Problem 5.15

Consider two long, straight wires, parallel to the z axis, spaced a distance d apart and carrying currents I in opposite directions. Describe the magnetic field $\mathbf{H}$ in terms of a magnetic potential Φ_M, with $\mathbf{H} = -\nabla\Phi_M$.

(a) If the wires are parallel to the z axis with positions $x = \pm d/2$ and $y = 0$, show that in the limit of small spacing, the potential is approximately that of a two-dimensional dipole,

$$\Phi_M \approx -\frac{Id\sin\phi}{2\pi\rho} + \mathcal{O}(d^2/\rho^2),$$

where ρ and ϕ are the usual polar coordinates.

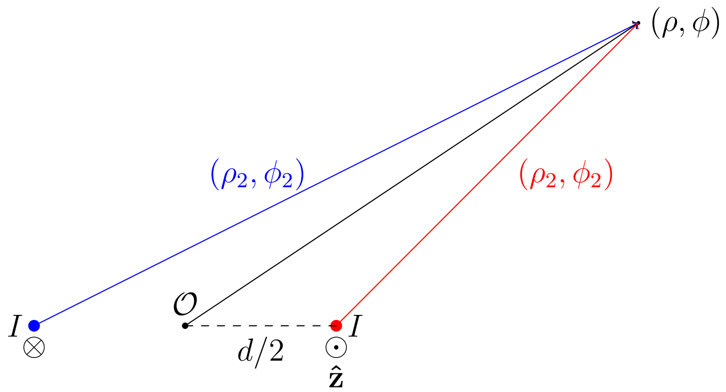
(b) The closely spaced wires are now centered in a hollow right circular cylinder of steel, of inner (outer) radius a (b) and magnetic permeability $\mu = \mu_r\mu_0$. Determine the magnetic scalar potential in the three regions: $0 < \rho < a$, $a < \rho < b$, and $\rho > b$. Show that the field outside the steel cylinder is a two-dimensional dipole field, as in part a, but with a strength reduced by the factor

$$F = \frac{4\mu_r b^2}{(\mu_r+1)^2b^2 - (\mu_r-1)^2a^2}.$$

Relate your result to Problem 5.14.

(c) Assuming that $\mu_r \gg 1$, and $b = a + t$, where the thickness is $t \ll b$, write down an approximate expression for F and determine its numerical value for $\mu_r = 200$ (typical of steel at 20 G), $b = 1.25\,\text{cm}$, $t = 3\,\text{mm}$. The shielding effect is relevant for reduction of stray fields in residential and commercial 60Hz, 110 or 220 V wiring.

Solution. The system has a translational symmetry along the z axis. Therefore, we only consider the 2D problem (i.e. let $z = 0$).



(a) We know that for a single wire (at the center, with current I in the z direction), the magnetic field is given by

$$\mathbf{B} = \frac{\mu_0 I}{2\pi\rho}\hat{\boldsymbol{\phi}}.$$

Using $\mathbf{B} = \mu_0 \mathbf{H}$ and $\mathbf{H} = -\nabla \Phi_M$, we see by inspection that

$$\Phi_M(\rho, \phi) = -\frac{I\phi}{2\pi}.$$

Now, for the given system we get

$$\Phi_M = \frac{I}{2\pi}\left(\phi_2 - \phi_1\right).$$

Observe that $\rho \sin\phi = \rho_1 \sin\phi_1 = \rho_2 \sin\phi_2$, where $\rho_{1,2} = \sqrt{\rho^2 + d^2/4 \mp \rho d \cos\phi}$. Therefore,

$$\Phi_M = \frac{I}{2\pi}\left[\arcsin\left(\frac{\rho}{\rho_2}\sin\phi\right) - \arcsin\left(\frac{\rho}{\rho_1}\sin\phi\right)\right].$$

In the limit of small spacing, we can expand the potential as a series in d/ρ. Using Mathematica, we obtain

$$\begin{aligned}\Phi_M &= \frac{I}{2\pi}\left[-\frac{d}{\rho}\sin\phi + \mathcal{O}\left(d^3/\rho^3\right)\right] \\ &= -\frac{Id\sin\phi}{2\pi\rho} + \mathcal{O}\left(d^3/\rho^3\right).\end{aligned}$$

(b) Again, we only consider a 2D problem, therefore we start with the general series solution (Equation 2.71)

$$\Phi_M(\rho, \phi) = a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty} a_n \rho^n \sin\left(n\phi + \mu_n\right) + \sum_{n=1}^{\infty} b_n \rho^{-n} \sin\left(n\phi + \nu_n\right)$$

Due to the assumed planar symmetry of the field in the $y = 0$ plane, $\mu_n = \nu_n = 0$. The potential can be expressed in the three regions [I. $(0 < r < a)$, II. $(a < r < b)$, and III. $(r > b)$] as

$$\begin{aligned}\Phi_M^{(\mathrm{I})} &= -\frac{Id\sin\phi}{2\pi\rho} + \sum_{n=1}^{\infty} \alpha_n \rho^n \sin(n\phi), \\ \Phi_M^{(\mathrm{II})} &= \gamma_0 \ln\rho + \sum_{n=1}^{\infty} \beta_n \rho^n \sin(n\phi) + \sum_{n=1}^{\infty} \gamma_n \rho^{-n} \sin(n\phi), \\ \Phi_M^{(\mathrm{III})} &= \delta_0 \ln\rho + \sum_{n=1}^{\infty} \delta_n \rho^{-n} \sin(n\phi).\end{aligned}$$

We now impose the boundary conditions

$$\left.\frac{\partial \Phi_M^{(\mathrm{III})}}{\partial \phi}\right|_{r=b} = \left.\frac{\partial \Phi_M^{(\mathrm{II})}}{\partial \phi}\right|_{r=b}, \qquad \left.\frac{\partial \Phi_M^{(\mathrm{II})}}{\partial \phi}\right|_{r=a} = \left.\frac{\partial \Phi_M^{(\mathrm{I})}}{\partial \phi}\right|_{r=a},$$
$$\mu_0 \left.\frac{\partial \Phi_M^{(\mathrm{III})}}{\partial r}\right|_{r=b} = \mu \left.\frac{\partial \Phi_M^{(\mathrm{II})}}{\partial r}\right|_{r=b}, \qquad \mu \left.\frac{\partial \Phi_M^{(\mathrm{II})}}{\partial r}\right|_{r=a} = \mu_0 \left.\frac{\partial \Phi_M^{(\mathrm{I})}}{\partial r}\right|_{r=a}.$$

Similar to Section 5.12 and Problem 5.14, the presence of $\sin\phi$ in the original potential (in part a) implies that only the $n = 1$ terms can be nonzero. In other words,

$$\begin{aligned}
\Phi_M^{(\mathrm{I})} &= -\frac{Id\sin\phi}{2\pi\rho} + \alpha_1\rho\sin\phi, \\
\Phi_M^{(\mathrm{II})} &= \beta_1\rho\sin\phi + \frac{\gamma_1}{\rho}\sin\phi, \\
\Phi_M^{(\mathrm{III})} &= \frac{\delta_1}{\rho}\sin\phi.
\end{aligned}$$

We now list each term in the boundary conditions. (We factor out the $\cos\phi$ and $\sin\phi$ terms since they will all cancel out in the end.)

$$\begin{aligned}
\left.\frac{\partial\Phi_M^{(\mathrm{III})}}{\partial\phi}\right|_{r=b} &= \frac{\delta_1}{b}, \\
\left.\frac{\partial\Phi_M^{(\mathrm{II})}}{\partial\phi}\right|_{r=b} &= \beta_1 b + \frac{\gamma_1}{b};
\end{aligned}$$

$$\begin{aligned}
\left.\frac{\partial\Phi_M^{(\mathrm{II})}}{\partial\phi}\right|_{r=a} &= \beta_1 a + \frac{\gamma_1}{a}, \\
\left.\frac{\partial\Phi_M^{(\mathrm{I})}}{\partial\phi}\right|_{r=a} &= -\frac{Id}{2\pi a} + \alpha_1 a;
\end{aligned}$$

$$\begin{aligned}
\mu_0\left.\frac{\partial\Phi_M^{(\mathrm{III})}}{\partial r}\right|_{r=b} &= -\mu_0\frac{\delta_1}{b^2}, \\
\mu\left.\frac{\partial\Phi_M^{(\mathrm{II})}}{\partial r}\right|_{r=b} &= \mu\beta_1 - \mu\frac{\gamma_1}{b^2};
\end{aligned}$$

$$\begin{aligned}
\mu\left.\frac{\partial\Phi_M^{(\mathrm{II})}}{\partial r}\right|_{r=a} &= \mu\beta_1 - \mu\frac{\gamma_1}{a^2}, \\
\mu_0\left.\frac{\partial\Phi_M^{(\mathrm{I})}}{\partial r}\right|_{r=a} &= \mu_0\frac{Id}{2\pi a^2} + \mu_0\alpha_1.
\end{aligned}$$

We obtain the following matrix equation

$$\begin{pmatrix} 0 & -b & -b^{-1} & b^{-1} \\ -a & a & a^{-1} & 0 \\ 0 & -\mu & \mu b^{-2} & -\mu_0 b^{-2} \\ -\mu_0 & \mu & -\mu a^{-2} & 0 \end{pmatrix}\begin{pmatrix}\alpha_1 \\ \beta_1 \\ \gamma_1 \\ \delta_1\end{pmatrix} = \begin{pmatrix} 0 \\ -Id/(2\pi a) \\ 0 \\ \mu_0 Id/(2\pi a^2)\end{pmatrix}.$$

And the coefficients can be easily solved:

$$\begin{aligned}
\alpha_1 &= \frac{(\mu_r^2 - 1)}{b^2(\mu_r+1)^2 - a^2(\mu_r-1)^2}\left(\frac{b^2}{a^2} - 1\right)\frac{Id}{2\pi}, \\
\beta_1 &= -\frac{\mu_r - 1}{b^2(\mu_r+1)^2 - a^2(\mu_r-1)^2}\frac{Id}{\pi}, \\
\gamma_1 &= -\frac{\mu_r + 1}{(\mu_r+1)^2 - (\mu_r-1)^2 a^2/b^2}\frac{Id}{\pi}, \\
\delta_1 &= -\frac{\mu_r}{(\mu_r+1)^2 - (\mu_r-1)^2 a^2/b^2}\frac{2Id}{\pi}.
\end{aligned}$$

Thus, the magnetic scalar potential is determined everywhere. Moreover, note that $\delta_1 \Big/ \left(-\dfrac{Id}{2\pi}\right) = F$. Therefore, the field outside is also a dipole field, reduced by a factor of F. This factor is analogous to the one found in Problem 5.14, but instead the steel cylinder shields the outside from the field inside.

(c) Let $b = a + t$, where $t \ll b$, and assume that $\mu_r \gg 1$, then

$$\begin{aligned}
F &= \frac{4\mu_r}{(\mu_r+1)^2 - (\mu_r-1)^2 a^2/b^2} \\
&\approx \frac{4}{\mu_r}\left(\frac{1}{1 - a^2/b^2}\right) \\
&= \frac{4}{\mu_r}\left(\frac{a}{2t} + \frac{3}{4} + \frac{t}{8a} - \frac{t^2}{16a^2} + \mathcal{O}(t^3)\right).
\end{aligned}$$

For $\mu_r = 200$ and $t/a \approx 0.3158$, we get $F \approx 0.04733$.

□