Problem 3.1

Two concentric spheres have radii a, b ($b > a$) and each is divided into two hemispheres by the same horizontal plane. The upper hemisphere of the inner sphere and the lower hemisphere of the outer sphere are maintained at potential V. The other hemispheres are at zero potential. Determine the potential in the region $a \leq r \leq b$ as a series in Legendre polynomials. Include terms at least up to $l = 4$. Check your solution against known results in the limiting cases $b \to \infty$, and $a \to 0$.

Solution.
Since the configuration has azimuthal symmetry, the potential can be expressed as a series:

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta).$$

Note that Φ is regular in the region $a \leq r \leq b$. We are given the boundary conditions

$$\begin{aligned}\Phi(a, 0 < \theta < \pi/2) =& V = \Phi(b, \pi/2 < \theta < \pi),\\ \Phi(b, 0 < \theta < \pi/2) =& 0 = \Phi(a, \pi/2 < \theta < \pi).\end{aligned}$$

Now,

$$\Phi(a, \theta) = \sum_{l=0}^{\infty} \left[A_l a^l + B_l a^{-(l+1)}\right] P_l(\cos\theta) = V_a(\theta),$$

where

$$V_a(\theta) = \begin{cases} V, & \text{for } 0 < \theta < \pi/2. \\ 0, & \text{for } \pi/2 < \theta < \pi. \end{cases}$$

Using the orthogonality relation,

$$\begin{aligned}\int_{-1}^{1} \mathrm{d}(\cos\theta) P_{l'}(\cos\theta)\Phi(a,\theta) &= \sum_{l=0}^{\infty} \left[A_l a^l + B_l a^{-(l+1)}\right] \frac{2}{2l+1}\delta_{ll'}\\ \frac{2}{2l+1}\left[A_l a^l + B_l a^{-(l+1)}\right] &= \int_{-1}^{1} \mathrm{d}(\cos\theta) P_l(\cos\theta) V_a(\theta)\\ &= V\int_{0}^{1} \mathrm{d}(\cos\theta) P_l(\cos\theta)\\ A_l a^l + B_l a^{-(l+1)} &= \frac{1}{2}(2l+1)V\int_{0}^{1} \mathrm{d}x P_l(x)\end{aligned}$$

Similarly,

$$\Phi(b, \theta) = \sum_{l=0}^{\infty} \left[A_l b^l + B_l b^{-(l+1)}\right] P_l(\cos\theta) = V_b(\theta),$$

where

$$V_b(\theta) = \begin{cases} 0, & \text{for } 0 < \theta < \pi/2. \\ V, & \text{for } \pi/2 < \theta < \pi. \end{cases}$$

Using the orthogonality relation,

$$\begin{aligned}\int_{-1}^{1} \mathrm{d}(\cos\theta) P_{l'}(\cos\theta)\Phi(b,\theta) &= \sum_{l=0}^{\infty} \left[A_l b^l + B_l b^{-(l+1)}\right] \frac{2}{2l+1}\delta_{ll'} \\ \frac{2}{2l+1}\left[A_l b^l + B_l b^{-(l+1)}\right] &= \int_{-1}^{1} \mathrm{d}(\cos\theta) P_l(\cos\theta) V_b(\theta) \\ &= V\int_{-1}^{0} \mathrm{d}(\cos\theta) P_l(\cos\theta) \\ A_l b^l + B_l b^{-(l+1)} &= \frac{1}{2}(2l+1)V\int_{-1}^{0} \mathrm{d}x P_l(x).\end{aligned}$$

Thus, we have the following matrix equation

$$\begin{pmatrix} a^l & a^{-(l+1)} \\ b^l & b^{-(l+1)} \end{pmatrix}\begin{pmatrix} A_l \\ B_l \end{pmatrix} = \frac{1}{2}(2l+1)V\begin{pmatrix} \int_0^1 \mathrm{d}x P_l(x) & \int_{-1}^0 \mathrm{d}x P_l(x)\end{pmatrix}^{\mathrm{T}}.$$

$$\begin{pmatrix} A_l \\ B_l \end{pmatrix} = \frac{1}{2}(2l+1)V\frac{1}{a^l b^{-(l+1)} - a^{-(l+1)}b^l}\begin{pmatrix} b^{-(l+1)} & -a^{-(l+1)} \\ -b^l & a^l \end{pmatrix}\begin{pmatrix} \int_0^1 \mathrm{d}x P_l(x) & \int_{-1}^0 \mathrm{d}x P_l(x)\end{pmatrix}^{\mathrm{T}}.$$

Using Mathematica to carry out these calculations, we find that the only nonzero terms up to $l = 4$ are

$$A_0 = \frac{V}{2},\ A_1 = \frac{3(a^2+b^2)}{4(a^3-b^3)}V,\ B_1 = -\frac{3a^2b^2(a+b)}{4(a^3-b^3)}V,\ A_3 = -\frac{7(a^4+b^4)}{16(a^7-b^7)}V,\ B_3 = \frac{7a^4b^4(a^3+b^3)}{16(a^7-b^7)}V.$$

Thus, the potential in the region $a \leq r \leq b$ is (up to order $l = 4$)

$$\begin{aligned}\Phi(r,\theta) = \frac{V}{2} &+ \frac{3}{4}V\left(\frac{a^2+b^2}{(a^3-b^3)}r - \frac{a^2b^2(a+b)}{(a^3-b^3)}\frac{1}{r^2}\right)P_1(\cos\theta) \\ &+ \frac{7}{16}V\left(-\frac{a^4+b^4}{(a^7-b^7)}r^3 + \frac{a^4b^4(a^3+b^3)}{(a^7-b^7)}\frac{1}{r^4}\right)P_3(\cos\theta) + \dots\end{aligned}$$

As $a \to 0$,

$$\Phi(r,\theta) \approx V\left(\frac{1}{2} - \frac{3r}{4b}P_1(\cos\theta) + \frac{7r^3}{16b^3}P_3(\cos\theta) + \dots\right).$$

Similarly, as $b \to \infty$,

$$\Phi(r,\theta) \approx V\left(\frac{1}{2} + \frac{3a^2}{4r^2}P_1(\cos\theta) - \frac{7a^4}{16r^4}P_3(\cos\theta) + \dots\right).$$

Note that this is similar to the configuration in Section 2.7. Recall that we obtained the potential outside a conducting sphere with a potential $+V$ ($-V$) on the upper (lower) hemisphere. Therefore, if we make the replacement $V \to V/2$, and add a constant potential $V/2$ (by superposition) in Equation 2.27 we obtain the limiting case $b \to \infty$. For the limiting case $a \to 0$, we simply replace $(a/r)^{l+1}$ by $(r/b)^l$ (Section 3.3), and include a minus sign beside the $P_l(\cos\theta)$ terms to account for the fact that the potential in the upper and lower hemispheres are reversed. $\square$