Problem 2.4

A point charge is placed a distance $d > R$ from the center of an equally charged, isolated, conducting sphere of radius R.

(a) Inside of what distance from the surface of the sphere is the point charge attracted rather than repelled by the charged sphere?

(b) What is the limiting value of the force of attraction when the point charge is located a distance $a(= d - R)$ from the surface of the surface of the sphere, if $a \ll R$?

(c) What are the results for parts (a) and (b) if the charge on the sphere is twice (half) as large as the point charge, but still the same sign?

Solution. The force on the point charge q (directed outwards) is given by Equation 2.9

$$F = \frac{1}{4\pi\epsilon_0}\frac{q}{d^2}\left[Q - \frac{qR^3(2d^2 - R^2)}{d(d^2 - R^2)^2}\right],$$

where Q is the charge on the conductor. Let $\kappa = Q/q$ and $r = d/R$. Then

$$F = \frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2}\left[\kappa - \frac{(2r^2 - 1)}{r(r^2 - 1)^2}\right].$$

For an equally charged conducting sphere, $\kappa = 1$

(a) We want to find the distance where $F = 0$. Clearly, this amounts to setting

$$1 - \frac{(2r^2 - 1)}{r(r^2 - 1)^2} = 0, \quad (r > 1)$$

which is equivalent to solving the quintic equation

$$r^5 - 2r^3 - 2r^2 + r + 1 = 0, \quad (r > 1).$$

This can be solved numerically; and the only real solution satisfying $r > 1$ is $r = 1.618$; i.e. inside a distance of $d = 1.618R$ the point charge is attracted rather than repelled by the charged sphere.

(b) In this case, the force is given by

$$F = \frac{1}{4\pi\epsilon_0}\frac{q^2}{(R + a)^2}\left[1 - \frac{(2r^2 - 1)}{r(r^2 - 1)^2}\right],$$

where $r = 1 + a/R$ and $\varepsilon := a/R \ll 1$. Taking the power series of F around $r = 1$, we obtain

$$\begin{aligned}
F &= \frac{1}{4\pi\epsilon_0}\frac{q^2}{(R + a)^2}\left[-\frac{1}{4\varepsilon^2} - \frac{1}{2\varepsilon} + \dots\right] \\
&\approx \frac{1}{4\pi\epsilon_0}\frac{q^2}{R^2}\left(-\frac{1}{4\varepsilon^2}\right) \\
&\approx -\frac{1}{16\pi\epsilon_0}\frac{q^2}{a^2}.
\end{aligned}$$

(c) For $\kappa = 2$, the quintic equation becomes

$$2r^5 - 4r^3 - 2r^2 + 2r + 1 = 0, \quad (r > 1);$$

and the only valid solution is $d = 1.428R$. For $\kappa = 1/2$, the quintic equation becomes

$$\frac{1}{2}r^5 - r^3 - 2r^2 + \frac{1}{2}r + 1 = 0, \quad (r > 1);$$

and the only valid solution is $d = 1.882R$. Note that if we take the power series of F around $r = 1$, the leading contribution is of order ε^{-2}. Since κ appears in the zeroth-order term, it will have no effect to the limiting value of the force of attraction.

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