MAT7500 W14 Lecture Notes

Bröcker and Jänich, Chapter VII: Embedding

The goal of this chapter is two-fold: to show every smooth m-manifold

(1) immerses in $\mathbb{R}^n$ when $n \geq 2m$ and
(2) embeds in $\mathbb{R}^n$ when $n \geq 2m+1$.

In fact, we will establish a good deal more: we will show that if $f\colon M \to \mathbb{R}^n$ is an smooth map, then there are immersions which are *arbitrarily close* to f in a suitable topology on $C^\infty(M;\mathbb{R}^n)$ (when $n \geq 2m$). A similar statement for approximating maps into $\mathbb{R}^n$ by ~~embeddings~~ **one-to-one immersions** also holds for $n \geq 2m+1$ (recall that if M is compact a one-to-one immersion is an embedding).

Digression: the jet space. Suppose M^m and N^n are smooth manifolds and choose points $x \in M, y \in N$. Consider smooth maps $f\colon M \to N$ such that $f(x) = y$. Define an equivalence relation on such maps by $f \sim g$ if and only if $T_x f = T_x g$. Let $J^1(M,N)_{x,y}$ be the set of such equivalence classes We usually denote an equivalence class by $[f]$, but if confusion arises we sometimes write $[f]_x$. . Set

$$J(M,N) := \bigcup_{(x,y)\in M\times N} J^1(M,N)_{x,y}\,.$$

Notice that a map $f\colon M \to N$ gives rise to a function

$$j^1 f\colon M \to J^1(M,N)$$

defined by $x \mapsto [f]_x$.

Lemma. *$J^1(M,N)$ has the structure of a smooth manifold of dimension $m+n+mn$.*

Proof. Let $U \subset M$ and $V \subset N$ be open. Then $J^1(U,V) \subset J^1(M,N)$, as U and V vary, define a subbasis for a topology.

If we choose U and V to charts, we will show that $J^1(U,V)$ is also a chart. Without loss in generality, we may assume $U \cong \mathbb{R}^m$ and $V \cong \mathbb{R}^n$. Then $J^1(U,V)$ is in bijection with $J^1(\mathbb{R}^m,\mathbb{R}^n)$. Define a function

$$J^1(\mathbb{R}^m,\mathbb{R}^n) \to \mathbb{R}^m \times \mathbb{R}^n \times \hom(\mathbb{R}^m,\mathbb{R}^n)$$

by

$$[f]_x \mapsto (x, f(x), T_x f)\,.$$

This is clearly well-defined and is a bijection. To complete the argument we need to know how this behaves with respect to transition maps. I will omit this part of the argument (it's not very difficult). $\square$

Consider the projection

$$J^1(M,N) \to M \times N$$

given by $[f] \mapsto (x, y)$ where $[f] \in J^1(M,N)_{x,y}$. It is also not a stretch of the imagination to conclude from the above argument that this gives a *vector bundle* over $M \times N$ of rank mn. However, we will not need to know this fact.

Notice that $f \in C^\infty(M,N)$ defines a continuous map

$$j^1 f \colon M \to J^1(M,N)$$

by sending x to $[f]_x$.

The C^1-topology. Given an open subset $V \subset J^1(M,N)$, let

$$M(V) := \{f \in C^\infty(M,N) |\, \mathrm{image}(j^1 f) \subset V\}\,.$$

Note that $M(V \cap W) = M(V) \cap M(W)$.

Definition. The Whitney C^1-topology on $C^\infty(M,N)$ is the topology having basis $\{M(V)\}_V$ where $V \subset J^1(M,N)$ varies throughout the open subsets.

In order to understand this definition, let's try to make it more concrete. Since manifolds are metrizable, let's choose a metric space structure d^1 on $J^1(M,N)$ compatible with its topology. Let $\delta\colon M \to (0,\infty)$ be any continuous map. Define

$$B_\delta(f) : \{g \in C^\infty(M,N) |\, d(j^1 f(x), j^1 g(x)) < \delta(x)\}$$

Then $B_\delta(f)$ is an open set and as δ varies, we obtain a neighborhood basis on f in the Whitney C^1-topology.

Intuitively, $d(j^1 f(x), j^1 g(x)) < \delta(x)$ means that $f(x)$ and $g(x)$ are "close" and moreover, $T_x f$ and $T_x g$ are "close."

Example. Consider the case and $N = \mathbb{R}^n$ and $M = U \subset \mathbb{R}^m$ is an open set. Then we have a canonical diffeomorphism

$$J^1(U,\mathbb{R}) \cong U \times \mathbb{R}^n \times \hom(\mathbb{R}^m, \mathbb{R}^n) \subset \mathbb{R}^{m+n+mn}\,,$$

And we can equip $J^1(U,\mathbb{R})$ with the max norm coming from $\mathbb{R}^{m+n+mn}$.

For example let $m = 1 = n$. Then $B_\delta(f)$ consists of the smooth functions $g\colon U \to \mathbb{R}$ such that

$$|f(x) - g(x)| + |f'(x) - g'(x)| < \delta(x)$$

for all $x \in U$. So f and g are "close" in the C^1-topology if and only if there are pointwise as well as in their derivatives "close" in a way controlled by δ.

The Whitney Immersion Theorem.

Let $I(M^m, \mathbb{R}^n) \subset C^\infty(M, \mathbb{R}^n)$ consist of all immersions $M \to \mathbb{R}^n$.

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[1] Urysohn's Metrization Theorem says that a topological space is separable and metrizable if and only if it is regular, Hausdorff and second-countable. So any manifold is metrizable.

Theorem. *Assume $n \geq 2m$. Then $I(M^m, \mathbb{R}^n) \subset C^\infty(M, \mathbb{R}^n)$ is dense in the C^1-topology.*

As a first step, we need to analyse certain subspaces of the space of matrices. Let V and W be vector spaces of dimensions m and n respectively, where $m < n$. If $S\colon V \to W$ is a linear transformation, we define

$$\operatorname{corank}(S) := m - \operatorname{rank}(S)\,.$$

Let $L^r(V, W) \subset \hom(V, W)$ be the subset of those S such that $\operatorname{corank}(S) = r$.

Proposition. *$L^r(V, W) \subset \hom(V, W)$ is a submanifold of codimension $(n - m + r)r$.*

Proof. Let $S \in L^r(V, W)$ and set $k = m - r = \operatorname{rank}(S)$. Choose bases of V and W so that the matrix of

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is such that A is a $k \times k$ invertible matrix. Choose an open neighborhood U of S in $\hom(V, W)$ such that for all S' in U,

$$S' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$$

is such that A' is invertible $k \times k$. The neighbhorhood U exists because the $\det A \neq 0$ implies $\det A' \neq 0$ for all matrices sufficiently close to A. Now consider the smooth map

$$f\colon U \to \hom(\mathbb{R}^{m-k}, \mathbb{R}^{n-k})$$

given by $f(U) = D' - C'(A')^{-1}B'$. If we fix A, B, C, then the map

$$g\colon\ \hom(\mathbb{R}^{m-k}, \mathbb{R}^{n-k}) \to \hom(\mathbb{R}^{m-k}, \mathbb{R}^{n-k})$$

given by $g(D) = f(S) = D - CA^{-1}B$ is a diffeomorphism. In particular, $Tg_D = Tf_S$ is surjective, so the map f is a submersion. Then

$$f^{-1}(0) = L^r(V, W) \cap U$$

is a submanifold of codimension $(n - k)(m - k)$. $\square$

Proposition. *Let $S_r \subset J^1(M, N)$ be the subset of $[f] \in J^1(M, N)$, $[f]\colon T_xM \to T_yN$ say, such that* $\operatorname{corank}(f) = r$. *Then S_r is a submanifold of codimension $r(n - m + r)$.*

Proof. It's enough to consider the case $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$. Then $J^1(\mathbb{R}^m, \mathbb{R}^n) \cong \mathbb{R}^m \times \mathbb{R}^n \times \hom(\mathbb{R}^m, \mathbb{R}^n)$. With respect to this diffeomorphism, we have $S_r \cong \mathbb{R}^m \times \mathbb{R}^n \times L^r(\mathbb{R}^m, \mathbb{R}^m)$. The previous proposition then gives the result. $\square$

As preparation for the proof for the Whitney Immersion Theorem we first establish the result in the following special case:

Proposition. *Let $f: U \to \mathbb{R}^n$ be a smooth map, $U \subset \mathbb{R}^m$ open and $n \geq 2m$. Let $\epsilon > 0$. Then there is an immersion $g: U \to \mathbb{R}^n$ such that $||f - g||_{C^1} < \epsilon$.*

Proof. A smooth map $f: U \to \mathbb{R}^n$ is an immersion iff an only if the map

$$U \to \hom(\mathbb{R}^m, \mathbb{R}^n) = \mathbb{R}^{mn}$$

given by $x \mapsto T_x f$ has image disjoint from $L^r(\mathbb{R}^m, \mathbb{R}^n)$ for $r = 1, 2, \ldots$. But the codimension of $L^r(\mathbb{R}^m, \mathbb{R}^n) \subset \hom(\mathbb{R}^m, \mathbb{R}^n)$ is $\geq m + 1$.

Hence for almost all $A \in \hom(\mathbb{R}^m \mathbb{R}^n)$, the the map given by $x \mapsto T_x f + A$ misses $L^r(\mathbb{R}^m, \mathbb{R}^n)$ for $r \geq 1$. Let $g: U \to \mathbb{R}^n$ be given by $g(x) = f(x) + Ax$. Then $Tg_x = T_x f + A$. $\square$

The above statement can be *localized.* We need only consider a special case. Let $B_s \subset \mathbb{R}^m$ be the ball of radius s centered at the origin.

Addendum. *With $U = B_3$, assume that $f \in C^\infty(B_3, \mathbb{R}^n)$. Then there is an $h \in C^\infty(B_3, \mathbb{R}^n)$ which is an immersion on B_1 such that $||h - f||_{C^1} < \epsilon$ and $h = f$ on $B_3 \setminus B_2$.*

Proof. Let g be as in the proposition. Choose a smooth map $\phi: B_3 \to [0, 1] \subset \mathbb{R}$ such that $\phi \equiv 1$ on B_1 and $\phi \equiv 0$ off B_2 (see the next Lemma for some details) Now define

$$h(x) = f(x) + \phi(x)(g(x) - f(x)) . \quad \square$$

Support Lemma. *There is a smooth function $\phi: \mathbb{R}^m \to \mathbb{R}$ such that $\phi \equiv 1$ on B_a, $\phi \equiv 0$ on $\mathbb{R}^m \setminus B_b$, and $\phi(x) \in (0, 1)$ for $a < ||x|| < b$.*

Proof. consider the function $f: \mathbb{R}^1 \to \mathbb{R}^1$ given by

$$f(x) = \begin{cases} e^{-1/x^2}, & x > 0 \\ 0 & x \leq 0. \end{cases}$$

Then f is smooth. Consider the function

$$g(x) = f(x - a) f(b - x) .$$

Then $g(x)$ smooth, is positive on (a, b), and is zero elsewhere.

Next, consider the function

$$h(x) = \frac{\int_{-\infty}^{x} g \, dx}{\int_{-\infty}^{\infty} g \, dx}$$

Then h is smooth, $h(x) = 0$ for $x < a$ and $h(x) = 1$ for $x > b$. Furthermore, $0 < h(x) < 1$ for $x \in (a, b)$.

Finally, define the function$\phi: \mathbb{R}^m \to \mathbb{R}$ by

$$\phi(x) = 1 - h(||x||) .$$

Then ϕ is smooth, and $\phi(x) = 1$ for $x \in B_a$, $\phi(x) = 0$ for $x \notin B_b$, and $0 < \phi(x) < 1$ for $a < ||x|| < b$. $\square$

We also need to relativize the addendum:

Proposition. *In the setting of previous proposition, assume $F \subset B_3$ is closed and $f_{|F}$ is already an immersion. Then we may find $h \in C^r(B_3, \mathbb{R}^n)$ which is an immersion on $\bar{B}_1 \cup F$, where $||h - f||_{C^1} < \epsilon$ and $h = f$ on $F \cup (B_3 \setminus B_2)$.*

Proof. (Sketch). Set $K = F \cap \bar{B}_2$. Then K is compact in B_3 and it suffices to prove the statement for K in place of F. Let $V \subset B_3$ be an open neighbhorhood of K whose closure is compact. Since being an immersion is an open condition, we can chose V so that f is an immersion on $\bar{V}$. Then

$$\bar{B}_1 \cup K \subset (\bar{B}_1 \setminus V) \cup \bar{V} .$$

Let $\xi \colon : B_3 \to [0,1]$ be a smooth function such that $\phi \equiv 1$ on $\bar{B}_1 \setminus V$ and $\xi \equiv 0$ on $K \cup (\bar{B}_3 \setminus B_2)$. Set

$$h = f + \xi(g - f) .$$

Then h satisfies the conclusion. □

As a final preparation will need to find a sufficiently nice atlas for M.

Lemma. *There exists a locally finite smooth atlas $\{(h_i, U_i)\}_{i\geq 1}$ for M such that*

(1) $U_i = h_i^{-1}(B_3)$;
(2) $U_i^{(1)} = h_i^{-1}(B_1)$ *is a covering of M.*

Proof. Every point $x \in M$ has a chart $h \colon (U, x) \to (\mathbb{R}^m, 0)$ whose image is B_3. If we define $U^{(1)} := h^{-1}(B_1)$ then $(h, U^{(1)})$ is still a chart at x. Assume first that M is compact. Then we can choose such a chart for each $x \in M$, and then appeal to compactness to select finitely many such charts. This gives the result when M is compact.

In the general case, we can choose a countable covering $\{C_i\}$ of M by compact subsets (local compactness guarantees this). Set $A_1 := C_1$, and inductively define A_j as a compact neighborhood of $A_{j-1} \cup C_j$. Then $A_j \subset \operatorname{int}(A_{j+1})$ and $\cup_i A_i = M$.

For each i, choose finitely many charts $h_{ik} \colon U_{ik} \to B_3$ for $1 \leq k \leq s$ (where s depends on i), such that $U_{ik} \subset \operatorname{int}(A_{i+2}) - A_{i-1}$. If we set $U^1_{ik} := h_{ik}^{-1}(B_1)$, then we can arrange it so that U^1_{ik} forms an open covering of $A_{i+1} \setminus \operatorname{int}(A_i)$. This is because, $A_{i+1} \setminus \operatorname{int}(A_i)$ is compact and has $\operatorname{int}(A_{i+2}) \setminus A_{i-1}$ as an open neighborhood. The proof is completed by reindexing. □

Proof of the Whitney Immersion Theorem. For $j \geq 1$, let

$$M_j = \bigcup_{i=1}^{j} U_i^{(1)} ,$$

where $U_i^{(1)}$ is as in the Lemma.

then $M_j \subset M$ is an open set and therefore a submanifold. Given f, we can assume by induction that it has been modified to a C^∞-map $f_j \colon M \to \mathbb{R}^n$ so that f_j is an immersion on $\bar{M}_j$ and

$$||f - f_i||_{C^1} \leq \tfrac{\epsilon}{2^i}$$

as functions on $U_i^{(1)}$ for $i = 1, \dots, n$. Set $U := U_{j+1} = U_{j+1}^{(3)}$. We will show how to modify f_j to f_{j+1} with $f_{j+1} = f_j$ off $U^{(2)}$.

By the last proposition, there's an $h \in C^\infty(U^{(3)}, \mathbb{R}^n)$ such that

(1) h is an immersion on $\bar{U}^{(1)} \cup (\bar{M}_j \cap \bar{U}^{(3)})$ (since $h = f_j$ on $\bar{M}_j \cap \bar{U}^{(3)}$);
(2) $h = f_j$ on $U^{(3)} \setminus U^{(2)}$;
(3) $||h - f_j||_{C^1} < \frac{\epsilon}{2^{j+1}}$ on $U^{(1)}$.

Define $f_{j+1} \in C^\infty(M, \mathbb{R}^n)$ by setting $f_{j+1} = h$ on $U^{(3)}$ and $f_{j+1} = f_j$ on $M \setminus U^{(2)}$. Then f_{j+1} is an immersion on $M_{j+1} = M_j \cup U^{(1)}$ and $||f_{j+1} - f_j|_{C^1} < \frac{\epsilon}{2^{j+1}}$ as functions on on $U^{(1)}$.

Repeating this construction gives defines a sequence $f_1, f_2, \dots$. We set

$$g(p) = \lim_j f_j(p)\,.$$

This is a pointwise limit, but at each $x \in M$ it's a limit of a finite sequence, by the local finiteness property. This implies the limit is C^∞-uniform on compact sets in M and g is smooth as well as an immersion. By construction $||f - g||_{C^1} < \epsilon$. $\square$.

One-to-One Immersions

A one-to-one immersion is not necessarily an embedding—we saw examples a while back. Here is another example:

Example. Let $f\colon \mathbb{R} \amalg \mathbb{R} \to \mathbb{R}^2$ be defined on the first summand by $x \mapsto (x, 0)$ and on the second by $y \mapsto (0, e^y)$. Then f is a one-to-one immersion but not an embedding, since the image of f is the union of the x axis with the positive y axis. This is clearly not a submanifold.

Theorem. *Let $f\colon M^m \to \mathbb{R}^n$ be a smooth map $n \geq 2m + 1$. Then for f can be arbitrarily approximated by a one-to-one immersion $g\colon M \to \mathbb{R}^n$. In other words, the one-to-one immersions are dense in all smooth maps.*

Proof. As usual, the "arbitary approximation" of the theorem means we are choosing a map $\delta\colon M \to \mathbb{R}$. By the Immersion Theorem, we can assume f is an immersion. By the Rank Theorem, f is a local embedding.

Choose a countable open covering $\{U_i\}$ of M such that $f\colon U_\alpha \to \mathbb{R}^n$ is an embedding for all α. We can assume that this covering is an atlas of the kind we used in the proof of the Immersion Theorem. Let $\phi_i\colon M \to \mathbb{R}$ be a support function for $U_i = U_i^{(3)}$, where ϕ_i is supported on $U_i^{(2)}$. Inductively define a sequence of immersions $g_i\colon M \to \mathbb{R}^n$ where $g_0 = f$ and

$$g_j(x) = g_{j-1}(x) + \phi_j(x) \cdot b_j$$

where $b_j \in \mathbb{R}^n$ is has sufficiently small norm (this guarantees that g_j is an immersion). In fact, we may choose b_j so that

$$||g_j(x) - g_{j-1}(x)|| < \frac{\delta(x)}{2^j}$$

This will guarantee that all the g_j as wekk as $g := \lim_j g_j$ are all immersions that lie in a prescribed neighborhood of f.

Let $N \subset M \times M$ be the set of points (x, y) such that $\phi_j(x) \neq \phi_j(y)$. Then N is an open set, so it's a manifold of dimension $2m$. We have a smooth map

$$G\colon N \to \mathbb{R}^n$$

given by

$$(x, y) \mapsto \frac{-((g_{j-1}(x) - g_{j-1}(y))}{\phi_j(x) - \phi_j(y)}$$

Since $2m < n$, Sard's theorem implies the image of G (which consists entirely of critical values) has measure zero. So we can choose b_j to not be in this image. Then

$$g_j(x) = g_j(y) \Longleftrightarrow g_{j-1}(x) - g_{j-1}(y) = -(\phi_j(x) - \phi_j(y))b_j\,.$$

Since b_j is a regular value for G, this can only happen if $(x, y) \notin N$. Hence,

$$g_j(x) = g_j(y) \Longleftrightarrow \phi_j(x) = \phi_j(y)\,,$$

and therefore $g_{j-1}(x) = g_{j-1}(y)$.

Suppose $g(x) = g(y)$. Then since $g(x) = g_j(x)$ for large j, it follows that $g_j(x) = g_j(y)$ for large j, and therefore $\phi_{j-1}(x) = \phi_{j-1}(y)$. By downward induction we get

$$\phi_j(x) = \phi_j(y) \quad \text{and} \quad g_j(x) = g_j(y) \quad \text{for all } j \geq 0.$$

In particular, it follows that $f(x) = f(y)$ so x and y lie in different chart domains (since f is an embedding on each U_j). If for example $x \in U_j^{(1)}$ and $y \notin U_j$ then

$$\phi_j(x) = 1 \neq 0 = \phi_j(y)$$

from which we get a contradiction. □

Embeddings

Let $E(M, \mathbb{R}^n) \subset C^\infty(M, \mathbb{R}^n)$ be the subset given by all smooth embeddings $M \to \mathbb{R}^n$. Assume $n \geq 2n + 1$. Then it's *not necessarily true* that this subset is dense. In the compact case it is true, so to construct a counterexample, we have to use non-compact manifolds. The text gives a sequence of exercises how to construct such a counterexample (Chap. 7, Ex. 8). However, I do not know how to do these exercises!

However, we do have the following:

Whitney Embedding Theorem. *Any smooth m-manifold embeds in $\mathbb{R}^n$ if $n \geq 2m + 1$.*

The proof of this result is based on the following fact which we prove later:

Lemma. (Existence of Proper Maps). *On any smooth manifold M there is a proper map $\rho\colon M \to \mathbb{R}$.*

Proof of Whitney Embedding Theorem. Since there is a proper map $\rho\colon M \to \mathbb{R}$, there is also a proper map $f\colon M \to \mathbb{R}^n$ (for example $(\rho, \rho, \ldots, \rho)$). Approximate this by a one-to-one immersion $g\colon M \to \mathbb{R}^n$ so that $||g - f||_{C^1} \leq 1$.

Let $K \subset \mathbb{R}^n$ be compact. Then $K \subset \bar{B}_s$ for s sufficiently large. Then $g^{-1}(K) \subset f^{-1}(\bar{B}_{s+1})$ is an inclusion of a closed subset of a compact space. Hence $g^{-1}(K)$ is compact. So g is a proper one-to-one immersion. But this implies that g is a smooth embedding by the Corollary to the Closed Map Lemma (cf. below).

The proof of the Proper Map Lemma is based on the existence of a *smooth partition of unity* which we now describe, but not prove.

Lemma. (Existence of Smooth Partition of Unity). *Let M be a smooth manifold equipped with open covering $\mathcal{U} := \{U_\alpha\}$. Then there exists a partition of unity $\{\psi_i\}$ subordinate to $\mathcal{U}$, such that $\psi_i\colon M \to [0, 1]$ is smooth for all i.*

Proof of Proper Map Lemma. Let $\mathcal{U} := \{U_\alpha\}$ be an open cover of M such that $\bar{U}_\alpha$ is compact. Let $\{\psi_i\}$ be a smooth partition of unity subordinate to $\mathcal{U}$, Set

$$\rho = \sum_{k=1}^{\infty} k\psi_k$$

Then ρ is smooth. If $\rho(x) \leq j$ then at least one of the first j-functions $\psi_1, \ldots \psi_j$ must be non-zero at x, Hence $\rho^{-1}([-j, j])$ is a subset of

$$\bigcup_{i=1}^{j} \{x | \psi_i(x) \neq 0\} = \bigcup_{i=1}^{j} \mathrm{supp}(\psi_i)$$

But $\mathrm{supp}(\psi_i)$ is a closed set in one of the $\bar{U}_\alpha$ (by the definition of partition of unity) so it is compact. Hence the displayed set is also compact. Also $\rho^{-1}([-j, j])$, which is a closed subset of a compact set is also compact. But every compact subset $K \subset \mathbb{R}$ is contained in some $[-j, j]$. So the closed set $\rho^{-1}(K)$ is a closed subset of the compact set $\rho^{-1}([-j, j])$. Hence $\rho^{-1}(K)$ is also compact. $\square$

Recall that a space Z is *locally compact* if every point $x \in Z$ is contained in a compact set C which contains an open set that contains x. Observe that if Z is Hausdorff and locally compact, then the condition amounts to the existence of an open set U around x so that its closure is compact (one calls U *precompact* in this instance). Note too that any smooth manifold is locally compact (indeed, if $x \in M$ and $h\colon (U, x) \to (U', 0)$ is a diffeomorphism, where U is a neighborhood of x and U' is a neighborhood of 0 in $\mathbb{R}^m$, then $V := h^{-1}(B_s)$ for s sufficiently small is a neighborhood of x having compact closure).

Closed Map Lemma. *Let $f\colon X \to Y$ be a proper continuous map, where Y is Hausdorff and locally compact. Then f is a closed map, i.e., the image of any closed set is closed.*

Proof. Let $K \subset X$ be closed. It's enough to show that $f(K)$ is closed. Let $y \in Y$ be a limit point for $f(K)$. Let U be a neighborhood of y such that the closure of U is compact and such that y is a limit point for $f(K) \cap \bar{U}$. As f is proper, $f^{-1}(\bar{U})$ is compact. Consequently, $K \cap f^{-1}(\bar{U})$ is also compact. Since f is continuous, $f(K \cap f^{-1}(\bar{U})) = f(K) \cap \bar{U}$ is also compact hence closed. Therefore $y \in f(K) \cap \bar{U} \subset f(K)$. So $f(K)$ is closed. $\square$

Corollary. *If $f\colon M \to N$ is a proper one-to-one immersion, then f is an embedding.*

Proof. By the Closed Map Lemma, $f\colon M \to N$ is closed map and $f\colon M \to f(M)$ is a continuous bijection. Hence f is a homeomorphism and the notes from Chapter 5 imply that f is also an embedding.

The Limit Set. The following criterion may be helpful in distinguishing between one-to-one immersions and embeddings into euclidean space.

The *limit set* of a map $f\colon M^m \to \mathbb{R}^n$ is

$$L(f) := \{y \in \mathbb{R}^n \,|\, y = \lim_{k\to\infty} f(x_k), \text{ where } \{x_k\} \text{ is a sequence in } M \text{ which having no limit point}\}$$

Lemma. *(1). If $f\colon M \to \mathbb{R}^p$ is smooth, then $f(M)$ is closed if and only if $L(f) \subset f(M)$.*
(2). A one-to-one immersion $f\colon M \to \mathbb{R}^n$ is an embedding if and only if $f(M) \cap L(f) = \emptyset$.

Proof. (1). Suppose $f(M)$ is closed, and let $y \in L(f)$. Then $y = \lim_i f(x_i)$. But $f(x_i) \in f(M)$. Therefore $\lim_i f(x_i) \in f(M)$ since $f(M)$ is closed.

Conversely, suppose $L(f) \subset f(M)$. Let y be a point in the closure of $f(M)$. Then for each n there is an $x_n \in M$ such that $f(x_n)$ lies in the ball of radius $1/n$ centered at y. Set $x = \lim_i x_i$ if the limit exists. Then

$$f(x) = \lim_i f(x_i) = y$$

which implies that $y \in f(M)$. If the limit $\lim_i x_i$ doesn't exist, then $y \in L(f) \subset f(M)$. Hence, in either case $y \in f(M)$, so $f(M)$ is closed.

(2). Suppose $f(M) \cap L(f) = \emptyset$. Let $C \subset M$ be a closed set. If $f(C)$ isn't closed, then there's a point $y \notin f(C)$ such that y is a limit point of $f(C)$ in $f(M)$. Hence, $y \in L(f) \cap f(M)$, so we obtain a contradiction.

Conversely, suppose f is an embedding $y \in f(M) \cap L(f)$ then $y = f(x)$. Write $y = \lim_i f(x_i)$, where the sequence $\{x_i\}$ has no limit point. Since $M \to f(M)$ is a diffeomorphism it follows that $x = \lim_i x_i$, giving a contradiction. $\square$

Example. We saw in Chapter I that the figure eight curve $f\colon (-1,1) \to \mathbb{R}^2$, given by

$$f(t) = (\sin \pi t, \sin 2\pi t)$$

is a one-to-one immersion. The limit point set $L(f)$ is given by $f(0) = (0,0)$. Consequently, $f((-1,1)) \cap L(f) \neq \emptyset$, so f is not an embedding.

Example. According to an exercise in the book (we will try to solve it in class) there is a smooth map $f\colon \mathbb{R} \to \mathbb{R}^3$ such that

$$\mathbb{Q}^3 \subset f([n,\infty))$$

for $n = 0, 1, \dots$. Then $\mathbb{Q}^3 \subset L(f) \cap f(\mathbb{R})$.

Let $\epsilon\colon \mathbb{R} \to (0,\infty)$ be a continuous map which tends to 0 when $t \to \pm\infty$. Suppose there were an embedding $g\colon \mathbb{R} \to \mathbb{R}^3$ such that $||f - g||_{C^1} < \epsilon$. Then it is straightforward to check that $L(g) = L(f)$ so $\mathbb{Q}^3 \subset L(g)$. Moreover, since g is an embedding we have $L(g) \cap g(\mathbb{R}^3) = \emptyset$. We infer that

$$\mathbb{R}^3 \setminus \mathbb{Q}^3 \subset g(\mathbb{R}) \subset \mathbb{R}^3$$

This implies $g(\mathbb{R}) \subset \mathbb{R}^3$ is everywhere dense. But this will contradict the submanifold property, since the latter inclusion corresponds locally to the standard inclusion of $\mathbb{R} \subset \mathbb{R}^3$ and the latter inclusion is nowhere dense. So we obtain a contradiction. We conclude that there's no embedding g that is ϵ-close to f in the C^1-topology on $C^\infty(\mathbb{R}, \mathbb{R}^3)$.

Hence, the embeddings do not form a dense subset, even though we are in the Whitney range $3 \geq 2 \cdot 1 + 1$. This should be contrasted with the statement that one-to-one immersions do form a dense subset. This example shows that there's a great deal of difference in the behavior of smooth embeddings as compared with one-to-one immersions.