Geometric horizon for a distorted static spherically symmetric black hole

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Abstract

Recently, an alternative definition of a black hole horizon was proposed; such geometric definition is afforded by curvature invariants—scalars formed from the contractions involving the curvature tensor and its derivatives. It has been shown that for any stationary horizon, one can construct certain combinations of the gradient of curvature invariants such that they have vanishing norm on the horizon; this allows us to define a geometric horizon. However, its role in dynamical spacetimes is still not yet fully understood. In this work, we study the geometric horizon of a static spherically symmetric spacetime under first-order metric perturbations. Specifically, we derive an equation for a perturbed geometric horizon—defined by the vanishing squared norm of the gradient of the Kretschmann scalar; this equation is valid for any static spherically symmetric background spacetime in dimension $d \geq 4$, as long as the gradient of the Kretschmann scalar is well-defined.

Keywords: Black hole, curvature invariant, geometric horizon, metric perturbations

1 Introduction

The standard definition of a black hole and its event horizon—the complement (if it exists) of the causal past of future null infinity and its boundary, respectively—makes it unwieldy for most practical uses. A classical alternative would be to use the Killing horizon, since for any stationary and asymptotically flat black hole spacetime, the event horizon is a Killing horizon [1]. A review of classical and modern alternative definitions of black hole horizons and their uses can be found in [2]. A more recent alternative definition was proposed using curvature invariants—scalars obtained from contractions involving the curvature tensor and its derivatives. Using curvature invariants to detect the event horizon is not new. For instance, it is known that the Karlhede invariant $R_{abcd;e}R^{abcd;e}$ vanishes and changes sign as one crosses any smooth static spherically symmetric event horizon [3, 4], where R_{abcd} is the Riemann curvature tensor and $R_{abcd;e}$ is its (covariant) derivative. In [5], they found a horizon detector for the Kerr black hole using two curvature invariants. This was later generalized by [4] for any stationary spacetime. In [4], they also proved that for any static spherically symmetric black hole spacetime, the gradient of any well-defined curvature invariant must have zero (squared) norm on the event horizon [4]. This allows us to define the event horizon as the surface defined by the zero set of the norm of the gradient of some curvature invariant I. This definition is quasilocal, unlike the traditional definition which is global and teleological in nature. In [4], they proposed the geometric horizon be used to detect the approximate location of the horizon of any nearly stationary spacetime. However, the role of the geometric horizon in dynamical spacetimes is not yet fully resolved [6, 7]. Thus, in this paper we derive an equation for the geometric horizon of a static spherically symmetric black hole spacetime under first-order perturbations. The formalism is fully covariant and does not assume the Einstein field equations, but only that the (static spherically symmetric) background spacetime must contain a smooth stationary horizon and that a curvature invariant must have a well-defined gradient there [4]. Moreover, the results are expected to hold for higher dimensions, as long as the background spacetime has local cohomogeneity one; the local cohomogeneity is defined as “the codimension of the maximal dimensional orbits of the isometry group of the local metric, ignoring the breaking of any of these local isometries by global considerations”, copied directly from [4]. Thus, the geometric horizon is far more general; however, we shall be mainly concerned with its application in black hole spacetimes.

2 Perturbations

Let g_{ab} be the metric of the background spacetime. Suppose we introduce a small perturbation to this spacetime such that the metric of the perturbed spacetime is now ${g'}_{ab}$. For first-order perturbations, we have

$$ {g'}_{ab} = g_{ab} + h_{ab}, \tag{2.1} $$

$$g'^{ab} = g^{ab} - h^{ab}, \tag{2.2}$$

where $h^{ab} = g^{ac}g^{bd}h_{cd}$. We simply ignore terms which involve more than one h_{ab}'s. Let ∇_c and ∇'_c be the metric connection of g_{ab} and g'_{ab}, respectively. Then their difference defines a $(1,2)$-tensor field [8]

$$(\nabla_a - \nabla'_a)\omega_b = C^c{}_{ab}\omega_c. \tag{2.3}$$

For first-order perturbations, a straightforward calculation shows that

$$C^c{}_{ab} = \frac{1}{2}g^{cd}\left(\nabla_a h_{bd} + \nabla_b h_{da} - \nabla_d h_{ab}\right). \tag{2.4}$$

From now on, perturbed objects shall be marked with $(\,'\,)$.

3 Geometric horizon

Let $I = \mathbf{I}(g)$ be any well-defined curvature invariant in the background spacetime such that $H = I_{;a}I^{;a}$ vanishes on the horizon. Then we expect that $H' = I'_{;a}I'^{;a}$ approximately vanishes on the horizon of the perturbed spacetime, where $I' = \mathbf{I}(g')$. Clearly,

$$H' = g'^{ab}I'_{;a}I'_{;b} \tag{3.1}$$

$$H + H^{(1)} = \left(g^{ab} - h^{ab}\right)\left(I_{;a} + I^{(1)}_{;a}\right)\left(I_{;b} + I^{(1)}_{;b}\right) \tag{3.2}$$

$$H^{(1)} = 2I^{;a}I^{(1)}_{;a} - h^{ab}I_{;a}I_{;b}, \tag{3.3}$$

where $H^{(1)}$ is the first-order correction to the geometric horizon detector of the background spacetime, and $I^{(1)}$ is the first-order correction to the curvature invariant I. Thus, we only need to determine $I^{(1)}$. We have made the assumption that the geometric horizon can be treated perturbatively. In [4], they claimed that for a distorted static black hole in four dimensions that has no spatial Killing vector fields on and outside the horizon, the wedge product of the gradient of three curvature invariants is needed to locate the horizon. However, we claim that for first-order perturbations, the gradient of one curvature invariant suffices. In other words, we assume that the cohomogeneity requirement in [4] applies only to the background spacetime.

4 Schwarzschild spacetime

The Schwarzschild spacetime is the simplest black hole spacetime, and describes a static spherically symmetric black hole in vacuum. In the Schwarzschild spacetime, using the ingoing Eddington-Finkelstein coordinates, we can easily verify that for static perturbations the horizon detector H' reduces to the equation for the Killing horizon. The ingoing Eddington-Finkelstein coordinates are horizon-penetrating and well-behaved near the horizon. Thus, they are better suited for horizon perturbations than the Schwarzschild coordinates, wherein the metric becomes singular at the event horizon. In the ingoing Eddington-Finkelstein coordinates, the Schwarzschild line element is given by

$$\mathrm{d}s^2 = -f\mathrm{d}v + 2\mathrm{d}v\mathrm{d}r + r^2\mathrm{d}\Omega^2, \tag{4.1}$$

where $f(r) := 1 - 2M/r$. Because the Schwarzchild spacetime is static and spherically symmetric, the curvature invariant must only depend on the radial coordinate. Thus,

$$H' = \left(g^{rr} - h^{rr}\right)I_{;r}I_{;r} + 2I_{;r}\left(g^{rv}I^{(1)}_{;v} + g^{rr}I^{(1)}_{;r}\right), \tag{4.2}$$

$$= g'^{rr}I_{;r}I_{;r} + 2I_{;r}\left(I^{(1)}_{;v} + fI^{(1)}_{;r}\right). \tag{4.3}$$

We can then expand r around $2M$ to find its first-order correction such that $H'(r') = 0$. A direct computation shows that we can ignore terms involving products of f and first-order corrections ($\sim h$), as they will not contribute to the first-order correction to r which defines the geometric horizon. And since we are only interested in the zero set of H', we may divide everything by $-\left(I_{;r}\right)^2$, assuming it does not vanish. In the end, we shall be able to redefine H' as

$$H' = g'_{vv} - 2I^{(1)}_{;v}/I_{;r}. \tag{4.4}$$

For static perturbations, $I^{(1)}_{;v} = 0$; thus H' simply reduces to the equation for the Killing horizon, as it should [4].

5 Kretschmann scalar

For black hole spacetimes, one of the easiest nontrivial curvature invariant is the Kretschmann scalar $R_{abcd}R^{abcd}$, a second-order algebraic invariant. We now derive the first-order correction to the Kretschmann scalar. The Riemann tensor is defined as

$$R_{abc}{}^{d}\omega_{d} = \left(\nabla_{a}\nabla_{b} - \nabla_{b}\nabla_{a}\right)\omega_{c}. \tag{5.1}$$

It is straightforward to show that

$$R'_{abc}{}^{d} = R_{abc}{}^{d} - \left(\nabla_{a}C^{d}{}_{bc} - \nabla_{b}C^{d}{}_{ac}\right). \tag{5.2}$$

We may further write

$$R'_{abcd} = R_{abcd} + h_{d}{}^{e}R_{abce} + \nabla_{b}C_{dac} - \nabla_{a}C_{dbc}, \tag{5.3}$$

$$R'^{abcd} = R^{abcd} - R^{ab}{}_{g}{}^{d}h^{cg} - R^{a}{}_{f}{}^{cd}h^{bf} - R_{e}{}^{bcd}h^{ae} + \nabla^{b}C^{dac} - \nabla^{a}C^{dbc}, \tag{5.4}$$

and contracting them gives K'. The first-order correction to the Kretschmann scalar is then given by

$$K^{(1)} = 2R^{abcd}\left(\nabla_{b}C_{dac} - \nabla_{a}C_{dbc}\right) + R^{abcd}\left(R_{abce}h_{d}{}^{e} - R_{abed}h_{c}{}^{e} - R_{aecd}h_{b}{}^{e} - R_{ebcd}h_{a}{}^{e}\right). \tag{5.5}$$

Finally, we use (2.4) and the symmetries of the Riemann tensor to further simplify

$$K^{(1)} = -2R^{abcd}\left(\nabla_{a}\nabla_{c}h_{bd} - \nabla_{b}\nabla_{c}h_{ad}\right) - R^{abcd}\left(R_{aecd}h_{b}{}^{e} + R_{ebcd}h_{a}{}^{e}\right). \tag{5.6}$$

Thus, we now have an equation for a geometric horizon detector

$$H^{(1)} = 2K^{;a}K^{(1)}_{;a} - h^{ab}K_{;a}K_{;b}. \tag{5.7}$$

Similar derivations can also be done for other curvature invariants. For example, one might use the Karlhede invariant, a third-order differential invariant; although this is a lot harder to simplify, since third-derivatives are now involved. Of course, we do not know how the geometric horizon defined by the Karlhede invariant differs from that of the Kretschmann scalar.

6 Discussion

We have derived an equation which may be used to approximately locate a geometric horizon of a perturbed static spherically symmetric spacetime; i.e. the horizon is given by the zero set of H', although we do not yet know how such a horizon might be used. This can be used for checking whether geometric horizons in generically perturbed black holes have physical meaning, e.g., if they correspond at all to event horizons. The requirement that the background spacetime be static and spherically symmetric is so that the local cohomogeneity is one, otherwise we would have to take wedge products of the gradient of curvature invariants. Again, we assumed that under small perturbations the local cohomogeneity requirement is only for the background spacetime; as such, only one curvature invariant is needed for an otherwise distorted spacetime with no symmetries. Nevertheless, this assumption was supported by the fact that we were able to show that for static perturbations of the Schwarzschild spacetime (in the ingoing Eddington-Finkelstein coordinates), the equation for the geometric horizon reduces to that of the Killing horizon. In the case where the gradient of the Kretschmann scalar is well-defined, a formula was provided for the geometric horizon defined by the vanishing of its (squared) norm; for now, we do not know how this horizon relates to other notions of horizons. It will be interesting to check this possible correspondence with explicit calculations. We shall leave this for future work. Finally, we stress that the results in Sections 3 and 5 are fully covariant and can be easily generalized to higher dimensions with the appropriate conditions.

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