I. QUANTUM TOMOGRAPHY

Show how to reconstruct the 2×2 density matrix of a mixed ensemble of spin 1/2 particles when the ensemble averages $[S_x]$, $[S_y]$, and $[S_z]$ are available.

Solution. Let the density matrix have the form

$$\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Suppose we know the ensemble averages $[S_x]$, $[S_y]$, and $[S_z]$; then together with the normalization condition $\mathrm{Tr}(\rho) = a + d = 1$, they suffice to determine the matrix elements. Now,

$$\begin{aligned}
[S_x] &= \mathrm{Tr}(\rho S_x) = \frac{\hbar}{2} \mathrm{Tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) = \frac{\hbar}{2} \mathrm{Tr}\left(\begin{bmatrix} b & a \\ d & c \end{bmatrix} \right) &= \frac{\hbar}{2}(b + c), \\
[S_y] &= \mathrm{Tr}(\rho S_y) = \frac{\hbar}{2} \mathrm{Tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \right) = \frac{i\hbar}{2} \mathrm{Tr}\left(\begin{bmatrix} b & -a \\ d & -c \end{bmatrix} \right) &= \frac{i\hbar}{2}(b - c), \\
[S_z] &= \mathrm{Tr}(\rho S_z) = \frac{\hbar}{2} \mathrm{Tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) = \frac{\hbar}{2} \mathrm{Tr}\left(\begin{bmatrix} a & -b \\ c & -d \end{bmatrix} \right) &= \frac{\hbar}{2}(a - d).
\end{aligned}$$

The matrix elements can be easily solved by inspection. We obtain

$$\begin{aligned}
a &= \frac{1}{2}\left(1 + \frac{2}{\hbar}[S_z]\right), \quad & d &= \frac{1}{2}\left(1 - \frac{2}{\hbar}[S_z]\right), \\
b &= \frac{1}{\hbar}\Big([S_x] - i[S_y]\Big), \quad & c &= \frac{1}{\hbar}\Big([S_x] + i[S_y]\Big).
\end{aligned}$$

$\square$

II. UNITARY TIME EVOLUTION

Prove that a pure state ρ cannot evolve into a mixed state under Schrödinger dynamics (unitary time evolution).

Solution. For an undisturbed ensemble, the change in ρ is governed solely by the time evolution of the state kets. Given $\rho(t_0)$ the time evolution of ρ is therefore

$$\begin{aligned}\rho(t) &= \sum_i w_i \left|\alpha^{(i)}(t)\right\rangle \left\langle\alpha^{(i)}(t)\right| = \sum_i w_i \mathcal{U}(t,t_0) \left|\alpha^{(i)}(t_0)\right\rangle \left\langle\alpha^{(i)}(t_0)\right| \mathcal{U}(t,t_0)^\dagger \\ &= \mathcal{U}(t,t_0)\left(\sum_i w_i \left|\alpha^{(i)}(t_0)\right\rangle \left\langle\alpha^{(i)}(t_0)\right|\right)\mathcal{U}(t,t_0)^\dagger = \mathcal{U}(t,t_0)\rho(t_0)\mathcal{U}(t,t_0)^\dagger,\end{aligned}$$

for some unitary operator $\mathcal{U}(t,t_0)$. Now,

$$\begin{aligned}\mathrm{Tr}\left(\rho(t)^2\right) &= \mathrm{Tr}\left(\mathcal{U}(t,t_0)\rho(t_0)\mathcal{U}(t,t_0)^\dagger\mathcal{U}(t,t_0)\rho(t_0)\mathcal{U}(t,t_0)^\dagger\right) \\ &= \mathrm{Tr}\left(\mathcal{U}(t,t_0)\rho(t_0)^2\mathcal{U}(t,t_0)^\dagger\right) = \mathrm{Tr}\left(\rho(t_0)^2\right),\end{aligned}$$

where we have used the cyclic property of the trace in the last equality. Thus, if $\rho(t_0)$ is a pure state then it will remain a pure state, since $\mathrm{Tr}\left(\rho^2\right) = 1$ for a pure ensemble only. □

III. LOWERING OPERATOR

Let $\mathbf{J}$ be an angular momentum operator.

(a) Prove that $\mathbf{J}^2 = J_z^2 + J_+ J_- - \hbar J_z$.

(b) Use this result to simplify $J_- |j, m\rangle$. You may choose the coefficient to be real.

Solution.

(a) We use the fact that $J_\pm = J_x \pm i J_y$. Clearly, $J_x = (J_+ + J_-)/2$ and $J_y = -i(J_x - J_y)/2$. Therefore,

$$
\begin{aligned}
\mathbf{J}^2 &= J_x^2 + J_y^2 + J_z^2 \\
&= J_z^2 + \frac{1}{4}\left((J_+ + J_-)^2 - (J_+ - J_-)^2\right) \\
&= J_z^2 + \frac{1}{2}\left(J_+ J_- + J_- J_+\right) \\
&= J_z^2 + J_+ J_- - \frac{1}{2}\left(J_+ J_- - J_- J_+\right) \\
&= J_z^2 + J_+ J_- - \frac{1}{2}\left[J_+, J_-\right] \\
&= J_z^2 + J_+ J_- - \hbar J_z.
\end{aligned}
$$

(b) From the previous equation, we have $J_+ J_- = \mathbf{J}^2 - J_z^2 + \hbar J_z$. Then

$$
\begin{aligned}
J_+ J_- |j, m\rangle &= (\mathbf{J}^2 - J_z^2 + \hbar J_z) |j, m\rangle \\
&= \left(j(j+1)\hbar^2 - m^2\hbar^2 + m\hbar^2\right) |j, m\rangle \\
&= \hbar^2 \left(j^2 + j - m^2 + m\right) |j, m\rangle \\
&= \hbar^2 \left((j+m)(j-m) + j + m\right) |j, m\rangle \\
&= \hbar^2 (j+m)(j-m+1) |j, m\rangle
\end{aligned}
$$

Now, observe that $J_\mp = J_\pm^\dagger$, since J_x and J_y are Hermitian. Therefore,

$$
J_+ J_- |j, m\rangle = J_-^\dagger J_- |j, m\rangle = \hbar^2 (j+m)(j-m+1) |j, m\rangle .
$$

Clearly,

$$
|c_{j,m}^-|^2 := \langle j, m| J_-^\dagger J_- |j, m\rangle = \hbar^2 (j+m)(j-m+1).
$$

Thus, choosing the coefficient to be real, and using the fact that $J_- |j, m\rangle$ must be the same as $|j, m-1\rangle$ up to normalization, we have

$$
J_- |j, m\rangle = c_{j,m}^- |j, m-1\rangle = \hbar\sqrt{(j+m)(j-m+1)}\, |j, m-1\rangle .
$$

□