Fundations

Advanced Programming and Algorithmic Design

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Algorithms and Computational Models

What is an Algorithm?

Definition (Algorithm)

Is a sequence of well-defined steps that transforms a set of inputs into a set of outputs in a finite amount of time

A function described by an algorithm is calculable.

A function *implementable* in a computational model is computable.

Functions, Computability and Calcolability

Are all the functions computable in any specific model?

If this is not the case

- are there calculable functions that are not computable?
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Why Is This Relevant For Us?

What if calculability will not be related to computability?

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What if calculability will not be related to computability?

Algorithms would not guarantee implementability!

Halting Problem

Let h be the function that establish whether any program p eventually ends its execution ($\downarrow$) on an input i or runs forever($\uparrow$)

$$h(p,i) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } p(i) \text{ never ends} \\ 1 & \text{otherwise} \end{cases}$$

Definition (Halting problem)

Can we implement h?

Computability of Halting Problem

For any computable function $f(a, b)$, define

$$g_f(i) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } f(i,i) = 0 \\ \uparrow & \text{otherwise} \end{cases}$$

Since f is computable, so it is g_f. Let G_f implement it.

Can h be one of the f's?

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Since f is computable, so it is g_f. Let G_f implement it.

Can h be one of the f's?

- If $f(\text{G\_f}, \text{G\_f}) = 0 \implies g_f(\text{G\_f}) = 0$ and $h(\text{G\_f}, \text{G\_f}) = 1$
- If $f(\text{G\_f}, \text{G\_f}) \neq 0 \implies g_f(\text{G\_f}) \uparrow$ and $h(\text{G\_f}, \text{G\_f}) = 0$

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Thus, $h \neq f$ for all computable f's and h is not computable.

Church-Turing Thesis

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Every *effectively* calculable function is a computable function.

calculability $\Longrightarrow$ computability

If we have an algorithm for f, then f can be formally computed

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If we have an algorithm for f, then f can be formally computed

It also means that:

- all the "reasonable" computational models are equivalent
- we can avoid "hard-to-be-programmed" models (e.g., Turing machine)

Random-Access Machine (RAM)

- variables to store data (no types)
- arrays
- integer and floating point constants
- algebraic functions: $+$, $-$, $/$, $*$, $\lfloor\cdot\rfloor$, $\lceil\cdot\rceil$
- assignments
- pointers (no pointer arithmetic)
- conditional and loop statements
- procedure definitions and recursion
- simple "reasonable" functions, e.g., the length of an array

Algorithms are defined as programs on RAM.

A Simple Algorithm

Input: An array A of numbers $< a_1, \ldots, a_n >$.
Output: The maximum among $a_1, \ldots, a_n$.

```
def find_max(A):
    max_value ← A[1]
    for i ← 2..|A|:
        if A[i] > max_value:
            max_value ← A[i]
        endif
    endfor

    return max_value
enddef
```

RAM is not Real Hardware!!!

RAM models real hardware, but it lacks

- real HW limitations s.a. finiteness
- memory hierarchy
- instruction execution time

Time Complexity

How to Measure Algorithm Efficiency?

What about execution time?

How to Measure Algorithm Efficiency?

What about execution time? (for what input?)

Algorithms are not programs

Assuming 1 time unit per instruction are not realistic because execution time depends on:

- CPU instruction sets
- CPU/Memory/Bus Clock
- language and compiler
- OS memory handling
- ...

How to Measure Algorithm Efficiency?

~~What about execution time?~~

Any other ideas?

How to Measure Algorithm Efficiency?

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Any other ideas?

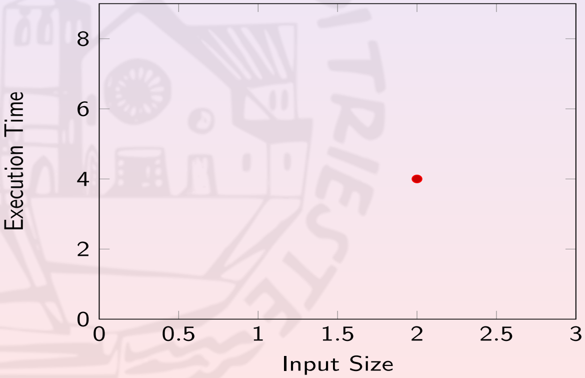
What about scalability?

Definition (Scalability)

Capacity for a system to handle input growth.

Growth Complexity

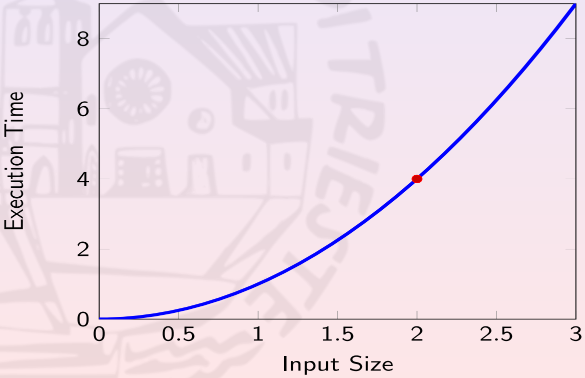
We do **not** measure the execution time for a given input



Growth Complexity

We do **not** measure the execution time for a given input

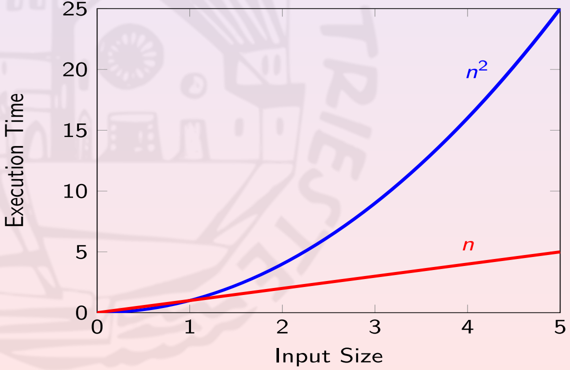
We **estimate** the relation between input size and execution time



Complexity Quiz!

Which growth is preferable between:

- n^2 and n?



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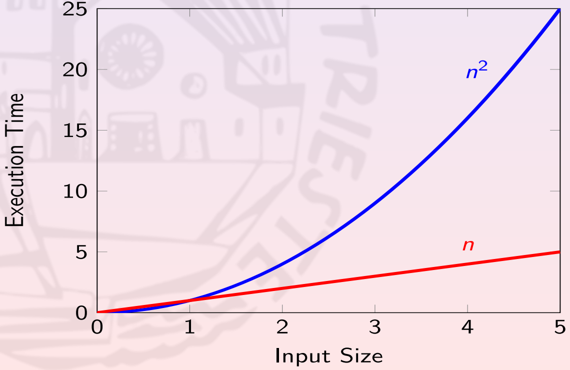

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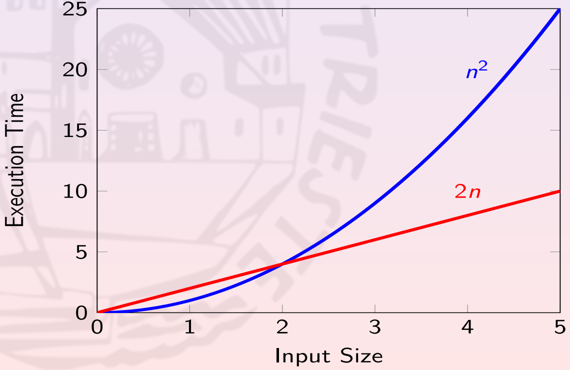

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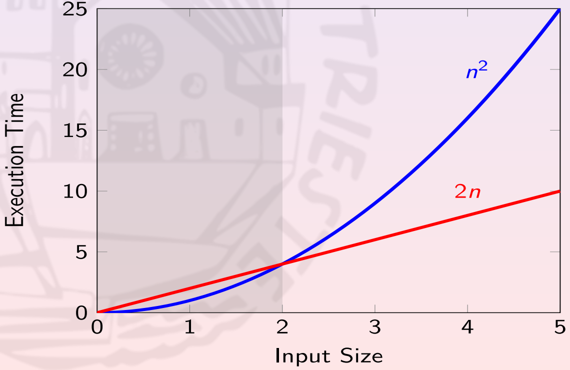

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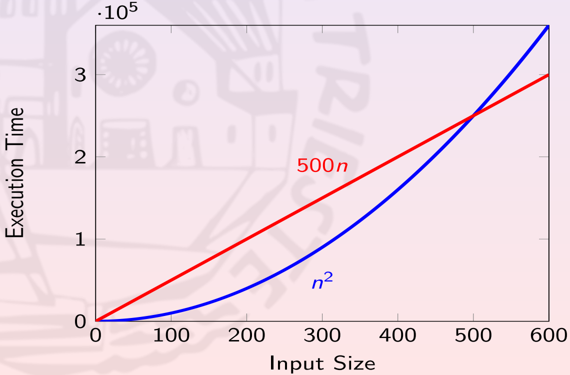
- n^2 and $500 * n$?



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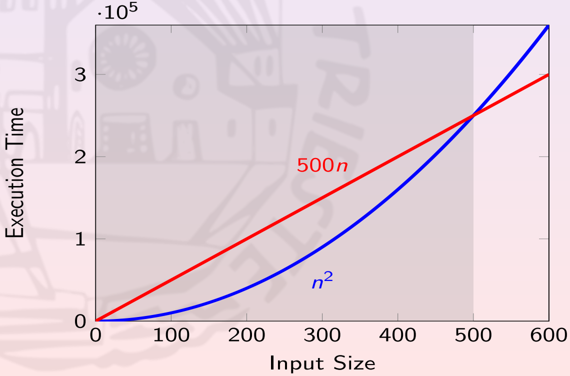

Asymptotic Time Complexity

Constants are not useful. We are looking at asymptotic behaviour.

We can abstract the single instruction execution time !!!

This intuition is also supported by linear time speedup theorem.

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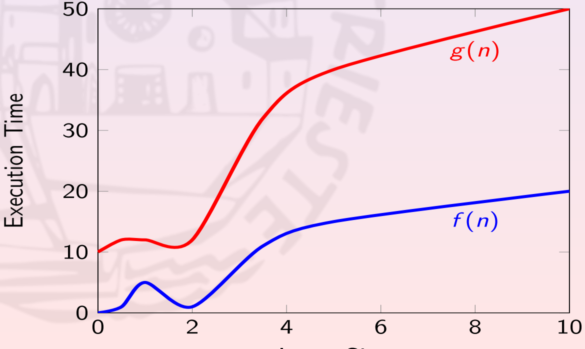
How to group all the functions that asymptotically are the same?

big O notation

$$O(f(n)) \stackrel{def}{=} \{g(n) \mid \exists c > 0 \exists n_0 > 0 \; m \geq n_0 \implies g(m) \leq c * f(m)\}$$

So, $g(n) \in O(f(n))$ iff



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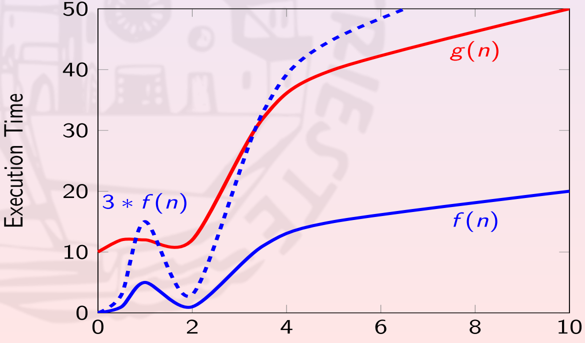

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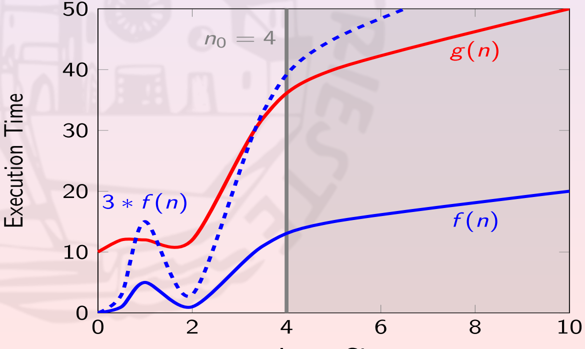


Some Useful Properties

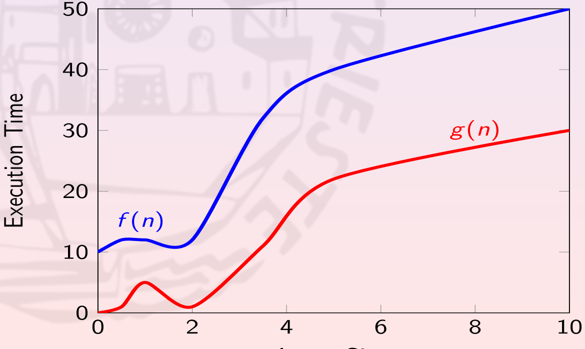
For any $c_1, c_2 \in \mathbb{N}$ and for any $k \in \mathbb{Z}$

- $f(n) \in O(f(n))$
- $O(f(n)) = O(c_1 * f(n) + k)$
- if $c_1 \geq c_2$, then $O(f(n)^{c_1} + k * f(n)^{c_2}) = O(f(n)^{c_1})$
- $O(f(n)^{c_1}) \subseteq O(f(n)^{c_1+c_2})$ es. $n \in O(n^2)$
- if $h(n) \in O(f(n))$ and $h'(n) \in O(g(n))$, then
  - $h(n) + h'(n) \in O(g(n) + f(n))$
  - $h(n) * h'(n) \in O(g(n) * f(n))$

big Ω notation

$$\Omega(f(n)) \stackrel{def}{=} \{g(n) | \exists c > 0 \exists n_0 > 0\ m \geq n_0 \implies c * f(m) \leq g(m)\}$$

So, $g(n) \in \Omega(f(n))$ iff



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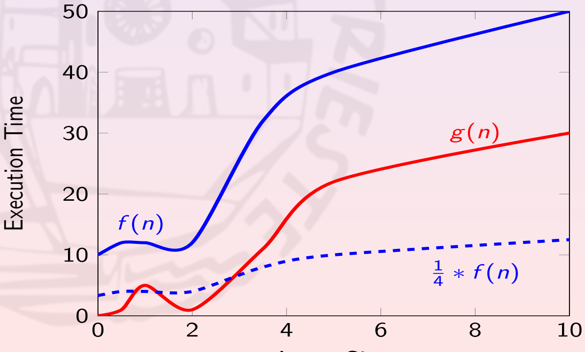

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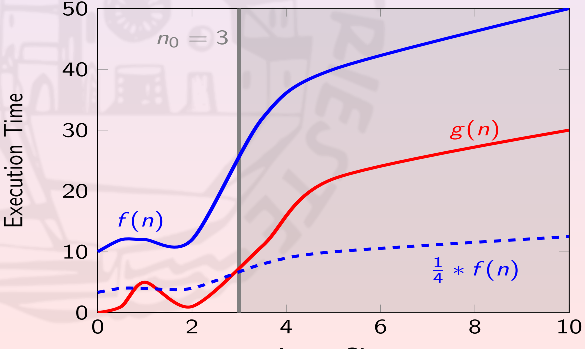

big Θ notation

$$\Theta(f(n)) \stackrel{def}{=} \{g(n) \mid \exists c_1, c_2 > 0 \exists n_0 > 0 \; m \geq n_0 \implies c_1 * f(m) \leq g(m) \leq c_2 * f(m)\}$$

Theorem

$$f(n) \in \Theta(g(n)) \iff f(n) \in O(g(n)) \cap \Omega(g(n))$$

Compute the Complexity of ...

Input: An array A of numbers $< a_1, \ldots, a_n >$.
Output: The maximum among $a_1, \ldots, a_n$.

```
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3       for i ← 2 upto len(A):
4           if A[i] > max_value:
5               max_value ← A[i]
6           endif
7       endfor
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9       return max_value
10  enddef
```

- 2 costs $O(1)$
- 4-6 cost $O(1)$
- 4-6 repeated $O(n)$ times
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$$O(1 + 1 * n \quad) =$$

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$O(1 + 1 * n + 1) = O(n)$

Some Useful Notions

Arrays and Lists (Abstract Data Types)

Arrays Are indexed collections of values fixed in length.

1	2	3	4	5	6	7	8	9	10
-4	0	1	2	5	6	7	11	12	13

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Single-Linked Lists Are sequences of values supporting `head` and `next` operations

$$\text{Head} \xrightarrow{\text{next}} \boxed{-4} \xrightarrow{\text{next}} \boxed{0} \xrightarrow{\text{next}} \boxed{1} \overset{\text{next}}{\dashrightarrow}$$

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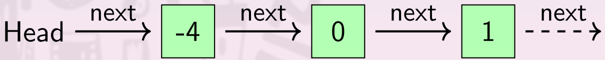
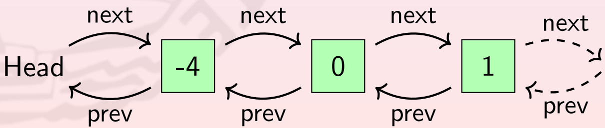


Double-Linked Lists Are sequences of values supporting `head`, `next` and `previous` operations



Queue and Stacks (Abstract Data Types)

Queues Are collections of values ruled according the FIFO policy. They support `head`, `is_empty`, `insert_back`, `extract_head` operations

Stacks Are collections of values ruled according the LIFO policy. They support `top`, `is_empty`, `insert_top`, `extract_top` operations

Graphs (Graph Theory)

Are pairs (V, E) where:

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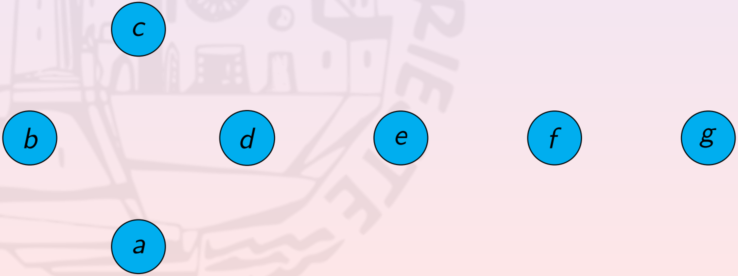
V is a set of nodes



Graphs (Graph Theory)

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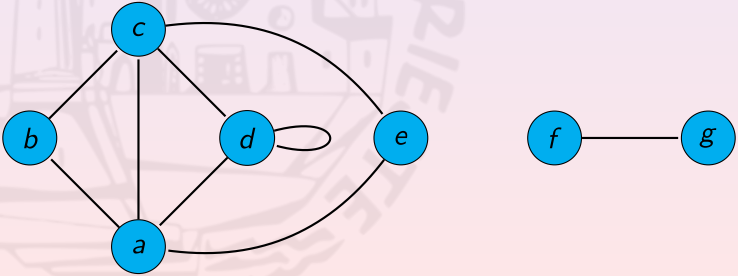

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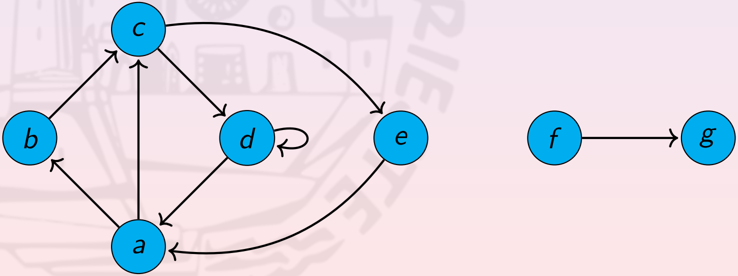
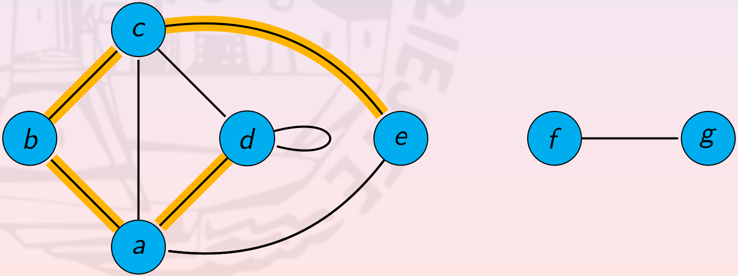
If the edges are (un)directed, the graph is (un)directed



Paths and Cycles

A path of length n between $a, b \in V$ is a sequence $e_1, \ldots, e_n$ s.t.

- e_1 involves a
- e_n involves b
- e_i and e_{i+1} involve a common node n_i



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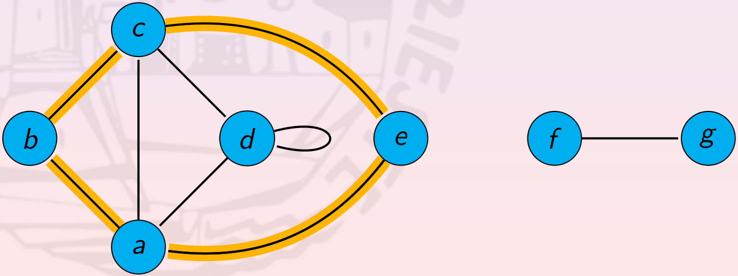
A cycle is a path whose initial and final node coincide.



Connected and Acyclic Graphs (Graph Theory)

A graph is connected if there is a path between every pairs of nodes

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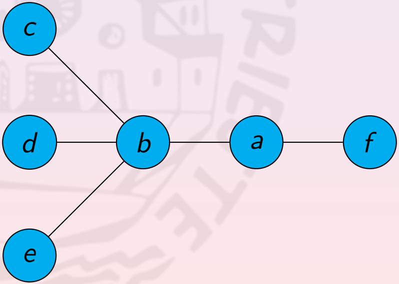
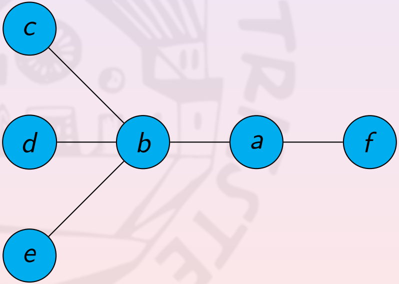
A tree is an connected and acyclic undirected graphs



Trees (Abstract Data Types)

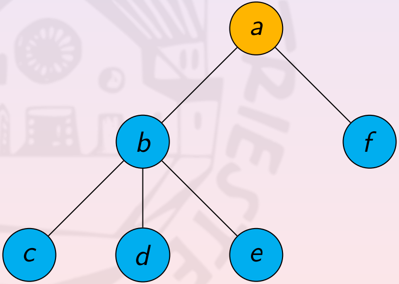
Organize data in a hierarchical finite tree (graph theory)



Trees (Abstract Data Types)

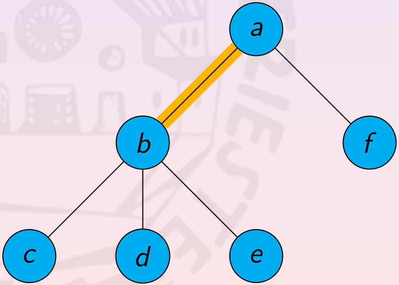
Organize data in a hierarchical finite tree (graph theory)

One of the nodes is the root of the graph



Tree Levels, Parents, Children and Siblings

The depth of a node is its distance from the root



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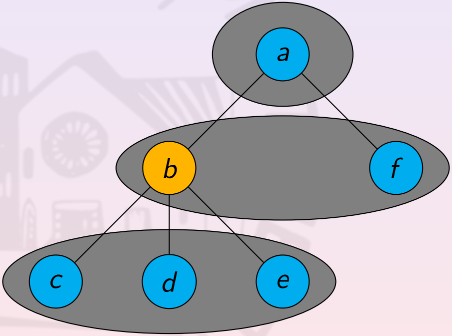
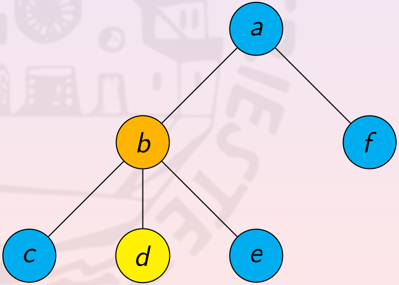


Tree Levels, Parents, Children and Siblings

The parent of a node is a node one step closer to the root



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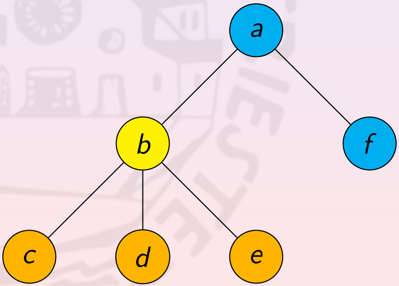

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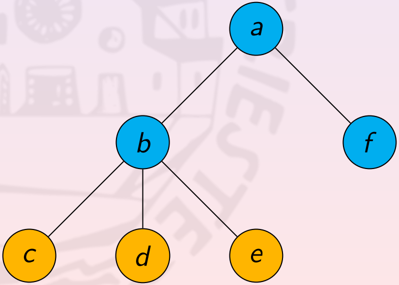
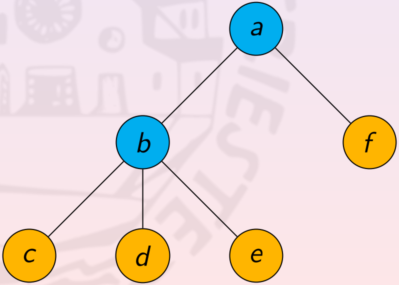
Two nodes are siblings if they have the same parent



Tree Leaves and Height

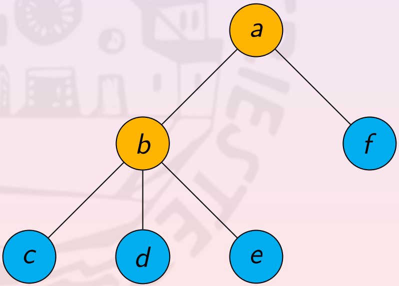
The leaves are nodes without children



Tree Leaves and Height

The leaves are nodes without children

The internal nodes have children

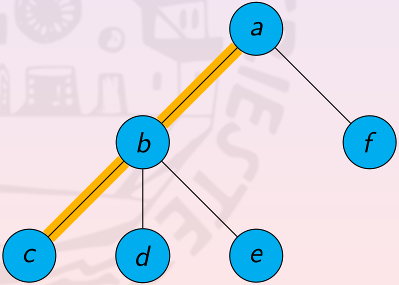


Tree Leaves and Height

The leaves are nodes without children

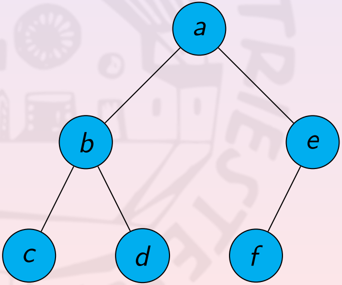
The internal nodes have children

The height of a tree is the max depth among those of its leaves



n-ary Tree and Completeness

Every node of a n-ary tree can have up to n children



n-ary Tree and Completeness

Every node of a *n*-ary tree can have up to *n* children

A *n*-ary tree is complete if the nodes in all the levels but the last one have *n* children