MAT7500 W14 LECTURE NOTES

BRÖCKER AND JÄNICH, CHAPTER VIII: DYNAMICAL SYSTEMS

The purpose of this section is to study *flows* on manifolds. A flow is really just a one parameter group of diffeomorphisms.

Here is the rough idea: Let $\mathrm{Diff}(M)$ be the set of diffeomorphisms $M \to M$. This has the structure of a group under composition:

(1) if f and g are diffeomorphisms, then so is $f \circ g$, and composition is associative;
(2) the identity map $\mathrm{id}\colon M \to M$ is the identity element for the multiplication.
(3) if f is a diffeomorphism, then so is f^{-1}, and this is the inverse to f in the group structure.

The idea is thata flow should be a "smooth" homomorphism $\phi\colon \mathbb{R} \to \mathrm{Diff}(M)$, where $\mathbb{R}$ is the group of real numbers under addition. The problem though is that we haven't even defined a topology on $\mathrm{Diff}(M)$ so we can't even talk about *continuous* homomorphisms $\mathbb{R} \to \mathrm{Diff}(M)$.

There is an alternative way to proceed: a functiom $\phi\colon \mathbb{R} \to \mathrm{Diff}(M)$ determines, and is determined by a function $\Phi\colon \mathbb{R} \times M \to M$, where

$$\Phi(t, x) := \phi(t)(x)$$

and we can makes sense of what it means for Φ to be smooth.

Definition. Let M be a smooth manifold. A smooth map

$$\Phi\colon \mathbb{R} \times M \to M$$

is said to be a *dynamical system* or *flow* if

(1) $\Phi(0, x) = x$,
(2) $\Phi(t, \Phi(s, x)) = \Phi(t + s, x)$.

Note that these two conditions amount to the statement that the corresponding function $\phi\colon \mathbb{R} \to \mathrm{Diff}(M)$ is a homomorphism.

Notation. We let $\Phi_t\colon M \to M$ denote the diffeomorphism given by $\Phi_t(x) := \Phi(t, x)$. Then the two axioms amount to

(1) $\Phi_0 = \mathrm{id}$, and
(2) $\Phi_t \circ \Phi_s = \Phi_{s+t}$

In particular,

$$\Phi_{-t} = (\Phi_t)^{-1}$$

This last condition is called *time reversibility.*

More generally, one has the notion of non-reversible flows, which given by smooth maps

$$\Phi\colon \mathbb{R}_{\geq 0} \times M \to M$$

satisfying the above axioms. This corresponds to a *monoid homomorphism*

$$\mathbb{R}_{\geq 0} \to \mathrm{Diff}(M, M)\,.$$

Given a (reversible) flow Φ, if we choose a point $x \in M$ we obtain a smooth curve

$$\alpha_x \colon \mathbb{R} \to M$$

given by $t \mapsto \Phi_t(x)$. This is called the *integral curve* of Φ which passes through x. The image of this curve, $\alpha_x(\mathbb{R})$ is called the *orbit* of x.

This gives rise to an equivalence relation on M: two points of M are related if and only if they lie on the same orbit.

Example. If Φ is a flow on M the tangent map the flow line $\alpha_x \colon \mathbb{R} \to M$ at $t = 0$ gives a homomorphism of vector spaces

$$T_0\alpha_x \colon \mathbb{R} = T_0\mathbb{R} \to T_xM\,.$$

Let us set

$$\dot{\alpha}_x(0) := T_0\alpha_x(1)\,.$$

Then $x \mapsto T_0\alpha_x(1)$ defines a smooth vector field on M. This is called the *velocity field* of the flow.

Assertion. *A flow line $\alpha_x \colon \mathbb{R} \to M$ must satisfy exactly one of the following conditions:*

(1) *a one-to-one immersion,*
(2) *a periodic immersion (the latter means theres a p such that $\alpha_x(t+p) = \alpha_x(t)$), or*
(3) *α_x is constant (one says in this case that x is a fixed point for the flow).*

Proof. The flow condition and the chain rule imply that

$$\dot{\alpha}_x(t) = T\Phi_t\dot{\alpha}_x(0)\,.$$

Since Φ_t is a diffeomorphism, either $\dot{\alpha}_x(t) \neq 0$ for all t (in which case α_x is an immersion) or $\dot{\alpha}_x(t) = 0$ for all t which implies α_x is constant. In the former case, if α_x isn't one-to-one, then there are $p, q \in \mathbb{R}$ such that $\alpha_x(p) = \alpha_x(q)$, $p < q$ and there is no point $p < r < q$ such that $\alpha_x(p) = \alpha_x(r)$. By the flow property, we obtain $\alpha_x(t) = \alpha_x(t + q - p)$ for all t.

Example. Suppose we think of M as representing a liquid. If we fix a starting time, and we fix a molecule in the liquid at that time then we can ask how a given molecule of the liquid moves. This describes an integral curve of a dynamical system on M.

Example. There are many ways to cook up a flow. For example, suppose that $f \colon M \to M$ is a diffeomorphism. Consider the following construction: form $M \times [0, 1]$ and identify $(x, 1)$ with $(f(x), 0)$. This is called the *mapping torus* of f. Denote it by T_f. It turns out to be a smooth manifold (but we cannot yet quite show that). It will be convenient in what follows to redefine T_f as the quotient space of $M \times \mathbb{R}$ in which we identify (x, t) with $(f(x), t - 1)$ for all $t \in \mathbb{R}$.

There is a canonical follow on T_f which is induced by the operation

$$(t, (x, s)) \mapsto (x, s + t)\,,$$

where $t \in \mathbb{R}$, $(x, s) \in T_f$. This type of flow is called a *suspension flow.* For a suspension flow, every integral curve is periodic of period one.

Example. Let $X\colon \mathbb{R} \to \mathbb{R}$ be a smooth vector field on $\mathbb{R}$ and fix $(t_0, x_0) \in \mathbb{R}^2$. Consider the initial value problem
$$\frac{df(t)}{dt} = X(f(t)), \qquad f(t_0) = x_0\,.$$
Then, according to the standard existence and uniqueness results, there's a solution $f(t)$ which is defined in an interval containing t_0. Since f depends on x, we may as well write $f = f(t,x)$. And the same results tell us there's an open rectangular box around (t_0, x_0) in $\mathbb{R}^2$ such that $f(t,x)$ is defined and smooth on this box So f describes a map
$$I \times J \to \mathbb{R}$$
where I is an open interval around t_0 and J is an open interval around x_0. This map describes a flow on its domain of definition—it's what we might call a *restricted flow*, i.e., $f(s+t,x) = f(s, f(t,x))$ whenever both sides are defined.

Definition. Let M be a smooth manifold. A *local flow* on M is a smooth map
$$\Phi\colon A \to M$$
from an open subset $A \subset \mathbb{R} \times M$, where A contains $0 \times M$ such that

(1) $\Phi(0,x) = x$,
(2) $\Phi(t, \Phi(s,x)) = \Phi(t+s, x)$ whenever defined,
(3) for each $x \in M$, the set $A \cap (\mathbb{R} \times x)$ is connected.

Notation. For a local flow $\Phi\colon A \to M$ we have a flow curve $\Phi_x\colon A \cap (\mathbb{R} \times x) \to M$. We shall denote the domain $A \cap (\mathbb{R} \times x)$ by the interval (a_x, b_x).

Integrability Theorem. *Locally, any smooth vector field on M arises as the velocity field of precisely one maximal local flow. If M is compact, this flow is a global one.*

Remark. The meaning of this statement will be made precise in the proof.

A preparation for the proof we will quote the standard existence and uniqueness theorems from the theory of ODEs:

Theorem. *Let $\Omega \subset \mathbb{R}^n$ be an open domain and let $f\colon \Omega \to \mathbb{R}^n$ be smooth. Then*

(Existence): *For each $x \in \Omega$ there is an open neighborhood $W \subset \Omega$, an $\epsilon > 0$ and a smooth map*
$$\phi\colon (-\epsilon, \epsilon) \times W \to \Omega$$
such that $\phi(0,x) = x$, and
$$\dot{\phi}(t,x) = f(\phi(t,x))$$
for all $(t,x) \in (-\epsilon, \epsilon) \times W$.

(Uniqueness): *Given two smooth curves*
$$\alpha\colon (a_0, a_1) \to \Omega, \qquad \alpha\colon (b_0, b_1) \to \Omega$$

whose domains contain 0 *such that* $\alpha(0) = x = \beta(0)$*. and* $\dot{\alpha}(t) = f(\alpha(t))$*,* $\dot{\beta}(t) = f(\beta(t))$ *for all* t *whenever defined. Then*

$$\alpha(t) = \beta(t)$$

on $(a_0, a_1) \cap (b_0, b_1)$*.*

Before proving the Integrability Theorem, we first paste the above into our manifold M using a chart. Let X be a smooth vector field on M and suppose (h, U) is a smooth chart. Set $U' = h(U) \subset \mathbb{R}^n$. Then using the diffeomorphism h, we can transfer X to a smooth Euclidean vector field $f \colon U' \to \mathbb{R}^n$ using the commutative diagram

$$\begin{tikzcd}
U \arrow[r, "X_{|U}"] \arrow[d, "h"', "\cong"] & TU \arrow[d, "\cong"', "Th"] \\
U' \arrow[r, "(\mathrm{id}, f)"'] & U' \times \mathbb{R}^n = TU'
\end{tikzcd}$$

i.e., $f(y) := T_{h^{-1}(y)}h(X(h^{-1}(y)))$. Then for a smooth curve $\alpha \colon (a, b) \to U$ we have that

$$\dot{\alpha}(t) = X(\alpha(t))$$

precisely when

$$\frac{dh \circ \alpha}{dt}(t) = f(h(\alpha(t)))\,.$$

We say in this case that α is a solution curve if and only if the equation $\dot{\alpha}(t) = X(\alpha(t)$ holds for all t.

Then the existence part of the ODE theorem shows that for each $x \in M$, there's a solution curve $\alpha_x \colon (a_x, b_x) \to M$ such that $\alpha_x(0) = x$. Furthermore, by uniqueness, any two such solutions agree on there domain of intersection. And if we take the union of all intervals of all solution curves, we obtain the *maximal* solution curve.

Proof of Integrability Theorem. Set

$$A := \bigcup_{x \in M} (a_x, b_x) \times x$$

where (a_x, b_x) is the domain of the maximal solution curve α_x determined by x.

Claim: $A \subset \mathbb{R} \times M$ is open and the function

$$\Phi \colon A \to M$$

given by $\Phi(t, x) = \alpha_x(t)$ is a maximal local flow whose velocity field is the given X.

To prove the claim, it suffices to establish:

(1) $\Phi(0, x) = x, \Phi(t, \Phi(s, x)) = \Phi(t + s, x)$;
(2) A is open;
(3) Φ is smooth.

Now, (1) follows directly from the statement that $\Phi\colon (a_x, b_x) \times x \to M$ is a solution curve α_x. So for the claim, we only need to establish (2) and (3).

For $x \in M$ let

$$J_x \subset [0, \infty)$$

be the set of $t \geq 0$ such that A contains an open neighborhood of $[0, t] \times x$ on which Φ is differentiable. It will be enough to show that $J_x = [0, b_x)$ and also that the corresponding statement holds for $t \leq 0$.

The definition of J_x implies that it is an open subset of $[0, b_x)$. Therefore, we will be done if we can show that J_x is non-empty and closed in $[0, b_x)$. The standard existence theorem in this situation shows that A contains an neighborhood of $0 \times M$ on which Φ is differentiable. In particular J_x isn't empty. Here are the details: For $p \in M$ we can find a neighborhood W, an $\epsilon > 0$ and a smooth map

$$\phi\colon (-2\epsilon, 2\epsilon) \times W \to M$$

such that $\phi|(-2\epsilon, 2\epsilon) \times q\colon (-2\epsilon, 2\epsilon) \to M$ is a solution curve with initial value $q \in W$. Then the existence of ϕ implies that A contains a neighborhood of $0 \times M$ on which Φ is smooth and by uniqueness $\Phi = \phi$ on $(-2\epsilon, 2\epsilon) \times W$.

We next show that J_x is closed in $[0, b_x)$. Suppose $\tau \in \overline{J}_x$. Then we want to show $\tau \in J_x$. Set $\Phi_\tau(x) = p$. Then if U is a sufficiently small neigbhorhood of $x \in M$, we have

$$[0, \tau - \epsilon] \times U \subset A$$

and Φ is defined and differentiable on a neighborhood of this set, where ϵ is as above and we impose the additional constraint $\tau - 2\epsilon > 0$.

Set

$$U' = \Phi_{\tau-\epsilon}^{-1} \phi_{-\epsilon}(W)$$

where W is the neighborhood of $p := \Phi_\tau(x)$ chosen above. Then U' is a neighborhood of x in M. Then Φ is defined and smooth in a neighborhood of $[0, \tau + \epsilon] \times U'$ (I think that's because $\Phi_s \Phi_{\tau-\epsilon}^{-1} = \Phi_{\epsilon-\tau+s}$ is defined for $\epsilon - \tau + s \leq 2\epsilon$, i.e., for $s \leq \tau + \epsilon$). In particular Φ is defined and smooth on a neighborhood of the subset $[0, \tau] \times x$.

Consider the smooth map

$$\begin{array}{rcl} \Psi\colon (\tau - 2\epsilon, \tau + 2\epsilon) \times U' & \to & M \\ (t, u) & \mapsto & \phi_{t-\tau} \circ \Phi_\tau(u)\,. \end{array}$$

Then by the uniqueness theorem, Ψ extends the solution curves defined by Φ on $[0, t - \epsilon] \times U'$. This shows that Φ extends to a smooth map on a neighbhorhood of $(0, \tau) \times x \cup (\tau - 2\epsilon, \tau + 2\epsilon) \times U'$. This implies $\tau \in J_x$. Hence, $J_x \subset [0, b_x]$ is closed. We have just concluded the proof of the claim.

The claim shows how to construct a maximal local flow to the given vector field. And the uniqueness of this local maximal local flow is a consequence of the claim since any other one having the same vector field must be a restriction of Φ since its flow lines are solution curves to the field and Φ has the maximal solution curves as flow lines. This establishes the uniqueness part of the Integrability Theorem.

We now show that the maximal flow of a velocity field is also a global flow when the manifold is compact. By compactness, the domain A of Φ contains a subset of the form $(-\epsilon, \epsilon) \times M$. Then A also contains $(-2\epsilon, 2\epsilon) \times M$ since we can define an extension by

$$\Phi(t,x) := \Phi(\tfrac{t}{2}, \Phi(\tfrac{t}{2}, x))\,.$$

As epsilon was arbitrary, we conclude that A contains $\mathbb{R} \times M$. So $A = \mathbb{R} \times M$ and the flow is global.

Remark. The argument we gave when M is compact to produce the global flow actually delivers slightly more: If $\alpha_x \colon (a_x, b_x) \to M$ is a solution curve of a vector field on M, $b_x < \infty$, and $K \subset M$ is a compact subset, then there's an $\epsilon > 0$ such that

$$\alpha_x(t) \notin K$$

for $t > b_x - \epsilon$.

For suppose $x \in K$, then we can choose ϵ so small that $[0, \epsilon] \times K \subset A$. The our argument shows that $\mathbb{R} \times K \subset A$. And that will imply that b_x is ∞ which contradicts our hypothesis.

On the other hand suppose $x \notin K$ and $\alpha_x(t) \subset K$ for all $t \in (b_x - \epsilon, b_x)$. Then a similar argument to the one we gave when M is compact shows that $\alpha_x((a_x, b_x) \subset K$. But this will also show that $b_x = \infty$, again giving a contradiction.

The Ehresmann Fibration Theorem

Consider a smooth map $p \colon E \to M$. If $x \in M$ we let $E_x := p^{-1}(x)$. We say that p *smooth fiber bundle*[1] if for any $x \in M$, there is an open neighborhood U of x and a diffeomorphism

$$U \times E_x \xrightarrow{\phi} p^{-1}(U)$$

such that $p \circ \phi \colon U \times E_x \to U$ coincides with the first factor projection. This condition says that p is locally like a product map (where by "locally" we mean in terms of points in M.

Clearly, a smooth fiber bundle is necessarily a smooth submersion. We will be interested in the extent to which the converse holds.

Theorem. (Ehresmann). *Let $p \colon E \to M$ is a proper submersion. Then p is a smooth fiber bundle.*

Proof. Choose a chart $h \colon (\mathbb{R}^m, 0) \xrightarrow{\cong} (U, x)$ where U is an open set of M. This allows us, by taking pullbacks, to assume $M = \mathbb{R}^m$. So we have a smooth proper submersion $p \colon E \to \mathbb{R}^m$.

Let $\frac{\partial}{\partial x_i} \colon \mathbb{R}^m \to \mathbb{R}^m$ denote the smooth vector field defined by the condition

$$\langle \tfrac{\partial}{\partial x_i}(y), v \rangle = dx_i(v)$$

[1] This is sometimes called a *locally trivial fibration* in the smooth category.

where $x_i \colon \mathbb{R}^m \to \mathbb{R}$ is a i-th coordinate projection $i = 1, \ldots, m$.

Claim 1: There are smooth vector fields $X_1, \ldots X_m$ on E such that

$$Tp \circ X_i = \tfrac{\partial}{\partial x_i} \qquad i = 1, \ldots, m\,.$$

To prove the claim, note that the rank theorem implies that at each $y \in E$ with $p(y) = x \in \mathbb{R}^m$ we can find a product neighborhood W an open neighborhood U of 0, an open neighborhood V of y inside E_x and a diffeomorphism $\psi \colon U \times V \to W$ such that $p \circ \psi$ is the first factor projection. Hence, p looks like a product map locally from the point of view of its domain. Clearly, the vector field $\frac{\partial}{\partial x_i}$ restricted to U lifts to $U \times V$ since $T_y(U \times V) = T_x U \oplus T_y E_x$. Using the diffeormorphism we see that $\frac{\partial}{\partial x_i}$ lifts to the neighborhood W of y. Doing this for every y in E and appealing to a partition of unity yields Claim 1.

Given Claim 1, we can appeal the the Integrability Theorem to find local flows $\Phi^1, \ldots \Phi^m$ whose associated velocity fields yield $X_1, \ldots X_m$.

Recall that $M = \mathbb{R}^m$ and $E_0 = p^{-1}(0)$. Then we define

$$\phi \colon \mathbb{R}^n \times E_0 \to p^{-1}(U)$$

by

$$((u_1, \ldots, u_m), y) \mapsto \Phi^1_{u_1} \circ \cdots \circ \Phi^m_{u_m}(y)$$

To complete the proof it suffices to establish the following.

Claim 2: The map ϕ is well-defined and a diffeomorphism. Moreover, $p \circ \phi$ is the first factor projection.

We first establish the last part. First observe

$$p \circ \Phi^i_{u_i}(y) = p(y) + u_i e_i$$

where e_i is the i-th standard basis vector of $\mathbb{R}^n$. To verify we think of both sides as a function of a single variable u_i. Note first that both sides agree by definition when $u_i = 0$. If we differentiate both sides with respect to u_i we see that the derivative both sides is e_i (since the vector field X_i lifts $\frac{\partial}{\partial x_i}$). Hence the two functions agree everywhere whenever they are defined.

Let $p_1 \colon \mathbb{R}^n \times E_0 \to \mathbb{R}^n$ be the projection. The last displayed equation implies that

$$p_1(u, y) = u = p \circ \phi(u, y)$$

because

$$\begin{aligned} p \circ \phi(u, y) &= p \circ \Phi^1_{u_1}(\Phi^2_{u_2} \cdots \circ \Phi^m_{u_m}(y)), \\ &= p(\Phi^2_{u_2} \cdots \circ \Phi^m_{u_m}(y)) + u_1 e_1, \\ &\vdots \\ &= p(y) + \sum_i u_i e_i, \\ &= u, \qquad \text{since } y \in E_0\,. \end{aligned}$$

This argument establishes the last sentence of Claim 2.

We now turn to the first sentence. Then we have that $p^{-1}(B_K) \subset E$ is compact where B_K is the closed ball of radius K (this uses the properness of p). The way ϕ is defined shows for all $u \in B_K$ all flow lines remain in $p^{-1}(B_K)$ (since the flows in E project to standard coordinate axis flows in $\mathbb{R}^n$). This shows that all the flow maps exist in the definition of ϕ.

Finally, it is clear that ϕ is smooth. The inverse smooth map for ϕ is given by the formula

$$\phi^{-1}(z) = (u, \Phi^n_{-u_m} \circ \cdots \circ \Phi^1_{-u_1}(z))$$

where $u := p(z)$. $\square$

Example. Consider the first factor projection $p\colon \mathbb{R}^2 \setminus 0 \to \mathbb{R}$. This is a (non-proper!) submersion, but it is not a smooth fiber bundle.

For if it were, there would exist an interval (a, b) containing 0 such that $(a, b) \times (\mathbb{R} \setminus 0)$ is diffeomorphic to $p^{-1}(a, b)$. It is easy to see that this isn't the case $(a, b) \times (\mathbb{R} \setminus 0)$ has two connected components whereas $p^{-1}(a, b)$ is connected.