Use the usual conventions for labeling angular momentum quantum numbers: For $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$, use m_i, m, j_i, j as quantum numbers for $J_{iz}, J_z, \mathbf{J}_i^2, \mathbf{J}^2$, respectively, and so on.

I. ELECTRON-POSITRON SYSTEM

Let $\mathbf{S}^-$ and $\mathbf{S}^+$ be the spin operators for an electron and positron, respectively. The Hamiltonian for the composite system is

$$H = A\mathbf{S}^- \cdot \mathbf{S}^+ + \frac{e\mathbf{B}\cdot(\mathbf{S}^- - \mathbf{S}^+)}{mc}.$$

(a) What are the four energy eigenvalues in the limit $A \ll eB/(\hbar mc)$ $(A \to 0)$?

(b) What are the four energy eigenvalues in the limit $A \gg eB/(\hbar mc)$ $(B \to 0)$?

(c) In which limit is the singlet spinor $(\chi_\uparrow^-\chi_\downarrow^+ - \chi_\downarrow^-\chi_\uparrow^+)/\sqrt{2}$ an energy eigenstate?

Solution. Without loss of generality, assume $\mathbf{B} = B\hat{\mathbf{z}}$. We may rewrite the Hamiltonian as

$$H = A(S_x^-S_x^+ + S_y^-S_y^+ + S_z^-S_z^+) + \frac{eB}{mc}\left(S_z^- - S_z^+\right).$$

(a) In the limit $A \to 0$, the Hamiltonian becomes

$$H_{A\to 0} = \frac{eB}{mc}\left(S_z^- - S_z^+\right).$$

The eigenvectors are simply all the combinations obtained by selecting one eigenvector for each S_z operator; i.e. $\chi_\uparrow^-\chi_\uparrow^+$, $\chi_\uparrow^-\chi_\downarrow^+$, $\chi_\downarrow^-\chi_\uparrow^+$, and $\chi_\downarrow^-\chi_\downarrow^+$. The energy eigenvalues are simply 0, $\hbar eB/(mc)$, $-\hbar eB/(mc)$, and 0, respectively.

(b) In the limit $B \to 0$, we must solve the eigenvalue problem. Note that we may write (in the $\chi^- \otimes \chi^+$ representation)

$$\begin{aligned} H_{B\to 0} = A\mathbf{S}^-\cdot\mathbf{S}^+ &= A(S_x^-S_x^+ + S_y^-S_y^+ + S_z^-S_z^+) = A(S_z^-S_z^+ + \frac{1}{2}S_+^-S_-^+ + \frac{1}{2}S_-^-S_+^+) \\ &= A\left[(S_z^-\otimes\mathbb{1}^+)(\mathbb{1}^-\otimes S_z^+) + \frac{1}{2}(S_+^-\otimes\mathbb{1}^+)(\mathbb{1}^-\otimes S_-^+) + \frac{1}{2}(S_-^-\otimes\mathbb{1}^+)(\mathbb{1}^-\otimes S_+^+)\right], \end{aligned}$$

where we have used $S_\pm = S_x \pm iS_y$ (this wasn't really necessary as we will use Mathematica anyway). Using Mathematica, we obtain

$$H_{B\to 0} = A\begin{pmatrix} \hbar^2/4 & 0 & 0 & 0 \\ 0 & -\hbar^2/4 & \hbar^2/2 & 0 \\ 0 & \hbar^2/2 & -\hbar^2/4 & 0 \\ 0 & 0 & 0 & \hbar^2/4 \end{pmatrix}.$$

We find that the eigenvalues are $A\hbar^2/4$ (with multiplicity 3) and $-3A\hbar^2/4$. The eigenvectors $\chi_\downarrow^-\chi_\downarrow^+$, $\chi_\uparrow^-\chi_\uparrow^+$, and $\left(\chi_\uparrow^-\chi_\downarrow^+ + \chi_\downarrow^-\chi_\uparrow^+\right)/\sqrt{2}$ all have eigenvalue $A\hbar^2/4$, while the remaining eigenvector is $\left(\chi_\uparrow^-\chi_\downarrow^+ - \chi_\downarrow^-\chi_\uparrow^+\right)/\sqrt{2}$ with eigenvalue $-3A\hbar^2/4$.

(c) It is clear that the singlet spinor $(\chi_\uparrow^-\chi_\downarrow^+ - \chi_\downarrow^-\chi_\uparrow^+)/\sqrt{2}$ is not an eigenvector of $H_{A\to 0}$. However, we see that it is an eigenvector of $H_{B\to 0}$, with negative energy eigenvalue since $A > 0$.

□

II. FERMI'S GOLDEN RULE

A perturbing potential $\lambda \mathbf{J}_1 \cdot \mathbf{J}_2$ is applied to a state of uncoupled spins with angular momentum quantum numbers $j_1 = j_2 = 1$. Use Fermi's golden rule to determine which transitions to states of total angular momentum j are allowed.

Solution. Let $H' = \lambda \mathbf{J}_1 \cdot \mathbf{J}_2$ be the perturbing Hamiltonian. Fermi's golden rule states that for a time-independent perturbing Hamiltonian the only allowed transitions must have the same energy. Let $|j_1, m_1; j_2, m_2\rangle$ denote the initial ket. Now consider the projection of $H' |j_1, m_1; j_2, m_2\rangle$ to an arbitrary state $|j_1', m_1'; j_2', m_2'\rangle$. (We can just ignore the original Hamiltonian H^0.)

$$
\begin{aligned}
\langle \alpha' | H' | \alpha \rangle :&= \langle j_1', m_1'; j_2', m_2' | H' | j_1, m_1; j_2, m_2 \rangle \\
&= \langle j_1', m_1'; j_2', m_2' | \lambda \mathbf{J}_1 \cdot \mathbf{J}_2 | j_1, m_1; j_2, m_2 \rangle \\
&= \lambda \langle j_1', m_1'; j_2', m_2' | (J_{1x} J_{2x} + J_{1y} J_{2y} + J_{1z} J_{2z}) | j_1, m_1; j_2, m_2 \rangle \\
&= \lambda \langle j_1', m_1'; j_2', m_2' | (J_{1z} J_{2z} + \frac{1}{2} J_{1+} J_{2-} + \frac{1}{2} J_{1-} J_{2+}) | j_1, m_1; j_2, m_2 \rangle \\
&= \lambda \langle j_1', m_1'; j_2', m_2' | J_{1z} J_{2z} | j_1, m_1; j_2, m_2 \rangle + \frac{1}{2} \lambda \langle j_1', m_1'; j_2', m_2' | J_{1+} J_{2-} | j_1, m_1; j_2, m_2 \rangle \\
&\qquad + \frac{1}{2} \lambda \langle j_1', m_1'; j_2', m_2' | J_{1-} J_{2+} | j_1, m_1; j_2, m_2 \rangle \\
&= \quad \lambda \hbar^2 m_1 m_2 \delta_{j_1}^{j_1'} \delta_{m_1}^{m_1'} \delta_{j_2}^{j_2'} \delta_{m_2}^{m_2'} \\
&\quad + \frac{1}{2} \lambda \hbar^2 \sqrt{(j_1 - m_1)(j_1 + m_1 + 1)} \sqrt{(j_2 + m_2)(j_2 - m_2 + 1)} \delta_{j_1}^{j_1'} \delta_{m_1+1}^{m_1'} \delta_{j_2}^{j_2'} \delta_{m_2-1}^{m_2'} \\
&\quad + \frac{1}{2} \lambda \hbar^2 \sqrt{(j_1 + m_1)(j_1 - m_1 + 1)} \sqrt{(j_2 - m_2)(j_2 + m_2 + 1)} \delta_{j_1}^{j_1'} \delta_{m_1-1}^{m_1'} \delta_{j_2}^{j_2'} \delta_{m_2+1}^{m_2'} \\
&= \quad \lambda \hbar^2 m_1 m_2 \delta_{1}^{j_1'} \delta_{m_1}^{m_1'} \delta_{1}^{j_2'} \delta_{m_2}^{m_2'} \\
&\quad + \frac{1}{2} \lambda \hbar^2 \sqrt{(1 - m_1)(m_1 + 2)} \sqrt{(1 + m_2)(2 - m_2)} \delta_{1}^{j_1'} \delta_{m_1+1}^{m_1'} \delta_{1}^{j_2'} \delta_{m_2-1}^{m_2'} \\
&\quad + \frac{1}{2} \lambda \hbar^2 \sqrt{(1 + m_1)(2 - m_1)} \sqrt{(1 - m_2)(m_2 + 2)} \delta_{1}^{j_1'} \delta_{m_1-1}^{m_1'} \delta_{1}^{j_2'} \delta_{m_2+1}^{m_2'}.
\end{aligned}
$$

Clearly, we see that the only allowed transitions must have $j_1' = j_2' = 1$. In that case

$$
\begin{aligned}
\langle \alpha' | H' | \alpha \rangle = &\quad \lambda \hbar^2 m_1 m_2 \delta_{m_1}^{m_1'} \delta_{m_2}^{m_2'} \\
&+ \frac{1}{2} \lambda \hbar^2 \sqrt{(1 - m_1)(m_1 + 2)} \sqrt{(1 + m_2)(2 - m_2)} \delta_{m_1+1}^{m_1'} \delta_{m_2-1}^{m_2'} \\
&+ \frac{1}{2} \lambda \hbar^2 \sqrt{(1 + m_1)(2 - m_1)} \sqrt{(1 - m_2)(m_2 + 2)} \delta_{m_1-1}^{m_1'} \delta_{m_2+1}^{m_2'}.
\end{aligned}
$$

The first term is nonzero if $m_1' = m_1$ and $m_2' = m_2$, as long as both m_1 and m_2 are nonzero. But we can ignore the first term since it corresponds to no transition. We require $m_1' + m_2' = m_1 + m_2$ for the remaining terms to be nonzero. Furthermore, m_1 and m_2 must increase/decrease by one while still preserving their sum; e.g. if m_1 increases by one then m_2 must decrease by one. Now, we enumerate each (m_1, m_2). Adopting the new notation $|m_1, m_2\rangle$, we find the following nonzero projections:

$$
\begin{aligned}
\langle\ \ 0,-1|H'|-1,\ \ 0\rangle &= \lambda\hbar^2,\\
\langle\ \ 0,\ \ 0|H'|-1,\ \ 1\rangle &= \lambda\hbar^2,\\
\langle-1,\ \ 0|H'|\ \ 0,-1\rangle &= \lambda\hbar^2,\\
\langle-1,\ \ 1|H'|\ \ 0,\ \ 0\rangle &= \lambda\hbar^2,\\
\langle\ \ 1,-1|H'|\ \ 0,\ \ 0\rangle &= \lambda\hbar^2,\\
\langle\ \ 1,\ \ 0|H'|\ \ 0,\ \ 1\rangle &= \lambda\hbar^2,\\
\langle\ \ 0,\ \ 0|H'|\ \ 1,-1\rangle &= \lambda\hbar^2,\\
\langle\ \ 0,\ \ 1|H'|\ \ 1,\ \ 0\rangle &= \lambda\hbar^2.
\end{aligned}
$$

Note that they all have the same energies. $\square$

III. SCHWINGER BOSONS

Let a and b refer to Schwinger bosons. Give the physical significance of the operators

$$K_+ = a^\dagger b^\dagger,$$
$$K_- = ab,$$

in terms of the corresponding spin representation. Give the nonvanishing matrix elements of $K_\pm$. [Why is this question from the textbook incomplete?]

Solution. Recall that we have the following nonzero commutation relations:

$$[a, a^\dagger] = 1 \quad \text{and} \quad [b, b^\dagger] = 1.$$

The number operators N_a and N_b are also defined as

$$N_a = a^\dagger a \quad \text{and} \quad N_b = b^\dagger b.$$

Let $|n_a, n_b\rangle$ be the eigenket of N_a and N_b with eigenvalue n_a and n_b, respectively. We obtain the following (Equation 3.95):

$$a^\dagger |n_a, n_b\rangle = \sqrt{n_a + 1}\, |n_a + 1, n_b\rangle\,, \quad a\,|n_a, n_b\rangle = \sqrt{n_a}\, |n_a - 1, n_b\rangle\,,$$
$$b^\dagger |n_a, n_b\rangle = \sqrt{n_b + 1}\, |n_a, n_b + 1\rangle\,, \quad b\,|n_a, n_b\rangle = \sqrt{n_b}\, |n_a, n_b - 1\rangle\,.$$

Then,

$$K_+ |n_a, n_b\rangle = \sqrt{(n_a + 1)(n_b + 1)}\, |n_a + 1, n_b + 1\rangle\,,$$
$$K_- |n_a, n_b\rangle = \sqrt{n_a n_b}\, |n_a - 1, n_b - 1\rangle\,.$$

Let A be an index which uniquely labels each $|n_a, n_b\rangle$. The matrix elements of $K_\pm$ in this representation is simply

$$(K_+)_{A'A} := \langle n'_a, n'_b|\, K_+ \,|n_a, n_b\rangle = \langle n_{a'}, n_{b'}|\, a^\dagger b^\dagger \,|n_a, n_b\rangle = \sqrt{(n_a + 1)(n_b + 1)}\delta_{a',a+1}\delta_{b',b+1},$$
$$(K_-)_{A'A} := \langle n'_a, n'_b|\, K_- \,|n_a, n_b\rangle = \langle n_{a'}, n_{b'}|\, ab \,|n_a, n_b\rangle = \sqrt{n_a n_b}\delta_{a',a-1}\delta_{b',b-1}.$$

Now, note that $j := (n_a + n_b)/2$ and $m := (n_a - n_b)/2$. Therefore, since K_+ increases both n_a and n_b by one, then j is also increased by one, whereas m remains the same. Similarly, K_- decreases both n_a and n_b by one, therefore j is decreased by one while m is again unchanged. Thus, we can see the physical significance of K_+ and K_-. They act as a raising and lowering operator for j, respectively. Rewriting $n_a = j + m$ and $n_b = j - m$, we obtain in the new representation (indexed by J)

$$\begin{aligned}(K_+)_{J'J} &:= \langle j', m'|\, K_+ \,|j, m\rangle = \langle j', m'|\, \sqrt{(j + m + 1)(j - m + 1)}\, |j + 1, m\rangle \\ &= \sqrt{(j + m + 1)(j - m + 1)}\delta_{j',j+1}\delta_{m',m}, \\ (K_-)_{J'J} &:= \langle j', m'|\, K_- \,|j, m\rangle = \langle j', m'|\, \sqrt{(j + m)(j - m)}\, |j - 1, m\rangle \\ &= \sqrt{(j + m)(j - m)}\delta_{j',j-1}\delta_{m',m}.\end{aligned}$$

□