I. Landé factor

Use the projection theorem to calculate the Landé factor g_J for the coupling of a uniform static magnetic field to an angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$.

Solution. Note that

$$\boldsymbol{\mu}_J = \boldsymbol{\mu}_L + \boldsymbol{\mu}_S = g_L \mu_B \mathbf{L} + g_S \mu_B \mathbf{S}.$$

By the projection theorem, we may write $\boldsymbol{\mu}_J = g_J \mu_B \mathbf{J}$. Therefore, taking the projection of $\boldsymbol{\mu}_J$ to $\mathbf{J}$ we get

$$\begin{aligned}
g_J \mathbf{J}^2 &= g_L \mathbf{L} \cdot \mathbf{J} + g_S \mathbf{S} \cdot \mathbf{J} \\
&= g_L \left(\mathbf{L}^2 + \mathbf{L} \cdot \mathbf{S}\right) + g_S \left(\mathbf{S} \cdot \mathbf{L} + \mathbf{S}^2\right) \\
&= g_L \left(\mathbf{L}^2 + \frac{1}{2}\left(\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2\right)\right) + g_S \left(\frac{1}{2}\left(\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2\right) + \mathbf{S}^2\right) \\
&= \frac{1}{2} g_L \left(\mathbf{J}^2 + \mathbf{L}^2 - \mathbf{S}^2\right) + \frac{1}{2} g_S \left(\mathbf{J}^2 - \mathbf{L}^2 + \mathbf{S}^2\right).
\end{aligned}$$

Using Equation 3.5.31

$$\begin{aligned}
g_J \hbar^2 j(j+1) &= \frac{1}{2} g_L \hbar^2 \left[j(j+1) + l(l+1) - s(s+1)\right] + \frac{1}{2} g_S \hbar^2 \left[j(j+1) - l(l+1) + s(s+1)\right] \\
g_J &= \frac{g_L \left[j(j+1) + l(l+1) - s(s+1)\right] + g_S \left[j(j+1) - l(l+1) + s(s+1)\right]}{2j(j+1)} \\
g_J &= g_L \frac{j(j+1) + l(l+1) - s(s+1)}{2j(j+1)} + g_S \frac{j(j+1) - l(l+1) + s(s+1)}{2j(j+1)}.
\end{aligned}$$

□

II. Problem 4.7 (Sakurai, 2nd ed.)

(a) Let $\psi(\mathbf{x}, t)$ be the wave function of a spinless particle corresponding to a plane wave in three dimensions. Show that $\psi^*(\mathbf{x}, -t)$ is the wave function for the plane wave with the momentum direction reversed.

(b) Let $\chi(\hat{\mathbf{n}})$ be the two-component eigenspinor of $\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$ with eigenvalue $+1$. Using the explicit form of $\chi(\hat{\mathbf{n}})$ (in terms of the polar and azimuthal angles β and α that characterize $\hat{\mathbf{n}}$), verify that $-i\sigma_2\chi^*(\hat{\mathbf{n}})$ is the two-component eigenspinor with the spin direction reversed.

Solution.

(a) The wave function of a plane wave is given by

$$\psi(\mathbf{x}, t) = \exp\left[i(\mathbf{k} \cdot \mathbf{x} - \omega t)\right] = \exp\left[\frac{i}{\hbar}(\mathbf{p} \cdot \mathbf{x} - Et)\right].$$

Reversing the momentum ($\mathbf{p} \to -\mathbf{p}$) we get

$$\exp\left[\frac{i}{\hbar}(-\mathbf{p} \cdot \mathbf{x} - Et)\right] = \exp\left[-\frac{i}{\hbar}\left(\mathbf{p} \cdot \mathbf{x} - E \cdot (-t)\right)\right] = \psi^*(\mathbf{x}, -t).$$

(b) From Equation 3.2.52, we have

$$\chi(\hat{\mathbf{n}}) = \begin{pmatrix} \cos(\beta/2)e^{-i\alpha/2} \\ \sin(\beta/2)e^{+i\alpha/2} \end{pmatrix}.$$

Now,

$$\begin{aligned} -i\sigma_2\chi^*(\hat{\mathbf{n}}) &= -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \cos(\beta/2)e^{-i\alpha/2} \\ \sin(\beta/2)e^{+i\alpha/2} \end{pmatrix}^* \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos(\beta/2)e^{+i\alpha/2} \\ \sin(\beta/2)e^{-i\alpha/2} \end{pmatrix} \\ &= \begin{pmatrix} -\sin(\beta/2)e^{-i\alpha/2} \\ \cos(\beta/2)e^{+i\alpha/2} \end{pmatrix}. \end{aligned}$$

We can check that this is equivalent to

$$\begin{aligned} &\exp(-i\sigma_3\alpha/2)\exp(-i\sigma_2\beta/2)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \left[\cos\frac{\alpha}{2} - i\sigma_3\sin\frac{\alpha}{2}\right]\left[\cos\frac{\beta}{2} - i\sigma_2\sin\frac{\beta}{2}\right]\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\alpha/2) - i\sin(\alpha/2) & 0 \\ 0 & \cos(\alpha/2) + i\sin(\alpha/2) \end{pmatrix}\begin{pmatrix} \cos(\beta/2) & -\sin(\beta/2) \\ \sin(\beta/2) & \cos(\beta/2) \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{+i\alpha/2} \end{pmatrix}\begin{pmatrix} -\sin(\beta/2) \\ \cos(\beta/2) \end{pmatrix} \\ &= \begin{pmatrix} -\sin(\beta/2)e^{-i\alpha/2} \\ \cos(\beta/2)e^{+i\alpha/2} \end{pmatrix}. \end{aligned}$$

$\square$

III. Problem 4.12 (Sakurai, 2nd ed.)

The Hamiltonian for a spin 1 system is given by

$$H = AS_z^2 + B(S_x^2 - S_y^2).$$

Solve this problem exactly to find the normalized energy eigenstates and eigenvalues. (A spin-dependent Hamiltonian of this kind actually appears in crystal physics.) Is this Hamiltonian invariant under time reversal? How do the normalized eigenstates you obtained transform under time reversal?

Solution.
Note that we can rewrite the Hamiltonian as

$$H = AS_z^2 + \frac{1}{2}B(S_+^2 + S_-^2).$$

Let $|1, m\rangle = |m\rangle$. Using the results from Section 3.5, we have

$$\begin{aligned} S_z |m\rangle &= m\hbar |m\rangle , \\ S_+ |m\rangle &= \sqrt{(1-m)(2+m)}\hbar |m+1\rangle , \\ S_- |m\rangle &= \sqrt{(1+m)(2-m)}\hbar |m-1\rangle . \end{aligned}$$

In the S_z basis, where $|+1\rangle = (1,0,0)^{\mathbf{T}}$, $|0\rangle = (0,1,0)^{\mathbf{T}}$, and $|-1\rangle = (0,0,1)^{\mathbf{T}}$, the operators have the following matrix form

$$\begin{aligned} S_z &= \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\ S_+ &= \sqrt{2}\hbar \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \\ S_- &= \sqrt{2}\hbar \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned}$$

Therefore, the Hamiltonian has the matrix form

$$H = A\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + B\hbar^2 \left[\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right] = \hbar^2 \begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ B & 0 & A \end{pmatrix}.$$

Using Mathematica, we find that the normalized eigenstates are $\psi_0 = |0\rangle$, $\psi_+ = (|-1\rangle + |+1\rangle)/\sqrt{2}$ and $\psi_- = (|-1\rangle - |+1\rangle)/\sqrt{2}$, with eigenvalues 0, $(A+B)\hbar^2$, and $(A-B)\hbar^2$, respectively. Since the (spin) angular momentum operator is odd under time reversal, and the Hamiltonian only consists of the square of each S_i and real numbers A and B, we have $\Theta\left(CS_i^2\right)\Theta^{-1} = C\Theta S_i^2\Theta^{-1} = C\Theta S_i\Theta^{-1}\Theta S_i\Theta^{-1} = C\left(-S_i\right)\left(-S_i\right) = CS_i^2$, where $C \in \mathbb{R}$; and thus, the Hamiltonian is invariant under time reversal. Using Equation 4.4.78,

$$\begin{aligned} \Theta\psi_0 &= \Theta |0\rangle = |0\rangle = \psi_0, \\ \Theta\psi_+ &= \Theta\left(|-1\rangle + |+1\rangle\right)/\sqrt{2} = \left(-|+1\rangle - |-1\rangle\right)/\sqrt{2} = -\psi_+, \\ \Theta\psi_- &= \Theta\left(|-1\rangle - |+1\rangle\right)/\sqrt{2} = \left(-|+1\rangle + |-1\rangle\right)/\sqrt{2} = +\psi_-. \end{aligned}$$

□