I. BASIS

Let the wavefunction for a state in the coordinate representation be $\psi(\mathbf{r})$. Expand this wavefunction using an orthornormal basis that consists of

i. functions $u_i(\mathbf{r})$ labeled by an integer index i,

ii. plane waves indexed by a wavevector $\mathbf{k}$,

iii. delta functions centered at position $\mathbf{r}'$,

iv. functions $w_\alpha(\mathbf{r})$ labeled by a real index α.

For each basis above use the orthonormality relations to express the expansion coefficients and amplitudes as overlap integrals (inner products).

Solution.

i. Let $|\psi\rangle \doteq \int \mathrm{d}^3r\, |\mathbf{r}\rangle \langle \mathbf{r}|\psi\rangle = \int \mathrm{d}^3r\, \psi(\mathbf{r})\, |\mathbf{r}\rangle$. Now,

$$|\psi\rangle = \sum_i |u_i\rangle \langle u_i|\psi\rangle \doteq \sum_i \int \mathrm{d}^3r\; |\mathbf{r}\rangle \langle \mathbf{r}|u_i\rangle \langle u_i|\psi\rangle = \int \mathrm{d}^3r \sum_i c_i\, u_i(\mathbf{r})\, |\mathbf{r}\rangle\,,$$

where $c_i := \langle u_i|\psi\rangle$ and $u_i(\mathbf{r}) := \langle \mathbf{r}|u_i\rangle$. Clearly, $\psi(\mathbf{r}) = \sum_i c_i\, u_i(\mathbf{r})$, with

$$c_i = \langle u_i|\psi\rangle \doteq \int \mathrm{d}^3r\, \langle u_i|\mathbf{r}\rangle \langle \mathbf{r}|\psi\rangle = \int \mathrm{d}^3r\, u_i^*(\mathbf{r})\psi(\mathbf{r}).$$

ii. Observe that

$$\begin{aligned}|\psi\rangle \doteq \int \mathrm{d}^3r\, \psi(\mathbf{r})\, |\mathbf{r}\rangle := \int \mathrm{d}^3r\, |\mathbf{r}\rangle \langle \mathbf{r}|\psi\rangle &= \int \mathrm{d}^3r \int \mathrm{d}^3k\, |\mathbf{r}\rangle \langle \mathbf{r}|\mathbf{k}\rangle \langle \mathbf{k}|\psi\rangle \\ &= \int \mathrm{d}^3r\, |\mathbf{r}\rangle \int \mathrm{d}^3k\, \langle \mathbf{r}|\mathbf{k}\rangle\, \psi(\mathbf{k}).\end{aligned}$$

Therefore, $\psi(\mathbf{r}) = \int \mathrm{d}^3k\, \langle \mathbf{r}|\mathbf{k}\rangle\, \psi(\mathbf{k})$, where $\psi(\mathbf{k}) := \langle \mathbf{k}|\psi\rangle$ and $\langle \mathbf{r}|\mathbf{k}\rangle = \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{r}}$. If we use the $\langle \mathbf{r}|\mathbf{k}\rangle$ plane waves as an orthonormal basis, we find that the expansion coefficients are simply $\psi(\mathbf{k}) = \int \mathrm{d}^3r\, \langle \mathbf{k}|\mathbf{r}\rangle\, \psi(\mathbf{r}) = \int \mathrm{d}^3r\, \langle \mathbf{r}|\mathbf{k}\rangle^*\, \psi(\mathbf{r})$, which we obtain by interchanging r and k in the previous arguments.

iii. Clearly, $\psi(\mathbf{r}) = \int \mathrm{d}^3r'\, \delta^3\left(\mathbf{r} - \mathbf{r}'\right) \psi(\mathbf{r}')$, where the expansion coefficients are simply given by $\psi(\mathbf{r}') = \int \mathrm{d}^3r\, \delta^3\left(\mathbf{r} - \mathbf{r}'\right) \psi(\mathbf{r})$.

iv. Again, let $|\psi\rangle \doteq \int \mathrm{d}^3r\,|\mathbf{r}\rangle\langle\mathbf{r}|\psi\rangle = \int \mathrm{d}^3r\,\psi(\mathbf{r})\,|\mathbf{r}\rangle$. Now,

$$|\psi\rangle = \int \mathrm{d}\alpha\,|w_\alpha\rangle\langle w_\alpha|\psi\rangle \doteq \int \mathrm{d}\alpha \int \mathrm{d}^3r\,|\mathbf{r}\rangle\langle\mathbf{r}|w_\alpha\rangle\langle w_\alpha|\psi\rangle = \int \mathrm{d}^3r\,|\mathbf{r}\rangle \int \mathrm{d}\alpha\,c_\alpha w_\alpha(\mathbf{r}),$$

where $c_\alpha := \langle w_\alpha|\psi\rangle$ and $w_\alpha(\mathbf{r}) = \langle\mathbf{r}|w_\alpha\rangle$. Clearly, $\psi(\mathbf{r}) = \int \mathrm{d}\alpha\,c_\alpha\,w_\alpha(\mathbf{r})$, with

$$c_\alpha = \langle w_\alpha|\psi\rangle \doteq \int \mathrm{d}^3r\,\langle w_\alpha|\mathbf{r}\rangle\langle\mathbf{r}|\psi\rangle = \int \mathrm{d}^3r\,w_\alpha^*(\mathbf{r})\psi(\mathbf{r}).$$

□

II. NORMALIZATION

Let the state vector for a physical state be $|\psi\rangle$. Demonstrate Parseval's relation using the coordinate basis $|\mathbf{r}\rangle$ and plane wave basis $|\mathbf{k}\rangle$.

Solution.

$$\begin{aligned}
\langle\psi|\psi\rangle &= \int \mathrm{d}^3r \int \mathrm{d}^3r'\,\langle\psi|\mathbf{r}\rangle\langle\mathbf{r}|\mathbf{r}'\rangle\langle\mathbf{r}'|\psi\rangle \\
&= \int \mathrm{d}^3r \int \mathrm{d}^3r'\,\psi^*(\mathbf{r})\psi(\mathbf{r}')\delta^3(\mathbf{r}-\mathbf{r}') \\
&= \int \mathrm{d}^3r\,\psi^*(\mathbf{r})\psi(\mathbf{r}) \\
\int \mathrm{d}^3r\,\psi^*(\mathbf{r})\psi(\mathbf{r}) &= \int \mathrm{d}^3r \int \mathrm{d}^3r' \int \mathrm{d}^3k \int \mathrm{d}^3k'\,\langle\psi|\mathbf{k}\rangle\langle\mathbf{k}|\mathbf{r}\rangle\langle\mathbf{r}|\mathbf{r}'\rangle\langle\mathbf{r}'|\mathbf{k}'\rangle\langle\mathbf{k}'|\psi\rangle \\
&= \int \mathrm{d}^3k \int \mathrm{d}^3k'\,\langle\psi|\mathbf{k}\rangle\langle\mathbf{k}|\mathbf{k}'\rangle\langle\mathbf{k}'|\psi\rangle \\
&= \int \mathrm{d}^3k \int \mathrm{d}^3k'\,\psi^*(\mathbf{k})\psi(\mathbf{k}')\delta^3(\mathbf{k}-\mathbf{k}') \\
&= \int \mathrm{d}^3k\,\psi^*(\mathbf{k})\psi(\mathbf{k}).
\end{aligned}$$

□

III. COMMUTATOR I

Evaluate the commutator $[\mathbf{r},\mathbf{p}]$, with $\mathbf{p} = -i\hbar\nabla$.

Solution. Note that in Cartesian coordinates,

$$\left[x_i, \frac{\partial}{\partial x_j}\right] = x_i \circ \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} \circ x_i = x_i\frac{\partial}{\partial x_j} - \left(\frac{\partial x_i}{\partial x_j} + x_i\frac{\partial}{\partial x_j}\right) = -\delta_{ij}.$$

Now, $\mathbf{r} = (x_1, x_2, x_3)$ and $\mathbf{p} = -i\hbar\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$. Therefore,

$$[x_i, p_j] = -i\hbar\left[x_i, \frac{\partial}{\partial x_j}\right] = i\hbar\delta_{ij}.$$

□

IV. COMMUTATOR II

Evaluate the commutator $[\mathbf{r}, \mathbf{L}]$, with $\mathbf{L} = \mathbf{r} \times \mathbf{p}$.

Solution. In Cartesian coordinates, $L_k = \epsilon_{kij} x_i p_j$.

$$[x_l, L_k] = \epsilon_{kij} [x_l, x_i p_j] = \epsilon_{kij} (x_l \circ x_i \circ p_j - x_i \circ p_j \circ x_l) .$$

Since $[x_i, x_l] = 0$,

$$[x_l, L_k] = \epsilon_{kij} (x_i \circ x_l \circ p_j - x_i \circ p_j \circ x_l) = \epsilon_{kij} x_i \circ [x_l, p_j] = i\hbar \delta_{lj} \epsilon_{kij} x_i = i\hbar \epsilon_{ilk} x_i .$$

□