Methods for Computation of the Alexander Polynomial

by

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AN ESSAY

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Contents

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List of Figures

1. Preliminaries

Knot theory is the field of mathematics that concerns itself with the properties of mathematical knots. Mathematical knots are not quite the same as typical knots. For example, the knot keeping shoelaces secure isn't a knot in the mathematical sense. The key feature of a mathematical knot which differentiates it from the knots of everyday life is that mathematical knots must be closed. In the classical case, one may think of it as taking a loop of string and manipulating it in ordinary three-dimensional space. It becomes immediately apparent that no matter how one tangles up the loop, it can be undone without cutting the string (imagine the string to be infinitesimally thick). This thought experiment hints at one of the most complicated problems of knot theory: is there a way to tell when two different knots can be transformed from one to the other without cutting them? Of course, knot theorists are concerned with much more than just the case of $\mathbb{R}^3$, and many results may be applied to $\mathbb{R}^n$ or even S^n.

As knot theory is mainly concerned with the qualitative properties of knots along with the effects of continuous actions on them, topology is naturally the field of mathematics that offers the most enticing tools. As mathematical knots are closed, one may consider them as loops, which leads the theory to being mainly a subset of algebraic topology. Naturally as the knots are subsets of manifolds, quite a few concepts from differential topology arise as well. It turns out, applying methods of algebraic topology to knots such as the fundamental group and integral homology is not enough to differentiate them. In 1928, the great mathematician J.W. Alexander showed that a highly useful and compact means of classifying three-dimensional knots could be found by analyzing the complement of the knot in $\mathbb{R}^3$ or S^3. He showed how to create a covering space for the complement of the knot whose homology groups actually possess a module structure. This module alone can be used to classify knots, but oftentimes it's very large and complicated. Alexander then used an algebraic technique to assign matrices to each module, and then used a few elementary matrix operations to arrive at a polynomial that can classify knots, which he naturally called the Alexander polynomial. The computation of the Alexander polynomial allows for an assignment of a polynomial with integer coefficients - which also allows for negative exponents - to each knot, up to knot type.

While the discovery of this method for obtaining the Alexander polynomial undoubtedly sparked a revolution in the developing field of knot theory, it is quite combinatorial in nature and it can be easy to lose sense of what is actually going on. Not only is the original definition quite abstract, but the calculation of the homology of the covering space of the complement of the knots can often be difficult. In the years since Alexander's breakthrough revelation several other - usually much simpler - methods of calculation have been determined. This paper will first go over some basics, then will delve into the simpler methods for calculation of the polynomial, and will conclude with the actual formulation first given by Alexander.

1.1. **Basic definitions.** Rolfsen's text, [8], is generally considered the best classical introduction to knot theory. As such, this paper is heavily influenced by it. In particular, the basic concepts illustrated in this section are based on [8, Chapter 1], as this is one of the foundations of the subject. To begin, Rolfsen gives multiple definitions for the mathematical notion of a knot.

Definition 1.1. A knot is a subspace K of a space X that is homeomorphic to a sphere S^p. A link is a subset of a space that is homeomorphic to the disjoint union of spheres

$$S^{p_1} \cup S^{p_2} \cup \ldots \cup S^{p_j}.$$

Another common method is to take knots as embeddings

$$K : S^p \to S^n.$$

Both of these formulations will be useful. K will represent either the map or its image; context will make clear which.

Definition 1.2. Two knots or links K, K' are equivalent if there's a self-homeomorphism h of the space X with

$$h(K) = K'.$$

This is an equivalence relation, and the equivalence class of K is called its type. Links must be assigned an ordering of the components which is preserved by this mapping. A stronger and more useful notion of equivalence can now be expressed.

Definition 1.3. The two knots K, K' are ambient isotopic if the previously defined homeomorphism h is the end map h_1 of an ambient isotopy, where an ambient isotopy is a homotopy

$$h_t : X \to X$$

with h_0 being the identity map and each h_t a homeomorphism.

They key difference between these two ideas of equivalence is that an ambient isotopy will deform the ambient space around the knot while the homeomorphism of the equivalence will not. In particular, all classical knots are equivalent to the trivial knot and thus each other, but clearly not ambient isotopic. Classical knots are an embedding $K : S^1 \to \mathbb{R}^3$ or S^3 where the image of K is polygonal, which means it's just the union of a finite number of line segments. As expected, classical links are disjoint unions of classical knots. This restriction makes sure that any knot or link is sufficiently "nice" and will behave as expected. The trivial knot in this case is the standard embedding of $S^1 \subset \mathbb{R}^3$ or S^3, while the trivial link is the disjoint union of n copies of S^1 contained in a single plane, where $n > 1$.

Definition 1.4. A knot invariant is a function $K \longmapsto f(K)$ that makes an assignment of an object $f(K)$ to each knot or link K that respects knot type. A large number of useful knot invariants have been crafted, measuring many different properties.

It seems natural to consider the complement of a knot K, $\mathbb{R}^3 - K$, a topological invariant. One may take a functor going from the category of topological spaces to some algebraic category using the composition:

$$K \longmapsto \mathbb{R}^3 - K \longmapsto F(\mathbb{R}^3 - K).$$

Unfortunately, homology and cohomology both have nothing to offer in this situation.

Theorem 1.5. *For any knot $K^p \subset S^n$,*

$$H_*(S^n - K^p) \cong H_*(S^{n-p-1}) \quad \text{and} \quad H^*(S^n - K^p) \cong H^*(S^{n-p-1}).$$

Which is a natural deduction from the well known:

Theorem 1.6. *If S is a subspace of S^n homeomorphic to S^k for some k with $0 \leq k < n$, then $\tilde{H}_i(S^n - S)$ is $\mathbb{Z}$ for $i = n - k - 1$ and 0 otherwise.*

This theorem is discussed and proven by Hatcher in [5, p. 169], and turns out to just be a simple consequence of Alexander duality. No matter how badly S^1 is embedded, the homology groups will be the same. Note that this is still the case for non-classical knots with local pathologies, the most famous example being the Alexander horned sphere. While homology and cohomology are useless in regards to the complement, the fundamental group proves to be instrumental.

Definition 1.7. For a knot or link $K^{n-2} \subset \mathbb{R}^n$, $\pi_1(\mathbb{R}^n - K)$ is called the group of K.

The first step in calculating the group of a knot is to draw its diagram, which happens to be the first step towards the Alexander polynomial as well. While the group of a knot is highly useful, it actually turns out not to be a knot invariant.

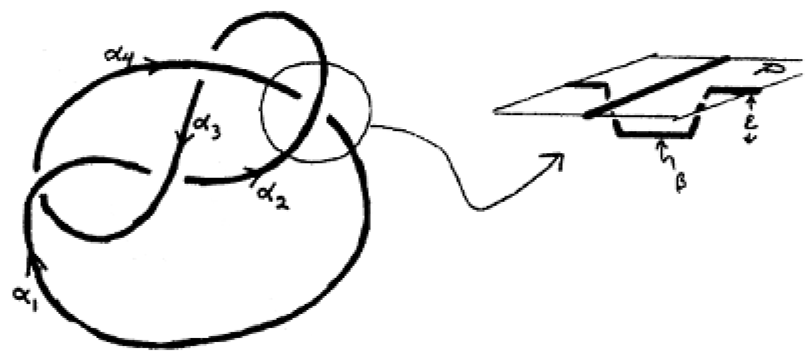


FIGURE 1. Close up of an intersection in the knot diagram for the figure-eight knot. [8]

1.2. **Knot diagrams.** The following method for creating a diagram of a classical knot is adapted from [8, p. 56]. Given a classical knot $K \subset \mathbb{R}^3$, one may form a two-dimensional diagram by projecting the knot onto a plane and indicating the layering of the arcs at each crossing. The diagram may then be used to go back to the three-dimensional knot, if desired. Since K is polygonal, it can be represented using a finite number of arcs. Divide K into exactly enough arcs such that there are no intersections on the plane, and call them $\alpha_1, \alpha_2, \ldots, \alpha_n$. Next, connect each of the α_j for $j = 1, 2, \ldots, n$ to α_{j-1} and α_{j+1} - with these connections being made modulo n - with small arcs that go slightly below the plane. Lastly, assign the arcs orientations that respect the ordering of their subscripts. This is fairly hard to describe in words, but Rolfsen provides an excellent example in [8, p. 56], shown slightly modified above in Figure 1. The zoomed in portion on the right details the construction at the intersections. The small arc under the plane is denoted β; it's some arbitrarily small distance below the plane P, denoted ϵ. These diagrams are highly useful, and it turns out they can be created for any arbitrary polygonal knot.

Definition 1.8. For a polygonal knot $K \subset \mathbb{R}^3$ and any plane P along with its orthogonal projection $p : \mathbb{R}^3 \to P$, P is <u>regular</u> with respect to K if for any $x \in P$, its preimage $p^{-1}(x)$ has an intersection with K of at most two points. If the intersection consists of two points, they cannot be a vertex of K (remember that K is just a union of line segments).

Making note of the following three facts concerning an arbitrary polygonal knot K and plane P, it becomes clear that a diagram may always be made:

- It only takes arbitrarily minute perturbations of P or K to ensure P to be regular for K.
- Denote the vertices of K as $v_0, \ldots, v_r$. For each knot $\exists \epsilon \in \mathbb{R}$ such that if one takes points $v_0', \ldots, v_r' \in \mathbb{R}^3$ with $|v_i - v_i'| < \epsilon$ for $i = 1, \ldots, r$, the polygon formed by these chosen points $K' = v_0'v_1' \ldots v_r'v_0'$ is a knot which is ambient isotopic to K.
- K is ambient isotopic to the knot projection previously explained when P is regular for K.

The idea is one can perturb K to be regular for P making adjustments small enough so that the resultant knot is ambient isotopic to K, then the projection will be an ambient isotopy. The condition that if $p^{-1}(x)$ for some x has two points then neither can be a vertex is necessary, for upon projecting down a vertex over an arc it would result in an erroneous intersection. When there are two vertices in the preimage, the resulting intersection would have more than three arcs. This discussion is based on Rolfsen's text, [8, p. 63], which goes into a bit more detail that isn't needed currently.

1.3. **Reidemeister moves.** It's obvious given the limited nature of projecting three dimensional objects onto a plane that there will be ambiguities inherent to knot diagrams. In particular, it's impossible in the general case to be certain when two diagrams actually represent the same knot. For example, the figure-eight knot has the three diagrams as shown in below in the image from [2] in Figure 2.

A case where it's possible to determine when two diagrams represent the same knot is when they both may be obtained from the other using a finite sequence of three distinct moves, which are shown in the diagram due to [1] featured in Figure 3 below. This idea is made precise by the following theorem.

Theorem 1.9. *Two diagrams represent knots which are ambient isotopic if and only if there exists a finite sequence of intermediary diagrams going from one to the other, where each diagram is obtained using exactly one Reidemeister move on the previous in the sequence.*

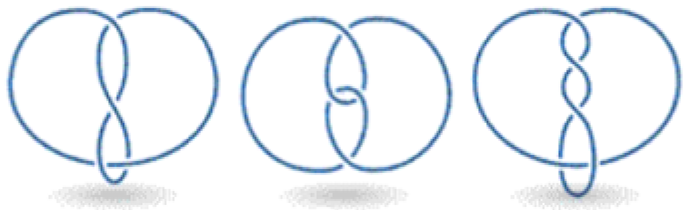


FIGURE 2. Diagrams of the figure-eight knot. [2]

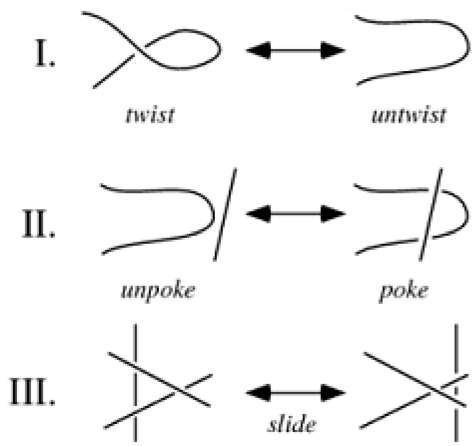


FIGURE 3. The three Reidemeister moves. [1]

It's interesting to note that the numbering for the types corresponds to how many arcs are involved. Many important invariants are defined using this concept. It's useful since one may check if a specific property is a knot invariant using only knot diagrams; no single Reidemeister move may alter the property if it is. This coverage of Reidemeister moves was inspired by the highly informative paper [3].

1.4. **Seifert surfaces.** Another useful construction associated with knots are Seifert surfaces, which are essentially just surfaces created by "filling in" the holes formed by the embedding in the larger space. To formalize this notion, several definitions are required. The following discussion follows [8, p. 120-1].

Definition 1.10. An n-dimensional <u>manifold</u> M^n is a metric space that can be covered with open sets that are homeomorphic to either $\mathbb{R}^n$ or $\mathbb{R}^n_+$. $\mathbb{R}^n_+$ is the half space of $\mathbb{R}^n$ obtained by restricting one of the components to be strictly nonnegative:

$$\mathbb{R}^n_+ = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}.$$

Points in M that possess neighborhoods homeomorphic to $\mathbb{R}^n$ make up the interior of M which is denoted $\mathring{M}$. The boundary of M is defined as

$$\partial M = M - \mathring{M}.$$

∂M is made up of those points in M with neighborhoods homeomorphic to $\mathbb{R}^n_+$. One calls a manifold closed if it's compact with $\partial M = \emptyset$ and open if $\partial M = \emptyset$ but it isn't compact.

Definition 1.11. A closed and connected n-manifold M is called orientable when $H_n(M) \neq 0$. When $\partial M \neq \emptyset$, it's called orientable if $H_n(M, \partial M) \neq 0$.

Definition 1.12. A subset $X \subset Y$ (where X and Y are manifolds) is bicollared when there's an embedding

$$b : X \times [-1, 1] \to Y \quad \text{with} \quad b(x, 0) = x, \quad \forall x \in X.$$

Similarly to knots, the map b itself or the image may be referred to as the bicollar. When X and Y are without boundary, the bicollar is actually a neighborhood of X in Y.

FIGURE 4. Two examples of Seifert surfaces. [6]

Definition 1.13. For some knot or link $K^n \subset S^{n+2}$, any connected, bicollared, and compact manifold $M^{n+1} \subset S^{n+2}$ with $\partial M = K$ is called a Seifert surface.

Seifert surfaces can oftentimes be very difficult to visualize. Two examples are shown in the diagram from [6] above in Figure 4. Seifert surfaces will be used later to construct cyclic coverings of knot complements. In this case, homology proves to be a very valuable tool, which is quite opposite to the homology of the complements themselves. The final item to consider for now is the cases where Seifert surfaces may be formed. It turns out that they may be constructed for any classical knot or link.

Theorem 1.14. *Any polygonal knot or link in $\mathbb{R}^3$ or S^3 has a corresponding polygonal Seifert surface.*

Proof. Let $L \subset \mathbb{R}^3$ be some classical knot or link. Perform a regular projection of L onto some plane P, taking care to give each component an orientation. Remove the over-crossing and under-crossing arcs at each intersection and replace them with "short cut" arcs in P as shown below in the slightly modified diagram from [8]. The arcs must respect the orientations originally assigned.

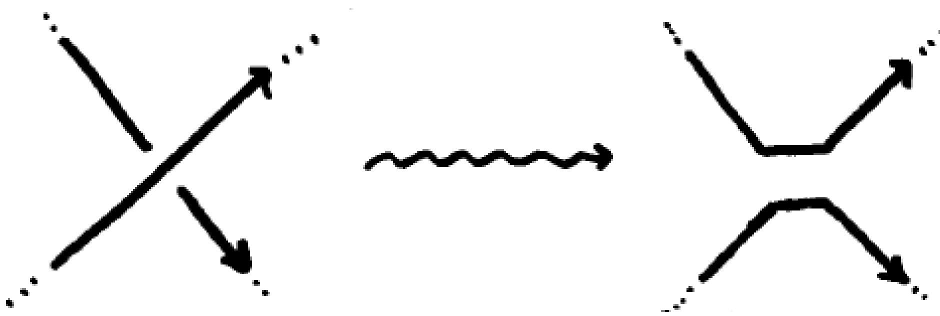


FIGURE 5. Creating closed arcs during Seifert surface construction. [8]

This leaves a disjoint collection of simple closed oriented arcs in P, with each bounding a disk with a bicollar. These disks may overlap, but one only needs to push them slightly off the plane one by one until they're disjoint. Since they have bicollars, one can prescribe orientations labeling one side of each disk with a "+" and the other with a "-". This assignment is arbitrary, but convention is such that the oriented boundary runs counterclockwise from the "+" side. Next the disks are connected at the initial intersections with half-twisted strips which yields a single 2-manifold M which clearly has $\partial M = L$. This step is visualized below in a diagram from [8] in Figure 6. When L is just a knot, M will be connected. Otherwise, the last step is to join up the disjoint components of M with tubes so that it's connected. ■

This proof based on [8, p. 120] was for the 3-dimensional case, but the theorem can actually be generalized to higher dimensions. First, another definition is necessary.

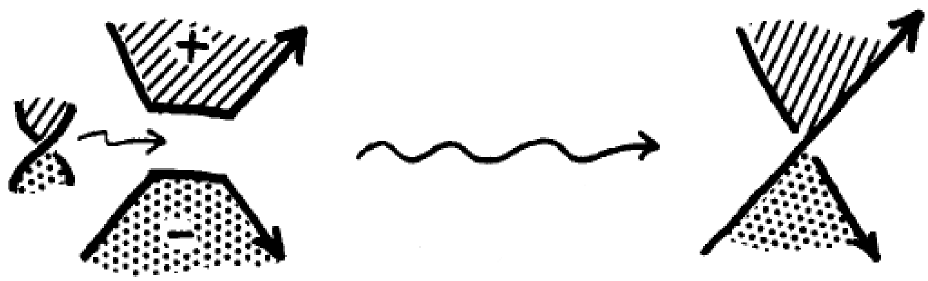


FIGURE 6. Connecting the disjoint disks during Seifert construction. [8]

Definition 1.15. For any submanifold $M^m \subset N^n$ with $\partial M = \partial N = \emptyset$, the tubular neighborhood of M is an embedding

$$t : M \times B^{n-m} \to N \quad \text{with} \quad t(x, 0) = x, \forall x \in M.$$

A submanifold is any subset of a manifold that is itself a manifold. One may view tubular neighborhoods as simply being the generalization of bicollars to higher dimensions.

Theorem 1.16. *Any polygonal knot or link $L^n \subset S^{n+2}$ that has a tubular neighborhood has a corresponding Seifert surface.*

This theorem is merely stated. The curious reader may check [8, p. 127] for a sketch of the proof. From the definition of the Seifert surface it's very clear that it can't be a knot invariant. All that's required is to have the appropriate type of manifold with the knot as its boundary, allowing for any number of different Seifert surfaces for any given knot. Despite of this, the collection of possible Seifert surfaces for a knot can be used to find a knot invariant called the genus. For any closed oriented surface F (in this case surface will refer to connected, bicollared, and compact 2-manifolds in S^3), it's a well-known result of algebraic topology that F will be homeomorphic to the 2-sphere with handles attached to its surface, the number of such handles is called the genus of F, denoted $g(F)$.

Definition 1.17. The minimum value of the genus as it ranges over the entire collection of possible Seifert surfaces for a given knot K is simply called the genus of K and denoted $g(K)$. It's a knot invariant and a method to calculate it for arbitrary polygonal knots exists, but is very difficult. While this is the case for arbitrary knots, it turns out the genus of a *constructed* orientable surface is trivial to calculate. It relies heavily on the Euler characteristic.

Definition 1.18. For a finite CW-complex X, the Euler characteristic is defined as the alternating sum

$$\chi(X) = k_0 - k_1 + k_2 - k_3 + \dots,$$

with k_n being the total number of n-cells in X. The means to relate the Euler characteristic of a surface and its genus is given in [7, p. 81] with the following theorem.

Theorem 1.19. *Seifert surfaces for classical knots or links will be 2-complexes thus for any such surface F,*

$$\chi(F) = k_0 - k_1 + k_2.$$

The genus and Euler characteristic are related by

$$\chi(F) = 2 - \mu(F) - 2g(F),$$

where $\mu(F)$ is the number of loops in ∂F. This will simply be the number of components in the given knot or link.

2. Skein relations

2.1. **The algorithm.**

The first and simplest method of obtaining the Alexander polynomial is now presented. This algorithm was created by Alexander to produce the polynomial. The presentation of the algorithm which follows was inspired by [3].

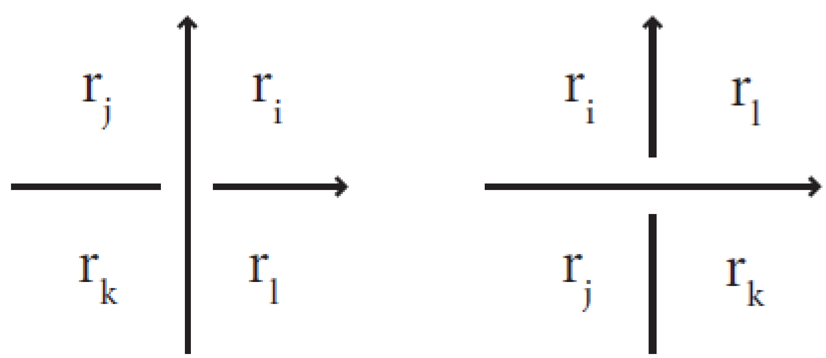


Figure 7. The two possibilities for knot diagram intersections. [3]

Algorithm 2.1. Take an oriented knot diagram of a knot or link. Label the intersections

$$x_1, \dots, x_n,$$

and call the $n+2$ regions cut out by these arcs in the plane

$$r_1, \ldots, r_{n+2}.$$

Construct a $n \times (n+2)$-matrix with the $(i,j)^{th}$ entry representing the the intersection x_i paired with region r_j. Upon examination of the production of knot diagrams, it becomes apparent that at each intersection only one of the two structures displayed above in the diagram from [3] in Figure 7 is possible. Determine the value to the $(i,j)^{th}$ entry by using the appropriate structure then assigning the following values:

$$r_i = t$$
$$r_j = -t$$
$$r_k = 1$$
$$r_l = -1.$$

In the case that the region doesn't touch the crossing, just assign the entry a 0. Delete a pair of columns that represent two adjacent regions, and finally take the determinant (deleting the two columns yields a square matrix). The result will be a polynomial in t, the Alexander polynomial $\triangle_K(t)$. This result is not unique in that the choice of columns to delete was arbitrary, but the differing results will be equal up to a factor of $\pm t^i, \forall i \in \mathbb{Z}$.

The next method of obtaining the Alexander polynomial is far more geometric in nature. Basically, one proceeds by examining the effects of taking a knot or link and "unknotting" it. The unknotting is achieved by manipulating the layering of the arcs found at each of the intersections on the knot diagram until a trivial knot or link occurs. Each of these unknotting actions yields a precise relation. The total action of these relations may be represented with a polynomial.

Definition 2.2. A Skein relation is some function on the three link diagrams depicted in the diagram from [7] below in Figure 8. These three link diagrams present the permutations of the layering of the two arcs that meet at any intersection in a knot diagram. It's possible for a line to pass under, pass over, or for the two lines not to meet. Explicitly,

$$F(L_-, L_0, L_+) = 0.$$

This is a bit of an abstract definition. It'll become clear with examples. But first, another necessary definition:

Definition 2.3. A Laurent polynomial over a field $\mathbb{F}$ is a linear combination of both positive and negative powers of the indeterminant where the coefficients are in $\mathbb{F}$. One may form a ring of Laurent polynomials

FIGURE 8. Labeling of the three types of intersections. [7]

by extending a classical polynomial ring to include inverses, e.g. $R[x]$ extends to $R[x, x^{-1}]$ for some ring R.

Algorithm 2.4. Take an oriented knot or link L and create a knot diagram for it. Pick an intersection, and determine if it's of the form L_-, L_0, or L_+. Using the Skein relation for the Alexander polynomial, which is

$$\triangle_{L_+}(t) - \triangle_{L_-}(t) - (t^{1/2} - t^{-1/2})\,\triangle_{L_0}(t) = 0$$

with $\triangle(\bigcirc) = 1$ ($\bigcirc$ stands for the trivial knot), solve for the link type of the intersection chosen. This is achieved by forming a tree with the original link diagram at the top, then adding a branch for each of the link diagrams created by changing the chosen intersection to each of the other two link types, then from each of those branches picking another intersection and performing the same procedure, until a trivial knot or link is obtained for each path from the top to the bottom. The final solution will be an integral Laurent polynomial with variable $t^{1/2}$, or

$$\triangle_L(t) \in \mathbb{Z}[t^{-1/2}, t^{1/2}].$$

This polynomial is precisely the Alexander polynomial of the knot or link.

2.2. **Examples.** The above description of the algorithm is a bit convoluted, but it's fairly simple in practice. This is now demonstrated by examining several examples, inspired by [3] and [7].

Example 2.5. The first link will be as simple as possible: the trivial link of two components, $\bigcirc\bigcirc$. This link has only one intersection which is of type L_0. Begin with the Skein relation, then take the intersection and change it to be of type L_+. This gives the link . Next change the intersection to type L_- which yields . The images for these two knots were found in [3]. These new links are inserted into the Skein relation in the appropriate terms. So, the formula becomes

$\triangle($$) - \triangle($$) - (t^{1/2} - t^{-1/2})\,\triangle\,(L_0) = 0.$

It's clear that both  and  are really just trivial knots (all that's needed is a twist, which is an ambient isotopy), and since it's given that $\triangle(\bigcirc) = 1$, the terms cancel and thus:

$$(t^{1/2} - t^{-1/2}) \triangle (L_0) = 0$$

$$\triangle(\bigcirc\bigcirc) = 0.$$

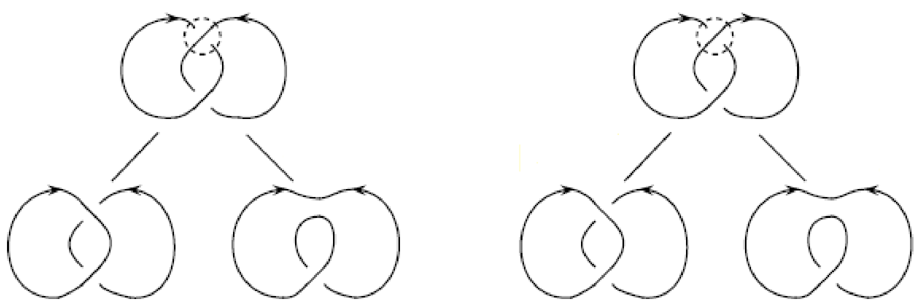


FIGURE 9. Skein tree of the Hopf link. [7]

Example 2.6. Next is the Hopf link  (this image was also found in [3]). The calculation here presents one with two choices of link. The diagram above from [7] in Figure 9 shows the process for either choice. Given orientations of the two components there will be two intersections; choosing one yields one of the two possible Skein trees as shown. The tree on the left comes from choosing the link of type L_+. Taking the left branch here is the result of changing the intersection to a L_- link and the right is for a L_0 link. It's clear that the bottom left link is just $\bigcirc\bigcirc$ and the bottom right is $\bigcirc$, so the Skein relation becomes:

$$\triangle L_+ - \triangle(\bigcirc\bigcirc) - (t^{1/2} - t^{-1/2}) \triangle (\bigcirc) = 0$$

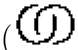
$$\triangle L_+ - 0 - (t^{1/2} - t^{-1/2})(1) = 0$$

$\triangle($$) = t^{-1/2} - t^{1/2}$

Choosing the link of type L_- gives the right Skein tree. Here the bottom left link comes from changing the intersection to one of type L_+, yielding $\bigcirc\bigcirc$. The bottom right link occurs with changing the link to L_0 , giving $\bigcirc$. The Skein relation is

$$\triangle(\bigcirc\bigcirc) - \triangle L_- - (t^{1/2} - t^{-1/2}) \triangle (\bigcirc) = 0$$

$$- \triangle L_{-} - (t^{1/2} - t^{-1/2})(1) = 0$$

$$\triangle($$  $$) = t^{1/2} - t^{-1/2}.$$

This is equivalent to the previous result since they're the same up to a factor of $-t^0 = -1$.

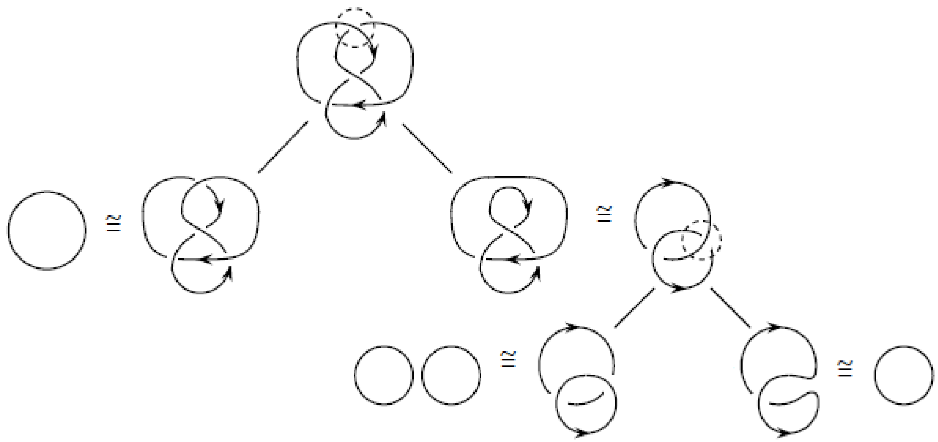
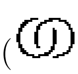


FIGURE 10. Skein tree of the figure eight knot. [7]

Example 2.7. The final example will be the familiar figure-eight knot, which is more complicated. Use the knot diagram - again from [7] - as shown above in Figure 10. Take the diagram on top, orient it appropriately, and pick the L_+ intersection as depicted. Changing this intersection to L_- gives the diagram on the bottom left which as shown is equivalent to ◯ using Reidemeister moves (it just takes two twists). Alternatively, changing the intersection to L_0 brings one down the right branch, to the Hopf link, which was just calculated. This example exhibits the recursive nature of these calculations. The Skein relation here is

$$\triangle L_{+} - \triangle L_{-} - (t^{1/2} - t^{-1/2}) \triangle (L_0) = 0.$$

Let the figure-eight knot be E. Going down the tree and performing the proper operations for each resultant trivial knot/link gives:

$$\triangle(E) = 1(\triangle(\bigcirc)) + (t^{1/2} - t^{-1/2})(1)(\triangle(\bigcirc\bigcirc)) - (t^{1/2} - t^{-1/2})(\triangle($$  $$)$$

$$= 1(1) + (t^{1/2} - t^{-1/2})(1)(0) - (t^{1/2} - t^{-1/2})(t^{1/2} - t^{-1/2})$$

$$= 1 - (t^{1/2} - t^{-1/2})^2$$

$$= t^{-1} + 3 - t$$

This procedure will work for any tame knot, due to the following theorem with proof based from [7, p. 62].

Theorem 2.8. *Any knot diagram for an arbitrary polygonal knot/link can be changed to the standard diagram of the trivial knot/link, respectively, by changing a finite number of intersection types. The minimum such number of changes required is called the* <u>*unknotting number*</u>.

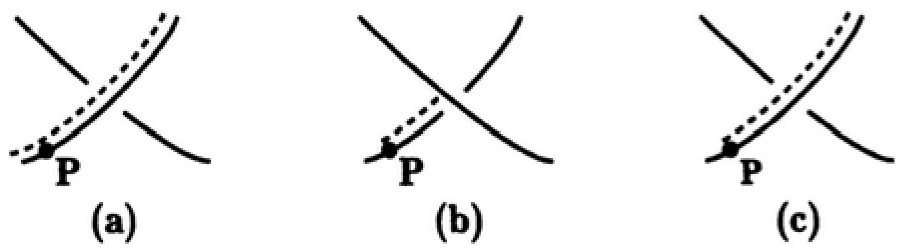


FIGURE 11. The first step of the unknotting theorem proof. [7]

Proof. Call the total number of intersections of the knot diagram n. The proof proceeds via induction on n. The base case $n = 0$ is the trivial knot, so clearly it holds here. Assume the theorem is true for any knot diagram with crossing number less than some arbitrary $n \in \mathbb{N}$. Now take some knot diagram with n intersections. Take some point P on the diagram which isn't an intersection. Start at P and go along the knot in the correct direction determined by its orientation. Upon reaching an intersection, if the current path takes one along an arc going over the intersection, keep going as shown in the diagram from [7] in Figure 11(a) above. If the current path takes one along the arc going below the intersection as in Figure 11(b), then change the intersection type as shown in Figure 11(c). After crossing enough of the intersections, the diagram will only have intersections where the arc being traversed is the upper arc. So then at some point along traversing the knot as shown below in the diagram again from [7] in Figure 12(a) one will reach some intersection A which has already been traversed as in Figure 12(b). This is just a "loop" starting with endpoint A and after the appropriate Reidemeister moves the loop can be eliminated as in Figure 12(c). The diagram obtained from this procedure has less than n intersections so then by the assumption it just needs finitely many intersection changes, so the argument is complete by induction. ■

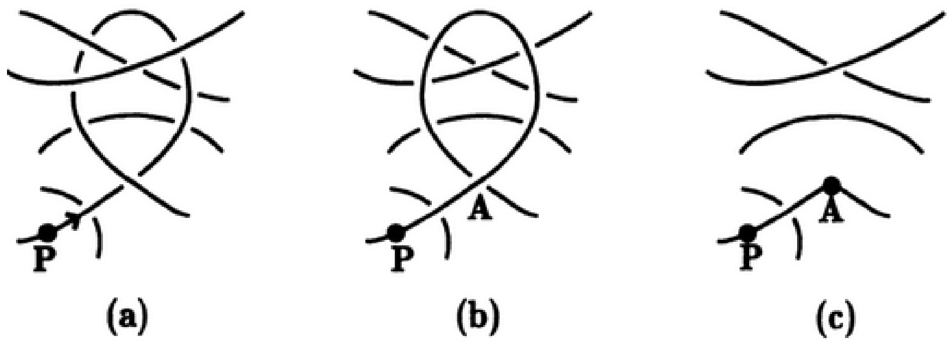


FIGURE 12. The second step of the unknotting theorem proof. [7]

3. SEIFERT METHODS

3.1. **The Seifert matrix.** Now a third method for calculating the Alexander polynomial is presented. This method is more complex than the previous two, but gives much more insight into what the calculation actually entails. In the process of the calculation the Seifert matrix will be formulated, which has far-reaching applications in knot theory beyond just the Alexander polynomial.

Definition 3.1. Let J and K be disjoint, oriented, and polygonal knots in S^3 or $\mathbb{R}^3$. For any intersection point on the diagram of $J \cup K$, there are the two possible configurations as shown below in the diagram from [2] in Figure 13. Add up the values of all the intersection points in the diagram which involve both J and K (any other intersection is just an intersection of either the regular projection of J or K), then divide by two. The value obtained is called the linking number of the link, and is a knot invariant. The linking number here is denoted $lk(J, K)$.

FIGURE 13. Signs of intersection types for calculation of linking number. [2]

Rolfsen introduces in [8, p. 132-3] seven other definitions of the linking number. Two of interest to the present discussion are:

- $[J] \subset H_1(S^3 - K)$ is the homology class of J in the complement of K. Since $H_1(S^3 - K) \cong \mathbb{Z}$ by Theorem 1.5 one can write $[J] = n\alpha$ where α is a generator of $\mathbb{Z}$ and then the linking number is just n.
- Since J is a loop in $S^3 - K$ it represents some element of $\pi_1(S^3 - K)$ with an appropriate basepoint. Here the fundamental group abelianizes to $\mathbb{Z}$ so J is carried to some integer. This integer is again the linking number.

It turns out that the various methods for calculating the linking number will be the same, up to sign. Generally any negative value will be converted to its corresponding natural number. While there is much to be said of the linking number as a knot invariant in its own right, in regards to the formulation of the Alexander polynomial it's used to obtain the incredibly useful Seifert matrix. The following discussion follows that found in [3].

Definition 3.2. Take L as some oriented link with n components with Seifert surface M of genus g. M is an orientable, compact, connected surface with n boundary components. For both simplicial or singular homology it'll be the case that

$$H_1(M;\mathbb{Z}) = \oplus_{2g+n-1}\mathbb{Z}$$

which has generators $[f_i]$, with the f_i being the loops shown below in the diagram found in [6] depicted in Figure 14. The goal here is to

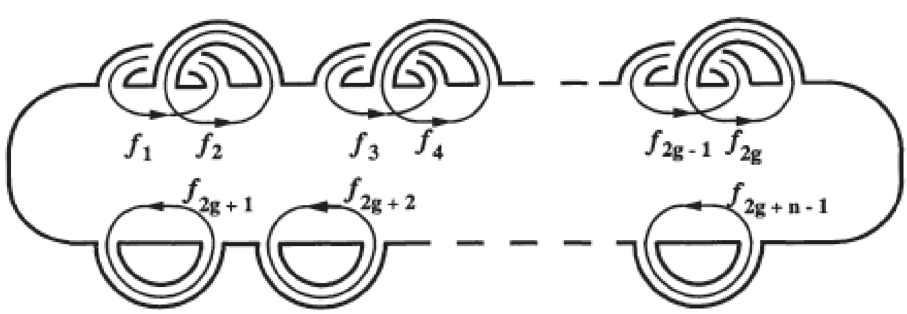


Figure 14. Homological generators for Skein surface of genus g for a link with n components. [6]

extract some essential information about the link from these homological generators. Each f_i is just a loop, so it seems natural to find the linking numbers of each set of pairs of them. This is the right idea, but it's entirely possible for some of these loops to intersect. Luckily, there is an easy way to ameliorate this. Since M is bicollared, the loops

f_i may be pushed "up" or "down". For any simple oriented curve f on M, create a new curve called f^+ by pushing f up some arbitrarily small distance ϵ. f^+ is parallel to f and lies in $M \times \{\epsilon\}$, it's called the positive push off of f. This ensures that all components in the links being observed will have empty intersections, but also changes none of their topological properties. Now, form a matrix S with

$$s_{ij} = lk(f_i, f_j^+) \quad \text{for} \quad i, j = 1, 2, \ldots, 2g + n - 1.$$

This is called the Seifert matrix of L.

It's clear from the previous calculation that if one forms the Seifert matrix of a knot with a Seifert surface of genus g then S will be a $2g \times 2g$ matrix. It's fairly obvious that the Seifert matrix isn't a knot invariant itself, since many different matrices can be formed for exactly the same knot by altering the genus of the Seifert surface used in each case. In addition, the basis of the homology groups are arbitrary. Despite of this, the Seifert matrix can be used to derive several important knot invariants. Notably,

Definition 3.3. For a polygonal knot or link L along with its Seifert matrix S that results from a prescribed Seifert surface M,

$$\triangle_L(t) \doteq det(S - tS^T)$$

is the Alexander polynomial of L. Here $\doteq$ represents that this expression is equal up to addition or subtraction of terms of the form $\pm t^n$.

3.2. **Cyclic coverings of the knot complement.** The Seifert surfaces of a knot can also be used to construct infinite or finite cyclic coverings of its complement, which leads to Alexander's original definition of the Alexander polynomial. The first singular homology group of the infinite cyclic cover can be represented by a module deemed Alexander's invariant, and Alexander's polynomial turns out to just be a generator of the presentation of this module. The previously described Seifert matrix will be shown to just be a representation of Alexander's invariant. The rest of this paper will focus on this original definition of the Alexander polynomial, which is the fourth and final method of calculation covered. This construction is wonderfully detailed in [8], and the discussion from this point forward is inspired by it. The first step will be to define and construct the covering spaces.

Definition 3.4. Take some Seifert surface M of a knot $K^n \subset S^{n+2}$. Define

$$N : \mathring{M} \times (-1, 1) \to S^{n+2}$$

which is an open bicollar of $\mathring{M}$ and thus

$$\mathring{M} = N(\mathring{M} \times 0).$$

Now take the following sets in N and S^{n+2}

$$N = N(\mathring{M} \times (-1, 1))$$

$$N^+ = N(\mathring{M} \times (0, 1))$$

$$N^- = N(\mathring{M} \times (-1, 0))$$

$$Y = S^{n+2} - M$$

$$X = S^{n+2} - K.$$

Take a countable number of copies of the triples

$$(N_i, N_i^+, N_i^-)$$

$$(Y_i, N_i^+, N_i^-)$$

for $i = 0, \pm 1, \pm 2, \ldots$ and denote their disjoint unions

$$\tilde{N} = \bigcup_{i=-\infty}^{\infty} N_i$$

$$\tilde{Y} = \bigcup_{i=-\infty}^{\infty} Y_i.$$

Take the quotient space obtained by gluing $\tilde{N}$ and $\tilde{Y}$ together using the identity map from each $N_i^+ \subset Y_i$ to $N_i^+ \subset N_i$ along with each $N_i^- \subset Y_i$ to $N_{i+1}^- \subset N_{i+1}$, as shown below by the diagram from [8] in Figure 15. The space obtained, call it $\tilde{X}$, is a path-connected open (n+2)-manifold. $\tilde{X}$ is the infinite cyclic cover of X. Rolfsen shows there's a deck transformation of $\tilde{X}$ with $Y_i \mapsto Y_{i+1}$ and $N_i \mapsto N_{i+1}$ that generates

$$Aut(\tilde{X}) \cong \mathbb{Z}.$$

It is also shown that $\tilde{X}$ is the universal cover abelian cover of X, in [8, p. 130].

The same general procedure can be used to create a finite cyclic covering space of X. Instead of taking countably many copies of the triples

$$(N_i, N_i^+, N_i^-)$$

$$(Y_i, N_i^+, N_i^-),$$

pick some integer k and take only that many copies. Then define $\tilde{Y}$ and $\tilde{N}$ in the same way, except this time the disjoint unions are finite. Create the quotient space in the same manner, but now glue

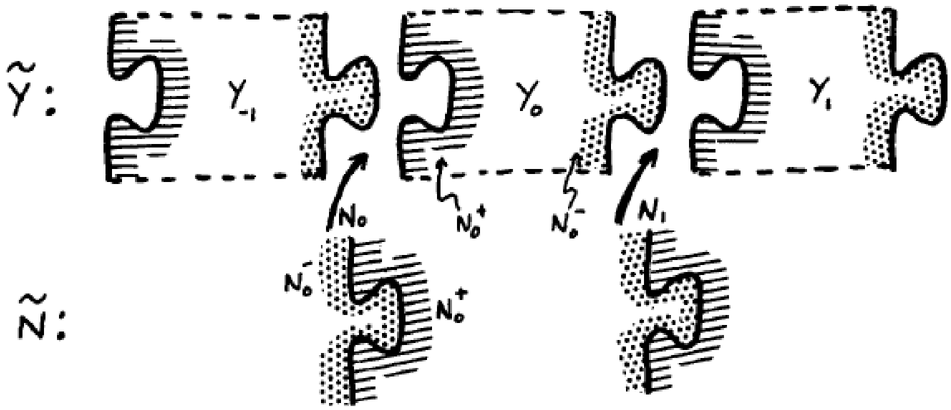


FIGURE 15. Illustration of the identification used for the infinite cyclic cover. [8]

$N_{k-1}^{-} \subset Y_{k-1}$ onto $N_0^{-} \subset N_0$ to obtain a cyclic cover of X with k sheets, call it $\tilde{X}_k$. Here, as one may expect,

$$Aut(\tilde{X}_k) \cong \mathbb{Z}/k.$$

3.3. **The Alexander invariant.** As stated in the previous section, the Alexander invariant is just the homology of the infinite cyclic covering space of the complement of a knot or link. It possesses a wealth of information about the knot or link in question. Not only does it allow for the computation of the Alexander polynomial, but it displays a module structure that allows for a more precise knot invariant. This is because for some higher-dimensional knots the Alexander polynomial fails to exist, and also there's some ambiguity over what its definition should be for links.

In order to describe the module structure of the Alexander invariant of a knot, the ring of finite Laurent polynomials will be utilized - let it be denoted Λ. It's important to note here that the units of this ring are just the polynomials of the form $\pm t^i$, $\forall i \in \mathbb{Z}$. Take any arbitrary polygonal knot $K^n \subset S^{n+2}$. As shown before it will always be possible to construct the infinite cyclic covering space $\tilde{X}$ of its complement,

$$X = S^{n+2} - K^n.$$

The Λ-module structure of $H_*(\tilde{X})$ integrates the actions of $\mathbb{Z}$ both as the group over which the coefficients of the polynomials are taken and the induced group of deck transformations of $\tilde{X}$.

There are two generators of this group of deck transformations (this is because $\mathbb{Z}$ has only two generators), so pick one and call it

$$\tau : \tilde{X} \to \tilde{X}.$$

Take some Laurent polynomial $p(t) \in \Lambda$ with

$$p(t) = c_{-r}t^{-r} + \ldots + c_0 + c_1 t + \ldots + c_s t^s$$

and some homology class $\alpha \in H_i(\tilde{X})$. Since τ is a deck transformation it clearly induces a homology isomorphism, call it

$$\tau_* : H_i(\tilde{X}) \to H_i(\tilde{X}).$$

Now the module structure of $H_i(\tilde{X})$ can be described via the following definition of the multiplication operation:

$$p(t)\alpha = c_{-r}\tau_*^{-r}\alpha + \ldots + c_0\alpha + c_1\tau_*\alpha + \ldots + c_s\tau_*^s\alpha.$$

It's obvious then that $p(t)\alpha$ is just a summation of homology classes with integer coefficients, since each τ_*^j is a homology isomorphism. Thus $p(t)\alpha$ is again a member of $H_i(\tilde{X})$. After a few tedious calculations it is proven that this multiplication in Λ makes the Alexander invariant of a knot a Λ-module, and that any two ambient isotopic knots have isomorphic Alexander invariants. This construction did involve a choice in that there are two possibilities for τ. The procedure can be made canonical by assigning fixed orientations to K^n and S^{n+2}, then requiring that the deck transformation generator chosen relates to some loop in X that when paired with K gives linking number 1. An example might be helpful at this point.

FIGURE 16. The lowly trefoil knot.

Example 3.5. Calculating the Alexander invariant of the trefoil knot will be relatively basic yet informative. This example follows along with Rolfsen's treatment in [8, p. 163-5]. For reference, the knot diagram of the trefoil is included above in Figure 16. Clearly the first step in determining the homology of the infinite cyclic cover of the trefoil's complement will be to construct the cover itself. This requires choosing a Seifert surface for the trefoil (recall that there are infinitely many possibilities). Let the Seifert surface be as depicted by Rolfsen in [8] below in Figure 17.

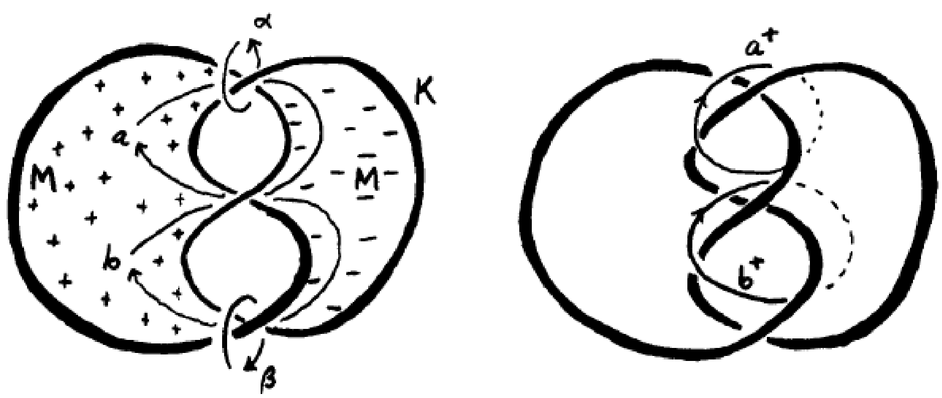


Figure 17. A Seifert surface for the trefoil knot along with homological generators of the complement and their positive pushoffs. [8]

Recall the construction from Definition 3.5. N, N^+, N^-, Y, and X shall represent the same sets as they did there. It's clear from basic homology theory that $H_1(\mathring{M})$ along with $H_1(Y)$ are free abelian groups where $H_1(\mathring{M})$ is generated by the cycles a and b and $H_1(Y)$ is generated by α and β as shown on the left half of Figure 17 above, an image from [8].

One can now imagine taking the cycles a and b and taking positive push offs to obtain the new cycles a^+ and b^+ as shown above in the right half of Figure 17. Since it's been established that

$$N : \mathring{M} \times (-1, 1) \to S^3$$

is an open bicollar of $\mathring{M}$, the <u>negative push offs</u> may be taken as well. The negative push off, as one may expect, is exactly the same as defined in Definition 3.2, but now a curve f will be pushed "down" by some arbitrarily small distance ϵ so that f is pushed to f^- which lies in $\mathring{M} \times \{-\epsilon\}$.

The diagram on the right in Figure 17 makes it clear that in $H_1(Y)$,

$$a^+ = -\alpha \tag{1}$$

$$b^+ = \alpha - \beta. \tag{2}$$

It only takes a bit of thinking, or perhaps jotting down a diagram, to see that the relations of the negative pushoffs gives, in $H_1(Y)$,

$$a^- = \beta - \alpha \tag{3}$$

$$b^- = -\beta. \tag{4}$$

The infinite cyclic cover of X is constructed as in Definition 3.4, using a quotient of infinitely many copies of Y and N. Rolfsen provides an insightful diagram of $\tilde{X}$ in [8], which is featured below in Figure 18. Here τ is a generator of the group of deck transformations of $\tilde{X}$.

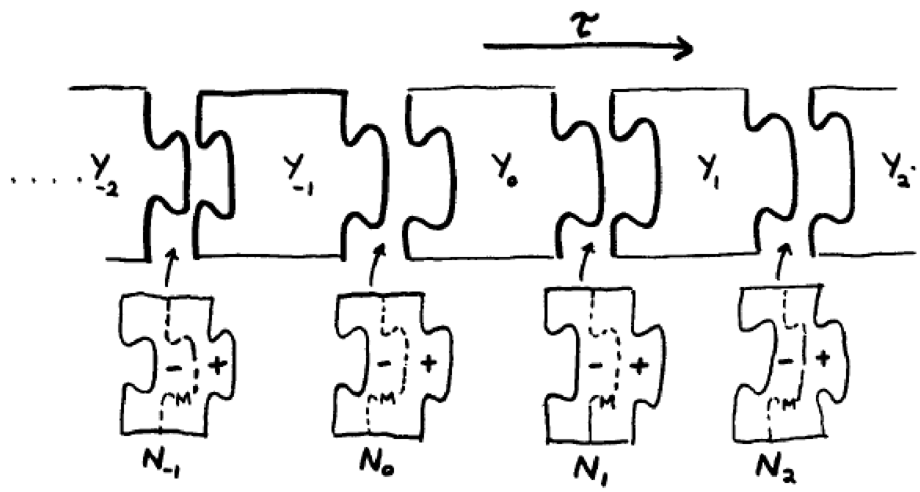


FIGURE 18. An infinite cyclic cover of the trefoil knot. [8]

When the module multiplication of the Alexander invariant was introduced the variable t was used to represent the application of the induced homology isomorphism $\tau_* : H_1(\tilde{X}) \to H_1(\tilde{X})$ to the generators. For example, the two generators of $H_1(Y_1)$ will just be $\tau_*(\alpha)$ and $\tau_*(\beta)$, which take the form $t\alpha$ and $t\beta$, respectively, where α and β are assigned as the homological generators of Y_0 (this is for simplicity; any Y_i can be taken as the "base" of the construction). Let

$$\tilde{Y} = \bigcup_{i=-\infty}^{\infty} Y_i,$$

as before. Clearly then the generators of $H_1(\tilde{Y})$ will just take the form of

$$t^j\alpha, t^j\beta, \quad \forall j \in \mathbb{Z}.$$

So then $H_1(\tilde{Y})$ is a free abelian group with these as its infinite generators. It was noted previously that elements of the form $\pm t^j$ are the units of Λ, which implies then that $H_1(\tilde{Y})$ may also be viewed as a free Λ-module generated by α and β.

Keeping this in mind, it's time to examine the relations created by the actual gluing process of Y_i to Y_{i+1} using N_{i+1} which creates $\tilde{X}$. To get an idea of what's occuring, first the gluing of Y_0 to Y_1 is discussed. Here looking at Figure 18 it becomes clear that

$$a_1^- = a_1^+ \tag{5}$$

$$b_1^- = b_1^+, \tag{6}$$

where a_1 and b_1 are just the generators for $H_1(\mathring{M})$ and $\mathring{M}$ refers to the copy of M contained in N_1. Obviously the positive pushoffs a_1^+ and b_1^+ are contained in N_1^+ and thus are identified with Y_1, which as mentioned before, has homological generators $t\alpha$ and $t\beta$. The negative pushoffs will be identified with Y_0 which has generators α and β. So applying equations (1)-(4) here, we see

$$\begin{aligned} a_1^+ &= -t\alpha \\ a_1^- &= \beta - \alpha \\ b_1^+ = t\alpha - t\beta &= t(\alpha - \beta) \\ b_1^- &= -\beta. \end{aligned}$$

Combining these results with equations (5) and (6) leads us to

$$\beta - \alpha = -t\alpha \tag{7}$$

$$-\beta = t(\alpha - \beta). \tag{8}$$

The only difference between this case and that of attaching Y_{i-1} to Y_i via N_i is that here the positive pushoffs are related to the generators $t^i\alpha$ and $t^i\beta$ while the negative pushoffs are related to $t^{i-1}\alpha$ and $t^{i-1}\beta$. So it's clear that equations (7) and (8) may be generalized to

$$t^{i-1}(\beta - \alpha) = -t^i\alpha \tag{9}$$

$$-t^{i-1}\beta = t^i(\alpha - \beta). \tag{10}$$

Equations (9) and (10) give an infinite number of relations which define $H_1(\tilde{X})$ as an abelian group:

$$H_1(\tilde{X}) \cong (\{t^i\alpha, t^i\beta\}; t^{i-1}(\beta - \alpha) = -t^i\alpha, -t^{i-1}\beta = t^i(\alpha - \beta), \forall i \in \mathbb{Z}).$$

But as mentioned before, the $\pm t^i$ terms are units when considering $H_1(\tilde{X})$ as a Λ-module, which leads to the module presentation:

$$H_1(\tilde{X}) \cong (\alpha, \beta; \quad \beta - \alpha = -t\alpha, -\beta = t(\alpha - \beta)). \tag{11}$$

Clearly equation (7) may be rewritten as

$$\beta = \alpha - t\alpha,$$

so β can be eliminated. Put this form of equation (7) into (8) to get

$$-(\alpha - t\alpha) = t(\alpha - (\alpha - t\alpha))$$

$$t^2\alpha - t\alpha + \alpha = 0$$
$$(t^2 - t + 1)\alpha = 0$$

giving the single relation

$$t^2 - t + 1 = 0.$$

The presentation thus simplifies to

$$H_1(\tilde{X}) \cong (\alpha; \quad t^2 - t + 1 = 0),$$

which is just another way of writing

$$H_1(\tilde{X}) \cong \Lambda/(t^2 - t + 1). \quad \blacksquare$$

The previous discussion only applies to knots. For some link, $L^n \subset S^{n+2}$ possessing k components (for $k > 1$), the Alexander invariant is obtained in a different manner since the universal abelian cover here isn't just the infinite cyclic cover. Since

$$X = S^{n+2} - L^n$$

is path-connected, the Hurewicz theorem tells us that the abelianization map for $\pi_1(X)$ will be the same as the Hurewicz homomorphism $\pi_1(X) \to H_1(X)$. Then by Theorem 1.6 along with Van Kampen it's clear that

$$H_1(X) \cong \mathbb{Z}^k,$$

so then since the covering space is universal abelian its group of deck transformations will be isomorphic to $\pi_1(X)$. Thus in this case the group of deck transformations will be some free abelian group on k generators. As in the case for knots, the module structure of the Alexander invariant will carry information of the action of the coefficient group for the polynomials $\mathbb{Z}$ along with the deck transformation automorphism group $\mathbb{Z}^k$. In this case the ring cannot just be Λ, instead it must be the ring of Laurent polynomials over k indeterminants that commute, say

$$x_1, \ldots, x_k.$$

This new ring will be denoted Λ_k. Now assign an ordering of the components of L along with orientations of each of the components and S^{n+2} itself. Proceed as in the previous case for knots, and it becomes apparent that the Alexander invariant for links with k components has a Λ_k-module structure.

Now that the structure of and method for obtaining the Alexander invariant for classical knots or links has been established, a bit of algebra must be discussed before reaching the Alexander polynomial. The following discussion will allow the extraction of a Laurent polynomial out of finitely-generated modules which will possess a great deal of information about the module. Naturally, these are the Alexander polynomials that are being worked towards.

Definition 3.6. Say A is some commutative ring with unit(s) and let M be some finitely-presented module on A,

$$M \cong (\alpha_1, \dots, \alpha_r; \rho_1, \dots, \rho_s).$$

Here the α_j are the generators and the ρ_j are relations which are just linear combinations of the generators, i.e.

$$\rho_i = a_{i1}\alpha_1 + \dots + a_{ir}\alpha_r$$

where each $a_{ij} \in A$. Clearly this setup may be represented in a simpler way by use of a matrix on the A-coefficients in the relators. Define the $s \times r$-matrix P by setting its $(i,j)^{th}$ component as a_{ij}. P is named the presentation matrix for M with respect to the presentation given. Clearly just being given P is enough to obtain the presentation which it was created from and thus P will determine the module up to A-isomorphism.

Giving any object a matrix representation is very powerful in that matrices are flexible objects and quite easy to manipulate. It turns out that the module corresponding to the presentation matrix P remains the same up to A-isomorphism for these eight operations, discussed in further detail in [8]:

(1) exchanging any two rows or any two columns
(2) adding an A-linear combination of rows to some row
(3) adding an A-linear combination of columns to some column
(4) multiplying through a row or column by some unit of A
(5) adding an extra column on the left of P and an extra row on the top of P, where the column is just 1 followed by 0s and the row is 1 followed by an arbitrary selection
(6) performing the inverse operation of that of number 5
(7) taking an A-linear combination of rows of P and adding it onto P
(8) deleting any row which is just an A-linear combination of the others.

It can be shown then that any two matrices with entries that are elements of the ring A correspond to presentations of isomorphic A-modules exactly when it only takes a finite sequence of these eight operations to go from one of the matrices to the other. This is very similar to the Reidemeister moves. This property allows one to take a presentation matrix and perform a plethora of matrix operations on it while maintaining the original presentation of the module at hand. In particular, this allows for the calculation of minors.

Definition 3.7. Say, as before, M is a module over the ring A with a $s \times r$ presentation matrix. Then the set of all $r \times r$ minors (by this it's meant all $r \times r$-matrices formed from P by deleting entire rows and columns) generates an ideal of A called the order ideal of M. When $s < r$ it's conventional to just assign it the trivial ideal.

From the previous discussion of the matrix operations that don't change the module presentation it's clear that both this ideal can be formulated and also the exact choice of P won't affect the order ideal. When M possesses a square presentation matrix then the order ideal will be principal and generated by the determinant of the matrix. This finally leads to the end of this formulation of the Alexander polynomial.

Definition 3.8. It was shown before that the Alexander invariant $H_1(\tilde{X})$ for some polygonal knot K is a module over the ring Λ which is finitely presentable. Thus using the previous discussion it can be assigned a presentation matrix, which is referred to as the Alexander matrix of K. This matrix will then have an order ideal, naturally referred to as the Alexander ideal of K. When this order ideal is principal, generators of the Alexander ideal will be the Alexander polynomial.

The Alexander matrix will always exist for polygonal knots. After the discussion previously of the eight operations one may perform on them while keeping the presentation the same, it's evident that the Alexander matrix cannot be a knot invariant. But, if one forms an equivalence relation on them where two Alexander matrices are equivalent if and only if they produce the same presentations of a module, then taking the equivalence classes gives a knot invariant. The Alexander ideal exists for all polygonal knots and will be an invariant. When the Alexander polynomial exists it'll be a knot invariant up to units of Λ. The previous discussion of Seifert matrices now fits into this description.

Theorem 3.9. *For a polygonal knot $K \subset S^3$ and a Seifert matrix S for it, Alexander matrices of K are given by either*

$$S^T - tS$$

$$S - tS^T.$$

It was shown before that any classical knot has a square Seifert matrix and thus using this theorem the Alexander matrix must be square so the Alexander ideal will be principal thus must have an Alexander polynomial. This leads to the same conclusion obtained after first defining the Seifert matrix, that the Alexander polynomial of a knot is just equal to

$$det(S^T - tS).$$

The theorem also immediately leads to an interesting result.

Corollary 3.10. For any classical knot, the Alexander polynomial $\triangle(t)$ has the following properties:

$$\triangle(t) \doteq \triangle(t^{-1})$$

$$\triangle(1) = \pm 1.$$

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