The concavity of Rényi entropy power

Giuseppe Savaré and Giuseppe Toscani

***Abstract*—We associate to the p-th Rényi entropy a definition of entropy power, which is the natural extension of Shannon's entropy power and exhibits a nice behaviour along solutions to the p-nonlinear heat equation in $\mathbb{R}^n$. We show that the Rényi entropy power of general probability densities solving such equations is always a concave function of time, whereas it has a linear behaviour in correspondence to the Barenblatt source-type solutions. This result extends Costa's concavity inequality for Shannon's entropy power to Rényi entropies.**

***Index Terms*—Entropy, information measure, information theory, Rényi entropy, nonlinear heat equation.**

I. Introduction

The p-th Rényi entropy of a random variable X with density f in $\mathbb{R}^n$ is defined by (see, e.g. Cover and Thomas [7] and Gardner [12])

$$\mathcal{H}_p(X) = \mathrm{H}_p(f) := \frac{1}{1-p} \log \int_{\mathbb{R}^n} f^p(x)\,\mathrm{d}x, \tag{1}$$

for $0 < p < +\infty$, $p \neq 1$.

Whenever $p > 1-2/n$, we consider the positive coefficients

$$\mu := 2 + n(p-1), \quad \nu := \frac{\mu}{n} = \frac{2}{n} + (p-1), \tag{2}$$

and we associate to the p-th Rényi entropy the entropy power (that we call *p-th Renyi entropy power* in the following) given by

$$\mathcal{N}_p(X) = \mathrm{N}_p(f) := \exp\big(\nu\,\mathcal{H}_p(X)\big). \tag{3}$$

The Rényi entropy for $p = 1$ is defined as the limit of $\mathcal{H}_p(X)$ as $p \to 1$. It follows directly from definition (1) that

$$\mathcal{H}_1(X) = \lim_{p\to 1} \mathcal{H}_p(X) = \mathcal{H}(X) = -\int_{\mathbb{R}^n} f(x)\log f(x)\,\mathrm{d}x.$$

Therefore, the Shannon's entropy can be identified with the Rényi entropy of index $p = 1$. In this case, the proposed Rényi entropy power of index $p = 1$, given by (3), coincides with Shannon's entropy power

$$\mathcal{N}(X) = \mathrm{N}(f) := \exp\left\{\frac{2}{n}\mathcal{H}(X)\right\}. \tag{4}$$

In 1985 Costa [6] proved that, if $u(\cdot,t)$, $t > 0$, are probability densities solving the heat equation

$$\frac{\partial}{\partial t} u = \Delta u, \tag{5}$$

posed in the whole space $\mathbb{R}^n$, then

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathrm{N}(u(\cdot,t)) \le 0. \tag{6}$$

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The authors acknowledge support by MIUR project "Optimal mass transportation, geometrical and functional inequalities with applications".

Manuscript received ; revised .

Inequality (6) is referred to as the *concavity of entropy power* theorem (see Riul [17] for an exhaustive list of references). The original proof of Costa has been simplified years later by Dembo [10], [11] with an argument based on the Blachman–Stam inequality [3]. Next, a direct proof of (6) in a strengthened form, with an exact error term, has been obtained by Villani [22]. The proof in [22] highlights a strong connection between the concavity of entropy power and some identities of Bakry and Emery [2], established through the so-called Γ_2 calculus as part of their famous work on logarithmic Sobolev inequalities and hyper-contractive diffusions. These connections, together with various consequences of the concavity of entropy power theorem, have been recently discussed in [19].

In this paper we show that the *concavity of entropy power* is a property which is not restricted to Shannon entropy power (4) in connection with the heat equation (5), but it holds for the p-th Rényi entropy power (3), if we put it in connection with the solution to the nonlinear heat equation

$$\frac{\partial}{\partial t} u = \Delta u^p, \tag{7}$$

posed in the whole space $\mathbb{R}^n$. The precise result is the following.

Theorem 1: Let $p > 1 - 2/n$ and let $u(\cdot,t)$ be probability densities in $\mathbb{R}^n$ solving (7) for $t > 0$. Then the p-th Rényi entropy power defined in (3) satisfies

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathrm{N}_p(u(\cdot,t)) \le 0 \quad \text{i.e. } t \mapsto \mathrm{N}_p(u(\cdot,t)) \text{ is concave.} \tag{8}$$

Like in the Shannon's case, inequalities (8) lied to sharp isoperimetric inequalities. It is remarkable that the range of the parameter p for which we can introduce the Rényi entropy power, coincides with the range for which there is mass conservation for the solution of (7) [4].

The rest of this paper is devoted to the proof of (8). Before entering into technical details, we will however explain in some details the physical reasons which in our opinion suggest to define the p-th Rényi entropy power in the form (3). To this aim, we will introduce in Section II some known facts about nonlinear heat equations (cfr. the book by Vazquez [21], which fully treats the case $p > 1$, and [4] for the case $p < 1$). Among other facts, this connection between Rényi entropies and the nonlinear heat equations allows to recover in a simple way a related p-th Fisher information recently considered in [14], [13].

II. Self-similar solutions and Rényi entropies

The relationship between Shannon's entropy power and the solution to the linear heat equation (5) can be fruitfully highlighted owing to the fundamental solution, representing heat release from a point source (here the origin $x = 0$ without loss of generality). The source-type solution of unit mass at time $t > 0$ is represented by the Gaussian density

$$M(x,t) := \frac{1}{(4\pi t)^{n/2}} \exp\left\{-\frac{|x|^2}{4t}\right\} \tag{9}$$

of variance equal to $2n$ [8]. Since the Shannon's entropy of $M(\cdot,t)$ equals

$$\mathrm{H}(M(\cdot,t)) = \frac{n}{2}\log(4\pi\,\mathrm{e}\,t),$$

it follows that the corresponding entropy power is a linear function of time, i.e.

$$\mathrm{N}(M(\cdot,t)) = 4\pi\,\mathrm{e}\,t, \qquad \text{hence} \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathrm{N}(M(t)) = 0.$$

In view of this remark, the concavity property of entropy power can be rephrased by saying that for all times $t > 0$ the fundamental source-type solution maximizes the second derivative of the Shannon's entropy power among all possible solutions to the heat equation.

This idea easily extends to the nonlinear heat equation (7). The corresponding fundamental solution was found around 1950 by Zel'dovich and Kompaneets and Barenblatt [21]. In the case $p > 1$ (see [4] for $p < 1$) the Barenblatt (also called self-similar or generalized Gaussian) solution departing from $x = 0$ takes the self-similar form (recall the definition of μ in (2))

$$M_p(x,t) := \frac{1}{t^{n/\mu}}\tilde{M}_p\left(\frac{x}{t^{1/\mu}}\right), \tag{10}$$

where

$$\tilde{M}_p(x) = \left(C - \kappa\,|x|^2\right)_+^{\frac{1}{p-1}}; \tag{11}$$

here $(s)_+ = \max\{s,0\}$, $\kappa := \frac{1}{2\mu}\,\frac{p-1}{p}$, and the constant C in (11) can be chosen to fix the mass of the source-type Barenblatt solution equal to one.

Notice that, if we consider the mass–preserving rescaling

$$\mathcal{R}_a : f(x) \to \mathcal{R}_a f(x) := a^{-n} f\left(x/a\right), \quad a > 0,\ x \in \mathbb{R}^n,$$

of a given nonnegative density f, a direct computation immediately yields

$$\mathrm{H}_p(\mathcal{R}_a f) = \mathrm{H}_p(f) + n\log a, \quad \mathrm{N}_p(\mathcal{R}_a f) = a^{\mu}\,\mathrm{N}_p(f). \tag{12}$$

An application of (12) to (10) with $a := t^{1/\mu}$ then shows that the p-th Rényi entropy power defined in (3) is a linear function of time

$$\mathrm{N}_p(M_p(t)) = \mathrm{N}_p(\tilde{M}_p)\,t, \quad \text{so that} \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathrm{N}_p(M_p(t)) = 0.$$

As before, the concavity property of Rényi entropy power would imply that for all times $t > 0$ the Barenblatt source-type solution maximizes the second derivative of the p-th Rényi entropy power. among all possible solutions to the nonlinear heat equation.

III. A NEW LYAPUNOV FUNCTIONAL

The proof of (8) requires to evaluate two time derivatives of the p-th Rényi entropy power, along the solution to the nonlinear heat equation (7). The first derivative of the entropy power can be easily obtained: in fact, if $u(\cdot,t)$ solves the nonlinear heat equation (7), integration by parts immediately leads to (cfr. Appendix B)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}_p(u(\cdot,t)) = \mathrm{I}_p(u(\cdot,t)), \quad t > 0, \tag{13}$$

where

$$\mathcal{I}_p(X) = \mathrm{I}_p(f) := \frac{1}{\int_{\mathbb{R}^n} f^p\,\mathrm{d}x}\int_{\{f>0\}} \frac{|\nabla f^p(x)|^2}{f(x)}\,\mathrm{d}x. \tag{14}$$

When $p \to 1$, identity (13) reduces to DeBruijn's identity, which connects Shannon's entropy functional with the Fisher information of a random variable with density

$$\mathcal{I}(X) = \mathrm{I}(f) := \int_{\mathbb{R}^n} \frac{|\nabla f(x)|^2}{f(x)}\,\mathrm{d}x, \tag{15}$$

via the heat equation. Analogous computations can be done here. Using identity (13) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{N}_p(u(\cdot,t)) = \nu\,\mathrm{N}_p(u(\cdot,t))\,\mathrm{I}_p(u(\cdot,t)).$$

Let us introduce the quantity

$$\Upsilon_p(u) := \mathrm{N}_p(u)\,\mathrm{I}_p(u), \quad \upsilon(t) := \Upsilon_p(u(\cdot,t)). \tag{16}$$

Since $2/n + p - 1 > 0$, the concavity of entropy power can be rephrased as the decreasing in time property of $t \mapsto \Upsilon_p(u(\cdot,t))$.

It is important to notice that by (12) and the scaling property of the p-th Fisher information

$$\mathrm{I}_p(\mathcal{R}_a f) = a^{-\mu}\,\mathrm{I}_p(f), \tag{17}$$

the functional Υ_p is invariant under the dilations $\mathcal{R}_\cdot$, i.e.

$$\Upsilon_p(\mathcal{R}_a f) = \Upsilon_p(f) \quad \text{for every } a > 0. \tag{18}$$

Property (18) allows to identify the long-time behavior of the function $\upsilon(t) = \Upsilon_p(u(\cdot,t))$. It is nonincreasing, and it will reach its infimum as time $t \to \infty$. The computation of the limit value uses in a substantial way the scaling invariance property. In fact, we can rescale $u(x,t)$ according to

$$U(x,t) = t^{-n/\mu}\,u(x\,t^{-1/\mu},t) = \mathcal{R}_{t^{1/\mu}}u(\cdot,t) \tag{19}$$

where μ is defined in (2) as usual, so that

$$\upsilon(t) = \Upsilon_p(u(\cdot,t)) = \Upsilon_p(U(\cdot,t)) \quad \text{for every } t > 0.$$

On the other hand, it is well-known that (cfr. for example [21])

$$\lim_{t\to\infty} U(x,t) = \tilde{M}_p(x), \tag{20}$$

the Barenblatt profile defined in (11). Therefore, denoting by $f(x) := u(x,0)$ the initial probability density, passing to the limit one obtains the (isoperimetric) inequality for the p-th Rényi entropy:

Theorem 2: If $p > n/(n+2)$ every smooth, strictly positive and rapidly decaying probability density f satisfies

$$\Upsilon_p(f) = \mathrm{N}_p(f)\,\mathrm{I}_p(f) \geq \mathrm{N}_p(\tilde{M}_p)\,\mathrm{I}_p(\tilde{M}_p) =: \gamma_{n,p}, \tag{21}$$

where the value of the strictly positive constant $\gamma_{n,p}$ is given (39) and (43) of Appendix A.

IV. Sobolev inequality revisited

Inequality (21) can be rewritten in a form more suitable to functional analysis. Let $f(x)$ be a probability density in $\mathbb{R}^n$. Then, if $p > n/(n+2)$

$$\int_{\mathbb{R}^n} \frac{|\nabla f^p(x)|^2}{f(x)} \,\mathrm{d}x \ge \gamma_{n,p} \left(\int_{\mathbb{R}^n} f^p(x) \,\mathrm{d}x \right)^{\frac{2+2n(p-1)}{n(p-1)}}. \quad (22)$$

If $n > 2$, the case $p = (n-1)/n$ is distinguished from the others, since it leads to

$$\frac{2+2n(p-1)}{n(p-1)} = 0, \quad \nu = \frac{1}{n},$$

and

$$\mathrm{N}_{1-1/n}(f) = \int_{\mathbb{R}^n} f^{1-1/n}(x) \,\mathrm{d}x.$$

In this case the concavity of $\mathrm{N}_{1-1/n}$ along (7) has been already known and has a nice geometric interpretation in terms of transport distances, see [15].

Note that the restriction $n > 2$ implies $(n-1)/n > n/(n+2)$. Hence, for $p = (n-1)/n$ we obtain that the probability density f satisfies the inequality

$$\int_{\mathbb{R}^n} \frac{|\nabla f^{(n-1)/n}(x)|^2}{f(x)} \,\mathrm{d}x \ge \gamma_{n,(n-1)/n}. \quad (23)$$

The substitution $f = g^{2^*}$, where $2^* = 2n/(n-2)$, yields

$$\int_{\mathbb{R}^n} \frac{|\nabla f^{(n-1)/n}(x)|^2}{f(x)} \,\mathrm{d}x = \left(\frac{2n-2}{n-2} \right)^2 \int_{\mathbb{R}^n} |\nabla g(x)|^2 \,\mathrm{d}x.$$

Therefore, for any given function $g \ge 0$ such that $g(x)^{2^*}$ is a probability density in $\mathbb{R}^n$, with $n > 2$, we obtain the inequality

$$\int_{\mathbb{R}^n} |\nabla g(x)|^2 \,\mathrm{d}x \ge \left(\frac{n-2}{2n-2} \right)^2 \gamma_{n,(n-1)/n}. \quad (24)$$

A careful computation (see Appendix A) gives

$$\gamma_{n,(n-1)/n} = n\pi \, \frac{2^2(n-1)^2}{n-2} \left(\frac{\Gamma(n/2)}{\Gamma(n)} \right)^{2/n},$$

and simple scaling argument finally shows that, if $g(x)^{2^*}$ has a mass different from 1, g satisfies the *Sobolev* inequality [1], [18]

$$\int_{\mathbb{R}^n} |\nabla g(x)|^2 \,\mathrm{d}x \ge \mathcal{S}_n \left(\int_{\mathbb{R}^n} g(x)^{2^*} \,\mathrm{d}x \right)^{2/2^*}, \quad (25)$$

where

$$\mathcal{S}_n = n(n-2)\pi \left(\frac{\Gamma(n/2)}{\Gamma(n)} \right)^{2/n}$$

is the sharp Sobolev constant. Hence, Sobolev inequality with the sharp constant is a consequence of the concavity of Rényi entropy power of parameter $p = (n-1)/n$, when $n > 2$.

In all the other cases, the concavity of Rényi entropy power leads to Gagliardo-Nirenberg type inequalities with sharp constants, like the ones recently studied by Del Pino and Dolbeault [9], and Cordero-Erausquin, Nazaret, and Villani, [5] with different methods.

V. Proof of the concavity of Rényi entropy power

Arguing as in [20], it is sufficient to consider the case of smooth, strictly positive and rapidly decaying probability densities.

For a given probability density u we set

$$\mathrm{E}_p(u) := \int \mathsf{e}_p(u(x)) \,\mathrm{d}x, \quad \text{where } \mathsf{e}_p(r) := \frac{1}{p-1} r^p.$$

Consequently, the p-th Rényi entropy of u can be written as

$$\mathrm{H}_p(u) := \frac{1}{1-p} \log \Big((p-1)\mathrm{E}_p(u) \Big).$$

Likewise, since $\mathsf{e}_p'(u) = q\, u^{p-1}$, $q := \frac{p}{p-1}$, we can write

$$\begin{aligned} \mathrm{F}_p(u) &:= \int \frac{|\mathrm{D}\, u^p|^2}{u} \,\mathrm{d}x = q^2 \int |\mathrm{D}\, u^{p-1}|^2\, u \,\mathrm{d}x \\ &= \int \big| \mathrm{D}\mathsf{e}_p'(u) \big|^2 u \,\mathrm{d}x. \end{aligned}$$

Recall also that

$$L_p(u) = u^p = u\, \mathsf{e}_p'(u) - \mathsf{e}_p(u), \quad L_p'(u) = u\, \mathsf{e}_p''(u).$$

The nonlinear heat equation (7) can be equivalently written as

$$\partial_t u - \nabla \cdot \Big(u\, \mathrm{D}\mathsf{e}_p'(u) \Big) = 0. \quad (26)$$

Using equation (26), and integrating by parts, we obtain (see Appendix B)

$$-\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{E}_p(u_t) = \mathrm{F}_p(u), \quad (27)$$

$$\begin{aligned} \mathrm{D}_p(u_t) &:= -\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{F}_p(u_t) \quad (28) \\ &= 2 \int u^p \Big(|\mathrm{D}^2 \mathsf{e}_p'(u)|^2 + (p-1)\big(\Delta \mathsf{e}_p'(u)\big)^2 \Big) \,\mathrm{d}x. \end{aligned}$$

For a given function ϕ_t which depends on time, and a positive constant σ

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \exp(\sigma\, \phi_t) = \exp(\sigma \phi_t) \Big(\sigma \phi_t'' + (\sigma \phi_t')^2 \Big).$$

Therefore

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \exp(\sigma \phi_t) \le 0 \qquad \Longleftrightarrow \qquad -\sigma \phi_t'' \ge (\sigma \phi_t')^2.$$

If $\phi_t := \mathrm{H}_p(u_t)$, where $u_t = u(\cdot, t)$, we have

$$\phi_t' = \frac{1}{p-1} \frac{-\mathrm{F}_p(u_t)}{\mathrm{E}_p(u_t)}, \quad (29)$$

and

$$\phi_t'' = \frac{1}{p-1} \frac{\mathrm{D}_p(u_t)\, \mathrm{E}_p(u_t) - \mathrm{F}_p(u_t)^2}{\mathrm{E}_p(u(t))^2}. \quad (30)$$

Hence we end up with the condition

$$\frac{\sigma}{p-1} \Big(\mathrm{D}_p(u_t)\, \mathrm{E}_p(u_t) - \mathrm{F}_p(u_t)^2 \Big) \ge \left(\frac{\sigma}{p-1} \right)^2 \mathrm{F}_p(u_t)^2,$$

i.e. (suppressing the index t)

$$\sigma \mathrm{D}_p(u) \int u^p \,\mathrm{d}x \ge \big(\sigma^2 + \sigma(p-1) \big) \mathrm{F}_p(u)^2.$$

Since $\sigma > 0$, the second derivative is non positive if

$$\mathrm{D}_p(u) \int u^p \,\mathrm{d}x \geq \big(\sigma + (p-1)\big)\big(\mathrm{F}_p(u)\big)^2. \tag{31}$$

Since

$$\mathrm{F}_p(u) = \int \mathrm{D}u^p \cdot \mathrm{De}'_p(u)\,\mathrm{d}x = -\int u^p\,\Delta\mathsf{e}'_p(u)\,\mathrm{d}x,$$

by Cauchy-Schwarz inequality we have

$$\mathrm{F}_p(u)^2 \leq \int u^p\,\mathrm{d}x \int u^p(\Delta\mathsf{e}'_p(u))^2\,\mathrm{d}x.$$

It follows that (31) holds if

$$\mathrm{D}_p(u) \geq (\sigma + (p-1)) \int u^p(\Delta\mathsf{e}'_p(u))^2\,\mathrm{d}x.$$

On the other hand, by the trace inequality, $|\mathrm{D}^2 f|^2 \geq \frac{1}{n}\big(\Delta f\big)^2$

$$\mathrm{D}_p(u) \geq 2\,(\frac{1}{n} + (p-1)) \int u^p(\Delta\mathsf{e}'_p(u))^2\,\mathrm{d}x,$$

and we end up with the condition

$$\sigma \leq \frac{2}{n} + p - 1 = \nu. \tag{32}$$

Choosing for σ the upper bound in (32) we conclude.

Appendix A
Computation of the constants

Let us recall here some useful formulas. The surface of the $n-1$ dimensional unit sphere $\mathbb{S}^{n-1}$ is given by $|\mathbb{S}^{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$. Let us first consider the case $p > 1$. If $a > 0$, using the integral representation of Beta function we have

$$\int_{\mathbb{R}^n} (1-|x|^2)_+^a\,\mathrm{d}x = |\mathbb{S}^{n-1}| \int_0^1 \rho^{n-1}(1-\rho^2)^a\,d\rho = \frac{2\pi^{n/2}}{\Gamma(n/2)}\int_0^1 t^{n/2-1}(1-t)^a\,dt = \frac{2\pi^{n/2}}{\Gamma(n/2)}B(\frac{n}{2}, a+1) = \pi^{n/2}\frac{\Gamma(a+1)}{\Gamma\left(\frac{n}{2}+a+1\right)}.$$

With this formula, we can evaluate quantities associated to the Barenblatt function (11). Indeed, if

$$A_p = \int_{\mathbb{R}^n} (1-|x|^2)_+^{1/(p-1)}\,\mathrm{d}x = \pi^{n/2}\frac{\Gamma\left(\frac{p+1}{p}\right)}{\Gamma\left(\frac{n}{2}+\frac{p+1}{p}\right)}, \tag{33}$$

we obtain a Barenblatt of mass equal to one, we denote by

$$\mathcal{B}_p(x) = (C_p - |x|^2)_+^{1/(p-1)}$$

if

$$C_p = A_p^{-\frac{2(p-1)}{n(p-1)+2}}. \tag{34}$$

Also [21]

$$\int_{\mathbb{R}^n} |x|^2\mathcal{B}_p(x)\,\mathrm{d}x = \frac{n(p-1)}{(n+2)p-n}C_p, \tag{35}$$

and, since

$$\int_{\mathbb{R}^n} \mathcal{B}_p(x)^p\,\mathrm{d}x = \int_{\mathbb{R}^n} (C_p - |x|^2)\mathcal{B}_p(x)\,\mathrm{d}x,$$

one obtains

$$\int_{\mathbb{R}^n} \mathcal{B}_p(x)^p\,\mathrm{d}x = \frac{2p}{(n+2)p-n}C_p. \tag{36}$$

Thanks to (35) and (36), we reckon the values of the p-Fisher information $\mathrm{I}_p(f)$ defined in (14) and of the Rényi entropy $\mathrm{H}_p(f)$, associated to $\mathcal{B}_p$

$$\mathrm{H}_p(\mathcal{B}_p) = \frac{1}{1-p}\log\frac{2p}{(n+2)p-n}C_p, \tag{37}$$

and

$$\mathrm{I}_p(\mathcal{B}_p) = n\frac{2p}{p-1}. \tag{38}$$

Hence, if $p > 1$ the value of the constant $\gamma_{n,p}$ is

$$\gamma_{n,p} = n\pi\frac{2p}{p-1}\cdot \cdot\left(\frac{\Gamma\left(\frac{p+1}{p}\right)}{\Gamma\left(\frac{n}{2}+\frac{p+1}{p}\right)}\right)^{2/n}\left(\frac{(n+2)p-n}{2p}\right)^{\frac{2+n(p-1)}{n(p-1)}}. \tag{39}$$

Analogous computations can be done in the case $p < 1$. In this case

$$A_p = \int_{\mathbb{R}^n} (1+|x|^2)^{1/(p-1)}\,\mathrm{d}x = \pi^{n/2}\frac{\Gamma\left(\frac{1}{1-p}-\frac{n}{2}\right)}{\Gamma\left(\frac{1}{1-p}\right)}, \tag{40}$$

while

$$\int_{\mathbb{R}^n} |x|^2\mathcal{B}_p(x)\,\mathrm{d}x = \frac{n(1-p)}{(n+2)p-n}C_p. \tag{41}$$

Note that, if $p < 1$, the second moment of the Barenblatt is bounded if and only if $p > n/(n+2)$. Therefore, the computations that follow are restricted to this domain of p. If this is the case, formula (36) still holds, while

$$\mathrm{I}_p(\mathcal{B}_p) = n\frac{2p}{1-p}. \tag{42}$$

Finally, if $n/(n+2) < p < 1$,

$$\gamma_{n,p} = n\pi\frac{2p}{1-p}\cdot \cdot\left(\frac{\Gamma\left(\frac{1}{1-p}-\frac{n}{2}\right)}{\Gamma\left(\frac{1}{1-p}\right)}\right)^{2/n}\left(\frac{(n+2)p-n}{2p}\right)^{\frac{2+n(p-1)}{n(p-1)}}. \tag{43}$$

Appendix B
Proof of identities (27) and (28)

(27) follows by a simple integration by parts:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}_p(u_t) = \int \mathsf{e}'_p(u)\,\partial_t u\,\mathrm{d}x = \int \mathsf{e}'_p(u)\,\nabla\cdot\Big(u\,\mathrm{De}'_p(u)\Big)\,\mathrm{d}x = -\int \mathrm{De}'_p(u)\cdot\mathrm{De}'_p(u)\,u\,\mathrm{d}x.$$

(28) is based on the Bochner identity or, equivalently, Bakry-Émery Γ-calculus (in their simplest Euclidean form), see [16] for analogous computations of the second derivative of E_p along geodesics in the Wasserstein space.

One has

$$\partial_t \mathsf{e}_p'(u) = \mathsf{e}_p''(u)\partial_t u = \mathsf{e}_p''(u)\,\nabla\cdot\Big(u\,\mathrm{D}\mathsf{e}_p'(u)\Big)$$
$$= u\,\mathsf{e}_p''(u)\,\Delta\mathsf{e}_p'(u) + \mathsf{e}_p''(u)\,\mathrm{D}u\cdot\mathrm{D}\mathsf{e}_p'(u)$$
$$= L_p'(u)\,\Delta\mathsf{e}_p'(u) + \big|\mathrm{D}\mathsf{e}_p'(u)\big|^2.$$

Recall the Bochner identity

$$2\Gamma_2(g) = \Delta|\mathrm{D}g|^2 - 2\mathrm{D}g\cdot\mathrm{D}\Delta g = 2|\mathrm{D}^2 g|^2,$$

and

$$(rL_p'(r) - L)' = rL_p''(r),$$

while

$$(rL_p'(r) - L)'\mathsf{e}_p''(r) = L_p''(r)L_p'(r).$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{F}_p(u_t) = \int\Big(\big|\mathrm{D}\mathsf{e}_p'(u)\big|^2\partial_t u + 2u\,\mathrm{D}\mathsf{e}_p'(u)\cdot\mathrm{D}\partial_t\mathsf{e}_p'(u)\Big)\,\mathrm{d}x$$
$$= \int\big|\mathrm{D}\mathsf{e}_p'(u)\big|^2\Delta u^p\,\mathrm{d}x$$
$$+ 2\int u\,\mathrm{D}\mathsf{e}_p'(u)\cdot\mathrm{D}\Big(L_p'(u)\,\Delta\mathsf{e}_p'(u) + \big|\mathrm{D}\mathsf{e}_p'(u)\big|^2\Big)\,\mathrm{d}x$$
$$= \int u^p\Delta\big|\mathrm{D}\mathsf{e}_p'(u)\big|^2\,\mathrm{d}x + 2\int uL_p'(u)\,\mathrm{D}\mathsf{e}_p'(u)\cdot\mathrm{D}\Delta\mathsf{e}_p'(u)\,\mathrm{d}x$$
$$+ 2\int\mathrm{D}L_p(u)\cdot\mathrm{D}L_p'(u)\,\Delta\mathsf{e}_p'(u)\,\mathrm{d}x - 2\int L_p(u)\Delta\big|\mathrm{D}\mathsf{e}_p'(u)\big|^2$$
$$= -\int L_p(u)\Delta\big|\mathrm{D}\mathsf{e}_p'(u)\big|^2\,\mathrm{d}x + 2\int L_p(u)\,\mathrm{D}\mathsf{e}_p'(u)\cdot\mathrm{D}\Delta\mathsf{e}_p'(u)\,\mathrm{d}x$$
$$+ 2\int\big(uL_p'(u) - L_p(u)\big)\,\mathrm{D}\mathsf{e}_p'(u)\cdot\mathrm{D}\Delta\mathsf{e}_p'(u)\,\mathrm{d}x$$
$$+ 2\int\mathrm{D}L_p(u)\cdot\mathrm{D}L_p'(u)\,\Delta\mathsf{e}_p'(u)\,\mathrm{d}x$$
$$= -\int L_p(u)\Gamma_2(\mathsf{e}_p'(u))\,\mathrm{d}x$$
$$- 2\int\big(uL_p'(u) - L_p(u)\big)\,\big(\Delta\mathsf{e}_p'(u)\big)^2\,\mathrm{d}x$$
$$= -2\int L_p(u)|\mathrm{D}^2\mathsf{e}_p'(u)|^2 + \big(uL_p'(u) - L_p(u)\big)\,\big(\Delta\mathsf{e}_p'(u)\big)^2\,\mathrm{d}x$$
$$= -2\int u^p\Big(|\mathrm{D}^2\mathsf{e}_p'(u)|^2 + (p-1)\big(\Delta\mathsf{e}_p'(u)\big)^2\Big)\,\mathrm{d}x.$$

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