Problem 2.7

Consider a potential problem in the half-space defined by $z \geq 0$, with Dirichlet boundary conditions on the plane $z = 0$ (and at infinity).

(a) Write down the appropriate Green function $G(\mathbf{x}, \mathbf{x}')$.

(b) If the potential on the plane $z = 0$ is specified to be $\Phi = V$ inside a circle of radius a centered at the origin, and $\Phi = 0$ outside that circle, find an integral expression for the potential at the point P specified in terms of cylindrical coordinates (ρ, ϕ, z).

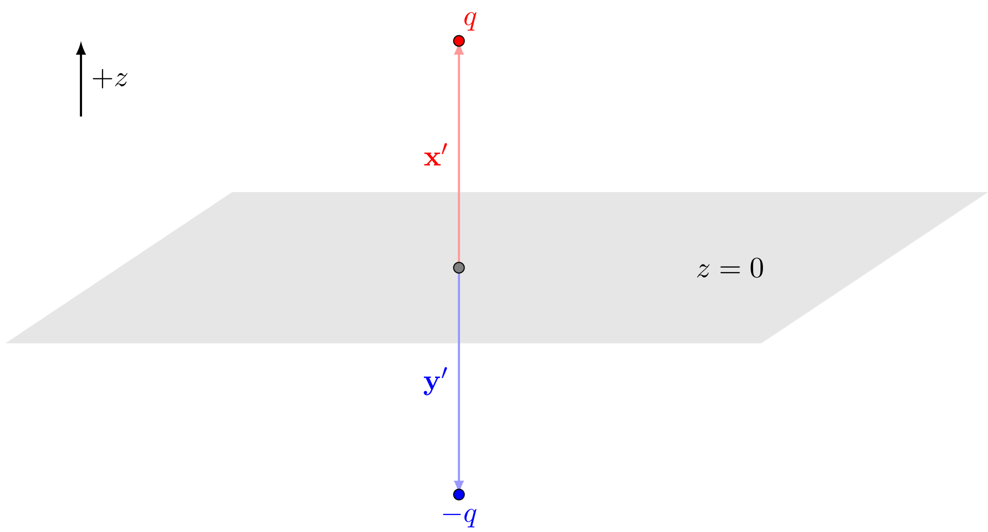
(c) Show that, along the axis of the circle ($\rho = 0$), the potential is given by

$$\Phi = V\left(1 - \frac{z}{\sqrt{a^2 + z^2}}\right).$$

(d) Show that at large distances ($\rho^2 + z^2 \gg a^2$) the potential can be expanded in a power series in $(\rho^2 + z^2)^{-1}$, and that the leading terms are

$$\Phi = \frac{Va^2}{2}\frac{z}{(\rho^2 + z^2)^{3/2}}\left[1 - \frac{3a^2}{4(\rho^2 + z^2)} + \frac{5(3\rho^2 a^2 + a^4)}{8(\rho^2 + z^2)^2} + \dots\right]$$

Solution.



(a) Suppose a point charge q is located at the point $\mathbf{x}' = (\rho', \phi', z')$ in the upper half-space. Clearly, in order for the potential to remain zero in the $z = 0$ plane a point charge $-q$ must be placed at the point $\mathbf{y}' = (\rho', \phi', -z')$. The Green function is simply

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} - \frac{1}{|\mathbf{x} - \mathbf{y}'|}.$$

Since $\mathbf{y}'$ is outside the region of interest, it is clear that the second term is harmonic in the region of interest. In cylindrical coordinates

$$\begin{aligned} G(\rho, \phi, z, \rho', \phi', z') = &\frac{1}{\sqrt{(\rho\cos\phi - \rho'\cos\phi')^2 + (\rho\sin\phi - \rho'\sin\phi')^2 + (z - z')^2}} \\ &- \frac{1}{\sqrt{(\rho\cos\phi - \rho'\cos\phi')^2 + (\rho\sin\phi - \rho'\sin\phi')^2 + (z + z')^2}}. \end{aligned}$$

Note that we can rewrite this as

$$G(\rho, \phi, z, \rho', \phi', z') = \frac{1}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi') + (z - z')^2}} - \frac{1}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi') + (z + z')^2}}.$$

(b) For Dirichlet boundary conditions, we use Equation 1.44,

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \rho(\mathbf{x}) G(\mathbf{x}, \mathbf{x}') \mathrm{d}^3x' - \frac{1}{4\pi} \oint_S \Phi(\mathbf{x}') \frac{\partial G}{\partial n'} \mathrm{d}a',$$

where $\mathcal{V}$ is the upper half-space, and S is the $z = 0$ plane and the boundary at infinity. Since there are no charges in $\mathcal{V}$, and $\Phi(\mathbf{x}')$ is only nonzero inside the circle of radius a centered at the origin in S, which we call C, the potential becomes

$$\Phi(\mathbf{x}) = -\frac{1}{4\pi} \int_C \Phi(\mathbf{x}') \frac{\partial G}{\partial n'} \mathrm{d}a' = -\frac{V}{4\pi} \int_0^{2\pi} \mathrm{d}\phi' \int_0^a \mathrm{d}\rho' \rho' \left.\frac{\partial G}{\partial n'}\right|_{z'=0}.$$

Now, $\hat{\mathbf{n}}' = -\hat{\mathbf{z}}'$. Therefore,

$$\begin{aligned}
\frac{\partial G}{\partial n'} = -\frac{\partial G}{\partial z'} &= -\frac{z - z'}{[\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi') + (z - z')^2]^{3/2}} \\
&\quad - \frac{z + z'}{[\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi') + (z + z')^2]^{3/2}} \\
\left.\frac{\partial G}{\partial n'}\right|_{z'=0} &= -\frac{2z}{[\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi') + z^2]^{3/2}}.
\end{aligned}$$

Thus,

$$\Phi(\mathbf{x}) = \frac{V}{4\pi} \int_0^{2\pi} \int_0^a \frac{2z\rho' \mathrm{d}\rho' \mathrm{d}\phi'}{[\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi') + z^2]^{3/2}}.$$

(c) Along the axis of circle ($\rho = 0$),

$$\begin{aligned}
\Phi(z) &= \frac{V}{4\pi} \int_0^{2\pi} \int_0^a \frac{2z\rho' \mathrm{d}\rho' \mathrm{d}\phi'}{(\rho'^2 + z^2)^{3/2}} \\
&= \frac{V}{4\pi} z \int_0^{2\pi} \mathrm{d}\phi' \int_0^a \frac{2\rho' \mathrm{d}\rho'}{(\rho'^2 + z^2)^{3/2}} \\
&= \frac{V}{4\pi} z \cdot 2\pi \cdot -2 \left.\frac{1}{\sqrt{\rho'^2 + z^2}}\right|_0^a \\
&= V\left(1 - \frac{z}{\sqrt{a^2 + z^2}}\right).
\end{aligned}$$

(d) Using our result from (b),

$$\begin{aligned}
\Phi(\mathbf{x}) &= \frac{V}{4\pi} \int_0^{2\pi} \int_0^a \frac{2z\rho' \mathrm{d}\rho' \mathrm{d}\phi'}{[\rho^2 + z^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi')]^{3/2}} \\
&= \frac{V}{2\pi} \frac{z}{(\rho^2 + z^2)^{3/2}} \int_0^{2\pi} \mathrm{d}\phi' \int_0^a \mathrm{d}\rho' \rho' \left[1 + \frac{\rho'^2}{\rho^2 + z^2} - \frac{2\rho\rho' \cos(\phi - \phi')}{\rho^2 + z^2}\right]^{-3/2}.
\end{aligned}$$

Let $x = \rho'^2/(\rho^2+z^2)$. Expanding the integrand as a series in x, we have

$$\begin{aligned}\Phi(\mathbf{x}) = &\frac{V}{2\pi}\frac{z}{(\rho^2+z^2)^{3/2}}\int_0^{2\pi}\mathrm{d}\phi'\int_0^a \mathrm{d}\rho'\rho' \\ &\times\left[1-\frac{3}{2}\left(1-2\frac{\rho}{\rho'}\cos(\phi-\phi')\right)\frac{\rho'^2}{\rho^2+z^2}+\frac{15}{8}\left(1-2\frac{\rho}{\rho'}\cos(\phi-\phi')\right)^2\left(\frac{\rho'^2}{\rho^2+z^2}\right)^2+\dots\right].\end{aligned}$$

Recall from our derivation of Equation 2.27 that

$$\int_0^{2\pi}\cos(\phi-\phi')\mathrm{d}\phi' = 0 \quad \text{and} \quad \int_0^{2\pi}\cos^2(\phi-\phi')\mathrm{d}\phi' = \pi.$$

Therefore,

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{Vz}{(\rho^2+z^2)^{3/2}}\int_0^a \mathrm{d}\rho'\rho'\left[1-\frac{3}{2}\frac{\rho'^2}{\rho^2+z^2}+\frac{15}{8}\left(1+2\frac{\rho^2}{\rho'^2}\right)\left(\frac{\rho'^2}{\rho^2+z^2}\right)^2+\dots\right] \\ &= \frac{Vz}{(\rho^2+z^2)^{3/2}}\left[\frac{a^2}{2}-\frac{3}{8}\frac{a^4}{\rho^2+z^2}+\frac{5}{16}\frac{a^6}{(\rho^2+z^2)^2}+\frac{15}{16}\frac{\rho^2a^4}{(\rho^2+z^2)^2}+\dots\right] \\ &= \frac{Va^2}{2}\frac{z}{(\rho^2+z^2)^{3/2}}\left[1-\frac{3}{4}\frac{a^2}{\rho^2+z^2}+\frac{5}{8}\frac{3\rho^2a^2+a^4}{(\rho^2+z^2)^2}+\dots\right]\end{aligned}$$

Note that if $\rho = 0$,

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{Va^2}{2z^2}\left(1-\frac{3}{4}\frac{a^2}{z^2}+\frac{5}{8}\frac{a^4}{z^4}+\dots\right) \\ &= V\left(\frac{1}{2}\frac{a^2}{z^2}-\frac{3}{8}\frac{a^4}{z^4}+\frac{5}{16}\frac{a^6}{z^6}+\dots\right) \\ &= V\left(1-\frac{1}{\sqrt{1+a^2/z^2}}\right) \\ &= V\left(1-\frac{z}{\sqrt{a^2+z^2}}\right),\end{aligned}$$

which is the result from (c).

□