GEODETIC POSITION COMPUTATIONS

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LECTURE NOTES
NO. 39

PREFACE

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GEODETIC POSITION COMPUTATIONS

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PREFACE

The purpose of these notes is to give the theory and use of some methods of computing the geodetic positions of points on a reference ellipsoid and on the terrain. Justification for the first three sections of these lecture notes, which are concerned with the classical problem of "computation of geodetic positions on the surface of an ellipsoid" is not easy to come by. It can only be stated that the attempt has been to produce a self contained package, containing the complete development of some representative methods that exist in the literature. The last section is an introduction to three dimensional computation methods, and is offered as an alternative to the classical approach. Several problems, and their respective solutions, are presented.

The approach taken herein is to perform complete derivations, thus staying away from the practice of giving a list of formulae to use in the solution of a problem. It is hoped that this approach will give the reader an appreciation for the foundation upon which the formulae are based, and in the end, the formulae themselves.

The notes evolved out of lecture notes prepared by E.J. Krakiwsky and from research work performed by D.B. Thomson over recent years at U.N.B. The authors acknowledge the use of ideas, contained in the lecture notes, of Professors Urho A. Uotila and Richard H. Rapp of the Department of Geodetic Science, The Ohio State University, Columbus, Ohio. Other sources used for important details are referenced within the text.

The authors wish to acknowledge the contribution made by the Surveying Engineering undergraduate class of 1975 to improving these notes by finding typographical errors. Mr. C. Chamberlain is particularly acknowledged for his constructive criticism, and assistance in preparing the manuscript for publication.

E.J. Krakiwsky

D.B. Thomson

February 14, 1974

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INTRODUCTION

The first three sections of these notes deal with the computation of geodetic positions on an ellipsoid. In chapter one, a review of ellipsoidal geometry is given in order that the development of further formulae can be understood fully. Common to all of the classical ellipsoidal computations is the necessity to reduce geodetic observations onto the ellipsoid, thus an entire chapter is devoted to this topic.

Two classical geometric geodetic computation problems are treated; they are called the <u>direct</u> and <u>inverse</u> geodetic problems. There are various approaches that can be adopted for solving these problems. Generally, they are classified in terms of "short", "medium", and "long" line formulae. Each of them involve different approximations which tend to restrict the interstation distance over which some formulae are useful for a given accuracy.

The last section of the notes deals with the computation of geodetic positions in three dimensions. First, the direct and inverse problems are developed, then two special problems -- those of azimuth and spatial distance intersections -- are dealt with. These solutions offer an alternative to the classical approach of geodetic position computations.

SECTION I: ELLIPSOIDAL GEOMETRY

1. The Ellipsoid of Rotation

Since an ellipsoid of rotation (reference ellipsoid) is generally considered as the best approximation to the size and shape of the earth, it is used as the surface upon which to perform terrestrial geodetic computations. Immediately below we study several geometric properties of an ellipsoid of rotation that are of special interest to geodesists. In particular, the radii of curvature of points on the suface of the ellipsoid, and some curves on that surface, are described.

1.1 Ellipsoidal Parameters

Figure 1 shows an ellipsoid of rotation. The parameters of a reference ellipsoid, which describe its size and shape, are:

i) the semi-major axis, a,

ii) the semi-minor axis, b.

The equation of any meridian curve (intersection of a meridian plane with the ellipsoid surface,(Figure 1), is

$$\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1 . \qquad (1)$$

The surface of an ellipsoid of rotation is given by

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1 . \qquad (1a).$$

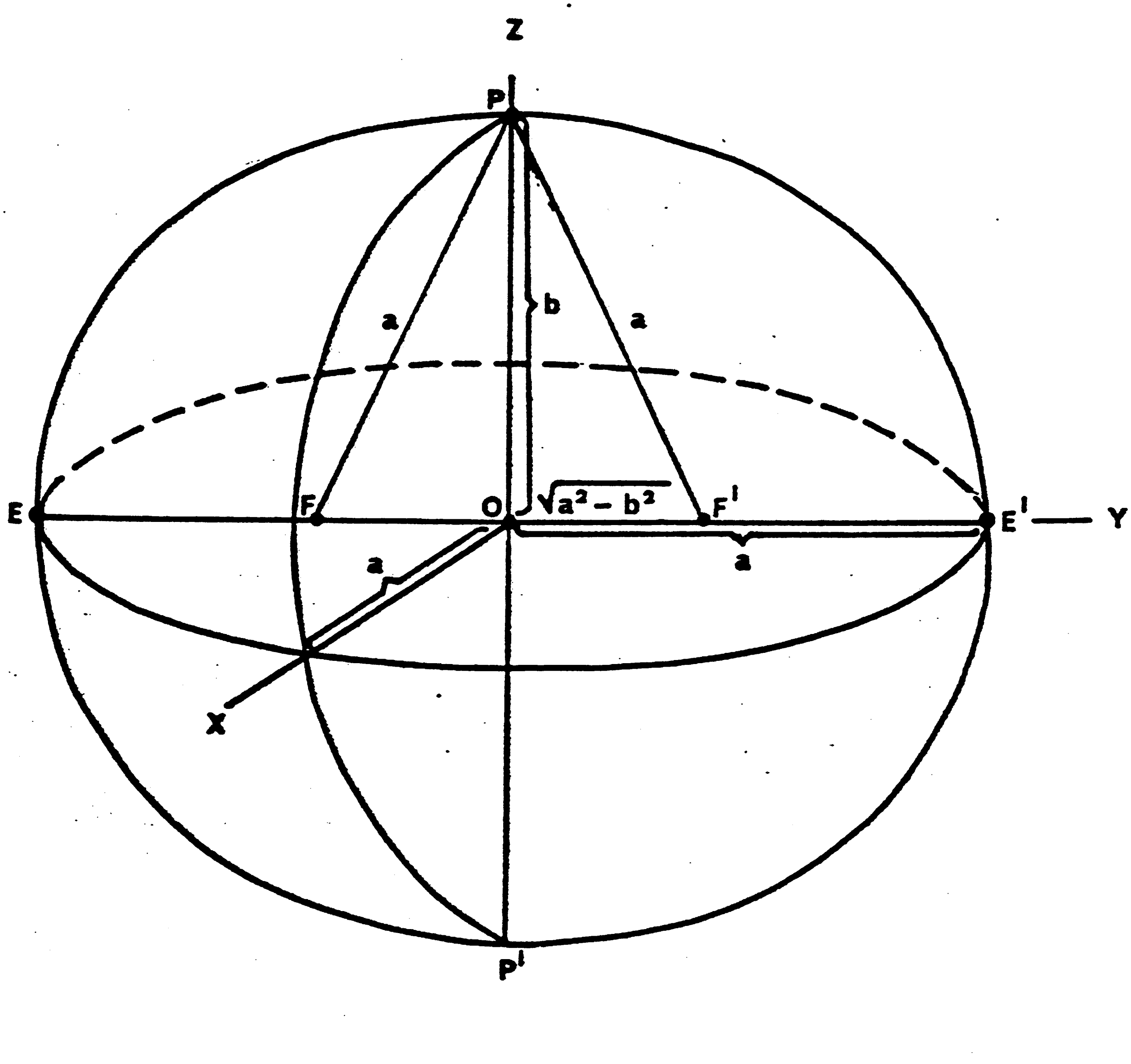


Figure 1

THE ELLIPSOID OF ROTATION

The points F and F' in Figure 1 are the focii of the meridian ellipse through points P, E', P', E. The focii are equidistant from the geometric centre (o) of the ellipse. The distances PF and PF' are equal to the semi-major axis a. This information is now used to help describe further properties of an ellipsoid.

The ellipsoidal (polar) flattening is given by

$$f = \frac{a-b}{a} . \tag{2}$$

Two other important properties, which are described for a meridian section of the ellipsoid are the first eccentricity

$$e^2 = \frac{a^2-b^2}{a^2} , \tag{3}$$

and the second eccentricity

$$e'^2 = \frac{a^2-b^2}{b^2} . \tag{4}$$

As an example of the magnitudes of these parameters for a geodetic reference ellipsoid, we present here the values for the Clarke 1866 ellipsoid, which is presently used for most North American geodetic position computations [Bomford, 1971, p 450]:

$$a = 6378206.4 \text{ m},$$

$$b = 6356583.8 \text{ m}.$$

Using (2),

$$f = 0.00339007 \ldots$$

which is often given in the form 1/f, which in this case is

$$1/f = 294.97869\ldots$$

Using (3) and (4) respectively, we get

$$e^2 = 0.00676865\ldots ,$$

$$e'^2 = 0.00681478\ldots$$

The four parameters a, b, e (or e') and f, and the relationships among them, are the principal ones used to develop further geodetic formulae.

1.2 Radii of Curvature

On the surface of an ellipsoid, an infinite number of planes can be drawn through a point on the surface which contains the normal at this point. These planes are known as normal planes. The curves of intersection of the normal planes and the surface of the ellipsoid are called normal sections. At each point, there are two mutually perpendicular normal sections whose curvatures are maximum and minimum, which are called the principal normal sections. These principal sections are the meridian and prime vertical normal sections, and their radii of curvature are denoted by M and N respectively (Figures 2 and 3). In Figure 2, it can be seen that the meridian radius of curvature increases from the equator to the pole, and the prime vertical radius of curvature behaves similarly (Figure 3). The reasons for this will be seen shortly onee the formulae for M and N have been developed.

1.2.1 Meridian Radius of Curvature

Consider a meridian section of an ellipsoid of rotation (Figure 4) given by

$$\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1. \tag{1}$$

The radius of curvature of this curve, at any point P, is given by

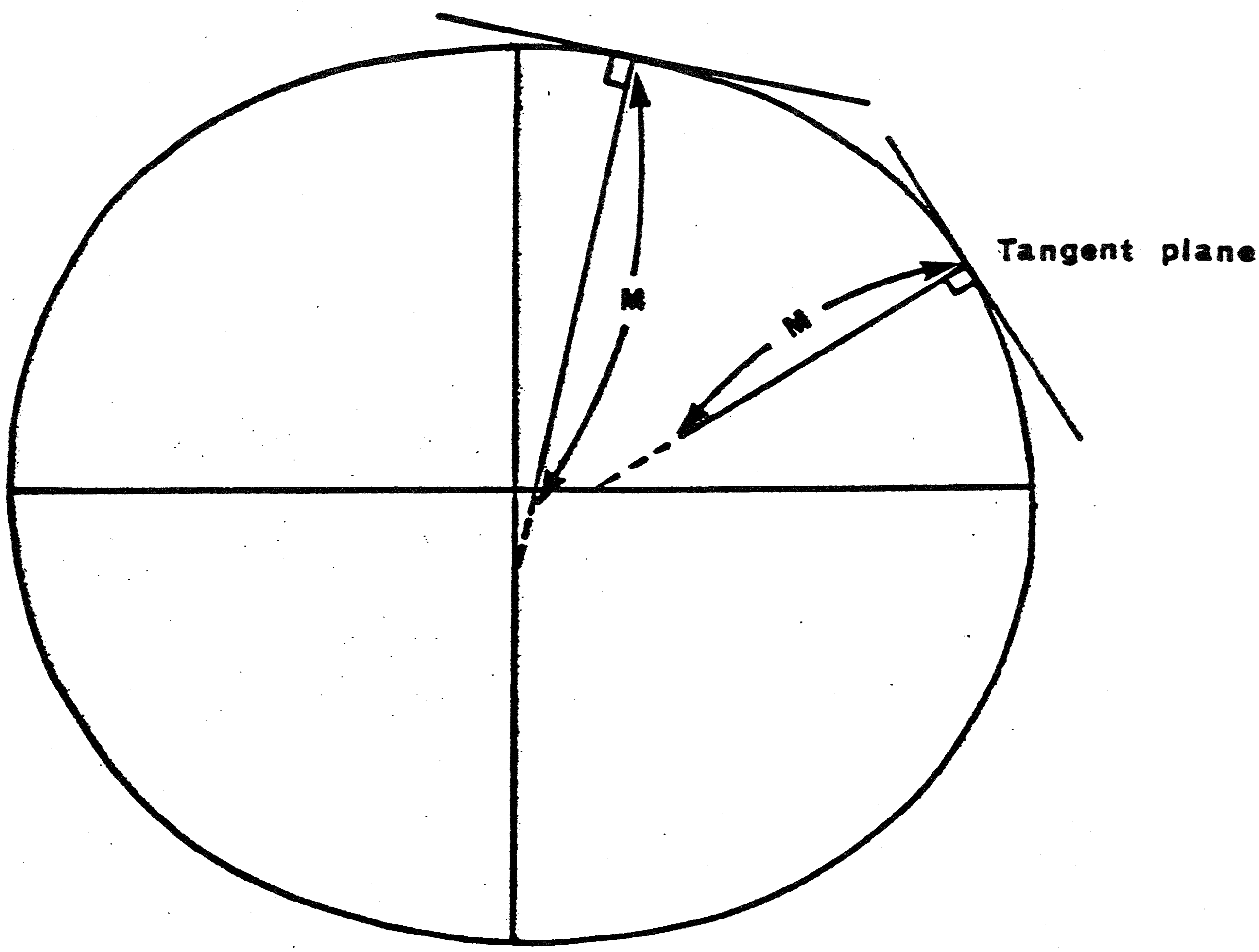


Figure 2

MERIDIAN NORMAL SECTION SHOWING THE MERIDIAN RADIUS OF CURVATURE (M)

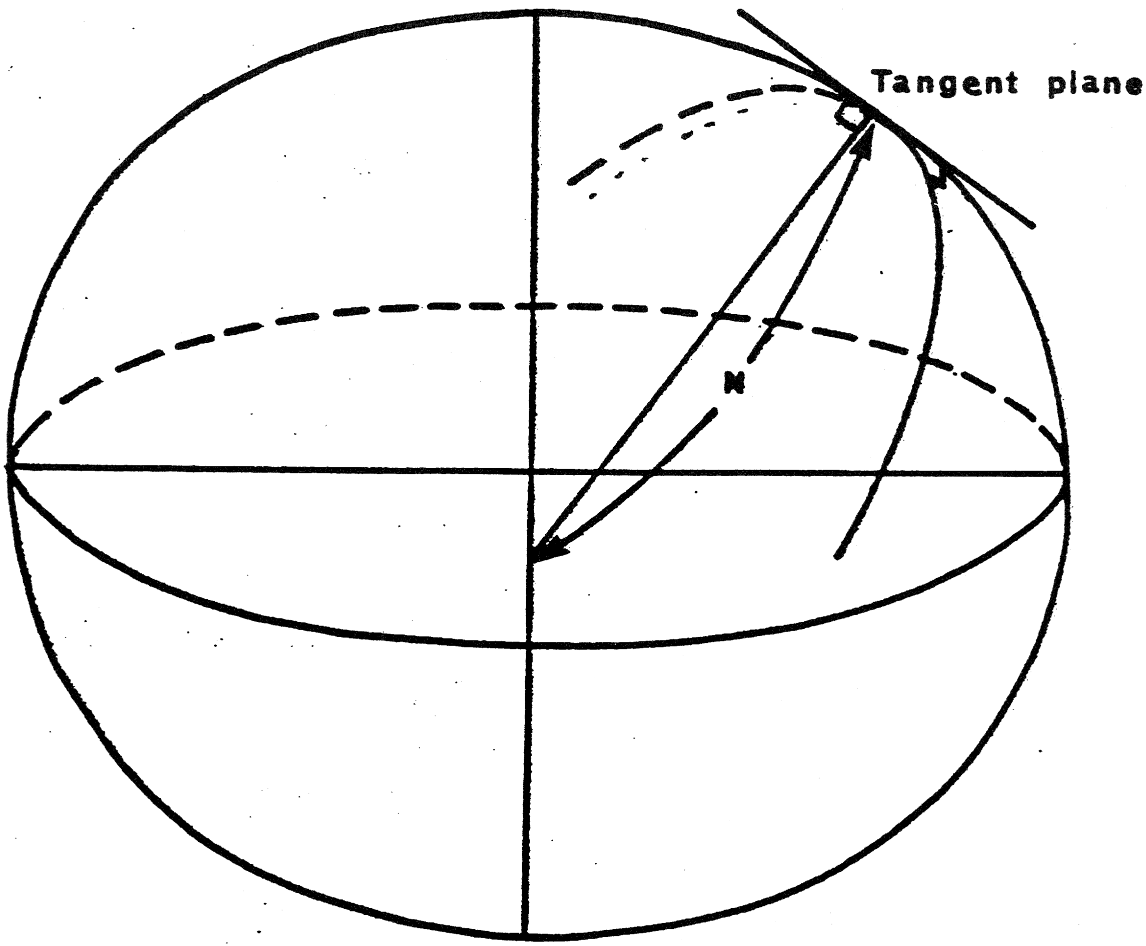


Figure 3

PRIME VERTICAL NORMAL SECTION SHOWING THE PRIME VERTICAL RADIUS OF CURVATURE (N)

[Philips,1957 , pp. 194-197]

$$M = \frac{\left(1 + \left(\frac{dz}{dx}\right)^2\right)^{3/2}}{\frac{d^2z}{dx^2}} \quad . \tag{5}$$

In the case of a meridian ellipse

$$\frac{dz}{dx} = - \frac{x}{z} \frac{b^2}{a^2} \quad , \tag{6}$$

and

$$\frac{d^2z}{dx^2} = - \frac{b^2}{a^2} \left(\frac{z - x \frac{dz}{dx}}{z^2}\right) \quad , \tag{7}$$

or

$$\frac{d^2z}{dx^2} = - \frac{b^2}{a^2z^2} \left(z + \frac{x^2}{z} \cdot \frac{b^2}{a^2}\right) \quad . \tag{7a}$$

From Figure 4, we can also see that the slope of the tangent to P is given by

$$\tan(90+\phi) = \frac{dz}{dx} = - \cot \phi \quad . \tag{8}$$

Equating (6) and (8) gives

$$-\cot \phi = - \frac{x}{z} \frac{b^2}{a^2} \tag{9}$$

or

$$\tan \phi = \frac{a^2}{b^2} \quad \frac{z}{x} \quad . \tag{9a}$$

Substituting

$$b = a(1-e^2)^{1/2} \tag{9b}$$

in (9a), yields

$$z = x(1-e^2) \tan \phi \quad . \tag{10}$$

Then, after replacing b and z in (1) with (9b) and (10) respectively, some simple manipulation results in

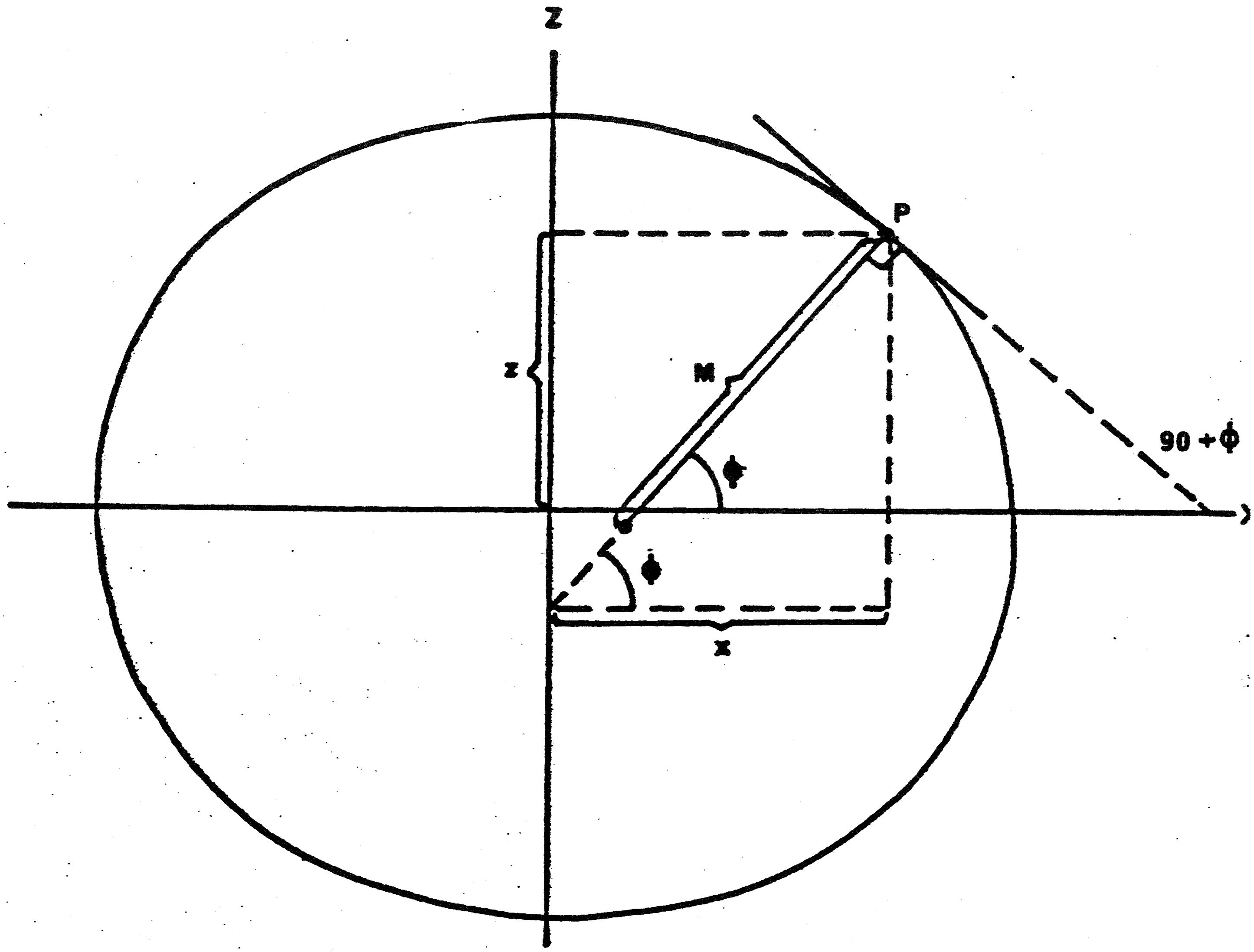


Figure 4

MERIDIAN RADIUS OF CURVATURE (M)

$$x = \frac{a \cos \phi}{(1-e^2 \sin^2 \phi)^{1/2}} \quad . \tag{11}$$

Substituting the above expression for x in equation (10) gives the formula

$$z = \frac{a(1-e^2)\sin \phi}{(1-e^2 \sin^2 \phi)^{1/2}} \quad . \tag{12}$$

Finally, replacing x and z in (6) and (7a), and placing these values in (5) for $\frac{dz}{dx}$ and $\frac{d^2z}{dx^2}$, the expression for the meridian radius of curvature becomes

$$\boxed{M = \frac{a(1-e^2)}{(1-e^2 \sin^2 \phi)^{3/2}} \quad .} \tag{13}$$

In equation (13), the only variable parameter is the geodetic latitude ϕ, thus at the equator ($\phi = 0°$),

$$M = a(1-e^2), \tag{13a}$$

and at the pole ($\phi = 90°$),

$$M = a/(1-e^2)^{1/2} \quad . \tag{13b}$$

The meridian radius of curvature increases in length as the point on the meridian moves from the equator to the pole.

1.2.2 Prime Vertical Radius of Curvature

From Figure 5,

$$\cos \phi = \frac{x}{N} , \tag{14}$$

or

$$N = \frac{x}{\cos \phi} \quad . \tag{14a}$$

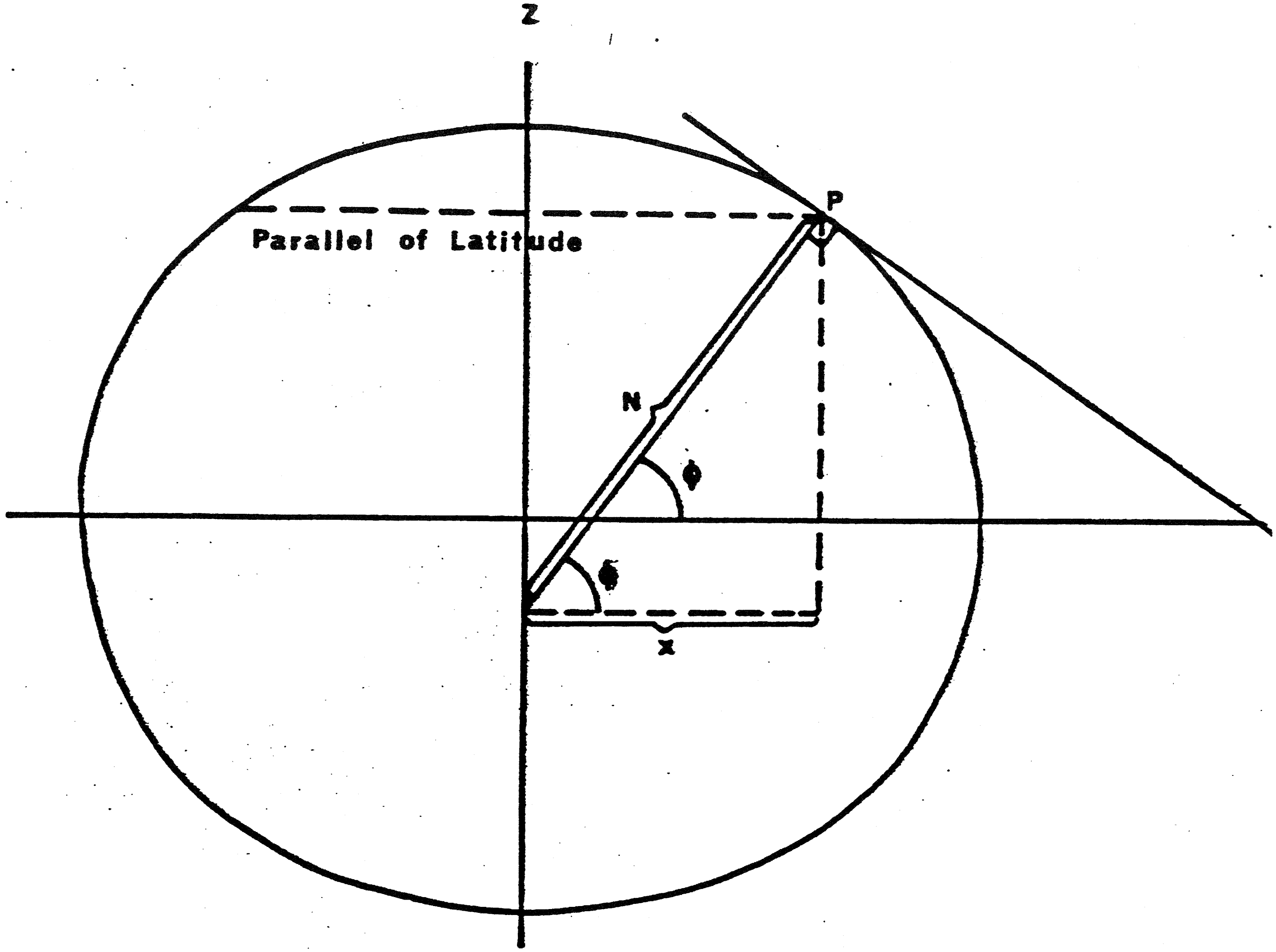


Figure 5

PRIME VERTICAL RADIUS OF CURVATURE (N)

Substituting the expression for x (11) in (14a) yields the expression for the radius of curvature in the prime vertical,

$$\boxed{N = \frac{a}{(1-e^2\sin^2\phi)^{1/2}} \ .} \tag{15}$$

Since the only variable parameter in (15) is ϕ, N then varies with ϕ. When $\phi = 0°$ (equator), N = a, and when $\phi = 90°$ (poles),

$$N = a/(1-e^2)^{1/2} = M \ . \tag{15a}$$

An important quantity that is used very often in geometric geodetic computations is the Gaussian Mean Radius of Curvature, which is given by

$$\boxed{R = \sqrt{MN} \ .} \tag{16}$$

In many instances, the mean radius is sufficiently accurate for position computations.

Another radius of curvature that may be needed from time to time is that of a parallel of latitude. Any parallel of latitude, viewed from the north pole of the ellipsoid (z axis), describes a circle. Its radius, as can be seen in Figure 5, is equal to the x-coordinate (in the meridian plane X-Z system). Then, from equation (14a), the radius of curvature of a parallel of latitude is given by

$$\boxed{R_\phi = N \cos \phi \ .} \tag{17}$$

It is easily seen that when $\phi = 0°$ (equator), $R_\phi = N$, thus $R_\phi = a$ (since N = a at $\phi = 0°$), and at either pole ($\phi = 90°$), $\cos \phi = 0$ and the radius disappears.

1.2.3 Radius of Curvature in Any Azimuth

As has been shown in Sections 1.2.1 and 1.2.2, the maximum and minimum radii of curvature of any point P on the surface of an ellipsoid of rotation lie in the meridian and prime vertical planes.
In some instances, geodetic computations require the radius of curvature in a plane other than the principal ones (Figure 6). The normal section in some azimuth α has a radius of curvature at any point P designated by R_α. It is solved for using Euler's Theorem [Lipschutz, 1969, pg. 196], and is called Euler's radius of curvature.

In Figure 6, the point P at which the radius R_α is required, is shown on the normal section PP'. Only a differential part of the normal section curve (ds) is shown, since the azimuth α of this small section is equivalent to the azimuth of a normal section of any length.

Euler's theorem is solved as follows. At the point P, we draw a tangent plane, and parallel to it, another plane (Figure 7) that intersects the surface of the ellipsoid. The latter plane, viewed along the normal through P, forms an ellipse in the plane BB' where the tangent plane intersects the ellipsoid surface. The elements of this "indicatrix" are shown in Figure 7. If we view this plane through the point P', in the azimuth α, the resulting section is Figure 8. Recall that the equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad . \tag{1}$$

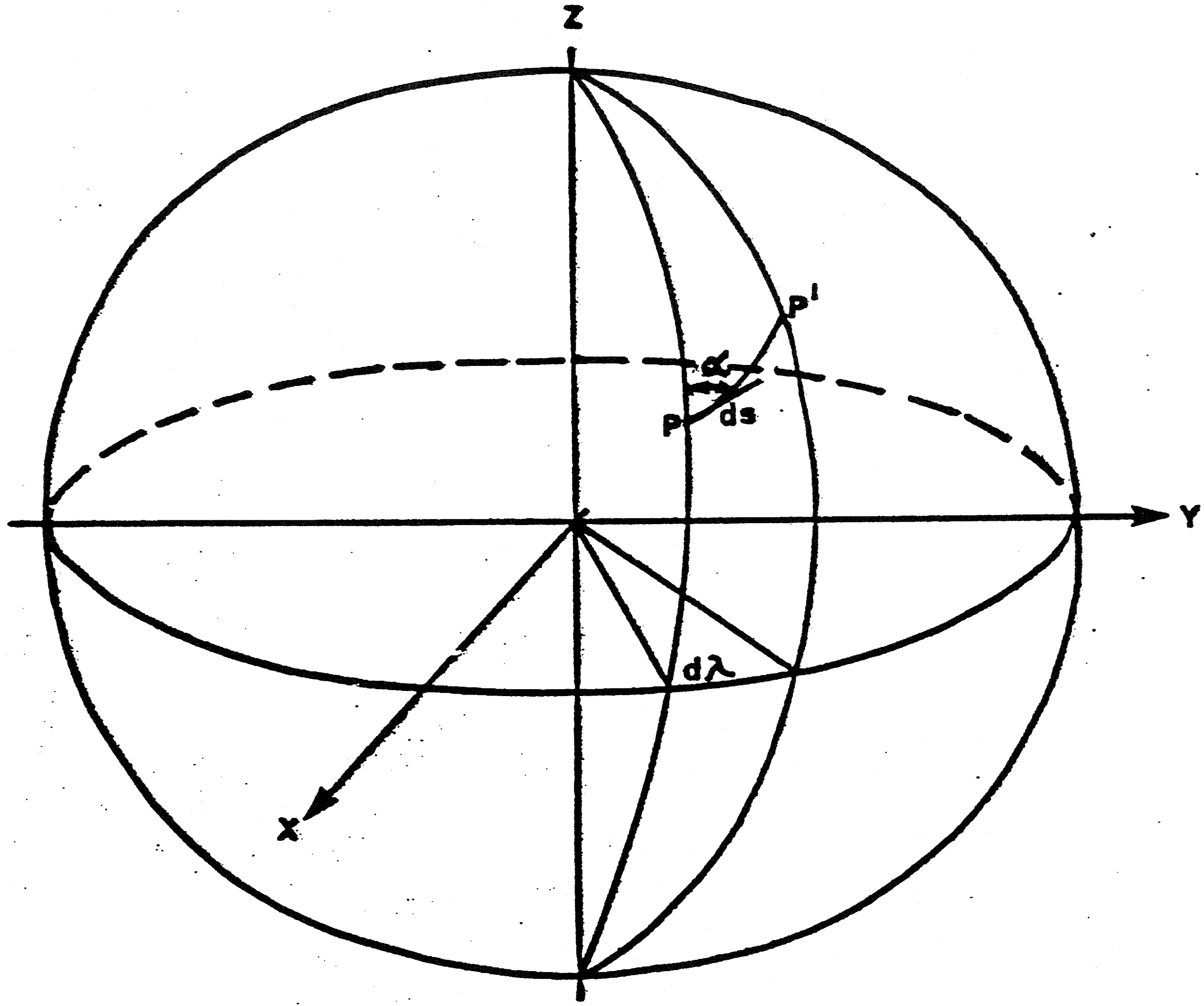


Figure 6

NORMAL SECTION AT ANY AZIMUTH α

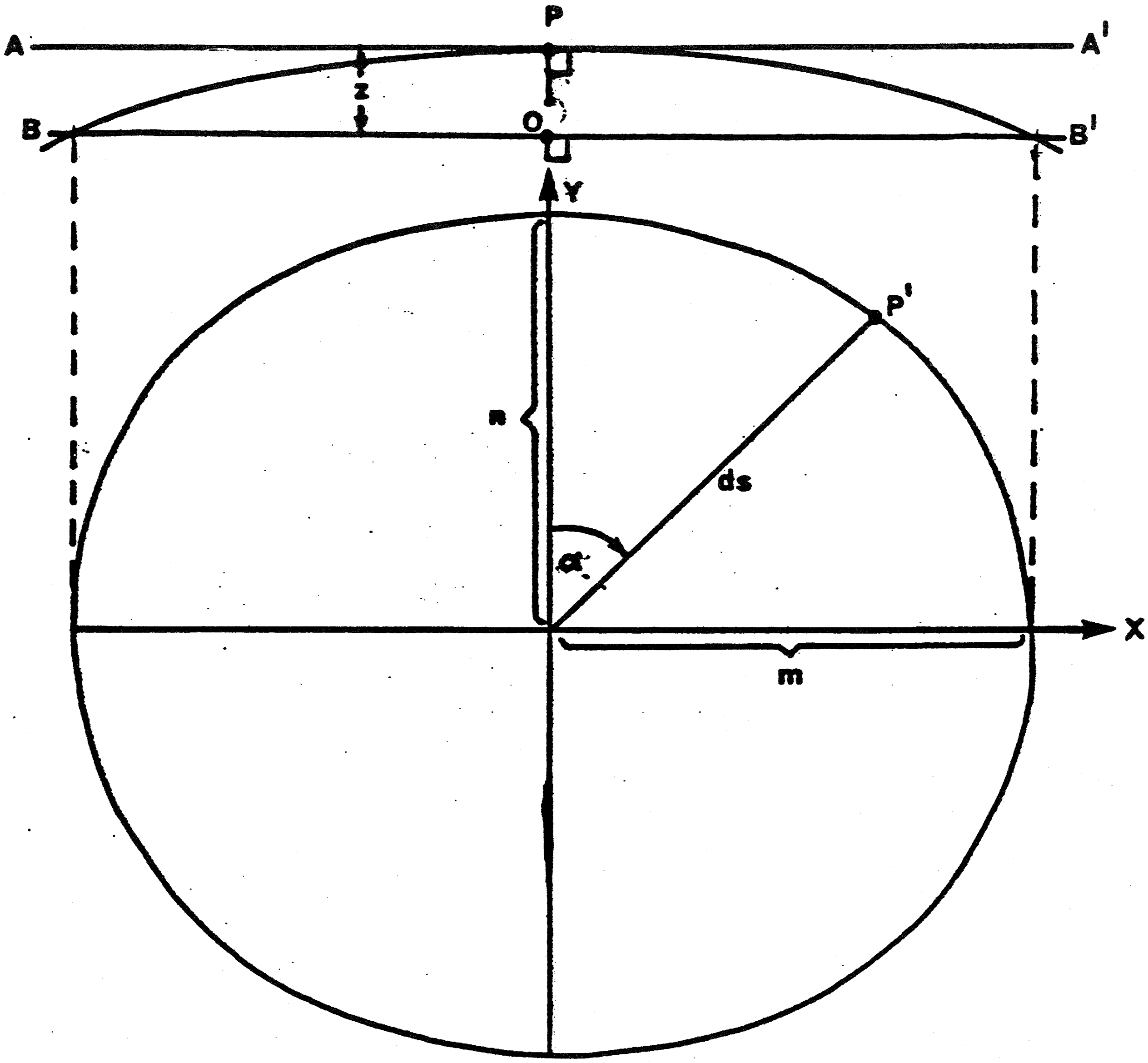


Figure 7

INDICATRIX FOR SOLUTION OF R_{α}

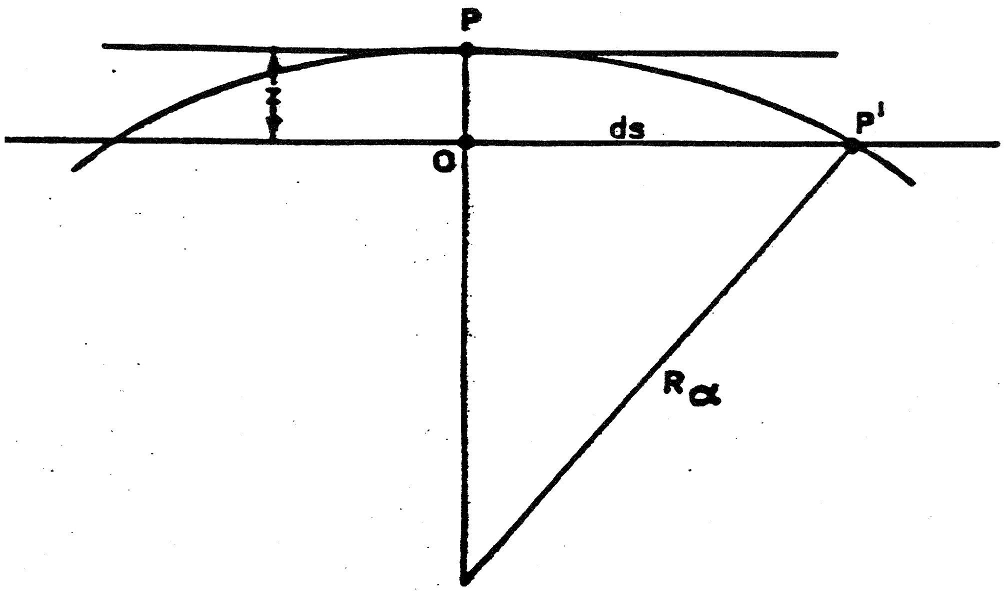


Figure 8

SECTION ALONG PP' (α) FOR SOLUTION OF R_{α}.

From Figure 7,

$$x = ds \sin \alpha$$
$$y = ds \cos \alpha \ , \qquad (17)$$

Then (1) becomes

$$\frac{ds^2 \sin^2\alpha}{m^2} + \frac{ds^2 \cos^2\alpha}{n^2} = 1. \qquad (18)$$

Using Figure 9, we can write

$$\sin \theta = \frac{z}{c} \ , \qquad (19)$$

and

$$\sin \theta = \frac{\frac{1}{2} c}{R_\alpha} \ , \qquad (20a)$$

which results in

$$z = \frac{c^2}{2R_\alpha} \ . \qquad (21)$$

Since PP' is a very small differential distance, then $C \simeq ds$, and we can write

$$z \simeq \frac{ds^2}{2R_\alpha} \ . \qquad (22)$$

When $\alpha = 0°$, s equals n and

$$z \simeq \frac{n^2}{2M} \ , \qquad (23)$$

and when $\alpha = 90°$, s equals m and

$$z \simeq \frac{m^2}{2N} \ . \qquad (24)$$

Combining (22) and (23), and (22) and (24) yields

$$n^2 = \frac{ds^2}{R_\alpha} M \qquad (25)$$

and

$$m^2 = \frac{ds^2}{R_\alpha} N \ . \qquad (26)$$

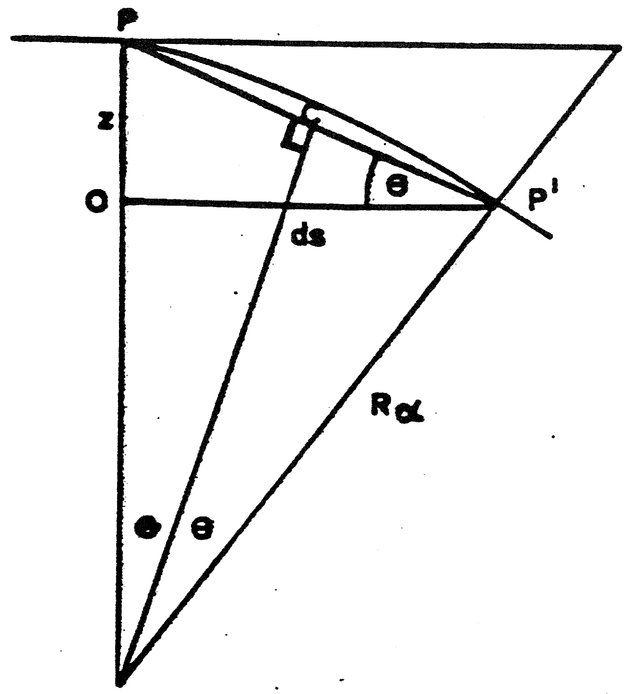


Figure 9

SOLUTION OF Z FOR SOLUTION OF R_α

Substituting n^2 and m^2 in (18) gives

$$\frac{R_\alpha \sin^2\alpha}{N} + \frac{R_\alpha \cos^2\alpha}{M} = 1 \, . \tag{27}$$

Finally, after rearranging the terms of (27), we get the expression for the Euler radius of curvature,

$$\boxed{R_\alpha = \frac{MN}{M \sin^2\alpha + N \cos^2\alpha} \, .} \tag{28}$$

1.3 <u>Curves on the Surface of an Ellipsoid</u>

There are two principal curves on the surface of an ellipsoid that are of special interest in geometric geodesy. They are the <u>normal section</u> and <u>geodesic</u> curves described below.

1.3.1 <u>The Normal Section</u>

In Section 1.2, the normal section was defined as the line of intersection of a normal plane (at a point P) and the surface of the ellipsoid. Consider two points on the surface of an ellipsoid (P_1 and P_2) which are on different meridians, and are at different latitudes. The normal section from P_1 to P_2 (direct normal section), is not coincident with the normal section from P_2 to P_1 (inverse normal section) (Figure 10).

The normal plane of the direct normal section, containing the points P_1, n_1 and P_2, contains the normal at P_1, and the inverse normal plane, $P_2n_2P_1$, contains the normal at P_2 and the point P_1. If the normal sections P_1P_2 and P_2P_1 were coincident, then the normals P_1n_1 and P_2n_2, in their

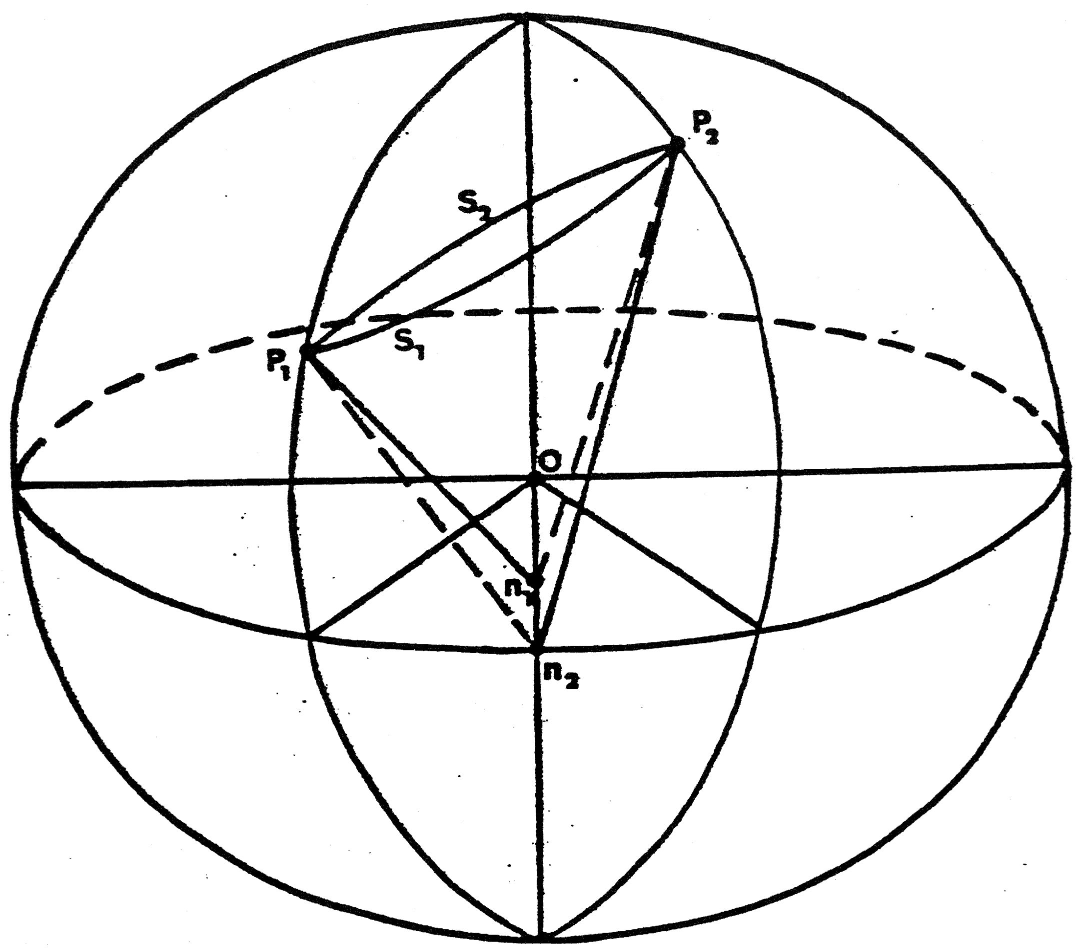


Figure 10

RECIPROCAL NORMAL SECTIONS

respective meridian planes, would intersect the minor axis at the same point. It can be shown that the intersection point z_n of any ellipsoidal normal section intersects the minor axis at [Zakatov, 1953; p. 39-40]

$$z_n = \frac{ae^2 \sin \phi_p}{(1-e^2 \sin^2 \phi_p)^{1/2}} \quad . \tag{29}$$

If two points have different longitudes, and $\phi_{p_1} < \phi_{p_2}$ (Figure 10), then $Z_{n_1} < Z_{n_2}$, and the normals $p_1 n_{p_1}$ and $p_2 n_{p_2}$ do not lie in the same plane. They are said to be <u>skew-normals</u>. However, if ϕ_{p_1} equals ϕ_{p_2}, the direct and inverse normal sections are coincident.

For two points on the same meridian, the ellipsoidal normals do not intersect at the same point on the minor axis. They are, however, in the same plane (the common meridian plane), thus the normal sections P_1P_2 and P_2P_1 are coincident.

The result of the aforementioned is that on the surface of the ellipsoid, the normal section does not give a unique line between two points. Thus, an ellipsoidal triangle is not uniquely defined by normal sections. In Figure 11, the direct normal section from A to B, AaB, is not coincident with the inverse normal section BbA.
Thus, the geodetic azimuth α_A does not refer to the same curve as does α_B. Similar problems exist for the azimuths A to C, B to C, etc.

We now look briefly at the magnitude of the separation between direct and inverse normal sections. In Figure 12, this separation is shown as the angle Δ. The formula for the solution of Δ is given by [Zakatov, 1953, p. 51]

$$\Delta'' = \rho''(\frac{1}{4} e^2 \sigma^2 \cos^2 \phi_m \sin 2\alpha_{p_{12}}) \ , \tag{30}$$

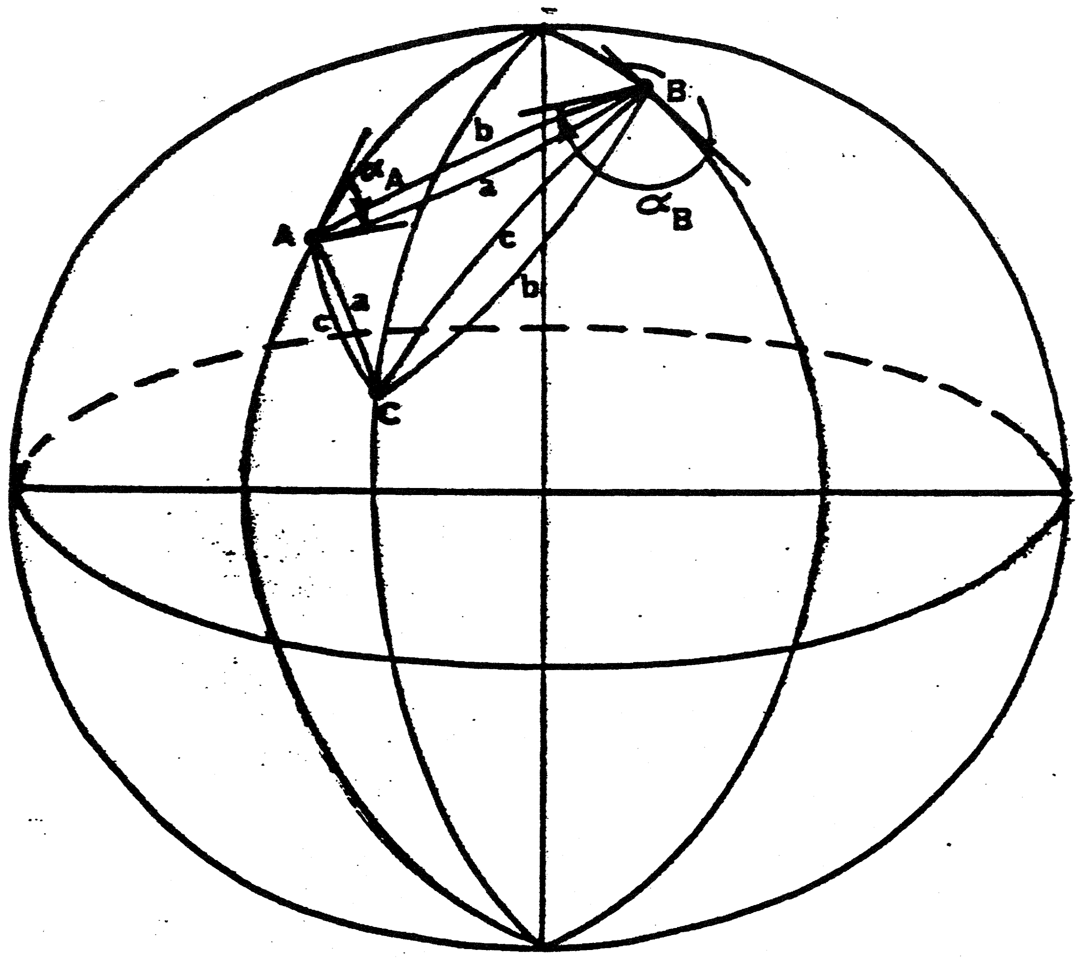


Figure 11

RECIPROCAL NORMAL SECTION TRIANGLE

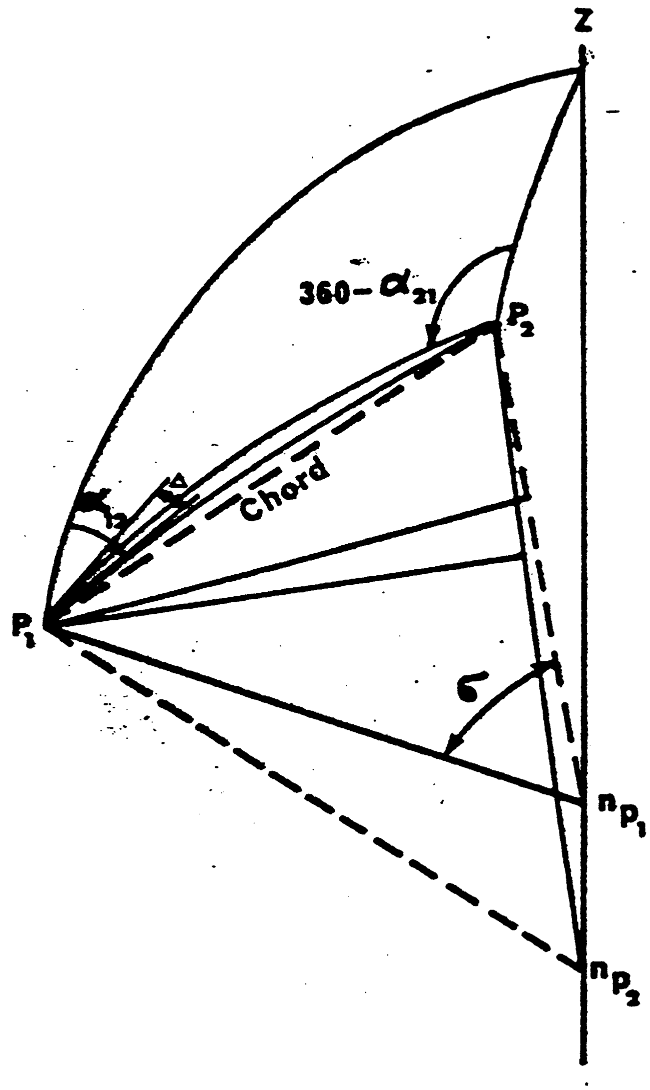


Figure 12

ANGULAR SEPARATION BETWEEN RECIPROCAL NORMAL SECTIONS where

$$\phi_m = \frac{\phi_{P_1} + \phi_{P_2}}{2} \qquad (31)$$

and

$$\sigma \simeq \frac{s}{N_m} \ ,$$

and

$$N_m = \frac{N_1 + N_2}{2} \ . \qquad (31a)$$

For example, a line P_1P_2, which is 200 km in length, and for maximum conditions ($\phi_m = 0°$ and $\alpha_{P_2} = 45°$), $\Delta = 0''.36$. Since most traverse or triangulation lines are shorter than this, and since the maximum situation will not always occur, the value of Δ is generally quite small, and in most instances, practically negligible.

1.3.2 The Geodesic

The geodesic, or geodetic line, between any two points on the surface of an ellipsoid, is the unique surface curve between the two points. At every point along the geodesic, the principal radius of curvature vector is coincident with the ellipsoidal normal. The geodesic (Figure 13), between two points P_1, P_2, is the shortest surface distance between these two points. The position of the geodesic with respect to the direct and inverse normal sections is shown in Figure 13.

To describe the geodesic mathematically, we will develop the differential equations for geodetic lines on a surface of rotation. The basic differential geometry required for this can be found in Phillips [1957] and Lipschutz [1969]. The general equation for a surface of rotation can be expressed as

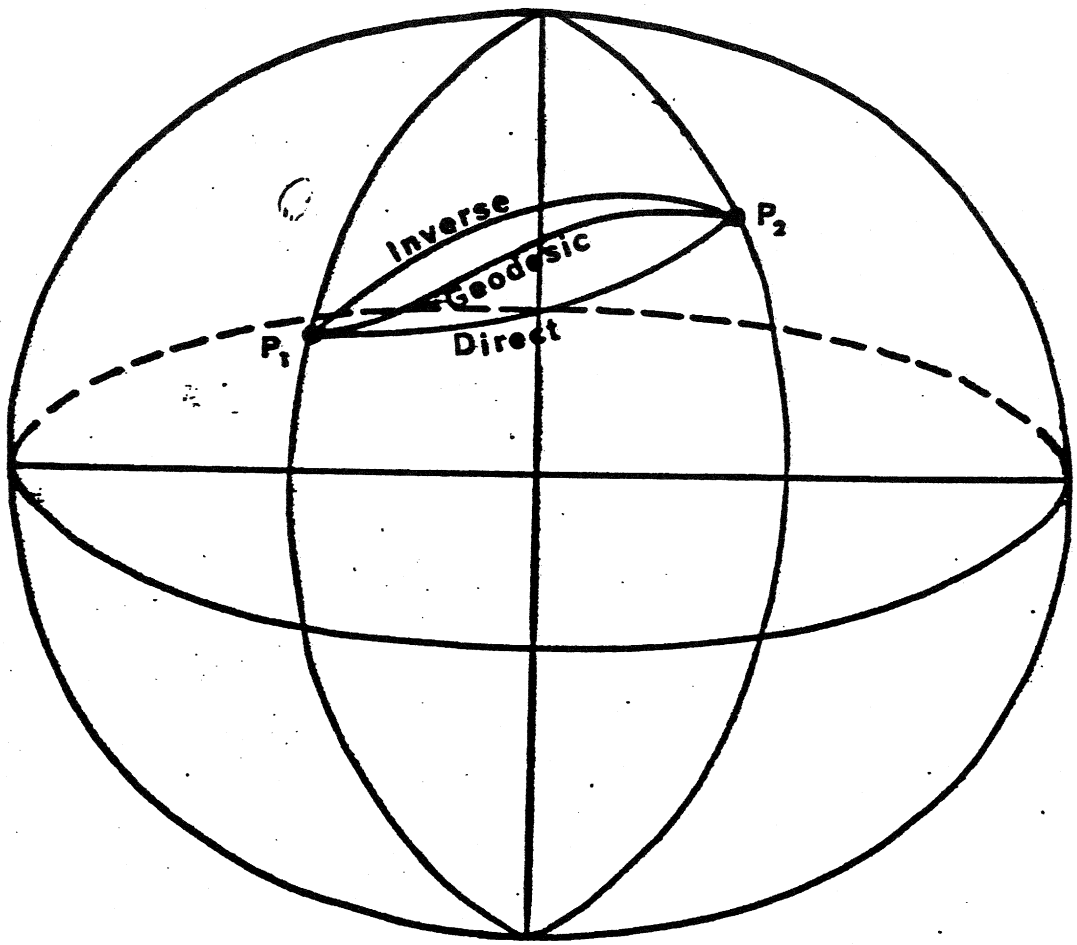


Figure 13

GEODESIC

$$F(x, y, z) = 0 \,. \tag{32}$$

The parametric equations for a geodesic on this surface are

$$\begin{aligned} x &= f_1(s) \,, \\ y &= f_2(s) \,, \\ z &= f_3(s) \,. \end{aligned} \tag{33}$$

The direction cosines of the normal to the surface are

$$\cos \beta_1 = \frac{\frac{\partial F}{\partial x}}{D} \; ; \quad \cos \beta_2 = \frac{\frac{\partial F}{\partial y}}{D} \; ; \quad \cos \beta_3 = \frac{\frac{\partial F}{\partial z}}{D} \; ; \tag{34}$$

where

$$D = \left(\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2 \right)^{1/2} \,. \tag{35}$$

The direction cosines of the principal normal to the curve (33) are

$$\cos \beta_{N_1} = R \frac{d^2x}{ds^2} \; ; \; \cos \beta_{N_2} = R \frac{d^2y}{ds^2} \; ;$$

$$\cos \beta_{N_3} = R \frac{d^2z}{ds^2} \quad ; \tag{36}$$

where R is the principal radius of curvature of the surface.

In the definition of the geodesic, it was stated that at every point on the curve, the normal to the surface and the principal radius vector (principal normal) are to be coincident. To satisfy this, we equate (34) and (36), which reduces to

$$\frac{\frac{\partial F}{\partial x}}{\frac{d^2x}{ds^2}} = \frac{\frac{\partial F}{\partial y}}{\frac{d^2y}{ds^2}} = \frac{\frac{\partial F}{\partial z}}{\frac{d^2z}{ds^2}} \,. \tag{37}$$

Since we are dealing with an ellipsoid of rotation, the surface of which can be represented by the equation

$$x^2 + y^2 + f(z) = 0 \ . \tag{38}$$

Then

$$\frac{\partial F}{\partial x} = 2x, \qquad \frac{\partial F}{\partial y} = 2y, \qquad \frac{\partial F}{\partial z} = f' \ (z) \ , \tag{39}$$

which when placed in (37) yields

$$y \frac{d^2x}{ds^2} - x \frac{d^2y}{ds^2} = 0 \ . \tag{40}$$

Integration of (40) yields

$$ydx - xdy = Cds \ , \tag{41}$$

where C is the constant of integration.

In Figure 14, the line PP' represents a differential part of a geodesic on the surface of the ellipsoid. Having the Cartesian coordinates of P (x, y, z), we can compute the coordinates of P', (x + dx, y + dy, z + dz), since ds is a very small distance. The coordinates of A (projection of P' into the plane of the parallel of latitude of P) are then x + dx, y + dy, z. The radius of this parallel is denoted by r. The area of triangle CPA is

$$\text{Area CPA} = \frac{1}{2} (ydx - xdy) \tag{42}$$

and the area of the sector PP''C is

$$\text{Area PP''C} = \frac{1}{2} \, rds \sin \alpha \ . \tag{43}$$

When ds is very small,

$$\text{Area CPA} = \text{Area PP''C} \ ,$$

thus

$$\frac{1}{2} (ydx - xdy) = \frac{1}{2} \, rds \sin \alpha \ , \tag{44}$$

and substituting (41) in (44) yields

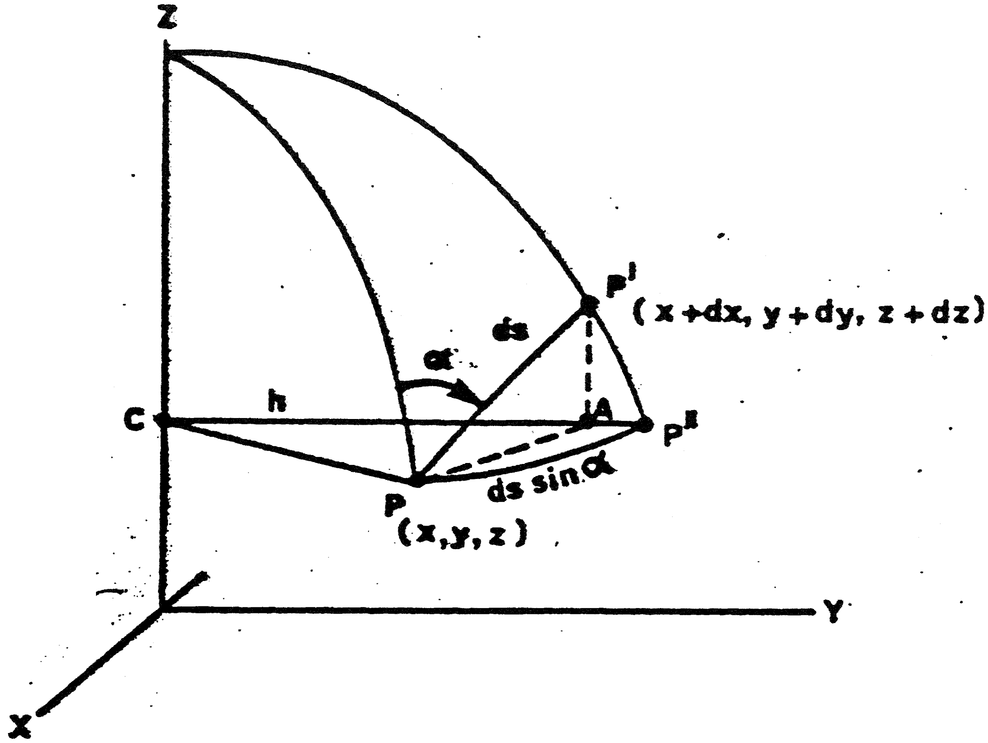


Figure 14

DIFFERENTIAL EQUATION OF A GEODESIC ON THE SURFACE OF AN ELLIPSOID OF ROTATION

$$C ds = r \sin \alpha ds \ , \tag{45}$$

or

$$r \sin \alpha = C \ . \tag{46}$$

Finally, substituting (17) in (46), we find that

$$\boxed{N \cos \phi \sin \alpha = C} \tag{47}$$

for any point along a geodesic on the surface of an ellipsoid of rotation.

In geometric geodetic computations, it is necessary to define our direct and inverse azimuths with respect to the same surface curve, and not with respect to the two normal sections. Thus we need the separation between the normal section and geodesic curves. The separation, stated here without proof, is given by [Zakatov, 1953, pp 41-45]

$$\delta = \frac{\Delta}{3} \tag{48}$$

where δ is the angle between the direct normal section and the geodesic at any point, and Δ is the angle between the reciprocal normal sections between two points. Further development of this, and the application of appropriate corrections, are given in 2.1.1.

Further, the distance s between two points on the surface of an ellipsoid is different if one uses a normal section rather than the geodesic. The difference is given by [Zakatov, 1953, p. 51]

$$\boxed{\Delta s = \frac{ae^4}{360} \sin^2 2\alpha_{12} \cos^4 \phi_m \sigma^5 \ ,} \tag{49}$$

which for a line 600 km in length amounts to approximately 9×10^{-6} m, which is obviously negligible for all practical purposes.

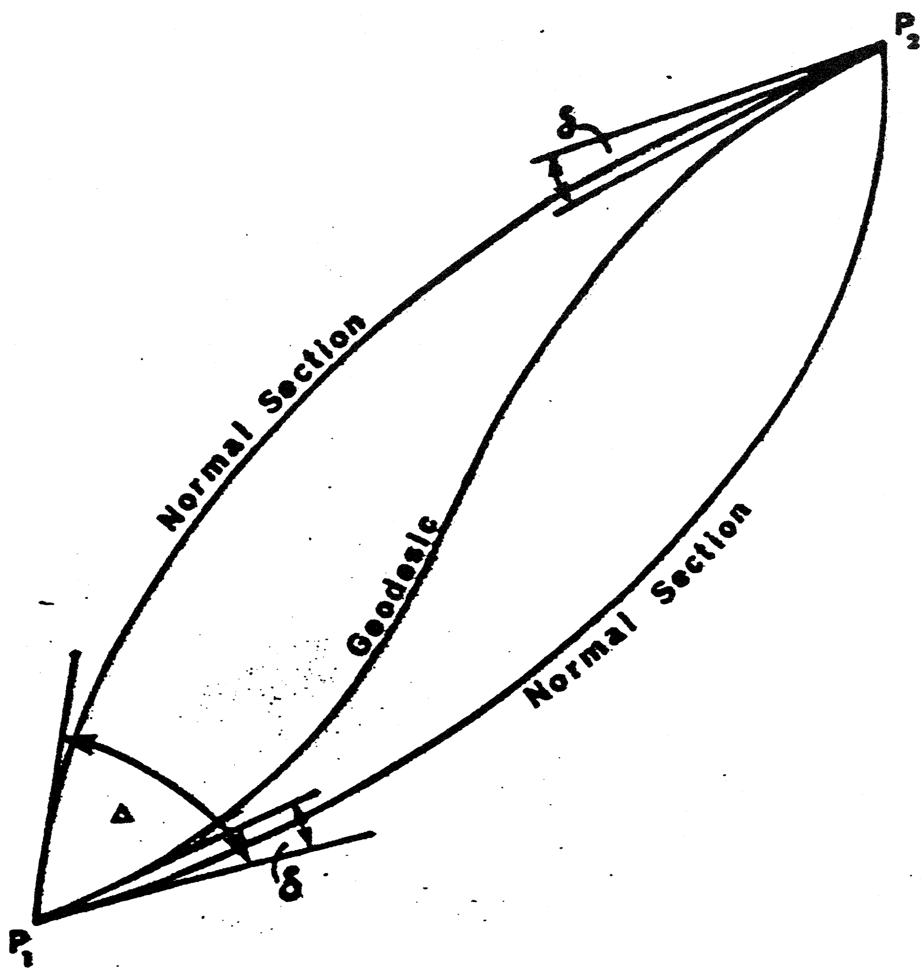


Figure 15

SEPARATION BETWEEN NORMAL SECTION AND GEODESIC

SECTION II. REDUCTION OF TERRESTRIAL GEODETIC OBSERVATIONS

2. Reduction to the Surface of the Reference Ellipsoid

Geodetic measurements (terrestrial directions, distances, zenith distances) are made on the surface of the earth. Classical computations of geodetic positions are made on the reference ellipsoid. Therefore, measurements must be reduced from the surface of the earth to the reference ellipsoid. When reducing measured quantities, there are two sets of effects to be considered - geometric effects and the effect of the variations in the earth's gravity field.

It should be noted that the reductions developed herein can be applied in an inverse fashion. That is, computed geodetic ellipsoidal quantities (distances, for instance) can be "reduced" up to the earth's surface (2.4).

2.1 Reduction of Horizontal Directions (or Angles)

When we measure directions on the surface of the earth, we level the instrument to ensure that the vertical axis is coincident with the local gravity vector. We know that the local gravity vector and the normal to the ellipsoid are not generally coincident. To refer directions to the ellipsoidal normal, a correction for the deflection of the vertical is needed.

Two other considerations are those of ellipsoidal geometry. First, the normals at two points on an ellipsoid are "skewed" to each other, thus when a target is above the ellipsoid, this point is not in the same plane as the normal projection of the target onto the ellipsoid.

The correction associated with this phenomenon is called the skew-normal correction. Secondly, we wish to have geodesic directions, and not normal section directions, thus a normal section-geodesic correction is needed.

2.1.1 Geometric Effects

Figure 16 shows the situation on the earth's surface for direction measurements, after the effects of gravity have been removed (2.1.2). In this figure, P_1' is the measuring station, which is on the normal P_1n_1. Point P_2' is the target at height h_2 above the ellipsoid point P_2. If $h_2 = 0$, the direction measured (shown here as an azimuth, i.e. $\alpha_{12} = d_{12} + z_{12}$, where z_{12} is the assumed known orientation parameter) would be between planes P_1zn_1, and $P_1P_2n_1$, that is α_{12}, the direct normal section azimuth. Since $h \neq 0$ in practice, the measured direction $\alpha_{12_{meas}}$ must be corrected. The reduction for this effect, called the <u>skew normal</u> or <u>height of target</u> reduction, must be applied.

From (29)

$$\overline{n_1n_2} = ae^2\,(\phi_2-\phi_1)\cos\phi_m\ , \tag{50}$$

and

$$(\phi_2-\phi_1) = \frac{s\cos\alpha_{12}}{M_m} \tag{51}$$

where $M_m = \dfrac{M_1 + M_2}{2}$, we get

$$\overline{n_1n_2} = ae^2\,\frac{s}{M_m}\cos\alpha_{12}\cos\phi_m\ , \tag{52}$$

where s is the arc length P_1P_2.

Now to derive the reduction δ_h, we proceed as follows. First, compute

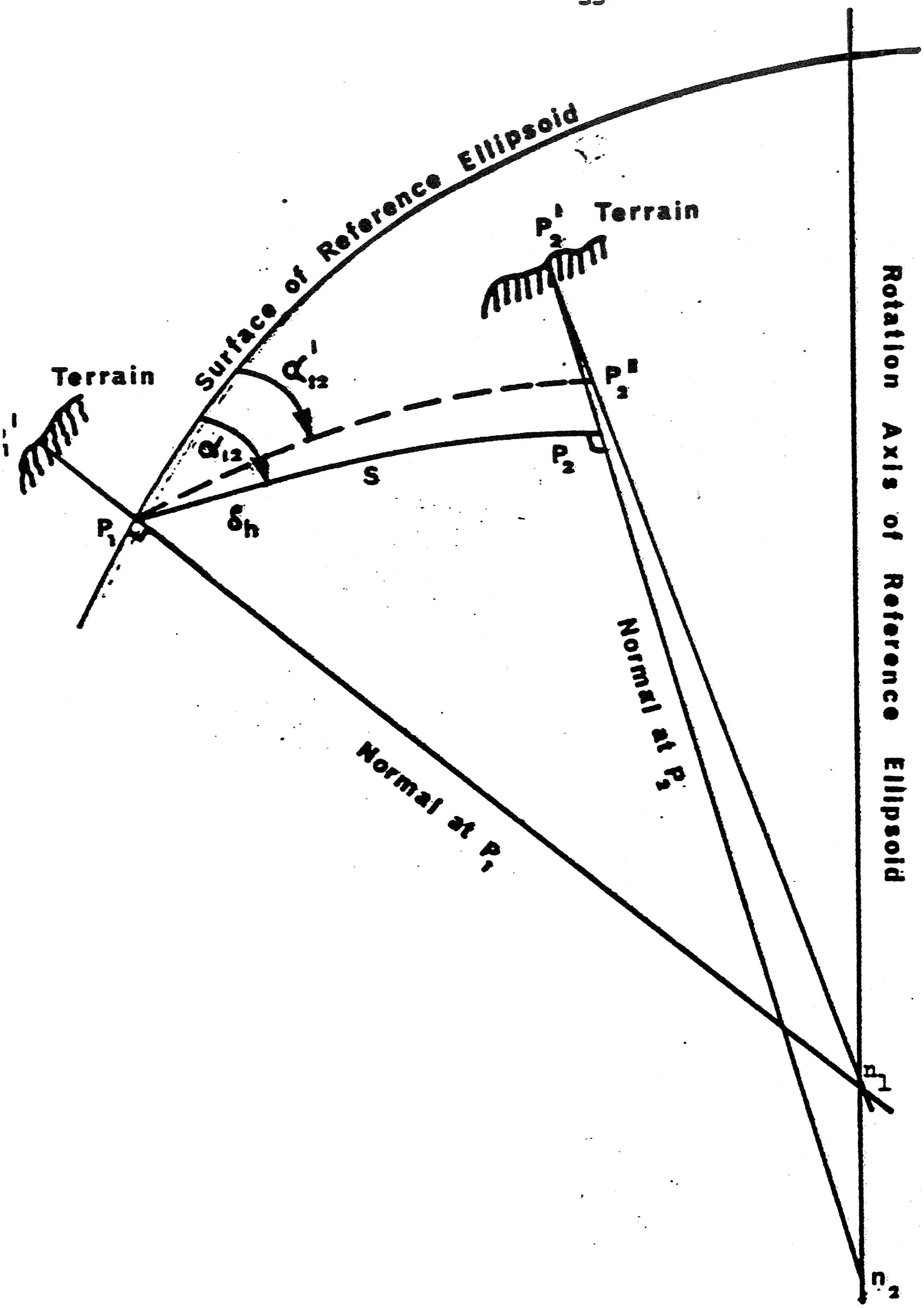


Figure 16

SKEW NORMAL CORRECTION

$$\overline{R_{n_1}} = \overline{n_1 n_2} \cos \phi_2$$

$$= ae^2 \frac{s}{M_m} \cos \alpha_{12} \cos^2 \phi_2 \qquad (53)$$

where ϕ_m has been replaced by ϕ_2 since the difference will give a negligible effect. Then, the angle at P_2' is given by

$$da = \frac{ae^2 s \cos \alpha_{12} \cos^2 \phi_2}{M_m \cdot \overline{P_2 R}} . \qquad (54)$$

Now, if we approximate the length $\overline{P_2R}$ by the semi-major axis a, (54) becomes

$$da = e^2 \frac{s}{M_m} \cos \alpha_{12} \cos^2 \phi_2 . \qquad (55)$$

We now compute $\overline{P_2P_2''}$ by using (55) as

$$\overline{P_2P_2''} = h_2 e^2 \frac{s}{M_m} \cos \alpha_{12} \cos^2 \phi_2 . \qquad (55a)$$

Then for triangle $P_1P_2P_2''$ we can write, (assuming a plane triangle)

$$\frac{\sin \delta_h}{\sin(\alpha_{21} - 180°)} = \frac{\overline{P_2P_2''}}{s} , \qquad (56)$$

which finally gives us, after some manipulation, the final formula for the skew-normal correction

$$\boxed{\delta_h'' = \rho'' \left(\frac{h_2}{M_m} e^2 \sin \alpha_{12} \cos \alpha_{12} \cos^2 \phi_2\right) .} \qquad (57)$$

When $\phi_2 = 45°$, and $h_2 = 200$ m, and 1000 m, δ_h equals 0".008 and 0".05, respectively. Obviously, there will be instances where the effect is significant, and must be taken into account. This is particularly true for higher order geodetic position computation work.

The second geometric effect to consider in direction measurement reduction is that of the difference between the normal section, to which we have now reduced our measurement, and the geodesic. This correction, which is derived simply by combining equations (30) and (48) is expressed as

$$\boxed{\delta''_g = \rho'' \left(\frac{e^2 s^2 \cos^2\phi_m \sin 2\alpha_{12}}{12\, N_m^2}\right) .} \tag{58}$$

where s is in metres.

When $\phi_m = 0''$, $\alpha_{12} = 45°$, and s = 200 km, 100 km and 50 km, δ_g is 0".12, 0".02 and 0".006. This effect could be significant and should be taken into account for geodetic work.

Some final points regarding these geometric effects are noted immediately below:

1) In equation (57), the ellipsoidal height h may be replaced by the orthometric height H with no significant effect on δ_h.

2) In most cases, δ_h and δ_g will be of approximately equal magnitude and opposite in sign. They should be computed, however, particularly for precise geodetic position computations.

3) Equations (57) and (58) are often expressed in other ways, all of which give equivalent results, but which may include further approximations. As an example, (57) may be expressed as [Bomford, 1971, p 122]

$$\boxed{\delta_h = \frac{h_2 e'^2}{2R} \sin 2\alpha_{12} \cos^2 \phi_m ,} \tag{59}$$

and (58) as [Bomford, 1971, p 124]

$$\boxed{\delta''_g = 0.028 \left(\frac{s_{km}}{100}\right)^2 \sin 2\alpha_{12} \cos^2\phi_m .} \tag{60}$$

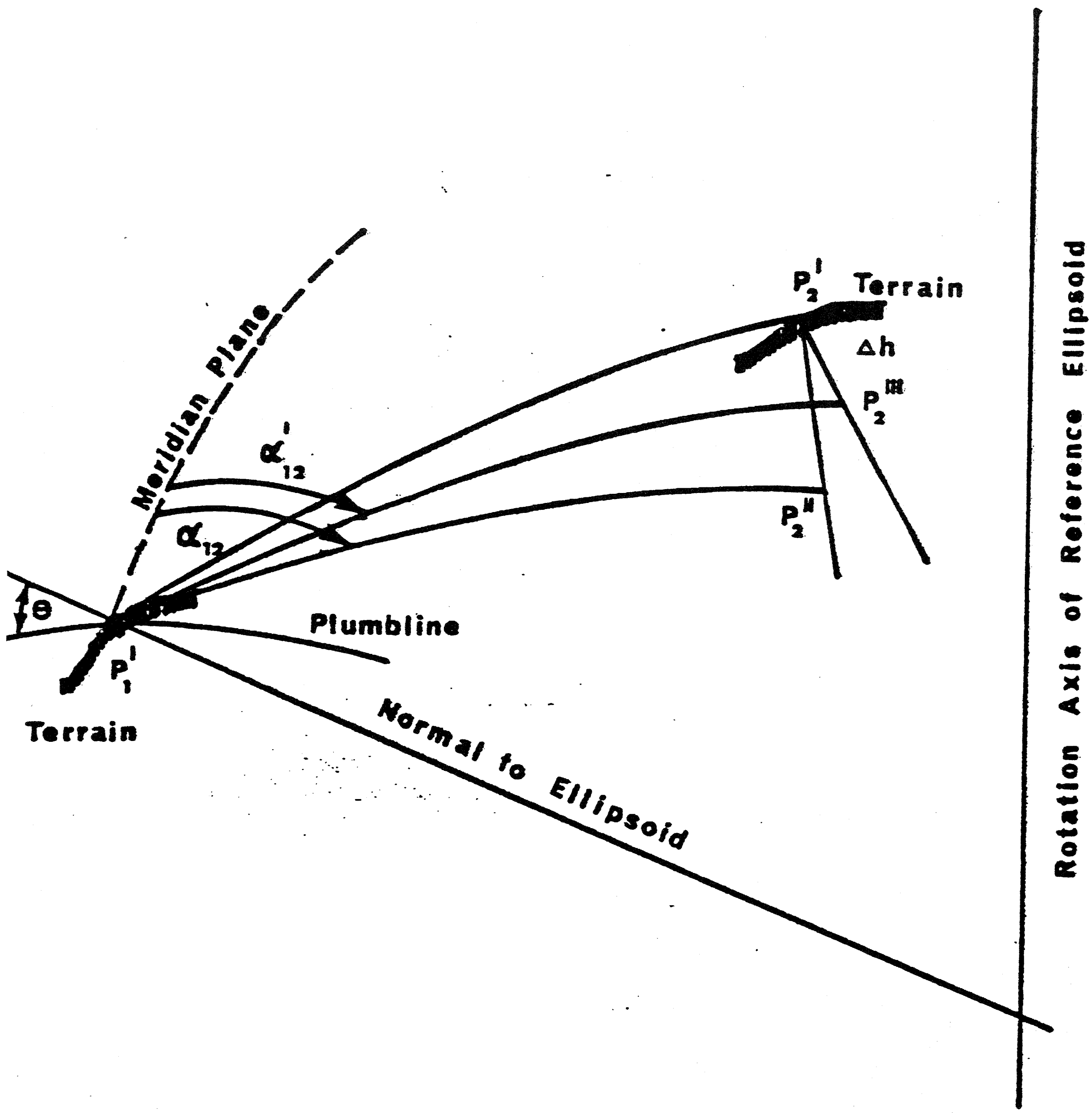


Figure 17

DEFLECTION OF THE VERTICAL CORRECTION

2.1.2 Gravimetric Effects

A theodolite is levelled with respect to the local gravity vector and not to the ellipsoid normal. A correction for the angle (deflection of the vertical) between the gravity vector and the ellipsoid normal is necessary. Figure 17 depicts the correction that must be applied. This topic is covered in depth in [Vanicek, 1972, pp 164-166]. We only state the reduction formula here as

$$\begin{aligned}\delta_\theta &= -\theta \cot z, \\ &= -(\xi_1 \sin \alpha_{12} - \eta_1 \cos \alpha_{12}) \cot z \quad ,\end{aligned} \tag{61}$$

where ξ is the meridian component of the deflection of the vertical, η is the prime vertical component of the deflection of the vertical, and z is the zenith distance. The effect of this reduction can vary from an insiginficant amount (if $\theta \approx 0$ or if $z = 90°$) to values of the magnitude 2" - 3" when for instance $\theta = 20''$ and $z = 80°$.

To apply this correction, and that required in 2.2, the deflections of the vertical at each point are required. These can be obtained in various ways. A rigorous approach is to observe the astronomic coordinates (Φ, Λ) at each station, which would be a difficult task. Alternately, one may utilize the results of a contemporary geoid computation technique [Vanicek and Merry, 1973], and compute ξ and η at each point.

2.2 Zenith Distances

The only effect on a zenith distance measurement is that of variations in the gravity field — that is, the deflections of the vertical. As in 2.1.3, we will only state the reduction formulae here as

$$z_R = z_m + (\xi_1 \cos \alpha_{12} + \eta_1 \sin \alpha_{12}) \ , \tag{62}$$

where z_m is the measured value of the zenith distance.

This topic is covered in [Vanicek, 1972, p 170, and Heiskanen and Moritz, 1967, pp 173-175], and will not be discussed further here.

2.3 Spatial Distances

In this section we treat the reduction of a measured spatial distance, on the surface of the earth, to the surface of the ellipsoid. After having made various instrumental and atmospheric corrections to the measured e.d.m. distance, we are left with a straight line spatial distance ℓ (Figure 18). This spatial distance is then reduced to the ellipsoid. The reduction is derived as follows.

First, compute

$$R = \frac{R_1 + R_2}{2} \ , \tag{63}$$

where R_1 and R_2 are the Euler radii of curvature (eqn. 28). Then, from triangle $P_1'P_2'O$, the cosine law yields

$$\ell^2 = (R+h_1)^2 + (R+h_2)^2 - 2(R+h_2)(R+h_1) \cos \psi, \tag{64}$$

where

$$\begin{aligned} h_1 &= H_1 + N_1 \ , \\ h_2 &= H_2 + N_2 \ , \end{aligned} \tag{65}$$

which are ellipsoidal heights, and are equal the sum of their respective orthometric heights (H_1 and H_2) and geoid heights (N_1 and N_2). Replacing

$$\cos \psi = 1 - \sin^2 \frac{\psi}{2} \tag{66}$$

in (64), and rearranging terms yields

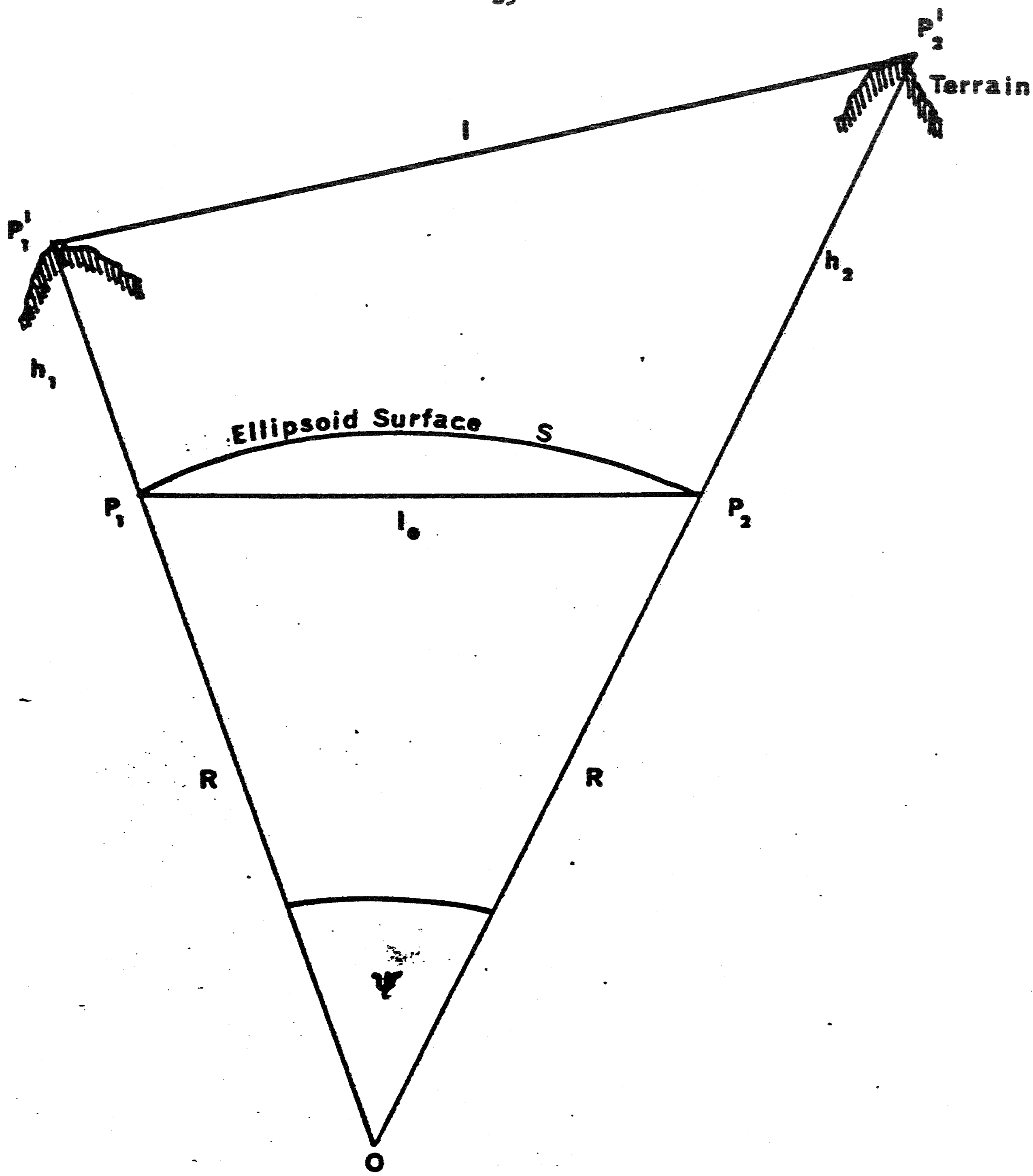


Figure 18

SPATIAL DISTANCE REDUCTION

$$\ell^2 = (h_2-h_1)^2 + 2R^2\ (1 + \frac{h_1}{R})(1 + \frac{h_2}{R})\sin^2 \frac{\psi}{2} . \qquad (67)$$

From triangle P_1P_2O, the cosine law and half-angle formulae yield

$$\ell_o = 2R \sin \frac{\psi}{2} , \qquad (68)$$

or

$$\psi = 2 \sin^{-1} \frac{\ell_o}{2R} . \qquad (68a)$$

Setting

$$h_2 - h_1 = \Delta h , \qquad (69)$$

(67) becomes

$$\ell^2 = \Delta h^2 + (1 + \frac{h_1}{R})(1 + \frac{h_2}{R})\ \ell_o^2 , \qquad (70)$$

which when rearranged is

$$\boxed{\ell_o = \left(\frac{\ell^2-\Delta h^2}{(1+\frac{h_1}{R})(1+\frac{h_2}{R})} \right)^{1/2} .} \qquad (71)$$

Now,

$$\boxed{S = R\psi = 2R \sin^{-1} \frac{\ell_o}{2R} .} \qquad (72)$$

Thus, using (71) and (72), we can reduce a spatial distance to the surface of the ellipsoid. These formulae are sufficiently rigorous for current geodetic work [Thomson and Vanicek, 1973].

Note that for a rigorous distance reduction the geoid height N is needed. There are various methods of computing N, one of which is that developed at U.N.B. [Vanicek and Merry, 1973].

No mention has been made here regarding precise base lines. The reason for this commission is that precise base lines are not being measured much any more, except for EDM instrument calibration for which reduction to the ellipsoid is not necessary.

Finally, it should be noted that there are many distance reduction formulae in use, some of which have been developed for specific reference ellipsoids, or regions of countries.

2.4 Reduction of Computed Geodetic Quantities to the Terrain

The situation often occurs in practice where computed geodetic quantities, namely distances and angles, must be measured on the terrain. These can not generally be compared directly with the computed values since the latter are usually given on the surface of the reference ellipsoid, thus they must be "reduced" to the terrain.

In order to reduce the required angles, one proceeds as follows. First, compute the directions (azimuths) between the points involved. Then, using equations (57), (58) and (61), compute the quantities δ''_h, δ''_g and δ''_θ respectively. These corrections are then applied to the computed direction α_{ij}, with signs opposite to those used for reduction to the ellipsoid, to obtain the direction that should be measured, α_{ij}^{meas}. Obviously, one would not be able to measure this direction (or angle) exactly since it, and the measurement taken, will have some standard deviations. A similar procedure is used for distance reduction. A simple rearrangement of terms in equation (72) yields

$$\ell_o = 2R \sin \frac{s}{2R} \quad , \tag{72a}$$

and similarly (71) gives

$$\ell = [\ell_o^2 \ (1 + \frac{h_1}{R})(1 + \frac{h_2}{R}) + \Delta h^2]^{1/2} \quad . \tag{71a}$$

Thus, we can compute the terrain spatial distance ℓ given the ellipsoidal distance s . Once again, as with the directions, it should be noted that both the computed spatial distance and the measured one will have some standard deviation meaning that an exact duplication of the computed distance by remeasurement will not be probable.

It has been shown that the reduction of geodetic angles and distances to the terrain is a straightforward process. Thus, when faced with the problem of checking measurements on the terrain which are given on the reference ellipsoid, some preliminary computations enables one to carry out the remeasurement task.

SECTION III. COMPUTATION OF GEODETIC POSITIONS ON THE REFERENCE ELLIPSOID.

3. Puissant's Formula - Short Lines

3.1 Introduction

These formula are named after the French mathematician who is credited with their development. Their derivation is based on a spherical approximation, thus they are generally considered to be correct to 1 ppm at 100 km, beyond which they break down rapidly (40 ppm at 250 km when $\phi = 60°$) [Bomford, 1971, p 134]. Thus, we say that Puissant's Formula is a "short" line formula.

3.2 Direct Problem

Given are the geodetic quantities ϕ_1, λ_1, s_{12} and α_{12} (Figure 19). We are required to compute the quantities ϕ_2, λ_2 and α_{21}.

In this derivation, we first compute ϕ_2. We obtain, for the spherical approximation, from spherical trigonometry (cosine law)

$$\sin \phi_2 = \sin \phi_1 \cos (\overline{P_1P_2}) + \cos \phi_1 \sin(\overline{P_1P_2}) \cos \alpha \qquad (73)$$

But $\overline{P_1P_2} \approx \frac{s_{12}}{N_1}$, and $\phi_2 = \phi_1 + d\phi$, and $\alpha = \alpha_{12}$ since it is stipulated that the meridians are in the same plane. Then

$$\sin(\phi_1 + d\phi) = \sin \phi_1 \cos \frac{s_{12}}{N_1} + \cos \phi_1 \sin \frac{s_{12}}{N_1} \cos \alpha_{12} \, . \qquad (74)$$

What is required now is to get an expression for $d\phi$. From equation (74), we can express the left hand side by

$$\sin (\phi_1 + d\phi) = \sin \phi_1 \cos d\phi + \cos \phi_1 \sin d\phi. \qquad (75)$$

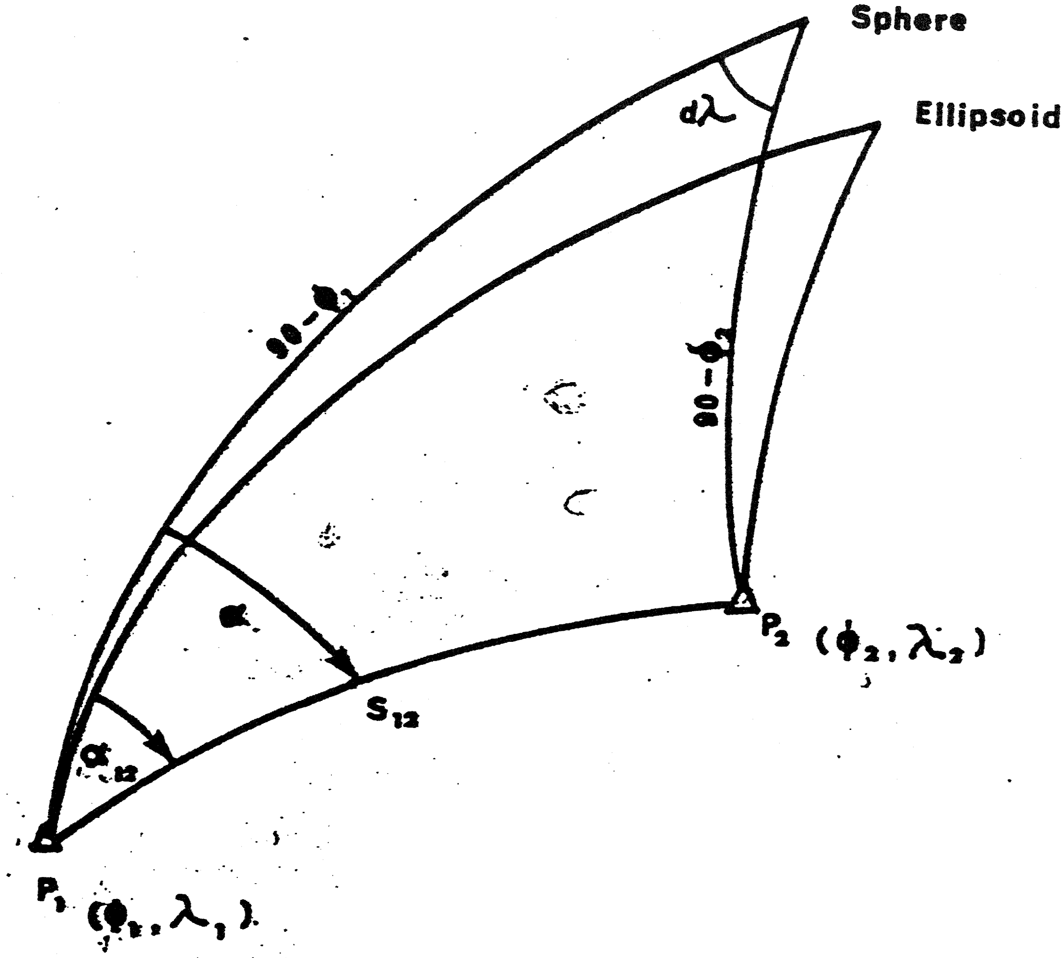


Figure 19

PUISSANT'S FORMULA FOR DIRECT PROBLEM

Expanding cos $d\phi$ and sin $d\phi$ in series (using the first two terms only), we write

$$\cos d\phi = 1 - \frac{d\phi^2}{2} \; \ldots \; ,$$

and (76).

$$\sin d\phi = d\phi - \frac{d\phi^3}{6} \ldots \; ,$$

then (75) becomes

$$(\sin \phi_1 + d\phi) = \sin \phi_1 - \sin \phi_1 \frac{d\phi^2}{2} + \cos \phi_1 d\phi - \cos_1 \phi \frac{d\phi^3}{6} + .. (77)$$

Taking the right hand side of (75), we expand cos $\frac{s_{12}}{N_1}$ and sin $\frac{s_{12}}{N_1}$ in a series (first two terms only):

$$\cos \frac{s_{12}}{N_1} = 1 - \frac{s^2}{2N_1^2} \ldots \quad ,$$

and (78)

$$\sin \frac{s_{12}}{N_1} = \frac{s_{12}}{N_1} - \frac{s^3}{6N_1^3} \ldots$$

Then (74) can be rewritten as

$$\sin \phi_1 + \cos \phi_1 d\phi - \sin \phi_1 \frac{d\phi^2}{2} - \cos \phi_1 \frac{d\phi^3}{6} + ..$$

$$= \sin \phi_1 + \frac{s_{12}}{N_1} \cos \alpha_{12} \cos \phi_1 - \frac{s_{12}^2}{2N_1^2} \sin \phi_1 -$$

$$- \frac{s_{12}^3}{6N_1^3} \cos \alpha_{12} \cos \phi_1 + \ldots \qquad (79)$$

After cancelling appropriate terms, and dividing (80) by cos ϕ_1, the expression for $d\phi$ is

$$d\phi = \frac{s_{12}}{N_1} \cos \alpha_{12} - \frac{s_{12}^2}{2N_1^2} \tan \phi_1 - \frac{s_{12}^3}{6N_1^3} \cos \alpha_{12} + \frac{d\phi^2}{2} \tan \phi_1 + \frac{d\phi^3}{6} + .. \qquad (80)$$

The above formula will obviously not yield the required solution since $d\phi$ appears on the right-hand side of the equation. To begin to solve this problem, we again use the spherical approximation and set

$$d\phi \approx \frac{s_{12}}{N_1} \cos \alpha_{12} \; . \tag{81}$$

Substituting (81) in (80) yields

$$d\phi = \frac{s_{12}}{N_1} \cos \alpha_{12} - \frac{s_{12}^2}{2N_1^2} \tan \phi_1 - \frac{s_{12}^3}{6N_1^3} \cos \alpha_{12} + $$
$$+ \frac{s_{12}^2}{2N_1^2} \cos^2 \alpha_{12} \tan \phi_1 + \frac{d\phi^3}{6} + \ldots \tag{82}$$

From (82) above, we can now get a more precise approximation for $d\phi$ (neglecting terms greater than the second power), namely

$$d\phi \approx \frac{s_{12}}{N_1} \cos \alpha_{12} - \frac{s_{12}^2}{2N_1^2} \tan \phi_1 \left(1 - \cos^2 \alpha_{12}\right) + \ldots \; , \tag{83}$$

which can be written more simply as

$$d\phi \approx \frac{s_{12}}{N_1} \cos \alpha_{12} - \frac{s_{12}^2}{2N_1^2} \tan \phi_1 \sin^2 \alpha_{12} + \ldots \; . \tag{84}$$

Squaring (84), and neglecting terms greater than the third power yields

$$d\phi^2 \approx \frac{s_{12}^2}{N_1^2} \cos^2 \alpha_{12} - \frac{s_{12}^3}{N_1^3} \cos \alpha_{12} \sin^2 \alpha_{12} \tan \phi_1 + \ldots \tag{85}$$

and further

$$d\phi^3 \approx \frac{s_{12}^3}{N_1^3} \cos^3 \alpha_{12} + \ldots \; . \tag{86}$$

Finally, substituting (85) and (86) in (80), and rearranging terms gives us

$$d\phi \approx \frac{s_{12}}{N_1}\cos \alpha_{12} - \frac{s_{12}^2}{2N_1^2}\tan \phi_1 - \frac{s_{12}^3}{6N_1^3}\cos \alpha_{12} + \frac{s_{12}^2}{2N_1^2}\cos^2\alpha_{12}\tan \phi_1 -$$

$$- \frac{s_{12}^3}{2N_1^3}\cos \alpha_{12}\sin^2\alpha_{12}\tan^2\phi_1 + \frac{s_{12}^3}{N_1^3}\cos^3\alpha_{12} + \ldots \qquad (87)$$

Collecting terms yields

$$d\phi \approx \frac{s_{12}}{N_1}\cos \alpha_{12} - \frac{s_{12}^2}{2N_1^2}\tan \phi_1\sin^2\alpha_{12} - \frac{s_{12}^3}{2N_1^3}\cos \alpha_{12}\sin^2\alpha_{12}\tan^2\phi_1 +$$

$$- \frac{s_{12}^3}{6N_1^3}\cos \alpha_{12}\sin^2\alpha_{12} + \ldots\ldots \quad . \qquad (88)$$

Further simplification is attained by setting

$$- \frac{s_{12}^3}{2N_1^3}\cos \alpha_{12}\sin^2\alpha_{12}\tan^2\phi_1 - \frac{s_{12}^3}{6N_1^3}\cos \alpha_{12}\sin^2\alpha_{12} =$$

$$= -\frac{s_{12}^3}{6N_1^3}\cos \alpha_{12}\sin^2\alpha_{12}(1 + 3\tan^2\phi_1) , \qquad (89)$$

which, when placed in (88) finally yields

$$d\phi \approx \frac{s_{12}}{N_1}\cos \alpha_{12} - \frac{s_{12}^2}{2N_1^2}\tan \phi_1\sin^2\alpha_{12} - \frac{s_{12}^3}{6N_1^3}\cos \alpha_{12}\sin^2\alpha_{12}(1+3\tan^2\phi_1)+.. \qquad (90)$$

Equation (90) is not a rigorous solution since the radius of curvature along the normal section P_1 to P_2 is taken to be a constant value N_1, when in fact it changes with latitude since $N = f_1(\phi)$ and $M = f_2(\phi)$ (equations (15) and (13) respectively). In order to take this change in curvature into account, we can write

$$d\phi = \frac{N_1}{M_m}\ (\text{right-hand side of (90)}) , \qquad (91)$$

where $1/M_m$ replaces $1/N_1$, and

$$M_m = \frac{M_1 + M_2}{2} \quad . \tag{92}$$

Since we do not know ϕ_2, we must use the approximation

$$M_2 = M_1 + dM_1 \tag{93}$$

in order to compute M_2. From (13), we compute

$$\frac{dM_1}{d\phi} = a(1-e^2)(-3/2)(1-e^2 \sin^2\phi_1)^{-5/2} (-2e^2 \sin \phi_1)\cos \phi_1, \tag{94}$$

which reduces to

$$\frac{dM_1}{d\phi} = M_1 \frac{3e^2 \sin \phi_1 \cos \phi_1}{(1-e^2 \sin^2 \phi_1)} \quad , \tag{95}$$

which when placed in (92) yields

$$M_m = \frac{M_1 + (M_1 + dM_1)}{2} = M_1 + \frac{dM_1}{2} \quad , \tag{96}$$

$$M_m = M_1 + \frac{dM_1}{d\phi} \left(\frac{d\phi''}{2\rho''}\right) \quad , \tag{96a}$$

$$M_m = M_1 + \frac{3}{2} M_1 \frac{e^2 \sin \phi_1 \cos \phi_1}{(1-e^2 \sin^2 \phi_1)} \left(\frac{d\phi''}{\rho''}\right) . \tag{97}$$

From (97), using the binomial series expansion gives

$$\frac{1}{M_m} = \frac{1}{M_1} \left(1 - \frac{3}{2} \frac{e^2 \sin \phi_1 \cos \phi_1}{(1-e^2 \sin^2 \phi_1)} \left(\frac{d\phi''}{\rho''}\right)\right) , \tag{98}$$

which when placed in (91) yields the final result

$$\boxed{d\phi'' = \left[\rho''\left(\frac{s_{12}\cos \alpha_{12}}{M_1} - \frac{s_{12}^2 \tan \phi_1 \sin^2\alpha_{12}}{2M_1N_1} - \frac{s_{12}^3 \cos \alpha_{12} \sin^2\alpha_{12}(1+3 \tan^2\phi_1)}{6M_1N_1^2} + ..\right)\left(1 - \frac{3e^2 \sin \phi_1 \cos \phi_1}{2(1-e^2 \sin^2\phi_1)} \left(\frac{d\phi''}{\rho''}\right)\right)\right] , \quad (99)}$$

where $d\phi''$ in the last term of (99) is computed using equation (90) (multiplied by ρ'').

Finally, we compute ϕ_2 by

$$\phi_2 = \phi_1 + d\phi \ . \tag{100}$$

The longitude of P_2 can be computed by

$$\lambda_2 = \lambda_1 + d\lambda \ . \tag{101}$$

From Figure 19, using a spherical approximation the sine law yields

$$\frac{\sin d\lambda}{\sin \frac{s_{12}}{N_2}} = \frac{\sin \alpha_{12}}{\sin(90-\phi_2)} \tag{102}$$

or

$$\sin d\lambda = \sin \frac{s_{12}}{N_2} \sin \alpha_{12} \sec \phi_2 \ . \tag{102a}$$

Now, approximating the sine terms on each side of (102a) by a trigonometric series, we can write (neglecting terms higher than the third power)

$$d\lambda - \frac{d\lambda^3}{6} + .. = (\frac{s_{12}}{N_2} - \frac{s_{12}^3}{6N_2^3} ..)(\sin \alpha_{12} \sec \phi_2) \tag{103}$$

or

$$d\lambda = \frac{s_{12}}{N_2} \sin \alpha_{12} \sec \phi_2 - \frac{s^3}{N_2^3} \sin \alpha_{12} \sec \phi_2 + \frac{d\lambda^3}{6} + .. \tag{103a}$$

Now, from the first two terms of (103a), (neglecting terms greater than the third power)

$$d\lambda^3 = \frac{s_{12}^3}{N_2^3} \sin^3 \alpha_{12} \sec^3 \phi_2 + \ldots \ , \tag{104}$$

which gives us

$$\boxed{d\lambda'' = \rho''[\frac{s_{12}}{N_2} \sin \alpha_{12} \sec \phi_2 (1 - \frac{s_{12}^2}{6N_2^2}(1 - \sin^2 \alpha_{12} \sec^2 \phi_2))],} \tag{105}$$

which when placed in (101) gives the solution for λ_2.

Although α_{21} is also a part of the direct problem, the derivation for its solution is given in the next section.

3.3 Inverse Problem

We are given the quantities ϕ_1, λ_1 of P_1, and ϕ_2, λ_2 of P_2 (Figure 20). The quantities required are s_{12}, α_{12} and α_{21}.

We begin by determining α_{21}. Using a spherical approximation

$$< P'P_2P_1 = 360-\alpha_{21} \tag{106}$$

and

$$\frac{1}{2}\,(<P'P_1P_2 + <P'P_2P_1) = \frac{1}{2}(\alpha_{12}+360-\alpha_{21}). \tag{107}$$

$$\alpha'_{12} - \alpha_{12} = d\alpha \tag{108}$$

or

$$\alpha'_{12} = d\alpha + \alpha_{12} \tag{108a}$$

where $d\alpha$ is the term which expresses the convergence of the meridians between points P_1 and P_2. Using Figure 20, we can write

$$\alpha_{21} = \alpha'_{12} + 180^\circ \tag{109}$$

and replacing α'_{12} by (108a) gives

$$\alpha_{21} = \alpha_{12} + d\alpha + 180^\circ \tag{109a}$$

Then, replacing α_{21} in (107) by (109a),

$$\frac{1}{2}\,(<P'P_1P_2 + <P'P_2P_1) = \frac{1}{2}\,(\alpha_{12}+360-\alpha_{12}-d\alpha-180^\circ) \tag{110}$$

or

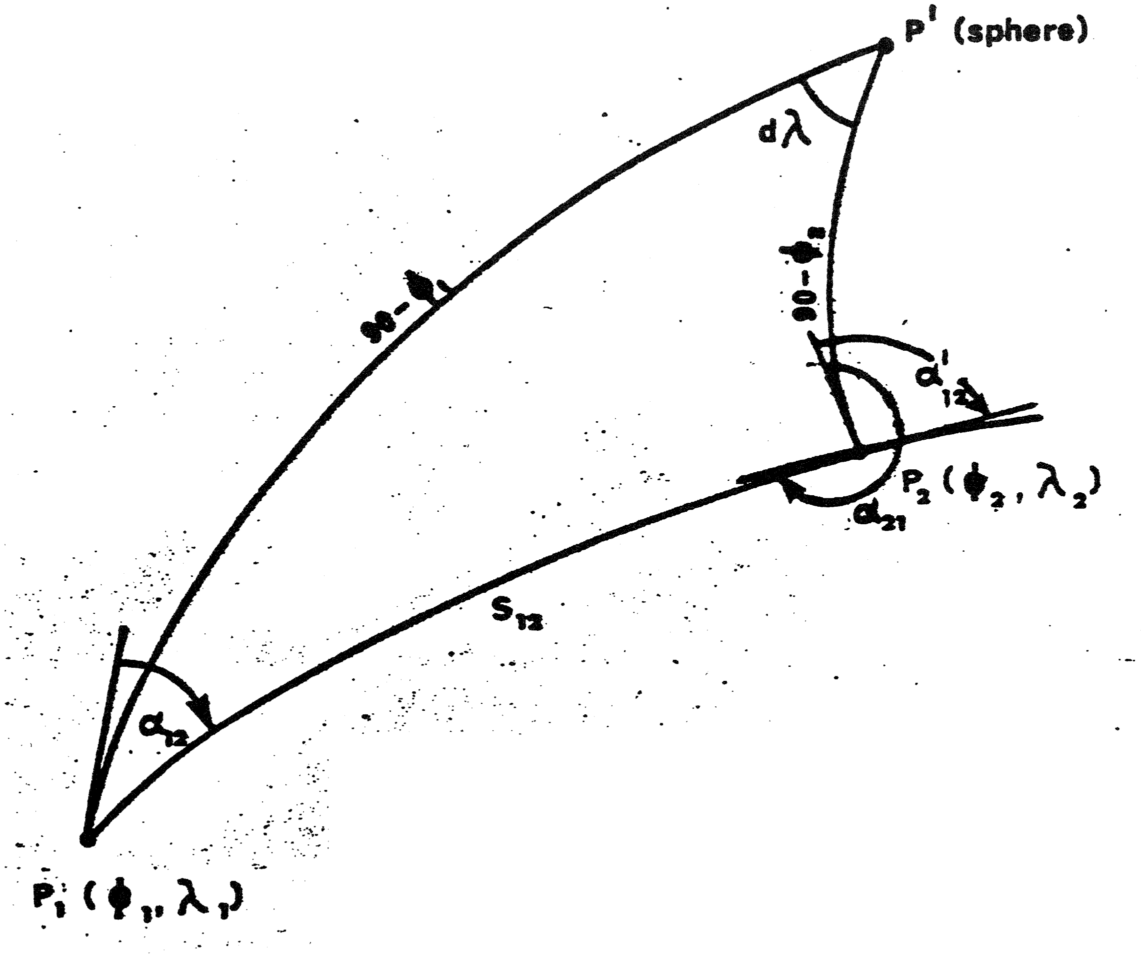


Figure 20

PUISSANT'S FORMULA FOR INVERSE PROBLEM

$$\frac{1}{2}\left(\angle P'P_1P_2 + \angle P'P_2P_1\right) = 90 - \frac{d\alpha}{2}. \qquad (110a)$$

Using spherical trigometry, the tangent law yields

$$\tan\left(90 - \frac{d\alpha}{2}\right) = \cot\frac{d\lambda}{2}\,\frac{\cos\frac{1}{2}[(90-\phi_2) - (90-\phi_1)]}{\cos\frac{1}{2}[(90-\phi_2) + (90-\phi_1)]} \qquad (111)$$

which reduces to (invert both sides of (111))

$$\tan\frac{d\alpha}{2} = \frac{\cos\left(90 - \frac{\phi_1+\phi_2}{2}\right)}{\cos\frac{1}{2}(\phi_1-\phi_2)}\tan\frac{d\lambda}{2} \qquad (112)$$

or

$$\tan\frac{d\alpha}{2} = \frac{\sin\frac{1}{2}(\phi_1+\phi_2)}{\cos\frac{d\phi}{2}}\tan\frac{d\lambda}{2} \quad . \qquad (112a)$$

Next we develop the tangent terms on both sides of (112a) which can be expressed by (neglecting terms greater than the third power)

$$\tan\frac{d\alpha}{2} = \sin\phi_m \sec\frac{d\phi}{2}\left(\frac{d\lambda}{2} + \frac{d\lambda^3}{24} + \ldots\right) \qquad (113)$$

and

$$\tan\frac{d\alpha}{2} = \frac{d\alpha}{2} + \frac{d\alpha^3}{24} + \ldots . \qquad (114)$$

which gives the final equation

$$\boxed{d\alpha'' = \rho''\left[d\lambda\sin\phi_m \sec\frac{d\phi}{2} + \frac{d\lambda^3}{12}\left(\sin\phi_m \sec\frac{d\phi}{2} - \sin^3\phi_m \sec^3\left(\frac{d\phi}{2}\right) + \ldots\right]\right.} \qquad (11$$

where ϕ_m is the mean latitude.

Replacing $d\alpha$ in (109a) by (115) gives us the required α_{21} once we have an expression for α_{12}.

The solution for α_{12} is as follows. Taking equation (99), and rearranging terms, we get

$$s_{12}\cos\alpha_{12} = \frac{d\phi''}{\rho''}\cdot\left(M_1/\left(1 - \frac{3e^2\sin\phi_1\cos\phi_1}{2(1-e^2\sin^2\phi_1)}\left(\frac{d\phi''}{\rho''}\right)\right)\right) +$$

$$+\frac{s_{12}^2\tan\phi_1\sin^2\alpha_{12}}{2N_1} + \frac{s_{12}^3\cos\alpha_{12}\sin^2\alpha_{12}(1+3\tan^2\phi_1)}{6N_1^2}; \qquad (116)$$

and using (105), a rearrangement of terms yields

$$s_{12}\sin\alpha_{12} = \frac{d\lambda''}{\rho''}\cdot\frac{N_2}{\sec\phi_2} + \frac{s_{12}^3}{6N_2^2}\sin\alpha_{12} - \frac{s_{12}^3}{6N_2^2}\sin^3\alpha_{12}\sec^2\phi_2 . \qquad (117)$$

Now, dividing (117) by (116) gives, after some manipulation of terms

$$\tan\alpha_{12} = \frac{(117)}{(116)} . \qquad (118)$$

Since α_{12} appears on the right hand side of (118), iteration is needed. First, begin by obtaining approximate values for α_{12} from (118) by using only the first term in the numerator and denominator and for s_{12} from (116) or (117), again using only the first term on the right hand side of the equations. More accurate values of α_{12} and s_{12} are obtained by using all terms in (118) and (116) or (117), respectively. Iterate until the changes in α_{12} and s_{12} are negligible. ($\Delta s \leq 0.001$ m and $\Delta\alpha_{12} < 0\overset{''}{.}001$).

3.4 Summary of Equations for the Solution of the Direct and Inverse Problems Using Puissant's Formulae

The following is an outline of the steps required for the solution of the direct problem using Puissant's formulae:

1. compute M_1 and N_1 using (13) and (15), respectively;
2. compute an approximate $d\phi''$ with (90).
3. solve for $d\phi''$ using (99), and ϕ_2 using (100);

4. compute N_2 with (15);

5. solve for $d\lambda''$ with (105) and λ_2 using (101);

6. using (115), compute $d\alpha''$ and finally α_{21} with (109a).

Similarly, we outline the steps required for the solution of the inverse problem as follows:

1. compute M_1 with (13), and N_1 and N_2 using (15);

2. compute α_{12} with (118);

3. compute $d\alpha''$ with (115), then α_{21} using (109a);

4. using either (116) or (117), compute s_{12}.

3.5 The Gauss Mid-Latitude Formulae

These formulae were first published in English in 1861. They are based on a spherical approximation of the earth and should only be used for points separated by less than 40 km at latitudes less than 80° [Allan et al, 1968]. The formulae are [Allan et al, 1968]

$$d\alpha'' = d\lambda'' \sin\phi_m , \tag{119}$$

$$d\phi'' = \rho'' \left(\frac{s_{12}\cos\alpha_m}{M_m}\right) \tag{120}$$

$$d\lambda'' = \rho'' \left(\frac{s_{12}\sin\alpha_m}{N_m\cos\phi_m}\right) . \tag{121}$$

where $$\alpha_m = \alpha_{12} + \frac{d\alpha}{2} . \tag{121a}$$

The similarities of the above formulae with the Puissant formulae are easily seen by comparing (119), (120), and (121) with the first terms of (115), (99), and (105) respectively.

In order to solve the direct problem with the mid-latitude formulae, an iterative procedure must be used. First, $d\phi''$ can be approximated using the measured azimuth in place of α_m, and M_1 can be used in place of M_m. Then, a first approximation of ϕ_2 is obtained using (100), a first approximation of $d\lambda$ via (121) and λ_2 by (101), thence $d\alpha$ is computed via (119). The iterative procedure can now be continued using successive approximate values for $d\phi$, $d\alpha$ (thus α_m and ϕ_m) until the desired limits have been reached. Finally, $d\lambda''$ is computed in order to obtain λ_2.

The inverse problem is computed without iteration since ϕ_m is immediately available. Using (119), $d\alpha$ is computed. Then, from (121) divided (120), one obtains tan α_m, thence α_{12} and α_{21} (121a). Finally, the distance s_{12} can be computed with either (120) or (121).

3.6 Other Short Line Formulae

There are many short line formulae in use. Some of these are included in [Bomford, 1971, pp. 133-139], and are called by names such as "Clarke's Approximate Formula" (1 ppm at < 150 km), and "Lilly's Approximate Formula" (15 m at 1000 km). All of these types of direct and inverse formulae (short lines) are based on spherical approximations and are not as rigorous as those such as Bessel's long line formula, developed in 4.

4. Bessel's Formulae - Long Lines

4.1 Introduction

The formulae for the direct and inverse geodetic problems developed below have been credited to Bessel [Jordon, 1962]. The derivation is based upon the geodesic on the ellipsoid. This fact distinguishes Bessel's formulae from formulae which are based on a spherical approximation (e.g. Puissant's), or even from formulae which are ellipsoidal based but use the normal section curve as the foundation for the derivation (e.g. Robbins, 1962).

The accuracy of the Bessel formulae is not limited by the separation between the two points in question nor by the location of the points on the earth. The accuracy is simply limited by the number of terms one wishes to retain in the series development of the various expressions.

The following derivation begins by developing the relationship between corresponding elements on the sphere and ellipsoid (not a spherical approximation but a rigorous treatment). The solution of an elliptical integral is then performed. Finally the direct and inverse problems are enunciated.

4.2 Fundamental Relationships

We begin by establishing some rigorous relationships between parameters on the sphere and parameters on the ellipsoid. In section (1.3.2), we developed the basic property of a geodesic (47), which on a sphere can be expressed as

$$\cos \beta \sin \alpha = \cos \beta_o \ , \tag{122}$$

where β is the reduced latitude [Krakiwsky and Wells, 1971, p 23], and β_o is called the "turning point" reduced latitude ($\alpha = 90°$). From Figure 21a, α on the reduced sphere is equal to α on the ellipsoid, as are β on the reduced sphere and β on the ellipsoid, thus we can write for both

$$\cos \beta \sin \alpha = \cos \beta_o \ . \tag{122a}$$

We now develop some differential relationships with the aid of Figure 21b. From the triangles in the spherical figures, we can write

$$a d\sigma \cos \alpha_{12} = a d\beta \ ,$$

and (123)

$$a d\sigma \sin \alpha_{12} = a \cos \beta' \, d\lambda,$$

where a is the radius of the reduced sphere (Figure 22), and $d\sigma$ is the angle subtended (at the origin of the sphere) by the normals at P and P'. Similarly, from the triangles in the ellipsoidal figure we can write

$$ds \cos \alpha_{12} = M d\phi$$

and (124)

$$ds \sin \alpha_{12} = N' \cos \phi' d\ell \ .$$

Dividing (124) by (123) yields

$$\frac{ds}{d\sigma} = \frac{M d\phi}{d\beta} = N' \frac{\cos \phi'}{\cos \beta'} \frac{d\ell}{d\lambda} \ . \tag{125}$$

From Figure 22 and equation (17)

$$N' \cos \phi' = a \cos \beta' \ , \tag{126}$$

which when substituted in (125) gives

$$\frac{ds}{d\sigma} = a \frac{d\ell}{d\lambda} \tag{127}$$

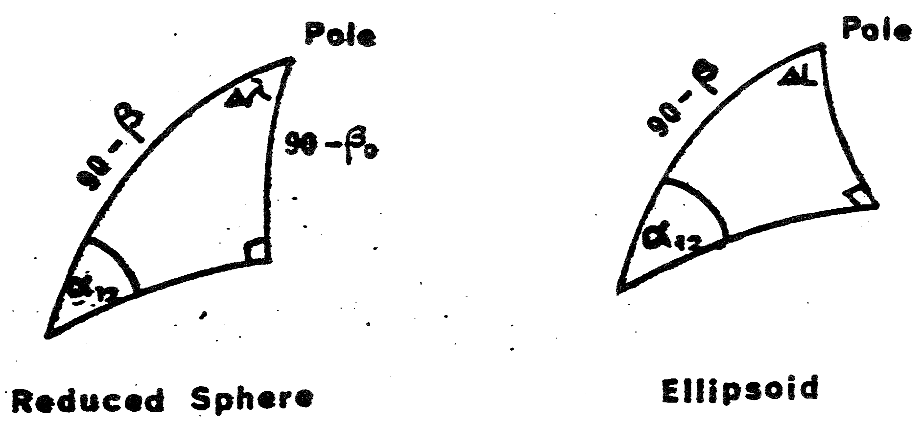
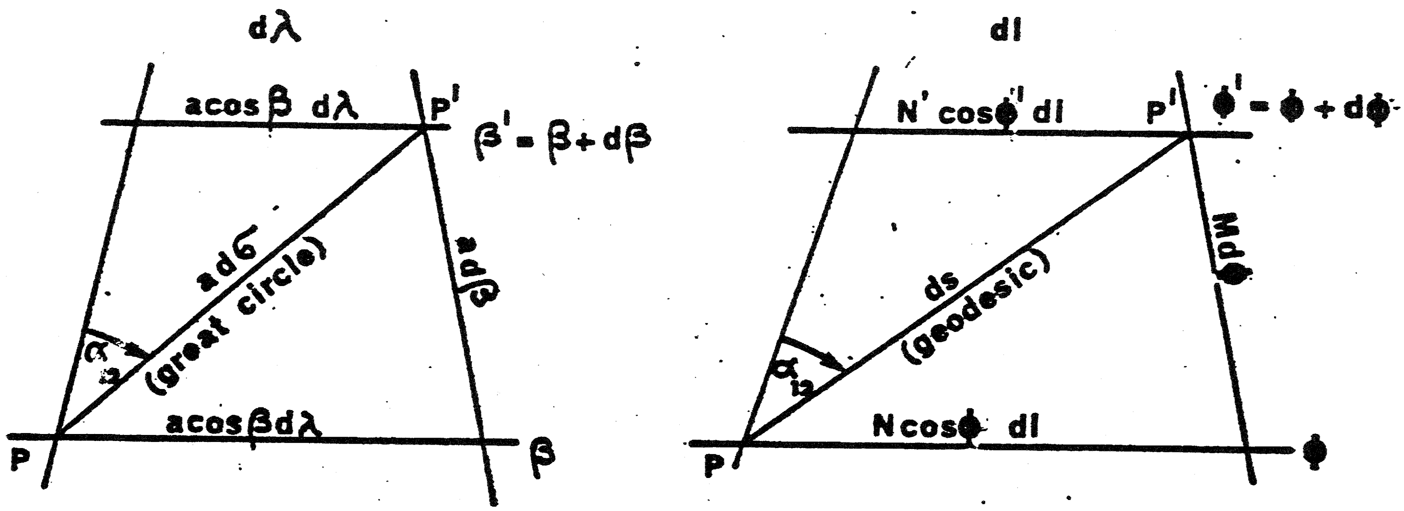


Figure 21a

FUNDAMENTAL RELATIONSHIPS FOR THE DEVELOPMENT OF BESSEL'S FORMULAE



Reduced Sphere

Ellipsoid

Figure 21b

FUNDAMENTAL RELATIONSHIPS FOR THE DEVELOPMENT OF BESSEL'S FORMULAE

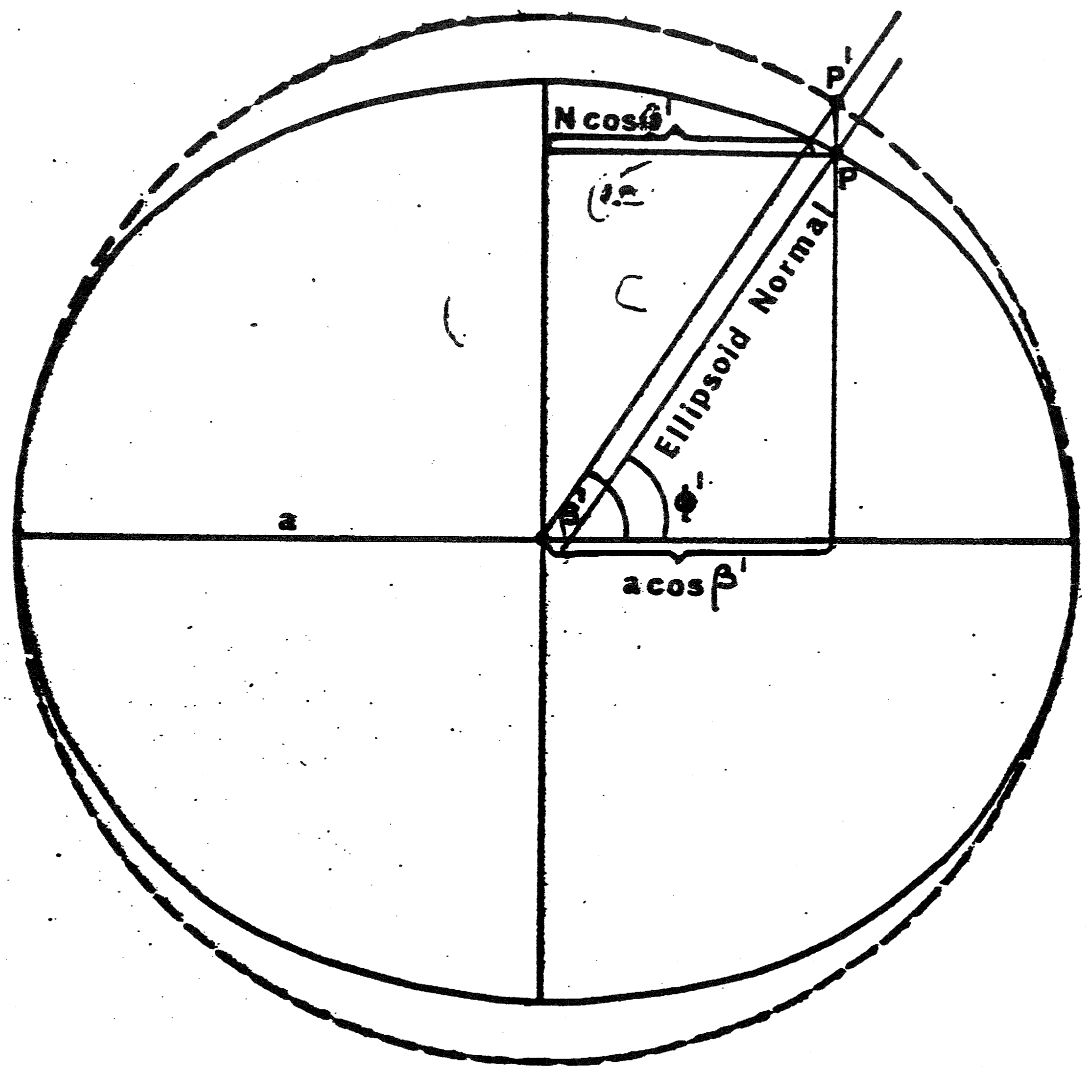


Figure 22

REDUCED SPHERE AND ELLIPSOID

or

$$\frac{d\ell}{d\lambda} = \frac{1}{a}\,\frac{ds}{d\sigma} \quad , \tag{127a}$$

which, from (125) yields

$$\frac{d\ell}{d\lambda} = \frac{M}{a}\,\frac{d\phi}{d\beta} \tag{127b}$$

Recalling that [Krakiwsky and Wells, 1971, p 28]

$$\tan \beta = (1-e^2)^{1/2} \tan \phi, \tag{128}$$

we can differentiate and get

$$\frac{d\beta}{\cos^2\beta} = (1-e^2)^{1/2}\,\frac{d\phi}{\cos^2\phi} \quad , \tag{129}$$

or

$$\frac{d\phi}{d\beta} = \frac{1}{(1-e^2)^{1/2}}\,\frac{\cos^2\phi}{\cos^2\beta} \quad , \tag{129a}$$

which when substituted in (127b) gives

$$\frac{d\ell}{d\lambda} = \frac{M}{a(1-e^2)^{1/2}}\,\frac{\cos^2\phi}{\cos^2\beta} \tag{130}$$

for any point on the ellipsoid.

Now, we want to get

$$d\ell/d\lambda = f(\beta) \ .$$

We begin by expressing

$$a \cos \beta = \frac{c}{V} \cos \phi \tag{131}$$

where

$$V = (1-e^2 \cos^2\beta)^{-1/2} \tag{132}$$

and(the curvature at the pole - equation (5a))

$$c = \frac{a^2}{b} \quad . \tag{133}$$

Squaring (123), and rearranging terms gives

$$\frac{d\ell}{d\lambda} = \frac{a}{Vc} \frac{1}{(1-e^2)^{1/2}} \tag{134}$$

where

$$M = \frac{C}{V^3} \quad . \tag{135}$$

A further reduction of (134), using (133), (3) and (131) finally yields

$$\frac{d\ell}{d\lambda} = \frac{1}{V} = \frac{1}{a}\frac{ds}{d\sigma} \quad . \tag{136}$$

Before proceeding further, we will derive (132). From (131)

$$\cos \phi = (\frac{b}{a}) \; V \cos \beta , \tag{137}$$

which when squared yields

$$\cos^2\phi = \frac{b^2}{a^2} \; V^2 \cos^2\beta \tag{137a}$$

or

$$\cos^2\phi = (1-e^2) \; V^2 \cos^2\beta \quad . \tag{137b}$$

Substituting (137b) in (137),

$$V^2 = 1+e'^2(1-e^2) \; V^2\cos^2\beta \quad , \tag{138}$$

which reduces to

$$V^2 \; [1-e'^2(1-e^2) \cos^2\beta] = 1 \; . \tag{138a}$$

Now, from equations (3) and (4),

$$(1-e^2) \; (1+e^2) = 1 \tag{139}$$

and

$$e^2 = e'^2 \, (1-e^2) \qquad (139a)$$

which when substituted in (138a) gives

$$V^2 \, (1-e^2\cos^2\beta) = 1 \qquad (140)$$

or

$$V = (1-e^2\cos^2\beta)^{-1/2} \, . \qquad (140a)$$

Returning back to the problem at hand we substitute (140a) in (136) we get

$$\frac{d\ell}{d\lambda} = (1-e^2 \cos^2\beta)^{1/2} \qquad (141)$$

and

$$\frac{ds}{d\sigma} = a(1-e^2\cos^2\beta)^{1/2} \, , \qquad (142)$$

respectively.

4.3 Solution of the Elliptic Integral

Next we solve (141) and (142), and we do so by integration. We begin by solving (142), to get a solution for $ds/d\sigma$. From Figure 23, we use the sine law of spherical trigonometry and obtain

$$\frac{\sin \alpha_{12}}{\sin(90°-\beta_o)} = \frac{\sin 90°}{\sin(90°-\beta_1)} \qquad (143)$$

or

$$\cos \beta_o = \sin \alpha_{12} \cos \beta_1 \, , \qquad (143a)$$

the fundamental property of a geodesic and great circle. Further, using Napiers rule of circular parts

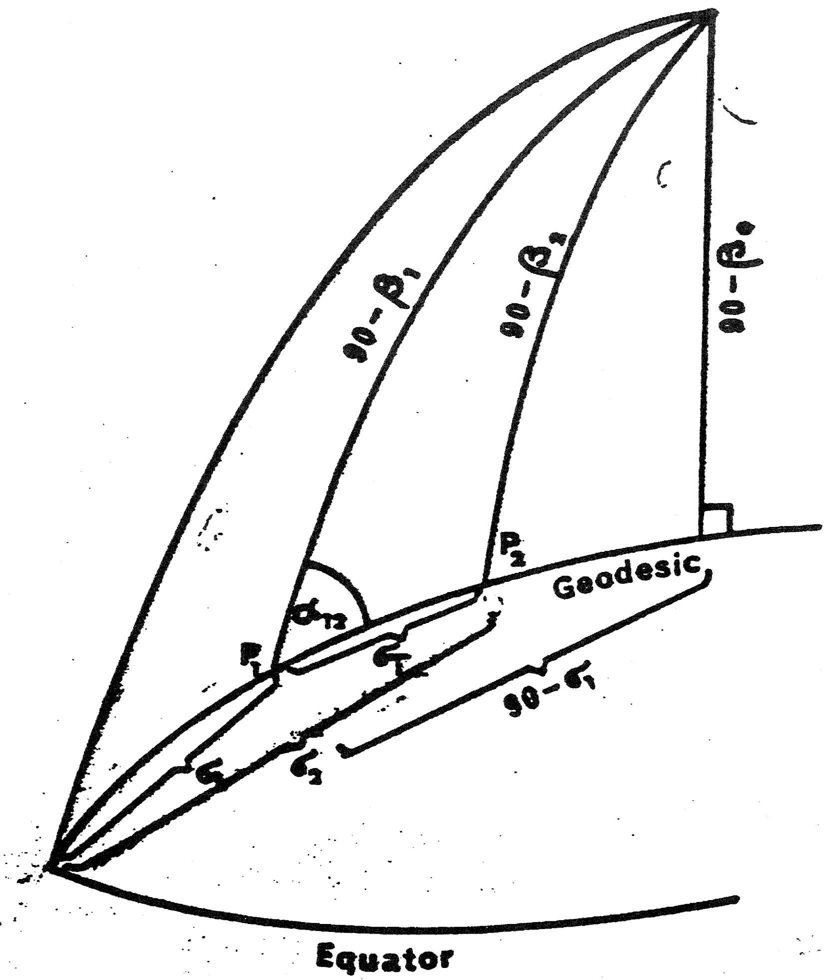


Figure 23

SOLUTION OF $\frac{ds}{d\sigma}$

$$\cos \alpha_{12} = \cot \sigma_1 \tan \beta_1 \tag{144}$$

or

$$\tan \sigma_1 = \frac{\tan \beta_1}{\cos \alpha_{12}} \quad , \tag{144a}$$

and another required relationship

$$\sin \beta_2 = \sin (\sigma_1 + \sigma_T) \sin \beta_o \ . \tag{145}$$

We generalize (145) for integration purposes (between points P_1 and P_2, Figure 23) as

$$\sin \beta = \sin (\sigma_1 + \sigma) \sin \beta_o \tag{145a}$$

so that σ is variable, reckoned from point P_1. Note that when $\sigma = \sigma_2$, $\beta = \beta_2$ and when $\sigma = 0$, $\beta = \beta_1$.

Rewriting (142) as

$$ds = a(1-e^2\cos^2\beta)^{1/2} \, d\sigma \ , \tag{146}$$

and then solving for $\cos^2\beta$ from 145a by

$$\cos^2\beta = 1-\sin^2 (\sigma_1+\sigma) \sin^2\beta_o \ , \tag{147}$$

in which we substitute $\sigma_1 = \sigma$, and $x = \sigma_1+\sigma$ (a new variable for integration), then $dx = d\sigma$ and we rewrite (147) as

$$\cos^2\beta = 1 - \sin^2 x \sin^2\beta_o \ , \tag{147a}$$

which finally gives

$$ds = a(1-e^2+e^2 \sin^2\beta_o \sin^2 x)^{1/2} \, dx \ . \tag{148}$$

From (3) and (4),

$$e^2 = \frac{e'^2}{1+e'^2} \quad \text{and} \quad 1-e^2 = \frac{1}{1-e'^2} \tag{149}$$

which when substituted in (148) gives

$$ds = a\left[\frac{1}{1+e'^2} + \frac{e'^2}{1+e'^2}\sin^2\beta_o\,\sin^2 x\right]^{1/2} dx \qquad (148a)$$

or

$$ds = \frac{a}{(1+e'^2)^{1/2}}\,(1+e'^2\sin^2\beta_o\sin^2 x)^{1/2} dx\,. \qquad (149)$$

Since

$$\frac{b}{a} = \frac{1}{(1+e'^2)^{1/2}}\;. \qquad (150)$$

and setting

$$k^2 = e'^2\,\sin^2\,\beta_o\;, \qquad (151)$$

(149) finally becomes

$$ds = b(1 + k^2\sin^2 x)\;dx\;. \qquad (152)$$

This expression is now integrated and evaluated for our particular parameters, which yields

$$s = b\int_{x=\sigma_1}^{x=\sigma_1+\sigma_T}(1+k^2\sin^2 x)^{1/2}dx\;. \qquad (153)$$

In mathematics this is known as an elliptical integral [Abramowitz and Segun, 1968, p 589]. The limits on $x(\sigma+\sigma_T)$ are

$$0 \leq \sigma \leq \sigma_T\;, \qquad (154)$$

then when

$$\sigma = 0,\; x = \sigma_1, \qquad (154a)$$

and when

$$\sigma = \sigma_T,\; x = \sigma_1+\sigma_T \qquad (154b)$$

Solving equation (153), we know that because k^2 is small, then

$$(1+k^2 \sin^2 x)^{1/2} = 1 + \frac{1}{2} k^2 \sin^2 x - \frac{1}{8} k^4 \sin^4 x + \frac{k^6}{16} \sin^6 x - \ldots \quad (155)$$

Using the trigonometric identities,

$$\sin^2 x = \frac{1}{2} (1-\cos 2x) \quad (156)$$

$$\sin^4 x = \quad \ldots\ldots$$

$$\vdots$$

etc.

and substituting in (155) gives

$$(1+k^2\sin^2 x)^{1/2} = [1 + \frac{k^2}{4} - \frac{3}{64} k^4 + \ldots] + [-\frac{1}{4} k^2 + \frac{1}{16} k^4 + \ldots]\cos x -$$

$$- \frac{k^4}{64} \cos 4x + \ldots \quad (155a)$$

Replacing

$$A = 1 + \frac{k^2}{4} - \frac{3}{64} k^4 + \ldots , \quad (157)$$

$$B = \frac{1}{4} k^2 - \frac{1}{16} k^4 + \ldots , \quad (157a)$$

$$C = \frac{k^4}{64} + \quad , \quad (157b)$$

$$D = \quad \ldots\ldots \quad (157c)$$

in (153) gives

$$\frac{s}{b} = A \int_{\sigma_1}^{\sigma_1+\sigma_T} dx - B \int_{\sigma_1}^{\sigma_1+\sigma_T} \cos 2x dx - C \int_{\sigma_1}^{\sigma_1+\sigma_T} \cos 4x dx - \ldots \quad (158)$$

Before carrying out the actual integration of (157), we consider the solution of general integral

$$\int_{\sigma_1}^{\sigma_1+\sigma_T} \cos nx dx = \frac{1}{n} \sin nx \Big|_{\sigma_1}^{\sigma_1+\sigma_T} \quad (159)$$

$$= \frac{1}{n} [\sin (\sigma_1+\sigma_T) - \sin n\sigma_1] \quad (159a)$$

Another substitution yields a better form, namely

$$\sin nx - \sin ny = 2 \cos \frac{n}{2}(x+y) \sin \frac{n}{2}(x-y) , \tag{160}$$

which when associated with our problem, we set

$$x = \sigma_1 + \sigma_T \quad ,$$

$$y = \sigma_1 \quad , \tag{161}$$

then

$$x + y = 2\sigma_1 + \sigma_T \quad , \tag{161a}$$

and

$$x - y = \sigma_T \quad .$$

Now, in (159a), the right hand side becomes

$$\sin n(\sigma_1+\sigma_T) - \sin n\sigma_1 = 2 \cos \frac{n}{2}(2\sigma_1+\sigma_T) \sin \frac{n}{2}\sigma_T \tag{162}$$

Now, evaluating (158), we get

$$\int_{\sigma_1}^{\sigma_1+\sigma_T} dx = \sigma_T \quad , \tag{163}$$

$$\int_{\sigma_1}^{\sigma_1+\sigma_T} \cos 2x dx = \cos (2\sigma_1+\sigma_T)\sin \sigma_T , \tag{163a}$$

$$\int_{\sigma_1}^{\sigma_1+\sigma_T} \cos 4x dx = \frac{1}{2}\cos(4\sigma_1+2\sigma_T)\sin 2\sigma_T \quad , \tag{163b}$$

etc.

Setting

$$\sigma_T = \sigma_2 - \sigma_1 \quad , \tag{164}$$

then

$$2\sigma_1 + \sigma_T = 2\sigma_1 + \sigma_2 - \sigma_1 \tag{164a}$$

or

$$2\sigma_1 + \sigma_T = \sigma_1 + \sigma_2 , \tag{164b}$$

and

$$2\sigma_m = \frac{\sigma_1 + \sigma_2}{2} \tag{164c}$$

or

$$2\sigma_m = 2\sigma_1 + \sigma_T \quad . \tag{164d}$$

When substituted in (163), the solution to (158) is

$$\boxed{\frac{s}{b} = A\sigma_T - B \cos \sigma_m \sin \sigma_T - \frac{C}{2} \cos 4\sigma_m \sin 2\sigma_T - \frac{D}{3} \cos 6\sigma_m \sin 3\sigma_T - \ldots} \tag{165}$$

From (164), we get a solution for σ_T as

$$\sigma_T = \frac{s}{Ab} + \frac{B}{A} \cos 2\sigma_m \sin \sigma_T + \frac{C}{2A} \cos 4\sigma_m \sin 2\sigma_T + \ldots \tag{166}$$

where

$$\begin{aligned} A &= 1 + \frac{k^2}{4} - \frac{3}{64} k^4 + \ldots , \\ B &= \frac{1}{4} k^2 - \frac{1}{16} k^4 + \ldots , \\ C &= \frac{k^4}{64} + \ldots , \\ D &= \ldots , \\ E &= \frac{5}{65536} k^8 , \\ k^2 &= e'^2 \sin^2 \beta_o \quad . \end{aligned} \tag{166a}$$

This represents the integration of the distance on the ellipsoid with respect to the distance on the sphere.

Now we turn our attention to the solution of $\frac{d\ell}{d\lambda}$ (141). Rewriting (141), we get

$$d\ell = (1-e^2 \cos^2\beta)^{1/2} \, d\lambda \tag{141a}$$

From Figure 24 ,

$$d\lambda \cos \beta = d\sigma \sin \alpha_{12} \tag{167}$$

or

$$d\lambda = \frac{\sin \alpha_{12}}{\cos \beta} \, d\sigma \tag{167a}$$

Applying the sine law (spherical trigonometry)

$$\frac{\sin \alpha_{12}}{\sin(90-\beta_o)} = \frac{\sin 90}{\sin(90-\beta)} \tag{168}$$

or

$$\sin \alpha_{12} = \frac{\cos \beta_o}{\cos \beta} \quad , \tag{168a}$$

which when substituted in (167a) yields

$$d\lambda = \frac{\cos \beta_o}{\cos^2\beta} \, d\sigma \quad . \tag{167b}$$

Substituting for $d\lambda$ in (141a), we get

$$d\ell = (1-e^2\cos^2\beta)^{1/2} \frac{\cos\beta_o}{\cos^2\beta} \, d\sigma \; . \tag{169}$$

Next we take $d\ell$ minus (167b) which gives

$$d\ell - d\lambda = \cos \beta_o \left[\frac{(1-e^2\cos^2\beta)^{1/2}}{\cos^2\beta} - \frac{1}{\cos^2\beta}\right] d\sigma \; . \tag{170}$$

Expanding $(1-e^2\cos^2\beta)^{1/2}$ in a series yields

$$(1-e^2\cos^2\beta)^{1/2} = 1 - \frac{e^2}{2} \cos^2\beta - \frac{e^4}{8} \cos^4\beta - \frac{e^6}{16} \cos^6\beta - \ldots \tag{171}$$

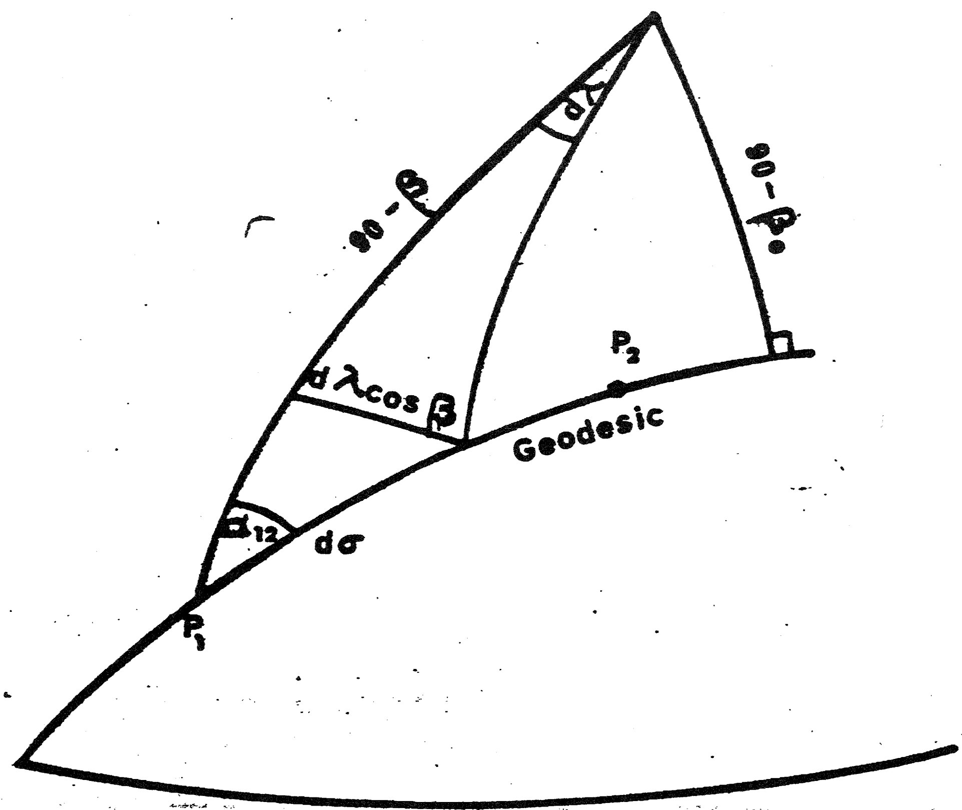


Figure 24

SOLUTION OF $\frac{dl}{d\lambda}$

which when divided by $\cos^2\beta$ gives

$$I = \frac{1}{\cos^2\beta} - \frac{e^2}{2} - \frac{e^4}{8}\cos^2\beta - \frac{e^6}{16}\cos^4\beta - \ldots \tag{171a}$$

Equation (170) is now

$$d\ell = d\lambda - \cos\beta_o\left[\frac{e^2}{2} + \frac{e^4}{8}\cos^2\beta + \frac{e^6}{16}\cos^4\beta + \ldots\right] d\sigma \tag{172}$$

or

$$d\ell = d\lambda - \frac{e^2}{2}\cos\beta_o\left[1 + \frac{e^2}{4}\cos^2\beta + \frac{e^4}{8}\cos^4\beta + \ldots\right]d\sigma \;. \tag{172a}$$

For the solution of (172a), we replace $\cos^2\beta$, $\cos^4\beta$, etc. by

$$\cos^2\beta = 1 - \sin^2\beta_o \sin^2 x \tag{173}$$

and

$$\cos^4\beta = 1 - 2\sin^2\beta_o \sin^2 x + \sin^4\beta_o \sin^4 x \;, \tag{173a}$$

(x is defined on page 64), which when placed in (172a) yields

$$d\ell = d\lambda - \frac{e^2}{2}\cos\beta_o\left[1 + \frac{e^2}{4}(1-\sin^2\beta_o\sin^2 x) + \frac{e^4}{8}(1-2\sin^2\beta_o\sin^2 x + \sin^4\beta_o \sin^4 x) + \ldots\right] dx \;. \tag{174}$$

The above expression is simplified and set up for integration in much the same manner as was done for the solution of $ds/d\sigma$. The results are as follows. The longitude difference on the ellipsoid is given by

$$L = \int_{L_1}^{L_2} d\ell \;, \tag{175}$$

and on the sphere by

$$\lambda = \int_{\lambda_1}^{\lambda_2} d\lambda \;. \tag{175a}$$

Then

$$L = \lambda - \frac{e^2}{2} \cos \beta_o [\int_{\sigma_1}^{\sigma_1+\sigma_T} (A'+B' \cos 2x+C'\cos 4x + \ldots) dx], \quad (176)$$

where

$$A' = 1 + \frac{e^2}{4} + \frac{e^4}{8} - \frac{e^2}{8} \sin^2\beta_o - \frac{e^4}{8} \sin^2\beta_o + \frac{3}{64} e^4\sin^4\beta_o + \ldots \quad (177)$$

$$B' = \frac{e^2}{8} \sin^2\beta_o + \frac{e^4}{8} \sin^2\beta_o - \frac{e^4}{16} \sin^4\beta_o + \ldots \quad , \quad (177a)$$

and

$$C' = \frac{e^4}{64} \sin^4\beta_o + \ldots \quad . \quad (177b)$$

$$D' = \ldots .$$

The result is then given by

$$L = \lambda - \frac{e^2}{2} \cos \beta_o [A'\sigma_T + B'\sin\sigma_T \cos 2\sigma_m + \frac{C'}{2} \sin 2\sigma_T \cos 4\sigma_m + \ldots$$

$$+ \frac{D'}{3} \sin^3\sigma \cos 6\sigma_m + \ldots] , \quad (178)$$

$$\boxed{(\lambda-L) = \frac{e^2}{2}\cos \beta_o [A'\sigma_T + B'\sin \sigma_T \cos 2\sigma_m + \frac{C'}{2} \sin 2\sigma_T \cos 4\sigma_m + \frac{D'}{3} \sin 3\sigma \cos 6\sigma_m + \ldots] . \quad (179)}$$

Now, with all the necessary relationships developed, we will turn our attention to the direct and inverse problems.

4.4 Direct Problem

Recall that for the direct problem we must know the geodetic coordinates ϕ_1, λ_1 of one point P_1, and the geodetic (geodesic) distance s_{12} and azimuth α_{12} to another point P_2, then we solve for ϕ_2, λ_2 of P_2

and α_{21}. The steps in the solution are as follows:

1. compute the reduced latitude β_1, using (128);

2. compute the azimuth of the geodesic at the equator, that is

$$\sin \alpha = \sin \alpha_{12} \cos \beta_1 \; ; \tag{122a}$$

3. compute the approximate spherical arc σ_o from (166) using only the first term (e.g. $\sigma_o = \frac{s}{bA}$), then compute σ_{i+1} by

$$\sigma_{i+1} = \sigma_o + \frac{B}{A} \cos 2\sigma_m \sin \sigma_i + \ldots ,$$

where the first iteration, $\sigma_i = \sigma_o$, and recall that

$$2\sigma_m = 2\sigma_1 + \sigma_i ,$$

in which σ_1 is solved for by (144a); this step is repeated until say $|\sigma_{i+1} - \sigma_i| \leq 0\overset{''}{.}00001$;

4. compute β_2 by (145), where β_o is computed using (143a);

5. compute ϕ_2 using (128);

6. compute the spherical longitude difference λ using the sine law (Figure 24), which gives

$$\sin \lambda = \frac{\sin \sigma \sin \alpha_{12}}{\cos \beta_2} \; ; \tag{180}$$

where the first approximation of σ is given by (181).
Then, using λ from (180), compute α, $\cos 2\sigma_m$, $\cos 4\sigma_m$, $\cos 6\sigma_m$ using (184), (185), (185a) and (185b), respectively; using (179), solve for $(\lambda-L)$; this step is then repeated, with $L = \lambda - (\lambda-L)$ (186), until say $|(\lambda-L)_i - (\lambda-L)_{i+1}| < 0\overset{''}{.}00001$; finally;

$$\lambda_2 = \lambda_1 + L.$$

7). the reverse azimuth is then computed via (186a) or (187a).

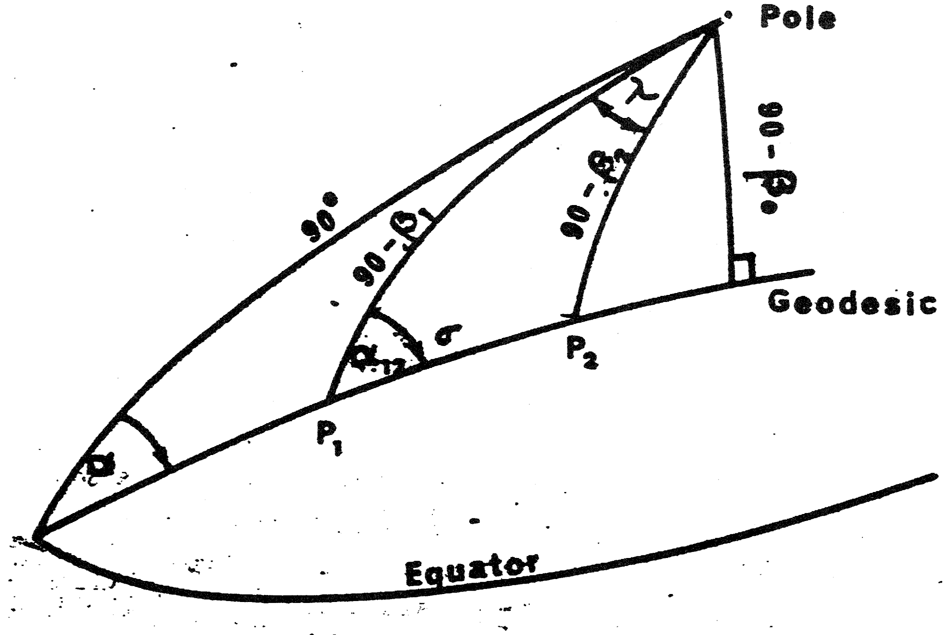


Figure 25

SOLUTION OF ARC LENGTH σ

4.5 Inverse Problem

In this problem we are given P_1 (ϕ_1, λ_1) and P_2 (ϕ_2, λ_2), from which we compute s_{12}, α_{12}, and α_{21}.

The first step is to compute β_1 and β_2 (reduced latitudes) using (128). Then, from the reduced sphere (Figure 25) we can compute the arc length ($\sigma = \sigma_T$) by using the cosine law of spherical trigonometry as

$$\cos \sigma = \sin \beta_1 \sin \beta_2 + \cos \beta_1 \cos \beta_2 \cos\lambda, \tag{181}$$

or

$$\sin \sigma = [(\sin \lambda \cos \beta_2)^2 + (\sin \beta_2 \cos \beta_1 - \sin \beta_1 \cos \beta_2 \cos \lambda)^2] \tag{181a}$$

Since this an iterative problem, (181) is solved first using $\lambda \approx L$ in the first approximation. We then compute

$$\sin \alpha_{12} = \frac{\sin \lambda \cos \beta_2}{\sin \sigma} . \tag{182}$$

To compute the azimuth of the geodesic at the equator, α, we combine (143a) and (182), which yields

$$\sin \alpha_{12} \cos \beta_1 = \sin \alpha \cos 0^\circ \tag{183}$$

or

$$\sin \alpha_{12} = \frac{\sin \alpha}{\cos \beta_1} , \tag{183a}$$

which when replaced in (182) yields

$$\sin \alpha = \frac{\cos \beta_1 \cos \beta_2 \sin \lambda}{\sin \sigma} . \tag{184}$$

Once again, $\sin \alpha$ is only a first approximation since $\lambda \approx L$.

Then compute

$$\cos 2\sigma_m = \cos\sigma - \frac{2\sin\beta_1 \sin\beta_2}{\cos^2\alpha} , \tag{185}$$

$$\cos 4\sigma_m = 2\cos^2 2\sigma_m - 1 , \tag{185a}$$

and

$$\cos 6\sigma_m = 4\cos^3 2\sigma_m - 3\cos 2\sigma_m \tag{185b}$$

We then use (179) to compute (λ-L). After completing this step, we compute

$$\lambda = L + (\lambda - L) , \tag{186}$$

and return to (181) and recompute quantities σ, α, $2\sigma_m$, $4\sigma_m$, $6\sigma_m$ using (181), (184), (185), (185a) and (185b), respectively. After recomputing (λ-L) using (179), we test $|(\lambda-L)_{i+1} - (\lambda-L)_i| \leq 0\overset{''}{.}00001$. When this test passes, we continue to compute α_{12}, α_{21} and s_{12}. The forward azimuth is computed using (183a), which is rewritten here as

$$\boxed{\sin\alpha_{12} = \frac{\sin\alpha}{\cos\beta_1} ,} \tag{186}$$

and

$$\boxed{\sin\alpha_{21} = \frac{\sin\alpha}{\cos\beta_2} .}$$

Alternately, the azimuths can be computed by

$$\boxed{\tan\alpha_{12} = \frac{\sin\lambda\cos\beta_2}{\sin\beta_2\cos\beta_1 - \cos\lambda\sin\beta_1\cos\beta_2}} \tag{187}$$

and

$$\boxed{\tan\alpha_{21} = \frac{\sin\lambda\cos\beta_1}{\sin\beta_2\cos\beta_1\cos\lambda - \sin\beta_1\cos\beta_2}} \tag{187a}$$

To complete the problem, the distance s_{12} is computed using (165).

4.6 Other Long Line Formulae

Many methods for the solution of the direct and inverse problems, for widely separated points on a reference ellipsoid, are available in the literature. As with the "short" and "medium" line formulae, they are generally given the names of their originators. Two of these, which have been used by the authors, are the methods of Rainsford [Rainsford, 1955] and Sodano [Sodano, 1963]. Rainsford's formulae are developed on the same principles as Bessel's. The major difference is that the coefficients of the longitude difference (179) are developed in terms of f, since they converge more rapidly than when given as a function of e^2. The main difference between Sodano's method, and those of Bessel and Rainsford, is that both the direct and inverse problems can be solved in a non-iterative fashion.

SECTION IV. COMPUTATION OF GEODETIC POSITIONS IN THREE DIMENSIONS

The geodetic position of a terrain point can be described mathematically in terms of a triplet of cartesian coordinates (x, y, z), referred to the average terrestrial, geodetic, local geodetic or local astronomic coordinate systems, or by geodetic latitude (ϕ), longitude (λ) and ellipsoidal height (h) referred to some reference ellipsoid. In the previous sections, which presented the classical two dimensional position computations, geodetic positions were described by only two coordinates, namely the geodetic latitude and longitude. The third component, the ellipsoidal height, was used only for the reduction of terrestrial measurements to the reference ellipsoid.

Computations of geodetic positions in three dimensions differ from the classical two dimensional approach in two significant ways. The first is that the latter has its basis in ellipsoidal geometry, while the former is based on three dimensional Euclidean principals and employs vector and matrix algebra. Secondly, the classical approach requires the use of geodesic distances and azimuths for rigorous computations, while straight line spatial distances (chords) and normal section three dimensional azimuths are used in three dimensional computations. Regarding the azimuth used herein, it should be noted that it refers to the normal section passing through the terrain points in question, and not that section which passes through the points projected on the reference ellipsoid. In view of the different treatment of observations in three dimensional position computations, no special chapter regarding them is presented. Instead, full explanations are given, where required, within the context of the development of the direct, inverse, azimuth intersection and spatial distance intersection problems.

5. DIRECT AND INVERSE PROBLEMS IN THREE DIMENSIONS

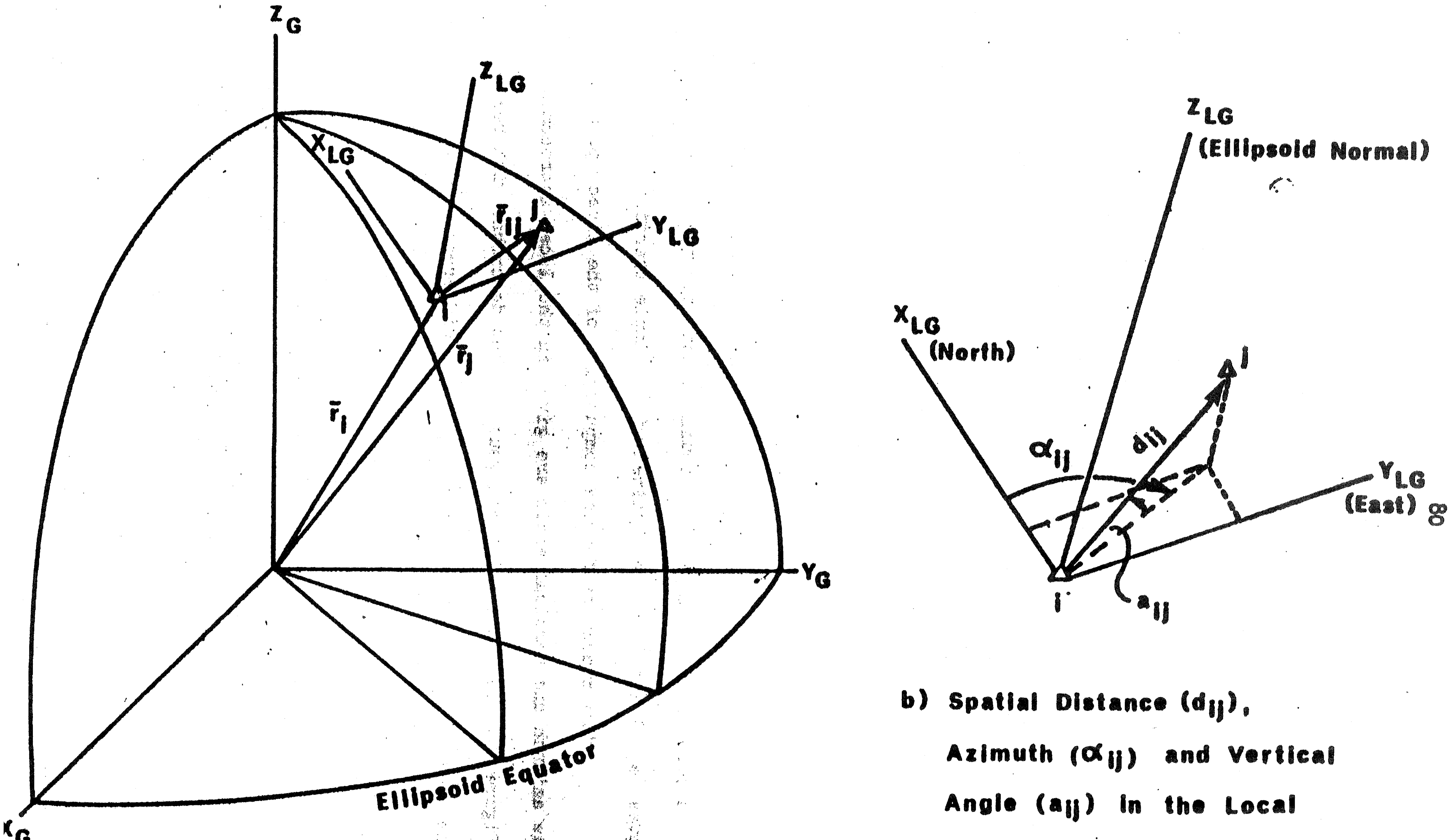
5.1 Direct Problem

The direct problem can be stated as: Given the coordinates (x_i, y_i, z_i) or (ϕ_i, λ_i, h_i) of a point i, and the terrestrial spatial distance, azimuth, and vertical angle (or height difference) to a second point j, compute the coordinates (x_j, y_j, z_j) or (ϕ_j, λ_j, h_j). Two cases of the direct problem may arise, depending on whether the azimuth and vertical angle are referred to the local geodetic (ellipsoid normal) or the local astronomic (gravity vertical) coordinate systems. We thus denote azimuths and vertical angles in the local geodetic system by α and a, and likewise the local astronomic system by A and v respectively (Figure 26).

The simplest method of solution of three dimensional problems is to use cartesian coordinates. If the coordinates which are required in the computations, are given by (ϕ, λ, h), a simple coordinate transformation [Krakiwsky and Wells, 1971] yields the cartesian coordinates. Similarly, if the results required are those of latitude, longitude and ellipsoidal height, then the cartesian coordinates are transformed to (ϕ, λ, h) after the position computations are completed [Krakiwsky and Wells, 1971].

The vector between two terrain points in a geodetic coordinate system is given by the expression

$$(\bar{r}_{ij})_G = \begin{bmatrix} x_j - x_i \\ y_j - y_i \\ z_j - z_i \end{bmatrix}_G = \begin{bmatrix} \Delta x_{ij} \\ \Delta y_{ij} \\ \Delta z_{ij} \end{bmatrix}_G . \tag{188}$$



a) Geodetic and Local Geodetic Systems

b) Spatial Distance (d_{ij}), Azimuth (α_{ij}) and Vertical Angle (a_{ij}) in the Local Geodetic System

Figure 26

DIRECT PROBLEM (LOCAL GEODETIC)

Now, the position vector of a point j, in the local geodetic system at i (Figure 26) is given by

$$\boxed{(\bar{r}_{ij})_{LG} = \begin{bmatrix} d_{ij} \cos a_{ij} \cos \alpha_{ij} \\ d_{ij} \cos a_{ij} \sin \alpha_{ij} \\ d_{ij} \sin a_{ij} \end{bmatrix}}, \qquad (189)$$

and $(\bar{r}_{ij})_G$ can be written

$$\boxed{(\bar{r}_{ij})_G = R_3\,(180-\lambda_i)\;R_2\,(90-\phi_i)\;P_2\,(\bar{r}_{ij})\;.} \qquad (190)$$

The reflection matrix, P_2, and the two rotation matrices, R_2 and R_3, transform the topocentric vector from the local geodetic system into the geodetic system. The position vector of the second point j, is obtained by vector addition as

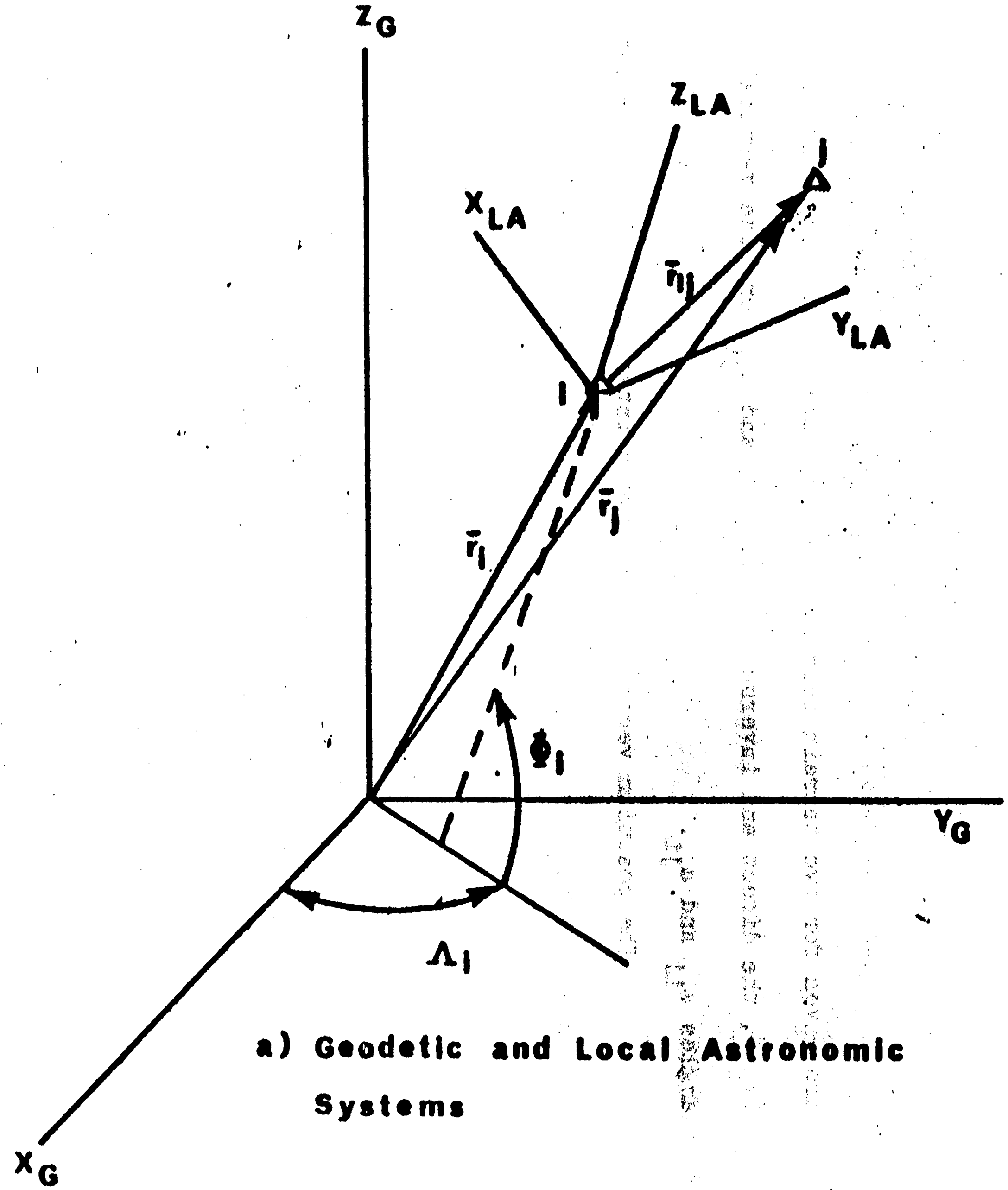
$$\boxed{(\bar{r}_j)_G = (\bar{r}_i)_G + (\bar{r}_{ij})_G} \qquad (191)$$

where (r_{ij}) is given by (190), and $(r_i)_G$ is the position vector of the given point i. As has been previously mentioned, the geodetic coordinates (ϕ_j, λ_j, h_j) can be obtained via a simple coordinate transformation.

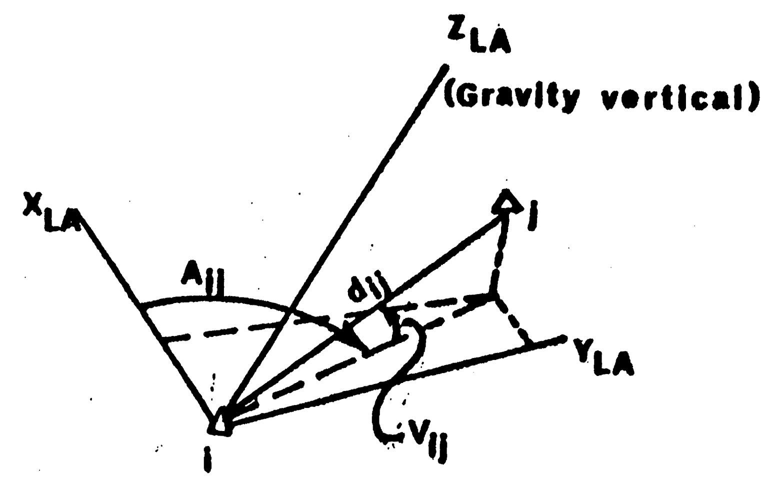
The procedure for the computation of the direct problem, when the azimuth and vertical angle are given in the local astronomic system (Figure 27), is completely analogous to that described with respect to the local geodetic system above. The only difference is in the expression used to compute the topocentric position vector $\bar{r}_{ij}$. In this case, it is given by

$$\boxed{(\bar{r}_{ij})_G = R_3\,(180-\Lambda_i)\;R_2\,(90-\Phi_i)\;P_2\,(\bar{r}_{ij})_{LA}}\;, \qquad (192)$$

where Φ_i and Λ_i are the astronomic latitude and longitude of the given point, and



a) Geodetic and Local Astronomic Systems



b) Spatial Distance (d_{ij}), Azimuth (A_{ij}) and Vertical Angle (V_{ij}) in the Local Astronomic System

Figure 27

$$\boxed{(\bar{r}_{ij})_{LA} = \begin{bmatrix} d_{ij} \cos v_{ij} \cos A_{ij} \\ d_{ij} \cos v_{ij} \sin A_{ij} \\ d_{ij} \sin v_{ij} \end{bmatrix}} \quad . \qquad (193)$$

Note that in this case (192) the position vector is rotated directly from the local astronomic system into the geodetic system. An alternative transformation is possible via the local geodetic system using the expression

$$\boxed{(\bar{r}_{ij})_G = R_3\,(180-\lambda_i)\; R_2\,(90-\phi_i)\; P_2 R_3(A_{ij}-\alpha_{ij})\; R_2(-\xi_i)\; R_1(\eta_i)(\bar{r}_{ij})_{LA} \qquad (194)}$$

In the above expression (194), A_{ij} and α_{ij} are the astronomic and geodetic azimuths respectively, and the quantities ξ_i and η_i are the two components of the deflection of the vertical at point i.

5.2 Inverse Problem

In this case, the triplets of coordinates (ϕ, λ, h) or (x, y, z) are given for two terrain points. Required are the spatial distance (d_{ij}), the direct and inverse azimuths α_{ij} and α_{ji}, and the vertical angles a_{ij} and a_{ji}.

The position vectors of the two points i and j in the geodetic system are given by

$$(\bar{r}_i)_G = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}_G = \begin{bmatrix} (N_i+h_i) \cos \phi_i \cos \lambda_i \\ (N_i+h_i) \cos \phi_i \sin \lambda_i \\ (N_i(1-e^2)+h_i) \sin \phi_i \end{bmatrix} , \qquad (195)$$

and

$$(\bar{r}_j)_G = \begin{bmatrix} x_j \\ y_j \\ z_j \end{bmatrix}_G = \begin{bmatrix} (N_j+h_j) \cos \phi_j \cos \lambda_j \\ (N_j+h_j) \cos \phi_j \sin \lambda_j \\ (N_j(1-e^2)+h_j) \sin \phi_j \end{bmatrix} . \qquad (196)$$

First, the difference vector $\bar{r}_{ij}$, in the geodetic system, is determined by

$$(\bar{r}_{ij})_G = (r_j)_G - (r_i)_G = \begin{bmatrix} x_j & - & x_i \\ y_j & - & y_i \\ z_j & & z_i \end{bmatrix}_G = \begin{bmatrix} \Delta x_{ij} \\ \Delta y_{ij} \\ \Delta z_{ij} \end{bmatrix}_G . \qquad (197)$$

Next, the above difference vector is rotated into the local geodetic coordinate system via an expression which is the inverse of (190), and is given by

$$(\bar{r}_{ij})_{LG} = P_2 R_2(\phi_i - 90)\, R_3(\lambda_i - 180)(\bar{r}_{ij})_G . \qquad (198)$$

Now, to determine the spatial distance, and the azimuth and vertical angle at i, we use the components of the vector $(r_{ij})_{LG}$ in the expressions

$$\boxed{d_{ij} = [\Delta x_{ij}^2 + \Delta y_{ij}^2 + \Delta z_{ij}^2]^{1/2}} , \qquad (199)$$

$$\boxed{\alpha_{ij} = \tan^{-1} [\frac{\Delta y_{ij}}{\Delta x_{ij}}]} , \qquad (200)$$

and

$$\boxed{a_{ij} = \sin^{-1} [\frac{\Delta z_{ij}}{d_{ij}}]} . \qquad (201)$$

The corresponding expressions for determining the azimuth, α_{ji}, and vertical angle, a_{ji}, in the local geodetic system at j are

$$(\bar{r}_{ji})_{LG} = P_2 R_2 (\phi_j - 90)\, R_3 (\lambda_j - 180)\, (r_{ji})_G , \qquad (202)$$

$$\boxed{\alpha_{ji} = \tan^{-1} [\frac{\Delta y_{ji}}{\Delta x_{ji}}]} , \qquad (203)$$

and

$$\boxed{a_{ji} = \sin^{-1} [\frac{\Delta z_{ji}}{d_{ij}}]} . \qquad (204)$$

6. Intersection Problems in Three Dimensions

The problem of determining the coordinates of a point on a plane using an intersection of two azimuths or distances from two known (coordinated) points is a straight-forward process [Faig, 1972]. This type of problem is not generally dealt with for computations on a reference ellipsoid. The intersection problem for the determination of the geodetic coordinates (ϕ, λ) can be dealt with quite simply using vector algebra. Two cases are presented herein, each of which requires information similar to that which would be required for rigorous two-dimensional computations.

6.1 Azimuth Intersection

The problem is defined as: Given the triplets of coordinates (ϕ_i, λ_i, h_i) and (ϕ_j, λ_j, h_j) for two terrain points i and j, and the terrain normal section azimuths α_{ik} and α_{jk} from the known points to the unknown point k, compute the geodetic coordinates ϕ_k and λ_k of the unknown point k. Note that the approximate ellipsoid height h_k is required for the computations.

In order to begin the solution, it is necessary to define a unit vector in any azimuth. This vector is denoted $\hat{t}_\alpha$, and is expressed in terms of the unit vectors $\hat{u}_x$ and $\hat{u}_y$, which are respectively the north and east directions of the local geodetic system (Figure 28). This is given by the equation

$$\hat{t}_\alpha = \hat{u}_x \cos \alpha + \hat{u}_y \sin \alpha, \tag{205}$$

where

$$\hat{u}_x = \begin{bmatrix} -\sin \phi \cos \lambda \\ -\sin \phi \sin \lambda \\ \cos \phi \end{bmatrix}_G , \tag{206}$$

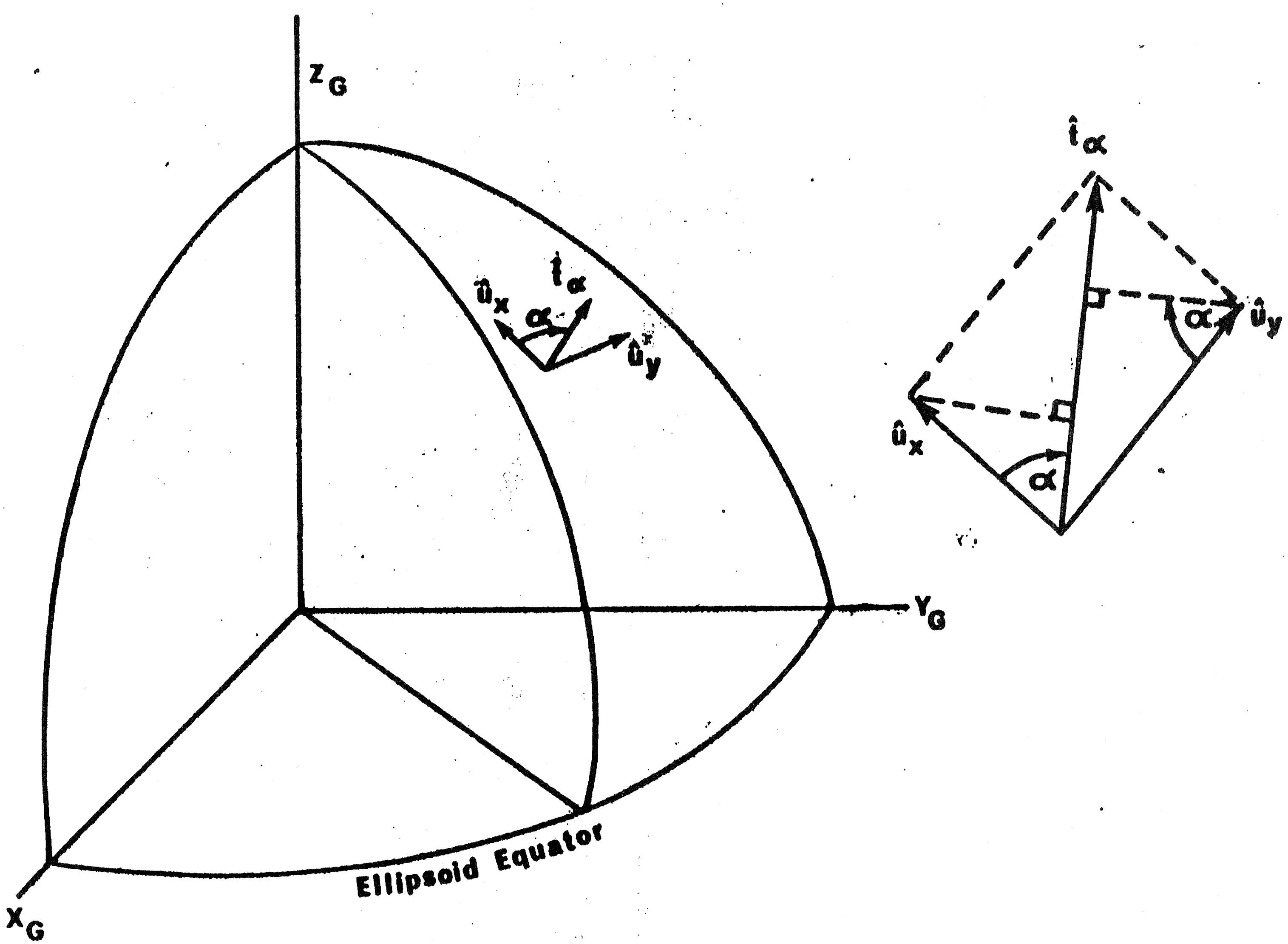


Figure 28

UNIT VECTORS IN THE LOCAL GEODETIC SYSTEM and

$$\hat{u}_y = \begin{bmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{bmatrix}_G . \qquad (207)$$

Using the expressions for $\hat{u}_x$ and $\hat{u}_y$, (205) can be rewritten as

$$\begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}_G = \begin{bmatrix} -\sin\phi\cos\lambda\cos\alpha - \sin\lambda\sin\alpha \\ -\sin\phi\sin\lambda\cos\alpha + \cos\lambda\sin\alpha \\ \cos\phi\cos\alpha \end{bmatrix}_G . \qquad (208)$$

Now, a unit vector perpendicular to ~~the~~ azimuth α is defined by

$$\hat{t}_{\alpha+90} = \hat{u}_x \cos(\alpha+90) + \hat{u}_y \sin(\alpha+90^\circ) . \qquad (209)$$

In order to solve for ϕ_k and λ_k, two equations must be formulated wherein these two quantities appear explicitly. First, two dot products are formed, each of which involves one vector in a plane defined by a pair of terrain points and the origin of the coordinate system, and a second vector that is in an azimuth at 90° to this plane (Figure 29). The two dot products are

$$(\bar{r}_k - \bar{r}_i) \cdot \hat{t}_{\alpha_{ik}+90^\circ} = 0 \quad , \qquad (210)$$

and

$$(\bar{r}_k - \bar{r}_j) \cdot \hat{t}_{\alpha_{jk}+90^\circ} = 0 \quad , \qquad (211)$$

where

$$\hat{t}_{\alpha_{ik}+90^\circ} = \begin{bmatrix} -\sin\phi_i\cos\lambda_i\cos(\alpha_{ik}+90) - \sin\lambda_i\sin(\alpha_{ik}+90) \\ -\sin\phi_i\sin\lambda_i\cos(\alpha_{ik}+90) + \cos\lambda_i\sin(\alpha_{ik}+90) \\ \cos\phi_i\cos(\alpha_{ik}+90) \end{bmatrix} \qquad (212)$$

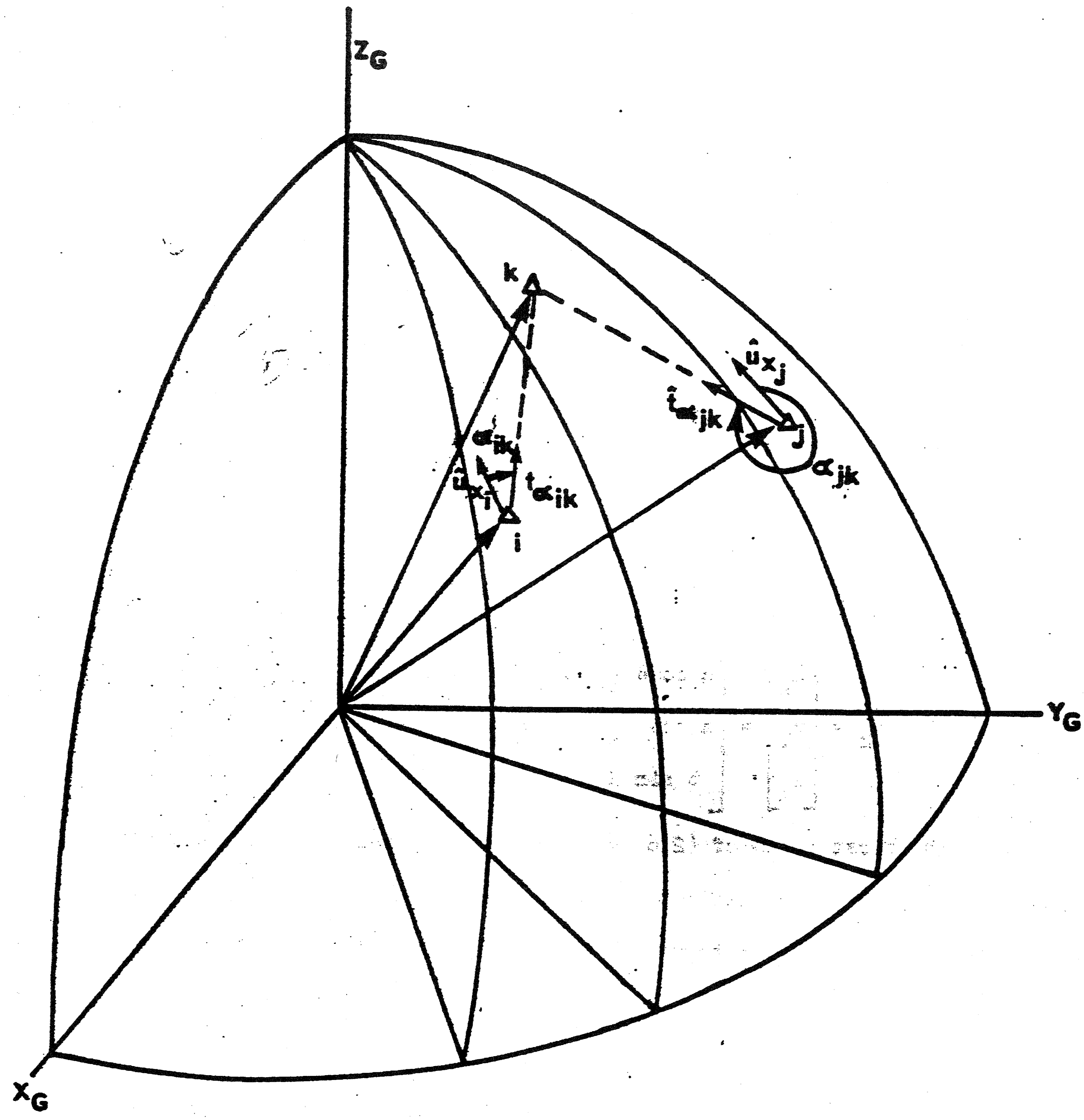


Figure 29

AZIMUTH INTERSECTION IN THREE DIMENSIONS

$$\hat{t}_{\alpha_{jk}+90°} = \begin{bmatrix} -\sin\phi_j \cos\lambda_j \cos(\alpha_{jk}+90) - \sin\lambda_j \sin(\alpha_{jk}+90) \\ -\sin\phi_j \sin\lambda_j \cos(\alpha_{jk}+90) + \cos\lambda_j \sin(\alpha_{jk}+90) \\ \cos\phi_j \cos(\alpha_{jk}+90) \end{bmatrix}, \tag{213}$$

$$(\bar{r}_k - \bar{r}_i) = \begin{bmatrix} x_k - x_i \\ y_k - y_i \\ z_k - z_i \end{bmatrix} = \begin{bmatrix} \Delta x_{ik} \\ \Delta y_{ik} \\ \Delta z_{ik} \end{bmatrix}, \tag{214}$$

and

$$(\bar{r}_k - \bar{r}_j) = \begin{bmatrix} x_k - x_j \\ y_k - y_j \\ z_k - z_j \end{bmatrix} = \begin{bmatrix} \Delta x_{jk} \\ \Delta y_{jk} \\ \Delta z_{jk} \end{bmatrix}. \tag{215}$$

In equations (214) and (215), the coordinates for i and j are taken as given constants, while those for k are given by three unknown functions [Krakiwsky and Wells, 1971]

$$\bar{r}_k = \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} = \begin{bmatrix} a \cos\beta_k \cos\lambda_k + h_k \cos\phi_k \cos\lambda_k \\ a \cos\beta_k \sin\lambda_k + h_k \cos\phi_k \sin\lambda_k \\ b \sin\beta_k + h_k \sin\phi_k \end{bmatrix}. \tag{216}$$

The first terms of (216) give the coordinates of k on the surface of the ellipsoid (defined by the semi-major and semi-minor axes a and b respectively) in terms of the reduced latitude, β_k, and geodetic longitude, λ_k. The second terms account for the fact that the terrain point k is located at an ellipsoid height h_k above the reference ellipsoid, and are expressed in terms of the geodetic latitude, ϕ_k, and longitude, λ_k.

Now, equations (210) and (211) can be rewritten as

$$f_1 = \Delta x_{ik} t_{x_i} + \Delta y_{ik} t_{y_i} + \Delta z_{ik} t_{z_i} = 0, \tag{217}$$

$$f_2 = \Delta x_{jk} t_{x_j} + \Delta y_{jk} t_{y_j} + \Delta z_{jk} t_{z_j} = 0 \ . \tag{218}$$

The unknwon quantities in the above equations are the coördinates of k, and in terms of these, (217) and (218) are non-linear. The next step in the solution is to approximate the equations (217) and (218) by a linear Taylor series using approximate values for the reduced latitude and longitude denoted by β_k^o and λ_k^o respectively. Thus

$$f_1 = f_1^o + \frac{\partial f_1}{\partial \beta_k} d\beta_k + \frac{\partial f_1}{\partial \lambda_k} d\lambda_k + \dots = 0 \ , \tag{219}$$

and

$$f_2 = f_2^o + \frac{\partial f_2}{\partial \beta_k} d\beta_k + \frac{\partial f_2}{\partial \lambda_k} d\lambda_k + \dots = 0 \ , \tag{220}$$

where

$$f_1^o = \Delta x_{ik}^o \ t_{x_i} + \Delta y_{ik}^o \ t_{y_i} + \Delta z_{ik}^o \ t_{z_i} \quad , \tag{221}$$

$$f_2^o = \Delta x_{jk}^o \ t_{x_j} + \Delta y_{jk}^o \ t_{y_j} + \Delta z_{jk}^o \ t_{z_j} \quad , \tag{222}$$

$$\Delta x_{ik}^o = a \cos \beta_k^o \cos \lambda_k^o + h_k^o \cos \phi_k^o \cos \lambda_k^o - x_i \ , \tag{223}$$

$$\Delta y_{ik}^o = a \cos \beta_k^o \sin \lambda_k^o + h_k^o \cos \phi_k^o \sin \lambda_k^o - y_i \ , \tag{224}$$

$$\Delta z_{ik}^o = b \sin \beta_k^o + h_k^o \sin \phi_k^o - z_i \quad , \tag{225}$$

$$\Delta x_{jk}^o = a \cos \beta_k^o \cos \lambda_k^o + h_k^o \cos \phi_k^o \cos \lambda_k^o - x_j , \tag{226}$$

$$\Delta y_{jk}^o = a \cos \beta_k^o \sin \lambda_k^o + h_k^o \cos \phi_k^o \sin \lambda_k^o - y_j \ , \tag{227}$$

$$\Delta z_{jk}^o = b \sin \beta_k^o + h_k^o \sin \phi_k^o - z_j \quad , \tag{228}$$

$$\begin{aligned} \frac{\partial f_1}{\partial \beta_k} &= t_{x_i} \ (-a \sin \beta_k^o \cos \lambda_k^o - h_k^o \sin \phi_k^o \cos \lambda_k^o) + \\ &+ t_{y_i} \ (-a \sin \beta_k^o \sin \lambda_k^o - h_k^o \sin \phi_k^o \sin \lambda_k^o) + \\ &+ t_{z_i} \ (b \cos \beta_k^o + h_k^o \cos \phi_k^o) \quad , \end{aligned} \tag{229}$$

$$\frac{\partial f_1}{\partial \lambda_k} = t_{x_i} \; (-a \cos \beta_k^o \sin \lambda_k^o - h_k^o \cos \phi_k^o \sin \lambda_k^o) +$$
$$+ t_{y_i} \; (a \cos \beta_k^o \cos \lambda_k^o + h_k^o \cos \phi_k^o \cos \lambda_k^o) \; . \tag{230}$$

Now, rewriting (229) and (230) as

$$\frac{\partial f_1}{\partial \beta_k} = t_{x_i} \; x_\beta + t_{y_i} \; y_\beta + t_{z_i} \; z_\beta \; , \tag{231}$$

$$\frac{\partial f_1}{\partial \lambda_k} = t_{x_i} \; x_\lambda + t_{y_i} \; y_\lambda \; , \tag{232}$$

and

$$\frac{\partial f_2}{\partial \beta_k} = t_{x_j} \; x_\beta + t_{y_j} \; y_\beta + t_{z_j} \; z_\beta \; , \tag{233}$$

$$\frac{\partial f_2}{\partial \lambda_k} = t_{x_j} \; x_\lambda + t_{y_j} \; y_\lambda \; . \tag{234}$$

It should be noted that in taking the partial derivatives, the geodetic latitude, ϕ_k, was taken as being synonomous with the reduced latitude, β_k. There is no loss in accuracy in subsequent computations due to this treatment. Additionally, an approximate value of h_k that is within 100 m of the true value is sufficient.

Rewriting (217) and (218), we get

$$\boxed{f_1^o + (x_\beta t_{x_i} + y_\beta t_{y_i} + z_\beta t_{z_i}) \; d\beta_k + (x_\lambda t_{x_i} + y_\lambda t_{y_i}) \; d\lambda_k = 0 \; ,} \tag{235}$$

$$\boxed{f_2^o + (x_\beta t_{x_j} + y_\beta t_{y_j} + z_\beta t_{z_j}) \; d\beta_k + (x_\lambda t_{x_j} + y_\lambda t_{y_j}) \; d\lambda_k = 0 \; .} \tag{236}$$

Equations (235) and (236) are solved in an iterative procedure until the corrections to β_k and λ_k are negligible ($< 0''.0001$). The value of the geodetic latitude, ϕ_k, is then solved for by [Krakiwsky and Wells, 1971]

$$\boxed{\phi_k = \tan^{-1}\left[\frac{a}{b}\tan\beta_k\right].} \tag{237}$$

6.2 <u>Spatial Distance Intersection</u>

The determination of the geodetic latitude (ϕ_k) and longitude (λ_k) of a terrain point, using two terrestrial spatial distances, is solved in a manner similar to that used for an azimuth intersection (6.1). Given are the two triplets of coordinates (ϕ_i, λ_i, h_i) and (ϕ_j, λ_j, h_j), and two terrestrial spatial distances, r_{ik} and r_{jk}, from the known points to the unknown point k. In addition, an approximate ellipsoid height, h_k^o, is required (within 100 m of the value of h_k is sufficient).

The key to the solution is the formation of two linear equations which are expressed in terms of the known and unknown parameters (Figure 30). We begin with the relationships

$$f_1 = [(x_k-x_i)^2 + (y_k-y_i)^2 + (z_k-z_i)^2]^{1/2} - r_{ik} = 0, \tag{238}$$

$$f_2 = [(x_k-x_j)^2 + (y_k-y_j)^2 + (z_k-z_j)^2]^{1/2} - r_{jk} = 0, \tag{239}$$

where (x_k, y_k, z_k) are given by (216). The above equations are nonlinear in terms of β_k and λ_k, thus they are approximated by a linear Taylor series expansion using approximate values for the reduced latitude, β_k^o, and geodetic longitude, λ_k^o. The linear form of equations (238) and (239) are given by

$$f_1 = f_1^o + \frac{\partial f_1}{\partial \beta_k}\, d\beta_k + \frac{\partial f_1}{\partial \lambda_k}\, d\lambda_k + \ldots = 0 , \tag{240}$$

and

$$f_2 = f_2^o + \frac{\partial f_2}{\partial \beta_k}\, d\beta_k + \frac{\partial f_2}{\partial \lambda_k}\, d\lambda_k + \ldots = 0 , \tag{241}$$

where

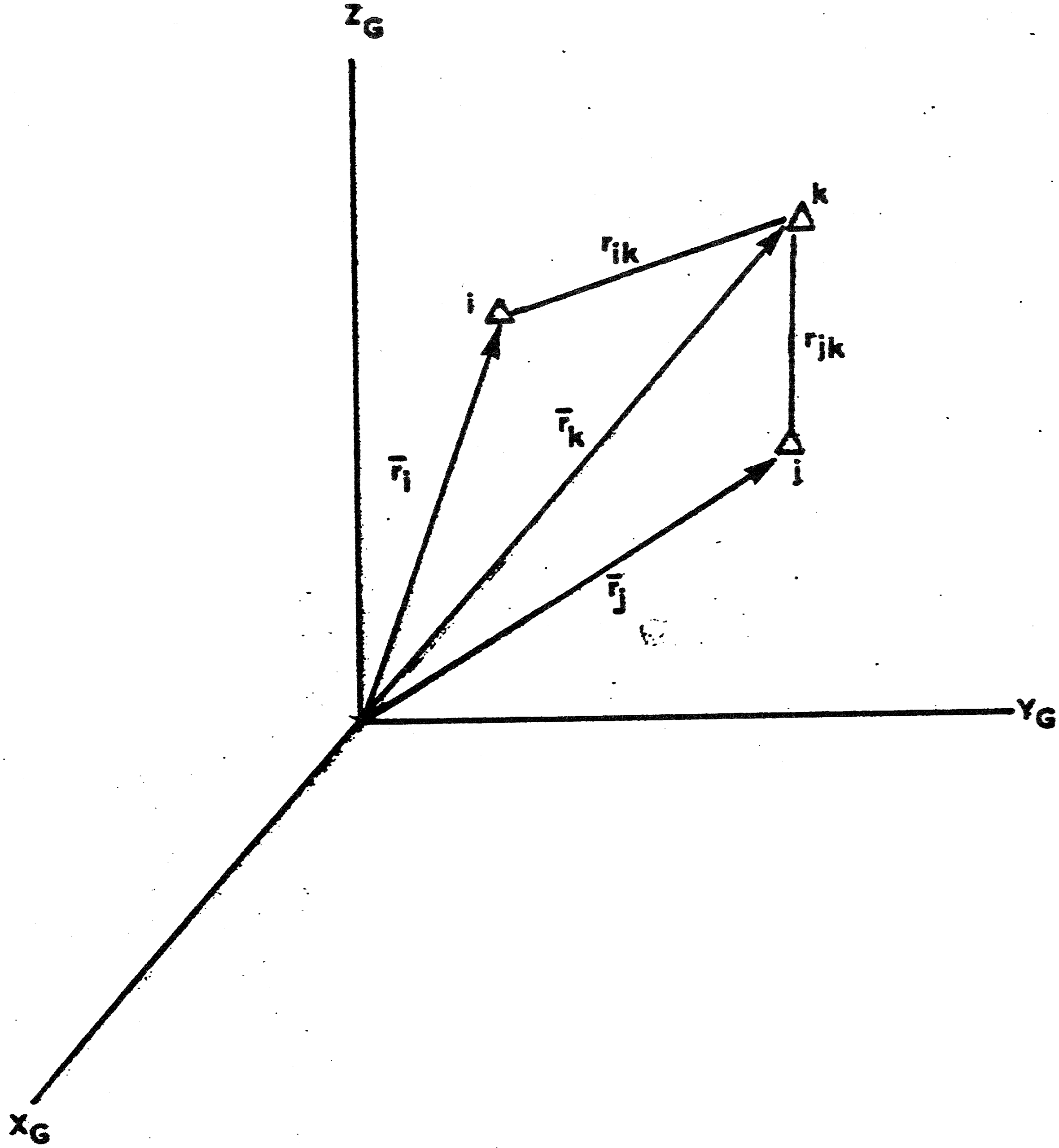


Figure 30

SPATIAL DISTANCE INTERSECTION IN THREE DIMENSIONS

$$f_1^o = r_{ik}^o - r_{ik} \quad , \tag{242}$$

$$f_2^o = r_{jk}^o - r_{jk} \quad , \tag{243}$$

$$\frac{\partial f_1}{\partial \beta_k} = \frac{1}{r_{ik}^o} \left[(x_k^o - x_i) \frac{\partial x_k}{\partial \beta_k} + (y_k^o - y_i) \frac{\partial y_k}{\partial \beta_k} + (z_k^o - z_i) \frac{\partial z_k}{\partial \beta_k} \right], \tag{244}$$

$$\frac{\partial f_1}{\partial \lambda_k} = \frac{1}{r_{ik}^o} \left[(x_k^o - x_i) \frac{\partial x_k}{\partial \lambda_k} + (y_k^o - y_i) \frac{\partial y_k}{\partial \lambda_k} + (z_k^o - z_i) \frac{\partial z_k}{\partial \lambda_k} \right], \tag{245}$$

$$\frac{\partial f_2}{\partial \beta_k} = \frac{1}{r_{jk}^o} \left[(x_k^o - x_j) \frac{\partial x_k}{\partial \beta_k} + (y_k^o - y_j) \frac{\partial y_k}{\partial \beta_k} + (z_k^o - z_j) \frac{\partial z_k}{\partial \beta_k} \right], \tag{246}$$

$$\frac{\partial f_2}{\partial \lambda_k} = \frac{1}{r_{jk}^o} \left[(x_k^o - x_j) \frac{\partial x_k}{\partial \lambda_k} + (y_k^o - y_j) \frac{\partial y_k}{\partial \lambda_k} + (z_k^o - z_j) \frac{\partial z_k}{\partial \lambda_k} \right]. \tag{247}$$

Now, the terms in equations (244)-(247) are derived from (216), and are given by

$$\frac{\partial x_k}{\partial \beta_k} = - a \sin \beta_k^o \cos \lambda_k^o - h_k^o \sin \phi_k^o \cos \lambda_k^o = x_\beta \quad , \tag{248}$$

$$\frac{\partial y_k}{\partial \beta_k} = - a \sin \beta_k^o \sin \lambda_k^o - h_k^o \sin \phi_k^o \sin \lambda_k^o = y_\beta \quad , \tag{249}$$

$$\frac{\partial z_k}{\partial \beta_k} = b \cos \beta_k^o + h_k^o \cos \phi_k^o = z_\beta \quad , \tag{250}$$

$$\frac{\partial x_k}{\partial \lambda_k} = - a \cos \beta_k^o \sin \lambda_k^o - h_k^o \cos \phi_k^o \sin \lambda_k^o = x_\lambda \quad , \tag{251}$$

$$\frac{\partial y_k}{\partial \lambda_k} = a \cos \beta_k^o \cos \lambda_k^o + h_k^o \cos \phi_k^o \cos \lambda_k^o = y_\lambda \quad , \tag{252}$$

$$\frac{\partial z_k}{\partial \lambda_k} = 0 \quad . \tag{253}$$

As in the case of the azimuth intersection, the geodetic latitude, ϕ_k, was taken as being synomomous with the reduced latitude β_k.

Now, (240) and (241) are rewritten for solution as

$$f_1 = f_1^o + \frac{1}{r_{ik}^o}[\Delta x_{ik}x_\beta + \Delta y_{ik}y_\beta + \Delta z_{ik}z_\beta]\, d\beta_k + \frac{1}{r_{ik}^o}[\Delta x_{ik}x_\lambda + \Delta y_{ik}y_\lambda]d\lambda_k, \tag{254}$$

$$f_2 = f_2^o + \frac{1}{r_{jk}^o}[\Delta x_{jk}x_\beta + \Delta y_{jk}y_\beta + \Delta z_{jk}z_\beta]d\beta_k + \frac{-1}{r_{jk}^o}[\Delta x_{jk}x_\lambda + \Delta y_{jk}y_\lambda]d\lambda_k. \tag{255}$$

The corrections $d\beta_k$ and $d\lambda_k$ are solved for using an iterative procedure. When the corrections become negligible (< 0".0001), the final values of β_k and λ_k are obtained, and ϕ_k is determined using (237).

7. CONCLUDING REMARKS

From first appearances, it would seem that the classical approach of geodetic position computations on the surface on an ellipsoid of rotation should be abandoned in favour of the three dimensional approach. The formulae for the latter are simpler to derive and impliment, and in the case of the direct and inverse problems, are given in a closed form. In addition, if the curvalinear coordinates (direct problem), or the ellipsoidal distance and normal section azimuths (inverse problem) are required, rigorous transformation formulae are available to obtain them [Krakiwsky and Wells, 1971; Section II].

The major hindrance to the use of the three dimensional approach lies in the geodetic observables, or the lack thereof. This is particularly true in the case of the direct problem, or any problem where the vertical angle (90°-zenith distance) is required. Due to refraction problems, the zenith distance can not be obtained to better than $\pm$ 1" which on a 10 km line yields a standard deviation in height of 10 cm [Heiskanen and Moritz, 1967]. This error would obviously affect the computations of the three dimensional coordinates (x, y, z) or (ϕ, λ, h) of a required point. The problem can be overcome by spirit levelling, but it is unlikely that these observations would be available in other than exceptional cases.

The two intersection problems that have been presented show how the three dimensional approach can be used to solve directly for curvalinear coordinates. It should be obvious that if sufficient observed information were available (eg. three spatial distances), the problems could be formulated and solved directly in terms of the three dimensional cartesian coordinates.

Finally, it should be noted that an equivalent amount of observed information is required for the classical and three dimensional approaches. The main difference is that for the ellipsoidal computations, (i.e. direct problem) the ellipsoidal height need not be known as accurately as for three dimensional computations. However, no matter which method is used, rigorous transformations will show that the results are equivalent. That is, the cartesian coordinates (x, y, z) will yield a set (ϕ, λ, h) in which the geodetic latitude (ϕ) and longitude (λ) are equal to those obtained from classical computations. Further, the spatial distances and terrain normal section azimuths, obtained from three dimensional computations (inverse problem) and rigorously reduced to the reference ellipsoid, are equal to the ellipsoidal distances and geodesic azimuths obtained from the inverse problem solved on the ellipsoid.

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