MATHEMATICAL MODELS FOR HORIZONTAL GEODETIC NETWORKS

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PREFACE

In order to make our extensive series of lecture notes more readily available, we have scanned the old master copies and produced electronic versions in Portable Document Format. The quality of the images varies depending on the quality of the originals. The images have not been converted to searchable text.

PREFACE

The purpose of these notes is to give the reader an appreciation for the mathematical aspects related to the establishment of horizontal geodetic networks by terrestrial methods. By terrestrial methods we mean utilizing terrestrial measurements (directions, azimuths, and distances). Vertical networks are not discussed and instrumentation is described only indirectly via the accuracy estimates assigned to the observations.

The approach presented utilizes the well known aspects of adjustment calculus which allow one to design and analyse geodetic networks. These notes do not provide extensive derivations. The relationships between models for the ellipsoid and models for a conformal mapping plane are also given.

These notes assume the reader to have a knowledge of differential and integral calculus, matrix algebra, and least squares adjustment calculus and statistics.

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1. INTRODUCTION

We begin these notes by first establishing the need for geodetic networks. Section 2 also serves as motivation for the mathematical developments in the remaining sections. In Sections 3 to 5, inclusive, the mathematical models for distance, azimuth, and direction observations on the ellipsoid are developed. It should be noted that the observations considered in these notes are assumed to be made on the ellipsoid; however, it is well known that terrain observations must be properly reduced to the ellipsoid [Bomford, 1971]. These models give the functional relationships between the observables and the unknown coordinates to be estimated. The models are also applied to examples to show how solutions of networks can be made. The technique of pre-analysis of networks is described in the final Section. This procedure allows one to design a network before actually making the observations.

2. THE NEED FOR GEODETIC NETWORKS

Traditionally, the emphasis placed on establishing geodetic networks was for the purpose of producing maps. The production of maps of varying scales still relies heavily on geodetic coordinates but at the same time accelerated activities in a wide variety of other disciplines have prompted, and also proven the need for these coordinates. The main areas where geodetic networks are or can be applied are the following [Krakiwksy and Vanicek, 1974]: mapping; boundary demarcation; urban management; engineering projects; hydrography; environmental management; ecology; earthquake-hazard assessment; space research. There are other, perhaps more indirect scientific areas of application like astronomy, various branches of geophysics, etc., which will not be dealt with here. To show how geodetic networks can be applied in the listed areas, let us single out the geodetic tasks associated with them.

Mapping

The need for a network of appropriately distributed points (geodetic control) of known horizontal and vertical positions has been demonstrated in Canada by the production of the 1:250 000 map series [Sebert, 1970]. Additional geodetic control of higher density and accuracy is now required by the federal government's 1:50 000 mapping programme, by the medium scale mapping programmes of the provinces [Roberts, 1966], by the large scale mapping programmes of the municipalities [Bogdan, 1972; McLellan, 1972] and by the special purpose mapping projects of private enterprise and the various levels of the government. The establishment of adequate geodetic control for the production of maps is clearly an important geodetic task.

Boundary Demarcation

The rigorous definition of Canada's international and provincial boundaries is of paramount importance; so are the boundaries of private land parcels [Roberts, 1960]. Recently much emphasis is also being placed on the speedy and accurate description of oil and gas concessions in the arctic and eastern continental shelf areas of Canada [Blackie, 1969; Crosby, 1969; Heise, 1971]. The positioning and staking-out of these boundaries is most economically done by relating them to a framework of points with known coordinate values - the geodetic network.

Urban Management

In the urban environment the "as built" locations of man's creations, such as underground utilities, need be defined and documented for future reference [Andrecheck, 1972]. The use of geodetic coordinates in the urban environment is clearly indicated in Hamilton [1973]. Hence another application of geodetic networks.

Engineering Projects

During the building of large structures, such as dams, bridges and buildings, it is useful to lay out the various components in predetermined locations. For this purpose various coordinate systems are used [Linkwitz, 1970]. The availability of control points is naturally desirable. As well, it is often necessary to know the a priori and a posteriori movements of the ground and water levels. In the case of dams, water tunnels and irrigation constructions, the exact knowledge of the equipotential surfaces is also needed. The determination of the movements and the location of the equipotential surfaces are geodetic undertakings.

Hydrography

It has been accepted that hydrographic surveys are essential to mapping Canada's coastal areas and to other continental shelf activities. The positioning of hydrographic ships, drilling vessels, and buoys with respect to a coordinate system is a requirement for any hydrographic survey [Blackie, 1973; Energy, Mines and Resources Canada, 1970]. This positioning again makes use of geodetic networks.

Environmental Management

It has been recognized that the establishment of environmental data banks as integrated information systems to serve in transportation, land use, community and social services, land titles extracts, assessment of tax data, population statistics, should be based on land parcels whose locations are to be uniquely defined in terms of coordinates [Konecny, 1969]. It is advisable that these coordinates be referred to a geodetic network.

Ecology

In the past decade, the necessity of studying the effects of human actions on the environment has been realized. One such effect is the man-made movements of the ground caused by underground removal of minerals or subsurface disposal of wastes [Van Everdingen and Freeze, 1971; Denman, 1972]. The detection and monitoring of these movements is clearly related to geodetic networks.

Earthquake-Hazard Assessment

Repeated geodetic measurements can give quantitative information about the creeping motion of the ground that allegedly precedes earthquakes [Canadian Geodynamics Subcommittee, 1972]. This information plays an important role in the development of mathematical models for earthquake-hazard assessment to help predict possible location and time and even approximate magnitude of earthquakes.

Space Research

The capability to predict orbits of the spacecraft is essential for any space research [NAS, 1969; NASA, 1972]. Prerequisites to this capability are an adequate geodetic coordinate system and a well defined external gravity field of the earth.

Having enumerated at least the principal practical applications of geodetic networks let us now turn to the development of the basic mathematical models used in horizontal geodetic network adjustments.

3. DISTANCE MATHEMATICAL MODEL

In this section we develop the mathematical model relating distances and coordinates. Coordinates are the geodetic latitude and longitude referred to a reference ellipsoid. Distances are the corresponding geodesic lengths between points on the ellipsoid. This means that observed distances must be reduced to the ellipsoid surface before they can be adjusted.

It is worthwhile to note that, if the geodesic distance between two points on the ellipsoid could be expressed in a closed form, as a function of the coordinates, then it would simply be a matter of linearization to obtain the linear form needed for the adjustment. But since the inverse problem on the ellipsoid does not have a mathematically expedient closed-form solution, an alternate approach must be taken. Two approaches are given below. The first is based on an ellipsoidal differential expression, while the second begins with a spherical approximation. The relationship of the ellipsoid model with the plane case model is also discussed, and an example using the ellipsoidal differential approach is worked out.

3.1 Ellipsoidal Differential Approach

The mathematical model for the geodesic distance, expressed as a function of two sets of coordinates, is symbolically written as

$$F_{ij} = S(\phi_i, \lambda_i, \phi_j, \lambda_j) - S_{ij} = 0, \qquad (3\text{-}1)$$

where the first term is a non-linear function of the coordinates, while the second term is the value for the geodesic distance. This non-linear model is approximated by a linear Taylor series. The resulting equation is

$$F_{ij} = F'_{ij} + dF_{ij} \tag{3-2}$$

$$= S(\phi_i^o, \lambda_i^o, \phi_j^o, \lambda_j^o) - S_{ij} + dS(\phi_i^o, \lambda_i^o, \phi_j^o, \lambda_j^o) - V_{S_{ij}} + \ldots = 0. \tag{3-3}$$

The first two terms represent the point of expansion, namely the value of the distance based on the approximate values of the coordinates $S(\phi_i^o, \lambda_i^o, \phi_j^o, \lambda_j^o)$, minus the observed value of the ellipsoid distance S_{ij}. The third term is the differential change in the distance due to differential changes in the coordinates and is described by the total differential [Helmert, 1880; Tobey,1928]

$$dS = \frac{\partial S}{\partial \phi_i} d\phi_i + \frac{\partial S}{\partial \lambda_i} d\lambda_i + \frac{\partial S}{\partial \phi_j} d\phi_j + \frac{\partial S}{\partial \lambda_j} d\lambda_j \ , \tag{3-4}$$

where

$$\frac{\partial S}{\partial \phi_i} = -M_i^o \cos \alpha_{ij}^o \ , \tag{3-5}$$

$$\frac{\partial S}{\partial \lambda_i} = N_j^o \sin\alpha_{ji}^o \ \cos \phi_j^o \ , \tag{3-6}$$

$$\frac{\partial S}{\partial \phi_j} = -M_j^o \cos\alpha_{ji}^o \ , \tag{3-7}$$

$$\frac{\partial S}{\partial \lambda_j} = -N_j^o \sin\alpha_{ji}^o \cos \phi_j^o, \tag{3-8}$$

M^o and N^o being the radii of curvature of the ellipsoid in the meridian and prime vertical planes, respectively, and α^o is the geodetic azimuth. Finally $V_{S_{ij}}$ is the correction to the observed length.

Substituting (3-4) into (3-3) we get an expression known as the observation equation,

$$
v_{s_{ij}} = - \frac{M_i^{\circ} \cos \alpha_{ij}^{\circ}}{\rho''} d\phi_i'' + \frac{N_j^{\circ} \sin \alpha_{ji}^{\circ} \cos \phi_j^{\circ}}{\rho''} d\lambda_i'' - \frac{M_j^{\circ} \cos \alpha_{ji}^{\circ}}{\rho''} d\phi_j''
$$

$$
- \frac{N_j^{\circ} \sin \alpha_{ji}^{\circ} \cos \phi_j^{\circ}}{\rho''} d\lambda_j'' + (S_{ij}^{\circ} - S_{ij}) \tag{3-9}
$$

$$
= a_{ij}\, d\phi_i'' + b_{ij}\, d\lambda_i'' + c_{ij}\, d\phi_j'' + d_{ij}\, d\lambda_j'' + S_{ij}^{\circ} - S_{ij} . \tag{3-10}
$$

This observation equation assumes the longitude to be positive east. Note that each term in the above has units of metres. This was achieved by changing the units for the corrections to the coordinates (3-4) from radians to arcseconds and the units of the coefficients from metres to metres divided by arcseconds (ρ'' = 206 264.806). Variances used to determine weights for the observed distances are given in metres squared. The importance of changing the units will become evident when faced with combining distance and direction information in one adjustment.

It is important to note that the distance S° is "approximate" in the sense that it is computed from approximate coordinates, and that the computation of its value must be made using formulae which are accurate to say better than one-tenth of the standard deviation of the observed length. In other words, the size of computational errors should be insignificant in comparison to observational errors. This implies the use of accurate inverse problem algorithms such as Bessel's method [Jordan, 1962]. This method leads to an iterative solution because of the non linearity of the observation equation (3-9).

3.2 Spherical Differential Approach

We begin with (3-1) as in the ellipsoid differential approach, that is

$$F_{ij} = S(\phi_i,\ \lambda_i,\ \phi_j,\ \lambda_j) - S_{ij} = 0 .$$

The first step is to replace the ellipsoid terms in the above with a spherical approximation - the radius of the sphere being equal to the Gaussian mean radius

$$R = \sqrt{\bar{M}\ \bar{N}} \quad , \qquad (3\text{-}11)$$

where $\bar{M}$ and $\bar{N}$ are the mean radii of curvature over the line in question.

The distance function is

$$\begin{aligned} S(\phi_i,\ \lambda_i,\ \phi_j,\ \lambda_j) &= R\theta \\ &= R \arccos\,[\sin\phi_i\ \sin\phi_j + \cos\phi_i\ \cos\phi_j\ \cos(\lambda_j - \lambda_i)], \qquad (3\text{-}12) \end{aligned}$$

where θ is the spherical angle in radians between the two points i and j. Note the intent of the above formula is not to compute a value of the distance, but to provide an expression for evaluating partial derivatives in the linearization as shown immediately below.

We again approximate (3-1) by a linear Taylor series, yielding

$$\begin{aligned} F_{ij} &= S(\phi_i^o,\ \lambda_i^o,\ \phi_j^o,\ \lambda_j^o) - S_{ij} + \\ &+ \left(\frac{\partial S}{\partial \phi_i}\, d\phi_i + \frac{\partial S}{\partial \lambda_i}\, d\lambda_i + \frac{\partial S}{\partial \phi_j}\, d\phi_j + \frac{\partial S}{\partial \lambda_j}\, d\lambda_j\right) - V_{S_{ij}} + .. = 0, \qquad (3\text{-}13) \end{aligned}$$

where

$$\frac{\partial S}{\partial \phi_i} = R^o \left[\frac{-\cos\phi_i^o\ \sin\phi_j^o + \sin\phi_i^o\ \cos\phi_j^o\ \cos\,(\lambda_j^o - \lambda_i^o)}{\sin\,\theta^o}\right], \qquad (3\text{-}14)$$

$$\frac{\partial S}{\partial \lambda_i} = R^o \left[\frac{-\cos\phi_i^o\ \cos\phi_j^o\ \sin\,(\lambda_j^o - \lambda_i^o)}{\sin\theta^o}\right], \qquad (3\text{-}15)$$

$$\frac{\partial S}{\partial \phi_j} = R \left[\frac{-\sin\phi_i^o \cos\phi_j^o + \cos\phi_i^o \sin\phi_j \cos(\lambda_j^o - \lambda_i^o)}{\sin \theta^o} \right], \tag{3-16}$$

$$\frac{\partial S}{\partial \lambda_j} = R \left[\frac{\cos\phi_i^o \cos\phi_j^o \sin(\lambda_j^o - \lambda_i^o)}{\sin \theta^o} \right]. \tag{3-17}$$

This is essentially the model used by Grant [1973]. Note that the units of the terms in the linearized model (3-13) may be changed as in (3-10). Again, the computation of the distance from approximate coordinates must be done rigorously and not, for example by (3-13). Bessel's method is one of the most reliable and accurate for this purpose. Again, for the same reasons as in Section 3.1, an iterative solution is required.

3.3 Relationship of the Ellipsoid Model with the Plane Model

We know that the mathematical model for a distance observation on the plane is

$$F_{ij} = [(x_j - x_i)^2 + (y_j - y_i)^2]^{1/2} - S_{ij} = 0, \tag{3-18}$$

and after linearization

$$F_{ij} = (S_{ij}^o - S_{ij}) - \frac{(x_j^o - x_i^o)}{S_{ij}^o} dx_i - \frac{(y_j^o - y_i^o)}{S_{ij}^o} dy_i + \frac{(x_j^o - x_i^o)}{S_{ij}^o} dx_j + \frac{(y_j^o - y_i^o)}{S_{ij}^o} dy_j - V_{S_{ij}} = 0, \tag{3-19}$$

or

$$V_{S_{ij}} = - \sin\alpha_{ij}^o \, dx_i - \cos\alpha_{ij}^o \, dy_i - \sin\alpha_{ji}^o \, dx_j - \cos\alpha_{ji}^o \, dy_j + S_{ij}^o - S_{ij} . \tag{3-20}$$

From Figure 3-1, we can obtain the relationship between differential elements on the ellipsoid and differential elements on the plane:

$$d\phi''_i = \rho'' \frac{dy_i}{M^o_i},$$

$$dy_i = \frac{M^o_i}{\rho''} \, d\phi''_i \,, \tag{3-21}$$

$$dy_j = \frac{M^o_j}{\rho''} \, d\phi''_j \quad . \tag{3-22}$$

$$d\lambda''_i = \rho'' \frac{dx_i}{N^o_i \cos \phi^o_i} \quad ,$$

$$dx_i = \frac{N^o_i \cos\phi^o_i}{\rho''} \, d\lambda''_i \,, \tag{3-23}$$

$$dx_j = \frac{N^o_j \cos \phi^o_j}{\rho''} \, d\lambda''_j \quad . \tag{3-24}$$

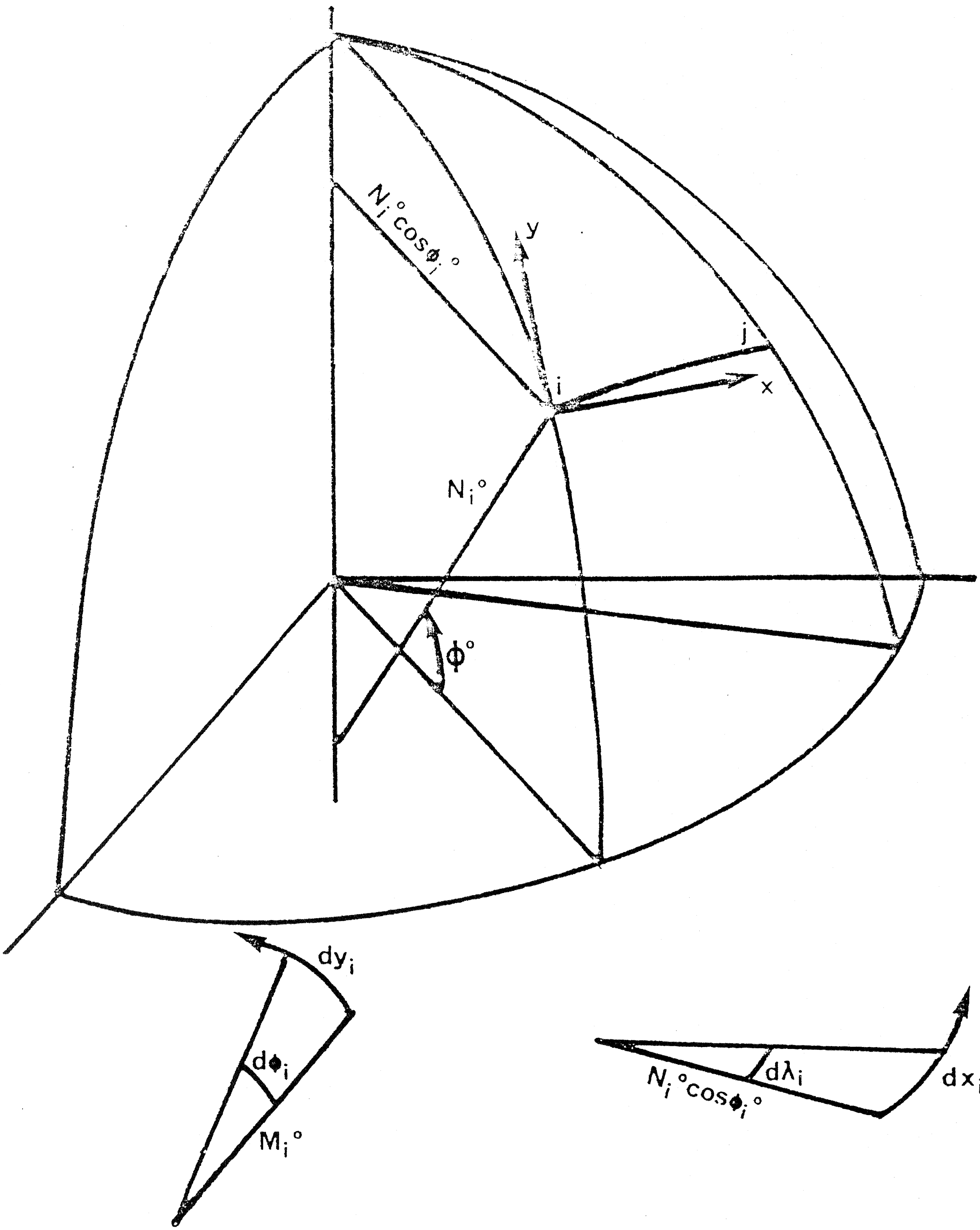
Substituting (3-21) to (3-24) into (3-20) yields an expression similar to (3-9) except for $d\lambda''_i$ term. It has been shown by Tobey [1928] that $N_i \cos\phi_i \sin\alpha_{ij}$ is approximately equal to $-N_j \cos\phi_j \sin\alpha_{ji}$, allowing us to deduce the ellipsoid model from the plane model.

3.4 Illustrative Example

We now extend the linearized mathematical model (3-9) to many distances between many stations. The observation equation in matrix form is

$$\underset{n,u}{A} \; \underset{u,1}{\hat{X}} + \underset{n,1}{W} = \underset{n,1}{\hat{V}} \,, \qquad \underset{n,n}{P} = \sigma^2_{on} \, \Sigma^{-1}_n \tag{3-25}$$

where n is the total number of distances observed and u the total number of coordinates of unknown stations. σ^2_o is the a priori variance factor and Σ the variance-covariance matrix for the observed distances.



$N_i^\circ \cos\phi_i^\circ$
y
j
i
x
N_i°
ϕ°
dy_i
$d\phi_i$
M_i°
$d\lambda_i$
$N_i^\circ \cos\phi_i^\circ$
dx_i

Figure 3-1

For example, a quadrilateral with two fixed points and all distances observed, (Figure 3-8)has the following observation equations in matrix form ,

$$\begin{bmatrix} c_{13} & d_{13} & 0 & 0 \\ 0 & 0 & c_{14} & d_{14} \\ c_{23} & d_{23} & 0 & 0 \\ 0 & 0 & c_{24} & d_{24} \\ a_{34} & b_{34} & c_{34} & d_{34} \end{bmatrix} \begin{bmatrix} d\phi_3 \\ d\lambda_3 \\ d\phi_4 \\ d\lambda_4 \end{bmatrix} + \begin{bmatrix} S^o_{13} - S_{13} \\ S^o_{14} - S_{14} \\ S^o_{23} - S_{23} \\ S^o_{24} - S_{24} \\ S^o_{34} - S_{34} \end{bmatrix} = \begin{bmatrix} v_{S_{13}} \\ v_{S_{14}} \\ v_{S_{23}} \\ v_{S_{24}} \\ v_{S_{34}} \end{bmatrix}, \tag{3-26}$$

where P is a diagonal matrix of the form

$$P_{5,5} = \sigma^2_o \begin{bmatrix} \sigma^2_{S_{13}} & & & & 0 \\ & \sigma^2_{S_{14}} & & & \\ & & \sigma^2_{S_{23}} & & \\ & & & \sigma^2_{S_{24}} & \\ 0 & & & & \sigma^2_{S_{34}} \end{bmatrix}^{-1} . \tag{3-27}$$

The above equations would now be used to perform an iterative least squares adjustment of the distances [Krakiwsky, 1975].

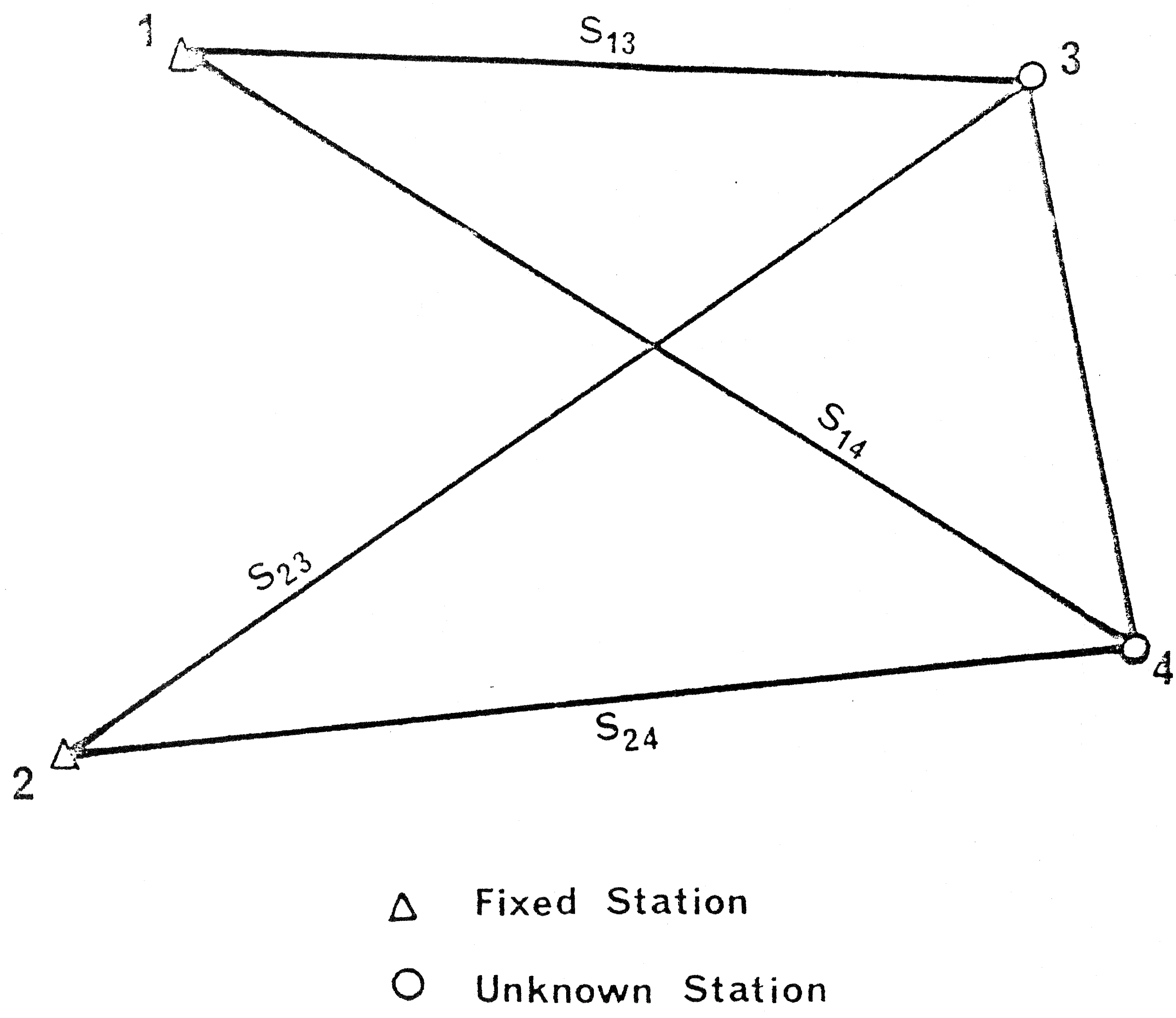


Figure 3-2

Quadrilateral with Two Fixed Points

4. AZIMUTH MATHEMATICAL MODEL

Azimuths are "absolute" directions which give orientation to a geodetic network. Azimuths must be referred to the ellipsoid as was the case for the distances described in the previous section. More specifically, the azimuth which we deal with here is the geodetic azimuth of the geodesic between two points in question.

4.1 Development of the Mathematical Model

The azimuth mathematical model is

$$F_{ij} = \alpha(\phi_i, \lambda_i, \phi_j, \lambda_j) - \alpha_{ij} = 0, \tag{4-1}$$

where the first term is a non-linear function for the azimuth in terms of the coordinates of two points i and j, while the second term is the value for the azimuth. This non-linear model is approximated by a linear Taylor series. The resulting equation is

$$\begin{aligned} F_{ij} &= F^o_{ij} + dF_{ij} \\ &= \alpha_{ij}(\phi^o_i, \lambda^o_i, \phi^o_j, \lambda^o_j) - \alpha_{ij} + d\alpha_{ij} - v_{\alpha_{ij}} + \ldots = 0. \end{aligned} \tag{4-2}$$

$\alpha_{ij}(\phi^o_i, \lambda^o_i, \phi^o_j, \lambda^o_j)$ is the value of the azimuth based on the approximate values of the coordinates. It must be computed accurately so as to introduce no error, or an error of magnitude much smaller than the standard deviation of the azimuth observation. α_{ij} is the observed value of the azimuth. The total differential is

$$d\alpha_{ij} = \frac{\partial\alpha_{ij}}{\partial\phi_i} d\phi_i + \frac{\partial\alpha_{ij}}{\partial\lambda_i} d\lambda_i + \frac{\partial\alpha_{ij}}{\partial\phi_j} d\phi_j + \frac{\partial\alpha_{ij}}{\partial\lambda_j} d\lambda_j , \tag{4-3}$$

and from differential ellipsoid geometry [Tobey, 1928]

$$d\alpha''_{ij} = \frac{M^o_i \sin \alpha^o_{ij}}{S^o_{ij}} \, d\phi''_i + \frac{M^o_j \sin \alpha^o_{ji}}{S^o_{ij}} \, d\phi''_j + \frac{N^o_j \cos \alpha^o_{ji} \cos\phi^o_j}{S^o_{ij}} \, (d\lambda''_i - d\lambda''_j) \ . \tag{4-4}$$

Note that longitudes are taken as positive east in the above.

Substituting (4-4) into (4-3) yields the observation equation

$$V_{\alpha_{ij}}'' = \frac{M^o_i \sin \alpha^o_{ij}}{S^o_{ij}} \, d\phi''_j + \frac{N^o_j \cos \alpha^o_{ji} \cos \phi^o_j}{S^o_{ij}} \, d\lambda''_i + \frac{M^o_j \sin\alpha^o_{ji}}{S^o_{ij}} \, d\phi''_j - \frac{N^o_j \cos \alpha^o_{ji} \cos\phi^o_j}{S^o_{ij}} \, d\lambda''_j + (\alpha^o_{ij} - \alpha^o_{ij})'' \ , \tag{4-5}$$

$$V_{\alpha_{ij}}'' = e_{ij} d\phi''_i + f_{ij} d\lambda''_i + g_{ij} d\phi''_j + (-f_{ij}) \, d\lambda''_j + (\alpha^o_{ij} - \alpha_{ij})'' \tag{4-6}$$

Note that the units of each term in the above is arcseconds. The coefficients are unitless thus the corrections to the coordinates have units of arcseconds. The variances used for determining the weights have units of arcseconds squared. Since the corrections to the coordinates also have units of arcseconds in the distance model, distances and azimuths can be combined in one solution.

4.2 Relationship of Ellipsoid Model to Plane Model

We know that the mathematical model on the plane for an azimuth is

$$F_{ij} = \text{acrtan}\left[\frac{x_j - x_i}{y_j - y_i}\right] - \alpha_{ij} = 0 \quad , \tag{4-7}$$

and after linearization

$$F_{ij} = (\alpha^{o}_{ij} - \alpha_{ij}) + \frac{(x^{o}_{j} - x^{o}_{i})}{S^{2}_{o_{ij}}} dy_{i} - \frac{(y^{o}_{j} - y^{o}_{i})}{S^{2}_{o_{ij}}} dx_{i}$$

$$- \frac{(x^{o}_{j} - x^{o}_{i})}{S^{2}_{o_{ij}}} dy_{j} + \frac{(y^{o}_{j} - y^{o}_{i})}{S^{2}_{o_{ij}}} dx_{j} - V_{\alpha_{ij}} + \dots = 0, \quad (4\text{-}8)$$

or

$$V''_{\alpha_{ij}} = \rho'' \frac{\sin \alpha^{o}_{ij}}{S^{o}_{ij}} dy_{i} - \rho'' \frac{\cos \alpha^{o}_{ij}}{S^{o}_{ij}} dx_{i} + \rho'' \frac{\sin\alpha^{o}_{ji}}{S^{o}_{ij}} dy_{j}$$

$$- \rho'' \frac{\cos \alpha^{o}_{ji}}{S^{o}_{ij}} dx_{j} + (\alpha^{o}_{ij} - \alpha_{ij})'' \quad . \quad (4\text{-}9)$$

If we substitute (3-21) to (3-24) into (4-9) we get

$$V''_{\alpha_{ij}} = \frac{M^{o}_{i} \sin \alpha^{o}_{ij}}{S^{o}_{ij}} d\phi''_{i} - \frac{N^{o}_{i} \cos \alpha^{o}_{ij} \cos\phi^{o}_{i}}{S^{o}_{ij}} d\lambda''_{i} + \quad (4\text{-}10)$$

$$+ \frac{M^{o}_{j} \sin\alpha^{o}_{ji}}{S^{o}_{ij}} d\phi''_{j} - \frac{N^{o}_{j} \cos \alpha^{o}_{ji} \cos \phi^{o}_{j}}{S^{o}_{ij}} d\lambda''_{j} + (\alpha^{o}_{ij} - \alpha_{ij})'' .$$

The above signs are for positive east longitude. To show the equivalence of the ellipsoid and plane models, we make the same assumption as that in Section 3.3, namely $N_i \cos\alpha_{ij} \cos\phi_i$ is approximately equal to $-N_j \cos\alpha_{ji} \cos\phi_j$.

4.3 Illustrative Example

Again we extend the mathematical model to include many azimuths between many stations. Consider the quadrilateral shown in Figure 3-2 to have azimuths observed between stations 1 and 3, and 3 and 4, in addition to all the distances. The corresponding observation equations in matrix form are:

$$\underset{7,4}{A}\;\underset{4,1}{\hat{X}} + \underset{7,1}{W} = \underset{7,1}{\hat{V}}, \qquad \underset{7,7}{P}, \qquad (4\text{-}11)$$

$$\left[\begin{array}{cccc} c_{13} & d_{13} & 0 & 0 \\ 0 & 0 & c_{14} & d_{14} \\ c_{23} & d_{23} & 0 & 0 \\ 0 & 0 & c_{24} & d_{24} \\ a_{34} & b_{34} & c_{34} & d_{34} \\ \hline g_{13} & -f_{13} & 0 & 0 \\ e_{34} & f_{34} & g_{34} & -f_{34} \end{array}\right] \begin{bmatrix} d\phi_3 \\ d\lambda_3 \\ d\phi_4 \\ d\lambda_4 \end{bmatrix} + \left[\begin{array}{c} S^o_{13} - S_{13} \\ S^o_{14} - S_{14} \\ S^o_{23} - S_{23} \\ S^o_{24} - S_{24} \\ S^o_{34} - S_{34} \\ \hline \alpha^o_{13} - \alpha_{13} \\ \alpha^o_{34} - \alpha_{34} \end{array}\right] = \left[\begin{array}{c} V_{S_{13}} \\ V_{S_{14}} \\ V_{S_{23}} \\ V_{S_{24}} \\ V_{S_{34}} \\ \hline V_{\alpha_{13}} \\ V_{\alpha_{34}} \end{array}\right], \qquad (4\text{-}12)$$

$$\underset{7,7}{P} = \sigma_o^2 \left[\begin{array}{c|c} \underset{5,5}{\Sigma^{-1}} & 0 \\ \hline 0 & \underset{2,2}{\Sigma^{-1}} \end{array}\right], \qquad \underset{2,2}{\Sigma} = \begin{bmatrix} \sigma^2_{\alpha_{13}} & 0 \\ 0 & \sigma^2_{\alpha_{34}} \end{bmatrix}.$$

The above equations would now be used to perform an iterative least squares adjustment of the distances and azimuths [Krakiwsky, 1975].

5. DIRECTION MATHEMATICAL MODEL

Direction observations are relative to the "zero" of the horizontal circle. The location of the zero relative to the north direction is an unknown "nuisance" parameter and must be solved for by the adjustment along with the unknown coordinates. The direction observations are usually arrived at from numerous sightings from a given point to other points. These directions are assumed to be referred to the ellipsoid surface.

The relationship between an azimuth α_{ij} and a direction d_{ij}, is given via the orientation unknown Z_i, namely (Figure 5-1)

$$\alpha_{ij} = d_{ij} + Z_i . \tag{5-1}$$

5.1 Development of Mathematical Model

The mathematical model for direction observations follows from (4-1)by substituting (5-1) for the azimuth, namely

$$F_{ij} = \alpha(\phi_i, \lambda_i, \phi_j, \lambda_j) - (d_{ij} + Z_i) = 0, \tag{5-2}$$

where the orientation unknown Z_i joins the coordinates as a quantity to be estimated. Linearization of (5-2) yields

$$F_{ij} = \alpha(\phi_i^o, \lambda_i^o, \phi_j^o, \lambda_j^o) - Z_i^o - d_{ij} + d\alpha_{ij} - dZ_i - V_{d_{ij}} + \ldots = 0 , \tag{5-3}$$

where all quantities have been previously defined except for Z_i^o which is the approximate value for the orientation unknown. It is usually obtained

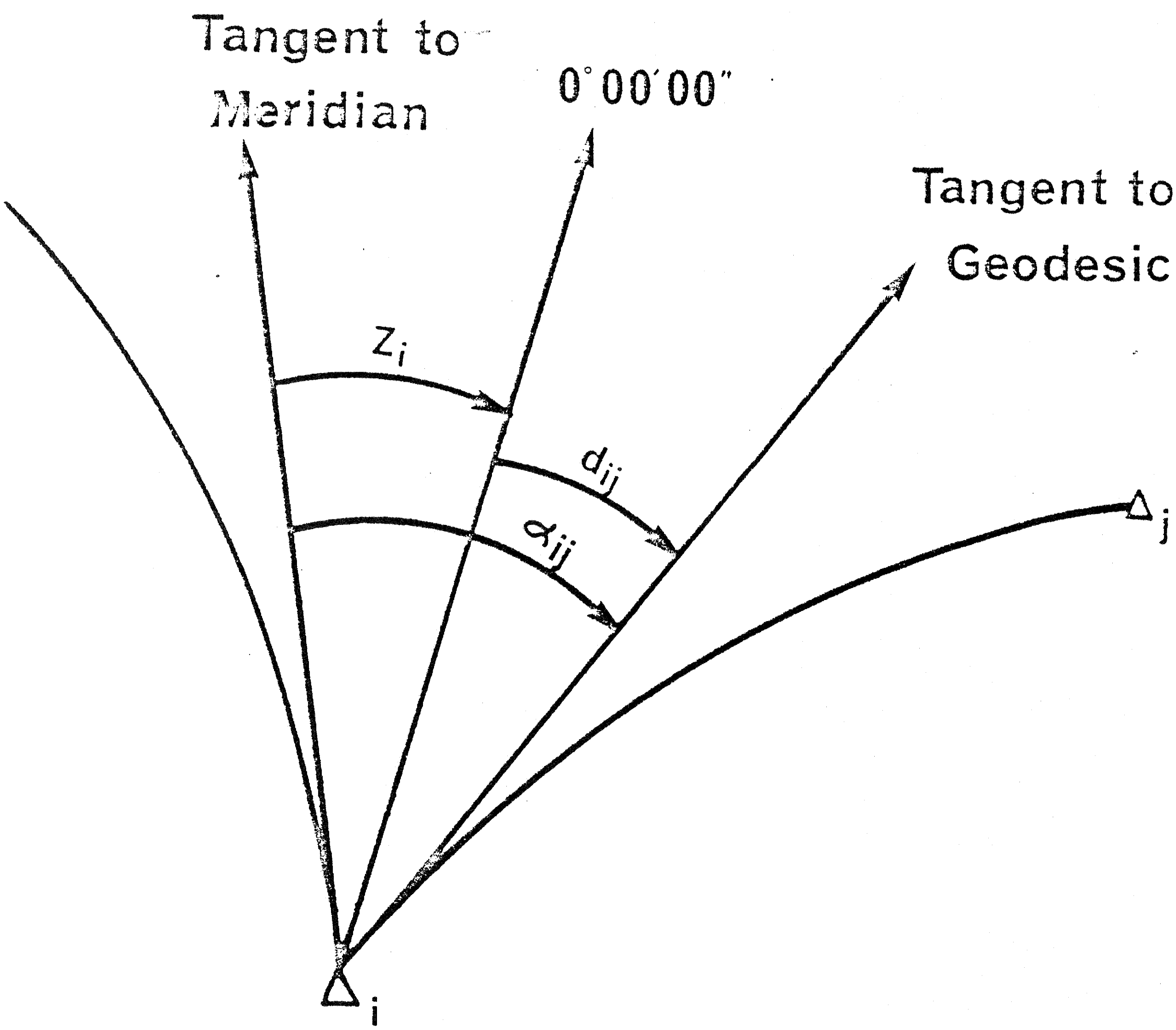


Figure 5-1

Orientation Unknown

by differencing the observed direction of a station sighted with the azimuth of the same line computed from the approximate coordinates. Several estimates at each station can be obtained with the mean being the approximate value, however only one value is necessary.

Substituting (4-4) into (5-3) yields the observation equation for a direction, namely

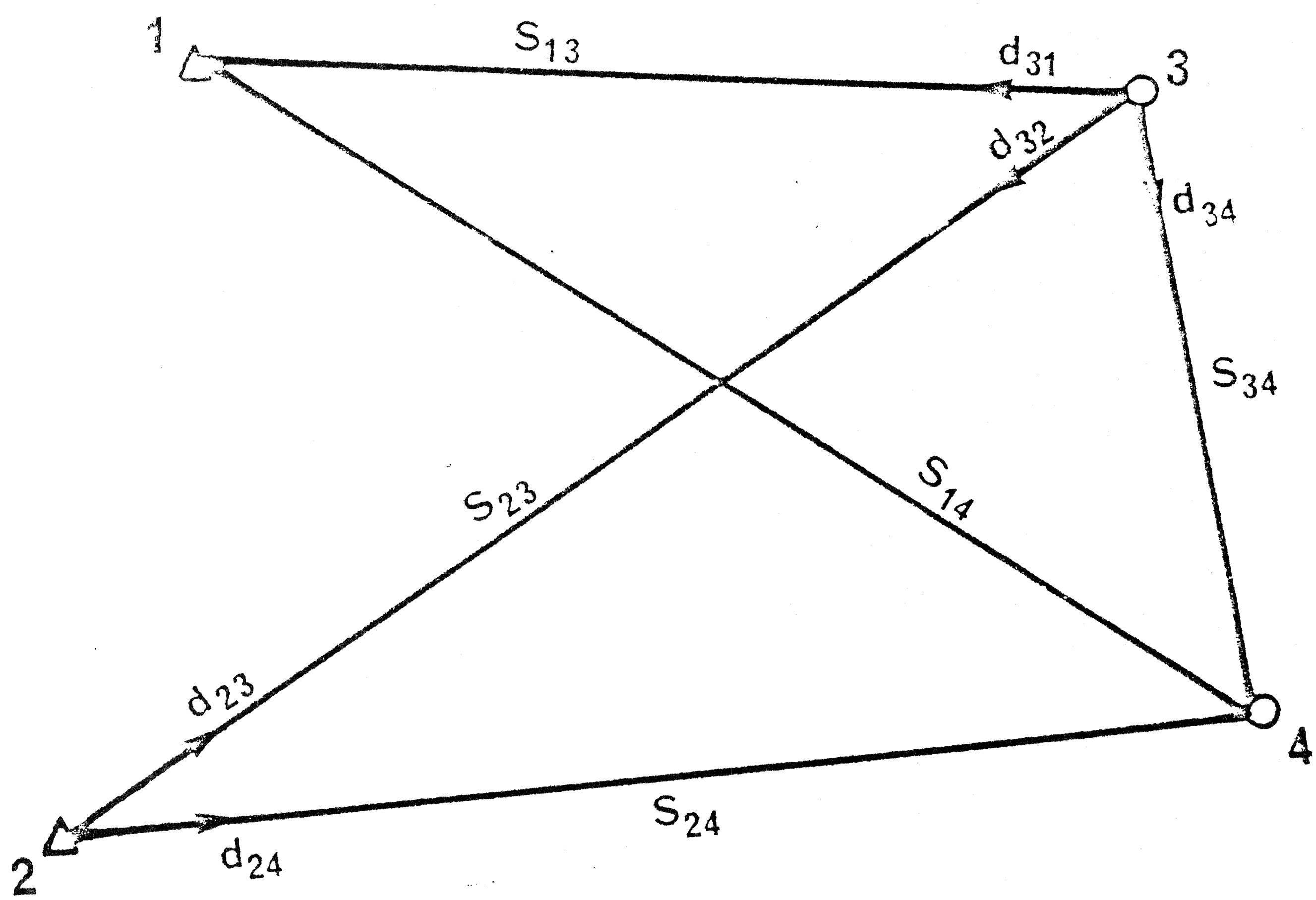
$$
\begin{aligned}
V''_{d_{ij}} = {} & \frac{M_i^o \sin \alpha_{ij}^o}{S_{ij}^o} \, d\phi''_i + \frac{N_j^o \cos \alpha_{ji}^o \cos \phi_j^o}{S_{ij}^o} \, d\lambda''_i \\
& + \frac{M_j^o \sin \alpha_{ji}^o}{S_{ij}^o} \, d\phi''_j - \frac{N_j^o \cos \alpha_{ji}^o \cos \phi_j^o}{S_{ij}^o} \, d\lambda''_j \\
& - dZ''_i + (\alpha_{ij}^o - d_{ij} - Z_i^o)'' ,
\end{aligned}
\tag{5-4}
$$

$$
\begin{aligned}
V''_{d_{ij}} = {} & e_{ij} \, d\phi''_i + f_{ij} \, d\lambda''_i + g_{ij} \, d\phi''_j + (-f_{ij}) \, d\lambda''_j \\
& - dZ''_i + (\alpha_{ij}^o - d_{ij} - Z_i^o)'' .
\end{aligned}
\tag{5-5}
$$

5.2 Illustrative Example

We now apply the direction mathematical model to the quadrilateral shown in Figure 5-2, where all distances have been observed plus the three directions at station 3 and two at station 2. The matrix form of the observation equation is

$$
\underset{10,6}{A} \; \underset{6,1}{\hat{X}} + \underset{10,1}{W} = \underset{10,1}{\hat{V}} , \qquad \underset{10,10}{P} , \tag{5-6}
$$



△ Fixed Station

O Unknown Station

d_{ij} → Direction

S_{ij} Distance

Figure 5-2

Direction and Distance Observations

$$\begin{bmatrix} c_{13} & d_{13} & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{14} & d_{14} & 0 & 0 \\ c_{23} & d_{23} & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{24} & d_{24} & 0 & 0 \\ a_{34} & b_{34} & c_{34} & d_{34} & 0 & 0 \\ \hline e_{31} & f_{31} & 0 & 0 & 0 & -1 \\ e_{32} & f_{32} & 0 & 0 & 0 & -1 \\ e_{34} & f_{34} & g_{34} & -f_{34} & 0 & -1 \\ g_{23} & -f_{23} & 0 & 0 & -1 & 0 \\ 0 & 0 & g_{24} & -f_{24} & -1 & 0 \end{bmatrix} \begin{bmatrix} d\phi_3 \\ d\lambda_3 \\ d\phi_4 \\ d\lambda_4 \\ \hline dZ_2 \\ dZ_3 \end{bmatrix} + \begin{bmatrix} S^{o}_{13} - S_{13} \\ S^{o}_{14} - S_{14} \\ S^{o}_{23} - S_{23} \\ S^{o}_{24} - S_{24} \\ S^{o}_{34} - S_{34} \\ \alpha^{o}_{31} - d_{31} - Z^{o}_{3} \\ \alpha^{o}_{32} - d_{32} - Z^{o}_{3} \\ \alpha^{o}_{34} - d_{34} - Z^{o}_{3} \\ \alpha^{o}_{23} - d_{23} - Z^{o}_{2} \\ \alpha^{o}_{24} - d_{24} - Z^{o}_{2} \end{bmatrix} = \begin{bmatrix} V_{S_{13}} \\ V_{S_{14}} \\ V_{S_{23}} \\ V_{S_{24}} \\ V_{S_{34}} \\ \hline V_{d_{31}} \\ V_{d_{32}} \\ V_{d_{34}} \\ V_{d_{23}} \\ V_{d_{24}} \end{bmatrix} \quad (5\text{-}7)$$

$$\underset{10,10}{P} = \sigma_o^2 \left[\begin{array}{c|c} \underset{5,5}{\Sigma^{-1}} & 0 \\ \hline 0 & \underset{5,5}{\Sigma^{-1}} \end{array}\right] \qquad \underset{5,5}{\Sigma} = \begin{bmatrix} \sigma^2_{d_{31}} & & & & 0 \\ & \sigma^2_{d_{32}} & & & \\ & & \sigma^2_{d_{34}} & & \\ & 0 & & \sigma^2_{d_{23}} & \\ & & & & \sigma^2_{d_{24}} \end{bmatrix} \quad (5\text{-}8)$$

Note the variance-covariance matrix for the distances has already been defined, while the new 5x5 matrix pertains to the direction observations. It is a diagonal matrix when all directions in all sets have been observed. The above equations would now be used to perform an iterative least squares adjustment of the distances and directions [Krakiwsky, 1975].

6. TECHNIQUE OF PRE-ANALYSIS

In this section we (i) describe and put the technique of pre-analysis in perspective, (ii) give the mathematical foundation upon which it is based, and (iii) outline the procedure of the technique and discuss the representation of results from a pre-analysis.

6.1 Description

By pre-analysis we mean the study of the design of geodetic networks. This is done prior to the establishment of the network in the field, thus no observations are necessary for performing a pre-analysis.

There are several aspects needing study, all of which are related to the accuracy of the network and thus to the economics. These aspects are:

(1) accuracy and distribution of the observations;

(2) roles played by the various kinds of observations (e.g. distances for scale, azimuths for orientation);

(3) geometry of the network (e.g. area net, chain net, figures of triangles, quadrilaterals, traverses);

(4) adjustment set-up (e.g. number and distribution of points held fixed, degrees of freedom).

We will concentrate on describing the technique for studying the above four aspects. No mention will be made on how to optimise field procedures and the like. These aspects belong to the realm of data acquisition. Optimisation of these and still other parameters not directly related to accuracy are, however, recognised as part of the entire problem and should be considered. The purpose of these notes is to describe the technique for optimisation of only those parameters related to accuracy, and which can be described within the method of least squares.

6.2 Mathematical Foundation

The technique of pre-analysis described herein is based on the method of least squares. To begin with we introduce the mathematical model relating the vector of unknown parameters X (coordinates) and the vector L of observables (distances, azimuths, directions), namely

$$F(\bar{X},\ \bar{L}) = 0. \qquad (6\text{-}1)$$

The above represents the set of equations, usually nonlinear, arising from a specific set of observations to be contained in the network. The linearization of the model by a linear Taylor's series yields the well-known equations for the parametric case

$$F(\bar{X},\ \bar{L}) = F(X^{o},\ L) + \left.\frac{\partial F}{\partial X}\right|_{X^{o},L} \hat{X} +, + \left.\frac{\partial F}{\partial L}\right|_{X^{o},L} \hat{V} = 0 \ , \qquad (6\text{-}2)$$

$$= W + A\hat{X} - \hat{V} = 0 \ . \qquad (6\text{-}3)$$

In the above, W is the point of expansion of F about the approximate values of the unknown coordinates (X^{o}) and the observed values of the observables (L). The remaining terms are the departures from the point of expansion resulting from corrections to the approximate coordinates, corrections to the observations and the nonlinearity of the mathematical model. The weight matrix corresponding to the observations L is

$$P = \sigma_{o}^{2} \quad \Sigma_{L}^{-1} \quad , \qquad (6\text{-}4)$$

where Σ_L is the variance covariance matrix of L and σ_o^2 is the a priori variance factor.

When the method of least squares is employed to get estimates for $\hat{X}$ and $\hat{V}$ in 6.3, the resulting equation is

$$\hat{X} = -(A^T PA)^{-1} A^T PW \ . \tag{6-5}$$

Its estimated variance-covariance matrix, and also that of the adjusted coordinates, $\bar{X} = X^o + \hat{X}$, is

$$\Sigma_{\hat{X}} = \sigma_o^2 \ (A^T PA)^{-1} \ , \tag{6-6}$$

when the a priori variance factor, σ_o^2, is assumed to be known. In the case where σ_o^2 is taken to be unknown, then the estimated variance-covariance matrix is given by

$$\hat{\Sigma}_{\hat{X}} = \hat{\sigma}_o^2 \ (A^T PA)^{-1} \ , \tag{6-7}$$

where $\hat{\sigma}_o^2$ is estimated from the adjustment as

$$\hat{\sigma}_o^2 = \frac{\hat{V}^T P\hat{V}}{df} \ , \tag{6-8}$$

and df = degrees of freedom. The estimate for V is obtained by substituting $\hat{X}$ into (6-3) and solving for $\hat{V}$.

There is more to the method of least squares, but let us stop here as we have recapitulated enough of the method to allow us to explain the fundamental equations upon which the technique of pre-analysis is based.

Since pre-analysis is essentially a design tool, no observations are made and thus there is no estimate for $\hat{X}$. Because W equals zero this leaves only one equation, namely (6-6) in the form

$$\Sigma_{\hat{X}} = \Sigma_{\bar{X}} = \sigma_o^2 (A^T P A)^{-1} , \tag{6-9}$$

$$= \sigma_o^2 (A^T \sigma_o^2 \Sigma_L^{-1} A)^{-1} , \tag{6-10}$$

$$= (A^T \Sigma_L^{-1} A)^{-1} . \tag{6-11}$$

Pre-analysis is based completely on the above equation and is performed by simply specifying the elements of the design matrix A and the variance-covariance matrix Σ_L.

6.3 Procedure

The inclusion or exclusion of certain elements in A and in Σ_L is in part the key to analysing the four aspects stated in Section 6.1. The presence of certain observables is accomplished by inserting a row of elements in the design matrix in columns corresponding to points between which observations are to be made. The accuracy is represented by a variance placed in the corresponding diagonal position of Σ_L. The geometry of the network is depicted through the numerical value of the elements in the design matrix. In the adjustment set-up the fixed points of the network are implicity represented in the design matrix by the absence of elements pertaining to points held fixed.

The following are the steps to be followed when performing a pre-analysis:

(1) Evaluate the coefficients of the particular linearized mathematical model (observation equation);

(2) Assign variances to observables;

(3) Place the coefficients of (1) into design matrix A and elements of (2) into Σ_L;

(4) Form the matrix product $A^T\Sigma_L^{-1}A$;

(5) Invert $(A^T\Sigma_L^{-1}A)$;

(6) Compute the standard two-dimensional confidence region for each point (relative to points held fixed), by solving the eigenvalue problem of

the corresponding 2 x 2 sub-matrix of $(A^T \Sigma_L^{-1} A)^{-1}$, [Mikhail, 1976];

(7) Compute the standard two-dimensional confidence region (relative) between each pair of points, by solving the eigenvalue problem of the corresponding relative 2 x 2 variance-covariance matrix between the two points;

(8) Increase the probabilities associated with (7) and (8) to a higher probability level (say 95%).

Some elaboration on items (7) and (8) is in order. To compute the standard relative variance-covariance matrix between a pair of points i and j we first formulate the mathematical model. In essence it is the coordinate differences

$$\Delta\phi_{ij} = \phi_j - \phi_i , \tag{6-12}$$

$$\Delta\lambda_{ij} = \lambda_j - \lambda_i , \tag{6-13}$$

that are of interest. In matrix form

$$\begin{bmatrix} \Delta\phi_{ij} \\ \Delta\lambda_{ij} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_i \\ \lambda_i \\ \phi_j \\ \lambda_j \end{bmatrix} , \tag{6-14}$$

where the coefficient matrix is denoted by G for further use. We wish to propagate the errors from the coordinates into the coordinate differences. This is achieved by

$$\underset{2,2}{\Sigma_{\Delta\phi\Delta\lambda}} = \underset{2,4}{G} \; \underset{4,4}{\Sigma_{\phi,\lambda}} \; \underset{4,2}{G^T} , \tag{6-15}$$

where

$$\underset{4,4}{\Sigma_{\phi,\lambda}} = \begin{bmatrix} \underset{2,2}{\Sigma_{\phi_i\lambda_i}} & \underset{2,2}{\Sigma_{i,j}} \\ \underset{2,2}{\Sigma_{j,i}} & \underset{2,2}{\Sigma_{\phi_j\lambda_j}} \end{bmatrix} \qquad (6\text{-}16)$$

is the variance-covariance information from $\Sigma_{\bar{x}} = (A^T \Sigma_L^{-1} A)^{-1}$.

To increase the probability level of the standard two-dimensional confidence region (about 38%) to a higher level, say 95%, we revert to the basics of multivariate statistics [Wells and Krakiwsky 1971, pp. 126-130]. The quadratic form for the parameters in the case of σ_o^2 known is

$$(\bar{X}-X)^T \Sigma_{\bar{X}}^{-1} (\bar{X}-X) \overset{d}{\rightarrow} \chi^2_{u,1-\alpha} \quad , \qquad (6\text{-}17)$$

where $\bar{X}$ is the least squares estimate, X is the true value of the parameters, u is the dimensionality of the problem, α is the desired confidence level, and $\chi^2_{u,1-\alpha}$ is a random variable with a chi-square distribution and degrees of freedom u. In other words, we can establish a confidence region for the deviations from the least squares estimate $\bar{X}$, of some other set X, based on the statistics of $\bar{X}$.

Returning to equation (6-17) we see that

$$(\bar{X} - X)^T \Sigma_{\bar{X}}^{-1} (\bar{X} - X) = \chi^2_{u,1-\alpha}$$

is an equation of a hyperellipsoid that defines the limits of the associated confidence region. Translating the origin of the coordinate system to $\bar{X}$ and assuming the dimensionality , u, to be 2, we get the equation of an ellipse

$$X^T \Sigma_{\bar{X}}^{-1} X = \chi^2_{2,1-\alpha} \tag{6-18}$$

or

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \sigma^2_{x_1} & \sigma_{x_1 x_2} \\ \sigma_{x_1 x_2} & \sigma^2_{x_2} \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \chi^2_{2,1-\alpha} \tag{6-19}$$

where x_1 and x_2 represents the coordinates (ϕ_i, λ_i) or the coordinate differences $(\Delta\phi_{ij}, \Delta\lambda_{ij})$.

An equation without cross product terms can be obtained via the eigenvalue problem, namely

$$[y_1 \; y_2] \begin{bmatrix} \sigma^2_{max} & 0 \\ 0 & \sigma^2_{min} \end{bmatrix}^{-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \chi^2_{2,1-\alpha} \tag{6-20}$$

where y_1 and y_2 are the transformed $x_1 x_2$ coordinates with respect to the rotated coordinate axes resulting from the eigenvalue problem.

To obtain the equation of the ellipse from the above, we write

$$\frac{y_1^2}{\sigma^2_{max} \cdot \chi^2_{2,1-\alpha}} + \frac{y_2^2}{\sigma^2_{min} \cdot \chi^2_{2,1-\alpha}} = 1, \tag{6-21}$$

which says that the semimajor (a) and semiminor (b) axes of the ellipse are

$$a = \sqrt{\chi^2_{2,1-\alpha}} \cdot \sigma_{max}$$

$$b = \sqrt{\chi^2_{2,1-\alpha}} \cdot \sigma_{min}$$

Thus, to obtain a confidence region with a certain probability level, the axes of the standard confidence region must be multiplied by a factor as shown above. Note for any 2 dimensional adjustment with $\alpha = .05$, $\sqrt{\chi^2_{2,1-.05}} = 2.45$.

For the case where σ_o^2 is unknown, a similar development for the semimajor and semiminor axes of the confidence ellipse yields

$$a = \sqrt{uF_{u,df,1-\alpha}} \quad \sigma_{max} , \tag{6-24}$$

$$b = \sqrt{uF_{u,df,1-\alpha}} \quad \sigma_{min} . \tag{6-25}$$

For any 2 dimensional adjustment with

df = 10 and $\alpha = 0.05$ we obtain

$$\sqrt{2F_{2,10,1-.05}} = 2.86 .$$

Also note that the above development applies to the examination of a single point. For "simultaneous ellipses", see, for example, Vanicek and Krakiwsky [in prep].

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