

Chain Matrix Multiplication

Advanced Programming and Algorithmic Design

Alberto Casagrande

Email: `acasagrande@units.it`

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The background of the slide features a large, faint watermark of the University of Trieste logo. The logo is circular and contains the text "UNIVERSITA' DEGLI STUDI DI TRIESTE" around the perimeter and "E SPLENDI" in the center. In the middle of the logo is a detailed illustration of a building with a prominent tower and a flag on top.

Problem Definition

Intuition for the Matrix-chain Multiplication Problem

Consider the matrices A_1, A_2, A_3

- A_1 having dimension 50×5
- A_2 having dimension 5×100
- A_3 having dimension 100×10

How many scalar multiplications does $A_1 \times A_2 \times A_3$ require?

Intuition for the Matrix-chain Multiplication Problem

Matrix product is associative i.e., $(A_1 \times A_2) \times A_3 = A_1 \times (A_2 \times A_3)$



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- if we compute $(A_1 \times A_2) \times A_3$

$$50 * 100 * 5 = 25000 \quad (\text{to compute } A_1 \times A_2)$$

$$50 * 10 * 100 = 50000 \quad (\text{to compute } (A_1 \times A_2) \times A_3)$$

- if we compute $A_1 \times (A_2 \times A_3)$

$$5 * 10 * 100 = 5000 \quad (\text{to compute } A_2 \times A_3)$$

$$50 * 10 * 5 = 2500 \quad (\text{to compute } A_1 \times (A_2 \times A_3))$$

75000 $((A_1 \times A_2) \times A_3)$ vs

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75000 $((A_1 \times A_2) \times A_3)$ **vs** **7500** $(A_1 \times (A_2 \times A_3))$

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Consider the **chain** of matrices $\langle A_1, \dots, A_n \rangle$ where

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Compute a **parenthesization** that minimizes the # of scalar products for the chain multiplication

A large, faint watermark of the University of Trieste logo is visible in the background. It features a circular emblem with a building and a figure, surrounded by the text "UNIVERSITA' DEGLI STUDI DI TRIESTE" and "FUND. 1222".

A Naïve Approach

Recursive Solution

We may try to search among all the possible parenthesizations

- if $n = 1$, the parenthesization is obvious
- if $n > 1$, the chain can be parenthesized as

$$(A_1 \times \dots A_k) \times (A_{k+1} \times \dots A_n)$$

for any $k \in [1, n - 1]$. Recursively produce the parenthesizations for $\langle A_1, \dots, A_k \rangle$ and $\langle A_{k+1}, \dots, A_n \rangle$

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How many parenthesizations has $\langle A_1, \dots, A_n \rangle$?

Counting Parenthesizations

$\langle A_1, \dots, A_n \rangle$ has

$$P(n) = \begin{cases} 1 & \text{if } n = 1 \\ \sum_{k=1}^{n-1} P(k) * P(n-k) & \text{if } n > 1 \end{cases}$$

different parenthesizations

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It can be proved that $P(n) \in \Omega(2^n)$

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Too many parenthesizations to be enumerated!!!
(if you don't believe it, try for $n = 8$)

Some Breakthrough Observations

- if $(A_1 \times \dots \times A_k) \times (A_{k+1} \times \dots \times A_n)$ is optimal for the chain,
 - the 1st part is optimal for $\langle A_1, \dots, A_k \rangle$
 - the 2nd part is optimal for $\langle A_{k+1}, \dots, A_n \rangle$
- many branches of the 'naïve' recursive approach perform the very same computation
 - e.g. for every parenthesization of $A_1 \times \dots \times A_k$, the parenthesizations for $A_{k+1} \times \dots \times A_n$

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Idea:

Recursively compute optimal parenthesizations and use dynamic programming



A Dynamic Programming Solution

Dynamic Programming Solution

Store the minimum # of products for all the sub-chains in m

Recursively, compute $m[i, j]$ as:

$$m[i, j] = \begin{cases} 0 & \text{if } i = j \\ \min_{k \in [i, j-1]} \{m[i, k] + m[k+1, j] + p_{i-1}p_kp_j\} & \text{if } i < j \end{cases}$$

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For each i, j also store in $s[i, j]$ the k that minimizes

$$m[i, k] + m[k+1, j] + p_{i-1}p_kp_j$$

i.e., the parenthesization for the current level

Dynamic Programming Solution: Example

Consider A_1 (3×5), A_2 (5×10), A_3 (10×2) and A_4 (2×3).

$A_1 \times A_2 \times A_3 \times A_4$

1	2	3	4	
0	?	?	?	1
	0	?	?	2
		0	?	3
			0	4

m

2	3	4	
?	?	?	1
	?	?	2
		?	3

s

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1	2	3	4	
0	?	?	?	1
	0	?	?	2
		0	?	3
			0	4

m

2	3	4	
?	?	?	1
	?	?	2
		?	3

s

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Consider A_1 (3×5), A_2 (5×10), A_3 (10×2) and A_4 (2×3).

$$((A_1) \times (A_2 \times A_3 \times A_4))$$

1	2	3	4	
0	?	?	?	1
	0	?	?	2
		0	?	3
			0	4

m

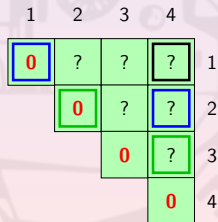
2	3	4	
?	?	?	1
	?	?	2
		?	3

s

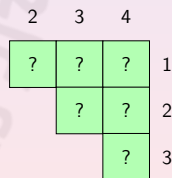
Dynamic Programming Solution: Example

Consider A_1 (3×5), A_2 (5×10), A_3 (10×2) and A_4 (2×3).

$$((A_1) \times ((A_2) \times (A_3 \times A_4)))$$



m

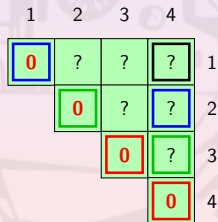


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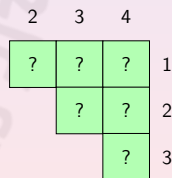
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$$((A_1) \times ((A_2) \times ((A_3) \times (A_4))))$$



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s

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Consider A_1 (3×5), A_2 (5×10), A_3 (10×2) and A_4 (2×3).

$$((A_1) \times ((A_2) \times ((A_3) \times (A_4))))$$

1	2	3	4	
0	?	?	?	1
	0	?	?	2
		0	60	3
			0	4

m

2	3	4	
?	?	?	1
	?	?	2
		3	3

s

Dynamic Programming Solution: Example

Consider A_1 (3×5), A_2 (5×10), A_3 (10×2) and A_4 (2×3).

$$((A_1) \times ((A_2) \times (A_3 \times A_4)))$$

1	2	3	4	
0	?	?	?	1
	0	?	210	2
		0	60	3
			0	4

m

2	3	4	
?	?	?	1
	?	2	2
		3	3

s

Dynamic Programming Solution: Example

Consider A_1 (3×5), A_2 (5×10), A_3 (10×2) and A_4 (2×3).

$$((A_1) \times ((A_2 \times A_3) \times (A_4)))$$

1	2	3	4	
0	?	?	?	1
	0	?	210	2
		0	60	3
			0	4

m

2	3	4	
?	?	?	1
	?	2	2
		3	3

s

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Consider A_1 (3×5), A_2 (5×10), A_3 (10×2) and A_4 (2×3).

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1	2	3	4	
0	?	?	?	1
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m

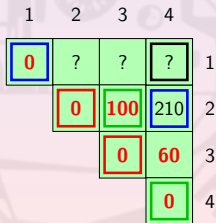
2	3	4	
?	?	?	1
	?	2	2
		3	3

s

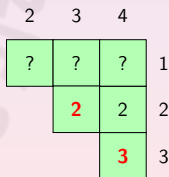
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1	2	3	4	
0	?	?	?	1
	0	100	130	2
		0	60	3
			0	4

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2	3	4	
?	?	?	1
	2	3	2
		3	3

s

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Consider A_1 (3×5), A_2 (5×10), A_3 (10×2) and A_4 (2×3).

$$((A_1) \times (A_2 \times A_3 \times A_4))$$

1	2	3	4	
0	?	?	175	1
	0	100	130	2
		0	60	3
			0	4

m

2	3	4	
?	?	1	1
	2	3	2
		3	3

s

Dynamic Programming Solution: Example

Consider A_1 (3×5), A_2 (5×10), A_3 (10×2) and A_4 (2×3).

$$((A_1 \times A_2) \times (A_3 \times A_4))$$

	1	2	3	4	
1	0	?	?	175	1
2		0	100	130	2
3			0	60	3
4				0	4

m

	2	3	4	
1	?	?	1	1
2		2	3	2
3			3	3

s

Dynamic Programming Solution: Example

Consider A_1 (3×5), A_2 (5×10), A_3 (10×2) and A_4 (2×3).

$$(((A_1) \times (A_2)) \times (A_3 \times A_4))$$

1	2	3	4	
0	150	?	175	1
	0	100	130	2
		0	60	3
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m

2	3	4	
1	?	1	1
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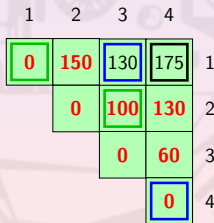
2	3	4	
1	?	1	1
	2	3	2
		3	3

s

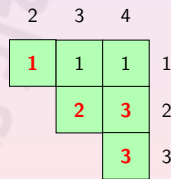
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m

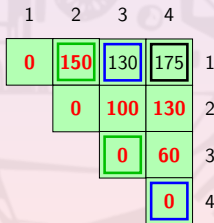


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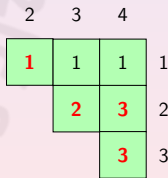
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m



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1	2	3	4	
0	150	130	175	1
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m

2	3	4	
1	1	1	1
	2	3	2
		3	3

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Dynamic Programming Solution: Example

Consider A_1 (3×5), A_2 (5×10), A_3 (10×2) and A_4 (2×3).

$$(A_1 \times A_2 \times A_3 \times A_4)$$

1	2	3	4	
0	150	130	148	1
	0	100	130	2
		0	60	3
			0	4

m

2	3	4	
1	1	3	1
	2	3	2
		3	3

s

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3			3	3

s

Dynamic Programming Solution: An Iterative Version

Both m and s can be computed iteratively from the shortest sub-chains to the longest one.

	1	2	3	4	
1	0	?	?	?	1
2		0	?	?	2
3			0	?	3
4				0	4

m

	2	3	4	
1	?	?	?	1
2		?	?	2
3			?	3

s

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	1	2	3	4	
1	0	150	?	?	1
2		0	100	?	2
3			0	60	3
4				0	4

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	2	3	4	
1	1	?	?	1
2		2	?	2
3			3	3

s

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Both m and s can be computed iteratively from the shortest sub-chains to the longest one.

	1	2	3	4	
1	0	150	130	?	1
2		0	100	130	2
3			0	60	3
4				0	4

m

	2	3	4	
1	1	1	?	1
2		2	3	2
3			3	3

s

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	1	2	3	4	
1	0	150	130	148	1
2		0	100	130	2
3			0	60	3
4				0	4

m

	2	3	4	
1	1	1	3	1
2		2	3	2
3			3	3

s

Dynamic Programming Solution: Code

```
def MatrixChain(P):  
    m ← allocate(1..n, 1..n)  
    s ← allocate(1..n-1, 2..n)  
    for i ← 1..n:  
        m[i, i] ← 0  
    for l ← 2..n:  
        for i ← 1..(n-l+1):  
            j ← i + l - 1  
            MatrixChainAux(P, m, s, i, j)  
        endfor  
    endfor  
  
    return (m, s)  
enddef
```

Dynamic Programming Solution: Code

```
def MatrixChainAux(P,m,s,i,j):  
    m[i,j] ← INFINITY  
    for k ← i..(j-1):  
        q ← m[i,k] + m[k+1,j] +  
            P[i-1]*P[k]+P[j]  
        if q < m[i,j]:  
            m[i,j] ← q  
            s[i,j] ← k  
        endif  
    endfor  
enddef
```

Dynamic Programming Solution: Complexity

The computation of $m[i, j]$ takes time:

$$\sum_{k=i}^{(j-1)} \Theta(1) = \Theta(j - i)$$

Since $i \in [1, n]$ and $j \in [i, n]$,

$$\begin{aligned} T_C(n) &= \sum_{i=1}^n \sum_{j=i}^n \Theta(j - i) = \Theta \left(\sum_{i=1}^n \left(\sum_{j=i}^n j \right) - n * i \right) \\ &= \Theta \left(\sum_{i=1}^n \frac{n * (n + 1)}{2} - \frac{i * (i + 1)}{2} - n * i \right) = \Theta(n^3) \end{aligned}$$