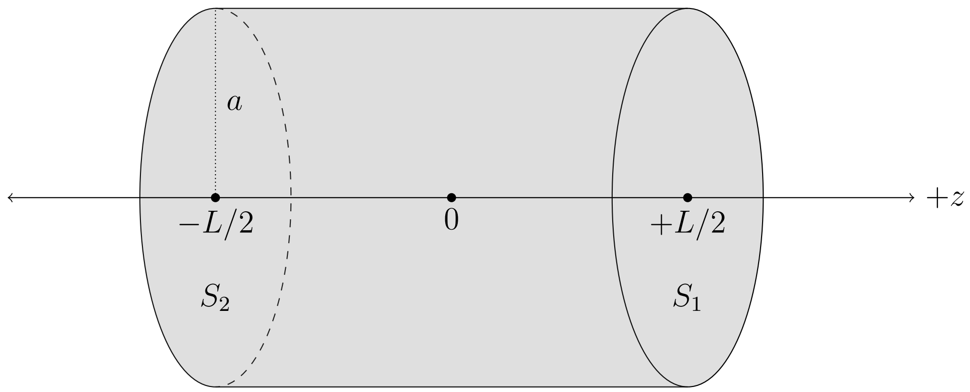
Problem 5.19

A magnetically "hard" material is in the shape of a right circular cylinder of length L and radius a. The cylinder has a permanent magnetization M_0, uniform throughout its volume and parallel to its axis.

(a) Determine the magnetic field $\mathbf{H}$ and magnetic induction $\mathbf{B}$ at all points on the axis of the cylinder, both inside and outside.

(b) Plot the ratios $\mathbf{B}/\mu_0 M_0$ and $\mathbf{H}/M_0$ on the axis as functions of z for $L/a = 5$.



Solution.

(a) Since there are no free currents, the magnetic field $\mathbf{H}$ is derivable from a scalar potential, $\mathbf{H} = -\nabla\Phi_M$. For a "hard" magnet, the scalar potential is given by Equation 5.100

$$\Phi_M(\mathbf{x}) = -\frac{1}{4\pi}\int_V \frac{\nabla'\cdot\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}^3x' + \frac{1}{4\pi}\oint_S \frac{\mathbf{n}'\cdot\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}a'.$$

Since the magnetization $\mathbf{M}(\mathbf{x}') = M_0\hat{\mathbf{z}}$ is uniform throughout the volume, $\nabla'\cdot\mathbf{M}(\mathbf{x}') = 0$ and the first integral vanishes. Moreover, since the magnetization is parallel to the axis of the cylinder only the surface integral along the two circular faces of the cylinder will be nonzero, because the normal vector $\mathbf{n}'$ on the curved surface is in the radial direction. Let S_1 and S_2 denote the upper ($z = +L/2$) and lower ($z = -L/2$) circular faces, respectively. The scalar potential is now

$$\begin{aligned}
\Phi_M(\mathbf{x}) &= \frac{1}{4\pi}\oint_S \frac{\mathbf{n}'\cdot\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}a' \\
&= \frac{1}{4\pi}\left[\int_{S_1}\frac{\mathbf{n}'\cdot\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}a' + \int_{S_2}\frac{\mathbf{n}'\cdot\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}a'\right] \\
&= \frac{1}{4\pi}\left[\int_{S_1}\frac{\hat{\mathbf{z}}\cdot(M_0\hat{\mathbf{z}})}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}a' + \int_{S_2}\frac{-\hat{\mathbf{z}}\cdot(M_0\hat{\mathbf{z}})}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}a'\right] \\
&= \frac{M_0}{4\pi}\left[\int_{S_1}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}a' - \int_{S_2}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}a'\right]
\end{aligned}$$

Due to the azimuthal symmetry, only the z component of $\mathbf{H}$ and $\mathbf{B}$ are nonzero. Thus, it suffices to evaluate the magnetic scalar potential only along the axis of the cylinder. Then H_z can be obtained by taking the derivative of the potential along the z direction. Now,

$$\begin{aligned}\Phi_M(z) &= \frac{M_0}{4\pi}\left[\int_0^{2\pi}\int_0^a \frac{s'\mathrm{d}s'\mathrm{d}\phi'}{\sqrt{s'^2+(z-L/2)^2}} - \int_0^{2\pi}\int_0^a \frac{s'\mathrm{d}s'\mathrm{d}\phi'}{\sqrt{s'^2+(z+L/2)^2}}\right]\\ &= \frac{M_0}{2}\left[\int_0^a \frac{s'\mathrm{d}s'}{\sqrt{s'^2+(z-L/2)^2}} - \int_0^a \frac{s'\mathrm{d}s'}{\sqrt{s'^2+(z+L/2)^2}}\right].\end{aligned}$$

Let $u_{1,2} = s'^2 + (z \mp L/2)^2$. Then $\mathrm{d}u_{1,2} = 2s'\mathrm{d}s'$. Therefore,

$$\begin{aligned}\Phi_M(z) &= \frac{M_0}{2}\left[\int_{u_1(0)}^{u_1(a)} \frac{\mathrm{d}u_1}{2\sqrt{u_1}} - \int_{u_2(0)}^{u_2(a)} \frac{\mathrm{d}u_2}{2\sqrt{u_2}}\right]\\ &= \frac{M_0}{2}\left[\sqrt{u_1}\Big|_{u_1(0)}^{u_1(a)} - \sqrt{u_2}\Big|_{u_2(0)}^{u_2(a)}\right]\\ &= \frac{M_0}{2}\left[\sqrt{a^2+(z-L/2)^2} - \sqrt{a^2+(z+L/2)^2} + \left|z+L/2\right| - \left|z-L/2\right|\right].\end{aligned}$$

Thus, we have the following piecewise smooth potential

$$\Phi_M(z) = \begin{cases} \frac{M_0}{2}\left(\sqrt{a^2+(z-L/2)^2} - \sqrt{a^2+(z+L/2)^2} + L\right), & z > +L/2 \\ \frac{M_0}{2}\left(\sqrt{a^2+(z-L/2)^2} - \sqrt{a^2+(z+L/2)^2} + 2z\right), & -L/2 < z < +L/2 \\ \frac{M_0}{2}\left(\sqrt{a^2+(z-L/2)^2} - \sqrt{a^2+(z+L/2)^2} - L\right), & z < -L/2 \end{cases}.$$

Now, we can obtain the magnetic field $\mathbf{H}$ along the axis. Again, only the z component is nonzero, and it is given by

$$H_{\text{out}}(z) = -\frac{\partial}{\partial z}\Phi_{M,\text{out}}(z) = \frac{M_0}{2}\left(\frac{z+L/2}{\sqrt{a^2+(z+L/2)^2}} - \frac{z-L/2}{\sqrt{a^2+(z-L/2)^2}}\right),$$

$$H_{\text{in}}(z) = -\frac{\partial}{\partial z}\Phi_{M,\text{in}}(z) = \frac{M_0}{2}\left(\frac{z+L/2}{\sqrt{a^2+(z+L/2)^2}} - \frac{z-L/2}{\sqrt{a^2+(z-L/2)^2}}\right) - M_0.$$

The magnetic induction $\mathbf{B}$ can be obtained from the relation

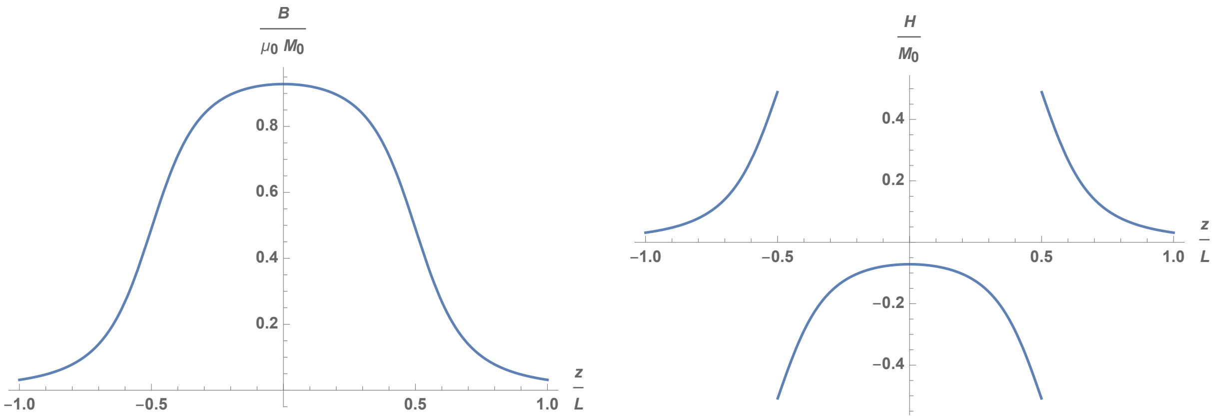
$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}.$$

We find

$$B_{\text{out}}(z) = -\frac{\partial}{\partial z}\Phi_{M,\text{out}}(z) = \frac{\mu_0 M_0}{2}\left(\frac{z+L/2}{\sqrt{a^2+(z+L/2)^2}} - \frac{z-L/2}{\sqrt{a^2+(z-L/2)^2}}\right),$$

$$B_{\text{in}}(z) = -\frac{\partial}{\partial z}\Phi_{M,\text{in}}(z) = \frac{\mu_0 M_0}{2}\left(\frac{z+L/2}{\sqrt{a^2+(z+L/2)^2}} - \frac{z-L/2}{\sqrt{a^2+(z-L/2)^2}}\right).$$

(b) Using Mathematica, we obtain the following plots for $L/a = 5$.



(a) $B/\mu_0 M_0$ as a function of z/L.

(b) H/M_0 as a function of z/L.

Figure 1: The behavior of the (z component of the) magnetic induction $\mathbf{B}$ and magnetic field $\mathbf{H}$ along the axis of the cylinder for $L/a = 5$.

□