Problem 1

If the translation symmetry in the Schwarzschild patch ($r > 2M$) is expressed as $(t, r) \to (t + c, r)$, derive the corresponding transformation in Kruskal-Szekeres coordinates. Show that this is a boost in the (U, V) or (T, X) plane that leaves $\mathrm{d}U\mathrm{d}V$ ($= \mathrm{d}T^2 - \mathrm{d}X^2$) and $r(U, V)$ invariant.

Solution. The coordinate transformations from (t, r) to (U, V) are given by

$$U = \mp e^{-u/4M} \quad \text{and} \quad V = e^{v/4M},$$

where

$$u = t - r^* \quad \text{and} \quad v = t + r^*,$$

with $r^* = r + 2M \ln |r/2M - 1|$. Clearly, if $t \to t + c$, then $u \to u + c$ and $v \to v + c$. Therefore, the corresponding transformations in (U, V) are

$$U \to Ue^{-c/4M} \quad \text{and} \quad V \to Ve^{c/4M}.$$

The product UV is invariant; and since r is an implicit function of the product UV, r is also invariant. Furthermore,

$$\mathrm{d}U\mathrm{d}V = -\frac{1}{16M^2}UV\mathrm{d}u\,\mathrm{d}v.$$

But, $\mathrm{d}u$ and $\mathrm{d}v$ are also invariant under $t \to t + c$. Thus, $\mathrm{d}U\mathrm{d}V$ is invariant under the transformation $t \to t + c$.

Problem 2

Show that the Kruskal coordinate V is an affine parameter along the event horizon. In other words, the null curve $x^\alpha(\lambda) = (U(\lambda), V(\lambda), \theta(\lambda), \phi(\lambda)) = (0, \lambda, \theta_0, \phi_0)$ is an affinely parametrized null geodesic.

Solution. The Schwarzschild metric is given by

$$g = -\frac{16M^3}{r} e^{-r/2M} (\mathrm{d}U\mathrm{d}V + \mathrm{d}V\mathrm{d}U) + r^2 \mathrm{d}\Omega^2,$$

where

$$e^{r/2M}\left(\frac{r}{2M} - 1\right) = -UV.$$

Consider the null curve $x^\alpha(\lambda) = (U_0, \lambda, \theta_0, \phi_0)$, parametrized by V. Clearly, the velocity of this curve is $u^\alpha = \delta^\alpha_V$. The acceleration is

$$a^\alpha = u^\alpha{}_{;\beta} u^\beta = u^\alpha{}_{,\beta} u^\beta + \Gamma^\alpha{}_{\beta\gamma} u^\gamma u^\beta = \delta^\alpha_{V,\beta}\delta^\beta_V + \Gamma^\alpha{}_{\beta\gamma}\delta^\gamma_V \delta^\beta_V = \Gamma^\alpha{}_{VV}.$$

We now compute $\Gamma^\alpha{}_{VV}$,

$$\Gamma^\alpha{}_{VV} = \frac{1}{2} g^{\alpha\beta}\left(g_{\beta V,V} + g_{V\beta,V} - g_{VV,\beta}\right) = g^{\alpha U} g_{UV,V}.$$

Thus, the only possible nonzero component of a^α is $a^V = \Gamma^V{}_{VV}$. Now,

$$\begin{aligned} g_{UV,V} &= \frac{\partial r}{\partial V}\frac{\partial}{\partial r} g_{UV} = \left(\frac{\partial V}{\partial r}\right)^{-1} \frac{\partial}{\partial r} g_{UV}, \\ &= \left(-\frac{re^{r/2M}}{4M^2 U_0}\right)^{-1} \frac{8M^2}{r}\left(1 + \frac{2M}{r}\right) e^{-r/2M}, \\ &= U_0 F(r). \end{aligned}$$

Along the event horizon, $U_0 = 0$. Thus, $a^\alpha = 0$ and V is an affine parameter.

Problem 3

Show that for a Killing vector field ξ^ν,

$$\nabla_\rho \nabla_\mu \xi^\nu = R^\nu{}_{\mu\rho\sigma}\xi^\sigma,$$

where $R^\nu{}_{\mu\rho\sigma}$ is the Riemann tensor.

Solution. By definition of Killing vectors,

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0. \tag{1}$$

Taking the derivative, we get

$$\nabla_\rho \nabla_\mu \xi_\nu + \nabla_\rho \nabla_\nu \xi_\mu = 0. \tag{2}$$

Permuting the indices, we have

$$\nabla_\rho \nabla_\mu \xi_\nu + \nabla_\rho \nabla_\nu \xi_\mu = 0, \tag{3}$$
$$\nabla_\mu \nabla_\nu \xi_\rho + \nabla_\mu \nabla_\rho \xi_\nu = 0, \tag{4}$$
$$\nabla_\nu \nabla_\rho \xi_\mu + \nabla_\nu \nabla_\mu \xi_\rho = 0. \tag{5}$$

Now, (3) + (4) − (5)

$$\nabla_\rho \nabla_\mu \xi_\nu + (\nabla_\rho \nabla_\nu - \nabla_\nu \nabla_\rho)\xi_\mu + (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\xi_\rho + \nabla_\mu \nabla_\rho \xi_\nu = 0. \tag{6}$$

By definition of the Riemann tensor,

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)\omega_\gamma = R_{\alpha\beta\gamma}{}^\delta \omega_\delta, \tag{7}$$

we get

$$\nabla_\rho \nabla_\mu \xi_\nu + \nabla_\mu \nabla_\rho \xi_\nu + R_{\rho\nu\mu}{}^\sigma \xi_\sigma + R_{\mu\nu\rho}{}^\sigma \xi_\sigma = 0. \tag{8}$$
$$2\nabla_\rho \nabla_\mu \xi_\nu + R_{\mu\rho\nu}{}^\sigma \xi_\sigma + R_{\rho\nu\mu}{}^\sigma \xi_\sigma + R_{\mu\nu\rho}{}^\sigma \xi_\sigma = 0. \tag{9}$$

Using the Bianchi identity

$$2\nabla_\rho \nabla_\mu \xi_\nu - R_{\nu\mu\rho}{}^\sigma \xi_\sigma + R_{\mu\nu\rho}{}^\sigma \xi_\sigma = 0. \tag{10}$$

Thus,

$$\nabla_\rho \nabla_\mu \xi^\nu = R^\nu{}_{\mu\rho\sigma}\xi^\sigma. \qquad \square$$

Problem 4

In Region II of the Kruskal manifold one may regard r as a time coordinate and introduce a new spatial coordinate x such that

$$\mathrm{d}s^2 = -\left(\frac{2M}{r} - 1\right)^{-1} \mathrm{d}r^2 + \left(\frac{2M}{r} - 1\right) \mathrm{d}x^2 + r^2 \mathrm{d}\Omega^2.$$

(a) Show that every timelike curve in Region II intersects the singularity at $r = 0$ within a proper time no greater than πM. For what curves is this bound attained?

(b) Show also that in constrast to Region I, particles "at rest" in the interior move along geodesics.

(c) Finally, let the proper time τ along one of these "at rest" geodesics be the time coordinate (instead of r). Consider the limiting behavior of the metric as one approaches the singularity at $r = 0$. Show that constant-τ slices exhibit anisotropic "Kasner-type" behavior in which two spatial directions contract, while the other spatial direction expands.

Solution.

(a) Let u be the velocity of a timelike curve in Region II. Then,

$$\begin{aligned} u^2 &= g_{rr} \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + g_{xx} \left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 + g_{\theta\theta} \left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 + g_{\phi\phi} \left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 \\ &= f^{-1} \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - f \left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 + r^2 \left(\frac{\mathrm{d}\Omega}{\mathrm{d}\tau}\right)^2 = -1. \end{aligned}$$

It follows that

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 \geq f^2 \left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 - f.$$

Also, since $\xi^\beta = \partial/\partial x = \delta^\alpha_x$ is a Killing vector,

$$\epsilon := g_{\alpha\beta} u^\alpha \xi^\beta = g_{xx} u^x = -f \frac{\mathrm{d}x}{\mathrm{d}\tau}$$

is a conserved quantity. Therefore,

$$\left(\frac{\mathrm{d}\tau}{\mathrm{d}r}\right)^2 \leq \frac{1}{\epsilon^2 - f} \leq -f^{-1}.$$

Finally,

$$\tau \leq \left| \int_{2M}^{0} \frac{\mathrm{d}r}{\sqrt{2M/r - 1}} \right| = 2M \int_0^1 \left(\frac{1}{u} - 1\right)^{-1/2} = \pi M.$$

This bound is attained for radial curves which start at $r = 2M$.

(b) Consider the curve $x^\alpha = (r(\lambda), x_0, \theta_0, \phi_0)$. Clearly, the (normalized) velocity is $u^\alpha = \delta^\alpha_r / \sqrt{g_{rr}}$, where $g_{rr} = f^{-1}$. Therefore,

$$u^\alpha{}_{;\beta} u^\beta = u^\alpha{}_{,\beta} u^\beta + \Gamma^\alpha{}_{\gamma\beta} u^\gamma u^\beta = \frac{1}{\sqrt{g_{rr}}} \frac{\partial}{\partial r} \left(\frac{1}{\sqrt{g_{rr}}}\right) \delta^\alpha_r + (g_{rr})^{-1} \Gamma^\alpha{}_{rr}.$$

Now,

$$\Gamma^{\alpha}{}_{rr} = \frac{1}{2} g^{\alpha\beta} \left(g_{\beta r,r} + g_{r\beta,r} - g_{rr,\beta} \right).$$

Clearly, the only nonzero components are

$$\Gamma^{r}{}_{rr} = \frac{1}{2} g^{rr} g_{rr,r} = -\frac{1}{2}\frac{f'}{f} = -\frac{M}{r^2}\frac{1}{f}.$$

Also,

$$\frac{1}{\sqrt{g_{rr}}}\frac{\partial}{\partial r}\left(\frac{1}{\sqrt{g_{rr}}}\right) = f^{1/2}\frac{\partial}{\partial r}\left(f^{-1/2}\right) = \frac{M}{r^2}.$$

Thus, $u^{\alpha}{}_{;\beta} u^{\beta} = 0$. In contrast to Region I, particles "at rest" in Region II move along geodesics.

(c) The line element is given by $\mathrm{d}s^2 = f^{-1}\mathrm{d}r^2 - f\mathrm{d}x^2 + r^2\mathrm{d}\Omega^2$. Suppose τ is the r-coordinate, then the line element of the constant-τ slices is $\mathrm{d}s^2 = -f\mathrm{d}x^2 + r^2\mathrm{d}\Omega^2$. As $r \to 0$, we have $-f \to +\infty$. Thus, two spatial directions contract, while the other spatial direction expands.