Problem 5.21

A magnetostatic field is due entirely to a localized distribution of permanent magnetization.

(a) Show that

$$\int \mathbf{B}\cdot\mathbf{H}\,\mathrm{d}^3x = 0$$

provided the integral is taken over all space.

(b) From the potential energy (5.72) of a dipole in an external field, show that for a continuous distribution of permanent magnetization the magnetostatic energy can be written as

$$W = \frac{\mu_0}{2}\int \mathbf{H}\cdot\mathbf{H}\,\mathrm{d}^3x = -\frac{\mu_0}{2}\int \mathbf{M}\cdot\mathbf{H}\,\mathrm{d}^3x$$

apart from an additive constant, which is independent of the orientation or position of the various constituent magnetized bodies.

Solution.

(a) Since there are no free currents, $\mathbf{H}$ is derivable from a potential. Using the product rule $\nabla\cdot(f\mathbf{A}) = \nabla f\cdot\mathbf{A} + f\nabla\cdot\mathbf{A}$, we get

$$\begin{aligned}\int \mathbf{B}\cdot\mathbf{H}\,\mathrm{d}^3x &= -\int \mathbf{B}\cdot\nabla\Phi_M\,\mathrm{d}^3x \\ &= -\int \left[\nabla\cdot(\Phi_M\mathbf{B}) - \Phi_M\nabla\cdot\mathbf{B}\right]\mathrm{d}^3x \\ &= -\int \nabla\cdot(\Phi_M\mathbf{B})\,\mathrm{d}^3x\,,\end{aligned}$$

where we used the fact that $\nabla\cdot\mathbf{B} = 0$. Now, by the Divergence theorem

$$\int \nabla\cdot(\Phi_M\mathbf{B})\,\mathrm{d}^3x = \oint (\Phi_M\mathbf{B})\,\mathrm{d}a.$$

Since the magnetostatic field is due entirely to a localized distribution, the magnetic induction $\mathbf{B}$ vanishes at infinity. Therefore, the surface integral (at infinity) is zero. Thus,

$$\int \mathbf{B}\cdot\mathbf{H}\,\mathrm{d}^3x = 0.$$

(b) We generalize Equation 5.72 to a continuous distribution of permanent magnetization. We take $\delta U = -\delta\mathbf{M}\cdot\mathbf{B}$. Provided that a linear relation exists between $\mathbf{M}$ and $\mathbf{B}$, we get

$$\begin{aligned}U &= -\frac{1}{2}\int \mathbf{M}\cdot\mathbf{B}\,\mathrm{d}^3x \\ &= -\frac{\mu_0}{2}\int \mathbf{M}\cdot(\mathbf{H}+\mathbf{M})\,\mathrm{d}^3x \\ &= -\frac{\mu_0}{2}\int \mathbf{M}\cdot\mathbf{H}\,\mathrm{d}^3x - \frac{\mu_0}{2}\int \mathbf{M}\cdot\mathbf{M}\,\mathrm{d}^3x.\end{aligned}$$

Note that $\mathbf{M} \cdot \mathbf{M}$ is a scalar, which is independent of the orientation—and whose integral is independent of the position of the various constituent magnetized bodies. Therefore, the second integral is constant, which we denote by $\mathcal{M}$. Thus,

$$\begin{aligned}
U &= -\frac{\mu_0}{2} \int \mathbf{M} \cdot \mathbf{H} \, \mathrm{d}^3 x + \mathcal{M} \,, \\
&= -\frac{\mu_0}{2} \int \left(\frac{1}{\mu_0} \mathbf{B} - \mathbf{H} \right) \cdot \mathbf{H} \, \mathrm{d}^3 x + \mathcal{M} \\
&= -\frac{1}{2} \int \mathbf{B} \cdot \mathbf{H} \, \mathrm{d}^3 x + \frac{\mu_0}{2} \int \mathbf{H} \cdot \mathbf{H} \, \mathrm{d}^3 x + \mathcal{M} \\
&= \quad \frac{\mu_0}{2} \int \mathbf{H} \cdot \mathbf{H} \, \mathrm{d}^3 x + \mathcal{M} \,.
\end{aligned}$$

$\square$