Gradient Flows and Diffusion Semigroups in Metric Spaces under Lower Curvature Bounds

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Abstract

We present some new results concerning well-posedness of gradient flows generated by λ-convex functionals in a wide class of metric spaces, including Alexandrov spaces satisfying a lower curvature bound and the corresponding L^2-Wasserstein spaces. Applications to the gradient flow of Entropy functionals in metric-measure spaces with Ricci curvature bounded from below and to the corresponding diffusion semigroup are also considered. These results have been announced during the workshop on "Optimal Transport: theory and applications" held in Pisa, November 2006.

Let (X, d) be a complete and separable metric space. A (constant speed, minimal) *geodesic* is a curve $\mathsf{x} : [0,1] \to X$ such that $\mathsf{d}(\mathsf{x}_s, \mathsf{x}_t) = |\dot{\mathsf{x}}|\,|s-t|$, $\forall\, s,t \in [0,1]$, $|\dot{\mathsf{x}}|$ denoting its (constant) metric velocity.

Definition 1 (λ-convexity) *A functional $\phi : X \to (-\infty, +\infty]$ is λ-convex, $\lambda \in \mathbb{R}$, if every couple of points $\mathsf{x}_0, \mathsf{x}_1 \in D(\phi) := \{u \in X : \phi(u) < +\infty\}$ can be connected by a geodesic x such that*

$$\phi(\mathsf{x}_t) \le (1-t)\phi(\mathsf{x}_0) + t\phi(\mathsf{x}_1) - \tfrac{1}{2}\lambda t(1-t)\mathsf{d}^2(\mathsf{x}_0, \mathsf{x}_1) \qquad \forall\, t \in [0,1]. \tag{1}$$

In contrast with the well known case when X is an Hilbert space [3], in arbitrary metric spaces λ-convexity is generally not sufficient to obtain the existence of a λ-contracting gradient flow, and it is a common belief that some "Riemannian-like" structure for X should also be required. When X is a *non positively curved* (NPC) Alexandrov space (i.e., the squared distance map $u \mapsto \frac{1}{2}\mathsf{d}^2(u,v)$ is 1-convex, see e.g. [4]), then a generation result reproducing the celebrated Crandall-Ligget argument has been proved by [7] and it has been refined in various directions in [1]. In this note we consider the case of spaces satisfying (in a suitable synthetic way) only a lower bound on the curvature. Besides Alexandrov spaces (considered by a completely different method in the unpublished [9] and, when X is compact and positively curved, in the recent [8]), our approach covers more general situations, as the Wasserstein space $\mathscr{P}_2(X)$, when the Riemannian manifold X has points with negative sectional curvature. In particular our conditions are preserved by the Wasserstein construction and avoid compactness of the sublevels of ϕ.

Let us recall the metric definition of gradient flow for a λ-convex functional (see [1, Chap. 4]).

Definition 2 (Gradient flow) *Let $\phi : X \to (-\infty, +\infty]$ be proper, l.s.c., and λ-convex. The gradient flow of ϕ with initial value $u_0 \in \overline{D(\phi)}$ is a locally Lipschitz curve $u : t \in (0, +\infty) \mapsto u_t \in D(\phi)$ such that*

$$\frac{d}{dt}\frac{1}{2}\mathsf{d}^2(u_t, v) + \frac{\lambda}{2}\mathsf{d}^2(u_t, v) \le \phi(v) - \phi(u_t) \quad \textit{for a.e. } t \in (0,+\infty), \quad \forall\, v \in D(\phi); \qquad \lim_{t\downarrow 0} u_t = u_0. \tag{2}$$

Existence of gradient flows will be proved by the so called *Minimizing Movements* variational scheme.

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Definition 3 (The "Minimizing Movements" approximation scheme) *A recursive minimizing sequence* $\{U^n_\tau\}_{n\in\mathbb{N}}$ *with step* $\tau > 0$ *and initial datum* $U^0 \in X$ *is any solution of the family of problems*

$$U^0_\tau := U^0, \qquad U^n_\tau \in \operatorname*{argmin}_V \Big(\tfrac{1}{2\tau}\mathsf{d}^2(U^{n-1}_\tau, V) + \phi(V)\Big) \quad n = 1, 2, \cdots. \tag{3}$$

A discrete solution $\overline{U}_\tau : [0,+\infty) \to X$ *is defined by setting* $\overline{U}_\tau(t) \equiv U^n_\tau$ *if* $t \in ((n-1)\tau, n\tau]$. *The variational scheme is* generically solvable *if there exists a minimizing sequence* $\{U^n_\tau\}_{n\in\mathbb{N}}$ *for every* U^0 *in a dense subset of* $D(\phi)$ *and for a vanishing sequence of time steps* τ *(depending on* U^0*).*

Definition 4 (Semi-concavity of the squared distance function) *We say that* X *is a* K*-SC (SemiConcave) space,* $\mathsf{K} \geq 1$*, if for every geodesic* x *and any* $y \in X$

$$\mathsf{d}^2(\mathsf{x}_t, y) \geq (1-t)\mathsf{d}^2(\mathsf{x}_0, y) + t\mathsf{d}^2(\mathsf{x}_1, y) - \mathsf{K}\, t(1-t)\mathsf{d}^2(\mathsf{x}_0, \mathsf{x}_1) \qquad \forall\, t \in [0,1]. \tag{4}$$

Examples
- PC SPACES: X is *Positively Curved* (PC) in the sense of ALEXANDROV iff it is K-SC with $\mathsf{K} = 1$.
- ALEXANDROV SPACES: if X is an Alexandrov space whose curvature is bounded from below by a negative constant $-\kappa$ and $D = \operatorname{diam}(X) < +\infty$, then X is K-SC with $\mathsf{K} = D\sqrt{\kappa}/\tanh(D\sqrt{\kappa})$. This class includes all bounded and complete Riemannian manifolds whose sectional curvature is bounded from below.
- PRODUCT AND L^2 SPACES: if (X_i, d_i) is a (even countable) collection of K-SC spaces, then $\boldsymbol{X} := \Pi_i X_i$ with the usual product distance is K-SC. If μ is a finite measure on some separable measure space Ω then $\mathscr{X} := L^2_\mu(\Omega; X)$ endowed with the distance $\mathsf{d}^2_{\mathscr{X}}(x, y) := \int_\Omega \mathsf{d}^2(x(\omega), y(\omega))\, d\mu(\omega)$
- WASSERSTEIN SPACE: $\mathscr{P}_2(X)$ is the set of all Borel probability measures μ on X with $\int_X \mathsf{d}^2(x, x_0)\, d\mu < +\infty$ for some $x_0 \in X$, endowed with the L^2-Wasserstein distance [11, 1]. $\mathscr{P}_2(X)$ is K-SC iff X is K-SC.

Definition 5 ((Upper) angles) *Let* x^1, x^2 *be two geodesics emanating from the same initial point* $\mathsf{x}_0 := \mathsf{x}^1_0 = \mathsf{x}^2_0$. *Their* upper angle $\sphericalangle_{\mathrm{u}}(\mathsf{x}^1, \mathsf{x}^2) \in [0, \pi]$ *is defined by*

$$\cos\big(\sphericalangle_{\mathrm{u}}(\mathsf{x}^1, \mathsf{x}^2)\big) := \liminf_{s,t\downarrow 0} \frac{\mathsf{d}^2(\mathsf{x}_0, \mathsf{x}^1_s) + \mathsf{d}^2(\mathsf{x}_0, \mathsf{x}^2_t) - \mathsf{d}^2(\mathsf{x}^1_s, \mathsf{x}^2_t)}{2\mathsf{d}(\mathsf{x}_0, \mathsf{x}^1_s)\mathsf{d}(\mathsf{x}_0, \mathsf{x}^2_t)}. \tag{5}$$

Definition 6 (Local Angle Condition (LAC)) *X satisfies the* local angle condition *(LAC) if for any triple of geodesics* x^i, $i = 1, 2, 3$, *emanating from the same initial point* x_0 *the corresponding angles* $\theta^{ij} := \sphericalangle_{\mathrm{u}}(\mathsf{x}^i, \mathsf{x}^j) \in [0, \pi]$ *satisfy one of the following* equivalent *conditions:*

1. $\theta^{12} + \theta^{23} + \theta^{31} \leq 2\pi$.
2. *There exists an Hilbert space* H *and vectors* $w^i \in H$ *such that* $\langle w^i, w^j\rangle_H = \cos(\theta^{ij})$ $\quad 1 \leq i, j \leq 3$.
3. *For every choice of* $\xi_1, \xi_2, \xi_3 \geq 0$ *one has* $\sum_{i,j=1}^3 \cos(\theta^{ij})\xi_i\xi_j \geq 0$.

Examples
- A BANACH SPACE X satisfies (LAC) iff X is a HILBERT SPACE.
- RIEMANNIAN MANIFOLDS AND ALEXANDROV SPACES with curvature bounded below satisfy (LAC).
- PRODUCT SPACES: $\boldsymbol{X} := \Pi_i X_i$ satisfies (LAC) iff each (X_i, d_i) does satisfy it.
- L^2 SPACES: The space $L^2_\mu(\Omega; X)$ satisfies (LAC) iff X satisfies it.
- WASSERSTEIN SPACE: The L^2-*Wasserstein space* $\mathscr{P}_2(X)$ satisfies (LAC) iff X does.
- Let $(\boldsymbol{e}_i)$ be an orthonormal basis of of $\mathbb{R}^4$ and let X be the cone $\{\sum_{i=1}^4 x_i\boldsymbol{e}_i : x_i \geq 0\} \subset \mathbb{R}^4$ with the distance $\mathsf{d}^2(\boldsymbol{x}, \boldsymbol{y}) := |\boldsymbol{x}|^2 + |\boldsymbol{y}|^2 - 2|\boldsymbol{x}|\,|\boldsymbol{y}|\cos\big(\frac{1}{3}\sqrt{2}\pi\big|\boldsymbol{x}/|\boldsymbol{x}| - \boldsymbol{y}/|\boldsymbol{y}|\big|\big)$. The geodesics $\mathsf{x}^{\boldsymbol{e}}_t := t\boldsymbol{e}$, $\boldsymbol{e} \in X$, $t \in [0,1]$, emanating from the origin satisfy (LAC) since $\sphericalangle_{\mathrm{u}}(\mathsf{x}^{\boldsymbol{e}}, \mathsf{x}^{\boldsymbol{f}}) \leq 2\pi/3$ but X is not an Alexandrov space since $\sum_{i,j=1}^4 \cos\big(\sphericalangle_{\mathrm{u}}(\mathsf{x}^{\boldsymbol{e}_i}, \mathsf{x}^{\boldsymbol{e}_j})\big) = -2 < 0$.

Main results

Let us recall that the *Metric Slope* of ϕ at $u \in D(\phi)$ is $|\partial\phi|u := \limsup_{v\to u} \big(\phi(u) - \phi(v)\big)^+/\mathsf{d}(u,v)$.

Theorem 7 (Generation result for gradient flows) *Let X be a K-SC space satisfying (LAC) and let $\phi : X \mapsto (-\infty,+\infty]$ be proper, l.s.c., and λ-convex. If (3) is generically solvable, then*

λ-contractive semigroup. *For every $u_0 \in \overline{D(\phi)}$ there exists a unique gradient flow $u := S[u_0]$ according to definition 2. The map $u_0 \mapsto S_t[u_0]$ is a λ-contracting continuous semigroup on $\overline{D(\phi)}$, i.e.*

$$S_{t+h}[u_0] = S_h[S_t[u_0]], \quad \mathsf{d}(S[u_0](t), S[v_0](t)) \le e^{-\lambda t}\mathsf{d}(u_0,v_0) \qquad \forall\, u_0,\, v_0 \in \overline{D(\phi)}. \tag{6}$$

Uniform error estimate. *For every time interval $[0,T]$ there exists a "universal" constant $C_{\mathsf{K},\lambda,T}$ (only depending on K,λ,T) such that for every discrete solution $\overline{U}_\tau$, $\tau \in (0,\frac{1}{2\lambda^-})$,*

$$\sup_{t\in[0,T]} \mathsf{d}^2(u_t,\overline{U}_\tau(t)) \le \begin{cases} C_{\mathsf{K},\lambda,T}\big(\phi(u_0) - \inf_X \phi\big)\cdot\sqrt{\tau} & \text{if } u_0 = U^0_\tau \in D(\phi), \\ C_{\mathsf{K},\lambda,T}|\partial\phi|^2(u_0)\cdot\tau & \text{if } u_0 = U^0_\tau \in D(|\partial\phi|). \end{cases} \tag{7}$$

Regularizing effect. *S_t maps $\overline{D(\phi)}$ into $D(|\partial\phi|) \subset D(\phi)$* for every $t > 0$, *$t \mapsto e^{\lambda t}|\partial\phi|(u_t)$ is nonincreasing, $t \mapsto \phi(u_t)$ is (locally semi-, if $\lambda < 0$) convex, and, when $\lambda \ge 0$,*

$$\phi(u_t) \le \phi(v) + \tfrac{1}{2t}\mathsf{d}^2(u_0,v), \qquad |\partial\phi|^2(u_t) \le |\partial\phi|^2(v) + \tfrac{1}{t^2}\mathsf{d}^2(u_0,v) \qquad \forall\, v \in X. \tag{8}$$

Energy identity. *The* right limits $|\dot u_{t+}| := \lim_{h\downarrow 0} \frac{\mathsf{d}(u_t,u_{t+h})}{h}$ *and* $\frac{d}{dt_+}\phi(u_t) := \lim_{h\downarrow 0} \frac{\phi(u_{t+h})-\phi(u_t)}{h}$ *exist for every $t \ge 0$, are finite if $t > 0$, and coincide with the corresponding left ones for $t \in (0,+\infty)\setminus\mathcal{C}$, $\mathcal{C}$ being at most countable. They satisfy the differential energy identity*

$$\tfrac{d}{dt_+}\phi(u_t) = -|\dot u_{t+}|^2 = -|\partial\phi|^2(u_t) \qquad \forall\, t \ge 0. \tag{9}$$

Asymptotic behavior. *If $\lambda > 0$, then ϕ admits a unique minimum point $\bar u$ and for $t \ge t_0 \ge 0$ we have*

$$\tfrac{\lambda}{2}\mathsf{d}^2(u_t,\bar u) \le \phi(u_t) - \phi(\bar u) \le \tfrac{1}{2\lambda}|\partial\phi|^2(u_t), \qquad \mathsf{d}^2(u_t,\bar u) \le \mathsf{d}^2(u_{t_0},\bar u)e^{-\lambda(t-t_0)}, \tag{10a}$$

$$\phi(u_t) - \phi(\bar u) \le \Big(\phi(u_{t_0}) - \phi(\bar u)\Big)e^{-2\lambda(t-t_0)}, \qquad |\partial\phi|(u_t) \le |\partial\phi|(u_{t_0})e^{-\lambda(t-t_0)}. \tag{10b}$$

Applications to metric-measure spaces. Let (X,d,γ) be a metric-measure space with $\gamma \in \mathscr{P}(X)$. On the Wasserstein space $\mathscr{X} = \mathscr{P}_2(X)$ we consider the Relative Entropy functional $\phi(\mu) = \mathrm{Ent}(\mu|\gamma) := \int_X \rho\log\rho\, d\gamma$ if $\mu = \rho\cdot\gamma \ll \gamma$, $\phi(\mu) := +\infty$ otherwise. According to [10, 6], X satisfies the **lower Ricci curvature bound** $\underline{\mathbb{C}\mathrm{urv}}(X,\mathsf{d},\gamma) \ge \lambda$ iff the functional $\phi = \mathrm{Ent}(\cdot|\gamma)$ is λ-geodesically convex in $\mathscr{P}_2(X)$. X is *non-branching* if two geodesics x,y with $\mathsf{x}_0 = \mathsf{y}_0$ and $\mathsf{x}_{\bar t} = \mathsf{y}_{\bar t}$ for some $\bar t \in (0,1)$ must coincide.

Theorem 8 (Markov semigroup and diffusion kernels) *Let us suppose that X is K-SC, satisfies (LAC) and $\underline{\mathbb{C}\mathrm{urv}}(X,\mathsf{d},\gamma) \ge \lambda$. There exists a unique λ-contracting gradient flow S_t generated by $\phi = \mathrm{Ent}(\cdot|\gamma)$ on $\mathscr{X}_\gamma := \big\{\mu \in \mathscr{P}_2(X) : \operatorname{supp}\mu \subset \operatorname{supp}\gamma\big\}$ as a limit of the (always solvable) Minimizing Movement scheme. S_t enjoys all the properties stated in Theorem 7 and, if X is* non-branching, *it satisfies the* linearity *condition*

$$S_t[\alpha\mu_0 + \beta\mu_1] = \alpha S_t[\mu_0] + \beta S_t[\mu_1] \qquad \forall\, \mu_0,\mu_1 \in \mathscr{P}_2(X), \quad \alpha,\beta \ge 0, \quad \alpha+\beta = 1. \tag{11}$$

γ is an invariant measure and the kernels $\nu_{x,t} := S_t[\delta_x] \ll \gamma$ satisfy the Chapman-Kolmogorov equation

$$\nu_{x,t+s}(E) = \int_X \nu_{y,t}(E)\, d\nu_{x,s}(y) \quad \forall\, E \in \mathscr{B}(X), \quad x \in \operatorname{supp}\gamma,\ t,s > 0; \qquad S_t[\mu] = \int_X \nu_{x,t}\, d\mu(x). \tag{12}$$

S_t can be uniquely extended to a Markov *(i.e. linear, order preserving, strongly continuous, contraction) semigroup $\mathscr{S}_t$ in $L^p(\gamma)$ such that $\mathscr{S}_t[\rho_0]\gamma = S_t[\rho_0\gamma]$ for every $\rho_0 \in L^p(\gamma)$ with $\rho_0\gamma \in \mathscr{P}_2(X)$.*

Remark 9 When $X = \mathbb{R}^d = \operatorname{supp}\gamma$, there exists a λ-convex potential $V : \mathbb{R}^d \to \mathbb{R}$ such that $\gamma = e^{-V}\mathcal{L}^d$ and the gradient flow $\mu_t = u_t\mathcal{L}^d$ generated by $\mathrm{Ent}(\cdot|\gamma)$ satisfies the Fokker-Planck equation [5, 2]

$$\partial_t u_t - \nabla \cdot \big(\nabla u_t + u_t \nabla V\big) = 0 \quad \text{in } \mathbb{R}^d \times (0, +\infty). \tag{13}$$

Stability of gradient flows. Let now consider a sequence $(X^k, \mathsf{d}^k, \gamma^k)$, $\gamma^k \in \mathscr{P}_2(X^k)$, of metric measure spaces converging to (X, d, γ) in the *measured* Gromov-Hausdorff distance. That means [10, 6] that a sequence of (separable and complete) coupling semidistances $\hat{\mathsf{d}}^k$ on the disjoint union $X^k \sqcup X$ exists s.t.

$$\lim_{k+\infty} \hat{\mathsf{d}}^k_W(\gamma^k, \gamma) = 0, \qquad \hat{\mathsf{d}}^k_W \text{ being the } L^2\text{-Wasserstein distance in } \mathscr{P}_2(X^k \sqcup X) \text{ induced by } \hat{\mathsf{d}}^k. \tag{14}$$

A sequence $\mu^k \in \mathscr{P}_2(X^k)$ converges to $\mu \in \mathscr{P}_2(X)$ if $\lim_{k+\infty} \hat{\mathsf{d}}^k_W(\mu^k, \mu) = 0$.

Theorem 10 *Let us assume that each metric space X^k is K-SC, it satisfies (LAC) and the lower Ricci bound $\mathbb{C}\mathrm{urv}(X^k, \mathsf{d}^k, \gamma^k) \geq \lambda$ for suitable constants λ, K independent of k. If $\mu^k_0 \in \mathscr{P}_2(X^k)$ is a sequence converging to $\mu_0 \in \mathscr{P}_2(X)$ with equibounded relative entropy $\sup_k \mathrm{Ent}(\mu^k|\gamma^k) < +\infty$, then for every $t \geq 0$ the solution $\mu^k_t := S^k_t[\mu^k_0]$ of the "Entropy gradient flow" given by Theorem 8 converges to the measure $\mu_t = S_t[\mu_0] \in \mathscr{P}_2(X)$ which is the (unique) Entropy gradient flow in $\mathscr{P}_2(X)$ with initial datum μ_0.*

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