Elec 405 - HW 4

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$g(u) = \mathcal{A}u \quad \forall u \in \mathbb{R}^{2n}$

Let $x \in V_1$

$T(x) = \begin{bmatrix} Re\{x\} \\ Im\{x\} \end{bmatrix}$

$f(x) = Ax$

$T(f(x)) = \begin{bmatrix} Re\{Ax\} \\ Im\{Ax\} \end{bmatrix} \qquad g(T(x)) = \begin{bmatrix} Re\{Ax\} \\ Im\{Ax\} \end{bmatrix} = \mathcal{A}\,T(x)$

$A = \begin{bmatrix} a_{11}+b_{11}i & \cdots & a_{1n}+b_{1,2n}i \\ a_{21}+b_{21}i & & \\ \vdots & & \end{bmatrix} \qquad X = \begin{bmatrix} x_{11}+x_{12}i \\ x_{21}+x_{22}i \\ \vdots \\ x_{n1}+x_{n2}i \end{bmatrix}$

$Ax = \begin{bmatrix} (a_{11}x_{11} - b_{11}x_{12}) + i(a_{11}x_{12} + b_{11}x_{11}) & \cdots \\ & \cdots \end{bmatrix}$

$\mathcal{A} \begin{bmatrix} x_{11} \\ \vdots \\ x_{n1} \\ x_{12} \\ \vdots \\ x_{n2} \end{bmatrix} \qquad \mathcal{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & -b_{11} & \cdots & -b_{1n} \\ \vdots & & & & & & \\ a_{n1} & \cdots & & a_{nn} & -b_{n1} & \cdots & -b_{nn} \\ b_{11} & b_{12} & \cdots & b_{1n} & a_{11} & \cdots & a_{1n} \\ \vdots & & & & & & \\ b_{n1} & \cdots & & b_{nn} & a_{n1} & \cdots & a_{nn} \end{bmatrix}$

where a_{ij} correspond to $Re(A_{ij})$ and $b_{ij} = Im(A_{ij})$

$\mathcal{A} = \begin{bmatrix} Re(A) & -Im(A) \\ Im(A) & Re(A) \end{bmatrix}$ where $Re(A), Im(A)$ are $\mathbb{R}^{n\times n}$ matrices

$\mathcal{A}$ is neither Hermitian, nor unitary, nor Skew-Hermitian.

e) $x, y \in \mathbb{C}^n \qquad \langle T(x), T(y)\rangle_{\mathbb{R}} \qquad \langle \cdot\cdot, \cdot\cdot \rangle_{\mathbb{C}}$

Let $x = \begin{bmatrix} x_{11}+x_{12}i \\ \vdots \\ x_{n1}+x_{n2}i \end{bmatrix}, \quad y = \begin{bmatrix} y_{11}+y_{12}i \\ \vdots \\ y_{n1}+y_{n2}i \end{bmatrix}$

$\langle T(x), T(y)\rangle = [y_{11} \cdots y_{n1} y_{12} \cdots y_{n2}] \begin{bmatrix} x_{11} \\ \vdots \\ x_{n1} \\ x_{12} \\ \vdots \\ x_{n2} \end{bmatrix} = x_{11}y_{11} + \cdots + x_{n1}y_{n1} + y_{12}x_{12} + \cdots + x_{n2}y_{n2}$

- a complex square matrix U is unitary if its conj. transpose U^* is also its inverse

$U^*U = UU^* = I$

(2) A is skew hermitian $\Rightarrow A^T = -A \rightarrow$ eigenvalues are purely imaginary $\rightarrow I+A$ has nonzero eig vals $\rightarrow$ invertible

$U^*U = ((I-A)(I+A)^{-1})^* ((I-A)(I+A)^{-1})$

$= \underbrace{((I+A)^{-1})^*}_{(I+A^*)^{-1}} \underbrace{(I-A)^*}_{(I-A^*)} (I-A)(I+A)^{-1}$

$\downarrow \qquad\qquad \downarrow$

$\cdots^{-1} \qquad (I+A)$

$$\left((I+A)^*\right)^{-1} \quad (I-A^*)$$
$$\downarrow \qquad\qquad \downarrow$$
$$(I-A)^{-1} \quad (I+A)$$

Let $I+A=W$, $I-A=V$

$U^*U = V^{-1}WVW^{-1} \qquad (W^{-1}V)^{-1}$

W and V commute because ($(I+A)(I-A) = I-A^2 = (I+A)(I-A)$)

$= V^{-1}VWW^{-1}$

$= II$

$= I$

Similarly $UU^* = VW^{-1}V^{-1}W$

Similarly $W^{-1}V^{-1} = (I+A)^{-1}(I-A)^{-1}$

$= \left((I-A)(I+A)\right)^{-1}$

$= (I-A^2)^{-1} = V^{-1}W^{-1}$

So $UU^* = VV^{-1}W^{-1}W$

$= I$

(3) P_v orthogonal

$U = I - 2P_v$ is unitary

Ans: $U^*U = (I-2P_v)^*(I-2P_v)$

$= (I-2P_v^*)(I-2P_v)$ since P_v is orth. proj. matrix $P_v = P_v^*$, $P_v = P_v^2$

$= I - 2P_v^* - 2P_v + 4P_v^*P_v$

$= I - 4P_v + 4P_v$

$= I$

Same argument holds for UU^*

$\Rightarrow U$ is a unitary matrix.

(4) $R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

a) eigenvalue decomposition of $R(\theta)$

1st find the eigenvalues of $R(\theta)$

$\det(\lambda I - R(\theta)) = 0 \Rightarrow \begin{bmatrix} \lambda-\cos\theta & -\sin\theta \\ \sin\theta & \lambda-\cos\theta \end{bmatrix} = V$

$\det(V) = (\lambda-\cos\theta)^2 - (-\sin^2\theta)$

$= \lambda^2 - 2\lambda\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1}$

$\det(V)=0 \Rightarrow \lambda^2 - 2\lambda\cos\theta + 1 = 0$

$\lambda = \dfrac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$

$= \cos\theta \pm \sqrt{\cos^2\theta - 1}$

$= \cos\theta \pm i\sin\theta$

$\Rightarrow \lambda_1 = e^{i\theta}$, $\lambda_2 = e^{-i\theta}$ $\Rightarrow \Lambda = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$

Finding corresponding eigenvectors (v_1, v_2):

Finding corresponding eigenvectors (v_1, v_2):

$$R(\theta)\cdot v_1 = \lambda_1 v$$

Let $v_1 = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix} = \begin{bmatrix} xe^{i\theta} \\ ye^{i\theta} \end{bmatrix}$$

$$x\cos\theta - y\sin\theta = x(\cos\theta + i\sin\theta)$$

$$y = -xi$$

$$x\sin\theta + y\cos\theta = y(\cos\theta + i\sin\theta)$$

$$x = yi$$
$$= -xii$$
$$= x \checkmark$$

$$v_1 = \begin{bmatrix} x \\ -xi \end{bmatrix} = x\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Let $v_2 = \begin{bmatrix} a \\ b \end{bmatrix}$

$$\begin{bmatrix} a\cos\theta - b\sin\theta \\ a\sin\theta + b\cos\theta \end{bmatrix} = (\cos\theta - i\sin\theta)\begin{bmatrix} a \\ b \end{bmatrix}$$

$$a\cos\theta - b\sin\theta = a\cos\theta - ai\sin\theta \quad *$$

$$b = ai$$

$$a\sin\theta + b\cos\theta = b\cos\theta - ib\sin\theta$$

(* substitute)

$$a = -ib$$
$$a = -iia = a \checkmark$$

$$v_2 = a\begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$V = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$$

find V^{-1} : $\det V = i + i = 2i$

$$V^{-1} = \frac{1}{2i}\begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} 0.5 & 0.5i \\ 0.5 & -0.5i \end{bmatrix}$$

So $$R(\theta) = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}\begin{bmatrix} 0.5 & 0.5i \\ 0.5 & -0.5i \end{bmatrix}$$

b) Is $R(\theta_1)R(\theta_2) = R(\theta_2)R(\theta_1)$?

$$R(\theta_1)R(\theta_2) = V\Lambda(\theta_1)\underbrace{V^{-1}V}_{I}\Lambda(\theta_2)V^{-1}$$

$$R(\theta_2)R(\theta_1) = V\Lambda(\theta_2)\underbrace{V^{-1}V}_{I}\Lambda(\theta_1)V^{-1}$$

The question is if $\Lambda(\theta_1)\Lambda(\theta_2)$ commute

$R(\theta_2)\,R(\theta_1)$ ... $\underbrace{\quad}_{I}$

The question is if $\Lambda(\theta_1)\,\Lambda(\theta_2)$ commute

Since Λ is a diagonal matrix $\rightarrow$ they commute

So, yes $\quad R(\theta_1)\,R(\theta_2) = R(\theta_2)\,R(\theta_1)$

c) $T(\theta, t) = \begin{bmatrix} R(\theta) & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & t_1 \\ \sin\theta & \cos\theta & t_2 \\ 0 & 0 & 1 \end{bmatrix}$

① find eigenvalues of $T(\theta, t)$

$\det(\lambda I - T(\theta, t)) = 0$

$$\det\left(\begin{pmatrix} \lambda-\cos\theta & -\sin\theta & t_1 \\ \sin\theta & \lambda-\cos\theta & t_2 \\ 0 & 0 & \lambda-1 \end{pmatrix}\right) = 0$$

$$= (\lambda-\cos\theta)\left|\begin{bmatrix} \lambda-\cos\theta & t_2 \\ 0 & \lambda-1 \end{bmatrix}\right| - (\lambda-\cos\theta)\left|\begin{bmatrix} \lambda-\cos\theta & t_1 \\ 0 & \lambda-1 \end{bmatrix}\right|$$

$$+ (\lambda-1)\left|\begin{bmatrix} \lambda-\cos\theta & -\sin\theta \\ \sin\theta & \lambda-\cos\theta \end{bmatrix}\right|$$

$$= \cancel{(\lambda-\cos\theta)\underbrace{(\lambda^2-\lambda\cos\theta-\lambda+\cos\theta)}_{A}} \cancel{- (\lambda-\cos\theta)\cdot A}$$

$$+ (\lambda-1)(\lambda^2 - 2\lambda\cos\theta + \cos^2\theta + \sin^2\theta)$$

$$= (\lambda-1)\underbrace{(\lambda^2 - 2\lambda\cos\theta + 1)}$$

$$= (\lambda-1)\quad(\lambda-e^{i\theta})(\lambda-e^{-i\theta}) \quad \checkmark \text{ from part a}$$

Then eigenvalues of T are λ_1, λ_2 of $R(\theta)$ and 1

$\lambda_3 \cdot v_3 = \underbrace{T(\theta, t)}_{\Downarrow} \cdot v_3$ where $v_3 \in \mathbb{R}^3$.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{so } t = 0, \quad \theta = 2\pi\cdot k \quad k \in \mathbb{Z}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta & t_1 \\ \sin\theta & \cos\theta & t_2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$x\cos\theta - y\sin\theta + zt_1 = x \rightarrow x(\cos\theta - 1) = y\sin\theta - zt_1$

$x\sin\theta + y\cos\theta + zt_2 = y$

$$x = \frac{y\sin\theta - zt_1}{\cos\theta - 1}$$

$$y(\cos\theta - 1) = -x\sin\theta - zt_2$$

$$y = \frac{-x\sin\theta - zt_2}{\cos\theta - 1}$$

plug in x: $\quad y = \dfrac{\dfrac{-y\sin^2\theta - zt_1\sin\theta}{\cos\theta - 1} - zt_2}{\cos\theta - 1}$

$y(\cos\theta - 1)$

$\cos\theta - 1$

$y(\cos\theta - 1)$

$$\theta = 0 \Rightarrow \begin{bmatrix} 1 & 0 & t_1 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{bmatrix} v_3 = v_3 \rightarrow v_3 = \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix} \text{ or } t = 0$$

$$\theta = \pi \quad \begin{bmatrix} -1 & 0 & t_1 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{bmatrix} v_3 = v_3 \rightarrow$$

d) Yes for $T(0, t_1)$ $T(0, t_2)$ because it will have eigenvalues all 1

5. Julia code

```julia
using LinearAlgebraAlgebra
using Random

function unitary_complex()
    n = 10
    # create a random Complex Matrix V
    V = randn((10, 10)) + randn((10, 10))*im
    # We will construct U from A via Gram Schmidt
    # such that columns of U are orthogonal
    U = V
    for i in 2:n
        v = V[:, i]
        v_3 = v
        for j in 1:i-1
            v_2 = U[:, j]
            v_3 -= dot(v_2, v) / dot(v_2, v_2) * v_2
        end
        U[:,i] = v_3
    end
    # normalize columns of U
    for i in 1:n
        U[:, i] /= dot(U[:, i], U[:, i]) ^.5
    end
    return U
end
```

```
unitary_complex (generic function with 1 method)
```


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