MAT7500 W14 Lecture Notes

Bröcker and Jänich: Chapter III: Vector Bundles

Motivation. We saw that for a manifold M, we have an assignment

$$p \mapsto T_pM$$

which sends a point of M to its tangent space. so what we have constructed is way of going from points of M to vector spaces. This assignment (in a sense later to be made precise) varies in a smooth way.

The goal of this chapter is to explain how this construction is an example of a more general notion of a vector bundle on a topological space. Roughly, a vector bundle is a "continuously varying family" of vector spaces. Our first step is to make this idea precise.

The Definition.

A *vector bundle* of rank d is a map of topological spaces $p\colon E \to X$ equipped with the following property: for each point $x \in X$ there is a neighborhood U of x and a homeomorphism

$$h\colon p^{-1}(U) \xrightarrow{\cong} U \times \mathbb{R}^d$$

called a *local trivialization* (or *bundle chart*) such that $p_1 \circ h = p$, where $p_1\colon U \times \mathbb{R}^d\colon U$ is first factor projection. That is, the diagram

$$\begin{tikzcd}
p^{-1}(U) \arrow[r] \arrow[d, "p"'] & U \times \mathbb{R}^d \arrow[d, "p_1"] \\
U \arrow[r, equal] & U
\end{tikzcd}$$

is required to commute. Furthermore, we require another condition, which runs as follows: for any two such pairs (U_α, h_α) and (U_β, h_β), set $U_{\alpha\beta} = U_\alpha \cap U_\beta$. The we will require that for each $x \in U_{\alpha\beta}$, the map $\mathbb{R}^d \to \mathbb{R}^d$ given by

$$v \mapsto p_2 \circ h_\beta h_\alpha^{-1}(x, v)$$

is a *linear isomorphism* of the vector space $\mathbb{R}^d$, where $p_2\colon U_{\alpha\beta} \times \mathbb{R}^d \to \mathbb{R}^d$ is second factor projection. In other words, the composition

$$x \times \mathbb{R}^d \xrightarrow{h_\alpha^{-1}} p^{-1}(x) \xrightarrow{h_\beta} x \times \mathbb{R}^d$$

is a vector space isomorphism.

Because of this additional condition, we see that $p^{-1}(x)$ *is equipped with the structure of a vector space in a preferred way* (since we can define vector addition using the map h_α is this is independent of which map that we are using).

If X is covered by open sets U_α attached to bundle charts (h_α, U_α), we call the collection $\mathfrak{B} = \{(h_\alpha, U_\alpha)\}$ a *bundle atlas.*

Remark. A single vector bundle $p\colon E \to X$ can have many atlases. The bundle atlas information is not a fixed part of the definition—i.e., the definition only requires that a bundle atlas exists but not necessarily chosen.

In a way similar to what we did in the case of smooth manifolds, we can always enlarge a bundle atlas to a *maximal bundle atlas.*

Terminology. One says that E is the *total space*, X is the *base space* and the inverse images $p^{-1}(x)$ for $x \in X$ are the *fibers* of $p\colon E \to X$. The map p is called bundle *projection.*

We sometimes denote the fiber at $x \in X$ by

$$E_x = p^{-1}(x)\,.$$

When $p\colon E \to X$ is understood, then we sometimes denote the bundle by its total space E.

In what follows, we set

$$h_{\alpha\beta} = h_\beta h_\alpha^{-1}\colon U_{\alpha\beta} \times \mathbb{R}^d \to U_{\alpha\beta} \times \mathbb{R}^d$$

then the adjoint of $h_{\alpha\beta}$ is the (continuous!) map

$$\hat{h}_{\alpha\beta}\colon U_{\alpha\beta} \to \mathrm{GL}_d(\mathbb{R})$$

given by $x \mapsto (v \mapsto p_2 h_{\alpha\beta}(x,v))$.

Example. (The Trivial Bundle). The projection $p_1\colon X \times \mathbb{R}^d \to X$ is a vector bundle of rank d. This is called the *trivial bundle.*

Example. (The Clutching Construction). Let $\hat{f}\colon S^{k-1} \to \mathrm{GL}_d(\mathbb{R})$ be a continous map (where $\mathrm{GL}_d(\mathbb{R}) \subset \mathbb{R}^{n^2}$ is a subspace). Let $S^k = D^k_+ \cup D^k_-$ be the decomposition of S^k into upper and lower hemispheres.

Define a space E to be the amalgmated union

$$(D^k_- \times \mathbb{R}^k) \cup_f (D^k_+ \times \mathbb{R}^k)$$

where $f\colon S^{k-1} \times \mathbb{R}^d \to S^{k-1} \times \mathbb{R}^d$ is the homeomorphism that is the adjoint of $\hat{f}$ (i.e., $f(x,v) = (x, \hat{f}(x)(v))$.

Let $p\colon E \to X$ be defined by first factor projection. We claim this gives a vector bundle of rank d. The idea in showing this isn't really difficult, but it is somewhat tedious.

To see it, let U_- be $S^k \setminus q$, where q is the north pole, and let $r\colon U_- \to S^{k-1}$ be the map defined by

$$r(x_1, \ldots, x_{k+1}) = \frac{(x_1, \ldots, x_k)}{\sqrt{\sum_i^k x_i^2}}$$

(this is an example of what is called a *deformation retraction*; if you don't know what this means, you can can learn more about it if you take MAT7510). Next define a homeomorphism

$$h_-\colon p^{-1}U_- = (D^k_- \times \mathbb{R}^k) \cup_f ((D^k_+ \setminus q) \times \mathbb{R}^k) \to U_- \times \mathbb{R}^k$$

by the formula

$$h_-(x,v) \quad = \quad \begin{cases} (x,v) & \text{for } x \in D^k_-; \\ (x, p_2 f^{-1}(r(x),v)) & \text{otherwise.} \end{cases}$$

The above defines a local trivialization of $p\colon E \to S^k$ on the open set $S^k \setminus q$. To complete the argument we must define a local trivialization in a neighborhood of q. This is given by setting $U_+ = \mathrm{int} D^k_+$ and defining $h_+\colon p^{-1}(U_+) \to U_+ \times \mathbb{R}^k$ by the identity map.

Remark. Although we do not as yet have the requisite tools to prove it, up to bundle isomorphism, it is true that any vector bundle $E \to S^k$ of rank d is given by the clutching construction applied to some map $\hat{f}\colon S^{k-1} \to \mathrm{GL}_d(\mathbb{R})$. Howver, see below for the $k = 1, d = 1$ case.

Example. (The Möbius Band). Let $f\colon S^0 \to \mathrm{GL}_1(\mathbb{R})$ be the inclusion (here $S^0 = \{\pm 1\}$ and $\mathrm{GL}_1(\mathbb{R}) = \mathbb{R} \setminus 0$. Take the clutching construction on this map. This gives a vector bundle $E \to S^1$ of rank 1. It is easy to see that E is the Möbius band, since E is a quotient space of $D^1 \times \mathbb{R}$ by identifying each point of the form $(-1,t)$ with the point $(+1,-t)$:

Fig.: The Möbius band as a vector bundle

Bundle Maps. Let $p\colon E \to X$ and $q\colon E' \to X$ be vector bundles over X. Then a *bundle homomorphism* is a map of spaces $f\colon E \to E'$ such that $q \circ f = p$ (this means f maps E_x to E'_x for every x), and furthermore and each map $f_x\colon E_x \to E'_x$ is required to be linear (we use the notation f_x to denote the restriction of f to the fiber at x).

We say that f has constant *rank* r when each $f_x\colon E_x \to E'_x$ has rank r.

A bundle homomorphism f is said to be a *monomorphism* (resp. *epimorphism*) if each f_x is injective (resp. surjective). It is a *bijection* if f is both a mono- and epi-morphism. It is an *isomorphism* when f is invertible, meaning that there's a bundle homomorphism $g\colon E' \to E$ such that $g \circ f$ and $f \circ g$ are the identity. Clearly, an isomorphism is, in particular, a bijection.

A vector bundle $p\colon E \to X$ is said to be trivializable if it is isomorphic to the trivial bundle $X \times \mathbb{R}^k \to X$.

If $p\colon E \to X$ is a vector bundle of rank k, then subspace $E' \subset E$ defines a *subbundle* of rank $\ell \le k$ if every point of X admits a bundle chart (= local trivialization) (h, U) for E such that $h(p^{-1}(U) \cap E') = U \times \mathbb{R}^\ell \subset U \times \mathbb{R}^k$. In this instance, (h, U) restricts to a bundle chart for E', so $E' \to X$ is a vector bundle of rank ℓ.

Example. Let $f\colon X \to \mathrm{GL}_k(\mathbb{R})$ be a continuous map. Then $\hat{f}\colon X \times \mathbb{R}^k \to X \times \mathbb{R}^k$, given by $\hat{f}(x, v) f(x)(v)$, is a bundle isomorphism from the trivial bundle to itself.

Example. (Restriction). If $A \subset X$ is a subspace, and $p\colon E \to X$ is a vector bundle of rank k, then so is $p^{-1}(A) \to A$. Henceforth, we write $E_{|A}$ for $p^{-1}(A)$.

Example. (Kernel subbundles). If $f\colon E \to E'$ is a bundle homomorphism, we set

$$K_x := \mathrm{kernel}(f_x\colon E_x \to E'_x) \subset E_x$$

and $K = \cup_{x \in X} K_x$. If we assume the rank of K_x is constant in x, then $K \subset E$ is a subbundle.

Example. (A Nontrivializable Vector Bundle). Let $\eta\colon E \to S^1$ denote the vector bundle of rank one given by the Möbius band. We will show that η is isn't trivializable.

Let $s\colon S^1 \to E$ be the map which sends a point $x \in S^1$ to the zero vector $0 \in E_x$. This map is one-to-one and a homeomorphism onto its image. It's an easy "paper cutting" exercise to show that $E \setminus s(S^1)$ is a connected space.

On the other hand, if there were a bundle isomorphism $h\colon E \xrightarrow{\cong} S^1 \times \mathbb{R}$, then $h \circ s\colon S^1 \to S^1 \times \mathbb{R}$ is necessarily the inclusion $i\colon S^1 \times 0 \subset S^1 \times \mathbb{R}$ and $S^1 \times \mathbb{R} \setminus S^1 \times 0$ is disconnected.

The homeomorphism h therefore restricts to a homeomorphism $E \setminus s(S^1) \cong S^1 \times \mathbb{R} \setminus S^1 \times 0$ from a connected space to a disconnected one. This gives a contradiction to the existence of h, so η is not trivializable.

The following result shows that a bundle homomorphism of fixed rank locally corresponds to a projection.

Rank Theorem. *If $f\colon E \to E'$ is a bundle homomorphism, of constant rank r, then there exist bundle charts (ϕ, U) for E and (ψ, U) for E', such that the diagram*

$$\begin{tikzcd}
E_{|U} \arrow[r, "f_{|U}"] \arrow[d, "\phi"', "\cong"] & E'|U \arrow[d, "\cong"', "\psi"] \\
U \times \mathbb{R}^k \arrow[r] & U \times \mathbb{R}^\ell
\end{tikzcd}$$

such that the bottom map is given by

$$(x, (v_1, \dots, v_k)) \mapsto (x, (v_1, \dots, v_r, 0, \dots, 0))$$

i.e., it's the inclusion of the projection onto the first r coordinates.

The proof will require an elementary lemma from linear algebra.

Lemma. *Suppose*

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is a decomposition of an $\ell \times k$ matrix into submatrices, where A is an $r \times r$ invertible matrix. Then S has rank r if and only if $D = CA^{-1}B$.

Proof. Set

$$T = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}.$$

Then T is invertible so S has rank r if and only if TS does. But TS is the matrix

$$\begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}$$

and it's clear that TS has rank r if and only if $D = CA^{-1}B$.

Corollary. *Let S be as in the previous lemma and suppose that S has rank r. Let V be the $k \times k$ matrix*

$$\begin{pmatrix} A^{-1} & -A^{-1}B \\ 0 & I \end{pmatrix}.$$

Then V is invertible and TSV is the $\ell \times k$ matrix of rank r

$$\begin{pmatrix} I_{r\times r} & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof of the Rank Theorem. Choose bundle charts (g, U) for E and (h, U) for E', then we have a commutative diagram

$$\begin{array}{ccc} E_{|U} & \xrightarrow{f_{|U}} & E'|U \\ g\downarrow\cong & & \cong\downarrow h \\ U \times \mathbb{R}^k & \xrightarrow{h\circ f_{|U} g\circ} & U \times \mathbb{R}^\ell . \end{array}$$

This diagram shows that we can assume at the outset that $E = U \times \mathbb{R}^k$, $E' = U \times \mathbb{R}^\ell$ are trivial, and $f\colon U \times \mathbb{R}^k \to U \times \mathbb{R}^\ell$ has constant rank r. Let $S_r(\ell \times k)$ be the set of $\ell \times k$ matrices of rank r, topologized as a subspace of $\mathbb{R}^{k\ell}$. Then f corresponds to a continuous map

$$\hat{f}\colon U \to S_r(\ell \times k)$$

by the formula $f(x, v) = (x, \hat{f}(x)v)$. If we fix for a moment any point $x \in U$, then by reordering the rows and columns of $\hat{f}(x)$ if necessary, we can assume that the $r \times r$ submatrix of $\hat{f}(x)$ given by the first r rows and columns is invertible. By continuity, this will also be true for all y sufficiently close to x. By replacing U be a smaller neighborhood if necessary, it suffices to assume that the $r \times r$ submatrix formed from the first r rows and columns of $\hat{f}(y)$ is invertible for all $y \in U$.

For each $y \in U$, let $T(y)$ and $V(y)$ be the invertible matrices constructed in the previous lemmas, so that $T(y)\hat{f}(y)V(y)$ is the matrix

$$\begin{pmatrix} I_{r\times r} & 0 \\ 0 & 0 \end{pmatrix}.$$

It is fairly clear that $T(y)$ and $V(y)$ are continuous functions of y. Set $\phi(y,v) = (y, V(y)^{-1}v)$ and $\psi(y,w) = (y, T(y)w)$. Then these define bundle charts satisfying the conclusions of the Rank Theorem $\square$.

Corollary. *If $f\colon E \to E'$ is a bundle monomorphism, then $f(E) \subset E'$ is a subbundle.*

Proposition. *If $f\colon E \to E'$ is a bundle bijection, then f is a bundle isomorphism.*

Proof. By the Rank Theorem, we have a commutative diagram

$$\begin{array}{ccc} E_{|U} & \xrightarrow{f_{|U}} & E'|U \\ {\scriptstyle\phi}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle\psi} \\ U \times \mathbb{R}^k & = & U \times \mathbb{R}^k \,. \end{array}$$

So $f_{|U}$ is a homeomorphism and $f_x\colon E_x \to E'_x$ is a linear isomorphism for each $x \in U$. In particular, f is a local homeomorphism and a bijection. This implies that f is a homeomorphism. The inverse $f^{-1}\colon E' \to E$ is clearly a bundle map, so f is a bundle isomorphism.

More General Morphisms. Suppose we are given a commutative diagram of spaces

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where the vertical maps are vector bundles. We say that F is a *linear map covering* f, if for every $x \in X$ the map of fibers $F_x\colon E_x \to E'_{f(x)}$ is a homomorphism of vector spaces.

In the special case when F_x is an isomorphism for all x, we say that F is a *bundle map* over f.

Pullbacks. Suppose that $p\colon E \to Y$ is a vector bundle of rank k and let $f\colon X \to Y$ be any map. The *pullback* (or *basechange*) of E to X is

$$f^*E := \{(x,e) |\, x \in Y, e \in E, f(x) = p(e)\}\,.$$

Note that first factor projection defines a map $q\colon f^*E \to X$. This defines a rank k vector bundle over X: for if $x \in X$, we can choose a bundle chart (ϕ, U) for E at $f(x)$. Then a bundle chart $h\colon q^{-1}(f^{-1}(U)) \to f^{-1}(U) \times \mathbb{R}^k$ is given by $h(x,e) = (x, \phi(f(x), e))$.

Note that the pullback construction is functorial up to canonical isomorphism: if $g\colon Z \to X$ is a map, then there's a canonical isomorphism of vector bundles

$$(f \circ g)^* E \;\cong\; g^* f^* E\,.$$

Remark. (Universality). Second factor projection defines a bundle map $f^*E \to E$ covering f. Suppose that $\phi\colon E' \to E$ is a linear map covering f, then there is a *unique* factorization of ϕ as

$$E' \xrightarrow{h} f^*E \to E$$

where h is a bundle homomorphism over X, and $f^*E \to E$ is the (canonical) bundle map covering f defined by second factor projection. The map h is defined by $h(e) = (p'(e), \phi(e))$, where $p'\colon E' \to X$ is bundle projection.

A special case of the pullback construction occurs when f is the inclusion of a subspace $A \subset X$. In this case f^*E coincides with the restriction $E_{|X}$.

Remark. In some sense, even in the case of a map f which isn't an inclusion, f^*E is still a kind of restriction (when suitably reinterpreted). To see this, note that the cartesian product

$$X \times E$$

is naturally a vector bundle over $X \times Y$. The restriction of this bundle along the *graph* of f, i.e.,

$$G_f\colon \{(y,x)|x = f(y)\} \subset Y \times X$$

gives the vector bundle $(X \times E)_{|G_f}$ On the other hand, G_f and X are canonically homeomorphic, via the map $X \to G_f$ given by $x \mapsto (x, f(x))$. If we identify these two spaces via the homeomorphism, then f^*E corresponds to $(X \times E)_{|G_f}$ (i.e., if we pullback the latter along $X \to G_f$ we obtain f^*E.

Whitney Sum (Fiber Product). if $p_1\colon E_1 \to X$ and $p_2\colon E_2 \to X$ are vector bundles, then the pullback of $E_1 \times E_2 \to X \times X$ along the diagonal map $X \to X \times X$ is a vector bundle over X. The fiber of this bundle at $x \in X$ is just $(E_1)_x \oplus (E_2)_x$. This is called the *Whitney sum* of E_1 and E_2. It is sometimes written as $E_1 \oplus E_2$.

Sections. If $p\colon E \to X$ is a vector bundle, then a *section* for E is a map $s\colon X \to E$ such that $p \circ s\colon X \to X$ is the identity. The *zero section* is the section defined by $s(x) = 0 \in E_x$. A section is said to be *nowhere zero* of each vector $s(x) \in E_x$ is non-trivial.

Smooth Vector Bundles. Suppose $p\colon E \to X$ is a vector bundle where X is a smooth manifold. Suppose there is bundle atlas $\mathfrak{B} = \{(h_\alpha, U_\alpha)\}$ such that the associated transition maps

$$\hat{h}_{\alpha\beta}\colon U_{\alpha\beta} \to \mathrm{GL}_k(\mathbb{R})$$

are smooth maps. In this case we say the $\mathfrak{B}$ is a *smooth bundle atlas*.

Prebundles. A pre-vector bundle over a space X of rank k is a triple $(E, p, \mathfrak{B})$ consisting of

(1) a *set* E;
(2) a surjective function $p\colon E \to X$;
(3) a vector space structure on each $E_x := p^{-1}(x)$ for $x \in X$;
(4) a set $\mathfrak{B} := \{(h_\alpha, U_\alpha\}$ consisting of a covering of open sets U_α of X and bijective functions
$$h_\alpha\colon p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$$
such that for each $x \in U_\alpha$ the induced function of vector spaces $h_{\alpha x}\colon E_x \to x \times \mathbb{R}^k$ is a linear isomorphism.
Furthermore, the transition maps
$$\hat{h}_{\alpha\beta}\colon U_{\alpha\beta} \to \mathrm{GL}_k(\mathbb{R})$$
are required to be continuous.

Of course, a vector bundle is just a pre-vector bundle with the additional condition that E has a topology making p continuous and the h_α a homeomorphism.

Conversely, there is a unique way to equip the set E in a pre-vector bundle with a topology making the pre-vector bundle into a vector bundle: a basis for this topology is gotten by taking the inverse images of open sets $h_\alpha^{-1}(V \times O)$ where V ranges through the open sets of X, O ranges through the open sets of $\mathbb{R}^k$ and α ranges through the indices of $\mathfrak{B}$.

Note. If X is a smooth manifold, and $(E, p, \mathfrak{B})$ is a *smooth* pre-vector bundle on X (meaning that the $\hat{h}_{\alpha\beta}$ are smooth maps), then the topology on E constructed above defines a smooth vector bundle structure.

The Tangent Bundle. Let M be a smooth manifold of dimension n and a smooth atlas $\mathcal{U} = \{(h_\alpha, U_\alpha)\}$. Set
$$TM = \bigcup_{x \in M} T_x M$$
Then we have a function $p\colon TM \to M$ sending a tangent vector $v \in T_x M$ to its initial point $x \in M$. In what follows we use the algebraic definition of $T_x M$ in terms of derivations $\mathcal{E}(x) \to \mathbb{R}$.

For each smooth chart $h\colon U \to \mathbb{R}^n$, we have the coordinate function $h_i\colon U \to \mathbb{R}$, $i = 1, \ldots, n$. Then the function
$$f\colon p^{-1}(U) \to U \times \mathbb{R}^n$$
defined by
$$f(x \in U, X \in T_x M) = (x, X(h_1), \ldots, X(h_n))$$
is a bijection (we say this in a previous lecture in Chapter 2). This defines a smooth pre-vector bundle structure on M of rank n. The associated smooth vector bundle is called the *tangent bundle.*

If $f\colon M \to N$ is a smooth map, then the tangent map construction of Chapter 2 gives a bundle map
$$Tf\colon TM \to TN$$
over f.

Definition. A vector field on a smooth manifold M is a section $s\colon M \to TM$.

Remark. In fact, we show below that TM is a smooth manifold (of dimension $2n$) in its own right. Hence, it makes sense to ask whether a vector field is smooth.

Lemma. *The tangent bundle TM is a smooth manifold of dimension $2n$.*

Proof. As above, the bundle charts $f\colon p^{-1}(U) \to U \times \mathbb{R}^n$ for TM are defined by $(x, X) \mapsto (x, X(h_1), \ldots X(h_n))$, where $h\colon U \to \mathbb{R}^n$ is a smooth chart for M. Then the composite

$$p^{-1}(U) \xrightarrow{f} U \times \mathbb{R}^n \xrightarrow{h} \mathbb{R}^n \times \mathbb{R}^n$$

is a smooth chart for TM with domain $p^{-1}(U)$. (It is trivial to check that TM is Hausdorff and second countable.) $\square$

Line Bundles Over S^1. The goal of this section is to give a sketch of the following statement:

Theorem. *Up to isomorphism, there are just two line bundles over S^1, the trivial bundle and the Möbius band bundle.*

In order to prove this, we need to develop a minimal amount of machinery.

Lemma. *Let $p\colon E \to X$ be a line bundle over X. Then p is trivializable if and only if p admits a nowhere zero section $s\colon X \to E$.*

Proof. A trivialization $X \times \mathbb{R} \xrightarrow{\cong} E$ determines a nowhere zero section $s\colon X \to E$ using the section $s_1\colon X \to X \times \mathbb{R}$ given by $s_1(x) = (x, 1)$. Conversely, a nowhere zero section s gives a trivialization $h\colon X \times \mathbb{R} \to E$ defined by $h(x, t) = ts(x)$. $\square$

Lemma. *Let $p\colon E \to [a, b]$ be a line bundle. Then p is trivializable.*

Proof. It suffices to consider the case $[a, b] = [0, 1]$. Let $c \in [0, 1]$ be the maximum value such that $E_{|[0,c]}$ is trivializable. If $c = 1$ there is nothing to prove, so assume that $c < 1$. Then $c > 0$ because the restriction of E to $[0, \delta)$ is trivializable for $\delta > 0$ sufficiently small. Choose a trivialization $E_{|[0,c]} \cong [0, c] \times \mathbb{R}$. In particular, we have a nowhere zero section $s\colon [0, c] \to E_{|[0,c]}$.

Let $\epsilon > 0$ be a number such that $E_{|[c-\epsilon,c+\epsilon]}$ is trivializable and choose a trivialization

$$\phi\colon E_{|[c-\epsilon,c+\epsilon]} \xrightarrow{\cong} [c - \epsilon, c + \epsilon] \times \mathbb{R}\,.$$

With respect to ϕ, the restriction of s to $[c - \epsilon, c]$ corresponds to a nowhere zero function $f\colon [c - \epsilon, c] \to \mathbb{R}$ (in the sense that $\phi \circ s_{|[c-\epsilon,c]} = (\mathrm{id}, f)$). Let $F\colon [c - \epsilon, c + \epsilon] \to \mathbb{R}$ be any continuous extension of f to a nowhere zero function (the extension exists by the Tietze Extension Theorem). Then F corresponds to a nowhere section $t\colon [c - \epsilon, c + \epsilon] \to E_{|[c-\epsilon,c+\epsilon]}$ that coincides with s on $[c - \epsilon, c]$. Hence, s and t together define a nowhere zero section $[0, c + \epsilon] \to E_{|[c-\epsilon,c+\epsilon]}$ which contradicts the maximality of c. Consequently, $c = 1$. $\square$

Suppose $p\colon E \to S^1$ is a line bundle. Writing $S^1 = D^1_+ \cup D^1_-$, $E_\pm = E_{|D^1_\pm}$ and using the previous lemma, we can choose trivializations

$$\phi_\pm\colon E_\pm \cong D^1_\pm \times \mathbb{R}\,.$$

We can recover E up to bundle isomorphism as follows: The clutching map $\hat{f}$ is defined using the adjoint of $f := \phi_+ \circ \phi_-^{-1}\colon S^0 \times \mathbb{R} \to S^0 \times \mathbb{R}$ in the sense that $f(x,t) = (x, \hat{f}(x)(t))$. In this case,

$$\hat{f}\colon S^0 \to \mathrm{GL}_1(\mathbb{R}) = \mathbb{R} \setminus 0 =: \mathbb{R}^\times$$

is a just a pair of 1×1-matrices, i.e., a choice nonzero of real numbers $r_\pm$ such that $\hat{f}((\pm 1, 0))(t) = r_\pm t$. Hence, we can completely specify E up to isomorphism by the ordered pair (r_-, r_+) and conversely, each such ordered pair determines a line bundle $E(r_-, r_+)$ over S^1 by means of the clutching construction.

Lemma. *(1). The ordered pair (r_-, r_+) and the ordered pair $(sgn(r_1), sgn(r_2))$ determine isomorphic line bundles.*

(2). The ordered pairs (r_-, r_+) and $(-r_-, -r_+)$ determine isomorphic line bundles.

Proof. (1). Let $s_\pm = \mathrm{sgn}(r_\pm)$. Choose a homeomorphism $h\colon D^1_+ \to [0,1]$ with $h((1,0)) = 0$. Define a bundle isomorphism $\phi\colon E(s_-, s_+) \to E(r_-, r_+)$ by the formula

$$\phi(x,t) \quad = \quad \begin{cases} (x,t) & \text{for} \quad (x,t) \in D^k_- \times \mathbb{R}; \\ (x, t((1-h(x))|r_-| + h(x)|r_+|) & \text{otherwise.} \end{cases}$$

(2). Define a bundle isomorphism $\phi\colon E(r_-, r_+) \to E(-r_-, -r_+)$ by the formula

$$\phi(x,t) \quad = \quad \begin{cases} (x,t) & \text{for} \quad (x,t) \in D^k_- \times \mathbb{R}; \\ (x,-t) & \text{otherwise.} \end{cases}$$

Proof of the theorem. The last lemma implies that any line bundle over S^1 is isomorphic to either the trivial bundle $E(1,1)$ or the Möbius bundle $E(-1,+1)$. We have already seen that these are non-isomorphic. □

The Tautological Bundle.

For non-negative integers k and n, let $I(k, n+k)$ denote the space of linear injections $\mathbb{R}^k \to \mathbb{R}^{n+k}$, in other words, the space of $(n+k) \times k$ matrices with real entries (topologized as a subspace of $\mathbb{R}^{k(n+k)}$.

Define an equivalence relation on this space by $f \sim g$ if and only if there is an element $A \in \mathrm{GL}_k(\mathbb{R})$ such that $g = fA$. The quotient space

$$G_k(\mathbb{R}^{n+k}) := I(k, n+k)/\sim$$

is called the *Grassmannian* of k-planes in $\mathbb{R}^{n+k}$. That's because an equivalence class $[f]$ is equivalent to specifying its image $f(\mathbb{R}^k) \subset \mathbb{R}^{n+k}$. So we can think of points of $G_k(\mathbb{R}^{n+k})$ as being k-dimensional subspaces $X \subset \mathbb{R}^{n+k}$.

In fact, $G_k(\mathbb{R}^{n+k})$ it can be shown that $G_k(\mathbb{R}^{n+k})$ has the structure of a smooth manifold of dimension nk (for what I want to say here, we don't need to know this last statement).

Let

$$F\colon G_k(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k} \to G_k(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k}$$

be the map given by

$$(X, v) \mapsto (X, p_{X^\perp}(v))\,,$$

where $p_{X^\perp}\colon \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ is orthogonal projection onto the orthogonal complement of X. Then F is a bundle map (of the trivial bundle to itself) over $G_k(\mathbb{R}^{n+k})$. For $X \in G_k(\mathbb{R}^{n+k})$ the kernel of $F_X\colon \{X\} \times \mathbb{R}^{n+k} \to \{X\} \times \mathbb{R}^{n+k}$ is given by the set of vectors $v \in X$, i.e., the vector space X itself. We denote this vector bundle by

$$\gamma^k\colon E \to G_k(\mathbb{R}^{n+k})\,.$$

It is a k-plane bundle called the canonical or tautological bundle over $G_k(\mathbb{R}^{n+k})$.

For example, consider the case $k = n = 1$. Then $G_1(\mathbb{R}^2)$ is the space of lines through the origin in $\mathbb{R}^2$. We can parametrize such lines by the angle they make with the X axis, with the condition that we identity the angle π with the angle zero. It follows that $G_1(\mathbb{R}^2)$ is homeomorphic to the circle S^1, or even better, it is really more closely identified with $\mathbb{R}P^1 = S^1/(x \sim -x)$ (which is of course homeomorphic to the circle). With respect to this identification, it's not difficult to show that $\gamma^1\colon E \to S^1$ in this case is isomorphic to the Möbius bundle (this will be one of your homework exercises)

More generally, if $k = 1$ and n is arbitary, then $G_1(\mathbb{R}^{n+1})$ is the space of lines in $\mathbb{R}^{n+1}$. It is true in this case that $G_1(\mathbb{R}^{n+1})$ is homeomorphic to $\mathbb{R}P^n$ (as you will see int he exercises). The bundle $\gamma^1\colon E \to \mathbb{R}P^n$ is called the *canonical line bundle*.