Problem 2.14

A variant of the preceding two-dimensional problem is a long hollow conducting cylinder of radius b that is divided into quarters, alternate segments being held at potential $+V$ and $-V$.

(a) Solve by means of the series solution (2.71) and show that the potential inside the cylinder is

$$\Phi(\rho,\phi) = \frac{4V}{\pi}\sum_{n=0}^{\infty}\left(\frac{\rho}{b}\right)^{4n+2}\frac{\sin[(4n+2)\phi]}{2n+1}.$$

(b) Sum the series and show that

$$\Phi(\rho,\phi) = \frac{2V}{\pi}\arctan\left(\frac{2\rho^2 b^2\sin 2\phi}{b^4-\rho^4}\right).$$

(c) Sketch the field lines and equipotentials.

Solution.

(a) The cylindrical symmetry of the problem allows us to consider only the potential on a cross section. Using (2.71), the ansatz is (since the other terms diverge at $\rho = 0$)

$$\begin{aligned}\Phi(\rho,\phi) &= a_0 + \sum_{n=1}^{\infty} a_n\rho^n\sin(n\phi+\alpha_n)\\ &= a_0 + \sum_{n=1}^{\infty}\rho^n\left[c_n\sin(n\phi)+d_n\cos(n\phi)\right].\end{aligned}$$

Now,

$$\Phi(b,\phi) = a_0 + \sum_{n=1}^{\infty} b^n\left[c_n\sin(n\phi)+d_n\cos(n\phi)\right] = V(\phi),$$

where

$$V(\phi) = \begin{cases} +V & \text{if } 0<\phi<\frac{\pi}{2} \text{ or } \pi<\phi<\frac{3\pi}{2}\\ -V & \text{if } \frac{\pi}{2}<\phi<\pi \text{ or } \frac{3\pi}{2}<\phi<2\pi\end{cases}.$$

Now, we use the following orthogonality relations

$$\begin{aligned}\int_0^{2\pi}\sin(m\phi)\sin(n\phi) &= \pi\delta_{mn},\\ \int_0^{2\pi}\cos(m\phi)\cos(n\phi) &= \pi\delta_{mn},\\ \int_0^{2\pi}\sin(m\phi)\cos(n\phi) &= 0.\end{aligned}$$

Clearly,

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi}V(\phi)\mathrm{d}\phi = 0.$$

$$\begin{aligned}
c_m &= \frac{1}{\pi b^m}\int_0^{2\pi} V(\phi)\sin(m\phi)\mathrm{d}\phi \\
&= -\frac{V}{m\pi b^m}\left[\cos(m\phi)\Big|_0^{\pi/2} - \cos(m\phi)\Big|_{\pi/2}^{\pi} + \cos(m\phi)\Big|_{\pi}^{3\pi/2} - \cos(m\phi)\Big|_{3\pi/2}^{2\pi}\right] \\
&= -\frac{V}{m\pi b^m}\left[\left(\cos\left(\frac{m\pi}{2}\right) - 1\right) - \left((-1)^m - \cos\left(\frac{m\pi}{2}\right)\right)\right. \\
&\qquad\qquad \left. + \left(\cos\left(\frac{3m\pi}{2}\right) - (-1)^m\right) - \left(1 - \cos\left(\frac{3m\pi}{2}\right)\right)\right] \\
&= -\frac{2V}{m\pi b^m}\left[\cos\left(\frac{m\pi}{2}\right) + \cos\left(\frac{3m\pi}{2}\right) - 1 - (-1)^m\right] \\
&= -\frac{2V}{m\pi b^m}\left[\cos\left(\frac{m\pi}{2}\right) + \cos\left(\frac{3m\pi}{2}\right) - 1 - (-1)^m\right]
\end{aligned}$$

Let $B(m)$ denote the expression in the square brackets. Note that $B(0) = 0$, $B(1) = 0$, $B(2) = -4$, and $B(3) = 0$. Also, $B(m+4) = B(m)$. Therefore, the only nonzero c_m coefficients are

$$c_m = \frac{8V}{m\pi b^m}, \quad m = 2, 6, 10, \ldots$$

Now,

$$\begin{aligned}
d_m &= \frac{1}{\pi b^m}\int_0^{2\pi} V(\phi)\cos(m\phi)\mathrm{d}\phi \\
&= \frac{V}{m\pi b^m}\left[\sin(m\phi)\Big|_0^{\pi/2} - \sin(m\phi)\Big|_{\pi/2}^{\pi} + \sin(m\phi)\Big|_{\pi}^{3\pi/2} - \sin(m\phi)\Big|_{3\pi/2}^{2\pi}\right] \\
&= \frac{2V}{m\pi b^m}\left[\sin\left(\frac{m\pi}{2}\right) + \sin\left(\frac{3m\pi}{2}\right)\right] \\
&= 0.
\end{aligned}$$

Therefore, the potential is

$$\begin{aligned}
\Phi(\rho,\phi) &= \sum_{n=1}^{\infty} c_n \rho^n \sin(n\phi) \\
&= \sum_{n=0}^{\infty} \frac{8V}{(4n+2)\pi b^{4n+2}}\rho^{4n+2}\sin[(4n+2)\phi] \\
&= \frac{4V}{\pi}\sum_{n=0}^{\infty}\left(\frac{\rho}{b}\right)^{4n+2}\frac{\sin[(4n+2)\phi]}{(2n+1)}.
\end{aligned}$$

(b) Note that

$$\begin{aligned}
\frac{4V}{\pi}\sum_{n=0}^{\infty}\left(\frac{\rho}{b}\right)^{4n+2}\frac{\sin[(4n+2)\phi]}{(2n+1)} &= \Im\left(\frac{4V}{\pi}\sum_{n=0}^{\infty}\left(\frac{\rho}{b}\right)^{4n+2}\frac{e^{i[(4n+2)\phi]}}{(2n+1)}\right) \\
&= \frac{8V}{\pi}\Im\left(\sum_{n=0}^{\infty}\frac{1}{(4n+2)}\left(\frac{\rho}{b}e^{i\phi}\right)^{4n+2}\right).
\end{aligned}$$

Using Mathematica,

$$\sum_{n=0}^{\infty} \frac{z^{4n+2}}{4n+2} = \frac{1}{2}\operatorname{arctanh}(z^2).$$

Therefore,

$$\Phi(\rho,\phi) = \frac{4V}{\pi}\Im\left(\operatorname{arctanh}(\frac{\rho^2}{b^2}e^{i2\phi})\right)$$

Also, since $\operatorname{arctanh} z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$ and $\Im\left(\ln z\right) = \arg z$,

$$\Phi(\rho,\phi) = \frac{2V}{\pi}\arg\left(\frac{b^2+\rho^2 e^{i2\phi}}{b^2-\rho^2 e^{i2\phi}}\right).$$

It remains to be shown that

$$\arg\left(\frac{b^2+\rho^2 e^{i2\phi}}{b^2-\rho^2 e^{i2\phi}}\right) = \arctan\left(\frac{2\rho^2 b^2 \sin 2\phi}{b^4-\rho^4}\right).$$

(c) The equipotential curves are determined by

$$\Phi(\rho,\phi) = \frac{2V}{\pi}\arctan\left(\frac{2\rho^2 b^2 \sin 2\phi}{b^4-\rho^4}\right) = \text{const.}$$

Using Mathematica, we can solve for $\rho(\phi)$ and create a parametric plot.

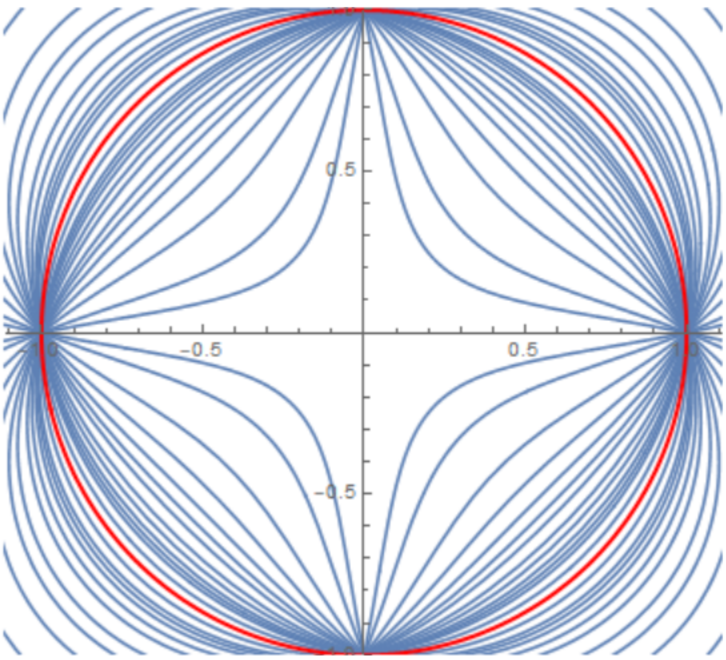

Figure 1: Equipotential curves (in blue). We only consider the curves inside the conductor, the surface of the conductor is given by $\rho/b = 1$ (in red).

The field lines are just curves which intersects the equipotential curves perpendicularly. We only provide a sketch below.

□

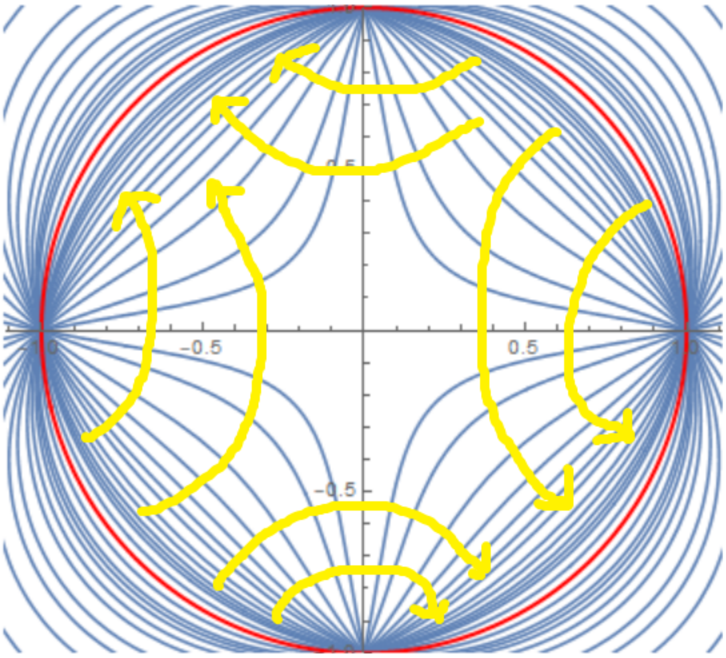

Figure 2: Field lines. Recall that the potential on the surface is $+V$ in first and third quadrant, and $-V$ in the remaining.