Bröcker and Jänich: Chapter V: Local and Tangential Properties

Motivation. Given a smooth map germ $f\colon (M^m, p) \to (N^n, q)$, it is reasonable to ask what it "looks like" in local coordinates. What is the simplest expression we can give for it?

Inverse Function Theorem. This theorem says a smooth map germ $f\colon (M^m, p) \to (N^n, f(p))$ is a local diffeomorphism at p if and only if the differential $T_p f\colon T_p M \to T_{f(p)} N$ is an isomorphism (of course, it is necessarily the case that $m = n$ for this to be possible).

The Rank Theorem for Maps. The *rank* of f at p, written $\mathrm{rk}_p f$, is the rank of the linear transformation $T_p f$, that is, the dimension of the image of $T_p f$.

Lemma. *The rank is locally non-increasing, i.e., there is a neighborhood U of p such that $\mathrm{rk}_x f \geq \mathrm{rk}_p f$ for all $x \in U$. That is, the rank function of f is lower semi-continuous.*

Proof. Suppose $\mathrm{rk}_p f = r$. We can assume that $f\colon (\mathbb{R}^m, p) \to (\mathbb{R}^m, q)$. Then $T_p f$ is identified with the Jacobian matrix $J_p f$. By rearranging rows an columns if necessary, we can assume that the $r \times r$ submatrix given by the first r rows and first r columns is invertible. Then as in the proof that we gave of the Rank Theorem for vector bundle maps, we can write $S := J_p f$ as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and since we are assuming $J_p f$ has rank r, if follows that A has rank r and $D = CA^{-1}B$. Recall the invertible matrix

$$T = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}$$

is such that

$$TS = \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}$$

Now, considering the matrices A, B, C and D above as matrices of functions defined in a sufficiently small neighborhood of $p \in M$ (i.e., $A = A(x)$, etc.), it follows that the matrix of functions TS (which has the same rank as S) has rank given by

$$\mathrm{rk}A + \mathrm{rk}(D - CA^{-1}B) = r + \mathrm{rk}(D - CA^{-1}B) \geq r\,. \quad \square$$

Example. Consider $f\colon (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ given by $f(x) = x^2$. Then $\mathrm{rk}_0 f = 0$ but $\mathrm{rk}_x f = 0$ for $x \neq 0$. This shows that the rank can change in an arbitrarily small neighborhood.

If however the rank doesn't change in a small neighborhood, then we have the following description.

Rank Theorem for Maps. *If $f\colon (M^m,p)\to(N^n,q)$ is a smooth map germ such that $\mathrm{rk}_x f = r$, for all x sufficiently near p, then there are chart germs $\phi\colon (M,p)\to(\mathbb{R}^m,0)$ and $\psi\colon (N,q)\to(\mathbb{R}^n,0)$ such that the following diagram commutes*

$$\begin{tikzcd}
(M,p) \arrow[r, "f"] \arrow[d, "\phi"'] & (N,q) \arrow[d, "\psi"] \\
(\mathbb{R}^m,0) \arrow[r, "P_r"'] & (\mathbb{R}^n,0)
\end{tikzcd}$$

in which $P_r(x_1,\dots,x_m)=(x_1,\dots,x_r,0,\dots,0)$ is the inclusion of the projection onto the first r coordinates.

Proof. Without any loss in generality, we can assume at the outset that $f\colon(\mathbb{R}^m,0)\to(\mathbb{R}^n,0)$. Consider the map germ $h\colon(\mathbb{R}^m,0)\to(\mathbb{R}^m,0)$ given by

$$(x_1,\dots,x_m)\mapsto(f_1(x),\dots f_r(x),x_{r+1},\dots,x_m)\,.$$

It is easy to check that dh_0 is invertible, so h is a local diffeomorphism near 0. Consider then the map germ

$$f\circ h^{-1}\colon(\mathbb{R}^m,0)\to(\mathbb{R}^n,0)$$

An elementary calculation shows that the latter has the form

$$(x_1,\dots x_m)\mapsto(x_1,\dots,x_r,g_{r+1}(x),g_{r+2}(x),\dots,g_n(x))\,.$$

where $g_i\colon(\mathbb{R}^m,0)\to(\mathbb{R},0)$ are smooth function germs. The Jacobian of the latter at any point x near the origin is a matrix of the form

$$\begin{pmatrix} I_r & 0 \\ ? & A(x)\end{pmatrix}$$

where $A(x)$ is the $(m-r)\times r$ matrix given by $(\frac{\partial g_i}{\partial x_j})$ for $r+1\le j\le m$. By assumption the rank of this matrix is always r for x sufficiently close to the origin. Hence $A(x)=0$ for such x, and the above matrix becomes

$$\begin{pmatrix} I_r & 0 \\ ? & 0\end{pmatrix}$$

Consider next the invertible map germ $k\colon(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ given by

$$(y_1,\dots,y_n)\mapsto(y_1,\dots,y_r,y_{r+1}-g_{r+1}(y_1,\dots,y_r,0,\dots,0),\dots,y_n-g_n(y_1,\dots,y_r,0,\dots,0)\,.$$

Then by direct calculation to composite $k\circ f\circ h^{-1}$ is the map germ

$$(x_1,\dots,x_m)\mapsto(x_1,\dots,x_r,g_{r+1}(x)-g_{r+1}(x_1,\dots,x_r,0\dots,0),\dots,g_n(x)-g_n(x_1,\dots,x_r,0\dots,0))\,.$$

But since $(\frac{\partial g_i}{\partial x_j})=0$ for $i>r$, it follows that $g_i(x)=g_i(x_1,\dots,x_r,0\dots,0)$ for $i>r$, so $k\circ f\circ h^{-1}$ is exactly the map P_r in the statement of the theorem.

Observation. If $\mathrm{rk}_p f$ is maximal, then it is necessarily locally constant, in which case $\mathrm{rk}_p f = m$ or n according as to whether $m<n$ or $m\ge n$.

Definition. A smooth map $f\colon M \to N$ is an *immersion at p* if $\mathrm{rk}_x f = m$ for all points x sufficiently near p. It is an immersion if $\mathrm{rk}_p f = m$ for all $p \in M$.

Similarly, f is a *submersion at p* if $\mathrm{rk}_x f = n$ for all x sufficiently near p, and it is an immersion if $\mathrm{rk}_p f = n$ for all $p \in M$.

An immediate consequence of the Rank Theorem for Maps are the following two results:

Local Immersion Theorem. *Suppose $f\colon M \to N$ is a smooth map which is an immersion at p. then there are local coordinates at p and $f(p)$, such that f expressed in these coordinates is given by the inclusion*

$$(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_m, 0, \ldots, 0)\,.$$

Local Submersion Theorem. *Suppose $f\colon M \to N$ is a smooth map which is a submersion at p. then there are local coordinates at p and $f(p)$, such that f expressed in these coordinates is given by the projection*

$$(x_1, \ldots, x_n, \ldots, x_m) \mapsto (x_1, \ldots, x_n)\,.$$

We also have the following consequences:

Corollary. *An immersion at p is the same thing as a local embedding at p.*

Recall that an embedding $f\colon M \to N$ is a one-to-one smooth map such that $f(M)$ is a smooth submanifold of N and $f\colon M \to f(M)$ is a diffeomorphism.

Corollary. *Suppose $f\colon M \to N$ is a one-to-one immersion such that $f\colon M \to f(M)$ is a homeomorphism. Then f is an embedding.*

Proof. (Sketch). By the local immersion theorem, near any point $p \in M$, we have that f takes the form $x \mapsto (x, 0, \ldots, 0)$ in local coordinates. In follows from this that $f(M)$ is a smooth submanifold of N. Furthermore, the map $f\colon M \to f(M)$ is local coordinates near any point p is given by the identity so f is a local diffeomorphism. Hence $f\colon M \to f(M)$ is a smooth homeomorphism which is a local diffeomorphism. This implies $f\colon M \to f(M)$ is a diffeomorphism. $\square$

Immersions Have Normal Bundles Too. If $f\colon M \to N$ is an immersion, then $Tf\colon TM \to TN$ is a linear map over f which is one-to-one at each $x \in M$. Hence we can form the *cokernel bundle*

$$TN/Tf(TM)\,.$$

This is called the normal bundle of the immersion f.

When is a Zero Set a Manifold?. Let us consider the following sort of problem: suppose that $g_1, \ldots, g_\ell\colon M \to \mathbb{R}$ are smooth functions on an m-manifold. Then

$$g = (g_1, \ldots, g_\ell)\colon M \to \mathbb{R}^\ell$$

is a smooth manifold. When is $g^{-1}(0)$ a smooth submanifold of M? We will give a partial answer this equation below.

Conversely, suppose that $P \subset M$ is a smooth manifold? Is there a smooth function $g\colon M \to \mathbb{R}^\ell$ for some ℓ such that $g^{-1}(0) = P$? In this case we would say that P is *cut out by the functions* $g_1, \ldots, g_\ell$. We will give a complete solution to this problem below.

Regular Values. A *regular value* for a smooth map $f\colon M \to N$ is a point $y \in N$ such that for all $p \in M$ with $f(p) = y$, the tangent map $T_p f\colon T_p M \to T_y N$ is surjective. If y is not a regular value it is called a *critical value* and the point p in this case is called a *critical point.*

Remark. In the special case when $N = \mathbb{R}^\ell$, the map f is of the form $(g_1, \dots, g_\ell)$ and the condition that $T_p f$ is surjective is equivalent to the statement that the linear functionals $T_p g_i\colon T_p M \to \mathbb{R}$ for $1 \le i \le \ell$ are linearly independent. For this reason, we say that the ℓ functions $g_1, \dots, g_\ell$ are *independent* at p.

Preimage Theorem. *If $y \in N$ is a regular value for f, then $f^{-1}(y) \subset M$ is a submanifold of codimension n.*

Proof. If $p \in f^{-1}(y)$, then the map f is a submersion at p. Hence there are local coordinates at p and y such that f expressed in these coordinates has the form of a projection map

$$P\colon (x_1, \dots, x_m) \mapsto (x_1, \dots, x_n)\,,$$

where the points p and y correspond to the origins in $\mathbb{R}^m$ and $\mathbb{R}^n$. In particular, near p we have that $f^{-1}(y)$ is given in coordinates by $P^{-1}(0)$ which of course is just $0 \times \mathbb{R}^{m-n}$. Doing this for each $p \in f^{-1}(y)$ shows that $f^{-1}(y)$ is a submanifold of dimension $m - n$. $\square$

The following two results address the problems raised before the statement of the Preimage Theorem.

Corollary. *If y is a regular value of $f\colon M \to N$, then the preimage manifold $f^{-1}(y)$ can be cut out by independent functions.*

Proof. Choose a diffeomorphism germ $h : (W, y) \to (U, 0)$ in which W is an open neighborhood of y and U is a neighborhood of the origin in $\mathbb{R}^n$. Then $g := h \circ f\colon f^{-1}(W) \to \mathbb{R}^n$ in which 0 is a regular value. Furthermore, $g^{-1}(0) = f^{-1}(y)$. $\square$

Corollary. *Any submanifold of a smooth manifold M is <u>locally</u> cut out by independent functions.*

Proof. Any submanifold $P \subset M$ of codimension ℓ is locally diffeomorphic to the inclusion $\mathbb{R}^{m-\ell} \times 0 \subset \mathbb{R}^m$ near any point $p \in M$ in which the origin of $\mathbb{R}^m$ corresponds to p. If we let $P\colon \mathbb{R}^m \to \mathbb{R}^\ell$ be the projection onto the last ℓ coordinates, an we use a choice of local diffeomorphism, the composition yields a smooth map germ $f\colon (M, p) \to (\mathbb{R}^\ell, 0)$ such tht 0 is a regular value and $f^{-1}(0)$ is an open neighborhood of p in P. $\square$

Remark. If y is a regular value for $f\colon M \to N$, we may apply the differential at $p \in f^{-1}(y)$ to the commutative diagram

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ f^{-1}(y) \arrow[u] \arrow[r] & y \arrow[u] \end{tikzcd}$$

to obtain a commutative diagram

$$\begin{array}{ccc} T_pM & \xrightarrow{T_pf} & T_yN \\ \uparrow & & \uparrow \\ T_pf^{-1}(y) & \longrightarrow & T_yy = 0\,. \end{array}$$

This diagram shows that $T_pf^{-1}(y)$ is contained in the kernel of T_pf. Since these vector spaces have the same dimension, we conclude that *the kernel of* T_pf *coincides with the tangent space* $T_pf^{-1}(y)$.

We also see from the above diagram that the normal bundle of $f^{-1}(y) \subset M$ is isomorphic to pullback along f of the normal bundle of $y \subset N$ (Let's make this an exercise). But the normal bundle of $y \subset N$ is a trivial bundle (any bundle over a point is trivializable). It follows that the normal bundle of $f^{-1}(y) \subset M$ is trivializable.

Applications. The regular value theorem above is a very strong result since it gives us an easy way to construct new manifolds from old ones. For example, Let $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ be the function given by $x \mapsto \sum x_i^2$. Then $1 \in \mathbb{R}$ is a regular value, so $f^{-1}(1) = S^n$ is a submanifold of $\mathbb{R}^{n+1}$ with trivial normal line bundle.

Here is an even more interesting example:

Example. Let $\mathrm{O}(n)$ be the set of real orthogonal $n \times n$ matrices. This is a subset of $\mathbb{R}^{n^2}$. We claim it is actually a submanifold of dimension $\frac{n(n-1)}{2}$.

To see this, let M_n be the set of all $n \times n$ matrices which we think of as the smooth manifold $\mathbb{R}^{n^2}$. Let S_n be the set of all symmetric $n \times n$ matrices, i.e., those matrices B such that $B = B^t$. Then S can be thought of as $\mathbb{R}^{n(n+1)/2}$.

Consider the smooth function $f\colon M_n \to S$ given by

$$A \mapsto AA^t\,.$$

We compute the differential of this map at a matrix A:

$$\begin{aligned} T_Af(B) &= \lim_{s\to 0} \frac{f(A+sB) - f(A)}{s} \\ &= \lim_{s\to 0} \frac{(A+sB)(A+sB)^t - AA^t}{s} \\ &= \lim_{s\to 0} \frac{AA^t + sBA^t + AsB^t + s^2BB^t - AA^t}{s} \\ &= \lim_{s\to 0} \frac{BA^t + AB^t + sBB^t}{s} \\ &= BA^t + AB^t \end{aligned}$$

Now, since S is a linear space, we can identify the tangent space T_IS with S and likewise $T_AM_n = M_n$. For any $A \in M_n$ satisfying $AA^t = I$, we wish to know whether $T_Af\colon M_n \to S$

is onto. To see this, let $C \in S$ be any element, and note that it can be written in the form $\frac{1}{2}C + \frac{1}{2}C^t$. Then we can solve for B in the equation $BA^t = \frac{1}{2}C$ (by setting $B = \frac{1}{2}CA$). Then

$$T_A f(B) = BA^t + AB^t = \tfrac{1}{2}C + \tfrac{1}{2}C^t = C$$

which shows that for any $C \in S$, the equation $T_A f(B) = C$ admits a solution. Hence, $I \in S$ is a regular value.

Consequently, $O(n) \subset f^{-1}(I)$ is a submanifold of M_n of codimension $n(n+1)/2$, i.e., it has dimension

$$n^2 - \tfrac{n(n+1)}{2} = \tfrac{n(n-1)}{2}\,.$$

Transversality. We saw that fort a smooth map $f\colon M \to N$, a condition for the preimage $f^{-1}(y)$ to be a manifold is that y is a regular point, i.e, for each $x \in f^{-1}(y)$ we have that $\mathrm{rk}_x f = n$. This is a special case of what is called a *transversality* condition.

More generally, we would like to know when $f^{-1}(L)$ is a submanifold of M if $L \subset N$ is a submanifold. Here is the relevant condition:

Definition. Let $f\colon M \to N$ be a smooth map, and $L \subset N$ be a submanifold of dimension ℓ. We say that f is *transverse* to L if for every $x \in f^{-1}(L)$ we have

$$\mathrm{image} T_x f + T_y L = T_y N\,.$$

(Notice that if $L = \{y\}$, then this is equivalent to saying $y \in N$ is regular.) We can restate the above by saying that the composition that

$$T_x M \xrightarrow{T_x f} T_y N \xrightarrow{\pi} T_y N/T_y L =: \nu_y L$$

is onto, where π is the projection and $\nu_y L$ the normal space to $L \subset N$ at y.

Observation. If $\dim M < \mathrm{codim} L$ then the condition is equivalent to the statement that $f^{-1}(L)$ is empty (since in this case $m + \ell < n$).

Theorem. *If $f\colon M \to N$ is transverse to L and $L \subset N$ has codimension k, then $f^{-1}(L) \subset M$ is a submanifold of codimension k, furthermore,*

$$\nu_{f^{-1}L} \cong f^* \nu_L$$

where ν_L is the normal bundle of L.

Proof. Let $f(x) = y$. Choose a coordinate neighborhood V of y, with $h\colon (V, y) \to (V', 0)$ a diffeomorphism such that $V' \subset \mathbb{R}^n$ an open set. By choosing V carefully, we can assume the restriction of h gives a diffeomormorphism

$$h\colon L \cap V \to \mathbb{R}^{n-k} \cap V'\,.$$

where $\mathbb{R}^{n-k} \subset \mathbb{R}^n$ is the inclusion of the last $n - k$ coordinates. Set $U = f^{-1}(V)$.

The transversality condition then implies that $0 \in \mathbb{R}^k$ is a regular value of the composite map

$$F\colon U \xrightarrow{f} V \underset{\cong}{\xrightarrow{h}} V' \xrightarrow{\pi} \mathbb{R}^k\,.$$

where $\pi\colon \mathbb{R}^n \to \mathbb{R}^k$ is projection onto the first k coordinates.

Hence, $f^{-1}(L) \cap U = f^{-1}(L \cap V) = F^{-1}(0)$ is a submanifold of U of codimension k. This implies $f^{-1}(L) \subset M$ is a smooth submanifold of codimension k, as the property of being a submanifold is a local condition.

Lastly, consider the composite map

$$T_xM \xrightarrow{T_xf} T_yN \xrightarrow{\pi} T_yN/T_yL$$

which is onto by the transversality condition. Since f maps $f^{-1}(L)$ to L, it follows that T_xf maps $T_xf^{-1}(L)$ to T_yL. We infer that $T_xf^{-1}(L)$ is contained in the kernel of the displayed composite, so it induces a linear map

$$T_xM/T_xf^{-1}(L) \to T_yN/T_yL$$

which is injective and is therefore an isomorphism. This means that the normal space to $f^{-1}(L)$ at x is canonically isomorphic to the normal space to L at y. This implies

$$\nu_{f^{-1}L} \cong f^*\nu_L\,.$$