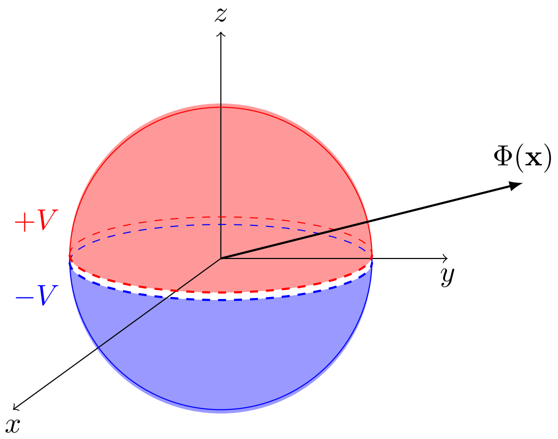
Deriving Equation 2.27



We start with Equation (2.19):

$$
\begin{aligned}
\Phi(\mathbf{x}) &= \frac{1}{4\pi}\int \Phi(a,\theta',\phi')\frac{a(x^2-a^2)}{(x^2+a^2-2ax\cos\gamma)^{3/2}}\mathrm{d}\Omega' \\
&= \frac{V}{4\pi}a(x^2-a^2)\int_0^{2\pi}\mathrm{d}\phi'\left(\int_0^{\pi/2}-\int_{\pi/2}^{\pi}\right)\sin\theta'\mathrm{d}\theta'(x^2+a^2-2ax\cos\gamma)^{-3/2} \\
&= \frac{V}{4\pi}a(x^2-a^2)\int_0^{2\pi}\mathrm{d}\phi'\left(\int_1^{0}-\int_{0}^{-1}\right)\mathrm{d}(-\cos\theta')(x^2+a^2-2ax\cos\gamma)^{-3/2} \\
&= \frac{V}{4\pi}a(x^2-a^2)\int_0^{2\pi}\mathrm{d}\phi'\left(\int_0^{1}+\int_{0}^{-1}\right)\mathrm{d}(\cos\theta')(x^2+a^2-2ax\cos\gamma)^{-3/2}
\end{aligned}
$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$. Making the substitutions $\theta' \to \pi-\theta'$ and $\phi' \to \phi'+\pi$ on the second integral, we have $\cos\gamma \to -\cos\gamma$ and

$$
-\int_{\pi/2}^{\pi}\sin\theta'\mathrm{d}\theta' \to \int_{\pi/2}^{0}\sin\theta'\mathrm{d}\theta' = -\int_0^1 \mathrm{d}(\cos\theta').
$$

Therefore,

$$
\begin{aligned}
\Phi(\mathbf{x}) &= \frac{V}{4\pi}a(x^2-a^2)\int_0^{2\pi}\mathrm{d}\phi'\int_0^1\mathrm{d}(\cos\theta')\left[(x^2+a^2-2ax\cos\gamma)^{-3/2}-(x^2+a^2+2ax\cos\gamma)^{-3/2}\right] \\
&= \frac{V}{4\pi}\frac{a(x^2-a^2)}{(x^2+a^2)^{3/2}}\int_0^{2\pi}\mathrm{d}\phi'\int_0^1\mathrm{d}(\cos\theta')\left[(1-2\alpha\cos\gamma)^{-3/2}-(1+2\alpha\cos\gamma)^{-3/2}\right],
\end{aligned}
$$

where $\alpha = \dfrac{ax}{x^2+a^2}$. Far away, we may assume $x \gg a$ or $\alpha \ll 1$. Note that

$$
\frac{1}{(1-x)^{3/2}} = 1 + \frac{3}{2}x + \frac{15}{8}x^2 + \frac{35}{16}x^3 + \dots
$$

Therefore, if we let $x = 2\alpha\cos\gamma$

$$
(1-2\alpha\cos\gamma)^{-3/2} - (1+2\alpha\cos\gamma)^{-3/2} = 6\alpha\cos\gamma + 35\alpha^3\cos^3\gamma + \dots
$$

Plugging this into the integral, note that

$$\begin{aligned}\int_0^{2\pi} \mathrm{d}\phi' \int_0^1 \mathrm{d}(\cos\theta')\cos\gamma &= \int_0^{2\pi} \mathrm{d}\phi' \int_0^1 \mathrm{d}(\cos\theta')\left[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')\right] \\ &= \quad \cos\theta \int_0^{2\pi} \mathrm{d}\phi' \int_0^1 \mathrm{d}(\cos\theta')\cos\theta' \\ &\quad + \sin\theta \int_0^{2\pi} \mathrm{d}\phi' \cos(\phi-\phi') \int_0^1 \mathrm{d}(\cos\theta')\sin\theta'.\end{aligned}$$

The second integral is zero, since $\int_0^{2\pi} \mathrm{d}\phi' \cos(\phi-\phi') = 0$. Thus,

$$\int_0^{2\pi} \mathrm{d}\phi' \int_0^1 \mathrm{d}(\cos\theta')\cos\gamma = \pi\cos\theta.$$

Also,

$$\begin{aligned}\cos^3\gamma = \quad & \cos^3\theta\cos^3\theta' + 3\cos^2\theta\cos^2\theta'\sin\theta\sin\theta'\cos(\phi-\phi') \\ & + 3\cos\theta\cos\theta'\sin^2\theta\sin^2\theta'\cos^2(\phi-\phi') + \sin^3\theta\sin^3\theta'\cos^3(\phi-\phi').\end{aligned}$$

Note that the integral of second term vanishes, just like before. Also,

$$\int_0^{2\pi} \cos^3(\phi-\phi')\mathrm{d}\phi' = \int_{-\phi}^{2\pi-\phi} \cos^3 u\mathrm{d}u = \int_{-\phi}^{2\pi-\phi} \cos u\left(1-\sin^2 u\right)\mathrm{d}u = -\int_{\sin(-\phi)}^{\sin(2\pi-\phi)} v^2\mathrm{d}v = 0;$$

so the integral of the last term vanishes. For the third term, note that

$$\int_0^{2\pi} \cos^2(\phi-\phi')\mathrm{d}\phi' = \frac{1}{2}\int_0^{2\pi}\left(1+\cos\left[2(\phi-\phi')\right]\right)\mathrm{d}\phi' = \pi.$$

Thus,

$$\begin{aligned}\int_0^{2\pi} \mathrm{d}\phi' \int_0^1 \mathrm{d}(\cos\theta')\cos^3\gamma &= 2\pi\cos^3\theta \int_0^1 \mathrm{d}(\cos\theta')\cos^3\theta' + 3\pi\cos\theta\sin^2\theta \int_0^1 \mathrm{d}(\cos\theta')\cos\theta'\sin^2\theta' \\ &= \frac{\pi}{2}\cos^3\theta + 3\pi\cos\theta\sin^2\theta \int_0^1 \mathrm{d}(\cos\theta')\left[\cos\theta' - \cos^3\theta'\right] \\ &= \frac{\pi}{2}\cos^3\theta + \frac{3\pi}{4}\cos\theta\sin^2\theta \\ &= \frac{\pi}{4}\cos\theta\left(2\cos^2\theta + 3\sin^2\theta\right) \\ &= \frac{\pi}{4}\cos\theta\left(3-\cos^2\theta\right).\end{aligned}$$

Finally, we obtain

$$\Phi(\mathbf{x}) = \frac{3Va^2}{2x^2}\left(\frac{x^3(x^2-a^2)}{(x^2+a^2)^{5/2}}\right)\cos\theta\left[1 + \frac{35}{24}\frac{a^2x^2}{(a^2+x^2)^2}\left(3-\cos^2\theta\right) + \ldots\right]$$

If we let $\beta = a^2/x^2$, we have

$$\Phi(\mathbf{x}) = \frac{3V}{2}\beta\frac{1-\beta}{(1+\beta)^{5/2}}\cos\theta\left[1 + \frac{35}{24}\frac{\beta}{(1+\beta)^2}\left(3-\cos^2\theta\right) + \ldots\right].$$

Using Mathematica to rewrite the series, we have

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{3V}{2}\cos\theta\left[\beta + \left(\frac{7}{8} - \frac{35}{24}\cos^2\theta\right)\beta^2 + \ldots\right] \\ &= \frac{3Va^2}{2x^2}\left[\cos\theta - \frac{7a^2}{12x^2}\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right) + \ldots\right].\end{aligned}$$