

# VARIATIONAL CONVERGENCE OF NONLINEAR DIFFUSION EQUATIONS: APPLICATIONS TO CONCENTRATED CAPACITY PROBLEMS WITH CHANGE OF PHASE

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SUMMARY. — We study a variational formulation for a Stefan problem in two adjoining bodies, when the heat conductivity of one of them becomes infinitely large. We study the “concentrated capacity” model arising in the limit, and we justify it by an asymptotic analysis, which is developed in the general framework of the abstract evolution equations of monotone type.

## 0. - INTRODUCTION.

Let us consider the heat conduction in two adjoining bodies  $\Omega_1, \Omega_2$  in the presence or not of a change of phase. If the thermal conductivity of  $\Omega_2$  along the normal direction to the common boundary  $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$  becomes infinitely large, a possible way to study the limit situation is to assume that the temperature in  $\Omega_2$  depends only on the coordinates on the surface  $\Gamma$  and to model the phenomenon by a system of two coupled parabolic (or elliptic, in the stationary case) equations in  $\Omega_1$  and on  $\Gamma$ .

This is a particular case of the wide class of the “concentrated capacity problems”, according to the name introduced by TICHONOV (1950) for the elliptic/parabolic boundary value problems, which involve second order tangential derivatives on the boundary. Among the many physical phenomena which can be modeled in this way, we recall the diffusion in fractured media [8], the plates and junctions in elastic multi-structures [9], the electric transmission through high conducting materials [33].

The interest of studying the Stefan model in a concentrated capacity was pointed out by RUBINSTEIN [36] and a mathematical formulation (allowing a change of phase in  $\Omega_2$ ) and many related results in some particular important geometrical situations have been given in FASANO, PRIMICERIO & RUBINSTEIN [20] (see also [37] and [2]).

In a recent series of papers [25], ..., [29] (for other references, comments and various related questions, see also [30]), MAGENES established very general uniqueness and existence results under various assumptions on the heat diffusion in  $\Omega_1$  (in the presence or not of a change of phase), on the topology of  $\Gamma$  (which may or may not coincide with a connected component of  $\partial\Omega_1$ ) and on the boundary conditions imposed on  $\partial\Omega_1 \setminus \bar{\Gamma}$ . The basic idea of these papers is to reduce the coupled system in  $\Omega_1, \Gamma$  to a unique evolution equation on  $\Gamma$ , where the heat conduction properties are described by a suitable choice of a Riemannian metric and the source term takes account of the heat exchange between  $\Omega_1$  and  $\Gamma$ ; this term is related to the solution itself via a non-local operator of Steklov–Poincaré type, which is also non-linear when a change of phase occurs in  $\Omega_1$ . It is clear that the study of this operator can be very complicated and requires fine parabolic estimates; in particular, when  $\Gamma$  has a border and Neumann boundary conditions are imposed on the remaining part  $\partial\Omega_1 \setminus \bar{\Gamma}$ , subtle technical difficulties arise (cf. [29]).

Our approach goes back to the original coupled problem in  $\Omega_1, \Omega_2$ ; we shall see that the natural variational formulation (which can be re-interpreted in the framework of abstract evolution equations as developed by BREZIS in [5], [6]) is well adapted to pass to the limit and the resulting problem preserves the same abstract structure. A quite general existence, uniqueness and convergence result is then given by applying the same abstract theory.

In this way we can determine how the resulting conductivity properties of  $\Gamma$  are influenced by its geometry and the corresponding properties of  $\Omega_2$ : in the simplest case of a constant conductivity along the “tangent directions”, we shall see that the Riemannian metric induced on  $\Gamma$  does not coincide with the standard one induced by the surrounding space (as it happens in the simplified planar case studied in [20], [2]) but it also depends on the principal curvatures of  $\Gamma$  and on the thickness of  $\Omega_2$ .

Our abstract theory is also applied to study another asymptotic limit which leads to equations in a concentrated capacity. Following the approach of [22], [33], who considered the linear case of the heat equation in a simple geometric situation, the global conductivity and the heat capacity blow up together to the shrinking of  $\Omega_2$  to  $\Gamma$ . When these two processes are suitable balanced, we obtain in the limit a concentrated capacity on  $\Gamma$ , without an explicit dependence on its geometry as above (for the modelization of different asymptotic behaviours of the conductivity, the capacity, and the thickness of the layer, see e.g. [34], [7], [1], [11], and the book [35]; another geometric situation, allowing self-contact domains, is studied in [38]).

We decided to develop the results we need in an abstract form, since it does not require more effort and can be employed in many different applications, such as porous media equations, homogenization of nonlinear diffusion equations (see [14]), problems where the concentrated capacity lies on manifolds of codimension higher than 1, etc.

From the abstract point of view, this possibility is equivalent to give an answer (as we try to do) to the following general question: what are the most general notions of convergence for *all* the data, which are compatible with the “type” of nonlinear diffusion equation we want to study. It is easy to conjecture that the general theory of the variational convergences (see [3], [12] and the references therein) plays a fundamental role here; in particular, the convergence in the sense of Mosco (cf. [31], [32]) seems very natural because of the convex structure of the problems.

The plane of the paper is the following: in the next section we introduce more precisely the asymptotic problems we shall deal with, starting from that of transmission in  $\Omega_1, \Omega_2$ ; our main results on the concentrated capacity models are then given in the second section. The abstract theory is presented in the third part of this paper; the fourth one contains the related proofs and the last one is devoted to detail the link between abstract and concrete situations; in the appendix we collect some useful properties of differential calculus on  $\Gamma$ , referring to [15], [16] for a very complete and detailed development of this argument.

The variational formulation of the problems and the links with the theory of evolution equations of monotone type are the common contribution of both authors; the variational convergence tools and the asymptotic analysis have been developed

by the first author.

# 1. - THE TRANSMISSION PROBLEM.

We are given two disjoint (strongly) Lipschitz open sets  $\Omega_1, \Omega_2$  of  $\mathbf{R}^N$  such that their common boundary

$$(1.1) \quad \bar{\Gamma} := \partial\Omega_1 \cap \partial\Omega_2$$

is the closure of a regular  $(N - 1)$ -submanifold  $\Gamma$  with border  $\Gamma' := \bar{\Gamma} \setminus \Gamma$  <sup>(1)</sup>; the remaining parts of the two boundaries are denoted by

$$(1.2) \quad \Gamma_i = \partial\Omega_i \setminus \bar{\Gamma}, \quad i = 1, 2,$$

and the exterior unit normal to  $\partial\Omega_i$  is denoted by  $\mathbf{n}_i$  (of course,  $\mathbf{n}_1 = -\mathbf{n}_2$  on  $\Gamma$ ); here and in the following we assume that the index  $i$  takes the integer values 1, 2.

We choose a pair of continuous functions  $\rho_i : \bar{\Omega}_i \mapsto \mathbf{R}$  and a pair of  $N \times N$  symmetric matrices  $A_i$ , continuously depending on  $x \in \bar{\Omega}_i$ . We assume that  $\rho_i, A_i$  satisfy in their domains

$$(1.3) \quad A_i = A_i^T, \quad \text{and} \quad \alpha \leq \rho_i \leq M, \quad \alpha |\mathbf{v}|^2 \leq A_i \mathbf{v} \cdot \mathbf{v} \leq M |\mathbf{v}|^2, \quad \forall \mathbf{v} \in \mathbf{R}^N,$$

where  $\alpha, M > 0$  are two fixed positive constants.  $\bar{\mathbf{n}}_i := A_i \mathbf{n}_i$  are the related conormal vectors.

As usual for the weak formulation of the Stefan problem, we introduce two monotone functions  $\beta_i : \mathbf{R} \mapsto \mathbf{R}$  satisfying

$$(1.4) \quad \beta_i(0) = 0, \quad \liminf_{|s| \rightarrow \infty} \frac{\beta_i(s)}{s} > 0,$$

and, for a constant  $c_\beta > 0$ ,

$$(1.5) \quad (\beta_i(s) - \beta_i(t)) (s - t) \geq c_\beta |\beta_i(s) - \beta_i(t)|^2, \quad \forall s, t \in \mathbf{R}.$$

Finally we fix a time interval  $]0, T[, T > 0$ , and we set

$$Q_i := \Omega_i \times ]0, T[, \quad \Sigma_i := \Gamma_i \times ]0, T[, \quad \Sigma := \Gamma \times ]0, T[, \quad \Sigma' := \Gamma' \times ]0, T[.$$

We shall consider the following “transmission” problem

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<sup>(1)</sup> The regularity of  $\Gamma$  (say  $C^2$ ) is not necessary to state and solve the next transmission problem, but it will be needed by the subsequent developments. In order to fix our ideas, we shall assume that  $\Gamma'$  is not empty; in the other case, some technical details become simpler.

1.1 PROBLEM  $TP$ . Given

$$f_i : Q_i \mapsto \mathbf{R}, \quad u_{0,i} : \Omega_i \mapsto \mathbf{R}$$

we look for

$$\theta_i, u_i : Q_i \mapsto \mathbf{R} \quad \text{with} \quad \theta_i = \beta_i(u_i)$$

which satisfy the parabolic differential equations

$$\rho_i \frac{\partial u_i}{\partial t} - \operatorname{div} (A_i \nabla \theta_i) = f_i, \quad \text{in } Q_i,$$

the transmission conditions

$$\theta_1 = \theta_2, \quad \frac{\partial \theta_1}{\partial \bar{\mathbf{n}}_1} = - \frac{\partial \theta_2}{\partial \bar{\mathbf{n}}_2} \quad \text{on } \Sigma,$$

the initial Cauchy conditions

$$u_i(x, 0) = u_{0,i}(x) \quad \text{in } \Omega_i,$$

and the lateral boundary conditions of variational type (i.e. Dirichlet, Neumann or mixed) on the remaining parts  $\Sigma_i$ ; in order to fix our ideas, we consider the Neumann case <sup>(2)</sup>

$$\frac{\partial \theta_i}{\partial \bar{\mathbf{n}}_i} = g_i \quad \text{on } \Sigma_i.$$

The following weak formulation is naturally associated to  $TP$  (see [29], [13]).

1.2 PROBLEM  $wTP$ . If

$$(1.6) \quad f_i \in L^2(Q_i), \quad g_i \in L^2(\Sigma_i), \quad u_{0,i} \in L^2(\Omega_i),$$

we say that  $\{(\theta_i, u_i)\}_{i=1,2}$  is a weak solution of  $TP$  if

$$(1.7) \quad u_i \in L^2(Q_i), \quad \theta_i \in L^2(0, T; H^1(\Omega_i)), \quad \text{with } \theta_i = \beta_i(u_i) \quad \text{a.e. in } Q_i,$$

$$(1.8) \quad \theta_1|_{\Sigma} = \theta_2|_{\Sigma} \quad \text{in the sense of traces,}$$

and

$$(1.9) \quad \begin{aligned} & \sum_i \int_{Q_i} \left\{ -\rho_i u_i \frac{\partial v_i}{\partial t} + A_i \nabla \theta_i \cdot \nabla v_i \right\} dx dt \\ &= \sum_i \left\{ \int_{\Omega_i} \rho_i u_{0,i} v_i(x, 0) dx + \int_{Q_i} f_i v_i dx dt + \int_{\Sigma_i} g_i v_i d\mathcal{H}^{N-1} dt \right\} \end{aligned}$$

for every couple of test functions  $v_i \in H^1(0, T; H^1(\Omega_i))$  with

$$(1.10) \quad v_1|_{\Sigma} = v_2|_{\Sigma} \quad \text{and} \quad v_i(\cdot, T) = 0 \quad \text{on } \Omega_i, \quad \text{in the sense of traces.}$$

We have

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<sup>(2)</sup> which is more complicated from the technical point of view (cf. [29]); the other (homogeneous) boundary conditions require only small changes.

1.3 PROPOSITION. *For every choice of the data  $f_i, g_i, u_{0,i}$  satisfying (1.6), there exists a unique solution of the previous problem. Moreover*

$$(1.11) \quad u_i : [0, T] \mapsto L^2(\Omega_i) \quad \text{is uniformly bounded and weakly continuous.}$$

1.4 REMARK. By applying various results on the weak maximum principles, on the abstract evolution equations or on the  $L^1$ -contraction semigroups, we could give several other existence and regularity results under different assumptions on the data <sup>(3)</sup>. We limit ourselves to this setting, since we are more interested to show how the fundamental structure of these equations is preserved by the limit process we shall introduce in a moment.  $\square$

The remaining part of this section is devoted to present two particular geometric situations, where the shape of  $\Omega_2$  and (some of) the data on  $\Omega_2$  are related to a perturbation parameter  $\varepsilon$  going to 0; the asymptotic behaviour of the corresponding solutions is the object of our investigation.

In order to describe the geometric model, we introduce the following definitions and assumptions.

1.5 DEFINITION. *For every  $x \in \mathbf{R}^N$  let  $d_\Gamma(x)$  be the distance of  $x$  from  $\Gamma$ ; we shall assume that*

$$(1.12) \quad d_\Gamma(x) := \inf_{y \in \Gamma} |x - y| \quad \text{is a function of class } C^2(\overline{\Omega_2}).$$

*In particular, this regularity implies that for every point  $x \in \Omega_2$  there exists a unique projection  $x_\Gamma$  on  $\Gamma$  satisfying*

$$(1.13) \quad |x - x_\Gamma| = d_\Gamma(x)$$

*so that we can define a  $C^1$  unit vector field  $\mathbf{n}(x)$ , normal to  $\Gamma$  (see [10], 2.5.4)*

$$(1.14) \quad \mathbf{n}(x) := \nabla d_\Gamma(x) = \frac{x - x_\Gamma}{d_\Gamma(x)} \quad (4).$$

*For every  $x \in \overline{\Omega_2}$  we call  $s_x$  the intersection of  $\Omega_2$  with the straight line passing through  $x$  and parallel to  $\mathbf{n}(x)$ :*

$$s_x := \{y \in \overline{\Omega_2} : \exists \lambda \in \mathbf{R}, y = x + \lambda \mathbf{n}(x)\}.$$

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<sup>(3)</sup> e.g. if the time derivative of  $g_i$  is a square integrable function on  $\Sigma_i$ , we have

$$\theta_i \in H_{loc}^1([0, T]; L^2(\Omega_i)) \cap L_{loc}^\infty([0, T]; H^1(\Omega_i)),$$

the analogous global result (i.e. near the origin) holding if  $\beta_i(u_{0,i}) \in H^1(\Omega_i)$ , with  $\beta_1(u_{0,1}) = \beta_2(u_{0,2})$  on  $\Gamma$ .

<sup>(4)</sup> Observe that  $\mathbf{n}(x)$  is defined also on  $\Gamma$ , where it coincides with  $\mathbf{n}_1(x) = -\mathbf{n}_2(x)$ .

1.6 REMARK. Even if the requirement (1.12) about the regularity of  $d_\Gamma$  is not necessary to prove the following Theorem 1.7 (the differentiability of  $d_\Gamma$  would be enough), we stated it to unify our assumptions. (1.12) is equivalent to assume that  $\Gamma$  is of class  $C^2$  and  $\Omega_2$  is contained in a suitable neighborhood of  $\Gamma$ , depending on its curvatures (see [15], 5.5, 5.6).  $\square$

First of all, we consider the case which originally motivated the introduction of the concentrated capacity models.

**Case I: blow up of the normal conductivity.**

For every  $\varepsilon > 0$  we perturb the problem  $TP$  by adding to the conductivity matrix  $A_2$  the normal term  $\varepsilon^{-1}\mathbf{n}\mathbf{n}^T$ ,  $\varepsilon > 0$ ; we obtain a family of transmission problems  $TP_I^\varepsilon$  where  $A_2$  is replaced by

$$(1.15) \quad A_2^\varepsilon(x) := A_2(x) + \frac{1}{\varepsilon}\mathbf{n}(x)\mathbf{n}^T(x).$$

If  $\{(\theta_i^\varepsilon, u_i^\varepsilon)\}_{\varepsilon>0}$  is the family of the corresponding solutions of  $TP_I^\varepsilon$ , we want to prove the existence of their limit  $(\theta_i, u_i)$  as  $\varepsilon$  goes to 0 and to characterize it.

To this aim it is natural to introduce the closed subspace  $H_{\mathbf{n}}^1(\Omega_2)$  of  $H^1(\Omega_2)$ , consisting of functions which are constant along  $s_x$  for  $\mathcal{H}^{N-1}$ -a.e.  $x \in \Gamma$ :

$$(1.16) \quad H_{\mathbf{n}}^1(\Omega_2) := \{v \in H^1(\Omega_2) : \mathbf{n}^T \cdot \nabla v = 0\}.$$

We also set

$$L_{\mathbf{n}}^2(\Omega_2) := \overline{H_{\mathbf{n}}^1(\Omega_2)}^{L^2(\Omega_2)}, \quad \Pi_{\mathbf{n}} := \text{orthogonal projection of } L^2(\Omega_2) \text{ on } L_{\mathbf{n}}^2(\Omega_2),$$

and we have

1.7 THEOREM. *Let  $(\theta_i^\varepsilon, u_i^\varepsilon)$ ,  $\varepsilon > 0$ , be the solution of the problems  $TP_I^\varepsilon$  previously defined; as  $\varepsilon$  goes to 0*

$$(1.17) \quad \theta_i^\varepsilon \rightarrow \theta_i \text{ strongly in } L^2(Q_i) \text{ and weakly in } L^2(0, T; H^1(\Omega_i))$$

*and, for every fixed  $t \in ]0, T]$*

$$(1.18) \quad u_1^\varepsilon(\cdot, t) \rightharpoonup u_1(\cdot, t), \quad \Pi_{\mathbf{n}} u_2^\varepsilon(\cdot, t) \rightharpoonup u_2(\cdot, t), \text{ weakly in } L^2(\Omega_i),$$

*the last convergence of (1.17) being also strong if  $u_{0,2}$  belongs to  $L_{\mathbf{n}}^2(\Omega_2)$ . Moreover,  $(\theta_i, u_i)$  is the unique solution of the following (weak) limit formulation.*



1.8 PROBLEM  $wLP_I$ . Find  $(\theta_i, u_i)$  satisfying (1.7), (1.8) and

$$(1.19) \quad u_2 \in L^2_{\mathbf{n}}(\Omega_2), \quad \theta_2 \in H^1_{\mathbf{n}}(\Omega_2) \quad \text{for a.e. } t \in ]0, T[,$$

such that (1.9) holds for every couple of test functions  $v_i \in H^1(0, T; H^1(\Omega_i))$  with (1.10) and

$$(1.20) \quad v_2 \in H^1_{\mathbf{n}}(\Omega_2) \quad \text{for a.e. } t \in ]0, T[.$$

We postpone to the next section the interpretation of this problem as the weak formulation of a system of two coupled evolution equations, one of which is set on the manifold  $\Gamma$ .

Now we focus our attention to the evident common structure of  $wTP_I^\varepsilon$  and  $wLP_I$ , in order to conjecture an abstract result.

Let us consider the Hilbert space

$$(1.21) \quad H := L^2_{\rho_1}(\Omega_1) \times L^2_{\rho_2}(\Omega_2) \quad (5)$$

whose elements we denote by  $U := (u_1, u_2)$ . On  $H$  is defined the (cyclically) monotone operator

$$(1.22) \quad \Lambda : H \mapsto H, \quad \Lambda(U) := [\beta_1(u_1), \beta_2(u_2)].$$

On the linear subspace of  $H$

$$(1.23) \quad V := \{\Theta = (\theta_1, \theta_2) \in H^1(\Omega_1) \times H^1(\Omega_2) : \theta_1|_\Gamma = \theta_2|_\Gamma\}$$

we define the (weakly) coercive bilinear forms

$$(1.24) \quad a_\varepsilon(\Theta, V) := \int_{\Omega_1} A_1 \nabla \theta_1 \cdot \nabla v_1 \, dx + \int_{\Omega_2} \left( A_2 \nabla \theta_2 \cdot \nabla v_2 + \frac{1}{\varepsilon} \frac{\partial \theta_2}{\partial \mathbf{n}} \frac{\partial v_2}{\partial \mathbf{n}} \right) dx,$$

and the time-dependent linear functionals  $L(t) \in V'$ ,  $t \in ]0, T[$

$$(1.25) \quad \langle L(t), V \rangle := \sum_i \left\{ \int_{\Omega_i} f_i(x, t) v_i \, dx + \int_{\Gamma_i} g_i(x, t) v_i \, d\mathcal{H}^{N-1}(x) \right\}.$$

The weak formulation  $wTP_I^\varepsilon$  consists in the search of

$$(1.26) \quad U^\varepsilon \in L^2(0, T; H), \quad \Theta^\varepsilon \in L^2(0, T; V)$$

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(5) i.e. the  $L^2$ -spaces with respect to the measures  $\rho_i \cdot \mathcal{L}$ ,  $\mathcal{L}$  being the usual Lebesgue measure on  $\mathbf{R}^N$ . By (1.3), they coincide (up to equivalent norms) with the usual  $L^2(\Omega_i)$ .

such that

$$(1.27) \quad \Theta^\varepsilon(t) = \Lambda U^\varepsilon(t) \quad \text{for a.e. } t \in ]0, T[$$

and (cf. (1.9))

$$(1.28) \quad \int_0^T \left\{ - (U^\varepsilon, V_t) + a_\varepsilon(\Theta^\varepsilon, V) \right\} dt = (U_0, V(0))_H + \int_0^T \langle L, V \rangle dt$$

for any choice of  $V \in H^1(0, T; V_\varepsilon)$  with  $V(T) = 0$ ; here  $U_0 := (u_{0,1}, u_{0,2})$

When  $\varepsilon = 0$  we simply have to define

$$(1.29) \quad V_0 := \left\{ \Theta = (\theta_1, \theta_2) \in H^1(\Omega_1) \times H_n^1(\Omega_2) : \theta_1|_\Gamma = \theta_2|_\Gamma \right\}$$

and

$$(1.30) \quad a_0(\Theta, V) := \int_{\Omega_1} A_1 \nabla \theta_1 \cdot \nabla v_1 dx + \int_{\Omega_2} A_2 \nabla \theta_2 \cdot \nabla v_2 dx$$

and to repeat the same requirements (1.26), ..., (1.28) <sup>(6)</sup>.

The possibility of this substitution in the limit is justified by the following basic fact:

**1.9 PROPOSITION.** *For  $\varepsilon > 0$  define  $V_\varepsilon := V$  and, for  $\varepsilon \geq 0$ ,*

$$(1.31) \quad \mathbf{a}_\varepsilon(\Theta) := \begin{cases} a_\varepsilon(\Theta, \Theta) & \text{if } \Theta \in V_\varepsilon, \\ +\infty & \text{if } \Theta \in H \setminus V_\varepsilon. \end{cases}$$

*Then as  $\varepsilon$  goes to 0,  $\mathbf{a}_\varepsilon$  converges to  $\mathbf{a}_0$  in the sense of Mosco (cf. [3], Thm. 3.20 and sect 3).  $\square$*

We shall see in the abstract setting of the third section that the combination of this convergence (which also allows to vary  $\Lambda, L$ , and  $U_0$  with respect to  $\varepsilon$ ) with a kind of uniform coercivity on the couple  $a_\varepsilon, \Lambda_\varepsilon$ , are the good assumptions to study in a general context the asymptotic behaviour of a family of nonlinear diffusion equations.

**Case II: blow up of the global conductivity when  $\Omega_2$  shrinks to  $\Gamma$ .**

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<sup>(6)</sup> Unlike  $V$ ,  $V_0$  is not dense in  $H$ ; this fact gives rise to non-uniqueness of the component  $U^0$  of the limit solution. Adding the further condition  $U^0 \in H_0 := \overline{V_0}^H$ , we overcome this difficulty, thanks to a compatibility property between  $\Lambda$  and (the orthogonal projection  $\Pi_0$  of  $H$  onto)  $H_0$ .

We consider a family of contractions in the direction of the vector field  $-\mathbf{n}(x)$  (see definition 1.5):

$$(1.32) \quad G^\varepsilon(x) := \varepsilon x + (1 - \varepsilon)x_\Gamma = x_\Gamma + \varepsilon d_\Gamma(x)\mathbf{n}(x), \quad 0 < \varepsilon \leq 1,$$

and we call  $\Omega_2^\varepsilon, s_x^\varepsilon$  the shrunked sets

$$\Omega_2^\varepsilon := G^\varepsilon(\Omega_2), \quad s_x^\varepsilon := G^\varepsilon(s_x).$$

As before, we have a family of problems  $TP_{II}^\varepsilon$ , where we also have to assign a varying set of data in  $\Omega_2^\varepsilon$

$$(1.33) \quad \rho_2^\varepsilon, A_2^\varepsilon, f_2^\varepsilon, u_{0,2}^\varepsilon, \quad (7)$$

while keeping fixed the remaining ones on  $\Omega_1$ .

The main assumption on the data of  $f_2^\varepsilon, u_{0,2}^\varepsilon$  is that they give rise, in the limit, to a suitable distribution on the lower dimensional manifold  $\Gamma$ . More precisely, we assume that there exist

$$(1.34) \quad \rho_\Gamma \in C^0(\Gamma), \quad A_\Gamma \in C^0(\Gamma; \mathbf{R}^{N \times N}); \quad f_\Gamma \in L^2(\Sigma), \quad u_{0,\Gamma} \in L^2(\Gamma),$$

such that

$$(1.35) \quad \begin{cases} \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \int_{Q_2^\varepsilon} |f_2^\varepsilon(x, t) - f_\Gamma(x_\Gamma, t)|^2 dx dt = 0, \\ \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \int_{\Omega_2^\varepsilon} |u_{0,2}^\varepsilon(x) - u_{0,\Gamma}(x_\Gamma)|^2 dx = 0, \end{cases}$$

and

$$(1.36) \quad \begin{cases} \lim_{\varepsilon \rightarrow 0} \sup_{x \in \Omega_2^\varepsilon} [\varepsilon |A_2^\varepsilon(x) - A_\Gamma(x_\Gamma)| + |\varepsilon \rho_2^\varepsilon(x) - \rho_\Gamma(x_\Gamma)|] = 0, \\ \text{with } A_\Gamma, \rho_\Gamma \text{ satisfying (1.3) on } \Gamma. \end{cases}$$

If  $\{(\theta_i^\varepsilon, u_i^\varepsilon)\}_{\varepsilon > 0}$  is the family of the solutions of the problems  $TP_{II}^\varepsilon$ , we want to characterize their limit, as  $\varepsilon \rightarrow 0$ , or, more precisely, the limit of

$$(\theta_1^\varepsilon, u_1^\varepsilon) \text{ in } Q_1, \quad \text{and} \quad (\bar{\theta}_2^\varepsilon, \bar{u}_2^\varepsilon) \text{ on } \Sigma,$$

where, for  $\mathcal{H}^{N-1}$ -a.e.  $x \in \Gamma$ ,  $\bar{\theta}_2^\varepsilon(x), \bar{u}_2^\varepsilon(x)$  are the mean values on  $s_x^\varepsilon$  of  $\theta_2^\varepsilon, u_2^\varepsilon$  respectively:

$$(1.37) \quad \bar{\theta}_2^\varepsilon(x) := \int_{s_x^\varepsilon} \theta_2^\varepsilon(y) d\mathcal{H}^1(y), \quad \bar{u}_2^\varepsilon(x) := \int_{s_x^\varepsilon} u_2^\varepsilon(y) d\mathcal{H}^1(y).$$

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(7) For the sake of simplicity, here we assume  $g_2^\varepsilon \equiv 0$ .

## 2. - THE LIMIT PROBLEMS IN THE CONCENTRATED CAPACITY.

Let us briefly recall some basic definitions on differential calculus and Sobolev spaces on  $\Gamma$ . The tangent and the normal spaces to  $\Gamma$  at the point  $x$  are defined by

$$\mathcal{T}_x := \{\mathbf{v} \in \mathbf{R}^N : \mathbf{n}(x) \cdot \mathbf{v} = 0\}, \quad \mathcal{N}_x := \{\mathbf{v} \in \mathbf{R}^N : \mathbf{v} = \lambda \mathbf{n}(x), \text{ for some } \lambda \in \mathbf{R}\},$$

the orthogonal projection on  $\mathcal{T}_x$  being given by

$$(2.1) \quad P_x : \mathbf{R}^N \mapsto \mathcal{T}_x, \quad P_x \mathbf{v} := [I - \mathbf{n}(x) \mathbf{n}^T(x)] \mathbf{v}.$$

The principal curvatures  $\kappa_1(x), \dots, \kappa_{N-1}(x)$  of  $\Gamma$  at  $x$  are the eigenvalues, besides 0, of the differential matrix of  $\mathbf{n}$  (see [21], p. 355)

$$(2.2) \quad S(x) := -D\mathbf{n}(x) = -D^2 d_\Gamma(x), \quad \in C^0(\overline{\Omega}_2).$$

If  $v^*$  is a regular extension to  $\Omega_2$  of a function  $v : \Gamma \mapsto \mathbf{R}$ , it is easy to check that the tangential gradient

$$\nabla_\Gamma v : x \in \Gamma \mapsto \mathcal{T}_x, \quad \nabla_\Gamma v(x) := P_x [\nabla v^*(x)],$$

is well defined and it is independent of the extension  $v^*$ .

A (regular) tangential vector field is a mapping

$$\mathbf{v} : \Gamma \mapsto \mathbf{R}^N \quad \text{such that} \quad \mathbf{v}(x) \in \mathcal{T}_x, \quad \forall x \in \Gamma.$$

For this kind of vector fields, we define the divergence on  $\Gamma$  as

$$(2.3) \quad \operatorname{div}_\Gamma \mathbf{v} := \operatorname{div} \mathbf{v}^* - \frac{\partial(\mathbf{v}^* \cdot \mathbf{n})}{\partial \mathbf{n}},$$

$\mathbf{v}^*$  being an extension of  $\mathbf{v}$  as before; also in this case, it is possible to check that  $\operatorname{div}_\Gamma \mathbf{v}$  does not depend on the extension  $\mathbf{v}^*$  (cf. [15], sect. 6). When  $\mathbf{v}$  is not tangential, it will be useful to define

$$(2.4) \quad \operatorname{div}_\Gamma \mathbf{v} := \operatorname{div}_\Gamma (P_x \mathbf{v}).$$

If  $\mathbf{v}$  is a regular tangential vector field and  $w$  is a regular function, the following Green's formula holds on  $\Gamma$

$$(2.5) \quad - \int_\Gamma \operatorname{div}_\Gamma \mathbf{v} w d\mathcal{H}^{N-1} = \int_\Gamma \mathbf{v} \cdot \nabla_\Gamma w d\mathcal{H}^{N-1} - \int_{\Gamma'} w(\mathbf{v} \cdot \mathbf{n}') d\mathcal{H}^{N-2}$$

where  $\mathbf{n}'(x)$  is the outward unit normal to  $\Gamma'$  in the tangent space  $\mathcal{T}_x$ .

2.1 REMARK. In this framework, the usual Laplace-Beltrami operator, induced by the Euclidean metric on  $\Gamma$ , has the simple form

$$\Delta_\Gamma v := \operatorname{div}_\Gamma(\nabla_\Gamma v). \quad \square$$

2.2 REMARK. The notions of  $\nabla_\Gamma, \operatorname{div}_\Gamma$  are usually given in an intrinsic way via local coordinates, which do not require any imbedding of  $\Gamma$  in an Euclidean space. We use this simpler (but, maybe, less elegant) approach, since it is more direct and it shows the strict relation with the differential-geometric properties of the ambient space.  $\square$

Now we introduce the usual (Hilbertian) Sobolev spaces on  $\Gamma$  (for the intrinsic definitions, see [4]).  $H^1(\Gamma)$  is the usual completion of  $C^1(\bar{\Gamma})$  <sup>(8)</sup> with respect to the norm induced by the scalar product

$$(u, v)_{H^1(\Gamma)} := \int_\Gamma \left[ u(x)v(x) + \nabla_\Gamma u(x) \cdot \nabla_\Gamma v(x) \right] d\mathcal{H}^{N-1}(x),$$

so that  $\nabla_\Gamma$  becomes a linear and continuous operator between  $H^1(\Gamma)$  and

$$(2.6) \quad L^2(\Gamma; \mathcal{T}(\Gamma)) := \left\{ \mathbf{v} \in L^2(\Gamma; \mathbf{R}^N) : \mathbf{v}(x) \in \mathcal{T}_x, \quad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \Gamma \right\}.$$

$H_0^1(\Gamma)$  is the closure in  $H^1(\Gamma)$  of the  $C^1$ -functions with compact support in  $\Gamma$  and  $H^{-1}(\Gamma)$  is its dual space. Via (2.5) we extend  $\operatorname{div}_\Gamma$  to a linear and continuous operator from  $L^2(\Gamma; \mathcal{T}(\Gamma))$  to  $H^{-1}(\Gamma)$ . If a vector field  $\mathbf{v} \in L^2(\Gamma; \mathcal{T}(\Gamma))$  with  $\operatorname{div}_\Gamma \mathbf{v} \in L^2(\Gamma)$  satisfies

$$(2.7) \quad - \int_\Gamma \operatorname{div}_\Gamma \mathbf{v} w d\mathcal{H}^{N-1} = \int_\Gamma \mathbf{v} \cdot \nabla_\Gamma w d\mathcal{H}^{N-1}, \quad \forall w \in H^1(\Gamma),$$

we say (in a formal way, but this argument could be made more precise, see [24]) that it has a vanishing normal component on  $\Gamma'$ , that is  $\mathbf{v} \cdot \mathbf{n}' = 0$ .

We can finally state our main results on the two problems of the previous section. For the sake of simplicity, we assume that

(G1) For every  $x \in \Gamma$ ,  $s_x$  is a segment of (regular) length  $\ell(x)$ , with

$$0 < \ell_0 \leq \ell(x) \leq \ell_1 < +\infty.$$

(G2)  $S(x)$  is bounded on  $\Gamma$  and there exists a constant  $\eta > 0$  such that

$$\det(I - \lambda S(x)) \geq \eta, \quad \forall x \in \Gamma, \quad 0 \leq \lambda \leq \ell(x).$$

(G3)  $g_2(x) = 0$  on  $\Sigma_2$ .

---

<sup>(8)</sup> Of course, when  $\mathcal{H}^{N-1}(\Gamma) = +\infty$ , we have to choose  $C^1$  functions with compact support in  $\bar{\Gamma}$ .

**Theorem I.** *Let us assume that (G1,2,3) hold together to (1.6), and let us denote by  $(\theta_i, u_i)$  the limits of the solutions of  $TP_I^\varepsilon$  given by Theorem 1.7. Let  $(\tilde{\theta}_2, \tilde{u}_2)$  be denote the traces on  $\Sigma$  of  $(\theta_2, u_2)$  <sup>(9)</sup>. Then  $\{(\theta_1, u_1), (\tilde{\theta}_2, \tilde{u}_2)\}$  is the unique weak solution of the following system of coupled equations*

$$\begin{cases} \theta_1 = \beta_1(u_1) & \text{in } Q_1, \\ \rho_1 \frac{\partial u_1}{\partial t} - \operatorname{div} (A_1 \nabla \theta_1) = f_1 & \text{in } Q_1, \\ \frac{\partial \theta_1}{\partial \bar{\mathbf{n}}_1} = g_1 & \text{on } \Sigma_1, \\ u_1(x, 0) = u_{0,1}(x) & \text{in } \Omega_1; \end{cases}$$

$$\theta_1 = \tilde{\theta}_2 \quad \text{on } \Sigma;$$

$$\begin{cases} \tilde{\theta}_2 = \beta_2(\tilde{u}_2) & \text{in } \Sigma, \\ \hat{\rho}_2 \frac{\partial \tilde{u}_2}{\partial t} - \operatorname{div}_\Gamma (\hat{A}_2 \nabla_\Gamma \tilde{\theta}_2) = \hat{f}_2 - \frac{\partial \theta_1}{\partial \bar{\mathbf{n}}_1} & \text{in } \Sigma, \\ \frac{\partial \tilde{\theta}_2}{\partial \bar{\mathbf{n}}'} = 0 & \text{on } \Sigma', \\ \tilde{u}_2(x, 0) = \hat{u}_{0,2}(x) & \text{in } \Gamma, \end{cases}$$

where  $\hat{f}_2, \hat{u}_{0,2}, \hat{\rho}_2$ , and  $\hat{A}_2$  can be explicitly computed from the corresponding values  $f_2, u_{0,2}, \rho_2, A_2$ , and from the matrix  $S(x)$ ;  $\bar{\mathbf{n}}'$  is the conormal vector  $\bar{\mathbf{n}}' := \hat{A}_2 \mathbf{n}'$ .

We shall give the general formulae in the last section; let us consider here the special case of  $N = 3$ ,  $A_2 = I$ . Using more familiar symbols, we call  $H$  and  $K$  the mean and the Gaussian curvatures of  $\Gamma$  (oriented by  $\mathbf{n}$ ) respectively

$$(2.8) \quad H(x) := \frac{1}{2} \operatorname{tr} S(x) = \frac{\kappa_1(x) + \kappa_2(x)}{2}, \quad K(x) := \kappa_1(x) \kappa_2(x).$$

Now for a generic point  $x \in \Gamma$  we introduce the standard parametrization of the segment  $s_x$

$$(2.9) \quad x_\lambda := x + \lambda \mathbf{n}(x), \quad 0 \leq \lambda \leq \ell(x),$$

and a deformation measure  $\mu_x$  on it

$$(2.10) \quad d\mu_x(\lambda) := \left[ 1 - 2H(x)\lambda + K(x)\lambda^2 \right] d\lambda = \det(1 - \lambda S(x)) d\lambda, \quad \lambda \in [0, \ell(x)].$$

---

<sup>(9)</sup> The trace operator maps continuously  $H_n^1(\Omega_2)$  onto  $H^1(\Gamma)$  and (it can be continuously extended by density from)  $L_n^2(\Omega_2)$  onto  $L^2(\Gamma)$ .

We have

$$(2.11) \quad \hat{f}_2(x, t) := \int_0^{\ell(x)} f_2(x_\lambda, t) d\mu_x(\lambda), \quad \hat{\rho}_2(x) := \int_0^{\ell(x)} \rho(x_\lambda) d\mu_x(\lambda),$$

$$(2.12) \quad \hat{u}_{0,2}(x) := \frac{1}{\hat{\rho}_2(x)} \int_0^{\ell(x)} u_{0,2}(x_\lambda) \rho_2(x_\lambda) d\mu_x(\lambda),$$

and finally

$$(2.13) \quad \hat{A}_2(x) := \int_0^{\ell(x)} (I - \lambda S(x))^{-2} d\mu_x(\lambda).$$

Thanks to (G1,2) and to the symmetry of  $S(x)$ ,  $\hat{\rho}_2$  and  $\hat{A}_2$  still verify on  $\Gamma$  a condition analogous to (1.3).

**2.3 REMARK.** This problem can be studied independently from the asymptotic approach (see [25],...[30]): in this case  $\hat{f}_2, \hat{u}_{0,2}, \hat{\rho}_2$  and  $\hat{A}_2$  are *a priori* given data and the weak formulation [29] has the same structure as in the previous section, formulae (1.21),..., (1.28). Here

$$(2.14) \quad H := L_{\rho_1}^2(\Omega_2) \times L_{\hat{\rho}_2}^2(\Gamma),$$

$$(2.15) \quad V_0 := \{\Theta = (\theta_1, \tilde{\theta}_2) \in H^1(\Omega_1) \times H^1(\Gamma) : \theta_1|_\Gamma = \tilde{\theta}_2\},$$

$$(2.16) \quad a_0(\Theta, V) := \int_{\Omega_1} A_1 \nabla \theta_1 \cdot \nabla v_1 dx + \int_\Gamma \hat{A}_2 \nabla_\Gamma \tilde{\theta}_2 \cdot \nabla_\Gamma \tilde{v}_2 d\mathcal{H}^{N-1}$$

and

$$(2.17) \quad \langle L(t), V \rangle := \int_{\Omega_1} f_1(x, t) v_1 dx + \int_{\Gamma_1} g_1(x, t) v_1 d\mathcal{H}^{N-1}(x) + \int_\Gamma \hat{f}_2(x, t) \tilde{v}_2 d\mathcal{H}^{N-1}(x)$$

Observe that a careful choice of the (couples of) test functions as in [29] allows us to give a precise meaning to each formula of the system of Theorem I in suitable Sobolev spaces (of negative order, if it is necessary). If  $\Gamma$  is  $C^\infty$ , we can use the standard distribution setting.  $\square$

Finally, we consider the case II (and the relative notation) of the previous section.

**Theorem II.** *Let us assume that (G1,2,3) hold together to (1.34) and (1.36); then we have*

$$\theta_1^\varepsilon \rightarrow \theta_1 \quad \text{strongly in } L^2(0, T; H^1(\Omega_1)), \quad \bar{\theta}_2^\varepsilon \rightarrow \bar{\theta}_2 \quad \text{strongly in } L^2(0, T; H^1(\Gamma)),$$

and, for every fixed  $t \in [0, T]$ ,

$$u_1^\varepsilon(\cdot, t) \rightharpoonup u_1(\cdot, t) \quad \text{weakly in } L^2(\Omega_1), \quad \bar{u}_2^\varepsilon(\cdot, t) \rightharpoonup \bar{u}_2(\cdot, t) \quad \text{weakly in } L^2(\Gamma).$$

Moreover,  $\{(\theta_1, u_1), (\bar{\theta}_2, \bar{u}_2)\}$  is the unique weak solution of the following coupled system

$$\begin{cases} \theta_1 = \beta_1(u_1) & \text{in } Q_1, \\ \rho_1 \frac{\partial u_1}{\partial t} - \operatorname{div}(A_1 \nabla \theta_1) = f_1 & \text{in } Q_1, \\ \frac{\partial \theta_1}{\partial \bar{\mathbf{n}}_1} = g_1 & \text{on } \Sigma_1, \\ u_1(x, 0) = u_{0,1}(x) & \text{in } \Omega_1; \\ \theta_1 = \bar{\theta}_2 & \text{on } \Sigma; \end{cases}$$

$$\begin{cases} \bar{\theta}_2 = \beta_2(\bar{u}_2) & \text{in } \Sigma, \\ \rho_\Gamma \frac{\partial \bar{u}_2}{\partial t} - \ell^{-1} \operatorname{div}_\Gamma(\ell A_\Gamma \nabla_\Gamma \bar{\theta}_2) = f_\Gamma - \ell^{-1} \frac{\partial \theta_1}{\partial \bar{\mathbf{n}}_1} & \text{in } \Sigma, \\ \frac{\partial \bar{\theta}_2}{\partial \bar{\mathbf{n}}'} = 0 & \text{on } \Sigma', \\ \bar{u}_2(x, 0) = u_{0,\Gamma}(x) & \text{in } \Gamma, \end{cases}$$

where  $\ell$  is the thickness of  $\Omega_2$  defined by (G1).

**2.4 REMARK.** Let us note that in the simplest case of constant coefficients of order of magnitude  $1/\varepsilon$ , i.e.

$$(2.18) \quad \rho_2^\varepsilon(x) := \rho_2/\varepsilon, \quad A_2^\varepsilon(x) := A_2/\varepsilon$$

we obtain in the limit

$$(2.19) \quad \rho_\Gamma(x) = \rho_2, \quad A_\Gamma(x) = A_2, \quad \forall x \in \Gamma,$$

without any deformation due to the curvature of  $\Gamma$ . In particular, when  $A_2 = I$  and  $\Omega_2$  has a uniform thickness, the usual Laplace operator in  $\Omega_2^\varepsilon$  induces the Laplace-Beltrami operator on  $\Gamma$  (see Remark 2.1).  $\square$



### 3. - THE ABSTRACT THEORY.

Let  $H$  be a (separable) Hilbert space  $H$  with scalar product  $(\cdot, \cdot)$  and norm  $|\cdot|$ ; on  $H$  we are given a l.s.c. convex and positive function

$$(3.1) \quad \phi : H \mapsto [0, +\infty], \quad \phi(0) = 0,$$

with domain  $D(\phi) := \{u \in H : \phi(u) < +\infty\}$ . We shall denote by  $\Lambda$  its subdifferential, defined as

$$(3.2) \quad \Lambda : H \mapsto 2^H, \quad w \in \Lambda u \Leftrightarrow (w, v - u) \leq \phi(v) - \phi(u), \quad \forall v \in D(\phi).$$

We consider a symmetric and positive bilinear form  $a : V_a \times V_a \mapsto \mathbf{R}$ , defined on a subspace (*not necessarily dense*)  $V_a$  of  $H$ . To the couple  $(V_a, a)$  is uniquely associated the generalized quadratic form  $\mathfrak{a} : H \mapsto [0, +\infty]$  (see [12], def. 11.7)

$$\mathfrak{a}(u) := \begin{cases} a(u, u) \geq 0 & \text{if } u \in V_a, \\ +\infty & \text{otherwise;} \end{cases}$$

we shall assume that  $V_a$  is complete with respect to the Hilbertian norm

$$(3.3) \quad \|v\|_{V_a}^2 := \mathfrak{a}(v) + |v|^2,$$

or, equivalently, that  $\mathfrak{a}$  is l.s.c. with respect to the  $H$ -topology ([12], 12.16). We call  $H_a$  the closure of  $V_a$  in  $H$  and  $H'_a \subset V'_a$  the dual spaces of  $H_a$  and  $V_a$  respectively <sup>(10)</sup>,  $\langle \cdot, \cdot \rangle$  being the duality pairing.

We want to study the problem

**Problem P**( $\mathfrak{a}, \phi; L, u_0$ ). *Given*

$$L \in L^2(0, T; V'_a) \quad \text{and} \quad u_0 \in H,$$

*find*  $\theta \in L^2(0, T; V_a)$  *and*  $u \in L^2(0, T; H)$  *such that*

$$(3.4) \quad \theta(t) \in \Lambda u(t), \quad \text{for a.e. } t \in ]0, T[,$$

*and*

$$(3.5) \quad \int_0^T \left\{ - (u, v_t) + a(\theta, v) \right\} dt = (u_0, v(0)) + \int_0^T \langle L, v \rangle dt,$$

---

<sup>(10)</sup> Since we shall deal with different couples of spaces  $V_a \subset H_a$  and in general  $H_a \neq H$ , we *do not identify* any space with its dual; on the contrary, the dense imbedding  $H'_a \subset V'_a$  is admissible, since it corresponds to the transpose of the continuous and dense inclusion of  $V_a$  in  $H_a$ .

for any choice of  $v \in H^1(0, T; V_a)$  with  $v(T) = 0$ .

Before stating our main results, let us make some remarks about this problem. First of all, if the more usual density hypothesis of  $V_a$  into  $H$  held, we could identify  $H_a = H$  with  $H' \subset V'_a$  and (3.5) would be the weak formulation of a Cauchy problem for an abstract differential equation of the type

$$(3.6) \quad \frac{d}{dt}(\Lambda^{-1}\theta) + A\theta \ni L, \quad (\Lambda^{-1}\theta)(0) \ni u_0,$$

where  $A : V_a \mapsto V'_a$  is the linear operator associated to  $a$  and  $\Lambda^{-1}$  is the inverse graph of  $\Lambda$ . Evolution problems of this type have been intensively studied; in particular DI BENEDETTO & SHOWALTER [17] (whose bibliography we refer to) gives a very general existence result, assuming  $V_a$  compactly embedded in  $H$  but allowing  $A$  to be a nonlinear (maximal monotone and bounded) operator.

In a particular but enlightening case, BREZIS [5] exploited the linearity and the *coercivity* of  $A$  to rewrite (3.6) as an evolution equation in  $V'_a$  governed by a subdifferential operator; thanks to the general theory of such equations (cf. [6]), this approach gives a more detailed insight of the solution of the problem.

Taking account of both these contributions, we decided to formulate the problem in a form which will be well adapted to study the dependence of the solution  $\theta$  on  $\mathbf{a}$  and  $\phi$ , avoiding compactness assumptions and allowing non-coercive bilinear forms  $a$ .

We stress that the general choice of a (possibly) non dense domain  $V_a$  is motivated by the limit procedure we shall perform. It is well known that the most natural notion of convergence for evolution problems related to convex functionals is that of Mosco (see the definition later on and the exposition of [3], based on [31], [32]). Since the density of the proper domain of these functionals is not preserved by the Mosco-convergence, we cannot assume this property without restricting the range of possible applications. A significant example is showed by Proposition 1.9.

In order to describe our assumption, we fix a continuous and positive bilinear form

$$b : H \times H \mapsto \mathbf{R}, \quad \mathbf{b}(u) := b(u, u) \geq 0, \quad \forall u \in H,$$

and we assume that

$$(A_\Lambda) \quad (\Lambda u - \Lambda v, u - v) \geq \mathbf{b}(\Lambda u - \Lambda v), \quad \forall u, v \in D(\Lambda),$$

$$(A_\mathbf{a}) \quad \exists \alpha > 0 : \quad \mathbf{a}(u) + \mathbf{b}(u) \geq \alpha|u|^2, \quad \forall u \in V_a,$$

and, on the data,

$$(A_{L, u_0}) \quad L \in L^2(0, T; V'_a), \quad u_0 \in D(\phi).$$

Denoting by  $J_a : H \mapsto H'_a$  the linear surjection

$$(3.7) \quad J_a : H \mapsto H'_a, \quad \langle J_a u, v \rangle := (u, v), \quad \forall u \in H, v \in H_a,$$

we have

**Theorem 1 (uniqueness).** *Let us assume that  $(A_\Lambda, A_a)$  hold and the data satisfy  $(A_{L,u_0})$ ; if  $(\theta^1, u^1), (\theta^2, u^2)$  are two solutions of the problem  $\mathbf{P}(\mathbf{a}, \phi; L, u_0)$ , then*

$$(3.8) \quad \theta^1 = \theta^2, \quad \text{and} \quad J_a u^1 = J_a u^2.$$

*Moreover  $\tilde{u} := J_a u^1 = J_a u^2$  belongs to  $H^1(0, T; V'_a)$  and satisfies the initial condition  $\tilde{u}(0) = J_a u_0$ .*

We can give some further information about the structure of the set  $U = U(\mathbf{a}, \phi; L, u_0)$  of the (not uniquely determined) components  $u$  of the solutions. ■

We associate to  $\phi$  the convex function

$$(3.9) \quad \tilde{\phi}_a : H'_a \mapsto [0, +\infty], \quad \tilde{\phi}_a(\tilde{v}) := \inf\{\phi(v) : J_a v = \tilde{v}\}, \quad D(\tilde{\phi}_a) = J_a[D(\phi)],$$

and we call  $K(\tilde{v})$ ,  $\tilde{v} \in H'_a$ , the set where the inf of (3.9) is achieved:

$$(3.10) \quad K(\tilde{v}) := \{v \in J_a^{-1}(\tilde{v}) : \phi(v) = \tilde{\phi}_a(\tilde{v})\}.$$

It is easy to check that  $K(\tilde{v})$  is a closed convex set; moreover it satisfies

$$(3.11) \quad \Lambda v \cap V_a \neq \emptyset \quad \Rightarrow \quad v \in K(J_a v).$$

We have

**3.1 PROPOSITION.** *Let  $(\theta, \tilde{u})$  be given as in the previous Theorem 1; then  $\tilde{\phi}_a(\tilde{u})$  is (essentially) bounded and a function  $u \in L^2(0, T; H)$  belongs to  $U(\mathbf{a}, \phi; L, u_0)$  if and only if*

$$(3.12) \quad u(t) \in K(\tilde{u}(t)), \quad \text{for a.e. } t \in ]0, T[.$$

*In particular, the set  $U$  of the solutions  $u$  of  $\mathbf{P}$  is a closed convex subset of  $L^2(0, T; H)$  satisfying*

$$u \in U \quad \Rightarrow \quad \phi(u(t)) = \tilde{\phi}_a(\tilde{u}(t)), \quad \text{for a.e. } t \in ]0, T[.$$

**3.2 REMARK.** The second request of  $(A_{L,u_0})$  could be replaced by the weaker

$$\tilde{\phi}_a(J_a u_0) < +\infty. \quad \square$$

A condition ensuring the existence is given by

**Theorem 2 (existence).** *With the same assumptions of the previous Theorem, let us suppose that*

$$(A_\phi) \quad \phi \text{ is coercive on } H: \quad \lim_{|v| \rightarrow \infty} \phi(v) = +\infty.$$

*Then there exists a solution of problem  $\mathbf{P}(\mathbf{a}, \phi; L, u_0)$  and  $U(\mathbf{a}, \phi; L, u_0)$  is a bounded subset of  $L^\infty(0, T; H)$ .* ■

3.3 REMARK. If  $(A_\phi)$  hold, then  $\tilde{\phi}_a$  is l.s.c. and for every  $\tilde{v} \in D(\tilde{\phi}_a)$  the set  $K(\tilde{v})$  is non empty and bounded in  $H$ . □

Let us denote by  $\Pi_a$  the orthogonal projection onto  $H_a$

$$(3.13) \quad \Pi_a : H \mapsto H_a, \quad (\Pi_a v - v, w) = 0, \quad \forall w \in H.$$

Since  $J_a$  restricted to  $H_a$  is the usual Riesz isomorphism between  $H_a$  and  $H'_a$  and  $J_a \circ \Pi_a = J_a$ , the knowledge of  $\tilde{u} \in H'_a$  is equivalent to the knowledge of the projection  $\Pi_a u \in H_a$ , which is therefore uniquely determined by the data of the problem.

Of course, if  $V_a$  is dense in  $H$ , also  $u$  is uniquely determined; one could add the further condition

$$(3.14) \quad u(t) \in H_a, \quad \text{for a.e. } t \in ]0, T[$$

in order to fix the solution, but this requirement may be not satisfied in general. Nevertheless, there is a simple *compatibility* condition, which allows (3.14):

**Corollary.** *Besides  $(A_{\Lambda, \mathbf{a}, \phi; L, u_0})$ , let us assume that*

$$(A_{comp}) \quad \forall v \in H : \quad \phi(\Pi_a v) \leq \phi(v).$$

*Then there exists a unique solution  $(\theta, u)$  of problem  $\mathbf{P}(\mathbf{a}, \phi; L, u_0)$  which satisfies (3.14), too; it is also continuous with respect to the weak topology of  $H$  and it satisfies the initial condition  $u(0) = \Pi_a u_0$ .*

3.4 REMARK. It is interesting to note that  $(A_{comp})$  is equivalent to

$$(3.15) \quad v \in H, \quad \Lambda v \in H_a \quad \Rightarrow \quad \Lambda v = \Lambda \Pi_a v \quad \square$$

Now we vary the functionals and the data according to a parameter  $\varepsilon$  going to 0 and we want to study the dependence of the solution  $(\theta, u)$  on  $\varepsilon$ . So we are given

$$\mathbf{a}_\varepsilon, \phi_\varepsilon, L_\varepsilon, u_{0,\varepsilon}, \quad \varepsilon \in [0, \varepsilon_0],$$

and we set correspondingly

$$V_\varepsilon := V_{a_\varepsilon}, \quad \Lambda_\varepsilon := \partial\phi_\varepsilon, \quad \tilde{\phi}_\varepsilon := \widetilde{(\phi_\varepsilon)_{a_\varepsilon}}, \quad J_\varepsilon := J_{a_\varepsilon}, \quad \text{and so on.}$$

If  $(\theta_\varepsilon, u_\varepsilon)$  is a solution of  $\mathbf{P}(\mathbf{a}_\varepsilon, \phi_\varepsilon; L_\varepsilon, u_{0,\varepsilon})$ ,  $\varepsilon \geq 0$ , we look for general conditions on the data in order to obtain the convergence of  $(\theta_\varepsilon, u_\varepsilon)$  to  $(\theta_0, u_0)$  with respect to a suitable topology. Of course we have to impose some kind of continuity property for the data at  $\varepsilon = 0$ ; we recall the definition of the convergence in the sense of Mosco (see e.g. [3], sect. 3.4).

**3.5 DEFINITION.** *Let  $\mathbf{H}$  be an Hilbert space; we say that a family of functions  $F_\varepsilon : \mathbf{H} \mapsto ]-\infty, \infty]$   $M$ -converges to  $F_0 : \mathbf{H} \mapsto ]-\infty, +\infty]$ , as  $\varepsilon$  goes to 0, if the following two conditions are satisfied:*

$$F_0(v) \leq \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(v_\varepsilon), \quad \text{for every family } v_\varepsilon \text{ weakly convergent to } v \text{ in } \mathbf{H},$$

$$\forall v \in \mathbf{H}, \forall \varepsilon > 0, \quad \exists v_\varepsilon \in \mathbf{H} : \lim_{\varepsilon \rightarrow 0} v_\varepsilon = v \quad \text{strongly in } \mathbf{H}, \quad F_0(v) = \lim_{\varepsilon \rightarrow 0} F_\varepsilon(v_\varepsilon).$$

In this case we write  $F_0 = M\text{-}\lim_{\varepsilon \rightarrow 0} F_\varepsilon$ .

It is well known [3] that this notion is well adapted to describe the convergence of convex variational functionals and it is strictly related to  $\Gamma$ -convergence and to the graph-convergence of the respective subdifferentials <sup>(11)</sup>. We shall assume that

$$(LIM_{\mathbf{a}, \Lambda, \phi}) \quad \begin{cases} \mathbf{a}_\varepsilon \text{ and } \phi_\varepsilon \text{ M-converge to } \mathbf{a}_0 \text{ and } \phi_0 \text{ respectively as } \varepsilon \rightarrow 0 \\ \mathbf{a}_\varepsilon, \Lambda_\varepsilon, \phi_\varepsilon \text{ satisfy } (A_{\mathbf{a}, \Lambda, \phi}) \text{ uniformly in } ]0, \varepsilon_0]. \end{cases}$$

In order to make precise the kind of convergence of the functionals  $L_\varepsilon$  (which belongs to varying dual spaces) we state the following

**3.6 DEFINITION.** *Let  $\mathbf{H}$  be an Hilbert space,  $\mathbf{a}_\varepsilon : \mathbf{H} \mapsto [0, +\infty]$  be a family of generalized quadratic forms  $M$ -converging to  $\mathbf{a}_0$ , and  $\mathbf{V}_\varepsilon$  be the domain of  $\mathbf{a}_\varepsilon$ . We say that a family of linear functionals  $\mathcal{L}_\varepsilon \in \mathbf{V}'_\varepsilon$  strongly converges to  $\mathcal{L}_0 \in \mathbf{V}'_{a_0}$  if*

$$\lim_{\varepsilon \rightarrow 0} \langle \mathcal{L}_\varepsilon, v_\varepsilon \rangle = \langle \mathcal{L}_0, v_0 \rangle,$$

for every choice of  $v_\varepsilon \in \mathbf{V}_\varepsilon$  such that

$$v_\varepsilon \rightharpoonup v_0 \quad \text{in } \mathbf{H} \quad \text{and} \quad \sup_{\varepsilon > 0} a_\varepsilon(v_\varepsilon) < +\infty.$$

---

<sup>(11)</sup> more precisely, a family of functionals  $F_\varepsilon$  (as in definition 3.5)  $M$ -converges to  $F_0$  if and only if it  $\Gamma$ -converges to  $F_0$  both in the *strong* and in the *weak* topology of  $\mathbf{H}$ . If the functionals  $F_\varepsilon$  are l.s.c., normalized (i.e.  $F_\varepsilon(0) = 0$ ), and convex, then they  $M$ -converge to  $F_0$  iff the subdifferentials  $\partial F_\varepsilon$   $G$ -converge to  $\partial F_0$ .

Accordingly to this definition, we shall assume that

$$(LIM_L) \quad L_\varepsilon \in L^2(0, T; V'_\varepsilon) \quad \text{strongly converges to} \quad L_0 \in L^2(0, T; V'_0), \quad (12)$$

and

$$(LIM_{u_0}) \quad J_\varepsilon u_{0,\varepsilon} \in V'_\varepsilon \quad \text{strongly converges to} \quad J_0 u_{0,0} \in V'_0, \quad \sup_{\varepsilon > 0} \phi_\varepsilon(u_{0,\varepsilon}) < +\infty.$$

We have

**Theorem 3.** *Let us assume that  $(LIM_{\mathbf{a}, \Lambda, \phi; L, u_0})$  hold and let us denote by  $(\theta_\varepsilon, u_\varepsilon)$  a solution of  $\mathbf{P}(\mathbf{a}_\varepsilon, \phi_\varepsilon; L_\varepsilon, u_{0,\varepsilon})$ . Then*

$$(3.16) \quad u_\varepsilon \text{ is uniformly bounded in } L^\infty(0, T; H),$$

$$(3.17) \quad \theta_\varepsilon \rightharpoonup \theta_0 \quad \text{in } L^2(0, T; H), \quad \lim_{\varepsilon \rightarrow 0} \int_0^T \mathbf{b}(\theta_\varepsilon - \theta_0) dt = 0,$$

and for every  $L^\infty(0, T; H)$ -weak\* cluster point  $u_0$  of  $u_\varepsilon$ , as  $\varepsilon \rightarrow 0$ ,  $(\theta_0, u_0)$  is a solution of  $\mathbf{P}(\mathbf{a}_0, \phi_0; L_0, u_{0,0})$ . Moreover, if  $(A_{comp})$  is satisfied for every  $\varepsilon \geq 0$  and  $u_\varepsilon$  is the weakly continuous solution belonging to  $H_\varepsilon$  (cf. the previous corollary), then

$$(3.18) \quad \Pi_0 u_\varepsilon(t) \rightharpoonup u_0(t) \quad \text{in } H, \quad \forall t \in [0, T].$$

Finally, if

$$(3.19) \quad u_{0,0} \in H_0, \quad \lim_{\varepsilon \rightarrow 0} \phi_\varepsilon(u_{0,\varepsilon}) = \phi_0(u_{0,0})$$

also holds, then

$$(3.20) \quad \lim_{\varepsilon \rightarrow 0} \int_0^T \mathbf{a}_\varepsilon(\theta_\varepsilon(t)) dt = \int_0^T \mathbf{a}_0(\theta_0(t)) dt.$$

**3.7 REMARK.** In order to obtain (3.20) it would sufficient the weaker (cf. Remark 3.2)

$$(3.21) \quad \lim_{\varepsilon \rightarrow 0} \tilde{\phi}_\varepsilon(J_\varepsilon u_{0,\varepsilon}) = \tilde{\phi}_0(J_0 u_{0,0}),$$

instead of  $(A_{comp})$  and (3.19).

---

<sup>(12)</sup> We are implicitly considering the usual extension (cf. [6], Prop. 2.16) of a quadratic form on  $V_\varepsilon \subset H$  to the time-dependent vectors of  $L^2(0, T; V_\varepsilon) \subset L^2(0, T; H)$ . This extension does not affect the  $M$ -convergence properties.

#### 4. - PROOFS OF THE ABSTRACT THEOREMS.

First of all we rewrite (3.5) as

$$(4.1) \quad \int_0^T \left\{ -(u, v_t) + a(\theta, v) + b(\theta, v) \right\} dt = (u_0, v(0)) + \int_0^T \left\{ \langle L, v \rangle + b(\theta, v) \right\} dt.$$

Now, replacing  $a$  by  $a + b$ , we can always assume that

$$(A_{\mathbf{a}\mathbf{b}}) \quad 0 \leq \mathbf{b}(u) \leq \mathbf{a}(u), \quad \mathbf{a}(u) \geq \alpha|u|^2, \quad \forall u \in V_a,$$

if we consider, instead of  $\mathbf{P}$ , the following more general formulation, depending on the parameter  $\lambda \in \mathbf{R}$ .

**Problem  $\mathbf{P}_\lambda(\mathbf{a}, \phi; L, u_0)$ .** *Given*

$$L \in L^2(0, T; V'_a) \quad \text{and} \quad u_0 \in H,$$

*find  $\theta \in L^2(0, T; V_a)$  and  $u \in L^2(0, T; H)$  such that*

$$(4.2) \quad \theta(t) \in \Lambda u(t) \quad \text{for a.e. } t \in ]0, T[$$

*and*

$$(4.3) \quad \int_0^T \left\{ -(u, v_t) + a(\theta, v) \right\} dt = (u_0, v(0)) + \int_0^T \left\{ \langle L, v \rangle + \lambda b(\theta, v) \right\} dt$$

*for any choice of  $v \in H^1(0, T; V_a)$  with  $v(T) = 0$ .*

We are going to prove the analogous of Theorem 1 for  $\mathbf{P}_\lambda$  under the hypotheses  $(A_{\mathbf{a}\mathbf{b}, \phi; L, u_0})$ .

We fix some notation.

**4.1 NOTATION.** *We denote by  $A : V_a \mapsto V'_a$  the linear isomorphism*

$$(4.4) \quad \hat{v} = Av \quad \Leftrightarrow \quad \langle \hat{v}, w \rangle = a(v, w), \quad \forall v, w \in V_a.$$

*When it is possible, the superscripts  $\tilde{\cdot}, \hat{\cdot}$  will denote the images in  $H'_a, V'_a$  of the correspondent vectors via  $J_a$  and  $A$ , respectively.  $|\cdot|_{\mathbf{a}}$  is the dual norm of  $H'_a$ :*

$$(4.5) \quad |\tilde{v}|_{\mathbf{a}} := \min\{|v| : J_a v = \tilde{v}\} = \min\{\langle \tilde{v}, w \rangle : w \in H_a, |w| \leq 1\}$$

*and  $a'(\cdot, \cdot)$  the dual scalar product on  $V'_a$ , with the associated norm  $\mathbf{a}'(\hat{v}) := a'(\hat{v}, \hat{v})$*

$$(4.6) \quad \mathbf{a}'(\hat{v}, \hat{w}) := \langle \hat{v}, w \rangle = \langle \hat{w}, v \rangle = a(v, w).$$

We extend  $\tilde{\phi}_a$  to  $V'_a$  by setting  $\tilde{\phi}_a(\tilde{v}) = +\infty$  if  $\tilde{v} \in V'_a \setminus H'_a$ , and we call

$$(4.7) \quad \tilde{\Lambda} : V'_a \mapsto 2^{V'_a}, \quad \tilde{\Lambda} := \partial_{a'} \tilde{\phi}_a,$$

its subdifferential with respect to the scalar product  $a'$ .  $b_a : V'_a \times V'_a \mapsto \mathbf{R}$  is the bilinear form

$$(4.8) \quad b_a(\hat{v}, \hat{w}) := b(v, w), \quad \forall \hat{v}, \hat{w} \in V'_a,$$

$B : H \mapsto H$  and  $B_a : V'_a \mapsto V'_a$  are the linear operators associated to  $b$  and  $b_a$ :

$$(4.9) \quad (Bv, w) := b(v, w), \quad \forall v, w \in H; \quad a'(B_a \hat{v}, \hat{w}) := b_a(\hat{v}, \hat{w}), \quad \forall \hat{v}, \hat{w} \in V'_a \quad \square$$

We briefly recall the (easy to prove) properties of  $K(\cdot)$ , we mentioned in the previous section:

**4.2 LEMMA.** *For every  $\tilde{v} \in D(\tilde{\phi}_a) \subset H'_a$  the set  $K(\tilde{v}) \subset H$  defined by (3.10) is closed and convex; it is also bounded and non-empty if  $(A_\phi)$  holds and in this case we have for every  $r \geq 0$*

$$(4.10) \quad \tilde{\phi}_a(\tilde{v}) \leq r \Rightarrow |\tilde{v}|_a \leq \omega(r), \quad \sup_{v \in K(\tilde{v})} |v| \leq \omega(r),$$

where  $\omega(\cdot)$  is the “modulus of continuity” of  $\phi$  at infinity

$$(4.11) \quad \omega : [0, +\infty[ \mapsto [0, +\infty[, \quad \omega(r) := \sup \{ |v| : v \in H, \phi(v) \leq r \}.$$

The first step consists in the following result

**4.3 THEOREM.** *The function  $\tilde{\phi}_a$  defined in (3.9) is proper, convex, and its subdifferential  $\tilde{\Lambda}$  with respect to the scalar product  $a'$  satisfies*

$$(4.12) \quad \hat{w} \in \tilde{\Lambda} \tilde{v} \Rightarrow w \in \Lambda v, \quad \forall v \in K(\tilde{v}),$$

and

$$(4.13) \quad w \in V_a, \quad w \in \Lambda v \Rightarrow \hat{w} \in \tilde{\Lambda} \tilde{v}, \quad v \in K(\tilde{v}),$$

where, following 4.1,  $\hat{w}$  and  $w$  are related by  $\hat{w} = Aw$ . Moreover, if  $(A_\phi)$  holds, then  $\tilde{\phi}_a$  is also l.s.c. in  $V'_a$  and coercive with respect to the norm of  $H'_a$ . In particular  $\tilde{\Lambda}$  is a maximal monotone operator in  $V'_a$ .

The proof is a series of simple checkings.



- $\tilde{\phi}_a$  is convex: if  $\tilde{u}, \tilde{v} \in D(\tilde{\phi}_a)$  and we choose, for a fixed  $\varepsilon > 0$ ,  

$$u \in J_a^{-1}(\tilde{u}), \quad v \in J_a^{-1}(\tilde{v}), \quad \text{with} \quad \phi(u) \leq \tilde{\phi}_a(\tilde{u}) + \varepsilon, \quad \phi(v) \leq \tilde{\phi}_a(\tilde{v}) + \varepsilon,$$
then for every  $\tau \in [0, 1]$  we have  $J_a(\tau u + (1 - \tau)v) = \tau \tilde{u} + (1 - \tau)\tilde{v}$  so that  

$$\tilde{\phi}_a(\tau \tilde{u} + (1 - \tau)\tilde{v}) \leq \phi(\tau u + (1 - \tau)v) \leq \tau \phi(u) + (1 - \tau)\phi(v) \leq \tau \tilde{\phi}_a(\tilde{u}) + (1 - \tau)\tilde{\phi}_a(\tilde{v}) + \varepsilon$$
Since  $\varepsilon > 0$  is arbitrary, we conclude.

- $\tilde{\phi}_a$  is proper:  $\tilde{\phi}_a(0) = 0$ .

- (4.12): by the definition of subdifferential we know

$$(4.14) \quad a'(\hat{w}, \tilde{z} - \tilde{v}) \leq \tilde{\phi}_a(\tilde{z}) - \tilde{\phi}_a(\tilde{v}), \quad \forall \tilde{z} \in D(\tilde{\phi}_a).$$

Let us set  $w := A^{-1}\hat{w} \in V_a$  and let us choose  $v \in K(\tilde{v})$ ; recalling (3.7) and (4.6), for every  $z \in H$ , with  $J_a z = \tilde{z}$ , we have

$$(4.15) \quad (w, z - v) = \langle w, \tilde{z} - \tilde{v} \rangle = a'(\hat{w}, \tilde{z} - \tilde{v}).$$

Combining with (4.14), since  $\tilde{\phi}_a(\tilde{v}) = \phi(v)$  and  $\tilde{\phi}_a(\tilde{z}) \leq \phi(z)$ , we deduce

$$(w, z - v) \leq \tilde{\phi}_a(\tilde{z}) - \tilde{\phi}_a(\tilde{v}) \leq \phi(z) - \phi(v) = \phi(z) - \phi(v), \quad \forall z \in H,$$

i.e.  $w \in \Lambda v$ .

- (4.13): we know that  $w \in V_a$  satisfies

$$(w, z - v) \leq \phi(z) - \phi(v), \quad \forall z \in H.$$

and, by (4.15), for every  $z \in H$  we have

$$(4.16) \quad a'(\hat{w}, \tilde{z} - \tilde{v}) \leq \phi(z) - \phi(v), \quad \tilde{z} := J_a z.$$

Choosing  $J_a z = \tilde{z} = \tilde{w}$  we get  $v \in K(\tilde{w})$ ; keeping  $\tilde{z}$  fixed and taking the infimum with respect to  $z \in J_a^{-1}(\tilde{z})$  we get

$$a'(\hat{w}, \tilde{z} - \tilde{v}) \leq \tilde{\phi}_a(\tilde{z}) - \phi(v), \quad \forall \tilde{z} \in H'_a;$$

recalling that  $\phi(v) \geq \tilde{\phi}_a(\tilde{v})$  we find

$$a'(\hat{w}, \tilde{z} - \tilde{v}) \leq \tilde{\phi}_a(\tilde{z}) - \tilde{\phi}_a(\tilde{v}), \quad \forall \tilde{z} \in H'_a.$$

- $\tilde{\phi}_a$  is coercive, if  $(A_\phi)$  holds: it follows from (4.10).
- $\tilde{\phi}_a$  is l.s.c. if  $(A_\phi)$  holds: we choose a sequence  $\tilde{v}_n \in V'_a$  converging to  $v$  with  $\tilde{\phi}_a(\tilde{v}_n) \leq r$ . By the previous Lemma, there exist  $v_n \in K(\tilde{v}_n) \subset H$  such that

$$J_a v_n = \tilde{v}_n, \quad \phi(v_n) = \tilde{\phi}_a(\tilde{v}_n) \leq r, \quad |v_n| \leq \omega(r).$$

We can extract a subsequence (still denoted by  $v_n$ ) weakly convergent to  $v$  in  $H$ . By the (weakly) lower semicontinuity of  $\phi$  we deduce  $\phi(v) \leq r$  and by the continuity and the linearity of  $J_a$  we have  $J_a v = \tilde{v}$ . We conclude that  $\tilde{\phi}_a(\tilde{v}) \leq \phi(v) \leq r$ . ■

Theorem 4.3 shows that we can associate to a solution  $(\theta, u)$  of  $\mathbf{P}_\lambda$  the new couple  $(\hat{\theta}, \hat{u})$ , with  $\hat{\theta} := A\theta$ ,  $\hat{u} := J_a u$ . This change of unknowns allows us to rewrite  $\mathbf{P}_\lambda$  as a perturbation of the evolution equation in  $V'_a$  associated to the subdifferential operator  $\tilde{\Lambda}$ .

4.4 THEOREM. Suppose that  $(\theta, u)$  is a solution of  $\mathbf{P}_\lambda(\mathbf{a}, \phi; L, u_0)$ . Then  $(\hat{\theta}, \tilde{u})$  solves the following abstract Cauchy problem  $\tilde{\mathbf{P}}_\lambda(\mathbf{a}, \phi; L, u_0)$

$$(4.17) \quad \begin{cases} \text{Find } \tilde{u} \in L^2(0, T; H'_a) \cap H^1(0, T; V'_a) \text{ and } \hat{\theta} \in L^2(0, T; V'_a) \text{ such that} \\ \tilde{u}_t(t) + \hat{\theta}(t) - \lambda B_a \hat{\theta}(t) = L(t), \quad \hat{\theta}(t) \in \tilde{\Lambda} \tilde{u}(t), \quad \text{for a.e. } t \in ]0, T[, \\ \tilde{u}(0) = \tilde{u}_0 = J_a u_0. \end{cases}$$

Conversely, if  $(\hat{\theta}, \tilde{u})$  is the solution of  $\tilde{\mathbf{P}}_\lambda(\mathbf{a}, \phi; L, u_0)$ , the corresponding  $\mathbf{P}$  is solved by every couple  $(\theta, u)$  satisfying

$$(4.18) \quad \theta := A^{-1} \hat{\theta} \quad \text{and} \quad u \in L^2(0, T; H) \quad \text{with} \quad u(t) \in K(\tilde{u}(t)), \quad \text{for a.e. } t \in ]0, T[.$$

PROOF. By setting  $\hat{v} := Av$  and recalling (4.9), we see that (4.3) is equivalent to

$$(4.19) \quad \int_0^T \{ -a'(\tilde{u}, \hat{v}_t) + a'(\hat{\theta}, \hat{v}) \} dt = a'(\tilde{u}_0, \hat{v}(0)) + \int_0^T a'(L + \lambda B_a \hat{\theta}, \hat{v}) dt$$

for any choice of  $\hat{v} \in H^1(0, T; V'_a)$  with  $\hat{v}(T) = 0$ . (4.19) is a weak formulation of the differential equation and the initial condition of (4.17). Applying (4.12) and (4.13), we conclude. ■

By the general theory of nonlinear evolution equation governed by subdifferential operators in Hilbert spaces (see [6], thm. 4.6) we deduce

4.5 COROLLARY. When  $(A_\phi)$  holds, together to  $(A_{\mathbf{ab}, \Lambda; L, u_0})$ , there exists a unique solution of problem  $\tilde{\mathbf{P}}_0(\mathbf{a}, \phi; L, u_0)$ . ■

In order to prove the uniqueness result of Theorem 1 we need the following a priori estimate.

4.6 LEMMA. Let  $(\hat{\theta}^j, \tilde{u}^j)$ ,  $j = 1, 2$ , be solutions of  $\tilde{\mathbf{P}}_0(\mathbf{a}, \phi; L^j + \lambda B_a \hat{\theta}^j, u_0^j)$  respectively, with

$$L^j, u_0^j \text{ satisfying } (A_{L, u_0}) \quad \text{and} \quad \hat{\theta}^j \in L^2(0, T; V'_a);$$

then, if  $(A_{\mathbf{ab}, \Lambda})$  hold and  $\gamma := \lambda^2 + 1$ , we have

$$(4.20) \quad \left. \begin{aligned} & \sup_{t \in [0, T]} e^{-\gamma t} \mathbf{a}'(\tilde{u}^1 - \tilde{u}^2) \\ & 2 \int_0^T e^{-\gamma t} \mathbf{b}_a(\hat{\theta}^1 - \hat{\theta}^2) dt \end{aligned} \right\} \leq \mathbf{a}'(\tilde{u}_0^1 - \tilde{u}_0^2) + \int_0^T e^{-\gamma s} \left[ \mathbf{b}_a(\hat{\theta}^1 - \hat{\theta}^2) + \mathbf{a}'(L^1 - L^2) \right] dt.$$

PROOF. We apply the standard monotonicity arguments to (4.17): taking the difference of the two equations satisfied by  $\tilde{u}^j$  and multiplying (with respect to the dual scalar product  $a'$ ) by the difference of the corresponding solutions and the weight  $2e^{-\gamma t}$ , we obtain

$$\begin{aligned} & \frac{d}{dt} \left\{ e^{-\gamma t} \mathbf{a}'(\tilde{u}^1 - \tilde{u}^2) \right\} + \gamma e^{-\gamma t} \mathbf{a}'(\tilde{u}^1 - \tilde{u}^2) + 2e^{-\gamma t} \mathbf{b}_a(\hat{\theta}^1 - \hat{\theta}^2) \leq \\ & 2\lambda e^{-\gamma t} \left[ b_a(\hat{\sigma}^1 - \hat{\sigma}^2, \tilde{u}^1 - \tilde{u}^2) + a'(L^1 - L^2, \tilde{u}^1 - \tilde{u}^2) \right] \leq \\ & e^{-\gamma t} \left[ \mathbf{b}_a(\hat{\sigma}^1 - \hat{\sigma}^2) + \lambda^2 \mathbf{b}_a(\tilde{u}^1 - \tilde{u}^2) + \mathbf{a}'(\tilde{u}^1 - \tilde{u}^2) + \mathbf{a}'(L^1 - L^2) \right] \leq \\ & e^{-\gamma t} \left[ \mathbf{b}_a(\hat{\sigma}^1 - \hat{\sigma}^2) + \mathbf{a}'(L^1 - L^2) + (\lambda^2 + 1) \mathbf{a}'(\tilde{u}^1 - \tilde{u}^2) \right] \end{aligned}$$

where we used the easy bound (cf.  $(A_{ab})$  and (4.8))

$$(4.21) \quad \mathbf{b}_a(\tilde{v}) = \mathbf{b}(A^{-1}\tilde{v}) \leq \mathbf{a}(A^{-1}\tilde{v}) = \mathbf{a}'(\tilde{v}), \quad \forall \tilde{v} \in V'_a.$$

Integrating in time we conclude. ■

We can now conclude the **proof of Theorem 1**. In fact, if  $(\theta^j, u^j)$ ,  $j = 1, 2$ , are two solutions of problem  $\mathbf{P}_\lambda(\mathbf{a}, \phi; L, u_0)$ , we know that

$$(4.22) \quad (\hat{\theta}^j, \tilde{u}^j) \quad \text{solves problem} \quad \tilde{\mathbf{P}}_0(\mathbf{a}, \phi; L + \lambda B_a \hat{\theta}^j, u_0).$$

By Lemma 4.6 we deduce

$$\tilde{u}^1 = \tilde{u}^2, \quad \mathbf{b}_a(\hat{\theta}^1 - \hat{\theta}^2) = 0,$$

and also, due to the positivity of  $b_a$ ,

$$B_a \hat{\theta}^1 = B_a \hat{\theta}^2.$$

Finally from the differential equation (4.17) we read

$$\hat{\theta}^1(t) = \hat{\theta}^2(t), \quad \text{for a.e. } t \in ]0, T[;$$

the thesis follows now by Theorem 4.4. ■

We establish another estimate.

4.7 LEMMA. *Let us assume  $(A_{\mathbf{a}\mathbf{b},\Lambda;L,u_0})$  and let  $(\theta, u)$  be a solution of problem  $\mathbf{P}_\lambda(\mathbf{a}, \phi; L, u_0)$ . We have*

$$(4.23) \quad \left. \begin{array}{l} \text{ess-sup}_{s \in ]0, T[} \phi(u(s)) \\ \frac{1}{2} \int_0^T \mathbf{a}(\theta) dt \end{array} \right\} \leq \phi(u_0) + \frac{1}{2} \int_0^T \mathbf{a}'_\varepsilon(L_\varepsilon) dt + \lambda \int_0^T \mathbf{b}(\theta) dt$$

PROOF. We take the  $a'$ -scalar product of the differential equation (4.17) with  $\hat{\theta}(t)$  and we integrate between 0 and  $s \leq T$ , obtaining

$$(4.24) \quad \int_0^s a'(\tilde{u}_t, \hat{\theta}) dt + \int_0^s \mathbf{a}'(\hat{\theta}) dt = \int_0^s a'(L + \lambda B_a \hat{\theta}, \hat{\theta}) dt.$$

We call  $\bar{\phi}_a$  the lower semicontinuous envelope of  $\tilde{\phi}_a$  in  $V'_a$  (cf. [18], Ch. I, 2.2, [12], Ch. 3):

$$(4.25) \quad \bar{\phi}_a(v) := \lim_{\rho \rightarrow 0} \inf \{ \tilde{\phi}_a(w) : \mathbf{a}'(w - v) \leq \rho^2 \}, \quad \forall v \in V'_a.$$

Of course  $\partial_{a'} \bar{\phi}_a$  is a maximal monotone operator which is related to  $\tilde{\Lambda}$  by ([18], p. 20)

$$(4.26) \quad \tilde{u} \in V'_a, \quad \tilde{\Lambda}(\tilde{u}) \neq \emptyset \quad \Rightarrow \quad \tilde{\Lambda}(\tilde{u}) = \partial_{a'} \bar{\phi}_a(\tilde{u}), \quad \bar{\phi}_a(\tilde{u}) = \tilde{\phi}_a(\tilde{u}).$$

Since in (4.24)  $\hat{\theta} \in \tilde{\Lambda}(\tilde{u}) = \partial_{a'} \bar{\phi}_a(\tilde{u})$ , a.e. in  $]0, T[$ , the first integral is equal to (see [17], Lemma 2.2)

$$\bar{\phi}_a(\tilde{u}(s)) - \bar{\phi}_a(\tilde{u}(0)), \quad \forall s \in ]0, T[.$$

Applying (4.26) again, we deduce that, for a.e.  $s \in ]0, T[$ ,

$$(4.27) \quad \tilde{\phi}_a(\tilde{u}(s)) + \int_0^s \mathbf{a}(\theta) dt = \bar{\phi}_a(u_0) + \int_0^s \left( \langle L, \theta \rangle + \lambda \mathbf{b}(\theta) \right) dt$$

and, by Schwarz inequality, we get (4.23), since for a.e.  $s \in ]0, T[$  it is  $u(s) \in K(\tilde{u}(s))$  and consequently  $\phi(u(s)) = \tilde{\phi}_a(\tilde{u}(s))$ . ■

Lemma 4.6 and 4.7 allow us to prove the following existence result.

4.8 THEOREM. *Let us assume that  $(A_{\mathbf{a}\mathbf{b},\Lambda;L,u_0})$  and  $(A_\phi)$  hold; then there exists a unique solution  $(\hat{\theta}, \tilde{u})$  of the perturbed problem  $\tilde{\mathbf{P}}_\lambda(\mathbf{a}, \phi; L, u_0)$ , which also satisfies*

$$(4.28) \quad \tilde{\phi}_a(\tilde{u}) \in L^\infty(0, T).$$

PROOF. We already noticed that if  $(A_\phi)$  holds besides the other assumptions  $(A_{\mathbf{a}\mathbf{b},\Lambda;L,u_0})$ , then  $\tilde{\mathbf{P}}_0$  admits a unique solution. Thanks to (4.22), it is natural to look for the solution of  $\tilde{\mathbf{P}}_\lambda$  as the limit of a standard fixed-point iterative technique. If we choose  $\hat{\theta}^0 \in L^2(0, T; V'_a)$  and we define by induction

$$(\hat{\theta}^{n+1}, \tilde{u}^{n+1}) := \text{the solution of } \tilde{\mathbf{P}}_0(\mathbf{a}, \phi; L + \lambda B_a \hat{\theta}^n, u_0),$$

it is easy to see that this sequence converges to the solution of  $\tilde{\mathbf{P}}_\lambda(\mathbf{a}, \phi; L, u_0)$ ; in fact, by Lemma 4.6,  $\tilde{u}^n$  and  $B_a \hat{\theta}^n$  are Cauchy sequences in  $L^\infty(0, T; V'_a)$  and  $L^2(0, T; V'_a)$  respectively, whereas

$$\int_0^T \mathbf{b}_a(\hat{\theta}^n) dt = \int_0^T \mathbf{b}(\theta^n) dt \quad \text{is uniformly bounded.}$$

The estimate (4.23) of Lemma 4.7 holds uniformly and proves that the sequence  $\hat{\theta}^n$  is bounded in  $L^2(0, T; V'_a)$  and therefore  $\tilde{u}^n$  is bounded in  $H^1(0, T; V'_a)$  by (4.17). From the uniqueness of the limit point of  $\tilde{u}^n$  in  $L^\infty(0, T; V'_a)$  and the standard monotonicity arguments, we can pass to the limit in (4.17); (4.28) follows by the first of (4.23). ■

Also the **proof of Theorem 2 and its Corollary** is almost completed; invoking Theorem 4.4, it remains to show that there exists  $u$  satisfying (4.18),  $\tilde{u}$  being given by the previous Theorem 4.8.

Since  $(A_\phi)$  holds, from (4.12) and Lemma 4.2 we know that

$$\theta(t) := A^{-1}\hat{\theta}(t) \in D(\Lambda^{-1}), \quad \text{for a.e. } t \in ]0, T[,$$

so that the minimal selection

$$u^*(t) \in \Lambda^{-1}(\theta(t)) : \quad |u^*(t)| = \min\{|v| : v \in \Lambda^{-1}\theta(t)\}$$

is surely measurable (cf. [6], Prop. 2.6(iii)); since  $\tilde{\phi}_a(\tilde{u})$  is bounded, taking account of (4.13) and of (4.10) we deduce that  $u^*$  belongs to  $L^\infty(0, T; H)$ . Finally, if  $(A_{comp})$  holds, then

$$u \in K(\tilde{u}) \quad \Rightarrow \quad \Pi_a u \in K(\tilde{u}),$$

so that the orthogonal projection on  $H_a$  of every solution  $u \in U(\mathbf{a}, \phi; L, u_0)$  is a solution again. Since  $H_a$  and  $H'_a$  are isomorphic via  $J_a$  and  $J_a \Pi_a u = \tilde{u}$  is uniquely determined and weakly continuous in  $H'_a$ , we conclude. ■

Now we study the limit behaviour of the solutions as stated in Theorem 3 and from now on we assume that  $(LIM_{\mathbf{a},\Lambda,\phi;L,u_0})$  hold.

First of all, we show some relations between  $M$ -convergence of  $\mathfrak{a}_\varepsilon$ , definition 3.6 and the pointwise convergence of the operators

$$(4.29) \quad \mathfrak{R}_\varepsilon : H \mapsto V_\varepsilon \subset H, \quad \mathfrak{R}_\varepsilon v := A_\varepsilon^{-1} J_\varepsilon v, \quad \forall v \in H,$$

which obviously satisfy

$$(4.30) \quad u = \mathfrak{R}_\varepsilon v \quad \Leftrightarrow \quad a_\varepsilon(u, w) = (v, w), \quad \forall w \in V_\varepsilon.$$

4.9 LEMMA.  $\{\mathfrak{R}_\varepsilon\}_{\varepsilon \geq 0}$  is a family of uniformly bounded symmetric operators satisfying, as  $\varepsilon \rightarrow 0$ ,

$$(4.31) \quad v_\varepsilon \rightarrow v_0 \quad \text{strongly in } H \quad \Rightarrow \quad \mathfrak{R}_\varepsilon v_\varepsilon \rightarrow \mathfrak{R}_0 v_0 \quad \text{strongly in } H,$$

and

$$(4.32) \quad v_\varepsilon \rightharpoonup v_0 \quad \text{weakly in } H \quad \Rightarrow \quad \mathfrak{R}_\varepsilon v_\varepsilon \rightharpoonup \mathfrak{R}_0 v_0 \quad \text{weakly in } H.$$

Moreover, if a family  $\hat{z}_\varepsilon \in V'_\varepsilon$  strongly converges to  $\hat{z}_0 \in V'_0$  as in definition 3.6, then

$$(4.33) \quad z_\varepsilon := A_\varepsilon^{-1} \hat{z}_\varepsilon \rightarrow \hat{z}_0 := A_0^{-1} z_0 \quad \text{in } H, \quad \mathfrak{a}'_\varepsilon(\hat{z}_\varepsilon) \rightarrow \mathfrak{a}'_0(\hat{z}_0).$$

PROOF. From (4.30) and  $(A_{\mathfrak{a}b})$  we have

$$\alpha |u|^2 \leq \mathfrak{a}_\varepsilon(u) \leq \frac{1}{\alpha} |v|^2.$$

Formulae (4.31) and (4.32) follow easily from [12], corollary 13.7(b) and def. 13.3.

Finally, if  $\hat{z}_\varepsilon \in V'_\varepsilon$  strongly converges to  $\hat{z}_0 \in V'_0$ , then, for every  $w_\varepsilon \rightharpoonup w$  in  $H$ , we have

$$(z_\varepsilon, w_\varepsilon) = a(z_\varepsilon, \mathfrak{R}_\varepsilon w_\varepsilon) = \langle \hat{z}_\varepsilon, \mathfrak{R}_\varepsilon w_\varepsilon \rangle \rightarrow \langle \hat{z}_0, \mathfrak{R}_0 w_0 \rangle = (z_0, w_0),$$

i.e.  $z_\varepsilon \rightarrow z_0$  strongly in  $H$ ; choosing  $w_\varepsilon := z_\varepsilon$  as test function, we get

$$\mathfrak{a}'_\varepsilon(\hat{z}_\varepsilon) = \langle \hat{z}_\varepsilon, z_\varepsilon \rangle \rightarrow \langle \hat{z}_0, z_0 \rangle = \mathfrak{a}'_0(\hat{z}_0) \quad \blacksquare$$

4.10 REMARK. Thanks to the linearity and to the uniform boundedness of  $\mathfrak{R}_\varepsilon$ , the previous properties also hold for vector valued functions; e.g., for every  $v_\varepsilon$  strongly converging to  $v_0$  in  $L^2(0, T; H)$  or  $H^1(0, T; H)$ ,

$$(4.34) \quad \mathfrak{R}_\varepsilon v_\varepsilon \rightarrow \mathfrak{R}_0 v_0 \quad \text{in } L^2(0, T; H) \text{ or } H^1(0, T; H), \text{ respectively} \quad \square$$

Now we prove that (3.5) is stable with respect to the weak limit of solutions.

4.11 PROPOSITION. *Let us assume that  $(LIM_{\mathbf{a},\Lambda,\phi;L,u_0})$  hold and let  $(u_\varepsilon, \theta_\varepsilon)$  be a family of solutions of  $\mathbf{P}_\lambda(\mathbf{a}_\varepsilon, \phi_\varepsilon; L_\varepsilon, u_{0,\varepsilon})$ ; then there exists a constant  $C$  independent of  $\varepsilon$  such that*

$$(4.35) \quad \|\phi(u_\varepsilon)\|_{L^\infty(0,T)} + \|u_\varepsilon\|_{L^\infty(0,T;H)} + \int_0^T \left( \mathbf{a}'\left(\frac{d}{dt}\tilde{u}_\varepsilon\right) + \mathbf{a}_\varepsilon(\theta_\varepsilon) + \mathbf{b}(\theta_\varepsilon) \right) dt \leq C.$$

Moreover, as  $\varepsilon \rightarrow 0$ , every couple of weak cluster points  $(\theta_0, u_0)$  of  $(\theta_\varepsilon, u_\varepsilon)$  in  $L^2(0, T; H)$  satisfies (3.5) for  $\mathbf{P}_\lambda(\mathbf{a}_0, \phi_0; L_0, u_{0,0})$ ; the corresponding  $(\hat{\theta}_0, \tilde{u}_0)$  satisfies the differential equation of (4.17) <sup>(13)</sup>.

PROOF. From definition 3.6,  $(LIM_{L,u_0})$ , and Lemma 4.9, we deduce that there exists  $C_{data} > 0$  such that

$$\phi_\varepsilon(u_{0,\varepsilon}) + \mathbf{a}'_\varepsilon(u_{0,\varepsilon}) + \int_0^T \mathbf{a}'_\varepsilon(L_\varepsilon) dt \leq C_{data}, \quad \forall \varepsilon \in [0, \varepsilon_0].$$

The uniformity assumption of  $(LIM_{\mathbf{a},\Lambda,\phi})$  and (4.20) entail that the integral in  $]0, T[$  of  $\mathbf{b}(\theta_\varepsilon)$  is uniformly bounded; by (4.23) of Lemma 4.7 we get (4.35).

Let us now assume that for a decreasing sequence  $\{\varepsilon_n\}_{n \in \mathbf{N}}$  converging to 0 we have

$$(4.36) \quad u_{\varepsilon_n} \rightharpoonup u_0, \quad \theta_{\varepsilon_n} \rightharpoonup \theta_0, \quad \text{in } L^2(0, T; H).$$

We fix

$$(4.37) \quad w \in H^1(0, T; H), \quad \text{with } w(T) = 0,$$

and we choose the test functions  $v_{\varepsilon_n} := \mathfrak{R}_{\varepsilon_n} w$  in the weak formulation (3.5) of  $\mathbf{P}_\lambda(\mathbf{a}_{\varepsilon_n}, \phi_{\varepsilon_n}; L_{\varepsilon_n}, u_{0,\varepsilon_n})$ , obtaining for every  $n \in \mathbf{N}$

$$(4.38) \quad \begin{aligned} & \int_0^T \left\{ -(u_{\varepsilon_n}, \mathfrak{R}_{\varepsilon_n} w_t) + (\theta_{\varepsilon_n}, w) \right\} dt \\ &= \langle \tilde{u}_{0,\varepsilon_n}, \mathfrak{R}_{\varepsilon_n} w(0) \rangle + \int_0^T \left\{ \langle L_{\varepsilon_n}, \mathfrak{R}_{\varepsilon_n} w \rangle + \lambda b(\theta_{\varepsilon_n}, \mathfrak{R}_{\varepsilon_n} w) \right\} dt. \end{aligned}$$

By Lemma 4.9 (cf. also 4.10) as  $n \rightarrow \infty$  we get

$$\int_0^T \left\{ -(u_0, \mathfrak{R}_0 w_t) + (\theta_0, w) \right\} dt = \langle \tilde{u}_{0,0}, \mathfrak{R}_0 w(0) \rangle + \int_0^T \left\{ \langle L_0, \mathfrak{R}_0 w \rangle + \lambda b(\theta_0, \mathfrak{R}_0 w) \right\} dt$$

for every  $w$  satisfying (4.37). Since  $\mathfrak{R}_0(H)$  is dense in  $V_0$  (see e.g. [12], prop. 12.17), we conclude. ■

<sup>(13)</sup> At this level, we don't say anything about  $\theta_0 \in \Lambda_0 u_0$ .

The previous Proposition does not say if  $\theta_0$  belongs to  $\Lambda_0 u_0$ ; in order to answer this basic question, we have to work a little bit more.

Let us denote by  $\mathcal{H}$  the Hilbert space

$$(4.39) \quad \mathcal{H} := L^2_\zeta(0, T; H) \quad \text{where } \zeta \text{ is the measure } d\zeta(t) := e^{-\gamma t} dt, \quad \gamma := \lambda^2 + 1,$$

and let us denote by  $\Lambda_\varepsilon$  again the canonical extension of  $\Lambda_\varepsilon$  to  $\mathcal{H}$  (see [6], 2.1.3 and 2.3.3). We introduce the multivalued operator  $\mathcal{W}_\varepsilon : \mathcal{H} \mapsto 2^\mathcal{H}$  with the same domain of  $\Lambda_\varepsilon$

$$(4.40) \quad w \in \mathcal{W}_\varepsilon(u) \quad \Leftrightarrow \quad \exists \theta \in \Lambda_\varepsilon u, \quad \text{such that } w = \theta + \gamma \mathfrak{R}_\varepsilon u - \lambda \mathfrak{R}_\varepsilon B\theta$$

4.12 LEMMA. *For every  $u, w \in \mathcal{H}$  such that  $w \in \mathcal{W}_\varepsilon u$ , there exists a unique  $\theta := \Theta_\varepsilon(w, u) \in \mathcal{H}$  such that*

$$(4.41) \quad \theta \in \Lambda_\varepsilon u \quad \text{and} \quad w = \theta + \gamma \mathfrak{R}_\varepsilon u - \lambda \mathfrak{R}_\varepsilon B\theta$$

as in the definition (4.40).

PROOF. From  $(A_\Lambda)$  we see that

$$\theta^1, \theta^2 \in \Lambda_\varepsilon u \quad \Rightarrow \quad \mathfrak{b}(\theta^1 - \theta^2) = 0 \quad \Rightarrow \quad B\theta^1 = B\theta^2,$$

so that

$$(4.42) \quad \Theta_\varepsilon(w, u) := w - \gamma \mathfrak{R}_\varepsilon u + \lambda \mathfrak{R}_\varepsilon B\bar{\theta}, \quad \forall \bar{\theta} \in \Lambda_\varepsilon u,$$

is well defined and belongs to  $\mathcal{H}$ . ■

We have

4.13 LEMMA.  $\mathcal{W}_\varepsilon$  is a maximal monotone operator in  $\mathcal{H}$ .

PROOF. The monotonicity is easy: if  $w^i \in \mathcal{W}_\varepsilon(u^i)$  with  $\theta^i := \Theta_\varepsilon(w^i, u^i)$ , we have

$$(4.43) \quad \begin{aligned} & \int_0^T (w^1 - w^2, u^1 - u^2) d\zeta \\ & \geq \int_0^T \left[ \gamma \mathfrak{a}'_\varepsilon(J_\varepsilon(u^1 - u^2)) + \mathfrak{b}(\theta^1 - \theta^2) - \lambda b(\theta^1 - \theta^2, \mathfrak{R}_\varepsilon(u^1 - u^2)) \right] d\zeta \\ & \geq \int_0^T \left[ \frac{\gamma}{2} \mathfrak{a}'_\varepsilon(J_\varepsilon(u^1 - u^2)) + \frac{1}{2} \mathfrak{b}(\theta^1 - \theta^2) \right] d\zeta \end{aligned}$$

In order to check the maximality, we fix  $v \in \mathcal{H}$  and we have to solve the equation

$$u + \mathcal{W}_\varepsilon u \ni v$$



We repeat the fixed point argument of Theorem 4.8, solving iteratively <sup>(14)</sup>

$$u^{n+1} + \theta^{n+1} + \gamma \mathfrak{R}_\varepsilon u^{n+1} = v + \lambda \mathfrak{R}_\varepsilon B \theta^n, \quad \theta^{n+1} \in \Lambda_\varepsilon u^{n+1} \quad \blacksquare$$

We already said that the  $M$ -convergence of a family of convex functionals implies the *graph* convergence of the respective subdifferentials ([3], thm. 3.66); here is the definition (see [3], 3.58):

4.14 DEFINITION. *Let  $\mathbf{H}$  be an Hilbert space and  $\mathcal{A}_\varepsilon$ ,  $\varepsilon \in [0, \varepsilon_0]$ , be a family of maximal monotone (multivalued) operators of  $\mathbf{H}$ . We say that  $\mathcal{A}_\varepsilon$   $G$ -converges to  $\mathcal{A}_0$  as  $\varepsilon \rightarrow 0$  if for every  $\theta_0, u_0 \in \mathbf{H}$  with  $\theta_0 \in \mathcal{A}_0 u_0$  there exist  $\bar{\theta}_\varepsilon \in \mathcal{A}_\varepsilon \bar{u}_\varepsilon$  such that*

$$\lim_{\varepsilon \rightarrow 0} \left[ \|\bar{\theta}_\varepsilon - \theta_0\|_{\mathbf{H}} + \|\bar{u}_\varepsilon - u_0\|_{\mathbf{H}} \right] = 0 \quad \square$$

4.15 LEMMA.  $\mathcal{W}_\varepsilon$   $G$ -converges to  $\mathcal{W}_0$  as  $\varepsilon \rightarrow 0$ .

PROOF. We fix  $w_0 \in \mathcal{W}_0(u_0)$  and choose  $\theta_0 := \Theta_0(w_0, u_0) \in \Lambda_0 u_0$  as suggested by Lemma 4.12. Since  $\Lambda_\varepsilon$   $G$ -converges to  $\Lambda_0$  as  $\varepsilon \rightarrow 0$ , we find by the definition

$$\bar{\theta}_\varepsilon \in \Lambda_\varepsilon \bar{u}_\varepsilon \quad \text{such that} \quad \lim_{\varepsilon \rightarrow 0} \left[ \|\bar{\theta}_\varepsilon - \theta_0\|_{\mathcal{H}} + \|\bar{u}_\varepsilon - u_0\|_{\mathcal{H}} \right] = 0.$$

Setting

$$(4.44) \quad \bar{w}_\varepsilon := \bar{\theta}_\varepsilon + \gamma \mathfrak{R}_\varepsilon \bar{u}_\varepsilon - \lambda \mathfrak{R}_\varepsilon B \bar{\theta}_\varepsilon \in \mathcal{W}_\varepsilon \bar{u}_\varepsilon$$

we deduce that  $\bar{w}_\varepsilon \rightarrow w_0$  strongly in  $\mathcal{H}$ , since the following strong convergences hold by Remark 4.10

$$\mathfrak{R}_\varepsilon \bar{u}_\varepsilon \rightarrow \mathfrak{R}_0 u_0, \quad \mathfrak{R}_\varepsilon B \bar{\theta}_\varepsilon \rightarrow \mathfrak{R}_0 B \theta_0 \quad \blacksquare$$

We have

4.16 PROPOSITION. *Let  $w_{\varepsilon_n} \in \mathcal{W}_{\varepsilon_n} u_{\varepsilon_n}$  be given in  $\mathcal{H} \times \mathcal{H}$ ,  $\{\varepsilon_n\}_{n \in \mathbf{N}}$  being a decreasing sequence going to 0, and let us assume that*

$$(4.45) \quad u_{\varepsilon_n} \rightharpoonup u_0, \quad w_{\varepsilon_n} \rightharpoonup w_0 \quad \text{in } \mathcal{H}, \quad \text{as } n \rightarrow \infty,$$

with

$$(4.46) \quad \limsup_{n \rightarrow \infty} \int_0^T (w_{\varepsilon_n}, u_{\varepsilon_n}) d\zeta \leq \int_0^T (w_0, u_0) d\zeta.$$

---

<sup>(14)</sup> This equation admits a unique solution for every  $n \in \mathbf{N}$ , thanks to the maximal monotonicity of  $\Lambda_\varepsilon$  and to the monotone and Lipschitz character of  $\mathfrak{R}_\varepsilon$  in  $\mathcal{H}$ .

Then  $w_0 \in \mathcal{W}_0 u_0$  and setting  $\theta_{\varepsilon_n} := \Theta_{\varepsilon_n}(w_{\varepsilon_n}, u_{\varepsilon_n})$  we have

$$(4.47) \quad \theta_{\varepsilon_n} \rightharpoonup \theta_0 := \Theta_0(w_0, u_0) \quad \text{in } \mathcal{H}, \quad \lim_{n \rightarrow \infty} \int_0^T \mathfrak{b}(\theta_{\varepsilon_n} - \theta_0) d\zeta = 0.$$

PROOF. The relation  $w_0 \in \mathcal{W}_0 u_0$  is a standard consequence of the maximal monotonicity and the graph convergence of  $\mathcal{W}_\varepsilon$ . Let us fix  $\theta_0 := \Theta_0(w_0, u_0) \in \Lambda_0 u_0$  and  $\bar{w}_\varepsilon, \bar{\theta}_\varepsilon, \bar{u}_\varepsilon$  as in the previous Lemma (we omit for simplicity the index  $n$ ); we obtain by (4.43)

$$0 \leq \int_0^T \left\{ \mathfrak{b}(\theta_\varepsilon - \bar{\theta}_\varepsilon) + \mathfrak{a}'(J_\varepsilon(u_\varepsilon - \bar{u}_\varepsilon)) \right\} d\zeta \leq C \int_0^T (w_\varepsilon - \bar{w}_\varepsilon, u_\varepsilon - \bar{u}_\varepsilon) d\zeta.$$

Splitting the right-hand scalar product and passing to the limit, we get by (4.45) and by (4.46)

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \left[ \mathfrak{b}(\theta_\varepsilon - \bar{\theta}_\varepsilon) + \mathfrak{a}'(J_\varepsilon(u_\varepsilon - \bar{u}_\varepsilon)) \right] d\zeta = 0.$$

Since  $\bar{\theta}_\varepsilon$  strongly converges to  $\theta_0$  in  $\mathcal{H}$ , we deduce the second part of (4.47) and consequently  $B\theta_\varepsilon \rightarrow B\theta_0$  in  $\mathcal{H}$ . By (4.45) and (4.32), we can pass to the limit in the equation

$$\theta_\varepsilon = w_\varepsilon - \gamma \mathfrak{R}_\varepsilon u_\varepsilon + \lambda \mathfrak{R}_\varepsilon B\theta_\varepsilon$$

and we conclude that

$$\theta_\varepsilon \rightharpoonup w_0 - \gamma \mathfrak{R}_0 u_0 + \lambda \mathfrak{R}_0 B\theta_0 = \Theta_0(w_0, u_0) = \theta_0 \quad \blacksquare$$

We conclude now the **proof of Theorem 3**. We consider a family  $(\theta_\varepsilon, u_\varepsilon)$  of solutions of  $\mathbf{P}_\lambda(\mathfrak{a}_\varepsilon, \phi_\varepsilon; L_\varepsilon, u_{0,\varepsilon})$  and we denote by  $(\theta_0, u_0)$  a weak limit point in  $\mathcal{H} \times \mathcal{H}$  of a suitable weakly convergent subsequence  $(\theta_{\varepsilon_n}, u_{\varepsilon_n})$ , whose existence is implied by Proposition 4.11. We define

$$w_\varepsilon := \theta_\varepsilon + \gamma \mathfrak{R}_\varepsilon u_\varepsilon - \lambda \mathfrak{R}_\varepsilon B\theta_\varepsilon, \quad \varepsilon \in [0, \varepsilon_0],$$

and we know that

$$w_\varepsilon \in \mathcal{W}_\varepsilon u_\varepsilon, \quad \theta_\varepsilon \in \Theta_\varepsilon(w_\varepsilon, u_\varepsilon), \quad \text{if } \varepsilon > 0.$$

Since (4.45) holds, we want to show that (4.46) is satisfied, too. We take the  $\mathfrak{a}'_\varepsilon$ -scalar product of the equation (4.17) with  $e^{-\gamma t} J_\varepsilon(u_\varepsilon(t))$ ; integrating we obtain

$$\frac{e^{-\gamma T}}{2} \mathfrak{a}_\varepsilon(\mathfrak{R}_\varepsilon u_\varepsilon(T)) + \int_0^T (w_\varepsilon, u_\varepsilon) d\zeta = \frac{1}{2} \mathfrak{a}'_\varepsilon(\tilde{u}_{0,\varepsilon}) + \int_0^T \langle L_\varepsilon, \mathfrak{R}_\varepsilon u_\varepsilon \rangle d\zeta,$$

which also holds for  $\varepsilon = 0$  by Proposition 4.11. Since  $\mathfrak{R}_{\varepsilon_n} u_{\varepsilon_n}$  is uniformly bounded in  $H^1(0, T; H)$ , it converges to  $\mathfrak{R}_0 u_0$  in the pointwise weak topology of  $H$ ; in particular

$$\liminf_{\varepsilon \rightarrow 0} \mathfrak{a}_{\varepsilon_n}(\mathfrak{R}_{\varepsilon_n} u_{\varepsilon_n}(t)) \geq \mathfrak{a}_0(\mathfrak{R}_0 u_0(t)), \quad \forall t \in [0, T].$$

Since  $\mathfrak{R}_{\varepsilon_n} u_{\varepsilon_n} \rightharpoonup \mathfrak{R}_0 u_0$  in  $L^2(0, T; H)$  we have by  $(LIM_L)$

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \langle L_{\varepsilon_n}, \mathfrak{R}_{\varepsilon_n} u_{\varepsilon_n} \rangle d\zeta = \int_0^T \langle L_0, \mathfrak{R}_0 u_0 \rangle d\zeta.$$

By Lemma 4.9, we deduce

$$\mathfrak{a}'_{\varepsilon_n}(u_{0, \varepsilon_n}) \rightarrow \mathfrak{a}'_0(u_{0,0})$$

and (4.46); by Proposition 4.16 we obtain that  $\theta_0 \in \Lambda u_0$  and by Proposition 4.11 we conclude that  $(\theta_0, u_0)$  is a solution of  $\mathbf{P}_0(\mathfrak{a}_0, \phi_0; L_0, u_{0,0})$ .

Theorem 1 and (4.47) entail (3.17).

The convergence (3.18) follows by the uniform boundedness of  $u_\varepsilon$  in  $H$ , the uniqueness of the (projection of) the limit by  $(A_{comp})$ , the pointwise convergence of  $\mathfrak{R}_\varepsilon u_\varepsilon$ , and the injectivity of  $\mathfrak{R}_0|_{H_0}$ .

In order to check (3.20), we first observe that the related assumptions of Theorem 3 surely imply (3.21), so that we assume this weaker condition. Let us recall that by the definition of  $M$ -convergence 3.5 we know

$$\liminf_{\varepsilon \rightarrow 0} \int_0^T \mathfrak{a}_\varepsilon(\theta_\varepsilon) dt \geq \int_0^T \mathfrak{a}_0(\theta_0) dt.$$

Therefore, it remains to check the opposite inequality for the “lim sup”. We rewrite (4.27), which now holds for every  $s \in [0, T]$  thanks to  $(A_\phi)$  (recall that  $\bar{\phi}_\varepsilon = \tilde{\phi}_\varepsilon$ , being  $\tilde{\phi}_\varepsilon$  l.s.c.)

$$\tilde{\phi}_\varepsilon(\tilde{u}_\varepsilon(T)) + \int_0^T \mathfrak{a}_\varepsilon(\theta_\varepsilon) dt = \tilde{\phi}_\varepsilon(u_{0, \varepsilon}) + \int_0^T \left( \langle L_\varepsilon, \theta_\varepsilon \rangle + \lambda \mathfrak{b}(\theta_\varepsilon) \right) dt.$$

Since

$$\lim_{\varepsilon \rightarrow 0} \tilde{\phi}_\varepsilon(\tilde{u}_{0, \varepsilon}) = \tilde{\phi}_0(\tilde{u}_{0,0})$$

by (3.21), and

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \left( \langle L_\varepsilon, \theta_\varepsilon \rangle + \lambda \mathfrak{b}(\theta_\varepsilon) \right) dt = \int_0^T \left( \langle L_0, \theta_0 \rangle + \lambda \mathfrak{b}(\theta_0) \right) dt$$

by  $(LIM_L)$  and (3.17), our thesis follows if we show that

$$(4.48) \quad \liminf_{\varepsilon \rightarrow 0} \tilde{\phi}_\varepsilon(\tilde{u}_\varepsilon(T)) \geq \tilde{\phi}_0(\tilde{u}_0(T)).$$

To this aim, we choose  $v_\varepsilon \in K(\tilde{u}_\varepsilon(T))$  and, up to a subsequence, we can assume that

$$v_\varepsilon \rightharpoonup v_0, \quad \mathfrak{R}_\varepsilon v_\varepsilon \rightharpoonup \mathfrak{R}_0 v_0 \quad \text{in } H.$$

On the other hand

$$\mathfrak{R}_\varepsilon v_\varepsilon = \mathfrak{R}_\varepsilon u_\varepsilon(T) \rightharpoonup \mathfrak{R}_0 u_0(T)$$

so that  $J_0 v_0 = J_0 u_0$ . We conclude

$$\liminf_{\varepsilon \rightarrow 0} \tilde{\phi}_\varepsilon(\tilde{u}_\varepsilon(T)) = \liminf_{\varepsilon \rightarrow 0} \phi_\varepsilon(v_\varepsilon) \geq \phi_0(v_0) \geq \tilde{\phi}_0(\tilde{u}_0(T)) \quad \blacksquare$$

## 5. - PROOFS OF THE THEOREMS OF SECT. 1 AND 2.

We begin by writing a Fubini-type formula for integrable functions on  $\Omega_2$  (cf. [16], sect. 3); the (sketches of the) proofs of this and other simple results are collected in the appendix.

5.1 NOTATION. We set (cf. (1.13) and (G2) of section 2)

$$(5.1) \quad R(x) := I - d_\Gamma(x)S(x_\Gamma), \quad r(x) := \det R(x)$$

and we denote by  $\mu$  the measure  $r \cdot \mathcal{H}^1$  and by  $\nu$  the measure  $r^{-1} \cdot \mathcal{L}$  on  $\Omega_2$ . For every segment  $s_x$ ,  $x \in \Gamma$ , and every  $\mathcal{H}^1$ -measurable function  $f$ , we have (cf. (2.9) and (2.10))

$$(5.2) \quad \int_{s_x} f d\mu := \int_{s_x} f(s)r(s) d\mathcal{H}^1(s) = \int_0^{\ell(x)} f(x_\lambda) d\mu_x(\lambda),$$

whereas for every  $L^1(\Omega_2)$  function  $g$  we have

$$(5.3) \quad \int_{\Omega_2} g d\nu := \int_{\Omega_2} g(x) \frac{dx}{r(x)}.$$

Observe that  $r$  is bounded on  $\Omega_2$  and greater than  $\eta > 0$  by (G2).  $\square$

5.2 LEMMA. Let  $f$  be a function of  $L^1(\Omega_2)$ ; then for  $\mathcal{H}^{N-1}$ -a.e.  $x \in \Gamma$ , the restriction  $f|_{s_x}$  is  $\mathcal{H}^1$ -measurable and

$$(5.4) \quad \int_{\Omega_2} f(x) dx = \int_\Gamma d\mathcal{H}^{N-1}(x) \int_{s_x} f d\mu,$$

$$(5.5) \quad \int_{\Omega_2} f d\nu = \int_\Gamma d\mathcal{H}^{N-1}(x) \int_{s_x} f d\mathcal{H}^1.$$

The second property we need is to characterize  $L^2_{\mathbf{n}}(\Omega_2)$  as the subspace of the  $L^2(\Omega_2)$ -functions which are constant along  $(\mathcal{H}^{N-1}$ -almost) every segments  $s_x$ ,  $x \in \Gamma$ .

5.3 LEMMA. The linear space  $C^1(\overline{\Omega}_2) \cap H^1_{\mathbf{n}}(\Omega_2)$  is dense in  $H^1_{\mathbf{n}}(\Omega_2)$  and

$$(5.6) \quad L^2_{\mathbf{n}}(\Omega_2) = \{u \in L^2(\Omega_2) : u|_{s_x} \text{ is constant for } \mathcal{H}^{N-1}\text{-a.e. } x \in \Gamma\}.$$

In particular, the projection  $\Pi_{\mathbf{n}}$  on  $L_{\mathbf{n}}^2(\Omega_2)$  with respect to the weighted scalar product of  $L_{\rho_2}^2(\Omega_2)$  is given by

$$(5.7) \quad \Pi_{\mathbf{n}} f(x) = \int_{\mathbb{S}_x} f d[\rho_2 \mu] := \frac{\int_{\mathbb{S}_x} f \rho_2 d\mu}{\int_{\mathbb{S}_x} \rho_2 d\mu}, \quad \text{for a.e. } x \in \Omega_2.$$

We begin now the PROOFS of the “concrete” theorems.

We observe that THEOREM 1.7 and the related existence result of PROPOSITION 1.3 are almost already explained in section 1: they follow by applying the abstract results 1-3 with the choices (1.21), ..., (1.25), and, for  $\varepsilon = 0$ , (1.29), (1.30). We limit us to point out the simple technical links.

- The definition of  $\phi$  is standard (see [5], ex. 2.8.1, 2.8.3): we introduce a primitive of  $\beta_i$

$$(5.8) \quad j_i : \mathbf{R} \mapsto [0, +\infty[, \quad j_i(s) := \int_0^s \beta_i(\tau) d\tau,$$

and we set for every  $U := (u_1, u_2) \in H$

$$(5.9) \quad \phi(U) := \sum_i \int_{\Omega_i} j_i(u_i(x)) \rho_i(x) dx \quad (15).$$

- $(A_\phi)$  follows from (1.4).
- $(A_{\Lambda, \mathbf{a}})$  hold for the quadratic form

$$(5.10) \quad \mathfrak{b}(\Theta) := c_\beta |\Theta|_H^2$$

thanks to (1.5) and to (1.3).

- $(A_{L, u_0})$  are trivially satisfied since  $D(\phi) = H$ .
- $(A_{comp})$  holds also for  $\varepsilon = 0$ . Of course, the projection  $\Pi_0$  of  $H$  on  $H_0$  is given by

$$\Pi_0(U) := (u_1, \Pi_{\mathbf{n}} u_2)$$

$\Pi_{\mathbf{n}}$  being given by (5.7); therefore, it is sufficient to prove that

$$\int_{\Omega_2} j_2(\Pi_{\mathbf{n}} u_2(x)) \rho_2(x) dx \leq \int_{\Omega_2} j_2(u_2(x)) \rho_2(x) dx$$

---

<sup>(15)</sup>  $\beta_i$  are Lipschitz functions thanks to (1.5), so that  $D(\phi)$  coincides with  $H$ . This assumption could be avoided, following the definitions of the quoted examples of [5].

for every  $u_2 \in L^2_{\rho_2}(\Omega_2)$ . By applying (5.4) and Jensen's inequality, we have

$$\begin{aligned} \int_{\Omega_2} j_2(\Pi_{\mathbf{n}} u_2(x)) \rho_2(x) dx &= \int_{\Gamma} d\mathcal{H}^{N-1}(x) \int_{\mathbb{S}_x} j_2\left(\int_{\mathbb{S}_x} u_2 d(\rho_2 \mu)\right) d(\rho_2 \mu) \leq \\ &\int_{\Gamma} d\mathcal{H}^{N-1}(x) \int_{\mathbb{S}_x} \left[ \int_{\mathbb{S}_x} j_2(u_2) d(\rho_2 \mu) \right] d(\rho_2 \mu) = \\ &\int_{\Gamma} d\mathcal{H}^{N-1}(x) \int_{\mathbb{S}_x} j_2(u_2) \rho_2 d\mu = \int_{\Omega_2} j_2(u_2(x)) \rho_2(x) dx. \end{aligned}$$

- $(LIM_{L, u_0})$  are trivial.
- $(LIM_{\mathbf{a}, \Lambda, \phi})$  is stated by Proposition 1.9; also the uniformity follows by the increasing property of  $\mathbf{a}_\varepsilon$ .
- (1.18) correponds to (3.18).
- (3.19) is satisfied if the initial datum  $u_{0,2}$  belongs to  $L^2_{\mathbf{n}}(\Omega_2)$ . In this case, (3.20) gives the strong convergence of  $\theta_i^\varepsilon$  in  $L^2(0, T; H^1(\Omega_i))$ , the convergence in  $L^2(Q_i)$  being ensured by (3.17) and (5.10). ■

The **proof of Theorem I** is also almost complete: the following two results show the equivalence between  $wLP_I$  and the weak formulation of the system of Theorem I given by Remark 2.3.

5.4 LEMMA. *Let us set*

$$(5.11) \quad L^2_R(\Omega_2; \mathbf{R}^N) := \{v \in L^2(\Omega_2; \mathbf{R}^N) : R(x)v(x) \in L^2_{\mathbf{n}}(\Omega_2; \mathbf{R}^N), \mathbf{n}(x) \cdot v(x) = 0\}.$$

*Then for every  $u \in H^1_{\mathbf{n}}(\Omega_2)$  we have  $\nabla u \in L^2_R(\Omega_2; \mathbf{R}^N)$ .*

5.5 LEMMA. *The trace operator  $u \mapsto \tilde{u} = u|_{\Gamma}$  is a linear isomorphism mapping  $H^1_{\mathbf{n}}(\Omega_2)$  onto  $H^1(\Gamma)$ ,  $L^2_{\mathbf{n}}(\Omega_2)$  onto  $L^2(\Gamma)$ , and  $L^2_R(\Omega_2; \mathbf{R}^N)$  onto  $L^2(\Gamma; \mathbf{R}^N)$ . Furthermore, it satisfies*

$$\nabla_{\Gamma}(\tilde{u}) = (\nabla u)|_{\Gamma}, \quad \forall u \in H^1_{\mathbf{n}}(\Omega_2)$$

and

$$(5.12) \quad \int_{\Omega_2} \rho_2 u v dx = \int_{\Gamma} \hat{\rho}_2 \tilde{u} \tilde{v} d\mathcal{H}^{N-1}, \quad \forall u, v \in L^2_{\mathbf{n}}(\Omega_2),$$

$$(5.13) \quad \int_{\Omega_2} A_2 \nabla \theta \nabla v dx = \int_{\Gamma} \hat{A}_2 \nabla_{\Gamma} \tilde{\theta} \cdot \nabla_{\Gamma} \tilde{v} d\mathcal{H}^{N-1}, \quad \forall \theta, v \in H^1_{\mathbf{n}}(\Omega_2),$$

$$\int_{\Omega_2} f_2(x, t) v(x) dx = \int_{\Gamma} \hat{f}_2(x, t) \tilde{v} d\mathcal{H}^{N-1}, \quad \forall v \in L^2_{\mathbf{n}}(\Omega_2),$$

where, for  $\mathcal{H}^{N-1}$ -a.e.  $x \in \Gamma$

(5.14)

$$\hat{f}_2(x, t) := \int_{\mathbb{S}_x} f(\cdot, t) d\mu, \quad \hat{\rho}_2(x) := \int_{\mathbb{S}_x} \rho_2 d\mu, \quad \hat{A}(x) := \int_{\mathbb{S}_x} R^{-1} A R^{-1} d\mu. \quad \blacksquare$$

We conclude this section with the **proof of Theorem II**, which we divide into three steps: first of all, we operate a rescaling of the variables in  $\Omega_2^\varepsilon$  in order to write a family of problems in the fixed domains  $\Omega_1, \Omega_2$ . Then we apply the abstract results in a similar way as in the previous proof; finally, we employ Lemma 5.5 to come back to the final formulation on  $\Omega_1, \Gamma$ .

STEP 1: RESCALING.

Following the notation of the case II, let us operate the change of variables

$$(5.15) \quad x := G^\varepsilon(z), \quad x \in \Omega_2^\varepsilon, \quad z \in \Omega_2.$$

For every function  $v(x)$  defined on  $\Omega_2^\varepsilon$ , we denote by  $v(z)$  again the composition  $v \circ G^\varepsilon$ , when no misunderstanding are possible. If  $\nu$  denotes the measure  $r^{-1}(z) dz$  on  $\Omega_2$  and  $R_\varepsilon, r_\varepsilon$  are given by

$$(5.16) \quad R_\varepsilon(z) := I - \varepsilon d_\Gamma(z) S(z_\Gamma), \quad r_\varepsilon(z) := \det R_\varepsilon(z),$$

standard computations show

5.6 LEMMA. *The change of variables (5.15) defines a linear isomorphism between  $L^2(\Omega_2^\varepsilon)$  and  $L^2(\Omega_2)$ ,  $H^1(\Omega_2^\varepsilon)$  and  $H^1(\Omega_2)$ ; the following formulae hold*

$$(5.17) \quad \int_{\Omega_2^\varepsilon} \rho_2^\varepsilon(x) u(x) v(x) dx = \int_{\Omega_2} \check{\rho}_2^\varepsilon(z) u(z) v(z) d\nu(z), \quad \forall u, v \in L^2(\Omega_2^\varepsilon),$$

(5.18)

$$\int_{\Omega_2^\varepsilon} A_2^\varepsilon(x) \nabla_x \theta(x) \cdot \nabla_x v(x) dx = \int_{\Omega_2} \check{A}_2^\varepsilon(z) \nabla_z \theta(z) \cdot \nabla_z v(z) d\nu(z) \quad \forall \theta, v \in H^1(\Omega_2^\varepsilon),$$

where  $\nabla_x$  and  $\nabla_z$  are the gradient with respect to the variables  $x$  and  $z$  respectively,

$$(5.19) \quad \int_{\Omega_2^\varepsilon} f_2^\varepsilon(x, t) v(x) dx = \int_{\Omega_2} \check{f}_2^\varepsilon(z, t) v(z) d\nu(z), \quad \forall v \in L^2(\Omega_2^\varepsilon),$$

where

$$(5.20) \quad \check{\rho}_2^\varepsilon(z) := \varepsilon r_\varepsilon(z) \rho_2^\varepsilon(z), \quad \check{f}_2^\varepsilon(z, t) := \varepsilon r_\varepsilon(z) f_2^\varepsilon(z, t),$$



and (in the rescaled variable  $z$ )

$$(5.21) \quad \check{A}_2^\varepsilon := \varepsilon r_\varepsilon P R R_\varepsilon^{-1} A_2^\varepsilon R_\varepsilon^{-1} R P + \frac{r_\varepsilon}{\varepsilon} N A_2^\varepsilon N.$$

Here,  $P = P_z$  is the tangent projection of (2.1), and  $N := N_z = \mathbf{n}(z)\mathbf{n}^T(z)$  is the normal one; recall that  $R$  and  $R_\varepsilon$  are symmetric matrices.

It is immediate to see that the new unknowns in the  $z$ -variable (coupled with the old ones in  $\Omega_1$ ) satisfy the same weak formulation of  $wPT$  in the fixed domains  $\Omega_1, \Omega_2$  if we replace  $\rho_2, A_2, f_2$  by the corresponding functions  $\check{\rho}_2^\varepsilon, \check{A}_2^\varepsilon, \check{f}_2^\varepsilon$ . We call  $RTP_{II}^\varepsilon$  this *rescaled* version of  $PT_{II}^\varepsilon$ .

STEP 2: APPLICATION OF THE ABSTRACT RESULTS.

Now we can apply the abstract machinery as before, setting (cf. with (1.21))

$$(5.22) \quad H := L_{\rho_1}^2(\Omega_1) \times L_\nu^2(\Omega_2),$$

$$(5.23) \quad a_\varepsilon(\theta, V) := \int_{\Omega_1} A_1 \nabla \theta_1 \cdot \nabla v_1, dx + \int_{\Omega_2} \check{A}_2^\varepsilon \nabla \theta_2 \cdot \nabla v_2 d\nu(z),$$

and, for  $\varepsilon > 0$ ,  $V_\varepsilon$  as in (1.23). Unlike case I, the operators  $\Lambda_\varepsilon$  are defined by

$$(5.24) \quad \theta \in \Lambda_\varepsilon(U) \quad \Leftrightarrow \quad \theta_1(x) = \beta_1(u_1(x)), \quad \theta_2(z) = \beta_2(u_2(z)/\check{\rho}_2^\varepsilon(z)),$$

and they are the subdifferentials in  $H$  of the convex functionals

$$(5.25) \quad \phi_\varepsilon(U) := \int_{\Omega_1} j_1(u_1(x)) dx + \int_{\Omega_2} \check{\rho}_2^\varepsilon(z) j_2(u_2(z)/\check{\rho}_2^\varepsilon(z)) d\nu(z),$$

where  $j_i$  are defined by (5.8). Finally, if

$$(5.26) \quad \begin{aligned} \langle L_\varepsilon(t), V \rangle &:= \int_{\Omega_1} f_1(x, t) v_1(x) dx + \int_{\Gamma_1} g_1(x, t) v_1(x) d\mathcal{H}^{N-1}(x) \\ &\quad + \int_{\Omega_2} \check{f}_2^\varepsilon(z, t) v_2(z) d\nu(z), \end{aligned}$$

then the same simple application of Lemma 5.6 gives the following statement.

5.7 LEMMA.  $(\theta_i^\varepsilon, u_i^\varepsilon)_{i=1,2}$  is the rescaled weak solution of  $RTP_{II}^\varepsilon$  if and only if the couple  $(\Theta^\varepsilon, U^\varepsilon)$  given by

$$(5.27) \quad U^\varepsilon := (u_1^\varepsilon, \check{\rho}_2^\varepsilon u_2^\varepsilon), \quad \Theta^\varepsilon := (\theta_1^\varepsilon, \theta_2^\varepsilon),$$

is the solution of  $\mathbf{P}(\mathbf{a}_\varepsilon, \phi_\varepsilon; L_\varepsilon, U_{0,\varepsilon})$  for the choices (5.22), ..., (5.26) and for the initial datum

$$U_{0,\varepsilon} := (u_{0,1}, \check{\rho}_2^\varepsilon u_{0,2}^\varepsilon). \quad \blacksquare$$

Now we observe that

$$(5.28) \quad \lim_{\varepsilon \rightarrow 0} r_\varepsilon(z) = 1, \quad \lim_{\varepsilon \rightarrow 0} R_\varepsilon(z) = I, \quad \lim_{\varepsilon \rightarrow 0} \check{\rho}_2^\varepsilon(z) = \check{\rho}_2(z) = \rho_\Gamma(z_\Gamma)$$

uniformly in  $\Omega_2$ . These limits lead us to set

$$(5.29) \quad \phi_0(U) := \int_{\Omega_1} j_1(u_1(x)) dx + \int_{\Omega_2} \check{\rho}_2(z) j_2(u_2(z)/\check{\rho}_2(z)) d\nu(z),$$

and, if  $V_0$  is defined as in (1.29),

$$\mathbf{a}_0(\Theta) := \int_{\Omega_1} A_1 \nabla \theta_1 \cdot \nabla \theta_1 dx + \int_{\Omega_2} \check{A}_2(z) \nabla \theta_2 \cdot \nabla \theta_2 d\nu(z) \quad \forall \Theta := (\theta_1, \theta_2) \in V_0,$$

where

$$(5.30) \quad \check{A}_2(z) := P_z R(z) A_\Gamma(z_\Gamma) R(z) P_z, \quad A_\Gamma \text{ given by (1.36).}$$

Similarly, we set

$$(5.31) \quad \check{f}_2(z, t) := f_\Gamma(z_\Gamma, t), \quad \check{u}_{0,2}(z) := u_{0,\Gamma}(z_\Gamma)$$

and we have

$$(5.32) \quad \begin{aligned} \langle L_0(t), v \rangle &:= \int_{\Omega_1} f_1(x, t) v_1(x) dx + \int_{\Gamma_1} g_1(x, t) v_1(x) d\mathcal{H}^{N-1}(x) + \\ &\quad \int_{\Omega_2} \check{f}_2(z, t) v_2(z) d\nu(z), \end{aligned}$$

and

$$(5.33) \quad U_{0,0} := (u_{0,1}, \check{\rho}_2(z) \check{u}_{0,2}(z)).$$

Again we can apply the abstract results of sect. 3; we omit the details, which are analogous to the previous calculations, thanks to the limits (5.28). We observe that the crucial role is played by the following natural result:

5.8 PROPOSITION. *For every  $0 \leq \varepsilon < 1$  let us define  $\mathbf{a}_\varepsilon, \phi_\varepsilon, L_\varepsilon, U_{0,\varepsilon}$  according to (5.22), ..., (5.26), and to (5.29), ..., (5.33), and let us assume that (1.34), ..., (1.36) hold. Then as  $\varepsilon$  goes to 0,  $\mathbf{a}_\varepsilon$  and  $\phi_\varepsilon$   $M$ -converge to  $\mathbf{a}_0$  and  $\phi_0$  in  $H$ , and  $L, U_{0,\varepsilon}$  strongly converges to  $L_0, U_{0,0}$  accordingly to the definition 3.6.*

PROOF. We limit us to consider the simpler case (2.18), (2.19).

- $\mathbf{a}_\varepsilon$   $M$ -converges to  $\mathbf{a}_0$  on  $H$ . We check the first condition of definition 3.5, the other one being trivial; furthermore, it is not restrictive to consider only the “ $\Omega_2$ -contribution” to  $\mathbf{a}_\varepsilon$  and  $\mathbf{a}_0$ , i.e.

$$(5.34) \quad \mathbf{a}_{2,\varepsilon}(\theta) := \int_{\Omega_2} \check{A}_2^\varepsilon(z) \nabla_z \theta(z) \cdot \nabla_z \theta(z) d\nu(z), \quad \forall \varepsilon \geq 0.$$

Let us given  $\theta^\varepsilon \in H^1(\Omega_2)$ ,  $\varepsilon > 0$ , with

$$\theta^\varepsilon \rightharpoonup \theta^0 \quad \text{in } L^2(\Omega_2), \quad \text{and} \quad \liminf_{\varepsilon \rightarrow 0} \mathbf{a}_{2,\varepsilon}(\theta^\varepsilon) < +\infty.$$

By (5.21) and (1.36) we get

$$\mathbf{a}_{2,\varepsilon}(\theta^\varepsilon) \geq \alpha\eta \int_{\Omega_2} |R_\varepsilon^{-1}(z)R(z)P_z \nabla_z \theta^\varepsilon(z)|^2 d\nu(z) + \frac{\alpha\eta}{\varepsilon^2} \int_{\Omega_2} \left| \frac{\partial \theta^\varepsilon}{\partial \mathbf{n}} \right|^2 d\nu(z).$$

Since  $R(z)$  has a uniformly bounded inverse and  $R_\varepsilon(z)$  is uniformly bounded, we deduce

$$\liminf_{\varepsilon \rightarrow 0} \|\theta^\varepsilon\|_{H^1(\Omega_2)} < +\infty, \quad \liminf_{\varepsilon \rightarrow 0} \int_{\Omega_2} \left| \frac{\partial \theta^\varepsilon}{\partial \mathbf{n}} \right|^2 d\nu(z) = 0,$$

so that  $\theta^0 \in H_{\mathbf{n}}^1(\Omega_2)$ ,  $\theta^\varepsilon \rightharpoonup \theta^0$  in  $H^1(\Omega_2)$ .

Since

$$\mathbf{v}^\varepsilon := \sqrt{r_\varepsilon} R_\varepsilon^{-1} R P_z \nabla_z \theta^\varepsilon \rightharpoonup R P_z \nabla_z \theta^0 = \mathbf{v}^0(z) \quad \text{in } L^2(\Omega_2; \mathbf{R}^N),$$

and

$$\mathbf{a}_{2,\varepsilon}(\theta^\varepsilon) \geq \int_{\Omega_2} A_\Gamma \mathbf{v}^\varepsilon(z) \cdot \mathbf{v}^\varepsilon(z) d\nu(z), \quad \text{for } \varepsilon > 0,$$

we deduce

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \mathbf{a}_{2,\varepsilon}(\theta^\varepsilon) &\geq \liminf_{\varepsilon \rightarrow 0} \int_{\Omega_2} A_\Gamma \mathbf{v}^\varepsilon(z) \cdot \mathbf{v}^\varepsilon(z) d\nu(z) \\ &\geq \int_{\Omega_2} A_\Gamma \mathbf{v}^0(z) \cdot \mathbf{v}^0(z) d\nu(z) = \mathbf{a}_{2,0}(\theta^0). \end{aligned}$$

- $\phi_\varepsilon$   $M$ -converges to  $\phi_0$  on  $H$ . As before, we can consider the behaviour of

$$\phi_{2,\varepsilon}(u) := \rho_\Gamma \int_{\Omega_2} r_\varepsilon(z) j_2(u^\varepsilon(z)/\rho_\Gamma r_\varepsilon(z)) d\nu(z),$$

where  $u^\varepsilon \rightharpoonup u^0$  in  $L^2(\Omega_2)$ . Being  $j_2$  of quadratic growth, we write

$$\phi_{2,\varepsilon}(u^\varepsilon) \geq \rho_\Gamma \int_{\Omega_2} j_2(u^\varepsilon / \rho_\Gamma r^\varepsilon(z)) d\nu(z) - C \|u^\varepsilon\|_{L^2(\Omega_2)}^2 \sup_{z \in \Omega_2} |r_\varepsilon(z) - 1|$$

for a suitable positive constant  $C$ , independent of  $\varepsilon$ . Since

$$u^\varepsilon / (\rho_\Gamma r^\varepsilon) \rightharpoonup u^0 / \rho_\Gamma \quad \text{in } L_\nu^2(\Omega_2),$$

we conclude by the convexity and the lower semicontinuity of  $j_2$ .

- The convergence of the data  $L_\varepsilon, U_\varepsilon$  are easy to check, since (1.34) and (1.35) entail

$$\check{f}_2^\varepsilon \rightarrow \check{f}_2 \quad \text{strongly in } L^2(Q_2), \quad \check{u}_{0,2}^\varepsilon \rightarrow \check{u}_{0,2} \quad \text{strongly in } L_\nu^2(\Omega_2) \quad \blacksquare$$

By the previous Proposition and Theorem 3, we deduce that the rescaled solutions  $(\theta_i^\varepsilon, u_i^\varepsilon)$  of Lemma 5.7 satisfy

$$(5.35) \quad \theta_i^\varepsilon \rightarrow \theta_i \quad \text{strongly in } L^2(0, T; H^1(\Omega_i))$$

and, if  $\Pi_n$  is now the orthogonal projection on  $L_n^2(\Omega_2)$  with respect to the scalar product of  $L_\nu^2(\Omega_2)$ ,

$$(5.36) \quad u_1^\varepsilon(\cdot, t) \rightharpoonup u_1(\cdot, t), \quad \Pi_n(\check{\rho}_2^\varepsilon(\cdot) u_2^\varepsilon(\cdot, t)) \rightharpoonup \check{\rho}_2(\cdot) u_2(\cdot, t), \quad \text{weakly in } L^2(\Omega_i),$$

for every  $t \in [0, T]$ , where  $\Theta := (\theta_1, \theta_2)$ ,  $U := (u_1, \check{\rho}_2 u_2)$  are the unique solution of  $\mathbf{P}(\mathbf{a}_0, \phi_0; L_0, U_{0,0})$  satisfying (3.14), i.e.  $\check{\rho}_2(\cdot) u_2(\cdot, t) \in L_n^2(\Omega_2)$  for a.e.  $t \in ]0, T[$ .

STEP 3: FORMULATION ON  $\Gamma$ .

It is clear that (5.35) implies <sup>(16)</sup>

$$\bar{\theta}_2^\varepsilon(x) = \int_{s_x} \theta_2^\varepsilon(z) d\mathcal{H}^1(z) \rightarrow \bar{\theta}_2(x) = \int_{s_x} \theta_2(z) d\mathcal{H}^1(z) = \theta_2|_\Gamma(x),$$

strongly in  $L^2(0, T; H^1(\Gamma))$ . Regarding  $u_2$ , we note that, (cf. Lemma 5.3)

$$\begin{aligned} \Pi_n(\check{\rho}_2^\varepsilon u_2^\varepsilon)(z) &= \frac{\int_{s_z} u_2^\varepsilon(s) \check{\rho}_2^\varepsilon(s) d\mathcal{H}^1(s)}{\int_{s_z} d\mathcal{H}^1(s)} = \int_{s_z} u_2^\varepsilon(s) \check{\rho}_2^\varepsilon(s) d\mathcal{H}^1(s) \\ &= \check{\rho}_2(z) \bar{u}_2^\varepsilon(z) + \int_{s_z} u_2^\varepsilon(s) [\check{\rho}_2^\varepsilon(s) - \check{\rho}_2(s)] d\mathcal{H}^1(s), \end{aligned}$$

---

<sup>(16)</sup> We neglect for simplicity the dependence on  $t$ . Observe that the mean value along the normal segments is not affected by the rescaling; its regularity on  $\Gamma$  depends on the regularity of the thickness  $\ell(x)$ .

so that by (5.36) and (5.28) we deduce

$$\lim_{\varepsilon \rightarrow 0} \Pi_{\mathbf{n}}(\check{\rho}_2^\varepsilon u_2^\varepsilon) = \lim_{\varepsilon \rightarrow 0} \check{\rho}_2 \bar{u}_2^\varepsilon, \quad \text{in the weak topology of } L^2(\Omega_2).$$

It follows that

$$\bar{u}_2(x) = \lim_{\varepsilon \rightarrow 0} \bar{u}_2^\varepsilon(x) = u_2|_\Gamma(x), \quad \text{weakly in } L^2(\Gamma).$$

In order to write the weak formulation of the coupled system, as suggested by Remark 2.3, we observe that (5.5) and Lemma 5.4, 5.5 entail

$$(5.37) \quad \int_{\Omega_2} \check{\rho}_2(z) u(z) v(z) d\nu(z) = \int_\Gamma \ell \rho_\Gamma u v d\mathcal{H}^{N-1}, \quad \forall u, v \in L_{\mathbf{n}}^2(\Omega_2),$$

$$(5.38) \quad \int_{\Omega_2} \check{A}_2(z) \nabla \theta(z) \nabla v(z) d\nu(z) = \int_\Gamma \ell A_\Gamma \nabla_\Gamma \theta \cdot \nabla_\Gamma v d\mathcal{H}^{N-1}, \quad \forall \theta, v \in H_{\mathbf{n}}^1(\Omega_2),$$

$$(5.39) \quad \int_{\Omega_2} \check{f}_2(z, t) v(z) d\nu(z) = \int_\Gamma \ell(x) f_\Gamma(x, t) v(x) d\mathcal{H}^{N-1}(x), \quad \forall v \in L_{\mathbf{n}}^2(\Omega_2) \quad \blacksquare$$

## 6. - APPENDIX.

We list here some useful differential identities; we recall that  $x_\lambda := x + \lambda \mathbf{n}(x)$  solves the Cauchy problem

$$(6.1) \quad x_0 = x, \quad \frac{dx_\lambda}{d\lambda} = \mathbf{n}(x_\lambda).$$

6.1 LEMMA. *For every point  $x \in \overline{\Omega}_2$  we have*

$$(6.2) \quad S(x)\mathbf{n}(x) = 0, \quad \mathbf{n}^T(x)S(x) = 0,$$

$$(6.3) \quad x_\lambda \in \overline{\Omega}_2 \Rightarrow S(x_\lambda)(I - \lambda S(x)) = S(x), \quad (I + \lambda S(x_\lambda)) = (I - \lambda S(x))^{-1}.$$

Moreover, setting

$$(6.4) \quad R(x) := (I + d_\Gamma(x)S(x))^{-1} = I - d_\Gamma(x)S(x_\Gamma), \quad r(x) := \det R(x),$$

we have

$$(6.5) \quad \frac{\partial r(x)}{\partial \mathbf{n}} = -(\operatorname{tr} S(x))r(x) = \operatorname{div} \mathbf{n}(x) r(x).$$

The PROOF follows by differentiating the identity

$$\mathbf{n}(x_\lambda) = \mathbf{n}(x), \quad \forall x, x_\lambda \in \Omega_2$$

and by the application of Liouville Theorem to (6.1). ■

6.2 LEMMA. *Let  $\Phi$  be the mapping*

$$(6.6) \quad \Phi : \Omega_2 \mapsto \mathbf{R}^N \times \mathbf{R}, \quad \Phi(x) := (x_\Gamma, d_\Gamma(x)).$$

Then

$$(6.7) \quad J\Phi(x)^2 := \det \left[ (D\Phi(x))^T D\Phi(x) \right] = \frac{1}{r^2(x)}.$$

PROOF. We shall see that (cf. (6.3))

$$(6.8) \quad (D\Phi(x))^T D\Phi(x) = (I + d_\Gamma(x)S(x))^2.$$

Since

$$Dx_\Gamma = I - \mathbf{n}(x)\mathbf{n}^T(x) + d_\Gamma(x)S(x) = P_x + d_\Gamma(x)S(x),$$

we have for every  $\mathbf{v} \in \mathbf{R}^N$

$$D\Phi(x)\mathbf{v} = (P_x\mathbf{v} + d_\Gamma(x)S(x)\mathbf{v}, N_x\mathbf{v}) \in \mathbf{R}^N \times \mathbf{R},$$

and

$$|D\Phi(x)\mathbf{v}|^2 = |P_x\mathbf{v}|^2 + |N_x\mathbf{v}|^2 + |d_\Gamma(x)S(x)\mathbf{v}|^2 + 2d_\Gamma(x)\mathbf{v}^T S(x)\mathbf{v} = |\mathbf{v} + d_\Gamma(x)S(x)\mathbf{v}|^2. \blacksquare$$

Since we are dealing with symmetric matrices, this is equivalent to (6.8). ■

### 6.3 PROOF OF LEMMA 5.2.

Let us observe that  $\Phi$  is a  $C^1$  diffeomorphism of  $\Omega_2$  onto  $\Phi(\Omega_2) \subset \Gamma \times \mathbf{R}$ , which satisfies

$$\Phi(x_\lambda) = (x, \lambda), \quad \forall x \in \Gamma, x_\lambda \in \Omega_2.$$

By the change of variable formula (see [19]) we write

$$\begin{aligned} \int_{\Omega_2} f(x) dx &= \int_{\Phi(\Omega_2)} f(\Phi^{-1}(z)) \frac{1}{J\Phi(\Phi^{-1}(z))} d\mathcal{H}^N(z) = \\ &= \int_{\Gamma} d\mathcal{H}^{N-1}(x) \int_0^{\ell(x)} f(x_\lambda) r(x_\lambda) d\lambda = \\ &= \int_{\Gamma} d\mathcal{H}^{N-1}(x) \int_{s_x} f(s) d\mu(s) \quad \blacksquare \end{aligned}$$

Our aim is now to show that  $L_{\mathbf{n}}^2(\Omega_2)$  is given by the functions of  $L^2(\Omega_2)$  which are constant along  $\mathcal{H}^{N-1}$ -a.e. segment  $s_x$ .

For  $x \in \overline{\Omega}_2$  let us define

$$\mathbf{b}(x) := \frac{\mathbf{n}(x)}{r(x)}, \quad \text{which satisfies} \quad \operatorname{div} \mathbf{b}(x) = 0 \quad \text{in } \Omega_2, \quad \mathbf{b}(x) = \mathbf{n}(x) \quad \text{on } \Gamma$$

by (6.5), and let us call  $s_x^+$  the part of  $s_x$  joining  $x$  with  $\Gamma_2$ . We have

6.4 COROLLARY. *Let  $v$  be a  $C^1$  function with compact support in  $\Omega_2$  and let us define*

$$(6.9) \quad \mathbf{v}(x) := -\mathbf{b}(x) \int_{s_x^+} v d\mu.$$

Then

$$\operatorname{div} \mathbf{v} = v.$$

PROOF. Being  $\mathbf{b}$  a divergence free vector field, we have

$$\operatorname{div} \mathbf{v}(x) = -\frac{1}{r(x)} \frac{\partial}{\partial \mathbf{n}} \int_{s_x^+} v d\mu = \frac{1}{r(x)} v(x) r(x) = v(x) \quad \blacksquare$$

6.5 COROLLARY. *Let  $u \in H_{\mathbf{n}}^1(\Omega_2)$ ; then*

$$u|_{s_x} = u(x_\Gamma) \quad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \Gamma.$$

PROOF. Let us fix  $v$  as in the previous corollary; then we calculate by the Green's formula

$$\begin{aligned} \int_{\Omega_2} u(x)v(x) dx &= \int_{\Omega_2} u(x)(\operatorname{div} \mathbf{v}(x)) dx = \\ &= - \int_{\Omega_2} \nabla u(x) \cdot \mathbf{v}(x) dx - \int_{\Gamma} u|_{\Gamma}(x) \mathbf{v}(x) \cdot \mathbf{n}(x) d\mathcal{H}^{N-1}(x), \end{aligned}$$

since  $\mathbf{v}$  vanishes on  $\Gamma_2$ . By definition of  $H_{\mathbf{n}}^1(\Omega_2)$ , we know that

$$\nabla u \cdot \mathbf{v} = \frac{1}{r} \nabla u \cdot \mathbf{n} = 0,$$

so that by (6.9)

$$\int_{\Omega_2} u(x)v(x) dx = \int_{\Gamma} u|_{\Gamma}(x) d\mathcal{H}^{N-1}(x) \int_{\mathbb{S}_x} v d\mu = \int_{\Omega_2} u(x_{\Gamma}) v(x) dx.$$

Since  $v$  is arbitrary, we conclude. ■

## 6.6 PROOF OF LEMMA 5.3.

Let  $u_n \in C^1(\overline{\Omega}_2)$  be a sequence converging to  $u \in H_{\mathbf{n}}^1(\Omega_2)$  in the strong topology of  $H^1(\Omega_2)$ . It is easy to check that

$$\bar{u}_n(x) = \Pi u_n(x) := \int_{\mathbb{S}_x} u_n d\mathcal{H}^1$$

is a  $C^1$  function in  $H_{\mathbf{n}}^1(\Omega_2)$ ; since the linear operator  $\Pi$  defined above is bounded in  $H^1(\Omega_2)$  and  $\Pi u = u$ , we conclude. ■

Lemma 5.4 and 5.5 follow easily; it is sufficient to work with  $C^1$  functions and to apply (6.4) to

$$u(x_{\lambda}) = u(x) \quad \Rightarrow \quad (I - \lambda S(x)) \nabla u(x_{\lambda}) = \nabla u(x).$$

We make explicit the last elementary computation for Lemma 5.6.

6.7 LEMMA. Let  $G^{\varepsilon}$  be defined as in (1.32),  $R_{\varepsilon}, r_{\varepsilon}$  as in (5.16); then

$$DG^{\varepsilon}(z)\mathbf{v} = R^{-1}(z)R_{\varepsilon}(z)P_z\mathbf{v} + \varepsilon N_z\mathbf{v}, \quad \det DG^{\varepsilon}(z) = \varepsilon \frac{r_{\varepsilon}(z)}{r(z)}.$$



PROOF. We know that

$$G^\varepsilon(z) = z - (1 - \varepsilon)d_\Gamma(z)\mathbf{n}(z),$$

so that by (6.3)

$$\begin{aligned} DG^\varepsilon(z)\mathbf{v} &= I - (1 - \varepsilon)\mathbf{n}(z)\mathbf{n}^T(z) + (1 - \varepsilon)d_\Gamma(z)S(z) = \\ &\varepsilon N_z\mathbf{v} + \left[ I + d_\Gamma(z)S(z) - \varepsilon d_\Gamma(z)S(z) \right] P_z\mathbf{v} \\ &\varepsilon N_z\mathbf{v} + (I + d_\Gamma(z)S(z)) \left[ I - \varepsilon d_\Gamma(z)(I + d_\Gamma(z)S(z))^{-1}S(z) \right] P_z\mathbf{v} = \\ &\varepsilon N_z\mathbf{v} + (I - d_\Gamma(z)S(z_\Gamma))^{-1} \left[ I - \varepsilon d_\Gamma(z)S(z_\Gamma) \right] P_z\mathbf{v} \quad \blacksquare \end{aligned}$$

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