MAT7500 W14 Lecture Notes

Bröcker and Jänich, Chapter XIII: Manifolds with Boundary

Motivation. Manifolds of dimension are locally modelled by opne sets in $\mathbb{R}^n$. In defining the notion of manifold with boundary, we replace $\mathbb{R}^n$ by upper half space:

$$\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_n \geq 0\}$$

Manifolds with Boundary. A space M is said to be a *topological n-manifold with boundary* it is Hausdorff, second countable and if each point $x \in M$ has a neighborhood homeomorphic to an open set of $\mathbb{R}^n_+$.

An atlas $\mathcal{U}$ of consisting of local charts $h_\alpha \colon U_\alpha \to V_\alpha$ where $U \subset M$ and $V \subset \mathbb{R}^n_+$ is said to be *smooth* if the transition maps $h_{\alpha\beta} := h_\beta h_\alpha^{-1}$ are smooth (in the sense that each point $x \in V_\alpha \cap V_\beta$ has a neighorhood W in $\mathbb{R}^n$ such that $h_{\alpha\beta}$ extends to a smooth map on W.

We say that M is a smooth n-manifold with boundary if it is a topological n-manifold with boundary that is equipped with a maximal smooth atlas.

Example. Of course $\mathbb{R}^n_+$ is a smooth n-manifold with boundary. Any smooth (ordinary) manifold can be considered a smooth manifold with boundary.

Lemma 1. *Suppose N^n is an ordinary manifold and let $f \colon N \to \mathbb{R}$ be a smooth function. Suppose that $a \in \mathbb{R}$ is a regular value. Then I claim that*

$$M := f^{-1}((-\infty, a])$$

has the structure of a smooth n-manifold with boundary.

Proof. Indeed, if $y \in M$ and $f(y) \neq a$, then there is a chart of y in N that is contained inside M. If $f(y) = a$, then note that $f^{-1}(a) \subset N$ is a codimension one submanifold. So locally $(N, f^{-1}(a))$ is diffeomorphic near y to $(\mathbb{R}^n, \mathbb{R}^n \times 0)$. But then it follows that such a local diffeomorphism locally models $(M, f^{-1}(a)$ as $(\mathbb{R}^n_+, \mathbb{R}^n \times 0)$.

Example. The closed unit n-disk D^n is a smooth n-manifold with boundary, since $x \mapsto (\sum x_i^2) - 1$ has the origin as a regular value (and $(\sum x_i^2) - 1 \leq 0$ defines D^n).

Definition/Notation. A boundary point $p \in M^n$ is a point which is mapped by a chart to a point $x \in \mathbb{R}^n_+$ such that $x_n = 0$. The set of boundary points is denoted by ∂M. It is a smooth $(n-1)$-manifold (without boundary). If $x_n \neq 0$, then p is called an interior point. The set of interior points is given by $M \setminus \partial M$. It is an n-manifold without boundary.

If M is an n-manifold with boundary, then its tangent space T_pM is defined just as when M is an ordinary manifold. It is a full n-dimensional vector space, even when p is a boundary point.

Suppose $f \colon X \to N$ is a smooth map from an m-manifold with boundary to an n-manifold, where $m > n$.

Lemma 2. *If $y \in N$ is a regular value for both f and the restriction $f_{|\partial X} \colon \partial X \to N$, then $f^{-1}(y) \subset X$ is a smooth $(m-n)$-manifold with boundary. Furthermore, the boundary $\partial f^{-1}(y)$ is equal to the intersection of $f^{-1}(y)$ with ∂X.*

Proof. Since we wish to check a local property, it is enough to prove the statement in the case of a map $f \colon \mathbb{R}^m_+ \to \mathbb{R}^n$. If $x \in f^{-1}(y)$ and x is an interior point, then there is a neighborhood of x that is diffeomorphic to $\mathbb{R}^{m-n}$, so for this x the manifold property holds.

If x is a boundary point, then we choose a smooth map $g \colon U \to \mathbb{R}^n$ where U is a neigborhood of x in $\mathbb{R}^m$ such that g coincides with f on $U \cap \mathbb{R}^m_+$. Consider the last coordinate projection

$$\pi \colon g^{-1}(y) \to \mathbb{R}$$

given by $x \mapsto x_n$. As the tangent map of g at a point $z \in \pi^{-1}(0)$ is given by the kernel of the map

$$T_z g =\colon T_z f \colon \mathbb{R}^m \to \mathbb{R}^n$$

and the assumption that $T_z f_{|\partial \mathbb{R}^m_+} = T_z f_{\mathbb{R}^{m-1} \times 0}$ is onto implies that this kernel cannot be entirely contained in $\mathbb{R}^{m-1} \times 0$.

Hence, the set $g^{-1}(y) \cap \mathbb{R}^n_+ = f^{-1}(y) \cap U$, which consists of all points $z \in g^{-1}(0)$ such that $\pi(z) \geq 0$, is a smooth manifold with boundary by Lemma 1. $\square$

Applications. Let X be a compact n-manifold with boundary.

Lemma 3. *There is no smooth map $f \colon X \to \partial X$ which restricts to the identity on ∂X.*

Proof. (Sketch). The argument will rely on the classification of compact 1-manfolds: *every nonempty compact connected 1-manifold N is diffeomorphic to the circle S^1 or the unit interval* $[0,1]$. We will not prove this statement, but it is not very difficult to establish (one way to do it is to construct a *Morse function* $f \colon N \to \mathbb{R}$ and study its critical point set).

We proceed by contradiction. Let $p \in \partial X$ be an regular value for the map $f \colon X \setminus \partial X \to \partial X$. Then p is automatically a $f_{|\partial X} = \mathrm{id}.$ so $A := f^{-1}(p)$ is a compact 1-manifold with boundary where $A \cap \partial X = \partial A$. Hence, A is a finite disjoint union of circles and closed unit intervals (up to diffeomorphism). Clearly, each circle is contained in the interior $X \setminus \partial X$. Therefore the cardinality of $A \cap \partial X$ is even. But

$$A \cap \partial X = f^{-1}(p) \cap \partial X = f^{-1}_{|\partial X}(p) = p$$

and we get a contradiction. $\square$

Brouwer Fixed Point Theorem, Smooth Version. *Any smooth map $f \colon D^n \to D^n$ possesses a fixed point.*

Proof. Proof by contradiction. Suppose f has no fixed point. Then let $g \colon D^n \to S^{n-1}$ given by $g(x)$ is the point that is nearer than x on the line through x and $f(x)$. Notice that if $x \in S^{n-1}$ then $g(x) = x$.

We need to verify that this is smooth. The line L through x and $f(x)$ is given by

$$L(x) = x + t(x - f(x)) \qquad t \in \mathbb{R}$$

So the intersection of L with S^{n-1} is given by the numbers t such that

$$(x + t(x - f(x)) \cdot (x + t(x - f(x)) = 1\,.$$

If we try to solve this equation, we find

$$g(x) = x + tu$$

where

$$u = \tfrac{x - f(x)}{||x - f(x)||} x \qquad t = -x \cdot u + (1 - x \cdot x + (x \cdot u)^2)^{1/2}\,.$$

Hence, g is smooth.

Since $g\colon D^n \to S^{n-1}$ is smooth and its restriction to S^{n-1} is the identity, we obtain a contradiction using Lemma 3.

Collar Neighborhoods. Let M be a manifold with boundar A *collar* is an smooth map

$$h\colon \partial M \times [0, \epsilon) \to M$$

which restricts to the identity $\partial M \times 0 \to \partial M$ and which is a diffeomorphism onto its image. The image of h is called a collar neighborhood. This is a kind of "half tubular neighborhood." We actually used a collar in the special case of defining connected sums.

Theorem. (Existence of Collars). *Every manifold with boundary has a collar.*

Proof. We will only consider the compact case. Here are the main ideas: first of all any manifold with boundary M^m has a tangent bundle (it's defined in essentially the same way that we defined it for ordinary manifolds, say, using smooth curves $\gamma\colon [0, \infty) \to M$). The tangent space to $p \in M$ a boundary point still has dimension m. The tangent space $T_x\partial M$ sits inside T_xM as a codimension one subspace. A tangent vector $v \in T_x\partial M$ is said to *point inward* if with respect to any (and therefore, all) local trivializations v corresponds to a vector in the interior of the upper half space $\mathbb{R}^m_+$. So T_xM has three kinds of non-trivial vectors in it: inward pointing vectors, outward pointing vectors and vectors lying in $T_x\partial X$. Clearly the property of pointing inward is a covex property. Using a partition of unity argument, there is then a vector field X on M sycg that $X_{|\partial M}$ points inward at every point.

For $x \in \partial M$, let $\alpha_x\colon [0, b_x) \to M$ be the solution curve. By compactness, there is an $\epsilon > 0$ such that $b_x \geq \epsilon$ for all x. Then the map

$$h\colon \partial M \times [0, \epsilon) \to M$$

defined by $h(x, t) = \alpha_x(t)$ defines a collar. □

Theorem. (Uniqueness of Collars). *If $h_0\colon \partial M \times [0, \epsilon_0) \to M$ and $h_1\colon \partial M \times [0, \epsilon_1) \to M$ are collars, then there is a $\delta > 0$ such that $\delta < \min \epsilon_i$ and an isotopy*

$$h_t\colon \partial M \times [0, \delta) \to M$$

such that H_0 coincides with h_0 and h_1 coincides with H_1.

Proof. We only give the idea in the compact case. The idea is that h_0 and h_1 give rise to inward pointing vector fields X_0 and X_1 which are both defined on some open neighbhorhood U of ∂M in M. Then the convexity property shows that $(1-t)X_0 + tX_0$ is also such a vector field. We integrate this to obtain a collar $h_t\partial M \times [0, \epsilon_t) \to M$. Finally, we may use compactness to find a δ which is less than ϵ_t for all $t \in [0, 1]$. □

Application: The Double. If M is a compact manifold with boundary, then we choose a collar $h\colon \partial M \times [0,\epsilon) \to M$. Without loss in generality we can assume $\epsilon = 1$ (by rescaling). Let U be the image of h. Set

$$D(M) = (M \setminus \partial M) \cup (M \setminus \partial M)/\sim$$

we identify each point of the form $h(x,t)$ in the left hand copy of $(M \setminus \partial M)$ with the point $h(x, 1-t)$ in the right hand copy. This clearly defines a smooth manifold (without boundary), called the *double* of M.

For example, if $M = D^n$ then $D(D^n) \cong S^n$. More generally, if $\partial M \cong S^{n-1}$ then $D(M) \cong M\sharp M$.

Collars in Sightly More Generality. If M is a manifold with boundary and $\partial_0 M$ denotes a connected component of ∂M, then a collar for ∂M restricts to a smooth map of

$$\partial_0 M \times [0,\epsilon) \to M$$

which we call a collar of $\partial_0 M$.

Gluing in General. Let M and N be smooth m-manifolds with boundary. Suppose that $\partial_1 M$ denotes a connected component of ∂M and $\partial_0 N$ denotes a connected component of ∂N Suppose further that

$$\phi\colon \partial_1 M \to \partial_0$$

is a diffeomorphism. Then we write

$$M \cup_\phi N$$

for the manifold obtained from M and N by gluing collars of $\partial_1 M$ and $\partial_0 N$ together along the evident diffeomorphism.

In detail, choose collars $h_1\colon \partial_1 M \times [0,1) \to M$ and $h_2\colon \partial_0 N \times [0,1) \to N$. Then $M \cup_\phi N$ is given by

$$(M \amalg N)/\sim$$

where we identify $h_1(x,t)$ with $h_2(x, 1-t)$. This operation is associative up to diffeomorphism.

Cobordism. Compact m-manifolds M_0 and M_1 without boundary are *cobordant* if there is a compact $(m+1)$-manifold with boundary, W, such that

$$\partial W = M_0 \amalg M_1 \,.$$

We call W a *cobordism* from M to N.

The notion of cobordism defines an equivalence relation on compact m-manifolds without boundary. Transitivity is a consequence of the gluing construction. Furthermore, the set of equivalence classes forms a group $\mathfrak{N}_m$, where addition is induced by disjoint union. This group is abelian. The 0-element is given by the empty manifold. Notice that a manifold represents the trivial element if and only if it bounds, i.e., it is a boundary of a compact $(m+1)$-manifold.

The operation of cartesian product defines an operation

$$\mathfrak{N}_m \times \mathfrak{M}_n \to \mathfrak{N}_{m+n}$$

making $\mathfrak{N}_* = \oplus_k \mathfrak{N}_k$ into a graded ring.

Example. The group $\mathfrak{N}_0$ is a cyclic group of order two, generated by a single point. The group $\mathfrak{N}_1$ is trivial, since any finite union of circles bounds.

Because of the example, $\mathfrak{N}_*$ is a $\mathbb{Z}_2$-algebra. Let's define a homomorphism of graded $\mathbb{Z}_2$-algebras

$$\psi\colon \mathbb{Z}_2[X_1, X_2, \dots] \to \mathfrak{N}_*$$

where X_j has degree j and we assume j isn't of the form $2^i - 1$. Here, the domain of ψ is a polynomial algebra on a countable number of variables. To define ψ, we only need to specify where to map X_j. We send it to the projective space $\mathbb{R}P^j$.

Thom's Theorem. *ψ isomorphism of graded rings. In particular, any compact manifold is cobordant to a product of projective spaces.*

Transversality Revisited. Thom's theorem is proved using both differential and algebraic topology. A key reduction in the proof is the statement that maps can be made transversal to submanifolds by means of a "small" perturbation:

Thom Transversality Theorem. *Let $f\colon M \to N$ be a map of compact smooth manifolds, and let $P \subset N$ be a submanifold. If $\epsilon > 0$ is given, then there is a smooth map $g\colon M \to N$ such that $||f - g||_{C^1} < \epsilon$ and g is transverse to P.*

Here are the two main steps of Thom's proof:

(1) There is a space MO such that $\pi_*(MO) \cong \mathfrak{N}_*$. This step uses the transversality theorem.
(2) Use techniques of algebraic topology to compute $\pi_*(MO)$.