I. CORRELATION FUNCTION (Problem 2.16)

Evaluate the correlation function

$$C(t) = \langle x(t)x(0)\rangle\,,$$

for the ground state of the harmonic oscillator $|0\rangle$. How does it decay with time?

Solution. From Equation 2.3.45a,

$$x(t) = x(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t.$$

Therefore,

$$C(t) = \langle 0|x(t)x(0)|0\rangle = \cos\omega t\,\langle 0|\,(x(0))^2\,|0\rangle + \frac{\sin\omega t}{m\omega}\,\langle 0|p(0)x(0)|0\rangle\,.$$

Clearly, the time dependence is controlled by the factors outside the expectation values; and we only need to determine the expectation values at $t = 0$. Thus, in order to simplify the notation we let $x(0) \to x$ and $p(0) \to p$. Recall from Equation 2.3.24 that

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger), \quad p = i\sqrt{\frac{m\hbar\omega}{2}}(-a + a^\dagger).$$

Therefore,

$$\begin{aligned}
C(t) &= \frac{\hbar\cos\omega t}{2m\omega}\,\langle 0|a^2 + aa^\dagger + a^\dagger a + a^{\dagger 2}|0\rangle + i\frac{\hbar\sin\omega t}{2m\omega}\,\langle 0| - a^2 - aa^\dagger + a^\dagger a + a^{\dagger 2}|0\rangle \\
&= \frac{\hbar\cos\omega t}{2m\omega}\,\langle 0|aa^\dagger|0\rangle - i\frac{\hbar\sin\omega t}{2m\omega}\,\langle 0|aa^\dagger|0\rangle \\
&= \frac{\hbar}{2m\omega}\,(\cos\omega t - i\sin\omega t)\,.
\end{aligned}$$

The correlation function oscillates and does not decay in time. □

II. DISPLACED OSCILLATOR (Problem 2.12)

At $t = 0$ let

$$|\psi(0)\rangle = e^{-ipl/\hbar} |0\rangle ,$$

where $|0\rangle$ is the ground state of a harmonic oscillator, p is the momentum operator, and l is a fixed quantity with dimensions of length. Evaluate the expectation value of the position operator $\langle x \rangle$ as a function of time.

Solution. From Equation 2.3.45,

$$\begin{aligned} x(t) &= x(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t, \\ p(t) &= -m\omega x(0)\sin\omega t + p(0)\cos\omega t. \end{aligned}$$

Therefore, in the Heisenberg picture,

$$\begin{aligned} \langle x(t)\rangle = \langle \psi(0)|x(t)|\psi(0)\rangle &= \cos\omega t \,\langle \psi(0)|x(0)|\psi(0)\rangle + \frac{\sin\omega t}{m\omega}\langle \psi(0)|p(0)|\psi(0)\rangle \\ &= \cos\omega t \,\langle 0|e^{ip(0)l/\hbar}x(0)e^{-ip(0)l/\hbar}|0\rangle + \frac{\sin\omega t}{m\omega}\langle 0|e^{ip(0)l/\hbar}p(0)e^{-ip(0)l/\hbar}|0\rangle . \end{aligned}$$

Using the Baker-Hausdorff lemma, for Hermitian operators $x(0)$ and $p(0)$, we have

$$\begin{aligned} e^{ip(0)l/\hbar}x(0)e^{-ip(0)l/\hbar} &= x(0) + \left(\frac{il}{\hbar}\right)[p(0), x(0)] + \left(\frac{i^2l^2}{2!\hbar^2}\right)[p(0),[p(0),x(0)]] + \dots \\ &= x(0) + \left(\frac{il}{\hbar}\right)[p(0), x(0)] \\ &= x(0) + l, \end{aligned}$$

and

$$e^{ip(0)l/\hbar}p(0)e^{-ip(0)l/\hbar} = p(0) + \left(\frac{il}{\hbar}\right)[p(0), p(0)] + \left(\frac{i^2l^2}{2!\hbar^2}\right)[p(0),[p(0),p(0)]] + \dots = p(0).$$

Thus,

$$\langle x(t)\rangle = \cos\omega t\,\langle 0|x(0)|0\rangle + \cos\omega t\,\langle 0|l|0\rangle + \frac{\sin\omega t}{m\omega}\langle 0|p(0)|0\rangle = l\cos\omega t.$$

$\square$

III. ELECTRON IN A MAGNETIC FIELD (Problem 2.39)

Let an electron move in a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$.

(a) Evaluate the commutator between the components of the kinematical momentum $[\Pi_x, \Pi_y]$.

(b) Obtain the energy spectrum

$$E_{kn} = \frac{\hbar^2 k^2}{2m} + \frac{|eB|\hbar}{mc}\left(n + \frac{1}{2}\right),$$

where $\hbar k$ is the eigenvalue of the p_z operator and n is a nonnegative integer.

Solution.

(a) For an electron in a magnetic field $\mathbf{\Pi} = \mathbf{p} - e\mathbf{A}/c$, where we take $\mathbf{A} = (-By, 0, 0)$ (from Equation 2.7.41). Therefore,

$$\begin{aligned}[\Pi_x, \Pi_y] &= \left[p_x - \frac{eA_x}{c}, p_y - \frac{eA_y}{c}\right] = [p_x, p_y] - \frac{e}{c}\left([A_x, p_y] + [p_x, A_y]\right) + \frac{e^2}{c^2}[A_x, A_y] \\ &= \frac{e}{c}\left([p_y, A_x] - [p_x, A_y]\right) = -\frac{i\hbar e}{c}\left(\partial_y A_x - \partial_x A_y\right) = \frac{i\hbar e}{c}\left(\nabla \times A\right)_z = \frac{i\hbar e}{c}B_z = \frac{i\hbar e}{c}B.\end{aligned}$$

(b) Consider the Hamiltonian

$$H = \frac{\mathbf{\Pi}^2}{2m} = \frac{\Pi_x^2 + \Pi_y^2 + \Pi_z^2}{2m}.$$

From the commutation relation, we can let $\Pi_x = yeB/c$ and $\Pi_y = p_y$, where $[y, p_y]$. Also, it is clear that $\Pi_z = p_z$. Therefore,

$$H = \frac{e^2B^2y^2}{2mc^2} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} = H_y + H_z,$$

where H_y corresponds to the Hamiltonian of a harmonic oscillator with $\omega = |eB|/mc$. Therefore,

$$E = \frac{\hbar^2 k^2}{2m} + \hbar\omega\left(n + \frac{1}{2}\right) = \frac{\hbar^2 k^2}{2m} + \frac{\hbar|eB|}{mc}\left(n + \frac{1}{2}\right).$$

□