SUPERPOSITION AND CHAIN RULE FOR BOUNDED HESSIAN FUNCTIONS

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Summary. – We study the properties of superposition of Bounded Hessian functions, establishing the validity of a second order chain rule. We prove rectifiability properties of all noncritical level sets of BH-functions by showing a geometric measure theory analogous of Dini's Theorem.

0. - Introduction and main results.

For any pair of smooth functions $u : \Omega \subset \mathbf{R}^N \mapsto \mathbf{R}^M$ and $f : \mathbf{R}^M \mapsto \mathbf{R}$ the composition is well defined, smooth and the classical chaine rules hold

$$(0.1) \qquad D[f \circ u](x) = Df(u(x))Du(x), \qquad x \in \Omega$$

$$(0.2) \qquad D^2[f \circ u](x) = D^2f(u(x))Du(x)Du(x) + Df(u(x))D^2u(x), \qquad x \in \Omega$$

where D denotes the distributional (and in such case classical, of course) derivatives.

When the regularity of both f and u is weakened, the definition of superposition may turn meaningless and, even if it is well posed in a suitable functional framework, the extension of the chain formula may be a more difficult question, requiring a finer analysis of the structure properties of the involved functions.

For instance, it is not so difficult to see that superposition with Lipschitz functions f operate on the Sobolev spaces $W^{1,p}(\Omega)$ ($p \geq 1$), Ω being a Lipschitz open set of $\mathbf{R}^N$, and also on $BV(\Omega)$, the space of (bounded variation) functions whose distributional

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derivative is a bounded measure on Ω. On the contrary, adapting (0.1) to these situations requires much more work and it was developed in various steps, as we shall see in a moment.

If we want to give at least a measure sense to the derivative of $f \circ u$ ([1]), further extensions of (0.1) (when f has only an L^p derivative, with $p < \infty$) are prevented by the fact that in general the superposition carries out of BV_{loc}, even if f is monotone and u is Lipschitz: for example ([2])

(0.3)
$$\begin{cases} f(t) := \text{sign}(t)\sqrt{|t|}, \quad u(x) := x^2(\sin \frac{1}{x})^2 \quad \text{for } x \in]0, \pi^{-1}[; \\ f \circ u(x) = \left|x \sin \frac{1}{x}\right|. \end{cases}$$

Here we are interested in the problems related to the second derivatives of a superposition. Of course, the simplest way to approach them is to reiterate the first differentiation formula, requiring at least a separate meaning to each factors $(Df) \circ u$ and Du; this implies that Df has to be Lipschitz by the previous remark and Du of bounded variation, that is (see [11], [21])

$$u \in BH(\Omega; \mathbf{R}^M) = \{v \in W^{1,1}(\Omega; \mathbf{R}^M) : D^2v \text{ is a matrix valued bounded measure}\}.$$

However, we can try to overcome these restrictions, avoiding to split the product.

A first result in this direction is given in [19] for the particular case $M = 1$ and

$$f(t) := \max(t, 0), \quad \text{for } t \in \mathbf{R},$$

whose derivative is of bounded variation but not absolutely continuous; in this case it is proved that

(0.4)
$$u \in BH(\Omega) \implies \max(u, 0) \in BH(\Omega)$$

with a uniform bound of the BH–norm.

On the other side, if $M \geq 2$ it is easy to see that the assumption $f \in W^{2,p}_{loc}(\mathbf{R}^M)$ with $p < M$ does not entail $f \circ u \in BH(\Omega)$, even for a C^∞ bounded function u. Taking for example

(0.5)
$$f(y) = |y|^\varepsilon, \quad \text{with } 0 < \varepsilon < 1, \qquad u(x) = x\mathbf{e}_1, \quad x \in]-1, 1[,$$

([1]) For a different point of view, see [6], [18]; in the Sobolev framework, see also [5].
([2]) Here and in the following

$$\text{sign}(t) := \begin{cases} t/|t| & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

then (the restriction of) f belongs to $W^{2,1}(B_1(0))$, the range of u is contained in the unit sphere $B_1(0)$, but $f \circ u(x) = |x|^\varepsilon$ is not a function of $BH(-1,1)$.

A possible way to unify these apparently different situations is to assume f Lipschitz and convex, recalling that in the one-dimensional case a BH-function can be always splitted into the difference of two Lipschitz convex ones (see section 2, lemma 2.5?). We shall prove

Theorem 1. *Let u be given in $BH(\Omega; \mathbf{R}^M)$ and let $f : \mathbf{R}^M \mapsto \mathbf{R}$ be a Lipschitz convex function, with*

$$(0.6) \qquad\qquad f(0) = 0, \qquad \text{if } \Omega \text{ is unbounded;}$$

then $f \circ u \in BH(\Omega)$.

Corollary. *Let u belong to $BH(\Omega)$ and let $f : \mathbf{R} \mapsto \mathbf{R}$ be a Lipschitz function satisfying* (0.6) *and*

$$(0.7) \qquad\qquad D^2 f \quad \text{is a bounded measure on } \mathbf{R}.$$

Then $f \circ u$ belongs to $BH(\Omega)$.

0.1 REMARK. We shall see that this result holds also for convex functions f defined in a general Banach space X, with $u \in BH(\Omega; X)$. Moreover, we notice that it is sufficient to assume f convex and Lipschitz on the convex hull generated by the essential range of u: if $\mathcal{K}$ is such a set, we extend f outside $\mathcal{K}$ by the usual formula (see, e.g., [13], Ch.3, Th.1)

$$f(x) = \inf_{v \in \mathcal{K}} \left[f(v) + L|x - v| \right], \qquad L \text{ being the Lipschitz constant of } f \text{ on } \mathcal{K}. \quad \square$$

0.2 REMARK. Choosing an appropriate norm on $BV(\Omega)$, we shall prove the estimate

$$(0.8) \quad \|D(f \circ u)\|_{BV(\Omega)} \leq 2L\|Du\|_{BV(\Omega)}, \qquad L \text{ being the Lipschitz constant of } f.$$

Even if in some particular case (e.g. $M = 1$, f convex and monotone: see [19]) it is possible to obtain a better constant, with respect to the norm chosen in (2.2) in general this estimate is sharp. ∎

Let us focus now the problem to write an explicit second order chain rule. For the first order one, STAMPACCHIA [20] stated it for a Lipschitz real function f, u belonging to $W^{1,p}(\Omega)$. He showed a proof in the case of a finite number of jump points for f' and the general case was proven by MARCUS & MIZEL [16] and can also be deduced by an unpublished result of SERRIN (1971).

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We notice that even in the quoted cases the two factors in the right–hand side of (0.1) cannot be splitted, and it is only their product that makes sense: denoting by $\mathcal{L}^N$ the usual Lebesgue measure on $\mathbf{R}^N$, $Df(u(x))$ is defined only $Du(x)\mathcal{L}^N(x)$–almost everywhere in Ω (which could mean nowhere in Ω).

If we allow u to have bounded variation, then in general Du is no longer absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^N$; anyway the usual Lebesgue decomposition holds true

$$(0.9) \qquad\qquad Du = D_a u + D_s u, \qquad D_a u = \nabla u \, \mathcal{L}^N,$$

where ∇u denotes the density of the absolutely continuous part with respect to $\mathcal{L}^N$.

Formula (0.1) has to be suitably modified in order to take account of the singular part of Du, more precisely of its restriction to the "jump set" S_u of u. We describe it by means of the "approximate limit" notion (see [14], [23], and the next section for the definition)

$$u_\mu(x) := \operatorname*{ap\,limsup}_{y\to x} u(y), \quad u_\lambda(x) := \operatorname*{ap\,liminf}_{y\to x} u(y),$$

which are finite for $\mathcal{H}^{N-1}$–a.e. point x and can be used to define the *precise representative* of u

$$u_*(x) := \frac{u_\mu(x) + u_\lambda(x)}{2}.$$

We set

$$S_u := \{x \in \mathbf{R}^N : u_\mu(x) > u_\lambda(x)\}, \qquad Ju := Du \lfloor S_u,$$

observing that outside S_u there exists the approximate limit $\tilde{u}(x) := \operatorname{ap\,lim}_{y\to x}$ and it coincides with the other three values $u_*(x)$, $u_\mu(x)$, $u_\lambda(x)$. It can be shown (see, e.g., [23] Th. 5.9.6) that S_u is countably $\mathcal{H}^{N-1}$, $(N-1)$-rectifiable ([14], 3.2.14) and there exists a unique Borel unit vector field $\mathbf{n}_u : S_u \mapsto \mathbf{R}^N$ such that

$$(0.10) \qquad\qquad Ju = (u_\mu - u_\lambda)\mathbf{n}_u \, \mathcal{H}^{N-1}\lfloor S_u. \ (^3)$$

$(^3)$ When $\mathbf{u} := (u_1, \dots, u_M)$ is vector valued, $S_{\mathbf{u}}$ is defined componentwise as $S_{\mathbf{u}} := \bigcup S_{u_i}$; it is possible to find a Borel couple $\mathbf{u}_+, \mathbf{u}_-$ of one-side traces of $\mathbf{u}$ on $S_{\mathbf{u}}$ with respect to a Borel unit normal $\mathbf{n}_{\mathbf{u}}$ such that

$$J\mathbf{u} = (\mathbf{u}_+ - \mathbf{u}_-) \otimes \mathbf{n}_{\mathbf{u}} \, \mathcal{H}^{N-1}\lfloor S_{\mathbf{u}}$$

$\mathbf{u}_+, \mathbf{u}_-$ and $\mathbf{n}_{\mathbf{u}}$ are defined up to interchanging the subscripts and the sign of $\mathbf{n}_{\mathbf{u}}$ and they satisfy

$$(u_i)_\mu = \max\big[(\mathbf{u}_+)_i, (\mathbf{u}_-)_i\big], \quad (u_i)_\lambda = \min\big[(\mathbf{u}_+)_i, (\mathbf{u}_-)_i\big].$$

For a Borel set $B \subset \Omega$ we have

(0.11)
$$\mathcal{H}^{N-1}(B) < +\infty \quad \Rightarrow \quad Du \lfloor B = D_s u \lfloor B = Ju \lfloor B,$$

so that the residual "Cantor part" of Du

(0.12)
$$Cu := D_s u - Ju$$

does not see the sets of finite $\mathcal{H}^{N-1}$ measure. In this framework, if $f \in C^1(\mathbf{R})$ and $u \in BV(\Omega)$, VOL'PERT [22] showed that $f \circ u \in BV(\Omega)$ with

(0.13)
$$D[f \circ u] = \widehat{f}'(u(x)) \, Du(x) \qquad \text{as Borel measures on } \Omega$$

where

(0.14)
$$\widehat{f}'(u(x)) := \int_0^1 f'\big(u_\lambda(x) + s(u_\mu(x) - u_\lambda(x))\big) \, ds =$$
$$\begin{cases} f'(u(x)) & \text{if } x \in \Omega \setminus S_u \\ \dfrac{f(u_\mu(x)) - f(u_\lambda(x))}{u_\mu(x) - u_\lambda(x)} & \text{if } x \in S_u. \end{cases}$$

Hence
$$J[f \circ u] = \big[f(u_\mu) - f(u_\lambda)\big]\nu_u \, \mathcal{H}^{N-1}\lfloor S_u.$$

DAL MASO, LE FLOCH & MURAT [10] proved (0.13) also for a Lipschitz f, provided that f' is replaced by its precise representative $(f')_*$ in the definition of $\widehat{f}'(u(x))$; finally AMBROSIO & DAL MASO [4] give a suitable interpretation of the same formula for vector valued functions u. A simple example of application of (0.13) can be given by choosing $u = \chi_{]0,1[}(x)$, $f(t) = t^2$; notice that in this case

$$\frac{d}{dx}u^2 \neq 2uu' \quad \text{pointwise, but} \quad \frac{d}{dx}u^2 = 2u_* u'.$$

We have now to consider how to extend the chain rule (0.2) to this measure valued framework; here the difficulties are sensibly different with respect to the first order case since

i. $D^2 f$ can have a singular part and there is not a standard way to define the composition of a measure with another function;

ii. in the vector–valued case (that is $M > 1$) the distributional derivative $D^2 f$ doesn't carry enough information to obtain the complete second derivative of $f \circ u$, even if $[D^2 f]_s \equiv 0$; at this end, one can consider, as in (0.5), $f(y) = |y|$, $y \in \mathbf{R}^M$, $u(x) = xe_1$, obtaining

$$\big[D^2 f(y)\big]_{ij} = \frac{\delta_{ij}|y|^2 - y_i y_j}{|y|^3} \in L^1_{loc}(\mathbf{R}^M), \quad D^2 u \equiv 0;$$
$$D^2 f(u(x))Du(x)Du(x) + Df(u(x))D^2 u(x) \equiv 0 \qquad \text{whereas} \quad D^2[f \circ u] = 2\delta_0.$$

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In this paper we limit ourselves to develop the question $i.$ in the case $M = 1$, $u \in BH(\Omega)$, f satisfying (0.6) and (0.7).

We say in advance that the regular part of $D^2(f \circ u)$,

$$\left[D^2(f \circ u) \right]_a := \nabla D(f \circ u)\,\mathcal{L}^N = \nabla^2(f \circ u)\,\mathcal{L}^N,$$

is as one may expect ([4])

$$(0.15) \qquad \nabla^2\big[f \circ u\big](x) = \ddot{f}(u(x))\nabla u(x) \otimes \nabla u(x) + \dot{f}(u(x))\nabla^2 u(x).$$

In order to understand (at least formally) how to treat the term $\ddot{f}(u)\nabla u \otimes \nabla u$ in the singular case, we have to rewrite it in the smooth case evaluating its integral (on a Borel set B) via the change of variables formula (see [13], 3.4.3).

We call $\boldsymbol{\xi}(x)$ the normalized gradient of u

$$\boldsymbol{\xi}(x) = \boldsymbol{s}(\nabla u(x)), \qquad \boldsymbol{s}(v) = \begin{cases} v/|v| & \text{if } |v| > 0, \\ 0 & \text{otherwise} \end{cases}$$

and, for couple of smooth functions u, f and every Borel set $B \subset \Omega$, we have

$$\int_B \ddot{f}(u(x))\nabla u(x) \otimes \nabla u(x)\,dx = \int_B \ddot{f}(u(x))|\nabla u(x)|^2\boldsymbol{\xi}(x) \otimes \boldsymbol{\xi}(x)\,dx =$$

$$(0.16) \qquad \int_{\mathbf{R}} \left\{ \int_{B \cap u^{-1}(t)} |\nabla u(x)|\boldsymbol{\xi}(x) \otimes \boldsymbol{\xi}(x)\,d\mathcal{H}^{N-1}(x) \right\} \ddot{f}(t)\,dt =$$

$$\int_{\mathbf{R}} \left\{ \int_{B \cap u^{-1}(t)} |\nabla u(x)|\boldsymbol{\xi}(x) \otimes \boldsymbol{\xi}(x)\,d\mathcal{H}^{N-1}(x) \right\} df''(t)$$

This representation has the advantage to avoid the superposition $f'' \circ u$, but requires a careful definition of the inside integral for *each* t-level set of a general $BH(\Omega)$–function u ([5]) since $f''(t)$ may countain a Dirac mass concentrated in an arbitrary point.

Nevertheless, we shall see that (0.16) holds also in the irregular case, provided all the occurrences of the various functions are substituted by the respective precise representatives and ξ is suitable defined on $S_{\nabla u}$. We have

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([4]) For a function f of one variable, we use the more familiar symbols

$$f',\ f'' \text{ for } Df,\ D^2 f \quad \text{and} \quad \dot{f},\ \ddot{f} \text{ for } \nabla f,\ \nabla^2 f.$$

We also define the tensor product $p \otimes q$ of a a couple p, q of vectors of $\mathbf{R}^N$ as the linear form

$$(p \otimes q)x = \langle q, x \rangle\, p, \qquad \forall\, x \in \mathbf{R}^N.$$

([5]) and not only for almost every t, as it is usual for the more familiar coarea formula

Theorem 2. *Let $u \in BH(\Omega)$ and $f : \mathbf{R} \mapsto \mathbf{R}$ be a Lipschitz function whose second derivative f'' is a bounded measure on $\mathbf{R}$. For every Borel set $B \subset \Omega$ we have*

$$(0.17) \quad D^2[f \circ u](B) = \int_{\mathbf{R}} \int_{B \cap u_*^{-1}(t)} |\nabla u|_* \, \boldsymbol{\xi} \otimes \boldsymbol{\xi} \, d\mathcal{H}^{N-1} \, df''(t) + \int_B \dot{f}_*(u_*(x)) \, dD^2u(x)$$

where

$$(0.18) \qquad \boldsymbol{\xi}(x) := \begin{cases} \boldsymbol{s}(\nabla u(x)) & \text{if } x \notin S_{\nabla u}, \\ \boldsymbol{n}_{\nabla u}(x) & \text{if } x \in S_{\nabla u}. \end{cases}$$

Let us make a few comments about this formula

- For the function f (and in general for a BH-function of one variable) $\dot{f}_* := (\dot{f})_*$ coincides at *every* point with the mean value of the left and right derivatives

$$\dot{f}(t_+) := \lim_{h \to 0^+} \frac{f(t+h) - f(t)}{h}, \quad \dot{f}(t_-) := \lim_{h \to 0^+} \frac{f(t) - f(t-h)}{h},$$

which exist everywhere. $\dot{f}_*$ is then a bounded and everywhere defined Borel function.

- Since $u \in BH(\Omega)$, its approximate limit $\tilde{u}$ exists at $\mathcal{H}^{N-1}$-a.e. point and coincides with u_*; thus, we can also use $\tilde{u}$ instead of u_* in (0.17). We recall that D^2u vanishes on the $\mathcal{H}^{N-1}$-negligible sets.

- The mapping $\Omega \ni x \mapsto |\nabla u(x)|$ is of bounded variation so that $|\nabla u|_*$ is finite $\mathcal{H}^{N-1}$-a.e.; in order to make it more explicit, we have to take account of the jump set $S_{\nabla u}$ of ∇u, which belongs to $BV(\Omega; \mathbf{R}^N)$. We have already said that for $\mathcal{H}^{N-1}$-a.e. point of $S_{\nabla u}$ it is possible to define two traces $\nabla_+ u := (\nabla u)_+$, $\nabla_- u := (\nabla u)_-$, related to the choice (unique up to the sign) of a geometric measure theory unit normal vector $\boldsymbol{n}_{\nabla u}$. It is easy to see that $S_{|\nabla u|} \subset S_{\nabla u}$ and

$$|\nabla u|_*(x) = \begin{cases} |\nabla u(x)| & \text{if } x \notin S_{\nabla u}, \\ (|\nabla_+ u(x)| + |\nabla_- u(x)|)/2 & \text{if } x \in S_{\nabla u}. \end{cases}$$

- Even if the sign of $\boldsymbol{\xi}(x)$ on $S_{\nabla u}$ depends on the choice of $\boldsymbol{n}_{\nabla u}$, the tensor product $\boldsymbol{\xi} \otimes \boldsymbol{\xi}$ is uniquely determined $\mathcal{H}^{N-1}$-a.e. and it corresponds to the linear projection onto the one-dimensional subspace generated by ∇u (outside $S_{\nabla u}$) or its jump (on $S_{\nabla u}$). In fact, by a general result of [1], it is possible to see that $\nabla_+ u, \nabla_- u, \boldsymbol{n}_{\nabla u}$ have the same direction $\mathcal{H}^{N-1}$-a.e. on $S_{\nabla u}$.

- Finally, we shall see that, for every Borel set B, the mapping

$$t \mapsto \mathcal{G}(B; t) = \int_{B \cap u_*^{-1}(t)} |\nabla u(x)|_* \, \boldsymbol{\xi}(x) \otimes \boldsymbol{\xi}(x) \, d\mathcal{H}^{N-1}(x)$$

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is well defined, uniformly bounded by $\|\nabla u\|_{BV(\Omega;\mathbf{R}^N)}$ and μ-measurable for every bounded measure μ on $\mathbf{R}$, in particular for f''. Other technical properties of it will be collected in the next sections.

Let us focus now some easy consequence of (0.17). First of all, we consider the one-dimensional case of an open interval $\Omega := (a, b)$ of $\mathbf{R}$. Since (the continuous representative of) u is a Lipschitz continuous function, we have $u_* \equiv u$ and

$$\dot{u}_*(x) = \frac{1}{2}\big(\dot{u}(x_+) + \dot{u}(x_-)\big), \quad |\dot{u}|_*(x) = \frac{1}{2}\big(|\dot{u}(x_+)| + |\dot{u}(x_-)|\big), \qquad \forall\, x \in (a, b),$$

an analogous formula holding for $\dot{f}_*$. Since $\mathcal{H}^{N-1} = \mathcal{H}^0$ is the counting measure, and $\boldsymbol{\xi} \otimes \boldsymbol{\xi}$ is 1 when $|\dot{u}| \neq 0$, (0.17) becomes

$$(0.19) \quad \big(f \circ u\big)''(B) = \int_{\mathbf{R}} \sum_{x \in B, u(x)=t} \frac{|\dot{u}(x_+)| + |\dot{u}(x_-)|}{2}\, df''(t) + \int_B \dot{f}_*(u(x))\, du''(x)$$

and we can split it according to the three component of D^2

$$\nabla^2 := \nabla\nabla, \qquad J^2 := J\nabla, \qquad C^2 := C\nabla.$$

Rewriting theorem 2 by mean of the simplified $1 - D$ notation (see note $(^2)$), we get the following statement

Corollary. *Let u be a BH-function of the interval $(a, b) \subset \mathbf{R}$ and f be as in theorem 2. Then $(f \circ u)''$ admits the decomposition $(f \circ u)\ddot{}\,\mathcal{L}^1 + J^2(f \circ u) + C^2(f \circ u)$ with*

$$(0.20) \qquad\qquad\qquad \big(f \circ u\big)\ddot{} = \ddot{f}(u)|\dot{u}|^2 + \dot{f}(u)\ddot{u}$$

$$(0.21) \qquad\qquad J^2(f \circ u) = |\dot{u}|_* \sum_{t \in S_{f'}} J^2 f(\{t\})\mathcal{H}^0\lfloor u^{-1}(t) + \dot{f}_*(u)\, J^2 u$$

and, for every Borel set $B \subset (a, b)$,

$$(0.22)\ \ C^2(f \circ u)(B) = \int_{\mathbf{R}} \sum_{x \in B, u(x)=t} \frac{|\dot{u}(x_+)| + |\dot{u}(x_-)|}{2}\, d\,C^2 f(t) + \int_B \dot{f}_*(u(x))\, dC^2 u(x).$$

0.3 REMARK. In the multidimensional case $\Omega \subset \mathbf{R}^N$, $N > 1$, we have an analogous representation, with (0.15) instead of (0.20) and (0.21) replaced by

$$(0.23) \qquad J^2(f \circ u) = |\nabla u|_* \boldsymbol{\xi} \otimes \boldsymbol{\xi} \left\{ \sum_{t \in S_{f'}} J^2 f(\{t\})\, \mathcal{H}^{N-1}\lfloor u_*^{-1}(t) \right\} + \dot{f}_*(u)\, J^2 u$$

In particular this formula shows that

$$(0.24) \qquad\qquad\qquad S_{\nabla(f \circ u)} \subset S_{\nabla u} \bigcup u_*^{-1}(S_{f'}) \quad \square$$

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Corollary. *If $u \in SBH(\Omega)$ (that is $Du \in SBV(\Omega)$ or equivalently $C^2u \equiv 0$) and $C^2f \equiv 0$, then $f \circ u \in SBH(\Omega)$.* $\square$

0.4 REMARK. Denoting by D^2_{ij} the usual distributional partial derivatives and by ξ_i the components of $\boldsymbol{\xi}$, we easily have:

$$(0.25) \quad D^2_{ij}[f \circ u](B) = \int_{\mathbf{R}} \int_{B \cap u_*^{-1}(t)} |\nabla u|_* \, \xi_i \xi_j \, d\mathcal{H}^{N-1} \, df''(t) + \int_B f'_*(u_*(x)) \, dD^2_{ij}u(x)$$

and the interesting expression for the Laplacian

$$(0.26) \quad \Delta[f \circ u](B) = \int_{\mathbf{R}} \int_{B \cap u_*^{-1}(t)} |\nabla u|_* \, d\mathcal{H}^{N-1} \, df''(t) + \int_B \dot{f}_*(u_*(x)) \, d\Delta u(x) \quad \square$$

Our arguments are essentialy based on two underlying ideas: the first one is the relationship between convex and BH–functions. It is well known (see [13], Th. 2 of sect. 6.3) that the Hessian of the difference of two convex function is a Radon measure on the intersection of their domains. We already noticed that, in dimension 1, also the converse is true, i.e.

$BH(a,b) =$ "functions which are difference of convex Lipschitz functions on (a,b)";

this property is deeply used in the proof of the chain rule even in higher dimensions by studying traces of second derivatives along 1 dimensional fibers.

The second basic fact, which also justifies this slicing method, consists in a geometric measure analogue of Dini's Theorem, stating the $\mathcal{H}^{N-1}, (N-1)$-rectifiability of *every* non critical level set of a $BH(\Omega)$-function u. This is another analogy with the properties of convex functions, whose level sets satisfy the thesis of the next theorem. Since this property is interesting itself, we conclude this section stating it precisely.

Theorem 3. *For a function $u \in BH(\Omega)$ and a real number $t \in \mathbf{R}$, let us set*

$$(0.27) \qquad\qquad L_u(t) := \big\{x \in \Omega : \ u_*(x) = t, \quad |\nabla u|_*(x) > 0\big\};$$

then $L_u(t)$ is countably $\mathcal{H}^{N-1}, (N-1)$-rectifiable and $\boldsymbol{\xi}(x)$ is an approximate unit normal to $L_u(t)$ at $\mathcal{H}^{N-1}$-a.e. x.

0.5 REMARK. Theorem 3 entails that the intersections of $L_u(t)$ with a generic Borel set $B \subset \Omega$ are (countably) rectifiable for any $t \in \mathbf{R}$; actually, the first internal integration of (0.17), (0.25), and (0.26) is performed on such sets. $\square$

The plane of the paper is the following: in the next section we collect the definitions and notation we shall use; the proof of theorem 1, which only requires the knowledge of the basic facts about the function of bounded variation, is given in section 2. Then we shall show the validity of the chain rule formula in the one dimensional case and in the last section we conclude the proof of theorem 2 in the multidimensional framework, by establishing the regularity result of theorem 3.

1. - Definitions, notation, and basic properties.

◇ *Preliminary notation.* From now on we fix a Lipschitz and bounded ([6]) open set $\Omega \subset \mathbf{R}^N$. For a given set $U \subset \mathbf{R}^N$ we denote by $\mathcal{H}^k(U)$ its k-dimensional Hausdorff measure and by $|U|$ or $\mathcal{L}^N(U)$ its Lebesgue (outer) measure. $B_\rho(x)$ is the open ball of radius ρ centered at x, $B_\rho := B_\rho(0)$. For every unit vector $\boldsymbol{n} \in \partial B_1$, we call $\pi_{\boldsymbol{n}}$ the hyperplane orthogonal to $\boldsymbol{n}$, separating $\mathbf{R}^N$ in the two demispaces

$$(1.1) \qquad H_+(\boldsymbol{n}) := \{x \in \mathbf{R}^N : \langle x, \boldsymbol{n} \rangle \geq 0\}, \quad H_-(\boldsymbol{n}) := \{x \in \mathbf{R}^N : \langle x, \boldsymbol{n} \rangle \leq 0\}.$$

We call $\mathbf{B}(\Omega)$ the σ-algebra of the Borel subset of Ω. $\mathbf{M}^{h,k}$ is the vector space of the $h \times k$ matrices, endowed with the standard euclidean norm. $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k$ is the canonical basis of $\mathbf{R}^k$; sometimes we will identify $\mathbf{R}^k$ with $\mathbf{M}^{k,1}$. For our purposes the "sign" function (and its vector analogue "$\boldsymbol{s}$") is defined by

$$\text{sign}(v) := \begin{cases} v/|v| & \text{if } v \neq 0, \\ 0 & \text{if } v = 0; \end{cases} \qquad \boldsymbol{s}(\boldsymbol{v}) := \begin{cases} \boldsymbol{v}/|\boldsymbol{v}| & \text{if } \boldsymbol{v} \neq 0, \\ 0 & \text{if } \boldsymbol{v} = 0, \end{cases} \quad \forall\, v \in \mathbf{R}, \ \boldsymbol{v} \in \mathbf{R}^N.$$

◇ *Vector measures and total variation.* A Borel matrix-valued measure is a countably additive map $\mu : \mathbf{B}(\Omega) \mapsto \mathbf{M}^{h,k}$; we associate to μ the real (signed) measures $\mu_{i,j}$ defined as

$$\mu_{i,j}(B) := \langle \mu(B)\boldsymbol{e}_j, \boldsymbol{e}_i \rangle, \quad \forall\, B \in \mathbf{B}(\Omega).$$

For every open set $U \subset \Omega$ we define the (total) variation measure of μ

$$|\mu|(U) = \sup\left\{ \sum_{i=1}^{h} \sum_{j=1}^{k} \int_U \phi_{i,j} \, d\mu_{i,j}, \quad \phi_{i,j} \in C^0_{comp}(U), \quad \sup_{x \in U} \sum_{i,j} |\phi_{i,j}(x)|^2 \leq 1 \right\}.$$

and we extend it to every set by

$$|\mu|(E) := \inf\left\{ |\mu|(U), \ E \subset U, \ U \text{ open subset of } \Omega. \right\}$$

It is possible to see that $|\mu|$ is a Radon measure on Ω ([13], p.5) with $|\mu|(\Omega) < +\infty$; we denote by $\mathcal{M}(\Omega; \mathbf{M}^{h,k})$ the Banach space of the Borel measures on Ω with values in $\mathbf{M}^{h,k}$, normed by their total variation evalued on Ω.

A subset E of Ω is μ-negligible, iff $|\mu|(E) = 0$; we recall that a set E is μ-measurable if there exist an F_σ-set $A \subset E$ and a G_δ-set $B \supset E$ such that $B \setminus A$ is μ-negligible ([17], 2.17 (c)), whereas a function $\psi : \Omega \mapsto \mathbf{R}$ is μ-measurable if there exists a Borel function $\phi : \Omega \mapsto \mathbf{R}$ which coincides with ψ for $|\mu|$-a.e. point of Ω ([17], 7.12, lemma 1).

([6]) for the sake of simplicity: the boundedness could be avoid with some minor technicalities.

◇ *Riesz Representation Theorem.* $\mathcal{M}(\Omega; \mathbf{M}^{h,k})$ can be identified with the space of the bounded linear operators $\mathcal{L}(C_0^0(\Omega); \mathbf{M}^{h,k})$ by the Riesz representation theorem ([13], 1.8, [17], 6.19): for every bounded linear

$$L : C_0^0(\Omega) \mapsto \mathbf{M}^{h,k}$$

there exists a unique measure $\mu = \mu_L$ such that

(1.2)
$$L(\phi) = \int_\Omega \phi \, d\mu, \quad \forall \phi \in C_0^0(\Omega).$$

The mapping $L \mapsto \mu_L$ is a linear isometry, since

(1.3)
$$\|L\|_{\mathcal{L}(C_0^0(\Omega); \mathbf{M}^{h,k})} = |\mu|(\Omega) = \|\mu\|_{\mathcal{M}(\Omega; \mathbf{M}^{h,k})}.$$

◇ *Approximate limits.* For any Borel function $u : \Omega \mapsto X$, X being a (separable) Banach space, the approximate limit of $u(y)$ as y goes to x, is characterized by

(1.4) $\quad \tilde{u}(x) = \underset{y \to x}{\text{ap lim}} \, u(y) \quad \Leftrightarrow \quad \lim_{\rho \to 0} \frac{\left| \{z \in B_\rho(x) : \|u(z) - \tilde{u}\|_X > \varepsilon\} \right|}{\rho^N} = 0, \quad \forall \varepsilon > 0.$

We know ([14], 2.9.13) that u coincides with $\tilde{u}$ at $\mathcal{L}^N$-a.e. point and, if u is also locally integrable in Ω, it holds ([14], 2.9.9)

(1.5)
$$\lim_{\rho \to 0} \fint_{B_\rho(x)} \|u(y) - \tilde{u}(x)\|_X \, dy = 0, \quad \text{for } \mathcal{L}^N\text{-a.e. } x \in \Omega$$

When $X = \mathbf{R}$, this notion can be also expressed by the approximate upper and lower limits: thay are Borel functions denoted by $u_\mu, u_\lambda : \Omega \mapsto [-\infty, +\infty]$ and defined for any $x \in \Omega$ as

$$u_\mu(x) := \inf \left\{ t \in [-\infty, +\infty] : \lim_{\rho \to 0} \rho^{-N} \left| \{y \in B_\rho(x) : u(y) > t\} \right| = 0 \right\}$$

and
$$u_\lambda(x) := \sup \left\{ t \in [-\infty, +\infty] : \lim_{\rho \to 0} \rho^{-N} \left| \{y \in B_\rho(x) : u(y) < t\} \right| = 0 \right\}$$

Outside the jump set

(1.6)
$$S_u := \{x \in \Omega : \ u_\lambda(x) < u_\mu(x)\}$$

the common (when it is finite) value of $u_\mu(x)$ and $u_\lambda(x)$ coincides with $\tilde{u}(x)$. We already stressed the importance for our aims of the *precise representative* $u_* : \Omega \mapsto \mathbf{R}$

$$u_*(x) := \begin{cases} \frac{u_\mu(x) + u_\lambda(x)}{2} & \text{if } -\infty < u_\lambda(x) \text{ and } u_\mu(x) < +\infty; \\ 0 & \text{otherwise.} \end{cases}$$

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When $X = \mathbf{R}^k$, we can consider the components of u along the canonical basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k$, and we define

$$(1.7) \qquad\qquad S_u := \bigcup_{j=1}^{k} S_{\langle u, \boldsymbol{e}_j \rangle}$$

$\diamond$ *Approximate differentiability.* Let $u : \Omega \mapsto \mathbf{R}^k$ be a Borel function, $x \in \Omega \setminus S_u$ such that $\tilde{u}(x) \in \mathbf{R}^k$; we say that u is approximate differentiable at x if there exists a matrix $\nabla u(x) \in \mathbf{M}^{N,k}$ $(^7)$ such that

$$\underset{y \to x}{\text{ap lim}} \frac{|u(y) - \tilde{u}(x) - {}^t\nabla u(x) \cdot (y - x)|}{|y - x|} = 0.$$

For a unit vector $\boldsymbol{\eta} \in \partial B_1$ we set $\nabla_{\boldsymbol{\eta}} u := {}^t\nabla u \cdot \boldsymbol{\eta}$; in particular $\nabla_j u := \nabla_{\boldsymbol{e}_j} u = {}^t\nabla u \cdot \boldsymbol{e}_j$.

$\diamond$ *Functions of Bounded Variation.* The space of the $\mathbf{R}^M$-valued function of bounded variation is defined by

$$(1.8) \qquad\qquad BV(\Omega; \mathbf{R}^M) := \left\{ \boldsymbol{v} \in L^1(\Omega; \mathbf{R}^M) : D\boldsymbol{v} \in \mathcal{M}(\Omega; \mathbf{M}^{M,N}) \right\}$$

where $D\boldsymbol{v}$ is the matrix of the distributional derivatives of v. We list here some basic properties of a function $\boldsymbol{v}$ of this class.

BV1. For $\mathcal{H}^{N-1}$-a.e. point $x \in \Omega \setminus S_{\boldsymbol{v}}$ there exists $\tilde{\boldsymbol{v}}(x) \in \mathbf{R}^k$, satisfying also (1.5) ([23], 5.9.6);

BV2. $S_{\boldsymbol{v}}$ is countably $(\mathcal{H}^{N-1}, N-1)$-rectifiable ([23], 5.9.6);

BV3. $\nabla \boldsymbol{v}$ exists a.e. in Ω and satisfies the Lebesge decomposition (0.9) ([14], 4.5.9(26));

BV4. for $\mathcal{H}^{N-1}$ almost every $x \in S_{\boldsymbol{v}}$ there exists $\boldsymbol{n}_{\boldsymbol{v}}(x) \in \partial B_1$, $\boldsymbol{v}_+(x), \boldsymbol{v}_-(x) \in \mathbf{R}^k$ (outer and inner trace, respectively, of $\boldsymbol{v}$ at x in the direction $\boldsymbol{n}_{\boldsymbol{v}}(x)$) such that ([23], 5.14.3)

$$\lim_{r \to 0} \fint_{B_r(x) \cap H_+} |\boldsymbol{v}(y) - \boldsymbol{v}_+(x)| \, dy = 0, \quad \lim_{r \to 0} \fint_{B_r(x) \cap H_-} |\boldsymbol{v}(y) - \boldsymbol{v}_-(x)| \, dy = 0$$

where $H_{\pm} = x + H_{\pm}(\boldsymbol{n}_{\boldsymbol{v}}(x))$ Moreover for $\mathcal{H}^{N-1}$-a.e. $x \in S_{\boldsymbol{v}}$ we have

$$\begin{cases} \langle \boldsymbol{v}, \boldsymbol{e}_j \rangle_\lambda(x) = \min \left\{ \langle \boldsymbol{v}_-(x), \boldsymbol{e}_j \rangle, \langle \boldsymbol{v}_+(x), \boldsymbol{e}_j \rangle \right\}, \\ \langle \boldsymbol{v}, \boldsymbol{e}_j \rangle_\mu(x) = \max \left\{ \langle \boldsymbol{v}_-(x), \boldsymbol{e}_j \rangle, \langle \boldsymbol{v}_+(x), \boldsymbol{e}_j \rangle \right\} \end{cases}$$

$(^7)$ If u is a regular function, ∇u is the transposed Jacobian matrix; when $k = 1$, since we identify $\mathbf{M}^{N,1}$ with $\mathbf{R}^N$, this notation is consistent with the usual gradient one.

When $\boldsymbol{v} = v$ is scalar valued, we have a canonical way to choose $\boldsymbol{n}_v$, so that $v_+ = v_\mu$, $v_- = v_\lambda$ on S_v.

BV5. The "jump measure" $J\boldsymbol{v} := D\boldsymbol{v} \lfloor S_{\boldsymbol{v}}$ can be expressed by

$$(1.9) \qquad J\boldsymbol{v} = (\boldsymbol{v}_+ - \boldsymbol{v}_-) \otimes \boldsymbol{n_v} \cdot \mathcal{H}^{N-1} \lfloor S_{\boldsymbol{v}}$$

and it satisfies (0.11); defining the Cantor part $C\boldsymbol{v}$ as in (0.12), we have

$$(1.10) \qquad \mathcal{H}^{N-1}(B) = 0 \quad \Rightarrow \quad |C\boldsymbol{v}|(B) = 0.$$

BV6. $\boldsymbol{v}$ admits a trace on $\partial\Omega$; more precisely, there exists a bounded linear operator

$$T : BV(\Omega; \mathbf{R}^k) \mapsto L^1(\partial\Omega; \mathbf{R}^k),$$

such that (see [23], 5.10.7, 5.14.4)

$$(1.11) \qquad \lim_{\rho \to 0} \fint_{B_\rho(x) \cap \Omega} |\boldsymbol{v}(y) - [T\boldsymbol{v}](x)| \, dy = 0 \quad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial\Omega.$$

Denoting by $\boldsymbol{v}^0$ the trivial extension of $\boldsymbol{v}$ outside Ω, $\boldsymbol{v}^0$ belongs to $BV(\mathbf{R}^N; \mathbf{R}^k)$ and ([23], 5.10.5)

$$(1.12) \qquad T\boldsymbol{v} = 2(\boldsymbol{v}^0)_* \quad \text{on } \partial\Omega.$$

When no misunderstanding are possible, we will often write $\boldsymbol{v}_{|\partial\Omega}$ or even $\boldsymbol{v}$ instead of $T\boldsymbol{v}$.

In order to give a precise meaning to (0.8), we fix now a norm on $BV(\Omega; \mathbf{R}^N)$, which is well adapted to our calculations. We split the derivarive of a function $\boldsymbol{v} \in BV(\Omega; \mathbf{R}^N)$ into its symmetric and antisymmetric part

$$SD\boldsymbol{v} := \frac{1}{2}(D\boldsymbol{v} + {}^t D\boldsymbol{v}), \quad AD\boldsymbol{v} := \frac{1}{2}(D\boldsymbol{v} - {}^t D\boldsymbol{v})$$

and we introduce the family $\boldsymbol{E}(\mathbf{R}^N)$ of the $N(N-1)/2$ unit vectors

$$(1.13) \qquad \boldsymbol{e}_{ii} := \boldsymbol{e}_i, \quad \boldsymbol{e}_{ij} := \frac{1}{\sqrt{2}}(\boldsymbol{e}_i + \boldsymbol{e}_j), \quad \text{if } i \neq j, \quad \boldsymbol{E}(\mathbf{R}^N) := \{\boldsymbol{e}_{ij}\}_{i,j=1,\dots,N}$$

Finally we consider the "component measures" of $SD\boldsymbol{v}$ and $AD\boldsymbol{v}$

$$SD_{ij}\boldsymbol{v} := \langle SD\boldsymbol{v} \cdot \boldsymbol{e}_{ij}, \boldsymbol{e}_{ij} \rangle = \langle D\boldsymbol{v} \cdot \boldsymbol{e}_{ij}, \boldsymbol{e}_{ij} \rangle,$$

$$AD_{ij}\boldsymbol{v} := \langle AD\boldsymbol{v} \cdot \boldsymbol{e}_i, \boldsymbol{e}_j \rangle = \frac{1}{2} \left(\langle D\boldsymbol{v} \cdot \boldsymbol{e}_i, \boldsymbol{e}_j \rangle - \langle D\boldsymbol{v} \cdot \boldsymbol{e}_j, \boldsymbol{e}_i \rangle \right)$$

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and we set

$$(1.14) \qquad \|\boldsymbol{v}\|_{BV(\Omega;\mathbf{R}^N)} := \sum_{i,j=1}^{N} \left(|SD_{ij}\boldsymbol{v}|(\Omega) + |AD_{ij}\boldsymbol{v}|(\Omega) + \int_{\partial\Omega} \langle \boldsymbol{v}, \boldsymbol{e}_{ij} \rangle \, d\mathcal{H}^{N-1} \right)$$

◇ *Bounded hessian functions.* We define

$$BH(\Omega) := \left\{ u \in W^{1,1}(\Omega) : D^2 u \in \mathcal{M}(\Omega; \mathbf{M}^{N,N}) \right\} = \left\{ u \in L^1(\Omega) : Du \in BV(\Omega; \mathbf{R}^N) \right\}.$$

We notice that the antisymmetric component AD^2u of function $u \in BH(\Omega)$ vanishes and the derivative Du is absolutely continuous with repect to $\mathcal{L}^N$: therefore we will identify it with the approximate differential ∇u, which exists at $\mathcal{H}^{N-1}$-a.e. point of Ω. Correspondingly we set

$$\nabla^2 u := \nabla(\nabla u), \quad J^2(u) := J(\nabla u), \quad C^2(u) := C(\nabla u)$$

and we can apply the properties detailed in the previous points BV1-5 to this framework:

1.1 PROPOSITION. *Let u be a $BH(\Omega)$-function; then for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$*

$$(1.15) \qquad u_\lambda(x) = u_\mu(x) = u_*(x) = \tilde{u}(x), \quad \lim_{r \to 0} \fint_{B_r(x)} |u(y) - u_*(x)| \, dy = 0.$$

Moreover, for every unit vector $\boldsymbol{\eta} \in \partial B_1$ and for $\mathcal{H}^{N-1}$-a.e. $x \notin S_{\nabla u}$, it holds

$$(1.16) \qquad \begin{cases} \left(\nabla_{\boldsymbol{\eta}} u \right)_\lambda(x) = \left(\nabla_{\boldsymbol{\eta}} u \right)_\mu(x) = \text{ap} \lim_{y \to x} \nabla_{\boldsymbol{\eta}} u(y) = \langle \nabla u(x), \boldsymbol{\eta} \rangle \\[2mm] \lim_{r \to 0} \fint_{B_r(x)} |\nabla u(y) - \nabla u(x)| \, dy = 0 \\[2mm] \fint_{B_r(x)} |u(y) - u_*(x) - \langle \nabla u(x), y - x \rangle| \, dy = o(r), \end{cases}$$

and, setting for $\mathcal{H}^{N-1}$-a.e. $x \in S_{\nabla u}$ (see BV4.)

$$\nabla_\pm u(x) := \left(\nabla u \right)_\pm(x), \quad \boldsymbol{n}(x) := \boldsymbol{n}_{\nabla u}(x), \quad H_\pm := x + H_\pm(\boldsymbol{n}_{\nabla u}),$$

it holds

$$(1.17) \qquad \begin{cases} \left(\nabla_{\boldsymbol{\eta}} u \right)_\lambda(x) = \min \left[\langle \nabla_+ u(x), \boldsymbol{\eta}(x) \rangle; \langle \nabla_- u(x), \boldsymbol{\eta}(x) \rangle \right] \\ \left(\nabla_{\boldsymbol{\eta}} u \right)_\mu(x) = \max \left[\langle \nabla_+ u(x), \boldsymbol{\eta}(x) \rangle; \langle \nabla_- u(x), \boldsymbol{\eta}(x) \rangle \right] \\[2mm] \lim_{r \to 0} \fint_{B_r(x) \cap H_\pm} |\nabla u(y) - \nabla_\pm u(x)| \, dy = 0 \\[2mm] \fint_{B_r(x) \cap H_\pm} |u(y) - u_*(x) - \langle \nabla_\pm u(x), y - x \rangle| \, dy = o(r). \end{cases}$$

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PROOF.   The property (1.15) follows by the Sobolev inclusion of $BH(\Omega)$ into $W^{1,1^*}(\Omega)$ ($1^* := N/(N-1)$), by the fine properties of this kind of functions ([13], 4.8), and by the relationships between the Hausdorff measures and the p-capacities ([13], 4.7.2; $p = 1^*$ in this case).

The previous point BV1. applied to $\nabla u \in BV(\Omega; \mathbf{R}^N)$ gives the first two equations of (1.16); analogously BV4. gives the first three formula of (1.17).

Finally, applying the same proof of [13], 6.1.1, we find the other two relations (^{8}). ∎

◇ *Functions of one variable.* When Ω is a bounded open interval $\mathcal{I} = (a, b) \subset \mathbf{R}$, many properties and definitions become easier or can be expressed in a different way.

First of all, we observe that for every function $u \in BV(\mathcal{I}; \mathbf{R}^k)$ we have (see [15], 1.30 and [7], sect. A.2)
(1.18)
$$|Du|(\mathcal{I}) = \sup_{0 < h < b-a} \frac{1}{h} \int_a^{b-h} |u(x+h) - u(x)| \, dx = \lim_{h \to 0} \frac{1}{h} \int_a^{b-h} |u(x+h) - u(x)| \, dx$$

and the integral terms of (1.18) are equivalent characterization of the *essential variation* of the function u (see [15] again). We take it as a definition and, when X is a Banach space and $u \in L^1(a, b; X)$, we set

(1.19)
$$\text{ess-}V_a^b(u) = \text{ess-}V_{\mathcal{I}}(u) := \sup_{0 < h < b-a} \frac{1}{h} \int_a^{b-h} \|u(x+h) - u(x)\|_X \, dx$$
$$= \lim_{h \to 0} \frac{1}{h} \int_a^{b-h} \|u(x+h) - u(x)\|_X \, dx.$$

Cooerepondingly we call ([7], Appendix)

$$BV(a, b; X) := \left\{ v \in L^1(a, b; X) : \text{ess-}V_a^b(v) < +\infty \right\} \; (^9)$$

Of course, $BV(a, b; X) \subset L^\infty(a, b; X)$, and this definition coincides with the previous one when $X = \mathbf{R}^k$.

(^{8}) Observe that they holds except for a $\mathcal{H}^{N-1}$-negligible set, whereas the analogous statement for BV-functions of [13] holds only $\mathcal{L}^N$-a.e.

(^{9}) When Ω is not an interval, we call Λ the (countable) collection of its connected components, and we set
$$\text{ess-}V_\Omega(v) := \sum_{\mathcal{I} \in \Lambda} \text{ess-}V_{\mathcal{I}}(v)$$

For a function $v \in L^1(\mathcal{I}; X)$ we denote by $[v]_h$ its Steklov averaging

$$[v]_h(t) = \fint_t^{t+h} v(\tau)\, d\tau, \qquad h \in\,]0, b-a[, \quad t \in [a, b-h].$$

If v is of bounded variation there exist the limits

$$\lim_{h \to 0^+} [v]_h(t) = v(t_+), \quad \lim_{h \to 0^+} [v]_h(t-h) = v(t_-)$$

at every point $t \in [a, b]$, with the natural meaning for $t = a$ or b. We have easily

$$S_v = \big\{ t \in (a, b) : v(t_+) \neq v(t_-) \big\}, \quad v(t_+) = v_+(t),\ v(t_-) = v_-(t) \text{ if we choose } \boldsymbol{n}_v = 1$$

Accordingly with (1.14), we set

(1.20) $$\|v\|_{BV(a,b;X)} := \text{ess-}V_a^b(v) + \|v(a_+)\|_X + \|v(b_-)\|_X$$

A function u will belong to $BH(a, b; X)$ if it is in $W^{1,1}(a, b; X)$ and its derivative is of bounded variation: $\dot{u} = u' \in BV(a, b; X)$. It is easy to see that u admits a continuous representative which is left and right differentiable at *every* point of (a, b).

$\diamond$ *"Fubinization" of a measure.* Given a unitary vector $\boldsymbol{\eta}$ of $\mathbf{R}^N$, we associate to a generic point $x \in \Omega$ the couple $(y, z) \in \pi^{\boldsymbol{\eta}} \times \mathbf{R}$ such that

$$x = y + z\boldsymbol{\eta},$$

and to every x-depending function u the family of real function $u_y : z \mapsto u_y(z) = u(y + z\boldsymbol{\eta}) = u(x)$ depending on the parameter y.

In order to make precise our notation, for a function $u : \Omega \subset \mathbf{R}^N \mapsto \mathbf{R}$, and a Borel set $E \subset \Omega$ we set

$$\begin{cases} E_y^{\boldsymbol{\eta}} = \{ z \in \mathbf{R} : y + z\,\boldsymbol{\eta} \in E \}, \quad \text{for } y \in \pi^{\boldsymbol{\eta}}, \qquad \tilde{\pi}^{\boldsymbol{\eta}} = \{ y \in \pi^{\boldsymbol{\eta}} : \Omega_y^{\boldsymbol{\eta}} \neq \emptyset \}, \\ u_y^{\boldsymbol{\eta}} : t \in \Omega_y^{\boldsymbol{\eta}} \mapsto u(x + t\,\boldsymbol{\eta}), \quad \text{for } y \in \tilde{\pi}^{\boldsymbol{\eta}}, \end{cases}$$

possibly suppressing the superscript $\boldsymbol{\eta}$, when no misunderstanding occours. The following results (see [3]) allows us to reconstruct a measure μ on Ω from its "sections" μ_y along the fibers Ω_y.

1.2 THEOREM. *Let us fix a unit vector $\eta \in R^N$ and assume that for $\mathcal{H}^{N-1}$-a.e. $y \in \tilde{\pi}^{\boldsymbol{\eta}}$ we are given a Borel measure μ_y on Ω_y in such a way that*

(FU_1) $\qquad y \mapsto \mu_y(E_y)$ *is an $\mathcal{H}^{N-1}$-measurable function,* $\quad \forall E \in \mathbf{B}(\Omega)$

and

$$(FU_2) \qquad \int_{\tilde{\pi}^{\boldsymbol{\eta}}} |\mu_y| \, (\Omega_y) \, d\mathcal{H}^{N-1}(y) < +\infty.$$

We can define a new measure μ denoted by

$$(1.21) \qquad \mu := \int_{\tilde{\pi}^{\boldsymbol{\eta}}} \mu_y \, d\mathcal{H}^{N-1}$$

such that for every Borel function $\phi : \Omega \mapsto \mathbf{R}$

$$(1.22) \qquad \int_{\Omega} \phi(x) \, d \left(\int_{\tilde{\pi}^{\boldsymbol{\eta}}} \mu_y \, d\mathcal{H}^{N-1} \right)(x) = \int_{\tilde{\pi}^{\boldsymbol{\eta}}} \left(\int_{\Omega_y} \phi_y(z) \, d\mu_y(z) \right) d\mathcal{H}^{N-1}(y)$$

Moreover we can express the total variation of (1.21) in the same form

$$(1.23) \qquad \left| \int_{\tilde{\pi}^{\boldsymbol{\eta}}} \mu_y \, d\mathcal{H}^{N-1} \right| = \int_{\tilde{\pi}^{\boldsymbol{\eta}}} |\mu_y| \, d\mathcal{H}^{N-1}$$

$\diamond$ *Fibers of BH-functions.* The one-dimensional fibers of BH-functions satisfy good properties as in the case of SBH ([9], thm.3) and we can adapt to the BH-setting the deep slicing result of [2], [3] for BV (see [9] for a similar extension). We have:

1.3 THEOREM. *Let $u \in BH(\Omega)$ and $\boldsymbol{\eta}$ a unitary vector of $\mathbf{R}^N$. For $\mathcal{H}^{N-1}$-a.e. $y \in \tilde{\pi}^{\boldsymbol{\eta}}$ we have*

$$(1.24) \qquad (u_*)_y \in BH(\Omega_y) \cap C^0(\overline{\Omega}_y)$$

and, setting

$$\dot{u}_y := (u_y)^{\cdot}, \quad \dot{u}_y(z_{\pm}) := \lim_{s \to z^{\pm}} \frac{u_*(y + s\boldsymbol{\eta}) - u_*(x + z\boldsymbol{\eta})}{s - z},$$

$$(1.25) \qquad \begin{cases} \left[(\nabla_{\boldsymbol{\eta}} u)_{\mu} \right]_y (z) = \max \left[\dot{u}_y(z_+), \dot{u}_y(z_-) \right] \\ \left[(\nabla_{\boldsymbol{\eta}} u)_{\lambda} \right]_y (z) = \min \left[\dot{u}_y(z_+), \dot{u}_y(z_-) \right] \end{cases}$$

$$(1.26) \qquad \left[S_{\nabla_{\boldsymbol{\eta}} u} \right]_y = S_{(\dot{u}_y)}$$

and

$$(1.27) \quad \left(T(\nabla_{\boldsymbol{\eta}} u) \right)_y = T\dot{u}_y, \quad \int_{\partial \Omega} |\nabla_{\boldsymbol{\eta}} u| \, |\langle \boldsymbol{\eta}, \boldsymbol{\nu} \rangle| \, d\mathcal{H}^{N-1} = \int_{\tilde{\pi}^{\boldsymbol{\eta}}} \sum_{z \in \partial \Omega_y} |\dot{u}_y(z)| \, d\mathcal{H}^{N-1}(y)$$

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where $\boldsymbol{\nu}$ is the exterior unit normal of $\partial\Omega$. Moreover if $\mathcal{D}$ is one of the operators D, ∇, C, J and we define

$$\mathcal{D}^2_{\boldsymbol{\eta\eta}} u := \langle \mathcal{D}^2 u \cdot \boldsymbol{\eta}, \boldsymbol{\eta} \rangle$$

we have

(1.28)
$$\mathcal{D}^2_{\boldsymbol{\eta\eta}} u = \int_{\tilde{\pi}^{\boldsymbol{\eta}}} \mathcal{D}\dot{u}_x \, d\mathcal{H}^{N-1}$$

PROOF.   Let us set

$$v := \langle \nabla u, \boldsymbol{\eta} \rangle = \nabla_{\boldsymbol{\eta}} u$$

which belongs to $BV(\Omega; \mathbf{R}^N)$ and satisfies obviously

$$\langle \mathcal{D}v, \boldsymbol{\eta} \rangle = \mathcal{D}^2_{\boldsymbol{\eta\eta}} u, \quad \mathcal{D} \in \{D, \nabla, J, C\}.$$

If we apply the theorem 3.2 of [3], we have for $\mathcal{H}^{N-1}$-a.e. $y \in \tilde{\pi}^{\boldsymbol{\eta}}$

(1.29)
$$\langle \mathcal{D}v, \eta \rangle = \int_{\tilde{\pi}^{\boldsymbol{\eta}}} \mathcal{D}v_y \, d\mathcal{H}^{N-1}, \quad \left[S_v \right]_y = S_{v_y}$$

and
(1.30)
$$\left(v_\mu \right)_y(z) = \max \left[v_y(z_+), v_y(z_-) \right], \quad \left[v_\lambda \right]_y(z) = \min \left[v_y(z_+), v_y(z_-) \right], \quad \forall z \in \Omega_y.$$

Formulae (1.24),…,(1.28) follow ([10]) if we show that for $\mathcal{H}^{N-1}$-a.e. $y \in \tilde{\pi}^{\boldsymbol{\eta}}$

(1.31)
$$v_y(z_\pm) = \dot{u}_y(z_\pm), \quad \forall z \in \Omega_y$$

But we can now apply the same result to u itself; since $\mathcal{H}^{N-1}(S_u) = 0$ we deduce that $(S_u)_y$ is empty for $\mathcal{H}^{N-1}$-a.e. $y \in \tilde{\pi}^{\boldsymbol{\eta}}$ and

$$\left(u_* \right)_y(z) = \left(u_\lambda \right)_y(z) = \left(u_\mu \right)_y(z) = u_y(z_\pm), \quad \forall z \in \Omega_y$$

i.e. $\left(u_* \right)_y(z)$ is the continuous representative of u_y. The same theorem implies $v_y(z) = \dot{u}_y(z)$ for $\mathcal{H}^1$-a.e. $z \in \Omega_y$. ∎

Finally we show that the $BH(\Omega)$-regularity can be recovered by the slicing procedure

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([10]) observe that the trace property (1.27) can be rewritten through the jump set of the trivial extension $(\nabla u)^0$ of ∇u: see BV6.

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1.4 THEOREM. *Let u be a function of $W^{1,1}(\Omega)$ such that for every unitary vector*
$\boldsymbol{\eta} \in \boldsymbol{E}(\mathbf{R}^N)$

$$\int_{\tilde{\pi}\boldsymbol{\eta}} \text{ess-}V_{\Omega_y^{\boldsymbol{\eta}}}(\dot{u}_y^{\boldsymbol{\eta}})\, d\mathcal{H}^{N-1}(y) = \int_{\tilde{\pi}\boldsymbol{\eta}} |D\dot{u}_y^{\boldsymbol{\eta}}|(\Omega_y^{\boldsymbol{\eta}})\, d\mathcal{H}^{N-1}(y) < +\infty.$$

Then $u \in BH(\Omega)$ and

$$(1.32) \qquad \|Du\|_{BV(\Omega,\mathbf{R}^N)} = \sum_{\boldsymbol{\eta} \in \boldsymbol{E}(\mathbf{R}^N)} \int_{\tilde{\pi}\boldsymbol{\eta}} \text{ess-}V_{\Omega_y^{\boldsymbol{\eta}}}(\dot{u}_y^{\boldsymbol{\eta}})\, d\mathcal{H}^{N-1}(y) + \int_{\partial\Omega} |\nabla_{\boldsymbol{\eta}} u|\, d\mathcal{H}^{N-1}$$

PROOF. By (1.14), and the definition of the total variation of a measure, we have
to bound the integrals

$$\int_{\Omega} u(x)\nabla_{\boldsymbol{\eta}\boldsymbol{\eta}}^2 \phi(x)\, dx = -\int_{\Omega} \nabla_{\boldsymbol{\eta}} u(x)\nabla_{\boldsymbol{\eta}} \phi(x)\, dx$$

in terms of the $L^\infty(\Omega)$-norm of a generic $C^2(\Omega)$ function ϕ with compact support. By
a standard localization procedure, we can adapt the result of [19] relative to $\Omega = \mathbf{R}^N$.
∎

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We first consider the case of a one-variable function u defined on a bounded open interval $\Omega = \mathcal{I} = (a, b) \subset \mathbf{R}$ with values in a Banach space X, which is the domain of a convex Lipschitz real function f.

We begin with a simple lemma:

2.1 LEMMA. *For every function $v \in BV(a, b; X)$ we have*

$$(2.1) \qquad \int_a^{b-h} \|v(x) - [v]_h(x)\|_X \, dx \le \frac{h}{2} \text{ess-}V_a^b(v).$$

PROOF.

$$\int_a^{b-h} \|v(x) - [v]_h(x)\|_X \, dx = \int_a^{b-h} \left\| \fint_0^h (v(x) - v(x+\tau)) \, d\tau \right\|_X dx \le$$

$$\int_a^{b-h} \fint_0^h \|(v(x) - v(x+\tau)\|_X \, d\tau \, dx = \fint_0^h \int_a^{b-h} \|v(x+\tau) - v(x)\|_X \, dx \, d\tau \le$$

$$\fint_0^h \text{ess-}V_a^b(v) \, \tau \, d\tau \le \frac{h}{2} \text{ess-}V_a^b(v) \quad \blacksquare$$

Recalling (1.20), we prove

2.2 THEOREM. *Let $u \in BH(\mathcal{I}; X)$ and $f : X \mapsto \mathbf{R}$ be a convex function with Lipshitz constant L. The mapping $x \in \mathcal{I} \mapsto f(u(x))$ belongs to $BH(\mathcal{I}; \mathbf{R})$, and we have the estimate:*

$$\left\| (f \circ u)' \right\|_{BV(\mathcal{I};\mathbf{R})} \le 2L \|u'\|_{BV(\mathcal{I};X)}.$$

PROOF. The function $x \in \mathcal{I} \mapsto f(u(x))$ is Lipschitz; denoting by X' the dual space of X, there exists a weakly*–measurable and essentialy bounded map $\theta : \mathcal{I} \mapsto X'$ such that

$$(2.2) \qquad \frac{d}{dx}(f(u(x)) = \langle \theta(x), \dot{u}(x) \rangle, \quad \theta(x) \in \partial f(u(x)) \qquad \text{for a.e. } x \in \mathcal{I}$$

where $\partial f : X \mapsto 2^{X'}$ denotes the subdifferential of f (see f.i. [7], [12]).

Let us now evaluate the essential variation of $v(x) := \langle \theta(x), \dot{u}(x) \rangle$:

$$\int_a^{b-h} \left| \langle \theta(x+h), \dot{u}(x+h) \rangle - \langle \theta(x), \dot{u}(x) \rangle \right| dt =$$

$$\int_a^{b-h} \left| \langle \theta(x+h) - \theta(x), [\dot{u}]_h(x) \rangle + \langle \theta(x+h), \dot{u}(x+h) - [\dot{u}]_h(x) \rangle - \right.$$

$$\left. \langle \theta(x), \dot{u}(x) - [\dot{u}]_h(x) \rangle \right| dt \le$$

$$\int_a^{b-h} \langle \theta(x+h) - \theta(x), [\dot{u}]_h(x) \rangle \, dt +$$

$$\int_a^{b-h} \left| \langle \theta(x+h), \dot{u}(x+h) - [\dot{u}]_h(x) \rangle - \langle \theta(x), \dot{u}(x) - [\dot{u}]_h(x) \rangle \right| dt$$

since

$$[\dot{u}]_h(x) = \frac{u(x+h) - u(x)}{h} \quad \text{and} \quad \frac{1}{h} \langle \theta(x+h) - \theta(x), u(x+h) - u(x) \rangle \ge 0,$$

by the monotonicity property of the subdifferential of f. We write the first positive integral into the form

$$\int_a^{b-h} \langle \theta(x+h), [\dot{u}]_h(x) - \dot{u}(x+h) \rangle \, dt - \int_a^{b-h} \langle \theta(x), [\dot{u}]_h(x) - \dot{u}(x) \rangle \, dt +$$

$$\int_{b-h}^b \langle \theta(x), \dot{u}(x) \rangle \, dt - \int_a^{a+h} \langle \theta(x), \dot{u}(x) \rangle \, dt$$

so that

$$\text{ess-}V_a^b(\dot{v}) \le$$

$$\limsup_{h \to 0^+} \frac{1}{h} \left\{ \int_a^{b-h} 2 \left[\langle \theta(x+h), [\dot{u}]_h(x) - \dot{u}(x+h) \rangle^+ + 2 \langle \theta(x), \dot{u}(x) - [\dot{u}]_h(x) \rangle^+ \right] dt + \right.$$

$$\left. \int_{b-h}^b \langle \theta(x), \dot{u}(x) \rangle \, dt - \int_a^{a+h} \langle \theta(x), \dot{u}(x) \rangle \, dt \right\} \le$$

$$L \left\{ 2 \, \text{ess-}V_a^b(\dot{v}) + \| \dot{u}(b_-) \|_X + \| \dot{u}(a_+) \|_X \right\} \quad \blacksquare$$

2.3 REMARK. When Ω is the union of disjoint intervals $\mathcal{I} \in \Lambda$, with

(2.3) $$\bar{I} \cap \bar{J} = \emptyset, \quad \forall \mathcal{I}, \mathcal{J} \in \Lambda$$

we obviously have

(2.4) $$\text{ess-}V_\Omega((f \circ u)') \le 2L \text{ess-}V_\Omega(\dot{u}) + L \sum_{x \in \partial \Omega} \| \dot{u}(x) \|_X \quad \square$$

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2.4 COROLLARY. *Let Ω be a Lipschitz bounded open set of $\mathbf{R}^N$, u be a function of $BH(\Omega; \mathbf{R}^M)$, and $f : \mathbf{R}^M \mapsto \mathbf{R}$ be convex with Lipschitz constant L. Then $f \circ u \in BH(\Omega)$ and for every unitary vector $\boldsymbol{\eta} \in \mathbf{R}^N$*

$$(2.5) \qquad \left| D^2_{\boldsymbol{\eta}\boldsymbol{\eta}}(f \circ u) \right|(\Omega) \leq 2L \left| D^2_{\boldsymbol{\eta}\boldsymbol{\eta}} u \right|(\Omega) + L \int_{\partial\Omega} |\nabla_{\boldsymbol{\eta}} u| \, |\langle \boldsymbol{\eta}, \boldsymbol{\nu} \rangle| \, d\mathcal{H}^{N-1}$$

where ν is the unit exterior normal to $\partial\Omega$. In particular (0.8) holds for the choice (1.14).

PROOF. Since $v := f \circ u$ belongs to $W^{1,1}(\Omega)$, by applying the characterization of theorem 1.4, we have to bound

$$\text{ess-}V_{\Omega^{\boldsymbol{\eta}}_y}(\dot{v}_y)$$

for $\mathcal{H}^{N-1}$-a.e. $y \in \tilde{\pi}^{\boldsymbol{\eta}}$, $\boldsymbol{\eta}$ being a generic unit vector of $\mathbf{R}^N$.

Being Ω Lipschitz, for $\mathcal{H}^{N-1}$-a.e. $y \in \tilde{\pi}^{\boldsymbol{\eta}}$, $\Omega^{\boldsymbol{\eta}}_y$ is a finite union of disjoint intervals $\mathcal{I} \in \Lambda^{\boldsymbol{\eta}}_y$ satisfying the analogous of (2.3). Since $v^{\boldsymbol{\eta}}_y = f \circ u^{\boldsymbol{\eta}}_y$, we apply (2.4), obtaining for $\mathcal{H}^{N-1}$-a.e. $y \in \tilde{\pi}^{\boldsymbol{\eta}}$

$$\text{ess-}V_{\Omega^{\boldsymbol{\eta}}_y}(\dot{v}_y) \leq 2L\text{ess-}V_{\Omega^{\boldsymbol{\eta}}_y}(\dot{u}^{\boldsymbol{\eta}}_y) + L \sum_{z \in \partial\Omega^{\boldsymbol{\eta}}_y} |\dot{u}^{\boldsymbol{\eta}}_y(z)|$$

Integrating with respect to $y \in \tilde{\pi}^{\boldsymbol{\eta}}$ and taking account of (1.28), (1.23), and (1.27), we get (2.5). We conclude by (1.32). ∎

We prove the corollary to the theorem 1 by a representation results for $BH(\mathbf{R})$-functions, which will turn useful in the next sections; a simple consequence will be the decomposition of such functions into the difference of two convex ones.

For a given Borel measure μ on $\mathbf{R}$ with $|\mu|(\mathbf{R}) < +\infty$ we set (see [18])

$$(2.6) \qquad I_2[\mu](x) = \frac{1}{2} \int_{\mathbf{R}} p_t(x) \, d\mu(t), \qquad p_t(x) = |x - t| - |t|$$

this operator allows us to reconstruct a function from its second derivative:

2.5 LEMMA. *If $f : \mathbf{R} \mapsto \mathbf{R}$ is a locally integrable function with $f'' \in \mathcal{M}(\mathbf{R})$, then f is Lipschitz and there exist $a, b \in \mathbf{R}$ such that*

$$(2.7) \qquad f(0) = a, \quad \frac{1}{2}\left[\dot{f}(+\infty) + \dot{f}(-\infty)\right] = b, \quad and \quad f(x) = a + b\,x + I_2[f''](x)$$

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2.6 REMARK. Let $f'' = \left(f''\right)^+ - \left(f''\right)^-$ be the usual Hahn decomposition of f'', $\left(f''\right)^\pm$ being positive finite Borel measures on $\mathbf{R}$. We obtain correpondingly $f = f_{conv} - f_{conc} + a + bx$ where a, b are given by the previous theorem and f_{conv}, f_{conc} are the convex functions

$$f_{conv}(x) = I_2\big[(f'')^+\big](x), \qquad f_{conc}(x) = I_2\big[(f'')^-\big](x) \quad \square$$

PROOF of the lemma. Let $g(x) = I_2[f''](x)$ and ϕ a smooth test function; we have

$$-2\int_{\mathbf{R}} g(x)\phi'(x)\, dx = -\int_{\mathbf{R}} \phi'(x)\left\{\int_{\mathbf{R}} p_t(x)\, df''(t)\right\} dx =$$

$$-\int_{\mathbf{R}} \left\{\int_{\mathbf{R}} \phi'(x)p_t(x)\, dx\right\} df''(t) =$$

$$\int_{\mathbf{R}} \left\{\int_{\mathbf{R}} \phi(x)\operatorname{sign}(x - t)\, dx\right\} df''(t) =$$

$$\int_{\mathbf{R}} \phi(x)\left\{\int_{\mathbf{R}} \operatorname{sign}(x - t)\, df''(t)\right\} dx =$$

$$\int_{\mathbf{R}} \phi(x)\Big[f'(x_-) - f'(-\infty) + f'(x_+) - f'(+\infty)\Big]\, dx$$

so that $g' = f' - b$ in the sense of distribution. Since $g(0) = 0$, we conclude. ■

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3. - The chain rule in one-dimensional case.

In this section we identify every function u of $BH(\mathcal{I}; \mathbf{R})$ with its continuous representative; hence u is Lipschitz and, recalling the properties detailed in the first section, we set

$$(3.1) \qquad \dot{u}(x_+) = \lim_{y \to x^+} \frac{u(y) - u(x)}{y - x}, \qquad \dot{u}(x_-) = \lim_{y \to x^-} \frac{u(y) - u(x)}{y - x}$$

with

$$[\dot{u}]_*(x) = \frac{\dot{u}(x_+) + \dot{u}(x_-)}{2}, \quad \dot{u}_\mu(x) = \max[\dot{u}(x_+), \dot{u}(x_-)], \quad \dot{u}_\lambda(x) = \min[\dot{u}(x_+), \dot{u}(x_-)]$$

$$t \notin S_{\dot{u}} \iff \dot{u}(x_-) = \dot{u}(x_+); \qquad t \in S_{\dot{u}} \implies \boldsymbol{n}_{\dot{u}}(x) = \operatorname{sign}(\dot{u}(x_+) - \dot{u}(x_-)).$$

In particular, the jump set of the derivative of a BH-function can be determined by the knowledge of the pointwise right and left derivatives (3.1). We have:

3.1 Lemma. *Let u be in $BH(\mathcal{I})$ and let us set*

$$v(x) := p_t(u(x)) = |u(x) - t| - |t|, \quad \textit{for a fixed } t \in \mathbf{R};$$

then $v \in BH(\mathcal{I})$ and v'' admits the following decomposition:

$$(3.2) \qquad\qquad\qquad \ddot{v} = \operatorname{sign}(u - t)\, \ddot{u},$$

$$(3.3) \quad C^2 v = \operatorname{sign}(u - t)\, C^2 u; \qquad J^2 v = \operatorname{sign}(u - t)\, J^2 u + 2|\dot{u}|_* \mathcal{H}^0 \lfloor \{x : u(x) = t\}$$

Proof. It is not too restrictive to consider the case $t = 0$. First we note that the continuity of u allows a precise definition of the open sets $\{x : u(x) > 0\}$, $\{x : u(x) < 0\}$ and of the closed one $\{x : u(x) = 0\}$; $\operatorname{sign}(u)$ is a Borel function. We know that v is also Lipschitz and

$$\begin{cases} v = u & \text{on } \{x : u(x) > 0\}, \\ v = -u & \text{on } \{x : u(x) < 0\}, \end{cases}$$

so that it remains to characterize $D\dot{v} \lfloor \{x : u(x) = 0\}$.

It is clear that if $u(x) = 0$ then

$$\begin{cases} \dot{u}(x_+) > 0 \Rightarrow \dot{v}(x_+) = \dot{u}(x_+) & \text{since } v(y) = u(y) \text{ in a right neighborhood of } x; \\ \dot{u}(x_+) < 0 \Rightarrow \dot{v}(x_+) = -\dot{u}(x_+) & \text{since } v(y) = -u(y) \text{ in a right neighborhood of } x; \\ \dot{u}(x_+) = 0 \Rightarrow \dot{v}(x_+) = 0 & \text{since } \lim_{y \to x^+} \left|\frac{v(y)}{y - x}\right| = \lim_{y \to x^+} \left|\frac{u(y)}{y - x}\right| = 0. \end{cases}$$

and consequently $\dot{v}(x_+) = |\dot{u}(x_+)|$; analogously, we have $\dot{v}(x_-) = -|\dot{u}(x_-)|$ so that the formula for $J^2 v = J\dot{v}$ is correct.

On the other hand,

$$\{x : u(x) = 0\} \subset S_{\dot{v}} \cup \{x : \dot{v}(x) = 0\}$$

$S_{\dot{v}}$ is surely negligible with respect to $|\ddot{v}|$ and $|C^2 v|$; the same holds for $\{x : \dot{v}(x) = 0\}$ by the Fleming-Rishel coarea formula (see, f.i. [2]). ∎

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In order to evaluate on test functions the measure associated to the second derivative of a composition, we introduce the following definition.

3.2 DEFINITION. *Let u be a fixed function of $BH(\mathcal{I})$; for every $B \in \mathbf{B}(\mathcal{I})$ we set*

$$(3.4) \qquad \mathcal{N}(B;t) := \mathcal{H}^0(B \cap u^{-1}\{t\}) = \mathcal{H}^0\{x \in B : u(x) = t\}$$

and for every bounded Borel function $\phi : \mathcal{I} \mapsto \mathbf{R}$ (everywhere defined) we set

$$(3.5) \qquad \mathcal{G}(\phi;t) := \sum_{\substack{x \in \mathcal{I} \\ u(x)=t}} \phi(x)|\dot{u}(x)|_* = \int_{x \in \mathcal{I}: u(x)=t} \phi \, dJ^2\big[p_t(u)\big] \quad \square$$

3.3 REMARK. The function $\mathcal{N}(B;\cdot)$ is the *multiplicity* of $u_{|B}$, which is denoted by $N(u_{|B}, t)$ in [14], 2.10.9 ([11]). Since u is continuous, we recall that $\mathcal{N}(B;\cdot)$ is a μ-measurable function, whenever μ is a Borel measure on $\mathbf{R}$ ([14], 2.10.10, 2.2.13) and the mapping

$$(3.6) \qquad B \in \mathbf{B}(\mathcal{I}) \mapsto \zeta(B) := \int_{\mathbf{R}} \mathcal{N}(B;t) \, d\mu(t)$$

is a Borel measure on $\mathcal{I}$. $\square$

We list here some useful properties of $\mathcal{G}$ we shall use:

3.4 PROPOSITION. *For every bounded Borel function $\phi : \mathcal{I} \mapsto \mathbf{R}$, the function $\mathcal{G}(\phi;\cdot)$ is bounded by*

$$(3.7) \qquad \sup_{t \in \mathbf{R}} |\mathcal{G}(\phi;t)| \leq \|\dot{u}\|_{BV(\mathcal{I})} \cdot \sup_{x \in \mathcal{I}} |\phi(x)|,$$

and it is μ-measurable, whenever μ is a Borel measure on $\mathbf{R}$ with $|\mu|(\mathbf{R}) < +\infty$. Moreover, the mapping

$$(3.8) \qquad B \in \mathbf{B}(\mathcal{I}) \mapsto \mu_G(B) := \int_{\mathbf{R}} \mathcal{G}(\chi_B;t) \, d\mu(t) = \int_{\mathbf{R}} \sum_{\substack{x \in B \\ u(x)=t}} |\dot{u}|_*(x) \, d\mu(t)$$

defines a finite Borel measure μ_G on $\mathcal{I}$, which satisfies $\mu_G = |\dot{u}|_* \cdot \zeta$, ζ given by (3.6).

([11]) For the sake of simplicity, we suppressed the occurence of u in our notation, since we will take u fixed in the following arguments.

PROOF.

- (3.7) follows from the definition (3.5) and theorem 2.2: denoting by $v := p_t(u)$, we have

$$|\mathcal{G}(\phi;t)| \leq \frac{1}{2} \int_{x \in \mathcal{I}: u(x)=t} |\phi| \, dJ^2 v \leq \frac{1}{2} \sup_{x \in \mathcal{I}} |\phi(x)| \cdot |J^2 v|\{x \in \mathcal{I}; u(x) = t\}.$$

  Since

$$|J^2 v|\{x \in \mathcal{I} : u(x) = t\} \leq |J\dot{v}|(\mathcal{I}) \leq \text{ess-}V_{\mathcal{I}}(\dot{v}) \leq \|\dot{v}\|_{BV(\mathcal{I})},$$

  we conclude recalling that

$$\|\dot{v}\|_{BV(\mathcal{I})} \leq 2\|\dot{u}\|_{BV(\mathcal{I})}.$$

- The μ-measurability of $t \mapsto \mathcal{G}(\phi;t)$ follows by standard approximation procedures: first of all, we can assume ϕ positive, since $\mathcal{G}(\phi;t)^{\pm} = \mathcal{G}(\phi^{\pm};t)$, the superscripts $^{\pm}$ denoting the positive and negative parts of the relative functions. Then we invoke [14], 2.3.3, to write

$$(3.9) \qquad \phi(x)|\dot{u}|_*(x) = \sum_{j=1}^{\infty} r_j \chi_{B_j}(x), \quad \forall\, x \in \mathcal{I}, \quad r_j \text{ positive numbers}, \; B_j \in \mathbf{B}(\mathcal{I}).$$

  Finally we split $\mathcal{G}(\phi;t)$ as

$$(3.10) \qquad \mathcal{G}(\phi;t) = \sum_{\substack{x \in \mathcal{I} \\ u(x)=t}} \sum_{j=1}^{\infty} r_j \chi_{B_j}(x) = \sum_{j=1}^{\infty} r_j \sum_{\substack{x \in \mathcal{I} \\ u(x)=t}} \chi_{B_j}(x) = \sum_{j=1}^{\infty} r_j \mathcal{N}(B_j;t)$$

  and we can apply remark 3.3.

- Also the property of μ_G follows from standard arguments. By the Hahn decomposition, we can assume μ positive; if we choose $B \in \mathbf{B}(\mathcal{I})$ and $\phi := \chi_B$ in (3.9) and (3.10), by [14] 2.4.8 and (3.6) we obtain

$$\mu_G(B) = \int_{\mathbf{R}} \mathcal{G}(\chi_B;t) \, d\mu(t) = \int_{\mathbf{R}} \sum_{j=1}^{\infty} r_j \mathcal{N}(B_j;t) \, d\mu(t) = \sum_{j=1}^{\infty} r_j \int_{\mathbf{R}} \mathcal{N}(B_j;t) \, d\mu(t) =$$

$$= \sum_{j=1}^{\infty} r_j \zeta(B_j) = \int_{\mathcal{I}} \chi_B(x)|\dot{u}|_*(x) \, d\zeta(x) = \int_B |\dot{u}|_*(x) \, d\zeta(x) \quad \blacksquare$$

When ϕ is more regular, we can give further information on $\mathcal{G}(\phi;t)$: first of all, we bound its essential variation.

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3.5 LEMMA. *Let us suppose that ϕ is also in $C_0^1(\mathcal{I})$; then $\mathcal{G}(\phi; \cdot)$ belongs to $BV(\mathbf{R})$ with*

$$(3.11) \qquad \text{ess-}V_{\mathbf{R}}(\mathcal{G}(\phi; \cdot)) \leq \text{ess-}V_{\mathcal{I}}(\phi \dot{u}) \leq \|\phi\|_{BV(a,b)} \cdot \|\dot{u}\|_{BV(a,b)}$$

PROOF.   If $\psi \in C^1(\mathbf{R})$, and $\dot{\psi}$ has compact support, an application of the co-area formula gives:

$$(3.12)$$
$$\int_{\mathbf{R}} \dot{\psi}(t) \mathcal{G}(\phi; t)\, dt = \int_{\mathbf{R}} \dot{\psi}(t) \left\{ \sum_{x:u(x)=t} \phi(x) |\dot{u}(x)|_* \right\} dt =$$
$$= \int_{\mathcal{I}} \dot{\psi}(u(x)) |\dot{u}(x)|^2 \phi(x)\, dx = \int_{\mathcal{I}} \frac{d}{dx} \big[\psi(u(x)) \big] \phi(x) \dot{u}(x)\, dx =$$
$$= -\int_{I} \psi(u(x))\, dD\big[\phi \dot{u}\big](x) \leq \text{ess-}V_{\mathcal{I}}(\phi \dot{u}) \cdot \sup_{t \in \mathbf{R}} |\psi(t)|,$$

since ϕ vanishes at the boundary of $\mathcal{I}$. ∎

$\mathcal{G}$ has another interesting property: it *coincides* with its precise representative $\mathcal{G}_*$. In order to prove this remarkable fact, we introduce a usual family of *symmetric* mollifiers with compact support (see e.g. [15], 1.14) $\{\rho_\varepsilon\}_{\varepsilon>0}$ on $\mathbf{R}$ and we define

$$(3.13) \qquad \mathcal{G}_\varepsilon(\phi; t) = [\mathcal{G}(\phi; \cdot) * \rho_\varepsilon](t) = \int_{\mathbf{R}} \rho_\varepsilon(t - \tau) \mathcal{G}(\phi; \tau)\, d\tau$$

The crucial step of our computations is given by the following

3.6 THEOREM. *Let us assume that $u \in BH(\mathcal{I})$ and $\phi \in C_0^1(\mathcal{I})$; then for every $t_0 \in \mathbf{R}$ we have:*

$$(3.14) \qquad \lim_{\varepsilon \to 0^+} \mathcal{G}_\varepsilon(\phi; t_0) = \mathcal{G}(\phi; t_0).$$

PROOF.   It is not too restrictive to assume $t_0 = 0$; we fix the primitive function $R_\varepsilon(t)$ of ρ_ε so that

$$R_\varepsilon(t) = \int_0^t \rho_\varepsilon(\tau)\, d\tau$$

and we choose $\psi := R_\varepsilon$ in (3.12), obtaining

$$\mathcal{G}_\varepsilon(\phi; 0) = \int_{\mathbf{R}} \rho_\varepsilon(t) \mathcal{G}(\phi; t)\, dt = -\int_{\mathcal{I}} R_\varepsilon(u(x))\, dD\big[\phi \dot{u}\big](x)$$

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Now we can pass to the limit, recalling that, by the symmetry of ρ_ε,

$$R_\varepsilon(u(x)) \to \frac{1}{2}\,\text{sign}(u(x)), \qquad \text{for } all \ x \in \mathcal{I}$$

and obtaining by the dominated convergence theorem

$$\lim_{\varepsilon \to 0} \mathcal{G}_\varepsilon(\phi; 0) = -\frac{1}{2}\int_{\mathcal{I}} \text{sign}(u(x))\,d\big[\phi\dot{u}\big]'(x).$$

We can substitute $\mathcal{I}$ in the previous integral by the union of the two open sets

$$\{x : u(x) > 0\} \text{ and } \{x : u(x) < 0\}$$

and we call Δ_+, Δ_- the countable collections of their connected components so that

(3.15)
$$-\int_{\mathcal{I}} \text{sign}(u(x))\,d\big[\phi\dot{u}\big]'(x) = \sum_{\mathcal{J} \in \Delta_-} D\big[\phi\dot{u}\big](\mathcal{J}) - \sum_{\mathcal{J} \in \Delta_+} D\big[\phi\dot{u}\big](\mathcal{J})$$

For each open interval $\mathcal{J} :=]\alpha, \beta[\in \Delta_+ \cup \Delta_-$ we have

$$\big[\phi\dot{u}\big]'(\mathcal{J}) = \phi(\beta)\dot{u}(\beta_-) - \phi(\alpha)\dot{u}(\alpha_+), \qquad u(\alpha) = u(\beta) = 0;$$

and

$$\begin{cases} \mathcal{J} \in \Delta_+ \ \Rightarrow \ \dot{u}(\beta_-) \le 0, \quad \dot{u}(\alpha_+) \ge 0, \\ \mathcal{J} \in \Delta_- \ \Rightarrow \ \dot{u}(\beta_-) \ge 0, \quad \dot{u}(\alpha_+) \le 0; \end{cases}$$

so that

$$\begin{cases} \mathcal{J} \in \Delta_+ \ \Rightarrow \ -\big[\phi\dot{u}\big]'(\mathcal{J}) = \phi(\beta)|\dot{u}(\beta_-)| + \phi(\alpha)|\dot{u}(\alpha_+)| \\ \mathcal{J} \in \Delta_- \ \Rightarrow \ \big[\phi\dot{u}\big]'(\mathcal{J}) = \phi(\beta)|\dot{u}(\beta_-)| + \phi(\alpha)|\dot{u}(\alpha_+)| \end{cases}$$

We call $\partial_+\Delta$ the set of the right boundary points of the intervals in $\Delta_+ \cup \Delta_-$ and similarly $\partial_-\Delta$; in this way (3.15) becomes

(3.16)
$$\sum_{x \in \partial_+\Delta} \phi(x)|\dot{u}(x_-)| + \sum_{x \in \partial_-\Delta} \phi(x)|\dot{u}(x_+)|.$$

Observe that

$$x \in \partial_+\Delta \setminus \partial_-\Delta \ \Rightarrow \dot{u}(x_+) = 0, \quad |\dot{u}(x_-)| = 2|\dot{u}(x)|_*$$
$$x \in \partial_-\Delta \setminus \partial_+\Delta \ \Rightarrow \dot{u}(x_-) = 0, \quad |\dot{u}(x_+)| = 2|\dot{u}(x)|_*$$

so that (3.16) can be splitted into

$$\sum_{x \in \partial_+\Delta \cap \partial_-\Delta} \phi(x)\Big[|\dot{u}(x_-)| + |\dot{u}(x_+)|\Big] + \sum_{x \in (\partial_+\Delta) \triangle (\partial_-\Delta)} 2\phi(x)|\dot{u}(x)|_* =$$

$$2 \sum_{x \in \partial_+\Delta \cup \partial_-\Delta} \phi(x)|\dot{u}(x)|_*$$

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Since
$$u(x) = 0, \quad |\dot{u}(x)|_* > 0 \ \Rightarrow \ x \in \partial_+\Delta \cup \partial_-\Delta$$
we conclude that
$$-\frac{1}{2}\int_{\mathcal{I}}\mathrm{sign}(u(x))\, d\big[\phi\dot{u}\big]'(x) = \sum_{\substack{x\in\mathcal{I}\\ u(x)=0}} \phi(x)|\dot{u}(x)|_* \quad \blacksquare$$

We can give the main application

3.7 THEOREM. *Let f be a Lipschitz real function with $f'' \in \mathcal{M}(\mathbf{R})$, $u \in BH(\mathcal{I})$, and $v = f \circ u$. We have*

$$(3.17) \qquad \int_{\mathcal{I}}\phi(x)\,dv''(x) = \int_{\mathbf{R}}\mathcal{G}(\phi;t)\,df''(t) + \int_{\mathcal{I}}\phi(x)(\dot{f})_*(u(x))\,du''(x)$$

for every C^1 function ϕ with compact support in $\mathcal{I}$.

PROOF.   We set
$$f_\varepsilon(x) = [f * \rho_\varepsilon](x), \quad v_\varepsilon = f_\varepsilon \circ u; \quad f_\varepsilon \in C^\infty(\mathbf{R}), \quad \ddot{f}_\varepsilon \in L^\infty(\mathbf{R}),$$

recalling that, as $\varepsilon \to 0$, $v_\varepsilon \to v$ uniformly on $\mathcal{I}$, and, by theorem 1,

$$(3.18) \qquad\qquad D^2 v_\varepsilon \rightharpoonup^* D^2 v \ \text{ in the sense of measures.}$$

Since $\dot{v}_\varepsilon = \dot{f}_\varepsilon(u)\dot{u}$ is the product of a C^1-Lipschitz function with a BV-one, we have

$$(3.19) \qquad\qquad v_\varepsilon'' = \ddot{f}_\varepsilon(u)\big(\dot{u}\big)^2 \cdot \mathcal{L}^1 + \dot{f}_\varepsilon(u)\,u'',$$

and, performing an integration,

$$\int_{\mathcal{I}}\phi(x)\,dv_\varepsilon''(x) = \int_{\mathcal{I}}\phi(x)\ddot{f}_\varepsilon(u(x))\big(\dot{u}(x)\big)^2\,dx + \int_{\mathcal{I}}\phi(x)\dot{f}_\varepsilon(u(x))\,du''(x).$$

By (3.18), the integral at the lefthand member converges to the lefthand integral of (3.17), and by the dominated convergence theorem, it is easy to see that

$$\int_{\mathcal{I}}\phi(x)\dot{f}_\varepsilon(u(x))\,du''(x) \ \to \ \int_{\mathcal{I}}\phi(x)f'_*(u(x))\,du''(x).$$

It remains to show that

$$\int_{\mathcal{I}}\phi(x)\ddot{f}_\varepsilon(u(x))\big(\dot{u}(x)\big)^2\,dx \ \to \ \int_{\mathbf{R}}\mathcal{G}(\phi;t)\,df''(t)$$

By the coarea formula we have

$$\int_{\mathcal{I}}\phi(x)\ddot{f}_\varepsilon(u(x))\big(\dot{u}(x)\big)^2\,dx = \int_{\mathbf{R}}\mathcal{G}(\phi;t)\ddot{f}_\varepsilon(t)\,dt$$

and by the usual properties of the (symmetric) convolution, this integral is equal to

$$\int_{\mathbf{R}}\mathcal{G}(\phi;t)[f'']_\varepsilon(t)\,dt = \int_{\mathbf{R}}\mathcal{G}_\varepsilon(\phi;t)\,df''(t).$$

Now we conclude by the dominated convergence theorem and by the previous theorem.
$\blacksquare$

In order to conclude the proof of theorem 1 in the one-dimensional case (i.e. formula (0.19)), we have only to note that the Borel measures $(f \circ u)''$, $(f'')_G$, (see (3.8)) and $\dot{f}_*(u) \cdot u''$ satisfies

$$\int_{\mathcal{I}} \phi \, d(f \circ u)'' = \int_{\mathcal{I}} \phi \, d(f'')_G + \int_{\mathcal{I}} \phi \, d(\dot{f}_*(u) \cdot u'')$$

for every $\phi \in C_0^1(\mathcal{I})$; by the usual density argument and Riesz representation theorem, we deduce

$$(3.20) \qquad\qquad\qquad (f \circ u)'' = (f'')_G + \dot{f}_*(u) \cdot u''.$$

The last step of this section is devoted to prove the splitting formulae (0.20),...,(0.22).

We observe that (0.21) follows easily from (0.19), by evaluating $(f \circ u)''$ on single points.

So we focus our attention on the other two formulae and we want to prove that

3.8 LEMMA. *Assume*

$$(3.21) \qquad u \in BH(\mathcal{I}), \quad f \text{ a Lipschitz real function with } f'' \in \mathcal{M}(\mathbf{R}), \quad v := f \circ u.$$

Then there exists a Borel set $A_v \subset \mathcal{I}$, such that

$$(3.22) \qquad\qquad\qquad |\mathcal{I} \setminus A_v| = 0$$

and

$$(3.23) \qquad v''(B \cap A_v) = \int_B \left(\ddot{f}(u(x))|\dot{u}(x)|^2 + \dot{f}(u(x))\ddot{u}(x) \right) dx, \quad \forall B \in \mathbf{B}(\mathcal{I})$$

PROOF.   We recall that there exist Borel sets $A_u \subset \mathcal{I}, A_f \subset \mathbf{R}$ such that

$$(3.24) \qquad |\mathcal{I} \setminus A_u| = 0, \quad u''(B \cap A_u) = \int_B \ddot{u}(x) \, dx, \quad \forall B \in \mathbf{B}(\mathcal{I})$$

and analogously

$$(3.25) \qquad |\mathbf{R} \setminus A_f| = 0, \quad f''(B \cap A_f) = \int_B \ddot{f}(t) \, dt, \quad \forall B \in \mathbf{B}(\mathbf{R}).$$

We define

$$Z := \{x \in \mathcal{I} : |\dot{u}|_*(x) = 0\}, \quad Z' := \mathcal{I} \setminus Z = \{x \in \mathcal{I} : |\dot{u}|_*(x) \neq 0\}$$

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and

(3.26)
$$A_v := A_u \cap (u^{-1}(A_f) \cup Z)$$

In order to check that $|\mathcal{I} \setminus A_v| = 0$, it is sufficient to see that

$$\mathcal{I} \setminus (u^{-1}(A_f) \cup Z) = u^{-1}(\mathbf{R} \setminus A_f) \cap Z'$$

is $\mathcal{L}^1$-negligible: this follows from [2], 3.1(iv). Now we choose $B \in \mathbf{B}(\mathcal{I})$ and we have

(3.27)
$$v''(B \cap A_v) = \int_{\mathbf{R}} \sum_{\substack{x \in B \cap A_v \\ u(x)=t}} |\dot{u}|_*(x) \, df''(t) + \int_{B \cap A_v} \dot{f}_*(u(x)) \, du''(x)$$

Since $A_v \subset A_u$ and $|\mathcal{I} \setminus A_v| = 0$, the second right-hand integral becomes

$$\int_{B \cap A_v} \dot{f}_*(u(x)) \, du''(x) = \int_{B \cap A_v} \dot{f}_*(u(x)) \ddot{u}(x) \, dx = \int_B \dot{f}_*(u(x)) \ddot{u}(x) \, dx,$$

where we can substitute f_* with f, since by the just quoted result of [2]

$$\ddot{u}(x) = 0 \quad \mathcal{L}^1 \text{ a.e. on } u^{-1}(\{f \neq f_*\}).$$

Finally we consider the first right-hand integral of (3.27). We observe that the integrand

$$\sum_{\substack{x \in B \cap A_v \\ u(x)=t}} |\dot{u}|_*(x)$$

surely vanishes if $t \notin A_f$, so that the integral becomes

(3.28)
$$\int_{\mathbf{R}} \sum_{\substack{x \in B \cap A_v \\ u(x)=t}} |\dot{u}|_*(x) \, df''(t) = \int_{A_f} \sum_{\substack{x \in B \cap A_v \\ u(x)=t}} |\dot{u}|_*(x) \ddot{f}(t) \, dt = \int_{\mathbf{R}} \sum_{\substack{x \in B \cap A_v \\ u(x)=t}} |\dot{u}|_*(x) \ddot{f}(t) \, dt$$

since $|\mathbf{R} \setminus A_f| = 0$. Another application of the change of variables formula gives

$$\int_{\mathbf{R}} \sum_{\substack{x \in B \cap A_v \\ u(x)=t}} |\dot{u}|_*(x) \ddot{f}(t) \, dt = \int_{B \cap A_v} \ddot{f}(u(x)) |\dot{u}(x)|^2 \, dx = \int_B \ddot{f}(u(x)) |\dot{u}(x)|^2 \, dx$$

where we used $|B \setminus A_v| = 0$. ∎

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4. - The regularity of the level sets and the general chain rule

First of all we prove theorem 3; we recall that we denoted by $L_u(t)$ the non-critical t-level set:

$$(4.1) \qquad L_u(t) := \big\{ x \in \Omega : \ u_*(x) = t, \quad |\nabla u|_*(x) > 0 \big\}.$$

4.1 Proposition. *Assume that $u \in BH(\Omega)$ and set $v := p_t(u) = |u - t| - |t|$ for a given $t \in \mathbf{R}$. Then*

$$(4.2) \qquad L_u(t) = S_{\nabla v} \cap \big\{ x \in \Omega : \ u_*(x) = t \big\}, \quad \textit{up to an } \mathcal{H}^{N-1}\textit{-negligible set,}$$

$$(4.3) \qquad \textit{for } \mathcal{H}^{N-1}\textit{-a.e. } x \in L_u(t), \ \boldsymbol{\xi}(x) \textit{ is an approximate unit normal to } L_u(t) \textit{ and}$$

$$(4.4) \qquad \nabla_{\pm} v(x) = \begin{cases} \pm |\nabla u(x)| \boldsymbol{\xi}(x) & \textit{if } x \in L_u(t) \setminus S_{\nabla u}, \\ \pm |\nabla_{\pm} u(x)| \boldsymbol{\xi}(x) & \textit{if } x \in L_u(t) \cap S_{\nabla u}; \end{cases}$$

$$(4.5) \qquad J^2 v = 2 |\nabla u|_* \, \boldsymbol{\xi} \otimes \boldsymbol{\xi} \cdot \mathcal{H}^{N-1} \quad \textit{on } L_u(t).$$

Proof.   It is not too restrictive to consider the case of the 0-level set, i.e. $t = 0$, $v := |u|$.

• Claim 1. *Let x be a point of approximate differentiability of u (i.e. (1.16) holds) and let $u_*(x)$ be 0. Then*

$$(4.6) \qquad x \in S_{\nabla v} \quad \Leftrightarrow \quad \nabla u(x) \neq 0.$$

Thanks to (1.16) we have

$$\fint_{B_r(x)} |u(y) - \langle \nabla u(x), y - x \rangle| \, dy = o(r).$$

If $\nabla u(x) = 0$ then we deduce

$$\fint_{B_r(x)} |v(y) - \langle 0, y - x \rangle| \, dy = \fint_{B_r(x)} |v(y)| \, dy = \fint_{B_r(x)} |u(y)| \, dy = o(r),$$

i.e. 0 is the approximate differential of v at x. When $\nabla u(x) \neq 0$ we consider $H_{\pm} := x + H_{\pm}(\boldsymbol{\xi}(x))$ and we observe that

$$y \in H_{\pm} \quad \Rightarrow \quad |\langle \nabla u(x), y - x \rangle| = |\nabla u(x)| \langle \pm \boldsymbol{\xi}(x), y - x \rangle = \langle \pm \nabla u(x), y - x \rangle.$$

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By the triangle inequality,

$$\fint_{B_r(x) \cap H_{\pm}} \left| v(y) - \langle \pm \nabla u(x), y - x \rangle \right| dy \leq \fint_{B_r(x) \cap H_{\pm}} \left| u(y) - \langle \nabla u(x), y - x \rangle \right| dy = o(r)$$

that is the choices

$$v_*(x) = 0, \quad \nabla_{\pm} v(x) := \pm \nabla u(x), \quad H_{\pm} := x + H_{\pm}(\boldsymbol{\xi}(x))$$

satisfies the last of (1.17) with respect to $v := |u|$ at x.

- We deduce that

  (4.7)
  $$L_u(0) \setminus S_{\nabla u} = S_{\nabla v} \setminus S_{\nabla u},$$

  and on this set (4.3),...,(4.5) are verified.

- CLAIM 2. *For $\mathcal{H}^{N-1}$-a.e. $x \in S_{\nabla u} \cap \{x \in \Omega : u_*(x) = 0\}$ we have $x \in S_{\nabla v}$ with*
  (4.8)
  $$\nabla_{\pm} v(x) = \pm |\nabla_{\pm} u(x)| \boldsymbol{\xi}(x), \quad \text{with respect to the approximate unit normal } \boldsymbol{\xi} = \boldsymbol{n}_{\nabla u}.$$

  Let $x \in S_{\nabla u}$, with $u_*(x) = 0$ and, by definition, $\boldsymbol{\xi}(x) := \boldsymbol{n}_{\nabla u}(x)$; thanks to (1.17) we have

  $$\fint_{B_r(x) \cap H_{\pm}} \left| u(y) - \langle \nabla_{\pm} u(x), y - x \rangle \right| dy = o(r),$$

  where $H_{\pm} := x + H_{\pm}(\boldsymbol{\xi}(x))$; since $\nabla_{\pm} u(x)$ have the same direction of $\boldsymbol{\xi}(x)$ [1] we can write as before

  $$y \in H_{\pm} \quad \Rightarrow \quad \left| \langle \nabla_{\pm} u(x), y - x \rangle \right| = |\nabla_{\pm} u(x)| \langle \pm \boldsymbol{n}(x), y - x \rangle.$$

  We conclude that

  $$\fint_{B_r(x) \cap H_{\pm}} \left| |u(y)| - |\nabla_{\pm} u(x)| \langle \pm \boldsymbol{\xi}(x), y - x \rangle \right| dy = o(r).$$

- By the previous claim we get

  $$S_{\nabla v} \cap \{x \in \Omega : u_*(x) = 0\} \cap S_{\nabla u} = L_u(t) \cap S_{\nabla u}$$

  and on this set (4.3),...,(4.5) hold. Taking account of (4.7), also (4.2) is proved. ∎

4.2 REMARK. Theorem 3 follows now by (4.2) and by (4.3). ☐

Now we consider the statement of theorem 2 and we verify it along every direction $\boldsymbol{\eta} \in \partial B_1$: theorem 1.3 and the one dimensional results of the previous section are the basic ingredients of this procedure.

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4.3 LEMMA. *Assume*

$$u \in BH(\Omega), \quad f \text{ a Lipschitz real function with } f'' \in \mathcal{M}(\mathbf{R}), \quad v := f \circ u, \quad \boldsymbol{\eta} \in \partial B_1$$

and

$$\phi : \Omega \mapsto \mathbf{R} \text{ a bounded Borel function.}$$

Then, for each operator $\mathcal{D}$ among D, ∇, J, C, we have

$$
\int_\Omega \phi \, \mathcal{D}^2_{\boldsymbol{\eta\eta}} v = \int_\Omega \phi(x) \dot{f}_*(u_*(x)) \, d\mathcal{D}^2 u(x) +
$$
(4.9)
$$
+ \int_{\tilde{\pi}\boldsymbol{\eta}} \left(\int_{\mathbf{R}} \sum_{\substack{z \in \Omega_y^{\boldsymbol{\eta}} \\ (u_*)_y(z) = t}} \phi_y(z) |\dot{u}_y|_*(z) \, d\mathcal{D}^2 f(t) \right) d\mathcal{H}^{N-1}(y)
$$

PROOF.   Since $v \in BH(\Omega)$, by (1.28) and the definition (1.22), we can write

$$
\int_\Omega \phi \, \mathcal{D}^2_{\boldsymbol{\eta\eta}} v = \int_{\tilde{\pi}\boldsymbol{\eta}} \left(\int_{\Omega_y^{\boldsymbol{\eta}}} \phi_y(z) \, d\mathcal{D}^2 v_y(z) \right) d\mathcal{H}^{N-1}(y),
$$

the function

(4.10)
$$
y \in \tilde{\pi}^{\boldsymbol{\eta}} \mapsto \int_{\Omega_y^{\boldsymbol{\eta}}} \phi_y(z) \, d\mathcal{D}^2 v_y(z)
$$

being $\mathcal{H}^{N-1}$-a.e. defined and $\mathcal{H}^{N-1}$-measurable. By theorem 1.3, for $\mathcal{H}^{N-1}$-a.e. $y \in \tilde{\pi}^{\boldsymbol{\eta}}$, $(u_*)_y$ is the continuous representative of a $BH(\Omega_y^{\boldsymbol{\eta}})$ function; since for $\mathcal{H}^{N-1}$-a.e. $y \in \tilde{\pi}^{\boldsymbol{\eta}}$ $v_y = f \circ (u_*)_y$, we can apply the splitting formulae (0.20),..., (0.22) in the one dimensional case and we get

$$
\int_{\Omega_y^{\boldsymbol{\eta}}} \phi_y(z) \, d\mathcal{D}^2 v_y(z) = \int_{\mathbf{R}} \sum_{\substack{z \in \Omega_y^{\boldsymbol{\eta}} \\ (u_*)_y(z) = t}} \phi_y(z) |\dot{u}_y|_*(z) \, d\mathcal{D}^2 f(t) +
$$
(4.11)
$$
+ \int_{\Omega_y^{\boldsymbol{\eta}}} \phi_y(z) \dot{f}_*((u_*)_y(z)) \, d\mathcal{D}^2 u_y(z)
$$

Since $\dot{f}_* \circ u_*$ is a bounded Borel function on Ω, by (1.28) and (1.22) we deduce that

(4.12)
$$
y \in \tilde{\pi}^{\boldsymbol{\eta}} \mapsto \int_{\Omega_y^{\boldsymbol{\eta}}} \phi_y(z) \dot{f}_*((u_*)_y(z)) \, d\mathcal{D}^2 u_y(z)
$$

is $\mathcal{H}^{N-1}$-measurable and
(4.13)
$$
\int_{\tilde{\pi}\boldsymbol{\eta}} \left(\int_{\Omega_y^{\boldsymbol{\eta}}} \phi_y(z) \dot{f}_*((u_*)_y(z)) \, d\mathcal{D}^2 u_y(z) \right) d\mathcal{H}^{N-1}(y) = \int_\Omega \phi(x) \dot{f}_*(u_*(x)) \, d\mathcal{D}^2 u(x).
$$

By difference, also

$$(4.14) \qquad y \in \tilde{\pi}^{\boldsymbol{\eta}} \mapsto \int_{\mathbf{R}} \sum_{\substack{z \in \Omega_y^{\boldsymbol{\eta}} \\ (u_*)_y(z)=t}} \phi_y(z) |\dot{u}_y|_*(z) \, d\mathcal{D}^2 f(t) \quad \text{is } \mathcal{H}^{N-1}\text{-measurable};$$

integrating on $\tilde{\pi}^{\boldsymbol{\eta}}$ and taking into account (4.13), we deduce (4.9). ∎

4.4 COROLLARY. *Let* $u \in BH(\Omega)$, $\boldsymbol{\eta} \in \partial B_1$, *and let* $\phi : \Omega \mapsto \mathbf{R}$ *be a bounded Borel function; then for every* $t \in \mathbf{R}$ *we have*

$$(4.15)$$
$$\int_{\tilde{\pi}^{\boldsymbol{\eta}}} \sum_{\substack{z \in \Omega_y^{\boldsymbol{\eta}} \\ (u_*)_y(z)=t}} \phi_y(z) |\dot{u}_y|_*(z) \, d\mathcal{H}^{N-1}(y) = \int_{L_u(t)} \phi(x) \big| \nabla u(x) \big|_* \langle \boldsymbol{\eta}(x), \boldsymbol{\xi}(x) \rangle^2 \, d\mathcal{H}^{N-1}(x).$$

PROOF. We apply the previous theorem with

$$v = f(u) := p_t(u), \quad \mathcal{D} = J, \quad \phi \text{ substituted by } \phi \cdot \chi_{L_u(t)};$$

then we compare the result with the explicit formula which follows by (4.5). ∎

In order to conclude the proof of theorem 2, we have simply to interchange the order of the two last integrals in (4.9) and to apply (4.15). This possibility is ensured by the applying the following

4.5 PROPOSITION. *Let* $u \in BH(\Omega)$, $\boldsymbol{\eta} \in \partial B_1$, $\phi : \Omega \mapsto \mathbf{R}$ *a bounded Borel function and* $\mu \in \mathcal{M}(\mathbf{R})$ *a Borel measure on* $\mathbf{R}$. *Then for* $\mathcal{H}^{N-1}$-*a.e.* $y \in \tilde{\pi}^{\boldsymbol{\eta}}$ *the mapping*

$$(4.16) \qquad (y,t) \in \tilde{\pi}^{\boldsymbol{\eta}} \times \mathbf{R} \mapsto \mathcal{G}(\phi; y, t) := \sum_{\substack{z \in \Omega_y^{\boldsymbol{\eta}} \\ (u_*)_y(z)=t}} \phi_y(z) |\dot{u}_y|_*(z)$$

is well defined and it is $\mathcal{H}^{N-1} \times \mu$-*measurable.*

We divide the PROOF in some steps. We can surely assume μ is positive.

• We denote by $\alpha^{\boldsymbol{\eta}}$ the set

$$(4.17) \qquad \alpha^{\boldsymbol{\eta}} := \big\{ y \in \tilde{\pi}^{\boldsymbol{\eta}} : (u_*)_y \in BH(\Omega_y^{\boldsymbol{\eta}}) \cap C^0(\Omega_y^{\boldsymbol{\eta}}) \big\}$$

which satisfies $\mathcal{H}^{N-1}(\tilde{\pi}^{\boldsymbol{\eta}} \setminus \alpha^{\boldsymbol{\eta}}) = 0$ and $[\mathcal{H}^{N-1} \times \mu](\alpha^{\boldsymbol{\eta}} \times \mathbf{R}) = 0$. By proposition 3.4,

$$(4.18). \qquad t \in \mathbf{R} \mapsto \mathcal{G}(\phi; y, t) \quad \text{is bounded and } \mu\text{-measurable, for every } y \in \alpha^{\boldsymbol{\eta}}.$$

We call μ_G (see (3.8)) the finite Borel measure on Ω

$$\mu_G(B) := \int_{\tilde{\pi}^{\eta}} \left(\int_{\mathbf{R}} \mathcal{G}(\chi_B; y, t)\, d\mu(t) \right) d\mathcal{H}^{N-1}(y), \quad \forall B \in \mathbf{B}(\Omega)$$

and for every set $E \subset \Omega$ we call

$$\begin{aligned}
\hat{E} :=& \{(y, t) \in \alpha^{\eta} \times \mathbf{R} : \mathcal{G}(\chi_E; y, t) > 0\} \\
=& \{(y, t) \in \alpha^{\eta} \times \mathbf{R} : \exists\, z \in E_y^{\eta},\ (u_*)_y(z) = t,\ |\dot{u}_y|_*(z) > 0\}.
\end{aligned}$$

- CLAIM 1. *Let us set*

$$\mathcal{G}_{\varepsilon}(\phi; y, t) := \int_{\mathbf{R}} \mathcal{G}(\phi; y, \tau)\rho_{\varepsilon}(t - \tau)\, d\tau$$

where $\{\rho_{\varepsilon}\}_{\varepsilon>0}$ is a family of symmetric mollifiers as in (3.13). Then $\mathcal{G}_{\varepsilon}$ is a Carathéodory's function, in particular it is $\mathcal{H}^{N-1} \times \mu$-measurable (see [12], VIII-1.3).

Choosing in (4.14)

$$f(\tau) := I_2[\rho_{\varepsilon}(t - \cdot)](\tau) = \frac{1}{2} \int_{\mathbf{R}} p_s(\tau)\rho_{\varepsilon}(t - s)\, ds,$$

so that $f''(\tau) = \rho_{\varepsilon}(t - \tau)$, we deduce that $y \mapsto \mathcal{G}_{\varepsilon}(\phi; y, t)$ is an $\mathcal{H}^{N-1}$-measurable function, for every $t \in \mathbf{R}$. On the other hand, $\mathcal{G}_{\varepsilon}$ is surely continuous in t, for $y \in \alpha^{\eta}$.

- CLAIM 2. *If ϕ belongs to $C^0_{comp}(\Omega)$, then $\mathcal{G}(\phi; y, t)$ is $\mathcal{H}^{N-1} \times \mu$-measurable.*

If $\phi \in C^1_{comp}(\Omega)$ the thesis follows by theorem 3.6, since

$$\lim_{\varepsilon \to 0} \mathcal{G}_{\varepsilon}(\phi; y, t) = \mathcal{G}(\phi; y, t), \quad \forall\, y \in \alpha^{\eta},\ \forall\, t \in \mathbf{R}.$$

When ϕ is in $C^0_0(\Omega)$, we can uniformly approximate it by a sequence ϕ^n of $C^1_{comp}(\Omega)$-functions: applying (3.7), we deduce that for every $y \in \alpha^{\eta}$

$$\lim_{n \to \infty} \sup_{t \in \mathbf{R}} |\mathcal{G}(\phi^n; y, t) - \mathcal{G}(\phi; y, t)| = 0.$$

- CLAIM 3. *Let E be a G_{δ} (or a F_{σ}) subset of Ω; then $\mathcal{G}(\chi_E; y, t)$ is $\mathcal{H}^{N-1} \times \mu$-measurable.*

It is sufficient to note that

(4.19) $$\lim_{n \to \infty} \mathcal{G}(\phi^n; y, t) = \mathcal{G}(\phi; y, t), \quad \forall\, y, t \in \alpha^{\eta} \times \mathbf{R}$$

if $\{\phi^n\}_{n \in \mathbf{N}}$ is a uniformly bounded and monotone family of Borel functions, pointwise converging to ϕ in Ω as $n \to \infty$. We recall that the characteristic function of each open subset U of Ω can be approximated in such a way by continuous functions compactly

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supported in U; the characteristic function of a G_δ-set is then obtained as a decreasing pointwise limit of characteristic functions of open sets.

- CLAIM 4. *Let E be a Borel subset of Ω; then $\mathcal{G}(\chi_E; y, t)$ is $\mathcal{H}^{N-1} \times \mu$-measurable.*

By the just quoted theorem 2.17 (c) of [17], there exist an F_σ subset A and a G_δ-subset B of Ω such that $A \subset E \subset B$ and $\mu_G(B \setminus A) = 0$. We shall see that

$$\mathcal{G}(\chi_A; y, t) = \mathcal{G}(\chi_E; y, t) = \mathcal{G}(\chi_B; y, t), \quad \mathcal{H}^{N-1} \times \mu\text{-a.e. in } \tilde{\pi}^{\boldsymbol{\eta}} \times \mathbf{R};$$

Since $\mathcal{G}(\phi; y, t)$ is monotone with respect to ϕ, it is sufficient to show that the set of the couples $(y, t) \in \alpha^{\boldsymbol{\eta}} \times \mathbf{R}$ such that

$$\mathcal{G}(\chi_B; y, t) - \mathcal{G}(\chi_A; y, t) = \mathcal{G}(\chi_{B \setminus A}; y, t) > 0$$

is $\mathcal{H}^{N-1} \times \mu$-negligible. But this is obvious, since $B \setminus A$ is a G_δ set, $\mathcal{G}(\chi_{B \setminus A}; y, t)$ is $\mathcal{H}^{N-1} \times \mu$-measurable, and

$$\int_{\tilde{\pi}^{\boldsymbol{\eta}} \times \mathbf{R}} \mathcal{G}(\chi_{B \setminus A}; y, t) \, d[\mathcal{H}^{N-1} \times \mu](y, t) = \mu_G(B \setminus A) = 0.$$

- Now we extend the previous point to every Borel function ϕ by the standard approximation procedure ([17], 1.14 (b), 1.17). $\blacksquare$

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