## Spin precession

We start with a Hamiltonian of a spin 1/2 system with magnetic moment $e\hbar/2m_ec$ (where $e < 0$ for the electron), subjected to an external uniform magnetic field $\mathbf{B}$ in the $z$-direction. We can write $H$ as

$$H = -\left(\frac{e}{m_ec}\right)\mathbf{S}\cdot\mathbf{B} = -\left(\frac{eB}{m_ec}\right)S_z.$$

Clearly, $[H, S_z] = 0$ and the $S_z$ eigenkets are also energy eigenkets, with corresponding energy eigenvalues

$$E_\pm = \mp\frac{e\hbar B}{2m_ec}, \quad \text{for } S_z\ \pm.$$

Define $\omega$ such that $E_+ - E_- = \hbar\omega$. For the electron, we have

$$\omega := -\frac{eB}{m_ec}.$$

Therefore, $H = \omega S_z$ and the time-evolution operator is

$$\mathcal{U}(t,0) = \exp\left(\frac{-i\omega S_z t}{\hbar}\right).$$

Now, given an arbirtary ket (at $t = 0$)

$$|\psi\rangle = a\,|+\rangle + b\,|-\rangle\,,$$

we see that at some later time $t$,

$$|\psi;t\rangle = a\exp\left(\frac{-i\omega t}{2}\right)|+\rangle + b\exp\left(\frac{+i\omega t}{2}\right)|-\rangle\,.$$

Thus,

$$\langle S_z\rangle_{\psi(t)} = \langle\psi;t|\,S_z\,|\psi;t\rangle = a^*a\left(+\frac{\hbar}{2}\right) + b^*b\left(-\frac{\hbar}{2}\right) = \frac{\hbar}{2}\left(a^2 - b^2\right),$$

and

$$\langle S_x\rangle_{\psi(t)} = a^*a\,\langle+|\,S_x\,|+\rangle + b^*b\,\langle-|\,S_x\,|-\rangle + a^*b\exp\left(+i\omega t\right)\langle+|\,S_x\,|-\rangle + b^*a\exp\left(-i\omega t\right)\langle-|\,S_x\,|+\rangle\,.$$

Using Equation (1.4.18a),

$$\langle S_x\rangle_{\psi(t)} = \frac{\hbar}{2}\left[a^*b\exp\left(+i\omega t\right) + b^*a\exp\left(-i\omega t\right)\right].$$