MAT7500 W14 Lecture Notes

Bröcker and Jänich: Chapter VI, 2/17/14

The content of Sard's Theorem is that regular values of a smooth map are easy to find. Here's the statement:

Theorem. *The set of critical values of a smooth map has Lebesgue measure zero.*

We will not prove this result, but we do need to make sense of it, and we also need to reformulate it for our consumption.

Wefirst need to make sense of the measure zero condition.

Recall that a subset $C \subset \mathbb{R}^n$ is said to have *measure zero* if for any $\epsilon > 0$, there is a sequence of n-cubes $W_i \subset \mathbb{R}^n$ such that

$$C \subset \bigcup_i W_i$$

and $\sum_i \mathrm{vol}\,(W_i) < \epsilon$. Note that a countable union of subsets of $\mathbb{R}^n$ of measure zero again has measure zero (exercise). Furthermore, we could have used open or closed cubes or open or closed balls to define the notion.

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Lemma. *If $U \subset \mathbb{R}^m$ is an open set, and $C \subset U$ has measure zero, then for any smooth map $f\colon U \to \mathbb{R}^m$, we have that $f(C)$ also has measure zero.*

This Lemma says that one cannot create a set of positive measure from one of measure zero when applying a smooth map from an open set of $\mathbb{R}^m$ to $\mathbb{R}^m$.

Let's prove it now.

Proof of Lemma. First of all, any such set U is a union of compact balls $B_n \subset B_{n+1}$. If we define $C_n = C \cap B_n$, then $C = \cup_n C_n$ and each C_n is contained in a compact ball and is also contained in U. So without loss in generality, we may assume that C is a subset of a compact ball. Let an $\epsilon > 0$ and let $\{W_i\}_{i\ge 1}$ be a covering of C by cubes $\mathrm{vol}(W_i) = 2a_i$, where $\sum_i a_i < \epsilon$ for a given choice of $\epsilon > 0$. Furthermore, we can assume that every W_i is contained in a larger compact ball $K \subset U$

By the Mean Value Theorem, we can write

$$f(x+h) - f(x) = R(x,h)\,,$$
$$||R(x,h)|| \le c|h|\,,$$

where $x, x+h \in K$ and $c > 0$ is some constant. Let W_i be any one of the cubes. Then $W_i = \{x|\, |x_j - y_j| \le a_i\}$ some fixed $y \in W_i$ and fixed Consequently,

$$||x - y|| \le \sqrt{m}a_i$$

for all $x \in W_i$ and

$$||f(x) - f(y)|| \le c\sqrt{m}a_i\,.$$

This implies $f(W_i)$ is a subset of a cube W_i' whose volume is

$$d_i := (2\sqrt{m}ca_i)^m$$

Hence

$$\sum_i \mathrm{vol}(W_i') = \sum_i d_i = (2c)^m \sum_i a_i^m < (2c)^m \epsilon^m\,. \quad \square$$

We are now ready to define what it means to be of measure zero in a manifold.

Definition. A subset $C \subset M^m$ has *measure zero* if for *every chart* (h, U), the subset $h(C \cap U) \subset \mathbb{R}^m$ has measure zero.

Remark. Every manifold M^m has a countable smooth atlas (h_α, U_α). It is really enough to check the condition using the charts in the atlas, i.e., we need only check that $h(C \cap U_\alpha) \subset \mathbb{R}^m$ has measure zero for all α. This is a direct consequence of the lemma above.

As I've mentioned already, we will not prove Sard's theorem. However, I want to single out an important consequence, which is a result of Brown.

Corollary. (A.B. Brown). *The set of regular values of a smooth map $f\colon M \to N$ is everywhere dense in M.*

By *everywhere dense*, I mean that given any point $z \in N$, and any open neighborhood U of z, there is a regular value $y \in U$.

Proof of Corollary. Let $C \subset N$ be the set of critical values of f. Then C has measure zero. Let $z \in N$ be any point. Then if $z \notin C$, there's nothing to prove. If $z \in C$, then we can choose a chart (h, U) for N at z, and we see $h(C \cap U) \subset \mathbb{R}^n$ has measure zero. This implies there exists a point $y \in U$ such that $h(y) \notin h(C \cap U)$. Hence $y \in U$ is a regular value. $\square$

Ideas of the Proof of Sard's Theorem. By introducing charts, it's enough to prove the special case when $f\colon U \to \mathbb{R}^p$, where $U \subset \mathbb{R}^n$ is an open set. The proof proceeds by induction on n. The case $n = 0$ is trivial. I will sketch the case $n = 1$. Then f parametrizes a smooth curve in $\mathbb{R}^p$. Let D denote its set of critical points.

Let $D_i \subset U \subset \mathbb{R}^1$ be the set of $x \in U$ such that $f^{(j)}(x) = 0$ for $j \le i$, where $f^{(j)}$ is the j-th derivative of f. Then $D_0 \supset D_1 \subset \cdots$.

Assertion. *Each of the following sets has measure zero:*

(1) $f(D \setminus D_1)$,
(2) $f(D_i \setminus D_{i+1})$, $\quad i \ge 1$, *and*
(3) $f(D_k)$ *has measure zero for* $k > n/p - 1$.

The assertion is a consequence of the so-called *Fubini Theorem* which says that $C \subset \mathbb{R}^n$ has measure zero when each slice $C_t = C \cap (\mathbb{R}^{n-1} \times \{t\})$ has measure zero in $\mathbb{R}^{n-1}$.

Given the assertion, Sard's Theorem follows since

$$f(D) = f(D \setminus D_1) \cup \bigcup_{i=1}^{k-1} f(D_i \setminus D_{i+1}) \cup f(D_k)\,.$$

Applications

(1). Let $f\colon M^m \to N^n$ be a smooth map, $m < n$. By Sard's Theorem, there is a regular value $y \in$. By dimensional reasons it follows that $f^{-1}(y)$ is empty. We conclude

Corollary. *If $f\colon M^m \to N^m$ is a smooth map, $m < n$. Then f cannot be onto.*

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(2). Call a smooth map $f\colon S^m \to S^n$ *null homotopic* if f admits a continuous extension $F\colon D^{m+1} \to S^n$

Corollary. *If $m < n$ than any smooth $f\colon S^m \to S^n$ is null homotopic.*

Proof. By our first application, we know that there's a point y such that $f(S^m) \subset S^n \setminus y$. Let $h\colon S^n \setminus y \to \mathbb{R}^n$ be stereographic projection (using y instead of the north pole to project from). Then $h \circ f\colon S^m \to \mathbb{R}^n$ admits a continuous extension to $g\colon D^{m+1} \to \mathbb{R}^n$ by means of the formula

$$x \mapsto ||x|| f(\tfrac{x}{||x||})$$

where the origin of D^{m+1} is interpreted as mapping to the origin of $\mathbb{R}^n$. Set $F = h^{-1}g\colon D^{m+1} \to S^n$. Then F is continuous and extends f. □

(3). Suppose $f\colon M \to \mathbb{R}^p$ is smooth and $N \subset \mathbb{R}^p$ is a submanifold. Let $\epsilon > 0$ be any number and suppose $v \in \mathbb{R}^p$ satisfies $||v|| < \epsilon$. Consider the map

$$g\colon M \to \mathbb{R}^p$$

given by $g(x) = f(x) + v$. We claim that for almost all such v, the map g is transverse to N.

To see this, consider the smooth map

$$h\colon M \times N \to \mathbb{R}^p$$

defined by $h(x, y) = y - f(x)$.

We now show h is transverse to $v \in \mathbb{R}^p$ iff and only if g is transverse to N. To see why this is true, we compute the tangent maps in each case. Note that $Tg_x\colon T_xM \to \mathbb{R}^p$ is the same as Tf_x. On the other hand $Th_{x,y}\colon T_xM \oplus T_yN \to \mathbb{R}^p$ is the map given by $(u, w) \mapsto Tf_x(u) - w$. Hence $Th_{x,y}$ is surjective precisely when any vector $w' \in \mathbb{R}^p$ can be written as

$$w' = Tf_x(u) - w$$

for some $u \in T_xM$ and $w \in T_yN$, where $y = f(x) + v$, i.e., $x \in g^{-1}(N)$, and

$$\text{image}(Tf_x) + T_yN = \mathbb{R}^p\,.$$

This gives the claim.

Now, according to Sard's theorem, $v \in \mathbb{R}^p$ is a regular value of h for almost every v. Therefore, g is transverse to L for almost every v, in particular for almost every v such that $||v|| < \epsilon$.

What this shows that for any smooth $f\colon M \to N$, there is an map $g\colon M \to \mathbb{R}^p$ which is "arbitarily close" to f (in a suitable sense) such that g is transverse to N. To make sense of "arbitrarily close" we would have to provide the set of all smooth maps $M \to N$ with a suitable topology (in fact there is one: it's called the *Whitney C^∞ Topology.*

(4). The Brouwer Fixed Point Theorem. Suppose $f\colon D^m \to D^m$ is a continuous map. By a *fixed point* for f we mean a point $x \in D^m$ such that $f(x) = x$.

Brouwer Fixed Point Theorem. *Every continuous map $f\colon D^m \to D^m$ has a fixed point.*

As it turns out, the proof of this result relies on Sard's Theorem. Although we do not yet have the tools to prove this in general, I will give a sketch in two cases: $m = 1, 2$.

Proof when $m = 1$. Suppose that $f\colon D^1 \to D^1$ has no fixed point. Define a map $g\colon D^1 \to S^0$ by mapping $x \in D^1$ to the point on S^0 given by the

$$g(x) = \frac{x - f(x)}{|x - f(x)|}\,.$$

Then g is continuous and if $x \in S^0$, then $g(x) = x$. So the image of g is disconnected. But D^1 is connected. This gives as contradiction, as the image of a connected set is connected.

Sketch Proof when $m = 2$. Suppose $f\colon D^2 \to D^2$ doesn't have a fixed point. Define a map $g\colon D^2 \to S^1$ by sending $x \in D^2$ to the intersection that the ray from $f(x)$ to x makes with S^1. Then g is continuous and $g(x) = x$ when $x \in S^1$. Hence, if $i\colon S^1 \to D^2$ is the inclusion we have $g \circ i = \mathrm{id}$.

Let $E \to S^1$ be the Möbius band bundle. Then as line bundles over S^1

$$E \cong (g \circ i)^* E = i^* g^* E$$

but g^*E is a bundle over D^2. It turns out that any bundle over a disk is trivializable (we haven't yet proved this, but the proof bears some similariliy with the argument I gave for bundles over $D^1 = [-1, 1]$. Hence i^*g^*E is also trivializable. This gives a contradiction, since E isn't trivializable.

Possible Proof when $m > 2$. All we need to do is show that there exists a nontrivial vector bundle $E \to S^{m-1}$. Then the argument we gave when $m = 2$ extends to all higher dimensions. However, it does require a fair amound of work to show that there is such a bundle.

Remark. When we actually prove the Brouwer Theorem, we will give a different argument.