{aligned}$$

Similarly, in the exterior region $r_< = r'$ and $r_> = r$. Therefore,

$$\begin{aligned} A_\phi(r,\theta) &= \frac{\mu_0}{4\pi}\sum_L \frac{P_L^1(\cos\theta)}{L(L+1)}\int J(r',\theta')\frac{r'^L}{r^{L+1}}P_L^1(\cos\theta')\mathrm{d}^3x' \\ &= \frac{\mu_0}{4\pi}\sum_L r^{-L-1} P_L^1(\cos\theta)\frac{1}{L(L+1)}\int J(r',\theta')r'^{L}P_L^1(\cos\theta')\mathrm{d}^3x' \\ &= -\frac{\mu_0}{4\pi}\sum_L \mu_L r^L P_L^1(\cos\theta). \end{aligned}$$

□