Problem 1

At $t = 0$, a particle sits at rest atop an axisymmetric dome in a uniform gravitational field g. The dome is described by $h = h(r)$, which is the vertical distance dropped from the dome's peak when the particle moves a distance r along the dome, away from the dome's peak ($r = 0$). If the particle's equation of motion is $\ddot{r} = 4/3r^{1/3}$, what is $h(r)$? Clearly, $r(t) = 0$ is a solution to the equation of motion. However, show that the particle can spontaneously start moving at an arbitrary time $t = T$. Briefly comment on what this result implies about Newtonian mechanics.

Solution. The Lagrangian for this system is given by

$$L = \frac{1}{2}m\dot{r}^2 + mgh.$$

Clearly, the equation of motion is $\ddot{r} = gh'(r)$. Thus,

$$h'(r) = \frac{4}{3g}r^{1/3} \Rightarrow h(r) = \frac{r^{4/3}}{g}.$$

the other solution to the equation $\ddot{r} = 4/3r^{1/3}$ is

$$r(t) = \frac{2\sqrt{2}}{27}(t - T)^3.$$

This violates the Newtonian principles of causality and determinism. $\square$

Problem 2

Consider the functionals

$$H[f] = \int G(x,y) f(y) \mathrm{d}y,$$
$$I[f] = \int_{-1}^{1} f(x) \mathrm{d}x,$$
$$J[f] = \int \left(\frac{\partial f}{\partial y} \right)^2 \mathrm{d}y.$$

Calculate the functional derivatives $\dfrac{\delta H[f]}{\delta f(z)}$, $\dfrac{\delta^2 I[f^3]}{\delta f(x_0)\delta f(x_1)}$, and $\dfrac{\delta J[f]}{\delta f(x)}$.

Solution.

(a) By definition,

$$\begin{aligned}
\frac{\delta H[f]}{\delta f(z)} &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(H[f(y) + \epsilon \delta(y-z)] - H[f(y)] \right) \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\int G(x,y) f(y) \mathrm{d}y + \epsilon \int G(x,y) \delta(y-z) \mathrm{d}y - \int G(x,y) f(y) \mathrm{d}y \right) \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\epsilon \int G(x,y) \delta(y-z) \mathrm{d}y \right) \\
&= \int G(x,y) \delta(y-z) \mathrm{d}y \\
&= G(x,z).
\end{aligned}$$

(b) Assume that $x_0, x_1 \in [-1,1]$, for if either one is outside the interval the functional derivative is obviously zero. Now,

$$\begin{aligned}
\frac{\delta I[f^3]}{\delta f(x_1)} &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(I[(f(x) + \epsilon \delta(x - x_1))^3] - I[f^3(x)] \right) \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\int_{-1}^{1} (f(x) + \epsilon \delta(x-x_1))^3 \mathrm{d}x - \int_{-1}^{1} f^3(x) \mathrm{d}x \right) \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(3\epsilon \int_{-1}^{1} f^2(x) \delta(x-x_1) \mathrm{d}x + \mathcal{O}(\epsilon^2) \right) \\
&= 3 \int_{-1}^{1} f^2(x) \delta(x-x_1) \mathrm{d}x \\
&= 3 f^2(x_1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\delta^2 I[f^3]}{\delta f(x_0)\delta f(x_1)} &= \frac{\delta}{\delta f(x_0)}\frac{\delta I[f^3]}{\delta f(x_1)} = \frac{\delta}{\delta f(x_0)}\left[3f^2(x_1)\right] \\
&= 3\frac{\delta}{\delta f(x_0)}\int_{-1}^{1} f^2(x)\delta(x-x_1)\mathrm{d}x \\
&= 3\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int_{-1}^{1}\left(f(x)+\epsilon\delta(x-x_0)\right)^2\delta(x-x_1)\mathrm{d}x - \int_{-1}^{1} f^2(x)\delta(x-x_1)\mathrm{d}x\right] \\
&= 3\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[2\epsilon\int_{-1}^{1} f(x)\delta(x-x_0)\delta(x-x_1)\mathrm{d}x + \mathcal{O}(\epsilon^2)\right] \\
&= 6\int_{-1}^{1} f(x)\delta(x-x_0)\delta(x-x_1)\mathrm{d}x \\
&= 6f(x_1)\delta(x_1-x_0).
\end{aligned}$$

(c) We have

$$\begin{aligned}
\frac{\delta J[f]}{\delta f(x)} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(J[f(y)+\epsilon\delta(y-x)] - J[f(y)]\right) \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int\left[\frac{\partial}{\partial y}\left(f(y)+\epsilon\delta(y-x)\right)\right]^2\mathrm{d}y - \int\left(\frac{\partial f}{\partial y}\right)^2\mathrm{d}y\right) \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int\left[\frac{\partial f}{\partial y}+\epsilon\delta'(y-x)\right]^2\mathrm{d}y - \int\left(\frac{\partial f}{\partial y}\right)^2\mathrm{d}y\right) \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(2\epsilon\int\frac{\partial f}{\partial y}\delta'(y-x)\mathrm{d}y + \mathcal{O}(\epsilon^2)\right) \\
&= 2\int\frac{\partial f}{\partial y}\delta'(y-x)\mathrm{d}y \\
&= -2\int\frac{\partial^2 f}{\partial y^2}\delta(y-x)\mathrm{d}y \\
&= -2\frac{\partial^2 f}{\partial y^2}(x).
\end{aligned}$$

Note that we performed an integration by parts in the sixth step.

□

Problem 3

Find the function f defined on $[0,1]$ with $f(0) = f(1) = 0$ which minimizes the functional

$$\int_0^1 \sqrt{H - f(x)}\sqrt{1 + f'(x)^2}\mathrm{d}x.$$

Can you do this without appealing to the Euler-Lagrange equation?

Solution. Note that $\sqrt{1 + f'(x)^2}\mathrm{d}x$ is just the infinitesimal arclength $\mathrm{d}s$. We can think of $f(x)$ as the height at point x. Thus, we can think of the $H - f(x)$ term as proportional to the potential energy. Therefore, we expect the function to be parabolic (as in projectile motion). In order to determine $f(x)$ exactly, we use Beltrami's identity: if $L(f, f')$ does not explicitly depend on x, then $f'\partial L/\partial f' - L$ is constant. Thus, we have

$$E = -\frac{\sqrt{H - f(x)}}{\sqrt{1 + f'(x)^2}}.$$

This can be solved using Mathematica, which gives us a quadratic function. Applying the boundary conditions $f(0) = f(1) = 0$, and imposing a symmetry condition $f'(0)^2 = f'(1)^2$ allows us to determine the constant

$$E^2 = \frac{1}{4}\left(2H - \sqrt{4H^2 - 1}\right)$$

Thus, the function which minimizes the functional is

$$f(x) = \frac{x(1-x)}{4E^2}.$$

□

Problem 4

For the harmonic oscillator, explain why the constant of motion $f(x, \dot{x}, t) = \arctan(\omega x/\dot{x}) - \omega t$ cannot be a Noether current.

Solution. The Lagrangian is given by

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2 x^2.$$

Therefore, a Noether current will have the form

$$\begin{aligned}
J &= \frac{\partial L}{\partial \dot{x}}\delta x - \left(\dot{x}\frac{\partial L}{\partial \dot{x}} - L\right)\delta t - F(x,t) \\
&= m\dot{x}\delta x - m\dot{x}^2\delta t + \left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2 x^2\right)\delta t \\
&= m\dot{x}\delta x - \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2\right)\delta t.
\end{aligned}$$

Thus, for $f(x, \dot{x}, t)$ to be a Noether current there must exist $\delta x = X(x,t)$ and $\delta t = T(x,t)$ such that

$$m\dot{x}X(x,t) - \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2\right)T(x,t) = f(x, \dot{x}, t).$$

Clearly, this cannot be satisfied since the LHS is at most quadratic in $\dot{x}$. $\square$

Problem 5

Let q_k be a cyclic coordinate of the Lagrangian $L(q, \dot{q}, t)$, so that its conjugate momentum p_k is a constant of motion. If $L \to \bar{L} + \mathrm{d}F(q,t)/\mathrm{d}t$, in general q_k will not be a cyclic coordinate of $\bar{L}$ and the conjugate momentum is not conserved. But then since the two Lagrangians differ by a total time derivative, the equations of motion for both Lagrangians have to be the same. How can the same equations of motion give rise to both conservation and nonconservation? Resolve this apparent paradox.

Solution. Since the Lagrangian differs by a $\mathrm{d}F(q,t)/\mathrm{d}t$ term, a conjugate momentum in L is not necessarily a conjugate momentum in $\bar{L}$. In general, they are related by

$$\begin{aligned}\frac{\partial L}{\partial \dot{q}_k} &= \frac{\partial \bar{L}}{\partial \dot{q}_k} + \frac{\partial}{\partial \dot{q}_k}\left(\frac{\mathrm{d}}{\mathrm{d}t}F(q,t)\right) \\ p_k &= \bar{p}_k + \frac{\partial}{\partial \dot{q}_k}\left(\frac{\mathrm{d}}{\mathrm{d}t}F(q,t)\right).\end{aligned}$$

□

Problem 6

The force acting on a charged particle of mass m and charge e is given by $\mathbf{F} = (e/c)\mathbf{v} \times \mathbf{B}$, where $\mathbf{B} = g\hat{\mathbf{r}}/r^2$. (This is the force from a magnetic monopole of magnetic charge g.)

(a) Find α and β for $\mathbf{Q} = \alpha\mathbf{r} \times \mathbf{v} + \beta\hat{\mathbf{r}}$ to be a constant of motion.

(b) Picking the z-axis parallel to $\mathbf{Q}$, prove that the particle is confined to move along a cone whose axis of symmetry is the z-axis.

Solution. Let $\mathbf{v} = \dot{\mathbf{r}}$. The equation of motion is clearly

$$m\ddot{\mathbf{r}} = \mathbf{F} = \frac{e}{c}\dot{\mathbf{r}} \times \mathbf{B} = \frac{ge}{cr^2}\dot{\mathbf{r}} \times \hat{\mathbf{r}}.$$

(a) Clearly,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{Q} &= \alpha\left(\dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}}\right) + \beta\dot{\hat{\mathbf{r}}} \\
&= \alpha\mathbf{r} \times \ddot{\mathbf{r}} + \beta\dot{\hat{\mathbf{r}}} \\
&= \alpha\mathbf{r} \times \left(\frac{ge}{mcr^2}\dot{\mathbf{r}} \times \hat{\mathbf{r}}\right) + \beta\dot{\hat{\mathbf{r}}} \\
&= \frac{\alpha ge}{mcr^2}\left(\dot{\mathbf{r}}\left(\mathbf{r} \cdot \hat{\mathbf{r}}\right) - \hat{\mathbf{r}}\left(\mathbf{r} \cdot \dot{\mathbf{r}}\right)\right) + \beta\dot{\hat{\mathbf{r}}} \\
&= \frac{\alpha ge}{mcr^2}\left(\left(\dot{r}\hat{\mathbf{r}} + r\dot{\hat{\mathbf{r}}}\right)r - r\hat{\mathbf{r}}\left(\hat{\mathbf{r}} \cdot \left(\dot{r}\hat{\mathbf{r}} + r\dot{\hat{\mathbf{r}}}\right)\right)\right) + \beta\dot{\hat{\mathbf{r}}} \\
&= \frac{\alpha ge}{mcr^2}\left(r\dot{r}\hat{\mathbf{r}} + r^2\dot{\hat{\mathbf{r}}} - r\hat{\mathbf{r}}\left(\dot{r} + r\hat{\mathbf{r}} \cdot \dot{\hat{\mathbf{r}}}\right)\right) + \beta\dot{\hat{\mathbf{r}}} \\
&= \frac{\alpha ge}{mcr^2}\left(r^2\dot{\hat{\mathbf{r}}} - r^2\hat{\mathbf{r}}\left(\hat{\mathbf{r}} \cdot \dot{\hat{\mathbf{r}}}\right)\right) + \beta\dot{\hat{\mathbf{r}}} \\
&= \frac{\alpha ge}{mc}\left(\dot{\hat{\mathbf{r}}} - \hat{\mathbf{r}}\left(\hat{\mathbf{r}} \cdot \dot{\hat{\mathbf{r}}}\right)\right) + \beta\dot{\hat{\mathbf{r}}} \\
&= \left(\frac{\alpha ge}{mc} + \beta\right)\dot{\hat{\mathbf{r}}} - \frac{\alpha ge}{mc}\hat{\mathbf{r}}\left(\hat{\mathbf{r}} \cdot \dot{\hat{\mathbf{r}}}\right)
\end{aligned}$$

This will only have a nontrivial zero when $\hat{\mathbf{r}} \cdot \dot{\hat{\mathbf{r}}} = 0$; in this case we can choose $\alpha = m$ and $\beta = -ge/c$.

(b) Let $\mathbf{Q} = Q\hat{\mathbf{z}}$, where Q is constant. Then

$$\begin{aligned}
\mathbf{Q} = Q\hat{\mathbf{z}} &= m\mathbf{r} \times \dot{\mathbf{r}} - \frac{ge}{c}\hat{\mathbf{r}} \\
&= mr\hat{\mathbf{r}} \times \left(\dot{r}\hat{\mathbf{r}} + r\dot{\hat{\mathbf{r}}}\right) - \frac{ge}{c}\hat{\mathbf{r}} \\
&= mr^2\hat{\mathbf{r}} \times \dot{\hat{\mathbf{r}}} - \frac{ge}{c}\hat{\mathbf{r}} \\
\theta &= \arccos\left(\frac{\mathbf{Q} \cdot \hat{\mathbf{r}}}{Q}\right) = \arccos\left(\beta/Q\right).
\end{aligned}$$

Thus, the particle is confined to move along the cone $\theta = \arccos\left(\beta/Q\right)$.

$\square$