ELEC505/ELEC405 Handout #3
Linear System Theory Monday, October 26, 2020

Homework Set #1
Due: Friday, November 6, 2020.

1. *Sampling and Matrix Notation*

   Let's represent a finite duration discrete time sequence with $\mathbf{x} \in \Re^n$, i.e., $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T$. Represent the following linear operations in the form $\mathbf{y} = \mathbf{A}\mathbf{x}$:

   (a) Let $\mathbf{y} \in \Re^{2n-1}$ be the output of the upsampling system defined by

   $$y_k = \begin{cases} x_{\frac{k+1}{2}} & k \text{ is odd} \\ 0 & k \text{ is even} \end{cases} \tag{1}$$

   for $k = 1, \dots, 2n-1$ and $\mathbf{y} = \begin{bmatrix} y_1 & y_2 & \dots & y_{2n-1} \end{bmatrix}^T$.

   (b) Assume n is even and $\mathbf{y} \in \Re^{\frac{n}{2}}$. We define $y_k = x_{2k}$ for $k = 1, \dots, \frac{n}{2}$.

   (c) Assume n is even and $\mathbf{y} \in \Re^{\frac{n}{2}}$. We define $y_k = \frac{x_{2k}+x_{2k-1}}{2}$ for $k = 1, \dots, \frac{n}{2}$.

2. *Vector Space of Polynomials*

   Let $\mathcal{P}_{n-1}$ represent the vectors space of polynomials with degree less than or equal to $n-1$. Therefore, each element in this vector space can be written as

   $$p(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_{n-1}x^{n-1}. \tag{2}$$

   As we discussed in class, we can form a one to one correspondence between $\mathcal{P}_{n-1}$ and $\Re^n$. Let $\mathbf{a} = \begin{bmatrix} c_0 & c_1 & \dots & c_{n-1} \end{bmatrix}^T$ be the vector in $\Re^n$ corresponding to $p(x)$ above, and let $\mathbf{b}$ be the vector corresponding to $\frac{dp(x)}{dx}$. Show that the relation between $\mathbf{b}$ and $\mathbf{a}$ can be written as

   $$\mathbf{b} = \mathbf{H}\mathbf{a} \tag{3}$$

   by explicitly writing $\mathbf{H}$.

3. *Affine functions*

   A function $f : \Re^n \to \Re^m$ is called affine if and only if for any $\mathbf{x}, \mathbf{y} \in \Re^n$ and any $\alpha, \beta \in \Re$ satisfying $\alpha + \beta = 1$, we have

   $$f(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y}) \tag{4}$$

   (Note that, if we didn't restrict $\alpha + \beta = 1$, this would be the definition of a linear function)

   (a) Suppose that $\mathbf{A} \in \Re^{m\times n}$ and $\mathbf{b} \in \Re^m$. Show that $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ is affine.

(b) Now the converse: Show that any affine function f can be represented as $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ for some $\mathbf{A} \in \Re^{m \times n}$ and $\mathbf{b} \in \Re^m$. (Hint: Show that $g(\mathbf{x}) = f(\mathbf{x}) - f(\mathbf{0})$ is linear).

Therefore, an affine function can be considered as a linear function shifted by an offset.

4. *Yet Another Proof of Cauchy-Schwarz...*

   (a) Suppose $a \geq 0$, $c \geq 0$ and for all $\lambda \in \mathbf{R}$, $a + 2b\lambda + c\lambda^2 \geq 0$. Show that $|b| \leq \sqrt{ac}$.

   (b) Given $\mathbf{u}, \mathbf{w} \in \mathbf{R}^n$, explain why $(\mathbf{u} + \lambda\mathbf{w})^T(\mathbf{u} + \lambda\mathbf{w}) \geq 0$ for all $\lambda \in \mathbf{R}$.

   (c) Apply a to the quadratic resulting from the expansion of the expression in (b), to get the Cauchy-Schwarz inequality:

$$|\mathbf{v}^T\mathbf{w}| \leq \sqrt{\mathbf{v}^T\mathbf{v}}\sqrt{\mathbf{w}^T\mathbf{w}}$$

.

5. *Inequality Proof*

   For $\mathbf{a}, \mathbf{b} \in \Re^n$, show that $\|\mathbf{a} + \mathbf{b}\| \geq \|\mathbf{a}\| - \|\mathbf{b}\|$, where $\|\cdot\|$ is the Euclidian norm.

6. *First steps in multivariable quadratic functions*

   Please write the multivariable quadratic function

$$f(x_1, x_2, x_3) = 2x_1^2 + 3x_1x_3 - 2x_3x_2 - x_2^2 + 4x_3^2 \tag{5}$$

   in the form $f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ where $\mathbf{x} \in \Re^3$. Either prove that $\mathbf{A}$ is unique and provide the corresponding $\mathbf{A}$, or show that it is non-unique by providing two alternative matrices for $\mathbf{A}$.

7. *On Gradients....*

   Let $f : \Re^n \to \Re$ be a differentiable multivariate function. Its gradient (at point $\mathbf{x}$) is defined as the vector

$$\nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}. \tag{6}$$

   where the partial derivatives are evaluated at $\mathbf{x}$.

   Find the explicit gradients of the following functions (using matrix notation only)

   (a) $f(\mathbf{x}) = \mathbf{a}^T\mathbf{x} + b$ for some $\mathbf{a} \in \Re^n$ and $b \in \Re$.

   (b) $f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ for some $\mathbf{A} \in \Re^{n \times n}$.

8. *What is a homework without a computational part?....*

Given a differentiable function $f(\mathbf{x})$, where $\mathbf{x} \in \Re^n$,

$$a(\mathbf{x}) = f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)^T(\mathbf{x} - \mathbf{x}_0) \tag{7}$$

is an affine approximation of $f(\mathbf{x})$ around $\mathbf{x}_0$.

Note that this affine function defines an hyperplane in $\Re^{n+1}$ in the standard form

$$\begin{bmatrix} \nabla f(\mathbf{x}_0) \\ -1 \end{bmatrix}^T \left(\begin{bmatrix} \mathbf{x} \\ a(\mathbf{x}) \end{bmatrix} - \begin{bmatrix} \mathbf{x}_0 \\ f(\mathbf{x}_0) \end{bmatrix} \right) = 0. \tag{8}$$

Now consider the function

$$f(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x} \tag{9}$$

defined over $\Re^2$, where

$$\mathbf{A} = \begin{bmatrix} 0.01 & 0.001 \\ 0.001 & 0.01 \end{bmatrix}. \tag{10}$$

Please plot the surfaces corresponding to this function and its affine approximation at the point $\mathbf{x}_0 = \begin{bmatrix} 25 & 0 \end{bmatrix}^T$ as show in Figure below:

Please submit your Matlab/Python/Julia code and your plot (Hint: Yes, you can shamelessly reuse your *drawhyperplane3D* function from your previous homework.) In case of Python/Julia notebooks, please submit a print out of your executed notebook.

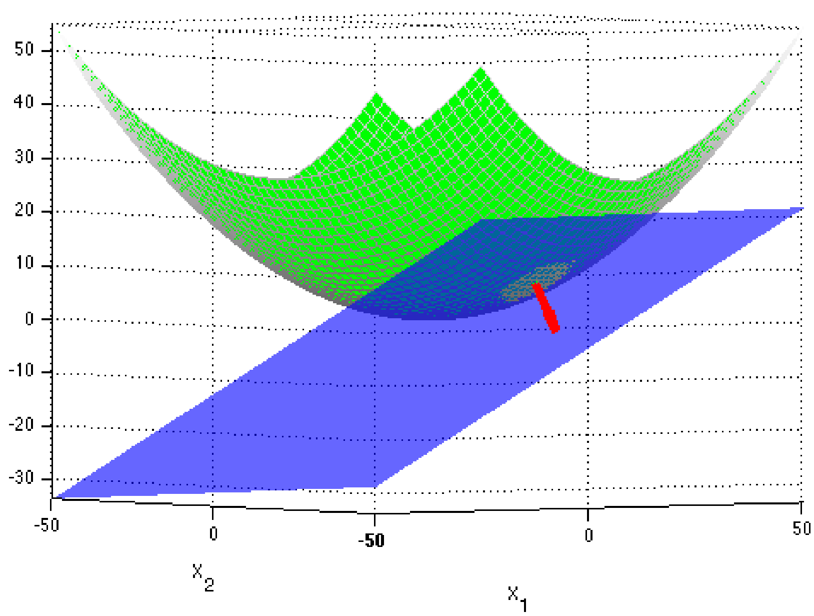

Figure 1: Affine approximation of a quadratic function.