Problem 5.35

An insulated coil is wound on the surface of a sphere of radius a in such a way as to produce a uniform magnetic induction B_0 in the z direction inside the sphere and dipole field outside the sphere. The medium inside and outside the sphere has a uniform conductivity σ and permeability μ.

(a) Find the necessary surface current density $\mathbf{K}$ and show that the vector potential describing the magnetic field has only an azimuthal component, given by

$$A_\phi = \frac{B_0 a^2}{2} \frac{r_<}{r_>^2} \sin\theta,$$

where $r_<$ ($r_>$) is the smaller (larger) of r and a.

(b) At $t = 0$ the current in the coil is cut off. [The coil's presence may be ignored from now on.] With the neglect of Maxwell's displacement current, the decay of the magnetic field is described by the diffusion equation (5.160). Using a Laplace transform and a spherical Bessel function expansion (3.113), show that the vector potential at times $t > 0$ is given by

$$A_\phi = \frac{3B_0 a}{\pi} \sin\theta \int_0^\infty e^{-\nu t k^2} j_1(k)\, j_1\left(\frac{kr}{a}\right) \mathrm{d}k,$$

where $\nu = 1/\mu\sigma a^2$ is a characteristic decay rate and $j_1(x)$ is the spherical Bessel function of order one. Show that the magnetic field at the center of the sphere can be written explicitly in terms of the error function $\Phi(x)$ as

$$B_z(0,t) = B_0 \left[\Phi\left(\frac{1}{\sqrt{4\nu t}}\right) - \frac{1}{\sqrt{\pi\nu t}} \exp\left(-\frac{1}{4\nu t}\right)\right].$$

(c) Show that the total magnetic energy at time $t > 0$ can be written as

$$W_m = \frac{6B_0^2 a^3}{\mu} \int_0^\infty e^{-2\nu t k^2} \left[j_1(k)\right]^2 \mathrm{d}k.$$

Show that at long times ($\nu t \gg 1$) the magnetic energy decays asymptotically as

$$W_m \to \frac{\sqrt{2\pi} B_0^2 a^3}{24\mu(\nu t)^{3/2}}.$$

(d) Find a corresponding expression for the asymptotic form of the vector potential (at fixed r, θ, and $\nu t \to \infty$) and show that it decays as $(\nu t)^{-3/2}$ as well. Since the energy is quadratic in the field strength, there seems to be a puzzle here. Show by numerical or analytic means that the behavior of the magnetic field at time t is such that, for distances small compared to $R = a(\nu t)^{1/2} \gg a$, the field is uniform with strength $(B_0/6\pi^{1/2})(\nu t)^{-3/2}$, and for distances large compared to R, the field is essentially the original dipole field. Explain physically.

Solution.

(a) In order to find $\mathbf{K}$, we use the boundary condition (Equation 5.87)

$$\hat{\mathbf{n}} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K}.$$

For a uniform linear medium we also have $\mathbf{B} = \mu \mathbf{H}$. Therefore, on the surface of the sphere we have

$$\mathbf{K} = \frac{1}{\mu} \hat{\mathbf{r}} \times (\mathbf{B}_{\text{out}} - \mathbf{B}_{\text{in}}) \Big|_{r=a}.$$

Note that the surface current density is only in the ϕ direction. Moreover, the wounding of the coil is uniform along the z direction, therefore by simple geometric arguments we note that $\Delta z \approx a \sin\theta \Delta\theta$. Thus, $\mathbf{K}(\theta) = \kappa \sin\theta\, \hat{\boldsymbol{\phi}}$, where κ is to be determined. Thus, we have a condition on the θ and ϕ components of the magnetic induction on the boundary,

$$\mu\kappa \sin\theta = B_{\text{out},\theta} - B_{\text{in},\theta} \quad \text{and} \quad B_{\text{out},\phi} = B_{\text{in},\phi} = 0.$$

And from the other boundary condition (Equation 5.76) we get $B_{\text{out},r} = B_{\text{in},r}$. Note that inside the sphere, by a change of coordinates, we obtain

$$\begin{aligned} B_{\text{in},r} &= B_0 \cos\theta, \\ B_{\text{in},\theta} &= -B_0 \sin\theta, \\ B_{\text{in},\phi} &= 0. \end{aligned}$$

Now we impose $\mathbf{B} = \nabla \times \mathbf{A}$ and the fact that only $A_{\text{in},\phi}$ is nonzero:

$$\begin{aligned} B_0 \cos\theta &= \frac{1}{r\sin\theta} \frac{\partial}{\partial\theta} \left(A_{\text{in},\phi} \sin\theta \right) \\ -B_0 \sin\theta &= -\frac{1}{r} \frac{\partial}{\partial r} \left(r A_{\text{in},\phi} \right). \end{aligned}$$

By inspection, we see that the solution is

$$A_{\text{in},\phi} = \frac{1}{2} B_0 r \sin\theta.$$

Now, outside the sphere we expect the vector potential of a dipole

$$\mathbf{A}_{\text{out}} = \frac{C}{r^2} \hat{\mathbf{z}} \times \hat{\mathbf{r}} = \frac{C}{r^2} \left(\cos\theta \hat{\mathbf{r}} - \sin\theta \hat{\boldsymbol{\theta}} \right) \times \hat{\mathbf{r}} = \frac{C}{r^2} \sin\theta\, \hat{\boldsymbol{\phi}}$$

Again, only $A_{\text{out},\phi}$ is nonzero and

$$\begin{aligned} B_{\text{out},r} &= \frac{1}{r\sin\theta} \frac{\partial}{\partial\theta} \left(A_{\text{out},\phi} \sin\theta \right) = \frac{2C}{r^3} \cos\theta \\ B_{\text{out},\theta} &= -\frac{1}{r} \frac{\partial}{\partial r} \left(r A_{\text{out},\phi} \right) = \frac{C}{r^3} \sin\theta. \end{aligned}$$

Imposing the boundary condition along the normal component, we get $C = a^3 B_0/2$. Then

$$A_{\text{out},\phi} = \frac{1}{2} B_0 \frac{a^3}{r^2} \sin\theta.$$

Thus, we have obtained the desired result for the vector potential $\mathbf{A}$ everywhere. Imposing the first boundary condition allows us to determine κ,

$$\kappa = \frac{1}{\mu \sin\theta} \left(B_{\text{out},\theta} - B_{\text{in},\theta}\right)\Big|_{r=a} = \frac{3B_0}{2\mu}.$$

Thus,

$$\mathbf{K} = \frac{3B_0}{2\mu} \sin\theta\, \hat{\boldsymbol{\phi}}.$$

(b) Using separation of variables for the diffusion equation (Equation 5.160), the time component $T(t)$ must satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t} T = -\frac{k^2}{\mu\sigma} T,$$

for some separation constant $-k^2$ to be determined by the spatial equations. Therefore, $T(t) = e^{-\nu t k^2 a^2}$. We also know that $\Theta(\theta) = \sin\theta$ and that there is no ϕ dependence due to azimuthal symmetry. Thus, we shall consider only the radial part for now. We may expand $A_\phi(r)$ using Equation 3.113,

$$A_\phi(r) = \int_0^\infty \tilde{A}_\phi(k) j_1(kr)\,\mathrm{d}k, \quad \text{where} \quad \tilde{A}_\phi(k) = \frac{2k^2}{\pi} \int_0^\infty r^2 A_\phi(r) j_1(kr)\,\mathrm{d}r.$$

Now,

$$\begin{aligned}
\tilde{A}_\phi(k) &= \frac{2k^2}{\pi} \left[\int_0^a r^2 A_{\text{in},\phi}(r) j_1(kr)\,\mathrm{d}r + \int_a^\infty r^2 A_{\text{out},\phi}(r) j_1(kr)\,\mathrm{d}r\right] \\
&= \frac{k^2}{\pi} B_0 \left[\int_0^a r^3 j_1(kr)\,\mathrm{d}r + \int_a^\infty a^3 j_1(kr)\,\mathrm{d}r\right] \\
&= \frac{3B_0}{\pi k^2} \left(-ka\cos(ka) + \sin(ka)\right) \\
&= \frac{3B_0 a^2}{\pi} \left(-\frac{\cos(ka)}{ka} + \frac{\sin(ka)}{k^2 a^2}\right) \\
&= \frac{3B_0 a^2}{\pi} j_1(ka).
\end{aligned}$$

Note that we can rewrite

$$A_\phi(r) = \int_0^\infty \tilde{A}_\phi(k) j_1(kr)\,\mathrm{d}k = \int_0^\infty \tilde{A}_\phi\left(\frac{k'}{a}\right) j_1\left(\frac{k' r}{a}\right) \frac{\mathrm{d}k'}{a}.$$

Since T also depends on k, plugging in the general solution $\tilde{A}_\phi(k'/a, r, \theta, t)$ we get

$$A_\phi = \frac{3B_0 a}{\pi} \sin\theta \int_0^\infty e^{-\nu t k'^2} j_1(k') j_1\left(\frac{k' r}{a}\right) \mathrm{d}k'.$$

Since k' is just a dummy variable, we can make the replacement $k' \to k$,

$$A_\phi = \frac{3B_0 a}{\pi} \sin\theta \int_0^\infty e^{-\nu t k^2} j_1(k) j_1\left(\frac{kr}{a}\right) \mathrm{d}k.$$

Now, the magnetic field at the center of the sphere can be obtained using $\mathbf{B} = \nabla \times \mathbf{A}$. In cylindrical coordinates, we have

$$B_z(0,t) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\, A_\phi\Big|_{r=\rho,\,\theta=\frac{\pi}{2}}\right)\Bigg|_{\rho=0}$$

if we restrict ourselves to the $z = 0$ plane. Plugging in our result for A_ϕ, we obtain

$$\begin{aligned}
B_z &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\phi) = \frac{\partial}{\partial\rho}A_\phi + \frac{1}{\rho}A_\phi \\
&= \frac{3B_0a}{\pi}\int_0^\infty e^{-\nu tk^2} j_1(k)\frac{\partial}{\partial\rho}j_1\left(\frac{k\rho}{a}\right)\mathrm{d}k + \frac{3B_0a}{\pi\rho}\int_0^\infty e^{-\nu tk^2} j_1(k)j_1\left(\frac{k\rho}{a}\right)\mathrm{d}k \\
&= \frac{3B_0a}{\pi}\int_0^\infty e^{-\nu tk^2} j_1(k)\left[\frac{\partial}{\partial\rho}j_1\left(\frac{k\rho}{a}\right) + \frac{1}{\rho}j_1\left(\frac{k\rho}{a}\right)\right]\mathrm{d}k \\
B_z(0,t) &= \frac{3B_0a}{\pi}\int_0^\infty e^{-\nu tk^2} j_1(k)\left(\frac{2k}{3a}\right)\mathrm{d}k \\
&= \frac{2B_0}{\pi}\int_0^\infty e^{-\nu tk^2} k j_1(k)\mathrm{d}k \\
&= \frac{2B_0}{\pi}\int_0^\infty e^{-\nu tk^2} k\left(-\frac{\cos k}{k} + \frac{\sin k}{k^2}\right)\mathrm{d}k \\
&= \frac{2B_0}{\pi}\int_0^\infty e^{-\nu tk^2}\left(\frac{\sin k}{k} - \cos k\right)\mathrm{d}k \\
&= B_0\left[\frac{2}{\pi}\int_0^\infty e^{-\nu tk^2}\frac{\sin k}{k}\mathrm{d}k - \frac{2}{\pi}\int_0^\infty e^{-\nu tk^2}\cos k\,\mathrm{d}k\right] \\
&= B_0\left[\Phi\left(\frac{1}{\sqrt{4\nu t}}\right) - \frac{1}{\sqrt{\pi\nu t}}\exp\left(-\frac{1}{4\nu t}\right)\right],
\end{aligned}$$

where we used Equation 5.175 on the first integral and Mathematica on the second integral.

(c) We start from Equation 5.149

$$W_m = \frac{1}{2}\int_{\mathbb{R}^3}\mathbf{J}\cdot\mathbf{A}\,\mathrm{d}^3x.$$

Using our result from part (a), we have $\mathbf{J} = \mathbf{K}e^{-\nu tk^2}\,\delta(r-a)$. Therefore, the volume integral over all space becomes a surface integral on the sphere of radius a (which we denote by S); i.e.

$$\begin{aligned}
W_m &= \frac{1}{2}\oint_S \mathbf{J}\cdot\mathbf{A}\,\mathrm{d}a \\
&= \frac{9B_0^2a}{4\pi\mu}\oint_S\int_0^\infty \sin^2\theta\, e^{-2\nu tk^2} j_1(k)j_1\left(\frac{kr}{a}\right)\mathrm{d}k\,\mathrm{d}a \\
&= \frac{9B_0^2a^3}{2\mu}\int_0^\pi \sin^3\theta\mathrm{d}\theta\int_0^\infty e^{-2\nu tk^2}\left[j_1(k)\right]^2\mathrm{d}k \\
&= \frac{6B_0^2a^3}{\mu}\int_0^\infty e^{-2\nu tk^2}\left[j_1(k)\right]^2\mathrm{d}k.
\end{aligned}$$

Using Mathematica, we can evaluate the integral $\int_0^\infty e^{-2\nu t k^2}\left[j_1(k)\right]^2 \mathrm{d}k$ and then perform a Maclaurin series expansion in the variable $1/(\nu t)$. We find

$$\int_0^\infty e^{-2\nu t k^2}\left[j_1(k)\right]^2 \mathrm{d}k = \frac{1}{72}\sqrt{\frac{\pi}{2}}\left(\frac{1}{\nu t}\right)^{3/2} + \mathcal{O}\left(\left(\frac{1}{\nu t}\right)^{5/2}\right).$$

Thus,

$$W_m \;\to\; \frac{6B_0^2 a^3}{\mu}\left[\frac{1}{72}\sqrt{\frac{\pi}{2}}\left(\frac{1}{\nu t}\right)^{3/2}\right] = \frac{\sqrt{2\pi}B_0^2 a^3}{24\mu(\nu t)^{3/2}}\,,$$

which is the desired result.

(d) Using the same procedure, we find

$$\int_0^\infty e^{-\nu t k^2} j_1(k) j_1\left(\frac{kr}{a}\right)\mathrm{d}k = \frac{\sqrt{\pi}\,r}{36\,a}\left(\frac{1}{\nu t}\right)^{3/2} + \mathcal{O}\left(\left(\frac{1}{\nu t}\right)^{5/2}\right).$$

Therefore,

$$A_\phi \;\to\; \frac{B_0\, r\sin\theta}{12\sqrt{\pi}(\nu t)^{3/2}}.$$

Note that $r\sin\theta = \rho$. Thus, the magnetic field for distances small compared to $R = a(\nu t)^{1/2}$ is

$$B_z = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\phi) = \frac{B_0}{6\sqrt{\pi}(\nu t)^{3/2}}.$$

For large distances of order R, the field is essentially that of the dipole; i.e. the potential is $A_\phi \sim B_0\sin\theta/r^2$.

□