Problem 1

Suppose that air resistance near the surface of the Earth is proportional to velocity. With the usual Cartesian coordinates (x, y) as generalized coordinates, (a) determine the dissipation function $\mathcal{F}$ for the air resistance on a projectile of mass m, and (b) write down the appropriate equations of motion using this $\mathcal{F}$. How does this compare with what you would have derived for the equations of motion starting instead from $\mathbf{F} = m\mathbf{a}$?

Solution.

(a) The air resistance is given by $\mathbf{F}_\mathrm{d} = -k\mathbf{v}$. Clearly, $h(v) = kv$. Therefore,

$$\mathcal{F} = \int_0^v h(v')\,\mathrm{d}v' = \frac{1}{2}kv^2 = \frac{1}{2}k\left(\dot{x}^2 + \dot{y}^2\right).$$

(b) The generalized forces are

$$Q_x = -\frac{\partial \mathcal{F}}{\partial \dot{x}} = -k\dot{x}, \quad \text{and} \quad Q_y = -\frac{\partial \mathcal{F}}{\partial \dot{y}} = -k\dot{y}.$$

The equations of motion are then simply

$$m\ddot{x} + k\dot{x} = 0, \quad \text{and} \quad m\ddot{y} + k\dot{y} + mg = 0;$$

exactly what we expect from using $\mathbf{F} = m\mathbf{a}$.

□

Problem 2

The electromagnetic field is invariant under a gauge transformation of the scalar and vector potential given by

$$\mathbf{A} \to \mathbf{A} + \nabla\psi(t, \mathbf{r}),\ \phi \to \phi - \frac{1}{c}\frac{\partial \psi}{\partial t},$$

where ψ is a differentiable function. What effect does this gauge transformation have on the Lagrangian of a particle moving in the electromagnetic field? Is the motion affected?

Solution. We follow the discussion on Jose and Saletan, Section 2.2.4. The Lagrangian is given by (Equation 2.49)

$$L = \frac{1}{2}m\dot{x}^2 - e\phi(\mathbf{x}, t) + \frac{e}{c}\dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}, t).$$

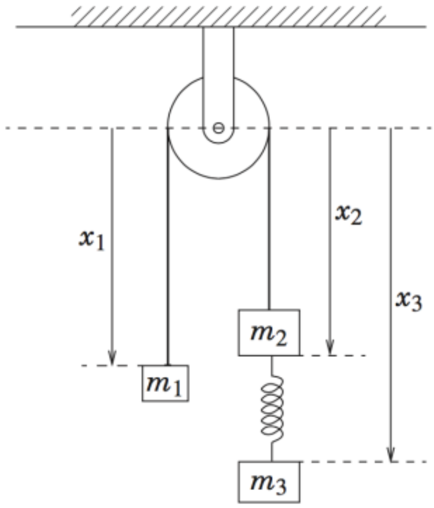
Under a gauge transformation, the Lagrangian becomes

$$L \to \frac{1}{2}m\dot{x}^2 - e\phi(\mathbf{x}, t) + e\dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}, t) + \frac{e}{c}\left(\frac{\partial \psi}{\partial t} + \dot{x} \cdot \nabla\psi(t, \mathbf{r})\right) \to L + \frac{e}{c}\frac{\mathrm{d}\psi}{\mathrm{d}t}.$$

Since the Lagrangian differs only by a total time derivative, the motion is not affected. □

Problem 3

Consider the system in the figure with all masses equal: $m_1 = m_2 = m_3 = m$. Assume that the pulley, string, and spring have negligible masses compared to the other masses. Let l be the natural length of the spring and k its spring constant. Assume as well that the string connecting m_1 and m_2 does not stretch. (a) How many degrees of freedom does this system have? (b) Adopting x_2 as one of your generalized coordinates, determine the system's Lagrangian and equations of motion. (c) What happens in the limit $k \to 0$? (d) Solve the equations of motion for $x_2(t)$, supposing that $x_2(t = 0) = 0$ and $x_3(t = 0) = l$. (e) For the conditions of (d), prove that the string always remains taut.



Solution.

(a) There are two degrees of freedom for this system; three generalized coordinates with one constraint: $x_1 + x_2 = d$.

(b) The Lagrangian is simply $L = T - V$, where

$$T = \frac{1}{2}\left(m_1\dot{x_1}^2 + m_2\dot{x_2}^2 + m_3\dot{x_3}^2\right) = \frac{1}{2}\left[(m_1 + m_2)\,\dot{x_2}^2 + m_3\dot{x_3}^2\right],$$

$$\begin{aligned} V &= -m_1 g x_1 - m_2 g x_2 - m_3 g x_3 + \frac{1}{2}k(x_3 - x_2)^2 \\ &= -m_1 g d - (m_2 - m_1)\,g x_2 - m_3 g x_3 + \frac{1}{2}k(x_3 - x_2 - l)^2. \end{aligned}$$

If all masses are equal then

$$L = \frac{1}{2}m\left(2\dot{x_2}^2 + \dot{x_3}^2\right) + mg(d + x_3) - \frac{1}{2}k(x_3 - x_2 - l)^2.$$

The equations of motion are

$$2m\ddot{x_2} - k(x_3 - x_2 - l) = 0, \text{ and } m\ddot{x_3} - mg + k(x_3 - x_2 - l) = 0.$$

(c) In the limit $k \to 0$, $\ddot{x_2} = 0$ and $\ddot{x_3} = g$. The masses m_1 and m_2 are stationary, while m_3 is in free fall.

(d) Adding the two equations of motion, we obtain $2\ddot{x_2} + \ddot{x_3} = g$. Applying the initial conditions and assuming both start from rest,

$$2x_2(t) + x_3(t) = \frac{1}{2}gt^2 + l.$$

(e) After some time, x_2 will reach its maximum value d. However, x_3 will continue to increase. Thus, the spring is always stretched.

$\square$

Problem 4

A Lagrangian for some physical system can be written as

$$L' = \frac{m}{2}\left(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2\right) - \frac{K}{2}\left(ax^2 + 2bxy + cy^2\right),$$

where a, b, and c are arbitrary constants but subject to the condition $b^2 - ac \neq 0$. What are the equations of motion? Examine the cases $\{a = 0 = c\}$ and $\{b = 0, c = -a\}$. What is the physical system described by the above Lagrangian? Show that the usual Lagrangian L for this system is related to L' by a point transformation $q_i = q_i(s_1, \ldots, s_n, t)$, $i = 1, \ldots, n$. What is the significance of the condition on $b^2 - ac$?

Solution. Using Lagrange's equation,

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{x}}\right) = \frac{\partial L'}{\partial x} \quad &\Longrightarrow \quad m\left(a\ddot{x} + b\ddot{y}\right) = -K\left(ax + by\right), \\
\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{y}}\right) = \frac{\partial L'}{\partial y} \quad &\Longrightarrow \quad m\left(b\ddot{x} + c\ddot{y}\right) = -K\left(bx + cy\right).
\end{aligned}$$

If $a = 0 = c$, then $m\ddot{y} = -Ky$, and $m\ddot{x} = -Kx$. If $b = 0$ and $c = -a$, then $m\ddot{x} = -Kx$ and $m\ddot{y} = -Ky$. In both cases we obtain the same equations of motion, that of two uncoupled harmonic oscillators. The physical system describes is that of two coupled harmonic oscillators. Let $u = ax + by$ and $v = bx + cy$. Then the equations of motion are $m\ddot{u} = -Ku$ and $m\ddot{v} = -Kv$. The usual Lagrangian for this system is

$$L = \frac{1}{2}m\left(\dot{u}^2 + \dot{v}^2\right) - \frac{1}{2}k\left(u^2 + v^2\right).$$

The significance of the condition on $b^2 - ac$ is to ensure that the point transformation is not degenerate, i.e. u is not parallel to v.

$\square$

Problem 5

Consider a particle that oscillates under the influence of gravity while constrained on a frictionless wire that's described parametrically by the cycloid

$$x = R\left(\theta - \sin\theta\right), \text{ and } y = R\left(1 - \cos\theta\right).$$

With θ as a generalized coordinate, what is its Lagrangian? Determine its period of oscillation without the small-amplitude approximation.

Solution. Clearly, $\dot{x} = R(1-\cos\theta)\dot{\theta}$ and $\dot{y} = R\sin\theta\,\dot{\theta}$. Therefore,

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = mR^2(1-\cos\theta)\dot{\theta}^2,$$

$$\text{and} \quad V = mgy = mgR\left(1-\cos\theta\right).$$

The Lagrangian is

$$\begin{aligned} L(\theta, \dot{\theta}) = T - V &= mR^2(1-\cos\theta)\dot{\theta}^2 - mgR\left(1-\cos\theta\right) \\ &= mR^2(1-\cos\theta)\dot{\theta}^2 + mgR(1+\cos\theta) - 2mgR \end{aligned}$$

Let $u(\theta) = \sqrt{1+\cos\theta}$. Then

$$\dot{u} = \frac{\mathrm{d}u}{\mathrm{d}\theta}\,\dot{\theta} = -\frac{1}{2}\sqrt{1-\cos\theta}\,\dot{\theta}.$$

Therefore,

$$L(u, \dot{u}) = 4mR^2\dot{u}^2 - mgRu^2 - 2mgR.$$

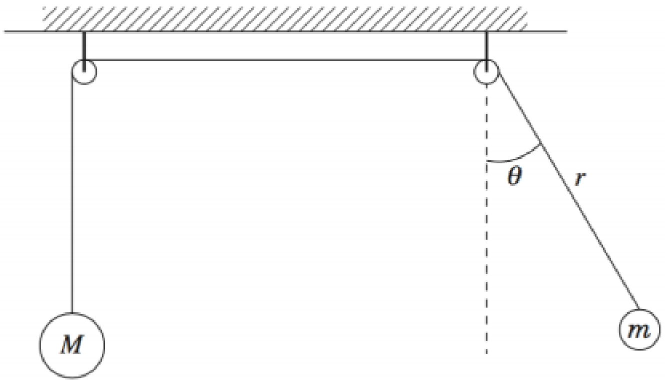
Thus, the equation of motion becomes

$$\ddot{u} + \frac{g}{4R}u = 0.$$

The period is clearly $T = 4\pi\sqrt{R/g}$. □

Problem 6

Consider the system in the figure below. Let $\mu = M/m$. Assuming again that the masses of the pulleys and string are negligible, write down the Lagrangian with r and θ as generalized coordinates. Determine the equations of motion. Study these equations of motion qualitatively and numerically with different values of μ, and different initial conditions. Provide plots of $(r(t), \theta(t))$, and a reasonably complete account of the system's possible behaviors. What do you notice when $\mu > 1$?



Define the coordinate x as the downward displacement of the mass M, such that $r + x$ is constant. The Lagrangian is simply

$$\begin{aligned} L &= \frac{1}{2}\left(m\dot{r}^2 + mr^2\dot{\theta}^2 + M\dot{x}^2\right) + Mgx + mgr\cos\theta \\ &= \frac{1}{2}\left(mr^2\dot{\theta}^2 + (m+M)\dot{r}^2\right) - gr\left(M - m\cos\theta\right). \end{aligned}$$

The equations of motion are

$$\begin{gathered} (m+M)\ddot{r} - mr\dot{\theta}^2 + g\left(M - m\cos\theta\right) = 0, \\ mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr\sin\theta = 0. \end{gathered}$$

We can rewrite these equations as

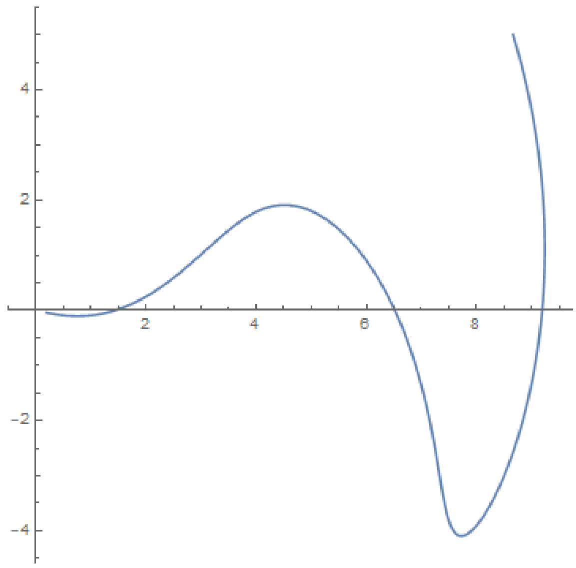
$$\begin{gathered} (1+\mu)\ddot{r} - r\dot{\theta}^2 + g\left(\mu - \cos\theta\right) = 0, \\ r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} + gr\sin\theta = 0. \end{gathered}$$

Consider the case where $\theta(0) = 0$ and $\dot{\theta}(0) = 0$ The equations of motion are just

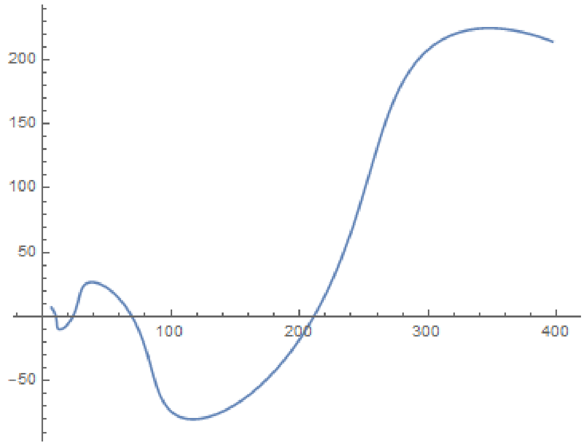
$$\ddot{r} = -g\left(\frac{M-m}{M+m}\right), \quad \text{and} \quad \ddot{\theta} = 0.$$

This is the equation for an Atwood machine. For the general case, we can solve this numerically. Some observations on the qualitative behavior of the system were made. We only consider the case where $\dot{r}(0) = 0$; i.e. only the angular displacement/speed are variable.

(i) If $\mu > 1$, in general $r \to 0$ with time, except when there is sufficient initial angular speed, then $r \to \infty$.



(a) $\mu > 1$. The pendulum bob swings as it ascends. The amplitude weakens as the length shortens. (The plot starts from right to left) Although a large enough initial angular speed might turn the behavior similar to (b).



(b) $\mu < 1$. The pendulum bob swings as it descends. The amplitude grows as the length (somewhat exponentially) increases. (The plot starts from left to right)

Figure 1: Qualitative behavior of (r, θ). The pendulum bobs are released from rest at some initial angle.

(ii) If $\mu = 1$ then r is fixed, unless there is an initial angular displacement/speed, then $r \to \infty$.

(iii) If $\mu < 1$ then $r \to \infty$ (somewhat exponentially) in time.

We consider the case $\mu = 1$. Clearly, the system will remain at rest when there is no initial angular displacement/speed. Is this a stable equilibrium? Assuming the equations are regular such that we can make a small-angle approximation, we have

$$2\ddot{r} - r\dot{\theta}^2 = 0,$$
$$r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} + gr\theta = 0.$$

The first equation tells us $2\ddot{r} = r\dot{\theta}^2 \geq 0$; thus, r can only increase and the equilibrium is unstable.