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A geometric horizon for a perturbed Schwarzschild spacetime

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ENDORSEMENT

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ABSTRACT

A geometric horizon for a perturbed Schwarzschild spacetime

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An alternative quasi-local definiton of the event horizon, through the use of curvature invariants, was recently proposed; and such geometric definition of a horizon can be generalized to dynamical black holes. One such geometric horizon for the Schwarzschild black hole is defined by the vanishing norm of the gradient of the Kretschmann scalar. In this work, we apply first-order perturbations to the Schwarzschild metric and derive a formula for the correction to the geometric horizon. We verify that this geometric horizon reduces to the Killing horizon for static perturbations. An invariant method to physically characterize and visualize spacetimes was also recently proposed, and we apply this method to a perturbed Schwarzschild spacetime.

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Table of Contents

Abstract **ii**

1 Introduction **1**
1.1 A rapid overview of tensors . . . 3

2 The Schwarzschild geometry **5**
2.1 The Schwarzschild solution . . . 5
2.2 Inside the black hole . . . 6

3 Curvature invariants **10**
3.1 Spacetime classification . . . 13
3.2 Physical characterization . . . 14
3.3 Geometric horizon . . . 16

4 Perturbations I: the basics **18**
4.1 Linear perturbations . . . 18
4.2 Schwarzschild as a product manifold . . . 19
4.3 The Preston-Poisson spacetime . . . 21
4.4 A preliminary investigation . . . 22

5 Perturbations II: a covariant formalism **28**
5.1 Metric perturbations revisited . . . 28
5.2 The ∇'_c connection . . . 30
5.3 The Riemann tensor . . . 31
5.4 The Kretschmann scalar . . . 33
5.5 Horizon detector . . . 35

6 Perturbations III: a practical formalism **37**
6.1 Some remarks . . . 37
6.2 Index gymnastics . . . 39
6.3 Decomposing $K_\clubsuit$ . . . 43
6.4 Decomposing $K_\spadesuit$ . . . 46
6.5 Tensor spherical harmonics . . . 47
6.6 Final form . . . 49

7 Conclusions and Recommendations **51**

References **53**

Chapter 1

Introduction

Black holes are arguably the most mysterious objects in the universe; and this shroud of mystery is what makes them all the more interesting. Black holes, once mere mathematical curiosities, are now taking the center stage of modern astrophysics and high energy physics. According to Strominger, black holes are the harmonic oscillators of the 21^{st} century [1]. This decade alone witnessed the first direct observation of gravitational waves and the first image of a black hole. Thus, black holes continue to be a rich source of problems for both theoretical and experimental research.

The defining feature of a black hole is its event horizon. However, the traditional definition of the event horizon is not always the most convenient [2]. Recently, it has been shown that we can construct certain scalars from curvature invariants that vanish on any stationary horizon [3]. The advantage is that we can locate the event horizon locally using curvature invariants, without the need to rely on the traditional nonlocal definition of the event horizon. Moreover, such a horizon described by the vanishing of some scalar constructed from curvature invariants can be generalized to dynamical spacetimes, which can be useful to numerical relativity simulations [4, 5]. Curvature invariants can also be used for spacetime classification and physical characterization [6]. Thus, in this work we shall also explore some applications of curvature invariants in general relativity.

Outline

Sections 2.1 - 4.3 serve as an extended introduction; the actual work starts in Section 4.4. In Chapter 2 we introduce the Schwarzschild spacetime—a black hole spacetime, which will serve as the background spacetime wherein we add the metric perturbations. We also introduce the event horizon and explain the need for horizon-penetrating coordinates, such as the ingoing Eddington-Finkelstein coordinates. In Chapter 3 we discuss curvature invariants and how they are useful in general relativity. In particular, we shall be interested in the geometric horizon, an alternative definition for a black hole horizon. In Chapter 4 we introduce the basics of first-order metric perturbation theory for the Schwarzschild spacetime. We also introduce one example of a perturbed Schwarzschild spacetime, which we shall examine using the tools in the previous chapter. In Chapters 5 and 6 we derive an expression for a perturbed geometric horizon detector.

Problem statement

The event horizon of a Schwarzschild black hole can be located using curvature invariants—the gradient of the Kretschmann scalar is null on the event horizon. This provides a local characterization of the event horizon. This can be generalized to provide an alternative definition for black hole horizons, which may be applied to dynamical spacetimes. We apply first-order metric perturbations and examine the behavior of this horizon.

Limitations

We shall only consider first-order perturbations of the Schwarzschild spacetime. This limits the applicability of our work, especially in astrophysics, as most black holes in the universe are rotating. Also, most of the literature on curvature invariants use the Newman-Penrose (or the Geroch–Held–Penrose) formalism, as it is more efficient for computations. However, since we plan to adopt the formalism of [7] for metric perturbations, we shall stick with the classical tensor-calculus formalism.

Notation and conventions

Geometric units ($c = G = 1$) are used throughout. The metric signature is $(-, +, +, +)$. The abstract index notation is assumed unless otherwise stated.

1.1 A rapid overview of tensors

We review some basic concepts from differential geometry—specifically tensors—in the context of general relativity for 'completeness'.[1] Everything here can be found in standard textbooks [8–11]; the goal is to establish some notation which we use.[2] Let $\mathcal{M}$ be a smooth manifold and define $\mathcal{F}$ as the ring of smooth functions on $\mathcal{M}$.[3] Given a point $p \in \mathcal{M}$, we can think of a (tangent) vector X^a at p as a map which takes a smooth function and returns its directional derivative along some C^1 curve evaluated at p; the tangent space V_p is the real vector space spanned by such vectors. The cotangent space V_p^*, whose elements are called covectors (one-forms), is the dual space of V_p, consisting of linear functionals on V_p. The duality means that we can also interpret vectors as linear functionals on V_p^*; this provides a canonical (natural) pairing between vectors and covectors. A (k, l)-tensor over V_p is a multilinear map which sends k covectors and l vectors to the reals. A vector is a $(1, 0)$-tensor and a covector is a $(0, 1)$-tensor. The universal property of the tensor product allows us to interpret (k, l)-tensors as elements of the tensor product space ${V_p}^{\otimes k} \otimes {V_p}^{*\otimes l}$.[4] A (smooth) vector field is a smooth section of the tangent bundle; i.e., a smooth assignment of a vector at each point in $\mathcal{M}$. Since vectors at a point map to the reals, vector fields map to smooth functions. We denote by $\mathcal{V}$ the free $\mathcal{F}$-module of vector fields, and $\mathcal{V}^*$ for covector fields. Similarly, a (k, l)-tensor field is a smooth section of some tensor bundle over the tangent bundle, and we denote by $\mathcal{T}^k_l$ the free $\mathcal{F}$-module of (k, l)-tensor fields. Often, we shall not discriminate between tensors at a point and tensor fields as the context should make things clear; we shall simply refer to both as tensors and use the same variables. The (pseudo-)metric tensor field, or simply the metric, is a symmetric nondegenerate bilinear map $g_{ab} : \mathcal{V} \times \mathcal{V} \to \mathcal{F}$. This provides an isomorphism $\flat : \mathcal{V} \to \mathcal{V}^* : X^b \mapsto X_a := g_{ab} X^b$. The inverse metric is defined as $g^{ab} : \mathcal{V}^* \times \mathcal{V}^* \to \mathcal{F}$ such that $g_{ab} X^a X^b = g^{ab} X_a X_b$.[5] This provides the map $\sharp : \mathcal{V}^* \to \mathcal{V} : \omega_b \mapsto \omega^a := g^{ab} \omega_b$, which is the inverse of $\flat$. Thus, the metric provides a

[1] This is simply not possible.

[2] One of the biggest problems among textbooks on differential geometry is the difference in notation. Ironically, in order to address this problem we shall introduce our own notation.

[3] We shall ignore global (topological) considerations on $\mathcal{M}$.

[4] The ordering of the vector spaces can also be made arbitrary.

[5] The inverse metric is not inverse in the sense that it is an inverse map, the reason will be revealed shortly. Another name for the inverse metric is the dual metric, since this is the canonical metric on the dual space.

canonical isomorphism between $\mathcal{V}$ and $\mathcal{V}^*$, called the musical isomorphism.[6] Clearly, the matrix of g^{ab} is the inverse of the matrix of g_{ab}; and this is why g^{ab} is called the inverse metric. Clearly, we see how the metric acts as an index raising and lowering operator.

[6]This means we can ignore distinctions between vectors and covectors.

Chapter 2

The Schwarzschild geometry

In this chapter we discuss the Schwarzschild geometry. The material covered here can be found in standard textbooks in general relativity [8–10, 12]. The primary goal of this chapter is to provide sufficient context and motivation for the things we shall encounter later on.

2.1 The Schwarzschild solution

Immediately after Einstein completed his general theory of relativity, Schwarzschild found an exact solution (to the field equations) which uniquely describes the gravitational field outside a static spherically symmetric mass in vacuum [9, 13]. The solution is usually expressed in the Schwarzschild coordinates, where the line element is given by

$$\mathrm{d}s^2 = -f\,\mathrm{d}t^2 + f^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{2.1}$$

where

$$f(r) = 1 - \frac{r_s}{r}. \tag{2.2}$$

The t coordinate can be interpreted as the time measured by an inertial observer infinitely far away, while the r coordinate can be interpreted as an areal radius, taken by measuring the circumference and dividing by 2π. By a Newtonian weak-field approximation, the parameter r_s is found to be $2M$, where M is the (ADM) mass of the body. For example, we can model the gravitational field of the Sun's exterior using the Schwarzschild metric. Note that f^{-1} becomes singular at the critical radius $r = r_s$, also known as the Schwarzschild radius, and the line element becomes ill-defined. This is not a problem for stars, since their radius is always larger than their Schwarzschild radius, and we only care about their exterior (for now). However, suppose an object lies beyond its Schwarzschild radius. We know that for light, $\mathrm{d}s^2 = 0$. If we only consider radial (i.e. $\mathrm{d}\Omega^2 = 0$) outgoing null (light) curves, upon integrating eq. (2.1) we get

$$t = r + r_s \ln{(r - r_s)}. \tag{2.3}$$

Thus, for a radially directed light ray to go from r_1 to r_2, we would measure a time

$$t_{1\to 2} = r_2 - r_1 + r_s \ln \frac{r_2 - r_s}{r_1 - r_s}. \tag{2.4}$$

Assuming $r_2 \gg r_1 > r_s$, if we take the limit $r_1 \to {r_s}^+$, we can easily see that $t_{1\to 2} \to +\infty$. Thus, it would take an infinite amount of time for light, or any matter, to escape from r_s (and beyond); furthermore, it can also be shown that as $r_1 \to {r_s}^+$, light emitted from r_1 would get infinitely red-shifted. Therefore, we cannot see directly in the vicinity of r_s, and the object would appear pitch-black.[1] [2] This object is known as a black hole.

Event horizon

The boundary of the black hole, defined by the Schwarzschild radius, is known as the event horizon. The event horizon is important because physical quantities such as the mass and angular momentum of a black hole are defined by integrating over the event horizon [14]. A black hole is formally defined as the complement of the causal past of future null infinity, and the event horizon is the boundary of such region [10]. However, this definition of a horizon is not always the most practical to use. Alternative notions of black hole horizons are reviewed in [2], and we shall discuss some of them in Section 3.3.

2.2 Inside the black hole

If it takes an infinite amount of time for light to escape from the event horizon, then it must also take an infinite amount of time for light from far away to reach the event horizon. Does that mean objects never cross the horizon? That is not the case;

[1]This is not entirely true, since black holes can be one of the brightest objects in the universe—quasars.

[2]This same object was predicted by Michell and Laplace as far back as the 18^{th} century using Newtonian mechanics, although their arrival at the exact expression for the Schwarzschild radius was a mere coincidence [13].

an observer can cross the event horizon in finite proper time. The Schwarzschild coordinates are simply bad coordinates near the event horizon. The idealized clock at infinity—the Schwarzschild coordinate time—ticks too fast when considering objects near the event horizon. In order to access the interior of the black hole, we must first do something about the coordinates. Recall that for radial null (light) curves, eq. (2.1) becomes

$$\mathrm{d}r = \pm f(r)\,\mathrm{d}t. \tag{2.5}$$

By a linear approximation around r_0, we can see that $r_0 + \Delta r \approx t_0 + f(r_0)\Delta t$. As $\lim_{r_0 \to r_s^+} f(r_0) = 0$, a small Δr corresponds to a large Δt. Suppose we include r_s to form the 'extended' Schwarzschild coordinates. At $r = r_s$, we see that t is no longer in one-to-one correspondence with r. Furthermore, $\mathrm{d}r/\mathrm{d}t = 0$ and light appears to stop. Thus, if we hope to extend the Schwarzschild metric and reach beyond r_s, we must replace t. A straightforward way to do this is to employ null coordinates (u, v), as one does in Minkowski spacetime, and perform an analytic continuation of the new coordinates. This allows us to maximally extend the Schwarzschild solution [15]. However, for most physical considerations we do not require the maximal extension. It is often enough to employ only one null coordinate to replace t.

Ingoing Eddington-Finkelstein coordinates

So far, we have only considered the exterior region $(r > r_s)$ of the Schwarzschild spacetime; we call this region as Region I. However, the Schwarzschild metric—expressed in the same Schwarzschild coordinates—also solves the Einstein field equations when $r \in (0, r_s)$. Although in this region, which we call Region II, the coordinates will have a new interpretation. Note that the Schwarzschild metric is regular in both Region I and Region II. By regular, this means the metric and its inverse is well-defined. Therefore, the problem lies at the event horizon. We know that the Schwarzschild coordinate time t fails at the event horizon. Thus, in order to penetrate the event horizon and reach the other region, we must replace t with a better coordinate. Due to physical constraints, we only consider extending Region I to include Region II, since going the other way is not possible for future-pointing causal curves. We wish to construct a new coordinate $v(t, r^*)$, which slows down t near the event horizon, in such a way that ingoing radial null curves are described by constant-v curves. Now,

$$\frac{\mathrm{d}v}{\mathrm{d}r} = 0 = \left(\frac{\partial v}{\partial t}\right)\left(\frac{\partial t}{\partial r}\right) + \left(\frac{\partial v}{\partial r^*}\right)\left(\frac{\partial r^*}{\partial r}\right). \tag{2.6}$$

Define $v = t + r^*$. Then, using eq. (2.6) we obtain the condition

$$\frac{\partial r^*}{\partial r} = -\frac{\partial t}{\partial r} = f^{-1}. \tag{2.7}$$

Integrating both sides with respect to r, similar to eq. (2.3), we get the Regge-Wheeler tortoise coordinate [3]

$$r^* = r + r_s \ln|r - r_s|. \tag{2.8}$$

Note that if $r \gg r_s$, then $v \approx t + r$, the Minkowski limit. In the new coordinates, the Schwarzschild line element becomes

$$\mathrm{d}s^2 = -f\,\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r + r^2\mathrm{d}\Omega^2. \tag{2.9}$$

We see that the metric is regular at $r = r_s$, so there is no singularity at r_s. Clearly, nothing stops us from setting the range of the new coordinates to be $v \in (-\infty, +\infty)$ and $r \in (0, \infty)$. These new coordinates now cover both Region I and Region II. Thus, we have extended Region I beyond the event horizon to include Region II.[4] The (v, r)-coordinates are called the ingoing Eddington-Finkelstein coordinates which we use in Chapter 4.

Singularities

Informally, a spacetime singularity is a breakdown in spacetime [16]. The singularity at r_s, which we encounter in the Schwarzschild metric expressed in Schwarzschild coordinates, is only a coordinate singularity; since the singularity was removed by using the ingoing Eddington-Finkelstein coordinates. A coordinate singularity is not a breakdown in spacetime, it is only apparently singular because of an unfortunate choice of coordinates; while a spacetime singularity is physical and irrelevant of the coordinates used. Does the Schwarzschild geometry contain a spacetime singularity? Since the Schwarzschild metric is also singular at $r = 0$, in

[3] The absolute value is not necessary if we are only interested in extending Region I to include Region II, and not the other way around.

[4] We can also do the same for Region II. However, note that if Region I and Region II are in their corresponding (t, r)-coordinates, we cannot map both of them simultaneously to the (v, r)-coordinates, since the tortoise coordinate r^* is not a bijective function of r; we can only extend each region separately.

both Schwarzschild and ingoing Eddington-Finkelstein coordinates, can we find a coordinate transformation such that the metric becomes regular at $r = 0$? It turns out that $r = 0$ is a curvature singularity—informally, a region of spacetime where the gravitational field is infinite. Since we know that the Riemann tensor measures the curvature of spacetime, which in turn tells us something about the gravitational field, we can use the Riemann tensor to find curvature singularities; note, however, that not all spacetime singularities are curvature singularities. In order to make sure that we do not end up with a coordinate singularity, we want something whose value is coordinate-independent, so we consider a scalar.[5] The Ricci scalar can be easily obtained from the Riemann tensor. However, a vacuum solution to the Einstein field equation, such as the Schwarzschild solution, is necessarily Ricci-flat, i.e. its Ricci tensor is everywhere zero. Thus, the Ricci scalar is always zero in the Schwarzschild spacetime. Now, consider the Kretschmann scalar K, obtained by a full contraction of the Riemann tensor with a copy of itself. A straightforward computation shows that

$$K = R^{abcd}R_{abcd} = \frac{48M^2}{r^6}. \tag{2.10}$$

Thus, we have a curvature singularity at $r = 0$. Since the Kretschmann scalar is essentially the 'square' of the Riemann tensor, it also provides a coordinate-invariant measure of the strength of the tidal force; clealy, we see that $\sqrt{K} \sim M/r^3$, consistent with the Newtonian view of the tidal force. The Kretschmann scalar is an example of a curvature invariant, which we discuss in the next chapter.

[5]In some cases, we could also use $\det g_{ab}$, but we need a good plot device.

Chapter 3

Curvature invariants

In this chapter, we introduce the curvature invariants and provide an overview of some of their uses in general relativity. Curvature invariants are of interest to both mathematicians and physicists. It is of interest to mathematicians mainly as a tool for spacetime classification [17, 18]. It is of interest to physicists as a tool for quasi-local characterizations of physical properties of black holes [6, 19, 20]. A fully rigorous discussion of invariant theory and spacetime classification requires highly specialized knowledge which is currently beyond our limitations, so we shall take most results on faith and only provide brief explanations. Thus, we keep equations to a minimum and mostly provide an informal qualitative discussion. So far we have encountered one curvature invariant—the Kretschmann scalar, which we used to show that the Schwarzschild spacetime has a curvature singularity at $r = 0$. Indeed, there is a lot more to these curvature invariants; and they remain active pursuits [4, 5, 21]. Thus, we review some of the classic and recent literature in order to motivate our main problem. First, we shall introduce another curvature invariant.

The Karlhede invariant

We now introduce another curvature invariant—the Karlhede invariant. Suppose we dive into a black hole, how would we know once we have crossed the event horizon? We know that this boundary is not a physical boundary which we can directly observe; as we mentioned before, the definition of the event horizon is not that practical. An observer is only limited to local calculations. Thus, is there a way of detecting the event horizon locally? Curvature invariants might provide some answers. In [22], they found out that the curvature invariant $\nabla_e R_{abcd} \nabla^e R^{abcd}$, vanishes on the event horizon of a Schwarzschild black hole. The Karlhede invariant is positive outside the event horizon and switches sign upon crossing the event horizon. Therefore, the Karlhede invariant provides a local means of locating the event horizon. Moreover, the Karlhede invariant also detects the event horizon of the Reissner-Nordström and the Taub-NUT black holes [22], and this can also be generalized to any smooth static spherically symmetric black holes [3]. However, this invariant fails to vanish on the event horizon of the Kerr black hole, instead it vanishes on the ergosurface (stationary limit). In fact, it has been shown that "the vanishing of the Karlhede invariant is not sufficient to locate an event horizon in non-spherically symmetric spacetimes" [23]. It was not until [6] that an invariant horizon detector for Kerr has been found, and this lead to a new definition of black hole horizon. We shall return to this in Section 3.3. So far, we have not yet defined curvature invariants; but, before we do, we first review some basic notions.

Scalars and tensors

For now, we shall view tensors in the classical sense—a tensor is an object with many components, which transform using a certain rule under a change of coordinates. One nice feature of scalars is that they are coordinate-independent: a scalar field simply assigns a value at each point in spacetime, and this value is the same no matter the chosen coordinatization (parametrization). This feature is obviously not shared by tensors; since, by definition, the components of a tensor are obtained by acting the tensor on coordinate basis co/vectors. Thus, we might say scalars are invariant—that which can be 'observed'.

Ideally, we wish to avoid objects which are coordinate-dependent. The trouble with coordinates is that they may complicate things; and sometimes it is not clear what is physical or not. To illustrate this, suppose we are given the Euclidean space $\mathbb{R}^3$. A straight line can be easily recognized in Cartesian coordinates, while it is not always easy for spherical coordinates. As another example, in Newtonian mechanics we know that the centrifugal and Coriolis forces are only fictitious forces caused by using a non-inertial reference frame.

However, we can never do away with coordinates, as they are essential for doing calculations. And although tensors have a nice geometric meaning, in general relativity—and pretty much all of physics—it usually suffices to exclusively work with their coordinate representation. For example, in deriving the Schwarzschild solution we first choose a coordinate system which exploits the spherical symmetry, and then solve for the components of the metric, constrained by the vacuum Einstein field equations; and most of the time one chart is all we need. In this case, we only ever need to know one coordinate representation of the Schwarzschild metric (and similarly for all tensors).

Metric and curvature

In general relativity, the metric is the fundamental object of study. Essentially, the metric contains all information on the geometry and the causal structure of the spacetime, which we denote by $(\mathcal{M}, g_{ab})$. However, the metric also invariably encodes information about the coordinates.[1] Sometimes, this is not what we want if we are only interested in the geometry of the spacetime. As we have seen before, the information encoded by the coordinates might obscure the physics encoded in the metric. In general, finding a coordinate transformation rule is rather difficult. For example, it took a decade before it was recognized by Lemaître that the Gullstrand-Painlevé solution was just a coordinate-transformation of the Schwarzschild solution. Thus, we are posed with the problem of equivalence for metrics.

Let $(\mathcal{M}, g_{ab})$ be the spacetime and consider a point $p \in \mathcal{M}$. Then we can always choose a coordinate system $\{x^i\}$ such that $g_{ij}(p) = \eta_{ij} = \mathrm{diag}(-1, 1, 1, 1)$ and $\partial_k g_{ij}(p) = 0$. However, we cannot set all the second derivatives of g_{ij} to zero; this evinces the 'curvature' of spacetime at p. The Riemann tensor is "the only tensor that can be constructed from the metric tensor and its first and second derivatives, and is linear in the second derivatives" [12]. The Riemann tensor does not encode information on the coordinates in the sense that the Riemann tensor is always zero for flat spacetimes no matter which coordinate system is used; one can say that the Riemann tensor is more 'physical'. A manifestation of the Riemann tensor is through geodesic deviation. A nonzero Riemann tensor means that there are tidal forces which cannot be eliminated by a change in coordinate system.

[1] Again, we stress the fact that we are treating tensors in the classical sense.

3.1 Spacetime classification

We return to the problem of equivalence for metrics. Given two metrics, each expressed in some coordinate system, how do we determine whether or not they are the same (up to diffeomorphisms[2]) [17]? First, it might be easier to determine how they are different. Let us assume that both of them cover the same region and that we can correctly label each point in our region with the proper coordinates.[3] Since we know that scalars are coordinate-invariant, then we need to construct a suitable set of scalars, which should be functions of the metric, and compare their values for each metric. If these scalars are different, then we are dealing with different spacetimes. Thus, the goal now is to construct such scalars.

Curvature invariants

Given an arbitrary tensor $T^{a_1\ldots a_k}{}_{b_1\ldots b_l} \in \mathcal{T}^k_l$, an easy way to construct a scalar from it is by its full contraction with a copy of itself, $T^{a_1\ldots a_k}{}_{b_1\ldots b_l} T_{a_1\ldots a_k}{}^{b_1\ldots b_l}$.[4] Now, all we need to do is to construct tensors from the metric. The full contraction of the metric with the inverse metric is trivial, since $g_{ab}g^{ab} = \delta^a{}_a = \dim \mathcal{M}$. Thus, we need to consider the next simplest tensor, the Riemann tensor—that which yields the Kretschmann scalar. We could also use other curvature tensors such as the Weyl tensor and the Ricci tensor. And if that is not enough, we can also take covariant derivatives of the Riemann tensor. Scalars obtained from the curvature tensors, their derivatives and Hodge dual are called curvature invariants. Invariants formed involving only polynomials of the Riemann tensor are called algebraic invariants, while those involving the derivatives of the Riemann tensor are called differential invariants [26, 27]. The order of a curvature invariant is the highest order derivative of the metric which appears in it. For our purposes, we consider only polynomial invariants—mostly quadratic monomial invariants, such as the Kretschmann and the Karlhede invariants.

Now that we have an idea of what curvature invariants are, we can go back to our question in the beginning of this section. How many curvature invariants do

[2]Isometry is usually implied through the pullback.

[3]The second assumption is a strong one! We cannot think of any physical way of realizing this. [24, 25]

[4]There are many other ways to construct scalars, but this requires the least creativity

we have to check before we can say that the two metrics are equivalent? Is there a basis for these curvature invariants? Clearly, we know that inequivalent invariants implies inequivalent metrics [17], does the inverse hold true? A result due to Cartan states that a metric can be uniquely characterized (up to isometry) by the Riemann tensor and a finite number of its covariant derivatives [28–30]. Thus, the problem essentially hinges on "whether we can construct the curvature tensors from the curvature invariants" [30]. It has been shown in [17] that spacetimes that do not belong to the Kundt class can be characterized by their curvature invariants. However, even though the metric is determined in principle, we cannot easily reconstruct it [30].

The problem of constructing a complete basis of algebraically independent curvature invariants is still an ongoing research [26, 31, 32]. A complete basis means that every curvature invariant can be generated by a polynomial of the elements of the set [26]. In [33], they showed that there are at most 14 independent real algebraic invariants of the Riemann tensor in a four-dimensional Lorentzian space. In an n-dimensional manifold $\mathcal{M}$, at most n scalar invariants can be functionally independent, i.e. independent functions on $\mathcal{M}$; while the number of algebraically independent scalar invariants, i.e. invariants not satisfying any polynomial relation is rather larger [28]. However, in this work we are not interested in the algebraic classification of spacetimes, our main interest is the applications of curvature invariants to the study of black holes.

3.2 Physical characterization

Curvature invariants can also be used for the physical characterization of known spacetimes. Kerr black holes are spinning black holes, to which the Schwarzschild black hole is the zero-spin limit.[5] In [6], they used curvature invariants for both local and global calculations of the mass and spin of a Kerr black hole. This is useful for numerical relativity simulations since "extracting information about the mass and angular momentum of black holes in numerical simulations requires finding the event horizon of the black hole, calculating the area and angular momentum of the horizon, then using the relationship between the area, mass, and angular

[5]Most black holes in the universe are believed to be spinning, since they are formed from collapsing stars. However, Kerr black holes are currently beyond our limits.

momentum in order to calculate the mass [6]." This invariant approach generalizes the method of [29].

The invariants of Abdelqader and Lake

In [6], they introduced seven curvature invariants constructed out of the Weyl tensor. In vacuum solutions such as Kerr, the Weyl tensor is equal to the Riemann tensor; so we will just use the Riemann tensor. The seven curvature invariants are given by

$$
\begin{aligned}
I_1 &= R_{abcd}R^{abcd}, \\
I_2 &= R_{abcd}R^{\star abcd}, \\
I_3 &= \nabla_e R_{abcd}\nabla^e R^{abcd}, \\
I_4 &= \nabla_e R_{abcd}\nabla^e R^{\star abcd}, \\
I_5 &= \nabla_a I_1 \nabla^a I_1, \\
I_6 &= \nabla_a I_2 \nabla^a I_2, \\
I_7 &= \nabla_a I_1 \nabla^a I_2,
\end{aligned}
$$

where $\star$ denotes the Hodge dual. Note that I_1 is the Kretschmann scalar and I_3 is the Karlhede invariant.

Syzygies

Not all of the curvature invariants introduced are algebraically independent. It was pointed out by [3], that the invariants satisfy the complex equation

$$\nabla_a \left(I_1 + iI_2\right) \nabla^a \left(I_1 + iI_2\right) = \frac{12}{5}\left(I_1 + iI_2\right)\left(I_3 + iI_4\right), \tag{3.1}$$

and another real equation which we omit. Thus, a total of three algebraic constraint equations are satisfied by the invariants; we call such constraint equations *syzygies*, i.e. a syzygy is an algebraic relation satisfied by some curvature invariants [6, 28]. Therefore, in the Kerr spacetime we have at most four independent curvature invariants, enough to determine the two free parameters, (M, a), and the two nontrivial coordinates (r, θ) [3, 6, 17].

Visualization

Curvature invariants can also be used to visualize spacetimes. Of course, since spacetime is four-dimensional we cannot have visualizations in the literal sense; but we can still have embedding diagrams and whatnot. For example, in [34], the Kretschmann scalar was used to visualize the Kerr-Newman spacetime . In [35–37], they used gradient flows of curvature invariants to visualize spacetime curvature. In [6], they proposed using an invariant constructed from a syzygy of the spacetime to visualize deviations from the spacetime, which shall provide an invariant and intuitive method to compare and visualize spacetimes.[6]

3.3 Geometric horizon

Perhaps one of the more important uses of curvature invariants is for locating black hole horizons. In Section 2.1, we gave the definition of the event horizon, which is the boundary of the causal past of future null infinity. However, this definition is not always the most practical. The event horizon is a nonlocal object and teleological[7] in nature, which makes it physically and philosophically problematic [2]. Other standard definitions of a black hole horizon include the Killing horizon—a null hypersurface whose normal is a Killing vector, for which it can be shown that the event horizon in a stationary asymptotically flat black hole spacetime must be a Killing horizon [38], and the apparent horizon, which involves finding trapped surfaces [2]. According to [39], "in eternally static solutions solutions such as the Schwarzschild solution and the Reissner-Nordström solution the event horizon is also an apparent horizon. However, in dynamical situations the apparent horizon need not be at the same location as the event horizon." Several alternative definitions of black hole horizons have been proposed, and a review is provided in [2].

In [6], they found an invariant (constructed out of the seven curvature invariants) which vanishes on the (inner and outer) horizon of Kerr. This was later generalized by [3], wherein they proved a theorem which states that for a spacetime of local cohomogeneity n that contains a stationary horizon and which has n scalar polyno-

[6]They proposed the 'Kerrness' invariant, which can be used to visualize deviations from Kerr. This was such a wasted opportunity, since they could have called it the 'Kerrvature' invariant.

[7]In other words, "we must know the global behaviour of the spacetime in order to determine the event horizon locally" [4].

mial curvature invariants whose gradients are well-defined there, the n-form wedge product has zero (squared) norm on the horizon.[8] They claim that the n-form wedge product "gives the precise location of any stationary horizon and should give the approximate location of a nearly stationary horizon. Hence it should be useful for numerical computations in general relativity [3]." This geometric definition of a horizon has since been generalized to include dynamical spacetimes [4, 5], which are of interest to numerical relativists [40]. We shall refer to such horizons as geometric horizons. In [21], they improved the theorem by using Cartan invariants. In [41], they constructed "a scalar polynomial curvature invariant that transforms covariantly under a conformal transformation from any spherically symmetric metric. This invariant has the additional property that it vanishes on the event horizon of any black hole that is conformal to a static spherical metric."

In this work, our primary goal is to use curvature invariants to examine perturbed Schwarzschild spacetimes; we are mostly interested in Sections 3.2 and 3.3. Due to the simplicity of the Schwarzschild spacetime, a lot of what we discussed reduce to much simpler things; and we shall reintroduce the things we need in Section 4.4. In the next chapter, we shall consider a specific example of a perturbed Schwarzschild spacetime. In particular, we shall use the invariant proposed by [6] to visualize its deviation from the Schwarzschild spacetime. We shall also investigate on the nature of the geometric horizon proposed by [3].

[8] The local cohomogeneity is defined as "the codimension of the maximal dimensional orbits of the isometry group of the local metric, ignoring the breaking of any of these local isometries by global considerations" [3].

Chapter 4

Perturbations I: the basics

Finding exact solutions to the Einstein field equations is rather difficult. Still, a lot of solutions have been found that there is even an entire book dedicated to them [28]. However, most of them are unphysical, and in astrophysics we mostly deal with Kerr—to which Schwarzschild is the static limit. When dealing with spacetimes with few symmetries finding an exact solution is almost impossible. Thus, we often rely on approximate solutions. In this chapter, we introduce some basic concepts on metric perturbations. We consider one perturbed Schwarzschild solution, to which we apply the invariant method of [6].

4.1 Linear perturbations

Let g_{ab} be the metric of the background spacetime. Suppose we introduce small perturbations to this spacetime such that the metric of the perturbed spacetime is now ${g'}_{ab}$. For small enough perturbations, we can linearly (in λ) approximate ${g'}_{ab}$ as

$$ {g'}_{ab} = g_{ab} + \lambda\, h_{ab} + \mathcal{O}(\lambda^2), \tag{4.1} $$

where h_{ab} is a symmetric $(0, 2)$-tensor field, and λ is some small parameter ($\lambda \ll 1$). We shall refer to h_{ab} as the perturbation, or the first-order correction to g_{ab}. As usual in first-order perturbation theory, we shall ignore all higher-order correction terms; in that case we can drop $\mathcal{O}(\lambda^2)$ from our expressions. Also, we can incorporate λ into h_{ab}, so that eq. (4.1) now reads

$$ {g'}_{ab} = g_{ab} + h_{ab}, \tag{4.2} $$

where the components of h_{ab} is a function of some small parameter. Since, both g_{ab} and ${g'}_{ab}$ are metrics, the following conditions must be satisfied:[1]

$$g^{ab}g_{bc} = \delta^a{}_c = {g'}^{ab}{g'}_{bc}. \tag{4.3}$$

In other words, the matrix of ${g'}^{ab}$ must equal the inverse matrix of ${g'}_{ab}$. Thus, the inverse metric ${g'}^{ab}$ can be derived by some matrix algebra

$$\begin{aligned} \mathbf{g}' &= \mathbf{g} + \mathbf{h}, \\ &= \mathbf{g}(\mathbf{1} + \mathbf{g}^{-1}\mathbf{h}), \end{aligned}$$

$$\begin{aligned} \mathbf{g}'^{-1} &= (\mathbf{1} + \mathbf{g}^{-1}\mathbf{h})^{-1}\mathbf{g}^{-1}, \\ &= (\mathbf{1} - \mathbf{g}^{-1}\mathbf{h})\mathbf{g}^{-1}, \\ &= \mathbf{g}^{-1} - \mathbf{g}^{-1}\mathbf{h}\mathbf{g}^{-1}. \end{aligned}$$

Clearly, we now have

$${g'}^{ab} = g^{ab} - h^{ab}, \tag{4.4}$$

where we have defined

$$h^{ab} = g^{ac}g^{bd}h_{cd}. \tag{4.5}$$

4.2 Schwarzschild as a product manifold

The spherical symmetry of the Schwarzschild spacetime allows its foliation into 2-spheres. In other words, we can decompose the spacetime manifold $\mathcal{M}$ as a product manifold $\mathfrak{M}^2 \times S^2$, where $\mathfrak{M}^2$ is a 2-dimensional submanifold and S^2 is the round 2-sphere. We can think of the manifold as $\mathfrak{M}^2$ with an S^2 attached at each point in $\mathfrak{M}^2$. We shall denote coordinates of $\mathfrak{M}^2$ by $\bar{x}^\alpha$, and S^2 by ω^A. We fix the standard spherical coordinate system, $\omega^A = \{\theta, \phi\}$, to S^2; while $\mathfrak{M}^2$ may be expressed in any coordinate system. By the Frobenius Theorem, the Schwarzschild line element can be expressed as

$$\mathrm{d}s^2 = \bar{g}_{\alpha\beta}(\bar{x})\,\mathrm{d}\bar{x}^\alpha \mathrm{d}\bar{x}^\beta + r(\bar{x})^2\,\Omega_{AB}(\omega)\,\mathrm{d}\omega^A \mathrm{d}\omega^B, \tag{4.6}$$

where $\bar{g}_{\alpha\beta}$ is the metric on $\mathfrak{M}^2$, and Ω_{AB} is the round metric on the unit sphere S^2 [7, 9]. Note that the radius r is a function on $\mathfrak{M}^2$.

[1] i.e., the raising operator composed with the lowering operator must be the identity operator.

We now write the general form of the metric perturbations. In matrix form,

$$[g_{ab}] = \left[\begin{array}{c:c} \bar{g}_{\alpha\beta} & \mathbf{0} \\ \hdashline \mathbf{0} & r^2\Omega_{AB} \end{array}\right] \quad \text{and} \quad [h_{ab}] = \left[\begin{array}{c:c} \bar{h}_{\alpha\beta} & h_{\alpha B} \\ \hdashline h_{A\beta} & r^2\tilde{h}_{AB} \end{array}\right].$$

Thus,

$$[g'_{ab}] = \left[\begin{array}{c:c} \bar{g}_{\alpha\beta} + \bar{h}_{\alpha\beta} & h_{\alpha B} \\ \hdashline h_{A\beta} & r^2(\Omega_{AB} + \tilde{h}_{AB}) \end{array}\right].$$

The Greek indices run from $\{0,1\}$, while the upper-case Latin indices run from $\{2,3\}$. The reason why we insisted on fixing the spherical coordinate system on S^2 is to write the perturbation fields in terms of spherical harmonic functions, which we introduce in the next section.

Tensor spherical harmonics

A detailed discussion can be found in [7]. In this section, we only consider objects on S^2.[2] The lowering and raising operator on S^2 is Ω_{AB} and Ω^{AB}, respectively. The Levi-Civita connection is denoted by $\tilde{\nabla}_C$ The scalar harmonics are the usual $Y^{lm}(\omega^A)$ in S^2, and they satisfy the eigenvalue equation

$$\left[\tilde{\Delta} + l(l+1)\right] Y^{lm} = 0, \tag{4.7}$$

where $\tilde{\Delta} = \Omega^{AB}\tilde{\nabla}_A\tilde{\nabla}_B$ is the Laplace-Beltrami operator on S^2. The vector and tensor harmonics come in two types: even-parity and odd-parity, and they are orthogonal to each other. For now, we shall only introduce the even-parity sector. The vector harmonics are defined as

$$Y^{lm}_A = \tilde{\nabla}_A Y^{lm}, \tag{4.8}$$

while the tensor harmonics are defined as

$$Y^{lm}_{AB} = \left[\tilde{\nabla}_A\tilde{\nabla}_B + \frac{1}{2}l(l+1)\Omega_{AB}\right] Y^{lm}. \tag{4.9}$$

It is easy to see that Y^{lm}_{AB} is trace-free, since contracting with Ω^{AB} yields eq. (4.7). The even-parity sector of the perturbation can be written as [3]

$$h_{\alpha\beta} = \sum_{lm} \underline{h}^{lm}_{\alpha\beta}(\bar{x})\, Y^{lm}, \tag{4.10}$$

$$h_{\alpha B} = \sum_{lm} j^{lm}_{\alpha}(\bar{x})\, Y^{lm}_B, \tag{4.11}$$

$$h_{AB} = r^2 \sum_{lm} \left(F^{lm}(\bar{x})\, \Omega_{AB} Y^{lm} + G^{lm}(\bar{x})\, Y^{lm}_{AB}\right), \tag{4.12}$$

[2] At this point, it should be clear that S^2 refers to the unit 2-sphere with the round metric.
[3] We use a slightly different notation from the one found in [7].

where $\underline{h}^{lm}_{\alpha\beta}$, j^{lm}_{α}, F^{lm}, and G^{lm} are objects on $\mathfrak{M}^2$.

4.3 The Preston-Poisson spacetime

Finding approximate solutions to the Einstein field equations is also not easy. We cannot simply plug in arbitrary functions on h_{ab} and expect the resulting spacetime to be physical. A handful of physical perturbed Schwarzschild solutions can be found in the literature [42, 43]. We only consider one convenient example.[4]

Black hole in a magnetic field

We now consider a physical example: the Preston-Poisson spacetime. The full details and derivations can be found in [44], we only provide a summary of the physical description. Everything in this section is paraphrased from [44]. Consider a black hole immersed in a uniform magnetic field. Imagine a giant solenoid producing a uniform magnetic field. An initially isolated Schwarzschild black hole is then inserted within the solenoid in a quasi-static and reversible process, in such a way that the black hole's surface area stays constant during the immersion. The black hole distorts the magnetic field within the structure, and the magnetic field distorts the geometry of the black hole. The perturbed black hole solution is a three parameter family of solutions characterized by the black hole mass M, the magnetic field strength B, and the Weyl curvature (tidal gravity) $\mathcal{E}$ due to the solenoid. The solution is obtained perturbatively through order B^2 and $\mathcal{E}$; note that B and $\mathcal{E}$ is small. The spacetime is stationary, axially symmetric, and only contains even-parity spherical-harmonic modes with $(l, m) = \{(0,0), (2,0)\}$. In the light-cone gauge of the ingoing Eddington-Finkelstein coordinates, the only nonzero components of the perturbation are $\underline{h}^{lm}_{vv}$, j^{lm}_{v}, F^{lm}, and G^{lm}. The nonzero components of the $l = 0$ sector of the even-parity perturbation are

$$F^{00} = -\frac{4}{9}\sqrt{\pi}B^2r^2, \tag{4.13}$$

$$\underline{h}^{00}_{vv} = -\frac{2}{9}\sqrt{\pi}B^2r(3r - 8M). \tag{4.14}$$

[4] This example is covenient because everything is provided, so we can just copy the perturbation functions.

For the $l = 2$ sector, they are

$$F^{20} = \frac{4}{9}\sqrt{\frac{\pi}{5}}B^2r^2, \tag{4.15}$$

$$\underline{h}^{20}_{vv} = \frac{4}{3}\sqrt{\frac{\pi}{5}}\left[-\frac{1}{3}B^2(3r^2 - 14Mr + 18M^2) + 3\mathcal{E}(r-2M)^2\right], \tag{4.16}$$

$$j^{20}_v = \frac{4}{3}\sqrt{\frac{\pi}{5}}\left[-\frac{1}{3}B^2r^2(r-3M) + \mathcal{E}r^2(r-2M)\right], \tag{4.17}$$

$$G^{20} = \frac{4}{3}\sqrt{\frac{\pi}{5}}\left[B^2M^2 + \mathcal{E}(r^2 - 2M^2)\right]. \tag{4.18}$$

Putting it all together, the nonzero components of the metric reads

$$\begin{aligned}
{g'}_{vv} &= -f - \frac{1}{9}B^2r(3r-8M) - \frac{1}{9}B^2(3r^2 - 14Mr + 18M^2)(3\cos^2\theta - 1) \\
&\quad + \mathcal{E}(r-2M)^2(3\cos^2\theta - 1), && (4.19)\\
{g'}_{vr} &= 1, && (4.20)\\
{g'}_{v\theta} &= \frac{2}{3}B^2r^2(r-3M)\sin\theta\cos\theta - 2\mathcal{E}r^2(r-2M)\sin\theta\cos\theta, && (4.21)\\
{g'}_{\theta\theta} &= r^2 - \frac{2}{9}B^2r^4 + \frac{1}{9}B^2r^4(3\cos^2\theta - 1) \\
&\quad + B^2M^2r^2\sin^2\theta + \mathcal{E}r^2(r^2 - 2M^2)\sin^2\theta, && (4.22)\\
{g'}_{\phi\phi} &= r^2\sin^2\theta - \frac{2}{9}B^2r^4\sin^2\theta + \frac{1}{9}B^2r^4\sin^2\theta(3\cos^2\theta - 1) \\
&\quad - B^2M^2r^2\sin^4\theta - \mathcal{E}r^2(r^2 - 2M^2)\sin^4\theta. && (4.23)
\end{aligned}$$

The Schwarzschild-Melvin spacetime

The Schwarzschild-Melvin spacetime is an exact solution to the Einstein-Maxwell equations. The solution describes a Schwarzschild black hole immersed in Melvin's magnetic universe [44–47]. This exact solution can be obtained from the perturbative Preston-Poisson solution when the tidal gravity is related to the magnetic field by

$$\mathcal{E} = \frac{1}{2}B^2. \tag{4.24}$$

The Schwarzschild-Melvin spacetime is "of interest to astrophysicists studying gravitational collapse in the presence of strong magnetic fields" [45].

4.4 A preliminary investigation

In Chapter 3, we learned how curvature invariants can serve as useful tools in probing spacetimes, primarily motivated by [6]. However, most of the discussion was abstract, so now we provide an example using the Preston-Poisson spacetime. Most of the calculations here are done using a computer algebra system.

Curvature invariants of Schwarzschild

Out of the seven curvature invariants introduced by [6], only three are nonzero in the Schwarzschild spacetime. These are

$$I_1 = K = R_{abcd}R^{abcd} = \frac{48M^2}{r^6}, \tag{4.25}$$

$$I_3 = \nabla_e R_{abcd} \nabla^e R^{abcd} = \frac{720M^2}{r^8} f, \tag{4.26}$$

$$I_5 = \nabla_a I_1 \nabla^a I_1 = \frac{82944M^4}{r^{14}} f. \tag{4.27}$$

Only two of these are independent, enough to determine M and r. Note that not only is I_5 functionally dependent on I_1, but it is also algebraically dependent on I_1 and I_3; the three curvature invariants satisfy the syzygy (from eq. (3.1))

$$I_5 = \frac{12}{5} I_1 I_3.$$

Thus, we can define the invariant

$$\chi = I_5 - \frac{12}{5} I_1 I_3, \tag{4.28}$$

which vanishes everywhere in the Schwarzschild spacetime.[5]. Thus, following the proposal of [6] we use this to 'visualize' the deviation of the Preston-Poisson spacetime from the Schwarzschild spacetime.

Visualizing the Preston-Poisson spacetime

Although the Schwarzschild-Melvin solution is exact, a naive computation takes too long.[6] Therefore, we consider only small B in order to perform a first-order approximation.[7] Using the perturbed metric in Section 4.3, the naive computation

[5]This is the Schwarzschild limit of the "Kerrness" invariant proposed in [6]

[6]The naive computation involves computing every component and performing every summation, with no regards to simplification or symmetries.

[7]If we do not do this, the 'naive' computation can easily take a whole day. Of course, our code may simply be badly optimized, but we leave that for another day. Alternatively, this problem might be addressed by using the Newman-Penrose formalism [48].

yields, ignoring $\mathcal{O}(B^4, \mathcal{E}^2, B^2\mathcal{E})$ terms,

$$I'_1 = \frac{48M^2}{r^6} + \frac{24M}{r^3}\left(1 + 3\cos 2\theta\right)\mathcal{E}, \tag{4.29}$$

$$\begin{aligned} I'_3 = &\frac{720M^2}{r^8} f - \frac{360M^2}{r^6} f^2 \left(1 + 3\cos 2\theta\right)\mathcal{E} \\ &+ \frac{24M^2}{r^8}\left(30M^2 - 42Mr + 11r^2 + \left(90M^2 - 78Mr + 19r^2\right)\cos 2\theta\right)B^2, \end{aligned} \tag{4.30}$$

$$\begin{aligned} I'_5 = &\frac{82944M^4}{r^{14}} f + \frac{41472M^3}{r^{13}} f\left(2M^2 - Mr + r^2\right)\left(1 + 3\cos 2\theta\right)\mathcal{E} \\ &+ \frac{13824M^4}{r^{14}}\left(6M^2 - 10Mr + 3r^2 + \left(18M^2 - 14Mr + 3r^2\right)\cos 2\theta\right)B^2. \end{aligned} \tag{4.31}$$

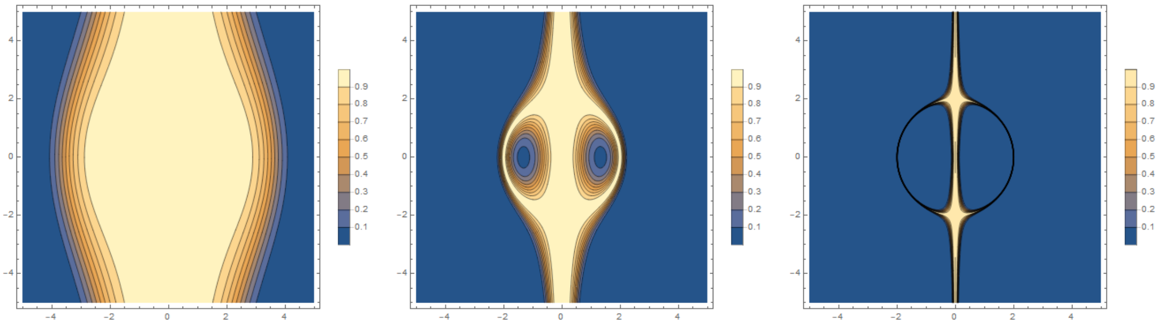
Note that I'_1 has no first-order correction (B^2) due to B, the correction is solely due to the tidal gravity $\mathcal{E}$; faithful to the interpretation of the Kretschmann scalar as a measure of the tidal force. Also, the correction to I'_1 is purely from $Y^{(2,0)}$. Now, we compute the invariant χ,

$$\chi = -\frac{4\sqrt{3}B^2}{5M} f r^3 \sin^2\theta. \tag{4.32}$$

Again, we are only working within first-order B^2 and $\mathcal{E}$. Note that χ does not depend on $\mathcal{E}$. This means that the invariant χ cannot distinguish between same-B subfamilies of the Preston-Poisson spacetime. As a consequence, the Schwarzschild-Melvin limit of the Preston-Poisson spacetime cannot be distinguished by χ alone. Nevertheless, we proceed with using χ to 'visualize' the Preston-Poisson spacetime. We define the invariant [6]

$$S = e^{-s\chi^2}. \tag{4.33}$$

The reason for the construction is simple, we want to 'measure', in a relative sense, deviations from the Schwarzschild spacetime. Such deviations manifest in the failure to satisfy the Schwarzschild syzygy. We take the square of χ to ensure that deviations only depend on the magnitude, and not the sign of χ. Then, we take the exponential of the product $-s\chi^2$, where $s > 0$ is an arbitrary sensitivity parameter, such that S runs from zero to one. When S is closer to one, then the spacetime is supposed to be more Schwarzschild-like. In the figures below, we plotted S, with varying sensitivity parameters, as a function of $(x, y) = r(\cos\theta, \sin\theta)$; a Cartesian coordinate transformation was used to better visualize the 'cross-section' of the black hole, and we set $B = 0.1M^{-1}$. Brighter regions correspond to larger values of S. Also, since χ is always zero when $r = 2M$ or $\theta = \{0\,, \pi\}$, then S is always one in those regions, we can see that by taking the limit as $s \to +\infty$. Again, the values of S have no physical meaning; they are meant to be compared in a relative sense. The contour plots only provide an intuitive picture of the spacetime. In [6], they proposed it as a tool to visualize spacetimes and aid numerical calculations.



(a) $s = 10$ (b) $s = 10^4$ (b) $s = 10^7$

The mass and areal radius

We can also check what happens if we naively calculate the mass and areal radius using the curvature invariants alone. The formulas are already provided in [6]. We obtain

$$
\begin{aligned}
M' &= M + \frac{1}{2}(6M^3 - 6M^2 r - 3Mr^2 + 2r^3)(1 + 3\cos 2\theta)\mathcal{E} \\
&\quad - \frac{1}{4}M\left(6M^2 - 10Mr + 3r^2 + (18M^2 - 14Mr + 3r^2)\cos 2\theta\right)B^2,
\end{aligned} \tag{4.34}
$$

$$
\begin{aligned}
r' &= r + \frac{r}{4M}(4M^3 - 4M^2 r - 2Mr^2 + r^3)(1 + 3\cos 2\theta)\mathcal{E} \\
&\quad - \frac{r}{12}\left(6M^2 - 10Mr + 3r^2 + (18M^2 - 14Mr + 3r^2)\cos 2\theta\right)B^2.
\end{aligned} \tag{4.35}
$$

We see that the measured mass M' and areal radius r' depend on M, r, B, and $\mathcal{E}$. Although we shall not attempt to do so here, it would be nice to have a physical characterization of the Preston-Poisson spacetime, in the sense that we can obtain the parameters M, r, B and $\mathcal{E}$ using the invariants alone, as was done for the Kerr spacetime in [6].

Event horizon

For a spherically symmetric spacetime, the local cohomogeneity is one; therefore, one curvature invariant is enough to locate the event horizon of the Schwarzschild spacetime [3]. Even though we see that I_3 and I_5 clearly vanishes when $r = r_s$, the theorem states that $\nabla_a I_1 \nabla^a I_1$, $\nabla_a I_3 \nabla^a I_3$, and $\nabla_a I_5 \nabla^a I_5$ must also vanish on the event horizon. In our perturbed spacetime, we make the naive replacement $I \to I'$. Since all three curvature invariants are well defined, we can use any one of them to locate the event horizon. In this case, we choose the simplest—I_1', whose gradient must be null on the horizon. By definition, the squared norm of the gradient of I_1' is precisely I_5'. Thus, the zero set of I_5' should define the horizon. Therefore, let us carefully examine I_5'. Since we are only interested in the zero set of I_5', we might as well divide everything by $-82944/r^{14}$; we are allowed to do this except at $r = 0$. After that, we obtain an equation for the horizon detector H' of the form

$$H' = g'_{vv} - \frac{r^3}{2M} f\,(1 + 3\cos 2\theta)\,\mathcal{E}. \tag{4.36}$$

Suppose we plug in the ansatz $r' = r_s + \lambda\, r_s^{(1)} + \mathcal{O}(\lambda^2)$, where $\lambda \sim B^2 \sim \mathcal{E}$. Observe that $f(r') \sim \lambda$. This means that any correction to the second term is necessarily $\sim \lambda^2$. Thus, we can drop the second term and let $H' = g'_{vv}$. This result also holds true if we use the gradient of I_3' or I_5'. This means that for the Preston-Poisson spacetime H' reduces to the same equation for locating the Killing horizon—the null hypersurface where the time-translation Killing vector $\partial/\partial v$ is null [44]. This follows from the theorem of [3], since the Preston-Poisson spacetime is static, and thus the event horizon is the Killing horizon, which H' locates. The event horizon of the Preston-Poisson spacetime, as it is found in [44], is described by

$$r_{\text{horizon}} = 2M\left(1 + \frac{2}{3}M^2 B^2 \sin^2\theta\right). \tag{4.37}$$

Note that r_{horizon} does not depend on $\mathcal{E}$ [44].

It is also worth noting that even though the Preston-Poisson spacetime is axially symmetric, the event horizon can be located using the gradient of only one curvature invariant. One might expect the event horizon detector to be the wedge product of the gradient of two curvature invariants, since for an axially symmetric spacetime the local cohomogeneity is two [3]; however, any such combination yields a result of order λ^2. Furthermore, according to [3], a general distorted static black hole in four dimensions that has no spatial Killing vector fields on and outside the horizon has local cohomogeneity three, and thus we need the wedge product of the three curvature invariants to detect the horizon. In other words, we need $\mathrm{d}I_1 \wedge \mathrm{d}I_3 \wedge \mathrm{d}I_5$ to locate the horizon of an arbitrary perturbed Schwarzschild black hole with no spatial symmetries. However, since the event horizon of the Preston-Poisson black hole can be approximately located using the gradient of only one curvature invariant, we are confident that this is also the case for any perturbed spacetime. In other words, we shall only rely on the local cohomogeneity of the backgroud spacetime. Thus, we no longer need to know the symmetries of the perturbed spacetime. In fact, by not requiring time-translation symmetry we can easily generalize this geometric horizon to nonstationary spacetimes.

If the horizon defined by H' ultimately reduces to the Killing horizon, then why bother with H'? Is there anything new provided by H'? The advantage of H' is that we no longer need to worry about finding Killing vectors, especially in coordinate systems where they are not as obvious. More importantly, [3, 4] proposed that H' can be generalized to dynamical spacetimes, as an alternative to the various definitions of dynamical horizons.

Chapter 5

Perturbations II: a covariant formalism

In this chapter, we revisit metric perturbations. Motivated by the questions which arose from our preliminary investigation in Section 4.4, we now wish to develop a proper framework for the perturbed geometric horizon. The problem with a naive computation, as done in the previous chapter, is that we fail to see which components of the perturbation h_{ab} dominate the first-order correction. We treated the computation of curvature invariants as a black box, which spits out correction terms upon adding small perturbations to the metric; that is what we wish to address, here and in the next chapter. The goal is to construct a formal first-order perturbation framework for curvature invariants. However, we shall see that a trade-off must be made. A proper treatment of perturbations, although conceptually clearer, requires a great deal of index gymnastics. We know that for the Schwarzschild spacetime, it is sufficient to consider only one curvature invariant, which we choose to be the Kretschmann scalar—the gradient of which has zero squared norm on the event horizon. Thus, we derive the first-order correction of the Kretschmann scalar.

5.1 Metric perturbations revisited

Recall that in Chapter 4, we introduced the metric perturbations. Let g_{ab} be the metric of the background spacetime and ${g'}_{ab}$ be the metric of the perturbed spacetime, then we have the following:

$$ {g'}_{ab} = g_{ab} + h_{ab}, \tag{5.1} $$

$$ g'^{ab} = g^{ab} - h^{ab}, \tag{5.2} $$

where we have defined

$$h^{ab} = g^{ac}g^{bd}h_{cd}. \tag{5.3}$$

The tensorial nature of $\nabla_a - \nabla'_a$

Suppose ∇_a and ∇'_a are both derivative operators,[1] we show that their difference defines a $(1,2)$-tensor field. This derivation can be found in [10]. Let ω_b be a covector field and consider the difference $\nabla_a(f\omega_b) - \nabla'_a(f\omega_b)$. Using the Leibniz rule,

$$\nabla_a(f\omega_b) - \nabla'_a(f\omega_b) = \nabla_a(f)\omega_b + f\nabla_a(\omega_b) - \nabla'_a(f)\omega_b - f\nabla'_a(\omega_b). \tag{5.4}$$

Now, by definition, $X^a\nabla_a f = Xf = X^a\nabla'_a f$, so the two derivative operators must agree on their action on any scalar field, i.e., $\nabla_a f = \nabla'_a f$, $\forall f \in \mathcal{F}$. Thus,

$$\nabla_a(f\omega_b) - \nabla'_a(f\omega_b) = f\left(\nabla_a\omega_b - \nabla'_a\omega_b\right). \tag{5.5}$$

By construction, the value of the derivative of a function at a point depends on how the value of the function changes as one moves away from that point. Therefore, we expect that the derivatives $\nabla_a\omega_b$ and $\nabla'_a\omega_b$, evaluated at a point $p \in M$, depend on how the value of ω_b changes in some neighborhood of p. However, we shall show that their difference only depends on the value of ω_b at p. Consider another covector field ω'_b and suppose that $\omega_b(p) = \omega'_b(p)$. Since $\omega'_b - \omega_b$ is also a covector field, it follows that there exist covector fields $\left\{\mu_b^{(\alpha)}\right\}$, which form a basis of $\mathcal{V}^*$, such that

$$\omega'_b - \omega_b = f_{(\alpha)}\,\mu_b^{(\alpha)}. \tag{5.6}$$

Clearly, each $f_{(\alpha)}$ vanishes at p. Now,

$$\begin{aligned}\nabla_a(\omega'_b - \omega_b) - \nabla'_a(\omega'_b - \omega_b) &= \nabla_a(f_{(\alpha)}\,\mu_b^{(\alpha)}) - \nabla'_a(f_{(\alpha)}\,\mu_b^{(\alpha)}) \\ &= f_{(\alpha)}\left(\nabla_a\mu_b^{(\alpha)} - \nabla'_a\mu_b^{(\alpha)}\right).\end{aligned} \tag{5.7}$$

Thus, we have $(\nabla_a - \nabla'_a)\omega'_b = (\nabla_a - \nabla'_a)\omega_b$ at p. Since ω'_b is arbitrary, in general $\omega'_b \neq \omega_b$ in the neighborhood of p. This implies that the evaluation at p, after the operator $\nabla_a - \nabla'_a$ acts on a covector field ω_b, only depends on the value of ω_b at p. Therefore, for each point $p \in M$ we can think of $\nabla_a - \nabla'_a$ as a map from covectors

[1] In our case, we take them to be the Levi-Civita connection of g_{ab} and g'_{ab}, respectively. However, they can be any derivative operator.

to $(0,2)$-tensors. Clearly, $\nabla_a - \nabla'_a$ is $\mathcal{F}$-linear, as we have shown in eq. (5.5), which means that we can associate with $\nabla_a - \nabla'_a$ a $(1,2)$-tensor field $C^c{}_{ab}$ such that

$$(\nabla_a - \nabla'_a)\omega_b = C^c{}_{ab}\omega_c. \tag{5.8}$$

By a similar argument, we expect that there exists a $(1,2)$-tensor field $S^c{}_{ab}$ such that

$$(\nabla_a - \nabla'_a)v^c = S^c{}_{ab}v^b. \tag{5.9}$$

To see how $C^c{}_{ab}$ and $S^c{}_{ab}$ are related, we use the fact that $v^c\omega_c \in \mathcal{F}$ and $\nabla_a f = \nabla'_a f$, $\forall f \in \mathcal{F}$. Therefore,

$$\begin{aligned}
(\nabla_a - \nabla'_a)(v^c\omega_c) = (\nabla_a - \nabla'_a)v^c\,\omega_c + v^c\,(\nabla_a - \nabla'_a)\omega_c = 0\,, \\
(\nabla_a - \nabla'_a)v^c\,\omega_c = -v^c\,(\tilde{\nabla}_a - \nabla_a)\omega_c\,, \\
S^c{}_{ab}v^b\,\omega_c = -v^c\,C^d{}_{ac}\omega_d\,, \\
S^c{}_{ab}v^b\omega_c = -C^d{}_{ac}v^c\omega_d = -C^c{}_{ab}v^b\omega_c\,, \\
S^c{}_{ab} = -C^c{}_{ab}.
\end{aligned} \tag{5.10}$$

We can continue this to obtain the action of $\nabla_a - \nabla'_a$ on an arbitrary (k,l)-tensor field. For any $T^{a_1\ldots a_k}{}_{b_1\ldots b_l} \in \mathcal{T}^k_l$, we have

$$\begin{aligned}
(\nabla_c - \nabla'_c)T^{a_1\ldots a_k}{}_{b_1\ldots b_l} = \quad & C^d{}_{cb_1}T^{a_1\ldots a_k}{}_{d\ldots b_l} \ldots + C^d{}_{cb_l}T^{a_1\ldots a_k}{}_{b_1\ldots d} \\
& - C^{a_1}{}_{cd}T^{d\ldots a_k}{}_{b_1\ldots b_l} \ldots - C^{a_k}{}_{cd}T^{a_1\ldots d}{}_{b_1\ldots b_l}\,.
\end{aligned} \tag{5.11}$$

Thus, we have

$$\begin{aligned}
\nabla'_c T^{a_1\ldots a_k}{}_{b_1\ldots b_l} = \nabla_c T^{a_1\ldots a_k}{}_{b_1\ldots b_l} & + C^{a_1}{}_{cd}T^{d\ldots a_k}{}_{b_1\ldots b_l} \ldots + C^{a_k}{}_{cd}T^{a_1\ldots d}{}_{b_1\ldots b_l} \\
& - C^d{}_{cb_1}T^{a_1\ldots a_k}{}_{d\ldots b_l} \ldots - C^d{}_{cb_l}T^{a_1\ldots a_k}{}_{b_1\ldots d}\,.
\end{aligned} \tag{5.12}$$

5.2 The ∇'_c connection

The Levi-Civita connection

From the Fundamental Theorem of Riemannian Geometry, for any metric there exists a unique torsion free metric connection, called the Levi-Civita connection.[2] Let ∇_c and ∇'_c be the Levi-Civita connection of g_{ab} and g'_{ab}, respectively. This means that

$$\nabla_c\, g_{ab} = \ 0 \ = \nabla'_c\, g'_{ab}. \tag{5.13}$$

[2]Unless otherwise stated, if we are given a metric the connection is automatically assumed to be the Levi-Civita connection.

Then,

$$(\nabla_c - \nabla'_c)\, {g'}_{ab} = {C^d}_{ca} {g'}_{db} + {C^d}_{cb} {g'}_{ad}\,, \tag{5.14}$$

$$\nabla_c {g'}_{ab} = {C^d}_{ca} {g'}_{db} + {C^d}_{cb} {g'}_{ad}\,. \tag{5.15}$$

From now on, we let g'^{ab} and ${g'}_{ab}$ be our index raising and lowering operators, respectively.[3] Permuting the indices of eq. (5.15), we get

$$\begin{aligned}
\nabla_c {g'}_{ab} &= C_{bca} + C_{acb}\,, \\
\nabla_b {g'}_{ca} &= C_{abc} + C_{cba}\,, \\
\nabla_a {g'}_{bc} &= C_{cab} + C_{bac}\,.
\end{aligned}$$

Combining the three equations, and taking advantage of the symmetry in the interchange of the last two indices of C_{cab},[4] we obtain

$$\nabla_c {g'}_{ab} + \nabla_b {g'}_{ca} + \nabla_a {g'}_{bc} = 2\,(C_{abc} + C_{bca} + C_{cab})\,. \tag{5.16}$$

The first two terms of the RHS of eq. (5.16) sum to $\nabla_c {g'}_{ab}$, and we can simplify eq. (5.16) to

$$C_{cab} = \frac{1}{2}\,(\nabla_a {g'}_{bc} + \nabla_b {g'}_{ca} - \nabla_c {g'}_{ab})\,. \tag{5.17}$$

Finally, we get

$${C^c}_{ab} = \frac{1}{2} g'^{cd}\,(\nabla_a h_{bd} + \nabla_b h_{da} - \nabla_d h_{ab})\,. \tag{5.18}$$

For first-order perturbations, this reduces to

$${C^c}_{ab} = \frac{1}{2} g^{cd}\,(\nabla_a h_{bd} + \nabla_b h_{da} - \nabla_d h_{ab})\,. \tag{5.19}$$

To summarize, given a metric g_{ab} and a perturbation h_{ab}, we can treat the connection ∇'_c of the metric ${g'}_{ab} = g_{ab} + h_{ab}$ as the connection ∇_c of g_{ab} plus some correction terms contained in the ${C^c}_{ab}$'s.

5.3 The Riemann tensor

The Riemann-Christoffel tensor, or simply the Riemann tensor, constructed from the metric g_{ab} can be written as[5]

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\,\omega_c = {R_{abc}}^d \omega_d. \tag{5.20}$$

[3] In other words, we let g' be the musical isomorphism.

[4] This can be shown by applying $\nabla - \nabla'$ to an exact form, and using the fact that $\nabla_a \nabla_b f = \nabla_b \nabla_a f$ for any scalar field f [10].

[5] Again, the Riemann tensor is actually a tensor field.

Similarly, we can construct the Riemann tensor for the metric ${g'}_{ab}$ using

$$\left(\nabla'_a \nabla'_b - \nabla'_b \nabla'_a\right) \omega_c = {R'}_{abc}{}^{d} \omega_d. \tag{5.21}$$

First, recall from eq. (5.8) that $\nabla'_b \omega_c = \nabla_b \omega_c - C^d{}_{bc}\omega_d$. Now,

$$\begin{aligned}
\left(\nabla'_a - \nabla_a\right) \nabla'_b \omega_c &= -C^d{}_{ab} \nabla'_d \omega_c - C^d{}_{ac} \nabla'_b \omega_d \\
&= -C^d{}_{ab}\left(\nabla_d \omega_c - C^e{}_{dc}\omega_e\right) - C^d{}_{ac}\left(\nabla_b \omega_d - C^e{}_{bd}\omega_e\right) \\
&= -C^d{}_{ab} \nabla_d \omega_c - C^d{}_{ac} \nabla_b \omega_d + \left(C^d{}_{ab} C^e{}_{dc} + C^d{}_{ac} C^e{}_{bd}\right) \omega_e. \qquad (5.22)
\end{aligned}$$

Simplifying further,

$$\begin{aligned}
\nabla'_a \nabla'_b \omega_c &= \nabla_a \nabla'_b \omega_c - C^d{}_{ab} \nabla_d \omega_c - C^d{}_{ac} \nabla_b \omega_d + \left(C^d{}_{ab} C^e{}_{dc} + C^d{}_{ac} C^e{}_{bd}\right) \omega_e \\
&= \nabla_a \left(\nabla_b \omega_c - C^d{}_{bc} \omega_d\right) - C^d{}_{ab} \nabla_d \omega_c - C^d{}_{ac} \nabla_b \omega_d \\
&\quad + \left(C^d{}_{ab} C^e{}_{dc} + C^d{}_{ac} C^e{}_{bd}\right) \omega_e \\
&= \nabla_a \nabla_b \omega_c - \nabla_a \left(C^d{}_{bc} \omega_d\right) - C^d{}_{ab} \nabla_d \omega_c - C^d{}_{ac} \nabla_b \omega_d \\
&\quad + \left(C^d{}_{ab} C^e{}_{dc} + C^d{}_{ac} C^e{}_{bd}\right) \omega_e \\
&= \nabla_a \nabla_b \omega_c - \nabla_a C^d{}_{bc} \omega_d - C^d{}_{bc} \nabla_a \omega_d - C^d{}_{ab} \nabla_d \omega_c - C^d{}_{ac} \nabla_b \omega_d \\
&\quad + \left(C^d{}_{ab} C^e{}_{dc} + C^d{}_{ac} C^e{}_{bd}\right) \omega_e. \qquad (5.23)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\nabla'_b \nabla'_a \omega_c &= \nabla_b \nabla_a \omega_c - \nabla_b C^d{}_{ac} \omega_d - C^d{}_{ac} \nabla_b \omega_d - C^d{}_{ba} \nabla_d \omega_c - C^d{}_{bc} \nabla_a \omega_d \\
&\quad + \left(C^d{}_{ba} C^e{}_{dc} + C^d{}_{bc} C^e{}_{ad}\right) \omega_e. \qquad (5.24)
\end{aligned}$$

Subtracting eq. (5.24) from eq. (5.23), we get

$${R'}_{abc}{}^{d} \omega_d = R_{abc}{}^{d} \omega_d - \left(\nabla_a C^d{}_{bc} - \nabla_b C^d{}_{ac}\right) \omega_d + \left(C^d{}_{ac} C^e{}_{bd} - C^d{}_{bc} C^e{}_{ad}\right) \omega_e. \tag{5.25}$$

Finally,

$${R'}_{abc}{}^{d} = R_{abc}{}^{d} - \left(\nabla_a C^d{}_{bc} - \nabla_b C^d{}_{ac}\right) + \left(C^e{}_{ac} C^d{}_{be} - C^e{}_{bc} C^d{}_{ae}\right). \tag{5.26}$$

For first-order perturbations, eq. (5.26) is simply

$$\begin{aligned}
{R'}_{abcd} &= R_{abcd} + h_d{}^e R_{abce} + \nabla_b C_{dac} - \nabla_a C_{dbc} \qquad (5.27) \\
&= {R^{(0)}}_{abcd} + {R^{(1)}}_{abcd}. \qquad (5.28)
\end{aligned}$$

where ${R^{(0)}}_{abcd} := R_{abcd}$ and ${R^{(1)}}_{abcd} := h_d{}^e R_{abce} + \nabla_b C_{dac} - \nabla_a C_{dbc}$.

Some remarks on index gymnastics

Recall that we have adopted ${g'}_{ab}$ as the index operator. However, note that for tensors involving h_{ab} it suffices to use g_{ab} as the index operator, since the correction terms brought about by an additional h_{ab} are automatically of second order (in λ). Thus, we need not use ${g'}_{ab}$ except for zero-order tensors. Also, since derivatives, by construction, commute with contractions and ∇_c is the metric connection of g_{ab}, we can interchange any pair of contracted indices; i.e., we raise the lowered index and lower the raised index.

The Ricci tensor

The Ricci curvature tensor is defined as $R_{ab} = R^c{}_{acb}$. Thus,

$$\begin{aligned}
{R'}_{ab} &= R_{ab} + \nabla_c C^c{}_{ab} - \nabla_a C^c{}_{cb} \\
&= R_{ab} + \frac{1}{2}\nabla_c \left(\nabla_a h_b{}^c + \nabla_b h^c{}_a - \nabla^c h_{ab}\right) - \frac{1}{2}\nabla_a \left(\nabla_c h_b{}^c + \nabla_b h^c{}_c - \nabla^c h_{cb}\right) \\
&= R_{ab} + \frac{1}{2}\left(\nabla_c \nabla_a h_b{}^c + \nabla_c \nabla_b h^c{}_a + \nabla_a \nabla^c h_{cb} - \nabla_c \nabla^c h_{ab} - \nabla_a \nabla_c h_b{}^c - \nabla_a \nabla_b h^c{}_c\right) \\
&= R_{ab} + \frac{1}{2}\left(\nabla^c \nabla_a h_{bc} + \nabla^c \nabla_b h_{ac} + \nabla_a \nabla^c h_{bc} - \nabla_c \nabla^c h_{ab} - \nabla_a \nabla^c h_{bc} - \nabla_a \nabla_b h^c{}_c\right) \\
&= R_{ab} + \nabla^c \nabla_{(a} h_{b)c} - \frac{1}{2}\left(\Delta h_{ab} + \nabla_a \nabla_b h\right), \qquad (5.29)
\end{aligned}$$

where $\Delta = \nabla_a \nabla^a$ is the Laplace-Beltrami operator associated with g_{ab}, $T_{(\cdots)}$ denotes the symmetrization of T, and $h = g^{ab} h_{ab}$ is the trace of h_{ab}.

The Ricci scalar

The Ricci scalar curvature is defined as $R = g^{ab} R_{ab} = R^a{}_a$. Thus,

$$\begin{aligned}
R' = {g'}^{ab} {R'}_{ab} &= \left(g^{ab} - h^{ab}\right) R_{ab} + g^{ab} \nabla^c \nabla_{(a} h_{b)c} - \frac{1}{2} g^{ab} \left(\Delta h_{ab} + \nabla_a \nabla_b h\right) \\
&= R - h^{ab} R_{ab} + \nabla_a \nabla_b h^{ab} - \Delta h. \qquad (5.30)
\end{aligned}$$

5.4 The Kretschmann scalar

Recall that the Kretschmann scalar is defined as $K = R^{abcd} R_{abcd}$. Then the perturbed Kretschmann scalar is given by $K' = {R'}^{abcd} {R'}_{abcd}$. Now,

$R'^{abcd} = g'^{ae}g'^{bf}g'^{cg}g'^{dh}R'_{efgh}$. This becomes

$$\begin{aligned}
R'^{abcd} &= R^{abcd} - R^{abc}{}_h h^{dh} - R^{ab}{}_g{}^d h^{cg} - R^a{}_f{}^{cd} h^{bf} - R_e{}^{bcd} h^{ae} \\
&\quad + R^{abc}{}_e h^{de} + \nabla^b C^{dac} - \nabla^a C^{dbc} \\
&= R^{abcd} - R^{ab}{}_g{}^d h^{cg} - R^a{}_f{}^{cd} h^{bf} - R_e{}^{bcd} h^{ae} + \nabla^b C^{dac} - \nabla^a C^{dbc} \qquad (5.31)\\
&= R^{(0)abcd} + R^{(1)abcd}, \qquad (5.32)
\end{aligned}$$

where $R^{(0)\,abcd} := R^{abcd}$, and
$R^{(1)\,abcd} = -R^{ab}{}_g{}^d h^{cg} - R^a{}_f{}^{cd} h^{bf} - R_e{}^{bcd} h^{ae} + \nabla^b C^{dac} - \nabla^a C^{dbc}$. Thus,

$$\begin{aligned}
K' = R'^{abcd}R'_{abcd} &= \left(R^{(0)abcd} + R^{(1)abcd}\right)\left(R^{(0)}{}_{abcd} + R^{(1)}{}_{abcd}\right) \\
&= K + R^{(0)abcd}R^{(1)}{}_{abcd} + R^{(0)}{}_{abcd}R^{(1)abcd} \\
&= K + R^{abcd}\left(h_d{}^e R_{abce} + \nabla_b C_{dac} - \nabla_a C_{dbc}\right) \\
&\quad + R_{abcd}\left(-R^{ab}{}_g{}^d h^{cg} - R^a{}_f{}^{cd} h^{bf} - R_e{}^{bcd} h^{ae} + \nabla^b C^{dac} - \nabla^a C^{dbc}\right) \\
&= K + 2R^{abcd}\left(\nabla_b C_{dac} - \nabla_a C_{dbc}\right) \\
&\quad + R^{abcd}\left(R_{abce}h_d{}^e - R_{abed}h_c{}^e - R_{aecd}h_b{}^e - R_{ebcd}h_a{}^e\right) \qquad (5.33)\\
&= K^{(0)} + K^{(1)}. \qquad (5.34)
\end{aligned}$$

where $K^{(0)} = K$, and
$K^{(1)} = 2R^{abcd}\left(\nabla_b C_{dac} - \nabla_a C_{dbc}\right) + R^{abcd}\left(R_{abce}h_d{}^e - R_{abed}h_c{}^e - R_{aecd}h_b{}^e - R_{ebcd}h_a{}^e\right)$.
In order to simplify things, we define

$$K^{(1)} = K_{\clubsuit} + K_{\spadesuit}, \qquad (5.35)$$

$$K_{\clubsuit} = 2R^{abcd}\left(\nabla_b C_{dac} - \nabla_a C_{dbc}\right), \qquad (5.36)$$

$$K_{\spadesuit} = R^{abcd}\left(R_{abce}h_d{}^e - R_{abed}h_c{}^e - R_{aecd}h_b{}^e - R_{ebcd}h_a{}^e\right). \qquad (5.37)$$

Note that we can still simplify eq. (5.36) using eq. (5.19).

$$\begin{aligned}
K_{\clubsuit} &= 2R^{abcd}\left(\nabla_b C_{dac} - \nabla_a C_{dbc}\right) \\
&= R^{abcd}\left[\nabla_b\left(\nabla_a h_{cd} + \nabla_c h_{da} - \nabla_d h_{ac}\right) - \nabla_a\left(\nabla_b h_{cd} + \nabla_c h_{db} - \nabla_d h_{bc}\right)\right] \\
&= R^{abcd}\left[\left(\nabla_b\nabla_a - \nabla_a\nabla_b\right)h_{cd} + \nabla_b\nabla_c h_{da} - \nabla_b\nabla_d h_{ac} - \nabla_a\nabla_c h_{db} + \nabla_a\nabla_d h_{bc}\right] \\
&= R^{abcd}\left(R_{bac}{}^e h_{ed} + R_{bad}{}^e h_{ce} + \nabla_b\nabla_c h_{da} - \nabla_b\nabla_d h_{ac} - \nabla_a\nabla_c h_{db} + \nabla_a\nabla_d h_{bc}\right) \\
&= R^{abcd}\left(R_{bace}h^e{}_d + R_{bade}h_c{}^e + \nabla_b\nabla_c h_{da} - \nabla_b\nabla_d h_{ac} - \nabla_a\nabla_c h_{db} + \nabla_a\nabla_d h_{bc}\right). \qquad (5.38)
\end{aligned}$$

A consequence of the symmetry of Riem

Recall that R^{abcd} is antisymmetric with respect to the interchange of the c and d indices. However, note that the first two terms inside the parenthesis of eq. (5.37), given by

$$R_{abce}h_d{}^e - R_{abed}h_c{}^e$$

is symmetric with respect to the interchange of the c and d indices. Thus, its contraction with R^{abcd} is zero. Therefore,

$$K_{\spadesuit} = -R^{abcd}\left(R_{aecd}h_b{}^e + R_{ebcd}h_a{}^e\right). \tag{5.39}$$

We can also conclude the same for first two terms inside the first parenthesis of eq. (5.38). Thus,

$$K_{\clubsuit} = R^{abcd}\left(\nabla_b\nabla_c h_{da} - \nabla_b\nabla_d h_{ac} - \nabla_a\nabla_c h_{db} + \nabla_a\nabla_d h_{bc}\right). \tag{5.40}$$

Note that this expression is unchanged under $(a \leftrightarrow b)$ and $(c \leftrightarrow d)$; therefore, we can simplify this further, and in many ways:

$$K_{\clubsuit} = 2R^{abcd}\left(\nabla_b\nabla_c h_{ad} - \nabla_b\nabla_d h_{ac}\right) = 2R^{abcd}\left(\nabla_a\nabla_d h_{bc} - \nabla_a\nabla_c h_{db}\right) = \dots \tag{5.41}$$

However, we shall continue to use eq. (5.40) in the next chapter, since it is more convenient. Finally, we may write the first-order correction to K as

$$K^{(1)} = 2R^{abcd}\left(\nabla_b\nabla_c h_{ad} - \nabla_b\nabla_d h_{ac}\right) - R^{abcd}\left(R_{aecd}h_b{}^e + R_{ebcd}h_a{}^e\right). \tag{5.42}$$

Note that this equation is coordinate-independent. In the next chapter, we shall work in a specific coordinate system to further decompose $K^{(1)}$.

5.5 Horizon detector

Let $\mathcal{I}$ be any well-defined curvature invariant in the Schwarzschild spacetime. Then the gradient of $\mathcal{I}$ is null on the event horizon. In other words, the invariant

$$\mathcal{H} = \nabla_a\mathcal{I}\nabla^a\mathcal{I} \tag{5.43}$$

vanishes on the event horizon. In our case, we choose $\mathcal{I}$ to be the Kretschmann scalar. Therefore, the curvature invariant $I_5 = \nabla_a K\nabla^a K$, locates the event horizon of the Schwarzschild spacetime, hence we shall rename it H. We would like to assume that the invariant $H' = \nabla'_a K' \nabla'^a K'$ also detects a horizon of the perturbed spacetime, although For scalars, $\nabla_a = \nabla'_a$, therefore

$$H' = g'^{ab} \nabla_a K' \nabla_b K'. \tag{5.44}$$

This is simply

$$\begin{aligned}
H' &= \left(g^{ab} - h^{ab}\right)\left(\nabla_a K + \nabla_a K^{(1)}\right)\left(\nabla_b K + \nabla_b K^{(1)}\right) \\
&= \left(g^{ab} - h^{ab}\right) \nabla_a K \nabla_b K + 2 g^{ab} \nabla_a K \nabla_b K^{(1)} \\
&= g^{ab} \nabla_a K \nabla_b K - h^{ab} \nabla_a K \nabla_b K + 2 g^{ab} \nabla_a K \nabla_b K^{(1)} \\
&= H - h^{ab} \nabla_a K \nabla_b K + 2 \nabla_a K \nabla^a K^{(1)}.
\end{aligned} \tag{5.45}$$

We emphasize that this equation is covariant (i.e. valid in any coordinate system). In the next chapter, we further develop the formalism in a specific coordinate system.

Chapter 6

Perturbations III: a practical formalism

In the previous chapter, we derived equations that are valid in any coordinate system. However, recall that in order to do meaningful physical calculations, we must work in a specific coordinate system. For linear perturbations, a convenient coordinate system was provided in Chapter 4 In this work, we only use the Schwarzschild and ingoing Eddington-Finkelstein coordinate systems. In this case, $\mathfrak{M}^2$ will be coordinatized by (t, r) or (v, r) such that the line element on $\mathfrak{M}^2$ is

$$\mathrm{d}s^2_{\mathfrak{M}^2} = -f\mathrm{d}t^2 + f^{-1}\mathrm{d}r^2\,, \tag{6.1}$$

$$= -f\mathrm{d}v^2 + 2\mathrm{d}v\,\mathrm{d}r\,. \tag{6.2}$$

Note that we are not limited to these two coordinate systems; we can use any coordinate system on $\mathfrak{M}^2$, even one where r is not one of the coordinates. Following [7], we introduce the covector

$$r_\alpha = \frac{\partial r}{\partial \bar{x}^\alpha}, \tag{6.3}$$

which is normal to the surfaces of constant-$r(\bar{x}^\alpha)$ surfaces. In the (t, r) and (v, r) coordinates $r_\alpha = (0, 1)$, and this lets us define $f = r_\alpha r^\alpha$ [7].

6.1 Some remarks

We now return to Section 5.5. As an illustration, we shall adopt the coordinates introduced in Section 4.2. We shall derive a formula for H' valid in the Schwarzschild (t, r) coordinates, and another valid in the ingoing Eddington-Finkelstein (v, r) coordinates. We know that in both the (t, r) and (v, r) coordinates, the only nonzero component of $\nabla_a K$ is $\nabla_r K$, and

$$H = f\nabla_r K \nabla_r K = \frac{82944 M^4}{r^{14}} f. \tag{6.4}$$

Clearly, we see how the vanishing at the horizon is dictated by none other than f.

Schwarzschild coordinates

In the (t, r)-coordinates,

$$\begin{aligned} H' &= H - h^{rr}\nabla_r K \nabla_r K + 2f\nabla_r K \nabla_r K^{(1)} \\ &= (\nabla_r K)^2 (f - h^{rr}) + 2f\nabla_r K \nabla_r K^{(1)}. \end{aligned} \tag{6.5}$$

Since we only care about the zero of this function, we may divide everything by $(\nabla_r K)^2$, which is never zero, and redefine H' as

$$H' = g'^{rr} + 2f \, \frac{\nabla_r K^{(1)}}{\nabla_r K}. \tag{6.6}$$

Thus,

$$H' = f\left(1 - f h_{rr} + 2\, \frac{\nabla_r K^{(1)}}{\nabla_r K}\right). \tag{6.7}$$

Ingoing Eddington-Finkelstein coordinates

In the (v, r)-coordinates,

$$\begin{aligned} H' &= H - h^{rr}\nabla_r K \nabla_r K + 2\nabla_r K \nabla_v K^{(1)} + 2f\nabla_r K \nabla_r K^{(1)} \\ &= (\nabla_r K)^2 (f - h^{rr}) + 2\nabla_r K \left(\nabla_v K^{(1)} + f\nabla_r K^{(1)}\right) \\ &= f - h^{rr} + 2\left(\frac{\nabla_v K^{(1)}}{\nabla_r K} + \frac{\nabla_r K^{(1)}}{\nabla_r K} f\right) \\ &= f - \left(h_{vv} + 2f h_{vr} + f^2 h_{rr}\right) + 2\left(\frac{\nabla_v K^{(1)}}{\nabla_r K} + \frac{\nabla_r K^{(1)}}{\nabla_r K} f\right) \\ &= -g'_{vv} + \left(2f h_{vr} + f^2 h_{rr}\right) + 2\left(\frac{\nabla_v K^{(1)}}{\nabla_r K} + \frac{\nabla_r K^{(1)}}{\nabla_r K} f\right). \end{aligned} \tag{6.8}$$

Suppose we only consider static perturbations, i.e., the components of h_{ab} are not functions of t or v. Then, the first-order correction to the horizon is essentially given by (we ignore the sign)

$$H' = g'_{vv}. \tag{6.9}$$

In order to prove this assertion, we plug in the ansatz $r' = r_s + \lambda\, r_s^{(1)} + \mathcal{O}(\lambda^2)$, similar to what we did in Section 4.4. Observe that $f(r') \sim \lambda$. If we multiply this to

$\nabla_r K^{(1)}$, which is also $\sim \lambda$, the result will be $\sim \lambda^2$. Similarly for the terms in the first parenthesis of eq. (6.8), since $h \sim \lambda$. Thus, H' reduces to equation for the Killing horizon, eq. (6.9). This is not surprising, since the perturbed spacetime is static; therefore H' locates the event horizon, as expected. Hence, any contribution of $K^{(1)}$ to H' must come from $\nabla_v K^{(1)}$, which is nonzero only when the perturbations are not static. And since we only know that the geometric horizon coincides with the event horizon for stationary spacetimes, we do not know what sort of dynamical horizon[1], if it even exists, H' defines. Can H' give the approximate location of a nearly stationary horizon as claimed in [3]? Also, we could have used any curvature invariant $\mathcal{I}$, how does the contribution of $\nabla_v K^{(1)}$ differ from the contribution of an arbitrary $\nabla_v \mathcal{I}^{(1)}$? What horizons do H' and $\mathcal{H}'$ define?

Moreover, in the Schwarzschild coordinates there can never be any correction to eq. (6.9). Why can there be no $\nabla_t K^{(1)}$ correction? Although this is not particularly troublesome, since the Schwarzschild coordinates are only valid outside the event horizon, so this might only manifest a failure of the coordinate system. After all, is this not the reason why we introduced horizon penetrating coordinates in Section 2.2? Unfortunately, the Preston-Poisson spacetime is static, so the event horizon is trivially given by eq. (6.9), and we cannot see the dynamical generalization of H'. Still, even in static perturbed Schwarzschild spacetimes H' can have some advantage, although it was not so obvious from our example. The Schwarzschild and ingoing Eddington-Finkelstein coordinate systems were already 'nice' to begin with. Equation 5.45 still has the advantage that it is valid in any coordinate system, and it can still be used when the traditional method of finding the Killing horizon is not so straightforward. Now, suppose we have a perturbed spacetime such that $\nabla_v K^{(1)}$ is not zero. What does the zero set of H' define? It might give a meaningful horizon, or it might not; we still do not know, but we shall derive a formula anyway.

6.2 Index gymnastics

We consider again the coordinate system in Section 4.2. The goal is to further decompose $K^{(1)}$ in terms of objects in $\mathfrak{M}^2$ and S^2, if possible. Recall that we take the

[1]Note that the *dynamical horizon* has a formal definition as a black hole horizon. However, we shall use the term for any horizon of a dynamical black hole spacetime.

line element to be of the form

$$\mathrm{d}s^2 = \bar{g}_{\alpha\beta}(x)\,\mathrm{d}x^\alpha \mathrm{d}x^\beta + r(x)^2\,\Omega_{AB}(\omega)\,\mathrm{d}\omega^A \mathrm{d}\omega^B, \tag{6.10}$$

where $\bar{g}_{\alpha\beta}(x)$ is the metric on $\mathfrak{M}^2$, and Ω_{AB} is the round metric on the unit sphere S^2 [7, 9]. Essentially, we wish to express concomitants[2] of g_{ab} in terms of concomitants of $\bar{g}_{\alpha\beta}$ and concomitants of Ω_{AB}, if possible. In other words, suppose $\{g_{ab}, \Gamma^a{}_{bc}, R_{abcd}, \ldots\}$ are objects in $\mathcal{M}$. Then we want to express them in terms of $\{\bar{g}_{\alpha\beta}, \bar{\Gamma}^\alpha{}_{\beta\gamma}, \bar{R}_{\alpha\beta\gamma\delta}, \ldots\}$, which are objects in $\mathfrak{M}^2$, and $\left\{\tilde{g}_{AB}, \tilde{\Gamma}^A{}_{BC}, \tilde{R}_{ABCD}, \ldots\right\}$, which are objects in S^2. Let $\mathfrak{M}^2$ be coordinatized by $\bar{x}^\alpha = \{\bar{x}^0, \bar{x}^1\}$, and S^2 by $\omega^A = \{\theta, \phi\}$. Here, the lower-case Latin indices no longer represent abstract indices, but denote components in the coordinate system $x^a = \{\bar{x}^0, \bar{x}^1, \theta, \phi\}$. In other words, lower-case Latin indices run from $\{0, 1, 2, 3\}$, lower-case Greek indices run from $\{0, 1\}$, and upper-case Latin indices run from $\{2, 3\}$. We shall use A, $\bar{A}$ and $\tilde{A}$ to denote objects on $\mathcal{M}$, $\mathfrak{M}^2$ and S^2, respectively. Note that $\tilde{g}_{AB} = \Omega_{AB}$ and $\mathbf{g} = \bar{\mathbf{g}} \bigoplus r^2\tilde{\mathbf{g}}$. As a trivial example, we write the metric g_{ab} in terms of $\bar{g}_{\alpha\beta}$ and $\tilde{g}_{AB}$. Each of the indices of the metric g_{ab} can either be from $\{0, 1\}$ or $\{2, 3\}$. Clearly,

$$\begin{aligned} g_{\alpha\beta} &= \bar{g}_{\alpha\beta}\,, \\ g_{AB} &= r^2\tilde{g}_{AB} = r^2\Omega_{AB}\,, \\ g_{\alpha B} &=\ 0\ = g_{A\beta}\,. \end{aligned}$$

We are now ready to derive some useful formulas.

Christoffel symbols

The Christoffel symbols are given by

$$\Gamma^c{}_{ab} = \frac{1}{2} g^{cd}\left(\partial_a g_{bd} + \partial_b g_{da} - \partial_d g_{ab}\right). \tag{6.11}$$

Each index of the Christoffel symbol can be anything from $\{0, 1, 2, 3\}$. There are six possibilities: (i) all indices are from $\{0, 1\}$, (ii) one lowered index is from $\{2, 3\}$ and the rest are from $\{0, 1\}$, (iii) the raised index is from $\{0, 1\}$ while the lowered indices are from $\{2, 3\}$, (iv) the raised index is from $\{2, 3\}$ while the lowered indices are from $\{0, 1\}$, (v) one lowered index is from $\{0, 1\}$ and the rest are from $\{2, 3\}$, (vi) all indices are from $\{2, 3\}$. As an example, we consider case (iii).

$$\Gamma^\gamma{}_{AB} = \frac{1}{2} g^{\gamma d}\left(\partial_A g_{Bd} + \partial_B g_{dA} - \partial_d g_{AB}\right). \tag{6.12}$$

[2]A concomitant is an object which is a function of the metric and its derivatives.

The indices γ, A, and B are fixed, while d is summed from $\{0,1,2,3\}$. Clearly, $g^{\gamma d}$ is only nonzero when d is from $\{2,3\}$, so we replace it by δ and only sum from $\{0,1\}$. In other words, we can replace $g^{\gamma d}$ by $g^{\gamma\delta} = \bar{g}^{\gamma\delta}$. Now,

$$\Gamma^{\gamma}{}_{AB} = \frac{1}{2}\bar{g}^{\gamma\delta}\left(\partial_A g_{B\delta} + \partial_B g_{\delta A} - \partial_\delta g_{AB}\right).$$

The first two terms in the parenthesis is clearly zero, since the metric has no such components. Therefore,

$$\Gamma^{\gamma}{}_{AB} = -\frac{1}{2}\bar{g}^{\gamma\delta}\partial_\delta g_{AB} = -\frac{1}{2}\bar{g}^{\gamma\delta}\frac{\partial}{\partial\bar{x}^\delta}\left(r^2\Omega_{AB}\right).$$

Since Ω_{AB} does not depend on $\bar{x}$, we can pull it out of the derivative. Therefore,

$$\Gamma^{\gamma}{}_{AB} = -rr^{\gamma}\Omega_{AB},$$

where $r^\gamma := \bar{g}^{\gamma\delta}r_\delta$. The other cases can be easily derived using the same approach; the only nonzero Christoffel symbols in $\mathcal{M}$ are

$$\Gamma^{\gamma}{}_{\alpha\beta} = \bar{\Gamma}^{\gamma}{}_{\alpha\beta}\,, \tag{6.13}$$

$$\Gamma^{C}{}_{AB} = \tilde{\Gamma}^{C}{}_{AB}\,, \tag{6.14}$$

$$\Gamma^{\gamma}{}_{AB} = -rr^{\gamma}\Omega_{AB}\,, \tag{6.15}$$

$$\Gamma^{C}{}_{A\beta} = \frac{r_\beta}{r}\delta^{C}{}_{A}\,. \tag{6.16}$$

Riemann tensor

Now that we have the Christoffel symbols, we can now compute for the components of the Riemann tensor, which are given by

$$R^{a}{}_{bcd} = \partial_c\Gamma^{a}{}_{bd} - \partial_d\Gamma^{a}{}_{bc} + \Gamma^{a}{}_{ce}\Gamma^{e}{}_{bd} - \Gamma^{a}{}_{de}\Gamma^{e}{}_{bc}. \tag{6.17}$$

Clearly,

$$\begin{aligned}
R^{\alpha}{}_{\beta\gamma\delta} &= \partial_\gamma\Gamma^{\alpha}{}_{\beta\delta} - \partial_\delta\Gamma^{\alpha}{}_{\beta\gamma} + \Gamma^{\alpha}{}_{\gamma e}\Gamma^{e}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\delta e}\Gamma^{e}{}_{\beta\gamma} \\
&= \partial_\gamma\Gamma^{\alpha}{}_{\beta\delta} - \partial_\delta\Gamma^{\alpha}{}_{\beta\gamma} + \Gamma^{\alpha}{}_{\gamma\epsilon}\Gamma^{\epsilon}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\delta\epsilon}\Gamma^{\epsilon}{}_{\beta\gamma} \\
&= \partial_\gamma\bar{\Gamma}^{\alpha}{}_{\beta\delta} - \partial_\delta\bar{\Gamma}^{\alpha}{}_{\beta\gamma} + \bar{\Gamma}^{\alpha}{}_{\gamma\epsilon}\bar{\Gamma}^{\epsilon}{}_{\beta\delta} - \bar{\Gamma}^{\alpha}{}_{\delta\epsilon}\bar{\Gamma}^{\epsilon}{}_{\beta\gamma} \\
&= \bar{R}^{\alpha}{}_{\beta\gamma\delta},
\end{aligned}$$

where $\bar{R}^{\alpha}{}_{\beta\gamma\delta}$ is the Riemann tensor on $\mathfrak{M}^2$. If we lower all the indices of the $R^{\alpha}{}_{\beta\gamma\delta}$, we simply get $R_{\alpha\beta\gamma\delta} = g_{\alpha e}R^{e}{}_{\beta\gamma\delta} = g_{\alpha\epsilon}R^{\epsilon}{}_{\beta\gamma\delta} = \bar{g}_{\alpha\epsilon}\bar{R}^{\epsilon}{}_{\beta\gamma\delta} = \bar{R}_{\alpha\beta\gamma\delta}$. Note that since $\mathfrak{M}^2$

is a 2-dimensional manifold, the Riemann tensor is completely determined by the Ricci scalar.[3] In fact, the Riemann tensor is given by

$$\bar{R}_{\alpha\beta\gamma\delta} = \frac{1}{2}\bar{R}\left(\bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma}\right) = \frac{2M}{r^3}\left(\bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma}\right), \tag{6.18}$$

which is a straightforward computation. Similarly,

$$\begin{aligned}
R^A{}_{BCD} &= \partial_C\Gamma^A{}_{BD} - \partial_D\Gamma^A{}_{BC} + \Gamma^A{}_{Ce}\Gamma^e{}_{BD} - \Gamma^A{}_{De}\Gamma^e{}_{BC} \\
&= \partial_C\Gamma^A{}_{BD} - \partial_D\Gamma^A{}_{BC} + \Gamma^A{}_{C\epsilon}\Gamma^\epsilon{}_{BD} + \Gamma^A{}_{CE}\Gamma^E{}_{BD} - \Gamma^A{}_{D\epsilon}\Gamma^\epsilon{}_{BC} - \Gamma^A{}_{DE}\Gamma^E{}_{BC} \\
&= \partial_C\tilde{\Gamma}^A{}_{BD} - \partial_D\tilde{\Gamma}^A{}_{BC} + \tilde{\Gamma}^A{}_{CE}\tilde{\Gamma}^E{}_{BD} - \tilde{\Gamma}^A{}_{DE}\tilde{\Gamma}^E{}_{BC} - f\left(\delta^A{}_C\Omega_{BD} - \delta^A{}_D\Omega_{BC}\right) \\
&= \tilde{R}^A{}_{BCD} - f\left(\delta^A{}_C\Omega_{BD} - \delta^A{}_D\Omega_{BC}\right).
\end{aligned}$$

Now, if we lower the indices of $R^A{}_{BCD}$ we get

$$\begin{aligned}
R_{ABCD} = g_{AE}R^E{}_{BCD} &= r^2\Omega_{AE}\left[\tilde{R}^E{}_{BCD} - f\left(\delta^E{}_C\Omega_{BD} - \delta^E{}_D\Omega_{BC}\right)\right] \\
&= r^2\left[\tilde{R}_{ABCD} - f\left(\Omega_{AC}\Omega_{BD} - \Omega_{AD}\Omega_{BC}\right)\right] \\
&= r^2(1-f)\tilde{R}_{ABCD} = 2Mr\tilde{R}_{ABCD},
\end{aligned}$$

where we used the fact that for the round sphere S^2,

$$\tilde{R}_{ABCD} = \frac{1}{2}\tilde{R}\left(\Omega_{AC}\Omega_{BD} - \Omega_{AD}\Omega_{BC}\right) = \left(\Omega_{AC}\Omega_{BD} - \Omega_{AD}\Omega_{BC}\right). \tag{6.19}$$

The only other nonzero component of the Riemann tensor has the form $R_{\alpha A\beta B}$, which we no longer derive. Thus, we have the following nonzero components of the Riemann tensor:

$$R_{\alpha\beta\gamma\delta} = \bar{R}_{\alpha\beta\gamma\delta}, \tag{6.20}$$

$$R_{ABCD} = 2Mr\tilde{R}_{ABCD}, \tag{6.21}$$

$$R_{\alpha B\gamma D} = -\frac{M}{r}\bar{g}_{\alpha\gamma}\Omega_{BD}. \tag{6.22}$$

Raising all their indices, we get

$$R^{\alpha\beta\gamma\delta} = \bar{R}^{\alpha\beta\gamma\delta}, \tag{6.23}$$

$$R^{ABCD} = \frac{2M}{r^7}\tilde{R}^{ABCD}, \tag{6.24}$$

$$R^{\alpha B\gamma D} = -\frac{M}{r^5}\bar{g}^{\alpha\gamma}\Omega^{BD}. \tag{6.25}$$

As a safety check, we can compute the Kretschmann scalar by summing over all the indices of $R_{abcd}R^{abcd}$. Note that we have to take into account the four possible permutations of the indices of $R^{\alpha A\beta B}$. It can then be easily be shown that the sum yields $48M^2/r^6$.

[3] which is the Gaussian curvature

6.3 Decomposing $K_\clubsuit$

We now begin the arduous process. Recall from eq. (5.40) that

$$K_\clubsuit = R^{abcd}\left(\nabla_b\nabla_c h_{da} - \nabla_b\nabla_d h_{ac} - \nabla_a\nabla_c h_{db} + \nabla_a\nabla_d h_{bc}\right).$$

Again, we shall suppose that the indices are summed indices and not abstract indices. First, note that the tensor expression inside the parenthesis is antisymmetric under the exchange of indices $(a \leftrightarrow b)$ and $(c \leftrightarrow d)$; and since the Riemann tensor also exhibits such symmetry, their product is symmetric. In other words, $K_\clubsuit$ is unchanged under the replacement $(a \leftrightarrow b)$ or $(c \leftrightarrow d)$. Then using eqs. (6.23) to (6.25), and taking into account the four permutations of $R^{\alpha B\gamma D}$, we can write

$$\begin{aligned} K_\clubsuit = {} & \bar{R}^{\alpha\beta\gamma\delta}\left(\nabla_\beta\nabla_\gamma h_{\delta\alpha} - \nabla_\beta\nabla_\delta h_{\alpha\gamma} - \nabla_\alpha\nabla_\gamma h_{\delta\beta} + \nabla_\alpha\nabla_\delta h_{\beta\gamma}\right) \\ & + \frac{2M}{r^7}\tilde{R}^{ABCD}\left(\nabla_B\nabla_C h_{DA} - \nabla_B\nabla_D h_{AC} - \nabla_A\nabla_C h_{DB} + \nabla_A\nabla_D h_{BC}\right) \\ & - \frac{4M}{r^5}\bar{g}^{\alpha\gamma}\Omega^{BD}\left(\nabla_B\nabla_\gamma h_{D\alpha} - \nabla_B\nabla_D h_{\alpha\gamma} - \nabla_\alpha\nabla_\gamma h_{DB} + \nabla_\alpha\nabla_D h_{B\gamma}\right). \end{aligned} \tag{6.26}$$

Using eqs. (6.18) and (6.19), we can simplify this to

$$\begin{aligned} K_\clubsuit = \bar{R}\Bigg\{ 2\Bigg[& \left(\bar{g}^{\alpha\gamma}\bar{g}^{\beta\delta} - \bar{g}^{\alpha\beta}\bar{g}^{\gamma\delta}\right)\nabla_\alpha\nabla_\beta h_{\gamma\delta} + \frac{1}{r^4}\left(\Omega^{AC}\Omega^{BD} - \Omega^{AB}\Omega^{CD}\right)\nabla_A\nabla_B h_{CD}\Bigg] \\ & + \frac{1}{r^2}\bar{g}^{\alpha\beta}\Omega^{AB}\left[\left(\nabla_\alpha\nabla_\beta h_{AB} + \nabla_A\nabla_B h_{\alpha\beta}\right) - \left(\nabla_\alpha\nabla_A + \nabla_A\nabla_\alpha\right)h_{\beta B}\right]\Bigg\}. \end{aligned} \tag{6.27}$$

Note that we cannot contract the indices, because the derivative operators are in $\mathcal{M}$.[4] Thus, we need to decompose $\nabla_a\nabla_b h_{cd}$.

Decomposing $\nabla_b h_{cd}$

We first decompose $\nabla_b h_{cd}$, which shall proceed the same way as the Christoffel symbols. Let $\bar{\nabla}_c$ and $\tilde{\nabla}_c$ be the metric connection on $\mathfrak{M}^2$ and S^2, respectively. Note that since h_{cd} is symmetric there are only three cases, which we multiply by two for ∇_c. Therefore, there are a total of six cases, similar to the Christoffel symbols. Consider case (i): $\nabla_\beta h_{\gamma\delta}$.

$$\begin{aligned} \nabla_\beta h_{\gamma\delta} &= \partial_\beta h_{\gamma\delta} - \Gamma^e{}_{\beta\gamma}h_{e\delta} - \Gamma^e{}_{\beta\delta}h_{\gamma e} \\ &= \partial_\beta h_{\gamma\delta} - \Gamma^\epsilon{}_{\beta\gamma}h_{\epsilon\delta} - \Gamma^E{}_{\beta\gamma}h_{E\delta} - \Gamma^\epsilon{}_{\beta\delta}h_{\gamma\epsilon} - \Gamma^E{}_{\beta\delta}h_{\gamma E} \\ &= \partial_\beta h_{\gamma\delta} - \bar{\Gamma}^\epsilon{}_{\beta\gamma}h_{\epsilon\delta} - \bar{\Gamma}^\epsilon{}_{\beta\delta}h_{\gamma\epsilon} \\ &= \bar{\nabla}_\beta h_{\gamma\delta}. \end{aligned}$$

[4]However, the tensor h_{ab} is not a problem.

Now, consider case (ii): $\nabla_\beta h_{\gamma D}$.

$$\begin{aligned}
\nabla_\beta h_{\gamma D} &= \partial_\beta h_{\gamma D} - \Gamma^e{}_{\beta\gamma} h_{eD} - \Gamma^e{}_{\beta D} h_{\gamma e} \\
&= \partial_\beta h_{\gamma D} - \Gamma^\epsilon{}_{\beta\gamma} h_{\epsilon D} - \Gamma^E{}_{\beta\gamma} h_{ED} - \Gamma^\epsilon{}_{\beta D} h_{\gamma\epsilon} - \Gamma^E{}_{\beta D} h_{\gamma E} \\
&= \partial_\beta h_{\gamma D} - \bar{\Gamma}^\epsilon{}_{\beta\gamma} h_{\epsilon D} - \frac{r_\beta}{r} \delta^E{}_D h_{\gamma E} \\
&= \bar{\nabla}_\beta h_{\gamma D} - \frac{r_\beta}{r} h_{\gamma D}.
\end{aligned}$$

Observe that $\bar{\nabla}_\beta$ acts on $h_{\gamma D}$ as if the latter was a covector on $\mathfrak{M}^2$. Similarly, $\delta^E{}_D$ acts on $h_{\gamma E}$ as if the latter is a covector on S^2. In general, if we have a tensor of the form $T^{\alpha_1\ldots\alpha_k A_1\ldots A_m}{}_{\beta_1\ldots\beta_l B_1\ldots B_n}$, then it is a (k,l)-tensor on $\mathfrak{M}^2$ and an (m,n)-tensor on S^2. In other words, a derivative operator on $\mathfrak{M}^2$ is completely oblivious to upper-case Latin indices, and vice-versa. Thus, we can treat $h_{\gamma\delta}$ as an $(0,2)$-tensor on $\mathfrak{M}^2$ and a scalar on S^2, $h_{\gamma D}$ as a covector on $\mathfrak{M}^2$ and a scalar on S^2, and h_{CD} as scalar on $\mathfrak{M}^2$ and an $(0,2)$-tensor on S^2. For brevity, we omit the derivations for the other cases and only state the final expressions for the six cases. They are

$$\nabla_\beta h_{\gamma\delta} = \bar{\nabla}_\beta h_{\gamma\delta}\,, \tag{6.28}$$

$$\nabla_\beta h_{\gamma D} = \bar{\nabla}_\beta h_{\gamma D} - \frac{r_\beta}{r} h_{\gamma D}\,, \tag{6.29}$$

$$\nabla_\beta h_{CD} = \bar{\nabla}_\beta h_{CD} - \frac{2r_\beta}{r} h_{CD}\,, \tag{6.30}$$

$$\nabla_B h_{\gamma\delta} = \tilde{\nabla}_B h_{\gamma\delta} - \frac{2}{r} r_{(\gamma} h_{\delta)B}\,, \tag{6.31}$$

$$\nabla_B h_{\gamma D} = \tilde{\nabla}_B h_{\gamma D} - \frac{r_\gamma}{r} h_{BD} + r r^\alpha h_{\alpha\gamma} \Omega_{BD}\,, \tag{6.32}$$

$$\nabla_B h_{CD} = \tilde{\nabla}_B h_{CD} + 2 r r^\alpha h_{\alpha(C}\,\Omega_{D)B}\,. \tag{6.33}$$

Decomposing $\nabla_a \nabla_b h_{cd}$

Now, we act another ∇_a on $\nabla_b h_{cd}$; there will be a total of twelve cases. Although not all of them shall be used, we write everything for completeness. First, note the following identity from [7]

$$\bar{\nabla}_\alpha \bar{\nabla}_\beta r = \frac{M}{r^2} \bar{g}_{\alpha\beta}, \tag{6.34}$$

where $\bar{\nabla}_\alpha r = r_\alpha$ Again, we omit the derivations for brevity. The twelve cases are

$$\nabla_\alpha \nabla_\beta h_{\gamma\delta} = \bar{\nabla}_\alpha \bar{\nabla}_\beta h_{\gamma\delta}\,, \tag{6.35}$$

$$\nabla_\alpha \nabla_\beta h_{\gamma D} = \bar{\nabla}_\alpha \bar{\nabla}_\beta h_{\gamma D} - \frac{2}{r} r_{(\alpha} \bar{\nabla}_{\beta)} h_{\gamma D} + \frac{2}{r^2} r_\alpha r_\beta h_{\gamma D} - \frac{M}{r^3} \bar{g}_{\alpha\beta} h_{\gamma D}\,, \tag{6.36}$$

$$\nabla_\alpha \nabla_\beta h_{CD} = \bar{\nabla}_\alpha \bar{\nabla}_\beta h_{CD} - \frac{4}{r} r_{(\alpha} \bar{\nabla}_{\beta)} h_{CD} + \frac{6}{r^2} r_\alpha r_\beta h_{CD} - \frac{2M}{r^3} \bar{g}_{\alpha\beta} h_{CD} \,, \tag{6.37}$$

$$\begin{aligned} \nabla_\alpha \nabla_B h_{\gamma\delta} = \bar{\nabla}_\alpha \tilde{\nabla}_B h_{\gamma\delta} - \frac{2}{r} \bar{\nabla}_\alpha h_{B(\gamma} r_{\delta)} + \frac{4}{r^2} r_\alpha r_{(\gamma} h_{\delta)B} \\ - \frac{2M}{r^3} \bar{g}_{\alpha(\gamma)} h_{\delta)B} - \frac{r_\alpha}{r} \tilde{\nabla}_B h_{\gamma\delta} \,, \end{aligned} \tag{6.38}$$

$$\begin{aligned} \nabla_\alpha \nabla_B h_{\gamma D} = \bar{\nabla}_\alpha \tilde{\nabla}_B h_{\gamma D} + \left(r \bar{\nabla}_\alpha h_{\beta\gamma} - r_\alpha h_{\beta\gamma} \right) r^\beta \Omega_{BD} + \frac{M}{r} h_{\alpha\gamma} \Omega_{BD} \\ - \frac{r_\gamma}{r} \bar{\nabla}_\alpha h_{BD} - \frac{M}{r^3} \bar{g}_{\alpha\gamma} h_{BD} + \frac{3}{r^2} r_\alpha r_\gamma h_{BD} - \frac{2r_\alpha}{r} \tilde{\nabla}_B h_{\gamma D} \,, \end{aligned} \tag{6.39}$$

$$\begin{aligned} \nabla_\alpha \nabla_B h_{CD} = \bar{\nabla}_\alpha \tilde{\nabla}_B h_{CD} + 2 r r^\beta \bar{\nabla}_\alpha h_{\beta(C} \Omega_{D)B} - 4 r_\alpha r^\beta h_{\beta(C} \Omega_{D)B} \\ + \frac{2M}{r} h_{\alpha(C} \Omega_{D)B} - \frac{3r_\alpha}{r} \tilde{\nabla}_B h_{CD} \,, \end{aligned} \tag{6.40}$$

$$\nabla_A \nabla_\beta h_{\gamma\delta} = \tilde{\nabla}_A \bar{\nabla}_\beta h_{\gamma\delta} - \frac{2}{r} \bar{\nabla}_\beta h_{A(\gamma} r_{\delta)} + \frac{4r_\beta}{r^2} r_{(\gamma} h_{\delta)A} - \frac{r_\beta}{r} \tilde{\nabla}_A h_{\gamma\delta} \,, \tag{6.41}$$

$$\begin{aligned} \nabla_A \nabla_\beta h_{\gamma D} = \tilde{\nabla}_A \bar{\nabla}_\beta h_{\gamma D} + \left(r \bar{\nabla}_\beta h_{\alpha\gamma} - r_\beta h_{\alpha\gamma} \right) r^\alpha \Omega_{AD} \\ - \frac{r_\gamma}{r} \bar{\nabla}_\beta h_{AD} + \frac{3}{r^2} r_\beta r_\gamma h_{AD} - \frac{2r_\beta}{r} \tilde{\nabla}_A h_{\gamma D} \,, \end{aligned} \tag{6.42}$$

$$\begin{aligned} \nabla_A \nabla_\beta h_{CD} = \tilde{\nabla}_A \bar{\nabla}_\beta h_{CD} + 2 r r^\alpha \bar{\nabla}_\beta h_{\alpha(C} \Omega_{D)A} \\ - 4 r^\alpha r_\beta h_{\alpha(C} \Omega_{D)A} - \frac{3r_\beta}{r} \tilde{\nabla}_A h_{CD} \,, \end{aligned} \tag{6.43}$$

$$\begin{aligned} \nabla_A \nabla_B h_{\gamma\delta} = \tilde{\nabla}_A \tilde{\nabla}_B h_{\gamma\delta} - \frac{4}{r} \tilde{\nabla}_{(A} h_{B)(\gamma} r_{\delta)} + \frac{2}{r^2} r_\gamma r_\delta h_{AB} \\ - r^\alpha r_{(\gamma} h_{\delta)\alpha} \Omega_{AB} + r r_\alpha \bar{\nabla}_\alpha h_{\gamma\delta} \Omega_{AB} \,, \end{aligned} \tag{6.44}$$

$$\begin{aligned} \nabla_A \nabla_B h_{\gamma D} = \tilde{\nabla}_A \tilde{\nabla}_B h_{\gamma D} + r r^\alpha \bar{\nabla}_\alpha h_{\gamma D} \Omega_{AB} - 2 f \Omega_{A(B} h_{D)\gamma} - 3 r^\alpha r_\gamma h_{\alpha(A} \Omega_{BD)} \\ - \frac{2r_\gamma}{r} \tilde{\nabla}_{(A} h_{B)D} + r r^\alpha \Omega_{D(B} \tilde{\nabla}_{A)} h_{\alpha\gamma} \,, \end{aligned} \tag{6.45}$$

$$\begin{aligned} \nabla_A \nabla_B h_{CD} = \tilde{\nabla}_A \tilde{\nabla}_B h_{CD} + 2 r^2 \Omega_{A(C} \Omega_{D)B} r^\alpha r^\beta h_{\alpha\beta} \\ + 2 r r^\alpha \Omega_{D(B} \tilde{\nabla}_{A)} h_{\alpha C} + 2 r r^\alpha \Omega_{C(B} \tilde{\nabla}_{A)} h_{\alpha D} \\ - 2 f \Omega_{A(C} h_{D)B} + \Omega_{AB} \left(r r^\alpha \bar{\nabla}_\alpha h_{CD} - 2 f h_{CD} \right) \,. \end{aligned} \tag{6.46}$$

Note that some of these cases are redundant, since we could have used the fact that $(\nabla_a \nabla_b - \nabla_b \nabla_a) \omega_c = R_{abc}{}^d \omega_d$.

Putting things together

We are now ready to simplify eq. (6.27). Note that

$$\left(\bar{g}^{\alpha\gamma}\bar{g}^{\beta\delta} - \bar{g}^{\alpha\beta}\bar{g}^{\gamma\delta}\right)\bar{\nabla}_\alpha\bar{\nabla}_\beta h_{\gamma\delta} = \bar{\nabla}_\alpha\bar{\nabla}_\beta h^{\alpha\beta} - \bar{\nabla}_\alpha\bar{\nabla}^\alpha {h_\beta}^\beta, \tag{6.47}$$

$$\begin{aligned}\left(\Omega^{AC}\Omega^{BD} - \Omega^{AB}\Omega^{CD}\right)&\nabla_A\nabla_B h_{CD} \\ &= \left(\tilde{\nabla}_A\tilde{\nabla}_B h^{AB} - \tilde{\nabla}_A\tilde{\nabla}^A {h_B}^B + f{h_A}^A\right) \\ &\quad + rr^\alpha\left(2\tilde{\nabla}_A {h_\alpha}^A - \bar{\nabla}_\alpha {h_A}^A\right) + 2r^2 r^\alpha r^\beta h_{\alpha\beta},\end{aligned} \tag{6.48}$$

$$\begin{aligned}\bar{g}^{\alpha\beta}\Omega^{AB}&\left[\left(\nabla_\alpha\nabla_\beta h_{AB} + \nabla_A\nabla_B h_{\alpha\beta}\right) - \left(\nabla_\alpha\nabla_A + \nabla_A\nabla_\alpha\right)h_{\beta B}\right] \\ &= \left(\tilde{\nabla}_A\tilde{\nabla}^A {h_\alpha}^\alpha + \bar{\nabla}_\alpha\bar{\nabla}^\alpha {h_A}^A - \bar{\nabla}_\alpha\tilde{\nabla}_A h^{\alpha A} - \tilde{\nabla}_A\bar{\nabla}_\alpha h^{\alpha A}\right) + \frac{2f}{r^2}{h_A}^A \\ &\quad - \frac{2}{r}\left(r^\alpha\bar{\nabla}_\alpha {h_A}^A + \frac{M}{r^2}{h_A}^A + M{h_\alpha}^\alpha\right) + 2rr^\alpha\left(\bar{\nabla}_\alpha {h_\beta}^\beta - 2\bar{\nabla}^\beta h_{\alpha\beta}\right).\end{aligned} \tag{6.49}$$

Thus,

$$\begin{aligned}K_\clubsuit = \bar{R}\Bigg\{&2\left[\left(\bar{\nabla}_\alpha\bar{\nabla}_\beta h^{\alpha\beta} - \bar{\Delta}h_{\bar{g}}\right) + \frac{1}{r^4}\left(\tilde{\nabla}_A\tilde{\nabla}_B h^{AB} - \tilde{\Delta}h_\Omega\right)\right] \\ &+ \frac{1}{r^2}\left[\bar{\Delta}h_\Omega + \tilde{\Delta}h_{\bar{g}} - \left(\bar{\nabla}_\alpha\tilde{\nabla}_A + \tilde{\nabla}_A\bar{\nabla}_\alpha\right)h^{\alpha A}\right] \\ &+ \frac{4f}{r^4}h_\Omega + \frac{4r_\alpha}{r^3}\left(\tilde{\nabla}_A h^{\alpha A} - \bar{\nabla}^\alpha h_\Omega\right) + \frac{4}{r^2}r^\alpha r^\beta h_{\alpha\beta} \\ &-\frac{1}{2}\bar{R}\left(h_{\bar{g}} + \frac{1}{r^2}h_\Omega\right) + \frac{2}{r}r^\alpha\left(\bar{\nabla}_\alpha h_{\bar{g}} - 2\bar{\nabla}^\beta h_{\alpha\beta}\right)\Bigg\},\end{aligned} \tag{6.50}$$

where $\bar{\Delta} := \bar{\nabla}_\alpha\bar{\nabla}^\alpha$ is the Laplace-Beltrami operator on $\mathfrak{M}^2$, $\tilde{\Delta} := \tilde{\nabla}_A\tilde{\nabla}^A$ on S^2. Also, $h_{\bar{g}} := \bar{g}^{\alpha\beta}h_{\alpha\beta}$ and $h_\Omega := \Omega^{AB}h_{AB}$ are the 'restricted' traces of h_{ab} on $\mathfrak{M}^2$ and S^2, respectively.

6.4 Decomposing $K_\spadesuit$

Fortunately, $K_\spadesuit$ is easier to decompose. It can be shown that [5]

$$K_\spadesuit = -\frac{3}{2}\bar{R}^2\left(h_{\bar{g}} + \frac{1}{r^2}h_\Omega\right). \tag{6.51}$$

[5] This was actually done by inspection.

Combining eqs. (6.50) and (6.51), we have the full first-order correction to the Kretschmann scalar

$$\begin{aligned}K^{(1)} = \bar{R}\Bigg\{ & 2\left[(\bar{\nabla}_\alpha \bar{\nabla}_\beta h^{\alpha\beta} - \bar{\Delta} h_{\bar{g}}) + \frac{1}{r^4}\left(\tilde{\nabla}_A \tilde{\nabla}_B h^{AB} - \tilde{\Delta} h_\Omega\right)\right] \\ & + \frac{1}{r^2}\left[\bar{\Delta} h_\Omega + \tilde{\Delta} h_{\bar{g}} - \left(\bar{\nabla}_\alpha \tilde{\nabla}_A + \tilde{\nabla}_A \bar{\nabla}_\alpha\right) h^{\alpha A}\right] \\ & + \frac{4f}{r^4} h_\Omega + \frac{4r_\alpha}{r^3}\left(\tilde{\nabla}_A h^{\alpha A} - \bar{\nabla}^\alpha h_\Omega\right) + \frac{4}{r^2} r^\alpha r^\beta h_{\alpha\beta} \\ & -2\bar{R}\left(h_{\bar{g}} + \frac{1}{r^2} h_\Omega\right) + \frac{2}{r} r^\alpha \left(\bar{\nabla}_\alpha h_{\bar{g}} - 2\bar{\nabla}^\beta h_{\alpha\beta}\right)\Bigg\}, \end{aligned} \tag{6.52}$$

We can further simplify $K_\clubsuit$ and $K_\spadesuit$. It is convenient to rewrite $h_{\alpha\beta} = \bar{h}_{\alpha\beta}$ and $h_{AB} = r^2 \tilde{h}_{AB}$, as introduced in Section 4.2, such that

$$[g'_{ab}] = \left[\begin{array}{c:c} \bar{g}_{\alpha\beta} + \bar{h}_{\alpha\beta} & h_{\alpha B} \\ \hdashline h_{A\beta} & r^2(\Omega_{AB} + \tilde{h}_{AB}) \end{array}\right].$$

We then define the traces $\bar{h} = \bar{g}^{\alpha\beta} \bar{h}_{\alpha\beta}$ and $\tilde{h} = \Omega^{AB} \tilde{h}_{AB}$. We can then derive some useful identities:

$$\bar{\nabla}_\alpha \left(r^2 \tilde{h}\right) = 2 r r_\alpha \tilde{h} + r^2 \bar{\nabla}_\alpha \tilde{h}, \tag{6.53}$$

$$\bar{\Delta}\left(r^2 \tilde{h}\right) = 2\tilde{h} + 4 r r^\alpha \bar{\nabla}_\alpha \tilde{h} + r^2 \bar{\Delta} \tilde{h}. \tag{6.54}$$

Now,

$$\begin{aligned}K^{(1)} = \bar{R}\Bigg\{ & 2\left[\left(\bar{\nabla}_\alpha \bar{\nabla}_\beta \bar{h}^{\alpha\beta} - \bar{\Delta}\bar{h}\right) + \frac{1}{r^2}\left(\tilde{\nabla}_A \tilde{\nabla}_B \tilde{h}^{AB} - \tilde{\Delta}\tilde{h}\right)\right] \\ & + \frac{1}{r^2}\left[r^2 \bar{\Delta}\tilde{h} + \tilde{\Delta}\bar{h} - \left(\bar{\nabla}_\alpha \tilde{\nabla}_A + \tilde{\nabla}_A \bar{\nabla}_\alpha\right) h^{\alpha A}\right] - 2\bar{R}\bar{h} \\ & + \frac{4}{r^2}\left[\frac{r_\alpha}{r}\tilde{\nabla}_A h^{\alpha A} + r^\alpha r^\beta \bar{h}_{\alpha\beta} - r r_\alpha \bar{\nabla}_\beta \bar{h}^{\alpha\beta} + \frac{1}{2} r r^\alpha \bar{\nabla}_\alpha \bar{h} - \frac{1}{2}\tilde{h}\right]\Bigg\}. \end{aligned} \tag{6.55}$$

6.5 Tensor spherical harmonics

Even-parity sector

In Section 4.2, we wrote the even-parity sector of the perturbation h_{ab} as

$$h_{\alpha\beta} = \bar{h}_{\alpha\beta} = \sum_{lm} \underline{h}^{lm}_{\alpha\beta}(\bar{x})\, Y^{lm}, \tag{6.56}$$

$$h_{\alpha B} = \sum_{lm} j^{lm}_\alpha(\bar{x})\, Y^{lm}_B, \tag{6.57}$$

$$h_{AB} = r^2 \tilde{h}_{AB} = r^2 \sum_{lm}\left(F^{lm}(\bar{x})\, \Omega_{AB} Y^{lm} + G^{lm}(\bar{x})\, Y^{lm}_{AB}\right). \tag{6.58}$$

From now on, we suppress the $\{l, m\}$ indices. Also, since we are only interested in the first-order correction to a scalar, there is no contribution from the odd-parity sector. In order to aid our simplification, we list some useful identities:

$$Y_A := \tilde{\nabla}_A Y, \tag{6.59}$$

$$Y_{AB} := \left(\tilde{\nabla}_A \tilde{\nabla}_B + \frac{1}{2} l(l+1)\Omega_{AB}\right) Y, \tag{6.60}$$

$$\Omega^{AB} Y_{AB} = 0, \tag{6.61}$$

$$\tilde{\nabla}_A Y^A = \tilde{\Delta} Y = -l(l+1)Y, \tag{6.62}$$

and

$$\begin{aligned}
\tilde{\nabla}_A \tilde{\nabla}_B Y^{AB} &= \tilde{\nabla}_A \tilde{\nabla}_B \left(\tilde{\nabla}^A \tilde{\nabla}^B + \frac{1}{2} l(l+1)\Omega^{AB}\right) Y \\
&= \tilde{\nabla}_A \tilde{\nabla}_B \tilde{\nabla}^A \tilde{\nabla}^B Y + \frac{1}{2} l(l+1) \tilde{\Delta} Y \\
&= \tilde{\nabla}_A \left(\tilde{\nabla}^A \tilde{\nabla}_B \tilde{\nabla}^B Y + \tilde{R}_B{}^{AB}{}_C \nabla^C Y\right) - \frac{1}{2} \tilde{\Delta}^2 Y \\
&= \frac{1}{2} \tilde{\Delta}^2 Y + \tilde{\nabla}_A \tilde{R}^A{}_C \tilde{\nabla}^C Y \\
&= \frac{1}{2} \tilde{\Delta}^2 Y + \frac{1}{2} \tilde{\nabla}_A \tilde{R}\, \tilde{\nabla}^A Y,
\end{aligned} \tag{6.63}$$

where $\tilde{\Delta}^2 := \tilde{\Delta}\tilde{\Delta}$, and we make use of the property

$$\tilde{\nabla}_A \tilde{R}^A{}_B = \frac{1}{2} \tilde{\nabla}_B \tilde{R}. \tag{6.64}$$

Because the scalar curvature of the round sphere is $\tilde{R} = 2$, we have

$$\tilde{\nabla}_A \tilde{\nabla}_B Y^{AB} = \frac{1}{2} \tilde{\Delta}^2 Y = \frac{1}{2} l^2 (l+1)^2 Y. \tag{6.65}$$

Also, note that

$$\tilde{h} = 2FY, \tag{6.66}$$

$$\bar{h} = \underline{h} Y. \tag{6.67}$$

Thus, we obtain

$$\begin{aligned}
K^{(1)}_{\text{even}} = \bar{R} \Bigg\{ & 2 \left[\left(\bar{\nabla}_\alpha \bar{\nabla}_\beta \underline{h}^{\alpha\beta} - \bar{\Delta} \underline{h}\right) + \frac{1}{r^2} l(l+1) \left(F + \frac{1}{2} l(l+1) G\right)\right] \\
& + \frac{4}{r^2} \left[r^\alpha r^\beta \underline{h}_{\alpha\beta} - \frac{1}{r} l(l+1) r_\alpha j^\alpha - r r_\alpha \bar{\nabla}_\beta \underline{h}^{\alpha\beta} + \frac{1}{2} r r^\alpha \bar{\nabla}_\alpha \underline{h} - F \right] \\
& + \frac{1}{r^2} \left[2r^2 \bar{\Delta} F + l(l+1) \left(2 \bar{\nabla}_\alpha j^\alpha - \underline{h}\right)\right] - 2 \bar{R} \underline{h} \Bigg\} Y.
\end{aligned} \tag{6.68}$$

Odd-parity sector

We previously claimed that the odd-parity sector has zero contribution to the first-order correction to the Kretschmann scalar. We now provide an outline of the proof. The odd-parity sector of the perturbation is given by[6]

$$\begin{aligned} h_{\alpha\beta} &= \bar{h}_{\alpha\beta} = 0\,, &(6.69)\\ h_{\alpha B} &= \sum_{lm} h^{lm}_{\alpha}(\bar{x})\, X^{lm}_{B}, &(6.70)\\ h_{AB} &= r^2\tilde{h}_{AB} = r^2\sum_{lm} H^{lm}(\bar{x}) X^{lm}_{AB}, &(6.71)\end{aligned}$$

where[7]

$$X^{lm}_{A} := -\varepsilon_{A}{}^{B}\tilde{\nabla}_{B}Y^{lm}, \tag{6.72}$$

$$\Omega^{AB}X_{AB} = 0, \tag{6.73}$$

$$X^{lm}_{AB} := -\frac{1}{2}\left(\varepsilon_{A}{}^{C}\tilde{\nabla}_{B} + \varepsilon_{B}{}^{C}\tilde{\nabla}_{A}\right)\tilde{\nabla}_{C}Y^{lm}. \tag{6.74}$$

Clearly, $\bar{h} = 0$ and $\tilde{h} = 0$. Thus, from eq. (6.55), the first-order correction to the Kretschmann scalar due to the odd-parity sector is

$$K^{(1)}_{\text{odd}} = \frac{2}{r^2}\bar{R}\left[H\tilde{\nabla}_{A}\tilde{\nabla}_{B}X^{AB} + \left(\frac{2}{r}r_{\alpha}h^{\alpha} - \bar{\nabla}_{\alpha}h^{\alpha}\right)\tilde{\nabla}_{A}X^{A}\right]. \tag{6.75}$$

It remains to be shown that $\tilde{\nabla}_{A}\tilde{\nabla}_{B}X^{AB}$ and $\tilde{\nabla}_{A}X^{A}$ are zero. The first one is easy, since $\tilde{\nabla}_{A}\tilde{\nabla}_{B}Y$ is symmetric while ε_{AB} is antisymmetric. The second one takes a lot more work, since we need to permute the derivative operators.

6.6 Final form

Hence, we have the final expression for the first-order correction to the Kretschmann scalar

$$\begin{aligned} K^{(1)} = K^{(1)}_{\text{even}} = \bar{R}\Bigg\{&2\left[\left(\bar{\nabla}_{\alpha}\bar{\nabla}_{\beta}\underline{h}^{\alpha\beta} - \bar{\Delta}\underline{h}\right) + \frac{1}{r^2}l(l+1)\left(F + \frac{1}{2}l(l+1)G\right)\right]\\ &+ \frac{4}{r^2}\left[r^{\alpha}r^{\beta}\underline{h}_{\alpha\beta} - \frac{1}{r}l(l+1)r_{\alpha}j^{\alpha} - rr_{\alpha}\bar{\nabla}_{\beta}\underline{h}^{\alpha\beta} + \frac{1}{2}rr^{\alpha}\bar{\nabla}_{\alpha}\underline{h} + F\right]\\ &+\frac{1}{r^2}\left[2r^2\bar{\Delta}F + l(l+1)\left(2\bar{\nabla}_{\alpha}j^{\alpha} - \underline{h}\right)\right]\Bigg\}Y - 2\bar{R}^2\underline{h}Y. \end{aligned} \tag{6.76}$$

[6] Again, we use a different notation from [7].

[7] ε_{AB} is the Levi-Civita tensor on S^2.

Note that these are summed over the $\{l, m\}$'s. And after all this work we only get to decompose the Kretschmann scalar. Higher-order differential invariants require more higher-order derivatives of h_{ab}.[8] Note that this equation is valid in any coordinate system in $\mathfrak{M}^2$.

In order to verify the correctness of our result, we apply it to our previous example, the Preston-Poisson spacetime. If we plug in the functions from Section 4.3, the result agrees with our naive computation in Section 4.4. We confirm that the $\{0, 0\}$ sector makes no contribution to $K^{(1)}$; the full first-order correction to the Kretschmann scalar is solely due to the $\{2, 0\}$ sector. But, this time we know the individual contributions of each function. However, this is no proof that our equation is correct, but it is a highly non-trivial check. One way to prove the correctness of our result is to plug in arbitrary perturbation functions.

[8] We can only shudder at the thought of decomposing the Karlhede invariant, which involves third-order derivatives.

Chapter 7

Conclusions and Recommendations

In Chapter 4, we examined the Preston-Poisson spacetime using an invariant method proposed by [6]. We found that the S invariant, which we used to visualize the Preston-Poisson spacetime, does not depend on the tidal parameter $\mathcal{E}$. We also verified that the geometric horizon proposed by [3] reduces to the equation for the Killing horizon. Intrigued by this result, we derived a covariant formula for a geometric horizon defined by the vanishing of the gradient of the Kretschmann scalar in Chapter 5. We do not yet know whether this corresponds to a physical horizon, or what would happen if we instead use other curvature invariants. In Chapter 6, we specifically derived a formula for the first-order correction to the geometric horizon, valid for any linear perturbation written in the formalism of [7]. We also showed that for static perturbations, the geometric horizon reduces to the Killing horizon as expected.

The formalism developed in this work may be generalized to higher-dimensional spacetimes. However, note that we made a convenient assumption that the geometric horizon can be treated perturbatively; i.e., we assumed that the local cohomogeneity requirement for the geometric horizon is that of the background spacetime. The geometric horizon conjecture states that "a spacetime horizon is always more algebraically special (in all of the orders of specialization) than other regions of spacetime" [5]. In other words, the geometric horizon depends on the Petrov type of the spacetime. However, in general a perturbed spacetime will not retain the *speciality character* of the background spacetime [48]. Thus, we must be careful in using the geometric horizon detector for general perturbed spacetimes. For the Preston-Poisson spacetime, this assumption is backed by evidence. In Chapter 4, we showed that only one curvature invariant is needed to locate the event (Killing) horizon of the Preston-Poisson spacetime; moreover, we found that the wedge product of any two curvature invariants are of second order. Therefore, we cannot use them to detect the horizon; contrary to what one might expect, since the Preston-Poisson spacetime is axially symmetric (i.e. local cohomogeneity 2).

Further work might include applying the geometric horizon equation to a dynamical perturbed Schwarzschild spacetime, such as a slowly collapsing shell, and studying its evolution. Since the vanishing of the Karlhede invariant also defines a geometric horizon, one might derive the first-order correction to the Karlhede invariant, similar to what we did in Chapters 4 - 6, and compare it to our results; although we expect the Karlhede invariant to be more cumbersome to decompose. If that is not enough, one can then obtain another geometric horizon defined by the vanishing of the norm of the gradient of the Karlhede invariant. It is also worth to pursue an equation for the first-order correction to a geometric horizon of the Kerr spacetime; however, a classical tensor calculus formalism might not be the most efficient for this task.

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