ELEC505/ELEC405 Handout #3
Linear System Theory Sunday, November 8, 2020

Homework Set #2

Due: Sunday, November 22 , 2020.

1. *Nearest Neighbor partitioning and halfspaces...*

   Suppose $\mathbf{a}, \mathbf{b} \in \mathbf{R}^n$ are two given points. Show that the set of points in $\mathbf{R}^n$ that are closer to $\mathbf{a}$ than $\mathbf{b}$ is a half space, i.e.:

$$\{\mathbf{x} \mid \| \mathbf{x} - \mathbf{a} \| \leq \| \mathbf{x} - \mathbf{b} \|\} = \{\mathbf{x} \mid \mathbf{c}^T \mathbf{x} \leq d\}$$

   for appropriate $\mathbf{c} \in \mathbf{R}^n$ and $d \in \mathbf{R}$. Provide $\mathbf{c}$ and d explicitly (in terms of $\mathbf{a}$ and $\mathbf{b}$), and draw a picture showing $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and the halfspace (for n=2, hand-drawing is fine let's give Matlab/Python/Julia stuff a break... you pick your favorite $\mathbf{a}, \mathbf{b}$.). Remember that the set $\{\mathbf{x} \mid \mathbf{c}^T \mathbf{x} = d\}$ (border) is called an hyperplane.

2. *Rank and Matrix Products....*

   For each of the following statements, either show that it is true or give a counterexample

   (a) If $\mathbf{AB}$ is full rank then $\mathbf{A}$ and $\mathbf{B}$ are full rank.

   (b) If $\mathbf{A}$ and $\mathbf{B}$ are full rank then $\mathbf{AB}$ is full rank.

   (c) If $\mathbf{A}$ and $\mathbf{B}$ have zero nullspace, then so does $\mathbf{AB}$.

   (d) If $\mathbf{A}$ and $\mathbf{B}$ are onto, then so is $\mathbf{AB}$.

3. *Fundamental Subspaces and Determinant....*

   Given $\mathbf{A} \in \mathbf{R}^{m \times n}$ show that

   (a) If $\mathcal{R}(\mathbf{A}) = \mathbf{R}^m$, then $\mathbf{A}\mathbf{A}^T$ is full rank.

   (b) If $\mathcal{N}(\mathbf{A}) = \{\mathbf{0}\}$ then $\mathbf{A}^T\mathbf{A}$ is full rank.

4. *Characteristic Polynomial of a Square Matrix....*

   Consider the characteristic polynomial $X(s) = det(sI - \mathbf{A})$ of the matrix $\mathbf{A} \in \mathbf{R}^{n \times n}$

   (a) Show that $X(s)$ is monic, which means that its leading coefficient is one: $X(s) = s^n + ...$

   (b) Show that s^{n-1} coefficient of $X(s)$ is given by $-\,\mathbf{Tr}\,\mathbf{A}$. (Remember $\mathbf{Tr}\,\mathbf{A}$ is the trace of the matrix $\mathbf{A}$ which is the sum of the diagonal entries of $\mathbf{A}$, i.e., $\mathbf{Tr}\,\mathbf{A} = \sum_{i=1}^n A_{ii}$.)

   (c) Show that the constant coefficient of $X(s)$ is given by $det(-A)$.

(d) Let $\lambda_1, ..., \lambda_n$ denote the eigenvalues of $\mathbf{A}$, so that

$$X(s) = s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0 = (s - \lambda_1)(s - \lambda_2)...(s - \lambda_n)$$

by equating coefficients show that $a_{n-1} = -\sum_{i=1}^{n} \lambda_i$ and $a_0 = \prod_{i=1}^{n}(-\lambda_i)$. Therefore conclude that $\mathbf{Tr}\,\mathbf{A} = \sum_{i=1}^{n} \lambda_i$ and $det\mathbf{A} = \prod_{i=1}^{n} \lambda_i$

5. *Schur's Theorem and Cayley-Hamilton Theorem...*

Schur's Theorem is as follows: Given $\mathbf{A} \in \mathbf{R}^{n\times n}$ with eigenvalues $\lambda_1, ..., \lambda_n$ in any prescribed order, there is a unitary matrix $\mathbf{U}$ such that

$$\mathbf{U}^*\mathbf{A}\mathbf{U} = \mathbf{T}$$

is upper triangular with diagonal entries $t_{ii} = \lambda_i$, $i = 1, ..., n$. Remember that a unitary matrix $\mathbf{U}$ has the property $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = I$ where $\mathbf{U}^*$ is the hermitian transpose of $\mathbf{U}$.

(a) (*Cayley-Hamilton Theorem*) Using Schur's Theorem above, prove the Cayley-Hamilton Theorem which states: Let $p_{\mathbf{A}}(t)$ be the characteristic polynomial of $\mathbf{A} \in \mathbf{R}^{n\times n}$, (i.e., $p_{\mathbf{A}}(t) = \det(t\mathbf{I} - \mathbf{A})$), then

$$p_{\mathbf{A}}(\mathbf{A}) = \mathbf{0}$$

Note: Don't just plug in A into determinant expression. First right the n^{th} characteristic polynomial in terms of t's powers and then plug in $\mathbf{A}$ to replace t. In other words, you should write a polynomial containing powers of $\mathbf{A}$ and show that this polynomial is equal to the zero <u>matrix</u>.

(b) Using the result of (a) and assuming

$$p_{\mathbf{A}}(t) = t^n + a_{n-1}t^{n-1} + ... + a_1 t + a_0$$

write each of $\mathbf{A}^n, \mathbf{A}^{n+1}, \mathbf{A}^{n+2}$ as polynomials of $\mathbf{A}$ with degree at most $n-1$. Assuming $\mathbf{A}$ is invertible write $\mathbf{A}^{-1}$ also as a polynomial of $\mathbf{A}$ of degree at most $n-1$.

6. *Theory in Action for Big Data!: Eigenvectors as community detectors*

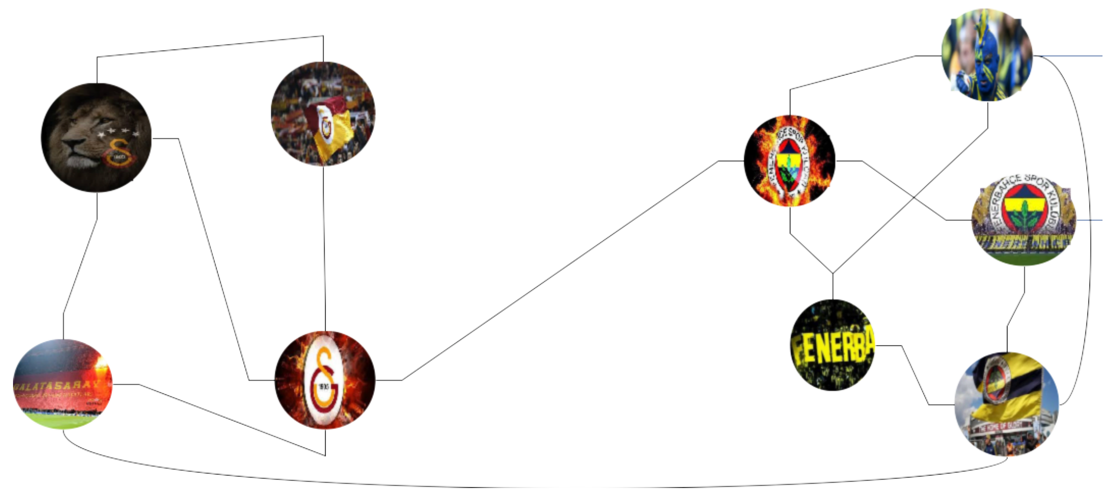

Figure 1: Two team fan communities and their facebook connectivity.

Now consider a population of $2N$ people,. Half of the population is part of community A, the other half is part of community B. All these guys have facebook accounts where they are potentially connected as illustrated in Figure 1. Two people in the same community are connected with probablility p and two people in different communities are connected with probability $q < p$. Connections are probabilistically independent.

We define the facebook connectivity matrix $\mathbf{C}$ as

$$C_{ij} = \begin{cases} 1 & \text{if } i, j \text{ connected,} \\ 0 & \text{otherwise} \end{cases}$$

It is a bit weird but we will assume that to be true for C_{ii} too... Anyway, we define $\mathcal{I}_A \subset \{1, \ldots, 2N\}$ as the indices of the people in community A, and $\mathcal{I}_B = \bar{\mathcal{I}_A}$.

(a) Assume that $\mathcal{I}_A = \{1, \ldots, N\}$ and $\mathcal{I}_B = \{N+1, \ldots, 2N\}$. Find the matrix $E(\mathbf{C})$, where $E(\cdot)$ is the expected value operator.

(b) What is the rank of $E(\mathbf{C})$ in the previous part?

(c) What are the eigenvalues and eigenvectors of $E(\mathbf{C})$ in part a? How would it change if $\mathcal{I}_A$ is arbitrary? How would you tell communities, i.e., $\mathcal{I}_A$ and $\mathcal{I}_B$ from eigenvalues, eigenvectors?

(d) Now going back to real life. We don't have p, q and $E(\mathbf{C})$ available. We have only $\mathbf{C}$ (It appears we can assume Facebook is willing to sell this information!). As a relief, the members of our geeky theoretical data science team claim (and prove) that for a relatively large population, eigenvectors of $\mathbf{C}$, corresponding to top eigenvalues, are fairly close to eigenvectors of $E(\mathbf{C})$. Using this information, here is the task:

- Take $N = 100$. Randomly generate $\mathcal{I}_A$ and $\mathbf{C}$. (Take $p = 0.6$, $q = 0.1$).

- Display **C** using imshow and print. Can you tell communities from this picture?
- Check out eigenvalues, eigenvectors? Are they helpful in detecting $\mathcal{I}_A$? How? Display relevant eigenvector(s) using stem function.
- How accurate is your result? (Run this test 1000 times and report the average percentage of correct identification of each person's community.)

Please include your plots as printouts attached to your handwritten solution with clear labeling including problem number.