EARTH-POLE WOBBLE

P. VANICEK

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EARTH - POLE WOBBLE

Petr Vaníček

Department of Surveying Engineering
University of New Brunswick
P.O. Box 4400
Fredericton, N.B.
Canada
E3B 5A3

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PREFACE

In order to make our extensive series of lecture notes more readily available, we have scanned the old master copies and produced electronic versions in Portable Document Format. The quality of the images varies depending on the quality of the originals. The images have not been converted to searchable text.

Preface

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P. Vaníček

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1) Euler's equations of rotation

1.1) Definition of coordinate systems

Let us consider a physical body B in a three dimensional space. In order to be able to describe its motion in the space let us define the following two Cartesian coordinate systems:

i) $S \equiv (C; x,y,z)$, fixed, rectangular, positive, centered on C with axes x,y,z;

ii) $S' \equiv (T; \xi,\eta,\zeta)$, rectangular, positive and "linked with B". Let T be, for simplicity, the center of gravity of B and let ξ,η,ζ - axes be oriented so far arbitrarily.

We shall be denoting the vectors expressed in S by Latin letters and the vectors expressed in S' by Greek letters. Besides, we are going to distinguish between free vectors

$$\vec{a} = \sum_{k=1}^{3} a_k \vec{e}_k \,, \qquad \vec{\alpha} = \sum_{k=1}^{3} \alpha_k \vec{\varepsilon}_k$$

and position (radius) vectors

$$\vec{r}_p = x_p \vec{e}_1 + y_p \vec{e}_2 + z_p \vec{e}_3$$

$$\vec{\rho}_p = \xi_p \vec{\varepsilon}_1 + \eta_p \vec{\varepsilon}_2 + \zeta_p \vec{\varepsilon}_3 \,.$$

Here, by $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ we denote the unit vectors in x,y,z - axes and $\vec{\varepsilon}_1$, $\vec{\varepsilon}_2$, $\vec{\varepsilon}_3$ the unit vectors in ξ,η,ζ - axes.

Since we are going to talk about dynamics, we shall have to consider all the problem from the point of view of time.

1.2 Dynamic system

When we let the individual vectors be time dependent we may regard the description of the individual points (and the body B) as a description of a dynamic system. When we talk about time variable vectors, we really talk about following functions:

$$\vec{a}(t) = \sum_{k=1}^{3} a_k(t)\ \vec{e}_k(t), \qquad \vec{\alpha}(t) = \sum_{k=1}^{3} \alpha_k(t)\ \vec{\varepsilon}_k(t),$$

and similarly

$$\vec{r}_p(t) = x_p\ (t)\ \vec{e}_1(t) + y_p(t)\ \vec{e}_2(t) + z_p(t)\ \vec{e}_3(t)$$

$$\vec{\rho}_p(t) = \xi_p(t)\ \vec{\varepsilon}_1(t) + \eta_p(t)\ \vec{\varepsilon}_2(t) + \zeta_p(t)\ \vec{\varepsilon}_3(t),$$

where all the individual quantities vary with time. The position vectors varying with time can be regarded, from the dynamic point of view, as motions.

Here, when talking about $\vec{e}_k(t)$, $\vec{\varepsilon}_k(t)$ we talk really about vectors varying with respect to the other coordinate system and not within the coordinate system they represent. Note that it is rather difficult to visualize the system S' linked with **B** if the body **B** is not rigid. We shall see later how this difficulty can be overcome.

1.3 Relative and absolute velocities

Let us assume now that all the time functions we shall be dealing with are smooth and have first and second derivatives with respect since, as we have said earlier, $\dot{\vec{e}}_k = \vec{0}$, $\overset{*}{\vec{\varepsilon}}_k = \vec{0}$ $(k = 1,2,3)$. The function $\dot{\vec{r}}$ is usually called absolute velocity (velocity with respect to the "absolute" system of coordinates S) and $\overset{*}{\vec{\rho}}$ is known as relative velocity (taken with respect to the relative system S').

The question often arises in dynamics as to what an absolute velocity a relative motion $\vec{\rho}$ has. In order to be able to answer this important question let us investigate the absolute (dot) time derivative of a vector expressed in S'. We have

$$\dot{\vec{\alpha}} = \sum_k \dot{\alpha}_k \vec{\varepsilon}_k + \sum \alpha_k \dot{\vec{\varepsilon}}_k = \sum_k \overset{*}{\alpha}_k \vec{\varepsilon}_k + \sum_k \alpha_k \dot{\vec{\varepsilon}}_k = \overset{*}{\vec{\alpha}} + \sum_k \alpha_k \dot{\vec{\varepsilon}}_k$$

where $\dot{\vec{\varepsilon}}_k \neq \vec{0}$ if S' "moves" with respect to S.

Taking the second term in the above equation we can write, denoting it by, say $\vec{\beta}$

$$\vec{\beta} = \sum_k \alpha_k \frac{d\vec{\varepsilon}_k}{dt}$$

and

$$\beta_1 = \vec{\beta}\cdot\vec{\varepsilon}_1 = \sum_k \alpha_k \frac{d\vec{\varepsilon}_k}{dt}\vec{\varepsilon}_1$$

$$\beta_2 = \vec{\beta}\cdot\vec{\varepsilon}_2 = \sum_k \alpha_k \frac{d\vec{\varepsilon}_k}{dt}\vec{\varepsilon}_2$$

$$\beta_3 = \vec{\beta}\cdot\vec{\varepsilon}_3 = \sum_k \alpha_k \frac{d\vec{\varepsilon}_k}{dt}\vec{\varepsilon}_3 \ .$$

On the other hand, we have

$$\frac{d(\vec{\varepsilon}_k\cdot\vec{\varepsilon}_k)}{dt} = 2\vec{\varepsilon}_k \cdot \frac{d\vec{\varepsilon}_k}{dt} = 0$$

because

$$\vec{\varepsilon}_k \cdot \vec{\varepsilon}_k = 1 \ .$$

Hence, we obtain for, say β_1:

$$\beta_1 = \alpha_2 \frac{d\vec{\varepsilon}_2}{dt}\vec{\varepsilon}_1 + \alpha_3 \frac{d\vec{\varepsilon}_3}{dt}\vec{\varepsilon}_1 \ .$$

But, we also can write

$$\frac{d\,(\vec{\varepsilon}_k \cdot \vec{\varepsilon}_j)}{dt} = \frac{d\vec{\varepsilon}_k}{dt}\,\vec{\varepsilon}_j + \vec{\varepsilon}_k\,\frac{d\vec{\varepsilon}_j}{dt} = 0 \qquad j \neq k$$

which again follows from

$$\vec{\varepsilon}_k \cdot \vec{\varepsilon}_j = 0 \qquad j \neq k\ .$$

Substituting this result into the equation for β_1 we obtain

$$\beta_1 = -\left(\frac{d\vec{\varepsilon}_1}{dt} \cdot \vec{\varepsilon}_2\right)\alpha_2 + \left(\frac{d\vec{\varepsilon}_3}{dt} \cdot \vec{\varepsilon}_1\right)\alpha_3$$

and analogously

$$\beta_2 = -\left(\frac{d\vec{\varepsilon}_2}{dt} \cdot \vec{\varepsilon}_3\right)\alpha_3 + \left(\frac{d\vec{\varepsilon}_1}{dt} \cdot \vec{\varepsilon}_2\right)\alpha_1$$

$$\beta_3 = -\left(\frac{d\vec{\varepsilon}_3}{dt} \cdot \vec{\varepsilon}_1\right)\alpha_1 + \left(\frac{d\vec{\varepsilon}_2}{dt} \cdot \vec{\varepsilon}_3\right)\alpha_2\ .$$

Denoting the scalars $\dot{\vec{\varepsilon}}_1 \cdot \vec{\varepsilon}_2$, $\dot{\vec{\varepsilon}}_2 \cdot \vec{\varepsilon}_3$, $\dot{\vec{\varepsilon}}_3 \cdot \vec{\varepsilon}_1$ by ω_3, ω_1, ω_2 respectively, we can rewrite the above equations as

$$\beta_1 = \omega_2\alpha_3 - \omega_3\alpha_2$$

$$\beta_2 = \omega_3\alpha_1 - \omega_1\alpha_3$$

$$\beta_3 = \omega_1\alpha_2 - \omega_2\alpha_1\ .$$

Defining a new vector $\vec{\omega}$ in S'

$$\vec{\omega} = \sum_k \omega_k\,\vec{\varepsilon}_k$$

these three scalar equations can be again represented in form of one vector equation:

$$\vec{\beta} = \vec{\omega} \times \vec{\alpha}$$

or

$$\vec{\beta} = \Omega\,\vec{\alpha}$$

where Ω is the antisymmetric tensor

$$\Omega = \left\| \begin{matrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{matrix} \right\|$$

belonging to ω.

We can then write finally for $\dot{\vec{\alpha}}$:

$$\dot{\vec{\alpha}} = \overset{*}{\vec{\alpha}} + \vec{\omega} \times \vec{\alpha} = \overset{*}{\vec{\alpha}} + \Omega\vec{\alpha} \quad .$$

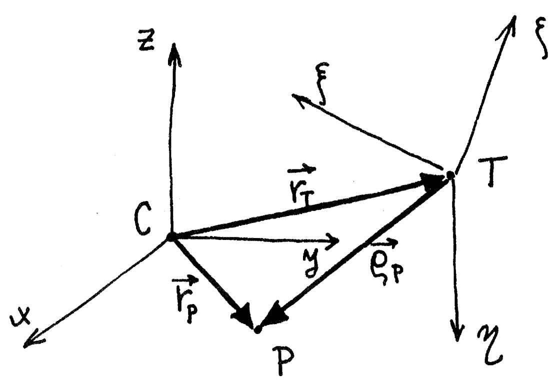
Note that applying this formula, valid for any free vector in S',

to $\vec{\omega}$ we get

$$\dot{\vec{\omega}} = \overset{*}{\vec{\omega}} + \vec{\omega} \times \vec{\omega} = \overset{*}{\vec{\omega}} \quad .$$

This result can now be utilized to answer the original question what is the absolute velocity of a point, whose motion is known in the relative sense only.

From the diagram we can write



$$\vec{r}_p = \vec{r}_T + \vec{\rho}_p \quad .$$

Hence

$$\dot{\vec{r}}_p = \dot{\vec{r}}_T + \dot{\vec{\rho}}_p \quad .$$

But from the formula for the absolute time derivative of a vector expressed in S' we have

$$\dot{\vec{\rho}}_p = \overset{*}{\vec{\rho}}_p + \vec{\omega} \times \vec{\rho}_p$$

so that

$$\dot{\vec{r}}_p = \dot{\vec{r}}_T + \overset{*}{\vec{\rho}}_p + \vec{\omega} \times \vec{\rho}_p .$$

Suppose now that the point P is a part of the physical body B . Then the last equation can be given the following physical interpretation. The absolute velocity of P (with respect to S; i.e., the absolute space around the body) is composed of three individual velocities:

i) $\dot{\vec{r}}_T$ - <u>the translation velocity of the body B</u> (or its center of gravity T which means the same) with respect to S;

ii) $\overset{*}{\vec{\rho}}_P$ - <u>the translation velocity of P within B</u> (with respect to T) that describes the expansion or contraction of B;

iii) $\vec{\omega} \times \vec{\rho}_P$, the term we shall try to interpret now.

Let us assume the body B to be for a moment rigid; i.e., $\overset{*}{\vec{\rho}}_P = \vec{0}$, and it does not have any translation velocity with respect to S; i.e., $\dot{\vec{r}}_T = \vec{0}$. Without any loss of generality, we can now place C to coincide with T and get

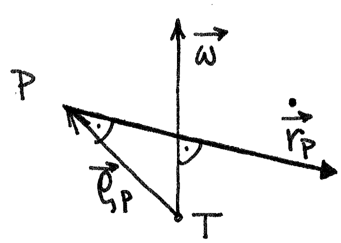
$$\vec{r}_T = \vec{0}$$

Hence, the absolute velocity of P will reduce to

$$\dot{\vec{r}}_P = \vec{\omega} \times \vec{\rho}_P,$$

and the only freedom left for the point P is rotation around T at a constant distance but in any direction.

This means that both $\vec{\omega}$ and $\vec{\rho}_P$ are perpendicular to $\dot{\vec{r}}_P$.



Moreover

$$v = |\dot{\vec{r}}| = |\vec{\omega}|\ |\vec{\rho}_P|\ \sin(\sphericalangle \vec{\omega}\vec{\rho}_P)$$

$$= \omega\rho_P \ \sin \widehat{\omega\rho}_P .$$

But $\rho_P \sin\widehat{\omega r}_P$ is the perpendicular distance of P from $\vec{\omega}$. From the elementary physics we know that the tangential velocity v of a particle rotating around a center of rotation with angular velocity ω' in a distance r' is given by

$$v = \omega' \ r' .$$

Therefore the vector $\vec{\omega}$ has to be interpreted as being always coincident with the instantaneous axis of rotation of B and having the absolute value given by the instantaneous angular velocity of B. It is consequently known as the <u>vector of rotation of B</u>. It is easily seen that its meaning does not change even for the general case of translating, non-rigid body. The third velocity in the gernal formula can be then understood as the <u>rotational velocity of B</u>.

1.4 Relative and absolute accelerations

It is not difficult to see how we go about defining the second time derivatives of the individual vectors with respect to their alternative systems of coordinates. Let us just say here that we can define $\ddot{\vec{a}}$, $\overset{**}{\vec{\alpha}}$, $\ddot{\vec{r}}$, $\overset{**}{\vec{\rho}}$ in the same way as we have done for the first derivatives. The last two quantities are known as <u>absolute and relative accelerations</u> respectively.

Same question arises here as how to express the absolute acceleration of a point (motion) known in S' only. We again write first

$$\ddot{\vec{\alpha}} = \overset{*\bullet}{\vec{\alpha}} + \dot{\vec{\omega}} \times \vec{\alpha} + \vec{\omega} \times \dot{\vec{\alpha}}$$

using the formula for $\dot{\vec{\alpha}}$ and taking its derivative with respect to S. Applying the same rule to $\overset{*}{\vec{\alpha}}$ as we have applied to $\vec{\alpha}$ we get

$$\overset{*\bullet}{\vec{\alpha}} = \overset{**}{\vec{\alpha}} + \vec{\omega} \times \overset{*}{\vec{\alpha}} \; .$$

(But $\overset{\bullet *}{\vec{\alpha}} + \overbrace{(\overset{*}{\vec{\alpha}} + \vec{\omega} \times \vec{\alpha})}^{*} = \overset{**}{\vec{\alpha}} + \overset{*}{\vec{\omega}} \times \vec{\alpha} + \vec{\omega} \times \overset{*}{\vec{\alpha}} = \overset{**}{\vec{\alpha}} + \dot{\vec{\omega}} \times \vec{\alpha} + \vec{\omega} \times \overset{*}{\vec{\alpha}} \neq \overset{*\bullet}{\vec{\alpha}}$)

On the other hand

$$\vec{\omega} \times \dot{\vec{\alpha}} = \vec{\omega} \times (\overset{*}{\vec{\alpha}} + \vec{\omega} \times \vec{\alpha}) = \vec{\omega} \times \overset{*}{\vec{\alpha}} + \vec{\omega} \times (\vec{\omega} \times \vec{\alpha}) .$$

Hence we obtain finally

$$\ddot{\vec{\alpha}} = \overset{**}{\vec{\alpha}} + 2\vec{\omega} \times \overset{*}{\vec{\alpha}} + \dot{\vec{\omega}} \times \vec{\alpha} + \vec{\omega} \times (\vec{\omega} \times \vec{\alpha}) .$$

Therefore, we can write for the absolute acceleration of P

$$\ddot{\vec{r}}_P = \ddot{\vec{r}}_T + \ddot{\vec{\rho}}_P = \ddot{\vec{r}} + \overset{**}{\vec{\rho}}_P + 2\vec{\omega} \times \overset{*}{\vec{\rho}}_P + \dot{\vec{\omega}} \times \vec{\rho}_P + \vec{\omega} \times (\vec{\omega} \times \vec{\rho}_P)$$

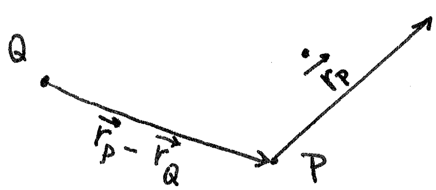
which is the formula we are looking for.

We may note that for a rigid body B, $\overset{*}{\vec{\rho}}_P = \overset{**}{\vec{\rho}}_P = \vec{0}$ and the formula becomes simpler. However, we may, of course, interpret the point P as moving with respect to B (whatever that means). Then the term $2\vec{\omega} \times \overset{*}{\vec{\rho}}_P$ is known as the Coriolis' acceleration. It can be seen that the Coriolis' acceleration is zero when P does not move with respect to B(S') or when it moves parallel to the instantaneous axis of rotation. The term $\dot{\vec{\omega}} \times \vec{\rho}_P$ does not have any fixed name in physics. C. Lanczos [1949], however, suggested to call it Euler's acceleration. It remains to be seen whether this name will be generally adopted or not.

1.5) Moment of motion

It is known from elementary mechanics that the vector quantity

$$\vec{M}_{PQ} = (\vec{r}_P - \vec{r}_Q) \times \dot{\vec{r}}_P \, m_P$$



where P,Q are two points and m_P is the mass attributed to P ("mass of P"), is the moment of motion of P with respect to Q. Here Q is regarded as immobile in S. We can easily see that if Q is chosen so that it coincides with C, the origin of S, we have specially:

$$\vec{M}_{PC} = \vec{r}_P \times \dot{\vec{r}}_P \, m_P .$$

We can define the moment of motion of a physical body B with respect to a point Q as the sum of the moments of motion of all the points in B. This may be written formally as

$$\vec{M}_{BQ} = \int_{P \, B} \{(\vec{r}_P - \vec{r}_Q) \times \dot{\vec{r}}_P \, m_P\} .$$

In particular, if B is an area integrable in Riemanian sense we may express the mass of any differential subarea dB as σdB , where σ is known as the density of B and can be regarded as a function of $\vec{\rho}_P$. Providing σ has in

B only finitely many surfaces of discontinuity, the above moment can be written

$$\vec{M}_{BQ} = \int_B \{(\vec{r} - \vec{r}_Q) \times \vec{r}\,\sigma\}\, dB.$$

The moment of motion of such a physical body, which we are going to assume always from now on, will be given by

$$\vec{M}_{BC} = \int_B (\vec{r} \times \dot{\vec{r}}\,\sigma)\, dB.$$

1.6) <u>Moment of force</u>

The vector

$$\vec{N}_{PQ} = (\vec{r}_P - \vec{r}_Q) \times \ddot{\vec{r}}_P\, m_P$$

defines the physical quantity known in mechanics as the <u>moment of force, acting on P, with respect to Q</u>. The force $\vec{f}_P$ acting on P is defined as

$$\vec{f}_P = \ddot{\vec{r}}_P\, m_P\ .$$

Rigorously, one should speak here only about "Newtonian forces" that are defined by the above formula.

As a special case, we get

$$\vec{N}_{PC} = \vec{r}_P \times \ddot{\vec{r}}_P\, m_P\ .$$

Analogously to § 1.5, we can define the <u>moment of force, acting on the physical body B</u>, with respect to Q as

$$\vec{N}_{BQ} = \int_B \{(\vec{r} - \vec{r}_Q) \times \ddot{\vec{r}}\,\sigma\}\, dB\ .$$

In particular

$$\vec{N}_{BC} = \int_B (\vec{r} \times \ddot{\vec{r}}\,\sigma)\, dB\ .$$

We can show that the moment of (Newtonian) force is an absolute time derivative of the moment of motion for both a point and a rigid physical body. To prove that, let us take for instance the moment of the rigid physical body B with repsect to a motionless point Q:

$$\vec{M}_{BQ} = \int_B \{(\vec{r} - \vec{r}_Q) \times \dot{\vec{r}}\, \sigma\}\, dB.$$

Its absolute time derivative reads:

$$\dot{\vec{M}}_{BQ} = \frac{d}{dt}\left(\int_B \vec{r} \times \dot{\vec{r}}\, \sigma\, dB - \int_B \vec{r}_Q \times \dot{\vec{r}}\, \sigma\, dB\right)$$

$$= \int_B \{\dot{\vec{r}} \times \dot{\vec{r}} + \vec{r} \times \ddot{\vec{r}}\}\, \sigma dB - \int_B \vec{r}_Q \times \ddot{\vec{r}}\, \sigma\, dB$$

since $\frac{d\sigma}{dt} = 0$ due to the rigidity of B. But here

$$\dot{\vec{r}} \times \dot{\vec{r}} = \vec{0}$$

and we end up with expression

$$\dot{\vec{M}}_{BQ} = \int_B \{(\vec{r} - \vec{r}_Q) \times \ddot{\vec{r}}\, \sigma\}\, dB$$

which is nothing else but $\vec{N}_{BQ}$ as defined earlier.

From now on we shall be talking only about rigid physical bodies for which $\overset{*}{\vec{\rho}}_P = \overset{**}{\vec{\rho}}_P = \vec{0}$, $P \in B$ and $\frac{d\sigma}{dt} = 0$.

1.7) Tensor of inertia, its use in formulating the moment of motion of a rigid body

Let us now express the moment of force acting on the rigid body B with respect to C using the vector of rotation of B. Substituting for $\dot{\vec{r}}$ in the expression for $\vec{M}_{BC}$ the final equation from § 3 we obtain

$$\vec{M}_{BC} = \int_B \{\vec{r} \times (\dot{\vec{r}}_T + \vec{\omega} \times \vec{\rho})\, \sigma\}\, dB.$$

The absolute position vector $\vec{r}$ can be also substituted for using the relation

$$\vec{r} = \vec{r}_T + \vec{\rho}$$

and we get

$$\vec{M}_{BC} = \int_B \{(\vec{r}_T + \vec{\rho}) \times (\dot{\vec{r}}_T + \vec{\omega} \times \vec{\rho})\ \sigma\}\ dB$$

$$= \int_B (\vec{r}_T \times \dot{\vec{r}}_T\ \sigma)\ dB + \int_B \{\vec{r}_T \times (\vec{\omega} \times \vec{\rho})\ \sigma\}\ dB +$$

$$+ \int_B \{\vec{\rho} \times \dot{\vec{r}}_T\ \sigma\}\ dB + \int_B \{\vec{\rho} \times (\vec{\omega} \times \vec{\rho})\ \sigma\}\ dB.$$

Here $\vec{r}_T$, $\dot{\vec{r}}_T, \vec{\omega}$ can be considered constant from the point of view of the integration because these quantities describe some properties of the whole body as such. Denoting the mass of B by $\mathfrak{U}$, i.e.,

$$\mathfrak{U} = \int_\beta \sigma\ d\beta,$$

we can hence rewrite the above equation as follows

$$\vec{M}_{BC} = \vec{r}_T \times \dot{\vec{r}}_T\ \mathfrak{U} + \vec{r}_T \times (\vec{\omega} \times \int_B \vec{\rho}\sigma dB) -$$
$$- \dot{\vec{r}}_T \times \int_B \vec{\rho}\sigma\ dB + \int_B \{\vec{\rho} \times (\vec{\omega} \times \vec{\rho})\ \sigma\}\ dB.$$

From elementary mechanics we know that the position vector of the center of gravity is given by

$$\vec{\rho}_T = \frac{1}{\mathfrak{U}} \int_B \vec{\rho}\sigma dB.$$

But due to our particular choice of S', $\vec{\rho}_T = \vec{0}$ and we have

$$\int_B \vec{\rho}\sigma dB = \vec{0}.$$

Hence we finally get

$$\vec{M}_{BC} = \vec{r}_T \times \dot{\vec{r}}_T\ \mathfrak{U} + \int_B \{\vec{\rho} \times (\vec{\omega} \times \vec{\rho})\ \sigma\}\ dB.$$

We have seen in § 1.3 that a vector (cross) product of two vectors can be also written as the product of the antisymmetric tensor belonging to the first vector, with the second vector. In order to be able to utilize this knowledge, let us rewrite the subintegral vector function of the last equation as

$$\vec{q} = \vec{\rho} \times (\vec{\omega} \times \vec{\rho})\sigma = -\vec{\rho} \times (\vec{\rho} \times \vec{\omega})\sigma \ .$$

Expressing the vector product in the brackets using the antisymmetric tensor R

$$R = \left\|\begin{matrix} 0, & -\zeta, & \eta \\ \zeta, & 0, & -\xi \\ -\eta, & \xi, & 0 \end{matrix}\right\|$$

we may write

$$\vec{q} = -\vec{\rho} \times R\vec{\omega}\sigma = -R(R\vec{\omega})\sigma$$
$$= -R^2\vec{\omega}\sigma \ .$$

For $R^2 = R \cdot R$ we obtain

$$R^2 = \left\|\begin{matrix} -\zeta^2-\eta^2, & \xi\eta, & \xi\zeta \\ \eta\xi, & -\xi^2-\zeta^2, & \eta\zeta \\ \zeta\xi, & \zeta\eta, & -\eta^2-\xi^2 \end{matrix}\right\|$$

as the reader can easily verify.

Substituting this result back into the integration we get

$$\int_B \vec{q}dB = -\int_B R^2\vec{\omega}\sigma dB = \int_B -R^2\sigma dB \cdot \vec{\omega}.$$

The integral in the last equation is known as the <u>tensor of inertia of B</u> (evaluated in S') and can be explicitly written as

$$\mathcal{J} = \left\|\begin{matrix} \int_B(\zeta^2+\eta^2)\sigma dB, & \int_B\xi\eta\sigma dB, & \int_B\xi\zeta\sigma dB \\ \int_B\xi\eta\sigma dB, & \int_B(\xi^2+\zeta^2)\sigma dB, & \int_B\eta\zeta\sigma dB \\ \int_B\xi\zeta\sigma dB, & \int_B\eta\zeta\sigma dB, & \int_B(\eta^2+\xi^2)\sigma dB \end{matrix}\right\| \ .$$

It is obviously symmetrical and is very often denoted thus

$$\mathcal{J} = \begin{Vmatrix} A, & -D, & -E \\ -D, & B, & -F \\ -E, & -F, & C \end{Vmatrix}$$

Its diagonal components (elements) are known as moments of inertia with respect to the individual axes ξ, η, ζ. The off-diagonal components are usually called products of inertia or deviation moments. We may note that for a differently selected S' we would have a different tensor of intertia. Actually, it is easily seen that the tensor of inertia can be determined for any point in or outside B and its components depend also on the choice of the direction of the individual axes. The developed tensor of inertia is known as the central tensor of inertia because it is related to the center of gravity.

Turning back to the original moment of motion we can now rewrite it as

$$\vec{M}_{BC} = \vec{r}_T \times \dot{\vec{r}}_T \mathfrak{M} + \mathcal{J}\vec{\omega}.$$

Comparing this equation with the equation of the moment of motion of a point we can see that the moment of motion of a rigid body is given by the sum of (i) the moment of motion of the center of gravity of B (with the mass of the whole body attributed to it);

(ii) the moment of motion of the body with respect to its own center of gravity. The last sentence can be easily verified by writing the equation above for $C \equiv T$:

$$\vec{M}_{BT} = \mathcal{J}\vec{\omega} .$$

1.8) Euler's equation in general form

We are now finally in a position to formulate the Euler's equation expressing the moment of force (acting on a rigid body) with respect to its center of gravity as a function of the vector of rotation of the body. We have shown in § 1.6 that $\overset{\circ}{\vec{M}}_{BQ} = \vec{N}_{BQ}$. In particular, we have

$$\overset{\circ}{\vec{M}}_{BC} = \vec{N}_{BC} \ .$$

Taking the formula for $\vec{M}_{BC}$ derived in § 1.7 and applying to it the rule developed in § 1.3 for evaluating the absolute time derivative of a vector expressed in S' (here we realize that $\mathcal{Y}\vec{\omega}$ is a vector expressed in S') we obtain

$$\vec{N}_{BC} = \vec{r}_T \times \ddot{\vec{r}}_T \mu + \mathcal{Y}\overset{*}{\vec{\omega}} + \vec{\omega} \times \mathcal{Y}\vec{\omega} \ .$$

Here, y is, of course, taken as time-independent since B is rigid.

Taking especially $C \equiv T$ we get

$$\vec{N}_{BT} = \mathcal{Y}\overset{*}{\vec{\omega}} + \vec{\omega} \times \mathcal{Y}\vec{\omega} \ .$$

This is the famous Euler's equation in its general form. It may be understood as describing the rotation $\vec{\omega}$ of a rigid body around the instantaneous axis of rotation going through its center of gravity T as a function of the mass distribution within B (expressed by means of $\mathcal{Y}$) and the moment of external forces (with respect to T) acting on the body.

Note that in § 1.3 we have derived:

$$\overset{\circ}{\vec{\omega}} = \overset{*}{\vec{\omega}} \ .$$

Hence the absolute and relative angular accelerations are identical. Thus the Euler's equation can be used to describe the rotation of B with respect to S also. However, we are not going to do this here.

1.9) Ellipsoid of inertia and principal axes of inertia

It is known from mechanics that the moments of inertia of a rigid body B with respect to all possible axes going through a point Q, when we interpret the reciprocals of their square roots as lengths on the appropriate axes, create an ellipsoidal surface centered on Q. This ellipsoidal surface is known as the ellipsoid of inertia belonging to Q of B. In a Cartesian system concentric with the ellipsoid, say $S'' \equiv (Q;X,Y,Z)$, the ellipsoidal surface is given by

$$AX^2 + BY^2 + CZ^2 - 2DXY - 2EXZ - 2FYZ = 1.$$

The shape, orientation and size of the ellipsoid of inertia vary from point to point. We sometimes talk about points where the ellipsoid degenerates to two-axis ellipsoid (rotational) or a sphere as speroidal or spherical points. If Q happens to coincide with the center of gravity then we talk about the central ellipsoid of inertia. It can be shown that the central ellipsoid of inertia is the largest of all - it has the largest possible volume.

In order to see what is the connection between the ellipsoid of inertia and the tensor of inertia let us consider the central ellipsoid of inertia so that both the ellipsoid and the tensor are related to the same point T. To begin with, we can show that the following equation is the equation of an ellipsoid:

$$\vec{\rho}\mathcal{Y}\vec{\rho} = 1$$

where $\vec{\rho} = \xi\vec{\varepsilon}_1 + \eta\vec{\varepsilon}_2 + \zeta\vec{\varepsilon}_3$ is a position-vector. We have

$$\mathcal{Y}\vec{\rho} = (A\xi - D\eta - E\zeta)\vec{\varepsilon}_1 + (-D\xi + B\eta - F\zeta)\vec{\varepsilon}_2 +$$
$$+ (-E\xi - F\eta + C\zeta)\vec{\varepsilon}_3$$

and multiplying this vector by $\vec{\rho}$ again we get:

$$\vec{\rho}\mathcal{J}\vec{\rho} = A\xi^2 - D\xi\eta - E\xi\zeta - D\eta\xi + B\eta^2 - F\eta\zeta - E\zeta\xi - F\zeta\eta + C\zeta^2$$

$$= A\xi^2 + B\eta^2 + C\zeta^2 - 2D\xi\eta - 2E\xi\zeta - 2F\eta\zeta = 1.$$

Hence the central tensor of inertia $\mathcal{J}$ describes an ellipsoid in the way shown above. Similar property holds true for the tensor of inertia at any other point.

1.10) Natural system of coordinates linked with a physical body, principal moments of inertia

It is known from analytical geometry that if the axes of the ellipsoid coincide with the axes of the Cartesian system used to describe it, its equation reads:

$$\left(\frac{X}{a}\right)^2 + \left(\frac{Y}{b}\right)^2 + \left(\frac{Z}{c}\right)^2 = 1 .$$

Let us then take our tensor of inertia $\mathcal{J}$, find its eigenvalues λ_1, λ_2, λ_3 from the well-known algebraic equation of third order

$$\det\ (\mathcal{J} - \lambda\mathcal{E}) = 0$$

where $\mathcal{E}$ is the identity matrix and λ the variable. Then we can evaluate the eigen-vectors $\vec{\varepsilon}_1'$, $\vec{\varepsilon}_2'$, $\vec{\varepsilon}_3'$ of $\mathcal{J}$ from the known equations

$$\mathcal{J}\vec{\varepsilon}_i' = \lambda_i\vec{\varepsilon}_i' \qquad i = 1,2,3$$

and take these -- they will create an orthogonal, positive vector basis in S'-- as new Cartesian axes. The new system can be denoted by $S'' \equiv (T;\xi',\eta',\zeta')$. The tensor $\mathcal{J}$ in this new coordinate system S'' will look thus:

$$\mathcal{J} = \begin{Vmatrix} \lambda_1, & 0, & 0 \\ 0, & \lambda_2, & 0 \\ 0, & 0, & \lambda_3 \end{Vmatrix} = \begin{Vmatrix} A', & 0, & 0 \\ 0, & B', & 0 \\ 0, & 0, & C' \end{Vmatrix} .$$

It is not difficult to see that the equation of the central ellipsoid of inertia in S'' will acquire the following form:

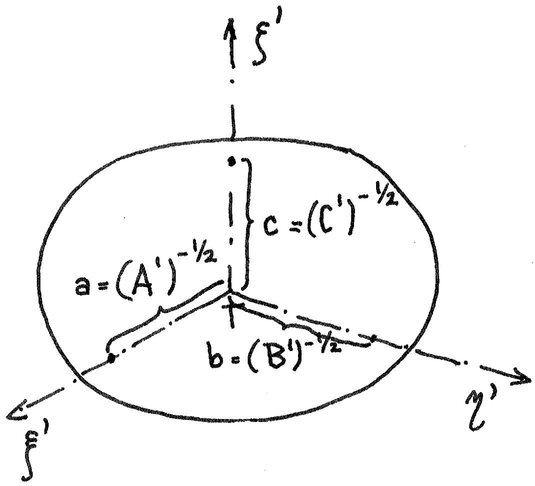
$$\vec{\rho}\boldsymbol{\mathcal{Y}}\vec{\rho} = \begin{Vmatrix} \xi' \\ \eta' \\ \zeta' \end{Vmatrix} \begin{Vmatrix} A', & 0, & 0 \\ 0, & B', & 0 \\ 0, & 0, & C' \end{Vmatrix} \begin{Vmatrix} \xi' \\ \eta' \\ \zeta' \end{Vmatrix}$$

$$= \begin{Vmatrix} \xi' \\ \eta' \\ \zeta' \end{Vmatrix} \begin{Vmatrix} A'\xi', & B'\eta', & C'\zeta' \end{Vmatrix} = A'\xi'^2 + B'\eta'^2 + C'\zeta'^2 = 1$$

This can be interpreted as follows--the eigenvectors of the tensor of inertia coincide with the three axes of the ellipsoid it describes. In addition, the diagonal elements A',B',C' of the diagonal form of the tensor of inertia, known in mechanics as the principal moments of inertia, are equal to the squares of the reciprocal values of the individual axes of the ellispoid. The principal moments of inertia have the property that one of them is the largest and one is the smallest of all possible moments of inertia of the point. Hence the directions with respect to which they are taken (the eigenvectors or the principal axes of inertia) have to coincide with the geometric axes of the ellipsoid of inertia. Since not only the directions of the axes but also their lengths are the same we can finally conclude that the ellipsoid

$$\vec{\rho}\boldsymbol{\mathcal{Y}}\vec{\rho} = 1$$

is the ellipsoid of inertia. In our particular case we shall be, of course, speaking about the principal central moments of inertia and central ellipsoid of inertia. We can also note, that the principal moments of inertia describe the inertial properties of B (with respect to the point where the tensor of inertia is evaluated) uniquely. The products of



inertia can then be regarded as virtual only and depending on the choice of S'. On the other hand, since we have to know 6 quantities to determine an ellipsoid uniquely we have to know either the 3 directions of the axes and their magnitudes or all 6 elements of the tensor of inertia.

The system S'' is known as the natural system of coordinates linked with B. It can be considered as natural because it is uniquely defined by the inertial properties of B alone and does not need the identification of a three points otherwise necessary to define a Cartesian system within B. We may note that it would also be natural to use this system for describing a non-rigid body.

1.11) Simplification of Euler's equation

Going now back to the Euler's equation (§ 1.18) we can see that there is nothing to stop us in formulating them in the natural system of coordinates. We recall, that the axes ξ, η, ζ in S' were oriented arbitrarily (see § 1.1). Hence we can specify their orientation now and understand, from now on, that S' is the natural system of coordinates.

Then we can write the Euler's equation as follows:

$$\left\|N_1, N_2, N_3\right\| = \left\|\begin{matrix} A,0,0 \\ 0,B,0 \\ 0,0,C \end{matrix}\right\| \left\|\begin{matrix} \overset{*}{\omega}_1 \\ \overset{*}{\omega}_2 \\ \overset{*}{\omega}_3 \end{matrix}\right\| + \left\|\begin{matrix} 0, & -\omega_3, & \omega_2 \\ \omega_3, & 0, & -\omega_1 \\ -\omega_2, & \omega_1, & 0 \end{matrix}\right\| \left(\left\|\begin{matrix} A,0,0 \\ 0,B,0 \\ 0,0,C \end{matrix}\right\| \left\|\begin{matrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{matrix}\right\| \right)$$

$$= \left\|A\overset{*}{\omega}_1, B\overset{*}{\omega}_2, C\overset{*}{\omega}_3\right\| + \left\|\begin{matrix} 0, & -\omega_3, & \omega_2 \\ \omega_3, & 0, & -\omega_1 \\ -\omega_2, & \omega_1, & 0 \end{matrix}\right\| \left\|\begin{matrix} A\omega_1 \\ B\omega_2 \\ C\omega_3 \end{matrix}\right\|$$

$$= \left\|A\overset{*}{\omega}_1, B\overset{*}{\omega}_2, C\overset{*}{\omega}_3\right\| + \left\|(C-B)\omega_2\omega_3, (A-C)\omega_1\omega_3, (B-A)\omega_1\omega_2\right\| .$$

This vector differential equation is usually written as a system of three scalar differential equations of first order for the components of the vector of rotation. They are known as the Euler's differential equations for the rotation of a rigid body and read

$$A \frac{d\omega_1}{dt} + (C - B)\ \omega_2\omega_3 = N_1$$

$$B \frac{d\omega_2}{dt} + (A - C)\ \omega_3\omega_1 = N_2$$

$$C \frac{d\omega_3}{dt} + (B - A)\ \omega_1\omega_2 = N_3 \ .$$

To recapitulate, we may say that the Euler's equations describe the rotation of a rigid body around an instantaneous axis of rotation going through its center of gravity as caused by the moment N of external Newtonian forces (with respect to T) in the natural coordinate system. We can note that each equation can be derived from another by cyclic exchange of the individual quantities. They are, needless to say, completely equivalent to the general form derived in § 1.8. They are sometimes referred to as the equations of a gyroscope. The system of Euler's equations is solvable for only a narrow family of special cases.

2) The earth-pole wobble

2.1) The rigid earth as a gyroscope

We can now apply the theory to the idealized earth. The earth may be, in the first approximation considered a huge rigid gyro spinning around its instantaneous axis of rotation, going through its center of gravity, with a period of one sidereal day. This is, of course, not the only motion the earth undergoes but only one of a whole series of motions (rotation around the sun, rotation around the center of our gallaxy, etc.). The daily spin can be studied from the point of view of the space around the earth. Studies of this kind reveal the precession and nutation of the earth axis of rotation. These motions are not, however, the object of our interest at this moment.

The purpose of our study is the axis of rotation (spin) as viewed from the point of view of the earth. We again recall that the vector $\vec{\omega}$ and its time derivatives in the Euler equation are taken with respect to the natural system of coordinates related to the body -- the earth in our case. Hence, the solution of the Euler's equation provides us directly with the vector of rotation as viewed from the earth.

In the first approximation, the earth can be regarded as a force-free gyro, i.e. as if no external forces exert any moments (with respect to its center of gravity) on it. Moreover, in the first approximation, two of its moments of inertia can be regarded equal since the earth is approximately rotationally symmetrical around its axis of rotation. Let us then call the semi-minor axis of the central ellipsoid of inertia (ellipsoid of rotation) by ζ and the other two axes, the orientation of which we do not have to specify, by ξ and η. The Euler's equations then can be written as

$$A\,\frac{d\omega_\xi}{dt} + (C - A)\,\omega_\eta\omega_\zeta = 0$$

$$A\,\frac{d\omega_\eta}{dt} + (A - C)\,\omega_\zeta\omega_\xi = 0$$

$$C\,\frac{d\omega_\zeta}{dt} = 0\ .$$

2.2) Solution to the Euler's equations for rigid earth, Euler's period

The Euler's equations describing the approximate rotation of the earth (as viewed from the earth) can now be solved. Solving the third equation first, we get

$$\omega_\zeta = \text{const.} = \mu\ .$$

Further, denoting the ratio $(C-A)/A$ by h we can rewrite the first two equations as

$$\dot{\omega}_\xi + h\mu\omega_\eta = 0$$
$$\dot{\omega}_\eta - h\mu\omega_\xi = 0\ .$$

To solve this system of linear differential equations of first order we transform it into two linear differential equations of second order

$$\ddot{\omega}_\xi + h^2\mu^2\omega_\xi = 0$$
$$\ddot{\omega}_\eta + h^2\eta^2\omega_\eta = 0$$

by differentiating the first (second) equation and substituting for $\dot{\omega}_\eta$ $(\dot{\omega}_\xi)$ into the second (first) equation. Note that we use the dot to describe the relative time derivative since $\dot{\vec{\omega}} = \overset{*}{\vec{\omega}}$, as we have seen in § 1.3. The second derivative should be understood relative as well.

Evidently, both equations are equations of simple harmonic motions. Solving the first equation, we get

$$\omega_\xi = \kappa \cos (h\mu t + \psi),$$

where κ and ψ are some integration constants. Substituting this result back into the first equation of first order above, one obtains

$$-\kappa \sin (h\mu t + \psi) \cdot h\mu + h\mu\omega\eta = 0 .$$

Hence

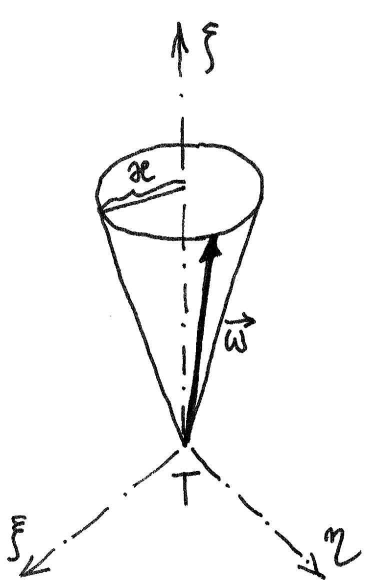
$$\omega_\eta = \kappa \sin (h\mu t + \psi).$$

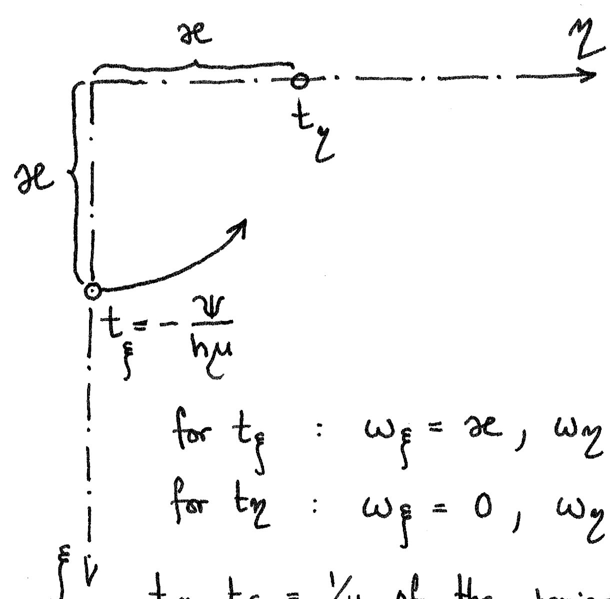
Needless to say that identical solutions ω_ξ, ω_η are obtained when we solve the equation for ω_η first.

We may note, to begin with, that the instantaneous angular velocity $|\vec{\omega}|$ of a rigid earth should be constant:

$$\omega^2 = \omega_\xi^2 + \omega_\eta^2 + \omega_\zeta^2 = \kappa^2 + \mu^2 = \text{const.}$$

Besides, we can see that the instantaneous axis of rotation $\vec{\omega}$ travels around the principal axis of inertia ζ in a circular cone. Both components $(\omega_\xi, \omega_\eta)$ have the same amplitude and a phase-lag of 90°. In addition, it can be seen that the motion is anticlockwise when viewed from the North:





for t_ξ : $\omega_\xi = \kappa$, $\omega_\eta = 0$;

for t_η : $\omega_\xi = 0$, $\omega_\eta = \kappa$

$t_\eta - t_\xi$ = 1/4 of the period

Finally, we can determine the period

$$P_E = 2\pi/(\mu h)$$

of the wobble, Taking $\mu = \omega_\zeta \approx 2\pi/(1$ sidereal day) -- based on the consideration that the overwhelming part of the actual angular velocity of the earth can be taken with respect to ζ (κ in angular units is very small indeed) -- and $h \simeq 1/305$ [Melchior, 1966] as determined from the precession and nutation of the earth, we get

$$P_E \sim 303 \text{ solar days.}$$

This period was first derived by Euler and under his name it is still known. It is the period of the earth-pole wobble (free nutation, free motion) of the rigid earth.

The value of κ can be determined from experimental data, ψ depends on the choice of time-origin.

2.3) Non-rigid earth, Chandler's period

It was established by Chandler [1891] that the fact that the actual period of the pole-wobble is longer than predicted by some 40%. The explanation for it was given by Newcombe [1892] as the non-rigidity of the earth. The value of the increase factor is given by [Tomaschek, 1957]

$$1/(1 - 1.07\ k)$$

where k is a function describing the ratio of the additional potential produced by a deformation to the potential of the deforming force. This function was first introduced by Love [1909] and became known as second Love's number. There is also the first Love's number h which we are not dealing with here. The theoretical development behind the above formula is somewhat involved and is left untouched in this treatise. Let us just mention that the value of k depends on the frequency of the deforming force--the earth responds elastically to short periodic stresses and plastically to long periodic stresses. The two Love's numbers together with two other functions describe fully the elastic properties of the earth.

There are basically two types of observations allowing us to determine the value of k. One is the earth-tides observations, second is the polar-wobble. Analyses of both indicate approximately the same value for k, namely 0.28 - 0.29 [Munk and Macdonald, 1960; Melchior, 1966; Jeffreys, 1970]. The corresponding period

$$P_C = P_E/(1 - 1.07\ k)$$

known as <u>Chandler's</u>, is then somewhere between 433 to 439 solar days.

2.4) <u>Excitation and damping of the wobble</u>

We have seen in the case of a rigid earth with no external forces that the amplitude of the wobble, κ, should remain constant. The fact that the earth is not rigid as well as the presence of external forces caused by the attraction of the celestial bodies should theoretically lead to the damping of the amplitude. It is a well-known principle in dynamics that wherever the energy of the dynamical system is dissipated the consequence is damping of the motion of the system. Here we can identify two sources of dissipation--tidal friction and internal friction. Once again, the mathematical tools in proving the above are too complicated to allow us to prove it here theoretically. Let us just state that the quantitative estimates of the parameters involved are so far very imprecise and not convincing [Jeffreys, 1970].

In spite of this theoretical prediction, the actually observed amount of wobble (amplitude) does not seem to decrease significantly over an extended period of time. The most sensible explanation for this disagreement is that besides damping, there exists another mechanism that excites the wobble. So far none of the existing theories (hypotheses) explains the excitation mechanism satisfactorily. The largest amount of credibility can be probably associated with the hypothesis that the earth pole wobble is somehow linked with the major tectonic earthquakes [Mansinha and Smylie, 1967].

2.5) Observations of the actual wobble

At the end of 19th century, the IAU decided to set up an international cooperative program--International Latitude Service (ILS)--to observe the actual wobble, determine its period and amplitude and thus add some valuable information to our knowledge of the earth. The observations began in 1899 simultaneously at 5 stations located on the same parallel $\phi = \pm 39° 08'$ (Mizusawa-Japan, Kitab-USSR, Carloforte-Italy, Gaithersburg-USA, Ukiah-USA). The network of the "latitude stations" has grown since to over 40 stations today distributed not only on the Northern but also Southern hemispheres, under the auspicies of two agencies--the ILS (also called IPMS--International Polar Motion Service) and BIH (Bureau International de l'Heure).

The latitude stations are continuously (more precisely--as often as they can) determining their instantaneous astronomic latitudes using a common set of stars and either PZT (Photographic Zenith Telescope) or Danjon's astrolabium for an instrument. The results of the observations are then sent to the respective international bodies to either Mizusawa or Paris. There the variations of latitude are adjusted and interpreted in terms of Cartesian coordinates X,Y describing the hodograph of the vector $\vec{\omega}$ in a plane tangent to the earth ellipsoid at the mean-pole, see e.g. [Krakiwsky and Wells, 1971]. These "coordinates of the instantaneous pole" are published periodically in two different modes i) predicted positions, extrapolated from the observed data; ii) actual positions, computed from the actual observed values.

Unfortunately, the adjustment procedure has changed several times during the existence of the international services [Munk and Macdonald, 1960]. Therefore, the published actual positions cannot be regarded homogeneous for the whole period of 72 years.

2.6) Results of analyses of the observed wobble

In order to determine the period P_C and the amplitude κ (and the phase-lag ψ) the data describing the actual positions of the pole have been analyses by scores of various scholars. The period can be determined from the data using one of the many methods for spectral analysis (technique designed to determine an unknown frequency or period of a given time-series). The numerical values for the Chandler's period vary with different authors from 420 to 440 days [Munk and Macdonald, 1960].

The mean amplitude for a certain span of data can then be determined using the least-squares approximation seeking the best-fitting periodic curves (for both constituents X and Y) with period P_C. The results indicate an average value of κ of the order of 0.2". This angular value corresponds to a displacement of the instantaneous pole of rotation on the surface of the earth of about $\pm$ 6.5 m.

The results invariably confirm the anti-clockwise polarity of the motion--as predicted by theory, see § 2.2. The sense of the motion can be determined from the phase-lags of the two constituents X,Y. Contrary to predictions, the hodograph is not completely circular, or put in other words, the circular "Chandlerian motion" (as the free nutation is sometimes known as) does not account fully for the actual "motion".

There are other components of the actual hodograph that need some attention. Three more motions can be distinguished in the actual data:

i) seasonal variation;

ii) secular variation;

iii) irregular fluctuations.

We shall now deal with these individually.

2.7) Seasonal variations

On the spectra of the constituents X,Y of the actual wobble, one can clearly see an annual peak indicating the presence of an annual motion. According to Orlov [1961], the magnitude of the annual component also varies with time within 40 to 120 msec of arc in both directions X and Y. The annual motion is then elliptical rather than circular and is again positive (anticlockwise).

The origin of this annual component is very probably linked with the annual atmospheric changes. However, the mechanism of how the changes influence the wobble is as yet unclear. The author's own hypothesis, based on somewhat limited experiments [Vaníček, 1971], is that the annual motion (or at least its large portion) is only virtual. In other words, it is not a real motion of the pole but reflects the annual variation of the local verticals of the observing station. The variation of the vertical is then inevitably interpreted as the variation of the local latitude and thus transmitted to the international center where they are reinterpreted in terms of polar wobble. The local verticals change irregularly from station to station--this is why they may be interpreted as part of the wobble; in case the changes were the same for all the stations, they would cancel out and would not be interpretable globally--due to the variations of the local equipotential surface. This, in turn, can be attributed to the variations of ground water and snow distribution as well as the other crustal tilts.

2.8) Long-term variations

Presence of long-term variations in the polar wobble has been long suspected, again on the basis of the spectral analyses testimony. For a long time though, the span of available observation data was thought to be too short to allow any quantitative estimate of any such variations.

Recently, Markowitz [1968] discovered a 24-year period using 60 years of data and a specially designed technique dealing with the 5 principal latitude stations. His discovery was given a confirmation by the author's finding [Vaníček, 1969], based on the results of spectral analyses of the BIH data. The theoretical explanation for this long periodic component is being sought in the mantle-core coupling [Busse, 1969]. For the final word we shall probably have to wait for some time.

Into the same category of motions falls the drift of the pole. The drift is thought to be (at least partly) due to the crustal displacement such as the continental drift, isostatic movements, etc. It is presently assumed to be of the order of 3.2 msec of arc per year [Markowitz, 1968] corresponding thus to the displacement of the pole on the earth surface by some 0.1 m per year. This motion, though has really nothing to do with the vector of rotation. It reflects the motion of the principal axis of inertia with respect to the observing stations. To be more precise, it reflects the motions of the observing stations with respect to the principal axis of inertia that has to be considered the only "fixed" axis of the earth--see § 1.10.

2.9) Irregular fluctuations

Besides the described, more or less, regular motions, we experience some irregular fluctuations in the actual data. Some of them can be attributed to certain combinations of random errors in the individual observations, some of them are probably due to as yet unknown global influences. Nothing definite can be said about their fluctuations yet.

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