Problem 2.2

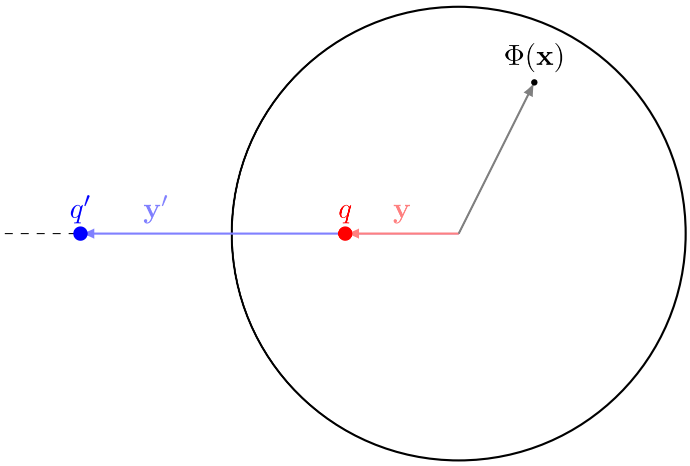
Using the method of images, discuss the problem of a point charge q inside a hollow, grounded, conducting sphere of inner radius a. Find

(a) the potential inside the sphere;

(b) the induced surface-charge density;

(c) the magnitude and direction of the force acting on q.

(d) Is there any change in the solution if the sphere is kept at a fixed potential V? If the sphere has a total charge Q on its inner and outer surfaces?



Solution. By symmetry, the image charge q' must lie on the line from the center of the sphere to q. Let the center of the sphere be the origin of our coordinate system. We wish to find the potential Φ in the region $|\mathbf{x}| < a$, where we also know that $\Phi(|\mathbf{x}| = a) = 0$. Let $\mathbf{x} = x\hat{\mathbf{n}}$ and $\mathbf{y} = y\hat{\mathbf{n}}'$. Clearly, $\mathbf{y}' = y'\hat{\mathbf{n}}'$. The potential is now given by

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|\mathbf{x}-\mathbf{y}|} + \frac{q'}{|\mathbf{x}-\mathbf{y}'|}\right) \\ &= \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|x\hat{\mathbf{n}}-y\hat{\mathbf{n}}'|} + \frac{q'}{|x\hat{\mathbf{n}}-y'\hat{\mathbf{n}}'|}\right).\end{aligned}$$

Imposing the boundary condition $\Phi(x = a) = 0$,

$$\Phi(\mathbf{x})\Big|_{x=a} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|a\hat{\mathbf{n}}-y\hat{\mathbf{n}}'|} + \frac{q'}{|a\hat{\mathbf{n}}-y'\hat{\mathbf{n}}'|}\right).$$

If we factor out a in both denominators, we obtain the trivial solution

$$\Phi(\mathbf{x})\Big|_{x=a} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{a|\hat{\mathbf{n}}-(y/a)\hat{\mathbf{n}}'|} + \frac{q'}{a|\hat{\mathbf{n}}-(y'/a)\hat{\mathbf{n}}'|}\right).$$

This is not a physical solution, since we are forced to conclude that $q' = -q$ and $y' = y$, which means the image charge is inside our region of interest. Thus, we factor y' in the second denominator instead.

$$\Phi(\mathbf{x})\Big|_{x=a} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{a|\hat{\mathbf{n}}-(y/a)\hat{\mathbf{n}}'|} + \frac{q'}{y'|(a/y')\hat{\mathbf{n}}-\hat{\mathbf{n}}'|}\right).$$

Clearly, the choices

$$\frac{q}{a} = -\frac{q'}{y'} \quad \text{and} \quad \frac{y}{a} = \frac{a}{y'}$$

make the potential zero. Therefore, we obtain

$$y' = \frac{a^2}{y} \quad \text{and} \quad q' = -\frac{a}{y}q.$$

(a) We now plug in these values into our first equation. The potential inside the sphere is given by

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|x\hat{\mathbf{n}} - y\hat{\mathbf{n}}'|} - \frac{a}{|xy\hat{\mathbf{n}} - a^2\hat{\mathbf{n}}'|}\right) \\ &= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{x^2+y^2-2xy\cos\gamma}} - \frac{a}{\sqrt{x^2y^2+a^4-2xya^2\cos\gamma}}\right).\end{aligned}$$

(b) The induced surface-charge density inside the sphere is given by

$$\sigma = -\epsilon_0 \frac{\partial\Phi}{\partial n}\bigg|_{x=a} = \epsilon_0 \frac{\partial\Phi}{\partial x}\bigg|_{x=a},$$

since the normal direction points inside the sphere. Now,

$$\begin{aligned}\frac{\partial\Phi}{\partial x}\bigg|_{x=a} &= \frac{q}{4\pi\epsilon_0}\left(-\frac{1}{2}\right)\frac{1}{(a^2+y^2-2ay\cos\gamma)^{3/2}}\left[(2a-2y\cos\gamma) - \frac{1}{a^2}\left(2ay^2 - 2a^2y\ \cos\gamma\right)\right] \\ &= \frac{q}{4\pi\epsilon_0}\frac{a}{(a^2+y^2-2ay\cos\gamma)^{3/2}}\left(\frac{y^2}{a^2}-1\right) \\ &= \frac{q}{4\pi\epsilon_0}\frac{a}{y^3\left(1+a^2/y^2-2(a/y)\cos\gamma\right)^{3/2}}\left(\frac{y^2}{a^2}\right)\left(1-\frac{a^2}{y^2}\right) \\ &= \frac{q}{4\pi\epsilon_0}\frac{1}{\left(1+a^2/y^2-2(a/y)\cos\gamma\right)^{3/2}}\left(\frac{1}{ay}\right)\left(1-\frac{a^2}{y^2}\right).\end{aligned}$$

Thus,

$$\sigma = \frac{q}{4\pi a^2}\frac{1}{\left(1+a^2/y^2-2(a/y)\cos\gamma\right)^{3/2}}\left(\frac{a}{y}\right)\left(1-\frac{a^2}{y^2}\right).$$

This is similar to Equation (2.5) sans a minus sign. However, since q is inside the sphere, $y < a$ and therefore σ is always negative; i.e.

$$\sigma = -\frac{q}{4\pi a^2}\frac{1}{\left(1+a^2/y^2-2(a/y)\cos\gamma\right)^{3/2}}\left(\frac{a}{y}\right)\left(\frac{a^2}{y^2}-1\right), \quad \frac{a}{y} > 1.$$

(c) Clearly, the force on q is directed to the left. The magnitude is

$$F = \frac{qq'}{4\pi\epsilon_0}\frac{1}{(y-y')^2} = \frac{q^2}{4\pi\epsilon_0}\left(\frac{a}{y}\right)\left(y-\frac{a^2}{y}\right)^{-2} = \frac{1}{4\pi\epsilon_0}\left(\frac{q^2}{a^2}\right)\left(\frac{a}{y}\right)^3\left(1-\frac{a^2}{y^2}\right)^{-2}.$$

(d) The only change in the solution is the potential, since the configuration of the induced charge is already in electrostatic equilibrium with the point charge. Also, any change in the force will violate Gauss's law. Therefore, the potential can only change by an addition of a constant. If the sphere is kept at fixed V, then $\Phi \to \Phi + V$. Since the potential must be continuous, and we know that the potential outside the sphere is that of a point charge at the center (whose total charge is the charge on the conductor plus the charge enclosed), if the sphere has a total charge Q on its inner and outer surface, then

$$\Phi \to \Phi + \frac{1}{4\pi\epsilon_0}\frac{Q+q}{b},$$

where b is the outer radius of the sphere.

$\square$