Problem 1

Consider two observers A and B who remain at fixed (r, θ, ϕ) in the Schwarzschild geometry. Let A have $r = r_A$ and B have $r = r_B$, where $r_B > r_A$. Now assume that A sends two photons to B separated by a coordinate time Δt as measured by A. Show that the proper time between the photons emitted by A, as measured by A, is $\Delta\tau_A = \sqrt{1 - 2M/r_A}\Delta t$. Calculate the ratio $\Delta\tau_B/\Delta\tau_A$, where $\Delta\tau_B$ is the proper time between the photons received by B, as measured by B.

Solution. We only consider the case where A and B are both in Region I, otherwise the question does not make sense. From the Schwarzschild line element, we have

$$(\Delta\tau)^2 = -(\Delta s)^2 = f(r)(\Delta t)^2 - \left(f(r)^{-1}(\Delta r)^2 + r^2(\Delta\theta)^2 + r^2\sin^2\theta(\Delta\phi)^2\right).$$

Since A and B have fixed (r, θ, ϕ), this reduces to

$$(\Delta\tau)^2 = f(r)(\Delta t)^2,$$
$$\Delta\tau = \sqrt{f(r)}\Delta t,$$

where we take the positive solution. Suppose A sends two photons to B, separated by a coordinate time Δt as measured by A. Then A measures a proper time

$$\Delta\tau_A = \sqrt{f(r_A)}\Delta t = \sqrt{1 - \frac{2M}{r_A}}\Delta t.$$

By definition of the coordinate time, B will also measure the same coordinate time Δt as A. Thus,

$$\Delta\tau_B = \sqrt{f(r_B)}\Delta t = \sqrt{1 - \frac{2M}{r_B}}\Delta t.$$

Finally, we have

$$\frac{\Delta\tau_B}{\Delta\tau_A} = \sqrt{\frac{1 - 2M/r_B}{1 - 2M/r_A}}.$$

□

Problem 2

Show that in Region II of the Kruskal manifold one may regard r as a time coordinate and introduce a new spatial coordinate x such that

$$\mathrm{d}s^2 = -\left(\frac{2M}{r} - 1\right)^{-1} \mathrm{d}r^2 + \left(\frac{2M}{r} - 1\right) \mathrm{d}x^2 + r^2 \mathrm{d}\Omega^2.$$

Hence show that every timelike curve in Region II intersects the singularity at $r = 0$ within a proper time no greater than πM. For what curves is this bound attained?

Solution. The Region II of the Kruskal manifold is covered by the ingoing Eddington-Finkelstein coordinates, where the line element is of the form

$$\mathrm{d}s^2 = -f\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\mathrm{d}\Omega^2.$$

Introduce new coordinates $x = v - r^*$ and $r = r$, where r^* is the tortoise coordinate. It is easy to show that $\mathrm{d}v = \mathrm{d}x + f^{-1}\mathrm{d}r$. Making the appropriate replacements, we have

$$\mathrm{d}s^2 = -f\mathrm{d}x^2 + f^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2.$$

Now, in Region II, $f < 0$, therefore

$$\mathrm{d}s^2 = -\left(-f\right)^{-1} \mathrm{d}r^2 + \left(-f\right) \mathrm{d}x^2 + r^2\mathrm{d}\Omega^2.$$

Thus, we can regard r as a time coordinate and x as a spatial coordinate such that

$$\mathrm{d}s^2 = -\left(\frac{2M}{r} - 1\right)^{-1} \mathrm{d}r^2 + \left(\frac{2M}{r} - 1\right) \mathrm{d}x^2 + r^2 \mathrm{d}\Omega^2.$$

Let u be the velocity of a timelike curve in Region II. Then,

$$\begin{aligned} u^2 &= g_{rr}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + g_{xx}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 + g_{\theta\theta}\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 + g_{\phi\phi}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 \\ &= f^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - f\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 + r^2\left(\frac{\mathrm{d}\Omega}{\mathrm{d}\tau}\right)^2 = -1. \end{aligned}$$

It follows that

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 \geq f^2\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 - f.$$

Also, since $\xi^\beta = \partial/\partial x = \delta^\alpha_x$ is a Killing vector,

$$\epsilon := g_{\alpha\beta}u^\alpha\xi^\beta = g_{xx}u^x = -f\frac{\mathrm{d}x}{\mathrm{d}\tau}$$

is a conserved quantity. Therefore,

$$\left(\frac{\mathrm{d}\tau}{\mathrm{d}r}\right)^2 \leq \frac{1}{\epsilon^2 - f} \leq -f^{-1}.$$

Finally,

$$\tau \leq \left|\int_{2M}^{0} \frac{\mathrm{d}r}{\sqrt{2M/r - 1}}\right| = 2M\int_0^1 \left(\frac{1}{u} - 1\right)^{-1/2} = \pi M.$$

This bound is attained for radial curves which start at $r = 2M$.

Problem 3

Show that in Kruskal coordinates, the timelike Killing vector k is given by

$$k = \frac{1}{4M}\left(V\frac{\partial}{\partial V} - U\frac{\partial}{\partial U}\right), \quad k^2 = -\left(1 - \frac{2M}{r}\right).$$

Solution. In the null Kruskal coordinates, we have

$$U = \mp e^{-u/4M} \quad \text{and} \quad V = e^{v/4M},$$

where

$$u = t - r^* \quad \text{and} \quad v = t + r^*,$$

with $r^* = r + 2M\ln|r/2M - 1|$. In Schwarzschild coordinates, the timelike Killing vector field is $k = \partial/\partial t$. In (u, v)-coordinates,

$$k = \left(\frac{\partial u}{\partial t}\right)\frac{\partial}{\partial u} + \left(\frac{\partial v}{\partial t}\right)\frac{\partial}{\partial v} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}.$$

Thus, in (U, V)-coordinates,

$$\begin{aligned}
k &= \left(\frac{\partial U}{\partial u} + \frac{\partial U}{\partial v}\right)\frac{\partial}{\partial U} + \left(\frac{\partial V}{\partial u} + \frac{\partial V}{\partial v}\right)\frac{\partial}{\partial V}, \\
&= \left(\frac{\partial U}{\partial u}\right)\frac{\partial}{\partial U} + \left(\frac{\partial V}{\partial v}\right)\frac{\partial}{\partial V}, \\
&= \frac{1}{4M}\left(\pm e^{-u/4M}\frac{\partial}{\partial U} + e^{v/4M}\frac{\partial}{\partial V}\right), \\
&= \frac{1}{4M}\left(V\frac{\partial}{\partial V} - U\frac{\partial}{\partial U}\right).
\end{aligned}$$

Now, the line element in (U, V)-coordinates is given by

$$\mathrm{d}s^2 = -\frac{32M^3}{r}e^{-r/2M}\mathrm{d}U\mathrm{d}V + r^2\mathrm{d}\Omega^2.$$

Thus,

$$\begin{aligned}
k^2 &= g(k, k) = \mathrm{d}s^2\left(k\right), \\
&= -\frac{32M^3}{r}e^{-r/2M}\mathrm{d}U(k)\mathrm{d}V(k), \\
&= -\frac{32M^3}{r}e^{-r/2M}\left(\frac{1}{4M}\right)^2(-UV), \\
&= -\frac{2M}{r}e^{-r/2M}(-UV), \\
&= -\frac{2M}{r}e^{-r/2M}e^{r/2M}\left(\frac{r}{2M} - 1\right), \\
&= -\left(1 - \frac{2M}{r}\right).
\end{aligned}$$

□

Problem 4

[Problem 2 in Section 2.6 of Poisson's Toolkit]

Consider the Friedmann-Robertson-Walker line element.

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\Omega^2\right).$$

The vector tangent to the congruence is $u^\alpha = \partial x^\alpha/\partial t$. In the chart $x^\alpha = (t, r, \theta, \phi)$,

$$u^\alpha = (1,0,0,0) = \delta^\alpha_t,$$
$$u_\alpha = g_{\alpha\beta}u^\beta = g_{\alpha\beta}\delta^\beta_t = g_{\alpha t} = -\delta^t_\alpha.$$

(a) Now,

$$u_{\alpha;\beta} = u_{\alpha,\beta} - \Gamma^\gamma{}_{\alpha\beta}u_\gamma = -\delta^t_{\alpha,\beta} + \Gamma^\gamma{}_{\alpha\beta}\delta^t_\gamma = \Gamma^t{}_{\alpha\beta}.$$

$$\begin{aligned}\Gamma^t{}_{\alpha\beta} &= \frac{1}{2}g^{t\gamma}\left(g_{\gamma\alpha,\beta} + g_{\gamma\beta,\alpha} - g_{\alpha\beta,\gamma}\right) = \frac{1}{2}g^{tt}\left(g_{t\alpha,\beta} + g_{t\beta,\alpha} - g_{\alpha\beta,t}\right), \\ &= \frac{1}{2}g^{tt}\left(\delta^t_\alpha g_{tt,\beta} + \delta^t_\beta g_{tt,\alpha} - g_{\alpha\beta,t}\right) = -\frac{1}{2}g^{tt}g_{\alpha\beta,t} = \frac{1}{2}g_{\alpha\beta,t}.\end{aligned}$$

Clearly, the only nonzero components are $\Gamma^t{}_{ii}$, where $x^i = (r,\theta,\phi)$. It is easy to show that

$$\Gamma^t{}_{ij} = \frac{\dot a}{a}h_{ij},$$

where h_{ij} is the spatial metric. Thus, $u_{\alpha;\beta}u^\beta = \Gamma^t{}_{\alpha\beta}\delta^\beta_t = \Gamma^t{}_{\alpha t} = 0$, and the congruence is geodesic.

(b) Let $B_{\alpha\beta} = u_{\alpha;\beta}$. Since $u_{\alpha;\beta} = \Gamma^t{}_{\alpha\beta}$, then $B_{\alpha\beta}$ is symmetric. Thus, $\omega_{\alpha\beta} = 0$. Also, since the only nonzero components of the Christoffel symbols are $\Gamma^t{}_{ij}$, and the matrix formed is diagonal, then $B_{\alpha\beta}$ is pure trace. Therefore, $\sigma_{\alpha\beta} = 0$. The trace of $B_{\alpha\beta}$ is simply

$$\Theta = g^{\alpha\beta}B_{\alpha\beta} = \frac{\dot a}{a}h_{ij}h^{ij} = 3\frac{\dot a}{a}.$$

The congruence is expanding, shear-free, and nonrotating.

(c) The Raychaudhuri equation is simply

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Theta = -\frac{1}{3}\Theta^2 - R_{\alpha\beta}u^\alpha u^\beta.$$

Now, $\mathrm{d}\Theta/\mathrm{d}\tau = \Theta_{,\alpha}u^\alpha = \dot\Theta$. Also, from the Einstein field equations, $R_{\alpha\beta} = 8\pi\left(T_{\alpha\beta} - \frac{1}{2}Tg_{\alpha\beta}\right)$. Therefore,

$$\begin{aligned}\dot\Theta &= -\frac{1}{3}\Theta^2 - 8\pi\left(T_{\alpha\beta} - \frac{1}{2}Tg_{\alpha\beta}\right)u^\alpha u^\beta, \\ 3\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot a}{a}\right) &= -3\left(\frac{\dot a}{a}\right)^2 - 8\pi\left(T_{tt} - \frac{1}{2}Tg_{tt}\right), \\ 3\frac{\ddot a}{a} - 3\left(\frac{\dot a}{a}\right)^2 &= -3\left(\frac{\dot a}{a}\right)^2 - 8\pi\left(\rho + \frac{1}{2}(-\rho + 3p)\right), \\ \frac{\ddot a}{a} &= -\frac{4}{3}\pi(\rho + 3p).\end{aligned}$$

$\square$

Problem 5

[Problem 3 in Section 2.6 of Poisson's Toolkit]

Consider the vector field u in Schwarzschild coordinates

$$u^\alpha \frac{\partial}{\partial x^\alpha} = \frac{1}{\sqrt{1-3M/r}} \left(\frac{\partial}{\partial t} + \sqrt{\frac{M}{r^3}} \frac{\partial}{\partial \theta} \right)$$

(a) The vector field is timelike:

$$g(u,u) = \frac{1}{1-3M/r} \left(g_{tt} + g_{\theta\theta} \frac{M}{r^3} \right) = \frac{1}{1-3M/r} \left(-f + \frac{M}{r} \right) = -1.$$

The geodesic equation is satisfied:

$$\begin{aligned} u^\alpha{}_{;\beta} u^\beta &= u^\alpha{}_{,\beta} u^\beta + \Gamma^\alpha{}_{\beta\gamma} u^\gamma u^\beta = u^\alpha{}_{,r} u^r + \Gamma^\alpha{}_{\beta\gamma} u^\beta u^\gamma = \Gamma^\alpha{}_{\beta\gamma} u^\beta u^\gamma, \\ &= \Gamma^\alpha{}_{tt} u^t u^t + 2\Gamma^\alpha{}_{t\theta} u^t u^\theta + \Gamma^\alpha{}_{\theta\theta} u^\theta u^\theta = \delta^\alpha_r \Gamma^r{}_{tt} u^t u^t + \delta^\alpha_r \Gamma^r{}_{\theta\theta} u^\theta u^\theta, \\ &= \frac{1}{1-3M/r} \delta^\alpha_r \left(\frac{M}{r^2} f + (-fr) \frac{M}{r^3} \right) = 0. \end{aligned}$$

The geodesics are constant-ϕ curves along the direction $\hat{\theta}$.

(b) Let $B_{\alpha\beta} = u_{\alpha;\beta}$. The expansion is given by

$$\begin{aligned} \Theta = u^\alpha{}_{;\alpha} &= u^\alpha{}_{,\alpha} + \Gamma^\alpha{}_{\alpha\gamma} u^\gamma = u^r{}_{,r} + \Gamma^\alpha{}_{\alpha\gamma} u^\gamma = \Gamma^\alpha{}_{\alpha\gamma} u^\gamma, \\ &= \Gamma^\alpha{}_{\alpha t} u^t + \Gamma^\alpha{}_{\alpha\theta} u^\theta = \Gamma^\phi{}_{\phi\theta} u^\theta = \cot\theta\, u^\theta = \cot\theta \sqrt{\frac{M/r^3}{1-3M/r}} \end{aligned}$$

The expansion is positive [negative] in the northern [southern] hemisphere since the constant-ϕ curves spread out [gather]; it is singular at the north and south poles since ϕ is not well-defined there.

(c) Now, $u_\alpha \mathrm{d}x^\alpha = g_{\alpha\beta} u^\beta \mathrm{d}x^\alpha = g_{tt} u^t \mathrm{d}t + g_{\theta\theta} u^\theta \mathrm{d}\theta = -f u^t \mathrm{d}t + r^2 u^\theta \mathrm{d}\theta$. Due to the symmetry of the Christoffel symbols, $\omega_{\alpha\beta} = \frac{1}{2}(u_{\alpha;\beta} - u_{\beta;\alpha}) = \frac{1}{2}(u_{\alpha,\beta} - u_{\beta,\alpha})$. Clearly, the only nonzero components are $\omega_{tr} = -\omega_{rt}$ and $\omega_{\theta r} = -\omega_{r\theta}$. They are

$$\omega_{tr} = \frac{1}{2} u_{t,r} = -\frac{M(r-6M)}{2r(r-3M)^2} \sqrt{1 - \frac{3M}{r}} \quad \text{and} \quad \omega_{\theta r} = \frac{1}{2} u_{\theta,r} = \frac{r-6M}{4Mr^3} \left(\frac{Mr^2}{r-3M} \right)^{3/2}.$$

Thus,

$$\omega_{\alpha\beta}\omega^{\alpha\beta} = 2\omega_{tr}\omega^{tr} + 2\omega_{\theta r}\omega^{\theta r} = 2g^{tt}g^{rr}{\omega_{tr}}^2 + 2g^{\theta\theta}g^{rr}{\omega_{\theta r}}^2 = -2{\omega_{tr}}^2 + \frac{2f}{r^2}{\omega_{\theta r}}^2 = \frac{M}{8r^3} \frac{(r-6M)^2}{(r-3M)^2}.$$

(d)

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Theta = \Theta_{,\alpha} u^\alpha = \Theta_{,\theta} u^\theta = -\csc^2\theta\, (u^\theta)^2 = -\frac{M\csc^2\theta}{r^2(r-3M)}.$$

Because Schwarzschild is Ricci-flat, we only need to calculate $B^{\alpha\beta}B_{\beta\alpha}$. Using Mathematica,

$$B^{\alpha\beta}B_{\beta\alpha} = \frac{M\csc^2\theta}{r^2(r-3M)} = -\frac{\mathrm{d}}{\mathrm{d}\tau}\Theta. \qquad \square$$