Problem 5.24

(a) For the perfectly conducting plane of Section 5.13 with the circular hole in it and the asymptotically uniform tangential magnetic field $\mathbf{H}_0$ on one side, calculate the added tangential magnetic field $\mathbf{H}^{(1)}$ on the side of the plane with $\mathbf{H}_0$. Show that its components for $\rho > a$ are

$$\begin{aligned} H_x^{(1)} &= \frac{2H_0a^3}{\pi}\frac{xy}{\rho^4\sqrt{\rho^2-a^2}}, \\ H_y^{(1)} &= \frac{2H_0a^3}{\pi}\frac{y^2}{\rho^4\sqrt{\rho^2-a^2}} + \frac{H_0}{\pi}\left[\frac{a}{\rho}\sqrt{1-\frac{a^2}{\rho^2}} - \arcsin\left(\frac{a}{\rho}\right)\right]. \end{aligned}$$

(b) Sketch the lines of surface current flow in the neighborhood of the hole on both sides of the plane.

Solution.

(a) Following the discussion in Section 5.13, the added potential was found to be (5.129)

$$\Phi^{(1)}(\mathbf{x}) = \frac{2H_0a^2}{\pi}\int_0^\infty \mathrm{d}k\, j_1(ka)e^{-k|z|}J_1(k\rho)\sin\phi.$$

Since we are only concerned with the magnetic field tangent to the surface, it suffices to know the potential at $z=0$. Then

$$\Phi^{(1)}(\rho,\phi) = \frac{2H_0a^2}{\pi}\sin\phi\int_0^\infty \mathrm{d}k\, j_1(ka)J_1(k\rho).$$

Using Mathematica, for $\rho > a$ we obtain

$$\begin{aligned} \Phi^{(1)}(\rho,\phi) &= \frac{2H_0a^2}{\pi}\sin\phi\left[\frac{\rho}{2a^2}\arcsin\left(\frac{a}{\rho}\right) - \frac{1}{2a}\sqrt{1-\frac{a^2}{\rho^2}}\right] \\ &= \frac{H_0}{\pi}\sin\phi\left[\rho\arcsin\left(\frac{a}{\rho}\right) - a\sqrt{1-\frac{a^2}{\rho^2}}\right]. \end{aligned}$$

Now, from $\mathbf{H} = -\nabla\Phi$ (restricted to the surface) we get

$$H_\rho^{(1)}(\rho,\phi) = -\frac{\partial}{\partial\rho}\Phi^{(1)}(\rho,\phi) = \frac{H_0}{\pi}\sin\phi\left[\frac{a}{\rho}\left(1+\frac{a^2}{\rho^2}\right)\left(1-\frac{a^2}{\rho^2}\right)^{-1/2} - \arcsin\left(\frac{a}{\rho}\right)\right],$$

and

$$H_\phi^{(1)}(\rho,\phi) = -\frac{1}{\rho}\frac{\partial}{\partial\phi}\Phi^{(1)}(\rho,\phi) = \frac{H_0}{\pi}\cos\phi\left[\frac{a}{\rho}\sqrt{1-\frac{a^2}{\rho^2}} - \arcsin\left(\frac{a}{\rho}\right)\right].$$

Finally, we get (by a change of basis)

$$H_x^{(1)} = H_\rho^{(1)}\cos\theta - H_\phi^{(1)}\sin\theta = \frac{H_0}{\pi}\cos\phi\sin\phi\frac{2a^3}{\rho^2\sqrt{\rho^2-a^2}} = \frac{2H_0a^3}{\pi}\frac{xy}{\rho^4\sqrt{\rho^2-a^2}}.$$

And

$$\begin{aligned}
H_y^{(1)} &= H_\rho^{(1)} \sin\theta - H_\phi^{(1)} \cos\theta \\
&= \frac{H_0}{\pi} \sin^2\phi \left[\frac{a}{\rho} \left(1 + \frac{a^2}{\rho^2}\right) \left(1 - \frac{a^2}{\rho^2}\right)^{-1/2} - \arcsin\left(\frac{a}{\rho}\right) \right] \\
&\quad + \frac{H_0}{\pi} \cos^2\phi \left[\frac{a}{\rho} \sqrt{1 - \frac{a^2}{\rho^2}} - \arcsin\left(\frac{a}{\rho}\right) \right] \\
&= \frac{H_0}{\pi} \sin^2\phi \frac{2a^3}{\rho^2\sqrt{\rho^2 - a^2}} + \frac{H_0}{\pi} \left[\frac{a}{\rho} \sqrt{1 - \frac{a^2}{\rho^2}} - \arcsin\left(\frac{a}{\rho}\right) \right] \\
&= \frac{2H_0 a^3}{\pi} \frac{y^2}{\rho^4\sqrt{\rho^2 - a^2}} + \frac{H_0}{\pi} \left[\frac{a}{\rho} \sqrt{1 - \frac{a^2}{\rho^2}} - \arcsin\left(\frac{a}{\rho}\right) \right].
\end{aligned}$$

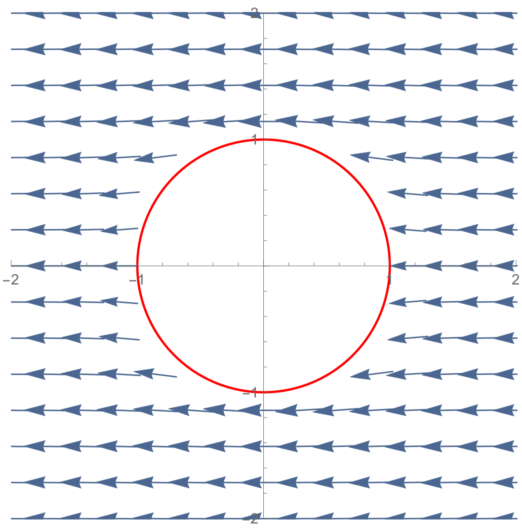
(b) The surface current density is given by $\mathbf{K} = \hat{\mathbf{n}} \times \mathbf{H}$. Therefore, on the side of the surface with $\mathbf{H}_0$ (i.e. above)

$$\mathbf{K}_{\text{above}} = \hat{\mathbf{z}} \times \mathbf{H}_{\text{above}} = \left(-H_y^{(0)} - H_y^{(1)}, H_x^{(1)}\right).$$

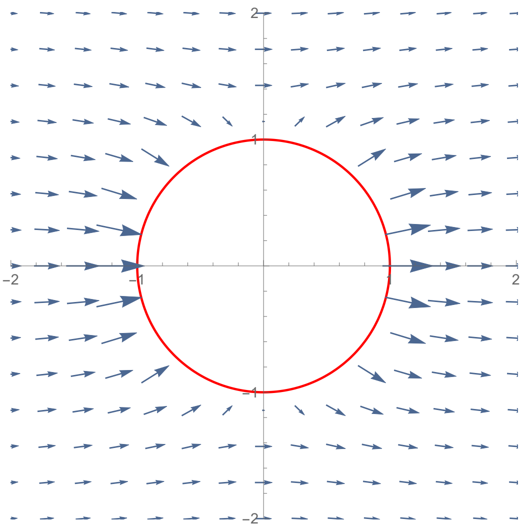
Similarly, the current density on the other side (below) is

$$\mathbf{K}_{\text{below}} = -\hat{\mathbf{z}} \times \mathbf{H}_{\text{below}} = \left(-H_y^{(1)}, H_x^{(1)}\right),$$

where we used the fact that $H_x^{(1)}$ and $H_y^{(1)}$ reverses sign below since they are odd in z. Using Mathematica we obtain the following plots (Figure 1).



(a) Above



(b) Below

Figure 1: Surface current density $\mathbf{K}$ (for $H_0 > 0$).

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