ELEC 405 - Homework 5

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Contents

1. Quadratic Forms and Hermitian Matrices

1.
$$A = \frac{1}{9}\begin{bmatrix} -2 & 2 & 8 \\ 2 & 7 & 10 \\ 8 & 10 & 4 \end{bmatrix} \tag{1}$$

2. Let $A = LDL^*$ where L is a lower triangular matrix and D is a diagonal matrix.

$$\begin{aligned}
L &= \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \\
D &= \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \\
A &= LDL^* \\
&= \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} d_1 & ad_1 & bd_1 \\ ad_1 & a^2d_1 + d_2 & abd_1 + cd_2 \\ bd_1 & abd_1 + cd_2 & b^2d_1 + c^2d_2 + d_3 \end{bmatrix}
\end{aligned} \tag{2}$$

$$\begin{aligned} d_1 &= -\frac{2}{9} & a^2d_1 + d_2 &= \frac{7}{9} \Rightarrow d_2 = 1 \\ ad_1 &= \frac{2}{9} \Rightarrow a = -1 & abd_1 + cd_2 &= \frac{10}{9} \Rightarrow c_2 = 2 \\ bd_1 &= \frac{8}{9} \Rightarrow b = -4 & b^2d_1 + c^2d_2 + d_3 &= \frac{4}{9} \Rightarrow d_3 = 0 \end{aligned} \tag{3}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -4 & 2 & 1 \end{bmatrix}, D = \begin{bmatrix} -\frac{2}{9} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4}$$

There is also another method: first getting LU decomposition of A and then extracting the diagonal elements to a diagonal matrix D, this method is also tried and can be seen in the Appendix.

The inertia of A is $(1, 1, 1)$ because A is star-congruent to D in (4) so they must have the same inertia.

3. No, A is indefinite because it has both negative and positive eigenvalues.
4. The eigenvalues determined by the Julia Notebook (see Appendix) are as follows:

$$-1.0000000000000004, -4.764101178639501e - 17, 1.9999999999999998$$

Dividing by 9 causes overflow in Julia and therefore the second eigenvalue is actually 0.

2. Matrix Factorization Coding Experience...

See the Appendix.

3. Random Vector Generation...

See the Appendix.

4. Positive matrices

Eigenvectors of a positive definite Hermitian matrix A form an orhonormal basis for $\mathcal{C}^n$. So that we can write x in terms of the eigen basis.

$$\begin{aligned} x &= \Sigma_{i=1}^n \beta_i v_i \\ \|x\|^2 &= \Sigma_{i=1}^n \beta_i \times \beta_i^* \|v_i\|^2 = \Sigma_{i=1}^n |\beta_i|^2 && (5) \\ x^*Ax &= \Sigma_{i=1}^n \Sigma_{i=1}^n \beta_i \times \beta_j v_i^* A v_j = \Sigma_{i=1}^n \Sigma_{i=1}^n \beta_i \times \beta_j \lambda_i v_i^* v_j \\ &= \Sigma_{i=1}^n \lambda_i |\beta_i|^2 && (6) \end{aligned}$$

$(6) \geq \lambda_{min} \times (5) \Rightarrow$

5. Quadratic approximations of multivariable functions

1. Gradient:

$$\nabla f([x_1 x_2]^T) = e^{-x_1^2 - x_2^2} \begin{bmatrix} -2x_1(4x_1^2 - 5) \\ (2 - 8x_1^2)x_2 \end{bmatrix} \tag{7}$$

Hessian:

$$H([x_1 x_2]^T) = e^{-x_1^2 - x_2^2} \begin{bmatrix} -32x_1^2 + 4(4x_1^2 - 1)x_1^2 + 8 - 2(4x_1^2 - 1) & 4x_1(4x_1^2 - 1)x_2 - 16x_1x_2 \\ 4x_1(4x_1^2 - 1)x_2 - 16x_1x_2 & 4(4x_1^2 - 1)x_2^2 - 2(4x_1^2 - 1) \end{bmatrix} \tag{8}$$

2. See the Appendix.
3. See the Appendix.

Appendices

Question 1

In [1]:
```julia
using LinearAlgebra
using Plots
```

In [2]:
```julia
A = [-2 2 8; 2 7 10; 8 10 4]
A
```

Out[2]:
```
3×3 Array{Int64,2}:
 -2   2   8
  2   7  10
  8  10   4
```

Method 1

In [3]:
```julia
l(a, b, c) = [1 0 0; a 1 0; b c 1]
d(a, b, c) = [a 0 0; 0 b 0; 0 0 c]
```

Out[3]:
```
d (generic function with 1 method)
```

In [4]:
```julia
L = l(-1, -4, 2)
D = d(-2/9, 1, 0)
L * D * L'
```

Out[4]:
```
3×3 Array{Float64,2}:
 -0.222222  0.222222  0.888889
  0.222222  0.777778  1.11111
  0.888889  1.11111   0.444444
```

In [5]:
```julia
A/9
```

Out[5]:
```
3×3 Array{Float64,2}:
 -0.222222  0.222222  0.888889
  0.222222  0.777778  1.11111
  0.888889  1.11111   0.444444
```

Method 2

Use LU Decomposition to construct LDU decomposition

In [6]:
```julia
E1 = [-9/2 0 0; 1 1 0; 4 0 1]
1/9*E1*A
```

Out[6]:
```
3×3 Array{Float64,2}:
 1.0  -1.0  -4.0
 0.0   1.0   2.0
 0.0   2.0   4.0
```

In [7]:
```julia
E2 = [1 0 0; 0 1 0; 0 -2 1]
U = E2*E1*A
U /= 9
```

Out[7]:
```
3×3 Array{Float64,2}:
 1.0  -1.0  -4.0
 0.0   1.0   2.0
 0.0   0.0   0.0
```

In [8]:
```
inv(E2*E1)
```

Out[8]:
```
3×3 Array{Float64,2}:
 -0.222222  0.0  0.0
  0.222222  1.0  0.0
  0.888889  2.0  1.0
```

$L = (E2 \times E1)^-1$
Then we need to extract the first diagonal of L into a diagonal matrix

In [9]:
```
L0 = [-2/9  0.0  0.0;
  2/9  1.0  0.0;
  8/9  2.0  1.0]
```

Out[9]:
```
3×3 Array{Float64,2}:
 -0.222222  0.0  0.0
  0.222222  1.0  0.0
  0.888889  2.0  1.0
```

Now we need to extract the $-\frac{2}{9}$ term to D, which will multiply the first column with $-\frac{2}{9}$ when $L \times D$ is applied

In [10]:
```
L = [1  0.0  0.0;
  -1  1.0  0.0;
  -4  2.0  1.0]
```

Out[10]:
```
3×3 Array{Float64,2}:
  1.0  0.0  0.0
 -1.0  1.0  0.0
 -4.0  2.0  1.0
```

In [11]:
```
D = [-2/9 0 0;
  0  1  0;
  0 0  0]
```

Out[11]:
```
3×3 Array{Float64,2}:
 -0.222222  0.0  0.0
  0.0       1.0  0.0
  0.0       0.0  0.0
```

In [12]:
```
L * D * L'
```

Out[12]:
```
3×3 Array{Float64,2}:
 -0.222222  0.222222  0.888889
  0.222222  0.777778  1.11111
  0.888889  1.11111   0.444444
```

In [13]:
```
A/9
```

Out[13]:
```
3×3 Array{Float64,2}:
 -0.222222  0.222222  0.888889
  0.222222  0.777778  1.11111
  0.888889  1.11111   0.444444
```

inertia

In [14]:
```
eig = eigvals(A/9)
```

Out[14]:
```
3-element Array{Float64,1}:
 -1.0000000000000004
 -4.764101178639501e-17
  1.9999999999999998
```

In [15]:
```
E = eigvecs(A/9)
```

Out[15]:
```
3×3 Array{Float64,2}:
 -0.666667   0.666667  -0.333333
 -0.333333  -0.666667  -0.666667
  0.666667   0.333333  -0.666667
```

In [62]:
```
e3 = E[:, 3]
```

Out[62]:
```
3-element Array{Float64,1}:
 -0.33333333333333326
 -0.6666666666666664
 -0.6666666666666669
```

In [63]:
```
eig[3]*e3
```

Out[63]:
```
3-element Array{Float64,1}:
 -0.6666666666666664
 -1.3333333333333326
 -1.3333333333333335
```

In [64]:
```
1/9*A*e3
```

Out[64]:
```
3-element Array{Float64,1}:
 -0.6666666666666667
 -1.3333333333333335
 -1.333333333333333
```

Question 2

Finding Cholesky Factorization of a Matrix

In [168]:
```
function cholesky_factorization(A)
    n = size(A)[1]
    α = sqrt(Complex(A[1]))
    if n == 1
        return α
    end
    v = A[2:n, 1]/α
    return [α (zeros(n-1)+im*zeros(n-1))'; v cholesky_factorization(A[2:n, 2:n] - v*v')]
end
```

Out[168]:
```
cholesky_factorization (generic function with 1 method)
```

In [146]:
```
n = 3
CL_real = [2 0 0 ; 1 3 0; 2 1 3]
C_real = CL_real * CL_real'
eigvals(C_real)
```

Out[146]:
```
3-element Array{Float64,1}:
  2.586414898264002
  6.690759634051093
 18.722825467684906
```

In [172]:
```
cholesky_factorization(C_real) == CL_real
```

Out[172]:
```
true
```

In [139]:
```
n = 3
CL = [2 0 0; 1+.5im 3 0; 2+im 1-4im 3]
C = CL * CL'
eigvals(C)
```

Out[139]:
```
3-element Array{Float64,1}:
  1.4818827284435265
  5.751134554403829
 38.01698271715264
```

Check if the factorization method is correct

In [170]:
```
cholesky_factorization(C)
```

Out[170]:
```
3×3 Array{Complex{Float64},2}:
 2.0+0.0im  0.0-0.0im  0.0-0.0im
 1.0+0.5im  3.0+0.0im  0.0-0.0im
 2.0+1.0im  1.0-4.0im  3.0+0.0im
```

In [171]:
```
cholesky_factorization(C) == CL
```

Out[171]:
```
3×3 BitArray{2}:
 1  1  1
 1  1  1
 1  1  1
```

In [173]:
```
# built-in cholesky
cholesky(C)
```

Out[173]:
```
Cholesky{Complex{Float64},Array{Complex{Float64},2}}
U factor:
3×3 UpperTriangular{Complex{Float64},Array{Complex{Float64},2}}:
 2.0+0.0im  1.0-0.5im  2.0-1.0im
     ⋅      3.0+0.0im  1.0+4.0im
     ⋅          ⋅      3.0+0.0im
```

Question 3

In [228]:
```
function generate_rand_vec(N, R)
    V = randn(3, N)
    R_L = cholesky_factorization(R)
    vectors = zeros(3, N) + zeros(3, N) * im
    for i in 1:N
        y_i = R_L * (V[:, i])
        # since mean is zero, we don't need to define x_i = y_i + m_x
        vectors[:, i] = y_i
    end
    return vectors
end
```

Out[228]:
```
generate_rand_vec (generic function with 1 method)
```

In [231]:
```
R = [28 15+9im 2+21im; 15-9im 48 15-11im; 2-21im 15+11im 30]
```

Out[231]:
```
3×3 Array{Complex{Int64},2}:
 28+0im   15+9im    2+21im
 15-9im   48+0im   15-11im
  2-21im  15+11im  30+0im
```

In [232]:
```
N = 100000
vectors = generate_rand_vec(N, R)
```

Out[232]:
```
3×100000 Array{Complex{Float64},2}:
 5.89581+0.0im       -3.40973+0.0im        …   -7.7226+0.0im
 5.64755-1.89508im   -5.08827+1.09598im      0.309757+2.48226im
 1.54956-2.97109im  -0.485671+0.656249im     0.173862+8.38381im
```

In [233]:
```
r = zeros(3, 3) + im*zeros(3, 3)
for j in 1:N
    v = vectors[:, j]
    r += v*v'
end
R_hat = r/N
R_hat
```

Out[233]:
```
3×3 Array{Complex{Float64},2}:
 28.0487+0.0im      15.0907+9.01565im  2.01648+20.9989im
 15.0907-9.01565im   48.317+0.0im       15.064-11.0908im
 2.01648-20.9989im   15.064+11.0908im  30.0612+0.0im
```

- It seems like it is not likely to find $\hat{R}$ without using for loops
- $\hat{R}$ is close to R.
- My random generator seems to be working fine.

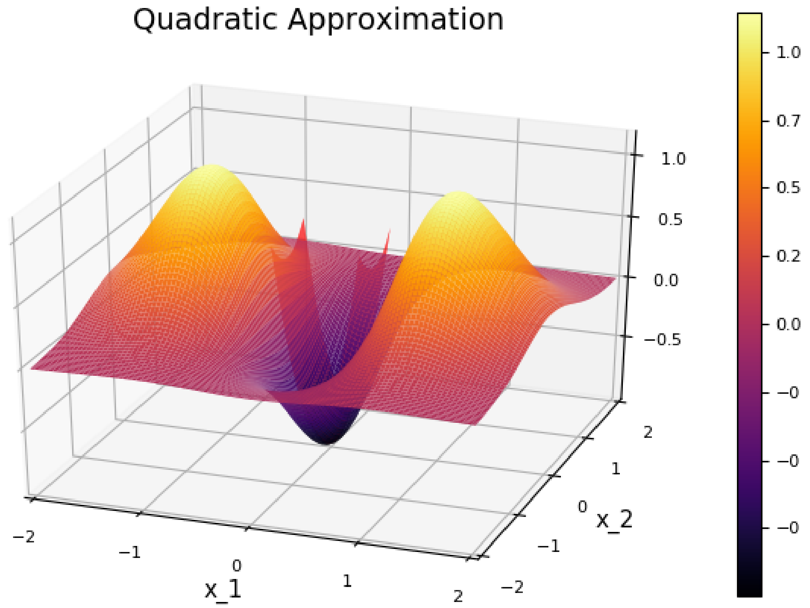
Question 5

In [39]:
```
f(x, y) = (4*x^2-1)*exp(-x^2-y^2)
f(x) = f(x[1], x[2])

ex(x_1, x_2) = exp(-x_1^2-x_2^2)
ex(x) = ex(x[1], x[2])

∇f(x_1, x_2) = [(10*x_1-8x_1^2)*ex(x_1,x_2); -2*x_2*ex(x_1, x_2)]
∇f(x) = ∇f(x[1], x[2])

H(x_1, x_2) = ex(x_1, x_2)*
    [(-32*x_1^2 + 4*(4 *x_1^2 - 1)*x_1^2 + 8 - 2 *(4* x_1^2 - 1))  (4 * x_1 * (4 * x_1^2 - 1) * x_2 - 16
*x_1 *x_2);
        (4 * x_1 * (4 * x_1^2 - 1)*x_2 - 16  *x_1 *x_2) (4 *(4* x_1^2 - 1) *x_2^2 - 2 *(4* x_1^2 - 1))]
H(x) = H(x[1], x[2])

f_q(x0, x) = f(x) + dot(∇f(x0), (x-x0)) + 0.5*dot(H(x0)*(x-x0), (x-x0))
f_q(x) = f_q([0; 0], x)
fq(x_1, x_2) = f_q([x_1; x_2])
```

Out[39]:
```
fq (generic function with 1 method)
```

In [45]:
```
camera = (20, 30)
pyplot()
x_1 = range(-2, stop=2, length=100)
x_2 = range(-2, stop=2, length=100)
p = plot(x_1, x_2, f, title="Quadratic Approximation",
        st =:surface, camera=camera, xlabel="x_1", ylabel="x_2")

xq_1 = range(-.4, stop=.4, length=100)
xq_2 = range(-.4, stop=.4, length=100)
plot!(p, xq_1, xq_2, fq, st =:surface, camera=camera, color=:red)
```

Out[45]:



Question 5.c

1. Find an appropriate point.
2. Plot it

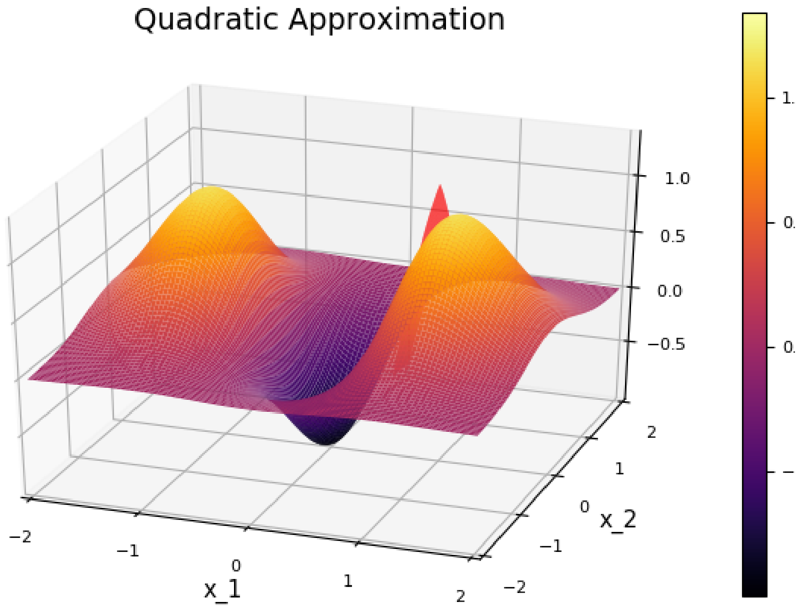
In [44]:

```
for i in x_1
    for j in x_2
        hess = H(i, j)
        if all(x -> x< 0, eigvals(hess)) && ∇f(i, j) != 0
            println(i, j)
            break
        end
    end
end
```

```
-1.5555555555555556-0.2222222222222222
-1.5151515151515151-0.3434343434343434
-1.4747474747474747-0.42424242424242425
-1.4343434343434343-0.5050505050505051
-1.393939393939394-0.5454545454545454
-1.3535353535353536-0.5858585858585859
-1.3131313131313131-0.6262626262626263
-1.2727272727272727-0.6262626262626263
-1.2323232323232323-0.6666666666666666
-1.1919191919191918-0.6666666666666666
-1.1515151515151516-0.6666666666666666
-1.1111111111111112-0.6666666666666666
-1.0707070707070707-0.6666666666666666
-1.0303030303030303-0.6666666666666666
-0.98989898989899-0.6666666666666666
-0.9494949494949495-0.6262626262626263
-0.9090909090909091-0.6262626262626263
-0.8686868686868687-0.5858585858585859
-0.8282828282828283-0.5454545454545454
-0.7878787878787878-0.5050505050505051
-0.7474747474747475-0.42424242424242425
-0.7070707070707071-0.3838383838383838
-0.6666666666666666-0.30303030303030304
-0.6262626262626263-0.2222222222222222
-0.5858585858585859-0.1414141414141414
-0.5454545454545454-0.06060606060606061
0.5454545454545454-0.06060606060606061
0.5858585858585859-0.1414141414141414
0.6262626262626263-0.2222222222222222
0.6666666666666666-0.30303030303030304
0.7070707070707071-0.3838383838383838
0.7474747474747475-0.42424242424242425
0.7878787878787878-0.5050505050505051
0.8282828282828283-0.5454545454545454
0.8686868686868687-0.5858585858585859
0.9090909090909091-0.6262626262626263
0.9494949494949495-0.6262626262626263
0.98989898989899-0.6666666666666666
1.0303030303030303-0.6666666666666666
1.0707070707070707-0.6666666666666666
1.1111111111111112-0.6666666666666666
1.1515151515151516-0.6666666666666666
1.1919191919191918-0.6666666666666666
1.2323232323232323-0.6666666666666666
1.2727272727272727-0.6262626262626263
1.3131313131313131-0.6262626262626263
1.3535353535353536-0.5858585858585859
1.393939393939394-0.5454545454545454
1.4343434343434343-0.5050505050505051
1.4747474747474747-0.42424242424242425
1.5151515151515151-0.3434343434343434
1.5555555555555556-0.2222222222222222
```

In [43]:
```
x0_approximation = [.7; .4]
fq_negative(x_1, x_2) = f_q(x0_approximation, [x_1; x_2])
camera = (20, 30)
pyplot()
x_1 = range(-2, stop=2, length=100)
x_2 = range(-2, stop=2, length=100)
p = plot(x_1, x_2, f, title="Quadratic Approximation",
        st =:surface, camera=camera, xlabel="x_1", ylabel="x_2")

xq_1 = range(x0_approximation[1]-.2, stop=x0_approximation[1]+.2, length=100)
xq_2 = range(x0_approximation[2]-.2, stop=x0_approximation[2]+.2, length=100)
plot!(p, xq_1, xq_2, fq_negative, st =:surface, camera=camera, color=:red)
```

Out[43]:



In [ ]: