Stability of confined fluid interface

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1 Abstract

2 Introduction

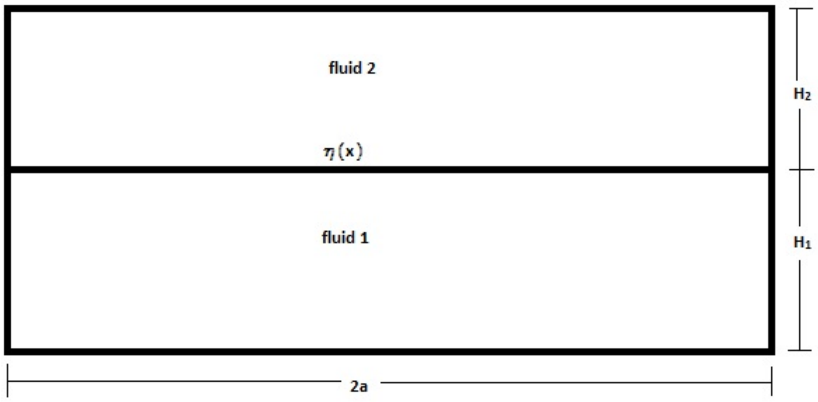


Figure 1: Schematic of the problem

As a case for study, a simple 2 fluid system is considered as shown in the figure. Since there is translational invariance with respect to time, we can assume a normal mode solution of the form $e^{\imath(\omega t)}$. Here, we consider only irrotational perturbations with the justification that, perturbations in vorticity would die away in the absence of viscosity. In the case of irrotational flow, use of a scalar potential $\phi(x, y)$ would prove to be useful and simple. The governing equations are derived with respect to the normal mode assumption and these equations will be solved using both analytical and numerical methods.

3 Analytical solution

3.1 Governing perturbation equations

The governing equations have alll been non-dimensionalized by the following scales. These will be the scales used throughout this report. Length scale: a, density scale: $\gamma a^2 g$, time scale: $\sqrt{\frac{a}{g}}$. The non dimensional equations are:

$$\frac{\partial^2 \hat{\phi}_j}{\partial x^2} + \frac{\partial^2 \hat{\phi}_j}{\partial y^2} = 0 \tag{1}$$

$$\frac{\partial \hat{\phi}_2}{\partial y}(x, h_2) = 0 \tag{2}$$

$$\frac{\partial \hat{\phi}_1}{\partial y}(x, -h_1) = 0 \tag{3}$$

$$\frac{\partial \hat{\phi}_j}{\partial x}(\pm 1, y) = 0 \tag{4}$$

$$\frac{\partial \hat{\phi}_j}{\partial y}(x,0) = \imath\omega\hat{\eta(x)} \tag{5}$$

$$\frac{d^2\hat{\eta}}{dx^2} = Ca\hat{\eta} + \imath\omega(\bar{\rho_1}\hat{\phi_1}(x,0) - \bar{\rho_2}\hat{\phi_2}(x,0)) \tag{6}$$

In the above, ω is the non-dimensional eigen value and $Ca, \bar{\rho_j}, h_j$ are non-dimensional parameters.

$$Ca = \frac{(\rho_1 - \rho_2)a^2 g}{\gamma} \tag{7}$$

$$\bar{\rho_j} = \frac{\rho_j a^2 g}{\gamma} \tag{8}$$

$$h_j = \frac{H_j}{a} \tag{9}$$

The quantities H_j and a are real y limits and x limits.

3.2 solution

The equation (1) is the only governing partial differential equation while the rest are all boundary conditions. This equation is bascically a laplace equation and the standard way to solve this is to use separation of variables. Since the boundary conditions are also linear and homogeneous, this methodology would prove to be very much more easier. The solution for the equation is assumed to be of the form:

$$\hat{\phi_j} = X_j(x)Y_j(y) \tag{10}$$

Substituting this into equation (1) would give:

$$\frac{X_j''(x)}{X_j(x)} + \frac{Y_j''(y)}{Y_j(y)} = 0 \tag{11}$$

Since it is a sum of functions of two different independent variables, for the sum to be zero, each of them must be a constant with opposite signs.

$$\frac{X_j''(x)}{X_j(x)} = -k^2 \text{ and } \frac{Y_j''(y)}{Y_j(y)} = k^2 \tag{12}$$

Upon applying the boundary condition (4),

$$X_j(x) = cos(k_n x) \text{ and } k_n = n\pi \quad n \in Natural\ numbers \tag{13}$$

Upon applying the boundary conditions (2) and (3),

$$\hat{\phi_j} = Acos(k_n x)\frac{e^{-k_n y} - e^{-2k_n h_j}e^{k_n y}}{k_n(1 + e^{-2k_n h_j})} \tag{14}$$

Where, h_j is h_2 or $-h_1$.The equation (5) will give the interface shape as:

$$\hat{\eta}(x) = \frac{\imath A}{\omega_n}cos(k_n x) \tag{15}$$

Substituting these solutions in equation (6), we get the dispertion relation as:

$$\omega_n = \pm\sqrt{\frac{Cak_n + k_n^3}{\bar{\rho_2}tanh(k_n h_2) + \bar{\rho_1}tanh(k_n h_1)}} \tag{16}$$

3.3 Discussion

For the system to have no instability, all admissible values of k_n must give a non-negative imaginary part of ω_n. This means,

$$k_n \geq \sqrt{-Ca} \tag{17}$$

Ca is bacically $\bar{\rho_1} - \bar{\rho_2}$. So, if the lower fluid is denser, the system is always stable. For the system to be always stable even with adverse density stratification, the above equation must be satisfied for all admissible values of $k_n = n\pi$.

$$Ca \leq \pi^2 \tag{18}$$

This is the point of bifurcation where, the system transitions into an unstable mode. So, if we were to contain an adverse density stratification, we would like to choose a container of size such that,

$$a \leq \pi\sqrt{\frac{\gamma}{(\rho_1 - \rho_2)g}} \tag{19}$$

3.4 Observation

The shape of the interface has a slope of zero at the boundaries. Generally, the dynamic contact angle when linearized is of the form:

$$\frac{d\hat{\eta}}{dx}(\pm 1) = C\frac{\partial \hat{\phi}_j}{\partial y}(\pm 1, 0) \tag{20}$$

On substituting the solution obtained above, we get,

$$C = 0 \tag{21}$$

This means, the physics of the problem doesn't allow the slope of the interface at the boundaries to change. This very observation made the author to not assume a normal mode of the form $e^{\imath(kx-\omega t)}$ and solve for the PDE numerically. More of this will be discussed in the next section. But, the separation of variables solution stands unquestionable. So, there is something in the physics of the problem which prohibits the contact angle from varying.

4 Numerical method

4.1 *Governing perturbation equations*

The schematic of the problem is as follows:
The governing equations are same as in the previous section. To simplify the numerical code, the following boundary condition is used. This is as discussed in the previous section a consequence of the governing equations.

$$\frac{d\hat{\eta}}{dx}(\pm 1) = 0 \tag{22}$$

4.2 *Discretization of the domain*

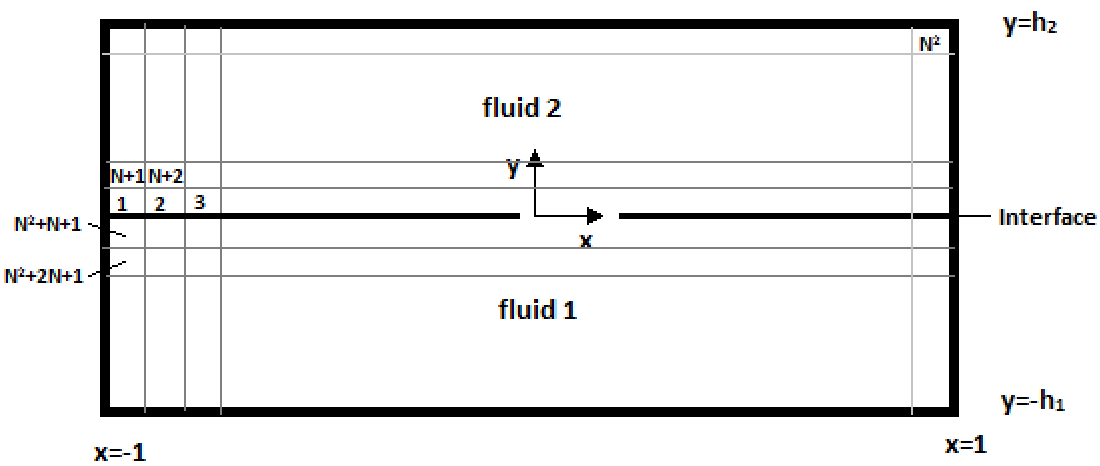


Figure 2: Schematic for discretization

The domain is disretized into grids as shown in the schematic. It is done such that the *fluid 1* and *fluid 2* regions are discretized into $N \times N$ grid points and the interface is discretized into N grid points. This means, the resulting unknowns will be $(2N^2 + N)$ resulting in a matrix equation with matrices of dimension$(2N^2 + N) \times (2N^2 + N)$.

The unknowns are all collected in a vector ψ.

$$\psi = \begin{pmatrix} \hat{\phi}_2(1) \\ \hat{\phi}_2(2) \\ \vdots \\ \hat{\phi}_2(N^2-1) \\ \hat{\phi}_2(N^2) \\ \hat{\eta}(N^2+1) \\ \hat{\eta}(N^2+2) \\ \vdots \\ \hat{\eta}(N^2+N-1) \\ \hat{\eta}(N^2+N) \\ \hat{\phi}_1(N^2+N+1) \\ \hat{\phi}_1(N^2+N+2) \\ \vdots \\ \hat{\phi}_1(2N^2+N-1) \\ \hat{\phi}_1(2N^2+N) \end{pmatrix} \tag{23}$$

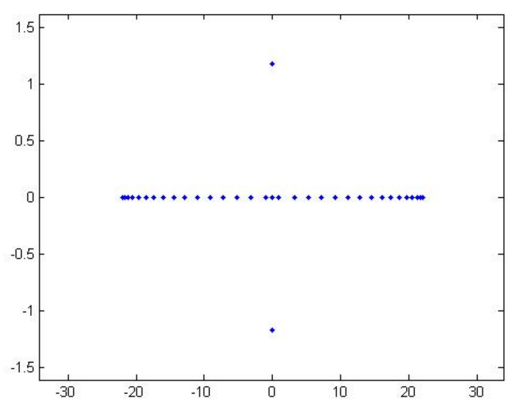
The governing equation and the boundary conditions in discretized form are now wriiten in the form:

$$A\psi = \omega B\psi \tag{24}$$

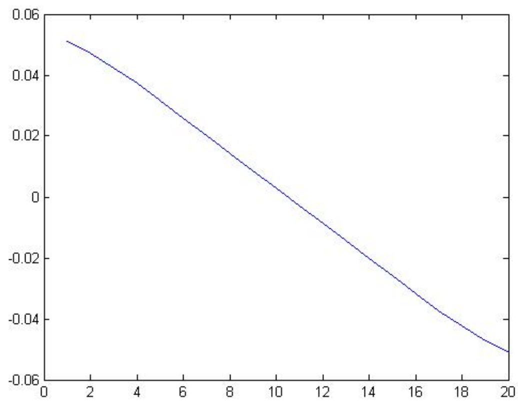
A and B are the corresponding coefficient matrices. This matrix equation must have non trivial solutions. This means, we must look for eigen values of the equation (24). The construction of the matrices A and B and computation of the eigen values has been done using MATLAB.

4.3 ***Results***

The predicted bifurcation of the system happens at $Ca = \pi^2$ i.e., $Ca \approx 9.87$. The following are the plots of the eigen values and the interface shape at values of Ca around the bifurcation point.



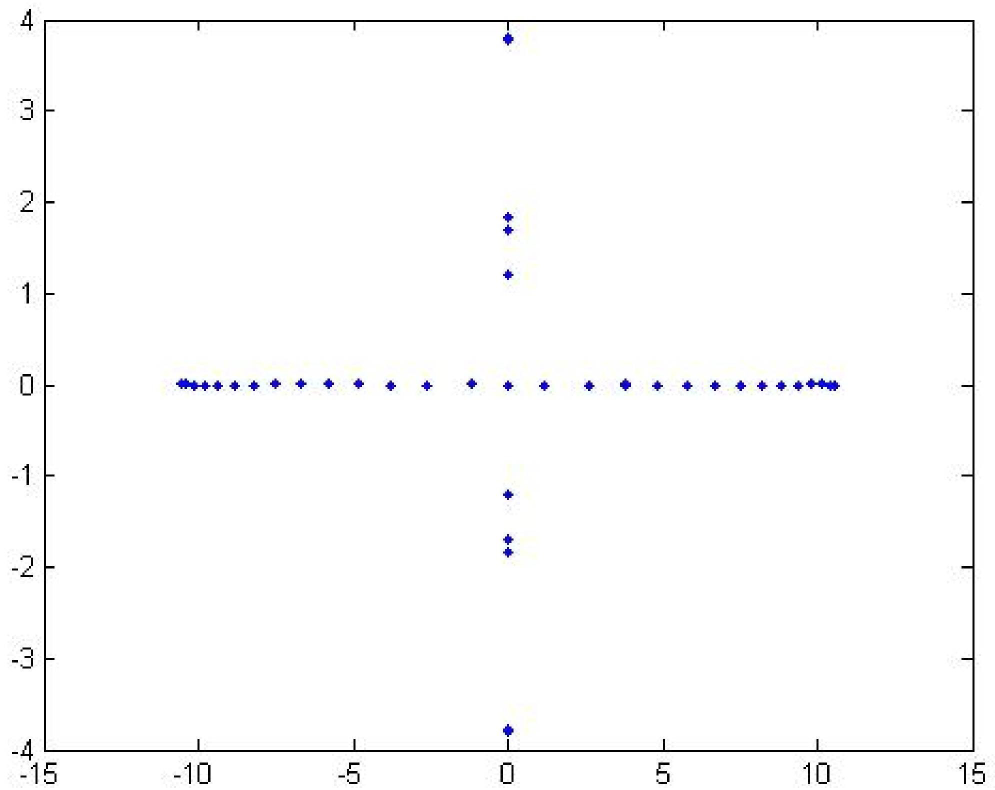
(a) Eigenvalue plot



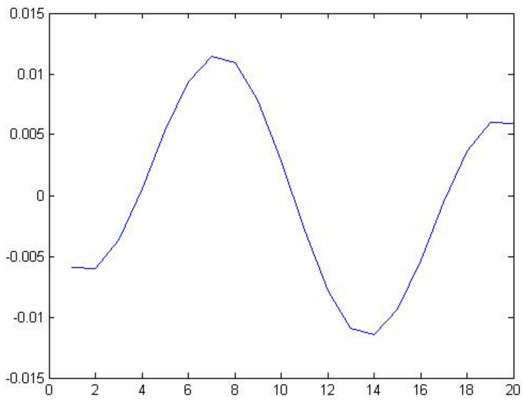
(b) Interface shape

Figure 3: $Ca = -\pi^2$

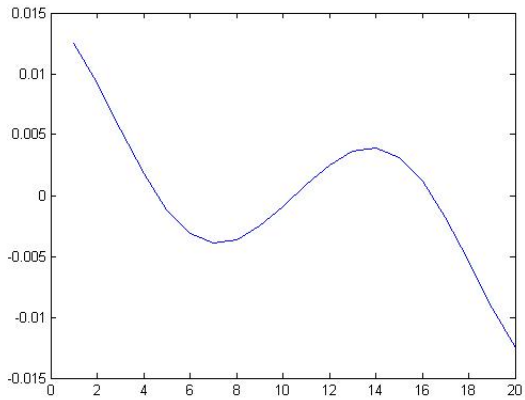
We can see that there is exactly only 1 mode with a negative imaginary part of the eigen value. This is because this is exactly at the bifurcation point.



(a) Eigenvalue plot



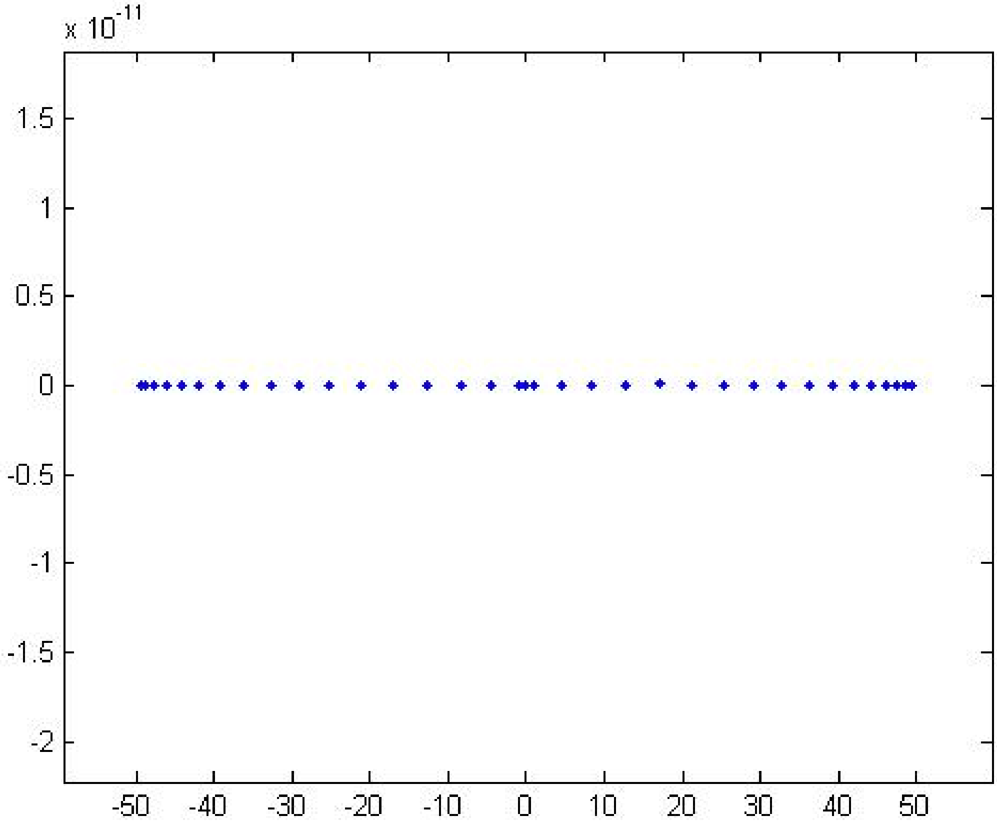
(b) Interface shape $n = 1$



(c) Interface shape $n = 2$

Figure 4: $Ca = -4\pi^2$

At a value of $Ca = -4\pi^2$, analytical calculations suggest a possible 2 modes with negative imaginary part for the eigen value. This is seen in the figure above. The farther most ones are spurious as the mode shape is very random and the same is true with one of the eigen values at around $0 - 2\imath$. This means that there are effectively ony 2 actual modes. This is consistent with the analytical solution.



(a) Eigenvalue plot

Figure 5: $Ca = -2$

As expected, at a value of Ca lower than the critical value, we see no imaginary part of the eigen value appearing. This is consistent with the analytical calculations.