Problem Set 6

Problem 1

Consider the two functions

$$S_a(q, P, t) = \frac{(q-P)^2}{2t},$$
$$S_b(q, P, t) = q\sqrt{2P} - Pt.$$

(a) Verify that both are solutions of the Hamilton-Jacobi equation for the free particle $H = p^2/2$.

(b) Verify that both generators yield the correct solution for $q(t)$ of the free particle.

(c) These generate two different sets of "trivial" canonical coordinates (Q_a, P_a) and (Q_b, P_b). What is the physical significance of each set?

Solution.

(a)

$$\frac{\partial S_a}{\partial t} + H(q, \frac{\partial S_a}{\partial q}, t) = -\frac{(q-P)^2}{2t^2} + \frac{1}{2}\left(\frac{q-P}{t}\right)^2 = 0,$$
$$\frac{\partial S_b}{\partial t} + H(q, \frac{\partial S_b}{\partial q}, t) = -P + \frac{1}{2}\left(\sqrt{2P}\right)^2 = 0.$$

(b)

$$p = \frac{\partial S_a}{\partial q} = \frac{q-P}{t} \implies q = pt + P,$$
$$p = \frac{\partial S_b}{\partial q} = \sqrt{2P} \implies P = \frac{p^2}{2} = E.$$

(c)

$$Q_a = \frac{\partial S_a}{\partial P} = -\frac{(q-P)}{t} = -p,$$
$$Q_b = \frac{\partial S_b}{\partial P} = \frac{q}{\sqrt{2P}} - t = \frac{q}{p} - t.$$

We see that $(Q_a, P_a) = (-p, q - pt)$ and $(Q_b, P_b) = (q/p - t, p^2/2)$. The (Q_a, P_a) are momentum-position coordinates, while (Q_b, P_b) are time-energy coordinates. We can see from (b) that P_a is the initial position, while P_b is the initial energy. Also, $-Q_a$ is the initial velocity, while $-Q_b$ is the initial time.

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Problem 2

Compute the Poisson brackets between the Cartesian components of the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. Show that $\{L_i, L_j\} = \epsilon_{ijk} L_k$. Show also that $\{L_i, H\} = 0$ for any central force field. What does this last statement imply?

Solution. It can be shown by a straightforward computation that

$$\begin{aligned}
\{L_i, L_j\} &= \{\epsilon_{ilk} r_l p_k, \epsilon_{jmn} r_m p_n\} = \epsilon_{ilk}\epsilon_{jmn}\{r_l p_k, r_m p_n\} \\
&= \epsilon_{ilk}\epsilon_{jmn}\left(\frac{\partial}{\partial r^\alpha}(r_l p_k)\frac{\partial}{\partial p^\alpha}(r_m p_n) - \frac{\partial}{\partial p^\alpha}(r_l p_k)\frac{\partial}{\partial r^\alpha}(r_m p_n)\right) \\
&= \epsilon_{ilk}\epsilon_{jmn}\left(\delta_{l\alpha} p_k r_m \delta_{n\alpha} - r_l \delta_{k\alpha}\delta_{m\alpha} p_n\right) \\
&= \epsilon_{i\alpha k}\epsilon_{jm\alpha} r_m p_k - \epsilon_{il\alpha}\epsilon_{j\alpha n} r_l p_n \\
&= -\epsilon_{\alpha ik}\epsilon_{\alpha jm} r_m p_k + \epsilon_{\alpha il}\epsilon_{\alpha jn} r_l p_n \\
&= -\left(\delta_{ij}\delta_{km} - \delta_{im}\delta_{kj}\right) r_m p_k + \left(\delta_{ij}\delta_{ln} - \delta_{in}\delta_{lj}\right) r_l p_n \\
&= -\delta_{ij}\delta_{km} r_m p_k + r_i p_j + \delta_{ij}\delta_{ln} r_l p_n - r_j p_i \\
&= r_i p_j - r_j p_i = \epsilon_{ijk}\epsilon_{klm} r_l p_m \\
&= \epsilon_{ijk} L_k.
\end{aligned}$$

We could have also used the Leibniz rule to simplify some steps. The Hamiltonian of a central force field has the form

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + V(r).$$

Now given an arbitrary p_l, we have

$$\begin{aligned}
\{L_i, p_l^2\} &= \epsilon_{ijk}\{r_j p_k, p_l^2\} \\
&= \epsilon_{ijk}\left(r_j\{p_k, p_l^2\} + \{r_j, p_l^2\} p_k\right) \\
&= \epsilon_{ijk}\left(r_j\{p_k, p_l\} p_l + r_j p_l \{p_k, p_l\} + \{r_j, p_l\} p_l p_k + p_l \{r_j, p_l\} p_k\right) \\
&= \epsilon_{ijk}\left(2\delta_{jl} p_l p_k\right) \\
&= 2\epsilon_{ijk} p_j p_k \\
&= 0,
\end{aligned}$$

since ϵ_{ijk} is antisymmetric while $p_j p_k$ is symmetric. Now,

$$\begin{aligned}
\{L_i, V(r)\} &= \epsilon_{ijk}\{r_j p_k, V(r)\} \\
&= \epsilon_{ijk}\left(\frac{\partial}{\partial r_\alpha}(r_j p_k)\frac{\partial}{\partial p_\alpha}V(r) - \frac{\partial}{\partial p_\alpha}(r_j p_k)\frac{\partial}{\partial r_\alpha}V(r)\right) \\
&= \epsilon_{ijk}\left(-\delta_{\alpha k} r_j \frac{\partial}{\partial r_\alpha}V(r)\right) \\
&= -\epsilon_{ijk} r_j \frac{\partial}{\partial r_k}V(r) \\
&= -\left(\mathbf{r} \times \nabla V(r)\right)_i \\
&= 0,
\end{aligned}$$

since $\mathbf{r}$ is always parallel to the force $\mathbf{F}(r) := -\nabla V(r)$ for a central potential. $\square$

Problem 3

Consider the transformation

$$Q = q + \frac{1}{2}gt^2, \quad P = p + mgt,$$

where g and m are constants. Is this transformation canonical? If no, explain why. If yes, find the new Hamiltonian $K(Q, P)$.

Solution. If the transformation is canonical then $\{Q, P\}_{(q,p)} = 1$. Now,

$$\{Q, P\} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = 1.$$

The transformation is canonical as expected, since the new coordinates are just translations of their respective old coordinates. Consider a generating function $F_2(q, P, t)$.

$$\begin{aligned}
\frac{\partial F_2}{\partial P} &= Q = q + \frac{1}{2}gt^2 \\
F_2 &= qP + \frac{1}{2}gt^2 P + F(q,t) \\
\frac{\partial F_2}{\partial q} &= P + \partial_q F(q,t) = p \\
\partial_q F(q,t) &= -mgt \\
F(q,t) &= -mgqt + G(t).
\end{aligned}$$

The new Hamiltonian is

$$\begin{aligned}
K &= H + \partial_t F_2 = H - mgq \\
K(Q,P,t) &= \hat{H}(Q,P) - mgQ + \frac{1}{2}mg^2t^2.
\end{aligned}$$

□

Problem 4

Consider the Hamiltonian

$$H = \frac{tp^2}{2} - \frac{q^3}{3t^4}.$$

(a) What is the corresponding Lagrangian? Write down the Euler-Lagrange equation for $q(t)$. (This should be a second-order nonlinear ordinary differential equation.)

(b) Show that the transformation $Q = t^s q$, $P = t^{-s} p$ is canonical for any s. What is the new Hamiltonian $K(Q, P)$?

(c) Choose s so that a constant of motion is easy to spot from $K(Q, P)$. Use this constant of motion to reduce the solution of the equations of motion to quadrature.

Solution.

(a) From Hamilton's equation we find $\dot{q} = \partial H/\partial p = tp$. The Lagrangian is given by

$$\begin{aligned} L(q, \dot{q}, t) &= p\dot{q} - H \\ &= \frac{tp^2}{2} + \frac{q^3}{3t^4} \\ &= \frac{\dot{q}^2}{2t} + \frac{q^3}{3t^4}. \end{aligned}$$

The Euler-Lagrange equation is simply

$$\begin{aligned} \frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) &= 0 \\ \frac{q^2}{t^4} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{q}}{t}\right) &= 0 \\ \frac{q^2}{t^4} - \frac{\ddot{q}}{t} + \frac{\dot{q}}{t^2} &= 0 \\ \ddot{q} - \frac{\dot{q}}{t} - \frac{q^2}{t^3} &= 0. \end{aligned}$$

(b) Observe that

$$\{Q, P\}_{(q,p)} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = t^s t^{-s} = 1.$$

Therefore, the transformation is canonical for any s. The new Hamiltonian is simply

$$K(Q, P, t) = H(q(Q, P, t), p(Q, P, t), t) + \frac{\partial F_2}{\partial t},$$

for a valid generator $F_2(q, P, t)$. We now solve for F_2,

$$\begin{aligned} \frac{\partial F_2}{\partial P} &= Q = t^s q \\ F_2 &= t^s qP + F(q, t) \\ \frac{\partial F_2}{\partial q} = t^s P + \partial_q F(q, t) = p + \partial_q F(q, t) &= p \\ \partial_q F(q, t) &= 0 \\ F(q, t) = G(t) F_2 &= t^s qP + G(t) \end{aligned}$$

The new Hamiltonian is now

$$K(Q,P,t) = \frac{1}{2}t^{2s+1}P^2 - \frac{1}{3}t^{-(3s+4)}Q^3 + st^{-1}QP.$$

(c) Note that for $s = -1$, we get

$$K(Q,P,t) = \left(\frac{1}{2}P^2 - \frac{1}{3}Q^3 - QP\right)t^{-1}.$$

Consider the function

$$F(Q,P) = \frac{1}{2}P^2 - \frac{1}{3}Q^3 - QP.$$

It can be easily checked that $\{F,K\} = 0$; therefore it is a constant of motion which we shall call $\tilde{F}$. Now,

$$\begin{aligned}\tilde{F} &= \frac{1}{2}t^2p^2 - \frac{1}{3}t^{-3}q^3 - qp \\ &= \frac{1}{2}\dot{q}^2 - \dot{q}\left(\frac{q}{t}\right) - \frac{1}{3}\left(\frac{q}{t}\right)^3.\end{aligned}$$

This can be solved numerically.

$\square$

Problem 5

The Hamiltonian for a charged particle in a uniform magnetic field $\mathbf{B} = B_0\hat{\mathbf{k}}$ is given by

$$H = \frac{1}{2m}\left[\left(p_x + \frac{1}{2}eB_0 y\right)^2 + \left(p_y - \frac{1}{2}eB_0 x\right)^2 + p_z^2\right].$$

(a) Solve the Hamilton-Jacobi equation for S.

(b) Determine the phase space transformation generated by S.

(c) Solve for $x(t)$ and $y(t)$. Interpret your solution.

Solution.

(a) The Hamilton-Jacobi equation takes the form

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial x} + \frac{1}{2}eB_0 y\right)^2 + \left(\frac{\partial S}{\partial y} - \frac{1}{2}eB_0 x\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2\right] + \frac{\partial S}{\partial t} = 0$$

Since H does not explicitly depend on time and z is a cyclic coordinate, we have

$$S = -Et + P_z z + W(x, y).$$

Plugging this in the equation, we get

$$\begin{aligned}\frac{1}{2m}\left[\left(\frac{\partial W}{\partial x} + \frac{1}{2}eB_0 y\right)^2 + \left(\frac{\partial W}{\partial y} - \frac{1}{2}eB_0 x\right)^2 + P_z^2\right] &= E \\ \left(\frac{\partial W}{\partial x} + \frac{1}{2}eB_0 y\right)^2 + \left(\frac{\partial W}{\partial y} - \frac{1}{2}eB_0 x\right)^2 &= 2mE - P_z^2\end{aligned}$$

If we impose

$$\frac{\partial W}{\partial y} = -\frac{1}{2}eB_0 x + P_y,$$

then

$$W(x, y) = -\frac{1}{2}eB_0 xy + P_y y + U(x).$$

Plugging this ansatz into the Hamilton-Jacobi equation, we get

$$\begin{aligned}(U'(x))^2 + (P_y - eB_0 x)^2 &= 2mE - P_z^2 \\ U'(x) &= \sqrt{2mE - P_z^2 - (P_y - eB_0 x)^2} \\ U(x) &= \int \sqrt{2mE - P_z^2 - (P_y - eB_0 x)^2}\mathrm{d}x.\end{aligned}$$

The generating function is therefore

$$S = -Et - \frac{1}{2}eB_0 xy + P_y y + P_z z + \int \sqrt{2mE - P_z^2 - (P_y - eB_0 x)^2}\mathrm{d}x.$$

(b) The phase space transformations are

$$\begin{aligned}
\beta_1 &= \frac{\partial S}{\partial E} = -t + 2m \int \frac{1}{\sqrt{2mE - P_z^2 - (P_y - eB_0 x)^2}} \mathrm{d}x \\
\beta_2 &= \frac{\partial S}{\partial P_z} = z + 2P_z \int \frac{1}{\sqrt{2mE - P_z^2 - (P_y - eB_0 x)^2}} \mathrm{d}x \\
\beta_3 &= \frac{\partial S}{\partial P_y} = y - 2 \int \frac{P_y - eB_0 x}{\sqrt{2mE - P_z^2 - (P_y - eB_0 x)^2}} \mathrm{d}x \\
p_x &= \frac{\partial S}{\partial x} = -\frac{1}{2} eB_0 y + \sqrt{2mE - P_z^2 - (P_y - eB_0 x)^2} \\
p_y &= \frac{\partial S}{\partial y} = -\frac{1}{2} eB_0 x + P_y \\
p_z &= \frac{\partial S}{\partial z} = P_z.
\end{aligned}$$

□