NUMERICAL SOLUTION OF THE VISCOUS BURGERS' EQUATION USING FTCS METHOD

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ABSTRACT

In this paper, one-dimensional non-linear advection diffusion equation is solved both in non-conservative form and conservative form using forward time centered space (FTCS) method in a periodic domain.

NOMENCLATURE

f_i^n	f at time n and x I
f	field quantity
t	time
x	spatial coordinate
Δt	time step
h	length between x_i and x_{i+1}

1. INTRODUCTION

One-dimensional non-linear advection diffusion equation, also known as Burgers' equation, has many application areas such as fluid mechanics, gas dynamics and traffic flow. The non-conservative form of the equation is given by

$$\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial x} = v\frac{\partial^2 f}{\partial x^2} \tag{1}$$

and conservative form is given by

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}f^2\right) = v\frac{\partial^2 f}{\partial x^2} \tag{2}$$

In this paper, both conservative and non-conservative forms are discretized, and the equations are solved using numerical methods. Then the solutions are compared.

2. DISCRETIZATION

The equations are approximated by using a forward in time approximation for the time derivative, and centered approximations for the first and second spatial derivatives. The equations are solved in a periodic domain of length 1 and v is taken as 0.01. MATLAB is used to solve the numerical equations. The initial conditions are shown in Eq. (3). Eqs. (7,9) are used in MATLAB calculations.

$$f(x, t = 0) = \sin(2\pi x) + 1.0 \tag{3}$$

2.1 Conservative Form

$$\frac{\partial f}{\partial t} = \frac{f_i^{n+1} - f_i^n}{\Delta t} \tag{4}$$

$$\begin{array}{rl} & f_{i+1} = f_i + h\frac{\partial f}{\partial x} + h^2\frac{\partial^2 f}{\partial x^2} + \cdots \\ + & f_{i-1} = f_i - h\frac{\partial f}{\partial x} + h^2\frac{\partial^2 f}{\partial x^2} + \cdots \\ \hline & f_{i+1} - f_{i-1} = 2h\frac{\partial f}{\partial x} + O(h^3) \end{array}$$

$$\frac{\partial f}{\partial x} = \frac{f_{i+1} - f_{i-1}}{2h} \tag{5}$$

$$\begin{array}{rl} & f_{i+1} = f_i + h\frac{\partial f}{\partial x} + h^2\frac{\partial^2 f}{\partial x^2} + \cdots \\ + & f_{i-1} = f_i - h\frac{\partial f}{\partial x} + h^2\frac{\partial^2 f}{\partial x^2} + \cdots \\ \hline & f_{i+1} + f_{i-1} = 2f_i + h^2\frac{\partial^2 f}{\partial x^2} \end{array}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{f_{i+1}^n - 2f_i^n + f_{i-1}^n}{h^2} \tag{6}$$

Plugging Eqs. (4,5,6) into Eq. (1).

$$\frac{f_i^{n+1} - f_i^n}{\Delta t} + f_i^n\frac{f_{i+1}^n - f_{i-1}^n}{2h} = v\frac{f_{i+1}^n - 2f_i^n - f_{i-1}^n}{h^2}$$

After rearrangement, we get

$$f_i^{n+1} = f_i^n + \left(v\frac{f_{i+1}^n - 2f_i^n - f_{i-1}^n}{h^2} - f_i^n\frac{f_{i+1}^n - f_{i-1}^n}{2h}\right)\Delta t \tag{7}$$

2.2 Non-Conservative Form

$$g = \frac{1}{2}f^2$$

$$\frac{\partial g}{\partial x} = \frac{g_{i+1} - g_{i-1}}{2h} = \frac{f_{i+1}^2 - f_{i-1}^2}{4h} \tag{8}$$

Plugging in Eqs. (4, 6,7) into Eq. (2).

$$\frac{f_i^{n+1} - f_i^n}{\Delta t} + \frac{f_{i+1}^2 - f_{i-1}^2}{4h} = v\frac{f_{i+1}^n - 2f_i^n - f_{i-1}^n}{h^2}$$

After rearrangement, we get

$$f_i^{n+1} = f_i^n + \left(v\frac{f_{i+1}^n - 2f_i^n - f_{i-1}^n}{h^2} - \frac{f_{i+1}^2 - f_{i-1}^2}{4h}\right)\Delta t \tag{9}$$

3. RESULTS AND DISCUSSION

The initial conditions are shown in Fig. (1). As time passes, magnitude of f gets smaller, and the solution becomes more like a straight line. The solutions at different times are shown in Figs. (2-5). 201 grid points are used in the numerical method.

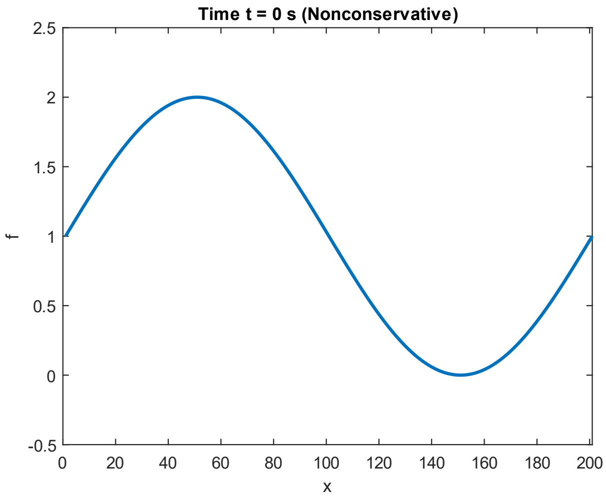

FIGURE 1: INITIAL CONDITIONS

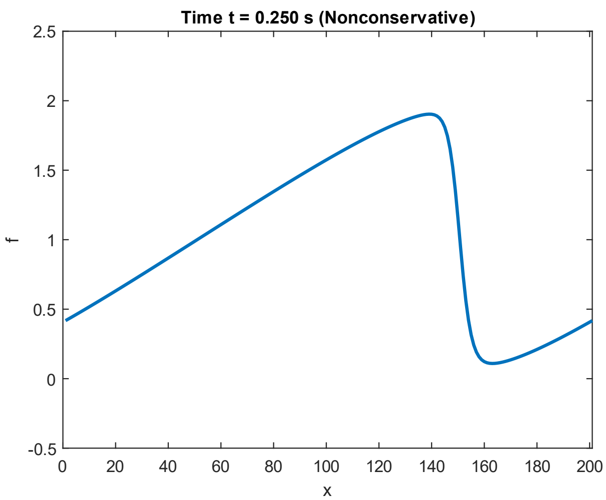

FIGURE 2: NONCONSERVATIVE SOLUTION AT TIME T=0.25S

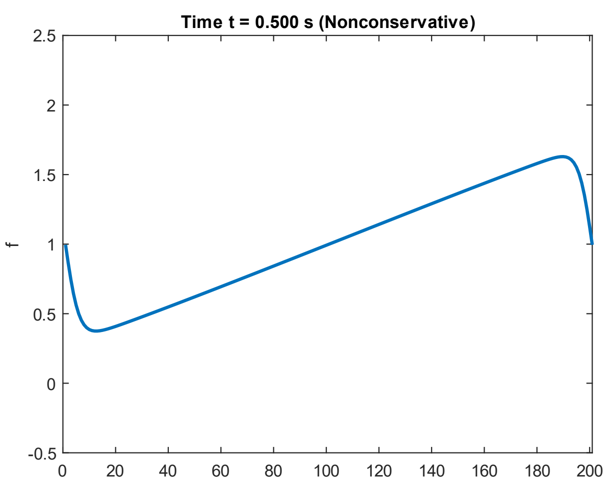

FIGURE 3: NONCONSERVATIVE SOLUTION AT TIME T=0.50S

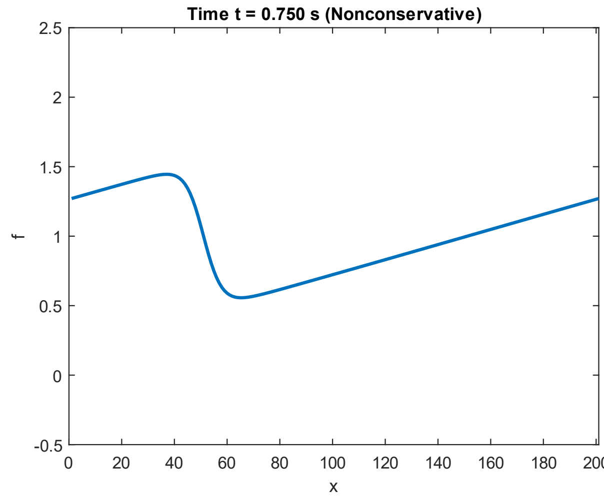

FIGURE 4: NONCONSERVATIVE SOLUTION AT TIME T=0.75S

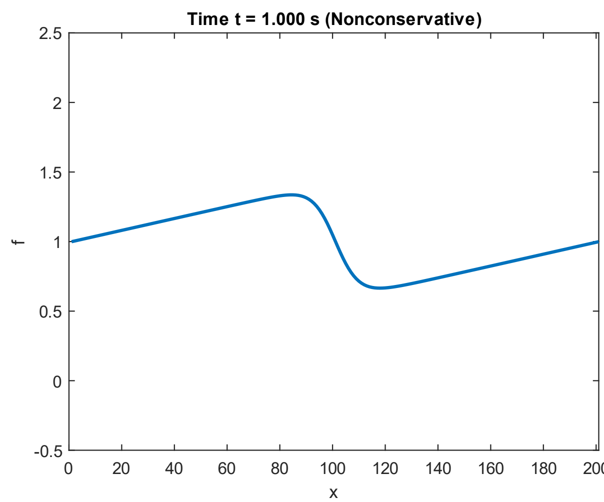

FIGURE 5: NONCONSERVATIVE SOLUTION AT TIME T=1 S

As shown in Figs. (6-8), the solution is not reliable for low number of grid points. Although, they are all solutions for the same problem, the results are significantly different.

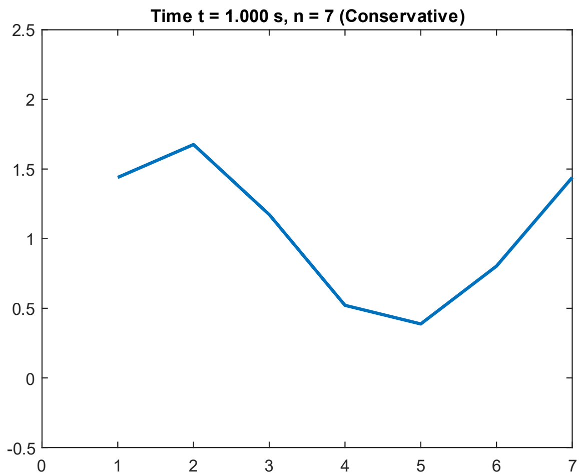

FIGURE 6: CONSERVATIVE SOLUTION AT TIME T=1 S WITH NUMBER OF GRID POINTS N = 7.

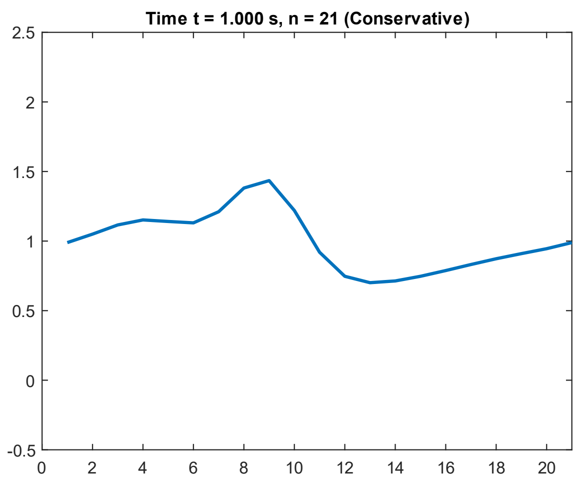

FIGURE 7: CONSERVATIVE SOLUTION AT TIME T=1 S WITH NUMBER OF GRID POINTS N = 21

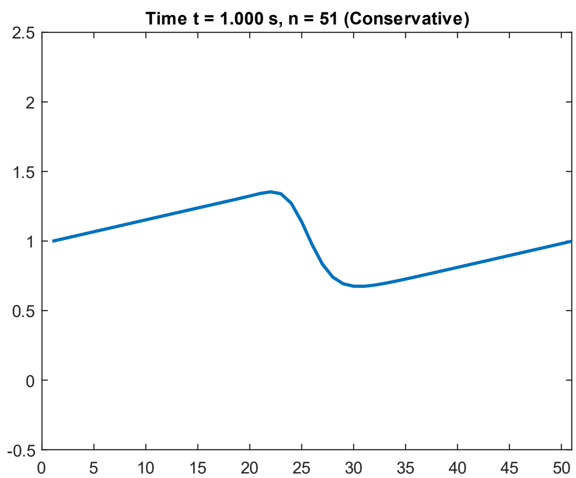

FIGURE 8: CONSERVATIVE SOLUTION AT TIME T=1 S WITH NUMBER OF GRID POINTS N = 51

Both non-conservative and conservative forms gave very similar results. To compare them, sum of absolute values of the difference between conservative and non-conservative solution at each grid point is calculated. As shown in Fig. (9), the difference between two solutions goes to zero as finer grids are used. The difference sharply decreases between n=10 and n=50, then slowly approaches to zero. Grid convergence is achieved after n≈230.

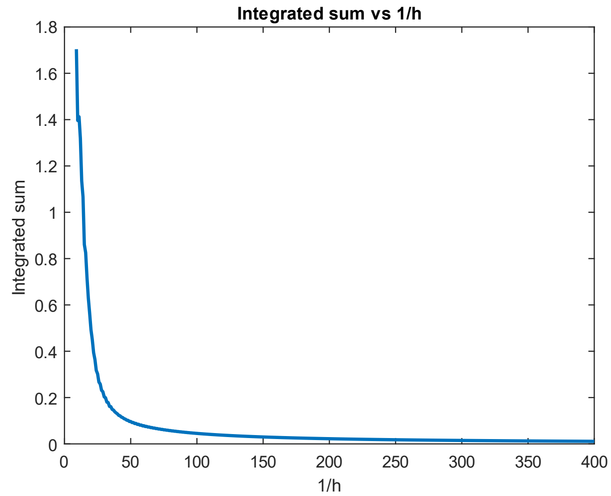

FIGURE 9: INTEGRATED SUM VS 1/H

The solutions are only stable when Δt was sufficiently small. Using finer grids requires smaller Δt. In Table (1), the number of grid points and maximum magnitude of time step for stable solution is shown. Those results are obtained from nonconservative solution.

n (number of grid points)	Δt (s)
101	0.00538
201	0.00128

TABLE 1: Number of grid points and corresponding maximum time step. (The solution may not be stable after t=1 s)

4. CONCLUSION

When solving Burgers' equation with numerical methods, using higher number of grids and smaller time step increases the accuracy of the solution. When the number of grids is too small (such as 10), the results obtained from the solution are not usable. Grid convergence is determined to be achieved around 230. There is a relationship between the number of grid points and maximum magnitude of time step to achieve stable solution. Increasing number of grids requires smaller time step for stable solution, which increases computation time.