Eigenvalues and Eigenvectors

Numerical Analysis

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Eigenvalues and Eigenvectors

Let $\mathrm{A} \in \mathbb{C}^{n\times n}$, the eigenvalue problem consists in finding a scalar λ (real or complex) and a nonnull vector $\mathbf{x}$ such that

$$\mathrm{A}\mathbf{x} = \lambda\mathbf{x} \tag{1}$$

λ is called an eigenvalue of A, while $\mathbf{x}$ is the associated eigenvector. The latter is not unique; indeed $\alpha\mathbf{x}$ with $\alpha \neq 0$, are also eigenvectors associated with λ. Should $\mathbf{x}$ be known, λ can be recovered by using the Rayleigh quotient

$$\mathbf{x}^H \mathrm{A}\mathbf{x}/\|\mathbf{x}\|^2.$$

The eigenvalues of A are the roots of the characteristic polynomial of A:

$$p_{\mathrm{A}}(\lambda) = \det(\mathrm{A} - \lambda\mathrm{I}).$$

A $n \times n$ matrix has exactly n eigenvalues (real or complex), not necessarily distinct. A matrix A$\in \mathbb{C}^{n\times n}$ is said to be diagonalizable if there exists a nonsingular matrix U$\in \mathbb{C}^{n\times n}$ such that $\mathrm{U}^{-1}\mathrm{A}\mathrm{U} = \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.

The power method

Let A with real entries and assume that its eigenvalues are ordered

$$|\lambda_1|>|\lambda_2| \geq |\lambda_3| \geq \ldots \geq |\lambda_n|. \qquad (2)$$

The power method is: Let $\mathbf{x}^{(0)} \in \mathbb{C}^n$ be given and set $\mathbf{y}^{(0)} = \mathbf{x}^{(0)}/\|\mathbf{x}^{(0)}\|$, for $k = 1, 2, ...$ compute

$$\mathbf{x}^{(k)} = \mathrm{A}\mathbf{y}^{(k-1)}, \quad \mathbf{y}^{(k)} = \frac{\mathbf{x}^{(k)}}{\|\mathbf{x}^{(k)}\|}, \quad \lambda^{(k)} = (\mathbf{y}^{(k)})^H \mathrm{A}\mathbf{y}^{(k)}$$

Until $|\lambda^{(k)} - \lambda^{(k-1)}| < \varepsilon|\lambda^{(k)}|$, where ε is the desired tolerance.

Convergence

Since we have assumed that the eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ of A are linearly independent, these eigenvectors form a basis for $\mathbb{C}^n$. Thus the vectors $\mathbf{x}^{(0)}$ and $\mathbf{y}^{(0)}$ can be written as

$$\mathbf{x}^{(0)} = \sum_{i=1}^{n} \alpha_i \mathbf{x}_i, \ \mathbf{y}^{(0)} = \beta^{(0)} \sum_{i=1}^{n} \alpha_i \mathbf{x}_i, \quad \text{with } \beta^{(0)} = 1/\|\mathbf{x}^{(0)}\| \text{ and } \alpha_i \in \mathbb{C}.$$

At the first step the power method gives

$$\mathbf{x}^{(1)} = \mathrm{A}\mathbf{y}^{(0)} = \beta^{(0)} \mathrm{A} \sum_{i=1}^{n} \alpha_i \mathbf{x}_i = \beta^{(0)} \sum_{i=1}^{n} \alpha_i \lambda_i \mathbf{x}_i \qquad \text{and, similarly,}$$

$$\mathbf{y}^{(1)} = \beta^{(1)} \sum_{i=1}^{n} \alpha_i \lambda_i \mathbf{x}_i, \qquad \beta^{(1)} = \frac{1}{\|\mathbf{x}^{(0)}\| \ \|\mathbf{x}^{(1)}\|}.$$

At a given step k we will have

$$\mathbf{y}^{(k)} = \beta^{(k)} \sum_{i=1}^{n} \alpha_i \lambda_i^k \mathbf{x}_i, \qquad \beta^{(k)} = \frac{1}{\|\mathbf{x}^{(0)}\| \cdots \|\mathbf{x}^{(k)}\|}$$

Convergence

And therefore

$$\mathbf{y}^{(k)} = \lambda_1^k \beta^{(k)} \left(\alpha_1 \mathbf{x}_1 + \sum_{i=2}^{n} \alpha_i \frac{\lambda_i^k}{\lambda_1^k} \mathbf{x}_i \right).$$

Since $|\lambda_i/\lambda_1| < 1$ for $i = 2, \ldots, n$, the vector $\mathbf{y}^{(k)}$ tends to align along the same direction as the eigenvector $\mathbf{x}_1$ when k tends to $+\infty$, provided $\alpha_1 \neq 0$.

Convergence

Example 1. Consider the family of matrices

$$A(\alpha) = \begin{bmatrix} \alpha & 2 & 3 & 13 \\ 5 & 11 & 10 & 8 \\ 9 & 7 & 6 & 12 \\ 4 & 14 & 15 & 1 \end{bmatrix}, \qquad \alpha \in \mathbb{R}.$$

We want to approximate the eigenvalue with largest modulus by the power method. When $\alpha = 30$, the eigenvalues of the matrix are given by $\lambda_1 = 39.396$, $\lambda_2 = 17.8208$, $\lambda_3 = -9.5022$ and $\lambda_4 = 0.2854$.
The method approximates λ_1 in 22 iterations with a tolerance $\varepsilon = 10^{-10}$ and $\mathbf{x}^{(0)} = \mathbf{1}^T$.

If $\alpha = -30$ the iterations are 708. The reason is that $|\lambda_2|/|\lambda_1| = 0.9704$ is close to unity: $\lambda_1 = -30.643$, $\lambda_2 = 29.7359$, $\lambda_3 = -11.6806$ and $\lambda_4 = 0.5878$.

The inverse power method

The inverse power method can find the smaller eigenvalue of a non-singular matrix A:
Let $\mathbf{x}^{(0)} \in \mathbb{C}^n$ be given and set $\mathbf{y}^{(0)} = \mathbf{x}^{(0)}/\|\mathbf{x}^{(0)}\|$, for $k = 1, 2, \ldots$ compute

$$\mathbf{x}^{(k)} = \mathrm{A}^{-1}\mathbf{y}^{(k-1)}, \quad \mathbf{y}^{(k)} = \frac{\mathbf{x}^{(k)}}{\|\mathbf{x}^{(k)}\|}, \quad \mu^{(k)} = (\mathbf{y}^{(k)})^H \mathrm{A}^{-1}\mathbf{y}^{(k)}$$

Until $|\mu^{(k)} - \mu^{(k-1)}| < \varepsilon|\mu^{(k)}|$, where ε is the desired tolerance.
If A admits n linearly independent eigenvectors, and if also the eigenvalue λ_n of minimum modulus is distinct from the others, then

$$\lim_{k\to\infty} \mu^{(k)} = 1/\lambda_n,$$

At each step k we have to solve a linear system of the form $\mathrm{A}\mathbf{x}^{(k)} = \mathbf{y}^{(k-1)}$.

The power method with shift

The power method with shift can find the eigenvalue of A near to a given number μ:
Define $A_\mu = A - \mu I$, whose eigenvalues are $\lambda(\mathrm{A}_\mu) = \lambda(\mathrm{A}) - \mu$.
In order to approximate λ_μ, we can at first approximate the eigenvalue of minimum length of A_μ
Let $\mathbf{x}^{(0)} \in \mathbb{C}^n$ be given and set $\mathbf{y}^{(0)} = \mathbf{x}^{(0)}/\|\mathbf{x}^{(0)}\|$, for $k = 1, 2, \ldots$ compute

$$\mathbf{x}^{(k)} = \mathrm{A}_\mu^{-1}\mathbf{y}^{(k-1)}, \quad \mathbf{y}^{(k)} = \frac{\mathbf{x}^{(k)}}{\|\mathbf{x}^{(k)}\|}, \quad \lambda_\mu^{(k)} = 1/(\mathbf{y}^{(k)})^H \mathrm{A}_\mu^{-1}\mathbf{y}^{(k)}$$

Until $|\lambda_\mu^{(k)} - \lambda_\mu^{(k-1)}| < \varepsilon|\lambda_\mu^{(k)}|$, where ε is the desired tolerance. 1 The searched eigenvalue of A is approximated by $\lambda = \lambda_\mu + \mu$.

Example 2. For the matrix A(30) of Example 1 we seek the eigenvalue closest to the value 17. We set $\mu = 17$ and apply the power method with shift with a tolerance $tol = 10^{-10}$ and initial guess $x0 = (1, 1, 1, 1)^T$. After 8 iterations the algorithm returns the value $\lambda = 17.82079703055703$. A less accurate knowledge of the *shift* would involve more iterations. For instance, if we set $\mu = 13$ the program returns the value $\lambda = 17.82079703064106$ after 19 iterations.

The value of the shift can be modified during the iterations, by setting $\mu = \lambda^{(k)}$. This yields a faster convergence; however the computational cost grows substantially since now at each iteration the matrix A_μ does change and the LU factorization has to be performed at each iteration.

How to compute the shift

We need to locate (more or less accurately) the eigenvalues of A in the complex plane.

Let A be a square matrix of dimension n. The Gershgorin circles $C_i^{(r)}$ and $C_i^{(c)}$ associated with its i-th row and i-th column are respectively defined as

$$C_i^{(r)} = \{z \in \mathbb{C} : \ |z - a_{ii}| \leq \sum_{j=1, j\neq i}^{n} |a_{ij}|\},$$

$$C_i^{(c)} = \{z \in \mathbb{C} : \ |z - a_{ii}| \leq \sum_{j=1, j\neq i}^{n} |a_{ji}|\}.$$

$C_i^{(r)}$ is called the i-th row circle and $C_i^{(c)}$ the i-th column circle.

All the eigenvalues of a given matrix A$\in \mathbb{C}^{n\times n}$ belong to the region of the complex plane which is the intersection of the two regions formed respectively by the union of the row circles and column circles.

Moreover, should m row circles (or column circles), with $1 \leq m \leq n$, be disconnected from the union of the remaining $n-m$ circles, then their union contains exactly m eigenvalues.

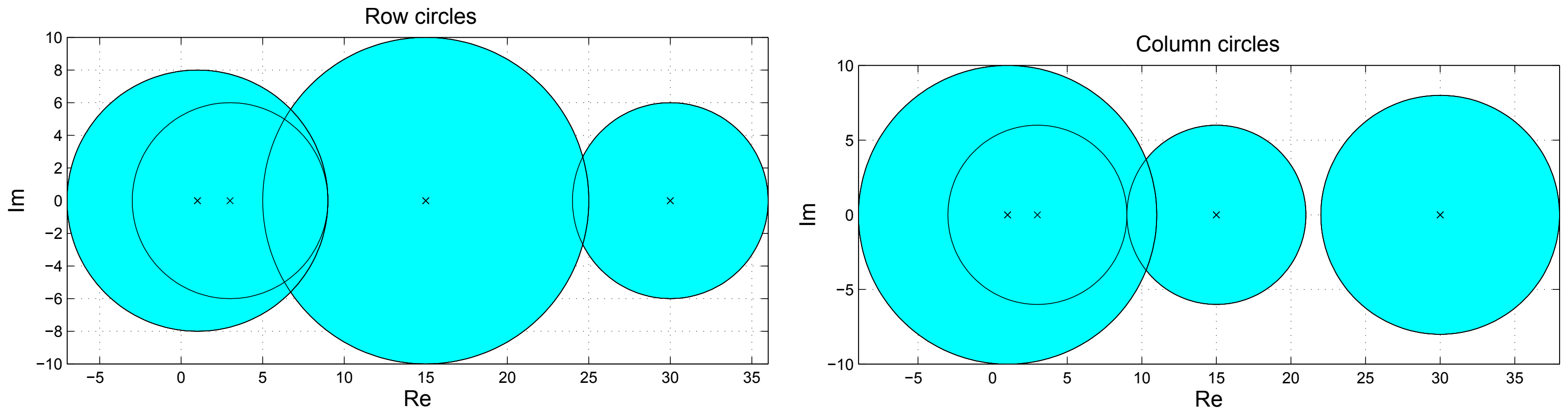
There is no guarantee that a circle should contain eigenvalues, unless it is isolated from the others. The information provided by Ghersghorin circles are in general quite coarse, thus the previous result can provide only a preliminary guess of the shift.

Note that from Proposition, all the eigenvalues of a strictly diagonally dominant matrix are non-null.

Example 3. Below we have plotted the Gershgorin circles associated with the matrix

$$A = \begin{bmatrix} 30 & 1 & 2 & 3 \\ 4 & 15 & -4 & -2 \\ -1 & 0 & 3 & 5 \\ -3 & 5 & 0 & -1 \end{bmatrix}.$$

The centers of the circles have been identified by a cross.



Row circles (*left*) and column circles (*right*) for the matrix of Example