ELEC 405 - Homework 3

Gul Sena Altintas - 64284

December 7, 2020

Contents

1. LU Decomposition

$$A = \begin{bmatrix} 6 & 30 & 36 \\ 2 & 10 & 4 \\ 5 & 4 & 2 \end{bmatrix}$$

First apply $r_1 \leftarrow \frac{1}{6}r_1$. Then apply $r_2 \leftarrow r_2 - 2r_1; r_3 \leftarrow r_3 - 5r_1$, then $r_2 \leftarrow -r_2/8; r_3 \leftarrow r_3/21$. We can describe these steps with a lower triangular matrix L_1 applied to A from the left and at this point $L_1 A$ is given in (1).

$$L_1 = \begin{bmatrix} \frac{1}{6} & 0 & 0 \\ \frac{2}{8\times 6} & \frac{-1}{8} & 0 \\ \frac{5}{21\times 6} & 0 & \frac{-1}{21} \end{bmatrix}$$

$$L_1 A = \begin{bmatrix} 1 & 5 & 6 \\ 0 & 0 & 1 \\ 0 & 1 & \frac{4}{3} \end{bmatrix} \tag{1}$$

Since we need to do a row exchange, we define the permutation matrix P such that $P = P^{-1} = P^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = P$. Now we need to redo the operations on PA:

$$PA = \begin{bmatrix} 6 & 30 & 36 \\ 5 & 4 & 2 \\ 2 & 10 & 4 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} \frac{1}{6} & 0 & 0 \\ \frac{5}{6\times 21} & -\frac{1}{21} & 0 \\ \frac{2}{6\times 8} & 0 & \frac{-1}{8} \end{bmatrix}$$

$$L_2PA = \begin{bmatrix} 1 & 5 & 6 \\ 0 & 1 & \frac{4}{3} \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

$$L_2^{-1} = \begin{bmatrix} 6 & 0 & 0 \\ 5 & -21 & 0 \\ 2 & 0 & -8 \end{bmatrix}$$

$$PA = LD$$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 5 & -21 & 0 \\ 2 & 0 & -8 \end{bmatrix} \begin{bmatrix} 1 & 5 & 6 \\ 0 & 1 & \frac{4}{3} \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

2. Orthogonality of Left-Right Eigenvectors

Let u_l be a left eigenvector of $A : u_l^T A = \mu u_l$ and u_r be a right eigenvector, $Au_r = \lambda u_r$. Consider the product $u_l^T Au_r = u_l^T(Au_r) = \lambda < u_l, u_r >= (u_l^T A)u_r = \mu < u_l, u_r >$. Since $\lambda \neq \mu < u_l, u_r >= 0$.

3. DFT as an Orthogonal Basis Change

1. Let $F^{n\times n}$ where the columns of F are f_{k+1} ($\begin{bmatrix} f_0 & f_1 & \dots & f_{N-1} \end{bmatrix}$). Let $a = e^{\frac{j2\pi(k)}{N}}$ then F looks like:

$$\begin{bmatrix} \frac{1}{N} & \frac{1}{N} & & \dots & \frac{1}{N} \\ \frac{1}{N} & \frac{1}{N}a & \frac{1}{N}a^2 & \dots & \frac{1}{N}a^{N-1} \\ \vdots & & & & \\ \frac{1}{N} & \frac{1}{N}(a^{N-1}) & & \dots & \frac{1}{N}(a^{N-1})^{N-1} \end{bmatrix} \tag{4}$$

Then we can rewrite the columns of F as a polynomial based on the powers of a where $p = \frac{1}{N}\sum_{n=0}^{N-1} a^n$:

$$\begin{bmatrix} p(a^0) & p(a) & \dots & p(a^{N-1}) \end{bmatrix} \tag{5}$$

Now, let's consider the inner product between columns of F

$$\begin{aligned} < f_k, f_m > &= \frac{1}{N^2}\sum_{l=0}^{N-1} a^{k\times l}a^{-m\times l} \\ &= \frac{1}{N^2}\sum_{l=0}^{N-1} a^{l\times(k-m)} \\ &\text{Let } p = k - m \\ &= \frac{1}{N^2}\sum_{l=0}^{N-1} a^{l\times p} \\ &\frac{1}{N^2}(1 + a^p + \dots + (a^p)^{N-1}) \\ &\frac{1}{N^2}\frac{1-(a^p)^N}{a - a^p} \\ &\text{Note that } a^{pN} = 1 \text{ when } p \neq 0 \\ &= 0 \\ &\text{Similarly when } k = m \text{ the result is } \frac{1}{N} \end{aligned} \tag{6}$$

Since $\|f_k\|$ is not equal to 1, ($\|f_0\| = \frac{1}{N}^{(N-1)/N}$)$\mathcal{F}$ doesn't form an orthonormal basis unless $N = 1$.

```
H = [1 -1 2 0; 0 1 -1 2; 2 0 1 -1; -1 2 0 1]
H
```

```
4×4 Array{Int64,2}:
  1  -1   2   0
  0   1  -1   2
  2   0   1  -1
 -1   2   0   1
```

```
eigvals(H)
```

```
4-element Array{Complex{Float64},1}:
 -0.9999999999999996 - 0.9999999999999993im
 -0.9999999999999996 + 0.9999999999999993im
  2.0000000000000004 + 0.0im
   4.000000000000001 + 0.0im
```

```
V = eigvecs(H)
```

```
4×4 Array{Complex{Float64},2}:
          0.5-0.0im          …  0.5+0.0im   0.5+0.0im
 -2.22045e-16+0.5im             0.5+0.0im  -0.5+0.0im
         -0.5+5.55112e-17im     0.5+0.0im   0.5+0.0im
  1.66533e-16-0.5im             0.5+0.0im  -0.5+0.0im
```

Figure 1: H and V

2. F is symmetric therefore our argument from above holds for FF^* but $\mathcal{F}$ is not unitary because the diagonal entries are $\frac{1}{N}$ (follows from (part 1 $< f_i, f_i >$). So the inverse of $\mathcal{F} = N \times \mathcal{F}$.

3. $\mathcal{F}$ for $n = 4$:

$$\begin{bmatrix} \frac{1}{N} & \frac{1}{N} & \frac{1}{N} & \frac{1}{N} \\ \frac{1}{N} & \frac{1}{N}e^{2j\pi/n} & \frac{1}{N}e^{4j\pi/n} & \frac{1}{N}e^{3j\pi/n} \\ \frac{1}{N} & \frac{1}{N}e^{4j\pi/n} & \frac{1}{N}e^{8j\pi/n} & \frac{1}{N}e^{6j\pi/n} \\ \frac{1}{N} & \frac{1}{N}e^{6j\pi/n} & \frac{1}{N}e^{12j\pi/n} & \frac{1}{N}e^{9j\pi/n} \end{bmatrix} \tag{7}$$

FFT of $x = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is

$$\begin{bmatrix} \frac{2}{N} \\ \frac{1}{N} + \frac{1}{N}e^{2j\pi/n} \\ \frac{1}{N} + \frac{1}{N}e^{4j\pi/n} \\ \frac{1}{N} + \frac{1}{N}e^{6j\pi/n} \end{bmatrix} \tag{8}$$

4.

$$H = \begin{bmatrix} h_0 & h_3 & h_2 & h_1 \\ h_1 & h_0 & h_3 & h_2 \\ h_2 & h_1 & h_0 & h_3 \\ h_3 & h_2 & h_1 & h_0 \end{bmatrix} \tag{9}$$

5. $Hf_k =$

6.

Figure 2: HV.jpg

4. Reflection through a hyperplane

Reflection of x with respect to a hyperplane H_z is actually $x + 2\times$(projection of x onto H_z -x). So $Qx = x + 2(Px - x) = 2Px - x$, Q should be of the form $2P - I$ where P is the orthogonal projection matrix to H_z. We can easily prove that Q is orthogonal: Linear transformation Qx preserves the length of x, ie $\|Qx\| = \|x\|$

$$\begin{aligned} x^T Q^T Q x &= x^T x \\ x^T Q^T Q x - x^T I^T I x &= 0 \\ x^T (Q^T Q - I) x &= 0 \qquad (10) \\ & \qquad (11) \end{aligned}$$

Note that when we apply Q twice to x, we get the original vector back, therefore $Q^T Q = I$ for any x and therefore Q is orthogonal.

5. Orthogonal matrices

1. Let U, V be two orthogonal matrices. Then we know that $UU^T = I = VV^T$. Now consider $(UV)(UV)^T = UVV^TU = UIU^T = UU^T = I$.
2. Suppose that $U \in \mathcal{R}^{2\times 2}$ is orthogonal. Let

$$\begin{aligned} U &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ UU^T = I &= \begin{bmatrix} a^2 + c^2 & ac + bd \\ ac + bd & b^2 + d^2 \end{bmatrix} \\ & \text{It must hold that } a^2 + c^2 = b^2 + d^2 = 1 \text{ (ie columns have unit length)} \\ & \text{and } ac + bd = 0 \text{ (ie columns are orthogonal)} \\ & \text{Let } a = \cos\alpha, c = \sin\alpha \text{ and } b = \sin\beta, d = \cos\beta \\ ab + cd &= \sin\alpha\sin\beta + \cos\alpha\cos\beta \\ &= \cos(\alpha - \beta) \\ \cos(\alpha - \beta) = 0 &\iff (\alpha - \beta) = k \times \frac{\pi}{2}, \text{k is odd and k} \in Z \\ \beta &= \alpha - k\frac{\pi}{2} \\ & \text{Rewriting U} \\ U &= \begin{bmatrix} \sin\alpha & \sin\beta - k\frac{\pi}{2} \\ \cos\alpha & \cos\beta - k\frac{\pi}{2} \end{bmatrix} \qquad (12) \end{aligned}$$

If $k\frac{\pi}{2} = (\frac{\pi}{2} \mod \pi)$ then $\sin\alpha - k\frac{\pi}{2} = -\cos\alpha$ and $\cos\alpha - k\frac{\pi}{2} = \sin\alpha$ then U is of the form:

$$\begin{bmatrix} \sin\alpha & -\cos\alpha \\ \cos\alpha & \sin\alpha \end{bmatrix} \qquad (13)$$

(13) represents a reflection. Take $x = \begin{bmatrix} \sin\alpha \\ \cos\alpha \end{bmatrix}$ on the unit circle U maps this vector to $\begin{bmatrix} \sin^2\alpha - \cos^2\alpha \\ 2\cos\alpha\sin\alpha \end{bmatrix} = \begin{bmatrix} \cos 2\alpha \\ \sin 2\alpha \end{bmatrix}$ which is reflection with respect to α. On the other hand, If $k\frac{\pi}{2} = (\frac{3\pi}{2} \mod \pi)$ then $\sin\alpha - k\frac{3\pi}{2} = \cos\alpha$ and $\cos\alpha - k\frac{3\pi}{2} = -\sin\alpha$ then U is of the form:

$$\begin{bmatrix} \sin\alpha & \cos\alpha \\ \cos\alpha & -\sin\alpha \end{bmatrix} \qquad (14)$$

Similarly, (14) is a rotation. Take $x = \begin{bmatrix} \sin\alpha \\ \cos\alpha \end{bmatrix}$ on the unit circle U maps this vector to $\begin{bmatrix} \sin^2\alpha + \cos^2\alpha \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ which is rotation by $-\alpha$. Therefore any orthogonal matrix in $\mathcal{R}^{2\times 2}$ is either a reflection or a rotation.