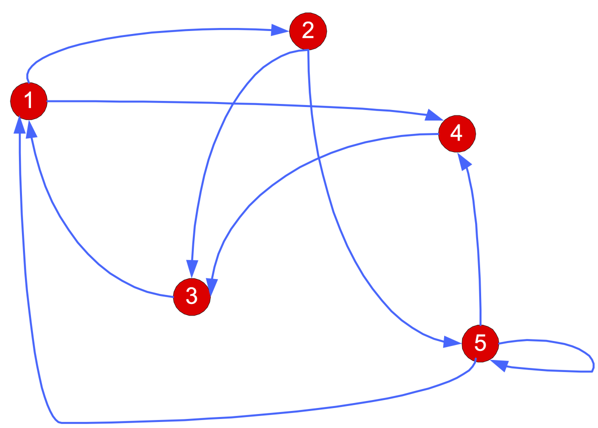
ELEC505/ELEC405 Handout #2
Linear System Theory Friday, October 16 , 2020

Homework Set #0
Due: Monday, October 26, 2020.

1. *An Application of Matrix Mutlitplication*

   The graph in Figure below shows the possible connections among the nodes. At each step you can move from one node to another connected node and you can not remain in a node if there is no self loop.



   According to this graph, for example, one can go from node-1 to node-2 and node-1 to node-4 in one step and node-1 to node-3 in two steps. We define the connectivity matrix **C** as follows:

$$C_{ij} = \begin{cases} 1 & \text{if you can go from node i to node j in one step} \\ 0 & \text{otherwise} \end{cases}$$

   (a) Provide the connectivity matrix **C** for the figure above.
   (b) Find $\mathbf{C}^2$. What does the i, j entry of $\mathbf{C}^2$ correspond to?
   (c) What does i, j entry of the $\mathbf{C}^k$ correspond to where $k \geq 1$ is an integer?

2. *simple plane geometry ....*

   We can represent a 2-vector with a point on a plane. (Similarly we can represent a 3-vector with a point in 3 dimensional space). This fact is useful in both directions: we can convert an algebraic problem involving our vectors into a geometrical problem or we can convert a geometrical problem into an algebraic problem...

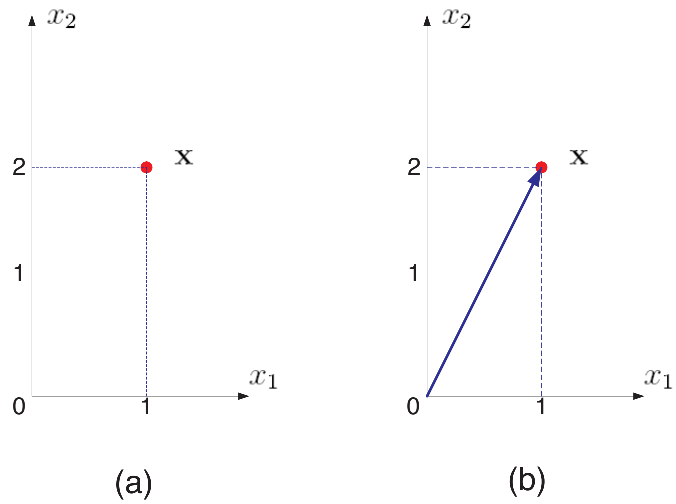


Figure 1: Graphical Representation of 2-vectors

For example, we can use a point on plane as shown in Figure 1-(a) to represent the 2-vector $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, where the first component of the vector $\mathbf{x}$ is the horizontal coordinate and the second component of the vector $\mathbf{x}$ is the vertical coordinate of the point.

Alternatively, we can represent the same vector with an arrow as shown in Figure 1-(b). The length of the arrow is called the *norm* of the vector. Therefore, for a 2-vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, the length of the arrow, i.e. the norm of the vector, is

$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2}$$

(a) What is the norm of the $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ vector?

(b) The distance between two points on the plane is the norm of the difference of the 2-vectors corresponding to these points. Therefore, if these two points are represented with vectors $\mathbf{y}$ and $\mathbf{z}$ respectively, then the distance between these two points is given by

$$\text{dist}(\mathbf{y}, \mathbf{z}) = \|\mathbf{y} - \mathbf{z}\|$$

What is the distance between the points $\mathbf{a} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$?

(c) The inner product that we defined in class can be used to find the angle between two vectors!. This is based on the fact that inner product of two 2-vectors can be written as

$$\begin{aligned} \langle \mathbf{x}, \mathbf{y} \rangle &= \mathbf{x}^T \mathbf{y} = x_1 y_1 + x_2 y_2 \\ &= \|\mathbf{x}\| \|\mathbf{y}\| \cos(\theta) \end{aligned}$$

where θ is the angle between the vectors. As a result, given the vectors, the angle between them can be found using

$$\begin{aligned} \theta &= \text{acos}\left(\frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \|\mathbf{y}\|}\right) \\ &= \text{acos}\left(\frac{x_1 y_1 + x_2 y_2}{\sqrt{x_1^2 + x_2^2} \sqrt{y_1^2 + y_2^2}}\right) \end{aligned}$$

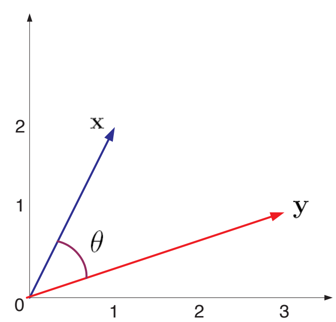


Figure 2: Inner product example

Given the vectors in Figure 2, find the angle between them.

(d) We define the following matrix

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Given the 2-vectors $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, we have the relation

$$\mathbf{b} = \mathbf{R}(\theta)\mathbf{a}.$$

Based on this relation, find b_1, b_2 in terms of a_1, a_2 and θ. What is the norm of $\mathbf{b}$, i.e., $\|\mathbf{b}\|$? What is the angle between $\mathbf{a}$ and $\mathbf{b}$?

For an example 2-vector $\mathbf{a}$ that you pick, find $\mathbf{b}$ for $\theta = \frac{\pi}{4}$ and $\frac{2\pi}{3}$. Draw these vectors on plane. What is the geometrical meaning of multiplication with $\mathbf{R}(\theta)$?

3. *Hperplane in a 2D-World*

   (a) Given a vector $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ define the set $A = \{\mathbf{x}|\mathbf{a}^T\mathbf{x} = 0, x \in \Re^2\}$. Draw this set on $\Re^2$. Please show your vector $\mathbf{a}$ as an arrow on the same plot.

   (b) Given the vector $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ define the set $B = \{\mathbf{x}|\mathbf{a}^T(\mathbf{x} - \mathbf{b}) = 0, x \in \Re^2\}$. Draw this set on $\Re^2$. Please show your vector $\mathbf{a}$ as an arrow placed on $\mathbf{b}$ on the same plot.

   (c) For $\mathbf{a}$ and $\mathbf{b}$ defined in previous parts, we define the set $H_- = \{\mathbf{x}|\mathbf{a}^T(\mathbf{x} - \mathbf{b}) \leq 0, x \in \Re^2\}$. Draw H_-.

   (d) Please write a MATLAB/Python/Julia function *drawhyperplane2D(a,b)* which will generate the plot in part b. An example execution result of drawhyperplane2D([-1;-1],[2,1]) is shown below. (You can choose a reasonable range for x_1). Use *axis equal* command in your code. You area allowed to use line drawing functions available such as MATLAB command *line*.

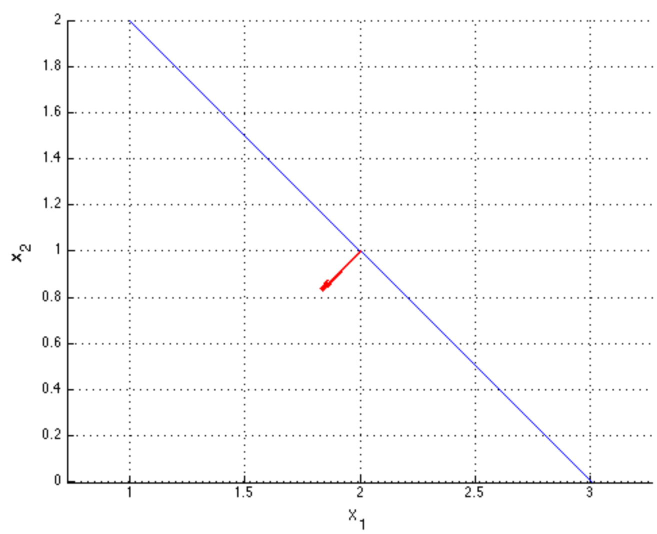


Figure 3: Sample execution drawhyperplane2D([-1;-1],[2,1])

   Submit your MATLAB/Python/Julia code (with clear comments) and the pdf/png of your plot for $\mathbf{a} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

4. *Hperplane in 3D-World*

   As a change to part b of previous problem, we define the hyperplane in 3-dimensions as, $B = \{\mathbf{x}|\mathbf{a}^T(\mathbf{x} - \mathbf{b}) = 0, x \in \Re^3\}$, for a given normal vector $\mathbf{a}$ and shift vector $\mathbf{b}$. Write a matlab function *drawhyperplane3D* to display such an hyperplane together with its normal vector similar to the figure shown below. Illustrate your code execution by providing the plot for $\mathbf{a} = \left[\ 2; -1; 2\ \right]$ and $\mathbf{b} = \left[\ 1; 2; 1\ \right]$. Please use an appropriate view angle (by *view*). Please provide your code (with comments).

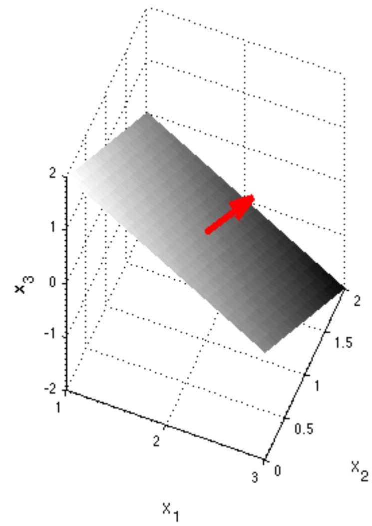


Figure 4: Sample execution drawhyperplane3D([1;1;1],[2;1;0])

5. *A Reminder on Solving Linear Systems of Equations*

   Given $\mathbf{Ax} = \mathbf{b}$, provide solutions for the following $\mathbf{A}, \mathbf{b}$ pairs

   (a)

$$\mathbf{A} = \begin{bmatrix} 2 & 13 & 5 & 6 \\ 0.5 & 3.25 & 5.25 & 2.5 \\ 1.0 & 10.5 & -0.5 & 9 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 16 \\ 7 \\ 3 \end{bmatrix}$$

   (b)

$$\mathbf{A} = \begin{bmatrix} 2 & 13 & 5 & 6 \\ 0.5 & 3.25 & 1.25 & 1.5 \\ 1.0 & 10.5 & -0.5 & 9 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 16 \\ 7 \\ 3 \end{bmatrix}$$

6. *Convolution Operator and Corresponding Matrices...*

   Suppose $\{x_k; k \in \mathbf{Z}\}$ and $\{y_k; k \in \mathbf{Z}\}$ are the input and the corresponding output sequences respectively of a causal linear time invariant system with impulse response $\{h_k; k \in \mathbf{Z}^+\}$, such that,

$$y_k = \sum_{i=0}^{\infty} h_i x_{k-i} \qquad\qquad k \in \mathbf{Z}.$$

(a) Suppose that x_k is a causal sequence, i.e.,

$$x_k = 0 \text{ for } k < 0.$$

We define the vectors,

$$\mathbf{x} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N) \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N) \end{bmatrix}$$

Find the matrix, $\mathbf{T}$ such that $\mathbf{y} = \mathbf{T}\mathbf{x}$. The matrix $\mathbf{T}$ describes a mapping from a chunk of input samples to to a chunk of output samples and it is a Toeplitz matrix.

(b) Suppose that x_k is a sequence, such that

$$x_k = 0 \text{ for } k > 0 \text{and for } k < -N.$$

We define the vectors,

$$\mathbf{x} = \begin{bmatrix} x(0) \\ x(-1) \\ \vdots \\ x(-N) \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N) \end{bmatrix} \tag{1}$$

Find the matrix, $\mathbf{H}$ such that $\mathbf{y} = \mathbf{H}\mathbf{x}$. The matrix $\mathbf{H}$ describes a mapping from a chunk of past input samples to to a chunk of output samples and it is a Hankel matrix.