ELEC505/ELEC405 Handout #4
Linear System Theory Wednesday, November 25, 2020

Homework Set #3
Due: Monday, December 7, 2020.

1. *LU Decomposition Exercise...*

   Find the L, U decomposition for

$$\begin{bmatrix} 6 & 30 & 36 \\ 2 & 10 & 4 \\ 5 & 4 & 2 \end{bmatrix}$$

2. *Orthogonality of Left-Right Eigenvectors*

   Let $\mathbf{A} \in \mathbf{R}^{n \times n}$ and if λ, μ are to distinct eigenvalues of $\mathbf{A}$, i.e. $\mu \neq \lambda$, then show that any left eigenvector of $\mathbf{A}$ corresponding to μ is orthogonal to any right eigenvector of $\mathbf{A}$ corresponding to λ.

3. *DFT as an Orthogonal Basis Change*

   Consider $\mathcal{C}^N$ the vector space of N dimensional complex vectors. We can define a basis $\mathcal{F}$=$\{\mathbf{f}_1, \ldots, \mathbf{f}_N\}$ where

$$\mathbf{f}_k = \begin{bmatrix} f_{k,1} \\ f_{k,2} \\ \vdots \\ f_{k,N} \end{bmatrix}, \quad f_{k,l} = \frac{1}{N} e^{\frac{j2\pi(k-1)(l-1)}{N}}. \tag{1}$$

   (a) Is $\mathcal{F}$ an orthogonal basis? Is it an orthonormal basis?

   (b) Define the matrix

$$\mathbf{F} = \begin{bmatrix} \mathbf{f}_1 & \mathbf{f}_2 & \ldots & \mathbf{f}_N \end{bmatrix}. \tag{2}$$

   Is $\mathbf{F}$ unitary? What is the inverse of $\mathbf{F}$?

   (c) Write down $\mathbf{F}$ for $N = 4$. What are the coordinates of $\mathbf{x} = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T$ corresponding to the basis $\mathcal{F}$? What is the FFT of $\mathbf{x}$?

   (d) Suppose $\{x_n : n = 0, \ldots 3\}, \{h_n : n = 0, \ldots 3\}$ and $\{y_n : n = 0, \ldots 3\}$ are discrete time sequences of length 4. We are also given that y_n is equal to the circular convolution of x_n and h_n, i.e.,

$$y_n = h_n \circledast x_n$$

   Defining

$$\mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix},$$

we would like to write the relation

$$\mathbf{y} = \mathbf{H}\mathbf{x} \tag{3}$$

Find $\mathbf{H}$ in terms of h_n, $n = 0, \ldots 3$.

(e) Check out the following multiplication:

$$\mathbf{H}\mathbf{f}_k \tag{4}$$

What is your comment about the result?

(f) Pick arbitrary values for $\{h_0, h_1, h_2, h_3\}$. Find the eigenvalues and eigenvectors of $\mathbf{H}$ using MATLAB/Python/Julia, find eigenvectors and eigenvalues (e.g., in Matlab)by typing

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[V D]=eig(H)
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where $\mathbf{V}$ is the matrix containing eigenvectors, and $\mathbf{D}$ is the matrix containing eigenvalues.) Compare $\mathbf{V}$ with $\mathbf{F}$, what is your comment.

(g) If we change the basis from the standard basis to $\mathcal{F}$, what would be the relation between $\breve{\mathbf{y}}$ and $\breve{\mathbf{x}}$, where $\breve{\mathbf{y}}$ and $\breve{\mathbf{x}}$ are the new coordinate vectors for $\mathbf{y}$ and $\mathbf{x}$ in Eq. (3) respectively, corresponding to basis $\mathcal{F}$. In other words find $\breve{\mathbf{H}}$ where $\breve{\mathbf{y}} = \breve{\mathbf{H}}\breve{\mathbf{x}}$. What is the effect of basis change as far as the mapping in (3) is concerned? Note that this comment is independent of the values of $\{h_0, h_1, \ldots, h_{N-1}\}$....

4. *Reflection through a hyperplane.*

Find the matrix $\mathbf{Q} \in \mathbf{R}^{n\times n}$ such that the reflection of $\mathbf{x}$ through the hyperplane $\{\mathbf{z} \mid \mathbf{a}^T\mathbf{z} = 0\}$ is given by $\mathbf{Q}\mathbf{x}$. Verify that matrix $\mathbf{Q}$ is orthogonal. (To reflect $\mathbf{x}$ through the hyperplane means the following: find the point $\mathbf{z}$ on the hyperplane closest to $\mathbf{x}$. Starting from $\mathbf{x}$ go in the direction $\mathbf{z} - \mathbf{x}$ through the hyperplane to a point on the opposite side, which has the same distance to $\mathbf{z}$ as $\mathbf{x}$ does.)

5. *Orthogonal matrices*

(a) Show that if $\mathbf{U}$ and $\mathbf{V}$ are orthogonal then so is $\mathbf{U}\mathbf{V}$.

(b) Suppose that $\mathbf{U} \in \mathbf{R}^{2\times 2}$ is orthogonal. Show that $\mathbf{U}$ is either a rotation or a reflection. Make clear how you decide whether a given orthogonal $\mathbf{U}$ is a rotation or reflection.