ELEC 405 - Homework 0

Gul Sena Altintas - 97968358

October 26, 2020

Contents

1. Application of Matrix Mutliplication

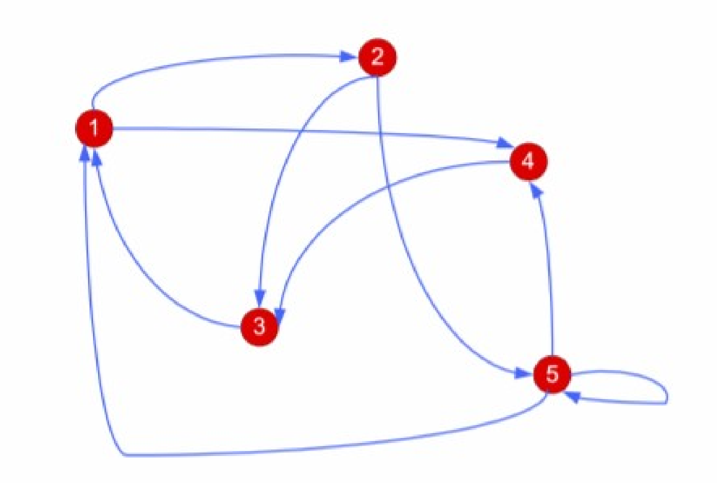


Figure 1: Graph

1. Connectivity matrix C:

C	1	2	3	4	5
1	0	1	0	1	0
2	0	0	1	0	1
3	1	0	0	0	0
4	0	0	1	0	0
5	1	0	0	1	1

Table 1: Connectivity Matrix

2. Find $\mathrm{C}^{\mathbf{2}}$. What does i, j entry of $\mathrm{C}^{\mathbf{2}}$ correspond to.
   Answer:
   $$\mathrm{C}^2_{ij} = \sum_{k=1}^{n} C_{ik} \times C_{kj} \tag{1}$$
   where n is the number of rows or columns -as C is a square matrix- ie. 5.
   $$\mathrm{C}^2 = \begin{bmatrix} 0 & 0 & 2 & 0 & 1 \\ 2 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 1 \end{bmatrix}$$
   Here, one can observe that the i, j entry of C^2 is actually the number of ways that one can go from node i to j in 2 steps.
3. What does i, j entry of the C^{k} correspond to where $k \geq 1$ is an integer?
   Answer: i, j the entry of the C^k correspond to number of ways one can go from node i to j in k steps. So, $C^k_{ij} = \sum_{l \in |n|} \cdots \sum_{t \in |n|} C_{il} \times C_{lm} \times \cdots \times C_{tj}$, where there are k summations. So for example, $C^3_{11} = 3$ because one can go from 1 to 1 in 3 steps using either one of the following combinations: $\{(1,2,3,1),(1,2,5),(1,4,3,1)\}$.

2. Simple plane geometry

1. Norm of $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \sqrt{1^2 + 2^2} = \sqrt{5}$
2. $$\begin{aligned} \mathrm{dist}(a, b) &= \|a - b\| \\ &= \| \begin{bmatrix} -3 \\ -2 \end{bmatrix} \| \\ &= \sqrt{(-3)^2 + (-2)^2} \\ &= \sqrt{13} \end{aligned}$$
3. $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, y = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.
   $$\begin{aligned} \theta &= acos(\frac{< x, y >}{\|x\|\|y\|}) \\ &= acos(\frac{1 \times 3 + 2 \times 1}{\sqrt{1^2 + 2^2} \times \sqrt{3^2 + 1^2}}) \\ &= acos(\frac{5}{5\sqrt{2}}) \end{aligned}$$

$$= \frac{\pi}{2}$$

4. $b = R(\theta)a$ where $R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, $a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$. Find b_1, b_2 in terms of a_1, a_2 and θ.

$$b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = R(\theta)a = \begin{bmatrix} a_1\cos\theta + a_2\sin\theta \\ -a_1\sin\theta + a_2\cos\theta \end{bmatrix}$$

3. Hyperplane in a 2D-World

1. $a = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. $A = \{x | a^T x = 0, x \in \mathcal{R}^2\}$.
   The set A corresponds to the line satisfying $x_1 + x_2 = 0$

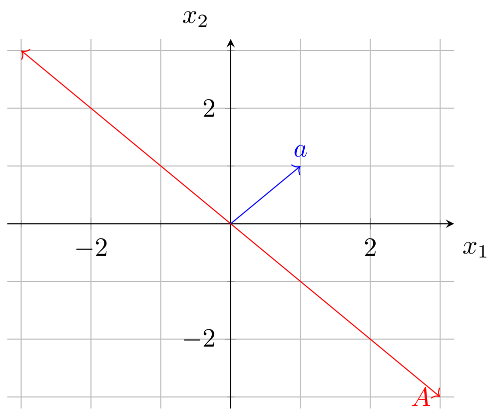


Figure 2: Draw a, A

2. $b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. $B = \{x | a^T(x - b) = 0, x \in \mathcal{R}^2\}$.
   The set B corresponds to the line satisfying $(x_1 - b_1) + (x_2 - b_2) = 0$, plugging b; $x_1 + x_2 = 3$.

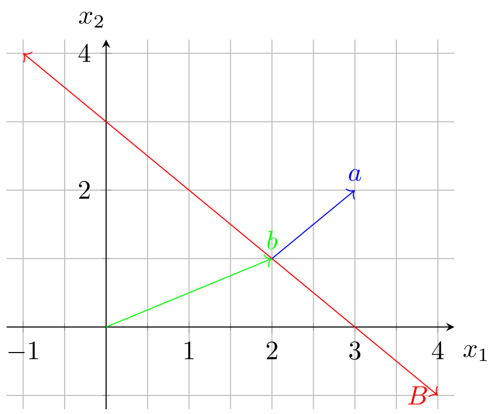


Figure 3: Draw b, B

3. $H\_ = \{x | a^T(x - b) \leq 0, x \in \mathcal{R}^2\}$.

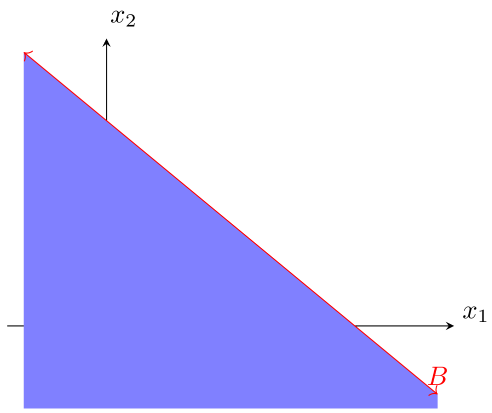


Figure 4: Draw $H\_$

4. See code in the Appendix.

4. Hyperplane in 3D World

See code in the Appendix.

5. A Reminder on Solving Linear Systems of Equations

Given $\mathbf{A}x = \mathbf{b}$, find x.

1. $\mathbf{A} = \begin{bmatrix} 2 & 13 & 5 & 6 \\ 0.5 & 3.25 & 5.25 & 2.5 \\ 1 & 10.5 & -0.5 & 9 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 16 \\ 7 \\ 3 \end{bmatrix}$

$$2x_1 + 13x_2 + 5x_3 + 6x_4 = 16 \tag{2}$$
$$0.5x_1 + 3.25x_2 + 5.25x_3 + 2.5x_4 = 7 \tag{3}$$
$$x_1 + 10.5x_2 - 0.5x_3 + 9x_4 = 3 \tag{4}$$

$$Eq(4) - 2 \times Eq(3) \Rightarrow 4x_2 - 11x_3 + 4x_4 = -11 \tag{5}$$
$$-Eq(2) + 4 \times Eq(3) \Rightarrow 16x_3 + 4x_4 = 12 \Rightarrow x_4 = 3 - 4x_3 \tag{6}$$
$$Eq(6) - Eq(5) \Rightarrow -4x_2 + 27x_3 = 23 \Rightarrow x_2 = 6.75x_3 - 5.75 \tag{7}$$
$$\text{Substituting 7 and 6 into 4} \Rightarrow x_1 + 70.875x_3 - 60.375 - 0.5x_3 + 27 - 36x_3 = 3 \tag{8}$$
$$x_1 = -34.375x_3 + 36.375$$
$$\text{Substituting all variables to Eq2} \Rightarrow -68.75x_3 - 72.75 + 87.75x_3 - 74.75 + 5x_3 + 18 - 24x_3 = 16$$
$$0 = 0 \tag{9}$$

Eq (9) implies that x_3 is a free variable so x is of the form:

$$x = \begin{bmatrix} -34.375x_3 + 36.375 \\ 6.75x_3 - 5.75 \\ x_3 \\ 3 - 4x_3 \end{bmatrix} \quad \text{for } x_3 \in \mathcal{R}$$

2. $\mathbf{A} = \begin{bmatrix} 2 & 13 & 5 & 6 \\ 0.5 & 3.25 & 1.25 & 1.5 \\ 1 & 10.5 & -0.5 & 9 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 16 \\ 7 \\ 3 \end{bmatrix}$

$$2x_1 + 13x_2 + 5x_3 + 6x_4 = 16 \tag{10}$$

$$0.5x_1 + 3.25x_2 + 1.25x_3 + 1.5x_4 = 7 \tag{11}$$
$$x_1 + 10.5x_2 - 0.5x_3 + 9x_4 = 3 \tag{12}$$

2×Eq(11)−Eq(10) => $0x_1 + 0x_2 + 0x_3 + 0x_3 + 0x_4 = 12$ ie $0 = 12$ ∴ this system is inconsistent and doesn't have any solutions.

6. Convolution Operator and Corresponding Matrices

$\{x_k : k \in \mathcal{Z}\}, \{y_k : k \in \mathcal{Z}\}, \{h_k : k \in \mathcal{Z}\}$ and

$$y_k = \sum_{i=0}^{\infty} h_i x_{k-i}, k \in \mathcal{Z} \tag{13}$$

1. x_k is a casual sequence, ie:

$$x_k = 0 \text{ for } k < 0$$

Define

$$x = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N) \end{bmatrix}, y = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N) \end{bmatrix}$$

Find T, Toeplitz matrix, such that $y = Tx$.

$$\begin{aligned} y(0) &= h_0 \times x(0) \\ y(1) &= h_0 \times x(1) + h_1 \times x(0) \\ y(2) &= h_0 \times x(2) + h_1 \times x(1) + h_2 \times x(0) \\ &\dots \\ y(N) &= h_0 \times x(N) + \dots + h_N \times x(0) \end{aligned}$$

$$\therefore T = \begin{bmatrix} h_0 & 0 & & \dots & 0 \\ h_1 & h_0 & 0 & \dots & 0 \\ & & \ddots & & \\ h_N & h_{N-1} & & \dots & h_0 \end{bmatrix} \tag{14}$$

2. x_k is a sequence such that:

$$x_k = 0 \text{ for } k > 0 \text{ and for } k < -N$$

Define

$$x = \begin{bmatrix} x(0) \\ x(-1) \\ \vdots \\ x(-N) \end{bmatrix}, y = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N) \end{bmatrix}$$

Find H, Hankel matrix, such that $y = Hx$.

$$\begin{aligned} y(0) &= h_0 \times x(0) + h_1 \times x_{-1} + \dots + h_N \times x_{-N} \\ y(1) &= h_1 \times x(0) + \dots + h_N \times x_{1-N} \\ &\dots \\ y(N) &= h_N \times x_0 \end{aligned}$$

$$\therefore H = \begin{bmatrix} h_0 & h_1 & & \dots & h_N \\ h_1 & & \dots & h_{N-1} & 0 \\ & & \ddots & & \\ h_N & 0 & & \dots & 0 \end{bmatrix} \tag{15}$$

A0

October 25, 2020

1 Elec 405 Assignment 0 - Julia Code

[1]:
```
using LinearAlgebra
```

1.1 Question 1

Trials using C matrix.

[11]:
```
C = [0 1 0 1 0; 0 0 1 0 1; 1 0 0 0 0; 0 0 1 0 0; 1 0 0 1 1]
C^3
```

[11]:
```
5×5 Array{Int64,2}:
 3  0  0  1  1
 1  2  1  3  1
 0  0  2  0  1
 0  1  0  1  0
 2  1  3  2  2
```

[12]:
```
C^5
```

[12]:
```
5×5 Array{Int64,2}:
 2  1  7  2  4
 8  2  3  5  4
 1  3  1  4  1
 3  0  0  1  1
 6  5  6  8  5
```

1.2 Question 3

[2]:
```
using Plots
```

[3]:
```
function draw_hyperplane_2D(a, b)
    pyplot()
    x=range(-3, stop=3, length=100)
    g(x) = (dot(a, b) - a[1]*x)/a[2]
    p= plot(x, g,
        title="2d hyperplane",
```

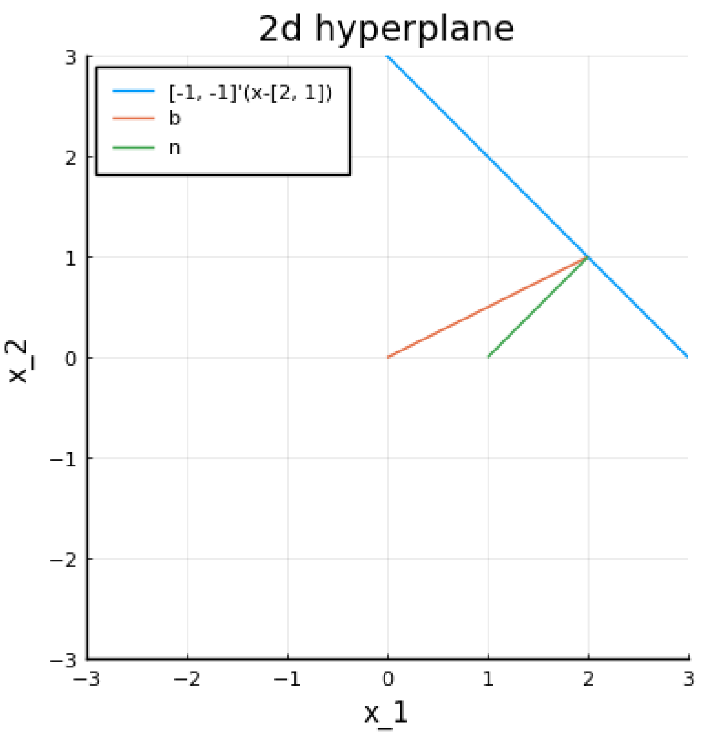
```
        label="$(a)'(x-$(b))",
        xlabel="x_1", ylabel="x_2",
        xlims=(-3, 3), ylims=(-3, 3),
        aspect_ratio=1)
    b_x1 = range(0, stop=b[1], length=100)
    b_x2 = range(0, stop=b[2], length=100)
    plot!(b_x1, b_x2, label="b")
    a_x1 = range(b[1], stop=a[1]+b[1], length=100)
    a_x2 = range(b[2], stop=a[2]+b[2], length=100)
    plot!(a_x1, a_x2, label="n")
    ylabel!(p, "x_2", label_position="left")
end
```

[3]: draw_hyperplane_2D (generic function with 1 method)

1.2.1 Homework example

[4]:
```
a = [-1; -1]
b = [2; 1]
draw_hyperplane_2D(a, b)
```

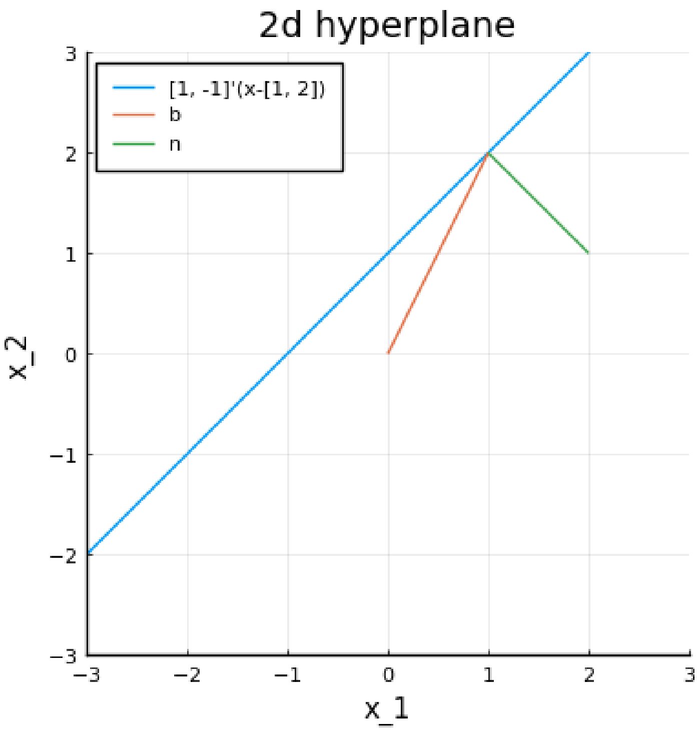
[4]:



1.2.2 Homework asked

[5]:
```
a = [1; -1]
b = [1; 2]
draw_hyperplane_2D(a, b)
```

[5]:



1.3 Question 4 *Hyperplane in 3D World*

[6]:
```
function draw_hyperplane_3D(a, b, camera=(60, 45))
    pyplot()
    x=range(-2, stop=2, length=100)
    y=range(-2, stop=2, length=100)
    g(x, y) = (dot(a, b) - dot(a[1: 2], [x; y]))/a[3]
    p= plot(x, y, g,
        title="Hyperplane in 3D",
        label="$(a)'(x-$(b))=0",
        xlabel="x_1", ylabel="x_2", zlabel="x_3",
        xlims= (-2, 2), ylims=(-2, 2), zlims=(-2, 2),
        st=:surface, camera=camera)

    # plot the normal vector, since b is on the plane plot the vector b to b+a
    a_x1 = range(b[1], stop=(b[1]+a[1]), length=100)
    a_x2 = range(b[2], stop=(b[2]+a[2]), length=100)
```

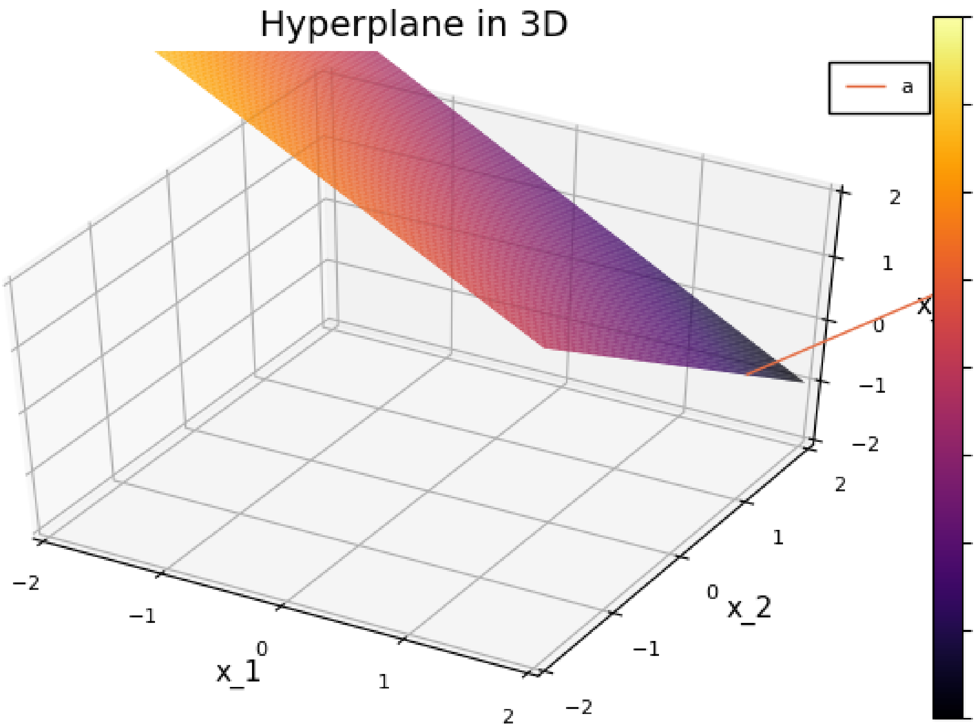
```
    a_x3 = range(b[3], stop=(b[3]+a[3]), length=100)
    plot!(a_x1, a_x2, a_x3, label="a")
end
```

[6]: draw_hyperplane_3D (generic function with 2 methods)

1.3.1 Homework example

[7]:
```
a = [1; 1; 1]
b = [2; 1; 0]
draw_hyperplane_3D(a, b, (30, 45))
```

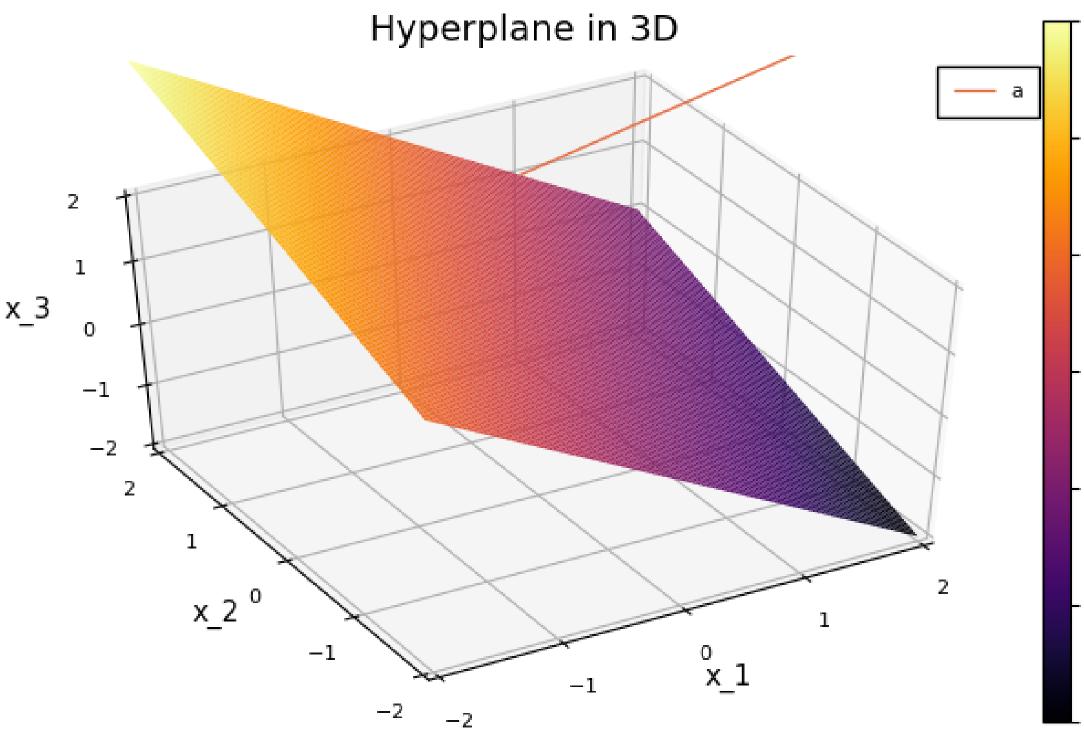
[7]:



1.3.2 Homework asked

[8]:
```
a = [2; -1; 2]
b = [1; 2; 1]
draw_hyperplane_3D(a, b, (-30, 45))
```

[8]:



1.4 Question 5

[9]:
```
A = [2 13 5 6; 0.5 3.25 5.25 2.5; 1 10.5 -.5 9]
b = [16; 7; 3]
x1 = A\b
x1
```

[9]:
```
4-element Array{Float64,1}:
  0.4251968503937001
  1.3092340730135996
  1.045812455261274
 -1.183249821045095
```

x_3 is a free variable, so that x can be written in terms of x_3 as below, $t(x)$ defines this function

[14]:
```
t(x) = [(72.75-68.75*x)/2; 6.75*x-5.75; x; 3-4*x]
```

[14]:
```
t (generic function with 1 method)
```

Below are some examples confirming for different x_3, $Ax = b$

[15]:
```
A*t(1)
```

[15]:
```
3-element Array{Float64,1}:
 16.0
  7.0
  3.0
```

[16]:
```
A*t(2)
```

[16]:
```
3-element Array{Float64,1}:
 16.0
  7.0
  3.0
```

[17]:
```
A*t(4)
```

[17]:
```
3-element Array{Float64,1}:
 16.0
  7.0
  3.0
```

[10]:
```
A2 = [2 13 5 6; 0.5 3.25 1.25 1.5; 1 10.5 -.5 9]
b = [16; 7; 3]
x2 = A2\b
x2
```

[10]:
```
4-element Array{Float64,1}:
  0.3152204649327587
  0.9434583895044879
  1.6171571367507307
 -0.7125505540393904
```

[13]:
```
A2*x2
```

[13]:
```
3-element Array{Float64,1}:
 16.705882352941174
  4.1764705882352935
  3.0000000000000036
```