

MULTISCALE MODELING FOR THE BIOELECTRIC ACTIVITY OF THE HEART *

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Abstract. The paper deals with the mathematical models for the electrical activity of the heart at the micro- and macroscopic level. By using the tools of the Γ -convergence theory, a rigorous mathematical derivation of the macroscopic model called “bidomain” directly from the microscopic properties of the tissue is presented.

Key words. Homogenization, Γ -convergence, degenerate evolution equations, Reaction-Diffusion systems, FitzHugh-Nagumo dynamic, cardiac electric field, bidomain model

AMS subject classifications. 35B27, 35K57, 35K65, 92C30, 93A30

1. Introduction. The aim of this work is to study, in the framework of the Γ -convergence theory, the asymptotic behaviour of the problem, modeling at the microscopic level (i.e. at the cellular level), the bioelectric activity of the heart.

The cardiac tissue is composed of an arrangement of elongated cells physically interconnected by specialized membrane structures of densely packed channels called *gap junctions*, see e.g. [29, 20]. The *gap junction* channels connect the cytoplasmatic compartments of adjacent cells and allow the intercellular flow of ionic currents. The intercellular communication between cardiac myocytes occurs in a end-to-end orientation and in a side-to-side apposition. At a cellular level the tissue can be viewed as composed by two conducting volumes, the intra- and extracellular spaces, separated by the cellular membrane. The two spaces are considered ohmic conducting media and since the junctional resistance between two adjoining cells is different from the myoplasm of either cell, the intracellular conductivity is space dependent. The microscopic mathematical model consists of a system of two partial differential equations of elliptic type and the unknown functions are the intra- and extracellular electric potentials. These equations are coupled by means of a distinctive evolutive boundary condition in the potential jump at the interface separating the two media, i.e. the cellular membrane. The problem, written in a non-dimensional form, contains a small parameter ε related to the microstructure.

In spite of the discrete cellular structure, it is well known that the cardiac tissue can be represented by a continuous model, called *bidomain model* which attempts to describe the averaged electric potentials and current flows inside (intra) and outside (extracellular space) the cardiac cells. Despite the widespread use of the macroscopic bidomain model, its rigorous derivation directly from the microscopic properties of the tissue is still lacking. Formally, the macroscopic equations can be obtained from the microscopic ones by multiple scale expansion and averaging. For instance, a first formal derivation, based on current balances and expressed by averages of integral identities, was obtained in [27]. By standard multiscale arguments of homogenization the same formal derivation can be found also in the Appendix of [15] or in [23, 24].

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We investigate the homogenization limit when $\varepsilon \rightarrow 0$ under the simplifying assumption that cardiac cells are arranged in a periodic box structure. In this work the micro- and the macroscopic structure of the cardiac tissue is studied by using the tools of the Γ -convergence theory and a rigorous mathematical derivation of the limit problem at the tissue level (i.e. the bidomain model) is presented.

The microscopic model of the cardiac tissue.

Intra- and extracellular regions. At a microscopic level the cardiac tissue Ω (a bounded Lipschitz open subset of \mathbb{R}^d , $d = 3$) is composed by a collection of elongated cardiac cells, connected end-to-end and/or side-to-side by junctions, surrounded by the extra-cellular fluid. The end-to-end contacts forms the long fibers structure of the cardiac muscle whereas the presence of lateral junctions establishes a connection between the elongated fibers.

We can consider the cardiac tissue Ω as composed of two *connected* regions,

the *intra-cellular one* (inside the cells) Ω_i^ε , separated from

the *extra-cellular one* (fluid outside the cells) Ω_e^ε

by a membrane surface Γ^ε ; thus $\Omega = \Omega_i^\varepsilon \cup \Omega_e^\varepsilon \cup \Gamma^\varepsilon$,

and $\Gamma^\varepsilon = \partial\Omega_i^\varepsilon \cap \partial\Omega_e^\varepsilon$ is the common part of the two boundaries of $\Omega_{i,e}^\varepsilon$.

Here $\varepsilon > 0$ is a small dimensionless parameter (whose precise definition in terms of the various physical constants will be discussed in the Appendix) which is proportional to the ratio between the “*micro*” scale of the length of the cells and the “*macro*” scale of the length of the cardiac fibers.

The periodic lattice of the cells. Following the standard approach of the homogenization theory, we are assuming that the cells are distributed according to an ideal periodic organization similar to a regular lattice of interconnected cylinders.

If $\mathbf{e}_1, \dots, \mathbf{e}_d$ is an orthogonal basis of \mathbb{R}^d , we denote by

$$E_i, \quad E_e := \mathbb{R}^d \setminus \overline{E_i}, \quad \text{with common boundary } \Gamma := \partial E_i \cap \partial E_e,$$

two reference open, *connected and periodic* subsets of \mathbb{R}^d with Lipschitz boundary, i.e. satisfying

$$E_{i,e} + \mathbf{e}_k = E_{i,e}, \quad k = 1, \dots, d. \quad (1.1)$$

The elementary periodicity region

$$Y := \left\{ \sum_{k=1}^d \alpha_k \mathbf{e}_k : 0 \leq \alpha_k < 1, \quad k = 1, \dots, d \right\} \quad (1.2)$$

with its intra- and extra-cellular part $Y_{i,e} = Y \cap E_{i,e}$ represents a reference unit volume box containing a single cell Y_i . The main geometrical assumption is that the physical intra- or extra-cellular regions are the ε -dilation of the reference lattices $E_{i,e}$, defined as

$$\varepsilon E_{i,e} = \{ \varepsilon \xi : \xi \in E_{i,e} \}, \quad \text{with } \varepsilon \Gamma := \{ \varepsilon \xi : \xi \in \Gamma \}, \quad (1.3)$$

and therefore the decomposition of the physical region Ω occupied by the heart into the intra- and extra-cellular domains $\Omega_{i,e}^\varepsilon$ (see e.g. Fig. 1.1) can be obtained simply by intersecting Ω with $\varepsilon E_{i,e}$, i.e.

$$\Omega_i^\varepsilon = \Omega \cap \varepsilon E_i, \quad \Omega_e^\varepsilon = \Omega \cap \varepsilon E_e, \quad \Gamma^\varepsilon = \Omega \cap (\partial\Omega_i^\varepsilon \cap \partial\Omega_e^\varepsilon) = \Omega \cap \varepsilon \Gamma. \quad (1.4)$$

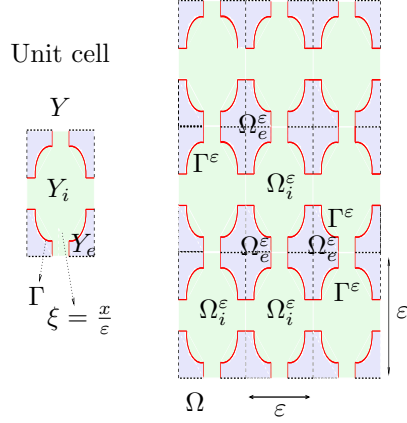


FIG. 1.1. Right. The ideal periodic geometry in a bidimensional section of the simplified 3-D periodic network of interconnected cells. Left. Unit cell in the microscopic variable $\xi = x/\varepsilon$.

Unknowns and equations. The electric properties of the tissue are described by the couple

$$\underline{u}^\varepsilon = (u_i^\varepsilon, u_e^\varepsilon), \quad u_{i,e}^\varepsilon : \Omega_{i,e}^\varepsilon \rightarrow \mathbb{R}, \quad \text{of intra- and extra-cellular potentials,} \quad (1.5)$$

each one admitting a trace $u_{i,e}^\varepsilon|_{\Gamma^\varepsilon}$ on Γ^ε , whose difference

$$v^\varepsilon := u_i^\varepsilon|_{\Gamma^\varepsilon} - u_e^\varepsilon|_{\Gamma^\varepsilon} : \Gamma^\varepsilon \rightarrow \mathbb{R} \quad \text{is the transmembrane potential,} \quad (1.6)$$

(in the following, we will simply write $v^\varepsilon = u_i^\varepsilon - u_e^\varepsilon$ on Γ^ε) and satisfies a *dynamic condition* on Γ^ε involving auxiliary functions

$$w^\varepsilon : \Gamma^\varepsilon \rightarrow \mathbb{R}^h, \quad \text{the so called gating (or recovery) variables.} \quad (1.7)$$

We denote by $\sigma_{i,e}^\varepsilon$ suitably rescaled symmetric conductivity matrices

$$\sigma_{i,e}^\varepsilon(x) = \sigma_{i,e}(x, \frac{x}{\varepsilon}), \quad (1.8)$$

obtained by continuous functions $\sigma_{i,e}(x, \xi) : \Omega \times E_{i,e} \rightarrow \mathbb{M}^{d \times d}$ satisfying the usual uniform ellipticity and periodicity conditions

$$\begin{aligned} \sigma|y|^2 &\leq \sigma_{i,e}(x, \xi)y \cdot y \leq \sigma^{-1}|y|^2 \\ \sigma_{i,e}(x, \xi + e_k) &= \sigma_{i,e}(x, \xi) \end{aligned} \quad \forall (x, \xi) \in \Omega \times E_{i,e}, \quad y \in \mathbb{R}^d, \quad (1.9)$$

for a given constant $\sigma > 0$; $\nu_{i,e}^\varepsilon$ are the exterior unit normals to the boundaries of $\Omega_{i,e}^\varepsilon$: observe that $\nu_i^\varepsilon = -\nu_e^\varepsilon$ on Γ^ε .

We can formulate the Reaction-Diffusion system satisfied by the vector $(u_i^\varepsilon, u_e^\varepsilon, w^\varepsilon)$, with $v^\varepsilon = u_i^\varepsilon - u_e^\varepsilon$ on Γ^ε , in the following way

$$\begin{aligned} -\operatorname{div}(\sigma_{i,e}^\varepsilon \nabla u_{i,e}^\varepsilon) &= 0 && \text{in } \Omega_{i,e}^\varepsilon \times (0, T), && (\text{quasi-stationary conduction}) && (P_{1a}^\varepsilon) \\ \left. \begin{aligned} -\sigma_i^\varepsilon \nabla u_i^\varepsilon \cdot \nu_i^\varepsilon \\ \sigma_e^\varepsilon \nabla u_e^\varepsilon \cdot \nu_e^\varepsilon \end{aligned} \right\} &= I_m^\varepsilon && \text{on } \Gamma^\varepsilon \times (0, T), && (\text{continuity equation}) && (P_{1b}^\varepsilon) \\ \varepsilon(\partial_t v^\varepsilon + I(v^\varepsilon, w^\varepsilon)) &= I_m^\varepsilon && \text{on } \Gamma^\varepsilon \times (0, T), && (\text{reaction surface condition}) && (P_2^\varepsilon) \\ \partial_t w^\varepsilon + r(v^\varepsilon, w^\varepsilon) &= 0 && \text{on } \Gamma^\varepsilon \times (0, T), && (\text{dynamic coupling}) && (P_3^\varepsilon) \end{aligned}$$

supplemented by the boundary and initial conditions

$$\begin{aligned} \sigma_{i,e}^\varepsilon \nabla u_{i,e}^\varepsilon \cdot \nu_{i,e} &= 0 && \text{on } (\partial\Omega_{i,e}^\varepsilon \setminus \Gamma^\varepsilon) \times (0, T), && (P_4^\varepsilon) \\ v^\varepsilon(\cdot, 0) &= v_0^\varepsilon && \text{on } \Gamma^\varepsilon, && (P_5^\varepsilon) \\ w^\varepsilon(\cdot, 0) &= w_0^\varepsilon && \text{on } \Gamma^\varepsilon, && (P_6^\varepsilon) \end{aligned}$$

where the coupling terms $I(v^\varepsilon, w^\varepsilon)$ (the *membrane ionic current*) and $r(v^\varepsilon, w^\varepsilon)$ depend on the particular model of the ionic flux through the cellular membrane chosen.

Here we are mainly concerning with the so called *FitzHugh–Nagumo model*, first introduced as a simplified membrane kinetic of the Hodgkin–Huxley equations for the transmission of nervous electric impulses (see e.g. [12, 18]), that requires only one scalar *recovery variable* w^ε (thus $h = 1$ in (1.7)). Therefore, I and r take the form:

$$I(v^\varepsilon, w^\varepsilon) := F(v^\varepsilon) + \Theta w^\varepsilon, \quad r(v^\varepsilon, w^\varepsilon) := \gamma w^\varepsilon - \eta v^\varepsilon, \quad (P_7^\varepsilon)$$

where Θ, γ, η are nonnegative constants, and

$$F \in C^1(\mathbb{R}) \quad \text{is a cubic-like function with} \quad \inf_{x \in \mathbb{R}} F'(x) > -\infty. \quad (1.11)$$

If $(u_i^\varepsilon, u_e^\varepsilon, w^\varepsilon)$ is a solution of this system, it is easy to check that $(u_i^\varepsilon + c, u_e^\varepsilon + c, w^\varepsilon)$ is still a solution, where $c = c(t)$ is an arbitrary family of additive time-dependent constants. We avoid the use of quotient spaces by fixing a reference open subdomain

$$\Omega_0 \subset\subset \Omega \quad \text{with} \quad \mathcal{L}^d(\partial\Omega_0) = 0, \quad \mathcal{L}^d(\Omega_0 \cap \Omega_e^\varepsilon) > 0, \quad (1.12)$$

(here \mathcal{L}^d denotes the usual Lebesgue measure on \mathbb{R}^d) and imposing that

$$\int_{\Omega_e^\varepsilon \cap \Omega_0} u_e^\varepsilon(x) dx = 0. \quad (P_8^\varepsilon)$$

We refer to the system $(P_{1a}^\varepsilon, P_{1b}^\varepsilon, \dots, P_8^\varepsilon)$ as the **(microscopic or cellular) Problem P^ε** .

Well posedness of P^ε and Energy estimate. The well posedness of Problem P^ε in suitable function spaces has been studied in [15], whose main result we will report in the next Section 4; here we only recall one of the basic a priori estimates, which involves the Energy-like functionals

$$\underline{a}^\varepsilon(\underline{u}^\varepsilon) := \sum_{i,e} \int_{\Omega_{i,e}^\varepsilon} \sigma_{i,e}^\varepsilon \nabla u_{i,e}^\varepsilon \cdot \nabla u_{i,e}^\varepsilon dx, \quad \underline{u}^\varepsilon = (u_i^\varepsilon, u_e^\varepsilon), \quad (1.13a)$$

$$b^\varepsilon(v^\varepsilon) := \varepsilon \int_{\Gamma^\varepsilon} |v^\varepsilon|^2 d\mathcal{H}^{d-1}, \quad b^\varepsilon(w^\varepsilon) := \varepsilon \int_{\Gamma^\varepsilon} |w^\varepsilon|^2 d\mathcal{H}^{d-1}, \quad (1.13b)$$

$$\phi^\varepsilon(v^\varepsilon) := \varepsilon \int_{\Gamma^\varepsilon} \varphi(v^\varepsilon) d\mathcal{H}^{d-1}, \quad (1.13c)$$

$$j^\varepsilon(v^\varepsilon) := \inf \left\{ \underline{a}^\varepsilon(\underline{u}^\varepsilon) : u_{i,e}^\varepsilon \in H^1(\Omega_{i,e}^\varepsilon), \quad u_i^\varepsilon - u_e^\varepsilon = v^\varepsilon \text{ on } \Gamma^\varepsilon \right\}, \quad (1.13d)$$

where φ is a positive, convex, primitive function of $x \mapsto F(x) + \lambda_F x$ for a sufficiently big $\lambda_F > -\inf_{x \in \mathbb{R}} F'(x)$ (see (4.1)) and \mathcal{H}^{d-1} denotes the usual $(d-1)$ -dimensional Hausdorff measure.

For every $v_0^\varepsilon, w_0^\varepsilon \in L^2(\Gamma^\varepsilon)$ (the L^2 space with respect to \mathcal{H}^{d-1}) with $j(v_0^\varepsilon) < +\infty$, there exists a unique variational solution $u_{i,e}^\varepsilon, w^\varepsilon, v^\varepsilon = u_i^\varepsilon - u_e^\varepsilon$ on Γ^ε , of problem P^ε satisfying the uniform energy bound

$$\sup_{t \in [0, T]} \left(\underline{a}^\varepsilon(\underline{u}^\varepsilon) + b^\varepsilon(w^\varepsilon) + \phi^\varepsilon(v^\varepsilon) \right) \leq C \left(j^\varepsilon(v_0^\varepsilon) + b^\varepsilon(w_0^\varepsilon) + \phi^\varepsilon(v_0^\varepsilon) \right), \quad (1.14)$$

for a constant $C = C(T, \lambda_F, \eta, \Theta)$ independent of ε .

Convergence to the solution of the macroscopic bidomain model..

One possible way to describe the asymptotic behavior of $u_{i,e}^\varepsilon, v^\varepsilon, w^\varepsilon$ as $\varepsilon \downarrow 0$ is to consider local averages. First of all we denote by $\beta_{i,e}, \beta$ the asymptotic local ratios (uniform in space) of the intra/extra cellular volumes and of the membrane surface area to the volume occupied by the tissue, i.e.

$$\begin{aligned}\beta_{i,e} &:= \lim_{\varepsilon \downarrow 0} \frac{\mathcal{L}^d(\Omega_{i,e}^\varepsilon \cap B_\rho(x))}{\mathcal{L}^d(\Omega \cap B_\rho(x))} = \frac{\mathcal{L}^d(Y_{i,e})}{\mathcal{L}^d(Y)}, \\ \beta &:= \lim_{\varepsilon \downarrow 0} \frac{\varepsilon \mathcal{H}^{d-1}(\Gamma^\varepsilon \cap B_\rho(x))}{\mathcal{L}^d(\Omega \cap B_\rho(x))} = \frac{\mathcal{H}^{d-1}(\Gamma \cap Y)}{\mathcal{L}^d(Y)},\end{aligned}\quad \forall \rho > 0, x \in \Omega. \quad (1.15)$$

Taking into account the *a priori* bound (1.14), we will introduce the following definition (where z^ε stays for either the transmembrane potential v^ε or the recovery variable w^ε).

DEFINITION 1.1 (A weak notion of convergence). *Let $u_{i,e}^\varepsilon \in L_{loc}^1(\Omega_{i,e}^\varepsilon), z^\varepsilon \in L_{loc}^1(\Gamma^\varepsilon), \varepsilon > 0$, be given families of functions. We say that $u_{i,e}^\varepsilon$ converges to $u_{i,e} \in L_{loc}^1(\Omega)$ and z^ε converges to $z \in L_{loc}^1(\Omega)$ as $\varepsilon \downarrow 0$ if for every test function $\zeta \in C_c^0(\Omega)$ we have*

$$\lim_{\varepsilon \downarrow 0} \int_{\Omega_{i,e}^\varepsilon} u_{i,e}^\varepsilon(x) \zeta(x) dx = \beta_{i,e} \int_{\Omega} u(x) \zeta(x) dx, \quad (1.16)$$

$$\lim_{\varepsilon \downarrow 0} \varepsilon \int_{\Gamma^\varepsilon} z^\varepsilon(x) \zeta(x) d\mathcal{H}^{d-1}(x) = \beta \int_{\Omega} z(x) \zeta(x) dx. \quad (1.17)$$

We say that a vector $(u_i^\varepsilon, u_e^\varepsilon, z^\varepsilon) \in L_{loc}^1(\Omega_i^\varepsilon) \times L_{loc}^1(\Omega_e^\varepsilon) \times L_{loc}^1(\Gamma^\varepsilon)$ is converging to $(u_i, u_e, z) \in (L_{loc}^1(\Omega))^3$ if each component is converging to the corresponding one according to (1.16) and (1.17).

REMARK 1.2 (Weak* convergence of the associated measures). *The above formulae correspond to consider the local weak* convergence in the sense of (signed) measures: to clarify this point, we introduce the reference positive measures*

$$\lambda_{i,e}^\varepsilon := \mathcal{L}^d|_{\Omega_{i,e}^\varepsilon}, \quad \lambda^\varepsilon := \varepsilon \mathcal{H}^{d-1}|_{\Gamma^\varepsilon}, \quad \lambda_{i,e} := \beta_{i,e} \mathcal{L}^d|_{\Omega}, \quad \lambda := \beta \mathcal{L}^d|_{\Omega}, \quad (1.18)$$

and the Radon measures

$$\tilde{u}_{i,e}^\varepsilon := u_{i,e}^\varepsilon \cdot \lambda_{i,e}^\varepsilon, \quad \tilde{z}^\varepsilon := z^\varepsilon \cdot \lambda^\varepsilon \quad (1.19)$$

whose densities are $u_{i,e}^\varepsilon$ and z respectively. The convergence introduced in Definition 1.1 is then equivalent to ask that

$$\tilde{u}_{i,e}^\varepsilon \rightharpoonup^* \tilde{u}_{i,e} := u_{i,e} \lambda_{i,e}, \quad \tilde{z}^\varepsilon \rightharpoonup^* \tilde{z} := z \cdot \lambda \quad (1.20)$$

in the local weak* topology of the space of Radon measures [4, Definition 1.58]. Since the microscopic problem is strictly related to the energy functionals (1.13a), (1.13b), it is natural to introduce the corresponding homogenized ones, which are defined by

$$\underline{a}(\underline{u}) := \sum_{i,e} \int_{\Omega} M_{i,e}(x) \nabla u_{i,e}(x) \cdot \nabla u_{i,e}(x) dx, \quad \underline{u} := (u_i, u_e), \quad (1.21a)$$

$$b(v) := \beta \int_{\Omega} |v(x)|^2 dx, \quad b(w) := \beta \int_{\Omega} |w(x)|^2 dx, \quad (1.21b)$$

$$\phi(v) := \beta \int_{\Omega} \varphi(v(x)) dx, \quad (1.21c)$$

$$j(v) := \inf \left\{ \underline{a}(\underline{u}) : \underline{u} = (u_i, u_e) \in H^1(\Omega) \times H^1(\Omega), u_i - u_e = v \text{ in } \Omega \right\}, \quad (1.21d)$$

where $M_i(x), M_e(x)$ are the symmetric and positive definite matrices obtained by solving the cellular problems for every $y \in \mathbb{R}^d$

$$M_{i,e}(x) y \cdot y := \min \left\{ \frac{1}{\mathcal{L}^d(Y)} \int_{Y_{i,e}} \sigma_{i,e}(x, \xi) (\nabla u(\xi) + y) \cdot (\nabla u(\xi) + y) d\xi : \right. \\ \left. u \in H_{loc}^1(\mathbb{R}^d), u \text{ } Y\text{-periodic} \right\}. \quad (1.22)$$

and satisfying the usual uniform ellipticity condition for a constant $\mu > 0$

$$\mu |y|^2 \leq M_{i,e}(x) y \cdot y \leq \mu^{-1} |y|^2, \quad \forall y \in \mathbb{R}^d, x \in \Omega. \quad (1.23)$$

We have now all the elements to state our main result:

THEOREM 1 (Convergence to the macroscopic problem P). *Let us suppose that, as $\varepsilon \downarrow 0$, the initial data $v_0^\varepsilon, w_0^\varepsilon \in L^2(\Gamma^\varepsilon)$ converge to $v_0, w_0 \in L^2(\Omega)$ according to (1.17), and the related energies satisfy*

$$\lim_{\varepsilon \downarrow 0} b^\varepsilon(v_0^\varepsilon) = b(v_0), \quad \lim_{\varepsilon \downarrow 0} b^\varepsilon(w_0^\varepsilon) = b(w_0), \quad \limsup_{\varepsilon \downarrow 0} \left(j^\varepsilon(v_0^\varepsilon) + \phi^\varepsilon(v_0^\varepsilon) \right) < +\infty. \quad (1.24)$$

Then there exists $\underline{u} = (u_i, u_e)$, $v = u_i - u_e$, w with

$$u_{i,e}, v, w \in C^0(0, T; H^1(\Omega)), \quad \partial_t v, \partial_t w \in L^2(0, T; L^2(\Omega)),$$

such that for every time $t \in [0, T]$

$$(u_{i,e}^\varepsilon, v^\varepsilon, w^\varepsilon) \rightarrow (u_{i,e}, v, w) \quad \text{as } \varepsilon \downarrow 0, \text{ according to Definition 1.1,} \quad (1.25)$$

with

$$\underline{a}^\varepsilon(\underline{u}^\varepsilon) = j^\varepsilon(v^\varepsilon) \rightarrow \underline{a}(\underline{u}) = j(v), \quad b^\varepsilon(v^\varepsilon) \rightarrow b(v), \quad b^\varepsilon(w^\varepsilon) \rightarrow b(w). \quad (1.26)$$

$(u_{i,e}, w)$ with $v = u_i - u_e$ is the (unique) variational solution of the **macroscopic R-D system**

$$\left. \begin{aligned} \operatorname{div}(M_i \nabla u_i) \\ - \operatorname{div}(M_e \nabla u_e) \end{aligned} \right\} = I_m \quad \text{in } \Omega \times (0, T), \quad (\text{continuity equation}) \quad (P_1)$$

$$\beta(\partial_t v + I(v, w)) = I_m \quad \text{in } \Omega \times (0, T), \quad (R\text{-}D \text{ condition}) \quad (P_2)$$

$$\partial_t w + r(v, w) = 0 \quad \text{in } \Omega \times (0, T), \quad (\text{dynamic coupling}) \quad (P_3)$$

supplemented by the boundary and initial conditions

$$M_{i,e} \nabla u_{i,e} \cdot \nu_{i,e} = 0 \quad \text{on } \partial\Omega \times (0, T), \quad (P_4)$$

$$v(\cdot, 0) = v_0 \quad \text{in } \Omega, \quad (P_5)$$

$$w(\cdot, 0) = w_0 \quad \text{in } \Omega; \quad (P_6)$$

again, the coupling terms I and r take the same form of (P_7^ε)

$$I(v, w) := F(v) + \Theta w, \quad r(v, w) := \gamma w - \eta v, \quad (P_7)$$

and a reference value for the potential u_e is determined by imposing

$$\int_{\Omega_0} u_e(x) dx = 0. \quad (P_8)$$

Thus we find the equations of the so called *bidomain model* (see e.g. [19, 13, 14, 30, 22]): it describes at a macroscopic level the averaged electric potentials and current flows inside (intra-) and outside (extra-cellular space) the cardiac cells, disregarding the discrete cellular structure and representing the cardiac tissue as the superposition of two interpenetrating and superimposed continua. In this representation Ω , the physical region occupied by the heart, coincides with the intra-, the extra- extracellular domains and at every point the two media are connected by a distributed cellular membrane on Ω . The two superposed conducting media are ohmic, i.e. their current densities are given by $\mathbf{j}_{i,e} = -M_{i,e} \nabla u_{i,e}$ with $M_{i,e}$ the conductivity tensors. Thus condition P_1 is the current conservation law

$$\operatorname{div}(\mathbf{j}_i + \mathbf{j}_e) = 0 \quad \text{and} \quad -\operatorname{div} \mathbf{j}_i = \operatorname{div} \mathbf{j}_e = I_m.$$

where I_m is the current per unit volume crossing the cellular membrane.

Convergence by extensions of the data. One could also consider a different approach to capture the asymptotic behavior of $u_{i,e}^\varepsilon, v^\varepsilon, w^\varepsilon$, by performing a preliminary extension of them to the whole Ω and considering the limit of such extensions in a suitable (ε -independent) function space.

THEOREM 2 (Weak convergence in H_{loc}^1 for extended solutions). *Let us assume that the initial data $v_0^\varepsilon, w_0^\varepsilon$ converge to v_0, w_0 according to (1.16), (1.17) of Definition 1.1, they satisfy the energy condition (1.24), and there exist extensions \tilde{w}_0^ε of w_0^ε which are bounded in $H^1(\Omega')$, for every $\Omega' \subset\subset \Omega$. Then there exist extensions $\tilde{u}_{i,e}^\varepsilon, \tilde{w}^\varepsilon$ of the microscopic solutions $u_{i,e}^\varepsilon, w^\varepsilon$ of problem \mathbf{P}^ε satisfying*

$$\sup_{t \in [0, T], \varepsilon > 0} \int_{\Omega'} \left(|\tilde{u}_{i,e}^\varepsilon|^2 + |\tilde{w}^\varepsilon|^2 + |\nabla \tilde{u}_{i,e}^\varepsilon|^2 + |\nabla \tilde{w}^\varepsilon|^2 \right) dx < +\infty \quad \forall \Omega' \subset\subset \Omega; \quad (1.28)$$

moreover, for every $\Omega' \subset\subset \Omega$ and $t \in [0, T]$, any family $\tilde{u}_{i,e}^\varepsilon, \tilde{w}^\varepsilon$ of such extensions will satisfy

$$\tilde{u}_{i,e}^\varepsilon \rightharpoonup u_{i,e}, \quad \tilde{w}^\varepsilon \rightharpoonup w \quad \text{weakly in } H^1(\Omega') \quad \text{as } \varepsilon \downarrow 0, \quad (1.29)$$

where $(u_{i,e}, w)$ is the solution of the macroscopic problem \mathbf{P} .

REMARK 1.3. As we will discuss in Section 2 the existence of admissible extensions $\tilde{u}_{i,e}^\varepsilon, \tilde{w}^\varepsilon$ satisfying the uniform bounds (1.28) follows by a general result of ACERBI et al. [1]; they also show that only local *a priori* bounds like (1.28) are available, due to the particular geometry of this problem: in fact, the boundary of $\Omega_{i,e}^\varepsilon$ could be quite irregular and one cannot find global extension operators which preserve the H^1 -norm.

Homogenization and Γ -convergence of the associated stationary problems. As we shall discuss in more detail in Section 5, the microscopic problems $(P_{1a}^\varepsilon, \dots, P_8^\varepsilon)$ can be considered as a sort of (perturbations of) gradient flows of ε -dependent energies with respect to a varying family of degenerate metrics, which are induced by nonnegative quadratic forms with a nontrivial kernel (the forms b^ε of (1.13b)).

The characterization of the asymptotic behaviour of the energy functionals, in the framework of Γ -convergence theory, is one of the crucial step of the proof of Theorem 1 and it is naturally related to a stationary homogenization problem, which is of independent interest. Here we state the stationary homogenization result in a simplified version, obtained by neglecting the role of the recovery variable w^ε .

We first introduce the family of convex functionals defined on $H^1(\Omega_i^\varepsilon) \times H^1(\Omega_e^\varepsilon)$

$$\mathcal{F}^\varepsilon(\underline{u}) := \frac{h}{2}b^\varepsilon(v - v_0^\varepsilon) + \frac{1}{2}a^\varepsilon(\underline{u}) + \phi^\varepsilon(v), \quad v := u_i - u_e \quad \text{on } \Gamma^\varepsilon, \quad (1.30)$$

where $h \geq 0$ is a given constant and $v_0^\varepsilon \in L^2(\Gamma^\varepsilon)$, and the limit functional defined in $(H^1(\Omega))^2$

$$\mathcal{F}(\underline{u}) := \frac{h}{2}b(v - v_0) + \frac{1}{2}a(\underline{u}) + \phi(v), \quad v := u_i - u_e \quad \text{in } \Omega \quad (1.31)$$

for a given $v_0 \in L^2(\Omega)$.

THEOREM 3 (Coercivity and Γ -convergence). *Let us suppose that $v_0^\varepsilon \in L^2(\Gamma^\varepsilon)$ converge to $v_0 \in L^2(\Omega)$ as $\varepsilon \downarrow 0$ according to (1.17), and satisfy*

$$\lim_{\varepsilon \downarrow 0} b^\varepsilon(v_0^\varepsilon) = b(v_0), \quad \limsup_{\varepsilon \downarrow 0} \left(j^\varepsilon(v_0^\varepsilon) + \phi^\varepsilon(v_0^\varepsilon) \right) < +\infty. \quad (1.32)$$

Then the following properties hold:

(a) *Compactness. If $\underline{u}^\varepsilon = (u_i^\varepsilon, u_e^\varepsilon) \in H^1(\Omega_i^\varepsilon) \times H^1(\Omega_e^\varepsilon)$ satisfies*

$$\liminf_{\varepsilon \downarrow 0} \mathcal{F}^\varepsilon(\underline{u}^\varepsilon) < +\infty, \quad \text{and} \quad \int_{\Omega_e^\varepsilon \cap \Omega_0} u_e^\varepsilon(x) dx = 0, \quad (1.33)$$

then there exists $\underline{u} \in H^1(\Omega) \times H^1(\Omega)$ satisfying

$$\int_{\Omega_0} u_e(x) dx = 0, \quad (1.34)$$

and a vanishing subsequence ε_n such that $\underline{u}^{\varepsilon_n}$ converges to \underline{u} according to Definition 1.1 as $n \rightarrow \infty$.

(b) *“liminf” inequality. For every family $\underline{u}^{\varepsilon_n}$ converging to \underline{u} according to Definition 1.1*

$$\liminf_{n \rightarrow \infty} \mathcal{F}^{\varepsilon_n}(\underline{u}^{\varepsilon_n}) \geq \mathcal{F}(\underline{u}). \quad (1.35)$$

(c) *“limsup” inequality. For every $\underline{u} = (u_i, u_e) \in (H^1(\Omega))^2$ satisfying (1.34) there exist $\underline{u}^\varepsilon \in H^1(\Omega_i^\varepsilon) \times H^1(\Omega_e^\varepsilon)$ converging to \underline{u} as in Definition 1.1 with $v^\varepsilon = u_i^\varepsilon - u_e^\varepsilon$ converging to $v = u_i - u_e$ as in (1.17) such that*

$$\int_{\Omega_e^\varepsilon \cap \Omega_0} u_e^\varepsilon(x) dx = 0, \quad \limsup_{\varepsilon \downarrow 0} \mathcal{F}^\varepsilon(\underline{u}^\varepsilon) \leq \mathcal{F}(\underline{u}). \quad (1.36)$$

Whenever a suitable weak (and metrizable) topology is introduced in the spaces of (signed) Radon measures (we postpone the discussion of this point to Section 3), the above result shows that \mathcal{F} is the Γ -limit of the coercive family of functionals \mathcal{F}^ε .

Well known results on Γ -convergence [17, Corollary 2.4] and a standard computation of the first variation of the functionals $\mathcal{F}^\varepsilon, \mathcal{F}$ [15] immediately yield

COROLLARY 1.4 (Homogenization of the stationary problems). *Under the same assumptions of Theorem 3, each functional \mathcal{F}^ε admits a unique minimizer*

$u^\varepsilon = (u_i^\varepsilon, u_e^\varepsilon) \in H^1(\Omega_i^\varepsilon) \times H^1(\Omega_e^\varepsilon)$ satisfying (1.33) with $v^\varepsilon = u_i^\varepsilon - u_e^\varepsilon \in H^{1/2}(\Gamma^\varepsilon)$ and characterized by the system

$$-\operatorname{div}(\sigma_{i,e}^\varepsilon \nabla u_{i,e}^\varepsilon) = 0 \quad \text{in } \Omega_{i,e}^\varepsilon, \quad (1.37a)$$

$$\left. \begin{aligned} -\sigma_i^\varepsilon \nabla u_i^\varepsilon \cdot \nu_i^\varepsilon \\ \sigma_e^\varepsilon \nabla u_e^\varepsilon \cdot \nu_e^\varepsilon \end{aligned} \right\} = I_m^\varepsilon \quad \text{on } \Gamma^\varepsilon, \quad (1.37b)$$

$$\varepsilon \left((h + \lambda_F) v^\varepsilon + F(v^\varepsilon) \right) = I_m^\varepsilon + \varepsilon h v_0^\varepsilon \quad \text{on } \Gamma^\varepsilon, \quad (1.37c)$$

$$\sigma_{i,e}^\varepsilon \nabla u_{i,e}^\varepsilon \cdot \nu_{i,e}^\varepsilon = 0 \quad \text{on } \partial\Omega_{i,e}^\varepsilon \setminus \Gamma^\varepsilon, \quad (1.37d)$$

$$\int_{\Omega_\varepsilon^\varepsilon \cap \Omega_0} u_e^\varepsilon(x) dx = 0. \quad (1.37e)$$

$(u_{i,e}^\varepsilon, v^\varepsilon)$ converge to $(u_{i,e}, v) \in (H^1(\Omega))^3$ as $\varepsilon \downarrow 0$ according to Definition 1.1; $u = (u_i, u_e)$ with $v = u_i - u_e$ is the (unique) minimizer of \mathcal{F} satisfying (1.34) and it is characterized by the system

$$\left. \begin{aligned} \operatorname{div}(M_i \nabla u_i) \\ -\operatorname{div}(M_e \nabla u_e) \end{aligned} \right\} = I_m \quad \text{in } \Omega, \quad (1.38a)$$

$$\beta \left((h + \lambda_F) v + F(v) \right) = I_m + \beta h v_0 \quad \text{in } \Omega, \quad (1.38b)$$

$$M_{i,e} \nabla u_{i,e} \cdot \nu_{i,e} = 0 \quad \text{on } \partial\Omega, \quad (1.38c)$$

$$\int_{\Omega_0} u_e(x) dx = 0. \quad (1.38d)$$

Plan of the paper. The paper is divided in two parts: the first one (Sections 2 and 3) is devoted to the stationary homogenization result stated in Theorem 3 and can be read independently of the remaining Sections. The next section contains some preliminary technical results related to the extension problem of functions defined on Γ^ε and $\Omega_{i,e}^\varepsilon$; we also present a natural generalization of the Riemann-Lebesgue lemma and some properties of the notion of convergence introduced in Definition 1.1; finally we discuss some applications to lower semicontinuity and approximation results for integral functionals defined on Γ^ε .

Section 3 will conclude the proof of Theorem 3 and its corollary. Here we combine very general results on Γ -convergence of non-coercive functionals [11] (related to elliptic problems with Neumann boundary conditions), with the more recent extension techniques of [1]. Our main contribution is to extend this framework to the homogenization of integral functionals defined on the interface between the two ε -domains.

The second part of the paper is more specifically devoted to the evolutive systems P^ε, P . The well posedness of their variational formulation and some preliminary accessory results are collected in Section 4.

In Section 5 we outline and carry out the main steps of the proof of Theorems 1 and 2. Here we adopt the point of view of the “Minimizing movement” approach to evolution equations, suggested by E. DE GIORGI in [16]: we perform an auxiliary “semi-implicit time discretization” of the microscopic R-D system, which consists in a recursive family of variational problems depending on the step size $\tau > 0$. The final macroscopic model will thus result in two limit procedures: a first one, in the time discretization, as the time step τ goes to 0 keeping ε fixed, and the second homogenization process as $\varepsilon \downarrow 0$.

Uniform approximation estimates for the discretized problem (which are strictly related to the *convexity* of the underlying functionals) allows to “invert” the order of the limits: we can therefore pass first to the limit as $\varepsilon \downarrow 0$ keeping τ fixed and in this way we obtain a family of *homogenized discrete macroscopic problems*; a final limit as $\tau \downarrow 0$ recovers the continuous form of the macroscopic problem. By this approach, the homogenization of the time dependent problem \mathbf{P}^ε is reduced to the homogenization of a finite sequence of stationary problems of elliptic type, which exhibit (up to lower order perturbation terms) the same structure we studied in Section 3.

Applications of Γ -convergence to evolution problems are well known for gradient flows of *convex* (or λ -*convex*) functionals in Hilbert spaces [6]: in that case the uniform convergence of the induced evolution semigroups can be deduced from the Γ -Mosco-convergence [26, 17] of the underlying functionals.

In our case things are more difficult, due to the degeneration of the parabolic structure of our systems (as was discussed in [15]): and to the lack of a “fixed” (i.e., independent of ε) Hilbert space, where the evolution can be settled.

Therefore, a general abstract result for studying the convergence of the present problem seems to be missing. Nevertheless, we tried to develop a general procedure (“uniform discretization estimates & Γ -convergence of the discretized variational problems”) to attack this kind of λ -convex but “degenerate” evolution problems: even if our arguments could have been presented in a more compact (but maybe more obscure) form, we decided to clarify their structure as much as possible, hoping that a better understanding of the main ideas of this approach could also be helpful for other applications in different contexts.

In section 6 we briefly sketch the rigorous derivation of the error estimates for the semi-implicit discretization of Problems \mathbf{P}^ε and \mathbf{P} : here we adapt to our setting the technique introduced by [28] (but see also [7, 31, 32]) to obtain optimal *a priori* estimates for gradient flows of convex functionals in Hilbert spaces.

Such kind of estimates could have also been derived by standard perturbation arguments from the general results of [9], for a fully implicit discretization scheme; here we chose a direct approach in order to have a precise control of the various constants involved (which should be *independent of ε*) and to keep the presentation simpler and almost self-contained.

The main advantage of the semi-implicit discretization (instead of an implicit one) relies in the variational structure of the problems, which should be solved at each time step: in fact, they are associated to the minimum of convex functionals.

For the sake of completeness, in Appendix A we briefly recall the derivation of the model at the cellular level from well established physical laws and we introduce its dimensionless form.

2. Notation, extension results, and related convergence properties.

2.1. Vector notation, function spaces and bilinear forms. In order to write the micro- and macro-scopic problems in a compact form, we introduce a vector notation, which will also be useful to deal with the evolution systems; thus

$$\underline{u}^\varepsilon, \underline{u} \quad \text{will denote the vectors } (u_i^\varepsilon, u_e^\varepsilon), (u_i, u_e), \quad (2.1)$$

and we will use underlined letters (as $\underline{u}, \underline{\phi}, \underline{\mathcal{V}}, \dots$) for vectors, functions, or spaces involving intra-extracellular couples. We set

$$\begin{aligned} \mathcal{H}^\varepsilon &:= L^2(\Gamma^\varepsilon), \quad \mathcal{V}^\varepsilon := \left\{ \underline{u} \in H^1(\Omega_i^\varepsilon) \times H^1(\Omega_e^\varepsilon) : v = u_i^\varepsilon - u_e^\varepsilon \in L^2(\Gamma^\varepsilon) \right\}, \\ \mathcal{H} &:= L^2(\Omega), \quad \mathcal{V} := H^1(\Omega) \times H^1(\Omega), \quad \mathcal{V}_{loc} := H_{loc}^1(\Omega) \times H_{loc}^1(\Omega), \end{aligned} \quad (2.2)$$

together with their closed subspaces

$$\mathcal{V}_0^\varepsilon := \left\{ \underline{u}^\varepsilon \in \mathcal{V}^\varepsilon : \int_{\Omega_\varepsilon \cap \Omega_0} u_e^\varepsilon(x) dx = 0 \right\}, \quad \mathcal{V}_0 := \left\{ \underline{u} \in \mathcal{V} : \int_{\Omega_0} u_e(x) dx = 0 \right\}. \quad (2.3)$$

REMARK 2.1. Since $\Omega_{i,e}^\varepsilon$ could have a very irregular boundary, we do not know if the traces $u_{i,e}^\varepsilon|_{\Gamma^\varepsilon}$ of $u_{i,e}^\varepsilon \in H^1(\Omega_{i,e}^\varepsilon)$ belong to $H^{1/2}(\Gamma^\varepsilon) \subset L^2(\Gamma^\varepsilon)$: a priori we only know $u_{i,e}^\varepsilon|_{\Gamma^\varepsilon} \in H_{loc}^{1/2}(\Gamma^\varepsilon) \subset L_{loc}^2(\Gamma^\varepsilon)$. Therefore, the integrability condition on $v = u_i^\varepsilon - u_e^\varepsilon$ on Γ^ε imposed in the definition (2.2) of $\mathcal{V}_0^\varepsilon$ is not redundant. We are now introducing some continuous and symmetric bilinear forms on $\mathcal{V}^\varepsilon, \mathcal{H}^\varepsilon, \mathcal{V}, \mathcal{H}$ which will play a crucial role in the following. Recalling (1.13a,b,c) and (1.21a,b,c), we set

$$\underline{a}^\varepsilon(\underline{u}, \underline{\hat{u}}) := \sum_{i,e} \int_{\Omega_{i,e}^\varepsilon} \sigma_{i,e}^\varepsilon \nabla u_{i,e} \cdot \nabla \hat{u}_{i,e} dx \quad \forall \underline{u}, \underline{\hat{u}} \in \mathcal{V}^\varepsilon, \quad (2.4)$$

$$\underline{a}(\underline{u}, \underline{\hat{u}}) := \sum_{i,e} \int_{\Omega} M_{i,e} \nabla u_{i,e} \cdot \nabla \hat{u}_{i,e} dx \quad \forall \underline{u}, \underline{\hat{u}} \in \mathcal{V}, \quad (2.5)$$

and we introduce the scalar products on $\mathcal{H}^\varepsilon, \mathcal{H}$

$$\begin{aligned} b^\varepsilon(w, \hat{w}) &:= \varepsilon \int_{\Gamma^\varepsilon} w \hat{w} d\mathcal{H}^{d-1} \quad \forall w, \hat{w} \in \mathcal{H}^\varepsilon, \\ b(w, \hat{w}) &:= \beta \int_{\Omega} w \hat{w} dx \quad \forall w, \hat{w} \in \mathcal{H}. \end{aligned} \quad (2.6)$$

We also set

$$\underline{b}^\varepsilon(\underline{u}, \underline{\hat{u}}) := \varepsilon \int_{\Gamma^\varepsilon} (u_i - u_e)(\hat{u}_i - \hat{u}_e) d\mathcal{H}^{d-1} = b^\varepsilon(B^\varepsilon \underline{u}^\varepsilon, B^\varepsilon \underline{\hat{u}}^\varepsilon) \quad \forall \underline{u}, \underline{\hat{u}} \in \mathcal{V}^\varepsilon, \quad (2.7)$$

$$\underline{b}(\underline{u}, \underline{\hat{u}}) := \beta \int_{\Omega} (u_i - u_e)(\hat{u}_i - \hat{u}_e) dx = b(\underline{B}\underline{u}, \underline{B}\underline{\hat{u}}) \quad \forall \underline{u}, \underline{\hat{u}} \in \mathcal{V}, \quad (2.8)$$

$$\underline{\phi}^\varepsilon(\underline{u}) := \varepsilon \int_{\Gamma^\varepsilon} \varphi(u_i - u_e) d\mathcal{H}^{d-1} = \phi^\varepsilon(B^\varepsilon \underline{u}) \quad \forall \underline{u} \in \mathcal{V}^\varepsilon, \quad (2.9)$$

$$\underline{\phi}(\underline{u}) := \beta \int_{\Omega} \varphi(u_i - u_e) dx = \phi(\underline{B}\underline{u}) \quad \forall \underline{u} \in \mathcal{V}, \quad (2.10)$$

where B^ε, B are the linear continuous operators

$$\begin{aligned} B^\varepsilon : \mathcal{V}^\varepsilon &\rightarrow \mathcal{H}^\varepsilon, \quad B^\varepsilon \underline{u} := u_i|_{\Gamma^\varepsilon} - u_e|_{\Gamma^\varepsilon} \quad \forall \underline{u} \in \mathcal{V}_0^\varepsilon, \\ B : \mathcal{V} &\rightarrow \mathcal{H}, \quad B \underline{u} := u_i - u_e \quad \forall \underline{u} \in \mathcal{V}_0. \end{aligned} \quad (2.11)$$

It is easy to check that the bilinear forms $\underline{a}^\varepsilon(\cdot, \cdot) + \underline{b}^\varepsilon(\cdot, \cdot)$ and $\underline{a}(\cdot, \cdot) + \underline{b}(\cdot, \cdot)$ are scalar products on $\mathcal{V}_0^\varepsilon$ and \mathcal{V}_0 ; they induce on $\mathcal{V}_0^\varepsilon$ and \mathcal{V}_0 the usual topology as closed subspaces of \mathcal{V}^ε and \mathcal{V} respectively. As in (1.13a,b,c) and (1.21a,b,c), we adopt the convention to write the associated quadratic forms as

$$\underline{b}^\varepsilon(\underline{u}) := \underline{b}^\varepsilon(\underline{u}, \underline{u}), \quad \underline{a}^\varepsilon(\underline{u}) := \underline{a}^\varepsilon(\underline{u}, \underline{u}), \quad \underline{b}(\underline{u}) := \underline{b}(\underline{u}, \underline{u}), \quad \underline{a}(\underline{u}) := \underline{a}(\underline{u}, \underline{u}). \quad (2.12)$$

2.2. Uniform bounds for extension operators. Let us now discuss some extension results for functions defined in $\Omega_{i,e}^\varepsilon, \Gamma^\varepsilon$, which will be applied to the notion of convergence introduced in Definition 1.1.

DEFINITION 2.2 (Extensions). *We say that $\tilde{u}^\varepsilon \in \mathcal{V}_{loc} = H_{loc}^1(\Omega) \times H_{loc}^1(\Omega)$ is an extension of $u^\varepsilon \in \mathcal{V}^\varepsilon$ if*

$$\tilde{u}_{i,e}^\varepsilon|_{\Omega_{i,e}^\varepsilon} = u_{i,e}^\varepsilon. \quad (2.13)$$

Analogously, we say that $\tilde{w}^\varepsilon \in H_{loc}^1(\Omega)$ is an extension of $w^\varepsilon \in H_{loc}^{1/2}(\Gamma^\varepsilon)$ if

$$\tilde{w}^\varepsilon|_{\Gamma^\varepsilon} = w^\varepsilon \quad \text{in the sense of traces.} \quad (2.14)$$

One of the technical difficulties in the present setting is to find suitable extension operators $T_{i,e}^\varepsilon$ of functions defined only on $\Omega_{i,e}^\varepsilon$ to the whole Ω which preserve uniform bounds of the L^2 and H^1 norms. Due to the possible irregular behaviour of the boundary of $\Omega_{i,e}$, only *local bounds* are available.

The following result proved by E. ACERBI ET AL. [1] is almost optimal. The crucial assumption, here, is that the sets $E_{i,e}$ are Lipschitz and connected; we are denoting by $\Omega(\delta)$, $\delta \geq 0$, the open subset of Ω defined by

$$\Omega(\delta) := \{x \in \Omega : d(x, \mathbb{R}^d \setminus \Omega) > \delta\}. \quad (2.15)$$

THEOREM 2.3. [1] *There exists linear and continuous extension operators $T_{i,e}^\varepsilon : H^1(\Omega_{i,e}^\varepsilon) \rightarrow H_{loc}^1(\Omega)$ and three constants $k_0, h_0, h_1 > 0$ independent of $\varepsilon > 0$ and Ω , such that for every $u \in H^1(\Omega_{i,e}^\varepsilon)$ we have*

$$T_{i,e}^\varepsilon u = u \quad \text{a.e. in } \Omega_{i,e}^\varepsilon, \quad (2.16)$$

$$\int_{\Omega(\varepsilon k_0)} |T_{i,e}^\varepsilon u|^2 dx \leq h_0 \int_{\Omega_{i,e}^\varepsilon} |u|^2 dx, \quad (2.17)$$

$$\int_{\Omega(\varepsilon k_0)} |\nabla T_{i,e}^\varepsilon u|^2 dx \leq h_1 \int_{\Omega_{i,e}^\varepsilon} |\nabla u|^2 dx. \quad (2.18)$$

As usual we set $\underline{T}^\varepsilon := (T_i^\varepsilon, T_e^\varepsilon) : \mathcal{V}^\varepsilon \rightarrow \mathcal{V}_{loc}$.

REMARK 2.4. *In general, it is not possible to construct a family of extension operators $T_{i,e}^\varepsilon : H^1(\Omega_{i,e}^\varepsilon) \rightarrow H^1(\Omega)$ satisfying (2.16), (2.17), (2.18) with $\Omega(\varepsilon k_0)$ replaced by Ω , since we do not have any control on the behavior of E^ε near $\partial\Omega$. For more details and an explicit counterexample we refer to [1].*

2.3. Generalized Riemann-Lebesgue lemma. Let us first recall a well known version of the Riemann-Lebesgue Lemma:

LEMMA 2.5 (**Generalized Riemann-Lebesgue Lemma**). *Let A be a bounded open subset of \mathbb{R}^d with $\mathcal{L}^d(\partial A) = 0$, and let $\zeta \in C^0(\overline{A})$. Then*

$$\lim_{\varepsilon \downarrow 0} \int_{A \cap E_{i,e}^\varepsilon} \zeta(x) dx = \beta_{i,e} \int_A \zeta(x) dx, \quad (2.19)$$

$$\lim_{\varepsilon \downarrow 0} \varepsilon \int_{A \cap \Gamma^\varepsilon} \zeta(x) d\mathcal{H}^{d-1}(x) = \beta \int_A \zeta(x) dx, \quad (2.20)$$

where the coefficients $\beta_{i,e}, \beta$ are defined in (1.15).

REMARK 2.6 (Weak* convergence in L^∞). (2.19) shows that the characteristic functions $\chi_{A \cap E_{i,e}^\varepsilon}$ of $A \cap E_{i,e}^\varepsilon$ are converging to $\beta_{i,e} \chi_A$ in the sense of distributions as $\varepsilon \downarrow 0$; since they are also uniformly bounded in $L^\infty(\Omega)$, an obvious weak*-compactness argument shows that

$$\chi_{A \cap E_{i,e}^\varepsilon} \rightharpoonup^* \beta_{i,e} \chi_A \quad \text{in } L^\infty(\Omega) \quad \text{as } \varepsilon \downarrow 0. \quad (2.21)$$

REMARK 2.7 (Weak* convergence in the space of measures). Lemma 2.5 also shows that the measures $\lambda_{i,e}^\varepsilon, \lambda^\varepsilon$ defined by (1.18) converge to $\lambda_{i,e}, \lambda$ respectively as $\varepsilon \downarrow 0$ in the weak* topology of the space of finite Radon measures on Ω . The next result reinforces Lemma 2.5, and shows that weak convergence in $H_{loc}^1(\Omega)$ implies the convergence in the sense of Definition 1.1. Since we will deal with functionals depending on the continuous parameter $\varepsilon > 0$ or on the discrete values of a suitable decreasing infinitesimal sequence $\{\varepsilon_n\}_{n \in \mathbb{N}}$, for notational convenience we will treat both the cases in the same way, by considering a general not empty set Λ of real numbers such that

$$\Lambda \subset (0, +\infty), \quad \inf \Lambda = 0. \quad (2.22)$$

Expressions like $\lim_{\varepsilon \downarrow 0, \varepsilon \in \Lambda}, \liminf_{\varepsilon \downarrow 0, \varepsilon \in \Lambda}$, etc..., have an obvious meaning as limits for ε going to 0 in Λ . Of course, when Λ contains an open interval $(0, \delta)$, $\delta > 0$, we will use the usual notation $\lim_{\varepsilon \downarrow 0}$.

PROPOSITION 2.8 (**Weak H_{loc}^1 -convergence yields 1.1**). *Let us suppose that z^ε weakly converge to z in $H_{loc}^1(\Omega)$ for $\varepsilon \downarrow 0, \varepsilon \in \Lambda$. Then for every continuous function $\zeta \in C_c^0(\Omega)$ we have*

$$\lim_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} \int_{\Omega_{i,e}^\varepsilon} z^\varepsilon(x) \zeta(x) dx = \beta_{i,e} \int_{\Omega} z(x) \zeta(x) dx, \quad (2.23)$$

$$\lim_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} \varepsilon \int_{\Gamma^\varepsilon} z^\varepsilon(x) \zeta(x) d\mathcal{H}^{d-1}(x) = \beta \int_A z(x) \zeta(x) dx. \quad (2.24)$$

In particular $z^\varepsilon|_{\Gamma^\varepsilon}$ and $z^\varepsilon|_{\Omega_{i,e}^\varepsilon}$ converge to z according to Definition 1.1.

Proof. By Rellich compactness theorem, we know that $z^\varepsilon \rightarrow z$ strongly in $L_{loc}^2(\Omega)$ as $\varepsilon \downarrow 0, \varepsilon \in \Lambda$. (2.23) thus follows directly from Remark 2.6 since

$$\chi_{\Omega_{i,e}^\varepsilon} \rightharpoonup^* \beta_{i,e} \quad \text{in } L^\infty(\Omega), \quad \text{and therefore} \quad z^\varepsilon \chi_{\Omega_{i,e}^\varepsilon} \rightharpoonup \beta_{i,e} z \quad \text{weakly in } L_{loc}^2(\Omega).$$

In order to prove (2.24), we first observe that for ε sufficiently small, we can find pluricellular regions (recall (1.2))

$$R^\varepsilon = \bigcup_{m=1}^{M^\varepsilon} \varepsilon(Y + \sum_{k=1}^d j_{m,k}^\varepsilon e_k) \quad \text{for some } \mathbf{j}_m^\varepsilon = (j_{m,1}^\varepsilon, \dots, j_{m,d}^\varepsilon) \in \mathbb{Z}^d,$$

and a regular open set A such that

$$\text{supp } \zeta \subset R^\varepsilon \subset A \subset \subset \Omega.$$

Poincaré inequality and a rescaling argument easily yield

$$\varepsilon \mathcal{H}^{d-1}(R^\varepsilon \cap \varepsilon \Gamma) = \beta \mathcal{L}^d(R^\varepsilon), \quad \varepsilon^2 \|z\|_{L^1(R^\varepsilon \cap \varepsilon \Gamma)}^2 \leq \varepsilon \beta \mathcal{L}^d(A) \|z\|_{L^2(R^\varepsilon \cap \varepsilon \Gamma)}^2, \quad (2.25)$$

$$\varepsilon \|z\|_{L^2(R^\varepsilon \cap \varepsilon \Gamma)}^2 \leq c_1 \left(\|z\|_{L^2(R_{i,e}^\varepsilon)}^2 + \varepsilon^2 \|\nabla z\|_{L^2(R_{i,e}^\varepsilon)}^2 \right). \quad (2.26)$$

Then, if $S := \sup_{x \in \Omega} |\zeta(x)|$, we have

$$\begin{aligned} \left| \varepsilon \int_{\Gamma^\varepsilon} (z^\varepsilon(x) - z(x)) \zeta(x) d\mathcal{H}^{d-1}(x) \right| &\leq \varepsilon S \|z^\varepsilon - z\|_{L^1(R^\varepsilon \cap \varepsilon \Gamma)} \leq \\ &\leq C \left(\|z^\varepsilon - z\|_{L^2(A)} + \varepsilon \|\nabla z^\varepsilon - \nabla z\|_{L^2(A)} \right) \end{aligned} \quad (2.27)$$

where $C := S\sqrt{c_1\beta\mathcal{L}^d(A)}$; since z^ε is bounded in $H^1(A)$ and converges to z in $L^2(A)$, (2.27) vanishes as $\varepsilon \downarrow 0$. Thus we simply have to prove (2.24) for $z^\varepsilon \equiv z$.

Choosing now another arbitrary function $\eta \in H^1(A) \cap C^0(\bar{A})$ and taking into account Lemma 2.5, we get

$$\begin{aligned} & \limsup_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} \left| \varepsilon \int_{\Gamma^\varepsilon} z(x) \zeta(x) d\mathcal{H}^{d-1}(x) - \beta \int_{\Omega} z(x) \zeta(x) dx \right| \\ & \leq \limsup_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} \left| \varepsilon \int_{\Gamma^\varepsilon} (z(x) - \eta(x)) \zeta(x) d\mathcal{H}^{d-1}(x) \right| + \left| \beta \int_{\Omega} (z(x) - \eta(x)) \zeta(x) dx \right| \\ & \leq (1 + \beta) Z \left(\varepsilon \|z - \eta\|_{L^1(R^\varepsilon \cap \Gamma)} + \|z - \eta\|_{L^1(R^\varepsilon)} \right) \leq C' \|z - \eta\|_{H^1(A)}, \end{aligned}$$

for a constant C' independent of z and η . Being η arbitrary and A regular, a standard density result yields (2.24). \square

COROLLARY 2.9. *Suppose that $w^\varepsilon \in H_{loc}^{1/2}(\Gamma^\varepsilon)$ converge to $w \in L_{loc}^1(\Omega)$ according to Definition 1.1 and let $\tilde{w}^\varepsilon \in H_{loc}^1(\Omega)$ be an extension of w^ε which is uniformly bounded in $H^1(A)$ for every open subset $A \subset \subset \Omega$. Then $w \in H_{loc}^1(\Omega)$ and $\tilde{w}^\varepsilon \rightharpoonup w$ in $H_{loc}^1(\Omega)$.*

Proof. By Proposition 2.8 w is the unique limit point of any weakly convergent subsequence of \tilde{w}^ε in $H_{loc}^1(\Omega)$. \square

2.4. Compactness properties. Recall that in (1.12) we introduced an open subset $\Omega_0 \subset \subset \Omega$ which induces closed subspaces of $H^1(\Omega_e^\varepsilon)$ and $H^1(\Omega)$ through the integral conditions (2.3). By Lemma 2.5 with $A := \Omega_0$ and $\zeta \equiv 1$ we have that

$$\lim_{\varepsilon \downarrow 0} \mathcal{L}^d(\Omega_0 \cap \varepsilon E_e) = \beta_e \mathcal{L}^d(\Omega_0), \quad \lim_{\varepsilon \downarrow 0} \varepsilon \mathcal{H}^{d-1}(\Omega_0 \cap \varepsilon \Gamma) = \beta \mathcal{L}^d(\Omega_0), \quad (2.28)$$

so that we can always assume that

$$\mathcal{L}^d(\Omega_0 \cap \varepsilon E_e) \geq \frac{\beta_e}{2} \mathcal{L}^d(\Omega_0), \quad \varepsilon \mathcal{H}^{d-1}(\Omega_0 \cap \varepsilon \Gamma) \geq \frac{\beta}{2} \mathcal{L}^d(\Omega_0) \quad \forall \varepsilon \in \Lambda. \quad (2.29)$$

LEMMA 2.10 (Uniform local H^1 bounds and compactness). *Let $u^\varepsilon \in \mathcal{V}_0^\varepsilon$, $\varepsilon \in \Lambda$, be a family of functions satisfying*

$$\sup_{\varepsilon \in \Lambda} \left(\underline{a}^\varepsilon(u^\varepsilon) + \underline{b}^\varepsilon(u^\varepsilon) \right) = S < +\infty. \quad (2.30)$$

Then for every open subset $A \subset \subset \Omega$ we have

$$\sup_{\varepsilon \in \Lambda} \|T^\varepsilon u^\varepsilon\|_{H^1(A)} < +\infty. \quad (2.31)$$

In particular, there exists an infinitesimal subsequence $\Lambda' = (\varepsilon_j)_{j \in \mathbb{N}} \subset \Lambda$ and a limit function $\underline{u} \in \mathcal{V}_0$ such that

$$\lim_{\varepsilon \downarrow 0, \varepsilon \in \Lambda'} T^\varepsilon u^\varepsilon = \underline{u} \quad \text{weakly in } \mathcal{V}_{loc} = H_{loc}^1(\Omega) \times H_{loc}^1(\Omega). \quad (2.32)$$

Proof. Let $\underline{u}^\varepsilon = (u_i^\varepsilon, u_e^\varepsilon) \in \mathcal{V}_0^\varepsilon$, $\tilde{\underline{u}}^\varepsilon := T^\varepsilon u^\varepsilon \in \mathcal{V}_{loc}$, and let us choose $\delta > 0$ sufficiently small such that (see (2.15) and, e.g., [33])

$$A \subset \Omega(\delta), \quad \Omega_0 \subset \Omega(2\delta), \quad \Omega(\delta) \text{ is Lipschitz.} \quad (2.33)$$

We can suppose that $(k_0 + \ell)\varepsilon < \delta$; by using the properties (2.16), (2.17), (2.18) of the extension operators $T_{i,e}^\varepsilon$, we get:

$$\|\nabla \check{u}_{i,e}^\varepsilon\|_{L^2(A)}^2 \leq \|\nabla \check{u}_{i,e}^\varepsilon\|_{L^2(\Omega(\delta))}^2 \leq h_1 \|\nabla u_{i,e}^\varepsilon\|_{L^2(\Omega_{i,e}^\varepsilon)}^2 \leq h_1 \sigma^{-1} \underline{a}^\varepsilon(\underline{u}^\varepsilon). \quad (2.34)$$

Poincaré inequality yields constants $c_{i,e}^\varepsilon$ (depending on $u_{i,e}^\varepsilon$) and c_P (depending only on Ω) satisfying

$$\|\check{u}_{i,e}^\varepsilon - c_{i,e}^\varepsilon\|_{L^2(\Omega(\delta))} \leq c_P \|\nabla \check{u}_{i,e}^\varepsilon\|_{L^2(\Omega(\delta))}. \quad (2.35)$$

Setting $\Omega_{0,e}^\varepsilon := \Omega_0 \cap \varepsilon E_e$, by the properties of the extension operator we know that

$$\int_{\Omega_{0,e}^\varepsilon} \check{u}_e^\varepsilon dx = 0, \quad (2.36)$$

so that

$$|c_e^\varepsilon| \mathcal{L}^d(\Omega_{0,e}^\varepsilon) \leq \int_{\Omega_{0,e}^\varepsilon} |c_e^\varepsilon - \check{u}_e^\varepsilon| dx \leq \mathcal{L}^d(\Omega_{0,e}^\varepsilon)^{1/2} \|\check{u}_e^\varepsilon - c_e^\varepsilon\|_{L^2(\Omega_{0,e}^\varepsilon)}$$

and therefore by (2.29) we have

$$|c_e^\varepsilon| \leq \left(\frac{\beta_e}{2} \mathcal{L}^d(\Omega_0) \right)^{-1/2} \|\check{u}_e^\varepsilon - c_e^\varepsilon\|_{L^2(\Omega(\delta))},$$

which, together with (2.35) and (2.34), shows that c_e^ε is uniformly bounded w.r.t. ε . In order to get an analogous bound for c_i^ε we will use the estimate:

$$\varepsilon \int_{\Gamma^\varepsilon} |u_i^\varepsilon - u_e^\varepsilon|^2 d\mathcal{H}^{d-1} = \underline{b}^\varepsilon(\underline{u}^\varepsilon) \leq S, \quad (2.37)$$

observing that by (2.33) and (2.26)

$$\begin{aligned} \varepsilon \int_{\Omega_0 \cap \varepsilon \Gamma} |u_{i,e}^\varepsilon - c_{i,e}^\varepsilon|^2 d\mathcal{H}^{d-1} &\leq c_1 \int_{\Omega(\delta) \cap \varepsilon E_{i,e}} \left((u_{i,e}^\varepsilon - c_{i,e}^\varepsilon)^2 + \varepsilon^2 |\nabla u_{i,e}^\varepsilon|^2 \right) dx \\ &\leq c_1 (c_P^2 + \varepsilon^2) \|\nabla \check{u}_{i,e}^\varepsilon\|_{L^2(\Omega(\delta))}^2 \leq c_2 := c_1 (c_P + \varepsilon^2) h_1 \sigma^{-1} S. \blacksquare \end{aligned}$$

Thus we get

$$\begin{aligned} \frac{\beta}{2} \mathcal{L}^d(\Omega_0) |c_i^\varepsilon - c_e^\varepsilon|^2 &\leq \varepsilon \int_{\Omega_0 \cap \varepsilon \Gamma} |c_i^\varepsilon - c_e^\varepsilon|^2 d\mathcal{H}^{d-1} \\ &\leq \varepsilon \int_{\Omega_0 \cap \varepsilon \Gamma} |c_i^\varepsilon - c_e^\varepsilon - (u_i^\varepsilon - u_e^\varepsilon) + (u_i^\varepsilon - u_e^\varepsilon)|^2 d\mathcal{H}^{d-1} \\ &\leq 3\varepsilon \int_{\Gamma^\varepsilon} |u_i^\varepsilon - u_e^\varepsilon|^2 d\mathcal{H}^{d-1} + 6c_2 = 3\underline{b}^\varepsilon(\underline{u}^\varepsilon) + 6c_2 \leq 3S + 6c_2. \end{aligned}$$

It follows that also c_i^ε is uniformly bounded with respect to ε , so that $\check{u}_{i,e}^\varepsilon$ are bounded in $H^1(A)$.

A standard diagonal argument yields (2.32) for some $\underline{u} \in \mathcal{V}$; the fact that \underline{u} belongs to \mathcal{V}_0 , too, follows from Proposition 2.8. \square

COROLLARY 2.11. *Let us consider $\underline{u} \in L_{loc}^2(\Omega) \times L_{loc}^2(\Omega)$ and let us suppose that $\underline{u}^\varepsilon \in \mathcal{V}_0^\varepsilon$, $\varepsilon \in \Lambda$ satisfy*

$$\sup_{\varepsilon \in \Lambda} \left(\underline{a}^\varepsilon(\underline{u}^\varepsilon) + \underline{b}^\varepsilon(\underline{u}^\varepsilon) \right) < +\infty. \quad (2.38)$$

$\underline{u} \in \mathcal{Y}_0$ and $\underline{u}^\varepsilon \rightarrow \underline{u}$ according to Definition 1.1 if and only if there exist extensions $\tilde{\underline{u}}^\varepsilon \in \mathcal{Y}_{loc}$ of $\underline{u}^\varepsilon$ such that

$$\tilde{\underline{u}}^\varepsilon \rightharpoonup \underline{u} \quad \text{weakly in } \mathcal{Y}_{loc} = H_{loc}^1(\Omega) \times H_{loc}^1(\Omega) \quad \text{as } \varepsilon \downarrow 0, \varepsilon \in \Lambda. \quad (2.39)$$

Moreover, if $\underline{u}^\varepsilon \rightarrow \underline{u}$ according to Definition 1.1, then every extension $\tilde{\underline{u}}^\varepsilon$ bounded in \mathcal{Y}_{loc} is weakly convergent to \underline{u} in \mathcal{Y}_{loc} , in particular we always have

$$\underline{T}^\varepsilon(\underline{u}^\varepsilon) \rightharpoonup \underline{u} \quad \text{weakly in } \mathcal{Y}_{loc} = H_{loc}^1(\Omega) \times H_{loc}^1(\Omega), \quad (2.40)$$

and setting $v^\varepsilon := B^\varepsilon \underline{u}^\varepsilon = u_i^\varepsilon|_{\Gamma^\varepsilon} - u_e^\varepsilon|_{\Gamma^\varepsilon}$ we have

$$v^\varepsilon \rightarrow u_i - u_e \quad \text{according to Definition 1.1.} \quad (2.41)$$

Proof. If $\tilde{\underline{u}}^\varepsilon$ is an extension of $\underline{u}^\varepsilon$ which is bounded in \mathcal{Y}_{loc} then any weak limit point $\tilde{\underline{u}}$ in \mathcal{Y}_{loc} should coincide with \underline{u} and belongs to \mathcal{Y}_0 by Proposition 2.8. Lemma 2.10 and (2.38) show that $\underline{T}^\varepsilon$ provides such an extension, so that the equivalence between the two notions of convergence is proved. \square

REMARK 2.12. Observe that for a general family $\underline{u}^\varepsilon \in \mathcal{Y}_0^\varepsilon$ converging to \underline{u} according to Definition 1.1, such that $v^\varepsilon = B^\varepsilon \underline{u}^\varepsilon$ converges to v as $\varepsilon \downarrow 0, \varepsilon \in \Lambda$, it may happen that $v \neq u_i - u_e$ in Ω . The above Corollary shows that this inconvenient can be avoided if \underline{u} satisfies the equibounded energy condition (2.38).

PROPOSITION 2.13 (**Compactness for the convergence 1.1**). *For $\varepsilon \in \Lambda$ let $\underline{u}^\varepsilon \in \mathcal{Y}_0^\varepsilon$ (resp. $w^\varepsilon \in L^2(\Gamma^\varepsilon)$) be satisfying (2.38) (resp. $\sup_{\varepsilon \in \Lambda} b^\varepsilon(w^\varepsilon) < +\infty$). Then there exists a decreasing vanishing subsequence $\Lambda' = (\varepsilon_j)_{j \in \mathbb{N}} \subset \Lambda$ and an element $\underline{u} \in \mathcal{Y}_0$ (resp. $w \in \mathcal{H}$) such that $\underline{u}^{\varepsilon_j}$ converges to \underline{u} as $j \rightarrow \infty$ according to Definition 1.1 (resp. $w^{\varepsilon_j} \rightarrow w$).*

Proof. The compactness of $\underline{u}^\varepsilon$ follows directly from Lemma 2.10 and Corollary 2.11. In the case of w^ε we observe that the total variation of the measures \tilde{w}^ε introduced in (1.19) are easily bounded by

$$|\tilde{w}^\varepsilon|(\Omega) = \varepsilon \int_{\Gamma^\varepsilon} |w^\varepsilon(x)| d\mathcal{H}^{d-1}(x) \leq C b^\varepsilon(w^\varepsilon)^{1/2}$$

thanks to (2.28). Therefore we can extract a subsequence $\Lambda' = (\varepsilon_j)_{j \in \mathbb{N}}$ and a limiting Radon measure \tilde{w} in Ω such that $\tilde{w}^\varepsilon \rightharpoonup^* \tilde{w}$, i.e.

$$\lim_{\varepsilon \rightarrow 0, \varepsilon \in \Lambda'} \varepsilon \int_{\Gamma^\varepsilon} w^\varepsilon(x) \zeta(x) d\mathcal{H}^{d-1}(x) = \int_{\Omega} \zeta(x) d\tilde{w}(x) \quad \forall \zeta \in C_c^0(\Omega). \quad (2.42)$$

On the other hand, keeping the same notation of (1.18), we have

$$w^\varepsilon = \frac{d\tilde{w}^\varepsilon}{d\lambda^\varepsilon}, \quad b^\varepsilon(w^\varepsilon) = \int_{\Omega} \left| \frac{d\tilde{w}^\varepsilon}{d\lambda^\varepsilon}(x) \right|^2 d\lambda^\varepsilon(x).$$

Since $\lambda^\varepsilon \rightharpoonup^* \lambda = \beta \mathcal{L}^d$, as showed by Remark 2.7, general lower semicontinuity results on integral functionals defined on measures [4, Thm. 2.34] show that for every convex function $\psi : \mathbb{R} \rightarrow [0, +\infty]$ with superlinear growth (in our case $\psi(s) := s^2$)

$$\liminf_{\varepsilon \downarrow 0, \varepsilon \in \Lambda'} \int_{\Omega} \psi \left(\frac{d\tilde{w}^\varepsilon}{d\lambda^\varepsilon}(x) \right) d\lambda^\varepsilon(x) < +\infty \quad \Rightarrow \quad \tilde{w} = w \cdot \lambda << \lambda \quad (2.43)$$

and

$$\int_{\Omega} \psi(w(x)) d\lambda(x) \leq \liminf_{\varepsilon \rightarrow 0, \varepsilon \in \Lambda'} \int_{\Omega} \psi\left(\frac{d\tilde{w}^\varepsilon}{d\lambda^\varepsilon}\right) d\lambda^\varepsilon. \quad (2.44)$$

It follows that $\tilde{w} = \beta w \mathcal{L}^d$ for $w \in \mathcal{H} = L^2(\Omega)$ and (2.42) yields (1.17). \square

2.5. Lower semicontinuity and convergence results for integral functionals on Γ^ε . Arguing as in the last part of the proof of Proposition 2.13 and taking into account (2.43) and (2.44), we have:

PROPOSITION 2.14 (Lower semicontinuity for the convergence 1.1). *Let $v^\varepsilon \in L^1_{loc}(\Gamma^\varepsilon)$, $\varepsilon \in \Lambda$, be converging to $v \in L^1_{loc}(\Omega)$ according to Definition 1.1 and let $\psi : \mathbb{R} \rightarrow [0, +\infty]$ be a convex, lower semicontinuous function with superlinear growth. We have*

$$\liminf_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} \int_{\Gamma^\varepsilon} \psi(v^\varepsilon(x)) d\mathcal{H}^{d-1}(x) \geq \beta \int_{\Omega} \psi(v(x)) dx. \quad (2.45)$$

When ψ is locally Lipschitz and v^ε are uniformly bounded with uniformly bounded extensions in $H^1_{loc}(\Omega)$, we can prove a convergence result.

LEMMA 2.15. *If $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is a locally Lipschitz function and*

$$v^\varepsilon \rightharpoonup v \quad \text{weakly in } H^1_{loc}(\Omega) \text{ as } \varepsilon \downarrow 0, \varepsilon \in \Lambda, \quad \sup_{\varepsilon \in \Lambda} \|v^\varepsilon\|_{L^\infty(\Omega)} = S < +\infty,$$

then

$$\lim_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} \int_{\Gamma^\varepsilon} \psi(v^\varepsilon) d\mathcal{H}^{d-1} = \beta \int_{\Omega} \psi(v) dx. \quad (2.46)$$

Proof. Up to a possible modification of ψ outside $[-S, S]$, it is not restrictive to assume that ψ is globally Lipschitz; since $\psi(v^\varepsilon) \rightharpoonup \psi(v)$ in $H^1_{loc}(\Omega)$, it is not restrictive to assume that ψ is the identity in (2.46).

If $\zeta \in C_c^0(\Omega)$ Proposition 2.8 yields

$$\begin{aligned} & \limsup_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} \left| \varepsilon \int_{\Gamma^\varepsilon} v d\mathcal{H}^{d-1} - \beta \int_{\Omega} v dx \right| \\ & \leq \limsup_{\varepsilon \downarrow 0} \varepsilon \int_{\Gamma^\varepsilon} |v^\varepsilon| |1 - \zeta| d\mathcal{H}^{d-1} + \beta \int_{\Omega} |v| |1 - \zeta| dx \\ & \leq S \limsup_{\varepsilon \downarrow 0} \varepsilon \int_{\Gamma^\varepsilon} |1 - \zeta| d\mathcal{H}^{d-1} + S\beta \int_{\Omega} |1 - \zeta| dx \\ & \leq 2S\beta \int_{\Omega} |1 - \zeta| dx. \end{aligned}$$

Taking the infimum of the last integral with respect to ζ we conclude. \square

Combining Proposition 2.14, Lemma 2.15, and the equivalence property stated by (2.41) of Corollary 2.11, we obtain

COROLLARY 2.16. *If $u^\varepsilon \in \mathcal{Y}_0^\varepsilon$, $\varepsilon \in \Lambda$, is a family satisfying the bounded energy condition (2.30) and converging to $u \in \mathcal{Y}_0$ according to Definition 1.1, then for every convex functional $\psi : \mathbb{R} \rightarrow [0, +\infty)$ we have*

$$\liminf_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} \int_{\Gamma^\varepsilon} \psi(u_i^\varepsilon - u_e^\varepsilon) d\mathcal{H}^{d-1} \geq \beta \int_{\Omega} \psi(u_i - u_e) dx. \quad (2.47)$$

Moreover, if $\sup_{\varepsilon \in \Lambda} \|u_i^\varepsilon - u_e^\varepsilon\|_{L^\infty(\Gamma^\varepsilon)} < +\infty$, then

$$\lim_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} \int_{\Gamma^\varepsilon} \psi(u_i^\varepsilon - u_e^\varepsilon) d\mathcal{H}^{d-1} = \beta \int_{\Omega} \psi(u_i - u_e) dx. \quad (2.48)$$

We conclude this section with a final auxiliary result.

LEMMA 2.17 (Bulk energy approximation). *Let $\psi : \mathbb{R} \rightarrow [0, +\infty)$ be a locally Lipschitz function such that $\psi' \geq 0$ in $(a, +\infty)$, $\psi' \leq 0$ in $(-\infty, -a)$ for some $a > 0$, and let $\underline{u} \in \mathcal{V}_0$ such that*

$$\int_{\Omega} \psi(u_i - u_e) dx < +\infty. \quad (2.49)$$

There exists a sequence $(u_k)_{k \in \mathbb{N}} \subset \mathcal{V}_0$ such that

$$\begin{aligned} u_{i,k} - u_{e,k} &\in L^\infty(\Omega), \quad \lim_{k \uparrow \infty} u_k \rightarrow \underline{u} \quad \text{strongly in } \mathcal{V}_0, \\ \lim_{k \uparrow \infty} \int_{\Omega} \psi(u_{i,k} - u_{e,k}) dx &= \int_{\Omega} \psi(u_i - u_e) dx. \end{aligned} \quad (2.50)$$

Proof. Recalling that $v := u_i - u_e$, we set

$$v_k := (v \wedge k) \vee (-k), \quad s = (u_i + u_e)/2, \quad \gamma_k := \frac{1}{\mathcal{L}^d(\Omega_0)} \int_{\Omega_0} (s(x) - v_k(x)/2) dx,$$

and

$$u_{i,k} := s - \gamma_k + v_k/2, \quad u_{e,k} := s - \gamma_k - v_k/2.$$

By construction $u_k \in \mathcal{V}_0$ and

$$u_{i,k} - u_{e,k} = v_k \in L^\infty(\Omega), \quad \lim_{k \uparrow \infty} v_k = v \quad \text{strongly in } H^1(\Omega).$$

In particular

$$\lim_{k \uparrow \infty} \gamma_k = \frac{1}{\mathcal{L}^d(\Omega_0)} \int_{\Omega_0} (s(x) - v(x)/2) dx = \frac{1}{\mathcal{L}^d(\Omega_0)} \int_{\Omega_0} u_e(x) dx = 0,$$

so that

$$u_k \rightarrow \underline{u} \quad \text{strongly in } \mathcal{V}_0 \quad \text{as } k \uparrow \infty.$$

Finally, since $k \mapsto \psi(v_k(x))$ is (definitively) nondecreasing and converges pointwise to $\psi(v(x))$, Beppo Levi's theorem yields

$$\lim_{k \uparrow +\infty} \int_{\Omega} \psi(v_k(x)) dx = \int_{\Omega} \psi(v(x)) dx.$$

□

3. Γ -convergence results.

3.1. Γ -convergence. For the reader's convenience, we include hereafter a few definitions and theorems used in the following [25, 17].

DEFINITION 3.1 (Γ -convergence). *Let (X, d) be a metric space and $\mathcal{F}^\varepsilon, \mathcal{F}, \varepsilon \in \Lambda$, be functionals from X into $[-\infty, +\infty]$. We say that $(\mathcal{F}^\varepsilon)_{\varepsilon \in \Lambda}$ $\Gamma(X)$ -converges to \mathcal{F} as $\varepsilon \downarrow 0, \varepsilon \in \Lambda$, i.e.*

$$\mathcal{F} = \Gamma(X)\text{-}\lim_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} \mathcal{F}^\varepsilon,$$

if for every $x \in X$ the following conditions are fulfilled:

$$\forall x^\varepsilon \in X : \lim_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} x^\varepsilon = x \quad \Rightarrow \quad \liminf_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} \mathcal{F}^\varepsilon(x^\varepsilon) \geq \mathcal{F}(x) \quad (3.1)$$

$$\exists \hat{x}^\varepsilon \in X : \lim_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} \hat{x}^\varepsilon = x, \quad \limsup_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} \mathcal{F}^\varepsilon(\hat{x}^\varepsilon) \leq \mathcal{F}(x). \quad (3.2)$$

REMARK 3.2. Notice that by (3.1) the “lim sup” in (3.2) is in fact a limit. We will sometimes use a slight variant of this property, when $\mathcal{F}^\varepsilon, \mathcal{F}$ admit the decomposition $\mathcal{F}^\varepsilon = \mathcal{F}_1^\varepsilon + \mathcal{F}_2^\varepsilon, \mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$, and $\mathcal{F}_1^\varepsilon, \mathcal{F}_2^\varepsilon$ will satisfy condition (3.1) with respect to $\mathcal{F}_1, \mathcal{F}_2$ respectively. In this case, every “optimal” family \hat{x}^ε for \mathcal{F}^ε satisfies

$$\hat{x}^\varepsilon \rightarrow x, \limsup_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} \mathcal{F}^\varepsilon(\hat{x}^\varepsilon) \leq \mathcal{F}(x) \quad \Rightarrow \quad \lim_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} \mathcal{F}_j^\varepsilon(\hat{x}^\varepsilon) = \mathcal{F}_j(x), \quad j = 1, 2. \quad (3.3)$$

THEOREM 3.3 (see [17], Corollary 2.4). *Let (X, d) a metric space, $\mathcal{F}^\varepsilon, \mathcal{F}, \varepsilon \in \Lambda$, be functionals from X into $(-\infty, +\infty]$ such that $\mathcal{F} = \Gamma(X)\text{-}\lim_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} \mathcal{F}^\varepsilon$, and let $x^\varepsilon \in X$ be a minimizer for \mathcal{F}^ε , i.e.*

$$\mathcal{F}^\varepsilon(x^\varepsilon) = \min \{ \mathcal{F}^\varepsilon(x) : x \in X \}.$$

*If the family $(x^\varepsilon)_{\varepsilon \in \Lambda}$ is **relatively compact in X** and x^0 is the unique minimizer for \mathcal{F} , then*

$$\lim_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} x^\varepsilon = x^0, \quad \lim_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} \mathcal{F}^\varepsilon(x^\varepsilon) = \mathcal{F}(x^0). \quad (3.4)$$

The following useful criterium [10, p. 97] allows to check (3.2) on a smaller \mathcal{F} -dense subset $D \subset X$.

PROPOSITION 3.4 (A density argument for Γ -lim sup). *Let $X, \mathcal{F}^\varepsilon, \mathcal{F}$ as in Definition 3.1 and let $D \subset X$ be satisfying*

$$\forall x \in X \quad \exists x^\varepsilon \in D : \lim_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} x^\varepsilon = x, \quad \lim_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} \mathcal{F}^\varepsilon(x^\varepsilon) = \mathcal{F}(x). \quad (3.5)$$

If

$$\forall x \in D, \quad \exists x^\varepsilon \in X : \lim_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} x^\varepsilon = x, \quad \limsup_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} \mathcal{F}^\varepsilon(x^\varepsilon) \leq \mathcal{F}(x), \quad (3.6)$$

then the “ Γ -lim sup” condition (3.2) for Γ -convergence is satisfied.

Combining the results of A. BRAIDES ([11], Theorem 14.8) and of E. ACERBI ET AL. [1] we obtain the following homogenization result for noncoercive integral functionals.

THEOREM 3.5. *Let us consider the family of integral functionals in $L^2_{loc}(\Omega)$*

$$a_{i,e}^\varepsilon(u) := \begin{cases} \int_{\Omega_{i,e}^\varepsilon} \sigma_{i,e}^\varepsilon(x) \nabla u(x) \cdot \nabla u(x) dx & \text{if } u \in L^2_{loc}(\Omega), u|_{\Omega_{i,e}} \in H^1(\Omega_{i,e}), \\ +\infty & \text{otherwise,} \end{cases} \quad (3.7)$$

where $\sigma_{i,e}^\varepsilon$ were introduced in (1.8) and (1.9), and let us define

$$a_{i,e}(u) := \begin{cases} \int_{\Omega} M_{i,e}(x) \nabla u(x) \cdot \nabla u(x) dx & \text{if } u \in H^1(\Omega), \\ +\infty & \text{if } u \in L^2_{loc}(\Omega) \setminus H^1(\Omega). \end{cases} \quad (3.8)$$

with $M_{i,e}$ defined as in (1.22). Then we have

$$a_{i,e}(u) = \Gamma(L^2_{loc}(\Omega))\text{-}\lim_{\varepsilon \downarrow 0} a_{i,e}^\varepsilon(u) = \Gamma(L^\infty(\Omega))\text{-}\lim_{\varepsilon \downarrow 0} a_{i,e}^\varepsilon(u). \quad (3.9)$$

3.2. Γ -convergence of \mathcal{F}^ε and proof of Theorem 3. In this paragraph we want to prove Theorem 3. The natural topology for this variational approach should be the one introduced by Definition 1.1. Therefore, in order to apply Γ -convergence technique, we have to imbed the domain of the functionals $\mathcal{F}^\varepsilon, \mathcal{F}$ in a fixed underlying metric space, whose converging sequences with equi-bounded energy coincide with those characterized by Definition 1.1.

To this aim, we consider the space of finite signed Radon measures on Ω [4, 1.57]

$$\mathcal{M} := \mathcal{M}(\Omega) = (C_c(\Omega))' = (C_0(\Omega))', \quad (3.10)$$

endowed with the dual norm

$$\|\mu\|_{\mathcal{M}} := |\mu|(\Omega), \quad (3.11)$$

and the (weaker) continuous distance

$$d(\mu, \nu) := \sup \left\{ \int_{\Omega} \zeta(x) d(\mu - \nu)(x) : \zeta \in C_c(\Omega) \cap \text{Lip}(\Omega), \right. \\ \left. \sup_{x \in \Omega} |\zeta(x)| \leq 1, \quad \text{Lip}(\zeta, \Omega) \leq 1 \right\}, \quad (3.12)$$

where $\text{Lip}(\Omega)$ (resp. $\text{Lip}(\zeta, \Omega)$) is the space of the Lipschitz real functions defined in Ω (resp. the Lipschitz constant of ζ).

Since $C_0(\Omega)$ is a separable Banach space, the dual unit ball of \mathcal{M} is weakly* compact and separable, so that the distance d induces the weak* topology of \mathcal{M} on each norm-bounded subset of \mathcal{M} ; in particular, (\mathcal{M}, d) is a separable metric space and norm-bounded sequences are relatively compact with respect to the weaker topology induced by the distance d .

We then identify vectors $\underline{u}^\varepsilon = (u_i^\varepsilon, u_e^\varepsilon) \in \mathcal{V}_0^\varepsilon, \underline{u} = (u_i, u_e) \in \mathcal{V}_0$ with measures $\tilde{\underline{u}}^\varepsilon = (\tilde{u}_i^\varepsilon, \tilde{u}_e^\varepsilon), \tilde{\underline{u}} = (\tilde{u}_i, \tilde{u}_e) \in \mathcal{M}^2$ as in (1.19) and (1.20), denoting by $\mathfrak{m}^\varepsilon : \mathcal{V}_0^\varepsilon \rightarrow \mathcal{M}^2, \mathfrak{m} : \mathcal{V}_0 \rightarrow \mathcal{M}^2$ the corresponding maps.

This operator allows to extend all the functionals on $\mathcal{V}_0^\varepsilon$ to \mathcal{M}^2 : e.g., in the case of \mathcal{F}^ε we set

$$\tilde{\mathcal{F}}^\varepsilon(\tilde{\underline{u}}) := \begin{cases} \mathcal{F}^\varepsilon(\underline{u}) & \text{if } \tilde{\underline{u}} = \mathfrak{m}^\varepsilon(\underline{u}), \text{ in } \mathcal{M}^2, \\ +\infty & \text{otherwise.} \end{cases} \quad (3.13)$$

We can thus consider coercivity and Γ -convergence of $\tilde{\mathcal{F}}^\varepsilon$ in \mathcal{M}^2 as $\varepsilon \downarrow 0$ which are in fact equivalent to statements (a),(b),(c) of Theorem 3. We split the proof of these properties in three steps. Recall that by (1.32) we can assume that $v_0^\varepsilon = B^\varepsilon u_0^\varepsilon$ with

$$\limsup_{\varepsilon \downarrow 0} \left(\underline{a}^\varepsilon(u_0^\varepsilon) + \underline{b}^\varepsilon(u_0^\varepsilon) \right) < +\infty, \quad (3.14)$$

v_0^ε converges to v_0 according to Definition 1.1, and $b^\varepsilon(v_0^\varepsilon) \rightarrow b(v_0)$; the functional \mathcal{F}^ε takes the form

$$\mathcal{F}^\varepsilon(u^\varepsilon) := \frac{h}{2} b^\varepsilon(v^\varepsilon - v_0^\varepsilon) + \frac{1}{2} \underline{a}^\varepsilon(u^\varepsilon) + \phi^\varepsilon(v^\varepsilon), \quad v^\varepsilon := B^\varepsilon u^\varepsilon. \quad (3.15)$$

(a) **Compactness.** It follows directly from Proposition 2.13, thanks to (1.33).

(b) **“liminf” inequality.** Suppose that $u^\varepsilon \in \mathcal{V}_0^\varepsilon$, $\varepsilon \in \Lambda$, converges to $u \in \mathcal{V}_0$ as $\varepsilon \downarrow 0$, $\varepsilon \in \Lambda$ according to Definition 1.1 and satisfies

$$\limsup_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} \left(\underline{a}^\varepsilon(u^\varepsilon) + \underline{b}^\varepsilon(u^\varepsilon) \right) < +\infty. \quad (3.16)$$

Corollary 2.11 and Theorem 3.5 yield

$$\liminf_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} \underline{a}^\varepsilon(u^\varepsilon) \geq \underline{a}(u), \quad (3.17)$$

whereas (2.47) of Corollary 2.16 and (3.14) show that

$$\liminf_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} b^\varepsilon(v^\varepsilon - v_0^\varepsilon) \geq b(u - u_0), \quad \liminf_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} \phi^\varepsilon(v^\varepsilon) \geq \phi(v), \quad v := Bu. \quad (3.18)$$

(c) **“limsup” inequality.** We introduce the set

$$D := \left\{ u = (u_i, u_e) \in \mathcal{V}_0 : Bu = v = u_i - u_e \in L^\infty(\Omega) \right\},$$

which satisfies the density condition (3.5) thanks to Lemma 2.17. By Proposition 3.4 it is then sufficient to prove the “limsup” inequality for $u \in D$.

By Theorem 3.5 we find a uniformly bounded family $\tilde{u}^\varepsilon = (\tilde{u}_i^\varepsilon, \tilde{u}_e^\varepsilon)$ converging to u in $L^\infty(\Omega) \times L^\infty(\Omega)$, whose restriction u^ε to $\Omega_i^\varepsilon \times \Omega_e^\varepsilon$ belongs to $\mathcal{V}_0^\varepsilon$ (we can add a vanishing constant to \tilde{u}_e^ε as in Lemma 2.17) such that

$$\lim_{\varepsilon \downarrow 0} \underline{a}^\varepsilon(u^\varepsilon) = \underline{a}(u).$$

The boundedness and the regularity of Ω shows that $\underline{b}^\varepsilon(u^\varepsilon)$ is bounded so that (3.16) holds. Therefore, a simple application of (2.48) of Corollary 2.16 yields

$$\lim_{\varepsilon \downarrow 0} b^\varepsilon(v^\varepsilon) = b(v), \quad \lim_{\varepsilon \downarrow 0} \phi^\varepsilon(v^\varepsilon) = \phi(v), \quad v^\varepsilon := B^\varepsilon u^\varepsilon, \quad v = Bu. \quad (3.19)$$

In order to conclude the proof we have to pass to the limit in the term $b^\varepsilon(v^\varepsilon - v_0^\varepsilon)$: we invoke the next lemma.

LEMMA 3.6. *For every couple $v_0^\varepsilon, v^\varepsilon \in \mathcal{H}^\varepsilon$ converging to v_0, v as in Definition 1.1, we have*

$$\left. \begin{aligned} \lim_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} b^\varepsilon(v_0^\varepsilon) &= b(v_0) \\ \limsup_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} b^\varepsilon(v^\varepsilon) &= S < +\infty \end{aligned} \right\} \Rightarrow \lim_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} b^\varepsilon(v_0^\varepsilon, v^\varepsilon) = b(v_0, v). \quad (3.20)$$

Proof. Let us recall that

$$\liminf_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} b^\varepsilon(z^\varepsilon) \geq b(z), \quad (3.21)$$

for every family $z^\varepsilon \in \mathcal{H}^\varepsilon$ converging to z according to Definition 1.1. For every positive scalar $\rho > 0$ we have

$$2b^\varepsilon(v_0^\varepsilon, v^\varepsilon) = 2b^\varepsilon(\rho^{-1}v_0^\varepsilon, \rho v^\varepsilon) = b^\varepsilon(\rho^{-1}v_0^\varepsilon + \rho v^\varepsilon) - \rho^{-2}b^\varepsilon(v_0^\varepsilon) - \rho^2b^\varepsilon(v^\varepsilon).$$

Taking the inferior limit as $\varepsilon \downarrow 0, \varepsilon \in \Lambda$ we get

$$\begin{aligned} \liminf_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} 2b^\varepsilon(v_0^\varepsilon, v^\varepsilon) &\geq b(\rho^{-1}v_0 + \rho v) - \rho^{-2}b(v_0) - \rho^2b(v) \\ &= 2b(v_0, v) + \rho^2(b(v) - S). \end{aligned}$$

Since $\rho > 0$ is arbitrary, we obtain

$$\liminf_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} b^\varepsilon(v_0^\varepsilon, v^\varepsilon) \geq b(v_0, v);$$

inverting sign to v_0^ε we get the thesis. \square

As a corollary, we also obtain the following accessory result:

COROLLARY 3.7. *Let $v^\varepsilon \in \mathcal{H}^\varepsilon$ be converging to $v \in \mathcal{H}$ according to Definition 1.1 and let us suppose that*

$$\lim_{\varepsilon \downarrow 0} b^\varepsilon(v^\varepsilon) = b(v), \quad \limsup_{\varepsilon \downarrow 0} j^\varepsilon(v^\varepsilon) < +\infty. \quad (3.22)$$

If $\underline{u}^\varepsilon = (u_i^\varepsilon, u_e^\varepsilon)$ is the unique solution of the minimum problem

$$\min \left\{ \underline{a}^\varepsilon(\underline{u}) : \underline{u} \in \mathcal{V}_0^\varepsilon, B^\varepsilon \underline{u} = v^\varepsilon \right\}, \quad (3.23)$$

then $\underline{u}^\varepsilon \rightarrow \underline{u}$ as $\varepsilon \downarrow 0$ according to Definition 1.1 and

$$\underline{a}^\varepsilon(\underline{u}^\varepsilon) = j^\varepsilon(v^\varepsilon) \rightarrow \underline{a}(\underline{u}) = j(v), \quad (3.24)$$

\underline{u} being the unique solution of

$$\min \left\{ \underline{a}(\underline{u}) : \underline{u} \in \mathcal{V}_0, B\underline{u} = v \right\}. \quad (3.25)$$

4. Variational formulation of the evolution problems. In this section we collect some basic notation and preliminary results on the variational formulation and the related well posedness of the micro- and macroscopic problems as discussed in [15].

4.1. Nonlinear terms and convex primitives. Recalling (1.11), from now on we set

$$\lambda_F := 1 + \left(\inf_{x \in \mathbb{R}} F'(x) \right)^-, \quad f(x) := F(x) + \lambda_F x, \quad (4.1)$$

$$\varphi(x) := \int_0^x f(\rho) d\rho = \frac{1}{2} \lambda_F x^2 + \int_0^x F(\rho) d\rho. \quad (4.2)$$

Observe that f is a strictly increasing C^1 function with $f' \geq 1$, so that φ is a strictly convex function with (at least) quadratic growth, thus satisfying

$$\varphi(x) \geq \varphi(0) = 0, \quad \varphi(x) \geq \frac{1}{2}|x|^2 \quad \forall x \in \mathbb{R}. \quad (4.3)$$

The conjugate function $\varphi^* : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\varphi^*(y) = \sup_{x \in \mathbb{R}} (yx - \varphi(x)) \quad (4.4)$$

is still a strictly convex function which satisfies

$$\varphi^*(y) \geq \varphi^*(0) = 0 \quad \forall y \in \mathbb{R}, \quad \lim_{|y| \rightarrow \infty} \frac{\varphi^*(y)}{|y|} = +\infty, \quad (4.5)$$

$$xy \leq \varphi(x) + \varphi^*(y), \quad xf(x) = \varphi(x) + \varphi^*(f(x)) \quad \forall x, y \in \mathbb{R}. \quad (4.6)$$

In particular, we have

$$|f(x)| \leq \varphi^*(f(x)) + \sup_{|z| \leq 1} \varphi(z) \quad \forall x \in \mathbb{R}, \quad (4.7)$$

and the “subdifferential inequality”

$$y = f(x) \quad \Leftrightarrow \quad y(z - x) \leq \varphi(z) - \varphi(x) \quad \forall z \in \mathbb{R}. \quad (4.8)$$

4.2. Vector notation for the complete system. Besides the couples $\underline{u}^\varepsilon = (u_i^\varepsilon, u_e^\varepsilon)$, $\underline{u} = (u_i, u_e)$, the micro- and macro-scopic evolution problems involve a third “recovery” variable w^ε, w . Thus the electric state of the heart will be determined by the three-component vectors

$$\mathbf{u}^\varepsilon := (\underline{u}^\varepsilon, w^\varepsilon), \quad \mathbf{u} := (\underline{u}, w). \quad (4.9)$$

As a general rule, boldface letters (as $\mathbf{u}, \phi, \mathbf{V}, \dots$) will occur when the 3-component vectors which determine the electric state of our systems are involved.

We will adopt the usual convention to identify functions $u = u(x, t)$ defined in the space-time cylinders $A \times (0, T)$ (A being some open subset of \mathbb{R}^d endowed with the Lebesgue measure \mathcal{L}^d or some Lipschitz hypersurface endowed with the $(d-1)$ -dimensional Hausdorff measure \mathcal{H}^{d-1}) with the time dependent function $u(\cdot, t)$ taking its values in suitable function subspaces of $L_{loc}^1(A)$ (we omit to indicate explicitly the dependence on the underlying measure \mathcal{L}^d or \mathcal{H}^{d-1}). Vector functions will therefore take their values in suitable product vector spaces, which we are now introducing.

As we said before in § 2.1, the electric potentials $\underline{u}^\varepsilon$ (resp. \underline{u}) will take their values in $\mathcal{V}_0^\varepsilon$ (resp. \mathcal{V}_0), whereas the recovery variables w^ε (resp. w) are valued into \mathcal{H}^ε (resp. \mathcal{H}). Therefore, we will also introduce the product spaces for the vector functions $\mathbf{u}^\varepsilon, \mathbf{u}$:

$$\mathcal{V}^\varepsilon := \mathcal{V}^\varepsilon \times \mathcal{H}^\varepsilon, \quad \mathcal{V}_0^\varepsilon := \mathcal{V}_0^\varepsilon \times \mathcal{H}^\varepsilon, \quad \mathcal{V} := \mathcal{V} \times \mathcal{H}, \quad \mathcal{V}_0 := \mathcal{V}_0 \times \mathcal{H}. \quad (4.10)$$

4.3. The variational formulation of the microscopic problem and its well posedness. As showed in [15], the variational formulation of Problem \mathbf{P}^ε can be easily obtained by performing the following steps:

1. choose test functions $\hat{\mathbf{u}} = (\hat{u}_i, \hat{u}_e, \hat{w}) \in \mathcal{V}_0^\varepsilon$;

2. multiply (P_{1a}^ε) by $\hat{u}_{i,e}$, integrate by parts using (P_{1b}^ε) and (P_4^ε) , and sum up taking (P_2^ε) into account;
3. take the b^ε -scalar product in \mathcal{H}^ε of (P_3^ε) with \hat{w} ;
4. sum up the results of the previous two steps.

In order to write the variational formulation in a compact form, we introduce the bilinear forms which are defined for every $\mathbf{u}, \hat{\mathbf{u}} \in \mathbf{V}^\varepsilon = \mathcal{V}^\varepsilon \times \mathcal{H}^\varepsilon$

$$\mathbf{b}^\varepsilon(\mathbf{u}, \hat{\mathbf{u}}) := \underline{b}^\varepsilon(u, \hat{u}) + b^\varepsilon(w, \hat{w}), \quad \mathbf{a}^\varepsilon(\mathbf{u}, \hat{\mathbf{u}}) := \underline{a}^\varepsilon(u, \hat{u}) + \gamma b^\varepsilon(w, \hat{w}), \quad (4.11)$$

$$\mathbf{g}^\varepsilon(\mathbf{u}, \hat{\mathbf{u}}) := \lambda_F \underline{b}^\varepsilon(u, \hat{u}) - \Theta b^\varepsilon(w, \underline{B}^\varepsilon \hat{u}) + \eta b^\varepsilon(\underline{B}^\varepsilon u, \hat{w}), \quad (4.12)$$

together with the related quadratic forms

$$\mathbf{a}^\varepsilon(\mathbf{u}) := \mathbf{a}^\varepsilon(\mathbf{u}, \mathbf{u}), \quad \mathbf{b}^\varepsilon(\mathbf{u}) := \mathbf{b}^\varepsilon(\mathbf{u}, \mathbf{u}), \quad (4.13)$$

the functionals

$$\phi^\varepsilon(\mathbf{u}) := \varepsilon \int_{\Gamma^\varepsilon} \varphi(u_i - u_e) d\mathcal{H}^{d-1}, \quad \psi^\varepsilon(\mathbf{u}) := \varepsilon \int_{\Gamma^\varepsilon} \varphi^*(f(u_i - u_e)) d\mathcal{H}^{d-1}, \quad (4.14)$$

the nonlinear form

$$\mathcal{F}^\varepsilon(\mathbf{u}, \hat{\mathbf{u}}) = \varepsilon \int_{\Gamma^\varepsilon} f(u_i - u_e)(\hat{u}_i - \hat{u}_e) d\mathcal{H}^{d-1} = \varepsilon \int_{\Gamma^\varepsilon} f(v) \hat{v} d\mathcal{H}^{d-1}, \quad (4.15)$$

which is well defined by (4.6) if $\psi^\varepsilon(\mathbf{u}), \phi^\varepsilon(\hat{\mathbf{u}}) < +\infty$. Adopting the point of view to consider $u_i^\varepsilon, u_e^\varepsilon, w^\varepsilon$ as time dependent functions with values in a “space dependent” functional space, as we discussed in § 4.2, Problem \mathbf{P}^ε can be solved by looking for the solution $\mathbf{u}^\varepsilon : (0, T) \rightarrow \mathbf{V}_0^\varepsilon$ of the abstract variational equation

$$\begin{cases} \frac{d}{dt} \mathbf{b}^\varepsilon(\mathbf{u}^\varepsilon, \hat{\mathbf{u}}) + \mathbf{a}^\varepsilon(\mathbf{u}^\varepsilon, \hat{\mathbf{u}}) + \mathcal{F}^\varepsilon(\mathbf{u}^\varepsilon, \hat{\mathbf{u}}) = \mathbf{g}^\varepsilon(\mathbf{u}^\varepsilon, \hat{\mathbf{u}}), \\ \mathbf{b}^\varepsilon(\mathbf{u}^\varepsilon(0), \hat{\mathbf{u}}) = \mathbf{b}^\varepsilon(\mathbf{u}_0^\varepsilon, \hat{\mathbf{u}}), \end{cases} \quad (4.16)$$

for any $\hat{\mathbf{u}} \in \mathbf{V}_0^\varepsilon$ with $\phi^\varepsilon(\hat{\mathbf{u}}) < +\infty$. Here the initial datum $\mathbf{u}_0^\varepsilon := (u_{0,i}^\varepsilon, u_{0,e}^\varepsilon, w_0^\varepsilon)$ satisfies

$$\mathbf{u}_0^\varepsilon \in \mathbf{V}_0^\varepsilon, \quad \phi^\varepsilon(\mathbf{u}_0^\varepsilon) < +\infty, \quad (4.17)$$

and it is related to v_0^ε by

$$\underline{B}^\varepsilon u_0^\varepsilon = v_0^\varepsilon, \quad \underline{a}^\varepsilon(u_0^\varepsilon) = j^\varepsilon(v_0^\varepsilon) = \min \left\{ \underline{a}^\varepsilon(\hat{u}) : \underline{B}^\varepsilon \hat{u} = v_0^\varepsilon \right\}. \quad (4.18)$$

The following theorem provides an existence result for the microscopic problem; see [15] for more details.

THEOREM 4.1. *Let us assume that \mathbf{u}_0^ε satisfies (4.17) and (4.18). Then, there exists a unique solution of the variational formulation (4.16) of Problem \mathbf{P}^ε*

$$\mathbf{u}^\varepsilon = (u_i^\varepsilon, u_e^\varepsilon, w^\varepsilon) \in C^0([0, T]; \mathbf{V}_0^\varepsilon), \quad v^\varepsilon := \underline{B}^\varepsilon u^\varepsilon,$$

with

$$\partial_t(\underline{B}^\varepsilon u^\varepsilon) = \partial_t v^\varepsilon, \quad \partial_t v^\varepsilon \in L^2(0, T; \mathcal{H}^\varepsilon), \quad (4.19)$$

$$\sup_{t \in (0, T)} \phi^\varepsilon(\mathbf{u}^\varepsilon) = \sup_{t \in (0, T)} \varepsilon \int_{\Gamma^\varepsilon} \varphi(v^\varepsilon) d\mathcal{H}^{d-1} < +\infty, \quad (4.20)$$

$$\int_0^T \psi^\varepsilon(\mathbf{u}^\varepsilon) dt = \varepsilon \int_0^T \int_{\Gamma^\varepsilon} \varphi^*(f(v^\varepsilon)) d\mathcal{H}^{d-1} dt < +\infty. \quad (4.21)$$

Moreover, the solution \mathbf{u}^ε satisfies the *a priori* estimates

$$\left. \begin{aligned} & \sup_{t \in [0, T]} \left(\mathbf{b}^\varepsilon(\mathbf{u}^\varepsilon) + \phi^\varepsilon(\mathbf{u}^\varepsilon) + \mathbf{a}^\varepsilon(\mathbf{u}^\varepsilon) \right) \\ & \int_0^T \left(b^\varepsilon(\partial_t v^\varepsilon) + b^\varepsilon(\partial_t w^\varepsilon) \right) dt \end{aligned} \right\} \leq C \left(\mathbf{b}^\varepsilon(\mathbf{u}_0^\varepsilon) + \phi^\varepsilon(\mathbf{u}_0^\varepsilon) + \mathbf{a}^\varepsilon(\mathbf{u}_0^\varepsilon) \right) \quad (4.22)$$

for a constant C independent of ε and of the initial datum; finally, at each time $t \in [0, T]$ $\mathbf{u}^\varepsilon(t)$ is the unique solution of the minimum problem

$$B^\varepsilon \mathbf{u}^\varepsilon(t) = v^\varepsilon(t), \quad \mathbf{a}^\varepsilon(\mathbf{u}^\varepsilon(t)) = j^\varepsilon(v^\varepsilon(t)) = \min \left\{ \mathbf{a}^\varepsilon(\hat{\mathbf{u}}^\varepsilon) : B^\varepsilon \hat{\mathbf{u}}^\varepsilon = v^\varepsilon(t) \right\}. \quad (4.23)$$

Theorem 4.1 and (4.22) relies on two *a priori* estimates which are interesting by themselves: we will briefly present their formal derivation after a short discussion on the abstract structure of the system (4.16). As we will see in the next paragraph, the main interest of this approach is that the macroscopic problem \mathbf{P} can be formulated in the same way.

4.4. The variational formulation of the macroscopic problem and its well posedness. The derivation of the variational formulation of this problem is completely analogous to the previous one; as before we introduce the bilinear forms on $\mathcal{V} \supset \mathcal{V}_0$

$$\mathbf{b}(\mathbf{u}, \hat{\mathbf{u}}) := \underline{b}(\mathbf{u}, \hat{\mathbf{u}}) + b(w, \hat{w}), \quad \mathbf{a}(\mathbf{u}, \hat{\mathbf{u}}) := \underline{a}(\mathbf{u}, \hat{\mathbf{u}}) + \gamma b(w, \hat{w}), \quad (4.24)$$

$$\mathbf{g}(\mathbf{u}, \hat{\mathbf{u}}) := \lambda_F \underline{b}(\mathbf{u}, \hat{\mathbf{u}}) - \Theta b(w, B\hat{\mathbf{u}}) + \eta b(B\mathbf{u}, \hat{w}), \quad (4.25)$$

together with the related quadratic forms

$$\mathbf{a}(\mathbf{u}) := \mathbf{a}(\mathbf{u}, \mathbf{u}), \quad \underline{b}(\mathbf{u}) := \underline{b}(\mathbf{u}, \mathbf{u}), \quad (4.26)$$

the functionals

$$\phi(\mathbf{u}) := \beta \int_{\Omega} \varphi(u_i - u_e) dx, \quad \psi(\mathbf{u}) := \beta \int_{\Omega} \varphi^*(f(u_i - u_e)) dx, \quad (4.27)$$

the nonlinear form

$$\mathcal{F}(\mathbf{u}, \hat{\mathbf{u}}) = \beta \int_{\Omega} f(u_i - u_e)(\hat{u}_i - \hat{u}_e) dx = \beta \int_{\Omega} f(v) \hat{v} dx, \quad (4.28)$$

which is well defined by (4.6) if $\psi(\mathbf{u}), \phi(\hat{\mathbf{u}}) < +\infty$. The solution $\mathbf{u} : (0, T) \rightarrow \mathcal{V}_0$ of the macroscopic problem thus satisfies the variational evolution equation

$$\begin{cases} \frac{d}{dt} \mathbf{b}(\mathbf{u}, \hat{\mathbf{u}}) + \mathbf{a}(\mathbf{u}, \hat{\mathbf{u}}) + \mathcal{F}(\mathbf{u}, \hat{\mathbf{u}}) = \mathbf{g}(\mathbf{u}, \hat{\mathbf{u}}), \\ \mathbf{b}(\mathbf{u}(0), \hat{\mathbf{u}}) = \mathbf{b}(\mathbf{u}_0, \hat{\mathbf{u}}), \end{cases} \quad (4.29)$$

for any $\hat{\mathbf{u}} \in \mathcal{V}_0$ with $\phi(\hat{\mathbf{u}}) < +\infty$. Again the initial datum $\mathbf{u}_0 := (u_{0,i}, u_{0,e}, w_0)$ satisfies

$$\mathbf{u}_0 \in \mathcal{V}_0, \quad \phi(\mathbf{u}_0) < +\infty, \quad (4.30)$$

and it is related to v_0 by

$$B\mathbf{u}_0 = v_0, \quad \mathbf{a}(\mathbf{u}_0) = j(v_0) = \min \left\{ \mathbf{a}(\hat{\mathbf{u}}) : B\hat{\mathbf{u}} = v_0 \right\}. \quad (4.31)$$

THEOREM 4.2. *Let us assume that \mathbf{u}_0 satisfies (4.30), (4.31). Then, there exists a unique solution of the variational formulation of Problem \mathbf{P}*

$$\mathbf{u} = (u_i, u_e, w) \in C^0([0, T]; \mathcal{V}_0), \quad v = Bu,$$

with

$$\partial_t(Bu) = \partial_t v, \quad \partial_t w \in L^2(0, T; \mathcal{H}), \quad (4.32)$$

$$\sup_{t \in (0, T)} \phi(\mathbf{u}) = \sup_{t \in (0, T)} \int_{\Omega} \varphi(v) dx < +\infty, \quad (4.33)$$

$$\int_0^T \psi(\mathbf{u}) dt = \int_0^T \int_{\Omega} \varphi^*(f(v)) dx dt < +\infty. \quad (4.34)$$

Moreover, the solution \mathbf{u} satisfies the a priori estimates

$$\left. \begin{aligned} & \sup_{t \in [0, T]} \left(\mathbf{b}(\mathbf{u}) + \phi(\mathbf{u}) + \mathbf{a}(\mathbf{u}) \right) \\ & \int_0^T \left(b(\partial_t w) + b(\partial_t v) \right) dt \end{aligned} \right\} \leq C \left(\mathbf{b}(\mathbf{u}_0) + \phi(\mathbf{u}_0) + \mathbf{a}(\mathbf{u}_0) \right) \quad (4.35)$$

for a constant C independent of the initial datum; at each time $t \in [0, T]$, $u(t)$ is the unique solution of the minimum problem

$$Bu(t) = v(t), \quad a(u(t)) = j(v(t)) = \min \left\{ a(\hat{u}) : B\hat{u} = v(t) \right\}. \quad (4.36)$$

4.5. Structural properties and a priori estimates. Now, we point out some distinctive properties of $\mathbf{a}^\varepsilon, \mathbf{b}^\varepsilon, \mathcal{F}^\varepsilon$, and ϕ^ε (see [15]): they are also valid for the macroscopic model, which corresponds to $\varepsilon = 0$.

NOTATION 4.3. In order to avoid tedious repetitions, whenever it is possible we will systematically include the case $\varepsilon = 0$ in our statements, simply by making the obvious identifications

$$\underline{u}^0 := \underline{u}, \quad w^0 := w, \quad v^0 := v, \quad \mathbf{u}^0 := \mathbf{u}, \quad \underline{a}^0 := \underline{a}, \quad b^0 := b, \quad \mathcal{V}^0 := \mathcal{V}, \dots \quad (4.37)$$

- A. \mathbf{b}^ε is continuous and *symmetric*, the associated quadratic form (still denoted by \mathbf{b}^ε) is *nonnegative* but its kernel has infinite dimension, so that (4.16) is a *degenerate evolution equation*.
- B. \mathbf{a}^ε is a continuous, symmetric and nonnegative bilinear form, too.
- C. The sum of the quadratic forms $\mathbf{a}^\varepsilon + \mathbf{b}^\varepsilon$ is coercive in $\mathcal{V}_0^\varepsilon$, thus providing an equivalent scalar product.
- D. The bilinear form \mathbf{g}^ε satisfies

$$\exists G \geq 0 : \quad |\mathbf{g}^\varepsilon(\mathbf{u}, \hat{\mathbf{u}})|^2 \leq G^2 \mathbf{b}^\varepsilon(\mathbf{u}) \mathbf{b}^\varepsilon(\hat{\mathbf{u}}) \quad \forall \mathbf{u}, \hat{\mathbf{u}} \in \mathcal{V}_0^\varepsilon, \quad (4.38)$$

where G is *independent of ε* ; in the present case we can choose $G = \lambda_F + \Theta + \eta$.

- E. The nonlinear form \mathcal{F} satisfies the subdifferential inequalities

$$\mathcal{F}^\varepsilon(\mathbf{u}, \mathbf{u}) = \psi^\varepsilon(\mathbf{u}) + \phi^\varepsilon(\mathbf{u}) \quad \forall \mathbf{u} \in \mathcal{V}_0^\varepsilon, \quad (4.39)$$

$$\mathcal{F}^\varepsilon(\mathbf{u}, \hat{\mathbf{u}} - \mathbf{u}) + \phi^\varepsilon(\mathbf{u}) \leq \phi^\varepsilon(\hat{\mathbf{u}}) \quad \forall \mathbf{u}, \hat{\mathbf{u}} \in \mathcal{V}_0^\varepsilon, \quad (4.40)$$

for the

$$\begin{aligned} &\text{convex and lower semicontinuous (l.s.c.) functional } \phi^\varepsilon, \\ &\text{with } \phi^\varepsilon(\mathbf{u}) \geq \mathbf{b}^\varepsilon(\mathbf{u}) \quad \forall \mathbf{u} \in \mathcal{V}_0^\varepsilon. \end{aligned} \quad (4.41)$$

REMARK 4.4 (Regularity of F). For what concerns the nonlinearity of the problems, we note that the regularity assumptions on F can be relaxed, so $F : \mathbb{R} \rightarrow \mathbb{R}$ can be a continuous function such that $F(0) = 0$ and

$$\exists \lambda_F \geq 0 : \quad (F(x) - F(y))(x - y) + (\lambda_F - 1)|x - y|^2 \geq 0 \quad \forall x, y \in \mathbb{R}. \quad (4.42)$$

Now we briefly show a formal derivation of the basic *a priori* estimates for the macroscopic problem (for the microscopic one, one can simply add the superscript ε to each occurrence of $\mathbf{a}, \mathbf{b}, \mathcal{F}, \phi, \mathbf{g}, \mathbf{u}, \dots$); we assume that $\mathbf{u} \in H^1(0, T; \mathcal{V}_0)$ and f has a linear growth, so that \mathcal{F} is a continuous form. The computations below can be made rigorous, e.g. by passing to the limit in the analogous stability estimates for a suitably discretized or regularized system, and will be studied in Section 6.

Recalling that

$$\frac{d}{dt} \mathbf{b}(\mathbf{u}(t), \hat{\mathbf{u}}) = \mathbf{b}(\mathbf{u}'(t), \hat{\mathbf{u}}), \quad \frac{1}{2} \frac{d}{dt} \mathbf{b}(\mathbf{u}(t)) = \mathbf{b}(\mathbf{u}'(t), \mathbf{u}) \quad \forall \hat{\mathbf{u}} \in \mathcal{V}_0,$$

choosing $\hat{\mathbf{u}} := \mathbf{u}(t)$ in (4.29) we get

$$\frac{1}{2} \frac{d}{dt} \mathbf{b}(\mathbf{u}(t)) + \mathbf{a}(\mathbf{u}(t)) + \mathcal{F}(\mathbf{u}(t), \mathbf{u}(t)) = \mathbf{g}(\mathbf{u}(t), \mathbf{u}(t)) \leq G \mathbf{b}(\mathbf{u}(t)), \quad (4.43)$$

so that a simple application of Gronwall lemma and the relation

$$\mathcal{F}(\mathbf{u}, \mathbf{u}) = \phi(\mathbf{u}) + \psi(\mathbf{u}), \quad (4.44)$$

yields

$$\max \left[\sup_{t \in [0, T]} \mathbf{b}(\mathbf{u}(t)), \int_0^T (\mathbf{a}(\mathbf{u}) + \phi(\mathbf{u}) + \psi(\mathbf{u})) dt \right] \leq e^{2GT} \mathbf{b}(\mathbf{u}_0). \quad (4.45)$$

Choosing now $\hat{\mathbf{u}} := \mathbf{u}'$ and observing that

$$\frac{d}{dt} \phi(\mathbf{u}(t)) = \mathcal{F}(\mathbf{u}(t), \mathbf{u}'(t)), \quad \frac{1}{2} \frac{d}{dt} \mathbf{a}(\mathbf{u}(t)) = \mathbf{a}(\mathbf{u}(t), \mathbf{u}'(t)),$$

the previous estimate and the Cauchy inequality yield

$$\begin{aligned} \mathbf{b}(\mathbf{u}'(t)) + \frac{d}{dt} \left(\frac{1}{2} \mathbf{a}(\mathbf{u}(t)) + \phi(\mathbf{u}(t)) \right) &= \mathbf{g}(\mathbf{u}(t), \mathbf{u}'(t)) \\ &\leq G \mathbf{b}(\mathbf{u}(t))^{1/2} \mathbf{b}(\mathbf{u}'(t))^{1/2} \leq \frac{1}{2} \mathbf{b}(\mathbf{u}'(t)) + \frac{G^2}{2} e^{2Gt} \mathbf{b}(\mathbf{u}_0). \end{aligned} \quad (4.46)$$

Integrating in time we get

$$\left. \begin{aligned} &\sup_{t \in [0, T]} \left(\frac{1}{2} \mathbf{a}(\mathbf{u}) + \phi(\mathbf{u}) \right) \\ &\frac{1}{2} \int_0^T \mathbf{b}(\mathbf{u}') dt \end{aligned} \right\} \leq \frac{1}{2} \mathbf{a}(\mathbf{u}_0) + \phi(\mathbf{u}_0) + \frac{G}{4} e^{2GT} \mathbf{b}(\mathbf{u}_0). \quad (4.47)$$

Combining (4.45) with (4.47) we obtain (4.35) with $C = \max(2, (\frac{G}{2} + 1)e^{2GT})$.

5. Proof of Theorems 1 and 2.

5.1. Outline. As we said in the Introduction, our approach (inspired by the so called “Minimizing movement” method introduced by E. De Giorgi [16, 3, 5]) combines general approximation results, yielding uniform error estimates for a semi-implicit Euler time-discretization of evolution equations like (4.16) and (4.29), with homogenization results for the discretized problems.

More precisely, we start in paragraph 5.2 by considering the approximation of \mathbf{P}^ε and \mathbf{P} in the time interval $[0, T]$ by the semi-implicit Euler method of time step $\tau = T/N > 0$: thus we will consider a uniform partition \mathcal{P}_τ of the time interval $[0, T]$ into N subintervals

$$\mathcal{P}_\tau := \{0 = t_0 < t_1 < \dots < t_{N-1} < t_N = T\}, \quad t_n := n\tau, \quad (5.1)$$

and we will replace the continuous problems by a sequence of discrete microscopic and macroscopic problems $\mathbf{P}_n^{\varepsilon, \tau}, \mathbf{P}_n^{0, \tau}$, $n = 1, \dots, N$, whose solutions $\mathbf{U}_n^{\varepsilon, \tau}, \mathbf{U}_n^{0, \tau}$ provide an approximation of $\mathbf{u}^\varepsilon(t), \mathbf{u}(t)$ for t in the time interval $(t_{n-1}, t_n]$.

Denoting by $\mathbf{U}^{\varepsilon, \tau}, \mathbf{U}^{0, \tau}$ the piecewise linear interpolant on \mathcal{P}_τ of the values $\mathbf{U}_n^{\varepsilon, \tau}, \mathbf{U}_n^{0, \tau}$, general error estimates for variational evolution problems with the structure discussed in 4.5 show that the error (measured in the natural energy norms) between the continuous solution \mathbf{u}^ε of \mathbf{P}^ε (resp. \mathbf{u} of \mathbf{P}) and the discrete solution $\mathbf{U}^{\varepsilon, \tau}$ (resp. $\mathbf{U}^{0, \tau}$) vanishes as $\tau \downarrow 0$ *uniformly with respect to ε* . We will devote Section 6 to prove these estimates.

Since the non symmetric parts $\mathbf{g}^\varepsilon, \mathbf{g}$ are discretized “explicitly”, it turns out (see §5.3) that each step of the discretization scheme (the problems $\mathbf{P}_n^{\varepsilon, \tau}$ and $\mathbf{P}_n^{0, \tau}$) involves the minimization of suitable functionals $\Phi_n^{\varepsilon, \tau}, \Phi_n^{0, \tau}$, which depend only ε, τ and on the discrete solutions $\mathbf{U}_{n-1}^{\varepsilon, \tau}, \mathbf{U}_{n-1}^{0, \tau}$ at the previous node of the partition \mathcal{P}_τ ; therefore $\mathbf{U}_n^{\varepsilon, \tau}$ and $\mathbf{U}_n^{0, \tau}$ are the unique minima of $\Phi_n^{\varepsilon, \tau}$ and $\Phi_n^{0, \tau}$ respectively.

By adapting the Γ -convergence results of Section 3, we shall see (Theorem 5.2) that each discrete value $\mathbf{U}_n^{\varepsilon, \tau}$ converges, as $\varepsilon \downarrow 0$ to $\mathbf{U}_n^{0, \tau}$, whenever τ is fixed. Since for each $\tau > 0$ the discrete interpolant $\mathbf{U}^{\varepsilon, \tau}$ is determined only by a *finite* number of vectors $\mathbf{U}_n^{\varepsilon, \tau}$, it follows that the discrete solution $\mathbf{U}^{\varepsilon, \tau}$ converges to the homogenized one $\mathbf{U}^{0, \tau}$ in the time interval $[0, T]$ as $\varepsilon \downarrow 0$.

Combining this result with the above uniform error estimate between $\mathbf{u}^\varepsilon, \mathbf{u}$ and $\mathbf{U}^{\varepsilon, \tau}, \mathbf{U}^{0, \tau}$, we will conclude that also the continuous solution \mathbf{u}^ε converges to \mathbf{u} at each time $t \in [0, T]$.

Thus, by introducing the auxiliary problems $\mathbf{P}_n^{\varepsilon, \tau}, \mathbf{P}_n^{0, \tau}$, the homogenization of time dependent evolution equations is reduced to the homogenization of a family of functionals depending on the parameters τ, n , which can be tackled by Γ -convergence arguments.

Let us reproduce the above argument in the following scheme:

$$\begin{array}{ccc} \mathbf{u}^\varepsilon := \left\{ \begin{array}{l} \text{Continuous solution of} \\ \text{the evolution problem } \mathbf{P}^\varepsilon \end{array} \right. & \begin{array}{c} \longleftrightarrow \\ \text{uniform error} \\ \text{estimates} \end{array} & \mathbf{U}^{\varepsilon, \tau} := \left\{ \begin{array}{l} \text{Discrete solution of the} \\ \text{variational problem } \mathbf{P}^{\varepsilon, \tau} \end{array} \right. \\ & & \downarrow \Gamma\text{-convergence} \\ \mathbf{u} := \left\{ \begin{array}{l} \text{Continuous solution of} \\ \text{the evolution problem } \mathbf{P} \end{array} \right. & \begin{array}{c} \longleftrightarrow \\ \text{uniform error} \\ \text{estimates} \end{array} & \mathbf{U}^{0, \tau} := \left\{ \begin{array}{l} \text{Discrete solution of the} \\ \text{variational problem } \mathbf{P}^{0, \tau} \end{array} \right. \end{array}$$

Let us now consider in more detail each step of the proof.

5.2. Time discretization. We look for a suitable approximation $\{U_n^{\varepsilon,\tau}\}_{n=0}^N \subset \mathcal{V}_0^\varepsilon$ of the values of u^ε on the grid \mathcal{P}_τ (5.1), which solves the following discrete problem $P^{\varepsilon,\tau}$

$$\left\{ \begin{array}{l} \text{Given } U_0^{\varepsilon,\tau} := u_0^\varepsilon \text{ find } U_1^{\varepsilon,\tau}, \dots, U_N^{\varepsilon,\tau} \in \mathcal{V}_0^\varepsilon \text{ which recursively solve} \\ b^\varepsilon\left(\frac{U_n^{\varepsilon,\tau} - U_{n-1}^{\varepsilon,\tau}}{\tau}, \hat{U}\right) + a^\varepsilon(U_n^{\varepsilon,\tau}, \hat{U}) + \mathcal{F}^\varepsilon(U_n^{\varepsilon,\tau}, \hat{U}) = g^\varepsilon(U_{n-1}^{\varepsilon,\tau}, \hat{U}) \quad (P^{\varepsilon,\tau}) \\ \text{for every choice of } \hat{U} \in D(\phi) \subset \mathcal{V}_0^\varepsilon. \end{array} \right.$$

In fact, due to the convexity of ϕ and to the coercivity of the quadratic form $a^\varepsilon + b^\varepsilon$ on $\mathcal{V}_0^\varepsilon$, it is easy to check that each step of $(P^{\varepsilon,\tau})$ is equivalent to the minimum problem

$$\text{find } U_n^{\varepsilon,\tau} \in \mathcal{V}_0^\varepsilon \text{ which attains the minimum } \min \left\{ \Phi_n^{\varepsilon,\tau}(U) : U \in \mathcal{V}_0^\varepsilon \right\}, \quad (P_n^{\varepsilon,\tau})$$

where

$$\Phi_n^{\varepsilon,\tau}(U) := \frac{1}{2\tau} b^\varepsilon(U - U_{n-1}^{\varepsilon,\tau}) + a^\varepsilon(U) + \phi^\varepsilon(U) - g^\varepsilon(U_{n-1}^{\varepsilon,\tau}, U). \quad (5.2)$$

We can proceed in a completely analogous way for the macroscopic problem, simply by setting $\varepsilon := 0$ and recalling notation 4.3. The corresponding discrete solutions $U^{\varepsilon,\tau}(t)$ and $U^{0,\tau}(t)$ are the piecewise linear interpolant of $\{U_n^{\varepsilon,\tau}\}_{n=0}^N$ and $\{U_n^{0,\tau}\}_{n=0}^N$ on the grid \mathcal{P}_τ , i.e.

$$U^{\varepsilon,\tau}(t) := (n - t/\tau)U_{n-1}^{\varepsilon,\tau} + (t/\tau - (n-1))U_n^{\varepsilon,\tau}, \quad \text{if } t \in ((n-1)\tau, n\tau]. \quad (5.3)$$

The next Theorem shows that the discrete solutions of the above schemes converge uniformly to the solutions of P^ε and P in the intrinsic energy norms induced by the bilinear forms $a^\varepsilon, b^\varepsilon$. We are denoting by $E^\varepsilon, E = E^0$ (recall notation 4.3) the (squared) errors

$$E^\varepsilon = \max_{t \in (0,T)} b^\varepsilon(u^\varepsilon(t) - U^{\varepsilon,\tau}(t)) + \int_0^T a^\varepsilon(u^\varepsilon(t) - U^{\varepsilon,\tau}(t)) dt. \quad (5.4)$$

THEOREM 5.1 (Uniform error estimates). *There exists a constant $C = C(G, T)$ independent of ε such that*

$$E^\varepsilon \leq C\tau \left(a^\varepsilon(u_0^\varepsilon) + b^\varepsilon(u_0^\varepsilon) + \phi^\varepsilon(u_0^\varepsilon) \right). \quad (5.5)$$

The proof of Theorem 5.1 is presented in Section 6: it is the extension to a semi-implicit discretization scheme of the arguments developed by [9]; in order to give a self-contained simpler proof, we decided to follow the general scheme introduced by [28], for the derivation of optimal *a priori* and *a posteriori* error estimates for evolution variational problems.

5.3. Discrete problems and the role of Γ -convergence. Let us now consider the convergence as $\varepsilon \downarrow 0$ of the solutions $U_n^{\varepsilon,\tau}$ of the discrete problem $P_n^{\varepsilon,\tau}$. The following crucial result provides the main induction step.

THEOREM 5.2 (Convergence of the discrete approximations). *Let us suppose that the vector $U_{n-1}^{\varepsilon,\tau} = (U_{n-1}^{\varepsilon,\tau}, W_{n-1}^{\varepsilon,\tau}) \in \mathcal{V}_0^\varepsilon$ converges to $U_{n-1}^{0,\tau}$ according to definition 1.1 as $\varepsilon \downarrow 0$, with*

$$\lim_{\varepsilon \downarrow 0} b^\varepsilon(U_{n-1}^{\varepsilon,\tau}) = b(U_{n-1}^{0,\tau}), \quad \limsup_{\varepsilon \downarrow 0} a^\varepsilon(U_{n-1}^{\varepsilon,\tau}) + \phi^\varepsilon(U_{n-1}^{\varepsilon,\tau}) < +\infty, \quad (5.6)$$

and let us define the functional $\Phi_n^{\varepsilon,\tau}, \Phi_n^{0,\tau}$ as in (5.2). Then the unique minimum $U_n^{\varepsilon,\tau} \in \mathcal{V}_0^\varepsilon$ of $\Phi_n^{\varepsilon,\tau}$ converges to the unique minimum $U_n^{0,\tau}$ of $\Phi_n^{0,\tau}$ as $\varepsilon \downarrow 0$ according to 1.1 and

$$\lim_{\varepsilon \downarrow 0} b^\varepsilon(U_n^{\varepsilon,\tau}) = b(U_n^{0,\tau}), \quad \lim_{\varepsilon \downarrow 0} a^\varepsilon(U_n^{\varepsilon,\tau}) = a(U_n^{0,\tau}), \quad \lim_{\varepsilon \downarrow 0} \phi^\varepsilon(U_n^{\varepsilon,\tau}) = \phi(U_n^{0,\tau}). \quad (5.7)$$

COROLLARY 5.3 (Convergence of the discrete solutions). *Let us suppose that the assumption (1.24) of Theorem 1 on the initial data $(v_0^\varepsilon, w_0^\varepsilon)$ and (v_0, w_0) holds and that $u_0^\varepsilon \in \mathcal{V}_0^\varepsilon$ is chosen as in (4.18). Then each vector $U_n^{\varepsilon,\tau}$, $n = 1, \dots, N$, of the discrete solution $U^{\varepsilon,\tau}$ converges to the corresponding one $U_n^{0,\tau}$ of the discrete solution $U^{0,\tau}$ as $\varepsilon \downarrow 0$ according to Definition 1.1.*

Thanks to Theorem 3.3, we will deduce Theorem 5.2 from a corresponding coercivity and Γ -convergence result for the functionals $\Phi_n^{\varepsilon,\tau}$. Observe that (5.7) is a consequence of the convergence of the energies $\Phi_n^{\varepsilon,\tau}(U_n^{\varepsilon,\tau}) \rightarrow \Phi_n^{0,\tau}(U_n^{0,\tau})$ given by (3.4) and Remark 3.2, thanks to the separate lower semicontinuity property (3.17).

As in § 3.2 we identify vectors $u^\varepsilon = (u_i^\varepsilon, u_e^\varepsilon, w^\varepsilon) \in \mathcal{V}_0^\varepsilon$, $u = (u_i, u_e, w) \in \mathcal{V}_0$ with measures $\tilde{u}^\varepsilon = (\tilde{u}_i^\varepsilon, \tilde{u}_e^\varepsilon, \tilde{u}^\varepsilon)$, $\tilde{u} = (\tilde{u}_i, \tilde{u}_e, \tilde{u}) \in \mathcal{M}^3$ through (1.19) and (1.20), denoting by $m^\varepsilon : \mathcal{V}_0^\varepsilon \rightarrow \mathcal{M}^3$, $m : \mathcal{V}_0 \rightarrow \mathcal{M}^3$ the corresponding maps. We also extend all the functionals on $\mathcal{V}_0^\varepsilon$ to \mathcal{M}^3 as we did in (3.13). We can thus consider the Γ -limit of $\tilde{\Phi}_n^{\varepsilon,\tau}$ in \mathcal{M}^3 as $\varepsilon \downarrow 0$, keeping τ fixed.

THEOREM 5.4 (Γ -convergence). *Let us fix $\tau > 0$ and let us suppose that the vectors $U_{n-1}^{\varepsilon,\tau} \in \mathcal{V}_0^\varepsilon$ satisfy the same assumption as in Theorem 5.2.*

If a family $\tilde{U}^\varepsilon = m^\varepsilon(U^\varepsilon)$, $U^\varepsilon \in \mathcal{V}_0^\varepsilon$, for $\varepsilon \in \Lambda$, satisfies

$$\limsup_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} \tilde{\Phi}_n^{\varepsilon,\tau}(\tilde{U}^\varepsilon) = \limsup_{\varepsilon \downarrow 0, \varepsilon \in \Lambda} \Phi_n^{\varepsilon,\tau}(U^\varepsilon) < +\infty, \quad (5.8)$$

then \tilde{U}^ε is relatively compact in \mathcal{M}^3 ; it converges to \tilde{U} in \mathcal{M}^3 if and only if $\tilde{U} = m(U)$ for some $U \in \mathcal{V}_0$ and U^ε converges to U according to Definition 1.1.

The functionals $\tilde{\Phi}_n^{\varepsilon,\tau}$, extensions of (5.2) satisfy

$$\Gamma(\mathcal{M}^3)\text{-}\lim_{\varepsilon \downarrow 0} \tilde{\Phi}_n^{\varepsilon,\tau} = \tilde{\Phi}_n^{0,\tau}. \quad (5.9)$$

The *proof* of Theorem 5.4 is completely analogous to the one of Theorem 3: if we write

$$U_{n-1}^{\varepsilon,\tau} = (\underline{U}_{n-1}^{\varepsilon,\tau}, W_{n-1}^{\varepsilon,\tau}), \quad U^\varepsilon = (\underline{U}^\varepsilon, W^\varepsilon), \quad \underline{U}^\varepsilon \in \mathcal{V}_0^\varepsilon, \quad W \in \mathcal{H}^\varepsilon,$$

we choose $h := \tau^{-1}$ and the term v_0^ε of (1.30) as $B^\varepsilon \underline{U}_{n-1}^{\varepsilon,\tau}$, then

$$\Phi_n^{\varepsilon,\tau}(U^\varepsilon) = \mathcal{F}^\varepsilon(\underline{U}^\varepsilon) + g_n^{\varepsilon,\tau}(U^\varepsilon), \quad (5.10)$$

where

$$\begin{aligned} g_n^{\varepsilon,\tau}(U^\varepsilon) &:= \frac{1}{2\tau} b^\varepsilon(W^\varepsilon - W_{n-1}^{\varepsilon,\tau}) + \frac{\gamma}{2} b^\varepsilon(W^\varepsilon) \\ &\quad - \lambda b^\varepsilon(\underline{U}_{n-1}^{\varepsilon,\tau}, \underline{U}^\varepsilon) + \Theta b^\varepsilon(W_{n-1}^{\varepsilon,\tau}, B^\varepsilon \underline{U}^\varepsilon) - \eta b^\varepsilon(B^\varepsilon \underline{U}_{n-1}^{\varepsilon,\tau}, W^\varepsilon). \end{aligned} \quad (5.11)$$

Thus the asymptotic behavior of $\Phi_n^{\varepsilon,\tau}$ can be easily deduced from Theorem 3 and Lemma 3.6.

REMARK 5.5. Usually, when one considers gradient flows of convex functionals \mathcal{F}^ε in a *fixed* Hilbert space, their asymptotic behaviour is determined by the Γ /Mosco convergence of Lyapunov functionals \mathcal{F}^ε (see e.g. [6]), and it is not necessary to take into account the dependence of τ .

Here also the underlying Hilbert spaces are changing in a singular way with respect to ε and the major novelty is the explicit presence of the mesh size τ in the minimizing functionals $\Phi_n^{\varepsilon,\tau}$, which reflects the metric of the functional spaces governing the gradient flows. In the present case, this metric is related to the quadratic forms \mathbf{b}^ε : therefore it is degenerate and it depends on ε .

5.4. Conclusion of the proof of Theorems 1 and 2. By Corollary 3.7, the a priori estimates (4.22) and (4.23), and the variational characterizations (4.23), (4.36), we simply have to show that $v^\varepsilon := B^\varepsilon u^\varepsilon$ and w^ε are converging to $v := Bu$ and w for every $t \in [0, T]$ according to Definition 1.1.

Let us consider the case of v^ε , the argument for w^ε being completely analogous. We set $V^{\varepsilon,\tau}(t) := B^\varepsilon U^{\varepsilon,\tau}(t)$, $\varepsilon \geq 0$: for every $\zeta \in C_c^0(\Omega)$ and every $t \in [0, T]$ (which will not be indicated explicitly) we have

$$\begin{aligned} \varepsilon \int_{\Gamma^\varepsilon} v^\varepsilon \zeta d\mathcal{H}^{d-1} - \beta \int_{\Omega} v \zeta dx &= \varepsilon \int_{\Gamma^\varepsilon} (v^\varepsilon - V^{\varepsilon,\tau}) \zeta d\mathcal{H}^{d-1} - \beta \int_{\Omega} (v - V^{0,\tau}) \zeta dx \\ &\quad + \varepsilon \int_{\Gamma^\varepsilon} V^{\varepsilon,\tau} \zeta d\mathcal{H}^{d-1} - \beta \int_{\Omega} V^{0,\tau} \zeta dx \end{aligned}$$

so that, if $Z := \sup_{\Omega} |\zeta|$ and $S^2 \geq \varepsilon \mathcal{H}^{d-1}(\Gamma^\varepsilon) + \beta \mathcal{L}^d(\Omega)$

$$\begin{aligned} \left| \varepsilon \int_{\Gamma^\varepsilon} v^\varepsilon \zeta d\mathcal{H}^{d-1} - \beta \int_{\Omega} v \zeta dx \right| &\leq SZb^\varepsilon (v^\varepsilon - V^{\varepsilon,\tau})^{1/2} + SZb(v - V^{0,\tau})^{1/2} \\ &\quad + \left| \varepsilon \int_{\Gamma^\varepsilon} V^{\varepsilon,\tau} \zeta d\mathcal{H}^{d-1} - \beta \int_{\Omega} V^{0,\tau} \zeta dx \right| \leq C\tau^{1/2} + r_\varepsilon \end{aligned}$$

where we applied the uniform estimates of Theorem 5.1 and the bounds on the initial data. Passing now to the limit as $\varepsilon \downarrow 0$ keeping τ fixed, we get

$$\limsup_{\varepsilon \downarrow 0} r_\varepsilon = \limsup_{\varepsilon \downarrow 0} \left| \varepsilon \int_{\Gamma^\varepsilon} v^\varepsilon \zeta d\mathcal{H}^{d-1} - \beta \int_{\Omega} v \zeta dx \right| \leq C\tau^{1/2}$$

thanks to Corollary 5.3. Finally letting $\tau \downarrow 0$ we get the desired convergence.

The corresponding property for the energy $b^\varepsilon(v^\varepsilon)$ follows by the same argument.

Let us eventually consider Theorem 2: if we show the existence of a suitable extension satisfying the uniform bound (1.28), then the thesis follows immediately from Theorem 1 and Corollary 2.11. Thanks to the *a priori* estimates (4.22) and Lemma 2.10, the choice $\tilde{u}_{i,e}^\varepsilon := T_{i,e}^\varepsilon u_{i,e}^\varepsilon$ will surely satisfy (1.28), therefore we should find an analogous extension for w^ε . Recalling that w^ε satisfies on Γ^ε the O.D.E.

$$\partial_t w^\varepsilon + \gamma w^\varepsilon = \eta v^\varepsilon,$$

so that

$$w^\varepsilon(x, t) = w_0^\varepsilon e^{-\gamma t} + \eta \int_0^t e^{-\gamma(t-s)} v^\varepsilon(x, s) ds \quad \forall x \in \Gamma^\varepsilon, t \in [0, T],$$

we can use the same formula to extend w^ε to Ω starting from the initial datum $\tilde{w}_0^\varepsilon \in H_{loc}^1(\Omega)$. Therefore, we set $\tilde{v}^\varepsilon(x, s) := \tilde{u}_i^\varepsilon - \tilde{u}_e^\varepsilon \in H_{loc}^1(\Omega)$ and correspondingly

$$\tilde{w}^\varepsilon(x, t) = \tilde{w}_0^\varepsilon e^{-\gamma t} + \eta \int_0^t e^{-\gamma(t-s)} \tilde{v}^\varepsilon(x, s) ds \quad \forall x \in \Omega, t \in [0, T], \quad (5.12)$$

and it is easy to see by differentiating under the integral sign that (1.28) is satisfied.

6. Uniform error estimates. In this section we are proving Theorem 5.1; since all the estimates will only depend on the structural assumptions of Section 4.5 and will therefore be independent of ε , for the sake of simplicity we will omit to indicate the explicit dependence on ε .

Thus, $\{\mathbf{U}_n^\tau\}_{n=0}^N$ is a discrete solution of the variational algorithm introduced in Section 5.2. We already denoted by \mathbf{U}^τ the piecewise linear interpolant of the discrete values, which can be expressed in the form

$$\mathbf{U}^\tau(t) := (1 - \ell(t))\mathbf{U}_{n-1}^\tau + \ell(t)\mathbf{U}_n^\tau \quad \text{if } t \in ((n-1)\tau, n\tau], \quad (6.1)$$

where ℓ is the piecewise linear (but discontinuous) function associated to the mesh \mathcal{P}_τ by

$$\ell(t) := \frac{t}{\tau} - (n-1) \quad \text{if } t \in ((n-1)\tau, n\tau].$$

The piecewise constant interpolant $\bar{\mathbf{U}}^\tau$ is defined by

$$\bar{\mathbf{U}}^\tau(t) := \mathbf{U}_n^\tau \quad \text{if } t \in ((n-1)\tau, n\tau]. \quad (6.2)$$

The basic quantity which will control our estimates is

$$\mathcal{E}(\mathbf{U}) := \frac{1}{2}\mathbf{a}(\mathbf{U}) + \phi(\mathbf{U}) \geq \mathbf{b}(\mathbf{U}). \quad (6.3)$$

We split the proof in several steps, denoting by C different constants which solely depend on G and T .

Discrete variational inequality. The discrete solution solves

$$\begin{aligned} \mathbf{b}\left(\frac{\mathbf{U}_n^\tau - \mathbf{U}_{n-1}^\tau}{\tau}, \mathbf{U}_n^\tau - \mathbf{V}\right) + \frac{1}{2}\mathbf{a}(\mathbf{U}_n^\tau - \mathbf{V}) + \mathcal{E}(\mathbf{U}_n^\tau) \\ \leq \mathcal{E}(\mathbf{V}) + \mathbf{g}(\mathbf{U}_{n-1}^\tau, \mathbf{U}_n^\tau - \mathbf{V}) \quad \forall \mathbf{V} \in D(\phi). \end{aligned} \quad (6.4)$$

This property follows from the well known (see e.g. [8]) variational characterization of the minima for a functional, like Φ_n^τ , which is the sum of a quadratic continuous form (involving $\mathbf{a}, \mathbf{b}, \mathbf{g}$) and a convex functional (ϕ in this case).

Stability estimates. There exists a constant $C = C(G, T)$ such that

$$\sum_{n=1}^N \tau \mathbf{b}\left(\frac{\mathbf{U}_n^\tau - \mathbf{U}_{n-1}^\tau}{\tau}\right) + \mathbf{a}(\mathbf{U}_n^\tau - \mathbf{U}_{n-1}^\tau) + \sup_{n=0, \dots, N} \mathcal{E}(\mathbf{U}_n^\tau) \leq C\mathcal{E}(\mathbf{u}_0). \quad (6.5)$$

We use a “discrete” version of the arguments of the formal *a priori* estimate of Section 4.5. We choose $\mathbf{V} := 0$ in (6.4); recalling the identity $2\mathbf{b}(x - y, x) = \mathbf{b}(x) - \mathbf{b}(y) + \mathbf{b}(x - y)$ and multiplying by 2τ we obtain

$$\begin{aligned} \mathbf{b}(\mathbf{U}_n^\tau) + \mathbf{b}(\mathbf{U}_n^\tau - \mathbf{U}_{n-1}^\tau) + 2\tau\mathbf{a}(\mathbf{U}_n^\tau) + 2\tau\phi(\mathbf{U}_n^\tau) \\ \leq \mathbf{b}(\mathbf{U}_{n-1}^\tau) + 2\tau\mathbf{g}(\mathbf{U}_{n-1}^\tau, \mathbf{U}_n^\tau) \\ = \mathbf{b}(\mathbf{U}_{n-1}^\tau) + 2\tau\left(\mathbf{g}(\mathbf{U}_{n-1}^\tau, \mathbf{U}_{n-1}^\tau) + \mathbf{g}(\mathbf{U}_{n-1}^\tau, \mathbf{U}_n^\tau - \mathbf{U}_{n-1}^\tau)\right) \\ \leq \mathbf{b}(\mathbf{U}_{n-1}^\tau) + 2\tau G^2(1 + \tau)\mathbf{b}(\mathbf{U}_{n-1}^\tau) + \mathbf{b}(\mathbf{U}_n^\tau - \mathbf{U}_{n-1}^\tau). \end{aligned} \quad (6.6)$$

A simple application of the Discrete Gronwall Lemma yields

$$\sup_{0 \leq m \leq N} \mathbf{b}(\mathbf{U}_m^\tau) \leq C_0 \mathbf{b}(\mathbf{u}_0), \quad \text{for } C_0 := e^{2G^2(1+\tau)T}. \quad (6.7)$$

Choosing $\mathbf{V} := \mathbf{U}_{n-1}^\tau$ in (6.4) and summing up for $n = 1$ to $m \leq N$ we get

$$\begin{aligned}
& \sum_{n=1}^m \left(\tau^{-1} \mathbf{b}(\mathbf{U}_n^\tau - \mathbf{U}_{n-1}^\tau) + \frac{1}{2} \mathbf{a}(\mathbf{U}_n^\tau - \mathbf{U}_{n-1}^\tau) \right) + \mathcal{E}(\mathbf{U}_m^\tau) \\
& \leq \mathcal{E}(\mathbf{u}_0) + \sum_{n=1}^m \mathbf{g}(\mathbf{U}_{n-1}^\tau, \mathbf{U}_n^\tau - \mathbf{U}_{n-1}^\tau) \\
& \leq \mathcal{E}(\mathbf{u}_0) + G \sum_{n=1}^m \left(\mathbf{b}(\mathbf{U}_{n-1}^\tau) \mathbf{b}(\mathbf{U}_n^\tau - \mathbf{U}_{n-1}^\tau) \right)^{1/2} \\
& \leq \mathcal{E}(\mathbf{u}_0) + \frac{G^2 \tau}{2} \sum_{n=1}^m \mathbf{b}(\mathbf{U}_{n-1}^\tau) + \frac{\tau^{-1}}{2} \sum_{n=1}^m \mathbf{b}(\mathbf{U}_n^\tau - \mathbf{U}_{n-1}^\tau)
\end{aligned} \tag{6.8}$$

so that choosing $m = N$ we get

$$\begin{aligned}
& \sum_{n=1}^N \left(\tau^{-1} \mathbf{b}(\mathbf{U}_n^\tau - \mathbf{U}_{n-1}^\tau) + \mathbf{a}(\mathbf{U}_n^\tau - \mathbf{U}_{n-1}^\tau) \right) \leq 2\mathcal{E}(\mathbf{u}_0) + C_0 G^2 T \mathbf{b}(\mathbf{u}_0), \\
& \sup_{1 \leq m \leq N} \mathcal{E}(\mathbf{U}_m^\tau) \leq \mathcal{E}(\mathbf{u}_0) + \frac{1}{2} C_0 G^2 T \mathbf{b}(\mathbf{u}_0).
\end{aligned}$$

A continuous version of the discrete variational inequalities. The piecewise linear interpolant \mathbf{U}^τ satisfies

$$\begin{aligned}
& \mathbf{b}\left(\frac{d}{dt} \mathbf{U}^\tau(t), \mathbf{U}^\tau(t) - \mathbf{V}\right) + \frac{1}{2} \mathbf{a}(\bar{\mathbf{U}}^\tau(t) - \mathbf{V}) + \mathcal{E}(\mathbf{U}^\tau(t)) \\
& \leq \mathcal{E}(\mathbf{V}) + \mathbf{g}(\mathbf{U}^\tau(t) - \tau \ell(t) \frac{d}{dt} \mathbf{U}^\tau(t), \mathbf{U}^\tau(t) - \mathbf{V}) + \mathcal{R}^\tau(t),
\end{aligned} \tag{6.9}$$

where

$$\mathcal{R}^\tau(t) := (1 - \ell) \left(\mathcal{E}(\mathbf{U}_{n-1}^\tau) - \mathcal{E}(\mathbf{U}_n^\tau) + \mathbf{g}(\mathbf{U}_{n-1}^\tau, \mathbf{U}_n^\tau - \mathbf{U}_{n-1}^\tau) \right),$$

and

$$\int_0^T \left(\mathbf{b}\left(\frac{d}{dt} \mathbf{U}^\tau(t)\right) + \tau^{-1} \mathbf{a}(\mathbf{U}^\tau(t) - \bar{\mathbf{U}}^\tau(t)) \right) dt \leq C \mathcal{E}(\mathbf{u}_0). \tag{6.10}$$

We check (6.9) simply by writing the “discrete” value of all the terms for $t \in (t_{n-1}, t_n)$, recalling that

$$\frac{d}{dt} \mathbf{U}^\tau(t) = \frac{\mathbf{U}_n^\tau - \mathbf{U}_{n-1}^\tau}{\tau} \quad \text{for } t \in (t_{n-1}, t_n). \tag{6.11}$$

Thus we have

$$\begin{aligned}
& \mathbf{b}\left(\frac{d}{dt} \mathbf{U}^\tau, \mathbf{U}^\tau - \mathbf{V}\right) = \mathbf{b}\left(\frac{\mathbf{U}_n^\tau - \mathbf{U}_{n-1}^\tau}{\tau}, (1 - \ell) \mathbf{U}_{n-1}^\tau + \ell \mathbf{U}_n^\tau - \mathbf{V}\right) \\
& = \mathbf{b}\left(\frac{\mathbf{U}_n^\tau - \mathbf{U}_{n-1}^\tau}{\tau}, \mathbf{U}_n^\tau - \mathbf{V}\right) - (1 - \ell) \tau^{-1} \mathbf{b}(\mathbf{U}_n^\tau - \mathbf{U}_{n-1}^\tau)
\end{aligned} \tag{6.12}$$

$$\begin{aligned}
& \mathcal{E}(\mathbf{U}^\tau) = \mathcal{E}((1 - \ell) \mathbf{U}_{n-1}^\tau + \ell \mathbf{U}_n^\tau) \leq (1 - \ell) \mathcal{E}(\mathbf{U}_{n-1}^\tau) + \ell \mathcal{E}(\mathbf{U}_n^\tau) \\
& = \mathcal{E}(\mathbf{U}_n^\tau) + (1 - \ell) \left(\mathcal{E}(\mathbf{U}_{n-1}^\tau) - \mathcal{E}(\mathbf{U}_n^\tau) \right)
\end{aligned} \tag{6.13}$$

$$\begin{aligned}
& \mathbf{g}(\mathbf{U}^\tau - \tau \ell \frac{d}{dt} \mathbf{U}^\tau, \mathbf{U}^\tau - \mathbf{V}) \\
& = \mathbf{g}((1 - \ell) \mathbf{U}_{n-1}^\tau + \ell \mathbf{U}_n^\tau - \ell(\mathbf{U}_n^\tau - \mathbf{U}_{n-1}^\tau), (1 - \ell) \mathbf{U}_{n-1}^\tau + \ell \mathbf{U}_n^\tau - \mathbf{V}) \\
& = \mathbf{g}(\mathbf{U}_{n-1}^\tau, \mathbf{U}_n^\tau - \mathbf{V}) - (1 - \ell) \mathbf{g}(\mathbf{U}_{n-1}^\tau, \mathbf{U}_n^\tau - \mathbf{U}_{n-1}^\tau)
\end{aligned} \tag{6.14}$$

(6.10) follows directly from (6.5) by (6.11) and

$$\bar{\mathbf{U}}^\tau(t) - \mathbf{U}^\tau(t) = (1 - \ell)(\mathbf{U}_n^\tau - \mathbf{U}_{n-1}^\tau) \quad \text{for } t \in (t_{n-1}, t_n). \tag{6.15}$$

An estimate for the remainder term.

$$\int_0^T |\mathcal{R}^\tau(t)| dt \leq C_1 \mathcal{E}(\mathbf{u}_0) \tau. \quad (6.16)$$

First of all, since \mathbf{U}_n^τ minimizes Φ_n^τ , we easily have

$$\mathcal{E}(\mathbf{U}_n^\tau) - \mathbf{g}(\mathbf{U}_{n-1}^\tau, \mathbf{U}_n^\tau) \leq \mathcal{E}(\mathbf{U}_{n-1}^\tau) - \mathbf{g}(\mathbf{U}_{n-1}^\tau, \mathbf{U}_{n-1}^\tau)$$

so that $\mathcal{R}^\tau(t) \geq 0$. Moreover

$$\begin{aligned} \int_0^T \mathcal{R}^\tau(t) dt &= \frac{1}{2} \tau \sum_{n=1}^N \left(\mathcal{E}(\mathbf{U}_{n-1}^\tau) - \mathcal{E}(\mathbf{U}_n^\tau) + \mathbf{g}(\mathbf{U}_{n-1}^\tau, \mathbf{U}_n^\tau - \mathbf{U}_{n-1}^\tau) \right) \\ &= \frac{1}{2} \tau \left(\mathcal{E}(\mathbf{u}_0) + \mathcal{E}(\mathbf{u}_0) + C_0 G^2 T \mathbf{b}(\mathbf{u}_0) \right) \leq C_1 \tau \mathcal{E}(\mathbf{u}_0), \end{aligned}$$

where we used, as in (6.8)

$$\begin{aligned} \sum_{n=1}^N \mathbf{g}(\mathbf{U}_{n-1}^\tau, \mathbf{U}_n^\tau - \mathbf{U}_{n-1}^\tau) &\leq \frac{G^2 \tau}{2} \sum_{n=1}^N \mathbf{b}(\mathbf{U}_{n-1}^\tau) + \frac{\tau^{-1}}{2} \sum_{n=1}^N \mathbf{b}(\mathbf{U}_n^\tau - \mathbf{U}_{n-1}^\tau) \\ &\leq \frac{C_0 G^2 T}{2} \mathbf{b}(\mathbf{u}_0) + \mathcal{E}(\mathbf{u}_0) + \frac{C_0 G^2 T}{2} \mathbf{b}(\mathbf{u}_0) \end{aligned}$$

A Gronwall-type estimate for the error. If \mathbf{U}^η , $\eta > 0$, is the discrete solution associated to the partition \mathcal{P}_η , we have

$$\sup_{t \in [0, T]} \mathbf{b}(\mathbf{U}^\tau(t) - \mathbf{U}^\eta(t)) + \int_0^T \mathbf{a}(\mathbf{U}^\tau(t) - \mathbf{U}^\eta(t)) \leq C(\tau + \eta) \mathcal{E}(\mathbf{u}_0). \quad (6.17)$$

Let ℓ^τ, ℓ^η be the interpolating functions corresponding to $\mathcal{P}_\tau, \mathcal{P}_\eta$. Choosing $\mathbf{V} := \mathbf{U}^\eta(t)$ in (6.9) and $\mathbf{V} := \mathbf{U}^\tau(t)$ in the analogous inequality written for \mathbf{U}^η , we obtain

$$\begin{aligned} &\frac{d}{dt} \mathbf{b}(\mathbf{U}^\tau - \mathbf{U}^\eta) + \mathbf{a}(\bar{\mathbf{U}}^\tau - \mathbf{U}^\eta) + \mathbf{a}(\mathbf{U}^\eta - \bar{\mathbf{U}}^\tau) \\ &\leq 2\mathbf{g}(\mathbf{U}^\tau - \mathbf{U}^\eta, \mathbf{U}^\tau - \mathbf{U}^\eta) - 2\mathbf{g}(\tau \ell^\tau \frac{d}{dt} \mathbf{U}^\tau - \eta \ell^\eta \frac{d}{dt} \mathbf{U}^\eta, \mathbf{U}^\tau - \mathbf{U}^\eta) + \mathcal{R}^\tau + \mathcal{R}^\eta \\ &\leq 3G\mathbf{b}(\mathbf{U}^\tau - \mathbf{U}^\eta) + 2G\tau^2 \mathbf{b}(\frac{d}{dt} \mathbf{U}^\tau) + 2G\eta^2 \mathbf{b}(\frac{d}{dt} \mathbf{U}^\eta) + \mathcal{R}^\tau + \mathcal{R}^\eta \end{aligned}$$

A direct application of Gronwall lemma, (6.16), and (6.10) yield

$$\begin{aligned} \sup_{t \in [0, T]} \mathbf{b}(\mathbf{U}^\tau(t) - \mathbf{U}^\eta(t)) &\leq \left(\int_0^T 2G\tau^2 \mathbf{b}(\frac{d}{dt} \mathbf{U}^\tau) + 2G\eta^2 \mathbf{b}(\frac{d}{dt} \mathbf{U}^\eta) + \mathcal{R}^\tau + \mathcal{R}^\eta dt \right) e^{3GT} \\ &\leq C\tau \mathcal{E}(\mathbf{u}_0). \end{aligned}$$

An analogous argument and (6.10) provides the integral bound for $\mathbf{a}(\mathbf{U}^\tau - \bar{\mathbf{U}}^\tau)$.

If now we pass to the limit as $\eta \downarrow 0$ we obtain the estimates of (5.5).

Appendix A. The derivation of the scaled Problem \mathbf{P}^ε . For completeness, in this appendix, we present the scaling used to obtain Problem \mathbf{P}^ε .

The basic equations modeling the electrical activity of the heart at the cellular level can be obtained as follows. Cardiac tissue consists of cells connected together

and surrounded by extracellular fluid. Let Ω_i^ε , Ω_e^ε be the intra- and extracellular ohmic conductive media, Γ^ε be the excitable membrane that separates Ω_i^ε and Ω_e^ε and let $\nu_i^\varepsilon, \nu_e^\varepsilon$ denote the unit exterior normals to the boundary of Ω_i^ε and Ω_e^ε respectively, satisfying $\nu_i^\varepsilon = -\nu_e^\varepsilon$ on Γ^ε . The electric behavior of the tissue is described by the intra- and extracellular potentials u_i^ε and u_e^ε and by their driven current densities $\mathbf{j}_{i,e}^\varepsilon = -\Sigma_{i,e} \nabla u_{i,e}^\varepsilon$. Due to the current conservation law, the normal current flux through the membrane Γ^ε is continuous $\nu_i^\varepsilon \cdot \mathbf{j}_i^\varepsilon = \nu_e^\varepsilon \cdot \mathbf{j}_e^\varepsilon$; hence we have

$$\Sigma_i \nabla u_i^\varepsilon \cdot \nu_i^\varepsilon + \Sigma_e \nabla u_e^\varepsilon \cdot \nu_e^\varepsilon = 0, \quad \text{on } \Gamma^\varepsilon \times (0, T)$$

where $\Sigma_{i,e}$ are the cellular conductivity matrices in the intra- and extracellular medium.

Since the only active source elements lie on the membrane Γ^ε , each flux equals the membrane current per unit area J_m , i.e.

$$J_m = \begin{cases} -\Sigma_i \nabla u_i^\varepsilon \cdot \nu_i^\varepsilon \\ \Sigma_e \nabla u_e^\varepsilon \cdot \nu_e^\varepsilon \end{cases} \quad \text{on } \Gamma^\varepsilon \times (0, T). \quad (\text{A.1})$$

The membrane current per unit area J_m consists of a capacitive and a ionic term (see [21]):

$$J_m := C_m \partial_t v^\varepsilon + I(v^\varepsilon, w^\varepsilon), \quad \text{on } \Gamma^\varepsilon, \quad (\text{A.2})$$

with C_m the surface capacitance of the membrane.

Moreover, disregarding the presence of applied current terms, we have that currents are conserved in Ω_i^ε and Ω_e^ε , then the intra- and extracellular potentials are solutions of

$$-\operatorname{div}(\Sigma_i \nabla u_i^\varepsilon) = 0, \quad \text{in } \Omega_i^\varepsilon \times (0, T) \quad -\operatorname{div}(\Sigma_e \nabla u_e^\varepsilon) = 0, \quad \text{in } \Omega_e^\varepsilon \times (0, T) \quad (\text{A.3})$$

with Neumann boundary conditions for u_i^ε , u_e^ε on the remaining part of the boundaries $\Gamma_{i,e}^\varepsilon = \partial\Omega_{i,e}^\varepsilon \setminus \Gamma^\varepsilon$.

We now want to rewrite problem (A.1-A.3) in a non dimensional form. To this end we note that we can consider two characteristic length scales: the microscopic scale, related to a typical dimension d_c of the cells (e.g. the cell diameter $15 - 20 \mu m$ or the length of the cell $100 \mu m$), and the macroscopic one determined by a suitable length constant of the tissue denoted by L . The cellular conductivity matrices Σ_i and Σ_e are symmetric positive definite matrices; let $\bar{\lambda} = \bar{\lambda}_i + \bar{\lambda}_e$ with $\bar{\lambda}_i$, $\bar{\lambda}_e$ the average eigenvalues on a cell element and let us consider

$$\sigma_{i,e} = \Sigma_{i,e} / \bar{\lambda}.$$

Here we assume that $v^\varepsilon = 0$, $w^\varepsilon = 0$ is the equilibrium point for problem (A.1,A.3), then we can define the macroscopic space scale along fibers L as:

$$L = \sqrt{d_c R_m \bar{\lambda}} \quad \text{with} \quad R_m^{-1} = \partial_v I(0, 0).$$

Now, we can convert the cellular problem into a non-dimensional form by scaling space and time with the macroscopic units of length $L = d_c / \varepsilon$ and with respect to the membrane constant $T = R_m C_m$ i.e. we perform the following space and time scaling

$$\hat{x} = x/L, \quad \hat{t} = t/T.$$

The dimensionless parameter ε is then a small parameter whose order of magnitude is the ratio of the two space scales macro and microscopic, i.e.

$$\varepsilon = d_c/L.$$

We take $\widehat{\mathbf{x}}$ to be the variable of the macro scale behavior and

$$\xi := \widehat{\mathbf{x}}/\varepsilon$$

to be the microscopic space variable measured in unit cell. For simplicity, in the sequel, we omit the superscripts $\widehat{}$ of the dimensionless variables.

Cardiac tissue exhibits a number of significant inhomogeneities in particular those related to cell-to-cell communications. The conductivity tensors are considered dependent both on the slow and on the fast variables, i.e. $\sigma_{i,e}(x, \frac{x}{\varepsilon})$. The latter dependence of the intracellular conductivity represents an attempt to include the effects of the gap junctions.

We then define the rescaled symmetric conductivity matrices

$$\sigma_{i,e}^\varepsilon(x) = \sigma_{i,e}(x, \frac{x}{\varepsilon})$$

obtained by the continuous functions $\sigma_{i,e}(x, \xi) : \Omega \times E_{i,e} \rightarrow \mathbb{M}^{d \times d}$ satisfying

$$\begin{aligned} 0 < \sigma|y|^2 \leq \sigma_{i,e}(x, \xi)y \cdot y \leq \sigma^{-1}|y|^2 \quad \forall (x, \xi) \in \Omega \times E_{i,e}, \quad y \in \mathbb{R}^d. \\ \sigma_{i,e}(x, \xi + e_k) = \sigma_{i,e}(x, \xi) \end{aligned} \quad (\text{A.4})$$

Finally, rescaling the equations (A.1)-(A.3) in the intra- and extracellular potentials we obtain:

$$\begin{aligned} -\operatorname{div}(\sigma_{i,e}^\varepsilon \nabla u_{i,e}^\varepsilon) &= 0 \quad \text{in } \Omega_{i,e}^\varepsilon \times (0, T), \\ -\sigma_i^\varepsilon \nabla u_i^\varepsilon \cdot \nu_i^\varepsilon &= \varepsilon (\partial_t v + I(v^\varepsilon, w^\varepsilon)) \quad \text{on } \Gamma^\varepsilon \times (0, T) \\ \sigma_e^\varepsilon \nabla u_e^\varepsilon \cdot \nu_e^\varepsilon &= \varepsilon (\partial_t v + I(v^\varepsilon, w^\varepsilon)) \quad \text{on } \Gamma^\varepsilon \times (0, T) \end{aligned}$$

that is problem \mathbf{P}^ε .

An homogenization process for different mathematical models, describing the response of biological tissues to electromagnetic fields and based on a completely different scaling, can be found in [2].

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