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Advanced Calculus

These are selected problems from my problem sets in Math 123.1 - Advanced Calculus I, which I took (in 2nd Sem. A.Y. 2017-2018) during my 4th year as a Physics undergrad. This course is the first of the two-semester analysis core taken by Math undergrads in UP Diliman. The main textbook we used was Bartle and Sherbert's Introduction to Real Analysis, 4th edition. I also took the second course which covered analysis in $\mathbb{R}^n$ (topology, differentiation, integration (measure via content), vector and normed spaces) and convergence of series; but I have not typed my solutions in LaTeX so I already lost them. **I make no claim that all my solutions are correct.**

Problem Set 1

Problem 1-a.

Consider the finite set $S = \{x_1, x_2, x_3\}$ where $x_1 < x_2 < x_3$. Show that S^{c} is open.

Proof.
Let $x \in S^{\mathsf{c}} = \mathbb{R} - S = (-\infty, x_1) \cup (x_1, x_2) \cup (x_2, x_3) \cup (x_3, +\infty)$.
If we choose $\epsilon = \min\{|x - x_1|, |x - x_2|, |x - x_3|\}$ then $N_\epsilon(x) \subseteq S^{\mathsf{c}}$. □

Problem 1-b.

Show that the set $S = \mathbb{Q}^{\mathsf{c}} \cap [-\sqrt{2}, \sqrt{2}]$ is not compact.

Proof.
Consider the set $G_n = (-2, -1/n) \cup (1/n, 2)$, where $n \in \mathbb{N}$.
Observe that, since $0 \in \mathbb{Q}$, $S \subseteq [-\sqrt{2}, 0) \cup (0, \sqrt{2}] \subseteq \bigcup_{n=1}^{\infty} G_n$. Therefore, the collection $\mathcal{G} = \{G_n\}$ is an open cover of S. Suppose $\mathcal{G}$ has a finite subcover, say $\mathcal{G}' = \{G_{n_i}\}_{i=1}^{m}$. Then $\bigcup_{i=1}^{m} G_{n_i} = (-2, -1/M) \cup (1/M, 2)$, where $M = \max\{n_i \big| i = 1, \ldots, m\}$. Since $1/M > 0$, by the density property of $\mathbb{Q}^{\mathsf{c}}$, $\exists r' \in \mathbb{Q}^{\mathsf{c}}$ such that $0 < r' < 1/M$. Clearly, $r' \in S$ but $r' \notin \bigcup_{i=1}^{m} G_{n_i}$ since $r' < 1/M$. Thus, $\mathcal{G}$ has no finite subcover. □

Problem 2.

Let $A \subset \mathbb{R}$ be a non-empty set that is bounded above. Suppose $s \in \mathbb{R}$ such that $\forall n \in \mathbb{N}$, $s + \frac{1}{n}$ is an upper bound of A and $s - \frac{1}{n}$ is not an upper bound of A. Show that $s = \sup A$.

Proof.
By the Completeness Axiom of $\mathbb{R}$, A has a supremum. Claim: $s = \sup A$.

Since $s < s + \frac{1}{n}$, $\forall n \in \mathbb{N}$, we only need to show that s is an upper bound of A and any $b < s$ is not an upper bound of A.

Suppose s is not an upper bound of A. Then $\exists x_1 \in A$ such that $x_1 > s$. Now, $x_1 - s > 0$ and by the Archimedian Property $\exists n_1 \in \mathbb{N}$ such that $0 < \frac{1}{n_1} < x_1 - s$. Upon rearrangement we find that $s + \frac{1}{n_1} < x_1$. This contradicts the fact that $s + \frac{1}{n_1}$ is an upper bound of A.

Suppose there exists a $b < s$ such that b is an upper bound of A. Now, $s - b > 0$ and by the Archimedian Property $\exists n_2 \in \mathbb{N}$ such that $0 < \frac{1}{n_2} < s - b$. Upon rearrangement we find that $b < s - \frac{1}{n_2}$. Since $s - \frac{1}{n_2}$ is not an upper bound of A, $\exists x_2 \in A$ such that $x_2 > s - \frac{1}{n_2}$. By transitivity, $x_2 > b$ and this contradicts our assumption that b is an upper bound of A.

□

Problem Set 2

Problem 1.
Use the definition of convergence to show that the following sequence converges,

$$\{x_n\} = \left\{\frac{n^2}{n^2 - n + 10}\right\}.$$

Proof.
Claim: $\lim x_n = 1$. Let $\epsilon > 0$. By the Archimedian Property, $\exists N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon} + \frac{1}{2}$. Choose $N > \max\{10, \frac{1}{\epsilon} + \frac{1}{2}\}$. Thus if $n \geq N$, we have

$$\begin{aligned}|x_n - 1| &= \left|\frac{n^2}{n^2 - n + 10} - 1\right| = \left|\frac{n - 10}{n^2 - n + 10}\right| = \frac{n - 10}{n^2 - n + 10} \\ &= \frac{(n - \frac{1}{2}) - \frac{19}{2}}{(n^2 - n + \frac{1}{4}) + \frac{39}{4}} < \frac{(n - \frac{1}{2})}{(n^2 - n + \frac{1}{4})} = \frac{1}{(n - \frac{1}{2})} \leq \frac{1}{(N - \frac{1}{2})} < \epsilon.\end{aligned}$$

□

Problem 2.
Consider the sequence $\{x_n\}$ that satisfies the recursive formulation

$$x_{n+1} = \frac{x_n}{2} + 2, \text{ for } n \in \mathbb{N},$$

with $x_1 = 8$. Use the Monotone Convergence Theorem to show that $\{x_n\}$ converges.

Proof.
We first show that $\{x_n\}$ is monotone. Observe that $\{x_n\}$ is strictly decreasing; i.e. $x_{n+1} < x_n$. We prove this by induction. Clearly, for $n = 1$, $x_2 = 6 < 8 = x_1$. Assume that for $n = k$, $x_{k+1} < x_k$. Then,

$$\begin{aligned}x_{k+1} &< x_k \\ \frac{x_{k+1}}{2} &< \frac{x_k}{2} \\ \frac{x_{k+1}}{2} + 2 &< \frac{x_k}{2} + 2 \\ x_{k+2} &< x_{k+1}.\end{aligned}$$

We now show that $\{x_n\}$ is bounded. Since $\{x_n\}$ is decreasing, $x_1 = 8$ is an upper bound. Also, note that $\forall n \in \mathbb{N}$,

$$\begin{aligned}x_1 = 8 &> 4 \\ \frac{x_1}{2} &> 2 \\ \frac{x_1}{2} + 2 &> 4 \\ x_2 &> 4 \\ &\vdots \\ x_n &> 4.\end{aligned}$$

Therefore, $\{x_n\}$ is bounded below by 4. Since $\{x_n\}$ is monotone and bounded, then it converges by the Monotone Convergence Theorem. □

Problem Set 3

Problem 1.
Let $\lfloor x \rfloor$ be the greatest integer function defined as

$$\lfloor x \rfloor = n, \quad \text{if } n \leq x < n+1.$$

Use the definition of a limit to show that

$$\lim_{x\to 3} \left\lfloor \frac{x}{2} \right\rfloor + |4-3x| = 6.$$

Proof.
Let $\epsilon > 0$. Choose $\delta = \min\left\{1, \frac{\epsilon}{3}\right\}$. Thus, if $0 < |x-3| < \delta$, we have

$$\begin{aligned}\left| \left\lfloor \frac{x}{2} \right\rfloor + |4-3x| - 6 \right| \leq |1 + |3x-4| - 6| = ||3x-4| - 5| = |3x-4-5| = |3x-9| \\ = 3|x-3| < 3\delta \leq \epsilon.\end{aligned}$$

□

Problem 2.
Use the Bolzano Intermediate Value Theorem to show that any polynomial of odd degree with real coefficients has at least one real root.

Proof.
Let $p : I(\subseteq \mathbb{R}) \to \mathbb{R}$ be a polynomial of odd degree $2n+1$, where $p(x) = a_{2n+1}x^{2n+1} + a_{2n}x^{2n} + \ldots + a_1 x + a_0$, with $n \in \mathbb{N}_0$, a_k's $\in \mathbb{R}$, and $a_{2n+1} \neq 0$. Clearly, p is a continuous function since $x \mapsto x$ and $x \mapsto$ const. are both continuous and we can construct p from their sums and products. Without loss of generality, take $a_{2n+1} > 0$. We may rewrite $p(x)$ as

$$p(x) = a_{2n+1}x^{2n+1}q(x), \text{ where } q(x) := 1 + \frac{a_{2n}}{a_{2n+1}}\frac{1}{x} + \ldots + \frac{a_1}{a_{2n+1}}\frac{1}{x^{2n}} + \frac{a_0}{a_{2n+1}}\frac{1}{x^{2n+1}},$$

provided that $x \neq 0$. Now,

$$\lim_{x\to-\infty} q(x) = 1 = \lim_{x\to+\infty} q(x),$$

since we can take the sum of the limit of each term in $q(x)$ and $\lim_{x\to\pm\infty} 1/x^m = 0$, $\forall m \in \mathbb{N}$. ($\forall \epsilon > 0$, choose $K = \sqrt[m]{1/\epsilon}$ such that if $|x| > K$ then $|1/x^m| < 1/K^m = \epsilon$.) Thus, for some $0 < \epsilon_1 < 1$, $\exists x_1 > 0$ such that if $x > x_1$ then

$$\begin{aligned}|q(x) - 1| = \left| \frac{a_{2n}}{a_{2n+1}}\frac{1}{x} + \ldots + \frac{a_1}{a_{2n+1}}\frac{1}{x^{2n}} + \frac{a_0}{a_{2n+1}}\frac{1}{x^{2n+1}} \right| < \epsilon_1 \\ -\epsilon_1 < \frac{a_{2n}}{a_{2n+1}}\frac{1}{x} + \ldots + \frac{a_1}{a_{2n+1}}\frac{1}{x^{2n}} + \frac{a_0}{a_{2n+1}}\frac{1}{x^{2n+1}} \\ 1 - \epsilon_1 < q(x),\end{aligned}$$

which implies that

$$p(x : x > x_1 > 0) > a_{2n+1}x^{2n+1}(1-\epsilon_1) > 0.$$

Similarly, for some $0 < \epsilon_2 < 1$, $\exists x_2 < 0$ such that if $x < x_2$ then $1 - \epsilon_2 < q(x)$, which implies that

$$p(x : x < x_2 < 0) < a_{2n+1}x^{2n+1}(1-\epsilon_2) < 0.$$

Thus, if we take $a < x_2$, $b > x_1$ we have $p(a) < 0 < p(b)$. Therefore, by the Intermediate Value Theorem, $\exists r \in [a, b]$ such that $p(r) = 0$. □

Problem Set 4

Problem 1.
Use the $\epsilon - \delta$ definition to show that

$$f(x) = \begin{cases} x^2 \cos(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

is differentiable at the origin. In addition, use Sequential Criterion to show that $f'(x)$ is not continuous at the origin.

Proof.

- Claim: $f'(0) = 0$.
  Let $\epsilon > 0$. Choose $\delta = \epsilon$. Thus, if $0 < |x - 0| < \delta$ then
  $$\left|\frac{f(x) - f(0)}{x - 0} - 0\right| = \left|\frac{f(x)}{x}\right| = \left|\frac{x^2 \cos(1/x)}{x}\right| = \left|x \cos \frac{1}{x}\right| = |x| \left|\cos \frac{1}{x}\right| \leq |x| < \delta = \epsilon.$$
  Thus, f is differentiable at $x = 0$.
- Note that, using the product rule and the chain rule for differentiation, $\forall c \neq 0$,
  $$\begin{aligned} f'(c) = \mathrm{D}\left[c^2 \cos \frac{1}{c}\right] &= \mathrm{D}\left[c^2\right] \cdot \cos \frac{1}{c} + c^2 \cdot \mathrm{D}\left[\cos \frac{1}{c}\right] \\ &= 2c \cos \frac{1}{c} + c^2 \left(-\sin \frac{1}{c}\right)\left(-\frac{1}{c^2}\right) = 2c \cos \frac{1}{c} + \sin \frac{1}{c}. \end{aligned}$$
  Consider the constant sequence $\{0\}$ which converges to zero. Clearly, $\{f'(0)\}$ also converges to zero. Consider another sequence $\{\frac{1}{(n+1/2)\pi}\}$. This sequence also converges to zero, since $0 \leq \frac{1}{(n+1/2)\pi} \leq \frac{1}{n}$ and we can use Squeeze Theorem. Now,
  $$\left\{f'\left(\frac{1}{(n+1/2)\pi}\right)\right\} = \left\{\frac{2}{(n+1/2)\pi} \cos\left[(n+1/2)\pi\right] + \sin\left[(n+1/2)\pi\right]\right\} = \{(-1)^n\}$$
  which is not a convergent sequence. Thus, by the Sequential Criterion $f'(x)$ is not continuous at $x = 0$.

$\square$

Problem 2.
Use IVT and Rolle's Theorem to show that the equation $x^3 - x^2 + 4x = 3$ has exactly one real root.

Proof.
Consider $f(x) = x^3 - x^2 + 4x - 3$ which is zero if x is a root of the given equation. Note that f is continuous and differentiable on $\mathbb{R}$ (and its subsets) since it is a polynomial. Also, $f(1) = 1 > 0$ and $f(-1) = -9 < 0$. Therefore, by IVT f has at least one zero in the interval $[-1, 1]$ and therefore in $\mathbb{R}$. We now prove by contradiction that f has exactly one zero in $\mathbb{R}$. Suppose $x_1, x_2 \in \mathbb{R}$, $x_1 \neq x_2$, are both zeros of f. Without loss of generality, assume $x_1 < x_2$. Since $f(x_1) = 0 = f(x_2)$, by Rolle's Theorem $\exists c \in (x_1, x_2)$ such that $f'(c) = 0$. However, we can make a crude 'estimate' for a lower bound of f' by minimizing it term-by-term. Clearly, $f'(x) = 3x^2 - 2x + 4$. Note that if $x \in (-1, 1)$ then $f'(x) \geq 3(0) - 2(1) + 4 = 2$ and if $x \in \mathbb{R} \backslash (-1, 1)$ then $x^2 > x$ and $f'(x) \geq 4$. Therefore, $f'(x) = 3x^2 - 2x + 4 > 0, \forall x \in \mathbb{R}$. This is a contradiction.

$\square$

Problem Set 5

Problem 1.
Let $f(x) = x^2$. Show that f is integrable on $[-1, 0]$ using the Integrability Criterion.

Proof.
Note that f is bounded on $[-1,0] =: I$ since $\forall x \in I,\ |f(x)| \leq 1$. Also, f is strictly decreasing on I. Let $\epsilon > 0$. Choose a partition on I, $\mathcal{P}_\epsilon = \{[x_{i-1}, x_i]\}_{i=1}^n$, such that $\forall i \in \{1, \ldots, n\}$, $\|\mathcal{P}_\epsilon\| = x_i - x_{i-1} = 1/n$, where $n > 1/\epsilon$. Clearly, $x_k = -1 + k/n$. Since f is strictly decreasing on I, $m_k = \inf\left\{f(x) \mid x \in [x_{k-1}, x_k]\right\} = f(x_k)$ and $M_k = \sup\left\{f(x) \mid x \in [x_{k-1}, x_k]\right\} = f(x_{k-1})$. Thus,

$$\begin{aligned}
U(f;\mathcal{P}_\epsilon) - L(f;\mathcal{P}_\epsilon) &= \sum_{k=1}^n (M_k - m_k)(x_k - x_{k-1}) = \sum_{k=1}^n \left(f(x_{k-1}) - f(x_k)\right)(x_k - x_{k-1}) \\
&= \sum_{k=1}^n (x_{k-1}^2 - x_k^2)\frac{1}{n} = \frac{1}{n}\sum_{k=1}^n \left(\left(-1 + \frac{k-1}{n}\right)^2 - \left(-1 + \frac{k}{n}\right)^2\right) \\
&= \frac{1}{n}\sum_{k=1}^n \left(1 - \frac{2(k-1)}{n} + \frac{k^2 - 2k + 1}{n^2} - 1 + \frac{2k}{n} - \frac{k^2}{n^2}\right) \\
&= \frac{1}{n}\sum_{k=1}^n \left(\frac{2}{n} - \frac{2k}{n^2} + \frac{1}{n^2}\right) = \frac{1}{n}\left(2 - \frac{2}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{1}{n}\right) \\
&= \frac{1}{n}\left(2 - 1 - \frac{1}{n} + \frac{1}{n}\right) = \frac{1}{n} < \epsilon.
\end{aligned}$$

□

Problem 2.
Let f be a continuous function with $f(x) \geq 0$, $\forall x \in [a, b]$. Use the indefinite integral of f,

$$F(x) = \int_a^x f, \quad x \in [a, b],$$

and the FTOC II to show that if $\int_a^b f = 0$ then $f(x) = 0$, $\forall x \in [a, b]$.

Proof.
Since f is continuous on $[a, b]$ then f is Darboux integrable on $[a, b]$. By the FTOC II, F is continuous on $[a, b]$ and differentiable on (a, b) with $F'(x) = f(x)$. Since $f(x) \geq 0$, $\forall x \in [a, b]$, then F is increasing on $[a, b]$. We know that $F(a) = \int_a^a f = 0$ and $F(b) = \int_a^b f = 0$. Therefore, if F is increasing on $[a, b]$ it can only be zero. It follows that $f(x) = F'(x) = 0$, $\forall x \in [a, b]$. □

Problem 3.
Consider the sequence $\{f_n(x)\}$, where

$$f_n(x) = x(1 - x^n).$$

(a) Show that for each n, $f_n(x)$ is bounded on $[0, 1]$.
(b) Determine if the sequence is uniformly convergent on $[0, 1]$ using the uniform norm.

Proof.
(a) Clearly, $\forall n$, f_n is continuous on $[0,1]$ since $f_n(x)$ is a polynomial of degree $n+1$. By the Boundedness Theorem, $\forall n$, f_n is bounded on $[0,1]$.
(b) We first show pointwise convergence. Let $x \in [0,1]$.
Case 1. $x \in [0,1)$. Then $f(x) = \lim f_n(x) = x$, since $\lim x^n = 0$ if $|x| < 1$.
Case 2. $x = 1$. Then $f_n(x) = 0$ and $f(x) = \lim f_n(x) = 0$.
Therefore, f_n converges pointwise to

$$f(x) = \begin{cases} x & \text{if } x \in [0,1), \\ 0 & \text{if } x = 1. \end{cases}$$

Now,

$$\begin{aligned} \|f_n - f\| = \sup \left\{ |f_n(x) - f(x)| \;\middle|\; x \in [0,1] \right\} &= \sup \left\{ \begin{array}{ll} |x(1-x^n) - x| & \big| \; x \in [0,1) \\ |x(1-x^n) - 0| & \big| \; x = 1 \end{array} \right\} \\ = \sup \left\{ \begin{array}{ll} x^{n+1} & \big| \; x \in [0,1) \\ 0 & \big| \; x = 1 \end{array} \right\} = 1. \end{aligned}$$

Thus, $\{f_n(x)\}$ is not uniformly convergent on $[0,1]$. $\square$

Problem Set 6

Problem 1.
Let $\{f_n\}$ be a sequence of functions on $D \subset \mathbb{R}$ to $\mathbb{R}$. Prove that the infinite series $\sum f_n$ is uniformly convergent on D if and only if $\forall \epsilon > 0$, $\exists M(\epsilon) \in \mathbb{N}$ such that if $m > n \geq M(\epsilon)$, then $|f_{n+1}(x) + \ldots + f_m(x)| < \epsilon$, $\forall x \in D$.

Proof.
($\Rightarrow$) Let $\sum f_n$ be uniformly convergent to f on D. Then the sequence of partial sums $\{s_n\}$, where $s_n(x) = f_1(x) + \ldots + f_n(x)$, $\forall x \in D$, converges uniformly to f. It follows that $\forall \epsilon > 0$, $\exists M(\epsilon) \in \mathbb{N}$ such that if $n \geq M(\epsilon)$

$$|s_n(x) - f(x)| < \frac{\epsilon}{2}, \quad \forall x \in D,$$

and similarly for $m > n \geq M(\epsilon)$. Thus,

$$\begin{aligned}|f_{n+1}(x) + \ldots + f_m(x)| &= |s_m(x) - s_n(x)| = |s_m(x) - f(x) + f(x) - s_n(x)| \\ &\leq |s_m(x) - f(x)| + |f(x) - s_n(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \quad \forall x \in D.\end{aligned}$$

($\Leftarrow$) Suppose that $\forall \epsilon > 0$, $\exists M(\epsilon) \in \mathbb{N}$ such that if $m > n \geq M(\epsilon)$, then $\forall x \in D$, $|f_{n+1}(x) + \ldots + f_m(x)| < \epsilon$. This means that $|s_m(x) - s_n(x)| < \epsilon$, $\forall x \in D$. It follows that for every $x \in D$, $\{s_n(x)\}$ is a Cauchy sequence of real numbers. Therefore, $\{s_n(x)\}$ is convergent to some value, which we define as $f(x)$, and is bounded. Thus, $\{s_n\}$ is a sequence of bounded functions on D which converges pointwise to some bounded function f on D. By the Cauchy Criterion for Uniform Convergence, $\{s_n\}$, i.e. $\sum f_n$, must converge uniformly to f. □

Problem 2.
We say that $\displaystyle\sum_{n=1}^{\infty} f_n$ is uniformly absolutely convergent on $D \subset \mathbb{R}$ if $\displaystyle\sum_{n=1}^{\infty} |f_n|$ is uniformly convergent on D. Prove that uniform absolute convergence implies uniform convergence.

Proof.
Suppose $\displaystyle\sum_{n=1}^{\infty} f_n$ is uniformly absolutely convergent on $D \subset \mathbb{R}$. By definition, $\displaystyle\sum_{n=1}^{\infty} |f_n|$ is uniformly convergent on D. By the Cauchy Criterion for Uniform Convergence, $\forall \epsilon > 0$, $\exists M(\epsilon) \in \mathbb{N}$, such that if $m > n \geq M(\epsilon)$, then

$$\Big| |f_{n+1}(x)| + \ldots + |f_m(x)| \Big| < \epsilon, \quad \forall x \in D.$$

Clearly, since

$$|f_{n+1}(x) + \ldots + f_m(x)| \leq |f_{n+1}(x)| + \ldots + |f_m(x)| = \Big| |f_{n+1}(x)| + \ldots + |f_m(x)| \Big| < \epsilon, \quad \forall x \in D,$$

it follows that $\displaystyle\sum_{n=1}^{\infty} f_n$ is uniformly convergent on D. □