Problem 3.15

Consider the following "spherical cow" model of a battery connected to an external circuit. A sphere of radius a and conductivity σ is embedded in a uniform medium of conductivity σ'. Inside the sphere there is a uniform (chemical) force in the z direction acting on the charge carriers; its strength as an effective electric field entering Ohm's law is F. In the steady state, electric fields exist inside and outside the sphere and surface charge resides on its surface.

(a) Find the electric field (in addition to F) and current density everywhere in space. Determine the surface-charge density and show that the electric dipole moment of the sphere is $p = 4\pi\epsilon_0\sigma a^3 F/(\sigma + 2\sigma')$.

(b) Show that the total current flowing out through the upper hemisphere of the sphere is

$$I = \frac{2\sigma\sigma'}{\sigma + 2\sigma'} \cdot \pi a^2 F.$$

Calculate the total power dissipation outside the sphere. Using the lumped circuit relations, $P = I^2 R_e = IV_e$, find the effective external resistance R_e and voltage V_e.

(c) Find the power dissipated within the sphere and deduce the effective internal resistance R_i and voltage V_i.

(d) Define the total voltage through the relation, $V_t = (R_e + R_i)I$ and show that $V_t = 4aF/3$, as well as $V_e + V_i = V_t$. Show that IV_t is the power supplied by the "chemical" force.

Solution.

(a) We start with the general series solution for each region. There are two regions of interest: the interior and exterior of the sphere; and we use a spherical coordinate system whose origin coincides with the center of the sphere. The system possesses azimuthal symmetry, and there are no charges in either region. Moreover, we require the potential to be finite at the origin and vanish at infinity. Therefore,

$$\Phi_{\text{in}}(r,\theta) = \sum_{l=0} A_l r^l P_l(\cos\theta),$$
$$\Phi_{\text{out}}(r,\theta) = \sum_{l=0} B_l r^{-(l+1)} P_l(\cos\theta).$$

Inside the sphere, we may assume that the electric field is also uniform in the z direction, i.e. $\mathbf{E}_{\text{in}} = E\hat{\mathbf{z}}$, since every point in the sphere experiences the same force. Now,

$$\mathbf{E}_{\text{in}} = -\nabla\Phi_{\text{in}} \quad \Longrightarrow \quad E = -\frac{\partial}{\partial z}\Phi_{\text{in}}.$$

Therefore, Φ_{in} must be linear in $z = r\cos\theta$; i.e., $\Phi_{\text{in}} = A_0 + A_1 r\cos\theta = -Er\cos\theta$, where we set $A_0 = 0$. We require the potential to be continuous on the surface; in particular

$$\Phi_{\text{in}}(a,\theta) = \Phi_{\text{out}}(a,\theta) \quad \Longrightarrow \quad -Ea\cos\theta = \frac{B_0}{a} + \frac{B_1}{a^2}\cos\theta + \mathcal{O}(\cos^2\theta).$$

Clearly, we may choose $B_1 = -Ea^3$ and $B_{m\neq 1} = 0$. Therefore,

$$\Phi_{\text{out}} = -\frac{Ea^3}{r^2}\cos\theta.$$

The surface-charge density Σ is given by applying the boundary condition

$$\begin{aligned}
\frac{\Sigma}{\epsilon_0} &= (\mathbf{E}_{\text{out}} - \mathbf{E}_{\text{in}}) \cdot \hat{\mathbf{r}} \Big|_{r=a} \\
&= -\frac{\partial}{\partial r}\Phi_{\text{out}}\Big|_{r=a} + \frac{\partial}{\partial r}\Phi_{\text{in}}\Big|_{r=a} \\
\Sigma(\theta) &= -3\epsilon_0 E \cos\theta.
\end{aligned}$$

It is also easy to verify that the tangential component of the electric field is continuous (we just take $\partial\Phi/\partial\theta$). Thus, the electric field inside and outside are given by

$$\begin{aligned}
\mathbf{E}_{\text{in}}(r,\theta) &= E\hat{\mathbf{z}} = E\left((\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} + (\hat{\mathbf{z}}\cdot\hat{\theta})\hat{\theta}\right) = E\left(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\theta}\right), \\
\mathbf{E}_{\text{out}}(r,\theta) &= -\nabla\Phi_{\text{out}} = -\frac{Ea^3}{r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}\right) = -\frac{Ea^3}{r^3}\left(3\cos\theta\hat{\mathbf{r}} - \hat{\mathbf{z}}\right).
\end{aligned}$$

In the steady state, $\mathbf{J}_{\text{in}} = \mathbf{J}_{\text{out}}$ on the surface. Therefore, we can obtain E,

$$\begin{aligned}
\hat{\mathbf{r}}\cdot\mathbf{J}_{\text{in}}\Big|_{r=a} &= \hat{\mathbf{r}}\cdot\mathbf{J}_{\text{out}}\Big|_{r=a} \\
\sigma\,\hat{\mathbf{r}}\cdot(\mathbf{E}_{\text{in}} + \mathbf{F})\Big|_{r=a} &= \sigma'\,\hat{\mathbf{r}}\cdot\mathbf{E}_{\text{out}}\Big|_{r=a} \\
\sigma\,(E+F)\cos\theta &= -2\sigma' E\cos\theta \\
E &= -\frac{\sigma}{\sigma + 2\sigma'}F.
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathbf{E}_{\text{in}}(r,\theta) &= -\frac{\sigma F}{\sigma + 2\sigma'}\hat{\mathbf{z}}, \\
\mathbf{E}_{\text{out}}(r,\theta) &= \frac{\sigma F}{\sigma + 2\sigma'}\frac{a^3}{r^3}\left(3\cos\theta\hat{\mathbf{r}} - \hat{\mathbf{z}}\right).
\end{aligned}$$

$$\begin{aligned}
\mathbf{J}_{\text{in}}(r,\theta) &= \sigma\left(\mathbf{E}_{\text{in}} + \mathbf{F}\right) = \frac{2\sigma\sigma' F}{\sigma + 2\sigma'}\hat{\mathbf{z}}, \\
\mathbf{J}_{\text{out}}(r,\theta) &= \frac{\sigma\sigma' F}{\sigma + 2\sigma'}\frac{a^3}{r^3}\left(3\cos\theta\hat{\mathbf{r}} - \hat{\mathbf{z}}\right).
\end{aligned}$$

$$\Sigma(\theta) = \frac{3\epsilon_0\sigma}{\sigma + 2\sigma'}F\cos\theta.$$

By inspection, we can see that Φ_{out} falls as $1/r^2$, similar to a dipole

$$\Phi_{\text{out}}(r,\theta) = \frac{\sigma F}{\sigma + 2\sigma'}\frac{a^3}{r^2}\cos\theta = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2},$$

where

$$\mathbf{p} := p\hat{\mathbf{z}} = \frac{4\pi\epsilon_0 a^3\sigma F}{\sigma + 2\sigma'}\hat{\mathbf{z}}.$$

(b) The current flowing out through the upper hemisphere S^+ is simply the surface-integral of $\mathbf{J}$. We choose $\mathbf{J}_{\text{in}}$ for convenience.

$$\begin{aligned}
I &= \int_{S^+} \mathbf{J}_{\text{in}} \cdot \mathrm{d}\mathbf{A} = \int_{S^+} \mathbf{J}_{\text{in}} \cdot \hat{\mathbf{r}}\, \mathrm{d}A \\
&= 2\pi a^2 \int_0^{\pi/2} \frac{2\sigma\sigma'}{\sigma + 2\sigma'} F\left(\hat{\mathbf{z}} \cdot \hat{\mathbf{r}}\right) \sin\theta \mathrm{d}\theta \\
&= 2\pi a^2 F \frac{2\sigma\sigma'}{\sigma + 2\sigma'} \int_0^{\pi/2} \cos\theta \sin\theta \mathrm{d}\theta \\
&= \pi a^2 F \frac{2\sigma\sigma'}{\sigma + 2\sigma'} \int_0^{\pi/2} \sin(2\theta) \mathrm{d}\theta \\
&= \pi a^2 F \frac{2\sigma\sigma'}{\sigma + 2\sigma'}.
\end{aligned}$$

The power dissipated outside the sphere is simply

$$\begin{aligned}
P_{\text{out}} &= \int_{V_{\text{out}}} \mathbf{J}_{\text{out}} \cdot \mathbf{E}_{\text{out}}\, \mathrm{d}V = \int_{V_{\text{out}}} \frac{\sigma^2\sigma' F^2}{(\sigma + 2\sigma')^2} \frac{a^6}{r^6} \left(3\cos^2\theta + 1\right) \mathrm{d}V \\
&= 2\pi \frac{\sigma^2\sigma' F^2}{(\sigma + 2\sigma')^2} \int_a^\infty \frac{a^6}{r^4} \mathrm{d}r \int_0^\pi \left(3\cos^2\theta + 1\right) \sin\theta \mathrm{d}\theta \\
&= \frac{8\pi a^3 \sigma^2 \sigma' F^2}{3(\sigma + 2\sigma')^2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
R_e &= P_{\text{out}}/I^2 = \frac{2}{3\pi a\sigma'}\,, \\
V_e &= P_{\text{out}}/I = \frac{4a\sigma F}{3(\sigma + 2\sigma')}.
\end{aligned}$$

(c) Similarly, within the sphere

$$\begin{aligned}
P_{\text{in}} &= \int_{V_{\text{in}}} \mathbf{J}_{\text{in}} \cdot \left(\mathbf{E}_{\text{in}} + \mathbf{F}\right) \mathrm{d}V = \frac{4\sigma\sigma'^2 F^2}{(\sigma + 2\sigma')^2} \cdot \frac{4\pi a^3}{3} = \frac{16\pi a^3 \sigma\sigma'^2 F^2}{3(\sigma + 2\sigma')^2}\,, \\
R_i &= P_{\text{in}}/I^2 = \frac{4}{3\pi a\sigma}\,, \\
V_i &= P_{\text{in}}/I = \frac{8a\sigma' F}{3(\sigma + 2\sigma')}.
\end{aligned}$$

(d) Finally, it is easy to see that

$$V_t = (R_e + R_i)I = V_e + V_i = \frac{4aF}{3}.$$

Also,

$$P_t = IV_t = \frac{8\pi a^3 \sigma\sigma' F^2}{3(\sigma + 2\sigma')} = P_{\text{in}} + P_{\text{out}}.$$

$\square$