Assignemnt #1
AS5435: Waves in fluids

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ME14B027

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1. **Acoustic Waves with gravity**:
   Base flow quantities are given by:

$$\vec{u}_b = 0 \tag{1}$$
$$\rho_b = \rho_0 \tag{2}$$
$$p_b = p_o - \rho_0 g z \tag{3}$$

   After simplification and linearizion of the Navier Stokes equations for a compressible fluid, we get the following equations and the state equation which govern the evolution of the perturbation flow quantities

$$\frac{\partial \rho}{\partial t} + \rho_0 \nabla.(\vec{u}) = 0 \tag{4}$$
$$\rho_0 \frac{\partial \vec{u}}{\partial t} = -\nabla p + \rho \vec{g} \tag{5}$$
$$\rho = \rho_0 \kappa_s p \tag{6}$$

   Simplifying the equations (4) to (6), we get the following equation in terms of density perturbation. It is obtained by taking the gradient of (5) and using (4) and (6) to eliminate $\vec{u}$ and p respectively.

$$\boxed{\frac{\partial^2 \rho}{\partial t^2} = \frac{1}{\kappa_s \rho_0} \nabla^2 \rho - \vec{g}.\nabla \rho} \tag{7}$$

   Here, $c = 1/\sqrt{\kappa_s \rho_0}$ is the phase velocity. Since (7) is a linear PDE, we can look for solutions of the form $e^{\iota(\vec{k}.\vec{r} - \omega t)}$. This will lead us to the following dispersion relation

$$\omega^2 = c^2 |\vec{k}|^2 + \iota(\vec{g}.\vec{k}) \tag{8}$$

2. The wave packet ϕ is defined by the gaussian spectrum $\hat{\phi}$ given by:

$$\hat{\phi}(k) = \hat{\phi}_0 \; e^{-\sigma^2 (k - k_0)^2} \tag{9}$$

$\phi(x,t)$ is obtained by performing an inverse fourier transformation on the $\hat{\phi}(k)$.

$$\phi(x,t) = \mathrm{Re}\left\{ \int_{-\infty}^{\infty} \hat{\phi}(k) e^{\iota(kx-\omega t)} dk \right\} \tag{10}$$

We can drop the real part operator for now and apply it later at the end of all the simplifications to recover only the real part of $\phi(x,t)$. Now, let us define $k' = k - k_0$ and $\omega' = \omega - \omega_0$.

$$\phi(x,t) = e^{\iota(k_0 x - \omega_0 t)} \int_{-\infty}^{\infty} \hat{\phi}_0 \; e^{-\sigma^2 k'^2} e^{\iota(k'x - \omega' t)} dk' \tag{11}$$

$$\phi(x,t) = e^{\iota(k_0 x - \omega_0 t)} \int_{-\infty}^{\infty} \hat{\phi}_0 \; e^{-\sigma^2 k'^2 + \iota x k' - \iota \omega' t} dk' \tag{12}$$

The frequency ω at any given wave number k is given by the dispersion relation which can be expanded in a taylor series expansion as

$$\omega(k) = \omega_0 + \left.\frac{d\omega}{dk}\right|_{k_0} (k - k_0) + \frac{1}{2} \left.\frac{d^2\omega}{dk^2}\right|_{k_0} (k - k_0)^2 + O(k'^3) \tag{13}$$

Approximating the dispersion relation to until the quadratic terms, we can simplify equation (12) into

$$\phi(x,t) = e^{\iota(k_0 x - \omega_0 t)} \int_{-\infty}^{\infty} \hat{\phi}_0 \; e^{-\sigma^2 k'^2 + \iota x k' - \iota\left(\left.\frac{d\omega}{dk}\right|_{k_0} k' + \frac{1}{2} \left.\frac{d^2\omega}{dk^2}\right|_{k_0} k'^2 \right) t} dk' \tag{14}$$

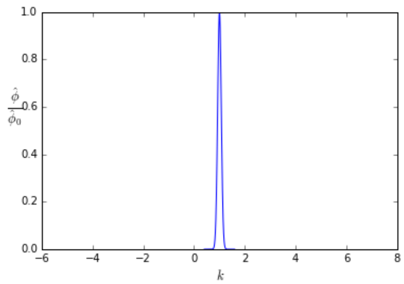
The above integral can be computed closed form by defining a new variable $v = \sigma'(t)\big(k' - \frac{\iota}{2\sigma'^2}(x - \left.\frac{d\omega}{dk}\right|_{k_0} t)\big)$ where $\sigma'(t)^2 = \sigma^2 + \frac{\iota}{2} \left.\frac{d^2\omega}{dk^2}\right|_{k_0} t$. The integral now simplifies to

$$\boxed{\phi(x,t) = \mathrm{Re}\left\{ \frac{\hat{\phi}_0 \sqrt{\pi}}{\sigma'(t)} \; e^{\frac{-\left(x - \left.\frac{d\omega}{dk}\right|_{k_0} t\right)^2}{4\sigma'(t)^2}} \; e^{\iota(k_0 x - \omega_0 t)} \right\}} \tag{15}$$

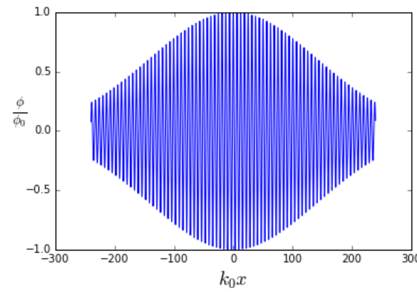
The system can be considered non-dispersive as long as the difference in the group velocities of close by wave numbers doesn't go beyond the rate at which the width of the wave packet is growing. One way to estimate this is to consider the time scale of growth of the spread itself. This is of the orders of $\frac{\sigma^2}{\omega_0''}$.

Plots:

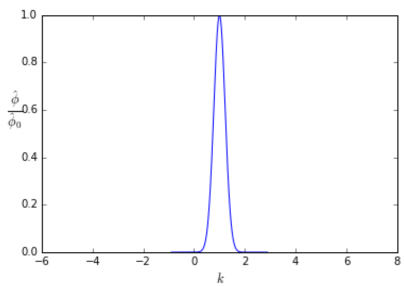
(a) Gaussian spectrum and initial wave form:



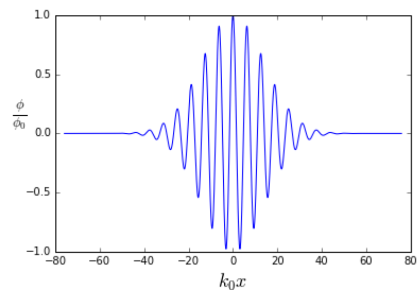
(a) $\sigma k_0 = 100$



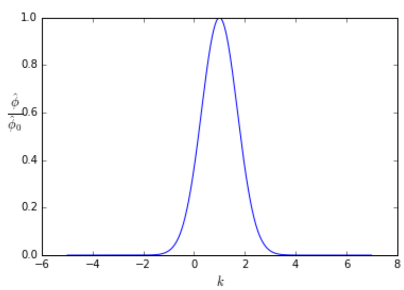
(b) $\sigma k_0 = 100$



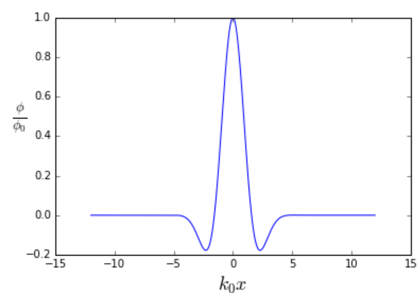
(c) $\sigma k_0 = 10$



(d) $\sigma k_0 = 10$



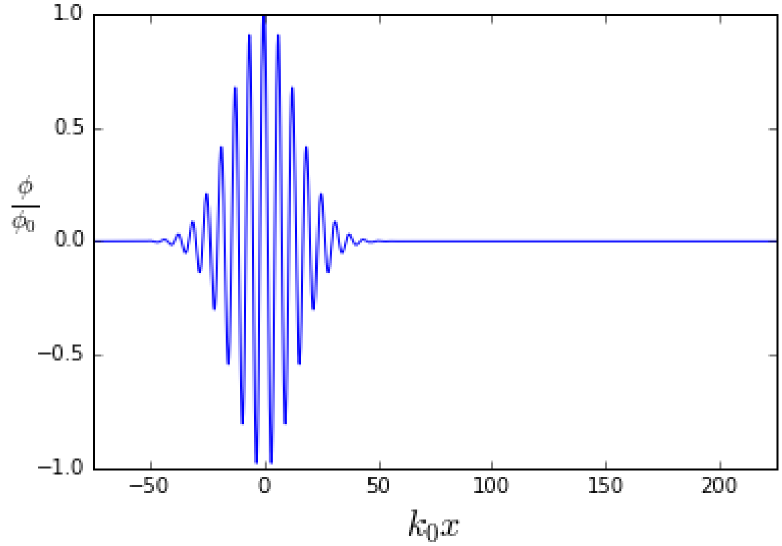
(e) $\sigma k_0 = 1$



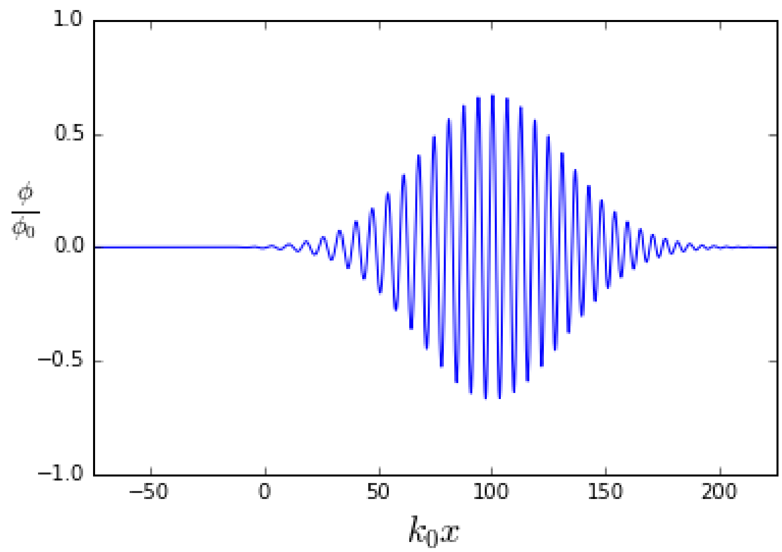
(f)$\sigma k_0 = 1$

Figure 1: Figures (a), (c) and (e) show the plot of the spectrum while (b), (d) and (f) are the plots of the corresponding wave packets at time $t = 0$. Clearly, as the parameter σ decreases, the spread of the wave packet decreases.

(b) Time evolution of the gaussian wave packet:



(a) $c_g k_0 t = 0$



(b) $c_g k_0 t = 100$

Figure 2: $\omega_0'' k_0 = 4c_g$ and $\sigma k_0 = 10$. It can be seen that as time progresses, the wave packet diffuses which is reflected in the reduction of the peak value of the envelope.

Code:

```
%matplotlib inline
import numpy as np
import matplotlib.pyplot as plt

#(i)
var = 0.01
k = np.arange(1-6*np.sqrt(var),1+6*np.sqrt(var),12*np.sqrt(var)/1000)
phi_h = np.exp(-(k-1)**2/var)

fig = plt.figure()
plt.plot(k,phi_h)
plt.xlabel(r'$\frac{k}{k_0}$',size=18)
plt.ylabel(r'$\frac{\hat{\phi}}{\hat{\phi}_0}$',rotation=0,size=18)
plt.xlim(-6,8)
plt.ylim(0,1)
plt.savefig('11.png')
plt.show()

x = np.arange(-24/np.sqrt(var),24/np.sqrt(var),48/(np.sqrt(var)*10000))
phi = np.exp(-(var*x)**2/4)*np.cos(x)
fig = plt.figure()
plt.plot(x,phi)
plt.xlabel(r'$k_0 x$',size=18)
plt.ylabel(r'$\frac{\phi}{\phi_0}$',rotation=0,size=18)
plt.savefig('12.png')
plt.show()

#(ii)
var = 0.1
k = np.arange(1-6*np.sqrt(var),1+6*np.sqrt(var),12*np.sqrt(var)/1000)
phi_h = np.exp(-(k-1)**2/var)

fig = plt.figure()
plt.plot(k,phi_h)
plt.xlabel(r'$\frac{k}{k_0}$',size=18)
plt.ylabel(r'$\frac{\hat{\phi}}{\hat{\phi}_0}$',rotation=0,size=18)
plt.xlim(-6,8)
plt.ylim(0,1)
plt.savefig('21.png')
plt.show()

x = np.arange(-24/np.sqrt(var),24/np.sqrt(var),48/(np.sqrt(var)*10000))
phi = np.exp(-(var*x)**2/4)*np.cos(x)
```

```
fig = plt.figure()
plt.plot(x,phi)
plt.xlabel(r'$k_0 x$',size=18)
plt.ylabel(r'$\frac{\phi}{\phi_0}$',rotation=0,size=18)
plt.savefig('22.png')
plt.show()

#(iii)
var = 1
k = np.arange(1-6*np.sqrt(var),1+6*np.sqrt(var),12*np.sqrt(var)/1000)
phi_h = np.exp(-(k-1)**2/var)

fig = plt.figure()
plt.plot(k,phi_h)
plt.xlabel(r'$\frac{k}{k_0}$',size=18)
plt.ylabel(r'$\frac{\hat{\phi}}{\hat{\phi}_0}$',rotation=0,size=18)
plt.xlim(-6,8)
plt.ylim(0,1)
plt.savefig('31.png')
plt.show()

x = np.arange(-12/np.sqrt(var),12/np.sqrt(var),24/(np.sqrt(var)*10000))
phi = np.exp(-(var*x)**2/4)*np.cos(x)
fig = plt.figure()
plt.plot(x,phi)
plt.xlabel(r'$k_0 x$',size=18)
plt.ylabel(r'$\frac{\phi}{\phi_0}$',rotation=0,size=18)
plt.savefig('32.png')
plt.show()

#Time t=0:
t = 0
var = 100
x=np.arange(-75,225,0.01)
phi = np.real((1/np.sqrt(1+2*1j*t/var))*np.exp(-(x-t)**2/(4*(var+2*1j*t))+1j*(x-
fig = plt.figure()
plt.plot(x,phi)
plt.ylim(-1,1)
plt.xlim(-75,225)
plt.xlabel(r'$k_0 x$',size=18)
plt.ylabel(r'$\frac{\phi}{\phi_0}$',rotation=0,size=18)
plt.savefig('0.png')
plt.show()

#Time t=100:
t = 100
```

```
var = 100
x=np.arange(-75,225,0.01)
phi = np.real((1/np.sqrt(1+2*1j*t/var))*np.exp(-(x-t)**2/(4*(var+2*1j*t))+1j*(x-
fig = plt.figure()
plt.plot(x,phi)
plt.ylim(-1,1)
plt.xlim(-75,225)
plt.xlabel(r'$k_0 x$',size=18)
plt.ylabel(r'$\frac{\phi}{\phi_0}$',rotation=0,size=18)
plt.savefig('100.png')
plt.show()
```