Problem 1

A projectile is launched from the surface of the Earth at polar angle θ (with respect to the North Pole) and pointed due East.

(a) If the launch angle with respect to the horizontal is α, show that the "horizontal" deflection of the projectile when it strikes the ground is $4V_0^3/g^2\omega\cos\theta\sin^2\alpha\cos\alpha$, where V_0 is the initial speed of the projectile and ω is the Earth's angular speed. What is the direction of this deflection?

(b) Let R be the range of the projectile if $\omega = 0$, calculate the $\mathcal{O}(\omega)$ correction to this range.

Solution. We start from the equation of motion (at Earth's surface)

$$\ddot{\mathbf{r}}(t) = \mathbf{g}' - 2\boldsymbol{\omega} \times \dot{\mathbf{r}}.$$

We choose a coordinate system where x, y, and z are in the direction of increasing θ, ϕ and r, respectively. We will only work up to linear order in ω. Therefore, $\mathbf{g}' = \mathbf{g} + \mathcal{O}(\omega^2) = -g\hat{\mathbf{z}} + \mathcal{O}(\omega^2)$. From now on we drop $\mathcal{O}(\omega^2)$ in our expressions. Let $\mathbf{r}(t) = \mathbf{r}_0(t) + \mathbf{r}_1(t)$. Then, $\ddot{\mathbf{r}}_0(t) = -\mathbf{g}$ and $\ddot{\mathbf{r}}_1(t) = -2\boldsymbol{\omega} \times \dot{\mathbf{r}}_0$. Now, we easily see that $\mathbf{r}_0(t) = (0, V_0 t\cos\alpha, V_0 t\sin\alpha - gt^2/2)$ and $\dot{\mathbf{r}}_0(t) = (0, V_0\cos\alpha, V_0\sin\alpha - gt)$. Therefore,

$$\begin{aligned}
\ddot{\mathbf{r}}_1(t) &= -2\boldsymbol{\omega} \times \dot{\mathbf{r}}_1(t) = -2\omega\hat{\mathbf{n}} \times (V_0\cos\alpha\,\hat{\mathbf{y}} + (V_0\sin\alpha - gt)\hat{\mathbf{z}}) \\
&= -2\omega V_0\cos\alpha\,\hat{\mathbf{n}} \times \hat{\mathbf{y}} - 2\omega(V_0\sin\alpha - gt)\hat{\mathbf{n}} \times \hat{\mathbf{z}} \\
&= 2\omega V_0\cos\alpha(\cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{z}}) - 2\omega(V_0\sin\alpha - gt)\hat{\mathbf{y}} \\
\mathbf{r}_1(t) &= \omega V_0 t^2\cos\alpha(\cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{z}}) + \left(\frac{1}{3}\omega g t^3 - \omega V_0 t^2\sin\alpha\right)\hat{\mathbf{y}}.
\end{aligned}$$

Thus,

$$\mathbf{r}(t) = \left(\omega V_0 t^2\cos\alpha\cos\theta,\ \frac{1}{3}\omega g t^3 - \omega V_0 t^2\sin\alpha + V_0 t\cos\alpha,\ (\omega V_0\cos\alpha\sin\theta - \frac{1}{2}g)t^2 + V_0 t\sin\alpha\right).$$

Suppose τ is the time it takes for the projectile to reach the ground. Then imposing $z(\tau) = 0$ and solving for τ, we obtain

$$\tau = \frac{2V_0\sin\alpha}{g - 2\omega V_0\cos\alpha\sin\theta}.$$

(a) The "horizontal" deflection is given by

$$\begin{aligned}
x(\tau) &= \omega V_0\tau^2\cos\alpha\cos\theta = \omega V_0\cos\alpha\cos\theta\left(\frac{2V_0\sin\alpha}{g - 2\omega V_0\cos\alpha\sin\theta}\right)^2 \\
&= \omega V_0\cos\alpha\cos\theta\left(\frac{2V_0\sin\alpha}{g} + \mathcal{O}(\omega)\right)^2 \\
&= \frac{4V_0^3}{g^2}\omega\cos\theta\sin^2\alpha\cos\alpha.
\end{aligned}$$

(b) The correction to the range is given by

$$\begin{aligned}
\delta R = y(\tau) - R &= \frac{1}{3}\omega g \tau^3 - \omega V_0 \tau^2 \sin\alpha + V_0 \tau \cos\alpha - \frac{V_0^2}{g}\sin 2\alpha \\
&= \frac{1}{3}\omega g \left(\frac{2V_0 \sin\alpha}{g}\right)^3 - \omega V_0 \sin\alpha \left(\frac{2V_0 \sin\alpha}{g}\right)^2 + \frac{V_0^2 \sin(2\alpha)}{g}\left(\frac{2\omega V_0 \cos\alpha \sin\theta}{g}\right) + \mathcal{O}(\omega^2) \\
&= -\frac{4\omega V_0^3 \sin^3\alpha}{3g^2} + \frac{2\omega V_0^3 \sin(2\alpha)\cos\alpha \sin\theta}{g^2} + \mathcal{O}(\omega^2) \\
&= \frac{4\omega V_0^3 \sin^3\alpha}{3g^2}\left(3\cot^2\alpha \sin\theta - 1\right) + \mathcal{O}(\omega^2).
\end{aligned}$$

$\square$

Problem 2

Show that the angular velocity in the lab/inertial frame (i.e. not the non-inertial body frame) are given in terms of the Euler angles by

$$\begin{aligned}
\omega_x &= \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi, \\
\omega_y &= \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi, \\
\omega_z &= \dot{\psi}\cos\theta + \dot{\phi}.
\end{aligned}$$

Solution. For infinitesimal rotations $\boldsymbol{\omega} = \boldsymbol{\omega_\phi} + \boldsymbol{\omega_\theta} + \boldsymbol{\omega_\psi}$. By definition,

$$(\boldsymbol{\omega_\phi})_x = \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix}, \quad (\boldsymbol{\omega_\phi})_\xi = \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix}, \quad (\boldsymbol{\omega_\phi})_{\xi'} = \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}.$$

Therefore,

$$\begin{aligned}
(\boldsymbol{\omega})_x &= (\boldsymbol{\omega_\phi})_x + \boldsymbol{D^{-1}}(\boldsymbol{\omega_\phi})_\xi + \boldsymbol{D^{-1}C^{-1}}(\boldsymbol{\omega_\phi})_{\xi'} \\
&= \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}.
\end{aligned}$$

Note that the inverse of a block diagonal matrix is obtained by inverting each block. We also know that the inverse of a rotation matrix is obtained by inverting the angle. Thus,

$$\begin{aligned}
(\boldsymbol{\omega})_x &= \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} \dot{\theta}\cos\phi \\ \dot{\theta}\sin\phi \\ 0 \end{bmatrix} + \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -\dot{\psi}\sin\theta \\ \dot{\psi}\cos\theta \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} \dot{\theta}\cos\phi \\ \dot{\theta}\sin\phi \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{\psi}\sin\theta\sin\phi \\ -\dot{\psi}\sin\theta\cos\phi \\ \dot{\psi}\cos\theta \end{bmatrix} \\
\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} &= \begin{bmatrix} \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi \\ \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi \\ \dot{\phi} + \dot{\psi}\cos\theta \end{bmatrix}.
\end{aligned}$$

□

Problem Set 5

Problem 3

An arbitrary rotation matrix can be described by a rotation by Φ about some axis $\hat{\mathbf{n}}$. Show that this rotation angle can be expressed in terms of the Euler angles by

$$\cos\frac{\Phi}{2} = \cos\frac{\phi+\psi}{2}\cos\frac{\theta}{2}.$$

Solution. We use the fact that the trace of a matrix is invariant under coordinate transformations. Thus, given a rotation by Φ about some axis $\hat{\mathbf{n}}$, we have

$$\begin{aligned}
\operatorname{Tr}\begin{bmatrix} \cos\Phi & \sin\Phi & 0 \\ -\sin\Phi & \cos\Phi & 0 \\ 0 & 0 & 1 \end{bmatrix} &= \operatorname{Tr}\begin{bmatrix} \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi & \sin\phi\cos\psi + \cos\phi\cos\theta\sin\psi & \sin\theta\sin\psi \\ -\cos\phi\sin\psi - \sin\phi\cos\theta\cos\psi & -\sin\phi\sin\psi + \cos\phi\cos\theta\cos\psi & \sin\theta\cos\psi \\ \sin\phi\sin\theta & -\cos\phi\sin\theta & \cos\theta \end{bmatrix} \\
1 + 2\cos\Phi &= \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi - \sin\phi\sin\psi + \cos\phi\cos\theta\cos\psi + \cos\theta \\
&= (\cos\phi\cos\psi - \sin\phi\sin\psi)(1+\cos\theta) + \cos\theta \\
&= \cos(\phi+\psi)(1+\cos\theta) + \cos\theta \\
2(1+\cos\Phi) &= (1+\cos(\phi+\psi))(1+\cos\theta) \\
\frac{1+\cos\Phi}{2} &= \frac{1+\cos(\phi+\psi)}{2}\frac{1+\cos\theta}{2} \\
\cos\frac{\Phi}{2} &= \cos\frac{\phi+\psi}{2}\cos\frac{\theta}{2}.
\end{aligned}$$

□

Problem 4

Consider a uniform mass density sphere of radius R and with a spherical cavity of radius $r < R$ that is tangent to its surface. Find the principal moments of inertia and determine its principal axes.

Solution. We first find the center of mass. Consider an arbitrary reference frame. Let ρ be the density, M be the mass, and V be the volume of the object. The center of mass is given by

$$\mathbf{r}_{\mathrm{CM}} = \frac{1}{M}\int_V \mathbf{r}\,\mathrm{d}m = \frac{\rho}{M}\int_V \mathbf{r}\,\mathrm{d}^3x = \frac{1}{V}\int_V \mathbf{r}\,\mathrm{d}^3x = \frac{1}{V}\left(\int_{V_R}\mathbf{r}\,\mathrm{d}^3x - \int_{V_r}\mathbf{r}\,\mathrm{d}^3x\right) = \frac{1}{V}\left(V_R\mathbf{r}_R - V_r\mathbf{r}_r\right),$$

where $\mathbf{r}_R$ is the center of mass of a uniform density sphere of radius V_R; and similarly defined for $\mathbf{r}_r$ and V_r. Therefore,

$$\mathbf{r}_{\mathrm{CM}} = \frac{R^3\mathbf{r}_R - r^3\mathbf{r}_r}{R^3 - r^3}.$$

If we choose a reference frame (R) whose origin is the center of the sphere of radius R, then $\mathbf{r}_R^{(R)} = 0$; and $\mathbf{r}_{\mathrm{CM}}^{(R)}$ is parallel to $\mathbf{r}_r^{(R)}$, both directed to the point of contact. Let $\hat{\mathbf{z}}$ be the appropriate unit vector in (R). Then

$$\mathbf{r}_{\mathrm{CM}}^{(R)} = -\frac{r^3(R-r)}{R^3 - r^3}\hat{\mathbf{z}} = -\frac{r^3}{R^2 + Rr + r^2}.$$

By inspection, we can guess that the symmetry axes are the z axis and any other axis in the x-y plane.

$$\begin{aligned} I_{zz}^{(R)} &= \int_V \rho(\mathbf{r}')(r'^2 - z^2)\mathrm{d}^3x = \int_{V_R}\rho(\mathbf{r}')(r'^2 - z^2)\mathrm{d}^3x - \int_{V_r}\rho(\mathbf{r}')(r'^2 - z^2)\mathrm{d}^3x \\ &= \frac{2}{5}M_R R^2 - \frac{2}{5}M_r r^2 = \frac{8\pi\rho}{15}\left(R^5 - r^5\right). \end{aligned}$$

$$\begin{aligned} I_{xx}^{(R)} = I_{yy}^{(R)} &= \int_V \rho(\mathbf{r}')(r'^2 - x^2)\mathrm{d}^3x = \int_{V_R}\rho(\mathbf{r}')(r'^2 - x^2)\mathrm{d}^3x - \int_{V_r}\rho(\mathbf{r}')(r'^2 - x^2)\mathrm{d}^3x \\ &= \frac{2}{5}M_R R^2 - \left(\frac{2}{5}M_r r^2 + M_r r_{\mathrm{CM}}^2\right) = \frac{8\pi\rho}{15}\left(R^5 - r^5\right) - \frac{4\pi\rho r^9}{3(R^2 + Rr + r^2)^2}. \end{aligned}$$

Now, in the center of mass frame, $I_{zz}^{(CM)} = I_{zz}^{(R)}$; and

$$\begin{aligned} I_{xx}^{(CM)} = I_{yy}^{(CM)} = I_{xx}^{(R)} - Mr_{\mathrm{CM}}^2 &= \frac{8\pi\rho}{15}\left(R^5 - r^5\right) - \frac{4\pi\rho r^9}{3(R^2 + Rr + r^2)^2} - \frac{4\pi\rho r^6(R^3 - r^3)}{3(R^2 + Rr + r^2)^2} \\ &= \frac{8\pi\rho}{15}\left(R^5 - r^5\right) - \frac{4\pi\rho r^6 R^3}{3(R^2 + Rr + r^2)^2}. \end{aligned}$$

□

Problem 5

Chapter 5, Derivation 6 (Goldstein, 3rd ed.)

(a) Show that the angular momentum of the torque-free symmetrical top rotates in the body coordinates about the symmetry axis with an angular frequency Ω. Show also that the symmetry axis rotates in space about the fixed direction of the angular momentum with the angular frequency

$$\dot{\phi} = \frac{I_3\omega_3}{I_1\cos\theta},$$

where ϕ is the Euler angle of the line of nodes with respect to the angular momentum as the space z axis.

(b) Using the results of Exercise 15, Chapter 4, show that ω rotates in space about the angular momentum with the same frequency $\dot{\phi}$, but that the angle θ' between $\boldsymbol{\omega}$ and $\mathbf{L}$ is given by

$$\sin\theta' = \frac{\Omega}{\dot{\phi}}\sin\theta'',$$

where θ'' is the inclination of $\boldsymbol{\omega}$ to the symmetry axis. Using the data given in Section 5.6, show therefore that Earth's rotation axis and the axis of angular momentum are never more than 1.5 cm apart on Earth's surface.

(c) Show from parts (a) and (b) that the motion of the force-free symmetrical top can be described in terms of the rotation of a cone fixed in the body whose axis is the symmetry axis, rolling on a fixed cone in space whose axis is along the angular momentum. The angular velocity vector is along the line of contact of the two cones. Show that the same description follows immediately from the Poinsot construction in terms of the inertia ellipsoid.

Solution.

(a) It is shown in Section 5.6 that for a symmetric top $I_1 = I_2 \neq I_3$. The solution to the Euler equations are

$$\omega_1 = A\cos(\Omega t), \quad \omega_2 = A\sin(\Omega t), \quad \omega_3 = \text{const.},$$

where $\Omega = (I_3 - I_1)\omega_3/I_1$. Clearly, in the body frame the inertia tensor is always diagonal as the principal axes are the body-frame axes. Therefore,

$$\mathbf{L} = (I_1\omega_1,\, I_1\omega_2,\, I_3\omega_3) = AI_1(\cos(\Omega t),\, \sin(\Omega t),\, \frac{I_3\omega_3}{AI_1}).$$

It is obvious that $\mathbf{L}$ rotates about the symmetry axis of I_3. Now, $\mathbf{L}$ is expressed in the body frame. Following Example 4.5 (Lemos), we can choose the two degenerate axes such that

$$\omega_1 = \dot{\theta}, \quad \omega_2 = \dot{\phi}\sin\theta, \quad \omega_3 = \dot{\phi}\cos\theta + \dot{\psi},$$

at a particular instant. In the inertial frame $\mathbf{L}$ is constant, which we choose to be in the z direction. Clearly, we have

$$L_1 = 0, \quad L_2 = L\sin\theta, \quad L_3 = L\cos\theta.$$

We see that $\dot{\theta} = 0$. Also,

$$\frac{I_3\omega_3}{\cos\theta} = \frac{L_3}{\cos\theta} = \frac{L_2}{\sin\theta} = \frac{I_1\omega_2}{\sin\theta} = I_1\dot{\phi} \quad \Longrightarrow \quad \dot{\phi} = \frac{I_3\omega_3}{I_1\cos\theta}.$$

(b) Using the results of Derivation 15 Chapter 4 (or Problem 2), we have

$$\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix} = \begin{bmatrix}\dot\theta\cos\phi + \dot\psi\sin\theta\sin\phi\\ \dot\theta\sin\phi - \dot\psi\sin\theta\cos\phi\\ \dot\phi + \dot\psi\cos\theta\end{bmatrix} = \begin{bmatrix}+\dot\psi\sin\theta\sin\phi\\ -\dot\psi\sin\theta\cos\phi\\ \dot\phi + \dot\psi\cos\theta\end{bmatrix}$$

Now, since $\dot\psi = \Omega$ and $\boldsymbol{\omega}$ rotates about the space z axis

$$\sin\theta' = \frac{\sqrt{\omega_x^2+\omega_y^2}}{|\boldsymbol{\omega}|} = \frac{\Omega\sin\theta}{|\boldsymbol{\omega}|}.$$

Similarly, $\boldsymbol{\omega}$ rotates about the symmetry axis by

$$\sin\theta'' = \frac{\sqrt{\omega_1^2+\omega_2^2}}{|\boldsymbol{\omega}|} = \frac{\dot\phi\sin\theta}{|\boldsymbol{\omega}|}$$

Thus, we see that

$$\sin\theta' = \frac{\Omega}{\dot\phi}\sin\theta''.$$

The distance of the two axes on Earth's surface is

$$d = R\sin\theta' = R\Omega\frac{I_1\cos\theta}{I_3\omega_3}\sin\theta'' = R\cos\theta\left(1-\frac{I_1}{I_3}\right)\sin\theta'' < 1.5\,\text{cm}.$$

(TO BE CONTINUED)

□

Problem 6

Chapter 5, Derivation 8 (Goldstein, 3rd ed.)
When the rigid body is not symmetrical, an analytic solution to Euler's equation for the torque-free motion cannot be given in terms of elementary functions. Show, however, that the conservation of energy and angular momentum can be used to obtain expressions for the body components of $\boldsymbol{\omega}$ in terms of elliptic integrals.

Solution. For torque-free motion, Euler's equations become

$$\begin{aligned}I_1\dot{\omega}_1 &= \omega_2\omega_3(I_2 - I_3),\\ I_2\dot{\omega}_2 &= \omega_3\omega_1(I_3 - I_1),\\ I_3\dot{\omega}_3 &= \omega_1\omega_2(I_1 - I_2);\end{aligned}$$

where $I_1 \neq I_2 \neq I_3$. The conservation of kinetic energy and angular momentum (as seen in the space frame) is given by

$$\begin{aligned}I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 &= 2E,\\ I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 &= L^2.\end{aligned}$$

We can therefore express ω_2 and ω_3 as a function of ω_1. Using Mathematica, we obtain

$$\begin{aligned}\omega_2 &= \sqrt{\frac{L^2 - 2EI_3 - I_1(I_1 - I_3)\omega_1^2}{I_2(I_2 - I_3)}},\\ \omega_3 &= \sqrt{\frac{L^2 - 2EI_2 - I_1(I_1 - I_2)\omega_1^2}{I_3(I_3 - I_2)}}.\end{aligned}$$

We can plug these in the first Euler equation to solve for ω_1 as a function of time. We find that ω_1 can be expressed as an inverse function of an elliptic integral of the first kind (see Landau and Lifshitz). This can be seen by rewriting $\omega_2 = a\sqrt{1 - b\,\omega_1^2}$ and $\omega_3 = c\sqrt{1 - d\,\omega_1^2}$. Thus, the first Euler equation can be written as

$$\begin{aligned}I_1\dot{\omega}_1 &= ac(I_2 - I_3)\sqrt{(1 - b\,\omega_1^2)(1 - d\,\omega_1^2)}\\ \mathrm{d}t &= \frac{I_1\mathrm{d}\omega_1}{ac(I_2 - I_3)\sqrt{(1 - b\,\omega_1^2)(1 - d\,\omega_1^2)}}.\end{aligned}$$

□

Problem 7

Chapter 5, Derivation 9 (Goldstein, 3rd ed.)
Apply Euler's equations to the problem of the heavy symmetrical top, expressing ω_t in terns of the Euler angles. Show that the two integrals of motion, Eqs. (5.53) and (5.54), can be obtained directly from Euler's equations in this form.

Solution. The Euler equations are

$$\begin{aligned} I_1\dot{\omega_1} - \omega_2\omega_3(I_1 - I_3) &= N_1 = -Mgl\sin\theta\cos\psi, \\ I_1\dot{\omega_2} - \omega_3\omega_1(I_3 - I_1) &= N_2 = \quad Mgl\sin\theta\sin\psi, \\ I_3\dot{\omega_3} &= N_3 = 0. \end{aligned}$$

Using our results previously derived in class (similar in procedure to Problem 2), the angular velocity in the body frame is related to the Euler angles by

$$\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} = \begin{pmatrix}\dot{\phi}\sin\psi\sin\theta + \dot{\theta}\cos\psi\\ \dot{\phi}\cos\psi\sin\theta - \dot{\theta}\sin\psi\\ \dot{\phi}\cos\theta + \dot{\psi}\end{pmatrix}.$$

It is obvious that one integral of motion is $I_3\omega_3 = I_3(\dot{\psi} + \dot{\phi}\cos\theta)$. Therefore the angular momentum component in the symmetry axis is conserved. Since the net torque is along ϕ, the angular momentum in the inertial z axis is also conserved. Suppose we choose the body frame such that $\psi = 0$ at this particular instant. Now, in the space frame

$$\begin{aligned} L_z &= I_3\omega_3\cos\theta + I_1\omega_2\sin\theta \\ &= I_3(\dot{\phi}\cos\theta + \dot{\psi})\cos\theta + I_1\dot{\phi}\sin^2\theta \\ &= (I_1\sin^2\theta + I_3\cos^2\theta)\dot{\phi} + I_3\dot{\psi}\cos\theta. \end{aligned}$$

□

Problem 8

Chapter 5, Exercise 28 (Goldstein, 3rd ed.)
Suppose that in a symmetrical top each element of mass has a proportionate charge associated with it, so that the e/m ratio is constant—the so-called charge symmetric top. If such a body rotates in a uniform magnetic field the Lagrangian, from (5.108), is

$$L = T - \boldsymbol{\omega}_l \cdot \mathbf{L}.$$

Show that T is a constant (which is a manifestation of the property of the Lorentz force that a magnetic field does no work on a moving charge) and find the other constants of motion. Under the assumption that ω_l is much smaller than the initial rotational velocity about the figure axis, obtain expressions for the frequencies and amplitudes of nutation and precession. From where do the kinetic energies of nutation and precession come?

Solution. By definition, $\boldsymbol{\omega}_l = -q\mathbf{B}/2m$, and therefore the Lagrangian is

$$L = \frac{1}{2}I_1(\omega_1^2 + \omega_2^2) + \frac{1}{2}I_3\omega_3^2 + \frac{q}{2m}\mathbf{B} \cdot \mathbf{L}$$

We adopt the same body axes as in Problem 5. Also, suppose the uniform magnetic field is in the z direction in the inertial frame. Then,

$$L = \frac{1}{2}I_1\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 + \frac{q}{2m}\left(\dot{\phi}I_1 B\sin^2\theta + (\dot{\phi}\cos\theta + \dot{\psi})I_3 B\cos\theta\right)$$

Note that ψ and ϕ are cyclic coordinates. Therefore, two integrals of motion are

$$\begin{aligned} p_\psi &= I_3(\dot{\psi} + \dot{\phi}\cos\theta) + \frac{qB}{2m}I_3\cos\theta, \\ p_\phi &= \dot{\phi}I_1\sin^2\theta - I_3\sin\theta(\dot{\psi} + \dot{\phi}\cos\theta) + \frac{qB}{2m}(I_1\sin^2\theta + I_3\cos^2\theta). \end{aligned}$$

(TO BE CONTINUED)

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