Geometric horizon for distorted black holes

Gabriel Sedrick S. Alkuino* and Michael Francis Ian G. Vega II

National Institute of Physics, University of the Philippines, Diliman, Quezon City 1101, Philippines
*Corresponding author: galkuino@nip.upd.edu.ph

Abstract

We previously proposed an equation for locating the geometric horizon of perturbed static spherically symmetric spacetimes using the Kretschmann scalar. The result was fully covariant, but not practical for dealing with perturbations. Here, we reformulate this geometric horizon detector into a more practical form, in terms of the spherical-harmonic coefficients of the metric perturbation that are the usual result of a perturbative calculation about spherically symmetric spacetimes. We apply our result to the Vaidya spacetime to demonstrate how the geometric horizon might differ from the traditional horizons found in the literature.

1 Introduction

Curvature invariants can be used to locate black hole horizons [1–4]. More specifically, it was found that spherically symmetric spacetimes the scalar $I_{,a}I^{,a}$, where I is any well-defined curvature invariant, can be used to locate the event horizon. In our previous article [5], we showed that under first-order perturbations of the metric, the geometric horizon is given by the zero set of

$$H' = g'_{vv} - 2I^{(1)}_{,v}/I_{,r}. \tag{1.1}$$

For static perturbations, this reduces to the Killing horizon; and we conjectured that this might be used to define a horizon for any arbitrary perturbed spacetime. Afterwards, we narrowed our focus to the geometric horizon defined using the Kretschmann scalar, K, as the curvature invariant. In general, any curvature invariant can be used; however, we suspect that the geometric horizon defined by each invariant will no longer coincide in the perturbative case. The reason for choosing the Kretschmann scalar is that it is the simplest invariant that will yield a nontrivial result. Then, the geometric horizon is completely prescribed once we find $K^{(1)}$. This was also derived in our previous article, but we found that the expression can be further simplified:

$$K^{(1)} = -2R^{acde}R^{b}{}_{cde}h_{ab} - 4R^{abcd}\nabla_a\nabla_c h_{bd}. \tag{1.2}$$

In this article, we simply continue the program. Note that the previous results are all fully covariant; i.e., they hold for any coordinate system. But, even though the results are completely general, they are not the most practical. In the next section, we specialize to the Schwarzschild spacetime; the formalism we shall develop is adopted from [6].

2 Spherical harmonics

Since we are dealing with spherically symmetric spacetimes, it is often useful to foliate the manifold by 2-spheres. In other words, we decompose the spacetime manifold $\mathcal{M}$ as a product manifold $\mathfrak{M}^2 \times S^2$, where S^2 is the 2-sphere with the round metric. We will denote coordinates of $\mathfrak{M}^2$ by $\bar{x}^\alpha$, and S^2 by ω^A. We fix the standard spherical coordinate system, $\omega^A = \{\theta, \phi\}$, to S^2; while $\mathfrak{M}^2$ may be expressed in any coordinate system. The Schwarzschild metric can be written as

$$g_{ab} = \bar{g}_{\alpha\beta}(\bar{x})\,\mathrm{d}\bar{x}^\alpha\mathrm{d}\bar{x}^\beta + r(\bar{x})^2\,\Omega_{AB}(\omega)\,\mathrm{d}\omega^A\mathrm{d}\omega^B, \tag{2.1}$$

where $\bar{g}_{\alpha\beta}$ is the metric on $\mathfrak{M}^2$, and Ω_{AB} is the round metric on the unit sphere S^2 [7]. And the perturbation can be written as

$$h_{ab} = \bar{h}_{\alpha\beta}(\bar{x})\,\mathrm{d}\bar{x}^\alpha\mathrm{d}\bar{x}^\beta + h_{\alpha A}\left(\mathrm{d}\bar{x}^\alpha\mathrm{d}\omega^A + \mathrm{d}\omega^A\mathrm{d}\bar{x}^\alpha\right) + r(\bar{x})^2\,\tilde{h}_{AB}(\omega)\,\mathrm{d}\omega^A\mathrm{d}\omega^B. \tag{2.2}$$

The general form of the perturbed metric is

$$[g'_{ab}] = \left[\begin{array}{c:c} \bar{g}_{\alpha\beta} + \bar{h}_{\alpha\beta} & h_{\alpha B} \\ \hdashline h_{A\beta} & r^2(\Omega_{AB} + \tilde{h}_{AB}) \end{array}\right].$$

The Greek indices run from $\{0, 1\}$, while the upper-case Latin indices run from $\{2, 3\}$. Again, due to the spherical symmetry of the background spacetime the perturbation can be conveniently expressed in terms of spherical harmonics [6]. The scalar harmonics are the usual $Y^{lm}(\omega^A)$ in S^2, and they satisfy the eigenvalue equation

$$\left[\tilde{\Delta} + l(l+1)\right] Y^{lm} = 0, \tag{2.3}$$

where $\tilde{\Delta} = \Omega^{AB}\tilde{\nabla}_A\tilde{\nabla}_B$ is the Laplace-Beltrami operator and $\tilde{\nabla}_C$ is the connection on S^2. The vector and tensor harmonics come in two types: even-parity and odd-parity, and they are orthogonal to each other. It will turn out that the odd-parity sector has no contribution to $K^{(1)}$, therefore, we shall only introduce the even-parity sector. The vector harmonics are defined as

$$Y^{lm}_A = \tilde{\nabla}_A Y^{lm}, \tag{2.4}$$

while the tensor harmonics are defined as

$$Y^{lm}_{AB} = \left[\tilde{\nabla}_A\tilde{\nabla}_B + \frac{1}{2}l(l+1)\Omega_{AB}\right] Y^{lm}. \tag{2.5}$$

The even-parity sector of the perturbation can be written as

$$\bar{h}_{\alpha\beta} = \sum_{lm} \underline{h}^{lm}_{\alpha\beta}(\bar{x})\, Y^{lm}, \tag{2.6}$$

$$h_{\alpha B} = \sum_{lm} j^{lm}_{\alpha}(\bar{x})\, Y^{lm}_B, \tag{2.7}$$

$$\tilde{h}_{AB} = \sum_{lm} \left(F^{lm}(\bar{x})\,\Omega_{AB}Y^{lm} + G^{lm}(\bar{x})\, Y^{lm}_{AB}\right), \tag{2.8}$$

where $\underline{h}^{lm}_{\alpha\beta}$, j^{lm}_{α}, F^{lm}, and G^{lm} are objects on $\mathfrak{M}^2$. Note the difference in our convention and in [6].

3 Index gymnastics

The goal now is to further decompose $K^{(1)}$ in terms of objects in $\mathfrak{M}^2$ and S^2. Essentially, we wish to express $\{g_{ab}, \Gamma^a{}_{bc}, R_{abcd}, \ldots\}_{\mathcal{M}}$ in terms of $\left\{\bar{g}_{\alpha\beta}, \bar{\Gamma}^{\alpha}{}_{\beta\gamma}, \bar{R}_{\alpha\beta\gamma\delta}, \ldots\right\}_{\mathfrak{M}^2}$ and $\left\{\tilde{g}_{AB}, \tilde{\Gamma}^{A}{}_{BC}, \tilde{R}_{ABCD}, \ldots\right\}_{S^2}$. And afterwards, express the result in terms of the spherical-harmonic coefficients defined in Section 2. From (1.2), we see that we only need to decompose R^{abcd} and $\nabla_a\nabla_b h_{cd}$. We find that the nonzero components of the Riemann tensor are

$$R^{\alpha\beta\gamma\delta} = \bar{R}^{\alpha\beta\gamma\delta}, \tag{3.1}$$

$$R^{ABCD} = \frac{2M}{r^7}\tilde{R}^{ABCD}, \tag{3.2}$$

$$R^{\alpha B\gamma D} = -\frac{M}{r^5}\bar{g}^{\alpha\gamma}\Omega^{BD}. \tag{3.3}$$

The tensor $\nabla_a\nabla_b h_{cd}$ requires a lot more work. We shall only state the results here, for the derivations see [8], which closely follows the procedure in [6, 9]. The twelve relevant cases are

$$\nabla_\alpha\nabla_\beta h_{\gamma\delta} = \bar{\nabla}_\alpha\bar{\nabla}_\beta h_{\gamma\delta}, \tag{3.4}$$

$$\nabla_\alpha\nabla_\beta h_{\gamma D} = \bar{\nabla}_\alpha\bar{\nabla}_\beta h_{\gamma D} - \frac{2}{r}r_{(\alpha}\bar{\nabla}_{\beta)}h_{\gamma D} + \frac{2}{r^2}r_\alpha r_\beta h_{\gamma D} - \frac{M}{r^3}\bar{g}_{\alpha\beta}h_{\gamma D}, \tag{3.5}$$

$$\nabla_\alpha\nabla_\beta h_{CD} = \bar{\nabla}_\alpha\bar{\nabla}_\beta h_{CD} - \frac{4}{r}r_{(\alpha}\bar{\nabla}_{\beta)}h_{CD} + \frac{6}{r^2}r_\alpha r_\beta h_{CD} - \frac{2M}{r^3}\bar{g}_{\alpha\beta}h_{CD}, \tag{3.6}$$

$$\begin{aligned}\nabla_\alpha\nabla_B h_{\gamma\delta} = {} & \bar{\nabla}_\alpha\tilde{\nabla}_B h_{\gamma\delta} - \frac{2}{r}\bar{\nabla}_\alpha h_{B(\gamma}r_{\delta)} + \frac{4}{r^2}r_\alpha r_{(\gamma}h_{\delta)B} \\ & - \frac{2M}{r^3}\bar{g}_{\alpha(\gamma)}h_{\delta)B} - \frac{r_\alpha}{r}\tilde{\nabla}_B h_{\gamma\delta},\end{aligned} \tag{3.7}$$

$$\nabla_\alpha\nabla_B h_{\gamma D} = \bar{\nabla}_\alpha\tilde{\nabla}_B h_{\gamma D} + \left(r\bar{\nabla}_\alpha h_{\beta\gamma} - r_\alpha h_{\beta\gamma}\right)r^\beta\Omega_{BD} + \frac{M}{r}h_{\alpha\gamma}\Omega_{BD}$$

$$
-\frac{r_\gamma}{r}\bar{\nabla}_\alpha h_{BD} - \frac{M}{r^3}\bar{g}_{\alpha\gamma}h_{BD} + \frac{3}{r^2}r_\alpha r_\gamma h_{BD} - \frac{2r_\alpha}{r}\tilde{\nabla}_B h_{\gamma D}\,, \tag{3.8}
$$

$$
\begin{aligned}
\nabla_\alpha\nabla_B h_{CD} = {} & \bar{\nabla}_\alpha\tilde{\nabla}_B h_{CD} + 2rr^\beta\bar{\nabla}_\alpha h_{\beta(C}\Omega_{D)B} - 4r_\alpha r^\beta h_{\beta(C}\Omega_{D)B} \\
& + \frac{2M}{r}h_{\alpha(C}\Omega_{D)B} - \frac{3r_\alpha}{r}\tilde{\nabla}_B h_{CD}\,,
\end{aligned} \tag{3.9}
$$

$$
\nabla_A\nabla_\beta h_{\gamma\delta} = \tilde{\nabla}_A\bar{\nabla}_\beta h_{\gamma\delta} - \frac{2}{r}\bar{\nabla}_\beta h_{A(\gamma}r_{\delta)} + \frac{4r_\beta}{r^2}r_{(\gamma}h_{\delta)A} - \frac{r_\beta}{r}\tilde{\nabla}_A h_{\gamma\delta}\,, \tag{3.10}
$$

$$
\begin{aligned}
\nabla_A\nabla_\beta h_{\gamma D} = {} & \tilde{\nabla}_A\bar{\nabla}_\beta h_{\gamma D} + \left(r\bar{\nabla}_\beta h_{\alpha\gamma} - r_\beta h_{\alpha\gamma}\right)r^\alpha\Omega_{AD} \\
& - \frac{r_\gamma}{r}\bar{\nabla}_\beta h_{AD} + \frac{3}{r^2}r_\beta r_\gamma h_{AD} - \frac{2r_\beta}{r}\tilde{\nabla}_A h_{\gamma D}\,,
\end{aligned} \tag{3.11}
$$

$$
\begin{aligned}
\nabla_A\nabla_\beta h_{CD} = {} & \tilde{\nabla}_A\bar{\nabla}_\beta h_{CD} + 2rr^\alpha\bar{\nabla}_\beta h_{\alpha(C}\Omega_{D)A} \\
& - 4r^\alpha r_\beta h_{\alpha(C}\Omega_{D)A} - \frac{3r_\beta}{r}\tilde{\nabla}_A h_{CD}\,,
\end{aligned} \tag{3.12}
$$

$$
\begin{aligned}
\nabla_A\nabla_B h_{\gamma\delta} = {} & \tilde{\nabla}_A\tilde{\nabla}_B h_{\gamma\delta} - \frac{4}{r}\tilde{\nabla}_{(A}h_{B)(\gamma}r_{\delta)} + \frac{2}{r^2}r_\gamma r_\delta h_{AB} \\
& - r^\alpha r_{(\gamma}h_{\delta)\alpha}\Omega_{AB} + rr_\alpha\bar{\nabla}_\alpha h_{\gamma\delta}\Omega_{AB}\,,
\end{aligned} \tag{3.13}
$$

$$
\begin{aligned}
\nabla_A\nabla_B h_{\gamma D} = {} & \tilde{\nabla}_A\tilde{\nabla}_B h_{\gamma D} + rr^\alpha\bar{\nabla}_\alpha h_{\gamma D}\Omega_{AB} - 2f\Omega_{A(B}h_{D)\gamma} - 3r^\alpha r_\gamma h_{\alpha(A}\Omega_{BD)} \\
& - \frac{2r_\gamma}{r}\tilde{\nabla}_{(A}h_{B)D} + rr^\alpha\Omega_{D(B}\tilde{\nabla}_{A)}h_{\alpha\gamma}\,,
\end{aligned} \tag{3.14}
$$

$$
\begin{aligned}
\nabla_A\nabla_B h_{CD} = {} & \tilde{\nabla}_A\tilde{\nabla}_B h_{CD} + 2r^2\Omega_{A(C}\Omega_{D)B}r^\alpha r^\beta h_{\alpha\beta} \\
& + 2rr^\alpha\Omega_{D(B}\tilde{\nabla}_{A)}h_{\alpha C} + 2rr^\alpha\Omega_{C(B}\tilde{\nabla}_{A)}h_{\alpha D} \\
& - 2f\Omega_{A(C}h_{D)B} + \Omega_{AB}\left(rr^\alpha\bar{\nabla}_\alpha h_{CD} - 2fh_{CD}\right)\,.
\end{aligned} \tag{3.15}
$$

Note that $h_{\alpha\beta} = \bar{h}_{\alpha\beta}$ and $h_{AB} = r^2\tilde{h}_{AB}$.

4 Kretschmann scalar

Putting things together, the first-order correction to the Kretschmann scalar (1.2) becomes

$$
\begin{aligned}
K^{(1)} = \bar{R}\Bigg\{ & 2\left[\left(\bar{\nabla}_\alpha\bar{\nabla}_\beta\bar{h}^{\alpha\beta} - \bar{\Delta}\bar{h}\right) + \frac{1}{r^2}\left(\tilde{\nabla}_A\tilde{\nabla}_B\tilde{h}^{AB} - \tilde{\Delta}\tilde{h}\right)\right] \\
& + \frac{1}{r^2}\left[r^2\bar{\Delta}\tilde{h} + \tilde{\Delta}\bar{h} - \left(\bar{\nabla}_\alpha\tilde{\nabla}_A + \tilde{\nabla}_A\bar{\nabla}_\alpha\right)h^{\alpha A}\right] - 2\bar{R}\bar{h} \\
& + \frac{4}{r^2}\left[\frac{r_\alpha}{r}\tilde{\nabla}_A h^{\alpha A} + r^\alpha r^\beta\bar{h}_{\alpha\beta} - rr_\alpha\bar{\nabla}_\beta\bar{h}^{\alpha\beta} + \frac{1}{2}rr^\alpha\bar{\nabla}_\alpha\bar{h} - \frac{1}{2}\tilde{h}\right]\Bigg\}.
\end{aligned} \tag{4.1}
$$

Now, using (2.6) - (2.8) the spherical-harmonic decomposition of $K^{(1)}$ is

$$
\begin{aligned}
K^{(1)} = \sum_{lm}\bar{R}\Bigg\{ & 2\left[\left(\bar{\nabla}^\alpha\bar{\nabla}^\beta\underline{h}^{lm}_{\alpha\beta} - \bar{\Delta}\underline{h}^{lm}\right) + \frac{1}{r^2}l(l+1)\left(F^{lm} + \frac{1}{2}l(l+1)G^{lm}\right)\right] \\
& + \frac{4}{r^2}\left[r^\alpha r^\beta\underline{h}^{lm}_{\alpha\beta} - \frac{1}{r}l(l+1)r^\alpha j^{lm}_\alpha - rr^\alpha\bar{\nabla}^\beta\underline{h}^{lm}_{\alpha\beta} + \frac{1}{2}rr^\alpha\bar{\nabla}_\alpha\underline{h}^{lm} - F^{lm}\right] \\
& + \frac{1}{r^2}\left[2r^2\bar{\Delta}F^{lm} + l(l+1)\left(2\bar{\nabla}^\alpha j^{lm}_\alpha - \underline{h}^{lm}\right)\right] - 2\bar{R}\underline{h}^{lm}\Bigg\}Y^{lm}.
\end{aligned} \tag{4.2}
$$

5 Example: Vaidya spacetime

Recall that the horizon defined by H' in (1.1) coincides with the Killing horizon if the perturbations are static, therefore we provide a simple example of a non-static spacetime. Consider the Vaidya metric in advanced Eddington-Finkelstein coordinates [10]. The line element is

$$\mathrm{d}s^2 = -\left(1 - \frac{2M(v)}{r}\right)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\mathrm{d}\Omega^2. \tag{5.1}$$

For the sake of illustration we assume the form $M(v) = M + \dot{M}v$, where $v\dot{M}/M \ll 1$. Then the only nonzero perturbation in (2.6) - (2.8) is $\bar{h}^{00}_{vv} = 2\dot{M}v/r$. Clearly, (4.1) becomes (note that $\bar{h} = 0$)

$$K^{(1)} = \bar{R}\left[2\,\bar{\nabla}^\alpha\bar{\nabla}^\beta \underline{h}^{00}_{\alpha\beta} + \frac{4}{r^2}\left(r^\alpha r^\beta \underline{h}^{00}_{\alpha\beta} - r r^\alpha \bar{\nabla}^\beta \underline{h}^{00}_{\alpha\beta}\right)\right] Y^{00} = \frac{96 M\dot{M}v}{r^6} = \frac{2\dot{M}v}{M}K. \tag{5.2}$$

We see that the horizon detector is essentially given by

$$H' = f - h_{vv} + 2\frac{\nabla_v K^{(1)}}{\nabla_r K} = f - \frac{2\dot{M}v}{r} - \frac{2r\dot{M}}{3M}; \tag{5.3}$$

And we find that the geometric horizon is defined by

$$r_{\mathrm{GH}}(v) \;=\; 2\left(M + \dot{M}v\right) + \frac{8}{3}M\dot{M} \;=\; 2M(v) + \frac{8}{3}M\dot{M}. \tag{5.4}$$

We do not yet know whether this horizon is physically useful or how it compares with other physically motivated horizons in more generic circumstances; it might also be worth checking whether the second term can be removed by a gauge transformation. This is left for future work.

6 Summary

We have derived a formula for the first-order correction to the Kretschmann scalar when the perturbations are expressed in the formalism provided by [6]. This gives us a complete formula for specifying a geometric horizon for any suitable perturbed spacetime. We have also provided a simple example to demonstrate that this horizon differs from the traditional ones found in the literature.

Acknowledgments

This research is supported by the University of the Philippines Diliman Office of the Vice Chancellor for Research and Development through project No. 191937 ORG.

References

[1] M. Abdelqader and K. Lake, Invariant characterization of the Kerr spacetime: Locating the horizon and measuring the mass and spin of rotating black holes using curvature invariants, *Phys. Rev. D* **91**, 084017 (2015).

[2] D. N. Page and A. A. Shoom, Local invariants vanishing on stationary horizons: a diagnostic for locating black holes, *Phys. Rev. Lett.* **114**, 141102 (2015).

[3] A. Coley and D. McNutt, Identification of black hole horizons using scalar curvature invariants, *Class. Quantum Gravity* **35**, 025013 (2017).

[4] A. A. Coley, D. D. McNutt, and A. A. Shoom, Geometric horizons, *Phys. Lett. B* **771**, 131 (2017).

[5] G. S. Alkuino and I. Vega, Michael Francis Ian, Geometric horizon for a distorted static spherically symmetric black hole, in *Proceedings of the Samahang Pisika ng Pilipinas 37* (Tagbilaran City, Philippines, 2019), SPP-2019-3C-01.

[6] K. Martel and E. Poisson, Gravitational perturbations of the Schwarzschild spacetime: A practical covariant and gauge-invariant formalism, *Phys. Rev. D* **71**, 104003 (2005).

[7] S. Carroll, *Spacetime and Geometry: An Introduction to General Relativity* (Addison Wesley, San Fransisco, CA, 2004).

[8] G. S. Alkuino, A geometric horizon for a perturbed Schwarzschild spacetime (unpublished bachelor's thesis) (University of the Philippines Diliman, Quezon City. 2019).

[9] R. M. Wald, *General Relativity* (Chicago University Press, Chicago, 1984).

[10] E. Poisson, *A Relativist's Toolkit* (Cambridge University Press, Cambridge, UK, 2004).