Problem 1

[Problem 1 in Section 3.13 of Poisson's Toolkit]

Consider the hypersurface in Schwarzschild spacetime defined by constant

$$T = t + 4M\left[\sqrt{r/2M} + \frac{1}{2}\ln\left(\frac{\sqrt{r/2M}-1}{\sqrt{r/2M}+1}\right)\right].$$

(a) We know that $n \propto \mathrm{d}T$. Now, a long computation yields $\mathrm{d}T = \mathrm{d}t + \sqrt{1-f}/f\,\mathrm{d}r$. This hypersurface is spacelike and $\mathrm{d}T$ is already normalized, so we take $n = -\mathrm{d}T = -\mathrm{d}t - \sqrt{1-f}/f\,\mathrm{d}r$. Let $T = T_0$. Then this surface is parametrized by $y^a = (r, \theta, \phi)$, where

$$t(r) = T_0 - 4M\left[\sqrt{r/2M} + \frac{1}{2}\ln\left(\frac{\sqrt{r/2M}-1}{\sqrt{r/2M}+1}\right)\right].$$

(b) The induced metric is given by

$$h_{ab} = g_{\alpha\beta}\, e^\alpha_a e^\beta_b = g_{\alpha\beta}\,\frac{\partial x^\alpha}{\partial y^a}\frac{\partial x^\beta}{\partial y^b}$$

Note that $e^\alpha_c = (\partial t/\partial y^c, \delta^a_c)$, where $x^\alpha = (t(r), y^a)$. It can be easily shown that $h_{ab} = \mathrm{diag}(1, r^2, r^2\sin^2\theta)$ and $h^{ab} = \mathrm{diag}(1, 1/r^2, \csc^2\theta/r^2)$.

(c) The extrinsic curvature is given by $K_{ab} = n_{\alpha;\beta}\, e^\alpha_a e^\beta_b$. Observe that

$$K_{ab} = n_{t;t}\frac{\partial t}{\partial y^a}\frac{\partial t}{\partial y^b} + 2n_{t;c}\,\frac{\partial t}{\partial y^a}\delta^c_b + n_{c;d}\,\delta^c_a\delta^d_b = n_{t;t}\frac{\partial t}{\partial y^a}\frac{\partial t}{\partial y^b} + 2n_{t;b}\,\frac{\partial t}{\partial y^a} + n_{a;b}.$$

The first two terms are only nonzero for K_{rr}, where we have

$$\begin{aligned} K_{rr} &= -\Gamma^r{}_{tt}n_r\frac{\partial t}{\partial r}\frac{\partial t}{\partial r} - 2\Gamma^t{}_{tr}\frac{\partial t}{\partial r} + n_{r,r} - \Gamma^r{}_{rr}n_r \\ &= n_{r,r} + (2\Gamma^t{}_{tr} - \Gamma^r{}_{rr})n_r - \Gamma^r{}_{tt}\,(n_r)^3 \\ &= \frac{1}{2}\sqrt{\frac{2M}{r^3}}. \end{aligned}$$

The other nonzero terms of K_{ab} are:

$$\begin{aligned} K_{\theta\theta} &= n_{\theta;\theta} = -\Gamma^r{}_{\theta\theta}n_r = -\sqrt{2Mr}, \\ K_{\phi\phi} &= n_{\phi;\phi} = -\Gamma^r{}_{\phi\phi}n_r = -\sqrt{2Mr}\sin^2\theta. \end{aligned}$$

Now, $K = h^{ab}K_{ab} = h^{rr}K_{rr} + h^{\theta\theta}K_{\theta\theta} + h^{\phi\phi}K_{\phi\phi} = -\dfrac{3}{2}\sqrt{\dfrac{2M}{r^3}}$.

(d) Since $\mathrm{d}T = \mathrm{d}t + \sqrt{1-f}/f\mathrm{d}r$, then

$$\begin{aligned}
\mathrm{d}s^2 &= -f\mathrm{d}t^2 + \frac{1}{f}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 = -f\left(\mathrm{d}T - \frac{\sqrt{1-f}}{f}\mathrm{d}r\right)^2 + \frac{1}{f}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \\
&= -f\mathrm{d}T^2 + 2\sqrt{1-f}\,\mathrm{d}T\,\mathrm{d}r + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \\
&= -\mathrm{d}T^2 + (1-f)\mathrm{d}T^2 + 2\sqrt{1-f}\,\mathrm{d}T\,\mathrm{d}r + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \\
&= -\mathrm{d}T^2 + \left(\sqrt{1-f}\,\mathrm{d}T + \mathrm{d}r\right)^2 + r^2\mathrm{d}\Omega^2 \\
&= -\mathrm{d}T^2 + \left(\sqrt{\frac{2M}{r}}\,\mathrm{d}T + \mathrm{d}r\right)^2 + r^2\mathrm{d}\Omega^2.
\end{aligned}$$

□

Problem 2

[Problem 2 in Section 3.13 of Poisson's Toolkit]

Note that the hypersurface can also be defined by $\Phi(z^A) := -(z^0)^2 + \sum_{i=1}^4 (z^i)^2 - a^2 = 0$.

(a) Since $n \propto \mathrm{d}\Phi$ we find that, upon normalization,

$$n_A \mathrm{d}z^A = \frac{1}{a}\left(-z^0 \mathrm{d}z^0 + \sum_{i=1}^{4} z^i \mathrm{d}z^i\right).$$

The tangent vectors are:

$$e_t = \begin{pmatrix} \cosh(t/a) \\ \sinh(t/a)\cos\chi \\ \sinh(t/a)\sin\chi\cos\theta \\ \sinh(t/a)\sin\chi\sin\theta\cos\phi \\ \sinh(t/a)\sin\chi\sin\theta\sin\phi \end{pmatrix}, \quad e_\chi = \begin{pmatrix} 0 \\ -a\cosh(t/a)\sin\chi \\ a\cosh(t/a)\cos\chi\cos\theta \\ a\cosh(t/a)\cos\chi\sin\theta\cos\phi \\ a\cosh(t/a)\cos\chi\sin\theta\sin\phi \end{pmatrix},$$

$$e_\theta = \begin{pmatrix} 0 \\ 0 \\ -a\cosh(t/a)\sin\chi\sin\theta \\ a\cosh(t/a)\sin\chi\cos\theta\cos\phi \\ a\cosh(t/a)\sin\chi\cos\theta\sin\phi \end{pmatrix}, \quad e_\phi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -a\cosh(t/a)\sin\chi\sin\theta\sin\phi \\ a\cosh(t/a)\sin\chi\sin\theta\cos\phi \end{pmatrix}.$$

(b) The induced metric is given by $g_{\alpha\beta} = \eta_{AB} e^A_\alpha e^B_\beta$. A long computation yields $g_{\alpha\beta} = \mathrm{diag}(-1, a^2\cosh^2(t/a), a^2\cosh^2(t/a)\sin^2\chi, a^2\cosh^2(t/a)\sin^2\chi\sin^2\theta)$. The spatial part defines a 3-sphere define with radius $r = a\cosh(t/a)$. The induced metric satisfies the vacuum Einstein field equations with cosmological constant $\Lambda = 3/a^2$.

(c) The extrinsic curvature is given by $K_{\alpha\beta} = n_{(A;B)} e^A_\alpha e^B_\beta$. By inspection, we see that $n_{(A;B)} = a^{-1}\eta_{AB}$. Thus, $K_{\alpha\beta} = a^{-1} g_{\alpha\beta}$. Since the ambient spacetime is flat, $R_{ABCD} = 0$. Thus, from the Gauss-Codazzi equations, the induced Riemann tensor is given by

$$R_{\alpha\beta\gamma\delta} = K_{\alpha\gamma}K_{\beta\delta} - K_{\alpha\delta}K_{\beta\gamma} = \frac{1}{a^2}\left(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\right). \qquad \square$$

Problem 3

[Problem 3 in Section 3.13 of Poisson's Toolkit]

(a) Since $\mathrm{d}s^2 = \mathrm{d}l^2 + r^2(l)\mathrm{d}\Omega^2 = \left(1 - \frac{2m(r)}{r}\right)^{-1} \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$, we have

$$\mathrm{d}l^2 = \left(1 - \frac{2m(r)}{r}\right)^{-1} \mathrm{d}r^2 \implies \left(1 - \frac{2m(r)}{r}\right) = \frac{\mathrm{d}r^2}{\mathrm{d}l^2} \implies m(r) = \frac{r}{2}\left[1 - \left(\frac{\mathrm{d}r}{\mathrm{d}l}\right)^2\right].$$

(b) At a moment of time symmetry,

$$m(r) = \int_0^r 4\pi r'^2 \rho(r')\mathrm{d}r' = 4\pi\rho \int_0^r r'^2 \mathrm{d}r' = \frac{4}{3}\pi\rho r^3.$$

Therefore, we can obtain the differential equation

$$\frac{\mathrm{d}l}{\mathrm{d}r} = \frac{1}{\sqrt{1 - 8\pi\rho r^2/3}}.$$

Solving for $l(r)$, and inverting, we have

$$r(l) = \sqrt{\frac{3}{8\pi\rho}} \sin\left(\sqrt{\frac{8\pi\rho}{3}}\, l + C\right).$$

Imposing the asymptotic condition and using $\sin x \approx x$. We get $C = 0$.

(c) Since $\sin x \leq 1$, $\forall x$, then $r(l) \leq \sqrt{3/(8\pi\rho)} =: r_{\text{max}}$, $\forall r$.

(d) Clearly, at $r = r_{\text{max}}$ we have $\mathrm{d}r/\mathrm{d}l = 0$. Thus, $2m(r_{\text{max}}) = r_{\text{max}}$. It is easy to see that $m(r_{\text{max}})$ is the maximum value of the mass function, since $(\mathrm{d}r/\mathrm{d}l)^2 \geq 0$.

(e) When $l \to \pi r_{\text{max}}$, we have $r \to 0$. Thus, the metric becomes singular.

Problem 4

Show that the surface gravity κ of the event horizon of a Kerr black hole of mass M and angular momentum J is given by

$$\kappa = \frac{\sqrt{M^4 - J^2}}{2M\left(M^2 + \sqrt{M^4 - J^2}\right)}.$$

Solution.
The surface gravity is defined by $2\kappa\xi_\alpha = \left(-\xi_\beta\xi^\beta\right)_{;\alpha}$, where $\xi^\alpha = t^\alpha + \Omega_H\phi^\alpha$. The squared norm of ξ^α is given by

$$\xi_\beta\xi^\beta = g_{\alpha\beta}\xi^\alpha\xi^\beta = -\frac{\rho^2\Delta}{\Sigma} + \frac{\Sigma}{\rho^2}\sin^2\theta\left(\Omega_H - \omega\right)^2,$$

where we used the second expression for the metric in eq. (5.44). Now,

$$\left(-\xi_\beta\xi^\beta\right)_{;\alpha} = \Delta\partial_\alpha\left(\frac{\rho^2}{\Sigma}\right) + \frac{\rho^2}{\Sigma}\partial_\alpha\Delta - \partial_\alpha\left(\frac{\Sigma}{\rho^2}\sin^2\theta\right)\left(\Omega_H - \omega\right)^2 - \frac{2\Sigma}{\rho^2}\sin^2\theta\left(\Omega_H - \omega\right)\partial_\alpha\left(\Omega_H - \omega\right).$$

At the horizon, $\omega = \Omega_H$ and $\Delta = 0$. Therefore,

$$\left(-\xi_\beta\xi^\beta\right)_{;\alpha} = \frac{\rho^2}{\Sigma}\partial_\alpha\Delta = \frac{\rho^2}{\Sigma}(2r - 2M)\Big|_{r=r_+}\partial_\alpha r = \frac{2\rho_+^2}{\Sigma_+}(r_+ - M)\delta^r{}_\alpha.$$

Note that r is the only nonzero component, so we find ξ_r. Since the Boyer-Lindquist coordinates are singular on the horizon, we use the ingoing congruence. In these coordinates, $\xi_r = g_{rv} + \Omega_H g_{r\psi} = 1 - a\Omega_H\sin^2\theta$. Thus,

$$\begin{aligned}
\kappa &= \frac{\rho_+^2\left(r_+ - M\right)}{\Sigma_+\left(1 - a\Omega_H\sin^2\theta\right)} = \frac{\left(r_+^2 + a^2\cos^2\theta\right)\left(r_+ - M\right)}{\left(r_+^2 + a^2\right)^2\left(1 - a^2\sin^2\theta/(r_+^2 + a^2)\right)} \\
&= \frac{\left(r_+^2 + a^2\cos^2\theta\right)\left(r_+ - M\right)}{\left(r_+^2 + a^2\right)\left(r_+^2 + a^2 - a^2\sin^2\theta\right)} = \frac{(r_+ - M)}{(r_+^2 + a^2)} \\
&= \frac{\sqrt{M^2 - a^2}}{2M^2 + 2M\sqrt{M^2 - a^2}} = \frac{\sqrt{M^4 - J^2}}{2M\left(M^2 + \sqrt{M^4 - J^2}\right)}.
\end{aligned}$$

□

Problem 5

Show using Hawking's area theorem that the maximum energy that can be extracted from the coalescence of two Schwarzschild black holes of equal mass is approximately 0.29 of the original mass.

Solution.
Consider two Schwarzschild black holes of equal mass M. Clearly, they both have area $A = 16\pi M^2$. Now, suppose the two black holes coalesce. Then the mass of the resulting black hole is $M' = 2M - \varepsilon$, where ε is the (extracted) radiated energy. The efficiency is then given by

$$\eta = \frac{E_{\text{out}}}{E_{\text{in}}} = \frac{\varepsilon}{2M} = 1 - \frac{M'}{2M}.$$

The resulting black hole may be spinning, so its area is bounded by its Schwarzschild limit; $A' \leq 16\pi M'^2$. By Hawking's area theorem, $A' \geq 2A = 32\pi M^2$. Thus, we have $M'^2 \geq 2M^2 \Rightarrow M' \geq \sqrt{2}M$. It follows that $\eta \leq 1 - 1/\sqrt{2} \approx 0.29$.

Problem 6

[Problem 4 in Section 5.7 of Poisson's Toolkit]

(a) Observe that

$$\left(\xi_{\alpha\beta}u^\alpha u^\beta\right)_{;\gamma} u^\gamma = \xi_{\alpha\beta;\gamma}u^\alpha u^\beta u^\gamma + \xi_{\alpha\beta}u^\alpha{}_{;\gamma}u^\beta u^\gamma + \xi_{\alpha\beta}u^\alpha u^\beta{}_{;\gamma}u^\gamma = \xi_{\alpha\beta;\gamma}u^\alpha u^\beta u^\gamma.$$

Now, permuting the indices gives the same quantity on the LHS, but not on the RHS. After summing the permutations and rearranging the RHS we get

$$\left(\xi_{\alpha\beta}u^\alpha u^\beta\right)_{;\gamma} u^\gamma = \xi_{(\alpha\beta;\gamma)}u^\alpha u^\beta u^\gamma = 0.$$

(b) This was verified using Mathematica.

(c) Clearly, $-\tilde{E} = g_{t\alpha}u^\alpha = g_{tt}\dot{t} + g_{t\phi}\dot{\phi}$ and $\tilde{L} = g_{\phi\alpha}u^\alpha = g_{\phi t}\dot{t} + g_{\phi\phi}\dot{\phi}$. We can write this as a matrix equation $\mathbf{x} = \mathbf{g}\cdot\mathbf{u}$, where $\mathbf{x} = (-\tilde{E}, A, B, \tilde{L})$, $\mathbf{g}$ is the matrix form of the Kerr metric, and A, B defined appropriately. Inverting this equation, we have $\mathbf{u} = \mathbf{g}^{-1}\cdot\mathbf{x}$. Thus,

$$\dot{t} = -g^{tt}\tilde{E} + g^{t\phi}\tilde{L} = \frac{1}{\rho^2\Delta}\left(\Sigma\tilde{E} - 2Mar\tilde{L}\right) = \frac{1}{\rho^2}\left(-a(a\tilde{E}\sin^2\theta - L) + (r^2+a^2)\frac{P}{\Delta}\right),$$

$$\dot{\phi} = -g^{\phi t}\tilde{E} + g^{\phi\phi}\tilde{L} = \frac{1}{\rho^2\Delta}\left(2Mar\tilde{E} + \frac{\Delta - a^2\sin^2\theta}{\sin^2\theta}\tilde{L}\right) = \frac{1}{\rho^2}\left(-(a\tilde{E} - \frac{\tilde{L}}{\sin^2\theta}) + \frac{aP}{\Delta}\right).$$

(d) First, note that $k_\alpha = (-1, \rho^2/\Delta, 0, a\sin^2\theta)$ and $l_\alpha = (-1, -\rho^2/\Delta, 0, a\sin^2\theta)$. Also, $\xi_{\alpha\beta}u^\alpha u^\beta = \left(\Delta k_{(\alpha}l_{\beta)} + r^2 g_{\alpha\beta}\right)u^\alpha u^\beta = \Delta k_\alpha u^\alpha l_\beta u^\beta + \epsilon r^2$, where $\epsilon = 0(-1)$ for null (timelike) geodesics. Therefore,

$$\begin{aligned}
\xi_{\alpha\beta}u^\alpha u^\beta = \mathcal{C} + (\tilde{L} - a\tilde{E})^2 &= \Delta\left(-\dot{t} + \frac{\rho^2}{\Delta}\dot{r} + a\sin^2\theta\dot{\phi}\right)\left(-\dot{t} - \frac{\rho^2}{\Delta}\dot{r} + a\sin^2\theta\dot{\phi}\right) + \epsilon r^2 \\
&= \Delta\left(\left(\dot{t} - a\sin^2\theta\dot{\phi}\right)^2 - \frac{\rho^4}{\Delta^2}\dot{r}^2\right) + \epsilon r^2 \\
\rho^4\dot{r}^2 &= \Delta^2\left(\dot{t} - a\sin^2\theta\dot{\phi}\right)^2 - \Delta\left[\mathcal{C} + (\tilde{L} - a\tilde{E})^2 - \epsilon r^2\right] \\
&= P^2 - \Delta\left[\mathcal{C} + (\tilde{L} - a\tilde{E})^2\right] + \epsilon\Delta r^2 = R + \epsilon\Delta r^2.
\end{aligned}$$

(e) Finally,

$$\begin{aligned}
\epsilon = g_{\alpha\beta}u^\alpha u^\beta &= g_{tt}\dot{t}^2 + 2g_{t\phi}\dot{t}\dot{\phi} + g_{\phi\phi}\dot{\phi}^2 + g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 \\
&= \dot{t}(g_{tt}\dot{t} + g_{t\phi}\dot{\phi}) + \dot{\phi}(g_{t\phi}\dot{t} + g_{\phi\phi}\dot{\phi}) + g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 \\
&= -\tilde{E}\dot{t} + \tilde{L}\dot{\phi} + \frac{\rho^2}{\Delta}\dot{r}^2 + \rho^2\dot{\theta}^2.
\end{aligned}$$

Now, it can be easily shown that

$$-\tilde{E}\dot{t} + \tilde{L}\dot{\phi} = -\frac{1}{\rho^2}\left(\frac{R}{\Delta} + \mathcal{C} + a^2\tilde{E}^2\cos^2\theta - \tilde{L}^2\cot^2\theta - \epsilon r^2\right).$$

Thus,

$$\begin{aligned}\rho^4\dot{\theta}^2 &= \mathcal{C} + \left(a^2\tilde{E}^2\cos^2\theta - \tilde{L}^2\cot^2\theta\right) + \epsilon(\rho^2 - r^2)\\ &= \mathcal{C} + \cos^2\theta\left(a^2\tilde{E}^2 - \frac{\tilde{L}^2}{\sin^2\theta}\right) + \epsilon a^2\cos^2\theta = \Theta + \epsilon a^2\cos^2\theta.\end{aligned}$$

□

Problem 7

[Problem 5 in Section 5.7 of Poisson's Toolkit]

(a) By definition, l^α is null with respect to $\eta_{\alpha\beta}$. Now, $g_{\alpha\beta}l^\alpha l^\beta = \eta_{\alpha\beta}l^\alpha l^\beta + Hl_\alpha l^\alpha l_\beta l^\beta = 0$.

(b) Denote the inverse metric by $g^{\alpha\beta} = \eta^{\alpha\beta} + \zeta^{\alpha\beta}$, where $\zeta^{\alpha\beta}$ is to be determined. Then, $g^{\alpha\beta}g_{\beta\gamma} = \left(\eta^{\alpha\beta} + \zeta^{\alpha\beta}\right)\left(\eta_{\beta\gamma} + Hl_\beta l_\gamma\right) = \delta^\alpha{}_\gamma$. Let η be the index operator, then we have $\eta^{\alpha\beta}Hl_\beta l_\gamma + \zeta^{\alpha\beta}\eta_{\beta\gamma} + H\zeta^{\alpha\beta}l_\beta l_\gamma = 0$. We can rewrite this as

$$\zeta^{\alpha\beta}\left(\eta_{\beta\gamma} + Hl_\beta l_\gamma\right) = -Hl^\alpha l^\beta \eta_{\beta\gamma} = -Hl^\alpha l^\beta\left(\eta_{\beta\gamma} + Hl_\beta l_\gamma\right),$$

where we added a term $-H^2 l^\alpha l^\beta l_\beta l_\gamma = 0$. Thus, $\zeta^{\alpha\beta} = -Hl^\alpha l^\beta$ and the inverse metric is $g^{\alpha\beta} = \eta^{\alpha\beta} - Hl^\alpha l^\beta$.

(c) Clearly, $g_{\alpha\beta}l^\beta = \eta_{\alpha\beta}l^\beta + Hl_\alpha l_\beta l^\beta = l_\alpha$ and $g^{\alpha\beta}l_\beta = \eta^{\alpha\beta}l_\beta - Hl^\alpha l^\beta l_\beta = l^\alpha$.

(d) The Christoffel symbols are given by

$$\Gamma^\gamma{}_{\alpha\beta} = \frac{1}{2}g^{\gamma\delta}\left(g_{\delta\beta,\alpha} + g_{\alpha\delta,\beta} - g_{\alpha\beta,\delta}\right).$$

Note that $g_{\alpha\beta,\gamma} = H_{,\gamma}l_\alpha l_\beta + Hl_{\alpha,\gamma}l_\beta + Hl_\alpha l_{\beta,\gamma}$. Thus,

$$\begin{aligned}\Gamma^\gamma{}_{\alpha\beta} = \frac{1}{2}g^{\gamma\delta}\big(&H_{,\alpha}l_\delta l_\beta + Hl_{\delta,\alpha}l_\beta + Hl_\delta l_{\beta,\alpha} + H_{,\beta}l_\alpha l_\delta + Hl_{\alpha,\beta}l_\delta + Hl_\alpha l_{\delta,\beta}\\ &-H_{,\delta}l_\alpha l_\beta - Hl_{\alpha,\delta}l_\beta - Hl_\alpha l_{\beta,\delta}\big).\end{aligned}$$

Since l^α satisfies the geodesic equation in flat spacetime, we can replace the partial derivatives of l^α to covariant derivatives. Also, $l^\alpha l_{\alpha|\beta} = 1/2\left(l^\alpha l_\alpha\right)_{|\beta} = 0$. Therefore, we can also replace $g^{\gamma\delta}$ by $\eta^{\gamma\delta}$, where $|$ denotes the flat covariant derivative. Thus,

$$\begin{aligned}\Gamma^\gamma{}_{\alpha\beta} = \frac{1}{2}\Big(&H_{,\alpha}l^\gamma l_\beta + Hl^\gamma{}_{|\alpha}l_\beta + Hl^\gamma l_{\beta|\alpha} + H_{,\beta}l_\alpha l^\gamma + Hl_{\alpha|\beta}l^\gamma + Hl_\alpha l^\gamma{}_{|\beta}\\ &-H^{,\gamma}l_\alpha l_\beta - Hl_\alpha{}^{|\gamma}l_\beta - Hl_\alpha l_\beta{}^{|\gamma}\Big).\end{aligned}$$

It follows that

$$\begin{aligned}l_\gamma\Gamma^\gamma{}_{\alpha\beta} &= -\frac{1}{2}H^{,\gamma}l_\gamma l_\alpha l_\beta = -\frac{1}{2}\dot{H}l_\alpha l_\beta\,,\\ l^\alpha\Gamma^\gamma{}_{\alpha\beta} &= \quad\frac{1}{2}H_{,\alpha}l^\alpha l^\gamma l_\beta = \quad\frac{1}{2}\dot{H}l^\gamma l_\beta\,.\end{aligned}$$

(e) Now, $l^\alpha{}_{;\beta}l^\beta = l^\alpha{}_{,\beta}l^\beta + \Gamma^\alpha{}_{\beta\gamma}l^\gamma l^\beta = l^\alpha{}_{|\beta}l^\beta + 1/2\ \dot{H}l^\alpha l_\beta l^\beta = 0$.

(f) The Ricci tensor is given by $R_{\alpha\beta} = R^\gamma{}_{\alpha\gamma\beta} = \Gamma^\gamma{}_{\alpha\beta,\gamma} - \Gamma^\gamma{}_{\alpha\gamma,\beta} + \Gamma^\gamma{}_{\gamma\delta}\Gamma^\delta{}_{\alpha\beta} - \Gamma^\gamma{}_{\beta\delta}\Gamma^\delta{}_{\alpha\gamma}$. Observe that $\Gamma^\gamma{}_{\gamma\delta} = 0$, and $\Gamma^\gamma{}_{\beta\delta}\Gamma^\delta{}_{\alpha\gamma}l^\alpha l^\beta = 0$. Therefore,

$$R_{\alpha\beta}l^\alpha l^\beta = \Gamma^\gamma{}_{\alpha\beta,\gamma}l^\alpha l^\beta = \partial_\gamma\left(\Gamma^\gamma{}_{\alpha\beta}l^\alpha l^\beta\right) - \Gamma^\gamma{}_{\alpha\beta}l^\alpha{}_{|\gamma}l^\beta - \Gamma^\gamma{}_{\alpha\beta}l^\alpha l^\beta{}_{|\gamma} = 0. \qquad \square$$