Problem 6.2

The charge and current densities for a single point charge q can be written formally as

$$\rho(\mathbf{x}', t') = q\,\delta[\mathbf{x}' - \mathbf{r}(t')], \quad \mathbf{J}(\mathbf{x}', t') = q\,\mathbf{v}(t')\,\delta[\mathbf{x}' - \mathbf{r}(t')],$$

where $\mathbf{r}(t')$ is the charge's position at time t' and $\mathbf{v}(t')$ is its velocity. In evaluating expressions involving the retarded time, one must put $t' = t_{\text{ret}} = t - R(t')/c$, where $\mathbf{R} = \mathbf{x} - \mathbf{r}(t')$ (but $\mathbf{R} = \mathbf{x} - \mathbf{x}'(t')$ inside the delta functions).

(a) As a preliminary to deriving the Heaviside-Feynman expressions for the electric and magnetic fields of a point charge, show that

$$\int \mathrm{d}^3x'\,\delta[\mathbf{x}' - \mathbf{r}(t_{\text{ret}})] = \frac{1}{\kappa}\,,$$

where $\kappa = 1 - \mathbf{v}\cdot\hat{\mathbf{R}}/c$. Note that κ is evaluated at the retarded time.

(b) Starting with the Jefimenko generalizations of the Coulomb and Biot-Savart laws, use the expressions for the charge and current densities for a point charge and the result of part (a) to obtain the Heaviside-Feynman expressions for the electric and magnetic fields of a point charge,

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left\{\left[\frac{\hat{\mathbf{R}}}{\kappa R^2}\right]_{\text{ret}} + \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\hat{\mathbf{R}}}{\kappa R}\right]_{\text{ret}} - \frac{1}{c^2}\frac{\partial}{\partial t}\left[\frac{\mathbf{v}}{\kappa R}\right]_{\text{ret}}\right\}$$

and

$$\mathbf{B} = \frac{\mu_0 q}{4\pi}\left\{\left[\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa R^2}\right]_{\text{ret}} + \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa R}\right]_{\text{ret}}\right\}.$$

(c) In our notation Feynman's expression for the electric field is

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left\{\left[\frac{\hat{\mathbf{R}}}{R^2}\right]_{\text{ret}} + \frac{[R]_{\text{ret}}}{c}\frac{\partial}{\partial t}\left[\frac{\hat{\mathbf{R}}}{R^2}\right]_{\text{ret}} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\hat{\mathbf{R}}\right]_{\text{ret}}\right\},$$

while Heaviside's expression for the magnetic field is

$$\mathbf{B} = \frac{\mu_0 q}{4\pi}\left\{\left[\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa^2 R^2}\right]_{\text{ret}} + \frac{1}{c\,[R]_{\text{ret}}}\frac{\partial}{\partial t}\left[\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa}\right]_{\text{ret}}\right\}.$$

Show the equivalence of the two sets of expressions for the fields.

Solution.

(a) Note that t_{ret} depends on $\mathbf{x}'$, such that

$$\int \mathrm{d}^3x'\,\delta[\mathbf{x}'-\mathbf{r}(t_{\text{ret}})] = \int \mathrm{d}^3x'\,\delta\left[\mathbf{x}'-\mathbf{r}\left(t-|\mathbf{x}-\mathbf{x}'|/c\right)\right].$$

We again use the property of the Dirac delta function

$$\delta\left(f(\mathbf{x}')\right) = \delta\left[\mathbf{x}'-\mathbf{r}\left(t-|\mathbf{x}-\mathbf{x}'|/c\right)\right] = \left|\frac{\mathrm{d}f}{\mathrm{d}\mathbf{x}'}(\mathbf{x_0})\right|^{-1}\delta(\mathbf{x}'-\mathbf{x}_0).$$

where $\mathbf{x}_0$ is the unique zero of $f(\mathbf{x}')$, which is $\mathbf{r}(t_{\text{ret}})$. Now,

$$\begin{aligned}
\delta\left[\mathbf{x}'-\mathbf{r}\left(t-|\mathbf{x}-\mathbf{x}'|/c\right)\right] &= \left|1-\frac{\partial\mathbf{r}}{\partial\mathbf{x}'}(\mathbf{x_0})\right|^{-1}\delta(\mathbf{x}'-\mathbf{x}_0)\\
&= \left|1-\frac{\partial\mathbf{r}}{\partial t}\cdot\nabla_{\mathbf{x}'}t\,(\mathbf{x_0})\right|^{-1}\delta(\mathbf{x}'-\mathbf{x}_0)\\
&= \left|1-\mathbf{v}\cdot\frac{1}{c}\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\right|^{-1}\delta(\mathbf{x}'-\mathbf{x}_0)\\
&= \left|1-\frac{1}{c}\mathbf{v}\cdot\hat{\mathbf{R}}\right|^{-1}\delta(\mathbf{x}'-\mathbf{x}_0)\\
&= \frac{1}{\kappa}\delta(\mathbf{x}'-\mathbf{x}_0),
\end{aligned}$$

where $\kappa = 1-\mathbf{v}\cdot\hat{\mathbf{R}}/c > 0$. Therefore,

$$\int \mathrm{d}^3x'\,\delta[\mathbf{x}'-\mathbf{r}(t_{\text{ret}})] = \frac{1}{\kappa}.$$

(b) The Jefimenko generalization of the Coulomb law is (6.55)

$$\mathbf{E}(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x'\left\{\frac{\hat{\mathbf{R}}}{R^2}\left[\rho(\mathbf{x}',t')\right]_{\text{ret}} + \frac{\hat{\mathbf{R}}}{cR}\left[\frac{\partial\rho(\mathbf{x}',t')}{\partial t'}\right]_{\text{ret}} - \frac{1}{c^2R}\left[\frac{\partial\mathbf{J}(\mathbf{x}',t')}{\partial t'}\right]_{\text{ret}}\right\}.$$

Using (6.57),

$$\mathbf{E}(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x'\left\{\frac{\hat{\mathbf{R}}}{R^2}\left[\rho(\mathbf{x}',t')\right]_{\text{ret}} + \frac{\hat{\mathbf{R}}}{cR}\frac{\partial}{\partial t}\left[\rho(\mathbf{x}',t')\right]_{\text{ret}} - \frac{1}{c^2R}\frac{\partial}{\partial t}\left[\mathbf{J}(\mathbf{x}',t')\right]_{\text{ret}}\right\}.$$

For point charges, $[\rho(\mathbf{x}',t')]_{\text{ret}} = q\delta[\mathbf{x}'-\mathbf{r}(t_{\text{ret}})]$ and $[\mathbf{J}(\mathbf{x}',t')]_{\text{ret}} = q\mathbf{v}(t_{\text{ret}})\delta[\mathbf{x}'-\mathbf{r}(t_{\text{ret}})]$. Then using our result from part (a), together with the fact that R does not explicitly depend on t (i.e. $\partial R/\partial t = 0$) before integration, and the fact that $\partial/\partial t$ commutes with the integration on x', we obtain

$$\mathbf{E}(\mathbf{x},t) = \frac{q}{4\pi\epsilon_0}\left\{\left[\frac{\hat{\mathbf{R}}}{\kappa R^2}\right]_{\text{ret}} + \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\hat{\mathbf{R}}}{\kappa R}\right]_{\text{ret}} - \frac{1}{c^2}\frac{\partial}{\partial t}\left[\frac{\mathbf{v}}{\kappa R}\right]_{\text{ret}}\right\}.$$

The Jefimenko generalization from the Biot-Savart law is (6.56)

$$\mathbf{B}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int \mathrm{d}^3x' \left\{ [\mathbf{J}(\mathbf{x}',t')]_{\text{ret}} \times \frac{\hat{\mathbf{R}}}{R^2} + \left[\frac{\partial \mathbf{J}(\mathbf{x}',t')}{\partial t'}\right]_{\text{ret}} \times \frac{\hat{\mathbf{R}}}{cR}\right\}.$$

Using the same arguments,

$$\begin{aligned}\mathbf{B}(\mathbf{x},t) &= \frac{\mu_0}{4\pi}\int \mathrm{d}^3x' \left\{ [\mathbf{J}(\mathbf{x}',t')]_{\text{ret}} \times \frac{\hat{\mathbf{R}}}{R^2} + \frac{\partial}{\partial t}\,[\mathbf{J}(\mathbf{x}',t')]_{\text{ret}} \times \frac{\hat{\mathbf{R}}}{cR}\right\} \\ &= \frac{\mu_0 q}{4\pi}\left\{\left[\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa R^2}\right]_{\text{ret}} + \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa R}\right]_{\text{ret}}\right\}.\end{aligned}$$

(c) Note that $[R]_{\text{ret}}$ is a function of t. Furthermore, observe that[1]

$$\kappa = \frac{\mathrm{d}t}{\mathrm{d}t'} = 1 + \frac{1}{c}\left[\frac{\partial R}{\partial t'}\right]_{\text{ret}} \qquad \text{and} \qquad \frac{1}{\kappa} = \frac{\mathrm{d}t'}{\mathrm{d}t} = 1 - \frac{1}{c}\frac{\partial}{\partial t}[R]_{\text{ret}}.$$

Therefore,

$$\begin{aligned}\mathbf{E} &= \frac{q}{4\pi\epsilon_0}\left\{\left[\frac{\hat{\mathbf{R}}}{\kappa R^2}\right]_{\text{ret}} + \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\hat{\mathbf{R}}}{\kappa R}\right]_{\text{ret}} - \frac{1}{c^2}\frac{\partial}{\partial t}\left[\frac{\mathbf{v}}{\kappa R}\right]_{\text{ret}}\right\} \\ &= \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{\kappa}\left[\frac{\hat{\mathbf{R}}}{R^2}\right]_{\text{ret}} + \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\hat{\mathbf{R}}}{\kappa R} - \frac{\mathbf{v}}{c\kappa R}\right]_{\text{ret}}\right\} \\ &= \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{\kappa}\left[\frac{\hat{\mathbf{R}}}{R^2}\right]_{\text{ret}} + \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\hat{\mathbf{R}}}{\kappa R} + \frac{1}{c\kappa R}\frac{\partial \mathbf{R}}{\partial t'}\right]_{\text{ret}}\right\}.\end{aligned}$$

Now, rewriting

$$\frac{1}{cR}\frac{\partial \mathbf{R}}{\partial t'} = \frac{1}{cR}\frac{\partial}{\partial t'}\left(R\,\hat{\mathbf{R}}\right) = \frac{1}{c}\frac{\partial \hat{\mathbf{R}}}{\partial t'} + \frac{\hat{\mathbf{R}}}{cR}\frac{\partial R}{\partial t'}\,,$$

[1]Equations 2.6 and 2.7 in Monaghan, J. J. "The Heaviside-Feynman expression for the fields of an accelerated dipole." Journal of Physics A: General Physics 1, no. 1 (1968): 112.

we get

$$\begin{aligned}
\mathbf{E} &= \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{\kappa}\left[\frac{\hat{\mathbf{R}}}{R^2}\right]_{\text{ret}} + \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\hat{\mathbf{R}}}{\kappa R} + \frac{1}{\kappa}\left(\frac{1}{c}\frac{\partial\hat{\mathbf{R}}}{\partial t'} + \frac{\hat{\mathbf{R}}}{cR}\frac{\partial R}{\partial t'}\right)\right]_{\text{ret}}\right\} \\
&= \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{\kappa}\left[\frac{\hat{\mathbf{R}}}{R^2}\right]_{\text{ret}} + \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\hat{\mathbf{R}}}{\kappa R} + \frac{1}{\kappa}\left(\frac{1}{c}\frac{\partial\hat{\mathbf{R}}}{\partial t'} + \frac{\hat{\mathbf{R}}}{R}(\kappa-1)\right)\right]_{\text{ret}}\right\} \\
&= \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{\kappa}\left[\frac{\hat{\mathbf{R}}}{R^2}\right]_{\text{ret}} + \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\hat{\mathbf{R}}}{R} + \frac{1}{\kappa c}\frac{\partial\hat{\mathbf{R}}}{\partial t'}\right]_{\text{ret}}\right\} \\
&= \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{\kappa}\left[\frac{\hat{\mathbf{R}}}{R^2}\right]_{\text{ret}} + \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\hat{\mathbf{R}}}{R} + \frac{1}{\kappa c}\frac{\partial t}{\partial t'}\frac{\partial\hat{\mathbf{R}}}{\partial t}\right]_{\text{ret}}\right\} \\
&= \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{\kappa}\left[\frac{\hat{\mathbf{R}}}{R^2}\right]_{\text{ret}} + \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\hat{\mathbf{R}}}{R} + \frac{1}{c}\frac{\partial\hat{\mathbf{R}}}{\partial t}\right]_{\text{ret}}\right\} \\
&= \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{\kappa}\left[\frac{\hat{\mathbf{R}}}{R^2}\right]_{\text{ret}} + \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\hat{\mathbf{R}}}{R}\right]_{\text{ret}} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\hat{\mathbf{R}}\right]_{\text{ret}}\right\}.
\end{aligned}$$

Finally, observe that

$$\begin{aligned}
\frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\hat{\mathbf{R}}}{R}\right]_{\text{ret}} &= \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\hat{\mathbf{R}}}{R^2}\cdot R\right]_{\text{ret}} \\
&= \frac{1}{c}\left[\frac{\hat{\mathbf{R}}}{R^2}\right]_{\text{ret}}\frac{\partial}{\partial t}[R]_{\text{ret}} + \frac{[R]_{\text{ret}}}{c}\frac{\partial}{\partial t}\left[\frac{\hat{\mathbf{R}}}{R^2}\right]_{\text{ret}} \\
&= \left(1-\frac{1}{\kappa}\right)\left[\frac{\hat{\mathbf{R}}}{R^2}\right]_{\text{ret}} + \frac{[R]_{\text{ret}}}{c}\frac{\partial}{\partial t}\left[\frac{\hat{\mathbf{R}}}{R^2}\right]_{\text{ret}}.
\end{aligned}$$

Thus,

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left\{\left[\frac{\hat{\mathbf{R}}}{R^2}\right]_{\text{ret}} + \frac{[R]_{\text{ret}}}{c}\frac{\partial}{\partial t}\left[\frac{\hat{\mathbf{R}}}{R^2}\right]_{\text{ret}} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\hat{\mathbf{R}}\right]_{\text{ret}}\right\},$$

which is the desired result.

Similarly,

$$\begin{aligned}
\mathbf{B} &= \frac{\mu_0 q}{4\pi}\left\{\left[\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa R^2}\right]_{\text{ret}} + \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa R}\right]_{\text{ret}}\right\} \\
&= \frac{\mu_0 q}{4\pi}\left\{\left[\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa R^2}\right]_{\text{ret}} + \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa}\cdot\frac{1}{R}\right]_{\text{ret}}\right\} \\
&= \frac{\mu_0 q}{4\pi}\left\{\left[\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa R^2}\right]_{\text{ret}} + \frac{1}{c[R]_{\text{ret}}}\frac{\partial}{\partial t}\left[\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa}\right]_{\text{ret}} + \left[\frac{\mathbf{v}\times\hat{\mathbf{R}}}{c\kappa}\right]_{\text{ret}}\frac{\partial}{\partial t}\left[\frac{1}{R}\right]_{\text{ret}}\right\} \\
&= \frac{\mu_0 q}{4\pi}\left\{\left[\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa R^2}\right]_{\text{ret}} + \frac{1}{c[R]_{\text{ret}}}\frac{\partial}{\partial t}\left[\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa}\right]_{\text{ret}} - \left[\frac{\mathbf{v}\times\hat{\mathbf{R}}}{c\kappa R^2}\right]_{\text{ret}}\frac{\partial}{\partial t}[R]_{\text{ret}}\right\} \\
&= \frac{\mu_0 q}{4\pi}\left\{\left[\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa R^2}\right]_{\text{ret}}\left(1 - \frac{1}{c}\frac{\partial}{\partial t}[R]_{\text{ret}}\right) + \frac{1}{c[R]_{\text{ret}}}\frac{\partial}{\partial t}\left[\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa}\right]_{\text{ret}}\right\} \\
&= \frac{\mu_0 q}{4\pi}\left\{\left[\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa^2 R^2}\right]_{\text{ret}} + \frac{1}{c[R]_{\text{ret}}}\frac{\partial}{\partial t}\left[\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa}\right]_{\text{ret}}\right\}.
\end{aligned}$$

$\square$