Problem 1.6

A simple capacitor is a device formed by two insulated conductors adjacent to each other. If equal and opposite charges are placed on the conductors, there will be a certain difference of potential between them. The ratio of the magnitude of the charge on one conductor to the magnitude of the potential difference is called the capacitance (in SI units it is measured in farads). Using Gauss's law, calculate the capacitance of

(a) two large, flat, conducting sheets of area A, separated by a small distance d;

(b) two concentric conducting spheres with radii a, b ($b > a$);

(c) two concentric conducting cylinders of length L, large compared to their radii a, b ($b > a$).

(d) What is the inner diameter of the outer conductor in an air-filled coaxial cable whose center conductor is a cylindrical wire of diameter 1 mm and whose capacitance is 3×10^{-11} F/m? 3×10^{-12} F/m?

Solution.

(a) Suppose we enclose a portion of a conducting sheet in a box such that the sheet divides the box equally and two faces of the box are parallel to the sheet. For small enough distances, we can approximate the conducting sheet as an infinite sheet, such that the electric field near the box is uniform and directed along the normal of the sheet. Therefore, only those two faces of the box parallel to the sheet have nonzero flux. Let A' be the surface area of one face of the box, and σ be the surface charge density. Then by Gauss's law,

$$\oint_S \mathbf{E} \cdot \hat{\mathbf{n}} \mathrm{d}a = \frac{Q_{\text{enc.}}}{\epsilon_0} \quad \Longrightarrow \quad 2EA' = \frac{\sigma A'}{\epsilon_0}.$$

Therefore $\mathbf{E} = \sigma/2\epsilon_0 \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the normal direction to the sheet. Now, suppose two parallel conducting sheets, one with total charge $+Q$ and the other $-Q$, are separated by a small distance d. Clearly, the electric field between them is $E = \sigma/\epsilon_0$, where $\sigma = Q/A$, directed along the normal of the two sheets and towards the negatively charged conductor. The potential difference between the two is simply

$$V = Ed = \frac{Qd}{\epsilon_0 A}.$$

Therefore, the capacitance is

$$C = \frac{Q}{V} = \frac{\epsilon_0 A}{d}.$$

(b) Suppose the inner sphere has total charge $+Q$. Clearly, by Gauss's law (using a concentric spherical shell with radius between a and b as the Gaussian surface),

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0} \frac{1}{r^2} \hat{\mathbf{r}}, \quad (a < r < b).$$

The potential difference between the two spheres is simply

$$V = \frac{Q}{4\pi\epsilon_0} \int_a^b \frac{1}{r^2} \mathrm{d}r = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{b} \right).$$

Therefore, the capacitance is

$$C = 4\pi\epsilon_0 \frac{ab}{b-a}.$$

(c) For $L >> b$, we can treat the cylinders as having infinite length. In this case the Gaussian surface is a concentric cylinder of length l situated between the two cylinders. Clearly, the only nonzero electric flux contribution is from the curved surface, wherein the electric field is constant by cylindrical symmetry; i.e., $E(2\pi s' l) = Q_{\text{enc.}}/\epsilon_0$. If $+Q$ be the total charge of the inner cylinder, then $Q_{\text{enc.}} = Q(l/L)$. Therefore,

$$\mathbf{E} = \frac{Q}{2\pi\epsilon_0 L}\frac{1}{s}\hat{\mathbf{s}}, \quad (a < s < b).$$

Then

$$V = \frac{Q}{2\pi\epsilon_0 L}\int_a^b \frac{1}{s}\mathrm{d}s = \frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{b}{a}\right),$$

and

$$C = \frac{2\pi\epsilon_0 L}{\ln(b/a)}.$$

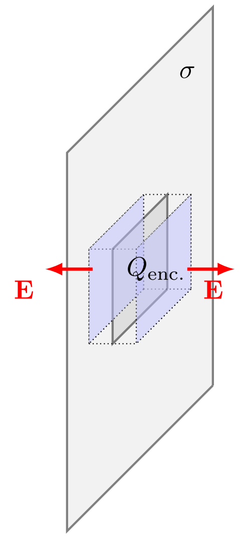
(d) Solving for b in the previous equation, we have

$$b = a\exp\left(\frac{2\pi\epsilon_0 L}{C}\right).$$

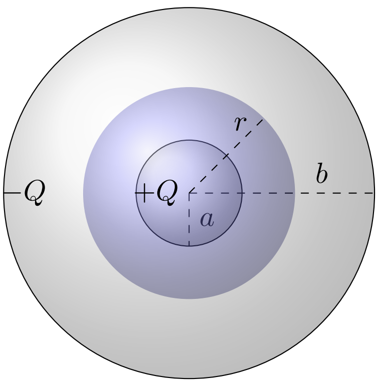
For $C/L = 3\times10^{-11}\,\mathrm{F/m}$, the inner diameter of the outer conductor is 6 mm. For $C/L = 3\times10^{-12}\,\mathrm{F/m}$, the inner diameter of the outer conductor is 100 km.

$\square$

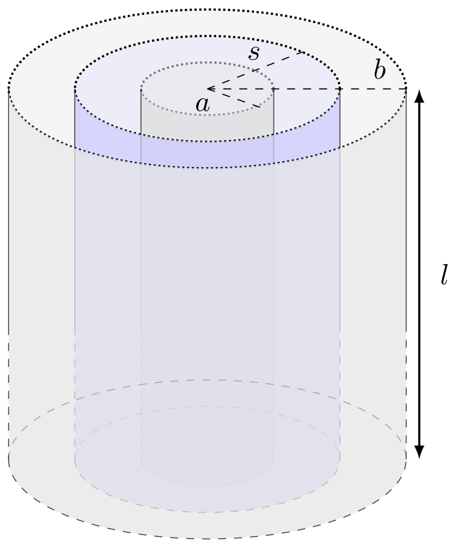
Figure 1: Gaussian surfaces for ...



...an infinite conducting sheet.



...a spherical capacitor.



...a cylindrical capacitor.