APPROXIMATION AND REGULARITY OF EVOLUTION VARIATIONAL INEQUALITIES

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Summary. In the framework of a Hilbert triple $\{V, H, V'\}$ we study the approximation and the regularity of parabolic variational inequalities, by a time discretization by means of the backward Euler scheme. Under suitable regularity hypotheses on the data, we prove that the order of convergence in $H^1(0,T;H)$ is $1/2$ and the solution belongs to $H^s(0,T;H)$, $\forall\, s < 3/2$. Moreover, in the case of a symmetric linear operator with $L^2(0,T;H)$ data, we prove the $H^{1/2}(0,T;V)$-regularity of the solution with the same error estimate in the "energy norm" of $L^2(0,T;V) \cap L^\infty(0,T;H)$.

0 Introduction.

Let $V \subset H$ be a couple of separable real Hilbert spaces, the inclusion being continuous and dense; the norms on V and H and the scalar product on H are denoted respectively by $\|\cdot\|$, $|\cdot|$ and $(\cdot,\cdot)$. We identify H with its dual H', so that the dual space V' is the completion of H with respect the dual norm:

$$\|h'\|_* = \sup_{\|v\|=1} (h', v)$$

and the relations:

$$V \subset H \equiv H' \subset V'$$

hold with continuous and dense imbeddings; $(\cdot,\cdot)$ can also be used for the pairing between V and V'.

Let A be a continuous and coercive linear operator from V to V':

$$\exists \alpha,\, M > 0 : \quad \|Av\|_* \le M\|v\|; \quad (Av, v) \ge \alpha\|v\|^2 \quad \forall\, v \in V \tag{0.1}$$

and let $\mathbb{K}$ be a closed convex subset of V with:

$$0 \in \mathbb{K} \tag{0.2}$$

We call K the closure of $\mathbb{K}$ in H and $\mathcal{P}$ the projection of H on K (with respect to the norm of H).

We want to study the approximation and the regularity of the solution $u(t)$ of the following problem:

0.1 Problem. *We are given an initial value $u_0 \in H$, a function $f(t)$ a.e. defined on the interval $]0,T[$ ($T \in]0,+\infty]$) with values in V', we ask for $u(t)$ such that:*

$$\begin{cases} u(t) \in \mathbb{K}, \quad a.e.\ in\ ]0,T[\\ (u'(t) + Au(t) - f(t), u(t) - v) \le 0 \quad \forall\, v \in \mathbb{K}, \quad a.e.\ in\ ]0,T[\\ u(0) = u_0 \end{cases} \tag{0.3}$$

This problem has been studied in several papers (see for instance [3], [4], [1]); our starting point is the work of Baiocchi [1] where the existence and the regularity of the solution of a weak form of Problem 0.1 were proved by studying the time discretization of the same problem by means of the backward Euler scheme; moreover, this technique gives an optimal estimate of the order of convergence in the space ([1]) Let us briefly recall the outline of this method. The natural function space for the datum f is the "sum" ([2])

$$S(0,T) = L^2(0,T;V') + L^1(0,T;H) = \{f = g + h;\ g \in L^2(0,T;V'),\ h \in L^1(0,T;H)\} \tag{0.6}$$

([1]) For a generic Hilbert space $\mathcal{H}$, $L^p(0,T;\mathcal{H})$ $(1 \le p \le +\infty)$ is the Banach space of the strongly Lebesgue–measurable functions f a.e. defined on $]0,T[$ with values in $\mathcal{H}$, such that $t \mapsto \|f(t)\|_{\mathcal{H}}$ belongs to $L^p(0,T)$; the L^p-norm of this function gives the norm in $L^p(0,T;\mathcal{H})$:

$$I(0,T) = L^2(0,T;V) \cap L^\infty(0,T;H) \tag{0.4}$$

([2]) We recall that $I(0,T) = \big(S(0,T)\big)'$, the pairing between $i(t) \in I(0,T)$ and $s(t) \in S(0,T)$ being given by:

$$\langle i, s\rangle = \int_0^T \big(i(t), s(t)\big)\, dt \tag{0.5}$$

In order to discretize this problem, we choose a stepsize $k > 0$ ([3]) and we set for $n \geq 0$:

$$J_{k,n} = [nk, (n+1)k[; \quad f_n^k = \frac{1}{k}\int_{J_{k,n}} f(t)\,dt \in V'; \quad u_{-1}^k = \mathcal{P}u_0 \tag{0.7}$$

Then we consider the sequence of equations in the unknown u_{n+1}^k, $n \geq -1$:

$$\begin{cases} u_{n+1}^k \in \mathbb{K}; \\ \left(u_{n+1}^k - u_n^k + k\,Au_{n+1}^k - k\,f_{n+1}^k, u_{n+1}^k - v\right) \leq 0 \quad \forall\, v \in \mathbb{K} \end{cases} \tag{0.8}$$

The Lions–Stampacchia theorem (see [9]) ensures us that, for all given $u_n^k \in V'$, there exists a unique solution u_{n+1}^k of (0.8), so that the sequence $\{u_n^k\}_{n\in\mathbb{N}}$ is well defined and it is contained in V, for $n \geq 0$.

We construct the continuous function $\hat{u}_k(t)$, linear on each $J_{k,n}$, such that $\hat{u}_k(nk) = u_n^k$,

$$\hat{u}_k(t) = (n+1-t/k)u_n^k + (t/k-n)u_{n+1}^k, \quad t \in J_{k,n}, \quad 0 \leq n < N \tag{0.9}$$

and we go to evaluate the error $u - \hat{u}_k$ in various norms, u being the solution of a weak form of problem 0.1.

In [1] is proved that, if $\{u_0, f\}$ satisfies compatibility and smoothness conditions (see Teorem 1.3), then $\hat{u}_k$ with its derivative remains bounded in $I(0,T)$ and converges to u in $I(0,T)$ with an $O(k)$-error.

We shall consider the error between the derivatives $u' - \hat{u}_k'$ in the norm of $L^2(0,T;H)$; our estimates gives a bound:

$$\|u' - \hat{u}_k'\|_{L^2(0,T;H)} = O(k^{1/2})$$

and the associated regularity $u \in H^s(0,T;H)$, $\forall\, s < 3/2$, and even a little more (see Theorem 1). Since u' can have jump discontinuities, also for smooth data, this is the maximal time regularity which one can expect.

We shall also consider the case of a symmetric operator A, with $\{u_0, f\} \in \mathbb{K} \times L^2(0,T;H)$. In this framework, it is well known that the solution u belongs to $H^1(0,T;H) \cap L^\infty(0,T;V)$ and is continuous with values in V. Also in this case we will prove that $\hat{u}_k'$ converges to u' in $L^2(0,T;H)$; moreover the solution u belongs to $H^{1/2}(0,T;H)$ and we can estimate the error in the "energy norm" of $I(0,T)$ as:

$$\|u - \hat{u}_k\|_{L^2(0,T;V)\cap L^\infty(0,T;H)} = o(k^{1/2})$$

Regularity results for the solution of 0.1 and related estimates of the rate of convergence of approximated solutions under "intermediate" assumptions on $\{u_0, f\}$ (in the same framework of [2]) are contained in a forthcoming paper.

We wish to thank Prof. C. Baiocchi for many useful discussions on this subject.

([3]) which will tend to 0; when $T < +\infty$, for the sake of semplicity, we may limit ourselves to the values of the type $k = T/N$, $N \in \mathbb{N}$.

1 Notations and main results.

We begin with a weak reformulation of Problem 0.1:

1.1 Problem. *Given* $u_0 \in H$ *and* $f \in S(0,T)$, *find* $u \in I(0,T) \cap C^0([0,T];H)$ *such that*

$$u(t) \in \mathbb{K}, \quad a.e.\ in\ ]0,T[$$

and the function:

$$t \in [0,T] \mapsto \frac{1}{2}|u(t)-v(t)|^2 + \int_0^t (v'(\tau)+Au(\tau)-f(\tau), u(\tau)-v(\tau))\ d\tau$$

is not increasing and bounded by $\frac{1}{2}|\mathcal{P}u_0 - v(0)|^2$, $\forall\, v \in H^1(0,T;V) \cap W^{1,\infty}(0,T;H)$, $v(t) \in \mathbb{K}$ *for* $t \in [0,T]$. (4)

1.2 Remark. This is really a weak form of Problem 0.1: assuming that the solution u of 1.1 is also absolutely continuous with values in H, the function:

$$\tau \mapsto (u'(\tau)+Au(\tau)-f(\tau), u(\tau)-v(\tau))\,; \quad v, v' \in I(0,T)$$

belongs to $L^1(0,t)$, $\forall t \in]0,T[$; moreover, thanks to the identity:

$$\frac{d}{d\tau}|u(\tau)-v(\tau)|^2 = 2(u'(\tau)-v'(\tau), u(\tau)-v(\tau)) \tag{1.2}$$

we get:

$$t \in [0,T] \mapsto \int_0^t (u'(\tau)+Au(\tau)-f(\tau), u(\tau)-v(\tau))\ d\tau \quad \text{is decreasing} \tag{1.3}$$

and:

$$|u(0)-v(0)| \le |\mathcal{P}u_0 - v(0)|$$

Since $\mathbb{K}$ is dense in K, we have $u(0) = \mathcal{P}u_0$; from (1.3) and the additional hypothesis $u_0 \in \mathrm{K}$ we deduce that u solves (0.3). □

The foundamental result we use is the following:

1.3 Theorem (Baiocchi [1]). *The operator:*

$$T_k : \{u_0, f\} \in H \times S(0,T) \ \mapsto \hat{u}_k \in L^2(0,T;V) \cap L^\infty(0,T;H) \tag{1.14}$$

is bounded and Lipschitz continuous, uniformly in k; the family $\{\hat{u}_k\}$ *converges as* $k \to 0^+$ *to the unique solution u of Problem* 1.1, *which also belongs to* $C([0,T];H)$ *and satisfies the initial condition:*

$$u(0) = \mathcal{P}u_0 \in \mathrm{K} \tag{1.15}$$

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(4) With $W^{1,p}(0,T;\mathcal{H})$, we denote the Banach space of the absolutely continuous functions in $L^p(0,T;\mathcal{H})$, with derivative in the same space; by induction we define also $W^{n,p}(0,T;\mathcal{H})$. As usual, the norms in this spaces are given by:

$$\|f\|^p_{W^{1,p}(0,T;\mathcal{H})} = \|f\|^p_{L^p(0,T;\mathcal{H})} + \|f'\|^p_{L^p(0,T;\mathcal{H})} \tag{1.1}$$

with the obvious changes when $p = \infty$. When $p = 2$ we are dealing with Hilbert spaces; in this case we use the notation $W^{n,2}(0,T;\mathcal{H}) = H^n(0,T;\mathcal{H})$ (see [3], [7] for more details).

in particular, the operator $T : \{u_0, f\} \mapsto u$ *is bounded and lipschitz continuous from* $H \times S(0,T)$ *to* $L^2(0,T;V) \cap L^\infty(0,T;H)$.
Furthermore, assuming:

$$f \in H^1(0,T;V') + BV(0,T;H)(^5), \quad \mathcal{P}u_0 \in \mathbb{K}, \quad A\mathcal{P}u_0 - f(0) \in H \tag{1.16}$$

the family $\{\hat{u}_k\}$ *is bounded and weakly** *converge to* u *in* $H^1(0,T;V) \cap W^{1,\infty}(0,T;H)$*;* $u \in H^1(0,T;V) \cap W^{1,\infty}(0,T;H)$ *is a strong solution of Problem* 0.1 *satisfying* (0.3) *and the estimate* ([6])*:*

$$\|u - \hat{u}_k\|_{L^2(0,T;V)\cap L^\infty(0,T;H)} \le C\,k\big[\|f\|_{H^1(0,T;V')+BV(0,T;H)} + |A\mathcal{P}u_0 - f(0)|\big] \tag{1.17}$$

1.4 Remark. In the same work is proved that u belongs to $H^{4/3-\epsilon}(0,T;H)$ ([7]), for any $\epsilon > 0$, if (1.16) holds true, while at the lowest level of regularity, that is $u_0 \in H$ and $f \in S(0,T)$, if $\mathbb{K}$ is a cone then u belongs to $B^{1/2}_{2\infty}(0,T;H)$. We don't use these informations. □

Our regularity results concerne the family of spaces $B^s_{2\infty}(0,T;\mathcal{H})$; the relationships between convergence estimates of the approximating solutions $\hat{u}_k$ and the regularity properties of u are showed by the following charcterization (see [8]):

1.5 Remark. Let s be in $]0,1[$ and $v \in L^2(0,T;\mathcal{H})$; then v belongs to $B^s_{2\infty}(0,T;\mathcal{H})$ iff there exists a measurable family of functions

$$k \in]0,\infty[\mapsto v_k \in H^1(0,T;\mathcal{H})$$

and a constant C independent of k such that:

$$\|v - v_k\|_{L^2(0,T;\mathcal{H})} \le C\,k^s; \quad \|v_k\|_{H^1(0,T;\mathcal{H})} \le \frac{C}{k^{1-s}} \tag{1.18}$$

Moreover, an equivalent norm on $B^2_{2\infty}(0,T;\mathcal{H})$ is given by the infimum of the constants C for which the estimate (1.18) holds for some $\{v_k\}$. □

Now we can state our results:

([5]) $BV(0,T;\mathcal{H})$ is the space of the function $h : [0,T] \mapsto \mathcal{H}$ of bounded variation; the variation of h is:

$$\|h\|_{BV(0,T;\mathcal{H})} = \sup \sum_{m=0}^{n} \|h(t_{m+1}) - h(t_m)\|_{\mathcal{H}}$$

where the supremum is taken over all the subdivisions of the interval $[0,T]$: $0 = t_0 < t_1 < \ldots < t_{n+1} = T$

([6]) As usual, we denote by C a constant only depending on α, M.

([7]) If $s \in]0,1[$, we consider the interpolation spaces (see [6], [8] and [5] for more details):

$$H^s(0,T;\mathcal{H}) = \big(L^2(0,T;\mathcal{H}), H^1(0,T;\mathcal{H})\big)_{s,2}$$

$$B^s_{2\infty}(0,T;\mathcal{H}) = \big(L^2(0,T;\mathcal{H}), H^1(0,T;\mathcal{H})\big)_{s,\infty}$$

Finally, if $s \in]1,2[$, $H^s(0,T;\mathcal{H})$ and $B^s_{2\infty}(0,T;\mathcal{H})$ are the spaces of the $H^1(0,T;\mathcal{H})$-functions, whose derivative belongs to $H^{s-1}(0,T;\mathcal{H})$ and $B^{s-1}_{2\infty}(0,T;\mathcal{H})$ respectively.

Theorem 1. *The operators T_k defined in* (1.14) *are uniformly bounded with values in $B^{1/2}_{2\infty}(0,T;H)$; the family $\{\hat{u}_k\}$ weakly* converges in $B^{1/2}_{2\infty}(0,T;H)$ to the solution u of problem* 1.1*, which satisfies:*

$$\begin{cases} \displaystyle\int_0^{T-h} |u(\tau+h)-u(\tau)|^2\,d\tau \le C\,h\big[\|f\|_{S(0,T)}+|u_0|\big] \quad \forall\, h\in[0,T] \\ \displaystyle\lim_{h\to0^+} h^{-1}\int_0^{T-h} |u(\tau+h)-u(\tau)|^2\,d\tau = 0 \end{cases} \tag{1.19}$$

Theorem 2. *Assume that* (1.16) *holds true; then the solution u of Problem* 1.1 *belongs to $B^{3/2}_{2\infty}(0,T;H)$. The family $\{\hat{u}_k\}$ is uniformly bounded in this space and strongly converges to u in $H^1(0,T;H)$ with the estimate:*

$$\|u'-\hat{u}'_k\|_{L^2(0,T;H)} \le C\,k^{1/2}\left[\|f\|_{H^1(0,T;V')+BV(0,T;H)}+|A\mathcal{P}u_0-f(0)|\right] \tag{1.20}$$

1.6 Remark. This regularity is optimal, at least in the family

$$\big(H^1(0,T;H),H^2(0,T;H)\big)_{\theta,q},\quad \theta\in]0,1[,\quad q\in[1,\infty]$$

even in the scalar case (say $V\equiv H\equiv\mathbb{R}$). Actually, [3] gives an example of solution u relative to smooth data with $u'\notin\big(L^2(0,T),H^1(0,T)\big)_{1/2,q}$, $\forall\, q<\infty$. $\square$

1.7 Remark. We recall that there hold the continuous and dense inclusions:

$$\forall\, r,s\in]0,1[,\ r<s,\ B^s_{2\infty}(0,T;\mathcal{H})\subset H^r(0,T;\mathcal{H}) \tag{1.21}$$

so that the regularity results announced in the introduction are justified. $\square$

1.8 Remark. The most intersting way to deduce a regularity result from an estimate of the rate of convergence is surely showed by remark 1.5. In this context, however, we cannot immediately use (1.20), since $\hat{u}_k$ does not belong to $H^2(0,T;H)$. In order to overcome this difficulties we shall introduce a piecewise quadratic approximating function $\tilde{u}_k$, which satisfies (1.20) and the bound

$$k^{1/2}\|\tilde{u}''_k\|_{L^2(0,T;H)}\le C\quad\square$$

1.9 Remark. One other characterization of the interpolation spaces in the case $T=+\infty$ is related to the estimate (1.19), and will be also useful in other circumstancies:

$$v\in B^{1/2}_{2\infty}(0,\infty;\mathcal{H}) \Leftrightarrow \begin{cases} v\in L^2(0,\infty;\mathcal{H}) \\ \displaystyle\sup_{h>0} h^{-1}\int_0^\infty \|v(t+h)-v(t)\|^2_{\mathcal{H}}\,dt<+\infty \end{cases} \tag{1.22}$$

The sum of L^2-norm of v and the square root of the last "sup" gives an equivalent norm on $B^s_{2\infty}(0,\infty;\mathcal{H})$. Analogously, we have for $H^{1/2}(0,\infty;\mathcal{H})$:

$$\|v\|^2_{H^{1/2}(0,\infty;H)} = \|v\|^2_{L^2(0,\infty;\mathcal{H})} + \int_0^\infty \frac{dh}{h^2}\int_0^\infty \|u(t+h)-u(t)\|^2\,dt \quad\square \tag{1.23}$$

In order to have shorter notations, we denote by $\mathcal{V}$ the subset of $H \times S(0,\infty)$ defined by (1.16), and we write:

$$\|\{u_0, f\}\|_{\mathcal{V}} = \|f\|_{H^1(0,\infty;V')+BV(0,\infty;H)} + |A\mathcal{P}u_0 - f(0)| \tag{1.24}$$

Theorem 3. *The operator $T : \{u_0, f\} \mapsto u$, $\mathcal{V} \mapsto H^1(0,\infty;H)$ is 1/2-Hölder continuous with respect to the $H \times S(0,\infty)$-metric on the bounded subset of $\mathcal{V}$.*

The last result requires the additional hypothesis on the symmetry of the operator A:

$$\forall\, v, w \in V : \ (Av, w) = (v, Aw) \tag{1.25}$$

We denote by $a(v,w)$ the bilinear form associated to A:

$$\forall\, v, w \in V, \quad a(v,w) = (Av, w)$$

$a(v,w)$ is a scalar product on V which induces an equivalent norm to $\|\cdot\|$.

It is well known that, if

$$f \in L^2(0,T;H)(^8) \quad \mathcal{P}u_0 \in \mathbb{K}, \tag{1.26}$$

then the solution u of problem 1.1 belongs to $H^1(0,T;H) \cap L^\infty(0,T;V)$, is continuous with values in $\mathbb{K}$ and satisfies the estimate:

$$\int_0^t |u'(\tau)|^2\, dt + a(u(t),u(t)) \le a(\mathcal{P}u_0, \mathcal{P}u_0) + \int_0^t |f(\tau)|^2\, d\tau, \qquad \forall\, t \in [0,T] \tag{1.27}$$

Therefore u is a strong solution.

Even if f does not satisfy any derivability hypothesis, one can expect an $O(\sqrt{k})$ error in the energy norm of $I(0,T)$: this estimate will be accomplished by a new smoothness result:

Theorem 4. *Assume that* (1.25) *and* (1.26) *hold true; then the functions $\hat{u}_k$ strongly converge to u in $H^1(0,T;H)$ as $k \to 0^+$; u belongs also to $H^{1/2}(0,T;V)$ and we have the estimates:*

$$\|u\|_{H^{1/2}(0,T;V)} \le C\left[\|f\|_{L^2(0,T;H)} + \|\mathcal{P}u_0\|\right] \tag{1.28}$$

$$\|u - \hat{u}_k\|_{I(0,T)} = o(k^{1/2}) \tag{1.29}$$

(^{8}) More generally, f can be chosen in $L^2(0,T;H) + BV(0,T;V')$; our proofs may be easily adapted to this case. On the other side, when $u_0 \in \mathbb{K}$ and $f \in BV(0,T;V')$, we can apply the intermediate convergence estimates of [1], and we can prove the last formula of the following theorem and the related regularity $u \in B^{1/2}_{2\infty}(0,T;V)$, whitout (1.25); so, we simplify our analysis considering only $L^2(0,T;H)$-data.

2 Proof of Theorem 1.

Preliminary remarks.

We limit ourselves to the case $T = +\infty$; otherwise we can extend the datum f to the whole interval $]0, +\infty[$ and consider the restriction to $]0, T[$ of the solution of the corresponding Cauchy problem: the extension and the restriction will preserve the regularity and the convergence estimates.

The regularity assumptions on $\{u_0, f\}$ are related to the behaviour of the quantity:

$$E_k = k^{1/2}\|u_1^k - u_0^k\| + |u_1^k - u_0^k| + |u_0^k - \mathcal{P}u_0| + \sup_{0\le\tau\le k} \|f(t+\tau) - f(t)\|_{S(0,\infty)} \tag{2.1}$$

A different way to formulate Theorem 1.3 is the following:

2.1 Lemma. *We have*

$$E_k \le C\left[\|f\|_{S(0,\infty)} + |u_0|\right], \qquad \lim_{k\to 0^+} E_k = 0 \tag{2.2}$$

and, if (1.16) *holds true,*

$$E_k \le C\,k\left[\|f\|_{H^1(0,\infty;V')+BV(0,\infty;H)} + |A\mathcal{P}u_0 - f(0)|\right] \tag{2.3}$$

Moreover the following estimate holds:

$$\|k\hat{u}_k'\|_{I(0,\infty)} \le C\,E_k; \qquad \|u - \hat{u}_k\|_{I(0,\infty)} \le C\,E_k \tag{2.4}$$

For the *proof* see [1]. ∎

It is useful to introduce the step functions $u_k(t), f_k(t)$ which take on $J_{k,n}$ the values u_{n+1}^k, f_{n+1}^k respectively (9).

On $J_{k,n}$ we have:

$$\hat{u}_k(t) = (n+1-t/k)u_n^k + (t/k - n)u_{n+1}^k$$

so that, introducing the k-periodic function $\ell_k(t)$ which takes the values t/k on $[0, k[$, the functions u_k and $\hat{u}_k$ are related by the following simple relations:

$$\hat{u}_k(t) = \ell_k(t)u_k(t) + (1-\ell_k(t))u_k(t-k); \qquad u_k(t) - \hat{u}_k(t) = k\,(1-\ell_k(t))\hat{u}_k'(t) \tag{2.5}$$

$$\hat{u}_k'(t) \equiv \frac{u_{n+1}^k - u_n^k}{k} = \frac{u_k(t) - u_k(t-k)}{k} \tag{2.6}$$

so that, if we are not interested to to the derivative of u, we can indifferently use u_k or $\hat{u}_k$, since (2.4) shows:

$$\|u_k - \hat{u}_k\|_{I(0,\infty)} \le C\,E_k, \qquad \|u_k - u\|_{I(0,\infty)} \le C\,E_k \tag{2.7}$$

Also f_k is strictly related to f via the E_k-estimate:

$$\|f - f_k\|_{S(0,\infty)} \le C\,E_k \tag{2.8}$$

u belongs to $B^{1/2}_{2\infty}(0, T; H)$.

We must estimate in $L^2(0,\infty; H)$ the difference $u(t+h) - u(t)$; we want to read this quantity on the dicretized equation and then to pass to the limit as $k \to 0^+$. The advantage is to use piecewise constant functions.

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(9) We extend this definition also for $t, n \le 0$ setting

$$u_n^k \equiv u_{-1}^k \qquad f_n^k \equiv f_0^k, \qquad n \le -1$$

and consequently:

$$u_k(t) \equiv \hat{u}_k(t) \equiv \mathcal{P}u_0,\ t < -k; \quad u_k(t) \equiv u_0^k,\ t \in [-k, 0[; \qquad f_k(t) \equiv f_0^k,\ t < 0$$

2.2 Lemma. *The function:*

$$h \mapsto \int_0^\infty |u_k(t+h) - u_k(t)|^2 \, dt \tag{2.9}$$

is linear on each $J_{k,n}$ and in pk takes the value:

$$\sum_{n=1}^{\infty} |u^k_{n+p} - u^k_n|^2 \tag{2.10}$$

Moreover, we have:

$$\sup_{h>0} h^{-1} \int_0^\infty |u_k(t+h) - u_k(t)|^2 \, dt = \sup_{p\in\mathbb{N}} p^{-1} \sum_{n=1}^{\infty} |u^k_{n+p} - u^k_n|^2 \tag{2.11}$$

Proof. Since u_k is constant on each $J_{k,n}$, if $h \in J_{k,p}$ we have:

$$\int_0^\infty |u_k(t+h) - u_k(t)|^2 \, dt = \sum_{n\geq 0} \int_{J_{k,n}} |u_k(t+h) - u_k(t)|^2 \, dt =$$
$$k(1-\ell_k(h)) \sum_{n\geq 1} |u^k_{n+p} - u^k_n|^2 + k\ell_k(h) \sum_{n\geq 1} \| u^k_{n+p+1} - u^k_n|^2 =$$

where $\ell_k(t)$ is the $k-$periodic function which on $J_{k,0}$ is t/k.

To show (2.11), let us suppose to know an uniform estimate of the type:

$$k \sum_{n=1}^{\infty} |u^k_{n+p} - u^k_n|^2 \leq C\, pk, \quad \forall\, p \in \mathbb{N},\ k > 0 \tag{2.12}$$

with C independent of p and k. For a generic $h \in J_{k,p}$ we get the bound:

$$\int_0^\infty |u_k(t+h) - u_k(t)|^2 \, dt \leq C(1-\ell_k(h))\, pk + C\ell_k(h)\,(p+1)k = C\, pk + C\, k\ell_k(h) = C\, h$$

since $k\,\ell_k(h) = h - p$. ∎

The proof of (2.12) is based on the following simple Lemma:

2.3 Lemma. *Let $\{v_n\}_{n\in\mathbb{N}}$ be a sequence in H; we have:*

$$\sum_{j=0}^{p-1} (v_{m+j+1} - v_{m+j}, v_{m+j+1} - v_m) = \frac{1}{2}|v_{m+p} - v_m|^2 + \frac{1}{2}\sum_{j=0}^{p-1} |v_{m+j+1} - v_{m+j}|^2 \tag{2.13}$$

Proof. We use the elementary identity:

$$2(y, y-x) = |x|^2 - |y|^2 + |x-y|^2, \qquad \forall\, x, y \in H \tag{2.14}$$

obtaining:

$$2\sum_{j=0}^{p-1} (v_{m+j+1} - v_{m+j}, v_{m+j+1} - v_m) =$$
$$2\sum_{j=0}^{p-1} (v_{m+j+1} - v_{m+j}, v_{m+j+1}) - 2\left(\sum_{j=0}^{p-1} v_{m+j+1} - v_{m+j}, v_m\right) =$$
$$\sum_{j=0}^{p-1} \left[|v_{m+j+1}|^2 - |v_{m+j}|^2 + |v_{m+j+1} - v_{m+j}|^2\right] + 2\,(v_m - v_{m+p}, v_m) =$$
$$\sum_{j=0}^{p-1} \left[|v_{m+j+1} - v_{m+j}|^2\right] + |v_m - v_{m+p}|^2 \quad ∎$$

2.4 THEOREM. *The solution u of 1.1 satisfies:*

$$\int_0^\infty |u(t+h)-u(t)|^2\,dt \le C\,h\big[\|f\|_{S(0,\infty)}+|u_0|\big]\,E_h \tag{2.15}$$

We choose $v=u_m^k$ in (0.8) written at $n=m+j$ and we sum for $j=0$ to $p-1$, obtaining:

$$|u_{m+p}^k-u_m^k|^2 \le 2k\left(\sum_{j=1}^{p} f_{m+j}^k - Au_{m+j}^k,\, u_{m+j}^k-u_m^k\right)$$

Finally, summing up with respect to m:

$$\begin{aligned}\sum_{m=1}^{\infty}|u_{m+p}^k-u_m^k|^2 \le& 2k\sum_{j=1}^{p}\int_0^\infty \big(f_k(t+jk)-Au_k(t+jk),u_k(t+jk)-u_k(t)\big)\,dt \le \\ & 2k\|f_k-Au_k\|_{S(0,\infty)}\sum_{j=1}^{p}\|u_k(t+jk)-u_k(t)\|_{I(0,\infty)} \le \\ & 2k\|f_k-Au_k\|_{S(0,\infty)}\sum_{j=1}^{p}\|u_k(t+jk)-u_k(t)\|_{I(0,\infty)} \le \\ & 2pk\|f_k-Au_k\|_{S(0,\infty)}\sup_{0\le\tau\le pk}\|u_k(t+\tau)-u_k(t)\|_{I(0,\infty)}\end{aligned}$$

and by the previous lemma:

$$\int_0^\infty |u_k(t+h)-u_k(t)|^2 \le C\,h\|f_k-Au_k\|_{S(0,\infty)}\sup_{0\le\tau\le h+k}\|u_k(t+\tau)-u_k(t)\|_{I(0,\infty)}$$

we get the first of the (1.19). Passing to the limit for $k\to 0^+$, we obtain:

$$\begin{aligned}\int_0^\infty |u(t+h)-u(t)|^2 \le& C\,h\|f-Au\|_{S(0,\infty)}\sup_{0\le\tau\le h}\|u(t+\tau)-u(t)\|_{I(0,\infty)} \le \\ \le& C\,h\big[\|f\|_{S(0,\infty)}+|u_0|\big]\,E_h\end{aligned}$$

and the last of the (1.19). ∎

3 Proof of Theorem 2.

The sketch of the proof is the following: first we exploit an important relation satisfied by u' (see the next Proposition) in order to obtain the strong convergence of $\hat{u}'_k$ in $L^2(0,T;H)$; then we construct from the sequence $\{u_n^k\}$ a more regular approximating function $\tilde{u}_k \in H^2(0,T;H)$ and we evaluate the growth of its norm when k goes to 0. Finally, we refine the first estimates, to give (1.20) and the related regularity for u.

3.1 Definition. *For every $w \in \mathbb{K}$, $\mathbb{K}_w$ will denote the closure in V of the cone:*

$$\{v \in V:\ \exists \lambda > 0,\ w + \lambda v \in \mathbb{K}\} = \bigcup_{\mu > 0} \mu(\mathbb{K} - w) \tag{3.1}$$

The importance of this concept is showed by the following:

3.2 Proposition ([3]). *Assume that* (1.16) *holds true; then for a.e.* $t \in [0,\infty[$ *we have:*

$$\begin{cases} u'(t) \in \mathbb{K}_{u(t)} \\ |u'(t)|^2 = (f(t) - Au(t), u'(t)) \\ \big(u'(t) + Au(t) - f(t), v\big) \geq 0, \quad \forall\, v \in \mathbb{K}_{u(t)} \end{cases} \tag{3.2}$$

3.3 Theorem. *Assuming* (1.16), *$\hat{u}'_k$ strongly converges to u' in $L^2(0,\infty;H)$.*

We know from Theorem 1.3 that

$$\hat{u}'_k \rightharpoonup^* u' \quad in\ I(0,\infty) \subset L^2(0,\infty;H) \tag{3.3}$$

so it remains to prove that ([10])

$$\limsup_{k \to 0^+} \|\hat{u}'_k\|^2_{L^2(0,\infty;H)} \leq \|u'\|^2_{L^2(0,\infty;H)} \tag{3.4}$$

The foundamental tool is to write (0.8) in the following continuous form thanks to (2.6):

$$\big(\hat{u}'_k(t) + Au_k(t) - f_k(t), u_k(t) - v\big) \leq 0, \qquad \forall\, v \in \mathbb{K}, \quad a.e.\ in\ ]0,+\infty[\tag{3.5}$$

Choosing $v = u_k(t-k)$ in this inequality and integrating over $[0,+\infty[$, we get:

$$\|\hat{u}'_k\|^2_{L^2(0,\infty;H)} \leq \int_0^\infty \big(f_k(t) - Au_k(t), \hat{u}'_k(t)\big)\, dt \tag{3.6}$$

By (2.8) and (2.7), $f_k - Au_k$ tends to $f - Au$ in $S(0,\infty)$ when k goes to 0, and by (3.3), ([2]) and (3.2) we conclude that:

$$\lim_{k \to 0^+} \|\hat{u}'_k\|_{L^2(0,\infty;H)} = \int_0^\infty \big(f(t) - Au(t), u'(t)\big)\, dt = \|u'\|_{L^2(0,\infty;H)} \quad \blacksquare$$

In order to obtain the stronger (1.20) we must find finer estimates on u', by means of the following variant of (3.2):

([10]) For a well known result on the uniformly convex spaces; on the other hand, in a Hilbert space $\mathcal{H}$ we can use (2.14).

3.4 NOTATION. *Given $v \in L^2(0,\infty;\mathcal{H})$ we denote by $[v]_h$ the Steklov averaging of v:*

$$[v]_h(t) = \frac{1}{h}\int_t^{t+h} v(\tau)\,d\tau, \qquad t \in \mathbb{R}^+, \quad h > 0 \tag{3.7}$$

3.5 PROPOSITION. *Let v be a function in $H^1(0,\infty;V)$, with $v(t) \in \mathbb{K}\ \forall t \in [0,+\infty[$ and u be the solution of Problem 0.1 with (1.16); then for a.e. $t > 0$ we have:*

$$\begin{aligned}&\big(u'(t) + Au(t) - f(t), u'(t) - [v']_h(t)\big) \le \\ &\quad \le \frac{1}{h}\Big\{\big(v'(t) + Av(t) - f(t), v(t) - u(t)\big) - \frac{1}{2}\frac{d}{dt}|u(t) - v(t)|^2 - \alpha\|u - v\|^2\Big\}\end{aligned} \tag{3.8}$$

Proof. Observe that:

$$[v']_h(t) = \frac{v(t+h) - v(t)}{h} = \frac{v(t+h) - u(t)}{h} + \frac{u(t) - v(t)}{h}$$

so that:

$$[v']_h(t) - \frac{u(t) - v(t)}{h} = \frac{v(t+h) - u(t)}{h} \in \mathbb{K}_{u(t)} \tag{3.9}$$

By proposition 3.2, we obtain:

$$\begin{aligned}&\big(u'(t) + Au(t) - f(t), u'(t) - [v']_h(t)\big) = \\ &= \Big(u'(t) + Au(t) - f(t), u'(t) - \frac{v(t+h) - u(t)}{h}\Big) - \\ &\qquad \Big(u'(t) + Au(t) - f(t), \frac{u(t) - v(t)}{h}\Big) \le \\ &\le \frac{1}{h}\big(u'(t) - v'(t) + Au(t) - Av(t), v(t) - u(t)\big) + \\ &\qquad + \frac{1}{h}\big(v'(t) + Av(t) - f(t), v(t) - u(t)\big) \quad \blacksquare\end{aligned}$$

At this point we are tempted to choose in (3.8) $h = k,\ v = \hat{u}_k$; before doing this substitution, we must study the behaviour of the term $[\hat{u}_k]_k$, so we introduce a new approximating function $\tilde{u}_k(t)$ which satisfies:

$$\tilde{u}_k'(t) = [\hat{u}_k']_k(t) \tag{3.10}$$

$\tilde{u}_k$ may be constructed by the piecewise quadratic real function $\psi_2(t)$ with support in $[-2,1]$:

$$\psi_2(t) = \begin{cases} \frac{(t+2)^2}{2} & \text{in } [-2,-1[\\ 1/2 - t(t+1) & \text{in } [-1,0[\\ \frac{(t-1)^2}{2} & \text{in } [0,1] \end{cases} \tag{3.11}$$

We use $\psi_2(t)$ to interpolate the values u_n^k in $H^2(0,\infty;H)$ ([11])

$$\tilde{u}_k(t) = \sum_{n=-1}^{\infty} u_n^k \psi_2(t/k - n) \tag{3.12}$$

([11]) By the same formula we may represent u_k and $\hat{u}_k$ choosing:

$$\psi_0(t) = \chi_{[-1,0[}(t); \qquad \psi_1(t) = \big(1 - |t|\big)^+$$

respectively. ψ_2 is constructed in such a way that:

$$\psi_2'(t) = \psi_1(t+1) - \psi_1(t); \qquad \int_{-\infty}^{+\infty} \psi_2(t)\,dt = 1$$

The consequences of this formula are simple relations between $\hat{u}_k$ and $\tilde{u}_k$, analogous to the ones of (2.5).

3.6 LEMMA. *We have the identities:*

$$\tilde{u}_k(t) = \frac{1}{2}u_k(t) + \frac{1}{2}\big[\ell_k(t)\hat{u}_k(t+k) + (1-\ell_k(t))\hat{u}_k(t)\big] \tag{3.13}$$

$$\tilde{u}_k'(t) = \frac{\hat{u}_k(t+k) - \hat{u}_k(t)}{k} = \ell_k(t)\hat{u}_k'(t+k) + (1-\ell_k(t))\hat{u}_k'(t) \tag{3.14}$$

$$\tilde{u}_k(t) - \hat{u}_k(t) = \frac{k}{2}\big[\ell_k(t)\tilde{u}_k'(t) + (1-\ell_k(t))\hat{u}_k'(t)\big] \tag{3.15}$$

$$\tilde{u}_k'(t) - \hat{u}_k'(t) = k\,\ell_k(t)\,\tilde{u}_k''(t) \tag{3.16}$$

Proof. Fix t in $J_{k,m}$ and consider $\tilde{u}_k(t)$ given by the sum (3.12): only the three terms relative to $n = m, m+1, m+2$ are different from 0, so that:

$$\begin{aligned}
&\tilde{u}_k(t) = \\
&\quad = u_m^k\frac{(t/k-m-1)^2}{2} + u_{m+1}^k[1/2-(t/k-m-1)(t/k-m)] + u_{m+2}^k\frac{(t/k-m)^2}{2} = \\
&\quad = \frac{(1-\ell_k)^2}{2}u_k(t-k) + [1/2+\ell_k(1-\ell_k)]u_k(t) + \frac{{\ell_k}^2}{2}u_k(t+k) = \\
&\quad = \frac{1}{2}u_k(t-k) + \frac{1}{2}\big[\ell_k(t)\hat{u}_k(t+k) + (1-\ell_k(t))\hat{u}_k(t)\big]
\end{aligned}$$

Taking the derivative and observing that ${\ell_k}'(t) = 1/k$ on each $J_{k,m}$, we get:

$$\begin{aligned}
\tilde{u}_k'(t) &= \frac{\hat{u}_k(t+k)-\hat{u}_k(t)}{2k} + \frac{\ell_k}{2k}\big[u_k(t+k)-u_k(t)\big] + \frac{1-\ell_k}{2k}\big[u_k(t)-u_k(t-k)\big] = \\
&= \frac{\hat{u}_k(t+k)-\hat{u}_k(t)}{k}
\end{aligned}$$

that is formula (3.14).

From (3.13) and (2.5), we have:

$$\begin{aligned}
\tilde{u}_k(t) - \hat{u}_k(t) &= \frac{1}{2}\big[u_k(t)-\hat{u}_k(t)\big] + + \frac{\ell_k(t)}{2}\big[\hat{u}_k(t+k)-\hat{u}_k(t)\big] = \\
&= \frac{k}{2}\big[(1-\ell_k)\hat{u}_k'(t) + \ell_k\tilde{u}_k'(t)\big]
\end{aligned}$$

and finally:

$$\tilde{u}_k'(t) - \hat{u}_k'(t) = \ell_k(t)\hat{u}_k'(t+k) - \ell_k(t)\hat{u}_k'(t) = k\ell_k(t)\hat{u}_k''(t) \quad \blacksquare$$

3.7 COROLLARY. $\tilde{u}_k$ *weakly* converges to* u *in* $H^1(0,\infty;V)\cap W^{1,\infty}(0,\infty;H)$ *and:*

$$\|u-\tilde{u}_k\|_{I(0,\infty)} \le C\,E_k, \qquad k\|\tilde{u}_k'\|_{I(0,\infty)} \le E_k \quad \blacksquare$$

Thanks to these identities, we give a first bound for the difference: $\|\tilde{u}_k'\|^2_{L^2(0,\infty;H)} - \|u'\|^2_{L^2(0,\infty;H)}$:

3.8 THEOREM. *If* (1.16) *holds true, we have:*

$$\|\tilde{u}_k'\|^2_{L^2(0,\infty;H)} - \|u'\|^2_{L^2(0,\infty;H)} \le C\,k\big[\|f\|_{H^1(0,\infty;V')+BV(0,\infty;H)} + |A\mathcal{P}u_0 - f(0)|\big] \tag{3.17}$$

Proof. Starting from (3.14) we get:

$$\begin{aligned}
&\int_0^\infty |\tilde{u}_k'(t)|^2\,dt = \\
&\quad = \frac{1}{3}\int_0^\infty |\hat{u}_k'(t+k)|^2\,dt + \frac{1}{3}\int_0^\infty |\hat{u}_k'(t)|^2\,dt + \frac{1}{3}\int_0^\infty \big(\hat{u}_k'(t+k), \hat{u}_k'(t)\big)\,dt \le \\
&\quad \le \int_0^\infty |\hat{u}_k'(t)|^2\,dt
\end{aligned} \tag{3.18}$$

and combining this formula with (3.6) and (3.2) we have ([12]):

$$\begin{aligned}
&\int_0^\infty |\tilde{u}_k'(t)|^2 - |u'(t)|^2\,dt \le \int_0^\infty |\hat{u}_k'(t)|^2 - |u'(t)|^2\,dt \le \\
&\quad \le \int_0^\infty \big(r_k(t), \hat{u}_k'(t)\big) - \big(r(t), u'(t)\big)\,dt = \\
&\quad = \int_0^\infty \big(r_k(t) - r(t), \hat{u}_k'(t)\big)\,dt + \int_0^\infty \big(r(t), \hat{u}_k'(t) - u'(t)\big)\,dt = \\
&\quad = \int_0^\infty \big(r_k(t) - r(t), \hat{u}_k'(t)\big)\,dt - \int_0^\infty \big(r'(t), \hat{u}_k(t) - u(t)\big)\,dt + \big(r(0), u_0^k - \mathcal{P}u_0\big) \le \\
&\quad \le C\,k\big[\|f\|_{\mathcal{H}(1,V')+BV(0,\infty;H)} + |A\mathcal{P}u_0 - f(0)|\big] \quad \blacksquare
\end{aligned}$$

In (3.5), we want to substitute the terms $u_k(t)$ with $\hat{u}_k(t)$; the following lemma gives a bound to the error we make:

3.9 LEMMA. *Let $v \in I(0,\infty)$ with $v(t) \in \mathbb{K}$ for a.e. $t \in [0,+\infty[$. We have:*

$$\int_0^\infty (\hat{u}_k'(t) + A\hat{u}_k(t) - f_k(t), \hat{u}_k(t) - v(t)) \le C\,{E_k}^2 + \|\hat{u}_k - v\|^2_{L^2(0,\infty;V)}$$

Proof. We split the integrand in the following way:

$$\begin{aligned}
&\big(\hat{u}_k'(t) + A\hat{u}_k(t) - f_k(t), \hat{u}_k(t) - v(t)\big) \le \\
&\quad \le \big(\hat{u}_k'(t) + Au_k(t) - f_k(t), \hat{u}_k(t) - v(t)\big) + (A\hat{u}_k(t) - Au_k(t), \hat{u}_k(t) - v(t)) \le \\
&\quad \le \big(\hat{u}_k'(t) + Au_k(t) - f_k(t), u_k(t) - v(t)\big) + \big(\hat{u}_k'(t) + Au_k(t) - f_k(t), \hat{u}_k(t) - u_k(t)\big) + \\
&\qquad + (A\hat{u}_k(t) - Au_k(t), \hat{u}_k(t) - v(t))
\end{aligned}$$

The first addendum in the last term is ≤ 0 by (3.5); in order to control the remaining ones, we recall that $\hat{u}_k(t) - u_k(t) = -k(1-\ell_k)\hat{u}_k'(t)$, obtaining:

$$(A\hat{u}_k(t) - Au_k(t), \hat{u}_k(t) - v(t)) \le M\,k\,(1-\ell_k)\|\hat{u}_k'(t)\|\,\|\hat{u}_k(t) - v(t)\| \le C\,{E_k}^2 + \|\hat{u}_k(t) - v(t)\|^2$$

([12]) In order to have shorter notations we will denote

$$r_k(t) = f_k(t) - Au_k(t); \qquad r(t) = f(t) - Au(t).$$

Remember that $\|r_k - r\|_{S(0,\infty)} \le C\,E_k$.

and:
$$\int_0^\infty (\hat{u}_k'(t) - r_k(t), \hat{u}_k(t) - v(t))\, dt \leq \frac{k}{2} \int_0^\infty (r_k(t) - \hat{u}_k'(t), \hat{u}_k'(t))\, dt$$

since r_k and $\hat{u}_k'$ are piecewise constant. Finally, observe that (3.5) is verified also for $t \in [-k, 0[$ and in particular we have (see previous note):

$$(r_k(t-k) - \hat{u}_k'(t-k), \hat{u}_k'(t)) \leq 0, \qquad for\ t \in [0, \infty[$$

With easy calculations:

$$\begin{aligned}
&\int_0^\infty \big(r_k(t) - \hat{u}_k'(t), \hat{u}_k'(t)\big)\, dt \leq \\
&\qquad \leq \int_0^\infty \big(r_k(t) - r_k(t-k) - [\hat{u}_k'(t) - \hat{u}_k'(t-k)], \hat{u}_k'(t)\big)\, dt \leq C\, \frac{{E_k}^2}{k} \quad \blacksquare
\end{aligned}$$

Now we can conclude our proof:

3.10 Theorem. *Assuming* (1.16), *the following estimate holds:*

$$\|u' - \tilde{u}_k'\|^2_{L^2(0,\infty;H)} \leq C\, k\big[\|f\|_{\mathcal{H}(1,V')+BV(0,\infty;H)} + |A\mathcal{P}u_0 - f(0)|\big] \tag{3.19}$$

Proof. We use the identity (2.14): thanks to Theorem 3.8 it remains to consider

$$\int_0^\infty \big(u'(t), u'(t) - \tilde{u}_k'(t)\big)\, dt$$

We use the proposition 3.5 via (3.10):

$$\begin{aligned}
&\int_0^\infty \big(u'(t) + Au(t) - f(t), u'(t) - \tilde{u}_k'(t)\big)\, dt \leq \\
&\quad \leq \frac{1}{k} \int_0^\infty \big(\hat{u}_k'(t) + A\hat{u}_k(t) - f_k(t), \hat{u}_k(t) - u(t)\big) + \big(f_k(t) - f(t), \hat{u}_k(t) - u(t)\big)\, dt + \\
&\qquad + \frac{1}{2k}\, |u(0) - \hat{u}_k(0)|^2
\end{aligned}$$

By the preceding lemma we deduce:

$$\begin{aligned}
&\int_0^\infty \big(u'(t), u'(t) - \tilde{u}_k'(t)\big)\, dt \leq \\
&\quad \leq \big(r(0), \mathcal{P}u_0 - u_0^k/2 - u_1^k/2\big) - \int_0^\infty \big(r'(t), u(t) - \tilde{u}_k(t)\big)\, dt + \\
&\qquad + \frac{1}{k} \int_0^\infty \big(r_k(t) - r(t), \hat{u}_k(t) - u(t)\big)\, dt + C\, \frac{{E_k}^2}{k} + \|\hat{u}_k - u\|^2_{L^2(0,\infty;V)} \leq \\
&\quad \leq C\, k\big[\|f\|_{\mathcal{H}(1,V')+BV(0,\infty;H)} + |A\mathcal{P}u_0 - f(0)|\big] \quad \blacksquare
\end{aligned}$$

3.11 Lemma. *There exist constants $C > 0$ such that:*

$$\left.\begin{aligned}
&\sqrt{3}\, \|\hat{u}_k' - \tilde{u}_k'\|_{L^2(0,\infty;H)} \\
=&k\, \|\tilde{u}_k''\|_{L^2(0,\infty;H)} \\
=&\|\hat{u}_k'(t+k) - \hat{u}_k'(t)\|_{L^2(0,\infty;H)}
\end{aligned}\right\} \leq C\, \frac{E_k}{\sqrt{k}} \tag{3.20}$$

Proof. We write (3.5) at the times t and $t+k$, choosing respectively $v = u_k(t+k)$ and $v = u_k(t)$; summing up, we get:

$$\big(\hat{u}_k'(t+k) - \hat{u}_k'(t), \hat{u}_k'(t+k)\big) \le \big(f_k(t+k) - f_k(t), \hat{u}_k'(t+k)\big), \qquad t \ge 0$$

integrating from 0 to $+\infty$ we obtain:

$$\int_0^\infty \big(\hat{u}_k'(t+k) - \hat{u}_k'(t), \hat{u}_k'(t+k)\big) \le \int_0^\infty \big(f_k(t+k) - f_k(t), \hat{u}_k'(t+k)\big) \le C\frac{{E_k}^2}{k}$$

Recalling (2.14), we get:

$$\begin{aligned}&\|\hat{u}_k'(t+k) - \hat{u}_k'(t)\|^2_{L^2(0,\infty;H)} = \\ &\quad = \int_0^\infty |\hat{u}_k'(t+k)|^2 - |\hat{u}_k'(t)|^2\, dt + 2\int_0^\infty \big(\hat{u}_k'(t+k) - \hat{u}_k'(t), \hat{u}_k'(t+k)\big) \le C\,\frac{{E_k}^2}{k}\end{aligned}$$

Since $\tilde{u}_k''$ is piecewise constant, by (3.16), we have:

$$\int_0^\infty |\tilde{u}_k'(t) - \hat{u}_k'(t)|^2\, dt = k^2 \int_0^\infty {\ell_k}^2(t)|\tilde{u}_k''(t)|^2\, dt = \frac{k^2}{3}\int_0^\infty |\tilde{u}_k''(t)|^2\, dt$$

that is (3.20). ■

3.12 Corollary. *The solution u belongs to $B^{3/2}_{2\infty}(0,\infty;H)$ and* (1.20) *holds true.*

Proof. (3.19) with (3.20) give immediately this regularity, if we prove the measurability of the family $\{\tilde{u}_k\}_{k>0}$. On the other hand, one can directly check that:

$$\begin{aligned}&\|u'(t+k) - u'(t)\|_{L^2(0,\infty;H)} \le \\ &\quad \le 2\|u' - \hat{u}_k'\|_{L^2(0,\infty;H)} + \|\hat{u}_k'(t+k) - \hat{u}_k'(t)\|_{L^2(0,\infty;H)} \le \\ &\quad \le C\,k\big[\|f\|_{\mathcal{H}(1,V')+BV(0,\infty;H)} + |A\mathcal{P}u_0 - f(0)|\big]\end{aligned}$$ ■

3.13 Remark. The whole family $\{\hat{u}_k\}$ is uniformly bounded in $B^2_{2\infty}(0,\infty;H)$. In fact, for $h > k$ we have:

$$\|\hat{u}_k'(t+h) - \hat{u}_k'(t)\|_{L^2(0,\infty;H)} \le 2\|\hat{u}_k'(t) - u'(t)\|_{L^2(0,\infty;H)} + \|u'(t+h) - u'(t)\|_{L^2(0,\infty;H)} \le C\,h$$

while for $h \le k$ the difference $\hat{u}_k'(t+h) - \hat{u}_k'(t)$ is equal to $[\hat{u}_k'(t+k) - \hat{u}_k'(t)]\chi_h(t)$ where χ_h is the characteristic function of $\cup_{n\ge1}[nk-h, nk[$. So:

$$\begin{aligned}\int_0^\infty \left|\hat{u}_k'(t+h) - \hat{u}_k'(t)\right|^2\, dt &= \sum_{n\ge1}\int_{nk-h}^{nk} \left|\hat{u}_k'(t+k) - \hat{u}_k'(t)\right|^2\, dt = \\ &= \sum_{n\ge1}\frac{h}{k}\int_{nk-k}^{nk} \left|\hat{u}_k'(t+k) - \hat{u}_k'(t)\right|^2\, dt = \\ &= \frac{h}{k}\int_0^\infty \left|\hat{u}_k'(t+k) - \hat{u}_k'(t)\right|^2\, dt \le C\,h\end{aligned}$$

since $\hat{u}_k'(t)$ is constant on each $J_{k,n-1} \supset [nk-h, nk[$. ■

4 Proof of Theorem 3.

We denote by $\mathcal{V}$ the subset of $H \times S(0,\infty)$ defined by (1.16), and we write:

$$\|\{u_0, f\}\|_{\mathcal{V}} = \|f\|_{H^1(0,\infty;V')+BV(0,\infty;H)} + |A\mathcal{P}u_0 - f(0)| \tag{4.1}$$

A interesting consequence of proposition 3.5 is the following:

4.1 Lemma. *Suppose that $u, v \in H^1(0,\infty;V) \cap W^{1,\infty}(0,\infty;H)$ are the solutions of problem 0.1 with respect to the data $\{u_0, f\}$, $\{v_0, g\} \in \mathcal{V}$ respectively. Then we have, $\forall\, h > 0$:*

$$\begin{aligned}
&\int_0^\infty \Big(u'(t) - v'(t) + Au(t) - Av(t) - [f(t) - g(t)], u'(t) - v'(t)\Big)\, dt \leq \\
&\qquad \leq \frac{|u_0 - v_0|^2}{h} + \frac{1}{h}\int_0^\infty 2\Big(f(t) - g(t), u(t) - v(t)\Big)\, dt + \\
&\qquad\quad + \int_0^\infty \Big(u'(t) + Au(t) - f(t), [v']_h(t) - v'(t)\Big)\, dt + \\
&\qquad\quad + \int_0^\infty \Big(v'(t) + Av(t) - g(t), [u']_h(t) - u'(t)\Big)\, dt
\end{aligned} \tag{4.2}$$

Proof. Starting from (3.8), we observe that the right hand member can be majorized by:

$$\frac{1}{h}\left\{\Big(g(t) - f(t), v(t) - u(t)\Big) + \frac{1}{2}\frac{d}{dt}|u(t) - v(t)|^2\right\}$$

In order to obtain (4.2), we change the role of u and v, we integrate from 0 to ∞ and we sum the two inequalities. ∎

Recalling that:

$$\frac{d}{dt}[v]_h(t) = [v']_h(t), \qquad \forall\, v \in H^1(0,\infty;\mathcal{H}),\ h > 0 \tag{4.3}$$

by simple calculations we deduce:

4.2 Corollary. *In the same hypothese of previous Lemma, u, v satisfy, $\forall\, h > 0$:*

$$\|u' - v'\|^2_{H^2(0,\infty;H)} \leq \begin{cases}
\Big[M\|u - v|_{I(0,\infty)} + \|f - g\|_{S(0,\infty)}\Big]\|u' - v'\|_{I(0,\infty)} + \\
\frac{1}{h}\Big[\|u - v\|_{I(0,\infty)}\|f - g\|_{S(0,\infty)} + |u_0 - v_0|^2\Big] + \\
\|f' - Au'\|_{S(0,\infty)}\|v - [v]_h\|_{I(0,\infty)} + \\
\|g' - Av'\|_{S(0,\infty)}\|u - [u]_h\|_{I(0,\infty)} + \\
|A\mathcal{P}u_0 - f(0)|\,|v(0) - [v]_h(0)| + |A\mathcal{P}v_0 - g(0)|\,|u(0) - [u]_h(0)| + \\
\int_0^\infty \big(u'(t), [v']_h(t) - v'(t)\big) + \big(v'(t), [u']_h(t) - u'(t)\big)\, dt
\end{cases} \tag{4.4}$$

We can conclude:

4.3 THEOREM. *The operator $T:\{u_0,f\}\mapsto u$, $\mathcal{V}\mapsto H^1(0,\infty;H)$ is 1/2-Hölder continuous with respect to the $H\times S(0,\infty)$-metric on the bounded subset of $\mathcal{V}$; more precisely, with the previous notations, it satisfies:*

$$\|u'-v'\|^2_{L^2(0,\infty;H)}\le C\left\{1+\|\{u_0,f\}\|^2_{\mathcal{V}}+\|\{v_0,g\}\|^2_{\mathcal{V}}\right\}\cdot\left[|u_0-v_0|+\|f-g\|_{S(0,\infty)}\right] \tag{4.5}$$

Proof. Let us consider (4.4); we observe that:

$$\|u'-v'\|_{I(0,\infty)},\ \|f'-Au'\|_{S(0,\infty)},\ \|g'-Av'\|_{S(0,\infty)},\ |A\mathcal{P}u_0-f(0)|,\ |A\mathcal{P}v_0-g(0)|$$

are bounded by $\|\{u_0,f\}\|_{\mathcal{V}}+\|\{v_0,g\}\|_{\mathcal{V}}$; furthermore, by the theorem 1.3, we have:

$$\|u-v\|_{I(0,\infty)}\le C\left[|u_0-v_0|+\|f-g\|_{S(0,\infty)}\right]$$

It remains to bound the terms $\|u-[u]_h\|_{I(0,\infty)}$, $|u(0)-[u]_h(0)|$, $\int_0^\infty\big(u'(t),[v']_h(t)-v'(t)\big)\,dt$ and those which are obtained changing u with v.

By well known result on approximation, we have:

$$\|u-[u]_h\|_{I(0,\infty)}\le h\,\|u'\|_{I(0,\infty)}\le C\,h\,\|\{u_0,f\}\|_{\mathcal{V}} \tag{4.6}$$

and:

$$|u(0)-[u]_h(0)|\le\frac{1}{h}\int_0^h|u(t)-u(0)|\,dt\le\frac{h}{2}\|u'\|_{L^\infty(0,h;H)}\le C\,h\,\|\{u_0,f\}\|_{\mathcal{V}} \tag{4.7}$$

Finally, we have:

$$\begin{aligned}\int_0^\infty&\big(u'(t),[v']_h(t)-v'(t)\big)\,dt=\\&=\int_0^\infty\big(u'(t)-v'(t),[v']_h(t)-v'(t)\big)+\big(v'(t),[v']_h(t)-v'(t)\big)\,dt\le\\&\le\frac14\|u'-v'\|^2_{L^2(0,\infty;H)}+\|v'-[v']_h\|^2_{L^2(0,\infty;H)}+\\&\qquad+\frac12\Big[\|[v']_h\|^2_{L^2(0,\infty;H)}-\|v'\|^2_{L^2(0,\infty;H)}-\|v'-[v']_h\|^2_{L^2(0,\infty;H)}\Big]\le\\&\le\frac14\|u'-v'\|^2_{L^2(0,\infty;H)}+\frac12\|v'-[v']_h\|^2_{L^2(0,\infty;H)}\end{aligned} \tag{4.8}$$

since:

$$\begin{aligned}\int_0^\infty|[v']_h(t)|^2\,dt&=\int_0^\infty\Big|\frac1h\int_0^h v'(t+\tau)\,d\tau\Big|^2\,dt\le\int_0^\infty\frac1h\int_0^h|v'(t+\tau)|^2\,d\tau\,dt\le\\&\le\frac1h\int_0^h d\tau\int_\tau^\infty|v'(t)|^2\,dt\le\int_0^\infty|v'(t)|^2\,dt\end{aligned}$$

The last term in (4.8) can be controlled by the $B^{1/2}_{2\infty}(0,\infty;H)$-norm of v' in the following way:

$$\begin{aligned}\|v'-[v']_h\|^2_{L^2(0,\infty;H)}&\le\int_0^\infty\frac1h\int_0^h|v'(t+\tau)-v'(t)|^2\,d\tau\,dt\le\\&\le\frac1h\int_0^h d\tau\int_0^\infty|v'(t+\tau)-v'(t)|^2\,dt\le\\&\le\frac1h\int_0^h\|v'\|^2_{B^{1/2}_{2\infty}(0,\infty;H)}\tau\,d\tau\le\frac{h}{2}\|v'\|^2_{B^{1/2}_{2\infty}(0,\infty;H)}\le\\&\le C\,h\,\|\{v_0,g\}\|^2_{\mathcal{V}}\end{aligned}$$

Combining this last result with (4.6), (4.7) and the preceding remarks, we get from (4.4): ([13])

$$\|u' - v'\|^2_{L^2(0,\infty;H)} \leq C\,\Delta\Big[a + b + \frac{\Delta}{h}\Big] + C\,ab\,h + \frac{1}{2}\|u' - v'\|^2_{L^2(0,\infty;H)} + \frac{a^2 + b^2}{2}\,h$$

Now choosing $h = \Delta = |u_0 - v_0| + \|f - g\|_{S(0,\infty)}$ we get (4.5). ∎

([13]) We set for the sake of semplicity:

$$\Delta = |u_0 - v_0| + \|f - g\|_{S(0,\infty)}, \qquad a = \|\{u_0, f\}\|_{\mathcal{V}}, \qquad b = \|\{v_0, g\}\|_{\mathcal{V}}$$

5 Proof of Theorem 4.

As in the preceding section we suppose $T = +\infty$; we begin with a lemma on the Hilbert transform:

5.1 DEFINITION. *Let v be a function in $L^2(0,\infty;\mathcal{H})$; the Hilbert transform Hv is the $L^2(-\infty,\infty;\mathcal{H})$-limit:*

$$Hv(t) = \lim_{\epsilon\to 0}\frac{1}{\pi}\int_{|\tau|\geq\epsilon}\frac{v(t-\tau)}{\tau}\,d\tau \tag{5.1}$$

This formula defines a linear isometry on $L^2(0,\infty;\mathcal{H})$ (see [10]).

5.2 LEMMA. *Let v be in $H^1(-\infty,\infty;H)$; then:*

$$\lim_{\epsilon\to 0^+}\int_\epsilon^\infty \frac{u(t+h)-2u(t)+u(t-h)}{h^2}\,dh = -\pi Hv'(t), \quad \text{in } L^2(-\infty,\infty;H)$$

Proof. We write the double difference in the integral as:

$$v(t+h)-2v(t)+v(t-h) = \int_0^h v'(t+\tau)-v'(t-\tau)\,d\tau$$

Before one passes to the limit in ϵ, it is possible to cahnge the order of integration, obtaining:

$$\begin{aligned}\int_\epsilon^\infty \frac{dh}{h^2}\int_0^h v'(t+\tau)-v'(t-\tau)\,d\tau &= \int_0^\infty d\tau[v'(t+\tau)-v'(t-\tau)]\int_{\epsilon\vee\tau}^\infty\frac{dh}{h^2} = \\ &= \int_0^\infty [v'(t+\tau)-v'(t-\tau)]\frac{d\tau}{\tau\vee\epsilon} = \\ &= -\int_{|\tau|\geq\epsilon}\frac{v'(t-\tau)}{\tau}\,d\tau + \frac{1}{\epsilon}\int_0^\epsilon v'(t+\tau)-v'(t-\tau)\,d\tau\end{aligned}$$

When $\epsilon\to 0$, the first term tends to $-\pi Hv'$ by (5.1), while the second tends to 0 in the $L^2(-\infty,\infty;H)$ norm. ■

Now we can prove the regularity result:

5.3 THEOREM. *If* (1.26) *and* (1.25) *hold true, the solution u of problem* 1.1 *belongs to $H^{1/2}(0,\infty;V)$ and satisfies:*

$$\int_0^\infty\frac{dh}{h^2}\int_0^\infty \|u(t+h)-u(t)\|^2\,dt \leq C\left[\|f\|^2_{L^2(0,\infty;H)}+\|\mathcal{P}u_0\|^2\right] \tag{5.2}$$

Proof. Choosing in (0.3) $v = u(t+h)$ and integrating in t on $[0,+\infty[$ we get:

$$\int_0^\infty a(u(t),u(t)-u(t+h))\,dt \leq \int_0^\infty (f(t)-u'(t),u(t)-u(t+h))\,dt, \quad \forall\, h>0 \tag{5.3}$$

Being a a symmetric form, we can apply (2.14):

$$\begin{aligned}\frac{1}{2}\int_0^h |||u(t)|||^2\,dt + \frac{1}{2}\int_0^\infty |||u(t+h)-u(t)|||^2\,dt \leq \\ \int_0^\infty (f(t)-u'(t),u(t)-u(t+h))\,dt, \quad \forall\, h>0\end{aligned}$$

From the other hand, choosing $v = u(t-h)$ ([14]) in (0.3) and repeating the same procedures we obtain:

$$-\frac{1}{2}\int_0^h |||u(t)|||^2\,dt + \frac{1}{2}\int_0^\infty |||u(t+h)-u(t)|||^2\,dt \leq \int_0^\infty (f(t)-u'(t), u(t)-u(t-h))\,dt$$

Summing with previous inequality we get:

$$\int_0^\infty |||u(t+h)-u(t)|||^2\,dt \leq \int_0^\infty (f(t)-u'(t), 2u(t)-u(t+h)-u(t-h))\,dt$$

Dividing by h^2 and integrating with respect to h from $\epsilon > 0$ to $+\infty$ we get:

$$\int_\epsilon^\infty \frac{dh}{h^2}\int_0^\infty |||u(t+h)-u(t)|||^2\,dt \leq \int_0^\infty \left(f(t)-u'(t), \int_\epsilon^\infty \frac{2u(t)-u(t+h)-u(t-h)}{h^2}\,dh\right)\,dt$$

When ϵ goes to 0, by previous Lemma the righthand member tends to

$$\pi\int_0^\infty (f(t)-u'(t), Hu'(t))\;dt \leq C[\|f\|^2_{L^2(0,\infty;H)} + \|\mathcal{P}u_0\|^2]$$

and we obtain (5.2). ∎

5.4 Remark. By the same calculations, we have:

$$\begin{cases} \sup\limits_{h>0} \dfrac{1}{h}\displaystyle\int_0^\infty \|u(t+h)-u(t)\|^2\,dt \leq C\,h\,[\|f\|^2_{L^2(0,\infty;H)} + \|\mathcal{P}u_0\|^2], \\ \lim\limits_{h\to 0^+} \dfrac{1}{h}\displaystyle\int_0^\infty \|u(t+h)-u(t)\|^2\,dt = 0 \end{cases}$$

On the other hand, these facts are consequences of the Semi-Groups and Interpolation Theory (see for instance [6]).

Now we can study the approximation of u by $\hat{u}_k$. Since the initial value $\mathcal{P}u_0$ belongs to $\mathbb{K}$, we can shift the functions $u_k, \tilde{u}_k, f_k$, so that $u_k(0) = u_0^k$, $\hat{u}_k(0) = \mathcal{P}u_0$ and $f_k(0) = f_0^k$; (3.5) is always verified.

The basic stability estimate can be easily obtained:

5.5 Proposition. *Assume* (1.26)*; then:*

$$\max\left[\alpha\|\hat{u}_k\|^2_{L^\infty(0,\infty;V)}, \alpha k\|\hat{u}_k'\|^2_{L^2(0,\infty;V)} + \|\hat{u}_k'\|^2_{L^2(0,\infty;H)}\right] \leq \|f\|^2_{L^2(0,\infty;H)} + |||\mathcal{P}u_0|||^2 \tag{5.4}$$

Moreover, the following integral relation holds true:

$$\int_0^T \left[|\hat{u}_k'(t)|^2 + \frac{k}{2}\,|||\hat{u}_k'(t)|||^2\right]dt + \frac{1}{2}|||\hat{u}_k(T)|||^2 \leq \frac{1}{2}|||\mathcal{P}u_0|||^2 + \int_0^T \left(f_k(t), \hat{u}_k'(t)\right)dt \tag{5.5}$$

([14]) We denote with u again the symmetric extension of u to the interval $]-\infty,+\infty[$.

Proof. Choosing $v = u_k(t-k)$ in (3.5) and integrating from 0 to T we have:

$$\int_0^T |\hat{u}_k'(t)|^2 + a(\hat{u}_k(t), \hat{u}_k'(t)) + a(u_k(t) - \hat{u}_k(t), \hat{u}_k'(t))\, dt \le \int_0^T \big(f_k(t), \hat{u}_k'(t)\big)\, dt$$

Recalling (2.5) we get:

$$\frac{1}{2}|||\hat{u}_k(T)|||^2 + \int_0^T |\hat{u}_k'(t)|^2 + k\,(1-\ell_k(t))|||\hat{u}_k'(t)|||^2\, dt \le \frac{1}{2}|||\mathcal{P}u_0|||^2 + \int_0^T \big(f_k(t), \hat{u}_k'(t)\big)\, dt$$

Finally, being $|||\hat{u}_k'(t)|||^2$ piecewise constant and positive, we get ([15]):

$$\int_0^T (1-\ell_k(t)|||\hat{u}_k'(t)|||^2\, dt \ge \frac{1}{2}\int_0^T |||\hat{u}_k'(t)|||^2\, dt$$

and (5.5); (5.4) follows from a simple application of a Schwartz inequality. ∎

Repeating the same arguments of Theorem 3.3 we have:

5.6 THEOREM. *$\hat{u}_k$ strongly converges to u in $H^1(0,\infty;H)$.*

Proof. We cannot use directly (3.2), but we have the integral formula:

$$\int_0^\infty |u'(t)|^2\, dt = \frac{1}{2}|||\mathcal{P}u_0|||^2 + \int_0^\infty \big(f(t), u'(t)\big)\, dt \tag{5.6}$$

In the discrete case, we consider (5.5), with $T=\infty$: since $\lim_{k\to 0}\|f_k - f\|_{L^2(0,\infty;H)} = 0$, we obtain:

$$\limsup_{k\to 0}\int_0^\infty |\hat{u}_k'(t)|^2\, dt \le \frac{1}{2}|||\mathcal{P}u_0|||^2 + \int_0^\infty \big(f(t), u'(t)\big)\, dt = \int_0^\infty |u'(t)|^2\, dt \quad ∎$$

5.7 COROLLARY.

$$\lim_{k\to 0} k\|\hat{u}_k'\|^2_{L^2(0,\infty;V)} = 0 \tag{5.7}$$

In order to prove (1.29), we give a discrete analogue of propositions 3.2 and 3.5, which hold without assuming the validity of (1.25):

5.8 THEOREM. *The discrete functions $u_k, \hat{u}_k$ satisfy:*

$$\big(\hat{u}_k'(t) + Au_k(t) - f_k(t), \hat{u}_k'(t) - w\big) \le 0, \qquad \forall\, w \in V : \hat{u}_k(t) + kw \in \mathbb{K} \tag{5.8}$$

Moreover, if $v \in H^1(0,\infty;H) \cap L^2(0,\infty;V)$, with $v(t) \in \mathbb{K}\ \forall\, t \in [0,\infty[$, we have, $\forall\, h \ge k$

$$\begin{aligned} \frac{d}{2\,dt}\Big[|||\hat{u}_k(t) - [v]_h(t)|||^2 + \frac{1}{h}|\hat{u}_k(t) - v(t)|^2\Big]|\hat{u}_k(t) - [v]_h(t)|^2 + \frac{1}{h}|||\hat{u}_k(t) - v(t)|||^2 \le \\ \le \big([v']_h(t) + A[v]_h(t) - f_k(t), [v']_h(t) - \hat{u}_k'(t)\big) + \\ + k(1-\ell_k(t))a(\hat{u}_k'(t), \hat{u}_k'(t) - [v']_h(t)) + \frac{1}{h}\big(v'(t) + Av(t) - f_k(t), v(t) - \hat{u}_k(t)\big) \end{aligned} \tag{5.9}$$

([15]) In general, if g is constant on each $J_{k,n}$ and $T = Nk$, we have:

$$\int_0^T (1-\ell_k(t))g(t)\, dt = \int_0^T \ell_k(t)g(t)\, dt = \frac{1}{2}\int_0^T g(t)\, dt$$

Proof. For (5.8), we have:

$$
\begin{aligned}
&\big(\hat u_k'(t)+Au_k(t)-f_k(t),\hat u_k'(t)-w\big)=\\
&\quad=\frac{1}{k}\big(\hat u_k'(t)+Au_k(t)-f_k(t),u_k(t)-u_k(t-k)-kw\big)=\\
&\quad=\frac{1}{k}\big(\hat u_k'(t)+Au_k(t)-f_k(t),\hat u_k(t)+u_k(t)-u_k(t-k)-[\hat u_k(t)+kw]\big)=\\
&\quad=\frac{1}{k}\big(\hat u_k'(t)+Au_k(t)-f_k(t),(1+\ell_k(t))u_k(t)-\ell_k(t)u_k(t-k)-[\hat u_k(t)+kw]\big)=\\
&\quad=\frac{1+\ell_k}{k}\big(\hat u_k'(t)+Au_k(t)-f_k(t),u_k(t)-\frac{\ell_k}{1+\ell_k}u_k(t-k)+\frac{1}{1+\ell_k}[\hat u_k(t)+kw]\big)\le 0
\end{aligned}
$$

since $\hat u_k(t)+kw$ belongs to $\mathbb{K}$ and

$$\frac{\ell_k(t)}{1+\ell_k(t)}u_k(t-k)+\frac{1}{1+\ell_k(t)}[\hat u_k(t)+kw]$$

is a convex combination of elements of $\mathbb{K}$.

At this point, the proof of (5.9) follows by the same calculations of (3.8); it remains the term:

$$k(1-\ell_k(t))\big(A\hat u_k'(t),\hat u_k'(t)-[v']_k(t)\big)=\big(Au_k(t)-A\hat u_k(t)\hat u_k'(t)-[v']_k(t)\big)$$

since we substituted in the right hand member of (5.8) $Au_k(t)$ by $A\hat u_k(t)$. ∎

5.9 Corollary. *Assume that* (1.25) *and* (1.26) *hold; we have:*

$$
\begin{aligned}
&\frac{1}{h}\left\{\frac{|v(T)-\hat u_k(T)|^2}{2}+\int_0^T |||v(t)-\hat u_k(t)|||^2\,dt\right\}+\\
&\int_0^T |[v]_h(t)-\hat u_k(t)|^2\,dt+|||[v]_h(T)-\hat u_k(T)|||^2\le\\
&\qquad\le\int_0^T\Big([v']_h(t)-f_k(t),[v']_h(t)-\hat u_k'(t)\Big)+\\
&\qquad+\frac{1}{2}\Big[|||[v]_h(T)|||^2-|||\hat u_k(T)|||^2-|||[v]_h(0)|||^2+|||\mathcal{P}u_0|||^2\Big]+\\
&\qquad+\int_0^T a(\hat u_k'(t),\hat u_k(t)-[v]_h(t))\,dt+\frac{1}{2k}|[v]_h(0)-\mathcal{P}u_0|^2+\\
&\qquad+\int_0^T\frac{1}{k}(v'(t)+Av(t)-f_k(t),v(t)-\hat u_k(t))+k|||\hat u_k'(t)|||^2-ka(\hat u_k'(t),[v]_h(t))\,dt
\end{aligned}
$$

Proof. Starting from (5.9), we integrate from 0 to T; the only term we modify is:

$$
\begin{aligned}
&\int_0^T a([v]_h(t), [v']_h(t) - \hat{u}_k'(t))\, dt = \\
&\quad = a([v]_h(T), [v]_h(T) - \hat{u}_k(T)) - a([v]_h(0), [v]_h(0) - \mathcal{P}u_0) + \\
&\qquad + \int_0^T a([v']_h(t), \hat{u}_k(t) - [v]_h(t))\, dt = \\
&\quad = a([v]_h(T), [v]_h(T) - \hat{u}_k(T)) - a([v]_h(0), [v]_h(0) - \mathcal{P}u_0) + \\
&\qquad + \int_0^T a([v']_h(t) - \hat{u}_k(t), \hat{u}_k(t) - [v]_h(t))\, dt + \int_0^T a(\hat{u}_k'(t), \hat{u}_k(t) - [v]_h(t))\, dt = \\
&\quad \le \frac{1}{2}\Big[|||[v]_h(T)|||^2 - |||\hat{u}_k(T)|||^2 - |||[v]_h(0)|||^2 + |||\hat{u}_k(0)|||^2 \Big] + \\
&\qquad + \int_0^T a(\hat{u}_k'(t), \hat{u}_k(t) - [v]_h(t))\, dt
\end{aligned}
$$

5.10 Theorem. *Assume that* (1.25) *and* (1.26) *hold; then we have:*

$$
\lim_{k \to 0^+} \|u - \hat{u}_k\|_{L^\infty(0,\infty;V)} = 0 \tag{5.10}
$$

and:

$$
\|u - \hat{u}_k\|^2_{L^\infty(0,\infty;H) \cap L^2(0,\infty;V)} \le C\, k \Big[\|\mathcal{P}u_0\|^2 + \|f\|_{L^2(0,\infty;H)} \Big]; \quad \lim_{k \to 0^+} \frac{1}{k} \|u - \hat{u}_k\|^2_{L^\infty(0,\infty;H) \cap L^2(0,\infty;V)} = 0
$$

Proof. We choose in 5.9 $h = k$, $v = u$ and recall that:

$$
\|[u']_k\|_{L^2(0,\infty;H)} \le \|u'\|_{L^2(0,\infty;H)}; \qquad \lim_{k \to 0^+} [u]_k = u \textit{ in } L^\infty(0,\infty;V) \cap H^1(0,\infty;H)
$$

and:

$$
\|u - [u]_k\|_{L^2(0,\infty;V) \cap L^\infty(0,\infty;H)} = o(k^{1/2})
$$

$$
\|u - [u]_k\|_{L^2(0,\infty;V)} = o(k^{1/2},
$$

References

6 References

[1] C. Baiocchi, *Discretizzazione di problemi parabolici*, to appear in Ricerche di Matematica.

[2] C. Baiocchi, F. Brezzi, *Optimal error estimates for linear parabolic problems under minimal regularity assumptions*, Calcolo **20** (1983), 143-176.

[3] J. Bergh, J. Löfström, *Interpolation Spaces* (1976), Springer Verlag, Berlin.

[4] G. Dahlquist, *A special problem for linear multistep methods*, BIT **3** (1963), 27-43.

[5] J. L. Lions, E. Magenes, *Non homogeneus boundary value problems and applications I* (1972), Springer Verlag, Berlin.

[6] J. L. Lions, J. Peetre, *Sur une classe d'espaces d'interpolation*, Inst. des Hautes Études Scientifiques Publ. Math. **19** (1964), 5-68.

[7] O. Nevanlinna, F. Odeh, *Multiplier techniques for linear multistep methods*, Numer. funct. anal. and optimiz. **3** (1981), 377-423.

[8] W. Rudin, *Real and Complex Analysis* (1974), McGraw-Hill, New York.

[9] G. Savaré, *Discretizzazioni $A(\Theta)-$stabili di equazioni differenziali astratte*, to appear in Calcolo.

[10] H. J. Stetter, *Analysis of discretization methods for ordinary differential equations* (1973), Springer Verlag, Berlin.

[11] F. Tomarelli, *Weak solutions for an abstract Cauchy problem of parabolic type*, Annali di Matematica Pura e Applicata IV **130** (1983), 93-123.

[12] F. Tomarelli, *Regularity theorems and optimal error estimates for linear parabolic Cauchy problems*, Numerische Mathematik **45** (1984), 23-50.

[13] O. B. Widlund, *A note on unconditionally stable linear multistep methods*, BIT **7** (1967), 65-70.