MAT7500 W14 Lecture Notes

Bröcker and Jänich: Chapter IV: Linear Algebra for Vector Bundles

Motivation. The main idea of this chapter is very simple: almost all the operations we do on vector spaces work for vector bundles. We already saw this in the case of the Whitney sum as being the analog of direct sum for vector bundles.

Some Vector Bundle Operations.

(1) Dual Bundle E^*. This is the bundle whose fiber at $x \in X$ is given by the vector space dual $(E_x)^*$.
(2) *Tensor Product.* If E and E' are vector bundles over X, then $E \otimes E'$ is the vector bundle over X whose fiber at x is $E_x \otimes E'_x$.
(3) *Quotients.* If $E' \subset E$ is a subbundle, we can form the quotient bundle E/E'. This is the bundle whose fiber at $x \in X$ is the quotient vector space E_x/E'_x.
(4) hom-*bundles.* We let $\hom(E, E')$ be the vector bundle over X whose fiber at $x \in X$ is $\hom(E, E')$
(5) *Exterior Powers.* The bundle $\Lambda^k E$ has fiber over x the vector space $\Lambda^k E_x$. When k is the rank of E, notice that $\Lambda^k E$ is a line bundle. This is called the *orientation* bundle of E.
(6) *Alternating Forms.* We let $\mathrm{Alt}^k E$ be the bundle whose fiber at x is the vector space of alternating k-forms on E_x. (Recall that an alternating k-form on a vector space V this is a function $f\colon V^{\oplus k} \to \mathbb{R}$ such that f is multilinear and alternating in the sense that if we switch two coordinates, the sign of f changes.)

(Note: $\mathrm{Alt}^k E$ is canonically isomorphic to $(\Lambda^k E)^*$.)

We will not go into the details as to why these all are vector bundles. The idea though is simple: they obviously give pre-vector bundles. Here's a general fact which makes all this work:

Fact. *If $V \mapsto F(V)$ is a continuous functor from vector spaces to vector spaces, and E is any vector bundle over X, then so is $F_\bullet E$, where the latter has fiber over $x \in X$ given by $F(E_x)$.*

In the above, a *continuous* functor is a functor in which the function between the hom-vector spaces $\hom(V, W) \to \hom(F(V), F(W))$ is a continuous map.

Remark. Some of the functors F of linear algebra are *contravariant.* This means that for a linear $V \to W$ there is a linear map the other way $F(W) \to F(V)$. For example, the operation $V \mapsto V^*$ is contravariant.

Definition. An *orientation* of a finite dimensional vector space V is an equivalence class of choice of basis $\mathfrak{b}$ for V. Two such bases $\mathfrak{b}, \mathfrak{b}'$ are related if the change of basis matrix has positive determinant. The orientation of V associated with the basis $\mathfrak{b}$ is denoted by $[\mathfrak{b}]$. A vector space equipped with orientation is said to be an *oriented vector space.*

If $f\colon V \to W$ is a linear map of vector spaces of the same dimension, and V and W are oriented, then the determinant $\det f \in \mathbb{R}$ is well-defined (see the exercises).

Definition. A vector bundle E of rank k over X is said to be otientable there exists a family

$$\{[\mathfrak{b}_x]\}_{x\in X}$$

in which $[\mathfrak{b}_x]$ is an orientation of E_x, such that for each $x \in X$ there is a local trivialization (h, U) at x such that $h_x \colon E_x \to \mathbb{R}^n$ has positive determinant.

If the above choices are given, we say that E is equipped with an orientation.

Alternative Definition. A vector bundle E is of rank k is *orientable* if the line bundle $\Lambda^k E$ is trivializable.

Second Alternative Definition. Let $s \colon X \to \Lambda^k E$ be the zero section. Define an equivalence relation on $\Lambda^k E \setminus s(X)$ by saying $x \sim y$ if and only if $y = \lambda x$ for some $\lambda > 0$. The quotient space with respect to this equivalence relation defines a two-fold covering space $\tilde{X} \to X$ (this is because a line has only two directions). This is called the *orientation cover* of E.

Then E is orientable if and only $\tilde{X} \to X$ is a trivial covering space (i.e., $\tilde{X} \cong X \times \mathbb{Z}_2$. A specific choice trivialization of this covering space amounts to a choice of orientation for E.

Example. The Möbius band E is a line bundle so it is its own orientation bundle. Since E is non-trivializable we see that E is non-orientable.

Obviously, a trivializable bundle is also orientable. Here is a fact we will not prove:

Fact. *A vector bundle E is orientable if and only if for all continuous maps $\gamma \colon S^1 \to X$ the pullback bundle $\gamma^* E$ over S^1 is trivializable.*

Oriented Manifolds. A manifold i M of dimension m is orientable if it's tangent bundle TM is an orientable vector bundle. We say that M is oriented if TM is provided with an orientation. In the exercises, you'll be asked to show that $\mathbb{R}P^2$ is not orientable.

Inner Product Bundles. Recall that one definition of a scalar product on a vector space V is given by specifying a certain element of $(V \otimes V)^*$, i.e., a linear map $\langle , \rangle \colon V \otimes V \to \mathbb{R}$, which is synmetric and positive definite.

If E is a vector bundle and s is a section of the associated bundle

$$(E \otimes E)^* \, ,$$

then we call s an *inner product structure* on E if each $s_x \in (E_x \otimes E_x)^*$ is an inner product. An inner product structure is sometimes called a *metric*.

If M is a manifold and $E \to M$ is a smooth vector bundle, then we say that an inner product structure on E is *smooth* if the section s is a smooth section (this makes sense since $(E \otimes E)^*$ is also a smooth bundle).

A smooth inner product structure on TM is called a *Riemannian metric*. In this case, we say that M is a Riemannian manifold.

Recall that if $W \subset V$ is a subvector space of an inner-product space, then an inner product on V induces one on W in an evident way. This will imply that if $E' \subset E$ is a subbundle, an E has an inner product structure, then so does E'.

Here is a lemma that I will not prove. It is relatively elementary.

Lemma. *With the notation as above, let $(E')^{\perp} \subset E$ be the union of the orthogonal complements $(E'_x)^{\perp} \subset E_x$. Then $(E')^{\perp} \subset E$ is a subbundle.*

Remark. If $W \subset V$ is a subvector space of an inner product space, then the composition $W^{\perp} \to V \to V/W$ is an isomorphism. So as a vector bundle $(E')^{\perp}$ is canonically isomorphic to the quotient bundle E/E'.

Definition. If $N \subset M$ is a submanifold, then the bundle $(TN)^{\perp} \subset TM_{|N}$ is called the *normal bundle* of N in M. It is a smooth bundle, and it's canonically isomorphic to the quotient bundle $TM_{|N}/TN$.

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The rest of the Chapter is concerned with the existence of inner product structures on a bundle. The answer is "yes, there is always an inner product structure" and the way to prove this is start with inner product structures on the parts of E which are trivial, and thereafter we try to glue things together. The key idea for the gluing argument is that of a *partition of unity.*

Partitions of Unity.

A a*partition of unity* on a space X is a set of continuous functions $\{\tau_\alpha\}_{\alpha \in J}$

$$\tau_\alpha \colon X \to [0,1]$$

such that

(1) for each $x \in X$ there is an open neighborhood U such that all but finitely many τ_α vanish on U;
(2) Furthermore, for every $x \in X$ we have $\sum_\alpha \tau_\alpha(x) = 1$.

We say that $\{\tau_\alpha\}$ is *subordinate* to an open covering $\mathcal{U}$ of X if the support of each τ_α is contained in at least one of the open sets $U \in \mathcal{U}$. By the *support* of a function $f \colon X \to \mathbb{R}$, we mean the set of points $x \in X$ such that $f(x) \neq 0$.

A result that is proved in many point-set topology courses (which we shall assume) is:

Theorem. *If X is a paracompact space, and $\mathcal{U}$ is an open cover of X, then there exists a partition of unity subordinate to it.*

Recall that a space is paracompact if every open cover of it has a *countable* subcover. It turns out that the converse to this theorem is true as well: a space which admits a partition of unity is necessarily paracompact. Manifolds are known to be paracompact (why?)

The Existence of Inner Product Structures.

Theorem. *If E is a vector bundle over a paracompact space X, then E admits an inner product structure.*

Proof. The proof will make use of the following fact: if V is a vector space with two different inner products u and v, then for any non-negative real numbers s, t the linear combination $su + tv$ is again an inner product.

Choose an open covering $\mathcal{U}$ of X such $E_{|U}$ is trivializable for each $U \in \mathcal{U}$. Choose a partition of unity $\{\tau_\alpha\}$ that is subordinate to $\mathcal{U}$ and choose for each α a $U_\alpha \in \mathcal{U}$ such that the support of τ_α is contained in U_α.

Since $E_{|U_\alpha}$ is trivializable, it can be equipped with an inner product structure w_α. Then the expression

$$\sum_\alpha \tau_\alpha w_\alpha$$

defines an inner product structure on E. $\square$