ELEC505/ELEC405 Handout #5
Linear System Theory Tuesday, December 22, 2020

Homework Set #5

Due: Wednesday, December 30, 2020.

1. *Quadratic Forms and Hermitian Matrices*

   Consider the quadratic form

$$f(\mathbf{x}) = \frac{1}{9}(-2x_1^2 + 7x_2^2 + 4x_3^2 + 4x_1x_2 + 16x_1x_3 + 20x_2x_3). \tag{1}$$

   (a) Find a symmetric matrix $\mathbf{A}$ so that $f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$.

   (b) Diagonalize the quadratic form using the $\mathbf{LDL}^T$ factorization, and determine the inertia of $\mathbf{A}$.

   (c) Is this a positive definite form?

   (d) Verify the inertia obtained above is correct by computing the eigenvalues of $\mathbf{A}$.

2. *Matrix Factorization Coding Experience...*

   Write your own Matlab/Python/Julia function to find Cholesky Factorization of a positive definite matrix.

3. *Random Vector Generation...*

   Write a MATLAB/Python/Julia function whose first couple of lines are provided as follows

```
function out=generaterandvec(N,R)
% Generates zero-mean complex Gaussian random vector sequence with the specified
% covariance matrix
% Usage
%  out=generaterandvec(N,R)
% Here
% N: Number of vectors to be generated
% R: The specified covariance matrix.
% out: Output matrix containing vector sequence. Assuming R is a pxp matrix
% out would be a pxN matrix
```

   in your function only specialized MATLAB/Python/Julia functions you can use are *randn* and *size*(*shape*). You can use your function for the previous question if you need. Generate $N = 100000$ random vectors with covariance

$$\mathbf{R} = \begin{bmatrix} 28 & 15+9i & 2+21i \\ 15-9i & 48 & 15-11i \\ 2-21i & 15+11i & 30 \end{bmatrix}. \tag{2}$$

Calculate the sample covariance matrix for the vector sequence generated using the formula

$$\hat{\mathbf{R}} = \frac{1}{N}\sum_{n=1}^{N} \mathbf{x}_i \mathbf{x}_i^*. \tag{3}$$

Can you find $\hat{\mathbf{R}}$ without using loops? Compare $\hat{\mathbf{R}}$ with the true covariance $\mathbf{R}$, are they close? does your random generator seem to work fine?

4. *Back to paper and pencil.... Positive matrices...*

   Let $\mathbf{A}$ be an $n \times n$ Hermitian matrix which is positive definite. Prove that there exists a $\alpha > 0$ such that for any $\mathbf{x} \in C^n$, $\mathbf{x}^*\mathbf{A}\mathbf{x} \geqslant \alpha\|\mathbf{x}\|^2$.

5. *Quadratic approximations of multivariable functions*

   For a given twice differentiable function $f(\mathbf{x})$ of $\mathbf{x} \in \Re^n$, we can obtain a quadratic approximation around $\mathbf{x}_0$, by truncating the Taylor series expansion:

$$f_Q(\mathbf{x}) \quad = \quad f((\mathbf{x_0})) + \langle \nabla f(\mathbf{x}_0), \mathbf{x} - \mathbf{x}_0 \rangle + \frac{1}{2}\langle \mathbf{H}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0), \mathbf{x} - \mathbf{x}_0 \rangle$$

   where

$$\nabla f(\mathbf{x}_0) = \begin{bmatrix} \frac{\partial f}{\partial x_1}|_{\mathbf{x}=\mathbf{x}_0} & \frac{\partial f}{\partial x_2}|_{\mathbf{x}=\mathbf{x}_0} & \cdots & \frac{\partial f}{\partial x_n}|_{\mathbf{x}=\mathbf{x}_0} \end{bmatrix}^T$$

   and $\mathbf{H}(\mathbf{x}_0)$ is the Hessian evaluated at $\mathbf{x}_0$, whose terms are given by

$$[\mathbf{H}(\mathbf{x}_0)]_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}|_{\mathbf{x}=\mathbf{x}_0},$$

   for $i, j = 1, \ldots, n$.

   Let $f(\begin{bmatrix} x_1 & x_2 \end{bmatrix}^T) = (4x_1^2 - 1)e^{-x_1^2 - x_2^2}$:

   (a) Find the expressions for the gradient vector and the Hessian matrix.

   (b) Find the expression for the gradient and Hessian at $\mathbf{x}_0 = \mathbf{0}$ and write down the quadratic approximation, and reproduce Figure 1 (showing f and f_Q together) in Matlab.

   (c) Pick a point in region $[-2, 2] \times [-2, 2]$, where gradient is non-zero and the Hessian is negative definite. Plot the figure similar to Figure 1. Please clearly indicate the point you picked.

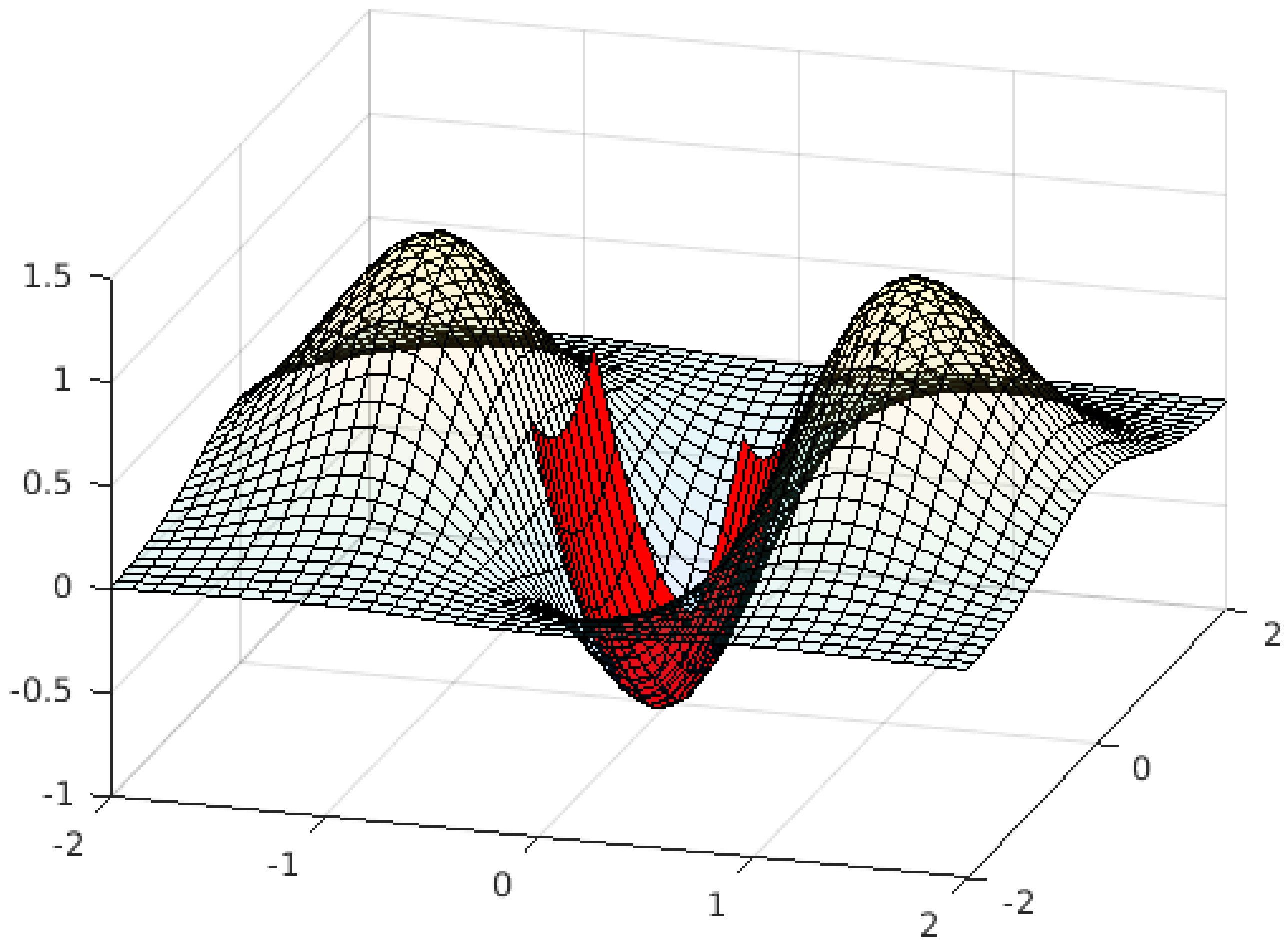

Figure 1: f and its quadratic approximation (red) around the origin