

A Posteriori Error Estimates for Variable Time-Step Discretizations of Nonlinear Evolution Equations

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Dedicated to Enrico Magenes on the occasion of his 75th birthday

Abstract

We study the backward Euler method with variable time steps for abstract evolution equations in Hilbert spaces. Exploiting convexity of the underlying potential or the angle-bounded condition, thereby assuming no further regularity, we derive novel a posteriori estimates of the discretization error in terms of computable quantities related to the amount of energy dissipation or monotonicity residual. These estimators depend solely on the discrete solution and data and impose no constraints between consecutive time steps. We also prove that they converge to zero with an optimal rate with respect to the regularity of the solution. We apply the abstract results to a number of concrete, strongly nonlinear problems of parabolic type with degenerate or singular character. © 2000 John Wiley & Sons, Inc.

Contents

1. Introduction	526
2. Abstract Setting and Examples	533
3. The Subdifferential Case	548
4. The Nonpotential Case	559
5. Applications	575
Bibliography	586

1 Introduction

For the sole purpose of motivating the new ideas in a simple setting, we start with ordinary differential equations in \mathbb{R}^d . We consider the Cauchy problem

$$(1.1) \quad \begin{cases} u'(t) + \mathfrak{F}(u(t)) = 0, & \forall t \in (0, T), \\ u(0) = u_0, \end{cases}$$

where $\mathfrak{F} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a given (nonlinear) vector field and $u_0 \in \mathbb{R}^d$.

We are interested in studying the error between the solution u of (1.1) and the approximate solution U defined by a variable-step implicit Euler scheme. More precisely, we choose a partition of the time interval $[0, T]$

$$(1.2) \quad \mathcal{P} := \{0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T\}$$

with variable step $\tau_n := t_n - t_{n-1}$, and we consider the sequence $\{U_n\}_{n=0}^N$ whose first term is given and the other ones are recursively defined for $1 \leq n \leq N$ by

$$(1.3) \quad \frac{U_n - U_{n-1}}{\tau_n} + \mathfrak{F}(U_n) = 0.$$

Hereafter, U is the piecewise linear interpolant of the values $\{U_n\}_{n=0}^N$ on the grid \mathcal{P} and $\tau := \max_{1 \leq n \leq N} \tau_n$ denotes the maximum of the time step sizes.

When \mathfrak{F} is continuous and satisfies the so-called *one-sided Lipschitz condition* ([22, (1.18)], [35, (6.2)]) with respect to the Euclidean scalar product $\langle \cdot, \cdot \rangle$; i.e., there exists $\lambda \in \mathbb{R}$ such that

$$(1.4) \quad \langle \mathfrak{F}(v) - \mathfrak{F}(w), v - w \rangle \geq -\lambda |v - w|^2, \quad \forall v, w \in \mathbb{R}^d,$$

then both (1.1) and (1.3), at least for sufficiently small τ such that $\lambda\tau < 1$, are uniquely solvable. Moreover, an a priori linear order of convergence is well-known for $E_n := |u(t_n) - U_n|$ provided $u'' \in L^1(0, T)$ ([21, section 7.3,a], [36]); for $\lambda > 0$,

$$(1.5) \quad E_n \leq e^{2\lambda t_n} (E_0 + 2\tau \|u''\|_{L^1(0, T)}) \quad \text{if } 2\lambda\tau \leq 1.$$

This estimate is derived upon evaluating local truncation errors and is optimal with respect to the time-stepping method. It is usually coupled with a time step selection based either on the evaluation of the local truncation error [31] or on duality techniques [27, 37], which require at least Lipschitz continuity of \mathfrak{F} and so $u'' \in L^\infty(0, T)$. The *monotone* case $\lambda = 0$

$$(1.6) \quad \langle \mathfrak{F}(v) - \mathfrak{F}(w), v - w \rangle \geq 0, \quad \forall v, w \in \mathbb{R}^d,$$

is perhaps the most important one in that a vector field \mathfrak{F} satisfying (1.4) with $\lambda > 0$ could be reduced to a monotone operator by adding a linear perturbation. For $\lambda \leq 0$, the error becomes [21, section 7.3,a]

$$(1.7) \quad E_n \leq e^{\lambda t_n/2} E_0 + \tau \|u''\|_{L^1(0, T)} \quad \text{if } \lambda\tau \geq -1.$$

So the initial error E_0 is no longer exponentially amplified if $\lambda = 0$ and even exhibits an exponential decay provided \mathfrak{F} is *strongly* monotone, i.e., $\lambda < 0$.

However, (1.5) and (1.7) are *nonoptimal* with respect to the regularity of the exact solution u , since they need an a priori estimate of $\|u''\|_{L^1(0,T)}$, whereas a linear rate of convergence could be expected for u' only bounded. This fact is undesirable not only when (1.1) is a dissipative ODE associated to an *irregular* vector field, for which u' may exhibit jump discontinuities, but also when (1.1) comes from a space discretization of a parabolic PDE. In this case the Lipschitz constant of the \mathfrak{F} 's grows to $+\infty$ as the mesh parameter tends to 0.

Our prime interest here is to obtain alternative error estimates that avoid this drawback; more precisely, we want to address five fundamental issues:

1. A posteriori estimates for $|u - U|$: computable quantities that depend solely on time steps, discrete solution, and data.
2. Optimal rates of convergence: a posteriori bounds converging to zero as $\tau \downarrow 0$ with an optimal rate with respect to the regularity of the solution u .
3. Minimal regularity: \mathfrak{F} is a (Lipschitz perturbation of a) monotone vector field that admits a lower semicontinuous *convex potential* ϕ or satisfies more general *angle-boundedness* conditions, but no other regularity assumptions (such as Lipschitz continuity) are made on \mathfrak{F} .
4. Uniform stability and explicit constants: The stability and error constants are uniform with respect to possible space discretizations (in particular, they do not depend on the space dimension d), and they are explicitly determined without need of solving any auxiliary, or dual, problem.
5. Variable time steps: no a priori constraints between consecutive time steps, which could just be tailored to the a posteriori error estimators alone.

We refer to (1) and (2) collectively as *optimal* a posteriori error estimates.

1.1 A Priori Error Estimates in Hilbert Spaces

Most contributions dealing with minimal regularity do not employ the finite dimension of \mathbb{R}^d but simply its Hilbert structure. This generality allows for direct applications to various (nonlinear) partial differential equations of (degenerate or singular) parabolic type as those considered later, and a multivalued \mathfrak{F} defined in a proper subdomain $D(\mathfrak{F})$ of a (functional) Hilbert space \mathcal{H} could also be accommodated within this approach. In many concrete cases, $D(\mathfrak{F})$ is related to some extra growth and regularity properties, which are necessary to define \mathfrak{F} on the functions of \mathcal{H} , whereas the possibility for dealing with multiple values allows a unified treatment of nondifferentiable constraints. In this framework, a seminal discovery was that the continuity of \mathfrak{F} can be replaced by the simple *solvability* of (1.3) for $\lambda\tau < 1$. Combining this assumption with the weak monotonicity (1.4) (suitably extended to Banach spaces), Crandall and Liggett [20] proved a fundamental and deep result in the case of a uniform partition \mathcal{P} (i.e., $\tau_n \equiv \tau$ for every n): U converges uniformly to u as $\tau \downarrow 0$. This was later extended to the case of a nonuniform \mathcal{P} in [19]; see also [18] for an interesting review of these and other related results and their implications to the well-posedness of (1.1) in a general Banach space. Moreover,

when \mathcal{P} is uniform and $u_0 \in D(\mathfrak{F})$, an a priori estimate of the error $|u - U|$ of order $O(\sqrt{\tau})$ is derived in [20], and of order $o(\sqrt{\tau})$ in [52]; Rulla showed in [52] that the latter is tight for general monotone operators.

However, improved a priori error estimates typical of linear problems [5, 59, 60] are available provided \mathfrak{F} is the gradient of a potential function ϕ . If we assume (1.6) for simplicity, then the underlying potential ϕ must be *convex*, and \mathfrak{F} can be characterized by means of the system of inequalities

$$z = \mathfrak{F}(w) \Leftrightarrow \langle z, v - w \rangle \leq \phi(v) - \phi(w), \quad \forall v \in \mathbb{R}^d.$$

The differential equations (1.1) and (1.3) are thus equivalent to the following evolution variational inequalities, where the explicit occurrence of \mathfrak{F} disappears:

$$(1.8) \quad \langle u'(t), u(t) - v \rangle + \phi(u(t)) - \phi(v) \leq 0, \quad \forall v \in \mathbb{R}^d,$$

$$(1.9) \quad \tau_n^{-1} \langle U_n - U_{n-1}, U_n - v \rangle + \phi(U_n) - \phi(v) \leq 0, \quad \forall v \in \mathbb{R}^d.$$

The variational inequalities (1.8) and (1.9) are convenient to study nonlinear problems governed by discontinuous functionals, such as obstacle problems. The lack of regularity of \mathfrak{F} , and so of u , is compensated by the convexity of ϕ .

An important first result was obtained by Baiocchi [4], who assumed ϕ to be the sum of a positive lower semicontinuous quadratic form and the indicator function of a convex set in a Hilbert space \mathcal{H} (obstacle problem). Baiocchi showed that if $u_0 \in D(\mathfrak{F})$, then the error $u - U$ in the natural energy norm is a priori of order $O(\tau)$. This result is optimal with respect to both the order and regularity of the solution u in that u' is only bounded and, even in one dimension, may exhibit jump discontinuities; both the truncation error analysis and duality argument make no sense in this context. Other error estimates and their relationships with regularity properties of u are studied in [53].

Rulla [52] has recently extended Baiocchi's result to generic lower semicontinuous convex functions ϕ upon exploiting fine operator properties of the Yosida approximation of \mathfrak{F} . The inequality approach has been examined by Savaré [54], who has derived similar error estimates to those of [52] under minimal regularity on an additional source term.

To recover a linear, and thus optimal, rate of convergence, all the above contributions rely on the mesh \mathcal{P} being uniform. Moreover, the resulting estimates are *not* really a posteriori and thus computable, because they involve the a priori Lipschitz regularity of the unknown solution u .

1.2 A Posteriori Error Estimation via Duality

Quasi-optimal a posteriori error estimates for variable time step discretizations are proved only for linear dissipative ODEs [37] and linear parabolic PDEs [28] or mildly nonlinear variants of them [29] under the constraint $\tau_{n-1} \leq a\tau_n$ between consecutive time steps. Suboptimal a posteriori error estimates are also available for degenerate parabolic problems [48, 49]. Their proof relies on duality techniques, which we now briefly recall.

Let \mathcal{R} be the residual, that is, the negative of the amount by which U misses being an exact solution

$$(1.10) \quad \mathcal{R} := -U' - \mathfrak{F}(U).$$

Adding (1.10) to (1.1), we obtain the error equation for $e := u - U$,

$$(1.11) \quad e' + \mathfrak{A}e = \mathcal{R},$$

where

$$\mathfrak{A} := \int_0^1 \nabla \mathfrak{F}(su + (1-s)U) ds.$$

We multiply (1.11) by $\varphi \in C^1(0, T)$ and integrate by parts over $(0, T)$ to get

$$(1.12) \quad \langle e(T), \varphi(T) \rangle = \langle e(0), \varphi(0) \rangle + \int_0^T \langle e, \varphi' - \mathfrak{A}^* \varphi \rangle + \int_0^T \langle \mathcal{R}, \varphi \rangle,$$

where \mathfrak{A}^* is the transpose of \mathfrak{A} . The a posteriori error estimate follows by selecting φ in (1.12) as the solution of the backward dual problem

$$(1.13) \quad \varphi' - \mathfrak{A}^* \varphi = 0 \text{ in } (0, T), \quad \varphi(T) = e(T),$$

and using *strong* stability properties of φ , such as a bound for φ' , for evaluating the initial error and the residual terms. In fact, we notice that, for a linear \mathfrak{F} , $\langle \mathcal{R}, \varphi \rangle = \langle \bar{U} - U, \varphi' \rangle$ where \bar{U} is the piecewise constant interpolant of $\{U_n\}_{n=1}^N$. This program may fail to apply to the present low regularity setting because

1. \mathfrak{F} may not be Lipschitz, and so \mathfrak{A} and \mathfrak{A}^* may be singular or even not well-defined;
2. (1.13) may make no sense or φ' may be unbounded;
3. φ is not computable because it depends on the unknown u (this implies no precise control of stability constants in the a posteriori error estimators);
4. if \mathfrak{A} were replaced by $\nabla \mathfrak{F}(U)$ to determine φ , then an additional term would arise involving further derivatives of the possibly singular $\nabla \mathfrak{F}$.

Our techniques below are not based on duality as those of Johnson et al. [28, 37] and Nochetto et al. [48, 49], and thus circumvent (1)–(4) altogether. They seem to capture the essence of dissipation at a more elementary level.

1.3 A Posteriori Error Estimation via Discrete Energy Dissipation

In the present paper, we resort to both the inequality and the operator approaches, and we establish a posteriori error estimates for variable time step discretizations (1.3) of (1.1), which are *optimal* with respect to both order and regularity solely and impose no constraints between consecutive time steps. We prove effective bounds for the local error arising at each step in terms of computable quantities, which are related to the amount of energy dissipation in the case \mathfrak{F} admits a convex potential; these quantities are not residuals. This allows for simple a posteriori error estimates and adaptive procedures for the choice of the time steps.

Besides the variational structure of (1.8) and (1.9), in the potential case our techniques exploit the Lyapunov properties of ϕ , which decreases along solution paths of both (1.1) and (1.3). In fact, it is known that ϕ satisfies the energy identity

$$(1.14) \quad |u'(t)|^2 + \frac{d}{dt}\phi(u(t)) = 0 \quad \text{a.e. } t \in (0, T)$$

and the discrete energy inequality

$$(1.15) \quad -\mathcal{E}_n := \left| \frac{U_n - U_{n-1}}{\tau_n} \right|^2 + \frac{\phi(U_n) - \phi(U_{n-1})}{\tau_n} \leq 0, \quad \forall 1 \leq n \leq N,$$

which follows directly from (1.9) upon choosing $v = U_{n-1}$. The discrete quantity $\mathcal{E}_n \geq 0$ in (1.15) is thus a measure of the deviation of $\{U_n\}_{n=0}^N$ from satisfaction of the conservation form (1.14). We now discuss the rather elementary derivation of the following a posteriori upper error bound for ODEs

$$(1.16) \quad \max_{t \in [0, T]} |u(t) - U(t)|^2 \leq |u_0 - U_0|^2 + \sum_{n=1}^N \tau_n^2 \mathcal{E}_n.$$

First of all, starting from (1.9) we write the *continuous* evolution variational inequality satisfied by U for all $t \in (t_{n-1}, t_n)$ and $v \in \mathbb{R}^d$

$$(1.17) \quad \langle U'(t), U(t) - v \rangle + \phi(U(t)) - \phi(v) \leq \mathcal{R}(t),$$

where $\mathcal{R}(t)$ is a residual term independent of the test function v given by

$$\mathcal{R}(t) := \langle U'(t), U(t) - U_n \rangle + \phi(U(t)) - \phi(U_n).$$

The second step consists in estimating $\mathcal{R}(t)$. Here we use the elementary identity

$$U(t) - U_n = \frac{t - t_n}{\tau_n} (U_n - U_{n-1}) = (t - t_n) U'(t)$$

and the convexity of ϕ to get the bound

$$\phi(U(t)) \leq \frac{t_n - t}{\tau_n} \phi(U_{n-1}) + \frac{t - t_{n-1}}{\tau_n} \phi(U_n).$$

Hence, in view of (1.15), $\mathcal{R}(t)$ can be bounded by

$$\mathcal{R}(t) \leq (t_n - t) \mathcal{E}_n.$$

Finally, choosing $v = U(t)$ in (1.8), $v = u(t)$ in (1.17), and adding the two inequalities, we see that the terms involving ϕ cancel out, and we end up with the error inequality

$$\frac{1}{2} \frac{d}{dt} |u(t) - U(t)|^2 \leq (t_n - t) \mathcal{E}_n \quad \text{a.e. } t \in (t_{n-1}, t_n).$$

Upon integration in time, this upper bound readily implies (1.16).

1.4 A Posteriori Error Estimation via Residual Monotone Terms

The discrete counterpart of the dissipation energy inequality

$$\frac{d}{dt}|u'(t)|^2 \leq 0 \quad \text{a.e. } t \in (0, T),$$

namely

$$(1.18) \quad \left| \frac{U_n - U_{n-1}}{\tau_n} \right|^2 \leq \left| \frac{U_{n-1} - U_{n-2}}{\tau_{n-1}} \right|^2 \quad \forall 1 \leq n \leq N,$$

can be derived as follows: Taking $v = U_n$ in (1.9) at the time step $n - 1$, with $U_{-1} := U_0 - \tau_0 \mathfrak{F}(U_0)$, we find

$$(1.19) \quad \tau_{n-1}^{-1} \langle U_{n-1} - U_{n-2}, U_{n-1} - U_n \rangle + \phi(U_{n-1}) - \phi(U_n) \leq 0.$$

By dividing (1.19) by τ_n and replacing it into (1.15), (1.18) follows from the elementary inequality $2\langle v - w, v \rangle \geq |v|^2 - |w|^2$ for $v, w \in \mathbb{R}^d$. Recalling (1.3), we also deduce a crucial bound of \mathcal{E}_n in terms of a residual monotone term

$$(1.20) \quad \mathcal{E}_n \leq \tau_n^{-1} \langle \mathfrak{F}(U_n) - \mathfrak{F}(U_{n-1}), U_n - U_{n-1} \rangle =: \mathcal{D}_n.$$

Inserting (1.20) into (1.16) yields another a posteriori error estimate

$$(1.21) \quad \max_{t \in [0, T]} |u(t) - U(t)|^2 \leq |u_0 - U_0|^2 + \sum_{n=1}^N \tau_n^2 \mathcal{D}_n.$$

Moreover, we prove in Section 3 an exponential decay of the error provided \mathfrak{F} is also *strongly* coercive.

We stress that \mathcal{D}_n does not depend on the existence of a potential ϕ for \mathfrak{F} . An amazing fact, fully developed in Section 4, is that (1.21) holds for a wider class of monotone operators than that of (sub)gradients, the so-called *angle-bounded* operators; they appear to possess the right structure for a posteriori error control. This generalization applies to 3-cyclically-monotone operators, to linear sectorial (unbounded) operators, and to the classical formulation of variational inequalities (e.g., of obstacle type) in a Hilbert triplet. On the other hand, it is hopeless to seek an estimate such as (1.21) for general monotone operators. In fact, we show below that (1.21) implies *optimal* a priori rates of convergence of order $O(\tau)$, which are excluded by the counterexample of Rulla [52, example 3].

1.5 Regularity and Rate of Convergence

We prove now that estimates (1.16) and (1.21) give rise to upper bounds of the error with respect to the largest time step τ . A simple first result is directly suggested by (1.15), which implies

$$0 \leq \tau_n \mathcal{E}_n \leq \phi(U_{n-1}) - \phi(U_n).$$

Therefore, we end up with a telescopic sum

$$\sum_{n=1}^N \tau_n^2 \mathcal{E}_n \leq \tau \sum_{n=1}^N (\phi(U_{n-1}) - \phi(U_n)) = \tau (\phi(U_0) - \phi(U_N)).$$

Without loss of generality, we may assume $\phi \geq 0$, for otherwise we may add an affine function, and we choose $U_0 := u_0$. We then obtain a rate of convergence of order $O(\sqrt{\tau})$

$$(1.22) \quad \max_{t \in [0, T]} |u(t) - U(t)|^2 \leq \phi(u_0) \tau.$$

On the other hand, making use of (1.3) in conjunction with (1.20), we can rewrite the residual monotone term as follows:

$$\mathcal{D}_n = \langle \mathfrak{F}(U_{n-1}) - \mathfrak{F}(U_n), \mathfrak{F}(U_n) \rangle.$$

By virtue of property $\mathcal{D}_n \geq 0$ and inequality $2\langle v - w, v \rangle \geq |v|^2 - |w|^2$, we then see that the sum on the right-hand side of (1.21) telescopes

$$2 \sum_{n=1}^N \tau_n^2 \mathcal{D}_n \leq \tau^2 \sum_{n=1}^N (|\mathfrak{F}(U_{n-1})|^2 - |\mathfrak{F}(U_n)|^2) \leq \tau^2 |\mathfrak{F}(U_0)|^2.$$

The following linear rate of convergence thus holds, again assuming $U_0 := u_0$:

$$(1.23) \quad \max_{t \in [0, T]} |u(t) - U(t)|^2 \leq \frac{1}{2} |\mathfrak{F}(u_0)|^2 \tau^2.$$

We observe that we have so far tacitly assumed that both \mathfrak{F} and ϕ are defined in the whole \mathbb{R}^d . However, even for scalar ODEs, the domains of \mathfrak{F} and ϕ satisfy $D(\mathfrak{F}) \subset D(\phi)$ and may not coincide, as the following example reveals:

$$\phi(w) := \begin{cases} -\sqrt{w} & \text{if } w \geq 0 \\ +\infty & \text{otherwise} \end{cases} \Rightarrow D(\phi) = [0, +\infty), \quad D(\mathfrak{F}) = (0, +\infty).$$

In the case of Hilbert spaces, the difference between $D(\phi)$ and $D(\mathfrak{F})$ plays a relevant role in that it reflects different regularity assumptions on the initial datum u_0 and thereby on the solution u . It follows that the difference between the rates of convergence of (1.22) and (1.23), which make sense provided $u_0 \in D(\phi)$ and $u_0 \in D(\mathfrak{F})$, respectively, is strictly related to the regularity of the exact solution u and thus to the approximability of u to a given order.

We point out that no extra regularity of u or U has been used (point (3)), as well as no constraints between consecutive time steps (point (5)). We also stress that the error estimates (1.16), (1.21), (1.22), and (1.23) can be applied directly to dissipative ODEs arising from the space discretization of (nonlinear) parabolic PDEs with mesh parameter $h > 0$ and initial datum $u_{0,h}$, because the monotonicity properties of the resulting discrete operators ϕ_h and \mathfrak{F}_h can be preserved by an adequate space discretization. However, the growth of $\phi_h(u_{0,h})$ and $|\mathfrak{F}_h(u_{0,h})|$ is usually different depending on the smoothness of u_0 even if the domains $D(\phi_h)$

and $D(\mathfrak{F}_h)$ coincide. The regularity of u_0 also dictates the size of the initial error $|u_0 - u_{0,h}|$ due to space discretization.

1.6 Adaptivity

The a posteriori error estimates (1.16) and (1.21) can be used in an adaptive strategy; note that the stability constants are 1 (point (4)). Given an error tolerance ε , the error control

$$\max_{t \in [0, T]} |u(t) - U(t)| \leq \varepsilon$$

is guaranteed upon *equidistributing* local discretization errors. In fact, assuming that U_0 satisfies $|u_0 - U_0|^2 \leq \varepsilon^2/2$, we then select the time step size τ_n so that

$$2T\tau_n\mathcal{E}_n \leq \varepsilon^2 \quad \text{or} \quad 2T\tau_n\mathcal{D}_n \leq \varepsilon^2.$$

1.7 Plan of the Paper

In view of numerous applications, we will derive our results in a general Hilbert framework. In Section 2 we briefly recall this abstract setting, along with some relevant examples and a novel discussion on additional coercivity properties of the convex potential ϕ and gradient field \mathfrak{F} . In Section 3 we prove and discuss our main results about a posteriori error estimation and ensuing optimal rates of convergence for *subdifferential* operators, including error estimates in (semi)norms other than the ambient one via coercivity. In Section 4 we show that *angle-bounded* operators are the natural class for a posteriori error estimation and present several important examples. We derive optimal a posteriori and a priori error estimates, thereby extending the results of Section 3. We also analyze Lipschitz perturbations and Hilbert triplets, which fit naturally within the setting of angle-bounded operators. In Section 5 we obtain new error estimates for several parabolic PDEs, mostly strongly nonlinear, for which the abstract theory applies.

2 Abstract Setting and Examples

In this section we recall definitions and basic existence results for the continuous problem, introduce several relevant examples, and discuss coercivity properties of ϕ and \mathfrak{F} for subdifferential operators. We present further examples and coercivity properties of angle-bounded operators in Section 4.

2.1 Continuous Problem and Existence Results

Maximal Monotone Operators

Let \mathcal{H} be a separable Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$, and let

$$\mathfrak{F} : \mathcal{H} \rightarrow 2^{\mathcal{H}}$$

be a (multivalued) operator with proper (and nonempty!) domain

$$D(\mathfrak{F}) := \{v \in \mathcal{H} : \mathfrak{F}(v) \neq \emptyset\}.$$

We will always use the symbol $\mathfrak{F}(v)$ to indicate any selection $v_* \in \mathfrak{F}(v)$, the same at every occurrence of $\mathfrak{F}(v)$. With this convention in mind, we will assume that \mathfrak{F} is *monotone*

$$\langle \mathfrak{F}(v) - \mathfrak{F}(w), v - w \rangle \geq 0, \quad \forall v, w \in D(\mathfrak{F}),$$

and *maximal*, i.e., for every $\varepsilon > 0$ and $v \in \mathcal{H}$ [9, proposition 2.2]

$$(2.1) \quad \text{the equation } w + \varepsilon \mathfrak{F}(w) \ni v \text{ admits a unique solution } w \in D(\mathfrak{F}).$$

These properties imply in particular that for every $w \in D(\mathfrak{F})$ the (nonempty) *set* $\mathfrak{F}(w)$ is closed and convex in \mathcal{H} ; its element of minimum norm is usually called the *minimal selection* of $\mathfrak{F}(w)$, and it is indicated by $\mathfrak{F}(w)^\circ$.

Subdifferentials

A particular important case arises when \mathfrak{F} is the *subdifferential* of a proper lower semicontinuous (l.s.c.) convex function ϕ with (nonempty, convex) domain $D(\phi)$

$$\phi : \mathcal{H} \rightarrow (-\infty, +\infty], \quad D(\phi) := \{v \in \mathcal{H} : \phi(v) < +\infty\}.$$

We say that $\mathfrak{F} = \partial\phi$. In this case [9, proposition 2.11] $D(\mathfrak{F})$ is densely included into $D(\phi)$; namely, $\overline{D(\mathfrak{F})} = \overline{D(\phi)}$ and \mathfrak{F} can be characterized by the inequalities

$$(2.2) \quad w_* \in \mathfrak{F}(w) \Leftrightarrow \langle w_*, v - w \rangle + \phi(w) - \phi(v) \leq 0, \quad \forall v \in D(\phi).$$

Evolution Equations: Strong, Weak, and “Energy” Solutions

Given

$$\text{an initial datum } u_0 \in \overline{D(\mathfrak{F})} \text{ and a function } f \in L^1(0, T; \mathcal{H}),$$

we will study suitable approximations of the Cauchy problem for the differential inclusion

$$(2.3) \quad u'(t) + \mathfrak{F}(u(t)) \ni f(t) \quad \text{a.e. } t \in (0, T)$$

subject to $u(0) = u_0$. To be precise, we have to distinguish between *strong* and *weak* solutions [9, section 3, definition 1].

DEFINITION 2.1 We say that $u \in C^0([0, T]; \mathcal{H})$ is a *strong* solution of (2.3) if u is also locally absolutely continuous in $(0, T)$ and satisfies (2.3) at almost every point; in particular, $u(t) \in D(\mathfrak{F})$ for a.e. $t \in (0, T)$. We say that $u \in C^0([0, T]; \mathcal{H})$ is a *weak* solution if u can be *uniformly* approximated by a sequence $\{u_k\}_{k \in \mathbb{N}}$ of strong solutions that solve (2.3) with respect to a family of data $\{f_k\}_{k \in \mathbb{N}}$ approaching f in $L^1(0, T; \mathcal{H})$; note that $u_k(0) \rightarrow u_0$ in \mathcal{H} .

When \mathfrak{F} is the subdifferential of ϕ defined by (2.2), then (2.3) can be equivalently written as an evolution variational inequality for a.e. $t \in (0, T)$,

$$(2.4) \quad \langle u'(t) - f(t), u(t) - v \rangle + \phi(u(t)) - \phi(v) \leq 0, \quad \forall v \in D(\phi).$$

In this case we will refer to u as an *energy solution*.

Basic Existence Results and Energy Solutions

Since they will be used extensively, we now recall some results of the theory of existence, regularity, perturbations, and (asymptotic) stability of solutions of (2.3) and (2.4) [9, section 3, theorems 4–7]. To this end we introduce the Banach space $BV(0, T; \mathcal{H})$ of functions $g : [0, T] \rightarrow \mathcal{H}$ with *bounded total variation* [9, definition A.2]

$$\text{Var } g := \sup_{0=r_0 < r_1 < \dots < r_J=T} \sum_{j=1}^J |g(r_j) - g(r_{j-1})| < +\infty.$$

We recall that if $g \in BV(0, T; \mathcal{H})$, then at every point $t_0 \in [0, T)$ there exists the *right limit* $g_+(t_0) := \lim_{t \downarrow t_0} g(t)$.

1. *Strong Lipschitz Solutions.* If $u_0 \in D(\mathfrak{F})$ and $f \in BV(0, T; \mathcal{H})$, then there exists a unique strong solution u of (2.3). Moreover, u is also Lipschitz-continuous, its range is included in $D(\mathfrak{F})$, and its right derivative $u'_+(t)$ exists at every point of $[0, T)$ and coincides with the minimal selection of $f_+(t) - \mathfrak{F}(u(t))$. In particular, at the initial time we have

$$(2.5) \quad u'_+(0) = (f_+(0) - \mathfrak{F}(u_0))^\circ.$$

2. *Stability and Weak Solutions.* The mapping

$$(u_0, f) \in D(\mathfrak{F}) \times BV(0, T; \mathcal{H}) \mapsto u,$$

defined by the previous statement, is Lipschitz-continuous with respect to the metric of $\mathcal{H} \times L^1(0, T; \mathcal{H})$ with values in $C^0([0, T]; \mathcal{H})$. In particular, for every $u_0 \in \overline{D(\mathfrak{F})}$ and $f \in L^1(0, T; \mathcal{H})$ there exists a unique weak solution of (2.3).

3. *Lipschitz Perturbations.* The previous results hold even if \mathfrak{F} is the sum of a maximal monotone operator and a Lipschitz perturbation defined in the same domain, or, equivalently, if $(\mathfrak{J}$ is the identity)

$$\exists L \geq 0 : \mathfrak{F} + L\mathfrak{J} \quad \text{is maximal monotone.}$$

4. *Energy Solutions and Regularizing Effects.* If \mathfrak{F} is the subdifferential of a lower semicontinuous convex function ϕ and u is a weak solution, then the function $t \mapsto \phi(u(t))$ belongs to $L^1(0, T)$, and it depends with continuity on the data $(u_0, f) \in \overline{D(\phi)} \times L^1(0, T; \mathcal{H})$; also $u(t) \in D(\phi)$ for a.e. $t \in (0, T)$. If $f \in L^2(0, T; \mathcal{H})$, then every weak solution u is a strong solution and belongs to $H^1_{\text{loc}}(0, T; \mathcal{H})$. The mapping $t \mapsto \phi(u(t))$ is locally absolutely continuous in $(0, T]$ and the following *energy identity* holds:

$$(2.6) \quad |u'(t)|^2 + \frac{d}{dt} \phi(u(t)) = \langle f(t), u'(t) \rangle \quad \text{a.e. } t \in (0, T).$$

In addition, $u \in H^1(0, T; \mathcal{H})$ if $u_0 \in D(\phi)$.

5. *Dissipation Inequality.* Finally, if f is absolutely continuous and u is a strong solution, we have

$$(2.7) \quad \frac{1}{2} \frac{d}{dt} |u'(t)|^2 \leq \langle f'(t), u'(t) \rangle \quad \text{in the distribution sense.}$$

Remark 2.2. Weak solutions can also be characterized as the solutions of a suitable integral formulation of (2.3) (see [9, lemma 3.1]). On the other hand, the *approximation approach* chosen in Definition 2.1 and the stability result of the previous theorem allow us to deal, at least initially, only with strong solutions and then to extend to the weak ones every estimate involving only quantities depending on u_0 and f that are *continuous* with respect to the $\mathcal{H} \times L^1(0, T; \mathcal{H})$ -metric. When \mathfrak{F} is a subdifferential, a different way to extend the notion of weak solutions, in order to deal with f in the Sobolev spaces $H^{-s}(0, T; \mathcal{H})$ of negative order $-s \in (-\frac{1}{2}, 0)$, has been exploited in [54].

2.2 Basic Examples of Subdifferential Type

We have already discussed in the introduction the case of dissipative ODEs in \mathbb{R}^d . We consider now other applications of the abstract framework provided \mathfrak{F} admits a convex potential ϕ ; we discuss nonpotential operators \mathfrak{F} and corresponding examples in Section 4.

Let Ω be a Lipschitz-connected open set of \mathbb{R}^m with outward unit normal \mathbf{n} ; $Q := \Omega \times (0, T)$ is the parabolic cylinder with lateral boundary $\Sigma := \partial\Omega \times (0, T)$.

Example 2.1: ODEs with Obstacles

We take the double obstacle potential in $\mathcal{H} := \mathbb{R}$

$$\phi(w) := \begin{cases} 0 & \text{if } w \in [-1, 1] \\ +\infty & \text{otherwise,} \end{cases} \quad D(\phi) := [-1, 1].$$

Correspondingly, the subdifferential \mathfrak{F} of ϕ is given by [9, p. 44]

$$\mathfrak{F}(w) := \begin{cases} (-\infty, 0] & \text{if } w = -1 \\ \{0\} & \text{if } -1 < w < 1 \\ [0, +\infty) & \text{if } w = 1, \end{cases} \quad D(\mathfrak{F}) := D(\phi).$$

ϕ is the indicator function $I_{[-1, 1]}$ of $[-1, 1]$, and \mathfrak{F} is the inverse of the *sign* graph. In this case, equation (2.3) becomes [9, remark 3.9]

$$\begin{cases} u'(t) = -f(t)^- & \text{if } u(t) = 1 \\ u'(t) = f(t) & \text{if } -1 < u(t) < 1 \\ u'(t) = f(t)^+ & \text{if } u(t) = -1. \end{cases}$$

The importance of this kind of ODE in phase transitions has been pointed out by Visintin [61] (see also [39]). They also arise in geometric motion of fronts (see [15, 47]).

Example 2.2: Linear Parabolic PDEs

Several examples follow:

Heat Equation. If

$$(2.8) \quad \mathcal{H} := L^2(\Omega), \quad \phi(w) := \frac{1}{2} \int_{\Omega} |\nabla w(x)|^2 dx, \quad D(\phi) := H_0^1(\Omega),$$

then the subdifferential \mathfrak{F} of ϕ is given by

$$(2.9) \quad \mathfrak{F}(w) := -\Delta w, \quad D(\mathfrak{F}) := \{v \in D(\phi) : \Delta v \in L^2(\Omega)\},$$

and (2.3) becomes the heat equation with homogeneous Dirichlet condition

$$\partial_t u - \Delta u = f \text{ a.e. in } Q, \quad u = 0 \text{ on } \Sigma.$$

Replacing $D(\phi)$ in (2.8) by $D(\phi) := H^1(\Omega)$ leads to

$$(2.10) \quad \mathfrak{F}(w) := -\Delta w, \quad D(\mathfrak{F}) := \{v \in D(\phi) : \Delta v \in L^2(\Omega), \partial_n v = 0 \text{ on } \partial\Omega\},$$

and to the heat equation with homogeneous Neumann condition

$$\partial_t u - \Delta u = f \text{ a.e. in } Q, \quad \partial_n u = 0 \text{ on } \Sigma.$$

Bi-Laplacian Operator. If Ω is $C^{1,1}$ or convex, and

$$(2.11) \quad \mathcal{H} := L^2(\Omega), \quad \phi(w) := \frac{1}{2} \int_{\Omega} |\Delta w(x)|^2 dx, \quad D(\phi) := H_0^2(\Omega),$$

then the subdifferential \mathfrak{F} of ϕ is the *bi-Laplacian*

$$(2.12) \quad \mathfrak{F}(w) := \Delta^2 w, \quad D(\mathfrak{F}) := \{v \in D(\phi) : \Delta^2 v \in L^2(\Omega)\}.$$

Now (2.3) is a fourth-order parabolic equation with zero essential conditions

$$\partial_t u + \Delta^2 u = f \text{ a.e. in } Q, \quad u = \partial_n u = 0 \text{ on } \Sigma.$$

Alternative boundary conditions can be obtained with different choices of $D(\phi)$ in (2.11): e.g., $D(\phi) := H^2(\Omega) \cap H_0^1(\Omega)$ and $D(\phi) := \{v \in H^2(\Omega) : \partial_n v = 0 \text{ on } \partial\Omega\}$ lead, respectively, to

$$D(\mathfrak{F}) := \{v \in D(\phi) : \Delta^2 v \in L^2(\Omega), \Delta v = 0 \text{ on } \partial\Omega\},$$

$$D(\mathfrak{F}) := \{v \in D(\phi) : \Delta^2 v \in L^2(\Omega), \partial_n(\Delta v) = 0 \text{ on } \partial\Omega\}.$$

Abstract Linear Equations: The Symmetric Case. Let \mathcal{V} be another Hilbert space

$$(2.13) \quad \mathcal{V} \subset \mathcal{H}, \text{ the inclusion being continuous and dense;}$$

let $\mathfrak{a} : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$ be a *positive, continuous, and weakly coercive* bilinear form

$$(2.14) \quad [v]^2 := \mathfrak{a}(v, v) \geq 0, \quad c_1 \|v\|_{\mathcal{V}}^2 \leq [v]^2 + |v|^2 \leq c_2 \|v\|_{\mathcal{V}}^2, \quad \forall v \in \mathcal{V},$$

for some constants $c_1, c_2 > 0$, and let $\mathfrak{A} : D(\mathfrak{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ be the linear operator associated to \mathfrak{a}

$$(2.15) \quad \begin{aligned} \langle \mathfrak{A}w, v \rangle &:= \mathfrak{a}(w, v), \quad \forall v \in \mathcal{V}, \\ D(\mathfrak{A}) &:= \left\{ v \in \mathcal{V} : \sup_{z \in \mathcal{V}, |z|=1} \mathfrak{a}(v, z) < +\infty \right\}. \end{aligned}$$

We consider, as a particular case of (2.3), the abstract *linear* evolution equation

$$(2.16) \quad u'(t) + \mathfrak{A}u(t) = f(t) \quad \text{in } (0, T).$$

If \mathfrak{a} is *symmetric*,

$$(2.17) \quad \mathfrak{a}(v, w) = \mathfrak{a}(w, v), \quad \forall v, w \in \mathcal{V},$$

then \mathfrak{A} is the subdifferential of [9, example 2.3.7]

$$(2.18) \quad \phi(w) := \begin{cases} \frac{1}{2}\mathfrak{a}(w, w) & \text{if } w \in \mathcal{V} \\ +\infty & \text{otherwise,} \end{cases} \quad D(\phi) := \mathcal{V},$$

and (2.16) is equivalent to (2.4).

We point out that (2.9) (or (2.10)) fits in this setting upon choosing

$$(2.19) \quad \mathcal{V} := H_0^1(\Omega), \quad \mathfrak{a}(v, w) := \int_{\Omega} \nabla v(x) \cdot \nabla w(x) dx,$$

(or $\mathcal{V} := H^1(\Omega)$) along with $\mathcal{H} := L^2(\Omega)$, and so does (2.12) provided $\mathcal{H} := L^2(\Omega)$, $\mathcal{V} := H_0^2(\Omega)$, and $\mathfrak{a}(v, w) := \int_{\Omega} \Delta v \Delta w dx$.

Example 2.3: Parabolic Variational Inequalities

We discuss several examples in turn.

Parabolic Obstacle Problems. Given $\mathcal{H} := L^2(\Omega)$ and an obstacle g satisfying $g \in H^1(\Omega)$ with $g \leq 0$ on $\partial\Omega$, let

$$\mathcal{K} := \{v \in H_0^1(\Omega) : v(x) \geq g(x) \text{ for a.e. } x \in \Omega\}$$

be the convex set of admissible functions, and let $I_{\mathcal{K}}$ be its indicator function

$$(2.20) \quad I_{\mathcal{K}}(w) := \begin{cases} 0 & \text{if } w \in \mathcal{K} \\ +\infty & \text{otherwise.} \end{cases}$$

Note that $I_{\mathcal{K}}$ is a *nonsmooth* lower semicontinuous convex function. Weaker assumptions on g in the framework of the capacity theory are allowed [40]. We consider

$$(2.21) \quad \phi(w) := \frac{1}{2} \int_{\Omega} |\nabla w(x)|^2 dx + I_{\mathcal{K}}(w), \quad D(\phi) := \mathcal{K},$$

so that the solution $u \in \mathcal{K}$ of (2.4) solves the variational inequality for a.e. $t \in (0, T)$ and all $v \in \mathcal{K}$

$$(2.22) \quad \int_{\Omega} (\partial_t u(u-v) + \nabla u \cdot (\nabla u - \nabla v)) dx \leq \int_{\Omega} f(u-v) dx.$$

We refer to (2.48) and Corollary 2.9 of Section 2.3 for this slightly stronger variational inequality including coercivity. By the regularity result of Brézis and Stampacchia [11, corollary II.3], if Ω is $C^{1,1}$ or convex and $g \in H^2(\Omega)$, then the previous variational inequality is equivalent to (2.3), where \mathfrak{F} has the simple expression

$$w^* \in \mathfrak{F}(w) \Leftrightarrow w^* \in -\Delta w + \partial I_{\mathcal{K}}(w), \quad D(\mathfrak{F}) := \mathcal{K} \cap H^2(\Omega).$$

Abstract Evolution Variational Inequalities. We observe that other kinds of constraints can be accommodated and an abstract theory can be developed within the same framework as Example 2.2 (see [4, 6, 8, 33, 42, 43, 53]). In fact, let \mathcal{V} , \mathfrak{a} , and \mathfrak{A} satisfy (2.13), (2.14), (2.15), and (2.17), and let

$$(2.23) \quad \begin{aligned} &\psi : \mathcal{V} \rightarrow (-\infty, +\infty] \text{ be convex and} \\ &\text{lower semicontinuous with respect to } \mathcal{V}, \end{aligned}$$

with nonempty domain $D(\psi)$. Note that ψ may fail to be lower semicontinuous with respect to \mathcal{H} . Let u be a solution of the evolution variational inequality

$$(2.24) \quad \langle u' - f, u - v \rangle + \mathfrak{a}(u, u - v) + \psi(u) - \psi(v) \leq 0, \quad \forall v \in D(\psi).$$

Upon choosing

$$\phi(w) := \frac{1}{2} \mathfrak{a}(w, w) + \psi(w), \quad D(\phi) := D(\psi),$$

and using that \mathfrak{a} is symmetric, we realize that (2.24) is a slightly stronger form than (2.4) in that it incorporates the coercivity term $\frac{1}{2} \mathfrak{a}(u - v, u - v)$ (see Section 2.3). In this case the associated \mathfrak{F} is given by

$$(2.25) \quad w_* \in \mathfrak{F}(w) \Leftrightarrow \mathfrak{a}(w, w - v) + \psi(w) \leq \psi(v) + \langle w_*, w - v \rangle, \quad \forall v \in D(\psi).$$

In particular, if \mathcal{V} and \mathfrak{a} are those in (2.19) and $\psi = I_{\mathcal{K}}$ is as in (2.20), then we are dealing with the previous obstacle problem (2.22).

A compatibility condition of ψ with respect to \mathfrak{A} , defined in (2.15), again simplifies the structure of \mathfrak{F} . If ψ is the restriction to \mathcal{V} of a convex function (still denoted by ψ) defined and lower semicontinuous in \mathcal{H} , and there exist $z \in \mathcal{H}$, $c \geq 0$, such that

$$(2.26) \quad \forall v \in D(\psi) : \quad w_\varepsilon + \varepsilon \mathfrak{A} w_\varepsilon = v + \varepsilon z \Rightarrow \psi(w_\varepsilon) \leq \psi(v) + c\varepsilon,$$

then the sum $\mathfrak{A} + \partial\psi$ is *maximal* monotone [9, proposition 2.17]. Since $\mathfrak{A}w + \partial\psi(w) \in \mathfrak{F}(w)$ for all $w \in D(\mathfrak{A}) \cap D(\partial\psi)$, we infer the following characterization of \mathfrak{F} :

$$(2.27) \quad \mathfrak{F} := \mathfrak{A} + \partial\psi, \quad D(\mathfrak{F}) := D(\mathfrak{A}) \cap D(\partial\psi).$$

Reaction-Diffusion Equations. Let \mathcal{H} , \mathcal{V} , and \mathfrak{a} be as in (2.19), and let ϕ be

$$\phi(w) := \int_{\Omega} \left(\frac{1}{2} |\nabla w|^2 + g(w) \right) dx, \quad D(\phi) := \{v \in H_0^1(\Omega) : g(v) \in L^1(\Omega)\},$$

where g is a continuous convex real function. Then [7, corollary 13], [12, lemma 3]

$$(2.28) \quad \begin{aligned} \mathfrak{F}(w) &:= -\Delta w + g'(w), \\ D(\mathfrak{F}) &:= \{v \in D(\phi) : -\Delta v, g'(v) \in L^2(\Omega)\}, \end{aligned}$$

because (2.26) is valid, and the PDE corresponding to (2.24) becomes

$$(2.29) \quad \partial_t u - \Delta u + g'(u) = f \quad \text{a.e. in } Q.$$

When g is not convex, but nevertheless $r \mapsto g'(r) + \lambda r$ with a constant $\lambda > 0$ is nondecreasing, then the λ -perturbation $w \mapsto -\Delta w + g'(w) + \lambda w$ of \mathfrak{F} in (2.28) is a subdifferential in $L^2(\Omega)$; the basic existence results are still applicable.

Allen-Cahn Equation. If g is the double-well potential $g(r) := (1 - r^2)^2$ for $r \in \mathbb{R}$, then (2.29) is the Allen-Cahn equation with regular potential upon proper rescaling [24]. We can also consider constraints: If $g(r) := I_{[-1,1]}(r) - r^2/2$ for $r \in \mathbb{R}$, then we get the Allen-Cahn equation with double obstacles ± 1 [15, 47].

Other choices of \mathcal{H} , \mathcal{V} , and \mathfrak{a} and of the smooth potential function g lead to important applications of reaction-diffusion systems (see, e.g., [16]).

Example 2.4: Quasi-Linear Parabolic PDEs

Let $\mathcal{H} := L^2(\Omega)$, $p > 1$,

$$(2.30) \quad \phi(w) := \int_{\Omega} G(x, \nabla w(x)) dx, \quad D(\phi) := L^2(\Omega) \cap W_0^{1,p}(\Omega).$$

Here $G(x, \xi) : \Omega \times \mathbb{R}^m \rightarrow \mathbb{R}$ is a Carathéodory function [26, chapter 8, definition 1.2] that is convex and continuously differentiable with respect to $\xi \in \mathbb{R}^m$ for a.e. $x \in \Omega$, and satisfies, together with its differential $\mathbf{a}(x, \xi) := \nabla_{\xi} G(x, \xi)$, the usual p -growth conditions (see [46, p. 29], [25, chapter II, section 1])

$$G(x, \xi) \geq \alpha_0 |\xi|^p - \alpha_1, \quad |\mathbf{a}(x, \xi)| \leq \alpha_2 (1 + |\xi|^{p-1}), \quad \forall \xi \in \mathbb{R}^m, \text{ a.e. } x \in \Omega,$$

where α_0 , α_1 , and α_2 are given positive constants. Then ϕ is convex and lower semicontinuous in \mathcal{H} , and its subdifferential is

$$(2.31) \quad \begin{aligned} \mathfrak{F}(w(\cdot)) &:= -\operatorname{div} \mathbf{a}(\cdot, \nabla w(\cdot)), \\ D(\mathfrak{F}) &:= \{v \in D(\phi) : \operatorname{div} \mathbf{a}(\cdot, \nabla v(\cdot)) \in L^2(\Omega)\}. \end{aligned}$$

The ensuing PDE reads

$$(2.32) \quad \partial_t u - \operatorname{div} \mathbf{a}(\cdot, \nabla u) = f \quad \text{a.e. in } Q.$$

p-Laplacian. The choice

$$(2.33) \quad G(x, \xi) := \frac{1}{p} |\xi|^p, \quad \mathbf{a}(x, \xi) = |\xi|^{p-2} \xi,$$

gives rise to the *p*-Laplacian $\mathfrak{F}(w) := -\operatorname{div}(|\nabla w|^{p-2} \nabla w)$ and associated evolution equation with homogeneous Dirichlet condition [8, 42]

$$\partial_t u - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = f \quad \text{a.e. in } Q, \quad u = 0 \quad \text{on } \Sigma.$$

Mean Curvature Operator for Cartesian Surfaces. We consider the case of *linear growth* $p = 1$, occurring, for example, for the area functional of Cartesian surfaces

$$G(\xi) := \sqrt{1 + |\xi|^2}, \quad \mathbf{a}(\xi) = -\frac{\xi}{\sqrt{1 + |\xi|^2}},$$

which, for $f \equiv 0$, leads to the vertical evolution of graphs by mean curvature

$$(2.34) \quad \partial_t u - \operatorname{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 0 \quad \text{a.e. in } Q.$$

For $u_0 \in W^{1,\infty}(\Omega)$ given, we supply the parabolic boundary condition

$$u(x, t) = u_0(x) \quad \text{for } t = 0 \text{ or } x \in \partial\Omega.$$

This problem does not admit, in general, a strong or weak solution that also satisfies the lateral boundary condition in the sense of traces in Sobolev spaces unless $\partial\Omega$ is a smooth hypersurface of nonnegative mean curvature; weaker conditions allowing less regular domains are also possible ([32, sections 15.6–15.9], [41]). There are two alternative ways to pose this problem. The integral of (2.30) in $W^{1,1}(\Omega)$ could be replaced by the corresponding relaxed version in $\operatorname{BV}(\Omega)$ that includes the total variation of w (see [26], [32, chapter 14]). Otherwise, it is possible to introduce the notion of *pseudosolution* (cf. [41, 58]) by formally substituting the lateral Dirichlet-type condition with the following nonlinear one:

$$\frac{\partial_n u}{\sqrt{1 + |\nabla u|^2}} + \operatorname{sign}(u - u_0) \ni 0 \quad \text{on } \Sigma,$$

where it is implicitly understood that

$$(2.35) \quad u \neq u_0 \Rightarrow |\partial_n u| = +\infty, \quad \left| \frac{\partial_n u}{\sqrt{1 + |\nabla u|^2}} \right| = 1.$$

This relaxed boundary condition (2.35) implies the stronger one $u = u_0$ under the extra geometric assumptions on $\partial\Omega$ that guarantee existence of a strong solution of (2.34) [41]; this justifies the prefix *pseudo*- attached to this more general kind of solution.

We adopt this second approach here and thus view (2.34) as the evolution inequality (2.4) in $\mathcal{H} := L^2(\Omega)$ for the *convex-area-type* functional

$$(2.36) \quad \begin{aligned} \phi(w) &:= \int_{\Omega} \sqrt{1 + |\nabla w|^2} dx + \int_{\partial\Omega} |w - u_0| dS, \\ D(\phi) &:= L^2(\Omega) \cap W^{1,1}(\Omega). \end{aligned}$$

In Section 5.3 we show how to overcome the lack of lower semicontinuity of ϕ with respect to the strong topology of $L^2(\Omega)$.

Example 2.5: Degenerate Parabolic PDEs

We first discuss a general setting that consists of *changing the pivot space* from \mathcal{H} to $\mathcal{V} \subset \mathcal{H}$. Let \mathcal{V} , \mathcal{H} , \mathfrak{a} , and \mathfrak{A} be as in (2.13)–(2.17). In addition, assume that

\mathfrak{a} is strongly coercive on \mathcal{V} ;

thus $[\cdot]$ is an equivalent norm for \mathcal{V} . Moreover, let $\phi : \mathcal{H} \rightarrow (-\infty, +\infty]$ be a lower semicontinuous convex function with $D(\phi) \subset \mathcal{V}$, and let $\mathfrak{B} : \mathcal{V} \rightarrow 2^{\mathcal{V}}$ be its subdifferential in \mathcal{V} with respect to the scalar product $\mathfrak{a}(\cdot, \cdot)$. Consequently,

$$(2.37) \quad w_* \in \mathfrak{B}(w) \Leftrightarrow \mathfrak{a}(w_*, v - w) + \phi(w) - \phi(v) \leq 0, \quad \forall v \in D(\phi),$$

defines \mathfrak{B} . To express the subdifferential \mathfrak{F} of ϕ in \mathcal{H} in terms of \mathfrak{A} and \mathfrak{B} , we proceed as follows: We first note that $w_* \in \mathfrak{F}(w)$ iff $w \in D(\phi)$ minimizes the functional $v \in \mathcal{V} \mapsto \phi(v) - \langle w_*, v \rangle = \phi(v) - \mathfrak{a}(\mathfrak{A}^{-1}w_*, v)$, where the last equality is due to (2.15). In view of (2.37), this yields

$$w \in D(\mathfrak{B}), \quad \mathfrak{A}^{-1}w_* \in \mathfrak{B}(w).$$

We now choose the ambient space \mathcal{H} , pivot space \mathcal{V} , and convex functional ϕ that yield degenerate parabolic PDEs. Other choices of \mathcal{H} , \mathcal{V} , and ϕ lead to a wide range of phase transition problems [17, 23, 39, 45, 56, 61]. Let

$$\mathcal{V} := L^2(\Omega), \quad \mathfrak{a}(v, w) := \int_{\Omega} v(x)w(x)dx, \quad \mathcal{H} := H^{-1}(\Omega),$$

where \mathcal{H} is endowed with the norm and scalar product

$$|w| := \|\nabla Gw\|_{L^2(\Omega)}, \quad \langle v, w \rangle :=_{H^{-1}(\Omega)} \langle v, Gw \rangle_{H_0^1(\Omega)}.$$

Here $-G$ denotes the inverse of the Laplace operator subject to a homogeneous Dirichlet boundary condition; hence $\mathfrak{A} = G^{-1} = -\Delta$ and $D(\mathfrak{A}) = H_0^1(\Omega)$.

Let $\beta : \mathbb{R} \mapsto \mathbb{R}$ be a continuous monotone function (β could also be a surjective monotone graph) such that

$$0 \in \beta(0), \quad \liminf_{|s| \rightarrow +\infty} \frac{\beta(s)}{s} > 0;$$

the latter could be weakened [7]. We next consider the convex primitive of β ,

$$j(r) := \int_0^r \beta(s)ds, \quad \forall r \in \mathbb{R}.$$

We introduce the lower semicontinuous convex functional

$$\phi(w) := \int_{\Omega} j(w(x)) dx, \quad D(\phi) := \{v \in L^2(\Omega) : j(v) \in L^1(\Omega)\},$$

whose (sub)differential operator in \mathcal{H} is given by

$$(2.38) \quad \mathfrak{F}(w) := -\Delta\beta(w), \quad D(\mathfrak{F}) := \{v \in D(\phi) : \beta(v) \in H_0^1(\Omega)\}.$$

We thus end up with the following equation in the sense of distributions:

$$(2.39) \quad \partial_t u - \Delta\beta(u) = f \text{ in } Q, \quad \beta(u) = 0 \text{ on } \Sigma.$$

We now discuss three specific choices of β .

Two-Phase Stefan Problem. The simplest problem corresponds to

$$(2.40) \quad \beta(r) := (r-1)^+ - r^-.$$

Porous Medium Equation. The typical choice for β is

$$(2.41) \quad \beta(r) := |r|^{p-2}r, \quad p > 2.$$

Hele-Shaw Problem. If $\beta := H^{-1}$ is the inverse of the Heaviside graph and $j := I_{[0,1]}$, $D(\phi) := L^\infty(\Omega; [0,1])$, then \mathfrak{F} in (2.38) turns out to be

$$w_* \in \mathfrak{F}(w) \Leftrightarrow Gw^* \in \beta(w) \quad \text{a.e. in } \Omega.$$

In terms of $v \in H^{-1}(u)$, (2.39) formally reads $\partial_t H(v) - \Delta v \ni f$, which is an elliptic-parabolic PDE.

2.3 Coercivity of a Subdifferential \mathfrak{F}

Before embarking on the analysis of the time discretization of (2.4), we explore the possibility of extracting more information from (2.3) than merely (2.4). We motivate the definitions with the linear symmetric case (2.16). After multiplying (2.16) by $u(t) - v$ for a generic $v \in \mathcal{V}$ and using the identity

$$2\mathfrak{a}(u, u-v) = \mathfrak{a}(u, u) - \mathfrak{a}(v, v) + \mathfrak{a}(u-v, u-v)$$

from (2.15) and (2.18), we arrive at

$$(2.42) \quad \langle u' - f, u-v \rangle + \phi(u) - \phi(v) + \phi(u-v) = 0.$$

If we compare (2.42) with (2.4), we gain control of the positive quadratic term

$$\phi(u-v) = \frac{1}{2} \mathfrak{a}(u-v, u-v).$$

We stress that this fact is not related to the linearity of this example but rather to the *coercivity* of \mathfrak{F} . We formalize this property in the following definitions:

DEFINITION 2.3 For every $w \in D(\mathfrak{F})$ and $v \in D(\phi)$, let $\sigma(w; v) \geq 0$ be

$$(2.43) \quad \sigma(w; v) := \phi(v) - \phi(w) - \sup_{w_* \in \mathfrak{F}(w)} \langle w_*, v - w \rangle.$$

If in addition $v \in D(\mathfrak{F})$, then let $\varrho(w, v) \geq 0$ be

$$(2.44) \quad \varrho(w, v) := \sigma(w; v) + \sigma(v; w) = \inf_{w_* \in \mathfrak{F}(w), v_* \in \mathfrak{F}(v)} \langle w_* - v_*, w - v \rangle.$$

Remark 2.4. Our purpose in dealing with σ is to obtain error estimates in (semi) norms other than that of \mathcal{H} in a concise and possibly sharp manner. Fine properties of σ related to one-sided derivatives of ϕ can be found in [51, section 23] and [14, theorem I-27]. Here we only observe that σ is a Borel function in its domain so that its composition with continuous (or Borel) time-dependent functions is still measurable, and the related integrals are well-defined. In fact, the separability of \mathcal{H} and the usual Yosida approximation technique ensure that \mathfrak{F} admits a sequence of Borel selections $\mathfrak{F}_k : D(\mathfrak{F}) \rightarrow \mathcal{H}$ such that $\mathfrak{F}(w) = \overline{\bigcup_k \{\mathfrak{F}_k(w)\}}$; therefore σ can be defined in terms of the supremum of a countable family of Borel functions.

Remark 2.5. We observe that, when \mathfrak{F} is single-valued, we have the *identities*, for all $w \in D(\mathfrak{F})$,

$$(2.45) \quad \sigma(w; v) = \phi(v) - \phi(w) - \langle \mathfrak{F}(w), v - w \rangle, \quad \forall v \in D(\phi),$$

$$(2.46) \quad \varrho(w, v) = \langle \mathfrak{F}(w) - \mathfrak{F}(v), w - v \rangle, \quad \forall v \in D(\mathfrak{F}).$$

Remark 2.6. In the general multivalued case, from (2.43) we can only say that

$$(2.47) \quad w_* \in \mathfrak{F}(w) \Leftrightarrow \langle w_*, v - w \rangle + \sigma(w; v) \leq \phi(v) - \phi(w), \quad \forall v \in D(\phi).$$

Hence, the solution u of both (2.3) and (2.4) satisfies the stronger inequality

$$(2.48) \quad \langle u'(t) - f(t), u(t) - v \rangle + \phi(u(t)) - \phi(v) + \sigma(u(t); v) \leq 0$$

for all $v \in D(\phi)$.

The following lower bounds of ϱ can easily be deduced for the previous examples: For the choices (2.9), (2.10), (2.21), and (2.28), we have

$$(2.49) \quad \varrho(w, v) \geq \int_{\Omega} |\nabla w - \nabla v|^2 dx, \quad \forall w, v \in D(\mathfrak{F}),$$

whereas for (2.12) we get

$$(2.50) \quad \varrho(w, v) \geq \int_{\Omega} |\Delta w - \Delta v|^2 dx, \quad \forall w, v \in D(\mathfrak{F}).$$

In general, if \mathfrak{F} is defined by (2.25) with \mathcal{V} , \mathfrak{a} , and ψ satisfying (2.13), (2.14), (2.17), and (2.23), then

$$(2.51) \quad \varrho(w, v) \geq \mathfrak{a}(w - v) = [w - v]^2.$$

The following (less immediate) property [25, chapter I, 4-(iii)]

$$\forall p \geq 2, \exists c_p > 0: (|r|^{p-2}r - |s|^{p-2}s)(r - s) \geq c_p |r - s|^p, \quad \forall r, s \in \mathbb{R},$$

leads to estimates for the p -Laplacian $\mathfrak{F}(w) = -\operatorname{div}(|\nabla w|^{p-2}\nabla w)$ (see (2.33))

$$(2.52) \quad \varrho(w, v) \geq c_p \int_{\Omega} |\nabla w - \nabla v|^p dx, \quad \forall w, v \in D(\mathfrak{F}),$$

and for the porous medium operator $\mathfrak{F}(w) = -\Delta(|w|^{p-2}w)$ (see (2.41))

$$(2.53) \quad \varrho(w, v) \geq c_p \int_{\Omega} |w - v|^p dx \quad \forall w, v \in D(\mathfrak{F}),$$

for $p \geq 2$.

We formalize the above lower bounds of ϱ in the following definition:

DEFINITION 2.7 We say that a function $[\cdot] : \mathcal{H} \rightarrow [0, +\infty]$ is a (*generalized*) *lower semicontinuous seminorm* on \mathcal{H} if it is proper, lower semicontinuous, convex, and positively homogeneous. The (multivalued) operator \mathfrak{F} is *p -coercive* with respect to $[\cdot]$ for a given $p \geq 2$ if

$$(2.54) \quad \varrho(w, v) \geq [w - v]^p, \quad \forall w, v \in D(\mathfrak{F}).$$

We stress that \mathfrak{F} could be p -coercive for $1 < p < 2$ only with respect to $[\cdot] \equiv 0$. An interesting and crucial fact is that (2.54) implies an analogous property for σ (see [8, (2.30)] for $p = 2$).

LEMMA 2.8 *Let \mathfrak{F} be p -coercive as in (2.54). Then*

$$(2.55) \quad \sigma(w; v) \geq \frac{1}{p} [w - v]^p, \quad \forall w \in D(\mathfrak{F}), \forall v \in D(\phi).$$

PROOF: Let us first observe that, for every choice of $w_* \in \mathfrak{F}(w)$ and $v \in D(\phi)$,

$$0 \leq \phi(v) - \phi(w) - \langle w_*, v - w \rangle = (\phi(v) - \langle w_*, v \rangle) - (\phi(w) - \langle w_*, w \rangle),$$

whence w is a minimum of $\phi(\cdot) - \langle w_*, \cdot \rangle$. Since ϱ does not change upon adding a linear term to ϕ , for proving (2.55) it suffices to show that

$$(2.56) \quad \phi(v) - \phi(w) \geq \frac{1}{p} [w - v]^p, \quad \forall v \in D(\phi),$$

whenever w is a minimum point of ϕ , i.e., $0 \in \mathfrak{F}(w)$. If ϕ is Gateaux-differentiable, and so $D(\mathfrak{F}) = D(\phi)$, (2.56) follows easily by evaluating the derivative of the function $s \mapsto \phi(w_s)$ with $w_s := w + s(v - w)$. In fact, the mean value theorem combined with (2.46) and (2.54) yields

$$\phi(v) - \phi(w) = \int_0^1 \frac{1}{s} \langle \mathfrak{F}(w_s), w_s - w \rangle ds = \int_0^1 \frac{1}{s} \varrho(w_s, w) ds \geq [w - v]^p \int_0^1 s^{p-1} ds.$$

Otherwise, we use a standard approximation argument. For $\varepsilon > 0$, let

$$(2.57) \quad \mathfrak{J}_\varepsilon := (I + \varepsilon \mathfrak{F})^{-1}, \quad \phi_\varepsilon := \phi \circ \mathfrak{J}_\varepsilon, \quad \mathfrak{F}_\varepsilon := \varepsilon^{-1}(I - \mathfrak{J}_\varepsilon),$$

be the resolvent of \mathfrak{F} and the Yosida approximations of ϕ and \mathfrak{F} , respectively. They satisfy [9, p. 28, theorem 2.2 and proposition 2.11]

$$\forall v \in \overline{D(\phi)}: \mathfrak{J}_\varepsilon(v) \rightarrow v, \quad \phi_\varepsilon(v) \uparrow \phi(v), \quad \text{as } \varepsilon \downarrow 0,$$

as well as $\mathfrak{J}_\varepsilon(w) \equiv w$, $\mathfrak{F}_\varepsilon(w) \equiv 0 = \mathfrak{F}(w)^\circ$ for every $\varepsilon > 0$ because $0 \in \mathfrak{F}(w)$. Since \mathfrak{F}_ε is continuous and is the (everywhere defined) Fréchet differential of ϕ_ε , we have as before

$$(2.58) \quad \phi_\varepsilon(v) - \phi_\varepsilon(w) = \int_0^1 \frac{1}{s} \langle \mathfrak{F}_\varepsilon(w_s), w_s - w \rangle ds.$$

Since $w_s = \mathfrak{J}_\varepsilon(w_s) + \varepsilon \mathfrak{F}_\varepsilon(w_s)$ and $\mathfrak{F}_\varepsilon(w_s) \in \mathfrak{F}(\mathfrak{J}_\varepsilon(w_s))$, (2.44) and (2.54) yield

$$\langle \mathfrak{F}_\varepsilon(w_s), w_s - w \rangle = \langle \mathfrak{F}_\varepsilon(w_s), \mathfrak{J}_\varepsilon(w_s) - w \rangle + \varepsilon |\mathfrak{F}_\varepsilon(w_s)|^2 \geq [\mathfrak{J}_\varepsilon(w_s) - w]^p.$$

We note that $w_s \in D(\phi)$ because $D(\phi)$ is convex, whence $\mathfrak{J}_\varepsilon(w_s) \rightarrow w_s$ as $\varepsilon \downarrow 0$. Inserting this inequality into (2.58), recalling the limit properties of ϕ_ε as $\varepsilon \downarrow 0$, and invoking the lower semicontinuity of $[\cdot]$ and Fatou's lemma, we obtain

$$\begin{aligned} \phi(v) - \phi(w) &= \lim_{\varepsilon \downarrow 0} (\phi_\varepsilon(v) - \phi_\varepsilon(w)) \geq \int_0^1 \frac{1}{s} \liminf_{\varepsilon \downarrow 0} [\mathfrak{J}_\varepsilon(w_s) - w]^p ds \\ &\geq \int_0^1 \frac{1}{s} [s(v - w)]^p ds. \end{aligned}$$

□

COROLLARY 2.9 *Up to the factor $1/p$, we can replace $\varrho(\cdot, \cdot)$ by the corresponding $\sigma(\cdot; \cdot)$ in the estimates (2.49), (2.50), (2.51), (2.52), and (2.53).*

Remark 2.10. The notion of p -coercivity as well as Lemma 2.8 can be *localized*: If we know that (2.54) holds for every w and v in $D(\mathfrak{F}) \cap K$, K being a convex set where ϕ is differentiable along segments, then (2.55) holds for every $w, v \in D(\mathfrak{F}) \cap K$. The same is true if K satisfies the *compatibility* condition

$$\mathfrak{J}_\varepsilon(K) \subset K, \quad \forall \varepsilon > 0 \text{ sufficiently small,}$$

where \mathfrak{J}_ε is the resolvent of \mathfrak{F} defined in (2.57). A simple example (which will turn out to be useful in the application to the p -Laplacian (2.33) with $1 < p < 2$) is provided by the sublevels of ϕ

$$K_R := \{v \in \mathcal{H} : \phi(v) \leq R\},$$

which are compatible with respect to \mathfrak{F} since $\phi(\mathfrak{J}_\varepsilon(v)) = \phi_\varepsilon(v) \leq \phi(v)$.

The case of the Stefan problem (2.38) and (2.40) does not fit into Lemma 2.8. However, since $0 \leq \beta'(r) \leq 1$ and $\mathcal{H} = H^{-1}(\Omega)$, we easily get

$$\begin{aligned} \varrho(w, v) &= \langle -\Delta(\beta(v) - \beta(w)), v - w \rangle = \int_{\Omega} (\beta(w) - \beta(v))(w - v) dx \\ &\geq \int_{\Omega} |\beta(w) - \beta(v)|^2 dx = [\mathfrak{F}(w) - \mathfrak{F}(v)]^2, \quad \forall w, v \in D(\mathfrak{F}), \end{aligned}$$

provided $[w] := \|Gw\|_{L^2(\Omega)}$ for all $w \in H^{-1}(\Omega)$ and G is defined in Example 2.5.

We would like to obtain an analogous bound for $\sigma(w; v)$, which will be important in deriving error estimates for temperature $\beta(u)$. This property can be recovered from the following general dual result:

LEMMA 2.11 *Let $[\cdot]$ be a lower semicontinuous seminorm on \mathcal{H} such that for a given $p \geq 2$,*

$$(2.59) \quad \varrho(w, v) \geq [\mathfrak{F}(w) - \mathfrak{F}(v)]^p, \quad \forall w, v \in D(\mathfrak{F}).$$

Then we have

$$(2.60) \quad \sigma(w; v) \geq \frac{1}{p} [\mathfrak{F}(w) - \mathfrak{F}(v)]^p, \quad \forall w, v \in D(\mathfrak{F}).$$

PROOF: Let us denote by ϕ_* the lower semicontinuous conjugate convex function

$$\phi_*(w_*) := \sup_{v \in D(\phi)} (\langle w_*, v \rangle - \phi(v)).$$

Its subdifferential \mathfrak{F}_* turns out to be the inverse of \mathfrak{F} [9, p. 41], i.e.,

$$\forall w, w_* \in \mathcal{H}: w_* \in \mathfrak{F}(w) \Leftrightarrow w \in \mathfrak{F}_*(w_*).$$

Therefore, from (2.2) we have $\langle w_*, v \rangle - \phi(v) \leq \langle w_*, w \rangle - \phi(w)$, whence

$$(2.61) \quad \phi_*(w_*) = \langle w_*, w \rangle - \phi(w), \quad \forall w_* \in D(\mathfrak{F}_*), \quad \forall w \in \mathfrak{F}_*(w_*).$$

We note that (2.59) can be equivalently written as

$$\langle w_* - v_*, \mathfrak{F}_*(w_*) - \mathfrak{F}_*(v_*) \rangle \geq [w_* - v_*]^p, \quad \forall w_*, v_* \in D(\mathfrak{F}_*),$$

and therefore the associated ϱ_* to \mathfrak{F}_* satisfies

$$\varrho_*(w_*, v_*) \geq [w_* - v_*]^p, \quad \forall w_*, v_* \in D(\mathfrak{F}_*).$$

By Lemma 2.8 and (2.43) we deduce

$$\phi_*(w_*) - \phi_*(v_*) - \langle v, w_* - v_* \rangle \geq \frac{1}{p} [w_* - v_*]^p$$

for all $w_*, v_* \in D(\mathfrak{F}_*)$ and $v \in \mathfrak{F}_*(v_*)$. Using (2.61) for v and w , we get (2.60). \square

Lemma 2.11 does not yet apply to the Stefan problem (2.40) because we need $v \in L^2(\Omega)$ instead of $v \in D(\mathfrak{F})$. Our next result assesses this issue.

LEMMA 2.12 *Let β satisfy $0 \leq \beta' \leq 1$ and let $\mathfrak{F}(\cdot) := -\Delta\beta(\cdot)$. Then*

$$\sigma(w; v) \geq \frac{1}{2} \|\beta(w) - \beta(v)\|_{L^2(\Omega)}, \quad \forall w \in D(\mathfrak{F}), \quad \forall v \in L^2(\Omega).$$

PROOF: Applying Lemma 2.11 with $p = 2$ and $[w] := \|Gw\|_{L^2(\Omega)}$, we obtain

$$\sigma(w; v) \geq \frac{1}{2} [\mathfrak{F}(w) - \mathfrak{F}(v)]^2 = \frac{1}{2} \|\beta(w) - \beta(v)\|_{L^2(\Omega)}^2, \quad \forall w, v \in D(\mathfrak{F}).$$

To show that this relation holds for all $v \in L^2(\Omega)$, we employ the Yosida approximation $v_\varepsilon := (I + \varepsilon \mathfrak{F})^{-1}v \in D(\mathfrak{F})$ of v , which satisfies $v_\varepsilon \rightarrow v$ in $H^{-1}(\Omega)$. On multiplying $v_\varepsilon - \varepsilon \Delta \beta(v_\varepsilon) = v$ by v_ε , we deduce $\|v_\varepsilon\|_{L^2(\Omega)} \leq \|v\|_{L^2(\Omega)}$ because $-\langle \Delta \beta(v_\varepsilon), v_\varepsilon \rangle \geq 0$ even for v_ε merely in $L^2(\Omega)$ [12, lemma 2]. Since $\|v_\varepsilon - v\|_{L^2(\Omega)}^2 = \|v_\varepsilon\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2 - 2\langle v_\varepsilon, v \rangle$, we infer the strong convergence

$$v_\varepsilon \rightarrow v, \quad \beta(v_\varepsilon) \rightarrow \beta(v), \quad \text{in } L^2(\Omega),$$

the latter being a consequence of the former. Hence we conclude that

$$\begin{aligned} \sigma(w; v) &= \phi(w) - \phi(v) - \int_{\Omega} \beta(w)(v - w) dx \\ &= \lim_{\varepsilon \rightarrow 0} \left(\phi(w) - \phi(v_\varepsilon) - \int_{\Omega} \beta(w)(v_\varepsilon - w) dx \right) = \lim_{\varepsilon \rightarrow 0} \sigma(w; v_\varepsilon) \\ &\geq \limsup_{\varepsilon \rightarrow 0} \frac{1}{2} \|\beta(w) - \beta(v_\varepsilon)\|_{L^2(\Omega)}^2 = \frac{1}{2} \|\beta(w) - \beta(v)\|_{L^2(\Omega)}^2. \end{aligned}$$

□

3 The Subdifferential Case

Notation

Let \mathcal{P} be the partition (1.2) of the time interval $[0, T]$ and

$$\tau_n := t_n - t_{n-1}, \quad \tau := \max_{1 \leq n \leq N} \tau_n.$$

For any sequence $\{W_n\}_{n=0}^N$ we define the piecewise constant functions \bar{W} and \underline{W} and the piecewise linear interpolant W on $(t_{n-1}, t_n]$ as

$$(3.1) \quad \bar{W}(t) := W_n, \quad \underline{W}(t) := W_{n-1}, \quad W(t) := (1 - \ell(t))W_n + \ell(t)W_{n-1},$$

for $1 \leq n \leq N$, where

$$\ell(t) := \frac{t_n - t}{\tau_n}.$$

In particular, $\bar{\tau}(t)$ takes the value τ_n on $(t_{n-1}, t_n]$. We also denote by $\{\delta W_n\}_{n=1}^N$ the discrete derivative of the sequence $\{W_n\}_{n=0}^N$

$$(3.2) \quad \delta W_n := \frac{W_n - W_{n-1}}{\tau_n} = \frac{\bar{W}(t) - \underline{W}(t)}{\bar{\tau}} = W'(t), \quad \forall t \in (t_{n-1}, t_n].$$

By $\delta\phi(W_n)$, $\delta\mathfrak{F}(W_n)$, and $\delta^2 W_n$ we denote the discrete derivatives of the sequences $\{\phi(W_n)\}_{n=0}^N$, $\{\mathfrak{F}(W_n)\}_{n=0}^N$, and $\{\delta W_n\}_{n=0}^N$ for $1 \leq n \leq N$, respectively, the latter provided δW_0 is defined.

The Discretization Scheme

Let $U_0 \in \overline{D(\phi)}$ and $\{F_n\}_{n=1}^N \subset \mathcal{H}$ be suitable approximations of the initial datum u_0 and of the function f over $\{(t_{n-1}, t_n]\}_{n=1}^N$, respectively. The discrete solution $\{U_n\}_{n=1}^N \subset D(\phi)$ is defined recursively by solving at every step $1 \leq n \leq N$ the variational inequality

$$(3.3) \quad \langle \delta U_n - F_n, U_n - v \rangle + \phi(U_n) - \phi(v) \leq 0, \quad \forall v \in D(\phi).$$

Inequality (3.3) is related to the minimum problem for the functional

$$W \in D(\phi) \mapsto \Phi_n(W; U_{n-1}) := \frac{1}{2} \tau_n^{-1} |W - U_{n-1}|^2 + \phi(W) - \langle F_n, W \rangle.$$

In fact, once U_{n-1} is given, U_n is a solution of (3.3) if and only if it minimizes $\Phi_n(W; U_{n-1})$, i.e.,

$$(3.4) \quad \Phi_n(U_n; U_{n-1}) = \min_{W \in \mathcal{H}} \Phi_n(W; U_{n-1}).$$

Thanks to the lower semicontinuity and convexity of ϕ , (3.4) admits a unique solution U_n . In view of (2.2), we infer that (3.3) is equivalent to the variational inclusion

$$(3.5) \quad \delta U_n + \mathfrak{F}(U_n) \ni F_n,$$

and thus $U_n \in D(\mathfrak{F})$ for all $1 \leq n \leq N$, irrespective of the choice of U_0 . Moreover, (2.47) yields the following *stronger* variational inequality:

$$(3.6) \quad \langle \delta U_n - F_n, U_n - v \rangle + \phi(U_n) - \phi(v) + \sigma(U_n; v) \leq 0, \quad \forall v \in D(\phi).$$

In particular, if $U_0 \in D(\mathfrak{F})$ and F_0 and τ_0 are defined, we set (see (2.5))

$$(3.7) \quad U_{-1} := U_0 - \tau_0(F_0 - \mathfrak{F}(U_0))^\circ, \quad \delta U_0 := \frac{U_0 - U_{-1}}{\tau_0},$$

so that (3.5) and (3.6) hold also for $n = 0$.

3.1 A Posteriori Error Analysis

Energy Dissipation and the Main a Posteriori Estimate

We now introduce a fundamental quantity, the amount of discrete *energy dissipation*.

DEFINITION 3.1 To every discrete solution $\{U_n\}_{n=0}^N$ of (3.3) we associate the error estimators

$$(3.8) \quad \mathcal{E}_n := \langle F_n - \delta U_n, \delta U_n \rangle - \delta \phi(U_n), \quad \forall 2 \leq n \leq N;$$

\mathcal{E}_1 is also defined provided U_0 belongs to $D(\phi)$.

These quantities are nonnegative (see Lemma 3.4 below), are computable, and will play a significant role in estimating the error $u - U$ a posteriori. Moreover, they are also strictly related to the variational formulation (3.4), because

$$\mathcal{E}_n = \tau_n^{-1} (\Phi_n(U_{n-1}; U_{n-1}) - \Phi_n(U_n; U_{n-1})) - \frac{1}{2} |\delta U_n|^2,$$

so that \mathcal{E}_n measures the *discrete speed of decay* of the functional Φ_n .

Let $E := \max(E_{\mathcal{H}}, E_{\sigma})$ be the error to be bounded and $E_{\mathcal{H}}$ and E_{σ} be given by

$$(3.9) \quad \begin{aligned} E_{\mathcal{H}} &:= \max_{t \in [0, T]} |u(t) - U(t)|, \\ E_{\sigma} &:= \left(2 \int_0^T (\sigma(u(t); U(t)) + \sigma(\bar{U}(t); u(t))) dt \right)^{\frac{1}{2}}. \end{aligned}$$

THEOREM 3.2 *Let u be the strong solution of (2.4), let $\{U_n\}_{n=0}^N$ be the discrete solution of (3.3) with $U_0 \in D(\phi)$, and let E and $\{\mathcal{E}_n\}_{n=1}^N$ be defined as in (3.9) and (3.8). The following a posteriori error estimate holds:*

$$(3.10) \quad E \leq \left(|u_0 - U_0|^2 + \sum_{n=1}^N \tau_n^2 \mathcal{E}_n \right)^{\frac{1}{2}} + \|f - \bar{F}\|_{L^1(0, T; \mathcal{H})}.$$

Remark 3.3. Theorem 3.2 is valid for *weak* solutions u provided σ in (3.9) is replaced by any coercivity function $\sigma_- : \overline{D(\phi)} \times \overline{D(\phi)} \rightarrow [0, +\infty]$ that is lower semicontinuous with respect to each argument separately and is dominated by σ on $D(\mathfrak{F}) \times D(\phi)$, i.e.,

$$\sigma_-(w; v) \leq \sigma(w, v), \quad \forall w \in D(\mathfrak{F}), \forall v \in D(\phi).$$

In fact, weak solutions are by definition uniform limits of the strong solutions u_k corresponding to approximations f_k of f in $L^1(0, T; \mathcal{H})$ (see Definition 2.1 and the basic existence results of Section 2). In addition, (3.10) is stable with respect to the uniform convergence of u_k and $L^1(0, T; \mathcal{H})$ convergence of f_k . In particular, if \mathfrak{F} is p -coercive with respect to a lower semicontinuous seminorm $[\cdot]$ as in Definition 2.7, by Lemma 2.8 it is possible to choose $\sigma_-(w; v) := [w - v]^p/p$.

We start the proof of Theorem 3.2 by establishing a simple relation between \mathcal{E}_n and $\sigma(U_n; U_{n-1})$, namely, the discrete counterpart of (2.6).

LEMMA 3.4 *Whenever it is defined, the energy dissipation quantity \mathcal{E}_n satisfies*

$$(3.11) \quad \tau_n^{-1} \sigma(U_n; U_{n-1}) \leq \mathcal{E}_n.$$

PROOF: Choosing $v = U_{n-1}$ in (3.6), we get

$$(3.12) \quad \tau_n \langle \delta U_n - F_n, \delta U_n \rangle + \tau_n \delta \phi(U_n) + \sigma(U_n; U_{n-1}) \leq 0,$$

which is (3.11) in disguise, namely, $\sigma(U_n; U_{n-1}) \leq \tau_n \mathcal{E}_n$. \square

Now we derive the continuous evolution variational inequalities satisfied by the discrete solution U , \bar{U} and by the error $u - U$.

LEMMA 3.5 *Let $\{U_n\}_{n=0}^N$ be the discrete solution of (3.3) with $U_0 \in D(\phi)$. The interpolants U and \bar{U} of $\{U_n\}_{n=0}^N$ defined in (3.1) satisfy a.e. in $(0, T)$*

$$(3.13) \quad \langle U' - \bar{F}, U - v \rangle + \phi(U) - \phi(v) + \sigma(\bar{U}; v) \leq \ell \bar{\tau} \bar{\mathcal{E}}, \quad \forall v \in D(\phi).$$

PROOF: In view of (3.1) and (3.2) we can write the stronger discrete variational inequality (3.6) a.e. in $(0, T)$ in the form

$$\langle U' - \bar{F}, U - v \rangle + \phi(U) - \phi(v) + \sigma(\bar{U}; v) \leq \mathcal{R}, \quad \forall v \in D(\phi),$$

where

$$\mathcal{R} := \langle U' - \bar{F}, U - \bar{U} \rangle + \phi(U) - \phi(\bar{U}).$$

Recalling again (3.1) and (3.2), we readily have the elementary identity

$$(3.14) \quad U - \bar{U} = -\ell(\bar{U} - \underline{U}) = -\ell\bar{\tau}U',$$

and, by the convexity of ϕ ,

$$\phi(U) - \phi(\bar{U}) \leq (1 - \ell)\phi(\bar{U}) + \ell\phi(\underline{U}) - \phi(\bar{U}) = -\ell(\phi(\bar{U}) - \phi(\underline{U})).$$

Therefore, we have $\mathcal{R} \leq -\ell\tau_n(\langle \delta U_n - F_n, \delta U_n \rangle + \delta\phi(U_n))$ in every interval $(t_{n-1}, t_n]$, and the assertion follows from the definition (3.8) of \mathcal{E}_n . \square

The following error inequality is an immediate consequence of Lemma 3.5:

LEMMA 3.6 *Let $U_0 \in D(\phi)$. The error $u - U$ satisfies a.e. in $(0, T)$*

$$(3.15) \quad \frac{1}{2} \frac{d}{dt} |u - U|^2 + \sigma(u; U) + \sigma(\bar{U}; u) \leq \ell\bar{\tau}\bar{\mathcal{E}} + \langle f - \bar{F}, u - U \rangle.$$

PROOF: We choose $v = U$ in (2.48) and $v = u$ in (3.13). Adding the two inequalities, we readily get (3.15). \square

Note that assuming $U_0 \in D(\phi)$ guarantees the validity of both Lemmas 3.5 and 3.6 all the way to $t = 0$; otherwise they would hold only on (τ_1, T) , because \mathcal{E}_n in (3.8) would be defined and (3.11) would hold only for $2 \leq n \leq N$.

Note also that $\phi(u)$ does not appear explicitly in the error equation (3.15), which is a key observation for arriving at a posteriori error estimates.

PROOF OF THEOREM 3.2: This is a consequence of (3.15) and the following Gronwall-like Lemma 3.7 (see also [4, theorem 3.4]) with $\lambda := 0$ and

$$a := |u - U|, \quad b^2 := 2(\sigma(u; U) + \sigma(\bar{U}; u)), \quad c^2 := 2\ell\bar{\tau}\bar{\mathcal{E}}, \quad d := |f - \bar{F}|.$$

\square

LEMMA 3.7 *Let $a, b, c, d : (0, T) \rightarrow [0, +\infty]$ be measurable functions, a^2 also being absolutely continuous on $[0, T]$. We assume they satisfy the differential inequality, depending on the parameter $\lambda \in \mathbb{R}$,*

$$\frac{d}{dt} a^2(t) + b^2(t) + 2\lambda a^2(t) \leq c^2(t) + 2d(t)a(t) \quad \text{a.e. in } (0, T).$$

Then we have

$$(3.16) \quad \max \left(\max_{t \in [0, T]} e^{\lambda t} a(t), \left(\int_0^T e^{2\lambda s} b^2(s) ds \right)^{\frac{1}{2}} \right) \leq \left(a^2(0) + \int_0^T e^{2\lambda t} c^2(t) dt \right)^{\frac{1}{2}} + \int_0^T e^{\lambda t} d(t) dt.$$

PROOF: It is not restrictive to assume that c^2 and d and therefore b^2 are integrable on $(0, T)$. We introduce the absolutely continuous functions

$$e(t) := e^{2\lambda t} a^2(t) + \int_0^t e^{2\lambda s} b^2(s) ds,$$

$$r(t) := \left(\left(a^2(0) + \int_0^t e^{2\lambda s} c^2(s) ds \right)^{\frac{1}{2}} + \int_0^t e^{\lambda s} d(s) ds \right)^2,$$

and observe that they satisfy the opposite differential inequalities

$$e'(t) \leq c^2(t) + 2d(t)\sqrt{e(t)}, \quad r'(t) \geq c^2(t) + 2d(t)\sqrt{r(t)}, \quad \text{a.e. in } (0, T).$$

Since $e(0) = r(0)$, by a standard comparison argument for differential inequalities, it follows that $e(t) \leq r(t)$ for all $t \in [0, T]$, which yields (3.16). \square

Remark 3.8. The estimate (3.16) seems to be almost optimal: If $b = c \equiv 0$, $\lambda = 0$, and $0 \leq d \in L^1(0, T)$, then (3.16) gives a sharp bound for the maximum of the function $a(t) := a_0 + \int_0^t d(s) ds$ with $a_0 \geq 0$, which obviously satisfies the ODE $\frac{d}{dt} a^2(t) = 2d(t)a(t)$ a.e. $t \in (0, T)$. If $d \equiv 0$, then (3.10) with $f \equiv 0$ coincides with the result (1.16), directly shown in the introduction.

Dissipation Inequalities and Residual Monotone Terms

When \mathfrak{F} is single-valued, we can obtain a characterization of (3.10) in terms of $\sigma(U_n; U_{n-1})$ or, provided that $U_0 \in D(\mathfrak{F})$, in terms of $\varrho(U_n, U_{n-1})$. We recall from (2.46) that

$$\varrho(U_n, U_{n-1}) = \langle \mathfrak{F}(U_n) - \mathfrak{F}(U_{n-1}), U_n - U_{n-1} \rangle.$$

COROLLARY 3.9 *If $U_0 \in D(\phi)$ and \mathfrak{F} is single-valued, then*

$$\mathcal{E}_n = \tau_n^{-1} \sigma(U_n; U_{n-1})$$

and, if $U_0 \in D(\mathfrak{F})$, the error E defined in (3.9) is bounded by

$$(3.17) \quad E \leq \left(|u_0 - U_0|^2 + \sum_{n=1}^N \tau_n \langle \mathfrak{F}(U_n) - \mathfrak{F}(U_{n-1}), U_n - U_{n-1} \rangle \right)^{\frac{1}{2}} + \|f - \bar{F}\|_{L^1(0, T; \mathcal{H})}.$$

PROOF: In view of (2.45) and (3.5), inequality (3.11) is in fact an identity provided \mathfrak{F} is single-valued; therefore $U_0 \in D(\mathfrak{F})$ and (2.44) yield

$$\tau_n \mathcal{E}_n = \sigma(U_n; U_{n-1}) \leq \varrho(U_n, U_{n-1}).$$

□

When \mathfrak{F} is not single-valued, Corollary 3.9 cannot be applied, because it might happen $\tau_n \mathcal{E}_n > \sigma(U_n; U_{n-1})$. In addition, there seems to be an indeterminacy for the choice of a suitable element in $\mathfrak{F}(U_n)$ and $\mathfrak{F}(U_{n-1})$ in (3.17). However, (3.5) suggests a canonical way to perform such a choice, that is,

$$(3.18) \quad F_n - \delta U_n \in \mathfrak{F}(U_n).$$

Therefore it is natural to replace the residual quantity

$$(3.19) \quad \langle \mathfrak{F}(U_n) - \mathfrak{F}(U_{n-1}), U_n - U_{n-1} \rangle$$

in (3.17) with

$$\langle (F_n - \delta U_n) - (F_{n-1} - \delta U_{n-1}), U_n - U_{n-1} \rangle = \tau_n^2 \langle \delta F_n - \delta^2 U_n, \delta U_n \rangle.$$

Due to the crucial role of this quantity in the subsequent estimates, we collect in the following definition the previous arguments:

DEFINITION 3.10 To every discrete solution $\{U_n\}_{n=0}^N$ of (3.5), we associate the error estimators

$$(3.20) \quad \mathcal{D}_n := \tau_n \langle \delta F_n - \delta^2 U_n, \delta U_n \rangle, \quad \forall 2 \leq n \leq N;$$

\mathcal{D}_1 is also defined provided $U_0 \in D(\mathfrak{F})$ and δU_0 is given by (3.7).

Since $U_n \in D(\mathfrak{F})$ for all $1 \leq n \leq N$, then \mathcal{D}_n is well-defined for $2 \leq n \leq N$, irrespective of the regularity of U_0 .

In view of (3.18), (3.19), and (2.44), we realize that $\mathcal{D}_n \geq 0$ and

$$(3.21) \quad \tau_n^{-1} \varrho(U_n, U_{n-1}) \leq \mathcal{D}_n$$

whenever it is defined, and that (3.21) is an identity provided \mathfrak{F} is single-valued. A discrete counterpart of the dissipation inequality (2.7) follows from $\mathcal{D}_n \geq 0$

$$\frac{1}{2} \delta |\delta U_n|^2 \leq \langle \delta^2 U_n, \delta U_n \rangle \leq \langle \delta F_n, \delta U_n \rangle.$$

The estimators \mathcal{E}_n and \mathcal{D}_n are related by an interesting property, which yields a generalization of Corollary 3.9.

LEMMA 3.11 If $U_0 \in D(\mathfrak{F})$, then

$$(3.22) \quad \mathcal{E}_n \leq \mathcal{D}_n, \quad \forall 1 \leq n \leq N.$$

PROOF: We write (3.6) for the unknown U_{n-1} , making use of (3.7) for $n = 1$, and choose $v = U_n$ as a test function to arrive at

$$-\tau_n \delta \phi(U_n) \leq \tau_n \langle \delta U_{n-1} - F_{n-1}, \delta U_n \rangle - \sigma(U_{n-1}; U_n).$$

Replacing this inequality in (3.12) yields

$$\tau_n \mathcal{E}_n \leq \tau_n^2 \langle \delta F_n - \delta^2 U_n, \delta U_n \rangle - \sigma(U_{n-1}; U_n) = \tau_n \mathcal{D}_n - \sigma(U_{n-1}; U_n).$$

□

Note that $U_0 \in D(\mathfrak{F})$ guarantees the validity of both (3.21) and (3.22) for all $1 \leq n \leq N$; otherwise they would hold only for $2 \leq n \leq N$.

COROLLARY 3.12 *If $U_0 \in D(\mathfrak{F})$, then the error E defined in (3.9) is bounded by*

$$(3.23) \quad E \leq \left(|u_0 - U_0|^2 + \sum_{n=1}^N \tau_n^2 \mathcal{D}_n \right)^{\frac{1}{2}} + \|f - \bar{F}\|_{L^1(0,T;\mathcal{H})}.$$

Remark 3.13. When \mathfrak{F} is 2-coercive with respect to a seminorm $[\cdot]$ as in (2.54), then we can give an alternative a posteriori error bound that allows for a more general functional setting for the source term f ; similar results are valid for $p > 2$. We will explore this idea in Section 4.5 within the context of a classical *Hilbert triplet*.

Exponential Decay

If \mathfrak{F} is also 2-coercive with respect to an equivalent norm of \mathcal{H} , i.e., \mathfrak{F} satisfies

$$(3.24) \quad \exists \alpha > 0: \varrho(w, v) \geq 2\alpha |w - v|^2, \quad \forall w, v \in D(\mathfrak{F}),$$

we gain an exponential decay of the various errors.

THEOREM 3.14 *Let \mathfrak{F} satisfy (3.24) and $U_0 \in D(\phi)$. The final error is bounded by*

$$(3.25) \quad |u(T) - U(T)| \leq \left(e^{-2\alpha T} |u_0 - U_0|^2 + \sum_{n=1}^N e^{-2\alpha(T-t_n)} \tau_n^2 \mathcal{E}_n \right)^{\frac{1}{2}} + \int_0^T e^{-\alpha(T-t)} |f(t) - \bar{F}(t)| dt.$$

PROOF: We first notice that (3.15) and Lemma 2.8 imply

$$\frac{d}{dt} |u - U|^2 + 2\alpha |u - U|^2 \leq 2\ell \bar{\tau} \bar{\mathcal{E}} + 2|f - \bar{F}| |u - U|.$$

The asserted estimate (3.25) then follows from Lemma 3.7 with

$$a := |u - U|, \quad b := 0, \quad c^2 := 2\ell \bar{\tau} \bar{\mathcal{E}}, \quad d := |f - \bar{F}|, \quad \lambda := \alpha.$$

□

3.2 Rate of Convergence and Regularity

In this section we demonstrate that the a posteriori estimators introduced in Section 3.1 converge to zero with an optimal rate and involve minimal regularity. No constraint between consecutive time steps is required.

Energy Solutions

First of all, we show that the a posteriori estimator $\sum_{n=1}^N \tau_n^2 \mathcal{E}_n$ in (3.10) possesses an order $O(\sqrt{\tau})$ under minimal regularity assumptions on the discrete data U_0 and F . This, in turn, yields an optimal $O(\sqrt{\tau})$ a priori rate of convergence for the error E provided $u_0 \in D(\phi)$ and $f \in L^2(0, T; \mathcal{H})$, which ensures the existence of a global energy solution.

THEOREM 3.15 *If $U_0 \in D(\phi)$ and $\phi \geq 0$, then the discrete estimators \mathcal{E}_n defined in (3.8) satisfy*

$$(3.26) \quad \sum_{n=1}^N \tau_n \mathcal{E}_n \leq \phi(U_0) + \frac{1}{4} \|\bar{F}\|_{L^2(0, T; \mathcal{H})}^2.$$

PROOF: Multiplying (3.8) by τ_n and adding from $n = 1$ to N , we readily get

$$(3.27) \quad \sum_{n=1}^N \tau_n \mathcal{E}_n \leq \phi(U_0) - \phi(U_N) + \int_0^T \langle \bar{F}, U' \rangle dt - \|U'\|_{L^2(0, T; \mathcal{H})}^2.$$

Since $\phi(U_N) \geq 0$, the Cauchy inequality yields the assertion. \square

Our next task is to establish a uniform estimate for the error E when $f \in L^2(0, T; \mathcal{H})$, even though such a regularity does not give any order of convergence for $\|f - \bar{F}\|_{L^1(0, T; \mathcal{H})}$ in (3.10). The stability constant may be further improved.

THEOREM 3.16 *If $U_0 := u_0 \in D(\phi)$, $F_n := \tau_n^{-1} \int_{t_{n-1}}^{t_n} f(t) dt$ for $1 \leq n \leq N$, and $F_0 := 0$, with $f \in L^2(0, T; \mathcal{H})$ and $\phi \geq 0$, then the solutions u of (2.4) and $\{U_n\}_{n=0}^N$ of (3.3) satisfy the following uniform estimate:*

$$(3.28) \quad E \leq \left(3\tau(\phi(u_0) + \|f\|_{L^2(0, T; \mathcal{H})}^2) \right)^{\frac{1}{2}},$$

where E is the global error defined in (3.9).

PROOF: In view of (3.10), (3.26), and the positivity of \mathcal{E}_n , to prove (3.28) it would remain to bound $\|f - \bar{F}\|_{L^1(0, T; \mathcal{H})}$. Since the sole regularity $f \in L^2(0, T; \mathcal{H})$ does not give any order of convergence for this term, we thus resort to the error equation (3.15) and argue as follows for all $1 \leq n \leq N$: Since \bar{F} is the L^2 projection of f in time over the piecewise constants, it satisfies

$$(3.29) \quad \|f - \bar{F}\|_{L^2(0, t_n; \mathcal{H})}^2 + \|\bar{F}\|_{L^2(0, t_n; \mathcal{H})}^2 = \|f\|_{L^2(0, t_n; \mathcal{H})}^2.$$

Let $\bar{R}(w)$ be the piecewise average of w over \mathcal{P} , whose value in the interval $(t_{n-1}, t_n]$ is $R_n := \tau_n^{-1} \int_{t_{n-1}}^{t_n} w(t) dt$. A simple but tedious calculation yields

$$\begin{aligned} \|u - U - \bar{R}(u - U)\|_{L^2(0, t_n; \mathcal{H})} &\leq \|u - \bar{R}(u)\|_{L^2(0, t_n; \mathcal{H})} + \|U - \bar{R}(U)\|_{L^2(0, t_n; \mathcal{H})} \\ &\leq \frac{1}{\sqrt{6}} \tau \|u'\|_{L^2(0, t_n; \mathcal{H})} + \frac{1}{2\sqrt{3}} \tau \|U'\|_{L^2(0, t_n; \mathcal{H})}. \end{aligned}$$

On integrating the error equation (3.15) over $(0, t_n)$ and using L^2 orthogonality in time, that is, $\int_{t_{n-1}}^{t_n} \langle f - \bar{F}, \bar{R}(u - U) \rangle = 0$, and the Cauchy inequality, we get from (3.27) and the positivity of \mathcal{E}_n

$$\begin{aligned} e(t_n) &:= |u(t_n) - U_n|^2 + 2 \int_0^{t_n} (\sigma(u; U) + \sigma(\bar{U}; u)) dt \\ &\leq \int_0^{t_n} 2\ell(t) \bar{\tau}(t) \bar{\mathcal{E}}(t) dt + 2 \|f - \bar{F}\|_{L^2(0, t_n; \mathcal{H})} \|u - U - \bar{R}(u - U)\|_{L^2(0, t_n; \mathcal{H})} \\ &\leq \tau \left(\phi(u_0) - \phi(U_n) - \|U'\|_{L^2(0, t_n; \mathcal{H})}^2 + \|\bar{F}\|_{L^2(0, t_n; \mathcal{H})} \|U'\|_{L^2(0, t_n; \mathcal{H})} \right. \\ &\quad \left. + \frac{\sqrt{2}}{\sqrt{3}} \|f - \bar{F}\|_{L^2(0, t_n; \mathcal{H})} \|u'\|_{L^2(0, t_n; \mathcal{H})} \right. \\ &\quad \left. + \frac{1}{\sqrt{3}} \|f - \bar{F}\|_{L^2(0, t_n; \mathcal{H})} \|U'\|_{L^2(0, t_n; \mathcal{H})} \right). \end{aligned}$$

On using (3.29) and the elementary inequality

$$(3.30) \quad 2ab \leq \varepsilon a^2 + \frac{b^2}{\varepsilon}, \quad \forall a, b \in \mathbb{R}, \varepsilon > 0,$$

with $\varepsilon = 1$ on the last three terms, we arrive at

$$e(t_n) \leq \tau \left(\phi(u_0) - \phi(U_n) + \frac{1}{2} \|f\|_{L^2(0, t_n; \mathcal{H})}^2 + \frac{1}{2} \|u'\|_{L^2(0, t_n; \mathcal{H})}^2 \right).$$

To prove the $H^1(0, T; \mathcal{H})$ stability of u , we integrate (2.6) in time and use the Cauchy inequality and the positivity of ϕ to arrive at

$$\|u'\|_{L^2(0, t_n; \mathcal{H})}^2 \leq \|f\|_{L^2(0, t_n; \mathcal{H})}^2 + 2\phi(u_0).$$

Therefore we get

$$(3.31) \quad e(t_n) \leq \tau (2\phi(u_0) - \phi(U_n) + \|f\|_{L^2(0, t_n; \mathcal{H})}^2).$$

To complete the proof of (3.28), it remains to estimate the error $|u(t) - U(t)|$ for all $t \in (t_{n-1}, t_n)$. To this end, we apply (3.10) from t_{n-1} to t_n and use (3.27) and (3.30)

with $\varepsilon = \frac{1}{2}$ and the Cauchy inequality to obtain

$$\begin{aligned} & \max_{t \in (t_{n-1}, t_n)} |u(t) - U(t)|^2 \\ & \leq \left((e(t_{n-1}) + \tau_n^2 \mathcal{E}_n)^{1/2} + \int_{t_{n-1}}^{t_n} |f - F_n| dt \right)^2 \\ & \leq \frac{3}{2} \left((e(t_{n-1}) + \tau(\phi(U_{n-1}) - \phi(U_n) + \frac{1}{4} \tau_n |F_n|^2)) + 3\tau_n \int_{t_{n-1}}^{t_n} |f - F_n|^2 dt \right). \end{aligned}$$

In view of (3.29) and (3.31) with t_{n-1} in place of t_n , we finally conclude

$$\max_{t \in (t_{n-1}, t_n)} |u(t) - U(t)|^2 \leq 3\tau \left(\phi(u_0) - \frac{1}{2} \phi(U_n) + \|f\|_{L^2(0, t_n; \mathcal{H})}^2 \right).$$

□

Strong Lipschitz Solutions

Estimate (3.23) turns out to be useful for proving a linear rate of convergence $O(\tau)$ for the error E provided $u_0 \in D(\mathfrak{F})$ and $f \in \text{BV}(0, T; \mathcal{H})$. To this end, we need a discrete counterpart of Lemma 3.7 that yields precise stability constants for a discrete system of inequalities (see also [4, theorem 3.3]); its proof is a discrete version of that of Lemma 3.7.

LEMMA 3.17 *Let $\{a_n\}_{n=0}^N$ and $\{b_n, c_n, d_n\}_{n=1}^N$ be nonnegative numbers. If*

$$(3.32) \quad 2a_n(a_n - a_{n-1}) + b_n^2 \leq c_n^2 + 2a_n d_n, \quad \forall 1 \leq n \leq N,$$

then

$$(3.33) \quad \max_{1 \leq n \leq N} a_n \leq \left(a_0^2 + \sum_{n=1}^N c_n^2 \right)^{\frac{1}{2}} + \sum_{n=1}^N d_n,$$

$$(3.34) \quad \left(\sum_{n=1}^N b_n^2 \right)^{\frac{1}{2}} \leq \left(a_0^2 + \sum_{n=1}^N c_n^2 \right)^{\frac{1}{2}} + \sqrt{2} \sum_{n=1}^N d_n.$$

PROOF: We set $R_0 := a_0$ and introduce the quantities for $1 \leq n \leq N$

$$C_n := \left(a_0^2 + \sum_{k=1}^n c_k^2 \right)^{\frac{1}{2}}, \quad D_n := \sum_{k=1}^n d_k, \quad R_n := C_n + D_n.$$

We first prove the inequality $a_n \leq R_n$ for all $1 \leq n \leq N$, which yields (3.33). From (3.32) we obviously deduce the following quadratic inequality for a_n :

$$a_n^2 - (a_{n-1} + d_n)a_n \leq \frac{1}{2}c_n^2.$$

This, together with $a_n \geq 0$, implies

$$(3.35) \quad 2a_n \leq a_{n-1} + d_n + ((a_{n-1} + d_n)^2 + 2c_n^2)^{\frac{1}{2}}.$$

It is not difficult to check that R_n satisfies the quadratic inequality

$$R_n^2 - (R_{n-1} + d_n)R_n = R_n(C_n - C_{n-1}) \geq C_n(C_n - C_{n-1}) \geq \frac{1}{2}c_n^2,$$

which, in view of $R_n \geq 0$, leads to

$$(3.36) \quad 2R_n \geq R_{n-1} + d_n + ((R_{n-1} + d_n)^2 + 2c_n^2)^{\frac{1}{2}}.$$

With the aid of (3.35), (3.36), and $a_0 = R_0$, a standard induction argument gives rise to the desired inequality $a_n \leq R_n$ for $1 \leq n \leq N$. To prove (3.34), we notice that (3.32) together with $a_n \leq R_n \leq R_N$ implies

$$b_n^2 \leq a_{n-1}^2 - a_n^2 + c_n^2 + 2R_N d_n,$$

which, upon summation from $n = 1$ to N , yields

$$\sum_{n=1}^N b_n^2 \leq C_N^2 + 2R_N D_N \leq (C_N + \sqrt{2}D_N)^2.$$

□

Theorem 3.18 below shows how to control the estimator $\sum_{n=1}^N \tau_n^2 \mathcal{D}_n$ in (3.23), and Corollary 3.20 expresses such bounds in terms of data.

THEOREM 3.18 *If $U_0 \in D(\mathfrak{F})$, the discrete estimators \mathcal{D}_n defined in (3.20) satisfy*

$$(3.37) \quad \left(\sum_{n=1}^N \mathcal{D}_n \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{2}} |\delta U_0| + \text{Var } \bar{F}.$$

PROOF: From (3.20) and the Cauchy inequality, we deduce

$$(3.38) \quad |\delta U_n| (|\delta U_n| - |\delta U_{n-1}|) + \mathcal{D}_n \leq \tau_n |\delta F_n| |\delta U_n|.$$

Since $\mathcal{D}_n \geq 0$ from (3.21), applying Lemma 3.17 with

$$a_n := |\delta U_n|, \quad b_n^2 := 2\mathcal{D}_n, \quad c_n := 0, \quad d_n := |F_n - F_{n-1}|,$$

the assertion (3.37) follows from the relation $\text{Var } \bar{F} = \sum_{n=1}^N \tau_n |\delta F_n|$. □

Remark 3.19. It is interesting to notice that Definition 3.10 of \mathcal{D}_n and the proof of Theorem 3.18 do not rely on the subdifferential structure of \mathfrak{F} but only on its monotonicity. This observation will be crucial in Section 4 in dealing with the more general class of angle-bounded operators. The proof of Theorem 3.20, instead, is based on Corollary 3.12 and thus on the existence of a convex potential.

THEOREM 3.20 *Let $U_0 := u_0 \in D(\mathfrak{F})$, $f \in \text{BV}(0, T; \mathcal{H})$, and $F_n := f_+(t_n)$. Then the solutions u of (2.4) and $\{U_n\}_{n=0}^N$ of (3.3) satisfy the following uniform estimate:*

$$(3.39) \quad E \leq \tau \left(\frac{1}{\sqrt{2}} |(f_+(0) - \mathfrak{F}(u_0))^\circ| + 2 \text{Var } f \right),$$

where E is the global error defined in (3.9).

PROOF: In view of Corollary 3.12 and Theorem 3.18, the estimate (3.39) follows from (3.7) and the properties $\|f - \bar{F}\|_{L^1(0,T;\mathcal{H})} \leq \tau \operatorname{Var} f$ and $\operatorname{Var} \bar{F} \leq \operatorname{Var} f$. \square

4 The Nonpotential Case

In this section we study (maximal) monotone operators \mathfrak{F} that are *not* the subgradient of a (convex) functional. We are driven by the validity of Theorem 3.18 irrespective of the existence of a potential ϕ (see Remark 3.19). On the other hand, the hyperbolic-like counterexample of Rulla [52, example 3] rules out the possibility of deriving a linear order of convergence $O(\tau)$ for monotone operators \mathfrak{F} without further restrictive properties.

In order to understand what kind of (hopefully minimal) abstract assumptions are reasonable on \mathfrak{F} , we rewrite the continuous and discrete problems (2.3) and (3.5) as equalities for a suitable selection in $\mathfrak{F}(u)$ and $\mathfrak{F}(\bar{U})$, namely,

$$u' + \mathfrak{F}(u) = f, \quad U' + \mathfrak{F}(\bar{U}) = \bar{F}.$$

Since we are interested in the error between u and U , it is natural to multiply the difference of the two equations by $u - U$ and use (3.1) to obtain

$$\begin{aligned} (4.1) \quad \frac{1}{2} \frac{d}{dt} |u - U|^2 &\leq \langle \mathfrak{F}(\bar{U}) - \mathfrak{F}(u), u - U \rangle + \langle f - \bar{F}, u - U \rangle \\ &\leq (1 - \ell) \langle \mathfrak{F}(\bar{U}) - \mathfrak{F}(u), u - \bar{U} \rangle \\ &\quad + \ell \langle \mathfrak{F}(\bar{U}) - \mathfrak{F}(u), u - \underline{U} \rangle + \langle f - \bar{F}, u - U \rangle. \end{aligned}$$

Since the first term on the right-hand side is ≤ 0 by virtue of the monotonicity of \mathfrak{F} , we are thus led to investigate the residual term

$$\langle \mathfrak{F}(\bar{U}) - \mathfrak{F}(u), u - \underline{U} \rangle.$$

4.1 Angle-Bounded Operators

We recall here the definition of *angle-bounded* operators, which was introduced by Brézis and Browder [10, section 1] and recently studied by Caffarelli [13, p. 573] in completely different contexts from ours, but turns out to be intimately related to the question of optimal approximability.

DEFINITION 4.1 A (multivalued) operator \mathfrak{F} from \mathcal{H} to $2^{\mathcal{H}}$ is said to be *angle-bounded* (or γ^2 -*angle-bounded*) if there exists a positive constant γ such that

$$(4.2) \quad \langle \mathfrak{F}(v) - \mathfrak{F}(w), w - z \rangle \leq \gamma^2 \langle \mathfrak{F}(v) - \mathfrak{F}(z), v - z \rangle, \quad \forall v, w, z \in D(\mathfrak{F}).$$

As usual, (4.2) must hold for every selection in $\mathfrak{F}(v), \mathfrak{F}(w), \mathfrak{F}(z)$, with that of $\mathfrak{F}(v)$ being the same on both sides of the inequality.

Note that angle-bounded operators are monotone: Take $z = v$ in (4.2). In many respects, angle-boundedness seems to be the right intermediate condition between monotonicity and existence of a convex potential. We elaborate upon this point further via several examples.

Example 4.1: Subdifferentials

We first check that if \mathfrak{F} is the subdifferential of a convex potential ϕ , then it is angle-bounded with $\gamma = 1$. Since

$$(4.3) \quad \begin{aligned} \langle \mathfrak{F}(v) - \mathfrak{F}(w), w - z \rangle = \\ \langle \mathfrak{F}(v) - \mathfrak{F}(z), v - z \rangle + \langle \mathfrak{F}(v), w - v \rangle + \langle \mathfrak{F}(w), z - w \rangle + \langle \mathfrak{F}(z), v - z \rangle, \end{aligned}$$

from (2.47) we deduce that, for every $v, w, z \in D(\mathfrak{F})$,

$$(4.4) \quad \begin{aligned} \langle \mathfrak{F}(v) - \mathfrak{F}(w), w - z \rangle &\leq \langle \mathfrak{F}(v) - \mathfrak{F}(z), v - z \rangle - \sigma(v; w) - \sigma(w; z) - \sigma(z; v) \\ &\leq \langle \mathfrak{F}(v) - \mathfrak{F}(z), v - z \rangle. \end{aligned}$$

The class of angle-bounded operators is thus larger than that of subdifferentials.

Example 4.2: k -Cyclically-Monotone Operators

From (4.3) we see that a necessary and sufficient condition for 1-angle-boundedness is the property

$$\langle \mathfrak{F}(v), w - v \rangle + \langle \mathfrak{F}(w), z - w \rangle + \langle \mathfrak{F}(z), v - z \rangle \leq 0, \quad \forall v, w, z \in D(\mathfrak{F}).$$

This class of operators is the case $k = 3$ of the k -cyclically-monotone operators, introduced by Rockafellar [50], [9, chapter II, section 7]. Following his definition, we say that \mathfrak{F} is *k -cyclically-monotone* if and only if for every choice of $v_1, v_2, \dots, v_k \in D(\mathfrak{F})$

$$\langle \mathfrak{F}(v_1), v_2 - v_1 \rangle + \langle \mathfrak{F}(v_2), v_3 - v_2 \rangle + \dots + \langle \mathfrak{F}(v_k), v_1 - v_k \rangle \leq 0,$$

and that \mathfrak{F} is *cyclically-monotone* if it is k -cyclically-monotone for every $k \in \mathbb{N}$. Since every k -cyclically-monotone operator, with $k \geq 3$, is also 3-cyclically-monotone, it is also 1-angle-bounded. Monotone operators correspond to 2-cyclically-monotone operators, which are not necessarily γ -angle-bounded. The most important link with the theory of subdifferentials of (lower semicontinuous) convex functions lies in their precise characterization as the (maximal) cyclically-monotone operators [50].

The correspondence between angle-bounded and cyclically-monotone operators is more transparent and complete provided $\mathfrak{F} = \mathfrak{A}$ is linear. Asplund [2] proved that \mathfrak{A} is k -cyclically-monotone for $k \geq 3$ if and only if \mathfrak{A} is γ_k^2 -angle-bounded with $2\gamma_k \cos(\pi/k) = 1$ (the 3-cyclically-monotone maps thus correspond to an angle of $\frac{\pi}{3}$ and $\gamma = 1$). For nontrivial linear operators, $\frac{1}{2}$ is thus the least admissible value of γ and corresponds to symmetric positive semidefinite operators.

Example 4.3: Linear Nonsymmetric Operators

Sectorial Operators. If $\mathfrak{F} = \mathfrak{A}$ is linear, then Definition 4.1 is equivalent to the *strong sector condition*

$$(4.5) \quad |\langle \mathfrak{A}v, w \rangle|^2 \leq 4\gamma^2 \langle \mathfrak{A}v, v \rangle \langle \mathfrak{A}w, w \rangle, \quad \forall v, w \in D(\mathfrak{A}).$$

To see this, we set $\tilde{v} = v - z$ and $\tilde{w} = w - z$ in (4.2) to get the equivalent formulation (we omit the tildes)

$$(4.6) \quad \langle \mathfrak{A}v, w \rangle \leq \gamma^2 \langle \mathfrak{A}v, v \rangle + \langle \mathfrak{A}w, w \rangle, \quad \forall v, w \in D(\mathfrak{A}).$$

Then we replace v by rv , $r \in \mathbb{R}$, and argue with the resulting quadratic inequality in r to realize that (4.5) is equivalent to (4.6). In addition, (4.5) can be rephrased in terms of the antisymmetric part of \mathfrak{A} as follows (see [1]):

$$(4.7) \quad |\langle \mathfrak{A}v, w \rangle - \langle \mathfrak{A}w, v \rangle| \leq 2\mu \langle \mathfrak{A}v, v \rangle^{\frac{1}{2}} \langle \mathfrak{A}w, w \rangle^{\frac{1}{2}} \leq \mu (\langle \mathfrak{A}v, v \rangle + \langle \mathfrak{A}w, w \rangle)$$

for a suitable $\mu \geq 0$. It turns out [10, proposition 11] that both (4.5) and (4.7) are equivalent to $\gamma^2 = (\mu^2 + 1)/4$.

Bilinear Forms. Every linear operator \mathfrak{A} , maximal monotone and strongly sectorial in the sense of (4.5) or (4.7), can be associated via (2.15) to a bilinear form \mathfrak{a} defined in a Hilbert space \mathcal{V} satisfying (2.14) and (2.13), respectively [44]. In this case, (4.7) becomes the natural generalization of (2.17)

$$(4.8) \quad \exists \mu \geq 0: \mathfrak{a}_a(v, w) \leq \mu \mathfrak{a}(v, v)^{\frac{1}{2}} \mathfrak{a}(w, w)^{\frac{1}{2}}, \quad \forall v, w \in \mathcal{V},$$

where \mathfrak{a}_a is the antisymmetric part of \mathfrak{a} ,

$$\mathfrak{a}_a(v, w) := \frac{1}{2} (\mathfrak{a}(v, w) - \mathfrak{a}(w, v)), \quad \forall v, w \in \mathcal{V}.$$

Such kind of bilinear forms obviously comprises the *strongly coercive* ones

$$\exists \alpha > 0: [v]^2 = \mathfrak{a}(v, v) \geq \alpha \|v\|_{\mathcal{V}}^2, \quad \forall v \in \mathcal{V}.$$

Second-Order Advection-Diffusion Problems. A significant example of a sectorial bilinear form in $\mathcal{H} := L^2(\Omega)$ and $\mathcal{V} := H^1(\Omega)$ is given by

$$(4.9) \quad \mathfrak{a}(v, w) := \int_{\Omega} (\nu^2 \nabla v \cdot \nabla w + \mathbf{b} \cdot \nabla v w + c v w) dx, \quad \forall v, w \in \mathcal{V},$$

where $\nu^2 > 0$, \mathbf{b} is a Lipschitz vector field defined in $\bar{\Omega}$, $c \in L^\infty(\Omega)$, and

$$-\frac{1}{2} \operatorname{div} \mathbf{b}(x) + c(x) \geq 0 \text{ a.e. } x \in \Omega, \quad \mathbf{b} \cdot \mathbf{n} \geq 0 \text{ on } \partial\Omega.$$

In this case

$$\begin{aligned} \mathfrak{A}w &:= -\nu^2 \Delta w + \mathbf{b} \cdot \nabla w + c w, \\ D(\mathfrak{A}) &:= \{v \in H^1(\Omega) : \Delta v \in L^2(\Omega), \partial_n v = 0 \text{ on } \partial\Omega\}, \end{aligned}$$

and (2.16) reads

$$(4.10) \quad \partial_t u - \nu^2 \Delta u + \mathbf{b} \cdot \nabla u + c u = f \text{ a.e. in } Q, \quad \partial_n u = 0 \text{ on } \Sigma.$$

The case of a nonhomogeneous Neumann condition $\partial_n u = g(x, t)$ is developed in Section 5.2. This requires Hilbert triplets, which are studied in Section 4.5.

Example 4.4: Abstract Variational Inequalities

Let \mathcal{V} , \mathfrak{a} , and ψ satisfy (2.13), (2.14), and (2.23), along with (4.8) instead of (2.17). The evolution variational inequality (2.24) still makes sense as well as the associated maximal monotone operator \mathfrak{F} defined by (2.25). We will prove in Lemma 4.15 that

$$(4.11) \quad \mathfrak{F} \text{ is } \gamma^2\text{-angle-bounded with } \gamma^2 := \max\left(1, \frac{1}{4}(1 + \mu^2)\right),$$

as well as $\mathfrak{F} = \mathfrak{A} + \partial\psi$ provided \mathfrak{A} in (2.15) and ψ in (2.23) satisfy the compatibility condition (2.26).

Example 4.5: Nonlinear Variational Boundary Conditions

We consider the PDE of (4.10), but with another lateral boundary condition,

$$(4.12) \quad \partial_t u - \nu^2 \Delta u + \mathbf{b} \cdot \nabla u + cu = f \text{ in } Q, \quad \partial_n u + \beta(u) \ni 0 \text{ on } \Sigma,$$

where $\beta : \mathbb{R} \rightarrow 2^{\mathbb{R}}$ is a maximal monotone real graph, which is thus [9, chapter 2, section 8] the subdifferential of a proper lower semicontinuous convex function $j : \mathbb{R} \rightarrow (-\infty, +\infty]$. Let us observe that when β is a continuous function, or, equivalently, j is differentiable, the lateral boundary condition in (4.12) is no longer a differential inclusion but a simpler nonlinear equation: In particular, $\beta(r) \equiv -g \in \mathbb{R}$ corresponds to the usual *Neumann* boundary condition

$$\partial_n u = g \quad \text{on } \Sigma.$$

On the other hand, the choice (see [34, section 3.2.2])

$$\beta(r) := \begin{cases} (-\infty, +\infty) & \text{if } r = g \\ \emptyset & \text{if } r \neq g, \end{cases}$$

corresponding to $j := I_{\{g\}}$, is associated to the *Dirichlet* boundary condition

$$u = g \quad \text{on } \Sigma,$$

whereas

$$\beta(r) := \begin{cases} \emptyset & \text{if } r < g \\ (-\infty, 0] & \text{if } r = g \\ \{0\} & \text{if } r > g, \end{cases}$$

corresponding to $j := I_{[g, +\infty)}$, gives rise to the (ambiguous) *Signorini* boundary condition (see, e.g., [6, example 7.9], [42, chapter 2, example 9.4], [33, chapter 6, section 2.1.1])

$$u \geq g, \quad \partial_n u \geq 0, \quad \partial_n u(u - g) = 0 \quad \text{on } \Sigma.$$

Upon introducing the convex functional

$$(4.13) \quad \psi(w) := \int_{\partial\Omega} j(w(x)) dS, \quad D(\psi) := \{v \in H^1(\Omega) : j(v) \in L^1(\partial\Omega)\},$$

which is lower semicontinuous with respect to $\mathcal{V} := H^1(\Omega)$, (4.12) admits the variational formulation (2.24). The related nonlinear operator \mathfrak{F} , defined in

$$D(\mathfrak{F}) := \{v \in H^1(\Omega) \cap D(\psi) : \Delta v \in L^2(\Omega), \partial_n v + \beta(v) \ni 0 \text{ on } \partial\Omega\},$$

cannot be directly expressed in terms of \mathfrak{A} and the subdifferential of ψ . To do so, we need the Hilbert triplets of Section 4.5.

Example 4.6: Angle-Bounded Vector Fields

This example is a direct generalization of Example 2.4. We consider (2.32) again, where now the Carathéodory vector field $\mathbf{a} : \Omega \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is not a gradient but is 2-coercive and angle-bounded instead: There exists positive constants α_0 , α_1 , α_2 , and γ such that for every choice of $x \in \Omega$, $\xi, \eta, \zeta \in \mathbb{R}^m$,

$$\begin{aligned} \alpha_0 |\xi|^2 - \alpha_1 &\leq \mathbf{a}(x, \xi) \cdot \xi \leq \alpha_2 (1 + |\xi|^2), \\ (\mathbf{a}(x, \xi) - \mathbf{a}(x, \eta)) \cdot (\eta - \zeta) &\leq \gamma^2 (\mathbf{a}(x, \xi) - \mathbf{a}(x, \zeta)) \cdot (\xi - \zeta). \end{aligned}$$

It is then easy to see that the operator (compare with (2.31))

$$\begin{aligned} \mathfrak{F}(w(\cdot)) &:= -\operatorname{div} \mathbf{a}(\cdot, \nabla w(\cdot)), \\ D(\mathfrak{F}) &:= \{v \in H_0^1(\Omega) : \operatorname{div} \mathbf{a}(\cdot, \nabla v(\cdot)) \in L^2(\Omega)\}, \end{aligned}$$

is γ^2 -angle-bounded in $\mathcal{H} := L^2(\Omega)$.

Remark 4.2. The concept of angle-boundedness for a linear operator \mathfrak{A} in a real Hilbert space is strictly related to the concept of *sectorial* operator introduced by Kato [38, chapter VI] in the complex framework. This complex structure shows, in particular, that the contraction semigroup generated by a *linear* angle-bounded operator is analytic (for a “real” proof, see [3]) and therefore has a regularizing effect on the initial data as in the case of a symmetric operator (and of a subdifferential in the nonlinear case). We do not know whether *nonlinear* angle-bounded operators exhibit a regularizing effect.

4.2 Coercivity of an Angle-Bounded Operator \mathfrak{F}

We now define and study a measure of the coercivity of an angle-bounded operator \mathfrak{F} that plays a role analogous to σ for subdifferentials; see Section 2.3. For every $v, w, z \in D(\mathfrak{F})$ and $v_* \in \mathfrak{F}(v)$, $w_* \in \mathfrak{F}(w)$, and $z_* \in \mathfrak{F}(z)$, we set

$$(4.14) \quad \sigma_\gamma(v, w, z) := \inf_{v_*, w_*, z_*} (\gamma^2 \langle v_* - z_*, v - z \rangle - \langle v_* - w_*, w - z \rangle).$$

Note that if $z = v$, we have $\sigma_\gamma(v, w, v) = \varrho(w, v)$ for all $w, v \in D(\mathfrak{F})$.

We observe that if \mathfrak{F} is a subdifferential, then (4.4) yields

$$\sigma_1(v, w, z) \geq \sigma(v; w) + \sigma(w; z) + \sigma(z; v), \quad \forall v, w, z \in D(\mathfrak{F}).$$

Consequently, if \mathfrak{F} is a subdifferential and is also 2-coercive with respect to a seminorm $[\cdot]$ as in (2.54), then Lemma 2.8 gives

$$\sigma_1(v, w, z) \geq \frac{1}{2} [v - w]^2 + \frac{1}{2} [w - z]^2 + \frac{1}{2} [z - v]^2.$$

A similar estimate holds for coercive operators that satisfy a suitable dual Lipschitz condition. To motivate this condition, let us consider a linear angle-bounded operator \mathfrak{A} and use the characterization (4.5). If we denote by $[\cdot]$ the lower semicontinuous extension of the Hilbert seminorm on $D(\mathfrak{A})$ and by $[\cdot]_*$ its conjugate

$$(4.15) \quad [w] := \langle \mathfrak{A}w, w \rangle^{\frac{1}{2}}, \quad \forall w \in D(\mathfrak{A}),$$

$$(4.16) \quad [w]_* := \sup_{v \in D(\mathfrak{A}): [v] \leq 1} \langle w, v \rangle, \quad \forall w \in D(\mathfrak{A}),$$

then (4.5) is equivalent to \mathfrak{A} being 2-coercive with respect to $[\cdot]$ (given by (4.15)) and Lipschitz with respect to the conjugate norm (4.16)

$$[\mathfrak{A}(v - w)]_* \leq 2\gamma[v - w], \quad \forall v, w \in D(\mathfrak{A}).$$

It is important to notice that a similar simple result holds also in the nonlinear framework; other interesting properties can be found in [3].

LEMMA 4.3 *Let $\mathfrak{F} : D(\mathfrak{F}) \rightarrow \mathcal{H}$ be 2-coercive with respect to a lower semicontinuous seminorm $[\cdot]$ as in (2.54), and let \mathfrak{F} satisfy the dual Lipschitz-type condition*

$$(4.17) \quad \exists L > 0: [\mathfrak{F}(v) - \mathfrak{F}(w)]_* \leq L[v - w], \quad \forall v, w \in D(\mathfrak{F}).$$

Then \mathfrak{F} is γ^2 -angle-bounded for $\gamma := L/2$, and, for every $\eta \geq L/\sqrt{2}$ ($= \sqrt{2}\gamma$), we have the extra coercivity-type estimate

$$(4.18) \quad \sigma_\eta(v, w, z) \geq \frac{1}{2} \max([v - w]^2, [w - z]^2), \quad \forall v, w, z \in D(\mathfrak{F}).$$

PROOF: We get from (2.54) and (4.17) with elementary calculations

$$\begin{aligned} \langle \mathfrak{F}(v) - \mathfrak{F}(w), w - z \rangle &= \langle \mathfrak{F}(v) - \mathfrak{F}(w), w - v \rangle + \langle \mathfrak{F}(v) - \mathfrak{F}(w), v - z \rangle \\ &\leq -[v - w]^2 + L[v - w][v - z] \\ &\leq \frac{L^2}{4}[v - z]^2 \leq \frac{L^2}{4} \langle \mathfrak{F}(v) - \mathfrak{F}(z), v - z \rangle. \end{aligned}$$

This shows that \mathfrak{F} is γ^2 -angle-bounded with $\gamma = L/2$. We can modify this derivation slightly to obtain

$$\begin{aligned} \langle \mathfrak{F}(v) - \mathfrak{F}(w), w - z \rangle &\leq -\frac{1}{2}[v - w]^2 + \frac{L^2}{2}[v - z]^2 \\ &\leq -\frac{1}{2}[v - w]^2 + \frac{L^2}{2} \langle \mathfrak{F}(v) - \mathfrak{F}(z), v - z \rangle. \end{aligned}$$

Likewise, a symmetric argument yields

$$\begin{aligned} \langle \mathfrak{F}(v) - \mathfrak{F}(w), w - z \rangle &= \langle \mathfrak{F}(z) - \mathfrak{F}(w), w - z \rangle + \langle \mathfrak{F}(v) - \mathfrak{F}(z), w - z \rangle \\ &\leq -[w - z]^2 + L[v - z][w - z] \\ &\leq -\frac{1}{2}[w - z]^2 + \frac{L^2}{2} \langle \mathfrak{F}(v) - \mathfrak{F}(z), v - z \rangle. \end{aligned}$$

Combining the last two inequalities, we get (4.18). □

Remark 4.4. If \mathfrak{A} is linear and γ^2 -angle-bounded, we can apply Lemma 4.3 and deduce the following coercivity estimate with respect to the induced seminorm $[\cdot]$ in (4.15) for every $\eta \geq \sqrt{2}\gamma$:

$$\sigma_\eta(v, w, z) \geq \frac{1}{2} \max([v - w], [w - z]), \quad \forall v, w, z \in D(\mathfrak{A}).$$

Remark 4.5. It is easy to see that the sum of two γ_i^2 -angle-bounded operators is still angle-bounded with $\gamma^2 := \max(\gamma_1^2, \gamma_2^2)$, and the resulting coercivity function σ_γ defined in (4.14) satisfies $\sigma_\gamma \geq \sigma_{\gamma_1} + \sigma_{\gamma_2}$. The same holds true for the nonlinear operators associated to (nonsymmetric) variational inequalities such as those in Example 4.4, but the proof is as follows: If ψ satisfies (2.23), then $\partial\psi : \mathcal{V} \rightarrow 2^{\mathcal{V}^*}$ is maximal monotone. At the same time, if (2.14) and (4.8) are valid, then the associated (linear) operator $\mathfrak{A} : \mathcal{V} \rightarrow \mathcal{V}^*$ is Lipschitz and coercive. The sum $\mathfrak{F}_\mathcal{V} = \mathfrak{A} + \partial\psi : \mathcal{V} \rightarrow 2^{\mathcal{V}^*}$ is thus maximal monotone and given by

$$(4.19) \quad w_* \in \mathfrak{F}_\mathcal{V}(w) \Leftrightarrow \mathfrak{a}(w, w - v) + \psi(w) \leq \psi(v) + \langle w_*, w - v \rangle, \quad \forall v \in D(\psi);$$

compare with (2.25). We infer that $\mathfrak{A} + \partial\psi$ is γ^2 -angle-bounded, with $\gamma^2 = \max\{1, (1 + \mu^2)/4\}$, because so is each summand with $\gamma^2 = (1 + \mu^2)/4$ and $\gamma^2 = 1$, respectively (see Examples 4.1 and 4.3). Likewise, we can argue with the coercivity function. Finally, if $\mathfrak{F} : \mathcal{H} \rightarrow 2^{\mathcal{H}}$ is the restriction of $\mathfrak{F}_\mathcal{V}$ to \mathcal{H} , namely,

$$(4.20) \quad \mathfrak{F}(w) := \mathfrak{F}_\mathcal{V}(w) \cap \mathcal{H}, \quad D(\mathfrak{F}) \subset D(\mathfrak{F}_\mathcal{V}) \subset D(\psi) \subset \mathcal{V},$$

then \mathfrak{F} is still γ^2 -angle-bounded with the same γ and coercivity function as $\mathfrak{F}_\mathcal{V}$, as well as maximal; we just remove certain pairs (w, w_*) in the graph of $\mathfrak{F}_\mathcal{V}$. Consequently, passing to the duality $\mathcal{V} - \mathcal{V}^*$ renders the above calculation feasible but does not imply that \mathfrak{F} is the sum of the restrictions of \mathfrak{A} and $\partial\psi$ to \mathcal{H} . The latter is true provided the compatibility condition (2.26) holds. These properties may be useful in studying Lipschitz and coercive perturbations of subdifferential operators in the sense of (4.17) and (2.54).

Remark 4.6. All the examples and remarks of this subsection deal only with the algebraic aspect of monotonicity and angle-boundedness. Of course, the related applications to evolution equations also require the fundamental *maximality* assumption (2.1). It is interesting to observe that a linear angle-bounded operator is maximal if and only if it (its graph) is closed and its domain is dense. On the contrary, a deeper investigation is required to check the *maximality* (2.1) of the sum of two angle-bounded operators, which in general is not a direct consequence of the maximality of each addendum (see, e.g., [9, chapter II, sections 6 and 9]). For instance, if \mathfrak{A} and ψ satisfy (2.13), (2.14), (2.15), (2.23), and (4.8), then a sufficient condition for $\mathfrak{F} := \mathfrak{A} + \partial\psi$ to be maximal monotone is the compatibility condition (2.26) [9, proposition 2.17].

4.3 Error Estimates

We observed in Remark 3.19 that the discrete residual (3.19) still makes sense in the nonpotential framework, provided we select (3.18) in $\mathfrak{F}(U_n)$ and $\mathfrak{F}(U_{n-1})$, and yields the estimator \mathcal{D}_n defined in (3.20). Moreover, these estimators \mathcal{D}_n satisfy Theorem 3.18. We now explore this idea and show both a posteriori and uniform a priori error estimates similar to those in Corollary 3.12 and Theorem 3.20.

We define the global error as in (3.9) by replacing E_σ with a corresponding term arising from σ_γ . Therefore we set $E_\gamma := \max(E_{\mathcal{H}}, E_{\sigma_\gamma})$, with $E_{\mathcal{H}}$ defined in (3.9) and

$$(4.21) \quad E_{\sigma_\gamma} := \left(2 \int_0^T ((1-\ell)\varrho(\bar{U}, u) + \ell\sigma_\gamma(\bar{U}, u, \underline{U})) dt \right)^{\frac{1}{2}}.$$

THEOREM 4.7 *Let u be the strong solution of (2.3) and $\{U_n\}_{n=0}^N$ be the solution of (3.5) with $U_0 \in D(\mathfrak{F})$. If \mathfrak{F} is γ^2 -angle-bounded, then*

$$(4.22) \quad E_\gamma \leq \left(|u_0 - U_0|^2 + \gamma^2 \sum_{n=1}^N \tau_n^2 \mathcal{D}_n \right)^{\frac{1}{2}} + \|f - \bar{F}\|_{L^1(0,T;\mathcal{H})}.$$

Moreover, Theorem 3.18 holds, i.e.,

$$\sum_{n=1}^N \tau_n^2 \mathcal{D}_n \leq \tau^2 \left(\frac{1}{\sqrt{2}} |\delta U_0| + \text{Var} \bar{F} \right)^2,$$

and if $U_0 := u_0$ and $F_n := f_+(t_n)$ with $f \in \mathbf{BV}(0, T; \mathcal{H})$, we have

$$(4.23) \quad E_\gamma \leq \tau \left(\frac{\gamma}{\sqrt{2}} |(f_+(0) - \mathfrak{F}(u_0))^\circ| + (1 + \gamma) \text{Var} f \right).$$

PROOF: In view of (4.14) and the fact that \mathfrak{F} is γ^2 -angle-bounded, we have

$$\langle \mathfrak{F}(\bar{U}) - \mathfrak{F}(u), u - \underline{U} \rangle + \sigma_\gamma(\bar{U}, u, \underline{U}) \leq \gamma^2 \langle \mathfrak{F}(\bar{U}) - \mathfrak{F}(\underline{U}), \bar{U} - \underline{U} \rangle.$$

On using (2.44), from (4.1) we deduce

$$(4.24) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} |u - U|^2 + (1-\ell)\varrho(\bar{U}, u) + \ell\sigma_\gamma(\bar{U}, u, \underline{U}) \\ & \leq |f - \bar{F}| |u - U| + \ell\gamma^2 \langle \mathfrak{F}(\bar{U}) - \mathfrak{F}(\underline{U}), \bar{U} - \underline{U} \rangle \\ & = |f - \bar{F}| |u - U| + \gamma^2 \ell \bar{\tau} \bar{\mathcal{D}}. \end{aligned}$$

Applying Lemma 3.7 with

$$\begin{aligned} a &:= |u - U|, \quad b^2 := 2((1-\ell)\varrho(\bar{U}, u) + \ell\sigma_\gamma(\bar{U}, u, \underline{U})), \\ c^2 &:= \gamma^2 2\ell \bar{\tau} \bar{\mathcal{D}}, \quad d := |f - \bar{F}|, \quad \lambda := 0, \end{aligned}$$

we prove (4.22). Finally, (4.23) follows from Theorem 3.18 as in Theorem 3.20. \square

Remark 4.8. The a posteriori estimate (4.22) is also valid for weak solutions (see Remark 3.3).

We conclude this section by showing that if \mathfrak{F} is a γ^2 -angle-bounded operator satisfying a finer bound like (4.18), then we can obtain a more precise result. So we assume that

$$(4.25) \quad \exists \eta \geq \gamma: \sigma_\eta(v, w, z) \geq \frac{1}{2} \max([v - w]^2, [w - z]^2), \quad \forall v, w, z \in D(\mathfrak{F}).$$

Choosing $z = v$ implies $\varrho(w, v) \geq [v - w]^2/2$ (compare with the usual 2-coercivity (2.54)). It follows from (4.21) that

$$(4.26) \quad E_{[\cdot]}^2 := \max \left(\int_0^T [u(t) - U(t)]^2 dt, \int_0^T [u(t) - \bar{U}(t)]^2 dt \right) \leq E_{\sigma_\eta}^2.$$

In fact, (4.25), (2.54), and the convexity of $[\cdot]$ coupled with (3.1) yield

$$(4.27) \quad \begin{aligned} & 2(1 - \ell)\varrho(\bar{U}, u) + 2\ell\sigma_\eta(\bar{U}, u, \underline{U}) \\ & \geq (1 - \ell)[u - \bar{U}]^2 + \ell \max([u - \bar{U}]^2, [u - \underline{U}]^2) \\ & = \max([u - \bar{U}]^2, (1 - \ell)[u - \bar{U}]^2 + \ell[u - \underline{U}]^2) \\ & \geq \max([u - \bar{U}]^2, [u - U]^2). \end{aligned}$$

COROLLARY 4.9 *Let u be a weak solution of (2.3) and $\{U_n\}_{n=0}^N$ be the solution of (3.5) with $U_0 \in D(\mathfrak{F})$. If \mathfrak{F} is γ^2 -angle-bounded and coercive in the sense of (4.25), then the error $E_{[\cdot]}$ defined in (4.26) satisfies*

$$(4.28) \quad E_{[\cdot]} \leq \left(|u_0 - U_0|^2 + \eta^2 \sum_{n=1}^N \tau_n^2 \mathcal{D}_n \right)^{\frac{1}{2}} + \|f - \bar{F}\|_{L^1(0, T; \mathcal{H})}.$$

PROOF: Since $\eta \geq \gamma$ and so \mathfrak{F} is η^2 -angle-bounded, we use (4.26) together with (4.22) to prove (4.28) for strong solutions. In view of the lower semicontinuity of the seminorm $[\cdot]$, (4.28) can also be extended to weak solutions. \square

COROLLARY 4.10 *Let \mathfrak{A} be a linear, closed, and densely defined operator satisfying (4.5), and let $[\cdot]$ be the Hilbert seminorm (4.15). If u is the weak solution of (2.3) and $\{U_n\}_{n=0}^N$ is the solution of (3.5) with $U_0 \in D(\mathfrak{A})$, then*

$$(4.29) \quad E_{\mathcal{H}} \leq \left(|u_0 - U_0|^2 + \gamma^2 \sum_{n=1}^N \tau_n [U_n - U_{n-1}]^2 \right)^{\frac{1}{2}} + \|f - \bar{F}\|_{L^1(0, T; \mathcal{H})},$$

$$(4.30) \quad E_{[\cdot]} \leq \left(|u_0 - U_0|^2 + 2\gamma^2 \sum_{n=1}^N \tau_n [U_n - U_{n-1}]^2 \right)^{\frac{1}{2}} + \|f - \bar{F}\|_{L^1(0, T; \mathcal{H})}.$$

Moreover, Theorem 3.18 holds, i.e.,

$$(4.31) \quad \sum_{n=1}^N \tau_n [U_n - U_{n-1}]^2 \leq \tau^2 \left(\frac{1}{\sqrt{2}} |\delta U_0| + \text{Var } \bar{F} \right)^2,$$

and if $U_0 := u_0$ and $F_n := f_+(t_n)$ with $f \in \text{BV}(0, T; \mathcal{H})$, then (4.23) holds and

$$(4.32) \quad \begin{aligned} E_{\mathcal{H}} &\leq \tau \left(\frac{\gamma}{\sqrt{2}} |f_+(0) - \mathfrak{A}u_0| + (1 + \gamma) \text{Var } f \right), \\ E_{[\cdot]} &\leq \tau (\gamma |f_+(0) - \mathfrak{A}u_0| + (1 + \sqrt{2}\gamma) \text{Var } f). \end{aligned}$$

PROOF: Since \mathfrak{A} is linear and single-valued, (3.21) and (4.15) yield

$$\mathcal{D}_n := \tau_n^{-1} \langle \mathfrak{A}U_n - \mathfrak{A}U_{n-1}, U_n - U_{n-1} \rangle = \tau_n^{-1} [U_n - U_{n-1}]^2.$$

We next apply Theorem 4.7 to deduce (4.29), and Lemma 4.3 (with $L = 2\gamma$ and $\eta = \sqrt{2}\gamma$) together with Corollary 4.9 to obtain (4.30). The remaining estimate (4.32) follows from (4.31) as in Theorem 3.20. \square

We remark that (4.30) requires the bigger coefficient $2\gamma^2$ instead of γ^2 because we also employ the coercivity (4.25) of \mathfrak{A} , which is not required to estimate the maximum of $|u(t) - U(t)|$ in (4.29).

Remark 4.11. In the case of a symmetric positive semidefinite linear operator \mathfrak{F} , for which $\gamma = \frac{1}{2}$, we could also apply the results of Section 3, because \mathfrak{F} is the subdifferential of a quadratic positive functional; this fact allows us to employ the finer estimators \mathcal{E}_n and the intermediate results of Theorems 3.15 and 3.16. However, since we can apply Corollary 4.10 with $\gamma = \frac{1}{2}$, both (4.29) and (4.30) give a slightly better result than (3.23), because $2\sigma(w; v) \geq [v - w]^2$.

4.4 Lipschitz Perturbations

Now we show how to deal with Lipschitz perturbations of an angle-bounded operator (in particular, of a subdifferential one).

We assume the existence of a Lipschitz operator $\mathfrak{G} : D(\mathfrak{F}) \rightarrow \mathcal{H}$, i.e.,

$$(4.33) \quad \exists L \geq 0: |\mathfrak{G}(v) - \mathfrak{G}(w)| \leq L|v - w|, \quad \forall v, w \in D(\mathfrak{F}),$$

such that the perturbed operator in $D(\mathfrak{F})$

$$\tilde{\mathfrak{F}} := \mathfrak{F} + \mathfrak{G} \quad \text{is maximal monotone and } \gamma^2\text{-angle-bounded.}$$

By property (iii) of the basic existence results of Section 2.1, the Cauchy problem (2.3) is well-posed again, and the discrete scheme (3.5) can be recursively solved if every step size τ_n satisfies $L\tau_n < 1$. If we set

$$g(t) := f(t) + \mathfrak{G}(u(t)), \quad G_n := F_n + \mathfrak{G}(U_n),$$

then (2.3), (3.5), and (4.1) can be rewritten in terms of the new operator $\tilde{\mathfrak{F}}$ with g and \bar{G} instead of f and \bar{F} , respectively. Moreover, if $U_0 \in D(\mathfrak{F})$, then the estimator \mathcal{D}_n defined in (3.20) becomes

$$(4.34) \quad \tilde{\mathcal{D}}_n := \tau_n \langle \delta F_n + \delta \mathfrak{G}(U_n) - \delta^2 U_n, \delta U_n \rangle, \quad \forall 1 \leq n \leq N,$$

with δU_0 defined as in (3.7), and satisfies $0 \leq \tau_n^{-1} \tilde{\varrho}(U_n, U_{n-1}) \leq \tilde{\mathcal{D}}_n$, where $\tilde{\varrho}$ is the coercivity function associated to $\tilde{\mathfrak{F}}$.

We need a discrete version of Lemma 3.7, a variant of Lemma 3.17.

LEMMA 4.12 *Let $\{a_n\}_{n=0}^N$ and $\{b_n, c_n, d_n\}_{n=1}^N$ be nonnegative numbers, let $\{-1 < \lambda_n \leq 0\}_{n=1}^N$ be given coefficients, and set*

$$\lambda := \min_{1 \leq n \leq N} \lambda_n, \quad s_n := \sum_{k=1}^n \lambda_k, \quad \bar{s}_n := \frac{s_n + s_{n-1}}{2}, \quad s_0 := 0.$$

If

$$(4.35) \quad 2a_n(a_n - a_{n-1}) + b_n^2 + 2\lambda_n a_n^2 \leq c_n^2 + 2a_n d_n, \quad \forall 1 \leq n \leq N,$$

then we have

$$(4.36) \quad \max \left(\max_{1 \leq n \leq N} e^{s_n/(1+\lambda)} a_n, \left(\sum_{n=1}^N e^{2\bar{s}_n/(1+\lambda)} b_n^2 \right)^{\frac{1}{2}} \right) \leq \left(a_0^2 + \sum_{n=1}^N e^{2\bar{s}_n} c_n^2 \right)^{\frac{1}{2}} + \sqrt{2} \sum_{n=1}^N e^{s_{n-1}} d_n.$$

PROOF: We introduce the positive quantities

$$p_0 := 1, \quad p_n := \prod_{k=1}^n (1 + \lambda_k),$$

$$\tilde{a}_n := p_n a_n, \quad \tilde{b}_n^2 := p_n p_{n-1} b_n^2, \quad \tilde{c}_n^2 := p_n p_{n-1} c_n^2, \quad \tilde{d}_n := p_{n-1} d_n,$$

and note that p_n is a discrete integrating factor. In fact, multiplying (4.35) by $p_n p_{n-1}$ converts (4.35) into

$$2\tilde{a}_n(\tilde{a}_n - \tilde{a}_{n-1}) + \tilde{b}_n^2 \leq \tilde{c}_n^2 + 2\tilde{a}_n \tilde{d}_n, \quad \forall 1 \leq n \leq N.$$

Then (4.36) readily follows from (3.33) and (3.34) upon realizing that

$$e^{\lambda_k/(1+\lambda)} \leq 1 + \lambda_k \leq e^{\lambda_k}$$

yields

$$\tilde{a}_n \geq e^{s_n/(1+\lambda)} a_n, \quad \tilde{b}_n \geq e^{\bar{s}_n/(1+\lambda)} b_n, \quad \tilde{c}_n \leq e^{\bar{s}_n} c_n, \quad \tilde{d}_n \leq e^{s_{n-1}} d_n.$$

□

THEOREM 4.13 *Let u be the weak solution of (2.3), let $\{U_n\}_{n=0}^N$ be the solution of (3.5) with $U_0 \in D(\mathfrak{F})$, and let $L\tau_n < 1$ for all $1 \leq n \leq N$. If $\tilde{\mathfrak{F}}$ is maximal monotone and γ^2 -angle-bounded, then*

$$(4.37) \quad E_{\mathcal{H}} \leq e^{LT} \left(\left(|u_0 - U_0|^2 + \gamma^2 \sum_{n=1}^N \tau_n^2 \tilde{D}_n \right)^{\frac{1}{2}} + \frac{L}{2} \sum_{n=1}^N \tau_n^2 |\delta U_n| + \|f - \bar{F}\|_{L^1(0,T;\mathcal{H})} \right).$$

Moreover, if $U_0 := u_0$ and $F_n := f_+(t_n)$, with $f \in \text{BV}(0, T; \mathcal{H})$, then

$$(4.38) \quad E_{\mathcal{H}} \leq \tau e^{LT} \left(\frac{1}{\sqrt{2}} \tilde{C} |(f_+(0) - \mathfrak{F}(u_0))^\circ| + (1 + \tilde{C}) \text{Var} f \right),$$

where $C_{\tau, T} := e^{L\tau(1+LT)/(1-L\tau)}$ and $\tilde{C} := C_{\tau, T}(\gamma + LT/\sqrt{2})$.

PROOF: We argue as in the proof of Theorem 4.7 but omit the coercivity terms. Since $\tilde{\mathfrak{F}}$ is γ^2 -angle-bounded, we get from (4.1), (4.33), and (3.14)

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |u - U|^2 &\leq \langle \tilde{\mathfrak{F}}(\bar{U}) - \tilde{\mathfrak{F}}(u), u - U \rangle + \langle g - \bar{G}, u - U \rangle \\ &\leq \ell \gamma^2 \langle \tilde{\mathfrak{F}}(\bar{U}) - \tilde{\mathfrak{F}}(\underline{U}), \bar{U} - \underline{U} \rangle + \langle f - \bar{F}, u - U \rangle \\ &\quad + \langle \mathfrak{G}(U) - \mathfrak{G}(\bar{U}), u - U \rangle + \langle \mathfrak{G}(u) - \mathfrak{G}(U), u - U \rangle \\ &\leq \gamma^2 \ell \bar{\tau} \tilde{D} + (|f - \bar{F}| + L \ell \bar{\tau} |U'|) |u - U| + L |u - U|^2. \end{aligned}$$

Applying Lemma 3.7 with

$$a := |u - U|, \quad b := 0, \quad c^2 := \gamma^2 2 \ell \bar{\tau} \tilde{D}, \quad d := |f - \bar{F}| + L \ell \bar{\tau} |U'|, \quad \lambda := -L,$$

we deduce the estimate (4.37) or, more precisely,

$$(4.39) \quad \begin{aligned} \max_{t \in [0, T]} e^{-Lt} |u(t) - U(t)| &\leq \left(|u_0 - U_0|^2 + \gamma^2 \sum_{n=1}^N \tau_n^2 e^{-2L t_{n-1}} \tilde{D}_n \right)^{\frac{1}{2}} \\ &\quad + \int_0^T e^{-Lt} |f - \bar{F}| dt + \frac{L}{2} \sum_{n=1}^N \tau_n^2 e^{-L t_{n-1}} |\delta U_n|. \end{aligned}$$

We proceed as in Theorems 3.18 and 3.20 to prove (4.38). In view of (4.33) and (4.34), we get

$$|\delta U_n| (|\delta U_n| - |\delta U_{n-1}|) + \tilde{D}_n - L \tau_n |\delta U_n|^2 \leq \tau_n |\delta F_n| |\delta U_n|.$$

Applying Lemma 4.12 with

$$a_n := |\delta U_n|, \quad b_n^2 := 2 \tilde{D}_n, \quad c_n := 0, \quad d_n := |F_n - F_{n-1}|, \quad \lambda_n := -L \tau_n,$$

and using that $t_n/(1 - L\tau) \leq t_{n-1} + \tau(1 + LT)/(1 - L\tau)$, we readily deduce

$$(4.40) \quad \begin{aligned} \max \left(\max_{1 \leq n \leq N} e^{-L t_{n-1}} |\delta U_n|, \left(2 \sum_{n=1}^N e^{-2L t_{n-1}} \tilde{D}_n \right)^{\frac{1}{2}} \right) &\leq \\ C_{\tau, T} \left(|\delta U_0| + \sqrt{2} \sum_{n=1}^N e^{-L t_{n-1}} |F_n - F_{n-1}| \right). \end{aligned}$$

The assertion (4.38) follows from (4.39) and (4.40). \square

4.5 Hilbert Triplets

Let \mathcal{V} be a Hilbert space with norm $\|\cdot\|$ continuously and densely embedded into \mathcal{H} , and let us adopt the usual convention of identifying \mathcal{H} with its dual \mathcal{H}^* . We can thus embed \mathcal{H} into \mathcal{V}^* , this inclusion also being dense, and consider the Hilbert triplet

$$\mathcal{V} \subset \mathcal{H} \equiv \mathcal{H}^* \subset \mathcal{V}^*.$$

Since the duality pairing between \mathcal{V} and \mathcal{V}^* is the (unique) extension by continuity of the \mathcal{H} -scalar product, we will use the same symbol for both bilinear forms. As in Examples 2.2 and 4.3, we are given a *positive, continuous, and weakly coercive* bilinear form $\mathfrak{a} : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$ satisfying (2.14) and (4.8), namely,

$$(4.41a) \quad [v]^2 := \mathfrak{a}(v, v) \geq 0,$$

$$(4.41b) \quad v \mapsto ([v]^2 + |v|^2)^{\frac{1}{2}} \quad \text{is an equivalent norm of } \mathcal{V},$$

$$(4.41c) \quad \exists \mu \geq 0 : \mathfrak{a}_a(v, w) := \frac{1}{2}(\mathfrak{a}(v, w) - \mathfrak{a}(w, v)) \leq \mu[v][w], \quad \forall v, w \in \mathcal{V}.$$

As in Example 2.3, we have a proper convex functional $\psi : \mathcal{V} \rightarrow (-\infty, +\infty]$, with domain $D(\psi)$, lower semicontinuous with respect to the (weak or strong) topology of \mathcal{V} .

Remark 4.14. We recall that if \mathfrak{a} is *continuous and strongly coercive*, i.e.,

$$(4.42) \quad \exists M, \alpha > 0 : \mathfrak{a}(v, w) \leq M\|v\|\|w\|, \quad [v]^2 \geq \alpha\|v\|^2, \quad \forall v, w \in \mathcal{V},$$

then (4.41a), (4.41b), and (4.41c) surely hold with $\mu := M/\alpha$; (4.42) is indeed the usual assumption in the stability and error analyses of variational inequalities. However, conditions (4.41a), (4.41b), and (4.41c), which are related to the strong sector condition (4.5) or (4.7) for the associated operator \mathfrak{A} defined in (2.15), lead to finer error estimates provided a precise bound for μ is known (e.g., $\mu = 0$ if \mathfrak{a} is symmetric).

Given a function $f : (0, T) \rightarrow \mathcal{V}^*$ and an initial datum $u_0 \in \overline{D(\psi)}^{\mathcal{H}}$, we study the evolution variational inequality (2.24), namely,

$$(4.43) \quad \langle u' - f, u - v \rangle + \mathfrak{a}(u, u - v) + \psi(u) - \psi(v) \leq 0, \quad \forall v \in D(\psi),$$

with the initial condition $u(0) = u_0$, and its discretization via the scheme

$$(4.44) \quad \langle \delta U_n - F_n, U_n - v \rangle + \mathfrak{a}(U_n, U_n - v) + \psi(U_n) - \psi(v) \leq 0, \quad \forall v \in D(\psi).$$

The latter depends on the choice of $U_0 \in \overline{D(\psi)}^{\mathcal{H}}$ and $F_n \in \mathcal{V}^*$ for $1 \leq n \leq N$.

It is convenient to recall the definitions (4.19) and (4.20) of $\mathfrak{F}_{\mathcal{V}} : \mathcal{V} \rightarrow 2^{\mathcal{V}^*}$ and its restriction $\mathfrak{F} : \mathcal{H} \rightarrow 2^{\mathcal{H}}$, namely,

$$(4.45) \quad \begin{aligned} w_* \in \mathfrak{F}_{\mathcal{V}}(w) &\Leftrightarrow \mathfrak{a}(w, w - v) + \psi(w) \leq \psi(v) + \langle w_*, w - v \rangle, \quad \forall v \in D(\psi), \\ \mathfrak{F}(w) &:= \mathfrak{F}_{\mathcal{V}}(w) \cap \mathcal{H}, \quad D(\mathfrak{F}) \subset D(\mathfrak{F}_{\mathcal{V}}) \subset D(\psi) \subset \mathcal{V}. \end{aligned}$$

The following result provides a crucial link with the previous theory:

LEMMA 4.15 *Let \mathfrak{a} be a bilinear form satisfying (4.41), let \mathfrak{A} be the associated linear operator in (2.15), and let \mathfrak{F} be the multivalued operator defined by (4.45). Then \mathfrak{F} is maximal monotone, γ^2 -angle-bounded, and η^2 -coercive in the sense of (4.25), with*

$$\gamma^2 := \max \left(1, \frac{1}{4}(1 + \mu^2) \right), \quad \eta^2 := \max \left(1, \frac{1}{2}(1 + \mu^2) \right).$$

In particular, if \mathfrak{a} is symmetric, i.e., $\mu = 0$ in (4.41c), then $\gamma = \eta := 1$ and \mathfrak{F} is the subdifferential of the functional

$$(4.46) \quad \phi(w) := \frac{1}{2}\mathfrak{a}(w, w) + \psi(w), \quad D(\phi) := D(\psi) \subset \mathcal{V}.$$

If \mathfrak{A} and ψ satisfy the compatibility condition (2.26), then

$$(4.47) \quad \mathfrak{F} := \mathfrak{A} + \partial\psi, \quad D(\mathfrak{F}) := D(\mathfrak{A}) \cap D(\partial\psi).$$

Finally, if

$$(4.48) \quad u_0 \in D(\mathfrak{F}), \quad f \in \text{BV}(0, T; \mathcal{H}), \quad F_n \in \mathcal{H}, \quad \forall 1 \leq n \leq N,$$

then u (respectively, $\{U_n\}_{n=0}^N$) is a solution of (4.43) (respectively, (4.44)) if and only if it is a strong solution of (2.3) (respectively, (3.5)).

PROOF: Since (4.41c) is (4.7) in disguise, it readily follows from the discussion in Example 4.3 that the linear operator \mathfrak{A} defined in (2.15) is γ_0^2 -angle-bounded with $\gamma_0^2 := (1 + \mu^2)/4$. Moreover, in view of Remark 4.4, (4.14), and (4.41a), we deduce that for $\eta_0^2 := (1 + \mu^2)/2$

$$\eta_0^2 \mathfrak{a}(v - z, v - z) - \mathfrak{a}(v - w, w - z) \geq \frac{1}{2} \max([v - w]^2, [w - z]^2).$$

According to Remark 4.5, the operator \mathfrak{F} defined in (4.45) is γ^2 -angle-bounded and η^2 -coercive with the asserted γ and η . In order to prove that \mathfrak{F} is maximal monotone, it only remains to verify (see (2.1)) that for every $z \in \mathcal{H}$ there exists $w \in D(\psi)$ such that $z - w \in \varepsilon \mathfrak{F}(w)$, or, equivalently, by (4.45),

$$\varepsilon \mathfrak{a}(w, w - v) + \langle w, w - v \rangle + \varepsilon \psi(w) - \varepsilon \psi(v) \leq \langle z, w - v \rangle, \quad \forall v \in D(\psi).$$

This is a consequence of the celebrated Lions-Stampacchia theorem [43].

If \mathfrak{a} is symmetric, then from (4.45) we infer that $w_* \in \mathfrak{F}(w)$ implies

$$\begin{aligned} \langle w_*, v - w \rangle &\leq \mathfrak{a}(w, v - w) + \psi(v) - \psi(w) \\ &\leq \left(\frac{1}{2} \mathfrak{a}(v, v) + \psi(v) \right) - \left(\frac{1}{2} \mathfrak{a}(w, w) + \psi(w) \right) = \phi(v) - \phi(w). \end{aligned}$$

Therefore w_* belongs also to the subdifferential of the functional ϕ defined by (4.46). Since \mathfrak{F} is maximal, we conclude that \mathfrak{F} coincides with $\partial\phi$.

Finally, (4.47) is the same as (2.27), and the equivalence of the differential equations follows from straightforward calculations. \square

In the framework of variational inequalities, it is sometimes useful to handle \mathcal{V}^* -valued source terms, so it is natural to consider f given by the sum of two different contributions. On the other hand, since α may not be strongly coercive with respect to \mathcal{V} , thereby making $[\cdot]$ only a seminorm on \mathcal{V} , it is convenient to introduce the domain of the conjugate norm $[\cdot]_*$, which is the Hilbert space \mathcal{V}_0^* defined by

$$\mathcal{V}_0^* := \left\{ v \in \mathcal{V}^* : [v]_* := \sup_{z \in \mathcal{V} : [z] \leq 1} \langle v, z \rangle < \infty \right\}, \quad \|v\|_{\mathcal{V}_0^*} := [v]_*.$$

Remark 4.16. If (4.42) holds, then

$$\mathcal{V}_0^* = \mathcal{V}^*, \quad [w]_* \leq \frac{1}{\sqrt{\alpha}} \|w\|_{\mathcal{V}^*}, \quad \forall w \in \mathcal{V}^*.$$

We now consider the splitting

$$(4.49) \quad f := g + h, \quad g \in L^2(0, T; \mathcal{V}_0^*), \quad h \in L^1(0, T; \mathcal{H}),$$

along with its discrete counterpart

$$(4.50) \quad F_n := G_n + H_n, \quad G_n \in \mathcal{V}_0^*, \quad H_n \in \mathcal{H}, \quad 0 \leq n \leq N.$$

If $U_0 \in D(\mathfrak{F}_{\mathcal{V}})$ and F_0 satisfy the *initial compatibility condition*

$$(4.51) \quad (F_0 - \mathfrak{F}_{\mathcal{V}}(U_0)) \cap \mathcal{H} \neq \emptyset,$$

then we can define $\delta U_0 := (F_0 - \mathfrak{F}_{\mathcal{V}}(U_0))^\circ$ and the discrete estimators \mathcal{D}_n for $1 \leq n \leq N$ as in (3.20). If α is *symmetric*, then we can also introduce the discrete estimators \mathcal{E}_n defined in (3.8) in terms of the functional ϕ of (4.46).

THEOREM 4.17 *Let u be the weak solution of (4.43) with $u_0 \in \overline{D(\psi)}^{\mathcal{H}}$ and f satisfying (4.49). Let $\{U_n\}_{n=0}^N$ be the discrete solution of (4.44) with $U_0 \in D(\mathfrak{F}_{\mathcal{V}})$ and F_n satisfying (4.50) and (4.51). If \mathcal{D}_n is defined in (3.20) for $1 \leq n \leq N$ and $\eta := \max(1, (1 + \mu^2)/2)$, then the errors $E_{\mathcal{H}}$ and $E_{[\cdot]}$ defined in (3.9) and (4.26) satisfy*

$$(4.52) \quad \max \left(E_{\mathcal{H}}, \frac{1}{\sqrt{2}} E_{[\cdot]} \right) \leq \left(|u_0 - U_0|^2 + 2 \|g - \bar{G}\|_{L^2(0, T; \mathcal{V}_0^*)}^2 + \eta^2 \sum_{n=1}^N \tau_n^2 \mathcal{D}_n \right)^{\frac{1}{2}} + \|h - \bar{H}\|_{L^1(0, T; \mathcal{H})}.$$

If α is symmetric, then the previous formula holds with $\eta := 1$ and \mathcal{E}_n instead of \mathcal{D}_n , where \mathcal{E}_n is defined by (3.8). Finally, if $U_0 := u_0 \in D(\mathfrak{F}_{\mathcal{V}})$, $(f_+(0) - \mathfrak{F}_{\mathcal{V}}(u_0)) \cap \mathcal{H} \neq \emptyset$, and $G_n := g(t_n)$, $H_n := h(t_n^+)$, with $g \in H^1(0, T; \mathcal{V}_0^)$, $h \in \text{BV}(0, T; \mathcal{H})$, then u and $\{U_n\}_{n=0}^N$ satisfy the following uniform estimate:*

$$(4.53) \quad \max \left(E_{\mathcal{H}}, \frac{1}{\sqrt{2}} E_{[\cdot]} \right) \leq \tau(\eta) (f_+(0) - \mathfrak{F}_{\mathcal{V}}(u_0))^\circ + (1 + \eta) \|g'\|_{L^2(0, T; \mathcal{V}_0^*)} + (1 + \sqrt{2}\eta) \text{Var } h.$$

PROOF: We argue as in Theorem 4.7. First we assume (4.48), so that u (respectively, U_n) is a strong solution of (2.3) (respectively, (3.5)), along with $U_0 \in D(\mathfrak{F})$. In view of Lemma 4.15 and (4.27), we thus write (4.24) as

$$\frac{d}{dt}|u-U|^2 + \max([u-\bar{U}]^2, [u-U]^2) \leq \eta^2 2\ell\bar{\tau}\bar{D} + 2\langle f - \bar{F}, u-U \rangle,$$

but need a novel estimate of the source term f to control a possibly \mathcal{V}_0^* -valued component g of f . We substitute f and \bar{F} by their decompositions (4.49) and (4.50), and use (4.16) together with Cauchy inequality to get

$$\begin{aligned} \frac{d}{dt}|u-U|^2 + \frac{1}{2} \max([u-\bar{U}]^2, [u-U]^2) \leq \\ \eta^2 2\ell\bar{\tau}\bar{D} + 2|h-\bar{H}||u-U| + 2[g-\bar{G}]_*^2. \end{aligned}$$

Consequently, to obtain (4.52) we apply Lemma 3.7 with $\lambda := 0$ and

$$\begin{aligned} a &:= |u-U|, \quad b^2 := \frac{1}{2} \max([u-\bar{U}]^2, [u-U]^2), \\ c^2 &:= \eta^2 2\ell\bar{\tau}\bar{D} + 2[g-\bar{G}]_*^2, \quad d := |h-\bar{H}|. \end{aligned}$$

By a standard approximation argument, we note that (4.52) also holds if u is a weak solution and g and G_n are really \mathcal{V}_0^* -valued.

When \mathfrak{a} is symmetric, Lemma 4.15 shows that \mathfrak{F} is a subdifferential, so that we can repeat the previous calculations starting from (3.15) instead of (4.24).

Finally, to demonstrate the rate of convergence (4.53), we notice that

$$\mathcal{D}_n \geq \tau_n^{-1} \varrho(U_n, U_{n-1}) \geq \tau_n^{-1} \langle \mathfrak{A}(U_n - U_{n-1}), U_n - U_{n-1} \rangle = \tau_n [\delta U_n]^2,$$

and thus use (4.16) and the Cauchy inequality to rewrite (3.38) as

$$\begin{aligned} 2|\delta U_n|(|\delta U_n| - |\delta U_{n-1}|) + \tau_n [\delta U_n]^2 + \mathcal{D}_n \leq \\ 2\tau_n |\delta H_n| |\delta U_n| + \tau_n [\delta G_n]_*^2 + \tau_n [\delta U_n]^2. \end{aligned}$$

We apply Lemma 3.17 with

$$a_n := |\delta U_n|, \quad b_n^2 := \mathcal{D}_n, \quad c_n^2 := \tau_n [\delta G_n]_*^2, \quad d_n := \tau_n |\delta H_n|,$$

thereby obtaining

$$\begin{aligned} \left(\sum_{n=1}^N \mathcal{D}_n \right)^{\frac{1}{2}} &\leq \left(|\delta U_0|^2 + \sum_{n=1}^N \tau_n [\delta G_n]_*^2 \right)^{\frac{1}{2}} + \sqrt{2} \operatorname{Var} \bar{H} \\ &\leq (|(f_+(0) - \mathfrak{F}_{\mathcal{V}}(u_0))^\circ|^2 + \|g'\|_{L^2(0,T;\mathcal{V}_0^*)}^2)^{\frac{1}{2}} + \sqrt{2} \operatorname{Var} h. \end{aligned}$$

Using the properties

$$\sqrt{2} \|g - \bar{G}\|_{L^2(0,T;\mathcal{V}_0^*)} \leq \tau \|g'\|_{L^2(0,T;\mathcal{V}_0^*)}, \quad \|h - \bar{H}\|_{L^1(0,T;\mathcal{H})} \leq \tau \operatorname{Var} h,$$

and the elementary inequality $(a^2 + b^2)^{1/2} \leq |a| + |b|$, we easily deduce the last assertion and thus conclude the proof. \square

5 Applications

We apply the abstract theory of Sections 3 and 4 to the examples of Sections 2.2 and 4.1 and obtain several novel and concrete error estimates in each case. Our chief purpose is to show the flexibility and range of applicability of our results. We do not claim completeness, nor do we state the results under minimal regularity of the data. We stick to the notation of previous sections, and further recall

- Ω is a Lipschitz-connected domain of \mathbb{R}^m , with outward unit normal \mathbf{n} ; $Q := \Omega \times (0, T)$ is the parabolic cylinder and $\Sigma := \partial\Omega \times (0, T)$ its lateral boundary;
- p_D^2 and p_N^2 are the first nonzero eigenvalues of the Laplacian in Ω with homogeneous Dirichlet and Neumann boundary conditions, respectively;
- the discrete partition of the time interval $[0, T]$ and the related piecewise constant and linear interpolants are defined at the beginning of Section 3.

5.1 Second-Order Linear Parabolic Equations

We consider, as in Examples 2.2 and 4.3, the parabolic problem

$$(5.1a) \quad \partial_t u - \nu^2 \Delta u + \mathbf{b} \cdot \nabla u + cu = f \quad \text{in } Q,$$

$$(5.1b) \quad u(\cdot, 0) = u_0(\cdot) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Sigma,$$

with elliptic operator \mathfrak{A} defined on $D(\mathfrak{A}) := \{v \in H_0^1(\Omega) : \Delta v \in \mathcal{H} := L^2(\Omega)\}$

$$(5.2) \quad \mathfrak{A}w := -\nu^2 \Delta w + \mathbf{b} \cdot \nabla w + cw.$$

The data $\nu^2 > 0$ constant, $\mathbf{b} \in W^{1,\infty}(\Omega)$, and $c \in L^\infty(\Omega)$ satisfy

$$(5.3) \quad |\mathbf{b}(x)| \leq b_0, \quad d(x) := -\frac{1}{2} \operatorname{div} \mathbf{b}(x) + c(x) \geq d_0^2 \geq 0, \quad \text{a.e. } x \in \Omega.$$

If $U_0 \in L^2(\Omega)$ and $F_n \in L^2(\Omega)$, the time discretization of (5.1) on the grid (1.2) is equivalent to seeking $U_n \in H_0^1(\Omega)$ for $1 \leq n \leq N$ such that

$$(5.4) \quad \tau_n^{-1}(U_n - U_{n-1}) - \nu^2 \Delta U_n + \mathbf{b} \cdot \nabla U_n + cU_n = F_n \quad \text{in } \Omega.$$

LEMMA 5.1 *The operator \mathfrak{A} in (5.2), with coefficients satisfying (5.3), is γ^2 -angle-bounded with $\gamma^2 := (1 + b_0^2/\nu^2(d_0^2 + \nu^2 p_D^2))/4$.*

PROOF: First of all, we observe that \mathfrak{A} satisfies, for all $w \in D(\mathfrak{A})$,

$$\langle \mathfrak{A}w, w \rangle = \int_{\Omega} (\nu^2 |\nabla w|^2 + d(x)|w|^2) dx \geq (\nu^2 p_D^2 + d_0^2) \|w\|_{L^2(\Omega)}^2,$$

because p_D is the best constant in the Poincaré inequality with homogeneous Dirichlet boundary condition: $p_D \|w\|_{L^2(\Omega)} \leq \|\nabla w\|_{L^2(\Omega)}$ for all $w \in H_0^1(\Omega)$. Moreover, the antisymmetric part of the bilinear form associated to \mathfrak{A} satisfies

$$\begin{aligned} |\langle \mathfrak{A}v, w \rangle - \langle \mathfrak{A}w, v \rangle| &\leq b_0 (\|\nabla v\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)} + \|\nabla w\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}) \\ &\leq \frac{2b_0}{\nu \sqrt{d_0^2 + \nu^2 p_D^2}} \langle \mathfrak{A}v, v \rangle^{\frac{1}{2}} \langle \mathfrak{A}w, w \rangle^{\frac{1}{2}}, \quad \forall v, w \in D(\mathfrak{A}). \end{aligned}$$

Therefore, (4.7) is valid with $\mu := b_0/\nu\sqrt{d_0^2 + \nu^2 p_D^2}$ and \mathfrak{A} is γ^2 -angle-valued with $\gamma^2 := (1 + \mu^2)/4$ according to [10, proposition 11]. \square

THEOREM 5.2 *Let u be the weak solution of (5.1) corresponding to data*

$$u_0 \in L^2(\Omega), \quad f \in L^1(0, T; L^2(\Omega)),$$

and (5.3). Let $\{U_n\}_{n=0}^N$ be the discrete solution of (5.4) with $U_0 \in H_0^1(\Omega)$ and $F_n \in L^2(\Omega)$, and let \bar{U} and U be its piecewise constant and linear interpolants, respectively. Then the estimator \mathcal{D}_n defined in (3.20) is given by

$$\mathcal{D}_n = \frac{1}{\tau_n} \int_{\Omega} (\nu^2 |\nabla U_n - \nabla U_{n-1}|^2 + d(x) |U_n - U_{n-1}|^2) dx,$$

and the following a posteriori error estimates hold:

$$(5.5) \quad \max(\|u - U\|_{L^\infty(0, T; L^2(\Omega))}, \nu \|\nabla u - \nabla U\|_{L^2(Q)}, \nu \|\nabla u - \nabla \bar{U}\|_{L^2(Q)}) \leq \left(\|u_0 - U_0\|_{L^2(\Omega)}^2 + 2\gamma^2 \sum_{n=1}^N \tau_n^2 \mathcal{D}_n \right)^{\frac{1}{2}} + \int_0^T \|f - \bar{F}\|_{L^2(\Omega)} dt.$$

Moreover, if

$$U_0 := u_0 \in H^2(\Omega) \cap H_0^1(\Omega), \quad \partial_t f \in L^1(0, T; L^2(\Omega)), \quad F_n(\cdot) := f(\cdot, t_n),$$

then the right-hand side of (5.5) can be bounded uniformly by

$$(5.6) \quad \tau \left(\gamma \|f(0) - \mathfrak{A}u_0\|_{L^2(\Omega)} + (1 + \sqrt{2}\gamma) \int_0^T \|\partial_t f\|_{L^2(\Omega)} dt \right).$$

PROOF: Since \mathfrak{A} is γ^2 -angle-bounded in view of Lemma 5.1, we apply the theory of Section 4, in particular Corollary 4.10, to conclude (5.5) and (5.6) even if (4.29) and (4.30) require $U_0 \in D(\mathfrak{A})$. In fact, since \mathcal{D}_n is continuous with respect to the norm $H_0^1(\Omega)$, by a simple approximation technique we may extend those estimates of Corollary 4.10 to $U_0 \in H_0^1(\Omega)$. \square

Remark 5.3. For a convection-dominated diffusion problem, the constant γ in (5.5) and (5.6) is proportional to the advection/diffusion ratio

$$\gamma = O(b_0/\nu) \text{ if } d_0 > 0, \quad \gamma = O(b_0/\nu^2) \text{ if } d_0 = 0.$$

There is, however, no exponential factor depending on b as long as $d_0 \geq 0$.

Remark 5.4. Since the linear operator \mathfrak{A} in (5.2) generates an analytic semigroup, optimal a posteriori error estimates in $L^\infty(0, T; L^2(\Omega))$ are derived in [30] using the smoothing effect of the dual problem. Our a posteriori estimator $\sum_{n=1}^N \tau_n^2 \mathcal{D}_n$ is different from that in [30], namely, $\max_{1 \leq n \leq N} \|U_n - U_{n-1}\|_{L^2(\Omega)}$, but both exhibit

optimal order and optimal regularity. To see this, differentiate (5.1a) and multiply by $\partial_t u$. In light of (5.3), we obtain

$$\begin{aligned} \frac{d}{dt} \|\partial_t u(t)\|_{L^2(\Omega)}^2 + 2\nu \|\nabla \partial_t u(t)\|_{L^2(\Omega)}^2 + 2\|\sqrt{d} \partial_t u(t)\|_{L^2(\Omega)}^2 \leq \\ 2\|\partial_t u(t)\|_{L^2(\Omega)} \|\partial_t f(t)\|_{L^2(\Omega)}. \end{aligned}$$

Making use of Lemma 3.7, we easily deduce the regularity bounds

$$\begin{aligned} \max \left(\max_{t \in [0, T]} \|\partial_t u(t)\|_{L^2(\Omega)}^2, 2 \int_0^T (\nu \|\nabla \partial_t u(t)\|_{L^2(\Omega)}^2 + \|\sqrt{d} \partial_t u(t)\|_{L^2(\Omega)}^2) dt \right) \leq \\ \left(\|f(0) - \mathfrak{A}u_0\|_{L^2(\Omega)} + \int_0^T \|\partial_t f(t)\|_{L^2(\Omega)} dt \right)^2. \end{aligned}$$

This elucidates the equivalence of both estimators alluded to before. In addition, we derive error estimates in $L^2(0, T; H_0^1(\Omega))$ via coercivity of \mathfrak{A} , which are not present in [30].

5.2 Nonlinear Variational Boundary Conditions

We consider the following parabolic problem associated to the linear PDE (5.1a) with nonlinear boundary condition (see [7, section 4, example 6] for a similar problem):

$$(5.7a) \quad \partial_t u - \nu^2 \Delta u + \mathbf{b} \cdot \nabla u + cu = 0 \quad \text{in } Q,$$

$$(5.7b) \quad u(\cdot, 0) = u_0(\cdot) \quad \text{in } \Omega, \quad \partial_n u + \beta(u) \ni g \quad \text{on } \Sigma,$$

where $\beta = \partial j : \mathbb{R} \rightarrow 2^{\mathbb{R}}$ is a maximal monotone real graph (see Example 4.5) and $g = g(x, t)$. We assume for simplicity that

$$(5.8) \quad |\mathbf{b}| \leq b_0, \quad c = \operatorname{div} \mathbf{b} = 0 \quad \text{in } \Omega, \quad \mathbf{b} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega.$$

When $2c - \operatorname{div} \mathbf{b}$ or $\mathbf{b} \cdot \mathbf{n}$ are strictly positive in a set of nonvanishing measure, we have a strictly coercive bilinear form \mathfrak{a} on $H^1(\Omega)$, and some estimates can be obtained in a slightly easier way as shown in Section 5.1. To formalize this problem, we introduce a second Hilbert space $\mathcal{V} := H^1(\Omega)$ as in Example 4.5, and the \mathcal{V} -lower semicontinuous convex functional (4.13): $\psi(w) := \int_{\partial\Omega} j(w(x)) dS$ if $w \in H^1(\Omega)$, $j(w) \in L^1(\partial\Omega)$. The problem (5.7) admits, for $c = 0$, the variational formulation (4.43) with bilinear form

$$\mathfrak{a}(v, w) := \int_{\Omega} (\nu^2 \nabla u \cdot \nabla v + \mathbf{b} \cdot \nabla uv) dx, \quad \forall v, w \in \mathcal{V}.$$

The symbol $\langle \cdot, \cdot \rangle$ denotes both the scalar product in $L^2(\Omega)$ and the duality between \mathcal{V}^* and \mathcal{V} , and $f : (0, T) \rightarrow \mathcal{V}^*$ is defined by

$$\langle f(t), v \rangle := \int_{\partial\Omega} g(x, t) v(x) dS(x), \quad \forall v \in \mathcal{V}.$$

To discretize this problem, we choose

$$(5.9) \quad U_0 \in H^1(\Omega) \text{ with } \partial_n U_0 + \beta(U_0) \ni G_0 \text{ on } \partial\Omega, \quad G_n \in L^2(\partial\Omega),$$

and solve the elliptic equations for $1 \leq n \leq N$ on the grid (1.2)

$$(5.10a) \quad \tau_n^{-1}(U_n - U_{n-1}) - \nu^2 \Delta U_n + \mathbf{b} \cdot \nabla U_n = 0 \quad \text{in } \Omega,$$

$$(5.10b) \quad \partial_n U_n + \beta(U_n) \ni G_n \quad \text{on } \partial\Omega.$$

The variational formulation of (5.10) can be written as (4.44) with

$$\langle F_n, v \rangle := \int_{\partial\Omega} G_n(x) v(x) dS(x), \quad \forall v \in \mathcal{V}.$$

Let κ^{-1} be the measure of Ω , $v_\Omega := \kappa \int_\Omega v$ be the average of v in Ω , and

$$i(t) := \int_{\partial\Omega} g(x, t) dS(x), \quad I(t) := \int_{\partial\Omega} G(x, t) dS(x).$$

The eigenvalue p_N is the best constant in Poincaré inequality $p_N \|w - w_\Omega\|_{L^2(\Omega)} \leq \|\nabla w\|_{L^2(\Omega)}$ for all $w \in \mathcal{V}$. Similarly, let $\theta > 0$ be the best constant in the inequality $\|w - w_\Omega\|_{L^2(\partial\Omega)} \leq \theta \|\nabla w\|_{L^2(\Omega)}$ for $w \in \mathcal{V}$.

THEOREM 5.5 *Let u be the weak solution of (5.7) with data satisfying (5.8) and*

$$u_0 \in L^2(\Omega), \quad g \in L^2(\Sigma).$$

Let $\{U_n\}_{n=0}^N$ be the discrete solution of (5.10), with coefficients satisfying (5.9), and let \bar{U} and U be its piecewise constant and linear interpolants. If \mathcal{D}_n is defined as in (3.20), then the following a posteriori error estimates hold:

$$\begin{aligned} \max \left(\|u - U\|_{L^\infty(0, T; L^2(\Omega))}, \frac{\nu}{\sqrt{2}} \|\nabla u - \nabla U\|_{L^2(\mathcal{Q})}, \frac{\nu}{\sqrt{2}} \|\nabla u - \nabla \bar{U}\|_{L^2(\mathcal{Q})} \right) \leq \\ \left(\|u_0 - U_0\|_{L^2(\Omega)}^2 + \frac{2\theta^2}{\nu^2} \|g - \bar{G}\|_{L^2(\Sigma)}^2 + \eta^2 \sum_{n=1}^N \tau_n^2 \mathcal{D}_n \right)^{\frac{1}{2}} + \sqrt{\kappa} \int_0^T |i - \bar{I}| dt, \end{aligned}$$

with $\eta^2 := \max(1, (1 + b_0^2/\nu^4 p_N^2)/2)$. Moreover, if

$$U_0 := u_0, \quad \partial_t g \in L^2(0, T; L^2(\Omega)), \quad G_n(\cdot) := g(\cdot, t_n),$$

then the right-hand side of the previous estimate can be bounded by

$$\tau \left(\eta \|\mathfrak{A} u_0\|_{L^2(\Omega)} + \frac{(1 + \eta)\theta}{\nu} \int_0^T \|\partial_t g\|_{L^2(\Omega)} dt + (1 + \sqrt{2}\eta) \sqrt{\kappa} \text{Var } i \right).$$

PROOF: Thanks to (5.8) and the Green formula, we see that

$$[v]^2 = \mathfrak{a}(v, v) = \nu^2 \int_\Omega |\nabla v|^2 dx \geq \nu^2 p_N^2 \|v - v_\Omega\|_{L^2(\Omega)}^2, \quad \forall v \in \mathcal{V}.$$

Moreover, since

$$\int_{\Omega} \mathbf{b} \cdot \nabla v \, dx = - \int_{\Omega} v \operatorname{div} \mathbf{b} \, dx + \int_{\partial\Omega} v \mathbf{b} \cdot \mathbf{n} \, dS = 0, \quad \forall v \in \mathcal{V},$$

we have

$$2\mathfrak{a}_a(v, w) = \int_{\Omega} (\mathbf{b} \cdot \nabla v(w - w_{\Omega}) - \mathbf{b} \cdot \nabla w(v - v_{\Omega})) \, dx \leq \frac{2b_0}{\nu^2 p_N} [v][w],$$

i.e., \mathfrak{a} satisfies (4.41c) with $\mu := b_0/\nu^2 p_N$. In particular, \mathfrak{F} is angle-bounded and satisfies (4.18) with the asserted η .

Our next task is to split the source term $\langle f, w \rangle = \int_{\partial\Omega} g w$ into components in \mathcal{V}^* and \mathcal{H} . To do so, we introduce the auxiliary function \tilde{g} defined up to an additive constant by

$$-\Delta \tilde{g} = -\kappa i \text{ in } \Omega, \quad \partial_n \tilde{g} = g \text{ on } \partial\Omega;$$

the existence of \tilde{g} is ensured by the compatibility condition

$$\int_{\Omega} -\kappa i \, dx + \int_{\partial\Omega} g \, dS = -i + i = 0.$$

Since

$$\int_{\Omega} \nabla \tilde{g} \cdot \nabla v \, dx = \int_{\partial\Omega} g v \, dS - \kappa i \int_{\Omega} v \, dx = \int_{\partial\Omega} g(v - v_{\Omega}) \, dS, \quad \forall v \in \mathcal{V},$$

we deduce that

$$\|\nabla \tilde{g}\|_{L^2(\Omega)} \leq \theta \|g\|_{L^2(\partial\Omega)}.$$

Therefore f admits the natural splitting $f = f_1 + f_2$ where

$$\langle f_1(t), v \rangle := \int_{\Omega} \nabla \tilde{g}(x, t) \cdot \nabla v(x) \, dx, \quad \langle f_2(t), v \rangle = \kappa i \int_{\Omega} v(x) \, dx,$$

and

$$[f_1]_* = \nu^{-1} \|\nabla \tilde{g}\|_{L^2(\Omega)} \leq \frac{\theta}{\nu} \|g\|_{L^2(\partial\Omega)}, \quad \|f_2\|_{L^2(\Omega)} = \kappa \|i\|_{L^2(\Omega)} = \sqrt{\kappa} |i|.$$

Since the same arguments are valid for F_n , it only remains to apply Theorem 4.17 to conclude the proof. \square

Remark 5.6. For an advection-dominated diffusion operator \mathfrak{A} , the stability constants in the a posteriori and a priori error estimates of Theorem 5.5 are proportional to the advection/diffusion ratio b_0/ν^2 , which is consistent with Remark 5.3.

5.3 Quasi-Linear Operators

We consider two relevant examples of quasi-linear operators, namely, the minimal surface operator and the p -Laplacian, both briefly discussed in Example 2.5.

Minimal Surface Operator

Let u be the *pseudo-solution* of the evolution minimal surface equation (2.34)

$$(5.11) \quad \partial_t u - \operatorname{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 0 \quad \text{in } Q,$$

subject to an initial and lateral boundary condition $u = u_0$. This definition involves the functional ϕ in $\mathcal{H} := L^2(\Omega)$ defined in (2.36) [41]

$$\phi(w) := \int_{\Omega} \sqrt{1 + |\nabla w|^2} dx + \int_{\partial\Omega} |w - u_0| dS, \quad D(\phi) := L^2(\Omega) \cap W^{1,1}(\Omega).$$

We consider the following minimization problem:

$$(5.12) \quad U_n := \arg \min_{V \in D(\phi)} \Phi_n(V; U_{n-1}),$$

where Φ_n is the quadratic perturbation of the area functional ϕ

$$\Phi_n(V; U_{n-1}) := \int_{\Omega} \left(\frac{1}{2\tau_n} |V - U_{n-1}|^2 + \sqrt{1 + |\nabla V|^2} \right) dx + \int_{\partial\Omega} |V - u_0| dS.$$

Equivalently, we consider the following elliptic variational inequality:

$$(5.13) \quad \int_{\Omega} \tau_n^{-1} (U_n - U_{n-1})(U_n - v) + \phi(U_n) - \phi(v) \leq 0, \quad \forall v \in D(\phi).$$

Since ϕ is not lower semicontinuous in $L^2(\Omega)$, it is not obvious that either (5.12) or (5.13) admits a solution. We could, in principle, substitute ϕ with its lower semicontinuous envelope

$$\bar{\phi}(u) := \inf \left\{ \liminf_{n \rightarrow +\infty} \phi(u_n) : u_n \in \mathcal{H}, \lim_{n \rightarrow +\infty} u_n = u \right\},$$

with proper domain $D(\bar{\phi}) = \operatorname{BV}(\Omega)$. Since $\bar{\phi}(v) \leq \phi(v)$, we could use $\bar{\phi}$ instead of ϕ to define the minimizing sequence U_n . However, in Theorem 5.7 we resort to a direct approach that avoids dealing with $\operatorname{BV}(\Omega)$. To do so, we introduce the (modified) descent speed \mathcal{E}_n of the functional Φ_n at time t_n

$$\mathcal{E}_n := -\frac{1}{2} |\delta U_n|^2 - \tau_n^{-1} (\Phi_n(U_n; U_{n-1}) - \Phi_n(U_{n-1}; U_{n-1})) = -|\delta U_n|^2 - \delta\phi(U_n).$$

THEOREM 5.7 *If $U_0 \in W^{1,\infty}(\Omega)$, then problem (5.12) is always uniquely solvable and it defines a family of uniformly bounded and locally Lipschitz functions*

$\{U_n\}_{n=1}^N$. We have the optimal a posteriori and a priori error estimates

$$\begin{aligned}
 \|u - U\|_{L^\infty(0,T;L^2(\Omega))}^2 &\leq \|u_0 - U_0\|_{L^2(\Omega)}^2 + \sum_{n=1}^N \tau_n^2 \mathcal{E}_n \\
 (5.14) \qquad &\leq \|u_0 - U_0\|_{L^2(\Omega)}^2 + \tau \|u_0 - U_0\|_{L^1(\partial\Omega)} \\
 &\quad + \tau \int_{\Omega} \sqrt{1 + |\nabla U_0(x)|^2} \, dx.
 \end{aligned}$$

Moreover, if the initial graph u_0 has a square-integrable curvature and $U_0 := u_0$, we obtain the following optimal a priori error estimate:

$$(5.15) \qquad \sum_{n=1}^N \tau_n^2 \mathcal{E}_n \leq \frac{1}{2} \tau^2 \int_{\Omega} \left| \operatorname{div} \frac{\nabla u_0}{\sqrt{1 + |\nabla u_0|^2}} \right|^2 \, dx.$$

PROOF: In order to apply the abstract theory of Section 3, we make the following simple observation: Lower semicontinuity of ϕ just guarantees the *existence* of both a strong continuous solution u of (5.11) and $\{U_n\}$ of (5.12), whereas *convexity* of ϕ alone is used in the error analysis. The existence of a strong solution u of (2.4) follows from [41, theorems 1.1 and 1.2]. Well-posedness of the minimum problem (5.12) follows from the gradient interior estimates of Bombieri, De Giorgi, and Miranda for the (time-independent) minimal surface equation and their extension to a more general class of equations with a bounded source term [57, theorem 5.1, example 5.2]. Such a boundedness is a consequence of the maximum principle, which yields a uniform $L^\infty(\Omega)$ bound for U_n .

Since ϕ is convex, we are now ready to apply the abstract theory of Section 3. We easily verify that the a posteriori estimate in (5.14) results from Theorem 3.2, whereas the a priori estimate is a consequence of Theorem 3.15. The linear rate (5.15) is a by-product of Theorem 3.18. \square

Remark 5.8. As a by-product of Remark 2.10 and Theorem 3.2, we can also prove *interior* estimates for $\nabla(u - U)$ similar to those in (5.14) and (5.15). In fact, for every $\Omega' \Subset \Omega$ there exists a constant $C' > 0$ depending on the $W^{1,\infty}(\Omega')$ -norms of u and U such that

$$\|\nabla u - \nabla U\|_{L^2(\Omega')}^2 + \|\nabla u - \nabla \bar{U}\|_{L^2(\Omega')}^2 \leq C' \left(\|u_0 - U_0\|_{L^2(\Omega)}^2 + \sum_{n=1}^N \tau_n^2 \mathcal{E}_n \right),$$

where $\Omega' := \Omega' \times (0, T)$. Moreover, if $\partial\Omega$ is a surface of strictly positive mean curvature (e.g., if Ω is strictly convex), this estimate holds up to the boundary; i.e., we can choose $\Omega' \equiv \Omega$. In particular, if $U_0 := u_0$ has square-integrable curvature, then we obtain an optimal *linear* a priori estimate.

Remark 5.9. A remarkable property of the linear rate of convergence stated in (5.15) is its validity even if the solutions u and U detach from the Dirichlet datum u_0 . This implies a regularity result of $\partial_t u$ and therefore of the curvature $\operatorname{div}(\nabla u / \sqrt{1 + |\nabla u|^2})$ past singularities.

Remark 5.10. If Ω is strictly convex, then the Dirichlet condition $U_n = u_0$ can be imposed strongly. In such a case, the a posteriori estimator becomes

$$\mathcal{E}_n \leq \mathcal{D}_n = \frac{1}{\tau_n} \int_{\Omega} \left(\frac{\nabla U_n}{\sqrt{1 + |\nabla U_n|^2}} - \frac{\nabla U_{n-1}}{\sqrt{1 + |\nabla U_{n-1}|^2}} \right) \nabla (U_n - U_{n-1}).$$

p -Laplacian

Given data $f \in L^1(0, T; L^2(\Omega))$ and $u_0 \in L^2(\Omega)$, we consider the evolution problem associated with the p -Laplacian ($p > 2$)

$$\partial_t u - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = f \text{ in } Q, \quad u(x, 0) = u_0(x) \text{ in } \Omega, \quad u(x, t) = 0 \text{ on } \Sigma.$$

This problem is equivalent to an evolution variational inequality (2.4) associated with the p -energy functional

$$\phi(v) = \int_{\Omega} \frac{1}{p} |\nabla v|^p, \quad \forall v \in D(\phi) := W_0^{1,p}(\Omega).$$

Given discrete data $U_0, \{F_n\}_{n=0}^N$, and the quadratic perturbation Φ_n of the p -energy functional ϕ

$$\Phi_n(V; U_{n-1}) := \int_{\Omega} \left(\frac{1}{2\tau_n} |V(x) - U_{n-1}(x)|^2 + \frac{1}{p} |\nabla V(x)|^p - F_n(x) V(x) \right) dx,$$

let U_n be the solution of the recursive minimization problem

$$U_n := \arg \min_{V \in D(\phi)} \Phi_n(V; U_{n-1}).$$

To state the error estimates, we note that the discrete estimator \mathcal{E}_n defined in (3.8) is the (modified) descent speed of the functional Φ_n at time t_n ,

$$\mathcal{E}_n = -\frac{1}{2} |\delta U_n|^2 - \tau_n^{-1} (\Phi_n(U_n; U_{n-1}) - \Phi_n(U_{n-1}; U_{n-1})),$$

and indicate by E the total discretization error of (2.52) with $c_p > 0$

$$E := \max \left(\|u - U\|_{L^\infty(0,T;L^2(\Omega))}, \frac{\sqrt{2}c_p}{p} (\|\nabla u - \nabla U\|_{L^p(Q)} + \|\nabla u - \nabla \tilde{U}\|_{L^p(Q)})^{p/2} \right).$$

THEOREM 5.11 *If $p \geq 2$, then the following optimal a posteriori and a priori error estimates are valid:*

$$\begin{aligned}
 (5.16) \quad E &\leq \left(\|u_0 - U_0\|_{L^2(\Omega)}^2 + \sum_{n=1}^N \tau_n^2 \mathcal{E}_n \right)^{\frac{1}{2}} + \|f - \bar{F}\|_{L^1(0,T;L^2(\Omega))} \\
 &\leq \left(\|u_0 - U_0\|_{L^2(\Omega)}^2 + \frac{\tau}{p} \int_{\Omega} |\nabla U_0(x)|^p dx + \frac{\tau}{4} \|\bar{F}\|_{L^2(Q)}^2 \right)^{\frac{1}{2}} \\
 &\quad + \|f - \bar{F}\|_{L^1(0,T;L^2(\Omega))}.
 \end{aligned}$$

Moreover, if $f \in L^2(Q)$ and we choose $F_n(x) := \tau_n^{-1} \int_{t_{n-1}}^{t_n} f(x, t) dt$ and $U_0 := u_0$, we obtain the uniform a priori error estimate

$$(5.17) \quad E \leq \sqrt{3\tau} \left(\frac{1}{p} \int_{\Omega} |\nabla U_0(x)|^p dx + \|f\|_{L^2(Q)}^2 \right)^{\frac{1}{2}}.$$

Finally, if the continuous data u_0, f and discrete data U_0, F_n satisfy

$$\operatorname{div}(|\nabla u_0|^{p-2} \nabla u_0) \in L^2(\Omega), \quad \partial_t f \in L^1(0, T; L^2(\Omega)),$$

and $U_0 := u_0$, $F_n := f(\cdot, t_n)$, then we get an a priori bound for \mathcal{E}_n

$$\begin{aligned}
 (5.18) \quad \left(\sum_{n=1}^N \tau_n^2 \mathcal{E}_n \right)^{\frac{1}{2}} &\leq \frac{\tau}{\sqrt{2}} \|\operatorname{div}(|\nabla u_0|^{p-2} \nabla u_0) + f(0)\|_{L^2(\Omega)} \\
 &\quad + \tau \|\partial_t f\|_{L^1(0,T;L^2(\Omega))},
 \end{aligned}$$

and we end up with the optimal a priori linear rate of convergence

$$(5.19) \quad E \leq \tau \left(\frac{1}{\sqrt{2}} \|\operatorname{div}(|\nabla u_0|^{p-2} \nabla u_0) + f(0)\|_{L^2(\Omega)} + 2 \|\partial_t f\|_{L^1(0,T;L^2(\Omega))} \right).$$

PROOF: Owing to the p -growth of $\Phi_n(V; U_{n-1})$ with respect to ∇V , we conclude that $\Phi_n(\cdot; U_{n-1})$ is lower semicontinuous in $\mathcal{H} = L^2(\Omega)$, and so the theory of Section 3 applies. In particular, (2.52) combined with Lemma 2.8 yields

$$\sigma(w; v) \geq \frac{c_p}{p} \int_{\Omega} |\nabla(w - v)|^p dx, \quad \forall w, v \in W_0^{1,p}(\Omega).$$

Consequently, the estimates in (5.16) follow from Theorems 3.2 and 3.15, whereas Theorem 3.16 yields (5.17). In light of (3.7), we finally deduce (5.18) from Theorem 3.18 and (5.19) from Corollary 3.20. \square

Remark 5.12. Taking into account Corollary 3.9, in the first inequality of (5.16) as well as in (5.18) we can substitute for \mathcal{E}_n with

$$\mathcal{D}_n = \tau_n^{-1} \int_{\Omega} (|\nabla U_n|^{p-2} \nabla U_n - |\nabla U_{n-1}|^{p-2} \nabla U_{n-1}) \cdot \nabla (U_n - U_{n-1}) dx.$$

Remark 5.13. If $1 < p < 2$, then we still obtain the above estimates for the error in $L^\infty(0, T; L^2(\Omega))$. We can also prove that there exists a constant C depending only on the norms of f, \bar{F} in $L^2(Q)$ and of $\nabla u_0, \nabla U_0$ in $L^p(\Omega)$ such that

$$\|\nabla u - \nabla U\|_{L^2(Q)} + \|\nabla u - \nabla \bar{U}\|_{L^2(Q)} \leq C \left(\|u_0 - U_0\|_{L^2(\Omega)}^2 + \sum_{n=1}^N \tau_n^2 \mathcal{E}_n \right)^{\frac{1}{2}}.$$

This easily follows from two simple observations. First, we point out that the coercivity term ϱ associated with the p -Laplacian

$$\varrho(v, w) = \int_{\Omega} (|\nabla v|^{p-2} \nabla v - |\nabla w|^{p-2} \nabla w) \cdot \nabla (v - w) dx$$

satisfies the following lower bound (cf., e.g., [55, section 2, example E3(ii)]):

$$\exists c_p > 0: \varrho(v, w) \geq c_p (\|\nabla v\|_{L^p(\Omega)} + \|\nabla w\|_{L^p(\Omega)})^{p-2} \|\nabla(v - w)\|_{L^p(\Omega)}^2.$$

Making use of Lemma 2.8 and the subsequent Remark 2.10, there exists a constant C depending *only* on the $L^p(\Omega)$ -norms of ∇v and ∇w such that the smaller coercivity term σ for the p -Laplacian satisfies

$$\sigma(v; w) \geq C \|\nabla(v - w)\|_{L^p(\Omega)}^2.$$

Finally, we observe that the $L^p(\Omega)$ norms of ∇u and ∇U are uniformly bounded in $(0, T)$ once we know an analogous bound for the initial data u_0, U_0 and for the $L^2(Q)$ -norms of f, \bar{F} . This is in turn an immediate consequence of (2.6) and its discrete counterpart (3.12).

5.4 Degenerate Operators

We conclude this section with the Stefan problem (2.39) and (2.40)

$$\partial_t u - \Delta \beta(u) = f, \quad \beta(r) := (r - 1)^+ - r^-,$$

which was formulated as a variational evolution inequality in $H^{-1}(\Omega)$ in Example 2.6.

We choose $U_0, u_0 \in L^2(\Omega)$, $f \in L^2(0, T; H^{-1}(\Omega))$, and a sequence of approximating $F_n \in H^{-1}(\Omega)$; at each step, we solve the semidiscrete problem

$$\delta U_n - \Delta \beta(U_n) = F_n \text{ in } H^{-1}(\Omega), \quad \beta(U_n) \in H_0^1(\Omega).$$

To carry out a *sharp* error analysis, we note that the residual estimator \mathcal{D}_n defined in (3.20) reads

$$\mathcal{D}_n := \tau_n^{-1} \int_{\Omega} (\beta(U_n) - \beta(U_{n-1}))(U_n - U_{n-1}) dx,$$

and that the total error E for enthalpy u and temperature $\beta(u)$ is

$$E := \max(\|u - U\|_{L^\infty(0, T; H^{-1}(\Omega))}, \|\beta(u) - \beta(U)\|_{L^2(Q)} + \|\beta(u) - \beta(\bar{U})\|_{L^2(Q)}),$$

where U is the piecewise linear interpolant of the discrete values U_n .

THEOREM 5.14 *If u and U are the continuous and discrete solutions of the Stefan problem with respect to the previous choice of data, then they satisfy the a posteriori error estimate*

$$(5.20) \quad E \leq \left(\|u_0 - U_0\|_{H^{-1}(\Omega)}^2 + \sum_{n=1}^N \tau_n^2 \mathcal{D}_n \right)^{\frac{1}{2}} + \|f - \bar{F}\|_{L^1(0, T; H^{-1}(\Omega))}.$$

Moreover, if $U_0 := u_0$ and we choose F_n as the mean value of f in the interval (t_{n-1}, t_n) , we get the optimal a priori bound

$$(5.21) \quad E \leq \sqrt{3\tau} \left(\frac{1}{2} \|\beta(u_0)\|_{L^2(\Omega)}^2 + \|f\|_{L^2(Q)}^2 \right)^{\frac{1}{2}}.$$

Finally, if the continuous data u_0, f satisfy

$$\beta(u_0) \in H_0^1(\Omega), \quad \partial_t f \in L^1(0, T; H^{-1}(\Omega)),$$

and $F_n := f(\cdot, t_n)$, we can bound the a posteriori estimator by

$$(5.22) \quad \left(\sum_{n=1}^N \tau_n^2 \mathcal{D}_n \right)^{\frac{1}{2}} \leq \frac{\tau}{\sqrt{2}} \|\nabla \beta(u_0)\|_{L^2(\Omega)} + \tau \|\partial_t f\|_{L^1(0, T; H^{-1}(\Omega))},$$

and we end up with the optimal a priori bound

$$E \leq \tau \left(\frac{1}{\sqrt{2}} \|\nabla \beta(u_0)\|_{L^2(\Omega)} + \frac{1}{\sqrt{2}} \|f(0)\|_{-1(\Omega)} + 2 \|\partial_t f\|_{L^1(0, T; H^{-1}(\Omega))} \right).$$

PROOF: In view of Lemma 2.12 we are in a position to apply to abstract results of Section 3. We readily see that Theorem 3.2 yields (5.20), Theorem 3.16 implies (5.21), Theorem 3.18 leads to (5.22), and finally Corollary 3.20 gives rise to (5.22). \square

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Bibliography

- [1] Amann, H. Über die Existenz und iterative Berechnung einer Lösung der Hammerstein’schen Gleichung. *Aequationes Math.* **1** (1968), 242–266.
- [2] Asplund, E. A monotone convergence theorem for sequences of nonlinear mappings. *Nonlinear functional analysis (Proc. Sympos. Pure Math., Vol. XVIII, Part 1, Chicago, Ill., 1968)*, 1–9. Amer. Math. Soc., Providence, R.I., 1970.
- [3] Baillon, J.-B.; Haddad, G. Quelques propriétés des opérateurs angle-bornés et n -cycliquement monotones. (French) *Israel J. Math.* **26** (1977), no. 2, 137–150.
- [4] Baiocchi, C. Discretization of evolution variational inequalities. *Partial differential equations and the calculus of variations*, Vol. I, 59–92. Progr. Nonlinear Differential Equations Appl., 1. Birkhäuser Boston, Boston, 1989.
- [5] Baiocchi, C.; Brezzi, F. Optimal error estimates for linear parabolic problems under minimal regularity assumptions. *Calcolo* **20** (1983), no. 2, 143–176.
- [6] Baiocchi, C.; Capelo, A. *Variational and quasivariational inequalities. Applications to free boundary problems*. Translated from the Italian by Lakshmi Jayakar. Wiley, New York, 1984.
- [7] Brézis, H. Monotonicity methods in Hilbert spaces and some applications to nonlinear partial differential equations. *Contributions to nonlinear functional analysis (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1971)*, 101–156. Academic Press, New York, 1971.
- [8] Brézis, H. Problèmes unilatéraux. *J. Math. Pures Appl. (9)* **51** (1972), 1–168.
- [9] Brézis, H. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*. (French) North-Holland Mathematics Studies, No. 5. Notas de Matemática (50). North-Holland, Amsterdam–London; American Elsevier, New York, 1973.
- [10] Brézis, H.; Browder, F. E. Nonlinear integral equations and systems of Hammerstein type. *Advances in Math.* **18** (1975), no. 2, 115–147.
- [11] Brézis, H. R.; Stampacchia, G. Sur la régularité de la solution d’inéquations elliptiques. (French) *Bull. Soc. Math. France* **96** (1968), 153–180.
- [12] Brézis, H.; Strauss, W. A. Semi-linear second-order elliptic equations in L^1 . *J. Math. Soc. Japan* **25** (1973), 565–590.
- [13] Caffarelli, L. A. The regularity of monotone maps of finite compression. *Comm. Pure Appl. Math.* **50** (1997), no. 6, 563–591.
- [14] Castaing, C.; Valadier, M. *Convex analysis and measurable multifunctions*. Lecture Notes in Mathematics, Vol. 580. Springer, Berlin–New York, 1977.

- [15] Chen, X.; Elliott, C. M. Asymptotics for a parabolic double obstacle problem. *Proc. Roy. Soc. London Ser. A* **444** (1994), no. 1922, 429–445.
- [16] Colli, P.; Savaré, G. Degenerate evolution systems modeling cardiac electric field at micro and macroscopic level. I.A.N. C.N.R., Pavia, no. 1007, 1996.
- [17] Colli, P.; Savaré, G. Time discretization of Stefan problems with singular heat flux. *Free boundary problems, theory and applications (Zakopane, 1995)*, 16–28. Pitman Res. Notes Math. Ser., 363. Longman, Harlow, 1996.
- [18] Crandall, M. G. Nonlinear semigroups and evolution governed by accretive operators. *Nonlinear functional analysis and its applications, Part 1 (Berkeley, Calif., 1983)*, 305–337. Proc. Sympos. Pure Math., 45, Part 1. Amer. Math. Soc., Providence, R.I., 1986.
- [19] Crandall, M. G.; Evans, L. C. On the relation of the operator $\partial/\partial s + \partial/\partial \tau$ to evolution governed by accretive operators. *Israel J. Math.* **21** (1975), no. 4, 261–278.
- [20] Crandall, M. G.; Liggett, T. M. Generation of semi-groups of nonlinear transformations on general Banach spaces. *Amer. J. Math.* **93** (1971), 265–298.
- [21] Crouzeix, M.; Mignot, A. L. *Analyse numérique des équations différentielles*. (French) [Numerical analysis of differential equations] Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree] Masson, Paris, 1984.
- [22] Dahlquist, G. Error analysis for a class of methods for stiff non-linear initial value problems. *Numerical analysis (Proc. 6th Biennial Dundee Conf., Univ. Dundee, Dun-Dee, 1975)*, 60–72. Lecture Notes in Mathematics, Vol. 506, Springer, Berlin, 1976.
- [23] Damlamian, A.; Kenmochi, N.; Sato, N. Subdifferential operator approach to a class of non-linear systems for Stefan problems with phase relaxation. *Nonlinear Anal.* **23** (1994), no. 1, 115–142.
- [24] de Mottoni, P.; Schatzman, M. Geometrical evolution of developed interfaces. *Trans. Amer. Math. Soc.* **347** (1995), no. 5, 1533–1589.
- [25] DiBenedetto, E. *Degenerate parabolic equations*. Universitext. Springer, New York, 1993.
- [26] Ekeland, I.; Temam, R. *Analyse convexe et problèmes variationnels*. (French) Collection Études Mathématiques. Dunod; Gauthier-Villars, Paris–Brussels–Montreal, 1974.
- [27] Eriksson, K.; Estep, D.; Hansbo, P.; Johnson, C. *Computational differential equations*. Cambridge University Press, Cambridge, 1996.
- [28] Eriksson, K.; Johnson, C. Adaptive finite element methods for parabolic problems. I. A linear model problem. *SIAM J. Numer. Anal.* **28** (1991), no. 1, 43–77.
- [29] Eriksson, K.; Johnson, C. Adaptive finite element methods for parabolic problems. IV. Nonlinear problems. *SIAM J. Numer. Anal.* **32** (1995), no. 6, 1729–1749.
- [30] Eriksson, K.; Johnson, C.; Larsson, S. Adaptive finite element methods for parabolic problems. VI. Analytic semigroups. *SIAM J. Numer. Anal.* **35** (1998), no. 4, 1315–1325 (electronic).
- [31] Gear, C. W. *Numerical initial value problems in ordinary differential equations*. Prentice-Hall, Englewood Cliffs, N.J., 1971.
- [32] Giusti, E. *Minimal surfaces and functions of bounded variation*. Monographs in Mathematics, 80. Birkhäuser, Basel–Boston, 1984.
- [33] Glowinski, R.; Lions, J.-L.; Trémolières, R. *Numerical analysis of variational inequalities*. Translated from the French. Studies in Mathematics and Its Applications, 8. North-Holland, Amsterdam–New York, 1981.
- [34] Grisvard, P. *Elliptic problems in nonsmooth domains*. Monographs and Studies in Mathematics, 24. Pitman (Advanced Publishing Program), Boston–London, 1985.
- [35] Hairer, E.; Wanner, G. *Solving ordinary differential equations. II. Stiff and differential-algebraic problems*. Second edition. Springer Series in Computational Mathematics, 14. Springer, Berlin, 1996.

- [36] Henrici, P. *Discrete variable methods in ordinary differential equations*. Wiley, New York–London, 1962.
- [37] Johnson, C. Error estimates and adaptive time-step control for a class of one-step methods for stiff ordinary differential equations. *SIAM J. Numer. Anal.* **25** (1988), no. 4, 908–926.
- [38] Kato, T. *Perturbation theory for linear operators*. Second edition. Grundlehren der Mathematischen Wissenschaften, Band 132. Springer, Berlin–New York, 1976.
- [39] Kenmochi, N. Systems of nonlinear PDEs arising from dynamical phase transitions. *Phase transitions and hysteresis (Montecatini Terme, 1993)*, 39–86. Lecture Notes in Mathematics, 1584. Springer, Berlin, 1994.
- [40] Kinderlehrer, D.; Stampacchia, G. *An introduction to variational inequalities and their applications*. Pure and Applied Mathematics, 88. Academic Press [Harcourt Brace Jovanovich], New York–London, 1980.
- [41] Lichnerowicz, A.; Temam, R. Pseudosolutions of the time-dependent minimal surface problem. *J. Differential Equations* **30** (1978), no. 3, 340–364.
- [42] Lions, J.-L. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. (French) Dunod; Gauthier-Villars, Paris, 1969.
- [43] Lions, J.-L.; Stampacchia, G. Variational inequalities. *Comm. Pure Appl. Math.* **20** (1967), 493–519.
- [44] Ma, Z. M.; Röckner, M. *Introduction to the theory of (nonsymmetric) Dirichlet forms*. Universitext. Springer, Berlin, 1992.
- [45] Magenes, E. Stefan problems in a concentrated capacity. *Advances in mathematical computation and applications*, 82–90. N.C.C., Novosibirsk, 1995.
- [46] Morrey, C. B., Jr. *Multiple integrals in the calculus of variations*. Die Grundlehren der mathematischen Wissenschaften, Band 130. Springer New York, New York, 1966.
- [47] Nochetto, R. H.; Paolini, M.; Verdi, C. Optimal interface error estimates for the mean curvature flow. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **21** (1994), no. 2, 193–212.
- [48] Nochetto, R. H.; Schmidt, A.; Verdi, C. Adaptive solution of phase change problems over unstructured tetrahedral meshes. *Grid generation and adaptive algorithms*, 163–181. The IMA Volumes in Mathematics and Its Applications, 113. Springer, New York, 1999.
- [49] Nochetto, R. H.; Schmidt, A.; Verdi, C. A posteriori error estimation and adaptivity for degenerate parabolic problems. *Math. Comp.* **69** (2000), no. 229, 1–24.
- [50] Rockafellar, R. T. Characterization of the subdifferentials of convex functions. *Pacific J. Math.* **17** (1966), 497–510.
- [51] Rockafellar, R. T. *Convex analysis*. Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970.
- [52] Rulla, J. Error analysis for implicit approximations to solutions to Cauchy problems. *SIAM J. Numer. Anal.* **33** (1996), no. 1, 68–87.
- [53] Savaré, G. Approximation and regularity of evolution variational inequalities. *Rend. Accad. Naz. Sci. XL Mem. Mat. (5)* **17** (1993), 83–111.
- [54] Savaré, G. Weak solutions and maximal regularity for abstract evolution inequalities. *Adv. Math. Sci. Appl.* **6** (1996), no. 2, 377–418.
- [55] Savaré, G. Regularity results for elliptic equations in Lipschitz domains. *J. Funct. Anal.* **152** (1998), no. 1, 176–201.
- [56] Savaré, G.; Visintin, A. Variational convergence of nonlinear diffusion equations: applications to concentrated capacity problems with change of phase. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* **8** (1997), no. 1, 49–89.
- [57] Temam, R. Solutions généralisées de certaines équations du type hypersurfaces minima. (French) *Arch. Rational Mech. Anal.* **44** (1971/72), 121–156.

- [58] Temam, R. Applications de l'analyse convexe au calcul des variations. (French) *Nonlinear operators and the calculus of variations (Summer School, Univ. Libre Bruxelles, Brussels, 1975)*, 208–237. Lecture Notes in Mathematics, Vol. 543, Springer, Berlin, 1976.
- [59] Thomée, V. *Galerkin finite element methods for parabolic problems*. Springer Series in Computational Mathematics, 25. Springer, Berlin, 1997.
- [60] Tomarelli, F. Regularity theorems and optimal error estimates for linear parabolic Cauchy problems. *Numer. Math.* **45** (1984), no. 1, 23–50.
- [61] Visintin, A. Stefan problem with phase relaxation. *IMA J. Appl. Math.* **34** (1985), no. 3, 225–245.

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