Matrix Multiplication

Advanced Programming and Algorithmic Design

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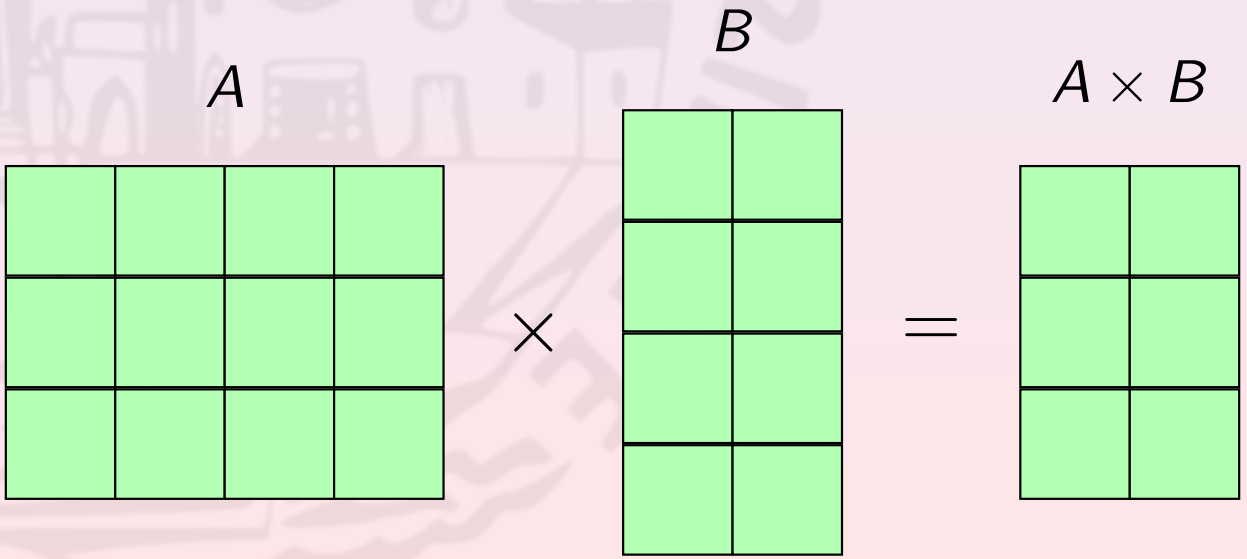
Matrix Multiplication

Definition (Row-Column Multiplication)

Let A be a $n \times m$ matrix and let B be a $m \times l$ matrix. $A \times B$ is a $n \times l$ matrix s.t.

$$(A \times B)[i,j] = \sum_k A[i,k] * B[k,j]$$



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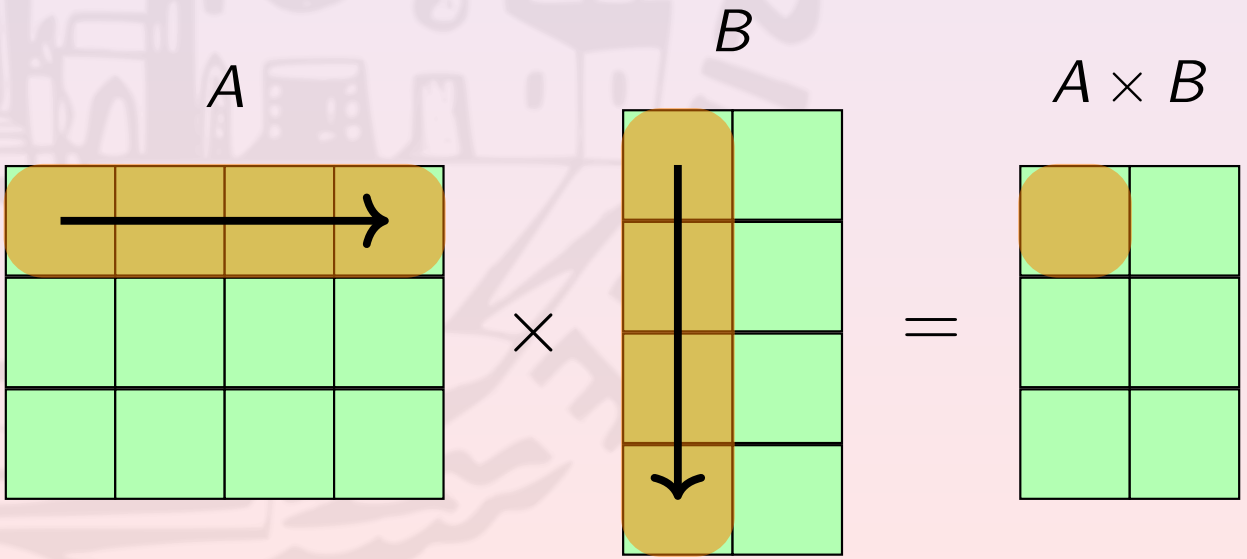

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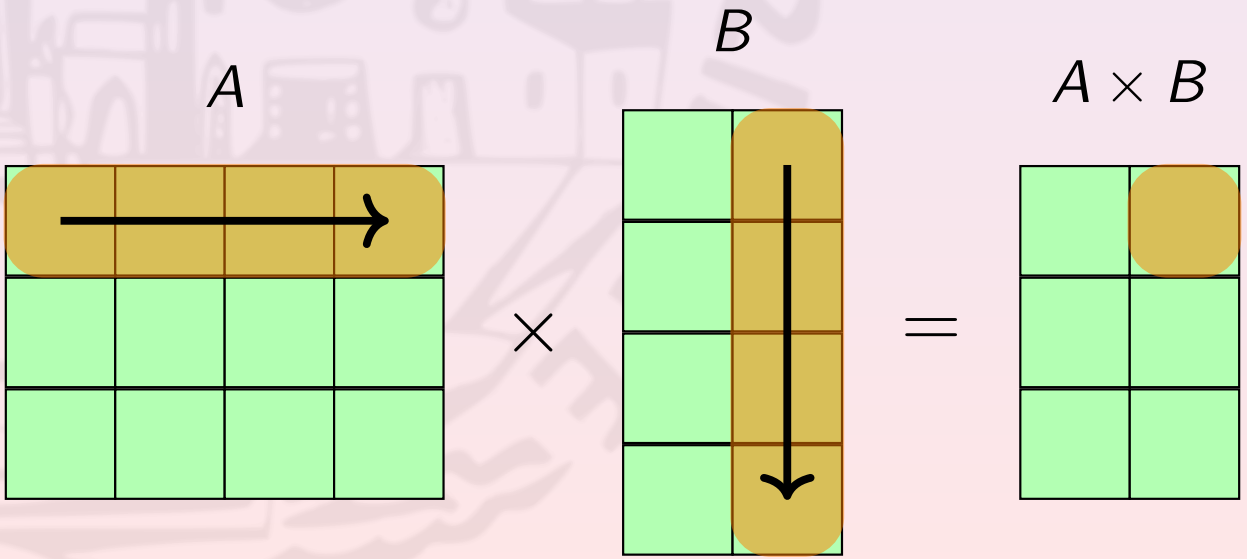

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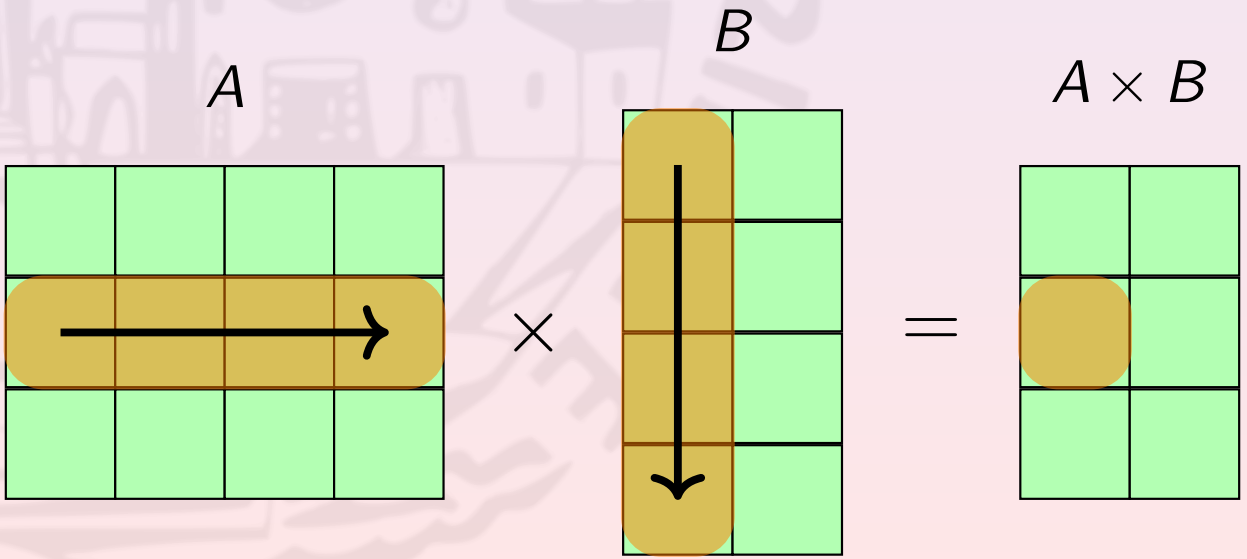

Problem Definition

Input: Two $n \times n$ matrices A and B
Output: The $n \times n$ matrix $A \times B$

E.g.,

A

1	-1	2
2	0	3
0	-1	2

,

B

4	-2	2
2	0	0
-1	3	0

$\Longrightarrow$

$A \times B$

0	-4	2
5	5	4
-4	6	0

Square matrices solution can easily be generalized

Naïve Solution: Code

```
def naive_mult(C, A, B):
    for i ← 1..rows(A):
        for j ← 1..cols(B):
            a ← 0
            for k ← 1..cols(A):
                a ← a + A[i,k] * B[k,j]
            endfor

            C[i,j] ← a
        endfor
    endfor

    return C
enddef
```

Naïve Solution: Complexity

The naïve solution mimes row-column definition

3 nested loops with indexes in $[1, n]$

The inner-block takes time $O(1)$

The overall execution takes time $\mathbf{\Theta(n^3)}$

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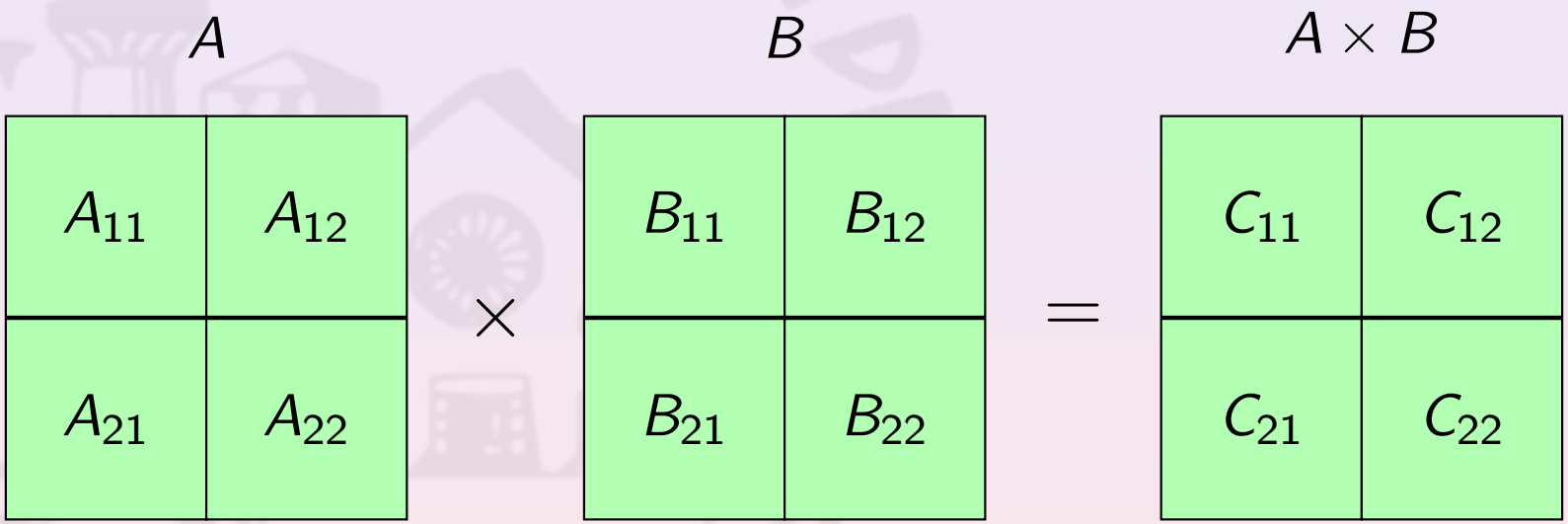
Can we find a better algorithm?

Divide-and-Conquer Strategy

What about splitting A and B in blocks?



where

$$C_{ij} = (A_{i1} \times B_{1j}) + (A_{i2} \times B_{2j})$$

Divide-and-Conquer Strategy (Cont'd)

$+$ is the elements-wise matrix sum (time complexity $\Theta(n^2)$)

$\times$ is the usual row-column multiplication

A_{ik} and B_{kj} are $\frac{n}{2} \times \frac{n}{2}$ matrices

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We can define a recursive algorithm:

- if $rows(A) < 2$, return `naive_mult(C, A, B)`
- for $i, j, k \in [12]$ recursively compute $D_{ijk} = A_{ik} \times B_{kj}$
- for $i, j \in [12]$ compute $C_{ij} = D_{ij1} + D_{ij2}$
- return C

Divide-and-Conquer Strategy: Complexity

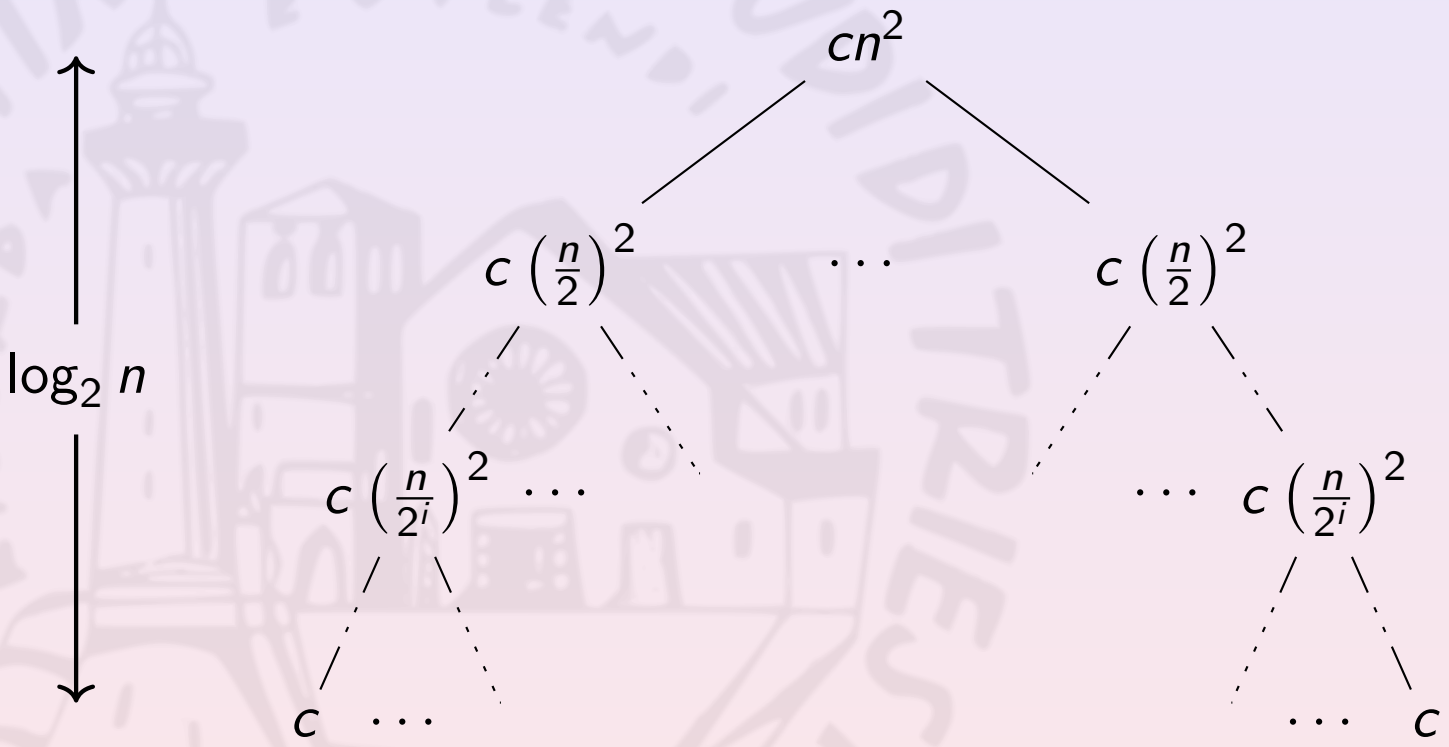
The recursive algorithm requires:

- 8 multiplications between $\frac{n}{2} \times \frac{n}{2}$ matrices
- 4 sums between $\frac{n}{2} \times \frac{n}{2}$ matrices

If T_M is the complexity of the algorithm

$$\begin{aligned} T_M(n) &= 8 * T_M(n/2) + 4 * \Theta(n^2) \\ &= 8 * T_M(n/2) + \Theta(n^2) \end{aligned}$$

Divide-and-Conquer Strategy: Complexity (Recursion Tree)



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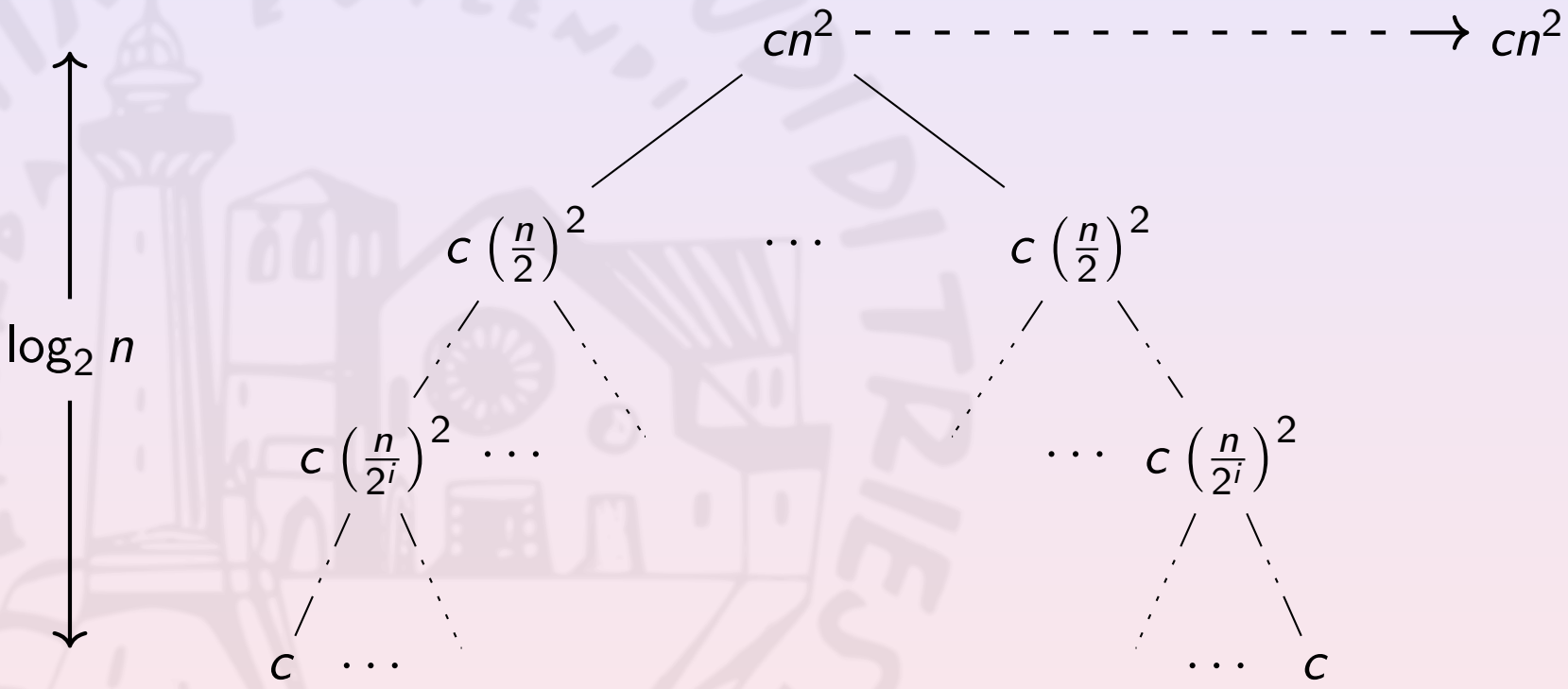

$$T_M(n) = cn^2\left(1 + \right)$$

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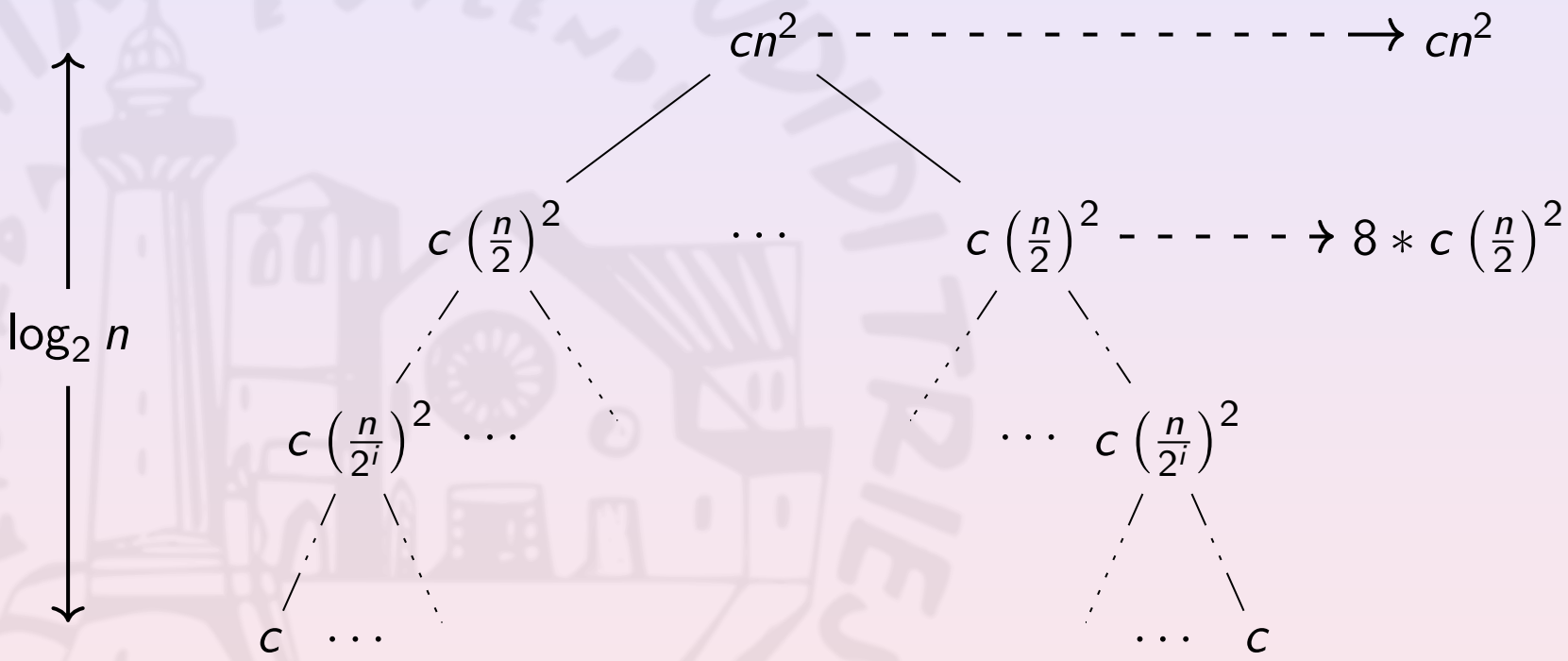

$$T_M(n) = cn^2\left(1 + 2 + \qquad\qquad\qquad\qquad\right)$$

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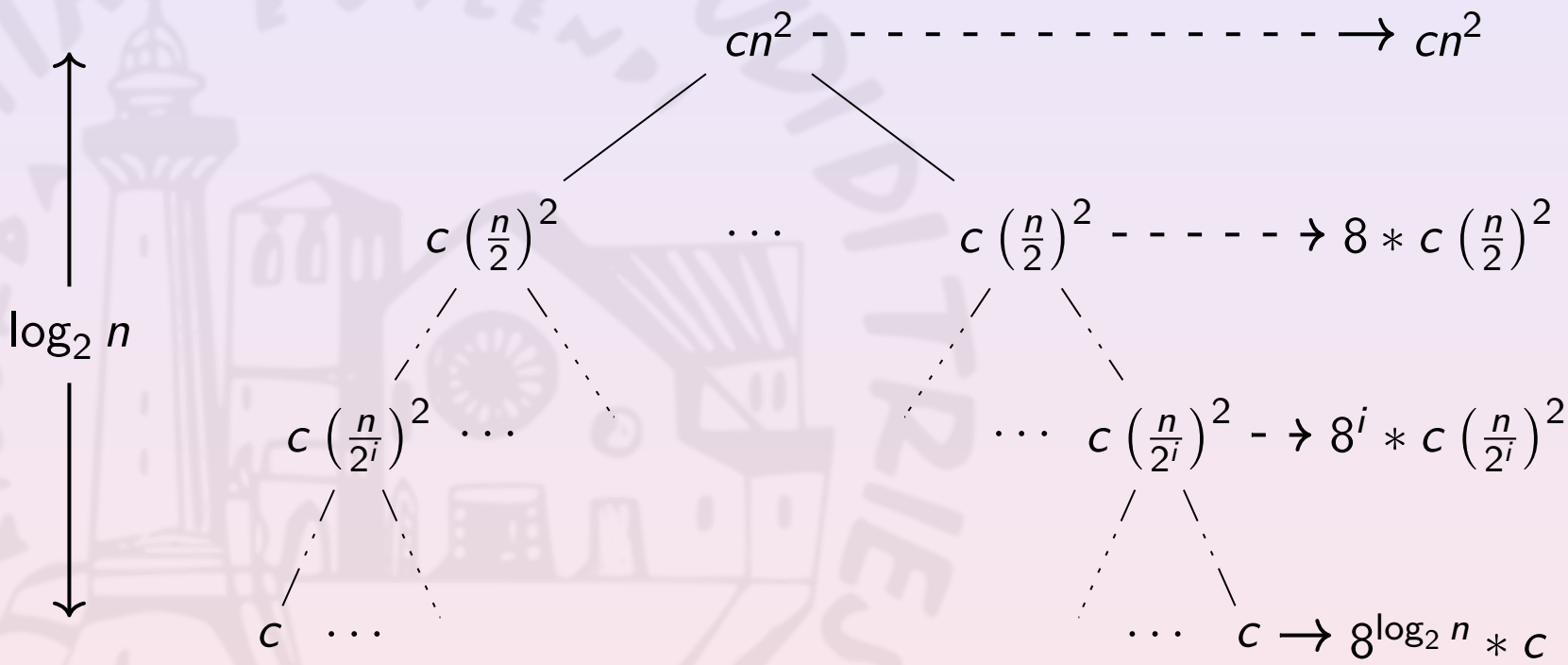
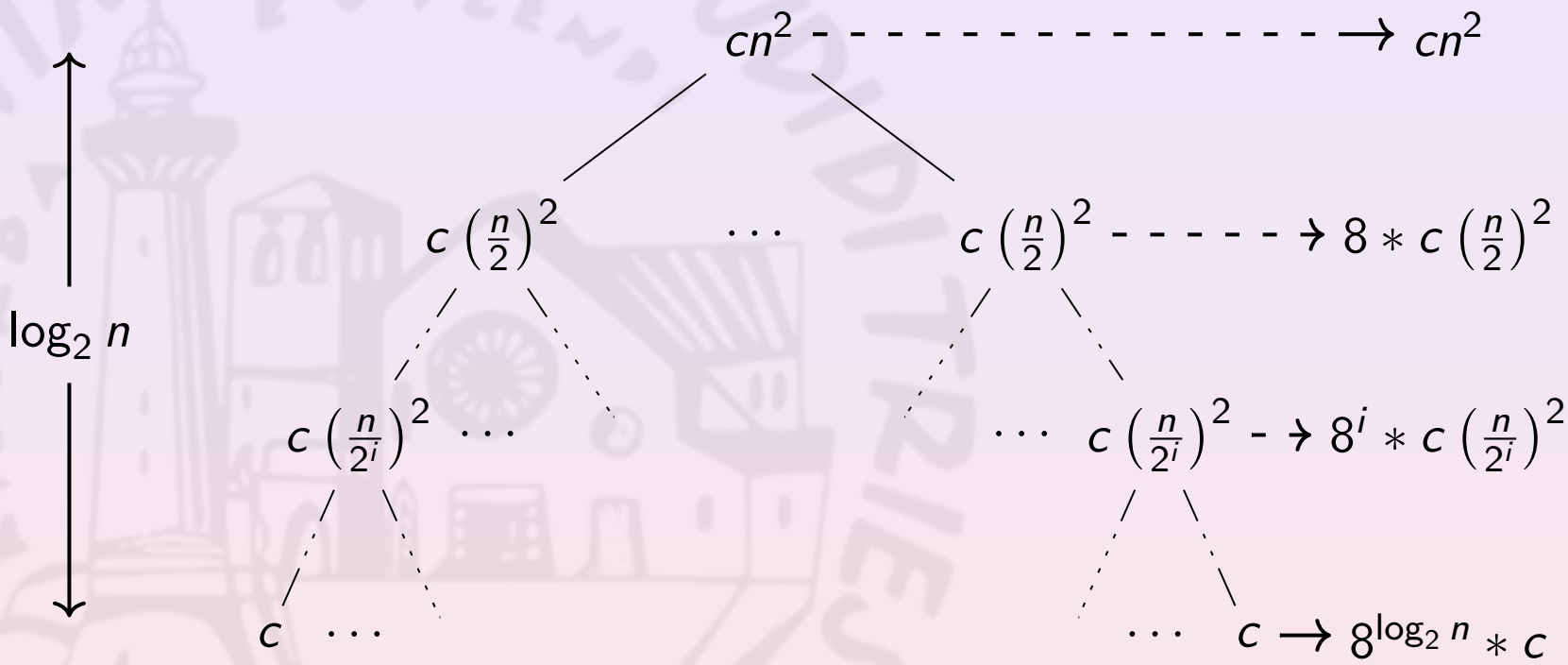


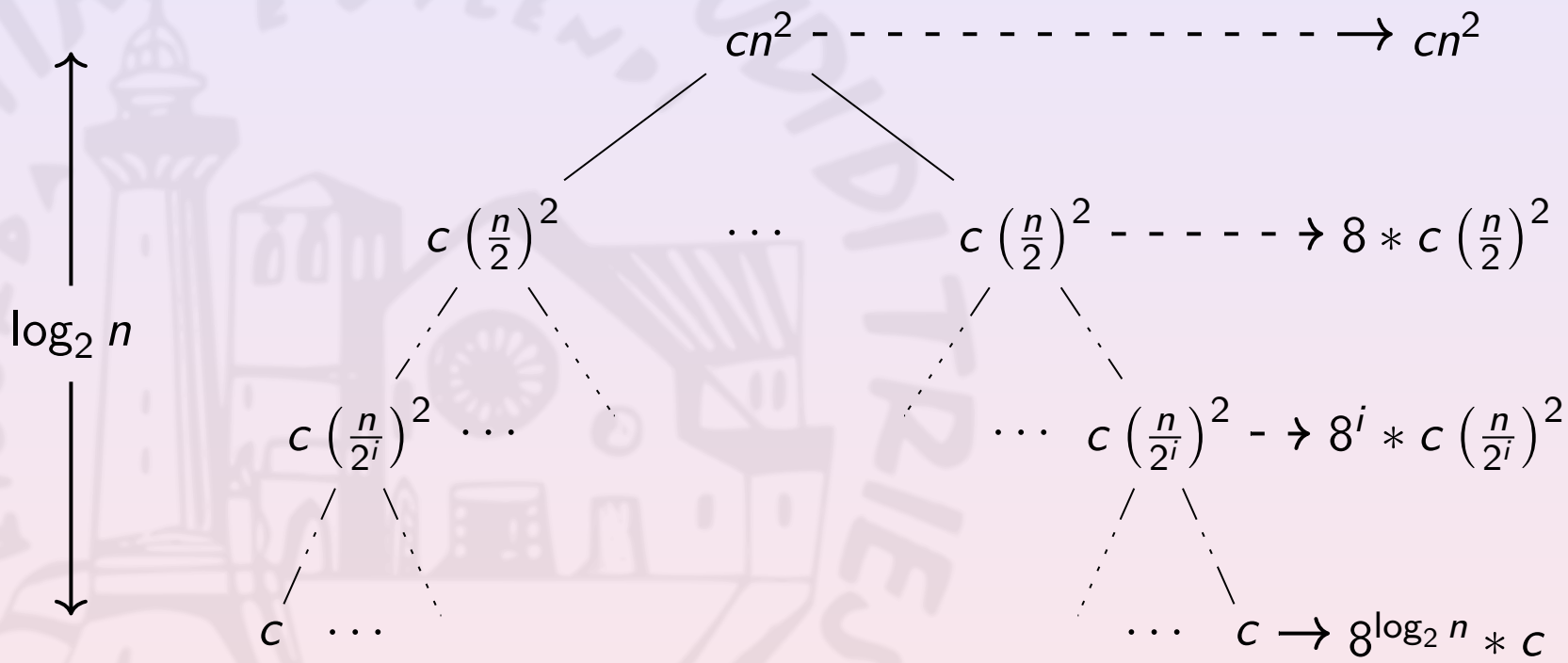
$$T_M(n) = cn^2\left(1 + 2 + \ldots + 2^i + \ldots + 2^{\log_2 n}\right)$$

Divide-and-Conquer Strategy: Complexity (Recursion Tree)



$$T_M(n) = cn^2\left(1 + 2 + \ldots + 2^i + \ldots + 2^{\log_2 n}\right)$$
$$= cn^2\left(2^{\log_2 n+1} - 1\right) = cn^2\left(2n - 1\right)$$

Divide-and-Conquer Strategy: Complexity (Recursion Tree)



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Some Further Thoughts

The divide-and-conquer approach had too many recursive calls

Can it be "rephrased" to decrease them?

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Can it be "rephrased" to decrease them?

Yes, it can!!!

Strassen's Algorithm

Sums ($\Theta(n^2)$)

$$S_1 = B_{12} - B_{22}$$
$$S_2 = A_{11} + A_{12}$$
$$S_3 = A_{21} + A_{22}$$
$$S_4 = B_{21} - B_{11}$$
$$S_5 = A_{11} + A_{22}$$
$$S_6 = B_{11} + B_{22}$$
$$S_7 = A_{12} - A_{22}$$
$$S_8 = B_{21} + B_{22}$$
$$S_9 = A_{11} - A_{21}$$
$$S_{10} = B_{11} + B_{12}$$

$\Longrightarrow$

Recursion Calls

$$P_1 = A_{11} \times S_1$$
$$P_2 = S_2 \times B_{22}$$
$$P_3 = S_3 \times B_{11}$$
$$P_4 = A_{22} \times S_4$$
$$P_5 = S_5 \times S_6$$
$$P_6 = S_7 \times S_8$$
$$P_7 = S_9 \times S_{10}$$

Strassen's Algorithm

Sums ($\Theta(n^2)$)

$$C_{11} = P_5 + P_4 - P_2 + P_6$$
$$C_{12} = P_1 + P_2$$
$$C_{21} = P_3 + P_4$$
$$C_{22} = P_5 + P_1 - P_3 - P_7$$

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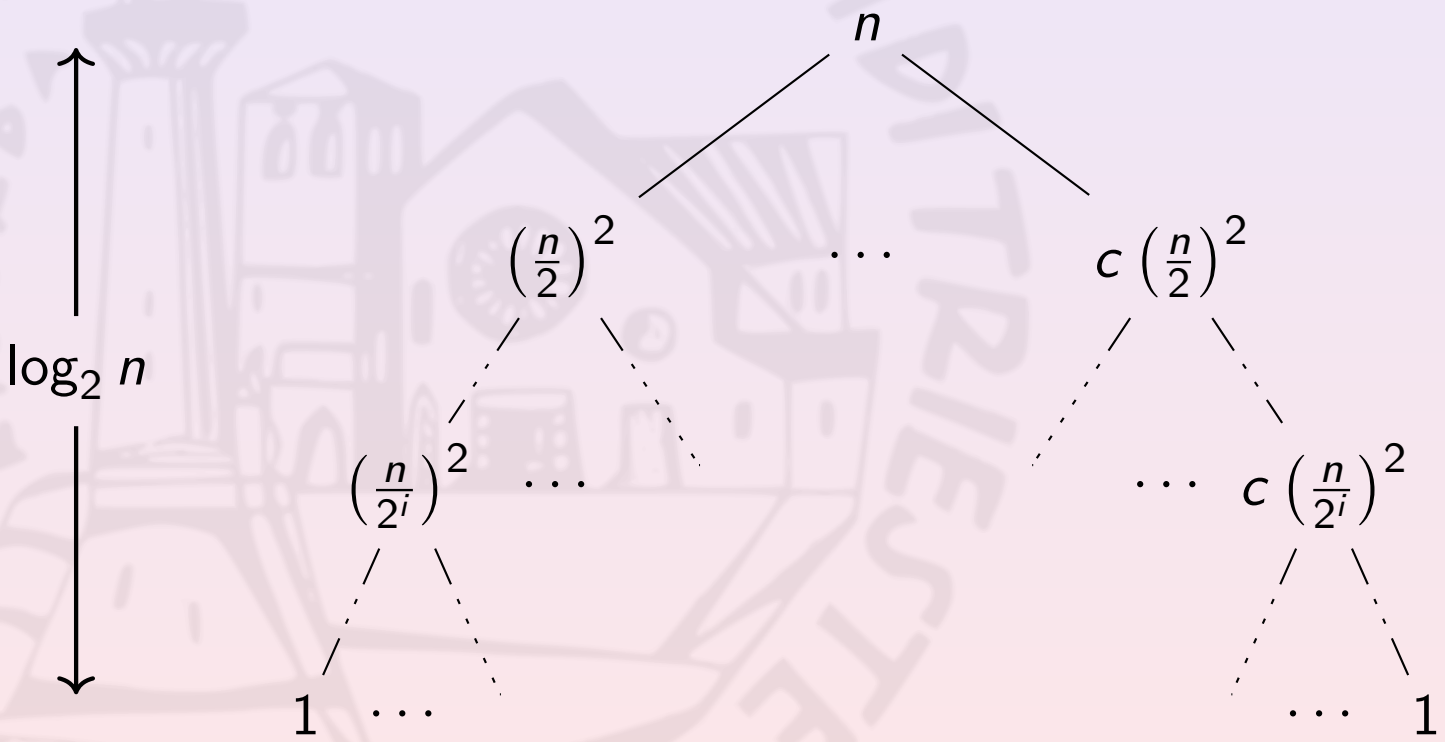
$$C_{22} = P_5 + P_1 - P_3 - P_7$$

The complexity equation is:

$$T_M(n) = 7 * T_M(n/2) + \Theta(n^2)$$

because the S's and the C's are computed by sums

Strassen's Algorithm: Complexity (Recursion Tree)



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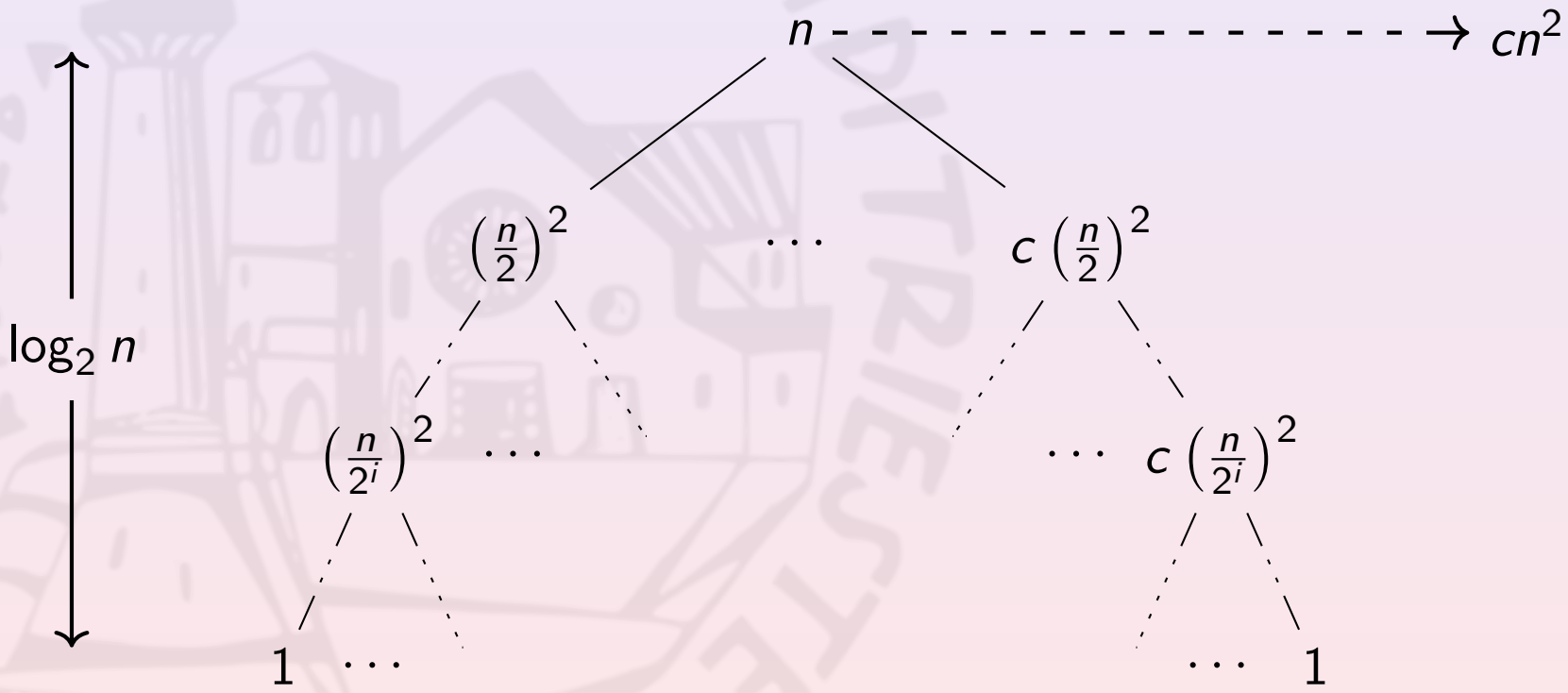

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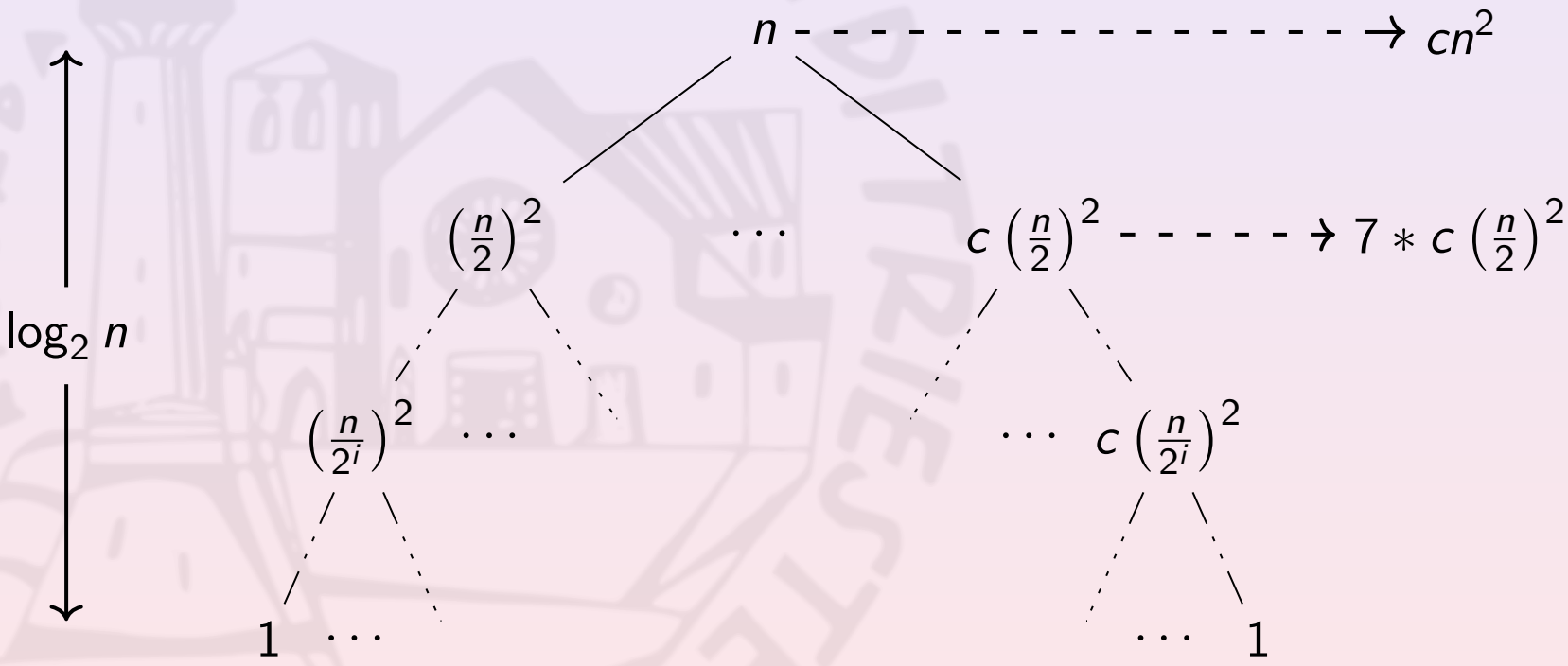

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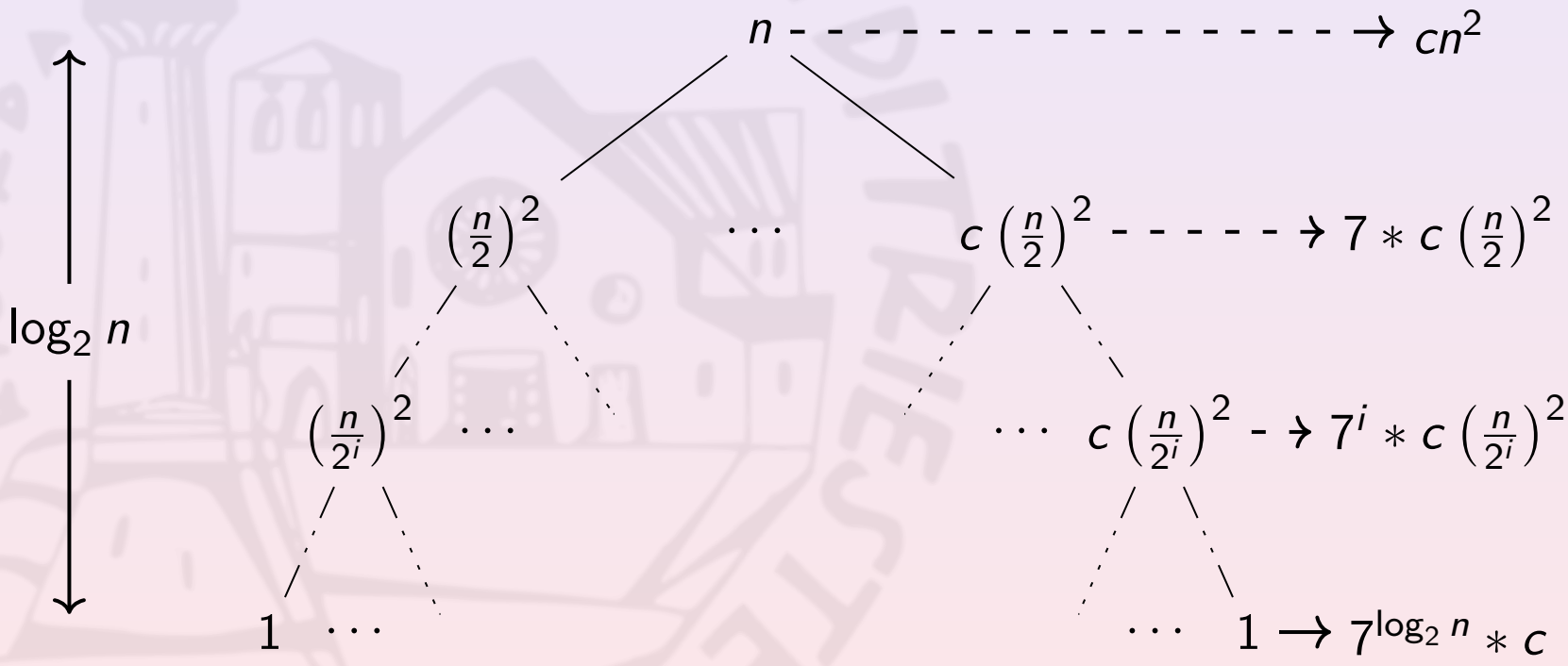

Strassen's Algorithm: Complexity (Cont'd)

$$T_M(n) = cn^2 \left(1 + \frac{7}{4} + \ldots + \left(\frac{7}{4}\right)^i + \ldots + \left(\frac{7}{4}\right)^{\log_2 n}\right)$$

$$= c'n^2 \left(\left(\frac{7}{4}\right)^{\log_2 n+1} - 1\right) \qquad c' = \frac{4}{3}c$$

$$= c'n^2 \left(\frac{7}{4}\right)^{\log_2 n+1} - c'n^2$$

$$= c''4^{\log_2 n} \left(\frac{7}{4}\right)^{\log_2 n} - c'n^2 \qquad c'' = \frac{7}{4}c'$$

$$= c''7^{\log_2 n} - c'n^2$$

$$= c''7^{\frac{\log_7 n}{\log_7 2}} - c'n^2 = c''n^{\log_2 7} - c'n^2$$

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