Problem 5.1

Starting with the differential expression

$$\mathrm{d}\mathbf{B} = \frac{\mu_0 I}{4\pi}\mathrm{d}\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}$$

for the magnetic induction at the point P with coordinate $\mathbf{x}$ produced by an increment of current $I\mathrm{d}\mathbf{l}'$ at $\mathbf{x}'$, show explicitly that for a closed loop carrying a current I the magnetic induction at P is

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\nabla\Omega,$$

where Ω is the solid angle subtentded by the loop at the point P. This corresponds to a magnetic scalar potential $\Phi_M = -\mu_0 I\Omega/4\pi$. The sign convention for the solid angle is that Ω is positive if the point P views the "inner" side of the surface spanning the loop, that is, if a unit normal $\hat{\mathbf{n}}$ to the surface is defined by the direction of current flow via the right-hand rule, Ω is positive if $\hat{\mathbf{n}}$ points away from the point P, and negative otherwise. This is the same convention as in Section 1.6 for the electric dipole layer.

Solution. Starting with the differential expression

$$\begin{aligned}
\mathrm{d}\mathbf{B} &= \frac{\mu_0 I}{4\pi}\mathrm{d}\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \\
\mathbf{B} &= \frac{\mu_0 I}{4\pi}\oint \mathrm{d}\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \\
&= \frac{\mu_0 I}{4\pi}\oint \mathrm{d}\mathbf{l}' \times \nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \\
B_i &= \frac{\mu_0 I}{4\pi}\oint \left[\mathrm{d}\mathbf{l}' \times \nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\right]\cdot\hat{\mathbf{x}}_i.
\end{aligned}$$

Now, using the cyclic property of $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$

$$\begin{aligned}
B_i &= \frac{\mu_0 I}{4\pi}\oint \mathrm{d}\mathbf{l}' \cdot \left[\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\times\hat{\mathbf{x}}_i\right] \\
&= \frac{\mu_0 I}{4\pi}\int \mathrm{d}a'\hat{\mathbf{n}}\cdot\nabla'\times\left[\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\times\hat{\mathbf{x}}_i\right],
\end{aligned}$$

where we used Stokes' theorem. Using the vector identity $\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\nabla\cdot\mathbf{B} - \mathbf{B}\nabla\cdot\mathbf{A} + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}$, we obtain

$$\begin{aligned}
B_i = \frac{\mu_0 I}{4\pi}\int \mathrm{d}a'\hat{\mathbf{n}}\cdot\Bigg[&\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}\nabla'\cdot\hat{\mathbf{x}}_i - \hat{\mathbf{x}}_i\nabla'^2\frac{1}{|\mathbf{x}-\mathbf{x}'|} \\
&+(\hat{\mathbf{x}}_i\cdot\nabla')\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|} - \left(\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}\cdot\nabla'\right)\hat{\mathbf{x}}_i\Bigg].
\end{aligned}$$

The first two terms clearly vanish, since $\hat{\mathbf{x}}_i$ is constant and we assume the point P is never on the loop. The last term also vanish because any derivative operator that acts on $\hat{\mathbf{x}}_i$ is zero.

Therefore,

$$
\begin{aligned}
B_i &= \frac{\mu_0 I}{4\pi} \int \mathrm{d}a' \hat{\mathbf{n}} \cdot \left[(\hat{\mathbf{x}}_i \cdot \nabla') \nabla' \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right] \\
&= \frac{\mu_0 I}{4\pi} \int \mathrm{d}a' (\hat{\mathbf{x}}_i \cdot \nabla') \hat{\mathbf{n}} \cdot \nabla' \frac{1}{|\mathbf{x} - \mathbf{x}'|} \\
&= \frac{\mu_0 I}{4\pi} \int \mathrm{d}a' \left(\frac{\partial}{\partial x_i} \right) \hat{\mathbf{n}} \cdot \nabla' \frac{1}{|\mathbf{x} - \mathbf{x}'|} \\
&= \frac{\mu_0 I}{4\pi} \frac{\partial}{\partial x_i} \int \mathrm{d}a' \hat{\mathbf{n}} \cdot \nabla' \frac{1}{|\mathbf{x} - \mathbf{x}'|} \\
&= \frac{\mu_0 I}{4\pi} \frac{\partial}{\partial x_i} \Omega.
\end{aligned}
$$

Thus,

$$
\mathbf{B} = \frac{\mu_0 I}{4\pi} \nabla \Omega.
$$

□