Characterization of a distorted Schwarzschild black hole using curvature invariants

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Abstract

One way to characterize spacetimes is by their curvature invariants. We expand upon the work of Abdelqader and Lake and apply their methods to a perturbed Schwarzschild spacetime—the Preston-Poisson spacetime. A set of curvature invariants may satisfy a syzygy from which one can construct a new dimensionless invariant which may provide an intuitive visualization of the deviation of two spacetimes. However, we point out some of the shortcomings of such an invariant. We also use the curvature invariants to obtain the approximate location of the horizon of the Preston-Poisson black hole.

Keywords: curvature invariants, syzygy, black holes

1 Introduction

Suppose we are given two metrics expressed in different coordinate systems. How do we determine if the two metrics pertain to different spacetimes and are not simply disguised by some coordinate transformation? One way to test is to use curvature invariants—scalars obtained from operations on the curvature tensor—as a means to test the nonequivalence of metrics, since inequivalent invariants imply inequivalent metrics [1]. However, the converse is not true and it has been shown by Coley et al. that Kundt class metrics can have identical invariants [1, 2]. In a paper by Abdelqader and Lake (AL), they proposed a way to visualize spacetimes based on their curvature invariants [3]. The idea is that some spacetimes have a maximum number of functionally independent curvature invariants, say n. If we wish to calculate m more curvature invariants then the $n+m$ invariants must satisfy some m independent functional relationships, each called a syzygy, in order to preserve n. Note that the syzygies are not unique and it is possible to construct different syzygies such that only m will be independent. The syzygies can then be used as an invariant measure since they are generally not satisfied by different spacetimes. In this case, for a spacetime which satisfies a certain syzygy, we can obtain a measure of how much another spacetime deviates from the previous one based on the syzygy. In the example by AL, they introduced seven curvature invariants which they computed for the Kerr spacetime. According to Page and Shoom (PS), since $n=4$ for the Kerr spacetime (corresponding to M, a, r, and θ in Boyer-Lindquist coordinates [4]) one can obtain three syzygies—one of which was used by AL to obtain a measure of the "Kerr"-ness of the Curzon-Chazy spacetime [3]. In this paper, we use the method introduced by AL in [3] to examine a perturbed Schwarzschild spacetime.

2 Curvature invariants of the Schwarzschild spacetime

For the Schwarzschild spacetime (and any static axisymmetric spacetime according to PS [4]), only three out of the seven curvature invariants introduced by AL are nonzero and they are

$$I_1 = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}, \tag{2.1}$$

$$I_3 = \nabla_\mu R_{\alpha\beta\gamma\delta}\nabla^\mu R^{\alpha\beta\gamma\delta}, \tag{2.2}$$

$$I_5 = \nabla_\mu I_1 \nabla^\mu I_1, \tag{2.3}$$

where we replace the Weyl tensor with the Riemann tensor to simplify later computations; we can do this since $C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta}$ in vacuum.

The curvature invariants must satisfy one syzygy from which a new dimensionless invariant can be constructed

$$\chi = \frac{-I_5 + \frac{12}{5}I_1 I_3}{I_1^{5/2}} \tag{2.4}$$

where the numerator is a syzygy which follows from the definition of the complex invariants and was derived in [3, 4]; i.e the numerator is always zero for the Schwarzschild spacetime. Whereas the denominator is only a scaling factor to make χ dimensionless [3]. Furthermore, we can visualize how much each point of some unknown spacetime deviates from Schwarzschild by constructing another invariant

$$K = e^{-s\chi^2}, \tag{2.5}$$

where s is an arbitrary sensitivity parameter [3]. Note that K does not provide an absolute measure as it only provides an intuitive picture of the spacetime region which can be useful when comparing two nearby points. Clearly, K approaches 1 when the spacetime is closer to Schwarzschild and approaches zero the larger the difference of the values in (2.1–2.3). (AL called K the "Kerrness" invariant as it provides a measure for deviations from the Kerr spacetime; in our present analysis we might call this as the "Schwarzschildness" invariant.) In [3], AL proposed to use K as a heat map which can provide an "invariant and intuitive method to compare and visualize spacetimes".

3 The Preston-Poisson spacetime

We now use AL's invariants, χ and K, to visualize a perturbed Schwarzschild spacetime. In particular, we consider the spacetime introduced by Preston and Poisson (PP) in [5]. Consider a black hole immersed in a uniform magnetic field, where the physical situation can be thought of as a black hole that was slowly inserted in a giant solenoid which produces a uniform magnetic field of strength B in the region of its linear extension [5]. The nonzero components of the Preston-Poisson metric, expressed in light-cone coordinates, and accurate through first order in B^2 and $\mathcal{E}$, are

$$\begin{aligned} g_{vv} =& -\left(1-\frac{2M}{r}\right) - \frac{1}{9}B^2\left(r(3r-8M) + (3r^2-14Mr+18M^2)(3\cos^2\theta - 1)\right) \\ &+ \mathcal{E}(r-2M)^2(3\cos^2\theta-1), \end{aligned} \tag{3.1}$$

$$g_{vr} = 1, \tag{3.2}$$

$$g_{v\theta} = \left(\frac{1}{3}B^2(r-3M) - \mathcal{E}(r-2M)\right) r^2 \sin\theta\cos\theta, \tag{3.3}$$

$$g_{\theta\theta} = r^2 + \frac{1}{3}B^2 r^4(\cos^2\theta - 1) + \left(B^2M^2 + \mathcal{E}(r^2-2M^2)\right) r^2\sin^2\theta, \tag{3.4}$$

$$g_{\phi\phi} = r^2\sin^2\theta + \frac{1}{3}B^2r^4(\cos^2\theta-1)\sin^2\theta - \left(B^2M^2 + \mathcal{E}(r^2-2M^2)\right) r^2\sin^4\theta, \tag{3.5}$$

where $\mathcal{E}$ is a parameter associated with the tidal field due to the solenoid and is of the same order of magnitude as B^2 [5, 6]. The perturbations will remain small as long as $r^2B^2 \ll 1$ and $r^2\mathcal{E} \ll 1$ [5, 6]. The invariant measure for the Preston-Poisson spacetime, $\chi_{\rm PP}$, can be computed using (2.4) and it is given by

$$\chi_{\rm PP} = \frac{1}{10\sqrt{3}M}\left(6M - 5r + (33r-78M)\cos 2\theta\right) r^2B^2 + \frac{3\sqrt{3}}{5M}(2M-r)(1+3\cos 2\theta)r^2\mathcal{E} + \mathcal{O}(B^4, \mathcal{E}^2). \tag{3.6}$$

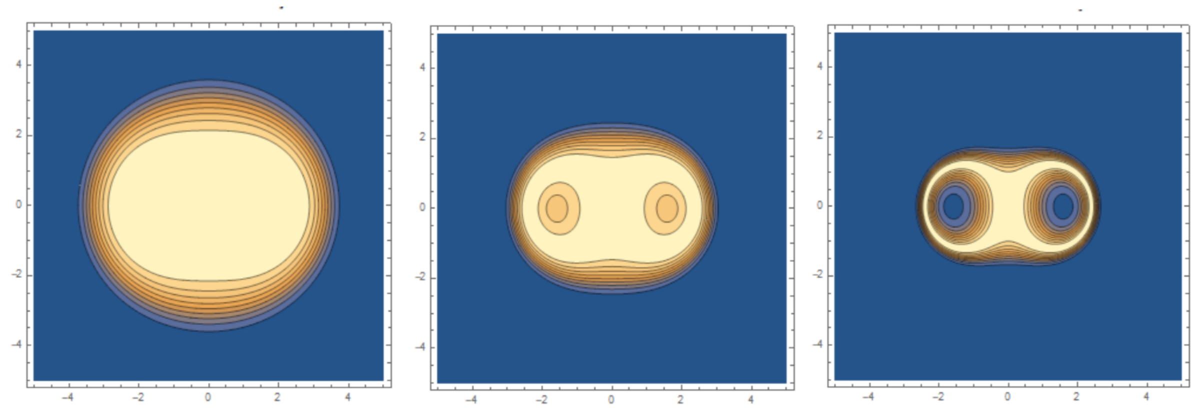
Note that the Preston-Poisson spacetime is no longer a vacuum spacetime and in general $C_{\alpha\beta\gamma\delta} \neq R_{\alpha\beta\gamma\delta}$; however, it does not matter that we replaced the Weyl tensor with the Riemann tensor since they both satsify the syzygy in the unperturbed case (Schwarzschild) and we need only concern ourselves with the relative deviations from the syzygy; i.e. the choice of invariants do not matter as long as they satisfy some syzygy in the unperturbed case. Also, for $\chi_{\rm PP}$ to become unit-less we express r in units of M, B in units of $1/M$, and $\mathcal{E}$ in units of $1/M^2$.

4 Special case: the Schwarzschild-Melvin spacetime

When $\mathcal{E} = \frac{1}{2}B^2$, the metric becomes that of the Schwarzschild-Melvin spacetime which describes a black hole immersed in Melvin's magnetic universe [5]. In this case (3.6) reduces to

$$\chi_{\rm SM} = \frac{1}{5\sqrt{3}M}\left(12M - 7r + 3(r-4M)\cos 2\theta\right) r^2B^2 + \mathcal{O}(B^4). \tag{4.1}$$

Hereafter, we drop the $\mathcal{O}$ symbol as all further computations will only be up to first order in B^2 and $\mathcal{E}$. Inserting (4.1) to (2.5), we may obtain a contour plot which can serve as a tool to locally visualize the Schwarzschild-Melvin spacetime. Figure 1 provides such an example. It is worth re-emphasizing that K does not provide a real physical measure of the spacetime deviation. There is already some arbitrariness in using $I_1^{5/2}$ in the denominator as a scaling factor and even more so that we can freely choose an s; this can be seen in Figure 1. As its name implies, we may always choose s to our preferred sensitivity in



(a) $s = 50$ (b) $s = 500$ (c) $s = 5000$

Figure 1: Contour plots for the K invariant as a function of r and θ with $B = 0.1M^{-1}$ for different sensitivity parameters. The contour lines correspond to $K = 0.1, 0.2, \ldots, 0.9$ and K ranges from 0 to 1. The darker (blue) regions corresponds to a lower K value. For a better visualization, a coordinate transformation $(x = r\sin\theta, y = r\cos\theta)$ was used. If we let $s \to +\infty$, we obtain the contour where χ_{SM} vanishes.

detecting deviations from our spacetime when comparing nearby points. We can see from Figure 1 that the Schwarzschild-Melvin spacetime deviates from the pure Schwarzschild spacetime in the asymptotic region. This is expected since the Schwarzschild spacetime is asymptotically flat while the Schwarzschild-Melvin spacetime becomes Melvin's magnetic universe in its asymptotic region. We have also obtained contour plots for the general Preston-Poisson spacetime with arbitrary $\mathcal{E}$; but, the results were not significant enough to warrant a separate discussion.

5 Using χ as an invariant measure

One problem with using χ alone as an invariant measure is that although Schwarzschild curvature invariants I_1, I_3, I_5 imply $\chi = 0$, the converse is not true. We can see from (4.1) that χ_{SM} vanishes when

$$r_{\chi_{\mathrm{SM}}}(\theta) := \frac{12M(\cos 2\theta - 1)}{3 - 7\cos 2\theta}, \tag{5.1}$$

whose contour looks like an '∞' symbol which can be seen in Figure 1 if we let $s \to +\infty$. This can be a problem if we are to use χ as a local measure of spacetime deviation. If we plug in (5.1) to (2.1–2.3) the curvature invariants we would obtain will not correspond to the Schwarzschild curvature invariants at $r_{\chi_{\mathrm{SM}}}$, yet they satisfy the Schwarzschild syzygy. Therefore, we must be careful when using χ as a local measure of spacetime deviation, since the vanishing of χ does not imply that the spacetime is Schwarzschild. This is unsurprising since by constructing χ we essentially reduced three numbers—I_1, I_3, and I_5—to one. We might try to use χ to characterize spacetimes based on their $\chi = 0$ contours. Since we know that χ is zero everywhere for the Schwarzschild spacetime and only for some contour in the Schwarzschild-Melvin spacetime, we might imagine this contour as some sort of an imprint of the Schwarzschild-Melvin spacetime. One nice thing about this is that we no longer need to invoke a sensitivity parameter. However, we are still faced with the problem of uniqueness. Clearly, since $r_{\chi_{\mathrm{SM}}}$ is independent of B then its contour is not unique; and this is not what we want when characterizing spacetimes.

6 Locating the horizon

With the invariants we have so far encountered, one might ask if there exists an invariant which can be used to locate the horizon; after all, the horizon is one of the more important things to know about when dealing with black hole spacetimes. It turns out that if we take the wedge product of the gradient of enough invariants then the norm of the wedge product vanishes at the horizon [4, 7–10]. This is summarized in a theorem by PS which states that "for a spacetime of local cohomogeneity n that contains a stationary horizon and which has n scalar polynomial curvature invariants $S^{(i)}$ whose gradients are well defined there, the n-form wedge product $W = \mathrm{d}S^{(1)} \wedge \mathrm{d}S^{(2)} \ldots \wedge \mathrm{d}S^{(n)}$ has zero squared norm on the horizon,

$$\|W\|^2 \equiv \frac{1}{n!}\delta^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\beta_n} g^{\beta_1\gamma_1}\ldots g^{\beta_n\gamma_n} S^{(1)}_{\alpha_1}\ldots S^{(n)}_{\alpha_n} S^{(1)}_{\gamma_1}\ldots S^{(n)}_{\gamma_n} \tag{6.1}$$

where the permutation tensor $\delta^{\alpha_1 \dots \alpha_n}_{\beta_1 \dots \beta_n}$ is $+1$ if $\alpha_1 \dots \alpha_n$ is an even permutation of $\beta_1 \dots \beta_n$, is -1 if odd, and is zero otherwise" [4]. Since the Preston-Poisson spacetime is a static axisymmetric spacetime and therefore of cohomogeneity 2 [4], we might expect to locate its horizon using any pair of our curvature invariants; i.e. $\mathrm{d}I_1 \wedge \mathrm{d}I_3$, $\mathrm{d}I_3 \wedge \mathrm{d}I_5$, and $\mathrm{d}I_1 \wedge \mathrm{d}I_5$. However, we would not find the horizon (at least in the first order of B^2 and $\mathcal{E}$) since the Preston-Poisson spacetime is treated as a perturbed Schwarzschild spacetime, a spherically symmetric spacetime and therefore of cohomogeneity 1—for which $\mathrm{d}I_1 \wedge \mathrm{d}I_3$, $\mathrm{d}I_3 \wedge \mathrm{d}I_5$, and $\mathrm{d}I_1 \wedge \mathrm{d}I_5$ are zero. For the Schwarzschild spacetime, it is the norm of $\mathrm{d}I_1$, $\mathrm{d}I_3$, and $\mathrm{d}I_5$ which vanishes at the horizon. If we compute the squared norm of $\mathrm{d}I_1$ for the Preston-Poisson spacetime we obtain

$$\|\mathrm{d}I_1\|^2 = -\frac{41472M^4}{r^{15}}(2M-r) - \frac{3456M^3}{r^{14}}(24M^3 - 24M^2r + 16Mr^2 - 5r^3)(1 + 3\cos 2\theta)\mathcal{E} \tag{6.2}$$

$$+\frac{1152M^3}{r^{14}}(36M^3 - 60M^2r + 16Mr^2 + r^3 + 3(36M^3 - 28M^2r + 4Mr^2 + r^3)\cos 2\theta)B^2, \tag{6.3}$$

and we find that this vanishes (up to order B^2 and $\mathcal{E}$) when

$$r_{\mathrm{horizon}}(\theta) = 2M\left(1 + \frac{2}{3}M^2B^2\sin^2\theta\right), \tag{6.4}$$

which is exactly what was obtained in [5]. Note that r_{horizon} does not depend on $\mathcal{E}$ as pointed out by PP.

7 Conclusions

The dimensionless invariant measure χ (or K) was proposed by AL as a tool to intuitively visualize spacetimes. In [3], they proposed to use χ as an aid to find out where a spacetime is approximately Kerr so that the other tools they proposed for numerical relativity calculations of approximate Kerr spacetimes can be trusted. However, we must be careful when using χ as we have shown that there are instances when χ is zero but the spacetimes are not similar. Furthermore, we were able to locate the horizon of a perturbed Schwarzschild black hole using the method proposed by PS [4].

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