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COMPACTNESS PROPERTIES FOR FAMILIES OF QUASISTATIONARY SOLUTIONS OF SOME EVOLUTION EQUATIONS

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ABSTRACT. The following typical problem occurs in passing to the limit in some phase field models: for two sequences of space–time dependent functions $\{\theta_n\}$, $\{\chi_n\}$ (representing, e.g., suitable approximations of the temperature and the phase variable) we know that the sum $\theta_n + \chi_n$ converges in some L^p-space as $n \uparrow +\infty$ and that the time integrals of a suitable "space" functional evaluated on θ_n, χ_n are uniformly bounded with respect to n. Can we deduce that θ_n and χ_n converge separately? LUCKHAUS (1990) gave a positive answer to this question in the framework of the two–phase Stefan problem with Gibbs–Thompson law for the melting temperature. PLOTNIKOV (1993) proposed an abstract result employing the original idea of Luckhaus and arguments of compactness and reflexivity type. We present a general setting for this and other related problems, providing necessary and sufficient conditions for their solvability: these conditions rely on general topological and coercivity properties of the functionals and the norms involved and do not require reflexivity.

1. INTRODUCTION.

In this paper we are mainly interested in the following type of problem, involving vector-valued functions defined on the time interval $]0,T[$, $T>0$:

Problem (P). Let us assume that A, B are two Banach spaces,

(P.1) $$L : A \to B \text{ is a } \textit{bounded linear operator,}$$

(P.2) $$\mathscr{F} : A \to [0,+\infty] \text{ is a } \textit{proper lower semi-continuous functional.}$$

We are given a sequence $u_n :]0,T[\to A$ of time dependent (strongly) measurable functions, such that

(P.3) $$\exists \lim_{n\uparrow+\infty} Lu_n =: \ell \quad \text{in } L^p(0,T;B) \quad \text{for some } p \in [1,+\infty];$$

moreover, we know an additional *a priori* estimate of the type

(P.4) $$S := \sup_{n\in\mathbb{N}} \int_0^T \mathscr{F}(u_n(t))\,dt < +\infty.$$

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Could we find $q \in [1, +\infty)$ and a suitable sub-sequence of u_n strongly converging in $L^q(0, T; A)$?

Of course, we are looking for some kind of general assumptions on $L, \mathscr{F}$ and, possibly, on the asymptotic behavior of $\{u_n\}_{n\in\mathbb{N}}$, which are sufficient to force the $L^q(0, T; A)$-convergence.

Before entering into the details of our analysis, let us mention some basic examples which better explain and motivate the question we are considering.

Example 1. We start from an important problem solved by S. Luckhaus [Luc90, Lemma 1,2] (see also [Vis96, VIII.3]) in the framework of the two phase Stefan-Gibbs-Thomson model for the melting temperature. In the cylindrical region

$$Q :=]0, T[\times \Omega, \quad \Omega \text{ being a bounded open subset of } \mathbb{R}^3, \tag{1.1}$$

we are considering two sequences of real functions (they are suitable approximations of the temperature and the phase variable) $\theta_n, \chi_n \in L^1(Q)$ which satisfy

$$|\chi_n(x,t)| = 1 \quad \text{a.e. in } Q, \quad \theta_n(\cdot, t) \in H^1_0(\Omega) \quad \text{for a.e. } t \in]0, T[\,, \tag{1.2}$$

$$\int_Q \left|\nabla_x \theta_n(x,t)\right|^2 dx\, dt \le S < +\infty, \tag{1.3}$$

and

$$\theta_n + \chi_n \to \ell \quad \text{strongly in } L^3(0, T; L^1(\Omega)) \text{ as } n \uparrow +\infty. \tag{1.4}$$

Luckhaus succeeded in proving that (1.2), (1.3), (1.4) imply the separated strong convergence in $L^1(Q)$ of θ_n and χ_n to θ and χ respectively; this result was a cornerstone of the existence proof of solutions of the two-phase Stefan problem.

Let us show that this situation can be rephrased with the notation of Problem (P). We choose

$$A := L^1(\Omega) \times L^1(\Omega), \quad B := L^1(\Omega), \quad p = 3,\ q = 1 \tag{1.5}$$

and we consider the couple $u_n := (\theta_n, \chi_n)$ as a time dependent function with values in A; so we set

$$L : u = (\theta, \chi) \in L^1(\Omega) \times L^1(\Omega) \to \theta + \chi \in L^1(\Omega), \tag{1.6}$$

and we introduce the functionals defined on $L^1(\Omega)$

$$\mathscr{F}_1(\theta) := \begin{cases} \int_\Omega |\nabla \theta(x)|^2\, dx & \text{if } \theta \in V := H^1_0(\Omega), \\ +\infty & \text{otherwise;} \end{cases} \tag{1.7}$$

$$\mathscr{F}_2(\chi) := \begin{cases} 0 & \text{if } |\chi(x)| = 1 \text{ a.e. in } \Omega, \\ +\infty & \text{otherwise.} \end{cases} \tag{1.8}$$

Finally we define

$$\mathscr{F}(u) := \mathscr{F}_1(\theta) + \mathscr{F}_2(\chi). \tag{1.9}$$

Luckhaus Theorem is then equivalent to say that in the context of (1.5), ..., (1.9) Problem (P) has an affirmative solution. Even if the original proof of this statement relies on careful capacity type estimates for Sobolev functions defined on Ω, it is natural to look for a more abstract principle behind this result.

Example 2. A first step in this direction was provided by P.I. PLOTNIKOV (see [PS93] and also [HS98a, HS98b] for other applications), who considered the case of two time dependent sequences

$$n \mapsto \theta_n, \chi_n \in L^p(0,T;B), \quad B \text{ being a } \textit{Banach space}, \tag{1.10}$$

such that

$$\theta_n + \chi_n \to \ell \quad \text{strongly in } L^p(0,T;B) \text{ as } n \uparrow +\infty. \tag{1.11}$$

Moreover, he supposed that

$$\theta_n(t) \in V, \quad \chi_n(t) \in K \quad \text{for a.e. } t \in]0,T[\,, \tag{1.12}$$

$$\int_0^T \|\theta_n(t)\|_V^p \, dt \le S < +\infty \quad \forall n \in \mathbb{N}, \tag{1.13}$$

where

$$V \hookrightarrow B \text{ is another Banach space, } K \text{ is a closed subset of } B, \tag{1.14}$$

satisfying the *compatibility condition*

$$\chi, \chi' \in K, \quad \chi - \chi' \in V \quad \Rightarrow \quad \chi = \chi'. \tag{1.15}$$

It is easy to identify the choices of p, B, V, K in (1.10, . . . 1.15) which correspond to (1.2,. . . , 1.4); [PS93] proved that if

$$p = 2,\ V, B \text{ are } \textit{reflexive},\ K \text{ is } \textit{compact}, \text{ the inclusion } V \subset B \text{ is } \textit{compact}, \tag{1.16}$$

then (1.10, . . . 1.15) entail the separated convergence of θ_n and χ_n in $L^2(0,T;B)$ as $n \uparrow +\infty$.

As before, setting $A := B \times B$, $L : u = (\theta, \chi) \to \theta + \chi$, $\mathscr{F}(u) = \mathscr{F}_1(\theta) + \mathscr{F}_2(\chi)$ with

$$\mathscr{F}_1(\theta) := \begin{cases} \|\theta\|_V^p & \text{if } \theta \in V, \\ +\infty & \text{otherwise,} \end{cases} \qquad \mathscr{F}_2(\chi) := \begin{cases} 0 & \text{if } \chi \in K, \\ +\infty & \text{otherwise,} \end{cases} \tag{1.17}$$

we are in the framework of Problem (P); the lower semi-continuity of $\mathscr{F}$ is ensured by the closeness of K and the reflexivity of V, which in particular entails its *relative completeness* in B

$$\left.\begin{aligned} v_n \in V, \quad \sup_n \|v_n\|_V < +\infty \\ v_n \to v \text{ strongly in } B \end{aligned}\right\} \Rightarrow \ v \in V, \quad \|v\|_V \le \liminf_{n\uparrow+\infty} \|v_n\|_V. \tag{1.18}$$

The latter property, which is equivalent to say that V coincides with its *Gagliardo completion* in B [Gag61], is in fact weaker than reflexivity or compactness, which are the crucial assumptions (1.16) of [PS93]; moreover, these assumptions are not necessary in order to give a positive answer to Problem (P), as Luckhaus Theorem shows. Our aim is to fill in the gap between these two results in the abstract context detailed before.

Example 3. With the notation of Problem (P), let us assume in particular that A is continuously embedded in B, L is the corresponding inclusion map, and $p = q$. We consider another Banach space $V \subset A$ and we define $\mathscr{F}$ as

$$\mathscr{F}(v) := \begin{cases} \|v\|_V^p & \text{if } v \in V, \\ +\infty & \text{if } v \in A \setminus V, \end{cases}$$

assuming that it is l.s.c. Therefore, for a given sequence $\{u_n\}_{n\in\mathbb{N}}$ which is uniformly bounded in $L^p(0,T;V)$, as in (P.4), and converges in $L^p(0,T;B)$ as in (P.3), we are asking if u_n converges in $L^p(0,T;A)$, too. It is not difficult to show that in this case the good condition is

$$\forall\, \varepsilon > 0,\ \exists\, C_\varepsilon > 0 : \quad \|v\|_A \le \varepsilon \|v\|_V + C_\varepsilon \|v\|_B \quad \forall\, v \in V. \tag{1.19}$$

Typically, two standard assumptions imply (1.19):

- V is compactly embedded in A [LM72, Ch. I, Th. 16.4],
- A is an interpolation space between V and B of class $\underline{\mathcal{K}}_\theta(V,B)$, for $\theta \in (0,1]$ [LP64, Chap. IV, Def. 1.1], i.e. there exists a constant $C > 0$ such that

  $$\|v\|_A \le C \|v\|_V^{1-\theta} \|v\|_B^\theta \quad \forall\, v \in V,$$

  or, equivalently, the real interpolation space $(V,B)_{\theta,1}$ is contained in A.

It is interesting to observe that (1.19) is in fact equivalent to say that the topologies induced by the norms of A and B coincide on V-bounded sets, i.e.

$$\left.\begin{array}{l} v_k \in V, \quad \|v_k\|_V \le S < +\infty \\ v_k \to v \text{ strongly in } B \end{array}\right\} \quad \Rightarrow \quad v_k \to v \text{ strongly in } A. \tag{1.20}$$

Plan of the paper. In the next section we will present a preliminary discussion about the reasonable assumptions, which seem to be necessary to solve Problem (P); afterward, in the same section, we will see that these assumptions are also sufficient and we will collect the main abstract results, whose proofs are detailed in § 3. Particular attention is devoted to the case of a "quasistationary" family u_n satisfying an a.e. minimality condition with respect to an other sequence of time-dependent functionals.

Applications are presented in the last section: they concern an improved version of Luckhaus and Plotnikov theorems and a convergence result for quasistationary solutions of phase field equations with Neumann boundary conditions: this last application refines, with different techniques, a recent result of R. Schätzle [Sch00].

2. Basic Assumptions and Main Results.

Preliminary discussion. Before stating the main theorems of this paper, we try to understand what kind of (hopefully minimal) assumptions could provide a reasonable answer to Problem (P).

Compactness. Let us start with a sequence of *constant* functions

$$u_n(t) := v_n \in A \quad \text{for } t \in]0,T[\,;$$

in this case (P.3) and (P.4) can be reformulated in terms of the constants v_n as

$$\exists\, \ell \in B : \lim_{n\uparrow+\infty} L v_n = \ell \text{ in } B, \quad \exists\, s < +\infty : \mathscr{F}(v_n) \le s \quad \forall\, n \in \mathbb{N}.$$

Since we are looking for a subsequence of v_n converging in A and we do not know other information on the sequence v_n and the values of ℓ, s, we are forced to impose a sort of *conditional compactness* to *all* the sublevels of $\mathscr{F}$. We adopt the following definition.

Definition 2.1. We say that a subset $K \subset A$ is *conditionally compact* w.r.t. L if every sequence $n \mapsto v_n \in K$ such that $\exists \lim_{n\uparrow+\infty} Lv_n$ in B has a (strongly) convergent sub-sequence.

So we are led up to assume that

$$\text{(H.1)} \qquad \begin{gathered} \textit{The sublevels } \Big\{v \in A : \mathscr{F}(v) \le s\Big\} \\ \textit{are conditionally compact w.r.t. } L, \quad \forall\, s \ge 0. \end{gathered}$$

Let us observe that the compactness of the sublevels trivially implies (H.1).

Coercivity. The second assumption regards the asymptotic behavior of $\mathscr{F}(v)$ as $\|v\|_A$ goes to $+\infty$. Consider, e.g., the case $q = 1$: we want to show that, roughly speaking, we need a super-linear growth of $\mathscr{F}$, at least in the directions not controlled by L.

In order to clarify this point, let us suppose that the super-linear growth fails along a diverging sequence $\{v_n\}_{n\in\mathbb{N}}$ in A, whose asymptotic direction belongs to $\operatorname{Ker} L$, i.e.

$$\text{(2.1)} \qquad \lim_{n\uparrow+\infty} \|v_n\|_A = +\infty, \quad \sup_{n\in\mathbb{N}} \frac{\mathscr{F}(v_n)}{\|v_n\|_A} < +\infty, \quad \lim_{n\uparrow+\infty} L\left(\frac{v_n}{\|v_n\|_A}\right) = 0.$$

Then we set

$$t_n := \|v_n\|_A^{-1}, \quad u_n(t) := \begin{cases} v_n & \text{if } 0 < t < t_n, \\ v_0 & \text{otherwise,} \end{cases}$$

and it is easy to see that, as $n \uparrow +\infty$,

$$\int_0^T \|Lu_n(t) - Lv_0\|\, dt \le t_n\big(\|Lv_n\|_B + \|Lv_0\|_B\big) \to 0,$$

$$\sup_{n\in\mathbb{N}} \int_0^T \mathscr{F}(u_n(t))\, dt \le \sup_{n\in\mathbb{N}} \Big(T\mathscr{F}(v_0) + t_n\mathscr{F}(v_n)\Big) < +\infty,$$

whereas

$$\lim_{n\uparrow+\infty} \int_0^T \|u_n(t) - v_0\|_A\, dt = 1.$$

As before, we introduce the crucial concept in the next definition, where we assume $q, p \in [1, +\infty)$.

Definition 2.2. We say that $\mathscr{F}$ is (q,p)-*conditionally coercive in* A w.r.t. L if

$$\text{(H.2)} \qquad \forall\, \varepsilon > 0\ \exists\, C_\varepsilon > 0 : \quad \|v\|_A^q \le \varepsilon\mathscr{F}(v) + C_\varepsilon(1 + \|Lv\|_B^p) \quad \forall\, v \in A.$$

Let us remark that, for $q = p = 1$, (H.2) is exactly *equivalent* to assume that sequences like (2.1) do not exist.

Non degeneracy. The last kind of questions regards the injectivity of L on the proper domain $D(\mathscr{F})$ of $\mathscr{F}$,

$$\text{(2.2)} \qquad D(\mathscr{F}) := \Big\{v \in A : \mathscr{F}(v) < +\infty\Big\}.$$

If it happens that

$$\exists\, v, w \in D(\mathscr{F}) : \quad v \ne w, \quad Lv = Lw,$$

then it is not difficult to exhibit a sequence of functions u_n whose range spans $\{v, w\}$, which do not have any convergent subsequence. Indeed we consider an orthogonal system $\{e_n\}_{n\in\mathbb{N}}$ in $L^2(0,T)$, constituted by functions assuming only the values $-1, 1$ (e.g., the standard Haar system) and we set

$$u_n(t) := \tfrac{1}{2}(v+w) + \tfrac{1}{2}(v-w)e_n(t).$$

Since

$$Lu_n(t) \equiv Lv = Lw \quad \text{is independent of } n,$$

and

$$\int_0^T \mathscr{F}(u_n(t))\,dt \le T\max\big\{\mathscr{F}(v), \mathscr{F}(w)\big\},$$

it is easy to see that (P.3) and (P.4) are satisfied, with $p = +\infty$. On the other hand, a subsequence $k \mapsto u_{n_k}$ converging in $L^1(0,T;A)$ would converge in every $L^q(0,T;A)$, $q < +\infty$; in particular, we should have

$$k \mapsto u_{n_k} \quad \text{is a Cauchy sequence in} \quad L^2(0,T;A),$$

but the orthogonality of the system $\{e_n\}_{n\in\mathbb{N}}$ yields for $n, m \in \mathbb{N}$, $n \neq m$,

$$\|u_n - u_m\|^2_{L^2(0,T;A)} = \tfrac{1}{4}\|v-w\|^2_A \int_0^T \big|e_n(t) - e_m(t)\big|^2\,dt = \tfrac{T}{2}\|v-w\|^2_A > 0.$$

We formalize the property we will need in the next definition.

Definition 2.3. We say that a subset $K \subset A$ is *non degenerate* w.r.t. L if

$$v, w \in K, \quad Lv = Lw \quad \Rightarrow \quad v = w. \tag{2.3}$$

Correspondingly, we will assume that

(H.3) *$D(\mathscr{F})$ is non degenerate w.r.t. L.*

Main results. The previous arguments show that properties (H.1), (H.2), and (H.3) are *necessary* in order to give a positive answer to Problem (P). The next Theorem says that they are also *sufficient*; moreover the convergence of the whole sequence u_n is provided.

Theorem 1. *Let us suppose that $L, \mathscr{F}$ satisfy* (P.1), (P.2), *and*

(H.1) *the sublevels of $\mathscr{F}$ are conditionally compact w.r.t. L,*
(H.2) *$\mathscr{F}$ is (q,p)-conditionally coercive in A w.r.t. L,*
(H.3) *$D(\mathscr{F})$ is non degenerate w.r.t. L.*

Then Problem (P) has an affirmative answer, i.e. every sequence of functions $u_n \in L^q(0,T;A)$ satisfying (P.3) *and* (P.4) *is strongly convergent in $L^q(0,T;A)$.*

We postpone the proof of Theorem 1 to the next section; now we want to refine this result, in order to have a better insight in the case (H.3) does not hold.

The possibly degenerate case. In some situation (H.3) could be false but nevertheless we could hope to overcome this difficulty by the knowledge of extra information on the asymptotic behavior of u_n. The idea is to distinguish between the role of the *a priori* estimate supplied by $\mathscr{F}$ and the *non degeneracy* condition, which should be satisfied only asymptotically by a suitable subset $\mathcal{U}(t)$ of the accumulation points of the sequences $\{u_n(t)\}_{n\in\mathbb{N}}$; this set is defined for a.e. $t \in]0,T[$ by

$$\begin{aligned}\mathcal{U}(t) := \Big\{ v \in A : &\text{ there exists an increasing subsequence } k \mapsto n_k \text{ s.t.}\\ &\lim_{k\uparrow+\infty} u_{n_k}(t) = v, \quad Lv = \ell(t), \quad \limsup_{k\uparrow+\infty} \mathscr{F}(u_{n_k}(t)) < +\infty \Big\}.\end{aligned} \tag{2.4}$$

By (P.2) we have

$$\mathcal{U}(t) \subset D(\mathscr{F}) \quad \text{for a.e. } t \in]0,T[\,.$$

Of course, if only (P.3) and (P.4) are available, together with (H.1) and (H.2), every point of $D(\mathscr{F})$ could belong to $\mathcal{U}(t)$ and (H.3) becomes necessary for the convergence, as we showed before.

However, we will show that (H.1) and (H.2) are sufficient to obtain some useful information on the pointwise asymptotic behavior of u_n: roughly speaking, for a.e. $t \in]0,T[$ the sequence $\{u_n(t)\}_{n\in\mathbb{N}}$ becomes arbitrarily closed to $\mathcal{U}(t)$ as $n \uparrow +\infty$. Any other information, which could entail that $\mathcal{U}(t)$ is reduced to a single point for a.e. $t \in]0,T[$, would provide the desired strong convergence in $L^q(0,T;A)$.

In order to measure the velocity of this asymptotic behavior, we recall the definition of the *distance* between a point $v \in A$ and a set $W \subseteq A$:

$$d(v,W) := \inf_{w\in W} \|v-w\|_A, \quad \text{with} \quad d(v,\varnothing) = +\infty. \tag{2.5}$$

Theorem 2. *Let us suppose that* (H.1) *and* (H.2) *hold, let* $u_n \in L^q(0,T;A)$ *be a sequence of functions satisfying* (P.3) *and* (P.4)*, and let*

$$\mathcal{U} : t \in]0,T[\, \mapsto \mathcal{U}(t) \subseteq A$$

be the time dependent multi-function whose values are the subsets of the accumulation points of $\{u_n(t)\}_{n\in\mathbb{N}}$ *defined for a.e.* $t \in]0,T[$ *by* (2.4)*. Then* $\mathcal{U}(t)$ *is non empty for a.e.* $t \in]0,T[$*, the maps* $t \mapsto \overline{\mathcal{U}(t)}$ *and* $t \mapsto d(u_n(t),\mathcal{U}(t))$ *are measurable, and*

$$\lim_{n\uparrow+\infty} \int_0^T d\big(u_n(t),\mathcal{U}(t)\big)^q \, dt = 0, \tag{2.6}$$

where d is defined by (2.5)*.*

Remark 2.4. (Measurability of $\mathcal{U}$). Since the functions u_n are almost separably valued [Yos80, V.4], we can find a negligible set $N \subset]0,T[$ such that

$$A_0 := \overline{\operatorname{span}\Big\{u_n(t) : n \in \mathbb{N},\ t \in]0,T[\setminus N\Big\}} \quad \text{is separable and complete.} \tag{2.7}$$

In particular,

$$\overline{\mathcal{U}(t)} \subset A_0 \quad \forall\, t \in]0,T[\setminus N,$$

and therefore a (possible) definition of the measurability of the multi-function $\overline{\mathcal{U}}$ is [CV77, Chap. III, § 2, Thm. 9]

$$\text{the function } t \in]0,T[\, \mapsto d(v,\mathcal{U}(t)) \quad \text{is measurable} \quad \forall\, v \in A. \tag{2.8}$$

Corollary 1. *Let us suppose that* (H.1) *and* (H.2) *hold, let* $u_n \in L^q(0,T;A)$ *be a sequence of functions satisfying* (P.3) *and* (P.4)*, and let* $t \in]0,T[\, \mapsto \mathcal{L}(t) \subset A$ *denote the multi-function*

$$\mathcal{L}(t) := L^{-1}\big(\ell(t)\big) \cap D(\mathscr{F}) \quad \textit{for a.e. } t \in]0,T[\,. \tag{2.9}$$

Then

$$\lim_{n\uparrow+\infty} \int_0^T d\big(u_n(t),\mathcal{L}(t)\big)^q \, dt = 0. \tag{2.10}$$

Proof. (2.10) is an immediate consequence of (P.3) and (2.6), which imply

$$\mathcal{U}(t) \subseteq \mathcal{L}(t) \quad \text{for a.e. } t \in]0,T[\,. \tag{2.11}$$

□

Corollary 2. *Let us suppose that* (H.1) *and* (H.2) *hold, let* $u_n \in L^q(0,T;A)$ *be a sequence of functions satisfying* (P.3) *and* (P.4). *Then* u_n *strongly converges in* $L^q(0,T;A)$ *if and only if*

$$\text{(H.3')} \qquad \mathcal{U}(t) \textit{ is non degenerate w.r.t. } L \textit{ for a.e. } t \in]0,T[\,.$$

Proof. By (2.11),

$$\mathcal{U}(t) \text{ is a singleton} \quad \Leftrightarrow \quad \mathcal{U}(t) \text{ is non degenerate w.r.t. } L. \tag{2.12}$$

□

Remark 2.5. Theorem 1 is an immediate consequence of the previous Corollary: since $\mathcal{U}(t) \subset D(\mathscr{F})$ for a.e. $t \in]0,T[$, (H.3) trivially implies (H.3').

Compactness for families of quasi-stationary problem. We have seen before that Theorems 1 and 2 are in some sense optimal, if we do not know any extra information on the sequence $\{u_n\}_{n\in\mathbb{N}}$; in particular, the non degeneracy of $D(\mathscr{F})$ is an essential requirement for the convergence of u_n.

On the other hand, if we know that the functions u_n satisfy suitable variational properties, it is possible to combine them with Theorem 2 in order to improve (2.6). Let us illustrate this feature by a simple example.

We consider another family of time-dependent functionals

$$\mathscr{G}_n : (t,v) \in]0,T[\,\times A \to \mathscr{G}_n(t,v) \in [-\infty,+\infty], \quad n \in \mathbb{N}, \tag{2.13}$$

which Γ-converges to $\mathscr{G}(t,\cdot)$ in A for a.e. $t \in]0,T[$ [Dal93, Chap. 4]

$$\mathscr{G}(t,\cdot) = \Gamma(A)\text{-}\lim_{n\uparrow+\infty} \mathscr{G}_n(t,\cdot) \quad \text{for a.e. } t \in]0,T[\,. \tag{2.14}$$

In order to measure "the degree of minimality" of $u_n(t)$ w.r.t. the functional $\mathscr{G}_n(t,\cdot)$, we introduce the quantity $\delta_n(t) \in [0,+\infty]$ defined by

$$\delta_n(t) := \sup_{v\in A} \Big(\mathscr{G}_n(t,u_n(t)) - \mathscr{G}_n(t,v)\Big) \quad \text{for a.e. } t \in]0,T[\,; \tag{2.15}$$

moreover, we will denote by $\mathcal{G}(t)$ the intersection of $\mathcal{L}(t)$ with the set of minimizers of $\mathscr{G}(t,\cdot)$

$$\mathcal{G}(t) := \Big\{v \in \mathcal{L}(t) : \mathscr{G}(t,v) = \min_{w\in A} \mathscr{G}(t,w)\Big\}. \tag{2.16}$$

We have the following result

Corollary 3. *Let* $\mathscr{F}, L, \mathscr{G}_n$ *satisfy* (P.1),(P.2),(2.13),(2.14)*; let* $u_n \in L^q(0,T;A)$ *be a sequence satisfying* (P.3),(P.4), *and let* δ_n *be defined by* (2.15). *If*

$$\lim_{n\uparrow+\infty} \delta_n(t) = 0 \quad \textit{a.e. in }]0,T[\,, \tag{2.17}$$

and (H.1), (H.2) *hold with the non degeneracy condition*

$$\text{(H.3'')} \qquad \mathcal{G}(t) \textit{ is non degenerate w.r.t. } L \textit{ for a.e. } t \in]0,T[\,,$$

then u_n *strongly converges in* $L^q(0,T;A)$ *as* $n \uparrow +\infty$.

Proof. By the general theory of Γ-convergence [Dal93, Cor. 7.20], (2.14), (2.15), and (2.4) yield

$$\mathcal{U}(t) \subseteq \mathcal{G}(t) \quad \text{for a.e. } t \in]0,T[\,.$$

Applying Corollary 2 we conclude. □

Now we make a few remarks on the assumptions of the above results and on the possibility of some natural extensions.

Further extensions and remarks.
Recession functional and coercivity. We can give an alternative description of condition (H.2) by means of the notion of *topological recession functional* (see e.g. [BBGT88]) For the sake of simplicity we are assuming that

$$p = q = 1, \quad A \textit{ is reflexive;} \tag{2.18}$$

it is easy to modify the results in the case $q > 1$ or when A is the dual of a separable Banach space.

Definition 2.6. The *weak sequential recession functional* of $\mathscr{F}$ (w.r.t. L) is the functional defined by

$$\mathscr{F}_\infty(v) := \inf\Big\{ \liminf_{n\uparrow+\infty} \lambda_n^{-1}\mathscr{F}(v_0+\lambda_n v_n) : \lambda_n \uparrow +\infty,\ v_n \rightharpoonup v,\ Lv_n \to Lv\Big\}$$

where $v, v_0 \in A$.

This definition is in fact independent of the choice of v_0 and gives raise to a positively homogeneous functional; in some sense, $\mathscr{F}_\infty(v)$ keeps trace of the asymptotic behavior of $\mathscr{F}(w)$ as w goes to infinity along the direction v.

Our interest in $\mathscr{F}_\infty$ relies in the following result, which shows that (H.2) is in fact equivalent to a sort of *compatibility* between $\mathscr{F}_\infty$ and $\operatorname{Ker} L$:

Proposition 2.7. *If* (2.18) *holds, then condition* (H.2) *is equivalent to*

$$v \in \operatorname{Ker} L \setminus \{0\} \quad \Rightarrow \quad \mathscr{F}_\infty(v) = +\infty. \tag{H.2'}$$

The lower semicontinuity of $\mathscr{F}$. We assumed that $\mathscr{F}$ is l.s.c. in order to simplify the exposition and to avoid subtle technicalities about measurability in (P.4).

Let us observe that it is always possible to replace $\mathscr{F}$ with its *lower semicontinuous envelope*

$$\overline{\mathscr{F}}(v) := \inf\Big\{ \liminf_{k\uparrow+\infty} \mathscr{F}(v_k) : v_k \to v\Big\} \quad \forall\, v \in A, \tag{2.19}$$

with proper domain

$$\begin{aligned} D(\overline{\mathscr{F}}) := \Big\{ & v \in A : \text{there exists a sequence } k \mapsto v_k \in A \text{ s.t.} \\ & \lim_{k\uparrow+\infty} v_k = v, \quad \sup_{k\in\mathbb{N}} \mathscr{F}(v_k) < +\infty \Big\}. \end{aligned} \tag{2.20}$$

$\overline{\mathscr{F}}$ is the greatest lower semicontinuous real functional defined on A which satisfies

$$\overline{\mathscr{F}}(v) \le \mathscr{F}(v) \quad \forall\, v \in A; \tag{2.21}$$

in particular, if the sequence u_n satisfies (P.4) (where the measurability of $\mathscr{F}(u_n)$ is implicitly assumed), then the same estimate holds with $\overline{\mathscr{F}}$ instead of $\mathscr{F}$. It is then natural to ask if (H.1), (H.2), and (H.3) are *stable* by taking the l.s.c. envelope of $\mathscr{F}$.

Proposition 2.8. *If a proper functional $\mathscr{F} : A \to [0,+\infty]$ satisfies* (H.1) *and* (H.2)*, then also $\overline{\mathscr{F}}$ satisfies them; in particular* (P.2) *is not required for Theorem 2 and its Corollary 2.*

Remark 2.9. It is not difficult to see that the right substitute of (H.3) is

$$D(\overline{\mathscr{F}}) \text{ is non degenerate,}$$

if we want Problem (P) to be always solvable for a generic proper functional $\mathscr{F}$.

Closed Operators. We could relax the *continuity* assumption (P.1) on L.

> *Theorems 1 and 2 hold for* closed linear operators $L : D(L) \subseteq A \to B$.

In this case should be intended that $u_n(t)$ belongs to the domain $D(L)$ of L for a.e. $t \in]0, T[$.

General measure spaces. The choice of functions defined on a time interval $]0, T[$ and of the usual Lebesgue measure is irrelevant:

> *Theorems 1 and 2 hold for a general* measure space $(I, \mathscr{M}, \mu)$, *μ being a positive σ-additive measure with $\mu(I) < +\infty$* [Rud87, Chap. I].

3. Proof of the Main Theorems

Two auxiliary results.

Notation 3.1. For every $\zeta \in B$ and every sequence $\boldsymbol{v} := \{v_n\}_{n\in\mathbb{N}}$ in A we set

$$\Lambda_{\boldsymbol{v},\zeta} := \Big\{ v \in D(\mathscr{F}) : \text{there exists an increasing subsequence}$$
$$k \mapsto n_k \quad \text{s.t.} \quad \lim_{k\uparrow+\infty} v_{n_k} = v, \quad Lv = \zeta, \quad \limsup_{k\uparrow+\infty} \mathscr{F}(v_{n_k}) < +\infty \Big\}. \tag{3.1}$$

Lemma 3.2. *If* (H.1) *holds, then for every $q \in [1, +\infty[$, $w \in A$, $\zeta \in B$, and every sequence $\boldsymbol{v} := \{v_n\}_{n\in\mathbb{N}}$ we have*

$$d(w, \Lambda_{\boldsymbol{v},\zeta})^q = \inf_{\varepsilon>0} \sup_{\sigma>0} \Big\{ \liminf_{n\uparrow+\infty} \Big(\|w - v_n\|_A^q + \varepsilon \mathscr{F}(v_n) + \sigma \|Lv_n - \zeta\|_B \Big) \Big\}. \tag{3.2}$$

Proof. Let us first prove the inequality

$$d(w, \Lambda_{\boldsymbol{v},\zeta})^q \geq \inf_{\varepsilon>0} \sup_{\sigma>0} \Big\{ \liminf_{n\uparrow+\infty} \Big(\|w - v_n\|_A^q + \varepsilon \mathscr{F}(v_n) + \sigma \|Lv_n - \zeta\|_B \Big) \Big\}; \tag{3.3}$$

in this case, it is not restrictive to assume that $\Lambda_{\boldsymbol{v},\zeta} \neq \varnothing$. Therefore, we fix $v \in \Lambda_{\boldsymbol{v},\zeta}$, $\varepsilon, \sigma > 0$, and a subsequence v_{n_k} as in (3.1), obtaining

$$\begin{aligned}
&\liminf_{n\uparrow+\infty} \Big(\|w - v_n\|_A^q + \varepsilon \mathscr{F}(v_n) + \sigma \|Lv_n - \zeta\|_B \Big) \\
&\qquad \leq \liminf_{k\uparrow+\infty} \Big(\|w - v_{n_k}\|_A^q + \varepsilon \mathscr{F}(v_{n_k}) + \sigma \|Lv_n - \zeta\|_B \Big) \\
&\qquad \leq \|w - v\|_A^q + \varepsilon \limsup_{k\uparrow+\infty} \mathscr{F}(v_{n_k}).
\end{aligned}$$

Being ε and σ arbitrary, we get

$$\inf_{\varepsilon>0} \sup_{\sigma>0} \Big\{ \liminf_{n\uparrow+\infty} \Big(\|w - v_n\|_A^q + \varepsilon \mathscr{F}(v_n) + \sigma \|Lv_n - \zeta\|_B \Big) \Big\} \leq \|w - v\|_A^q.$$

Finally, taking the infimum with respect to $v \in \Lambda_{\boldsymbol{v},\zeta}$ we get (3.3).

In order to prove the opposite inequality, for a fixed $\varepsilon > 0$ let us denote by s_ε the real number

$$s_\varepsilon := \sup_{\sigma>0} \Big\{ \liminf_{n\uparrow+\infty} \Big(\|w - v_n\|_A^q + \varepsilon \mathscr{F}(v_n) + \sigma \|Lv_n - \zeta\|_B \Big) \Big\};$$

it is not restrictive to assume $s_\varepsilon < +\infty$.

By a standard diagonal argument, there exists an increasing sequence $k \mapsto n_k \in \mathbb{N}$ such that

$$s_\varepsilon = \lim_{k\uparrow+\infty} \Big(\|w - v_{n_k}\|_A^q + \varepsilon \mathscr{F}(v_{n_k}) + k\|Lv_{n_k} - \zeta\|_B \Big);$$

in particular

$$\limsup_{k\uparrow+\infty} \mathscr{F}(v_{n_k}) < +\infty, \quad \lim_{k\uparrow+\infty} Lv_{n_k} = \zeta.$$

By the compactness assumption (H.1) we can extract a further subsequence, say $v_{n'_k}$, such that

$$\lim_{k\uparrow+\infty} v_{n'_k} = v_\infty \in \Lambda_{\boldsymbol{v},\zeta}, \quad \|w - v_\infty\|_A = \liminf_{k\uparrow+\infty} \|w - v_{n_k}\|_A.$$

Therefore, we have

$$d(w, \Lambda_{\boldsymbol{v},\zeta})^q \le \|w - v_\infty\|_A^q \le \liminf_{k\uparrow+\infty} \|w - v_{n_k}\|_A^q \le s_\varepsilon.$$

Since ε is arbitrary, we conclude. □

Lemma 3.3. *If* (H.1) *and* (H.2) *hold, then for every $\zeta \in B$ and every sequence $\boldsymbol{v} := \{v_n\}_{n\in\mathbb{N}}$ in A, such that the set $\Lambda_{\boldsymbol{v},\zeta}$ defined by* (3.1) *is not empty, we have*

$$\forall\, \varepsilon > 0\ \exists\, C_\varepsilon > 0 : d(v_n, \Lambda_{\boldsymbol{v},\zeta})^q \le \varepsilon\big(1 + \mathscr{F}(v_n)\big) + C_\varepsilon \|Lv_n - \zeta\|_B^p \quad \forall\, n \in \mathbb{N}. \tag{3.4}$$

Proof. Let us argue by contradiction and let us suppose (3.4) is not true; then we could find an $\varepsilon_0 > 0$ and an increasing subsequence $k \mapsto n_k$ s.t.

$$d(v_{n_k}, \Lambda_{\boldsymbol{v},\zeta})^q \ge \varepsilon_0\big(1 + \mathscr{F}(v_{n_k})\big) + k\|Lv_{n_k} - \zeta\|_B^p \quad \forall\, k \in \mathbb{N}. \tag{3.5}$$

We distinguish two cases:

- v_{n_k} *is bounded in A*: in this case

  $$\sup_{k\in\mathbb{N}} \mathscr{F}(v_{n_k}) < +\infty, \quad \lim_{k\uparrow+\infty} \|Lv_{n_k} - \zeta\|_B = 0,$$

  so that by (H.1) $k \mapsto v_{n_k}$ has at least one accumulation point, which *a fortiori* belongs to $\Lambda_{\boldsymbol{v},\zeta}$; this contradicts (3.5) which forces $d(v_{n_k}, \Lambda_{\boldsymbol{v},\zeta})$ to be greater than $\varepsilon_0 > 0$.
- v_{n_k} *is unbounded in A*: we fix $v_\infty \in \Lambda_{\boldsymbol{v},\zeta}$, and we observe that

  $$\|v_{n_k}\|_A^q \ge \tfrac{1}{2}\|v_{n_k}\|_A^q + \tfrac{1}{2^q} d(v_{n_k}, \Lambda_{\boldsymbol{v},\zeta})^q - \tfrac{1}{2}\|v_\infty\|_A^q.$$

  By (3.5) we obtain

  $$\begin{aligned} \|v_{n_k}\|_A^q &\ge \tfrac{1}{2}\|v_{n_k}\|_A^q + \tfrac{\varepsilon_0}{2^q}\mathscr{F}(v_{n_k}) - \tfrac{1}{2}\|v_\infty\|_A^q \\ &\ge \tfrac{\varepsilon_0}{2^q}\mathscr{F}(v_{n_k}) + \rho_k\Big(1 + \|Lv_{n_k}\|_B^p\Big) \end{aligned} \tag{3.6}$$

  where

  $$\rho_k := \tfrac{1}{2}\Big(1 + \|Lv_{n_k}\|_B^p\Big)^{-1}\Big(\|v_{n_k}\|_A^q - \|v_\infty\|_A^q\Big).$$

  (3.6) contradicts (H.2) since $\limsup_{k\uparrow+\infty} \rho_k = +\infty$.

□

Proof of Theorem 2.

Preliminaries. Now we can prove Theorem 2 in the case of a general measure space $(I, \mathscr{M}, \mu)$ as in paragraph 2; therefore, we will assume that u_n is a sequence of functions in $L^q(I; A)$ with

$$\exists\, \ell := \lim_{n\uparrow+\infty} Lu_n \quad \text{in } L^p(I;B) \quad \text{for some } p \in [1,+\infty), \tag{3.7}$$

and

$$S := \sup_{n\in\mathbb{N}} \int_I \mathscr{F}(u_n(t))\, d\mu(t) < +\infty; \tag{3.8}$$

(H.2) yields in particular that

$$S' := \sup_{n\in\mathbb{N}} \int_I \|u_n(t)\|_A^q\, d\mu(t) < +\infty. \tag{3.9}$$

Since the functions u_n are μ-almost separably valued [Yos80, V.4], it is not restrictive to assume that

$$A \quad \textit{is separable}. \tag{3.10}$$

Claim 1. $\mathcal{U}(t) \neq \varnothing$ for μ-a.e. $t \in I$.
Thanks to (3.7) there exists an increasing sequence $k \mapsto n_k$ such that

$$\lim_{k\uparrow+\infty} Lu_{n_k}(t) = \ell(t) \quad \text{strongly in } B, \text{ for } \mu\text{-a.e. } t \in I.$$

(3.8) and Fatou's Lemma yield

$$\int_I \Big(\liminf_{k\uparrow+\infty} \mathscr{F}(u_{n_k}(t)) \Big)\, dt \le S < +\infty,$$

and, in particular,

$$\liminf_{k\uparrow+\infty} \mathscr{F}(u_{n_k}(t)) < +\infty \quad \text{for } \mu\text{-a.e. } t \in I.$$

Taking account of the conditional compactness assumption (H.1), we deduce that the sets $\mathcal{U}(t)$ defined by (2.4) are not empty for every t in a measurable subset $I' \subseteq I$ with $\mu(I \setminus I') = 0$.

Claim 2. If $t \in I \mapsto w(t) \in A$ is (strongly) measurable, then the map

$$t \in I \mapsto d(w(t), \mathcal{U}(t)) \quad \textit{is measurable.}$$

The thesis is a simple consequence of Lemma 3.2, since for every $t \in I'$

$$\mathcal{U}(t) = \Lambda_{\{u_n(t)\}, \ell(t)};$$

in particular,

$$\begin{aligned} &d(w(t), \mathcal{U}(t)) = \\ &= \inf_{\varepsilon>0} \sup_{\sigma>0} \Big\{ \liminf_{n\uparrow+\infty} \Big(\|w(t) - u_n(t)\|_A + \varepsilon \mathscr{F}(u_n(t)) + \sigma \|Lu_n(t) - \ell(t)\|_B \Big) \Big\}. \end{aligned}$$

Observe that this property implies the measurability of the multifunction $t \mapsto \overline{\mathcal{U}(t)}$ (see Remark 2.4).

Claim 3. The functions $t \in I \mapsto \|u_n(t)\|_A^q$ *are equiintegrable.*
Since $Lu_n \to \ell$ in $L^p(I;B)$, it is easy to see that

$$\lim_{\mu(J)\downarrow 0} \int_J \|Lu_n(t)\|_B^p \, d\mu(t) = 0 \quad \text{uniformly w.r.t. } n.$$

Therefore, by (H.2)

$$\begin{aligned}\limsup_{\mu(J)\downarrow 0} \int_J \|u_n(t)\|_A^q \, d\mu(t) \le \varepsilon\Big(\sup_n \int_I \mathscr{F}(u_n(t))\, d\mu(t)\Big) \\ + C_\varepsilon \limsup_{\mu(J)\downarrow 0} \int_J \Big(1 + \|Lu_n(t)\|_B^p\Big)\, d\mu(t) \le \varepsilon S.\end{aligned}$$

Since ε is arbitrary we deduce that

$$\lim_{\mu(J)\downarrow 0} \int_J \|u_n(t)\|_A^q \, d\mu(t) = 0 \quad \text{uniformly w.r.t. } n.$$

Claim 4. The functions $t \mapsto d(u_n(t), \mathcal{U}(t))^q$ *are equiintegrable.*
We observe that

$$d(u_n(t), \mathcal{U}(t))^q \le 2^{q-1}\Big(\|u_n(t)\|_A^q + d(0, \mathcal{U}(t))^q\Big); \tag{3.11}$$

taking account of the previous Claim, it will be sufficient to prove that

$$\int_I d(0, \mathcal{U}(t))^q \, d\mu(t) < +\infty.$$

Starting from (3.2) and recalling (3.7), (3.8), and (3.9), Beppo Levi's Theorem and Fatou's Lemma yield

$$\begin{aligned}&\int_I d(0, \mathcal{U}(t))^q \, d\mu(t) \le \\ &\le \int_I \lim_{\sigma\uparrow+\infty} \liminf_{n\uparrow+\infty} \Big(\|u_n(t)\|_A^q + \mathscr{F}(u_n(t)) + \sigma\|Lu_n(t) - \ell(t)\|_B\Big)\, d\mu(t) = \\ &= \lim_{\sigma\uparrow+\infty} \int_I \liminf_{n\uparrow+\infty} \Big(\|u_n(t)\|_A^q + \mathscr{F}(u_n(t)) + \sigma\|Lu_n(t) - \ell(t)\|_B\Big)\, d\mu(t) \le \\ &\le \limsup_{\sigma\uparrow+\infty} \Big(S' + S + \sigma \lim_{n\uparrow+\infty} \int_I \|Lu_n(t) - \ell(t)\|_B \, d\mu(t)\Big) \le S' + S < +\infty.\end{aligned}$$

Claim 5. $\lim_{n\uparrow+\infty} \int_I d(u_n(t), \mathcal{U}(t))^q \, d\mu(t) = 0.$
For fixed $\varepsilon, M > 0$ we denote by $I_{\varepsilon,M}$ the subset of I

$$I_{\varepsilon,M} := \Big\{t \in I' : d(u_n(t), \mathcal{U}(t))^q \le \varepsilon\big(1 + \mathscr{F}(u_n(t))\big) + M\|Lu_n(t) - \ell(t)\|_B^p \ \ \forall\, n \in \mathbb{N}.\Big\}.$$

It is obvious that

$$\varepsilon \le \varepsilon', \quad M \le M' \quad \Rightarrow \quad I_{\varepsilon,M} \subseteq I_{\varepsilon',M'}.$$

By Lemma 3.3 we know that

$$\bigcup_{M>0} I_{\varepsilon,M} = I', \quad \forall\, \varepsilon > 0;$$

therefore,

$$\lim_{M\uparrow+\infty} \mu(I \setminus I_{\varepsilon,M}) = 0. \tag{3.12}$$

We obtain

$$\limsup_{n\uparrow+\infty}\int_I d(u_n(t),\mathcal{U}(t))^q\,d\mu(t)\le\limsup_{n\uparrow+\infty}\int_{I\setminus I_{\varepsilon,M}} d(u_n(t),\mathcal{U}(t))^q\,d\mu(t)+$$
$$+\varepsilon(1+S)+M\limsup_{n\uparrow+\infty}\int_{I_{\varepsilon,M}}\|Lu_n(t)-\ell(t)\|^p\,d\mu(t)$$
$$\le\limsup_{n\uparrow+\infty}\int_{I\setminus I_{\varepsilon,M}} d(u_n(t),\mathcal{U}(t))^q\,d\mu(t)+\varepsilon(1+S).$$

When $M\uparrow+\infty$, by the previous claim and (3.12), we get

$$\limsup_{n\uparrow+\infty}\int_I d(u_n(t),\mathcal{U}(t))^q\,d\mu(t)\le\varepsilon(1+S).$$

Finally, we choose ε arbitrarily small and we conclude. □

Proof of Proposition 2.7. Let us first prove that (H.2) implies (H.2'): this implication does not require A to be reflexive.

We fix

$$v_0\in A,\quad v\in\operatorname{Ker}L,\quad v\neq 0,$$

and two sequences $n\mapsto\lambda_n\in\mathbb{R}$, $n\mapsto v_n\in A$ such that

$$\lambda_n\uparrow+\infty,\quad v_n\rightharpoonup v,\quad Lv_n\to Lv\quad\text{as } n\uparrow+\infty. \tag{3.13}$$

Thanks to (H.2) we obtain

$$\varepsilon\mathscr{F}(v_0+\lambda_n v_n)\ge\lambda_n\|v_n\|_A-\|v_0\|_A-C_\varepsilon\Big(1+\|Lv_0\|_B+\lambda_n\|Lv_n\|_B\Big),$$

and dividing both members by λ_n,

$$\varepsilon\lambda_n^{-1}\mathscr{F}(v_0+\lambda_n v_n)\ge\|v_n\|_A-C_\varepsilon\|Lv_n\|_B-\lambda_n^{-1}\Big(\|v_0\|_A+C_\varepsilon(1+\|Lv_0\|_B)\Big).$$

Passing to the limit as $n\uparrow+\infty$ and taking into account (3.13) and

$$\liminf_{n\uparrow+\infty}\|v_n\|_A\ge\|v\|_A>0,\qquad\lim_{n\uparrow+\infty}\|Lv_n\|_B=\|Lv\|_B=0,$$

we get

$$\liminf_{n\uparrow+\infty}\lambda_n^{-1}\mathscr{F}(v_0+\lambda_n v_n)\ge\varepsilon^{-1}\|v\|_A.$$

Since ε can be chosen arbitrarily small, we deduce (H.2').

Let us prove now the opposite implication; we argue by contradiction and we suppose that there exists $\varepsilon>0$ and a sequence $\{w_n\}_{n\in\mathbb{N}}$ in A such that

$$\|w_n\|_A\ge\varepsilon\mathscr{F}(w_n)+n\Big(1+\|Lw_n\|_B\Big). \tag{3.14}$$

We set $\lambda_n:=\|w_n\|_A$, $v_n:=\lambda_n^{-1}w_n$, and we observe that (3.14) yields

$$\lim_{n\uparrow+\infty}\lambda_n=+\infty,\quad\varepsilon\lambda_n^{-1}\mathscr{F}(\lambda_n v_n)+n\|Lv_n\|\le 1. \tag{3.15}$$

Therefore we can extract a subsequence (still denoted by v_n) such that

$$v_n\rightharpoonup v,\quad Lv_n\to 0=Lv\quad\text{as } n\uparrow+\infty;$$

by Definition 2.6 of recession functional we get

$$\varepsilon\mathscr{F}_\infty(v)\le 1,$$

which contradicts (H.2') since $v\in\operatorname{Ker}L$. □

Proof of Proposition 2.8. Let us first check (H.1): we take a sequence $n \mapsto v_n$ such that

$$Lv_n \to \ell \in B, \quad \sup_{n\in\mathbb{N}} \overline{\mathscr{F}}(v_n) = s < +\infty. \tag{3.16}$$

By definition (2.19) of $\overline{\mathscr{F}}$, there exists another sequence $n \mapsto w_n \in D(\mathscr{F})$ such that

$$\lim_{n\uparrow+\infty} \|w_n - v_n\|_A = 0, \quad \sup_{n\in\mathbb{N}} \mathscr{F}(w_n) \le s + 1. \tag{3.17}$$

By (3.16) and the first limit of (3.17), $Lw_n \to \ell$ in B. Therefore, by (H.1), we can find a subsequence $k \mapsto w_{n_k}$ strongly converging in A; (3.17) yields the convergence of $k \mapsto v_{n_k}$.

In order to check that $\overline{\mathscr{F}}$ satisfies (H.2), it is sufficient to notice that the map

$$v \mapsto \|v\|_A^q - C_\varepsilon(1 + \|Lv\|_B^p) \tag{3.18}$$

is continuous in A and it is bounded from above by $\varepsilon\mathscr{F}$; by the extremal property of $\overline{\mathscr{F}}$, (3.18) is less than $\varepsilon\overline{\mathscr{F}}$, too. □

Proof of Theorem 2 for closed operators. We replace A with the Banach space $D(L)$ endowed with the graph norm

$$\|v\|_{D(L)} := \|v\|_A + \|Lv\|_B,$$

which makes L continuous; we will denote by $\mathscr{F}$ again its restriction to $D(L)$ and we simply have to check that (H.1) and (H.2) are satisfied in this setting, too.

Since a sequence $\{v_n\}$ in $D(L)$ converges in $D(L)$ iff it converges in A and Lv_n converges in B, (H.1) is immediate; (H.2) is also obvious, since the L-part of the norm of $D(L)$ is already controlled by the right-hand side of (H.2). □

4. Applications.

An abstract version of Luckhaus Theorem. First of all, we will consider the situation of Example 2; so we are assuming that

(4.1) $\quad B$ is a Banach space, $\mathscr{F}_1, \mathscr{F}_2 : B \to [0, +\infty]$ are proper l.s.c. functionals.

Theorem 3. *Let $1 \le q \le p < +\infty$ and let $\theta_n, \chi_n :]0, T[\to B$ be two sequences of (strongly) measurable functions satisfying,*

$$\int_0^T \mathscr{F}_1(\theta_n(t))\,dt + \int_0^T \mathscr{F}_2(\chi_n(t))\,dt \le S < +\infty \quad \forall\, n \in \mathbb{N},$$

and

$$\theta_n + \chi_n \to \ell \quad \textit{strongly in } L^p(0,T;B).$$

If

(4.2) *the sublevels of $\mathscr{F}_1$ or $\mathscr{F}_2$ are compact in B,*

$$\lim_{\|v\|_B\uparrow+\infty} \frac{\mathscr{F}_1(v)}{\|v\|_B^q} = +\infty \quad \textit{or} \quad \lim_{\|v\|_B\uparrow+\infty} \frac{\mathscr{F}_2(v)}{\|v\|_B^q} = +\infty, \tag{4.3}$$

then, denoting by

$$\mathcal{L}(t) := \Big\{(\theta,\chi) \in D(\mathscr{F}_1) \times D(\mathscr{F}_2) : \theta + \chi = \ell(t)\Big\},$$

we have

$$\lim_{n\uparrow+\infty} \int_0^T \inf_{(\theta,\chi)\in\mathcal{L}(t)} \Big\{\|\theta_n(t) - \theta\|_B^q + \|\chi_n(t) - \chi\|_B^q\Big\}\,dt = 0. \tag{4.4}$$

In particular, if $\mathscr{F}_1, \mathscr{F}_2$ *satisfy*

$$\left.\begin{array}{c}\theta, \theta' \in D(\mathscr{F}_1), \quad \chi, \chi' \in D(\mathscr{F}_2) \\ \theta + \chi = \theta' + \chi'\end{array}\right\} \quad \Rightarrow \quad \theta = \theta', \quad \chi = \chi', \tag{4.5}$$

then

$$\exists\, \theta, \chi \in L^q(0,T;B): \quad \theta_n \to \theta, \quad \chi_n \to \chi \quad \textit{strongly in } L^q(0,T,B). \tag{4.6}$$

Proof. As we did in Example 2, we set

$$A := B \times B, \quad \|(\theta,\chi)\|_A := \max\big(\|\theta\|_B, \|\chi\|_B\big), \quad L: u = (\theta,\chi) \to \theta + \chi \tag{4.7}$$

$$\mathscr{F}(u) := \mathscr{F}_1(\theta) + \mathscr{F}_2(\chi), \tag{4.8}$$

and we apply Theorem 2 with its corollary; therefore, we have to check that (H.1) and (H.2) are satisfied.

Control of (H.1). We take a sequence

$$u_n := (\theta_n, \chi_n) \in A \quad \text{such that } \theta_n + \chi_n \to \zeta \quad \text{in } B \tag{4.9}$$

and

$$\mathscr{F}_1(\theta_n) + \mathscr{F}_2(\chi_n) \le s < +\infty \quad \forall\, n \in \mathbb{N}. \tag{4.10}$$

Let us suppose that in (4.2), e.g., the sublevels of $\mathscr{F}_1$ are compact; then we can extract a subsequence $k \mapsto \theta_{n_k}$ converging in B, so that (4.9) yields the convergence of χ_{n_k} in B and u_{n_k} in A, too.

Control of (H.2). We can suppose that the first limit of (4.3) holds; it is easy to see that it is equivalent to

$$\forall\, \varepsilon > 0\ \exists\, c_\varepsilon \ge 0: \quad \|\theta\|_B^q \le \varepsilon \mathscr{F}_1(\theta) + c_\varepsilon \quad \forall\, \theta \in B. \tag{4.11}$$

Therefore, we get

$$\begin{aligned}\|u\|_A^q &\le \max\big(\|\theta\|_B^q, \|\chi\|_B^q\big) \le 2^{q-1}\big(\|\theta\|_B^q + \|\theta+\chi\|_B^q\big) \le \\ &\le 2^{q-1}\Big(\varepsilon\mathscr{F}_1(\theta) + 2c_\varepsilon + \|\theta+\chi\|_B^p\Big) \le 2^{q-1}\Big(\varepsilon\mathscr{F}(u) + c_\varepsilon + \|Lu\|_B^p\Big)\end{aligned}$$

so that (H.2) is satisfied. In order to prove that (4.5) implies (4.6), we apply Theorem 1 and we observe that (4.5) is equivalent to (H.3) in the framework of (4.7, 4.8). □

If we particularize the choice of $\mathscr{F}_1$ and $\mathscr{F}_2$ we easily get a refinement of Luckhaus - Plotnikov Theorems.

Corollary 4. *In the framework of* (1.10, …, 1.14), (1.18), *with* $1 \le q \le p < +\infty$, *let us assume that one of the following conditions holds*

i. K *is* compact *in* A

ii. K *is bounded and the inclusion of* V *in* B *is compact;*

iii. the inclusion of V *in* B *is compact and* $q < p$.

Then, denoting by

$$\mathcal{L}(t) := \Big\{(\theta,\chi) \in V \times K : \theta + \chi = \ell(t)\Big\},$$

we have

$$\lim_{n\uparrow+\infty} \int_0^T \inf_{(\theta,\chi)\in\mathcal{L}(t)} \Big\{\|\theta_n(t) - \theta\|_B^q + \|\chi_n(t) - \chi\|_B^q\Big\}\, dt = 0. \tag{4.12}$$

In particular, if (1.15) *holds too, then*

$$\exists\,\theta,\chi \in L^q(0,T;B): \quad \theta_n \to \theta, \quad \chi_n \to \chi \quad \textit{strongly in } L^q(0,T,B). \tag{4.13}$$

Proof. It is sufficient to apply Theorem 3 with $\mathscr{F}_1, \mathscr{F}_2$ given by (1.17). □

Remark 4.1. It is easy to obtain a simple variant of the previous Corollary in the case of a time-dependent family of closed sets $K(t)$ (see [HS98a, HS98b]): if (1.12) is replaced by

$$\theta_n(t) \in V, \quad \chi_n(t) \in K(t) \subset K \quad \text{for a.e. } t \in]0,T[\,, \tag{4.14}$$

where $\{K(t)\}_{t\in]0,T[}$ is a family of closed nonempty sets which satisfy

$$\chi,\chi' \in K(t), \quad \chi-\chi' \in V \quad \Rightarrow \quad \chi=\chi' \quad \text{for a.e. } t \in]0,T[\,,$$

then we can deduce (4.13), under the same other assumptions of the previous corollary. It is sufficient to apply Corollary 2, taking into account that in this case

$$\mathcal{U}(t) \subseteq V \times K(t) \quad \text{for a.e. } t \in]0,T[\,.$$

Quasistationary phase field equations with Neumann boundary conditions. In this last paragraph, we will apply Corollary 3 to refine the proof of [Sch00] about the convergence of solutions of quasistationary phase field equations with Neumann boundary conditions to solutions of the Stefan problem with the Gibbs-Thomson law.

Following [Sch00], we define Ω, Q as in (1.1), $W : s \in \mathbb{R} \mapsto (s^2-1)^2$, $\varepsilon > 0$,

$$G_\varepsilon : (\chi, w) \in H^1(\Omega) \times L^2(\Omega) \mapsto \int_\Omega \Big(\varepsilon|\nabla\chi|^2 + \tfrac{1}{\varepsilon}W(\chi) + \tfrac12\chi^2 - w\chi\Big)\,dx.$$

For $f \in L^2(Q), w \in L^2(\Omega), \delta_\varepsilon > 0$, we look for the functions $\theta_\varepsilon, \chi_\varepsilon, w_\varepsilon$, which solve

$$w_\varepsilon = \chi_\varepsilon + \theta_\varepsilon, \tag{4.15}$$

$$\begin{cases} \partial_t w_\varepsilon - \Delta\theta_\varepsilon = f & \text{in } Q,\\ \partial_\nu \theta_\varepsilon = 0 & \text{on } \partial\Omega\times\,]0,T[,\\ w_\varepsilon(x,0) = \underline{w}(x) & \text{in } \Omega; \end{cases} \tag{4.16}$$

$$\begin{cases} -2\varepsilon\Delta\chi_\varepsilon + \frac{1}{\varepsilon}W'(\chi_\varepsilon) = \theta_\varepsilon & \text{in } Q,\\ \partial_\nu\chi_\varepsilon = 0 & \text{on } \partial\Omega\times\,]0,T[\,; \end{cases} \tag{4.17}$$

$$G_\varepsilon(\chi_\varepsilon(\cdot,t), w_\varepsilon(\cdot,t)) \le G_\varepsilon(v, w_\varepsilon(\cdot,t)) + \delta_\varepsilon \quad \forall\, v \in H^1(\Omega), \text{ a.e. in }]0,T[\,. \tag{4.18}$$

[Sch00, Theorem 2.2] provides the existence of $(\theta_\varepsilon, \chi_\varepsilon, w_\varepsilon)$ for every $\varepsilon > 0$, together with $\delta_\varepsilon \downarrow 0$ as ε goes to 0, the *a priori* estimates

$$\|\partial_t w_\varepsilon\|_{L^2(0,T;H^{-1}(\Omega))}, \ \|\theta_\varepsilon\|_{L^2(0,T;H^1(\Omega))} \le S, \tag{4.19}$$

$$\int_\Omega \Big(|\chi_\varepsilon(x,t)|^4 + |\nabla\chi_\varepsilon(x,t)|\Big)\,dx \le S \quad \text{for a.e. } t \in]0,T[\,, \tag{4.20}$$

and the compactness property

$$\exists\,\varepsilon_n \downarrow 0: \quad w_{\varepsilon_n} \to w \quad \text{strongly in } L^2(Q) \text{ and a.e. in } Q. \tag{4.21}$$

Let us recall [Giu84] that $BV(\Omega)$ denotes the Banach space of the functions $v \in L^1(\Omega)$ with finite total variation

$$\int_\Omega |Dv| := \sup\Big\{ \int_\Omega v(x)\,\mathrm{div}\,\phi(x)\,dx : \phi \in C_0^\infty(\Omega;\mathbb{R}^N),\ |\phi(x)| \le 1 \text{ in } \Omega\Big\} < +\infty.$$

We also introduce the subset of $BV(\Omega)$

$$BV(\Omega;\{-1,1\}) := \Big\{v \in BV(\Omega) : |v(x)| = 1 \text{ for a.e. } x \in \Omega\Big\}$$

and we can prove the following result:

Theorem 4. *Let us assume that the function $\rho : [0,T] \to \mathbb{R}$, defined by*

$$\rho(s) := \int_\Omega \underline{w}(x)\,dx + \int_0^s \int_\Omega f(x,t)\,dx\,dt,$$

satisfies

$$\rho(s) \neq 0 \quad \text{for a.e. } s \in \,]0,T[\,. \tag{4.22}$$

Then there exist the limits

$$\lim_{n\to+\infty} \theta_{\varepsilon_n} = \theta, \qquad \lim_{n\to+\infty} \chi_{\varepsilon_n} = \chi \quad \text{strongly in } L^2(Q), \tag{4.23}$$

and they are solutions of the quasistationary Stefan problem with Neumann b.c.

$$w = \chi + \theta, \tag{4.24}$$

$$\begin{cases} \partial_t w - \Delta\theta = f & \text{in } Q,\\ \partial_\nu \theta = 0 & \text{on } \partial\Omega \times \,]0,T[\,,\\ w(x,0) = \underline{w}(x) & \text{in } \Omega; \end{cases} \tag{4.25}$$

$$x \mapsto \chi(t,x) \in BV(\Omega;\{-1,1\}) \quad \text{for a.e. } t \in \,]0,T[\,, \tag{4.26}$$

$$G(\chi(\cdot,t), w(\cdot,t)) \le G(v, w(\cdot,t)) \quad \forall\, v \in BV(\Omega;\{-1,1\}), \tag{4.27}$$

where

$$G(v,w) := \begin{cases} \frac{4}{3}\int_\Omega |Dv| - \int_\Omega w(x)v(x)\,dx & \text{if } v \in BV(\Omega;\{-1,1\});\\ +\infty & \text{otherwise.}\end{cases} \tag{4.28}$$

Proof. The Theorem follows by standard convergence arguments if we prove (4.23). Let us denote by I' the subset of the points $t \in \,]0,T[$ where the sequence $n \mapsto w_{\varepsilon_n}(\cdot,t)$ converges in $L^2(\Omega)$ and $\rho(t) \neq 0$; by (4.22) $]0,T[\,\setminus I'$ is negligible.

We apply Corollary 3 with $p = q = 2$,

$$B := L^2(\Omega), \quad A := B \times B, \quad L : u = (\theta,\chi) \mapsto \theta + \chi$$
$$\mathscr{F}(u) := \int_\Omega |\theta(x)|^2 + |\nabla\theta(x)|^2 + |\chi(x)|^4\,dx,$$
$$\mathscr{G}_n(t,u) := G_{\varepsilon_n}(\chi, w_{\varepsilon_n}(\cdot,t)) \quad \text{for } u = (\theta,\chi),$$

where we adopted the usual convention of setting $\mathscr{F}(u) = +\infty$ if $\theta \notin H^1(\Omega)$ or $\chi \notin L^4(\Omega)$, and $\mathscr{G}_n(t,u) = +\infty$ if $\chi \notin H^1(\Omega)$. Therefore, we are in the situation of Problem (P) and arguing as in Theorem 3 and its Corollary, we can easily check (H.1) and (H.2).

In order to complete the argument, we have to study the minimum set $\mathcal{G}(t)$ of the $\Gamma(L^2(\Omega))$-limit $\mathscr{G}$ of $\mathscr{G}_n$. The result of [Mod86] and a standard perturbation argument [Dal93] show that this functional is

$$\mathscr{G}(t,u) := G(\chi, w(\cdot,t)) + \tfrac{1}{2}|\Omega| \quad \forall\, t \in I'.$$

If $u_i := (\theta_i, \chi_i)$, $i = 1, 2$, belong to $\mathcal{G}(t)$ (defined by (2.16)) for some $t \in I'$, then

$$\chi_1 - \chi_2 = \theta_2 - \theta_1 \in H^1(\Omega). \tag{4.29}$$

Since $\mathcal{G}(t) \subset BV(\Omega; \{-1, 1\})$, then (4.29) yields

$$\chi_1 = \chi_2 \quad \text{or } \chi_1 = -\chi_2 \equiv \pm 1.$$

Since

$$\mathscr{G}(t, u_1) = \mathscr{G}(t, u_2),$$

the last case implies that

$$\int_\Omega w(x,t)\,dx = -\int_\Omega w(x,t)\,dx = 0.$$

On the other hand, passing to the limit into (4.16) and recalling (4.22), we get $\int_\Omega w(x,t)\,dx = \rho(t) \neq 0$ for every $t \in I'$: we conclude that $\mathcal{G}(t)$ is non degenerate w.r.t. L for every $t \in I'$. □

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