Gradient Flows of Probability Measures

Luigi Ambrosio*
Scuola Normale Superiore, Pisa

Giuseppe Savaré†
Dipartimento di Matematica, Università di Pavia

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*l.ambrosio@sns.it
†giuseppe.savare@unipv.it

Introduction

In a finite-dimensional smooth setting, the gradient flow of a function $\phi : \mathbb{M}^d \to \mathbb{R}$ defined on a Riemannian manifold $\mathbb{M}^d$ simply means the family of solutions $u : \mathbb{R} \to \mathbb{M}^d$ of the Cauchy problem associated to the differential equation

$$\frac{d}{dt}u(t) = -\nabla\phi(u(t)) \quad \text{in } T_{u(t)}\mathbb{M}^d, \quad t \in \mathbb{R}; \qquad u(0) = u_0 \in \mathbb{M}^d. \tag{0.1}$$

Thus, at each time $t \in \mathbb{R}$ equation (0.1), which is imposed in the tangent space $T_{u(t)}\mathbb{M}^d$ of $\mathbb{M}^d$ at the moving point $u(t)$, simply prescribes that the velocity vector $\boldsymbol{v}_t := \frac{d}{dt}u(t)$ of the curve u equals the opposite of the gradient of ϕ at $u(t)$.

The extension of the theory of gradient flows to suitable (infinite-dimensional) abstract/functional spaces and its link with evolutionary PDE's is a wide subject with a long history.

One of its first main achievement, going back to the pioneering papers by Komura [61], Crandall-Pazy [33], Brézis [21] (we refer to the monograph [22]), concerns an Hilbert space H and nonlinear contraction semigroups generated by a proper, convex, and lower semicontinuous functional $\phi : H \to (-\infty, +\infty]$. Since in general ϕ admits only a subdifferential $\partial\phi$ in a (possibly strict) subset $D(\partial\phi) \subset D(\phi) := \{u \in H : \phi(u) < +\infty\}$ and each tangent space of H can be identified with H itself, it turns out that (0.1) should be rephrased as a *subdifferential inclusion* on the positive real line

$$u'(t) \in -\partial\phi(u(t)), \quad t > 0; \qquad u(0) = u_0 \in \overline{D(\phi)}, \tag{0.2}$$

and it provides a general framework for studying existence, uniqueness, stability, asymptotic behavior, and regularizing properties of many PDE's of parabolic type.

The possibility to work in a more general metric space (E, d) and/or with nonsmooth perturbations of a convex functional $\phi : E \to (-\infty, +\infty]$ has been exploited by E. DeGiorgi and his collaborators in a series of papers originating from [37] and culminating in [64] (see also the presentation of [6] and our recent book [9]). One of the nice features of this approach is the so called "Minimizing Movement" approximation scheme [36]: it suggests a general variational procedure to approximate and construct gradient flows by a recursive minimization algorithm. For, one introduces a uniform partition $0 < \tau < 2\tau < \cdots < n\tau < \cdots$ of the positive real line, $\tau > 0$ being the step size, and starting from the initial value $U_\tau^0 := u_0$ one looks for a suitable approximation U_τ^n of u at the time $n\tau$ by iteratively solving the minimum problems

$$\min_{U \in E} \phi(U) + \frac{1}{2\tau}d^2(U, U_\tau^{n-1}). \tag{0.3}$$

Under general lower semicontinuity and coercivity assumptions, a minimizer U_τ^n of (0.3) exists so that a piecewise constant interpolant U_τ taking the value U_τ^n in each interval $((n-1)\tau, n\tau]$ can be constructed. Limit points (possibly after extracting a suitable subsequence) of $U_\tau(t)$ as $\tau \downarrow 0$ can be considered as good candidates for gradient flows of ϕ and in many circumstances it is in fact possible to give differential characterizations of their trajectories.

One of the most striking application of this variational point of view has been introduced by OTTO [57, 74] (also in collaboration with JORDAN and KINDERLEHRER): he showed that the Fokker-Planck equation

$$\partial_t u - \nabla \cdot (\nabla u + u \nabla V) = 0 \quad \text{in } \mathbb{R}^d \times (0, +\infty), \tag{0.4}$$

and nonlinear diffusion equations of porous media type

$$\partial_t u - \Delta \beta(u) = 0 \quad \text{in } \mathbb{R}^d \times (0, +\infty), \tag{0.5}$$

can be interpreted as gradient flows, in the metric space $E := \mathscr{P}_2(\mathbb{R}^d)$ of Borel probability measures in $\mathbb{R}^d$ with finite quadratic moment, of suitable integral functionals of the type

$$\phi(\mu) := \int_{\mathbb{R}^d} F(\rho(x))\, d\gamma(x), \quad \rho := \frac{d\mu}{d\gamma} \tag{0.6}$$

for a suitable choice of the nonlinearity F and of the reference measure γ in $\mathbb{R}^d$. Here the solutions u_t of (0.4) and (0.5) yield a corresponding family of evolving measures $\mu_t \in \mathscr{P}_2(\mathbb{R}^d)$ through the identification $\mu_t = u_t \mathscr{L}^d$.

One of the main novelties of OTTO's approach relies in the particular *distance* d on $\mathscr{P}_2(\mathbb{R}^d)$ which should be used to recover the above mentioned PDE's in the limit: it is the so called KANTOROVICH-RUBINSTEIN-WASSERSTEIN distance between two measures μ, $\nu \in \mathscr{P}_2(\mathbb{R}^d)$, defined as

$$\begin{aligned} W_2^2(\mu,\nu) := \min \Big\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 \, d\boldsymbol{\gamma}(x,y) : \\ \boldsymbol{\gamma} \in \mathscr{P}(\mathbb{R}^d \times \mathbb{R}^d),\ \pi^1_\# \boldsymbol{\gamma} = \mu,\ \pi^2_\# \boldsymbol{\gamma} = \nu \Big\}. \end{aligned} \tag{0.7}$$

The minimum in (0.7) is thus evaluated on all probability measures $\boldsymbol{\gamma}$ on the product $\mathbb{R}^d \times \mathbb{R}^d$ whose marginals $\pi^1_\# \boldsymbol{\gamma}$, $\pi^2_\# \boldsymbol{\gamma}$ are μ and ν respectively; π^1, π^2 : $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ denote the canonical projections on the first and the second factor.

By applying the "Minimizing Movement" scheme in $\mathscr{P}_2(\mathbb{R}^d)$ with the above choice (0.6) of ϕ and with $d := W_2$, it is in fact possible to show that its discrete trajectories converge to the solution of a suitable evolution PDE's. Moreover, OTTO introduced a formal "Riemannian" structure in the space $\mathscr{P}_2(\mathbb{R}^d)$ in order to guess first, and then prove rigorously the form of the limit PDE's and their gradient flow structure like in (0.1).

The aim of this paper is to present, in a simplified form, the general and rigorous theory developed in our book [9] (written with N. GIGLI), giving quite general answers to the following questions:

1) Give a rigorous meaning to the concept of gradient flow in $\mathscr{P}_2(\mathbb{R}^d)$.

2) Find general conditions on ϕ in order to guarantee the convergence of the "Minimizing Movement" scheme in $\mathscr{P}_2(\mathbb{R}^d)$.

3) Characterize the limit trajectories and study their properties, applying them to classes of specific and relevant examples.

In comparison with [9], the simplification comes from the fact that we mostly restrict ourselves to absolutely continuous measures, in finite-dimensional spaces, while in [9] none of these restrictions is present.

Concerning the first point, it is clear from the heuristic arguments of Otto and from (0.1) that one should make precise:

1a) the notion of **velocity vector field** of a curve $(\mu_t)_{t\in(0,T)}$ of measures in $\mathscr{P}_2(\mathbb{R}^d)$,

1b) the notion of **tangent space** $\mathrm{Tan}_\mu\mathscr{P}_2(\mathbb{R}^d)$ of $\mathscr{P}_2(\mathbb{R}^d)$ at a given measure μ,

1c) the notion of **gradient** of a functional ϕ (like (0.6)) at μ.

The investigations about velocity and tangent space are in fact strictly related to a deep analysis of the *continuity equation*

$$\partial_t\mu_t + \nabla\cdot(\boldsymbol{v}_t\mu_t) = 0 \quad \text{in } \mathbb{R}^d\times(0,T).$$

It is carried out in Section 2.6 after some basic preliminaries of measure theory (recalled in Section 1), a brief outline on optimal transportation and Wasserstein distance (presented in Sections 2.1-2.4), and a more detailed review on the classical representation formulas for solutions of the continuity equation, which is discussed in Section 2.5. Starting form the general definition of absolutely continuous curves in a (arbitrary) metric space, we will show that every absolutely continuous family of measures $(\mu_t)_{t\in(0,T)}$ in $\mathscr{P}_2(\mathbb{R}^d)$ satisfies the continuity equation

$$\partial_t\mu_t + \nabla\cdot(\boldsymbol{v}_t\mu_t) = 0 \quad \text{in the distribution sense of } \mathscr{D}'\left(\mathbb{R}^d\times(0,T)\right), \tag{0.8}$$

for a suitable **Borel velocity vector field** $\boldsymbol{v}_t \in L^2(\mu_t;\mathbb{R}^d)$ satisfying

$$\mathrm{Length}_a^b(\mu_t) = \int_0^T \Big(\int_{\mathbb{R}^d} |\boldsymbol{v}_t(x)|^2\,d\mu_t(x)\Big)^{1/2} dt \qquad \forall 0\le a<b\le T. \tag{0.9}$$

Furthermore, (0.8) and (0.9) uniquely determine $\boldsymbol{v}_t$ in $L^2(\mu_t;\mathbb{R}^d)$ up to a negligible set of times.

Since $\mathscr{P}_2(\mathbb{R}^d)$ is a length space (i.e. the infimum of the distance between any two points is the infimum of the lengths of all curves connecting the two points), one recovers also the Benamou-Brenier formula [15]

$$\begin{aligned} W_2(\mu,\nu) = \min\Big\{ \int_0^1 \Big(\int_{\mathbb{R}^d} |\boldsymbol{v}_t(x)|^2\,d\mu_t(x)\Big)^{1/2} dt :\\ \mu_t\in AC((0,1);\mathscr{P}_2(\mathbb{R}^d)) \quad \text{satisfies (0.8)}, \quad \mu_0=\mu,\ \mu_1=\nu\Big\}. \end{aligned} \tag{0.10}$$

Recalling the usual definition of the Riemannian distance on a manifold, we can thus consider $\boldsymbol{v}_t$ as the **velocity vector** of the curve (μ_t) and the squared $L^2(\mu_t;\mathbb{R}^d)$-norm as the metric tensor in $\mathscr{P}_2(\mathbb{R}^d)$.

It turns out that in general the set spanned by all the possible velocity vector field of a curve through a measure μ is a *proper* subset of $L^2(\mu_t;\mathbb{R}^d)$. For, $\boldsymbol{v}_t$ can be strongly approximated in $L^2(\mu_t;\mathbb{R}^d)$ by gradients of smooth functions (and this approximability property is equivalent to (0.9)); moreover, gradients of smooth functions are always velocity vectors (in the above sense) of smooth curves. These facts suggests the definition of the **tangent space** as

$$\mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d) := \overline{\Big\{\nabla\varphi : \varphi \in C_c^\infty(\mathbb{R}^d)\Big\}}^{L^2(\mu_t;\mathbb{R}^d)}. \tag{0.11}$$

One of the important byproducts of this analysis is the formula

$$\frac{d}{dt} W_2^2(\mu_t,\nu) = 2\int_{\mathbb{R}^d} \langle \boldsymbol{v}_t, \boldsymbol{t}_{\mu_t} - \boldsymbol{i}\rangle \, d\mu_t \qquad \text{for a.e. } t \tag{0.12}$$

for the squared Wasserstein distance from a given measure ν. Here $\boldsymbol{t}_{\mu_t}^\nu$ are the optimal transport maps between μ_t and ν (provided they exist, as it happens whenever μ_t are absolutely continuous) and $\boldsymbol{i}$ is the identity map.

Concerning 1c), any reasonable definition of gradient in infinite dimensional spaces should be sufficiently general to fit with various classes of non smooth functionals. For easy of exposition, in this paper we decided to focus our attention on the case of *geodesically convex* (or, more generally, λ-convex) functionals (we refer to [9] for more general results). Geodesics in $\mathscr{P}_2(\mathbb{R}^d)$ play a crucial role and their characterization is briefly discussed in Section 2.3. Section 3 is thus devoted to the analysis of convex functionals in $\mathscr{P}_2(\mathbb{R}^d)$ and to some particularly important examples, discovered by McCann [67].

Having at our disposal a nice Hilbertian structure at the level of each tangent space and a significant notion of convexity, it is natural to develop a **subdifferential theory** modeled on the well known linear one. We deal with this program in Section 4: first of all we define the (Fréchet) subdifferential $\partial\phi(\mu)$ of ϕ at a measure μ. Even if it is a multivalued map, it is possible to perform a natural *minimal selection* $\partial\phi^\circ(\mu)$ among its values, which enjoys nice features and always belongs to the tangent space $\mathrm{Tan}_\mu\mathscr{P}_2(\mathbb{R}^d)$. Sections 4.2–4.4 present the basic calculus properties of the subdifferential: they precisely reproduce the analogous ones of the linear framework and justify the interest for this notion. Section 4.5 contains the main characterizations of the subdifferential of the most relevant functionals (internal, potential and interaction energies, and the negative squared Wasserstein distance).

Combining all these notions, we end up with the rigorous definition of the gradient flow of a functional ϕ in Section 5: it always has the structure of the continuity equation

$$\partial_t\mu_t + \nabla\cdot(\boldsymbol{v}_t\mu_t) = 0 \quad \text{in } \mathscr{D}'(\mathbb{R}^d \times (0,T)), \tag{0.13}$$

which defines the **velocity** of μ_t, coupled with the **nonlinear condition**

$$\boldsymbol{v}_t = -\partial^\circ\phi(\mu_t) \in \mathrm{Tan}_{\mu_t}\mathscr{P}_2(\mathbb{R}^d) \quad \text{for } \mathscr{L}^1\text{-a.e. } t\in(0,T), \tag{0.14}$$

linking $\boldsymbol{v}_t$ to μ_t through the functional ϕ. When ϕ has the structure of (0.6) and $\mu_t = \rho_t \gamma$, (0.14) is equivalent (in a suitable weak sense) to

$$\boldsymbol{v}_t = -\nabla F'(\rho_t). \tag{0.15}$$

The remaining part of the section is devoted to study **the main properties of the gradient flows**, obtained independently from the existence issue, i.e. directly from the definition. We conclude the section providing an answer to the second question we raised before, i.e. **the construction of the gradient flow by means of the variational approximation scheme.**

Even in this case, (λ-geodesic) convexity plays a crucial role and we are able to obtain the same well known results of the theory in flat linear spaces. Here we only mention the generation of a contracting and regularizing semigroup satisfying, when $\lambda > 0$, nice asymptotic convergence estimates. In comparison with other papers ([29] and [76], for the porous medium equation on Riemannian manifolds), where similar goals are pursued, our approach is totally independent of the specific form of the functional ϕ and of the PDE that it induces: it is ultimately based on the one hand on monotonicity inequalities (ensured by the λ-convexity of ϕ), and on the other hand on (0.12), whose validity is a purely geometrical fact. Furthermore, as shown in [9], it extends also to the case when $\mathbb{R}^d$ is replaced by a separable Hilbert space and/or singular (e.g. concentrated) measures are allowed.

The last section illustrates our main examples and applications. A particular emphasis is devoted to the linear Fokker-Planck equation (0.4) associated to a convex potential V with arbitrary growth at infinity: as showed by Otto, it is the gradient flow in $\mathscr{P}_2(\mathbb{R}^d)$ of the relative entropy functional

$$\phi(\mu) := \int_{\mathbb{R}^d} \rho(x) \log \rho(x)\, d\gamma(x), \quad \rho = \frac{d\mu}{d\gamma}, \tag{0.16}$$

with respect to the invariant measure $\gamma := e^{-V}\mathscr{L}^d$. In this case the Wasserstein approach provides a linear semigroup in the space of measures (a Dirac mass concentrated in a point where the potential is finite is always allowed as an initial datum), which easily gives nice representation formulae for the solution. The restriction of the semigroup on absolutely continuous measures w.r.t. γ coincides with the Markov semigroup generated by the natural Dirichlet form associated to γ.

Applications to the case of nonlinear diffusion equations and to more complicated differential-integral equations are also considered.

Notation

$B_r(x)$	Open ball of radius r centered at x in a metric space
$\mathscr{B}(X)$	Borel sets in a separable metric space X
$C^0_b(X)$	Space of continuous and bounded real functions defined on X
$C^\infty_c(\mathbb{R}^d)$	Space of smooth real functions with compact support in $\mathbb{R}^d$
$\mathscr{P}(X)$	Probability measures in a separable metric space X
$\mathscr{P}_2(X)$	Probability measures with finite quadratic moment, see (1.3)
$\mathscr{L}^d$	The Lebesgue measure in $\mathbb{R}^d$
$\mathscr{P}^a_2(\mathbb{R}^d)$	Measures in $\mathscr{P}_2(\mathbb{R}^d)$ absolutely continuous w.r.t. $\mathscr{L}^d$
$L^p(\mu;\mathbb{R}^d)$	L^p space of μ-measurable $\mathbb{R}^d$-valued maps
$\operatorname{supp}\mu$	Support of μ, see (1.1)
$\boldsymbol{r}_\#\mu$	Push-forward of μ through $\boldsymbol{r}$, see (1.4)
π^i	Projection operators on a product space $\boldsymbol{X}$, see (1.8)
$\Gamma(\mu^1,\mu^2)$	2-plans with given marginals μ^1, μ^2
$\Gamma_o(\mu^1,\mu^2)$	Optimal 2-plans with given marginals μ^1, μ^2
$W_2(\mu,\nu)$	2-th Wasserstein distance between μ and ν, see (2.6)
$\boldsymbol{i}$	Identity map
$\boldsymbol{t}^\nu_\mu$	Optimal transport map between μ and ν given by Theorem 2.3
$\operatorname{Tan}_{\mu_t}\mathscr{P}_2(\mathbb{R}^d)$	Tangent bundle to $\mathscr{P}_2(\mathbb{R}^d)$, see (2.42)
$\mu^{1\to 2}_t$	Geodesic curve connecting μ^1 ro μ^2, see (3.1)
$\lvert u'\rvert(t)$	Metric derivative of $u:(a,b)\to E$, see (2.2)
$AC^p\left((a,b);E\right)$	Absolutely continuous $u:(a,b)\to E$ with $\lvert u'\rvert\in L^p(a,b)$, see (2.3)
$D(\phi)$	Proper domain of a functional ϕ, see (4.1)
$\operatorname{Lip}(\phi,A)$	Lipschitz constant of the function ϕ in the set A
$\partial\phi(v)$	Fréchet subdifferential of ϕ in Hilbert (4.2)
	or Wasserstein spaces, see Definition 4.1 and (4.20)
$\lvert\partial\phi\rvert(v)$	Metric slope of ϕ, see Definition (4.4) and (4.29)
$\partial^\circ\phi(\mu)$	Minimal selection in the subdifferential, see Lemma 4.10
$\overline{M}_\tau(t)$	Piecewise constant interpolation of M^n_τ, see (5.54)
$MM(\Phi;u_0)$	Minimizing movement of ϕ, see the definition before (5.55)

1 Notation and measure-theoretic results

In this section we recall the main notation used in this paper and some basic measure-theoretic terminology and results. Given a separable metric space (X, d), we denote by $\mathscr{P}(X)$ the set of probability measures $\mu : \mathscr{B}(X) \to [0, 1]$, where $\mathscr{B}(X)$ is the Borel σ-algebra. The support of $\mu \in \mathscr{P}(X)$ is the closed set

$$\operatorname{supp}(\mu) := \Big\{ x \in X : \mu(B_r(x)) > 0 \quad \forall\, r > 0 \Big\}. \tag{1.1}$$

When X is a Borel subset of an euclidean space $\mathbb{R}^d$, we set

$$\mathsf{m}_2(\mu) := \int_X |x|^2 \, d\mu,$$

we often make the identification

$$\mathscr{P}(X) = \Big\{ \mu \in \mathscr{P}(\mathbb{R}^d) : \mu(\mathbb{R}^d \setminus X) = 0 \Big\}, \tag{1.2}$$

and we denote by $\mathscr{P}_2(X)$ the subspace of $\mathscr{P}(X)$ made by measures with finite quadratic moment:

$$\mathscr{P}_2(X) := \{ \mu \in \mathscr{P}(X) : \ \mathsf{m}_2(\mu) < \infty \} \,. \tag{1.3}$$

We denote by $\mathscr{L}^d$ the Lebesgue measure in $\mathbb{R}^d$ and set

$$\mathscr{P}_2^a(X) := \left\{ \mu \in \mathscr{P}_2(X) : \ \mu \ll \mathscr{L}^d \right\},$$

whenever $X \in \mathscr{B}(\mathbb{R}^d)$.

1.1 Transport maps and transport plans

If $\mu \in \mathscr{P}(X_1)$, and $\boldsymbol{r} : X_1 \to X_2$ is a Borel (or, more generally, μ-measurable) map, we denote by $\boldsymbol{r}_\# \mu \in \mathscr{P}(X_2)$ the *push-forward of μ through $\boldsymbol{r}$*, defined by

$$\boldsymbol{r}_\# \mu(B) := \mu(\boldsymbol{r}^{-1}(B)) \qquad \forall\, B \in \mathscr{B}(X_2). \tag{1.4}$$

More generally we have

$$\int_{X_1} f(\boldsymbol{r}(x)) \, d\mu(x) = \int_{X_2} f(y) \, d\, \boldsymbol{r}_\# \mu(y) \tag{1.5}$$

for every bounded (or $\boldsymbol{r}_\# \mu$-integrable) Borel function $f : X_2 \to \mathbb{R}$. It is easy to check that

$$\nu \ll \mu \quad \Longrightarrow \quad \boldsymbol{r}_\# \nu \ll \boldsymbol{r}_\# \mu \qquad \forall\, \mu,\, \nu \in \mathscr{P}(X_1). \tag{1.6}$$

Notice also the natural composition rule

$$(\boldsymbol{r} \circ \boldsymbol{s})_\# \mu = \boldsymbol{r}_\# (\boldsymbol{s}_\# \mu) \quad \text{where } \boldsymbol{s} : X_1 \to X_2,\ \boldsymbol{r} : X_2 \to X_3,\ \mu \in \mathscr{P}(X_1). \tag{1.7}$$

We denote by π^i, $i = 1, 2$, the projection operators defined on a product space $\boldsymbol{X} := X_1 \times X_2$, defined by

$$\pi^1 : (x_1, x_2) \mapsto x_1 \in X_1, \qquad \pi^2 : (x_1, x_2) \mapsto x_2 \in X_2. \tag{1.8}$$

If $\boldsymbol{X}$ is endowed with the canonical product metric and the Borel σ-algebra and $\boldsymbol{\mu} \in \mathscr{P}(\boldsymbol{X})$, the *marginals* of $\boldsymbol{\mu}$ are the probability measures

$$\mu^i := \pi^i_{\#}\boldsymbol{\mu} \in \mathscr{P}(X_i), \quad i = 1,\, 2. \tag{1.9}$$

Given $\mu^1 \in \mathscr{P}(X_1)$ and $\mu^2 \in \mathscr{P}(X_2)$ the class $\Gamma(\mu^1, \mu^2)$ of transport plans between μ^1 and μ^2 is defined by

$$\Gamma(\mu^1, \mu^2) := \Big\{\boldsymbol{\mu} \in \mathscr{P}(X_1 \times X_2) : \ \pi^i_{\#}\boldsymbol{\mu} = \mu^i, \ i = 1, 2\Big\}. \tag{1.10}$$

Notice also that

$$\Gamma(\mu^1, \mu^2) = \{\mu^1 \times \mu^2\} \quad \text{if either } \mu^1 \text{ or } \mu^2 \text{ is a Dirac mass.} \tag{1.11}$$

To each couple of measures $\mu^1 \in \mathscr{P}(X_1), \mu^2 = \boldsymbol{r}_{\#}\mu^1 \in \mathscr{P}(X_2)$ linked by a Borel transport map $\boldsymbol{r} : X_1 \to X_2$ we can associate the transport plan

$$\boldsymbol{\mu} := (\boldsymbol{i} \times \boldsymbol{r})_{\#}\mu^1 \in \Gamma(\mu^1, \mu^2), \quad \boldsymbol{i} \text{ being the identity map on } X_1. \tag{1.12}$$

If $\boldsymbol{\mu}$ is representable as in (1.12) then we say that $\boldsymbol{\mu}$ is *induced* by $\boldsymbol{r}$. Each transport plan $\boldsymbol{\mu}$ concentrated on a $\boldsymbol{\mu}$-measurable graph in $X_1 \times X_2$ admits the representation (1.12) for some μ^1-measurable map $\boldsymbol{r}$, which therefore transports μ^1 to μ^2 (see, e.g., [7]).

1.2 Narrow convergence

Conformally to the probabilistic terminology, we say that a sequence $(\mu_n) \subset \mathscr{P}(X)$ is *narrowly* convergent to $\mu \in \mathscr{P}(X)$ as $n \to \infty$ if

$$\lim_{n\to\infty} \int_X f(x)\, d\mu_n(x) = \int_X f(x)\, d\mu(x) \tag{1.13}$$

for every function $f \in C^0_b(X)$, the space of continuous and bounded real functions defined on X.

Theorem 1.1 (Prokhorov, [40, III-59]) *If a set $\mathcal{K} \subset \mathscr{P}(X)$ is* tight, *i.e.*

$$\forall\, \varepsilon > 0 \quad \exists\, K_\varepsilon \text{ compact in } X \text{ such that} \quad \mu(X \setminus K_\varepsilon) \le \varepsilon \quad \forall\, \mu \in \mathcal{K}, \tag{1.14}$$

then $\mathcal{K}$ is relatively compact in $\mathscr{P}(X)$.

When one needs to pass to the limit in expressions like (1.13) w.r.t. *unbounded or lower semicontinuous* functions f, the following two properties are quite useful. The first one is a lower semicontinuity property:

$$\liminf_{n\to\infty} \int_X g(x)\, d\mu_n(x) \ge \int_X g(x)\, d\mu(x) \tag{1.15}$$

for every sequence $(\mu_n) \subset \mathscr{P}(X)$ narrowly convergent to μ and any l.s.c. function $g : X \to (-\infty, +\infty]$ bounded from below: it follows easily by a monotone approximation argument of g by continuous and bounded functions. Changing g in $-g$ one gets the corresponding "lim sup" inequality for upper semicontinuous functions bounded from above. In particular, choosing as g the characteristic functions of open and closed subset of X, we obtain

$$\liminf_{n\to\infty} \mu_n(G) \geq \mu(G) \quad \forall\, G \text{ open in } X, \tag{1.16}$$

$$\limsup_{n\to\infty} \mu_n(F) \leq \mu(F) \quad \forall\, F \text{ closed in } X. \tag{1.17}$$

The statement of the second property requires the following definitions: we say that a Borel function $g : X \to [0, +\infty]$ is *uniformly integrable* w.r.t. a given set $\mathcal{K} \subset \mathscr{P}(X)$ if

$$\lim_{k\to\infty} \int_{\{x:g(x)\geq k\}} g(x)\, d\mu(x) = 0 \quad \text{uniformly w.r.t. } \mu \in \mathcal{K}. \tag{1.18}$$

In the particular case of $g(x) := d(x, \bar{x})^p$, for some (and thus any) $\bar{x} \in X$ and a given $p > 0$, i.e. if

$$\lim_{k\to\infty} \int_{X\setminus B_k(\bar{x})} d^p(\bar{x}, x)\, d\mu(x) = 0 \quad \text{uniformly w.r.t. } \mu \in \mathcal{K}, \tag{1.19}$$

we say that the set $\mathcal{K} \subset \mathscr{P}(X)$ *has uniformly integrable p-moments*. The following lemma (see for instance Lemma 5.1.7 of [9] for its proof) provides a characterization of p-uniformly integrable families, extending the validity of (1.13) to unbounded but with p-growth functions, i.e. functions $f : X \to \mathbb{R}$ such that

$$|f(x)| \leq A + B\, d^p(\bar{x}, x) \qquad \forall x \in X, \tag{1.20}$$

for some A, $B \geq 0$ and $\bar{x} \in X$.

Lemma 1.2 *Let $(\mu_n) \subset \mathscr{P}(X)$ be narrowly convergent to $\mu \in \mathscr{P}(X)$. If $f : X \to \mathbb{R}$ is continuous, $g : X \to (-\infty, +\infty]$ is lower semicontinuous, and $|f|$ and g^- are uniformly integrable w.r.t. the set $\{\mu_n\}_{n\in\mathbb{N}}$, then*

$$\liminf_{n\to\infty} \int_X g(x)\, d\mu_n(x) \geq \int_X g(x)\, d\mu(x) > -\infty, \tag{1.21a}$$

$$\lim_{n\to\infty} \int_X f(x)\, d\mu_n(x) = \int_X f(x)\, d\mu(x). \tag{1.21b}$$

Conversely, if $f : X \to [0, \infty)$ is continuous, μ_n-integrable, and

$$\limsup_{n\to\infty} \int_X f(x)\, d\mu_n(x) \leq \int_X f(x)\, d\mu(x) < +\infty, \tag{1.22}$$

then f is uniformly integrable w.r.t. $\{\mu_n\}_{n\in\mathbb{N}}$.
In particular, a family $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathscr{P}(X)$ has uniformly integrable p-moments iff (1.21b) *holds for every continuous function $f : X \to \mathbb{R}$ with p-growth.*

1.3 The change of variables formula

Let $\boldsymbol{r} : A \subset \mathbb{R}^d \to \mathbb{R}^d$ be a function, with A open. Then, denoting by $\Sigma_{\boldsymbol{r}} = D(\nabla \boldsymbol{r})$ the Borel set where $\boldsymbol{r}$ is differentiable, there is a sequence of sets $\Sigma_n \uparrow \Sigma_{\boldsymbol{r}}$ such that $\boldsymbol{r}|_{\Sigma_n}$ is a Lipschitz function for any n (see [45, 3.1.8]). Therefore the well-known area formula for Lipschitz maps (see for instance [44, 45]) extends to this general class of maps and reads as follows:

$$\int_{\Sigma_{\boldsymbol{r}}} h(x)|\det \nabla \boldsymbol{r}|(x)\,dx = \int_{\mathbb{R}^d} \sum_{x \in \Sigma_{\boldsymbol{r}} \cap \boldsymbol{r}^{-1}(y)} h(x)\,dy \tag{1.23}$$

for any Borel function $h : \mathbb{R}^d \to [0, +\infty]$. This formula leads to a simple rule for computing the density of the push-forward of measures absolutely continuous w.r.t. $\mathscr{L}^d$.

Lemma 1.3 (Density of the push-forward) *Let $\rho \in L^1(\mathbb{R}^d)$ be a nonnegative function and assume that there exists a Borel set $\Sigma \subset \Sigma_{\boldsymbol{r}}$ such that $\boldsymbol{r}|_\Sigma$ is injective and the difference $\{\rho > 0\} \setminus \Sigma$ is $\mathscr{L}^d$-negligible. Then $\boldsymbol{r}_\#\big(\rho\mathscr{L}^d\big) \ll \mathscr{L}^d$ if and only if $|\det \nabla \boldsymbol{r}| > 0$ $\mathscr{L}^d$-a.e. on Σ and in this case*

$$\boldsymbol{r}_\#\big(\rho\mathscr{L}^d\big) = \frac{\rho}{|\det \nabla \boldsymbol{r}|} \circ \boldsymbol{r}^{-1}|_{\boldsymbol{r}(\Sigma)} \mathscr{L}^d.$$

Proof. If $|\det \nabla \boldsymbol{r}| > 0$ $\mathscr{L}^d$-a.e. on Σ we can put $h = \rho\chi_{\boldsymbol{r}^{-1}(B)\cap\Sigma}/|\det \nabla \boldsymbol{r}|$ in (1.23), with $B \in \mathscr{B}(\mathbb{R}^d)$, to obtain

$$\int_{\boldsymbol{r}^{-1}(B)} \rho\,dx = \int_{\boldsymbol{r}^{-1}(B)\cap\Sigma} \rho\,dx = \int_{B \cap \boldsymbol{r}(\Sigma)} \frac{\rho(\boldsymbol{r}^{-1}(y))}{|\det \nabla \boldsymbol{r}(\boldsymbol{r}^{-1}(y))|}\,dy.$$

Conversely, if there is a Borel set $B \subset \Sigma$ with $\mathscr{L}^d(B) > 0$ and $|\det \nabla \boldsymbol{r}| = 0$ on B, the area formula gives $\mathscr{L}^d(\boldsymbol{r}(B)) = 0$. On the other hand

$$\boldsymbol{r}_\#(\rho\mathscr{L}^d)(\boldsymbol{r}(B)) = \int_{\boldsymbol{r}^{-1}(f(B))} \rho\,dx > 0$$

because at $\mathscr{L}^d$-a.e. $x \in B$ we have $\rho(x) > 0$. Hence $\boldsymbol{r}_\#(\rho\mathscr{L}^d)$ is not absolutely continuous with respect to $\mathscr{L}^d$. $\square$

By applying the area formula again we obtain the rule for computing integrals of the densities:

$$\int_{\mathbb{R}^d} F\left(\frac{\boldsymbol{r}_\#(\rho\mathscr{L}^d)}{\mathscr{L}^d}\right) dx = \int_{\mathbb{R}^d} F\left(\frac{\rho}{|\det \nabla \boldsymbol{r}|}\right) |\det \nabla \boldsymbol{r}|\,dx \tag{1.24}$$

for any Borel function $F : [0, +\infty) \to [0, +\infty]$ with $F(0) = 0$. Notice that in this formula the set $\Sigma_{\boldsymbol{r}}$ does not appear anymore (due to the fact that $F(0) = 0$ and $\rho = 0$ out of Σ), so it holds provided $\boldsymbol{r}$ is differentiable $\rho\mathscr{L}^d$-a.e., it is $\rho\mathscr{L}^d$-essentially injective (i.e. there exists a Borel set Σ such that $\boldsymbol{r}|_\Sigma$ is injective and $\rho = 0$ $\mathscr{L}^d$-a.e. out of Σ) and $|\det \nabla \boldsymbol{r}| > 0$ $\rho\mathscr{L}^d$-a.e. in $\mathbb{R}^d$.

We will apply mostly these formulas when $\boldsymbol{r}$ is the gradient of a convex function $g : \Omega \to \mathbb{R}$, Ω being an open subset of $\mathbb{R}^d$. In this specific case it is well known that the (multivalued) subdifferential $\partial g(x)$ of g (we will recall its definition at the beginning of Section 4) is non empty for every $x \in \Omega$ and it is reduced to a single point $\nabla g(x)$ when g is differentiable at x: this happen for $\mathscr{L}^d$-a.e. $x \in \Omega$.

In the following result (see for instance [4, 44]) we are considering an arbitrary Borel selection $\boldsymbol{r} : \Omega \to \mathbb{R}^d$ such that

$$\boldsymbol{r}(x) \in \partial g(x) \qquad \text{for every } x \in \Omega. \tag{1.25}$$

Theorem 1.4 (Aleksandrov) *Let $\Omega \subset \mathbb{R}^d$ be a convex open set and let $g : \Omega \to \mathbb{R}$ be a convex function. Then g is a locally Lipschitz function, (every extension $\boldsymbol{r}$ satisfying* (1.25) *of) ∇g is differentiable at $\mathscr{L}^d$-a.e. point of Ω, its gradient $\nabla^2 g(x)$ is a symmetric matrix, and g has the second order Taylor expansion*

$$g(y) = g(x) + \langle \nabla g(x), y - x\rangle + \frac{1}{2}\langle \nabla^2 g(x), y - x\rangle + o(|y-x|^2) \quad \text{as } y \to x \tag{1.26}$$

for $\mathscr{L}^d$-a.e. $x \in \Omega$.

Notice that ∇g is also monotone

$$\langle \nabla g(x_1) - \nabla g(x_2), x_1 - x_2\rangle \geq 0 \qquad x_1,\, x_2 \in D(\nabla g),$$

and that the above inequality is strict if g is *strictly* convex: in this case, it is immediate to check that ∇g is injective on $D(\nabla g)$, and that $|\det \nabla^2 g| > 0$ on the differentiability set of ∇g if g is *uniformly* convex.

2 Metric and differentiable structure of the Wasserstein space

In this section we look at $\mathscr{P}_2(\mathbb{R}^d)$ first from the metric and then from the differentiable viewpoints.

2.1 Absolutely continuous maps and metric derivative

Let (E, d) be a metric space.

Definition 2.1 (Absolutely continuous curves) *Let $I \subset \mathbb{R}$ be an interval and let $u : I \to E$. We say that u is absolutely continuous if there exists $m \in L^1(I)$ such that*

$$d(u(s), u(t)) \leq \int_s^t m(\tau)\, d\tau \qquad \forall s,\, t \in I,\ s \leq t. \tag{2.1}$$

Any absolutely continuous curve is obviously uniformly continuous, and therefore it can be uniquely extended to the closure of I. It is not difficult to show (see for instance Theorem 1.1.2 in [9] or [11]) that the *metric derivative*

$$|u'|(t) := \lim_{h\to 0} \frac{d(u(t+h), u(t))}{|h|} \tag{2.2}$$

exists at $\mathscr{L}^1$-a.e. $t \in I$ for any absolutely continuous curve $u(t)$. Furthermore, $|u'| \in L^1(I)$ and is the minimal m fulfilling (2.1) (i.e. $|u'|$ fulfills (2.1) and $m \geq |u'|$ $\mathscr{L}^1$-a.e. in I for any m with this property). For $p \in [1, +\infty]$ we also set

$$AC^p(I; E) := \{u : I \to E : \ u \text{ is absolutely continuous and } |u'| \in L^p(I)\}. \tag{2.3}$$

2.2 The quadratic optimal transport problem

Let X, Y be complete and separable metric spaces such that and let $c : X \times Y \to [0, +\infty]$ be a Borel cost function. Given $\mu \in \mathscr{P}(X)$, $\nu \in \mathscr{P}(Y)$ the optimal transport problem, in Monge's formulation, is given by

$$\inf \left\{ \int_X c(x, \boldsymbol{t}(x))\, d\mu(x) : \ \boldsymbol{t}_\# \mu = \nu \right\}. \tag{2.4}$$

This problem can be ill posed because sometimes there is no transport map $\boldsymbol{t}$ such that $\boldsymbol{t}_\# \mu = \nu$ (this happens for instance when μ is a Dirac mass and ν is not a Dirac mass). Kantorovich's formulation

$$\min \left\{ \int_{X \times Y} c(x, y)\, d\boldsymbol{\gamma}(x, y) : \ \boldsymbol{\gamma} \in \Gamma(\mu, \nu) \right\} \tag{2.5}$$

circumvents this problem (as $\mu \times \nu \in \Gamma(\mu, \nu)$). The existence of an optimal transport plan, when c is l.s.c., is provided by (1.15) and by Theorem 1.1, taking into account that $\Gamma(\mu, \nu)$ is tight (this follows easily by the fact that the marginals of the measures in $\Gamma(\mu, \nu)$ are fixed, and by the fact that according to Ulam's theorem any finite measure in a complete and separable metric space is tight, see also Chapter 6 in [9] for more general formulations).

The problem (2.5) is truly a weak formulation of (2.4) in the following sense: if c is bounded and continuous, and if μ has no atom, then the "min" in (2.5) is equal to the "inf" in (2.4), see [47], [7]. This result can also be extended to classes of unbounded cost functions, see [79].

In the sequel we consider the case when $X = Y$ and $c(x, y) = d^2(x, y)$, where d is the distance in X, and denote by $\Gamma_o(\mu, \nu)$ the *optimal plans* in (2.5) corresponding to this choice of the cost function. In this case we use the minimum value to define the Kantorovich-Rubinstein-Wasserstein distance

$$W_2(\mu, \nu) := \left(\int_{X \times X} d^2(x, y)\, d\boldsymbol{\gamma} \right)^{1/2} \qquad \boldsymbol{\gamma} \in \Gamma_o(\mu, \nu). \tag{2.6}$$

Theorem 2.2 *Let X be a complete and separable metric space. Then W_2 defines a distance in $\mathscr{P}_2(X)$ and $\mathscr{P}_2(X)$, endowed with this distance, is a complete and separable metric space. Furthermore, for a given sequence $(\mu_n) \subset \mathscr{P}_2(X)$ we have*

$$\lim_{n\to\infty} W_2(\mu_n, \mu) = 0 \iff \begin{cases} \mu_n \text{ narrowly converge to } \mu, \\ (\mu_n) \text{ has uniformly integrable 2-moments.} \end{cases} \tag{2.7}$$

Proof. We just prove that W_2 is a distance. The complete statement is proved for instance in Proposition 7.1.5 of [9] or, in the locally compact case, in [86].

Let μ, ν, $\sigma \in \mathscr{P}_2(X)$ and let $\boldsymbol{\gamma} \in \Gamma_o(\mu,\nu)$ and $\boldsymbol{\eta} \in \Gamma_o(\nu,\sigma)$. General results of probability theory (see the above mentioned references) ensure the existence of $\boldsymbol{\lambda} \in \mathscr{P}(X \times X \times X)$ such that

$$(\pi^1, \pi^2)_\# \boldsymbol{\lambda} = \boldsymbol{\gamma}, \qquad (\pi^2, \pi^3)_\# \boldsymbol{\lambda} = \boldsymbol{\eta}.$$

Then, as

$$\pi^1_\# (\pi^1, \pi^3)_\# \boldsymbol{\lambda} = \pi^1_\# \boldsymbol{\lambda} = \pi^1_\# \boldsymbol{\gamma} = \mu, \qquad \pi^2_\# (\pi^1, \pi^3)_\# \boldsymbol{\lambda} = \pi^3_\# \boldsymbol{\lambda} = \pi^2_\# \boldsymbol{\eta} = \sigma,$$

we obtain that $(\pi^1, \pi^3)_\# \boldsymbol{\lambda} \in \Gamma(\mu, \sigma)$, hence

$$W_2(\mu,\nu) \le \left(\int_{X\times X} d^2(x_1,x_2)\, d(\pi^1,\pi^3)_\# \boldsymbol{\lambda} \right)^{1/2} = \|d(x_1,x_3)\|_{L^2(\boldsymbol{\lambda})}.$$

As $d(x_1,x_3) \le d(x_1,x_2) + d(x_2,x_3)$ and

$$\|d(x_1,x_2)\|_{L^2(\boldsymbol{\lambda})} = \|d(x_1,x_2)\|_{L^2(\boldsymbol{\gamma})} = W_2(\mu,\nu),$$

$$\|d(x_2,x_3)\|_{L^2(\boldsymbol{\lambda})} = \|d(x_2,x_3)\|_{L^2(\boldsymbol{\eta})} = W_2(\nu,\sigma),$$

the triangle inequality $W_2(\mu,\sigma) \le W_2(\mu,\nu) + W_2(\nu,\sigma)$ follows by the standard triangle inequality in $L^2(\boldsymbol{\lambda})$. □

In the Euclidean case $X = \mathbb{R}^d$, notice that, thanks to Lemma 1.2, the uniform integrability of $|x|^2$ with respect to $\{\mu_n\}_{n\in\mathbb{N}}$ is equivalent, assuming the narrow convergence of μ_n to μ, to the convergence of $\mathsf{m}_2(\mu_n)$ to $\mathsf{m}_2(\mu)$. Both conditions in the right hand side of (2.7) can be summarized, still thanks to the same lemma, by saying that (1.21b) holds for any continuous function f with at most quadratic growth.

Working with Monge's formulation the proof above is technically easier, as an admissible transport map between μ and σ can be obtained just composing transport maps between μ and ν with transport maps between ν and σ. However, in order to give a complete proof one needs to know either that optimal plans are induced by maps, or that the infimum in Monge's formulation coincides with the minimum in Kantorovich's one, and none of these results is trivial, even in Euclidean spaces.

Although in many situations that we consider in this paper the optimal plans are induced by maps, still the Kantorovich formulation of the optimal transport problem is quite useful to provide estimates *from above* on W_2. For instance:

$$W_2^2(\mu,\nu) \le \int_X d^2(\boldsymbol{t}(x),\boldsymbol{s}(x))\,d\sigma(x) \quad \text{whenever } \boldsymbol{t}_\#\sigma=\mu,\ \boldsymbol{s}_\#\sigma=\nu. \tag{2.8}$$

This follows by the fact that $(\boldsymbol{t},\boldsymbol{s})_\#\sigma\in\Gamma(\mu,\nu)$ and by the identity

$$\int_X d^2(\boldsymbol{t}(x),\boldsymbol{s}(x))\,d\sigma(x) = \int_{X\times X} d^2(x,y)\,d(\boldsymbol{t},\boldsymbol{s})_\#\sigma.$$

2.3 Geodesics in $\mathscr{P}_2(\mathbb{R}^d)$

Let (E,d) be a metric space. Recall that a constant speed geodesic $\gamma:[0,T]\to E$ is a map satisfying

$$d(\gamma(s),\gamma(t)) = \frac{(t-s)}{T}d(\gamma(0),\gamma(T)) \qquad \text{whenever } 0\le s\le t\le T.$$

Actually only the inequality $d(\gamma(s),\gamma(t))\le T^{-1}(t-s)d(\gamma(0),\gamma(T))$ needs to be checked for all $0\le s\le t\le T$. Indeed, if the strict inequality occurs for some $s<t$, then the triangle inequality provides

$$\begin{aligned} d(\gamma(0),\gamma(T)) &\le d(\gamma(0),\gamma(s))+d(\gamma(s),\gamma(t))+d(\gamma(t),\gamma(T)) \\ &< \frac{1}{T}(s+(t-s)+(T-t))d(\gamma(0),\gamma(T)) = d(\gamma(0),\gamma(T)), \end{aligned}$$

a contradiction.

Using this elementary fact one can show that, for any choice of $\mu,\nu\in\mathscr{P}_2(\mathbb{R}^d)$, and $\boldsymbol{\gamma}\in\Gamma_o(\mu,\nu)$, the map

$$\mu_t := \big((1-t)\pi^1+t\pi^2\big)_\#\boldsymbol{\gamma} \qquad t\in[0,1] \tag{2.9}$$

is a constant speed geodesic. Indeed,

$$\boldsymbol{\gamma}_{st} := \big(((1-s)\pi^1+s\pi^2),((1-t)\pi^1+t\pi^2)\big)_\#\boldsymbol{\gamma}\in\Gamma(\mu_s,\mu_t)$$

and this plan provides the estimate

$$W_2(\mu_s,\mu_t)\le(t-s)W_2(\mu,\nu), \tag{2.10}$$

as

$$\begin{aligned} \int_{\mathbb{R}^d\times\mathbb{R}^d}|x_1-x_2|^2\,d\boldsymbol{\gamma}_{st} &= \int_{\mathbb{R}^d\times\mathbb{R}^d}|(1-s)x_1+sx_2-(1-t)x_1-tx_2|^2\,d\boldsymbol{\gamma} \\ &= (s-t)^2\int_{\mathbb{R}^d\times\mathbb{R}^d}|x_1-x_2|^2\,d\boldsymbol{\gamma}. \end{aligned}$$

It has been proved in Theorem 7.2.2 of [9] that *any* constant speed geodesic joining μ to ν can be built in this way. We discuss additional regularity properties of the geodesics in the next section. Here we just mention that, in the case when $\boldsymbol{\gamma}$ is induced by a transport map $\boldsymbol{t}$ (i.e. $\boldsymbol{\gamma} = (\boldsymbol{i}, \boldsymbol{t})_{\#}\mu$), then (2.9) reduces to

$$\mu_t = \left((1-t)\boldsymbol{i} + t\boldsymbol{t}\right)_{\#}\mu \qquad t \in [0,1]. \tag{2.11}$$

2.4 Existence of optimal transport maps

The following basic result of [60, 20, 48] provides existence and uniqueness of the optimal transport map in the case when the initial measure μ belongs to $\mathscr{P}_2^a(\mathbb{R}^d)$.

Theorem 2.3 (Existence and uniqueness of optimal transport maps) *For any $\mu \in \mathscr{P}_2^a(\mathbb{R}^d)$, $\nu \in \mathscr{P}_2(\mathbb{R}^d)$ Kantorovich's optimal transport problem* (2.5) *with $c(x,y) = |x-y|^2$ has a unique solution $\boldsymbol{\gamma}$. Moreover:*

(i) $\boldsymbol{\gamma}$ is induced by a transport map $\boldsymbol{t}$, i.e. $\boldsymbol{\gamma} = (\boldsymbol{i}, \boldsymbol{t})_{\#}\mu$. In particular $\boldsymbol{t}$ is the unique solution of Monge's optimal transport problem (2.4).

(ii) The map $\boldsymbol{t}$ coincides μ-a.e. with the gradient of a convex function φ : $\mathbb{R}^d \to (-\infty, +\infty]$, whose finiteness domain $D(\varphi)$ has non empty interior and satisfies

$$\mu\big(\mathbb{R}^d \setminus D(\varphi)\big) = \mu\big(\mathbb{R}^d \setminus D(\nabla\varphi)\big) = 0. \tag{2.12}$$

(iii) If $\nu = \rho' \mathscr{L}^d \in \mathscr{P}_2^a(\mathbb{R}^d)$ as well, and $\boldsymbol{s}$ is the optimal transport map between ν and μ, then

$$\boldsymbol{s} \circ \boldsymbol{t} = \boldsymbol{i} \ \mu\text{-a.e. in } \mathbb{R}^d \qquad \text{and} \qquad \boldsymbol{t} \circ \boldsymbol{s} = \boldsymbol{i} \ \nu\text{-a.e. in } \mathbb{R}^d.$$

In particular $\boldsymbol{t}$ is μ-essentially injective, i.e. there exists a μ-negligible set $\mathscr{N} \subset \mathbb{R}^d$ such that, setting $\Omega = \mathbb{R}^d \setminus \mathscr{N}$, $\boldsymbol{t}|_\Omega$ is injective. Finally

$$\rho' := \frac{\rho}{\det \nabla^2 \varphi} \circ (\boldsymbol{t}|_\Omega)^{-1} \quad \nu\text{-a.e. in } \mathbb{R}^d.$$

Proof. Since $(\boldsymbol{i}, \boldsymbol{t})_{\#}\mu$ and $(\boldsymbol{s}, \boldsymbol{i})_{\#}\nu$ are both optimal plans between μ and ν, they coincide. Testing this identity between plans on $|\boldsymbol{s}(\boldsymbol{t}(x)) - x|$ (resp. $|\boldsymbol{t}(\boldsymbol{s}(y)) - y|$) we obtain that $\boldsymbol{s} \circ \boldsymbol{t} = \boldsymbol{i}$ μ-a.e. in $\mathbb{R}^d$ (resp. $\boldsymbol{t} \circ \boldsymbol{s} = \boldsymbol{i}$ ν-a.e. in $\mathbb{R}^d$):

$$\begin{aligned}
\int_{\mathbb{R}^d} |x - \boldsymbol{s}(\boldsymbol{t}(x))| \, d\mu(x) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - \boldsymbol{s}(y)| \, d(\boldsymbol{i}, \boldsymbol{t})_{\#}\mu \\
&= \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - \boldsymbol{s}(y)| \, d(\boldsymbol{s}, \boldsymbol{i})_{\#}\nu \\
&= \int_{\mathbb{R}^d} |\boldsymbol{s}(y) - \boldsymbol{s}(y)| \, d\nu(y) = 0.
\end{aligned}$$

The formula for the density of ν with respect to $\mathscr{L}^d$ follows by Lemma 1.3, taking into account the μ-essential injectivity of $\boldsymbol{t}$. $\square$

In the following we shall denote by $\boldsymbol{t}_\mu^\nu$ the unique optimal map given by Theorem 2.3. Notice that $\boldsymbol{t} = \nabla\varphi$ is uniquely determined only μ-a.e., hence φ is not uniquely determined, not even up to additive constants, unless $\mu = \rho\mathscr{L}^d$ with $\rho > 0$ $\mathscr{L}^d$-a.e. in $\mathbb{R}^d$. However, the existence proof (at least the one achieved through a duality argument), yields some "canonical" φ, given by the duality formula

$$\varphi(x) = \sup_{y\in\operatorname{supp}\nu} \langle x, y\rangle - \psi(y) \qquad x \in \mathbb{R}^d \tag{2.13}$$

for a suitable function $\psi : \operatorname{supp}\nu \to (-\infty, +\infty]$. This explicit expression is sometimes technically useful: for instance, it shows that when $\operatorname{supp}\nu$ is bounded we can always find a globally convex and Lipschitz map φ whose gradient is the optimal transport map.

The following result shows that optimal maps along geodesics enjoys nicer properties (see also [17]).

Theorem 2.4 (Regularity in the interior of geodesics) *Let $\mu,\ \nu \in \mathscr{P}_2(\mathbb{R}^d)$ and let*

$$\mu_t := \big((1-t)\pi^1 + t\pi^2\big)_\# \boldsymbol{\gamma}$$

be a constant speed geodesic induced by $\boldsymbol{\gamma} \in \Gamma_o(\mu,\nu)$. Then the following properties hold:

(i) For any $t \in [0,1)$ there exists a unique optimal plan between μ_t and μ, and this plan is induced by a map $\boldsymbol{s}_t$ with Lipschitz constant less than $1/(1-t)$.

(ii) If $\mu = \rho\mathscr{L}^d \in \mathscr{P}_2^a(\mathbb{R}^d)$ then $\mu_t \in \mathscr{P}_2^a(\mathbb{R}^d)$ for all $t \in [0,1)$.

Proof. (i) The necessary optimality conditions at the level of plans (see for instance §6.2.3 of [9], or [86]) imply that the support of $\boldsymbol{\gamma}$ is contained in the graph

$$\{(x,y) :\ y \in \Gamma(x)\}$$

of a monotone operator $\Gamma(x)$. On the other hand, the same argument used in the proof of (2.10) shows that the plan $\boldsymbol{\gamma}_t := \big(\pi^1, (1-t)\pi^1 + t\pi^2\big)_\# \boldsymbol{\gamma}$ is optimal between μ and μ_t. The support of $\boldsymbol{\gamma}_t$ is contained in the graph of the monotone operator $(1-t)I + t\Gamma$, whose inverse

$$\Gamma^{-1}(y) := \big\{x \in \mathbb{R}^d :\ y \in \Gamma(x)\big\}$$

is single-valued and $1/(1-t)$-Lipschitz continuous. Therefore the graph of Γ^{-1} is the graph of a $1/(1-t)$-Lipschitz map $\boldsymbol{s}_t$ pushing μ_t to μ. The uniqueness of this map, even at the level of plans, is proved in Lemma 7.2.1 of [9].

(ii) If $A \in \mathscr{B}(\mathbb{R}^d)$ is $\mathscr{L}^d$-negligible, then $\boldsymbol{s}_t(A)$ is also $\mathscr{L}^d$-negligible, hence μ-negligible. The identity $\boldsymbol{s}_t \circ \boldsymbol{t}_t = \boldsymbol{i}$ μ-a.e. then gives

$$\mu_t(A) = \mu(\boldsymbol{t}_t^{-1}(A)) \le \mu(\boldsymbol{s}_t(A)) = 0.$$

This proves that $\mu_t \ll \mathscr{L}^d$. □

2.5 The continuity equation with locally Lipschitz velocity fields

In this section we collect some results on the continuity equation

$$\partial_t \mu_t + \nabla \cdot (\boldsymbol{v}_t \mu_t) = 0 \qquad \text{in } \mathbb{R}^d \times (0,T), \tag{2.14}$$

which we will need in the sequel. Here μ_t is a Borel family of probability measures on $\mathbb{R}^d$ defined for t in the open interval $I := (0,T)$, $\boldsymbol{v} : (x,t) \mapsto \boldsymbol{v}_t(x) \in \mathbb{R}^d$ is a Borel velocity field such that

$$\int_0^T \int_{\mathbb{R}^d} |\boldsymbol{v}_t(x)| \, d\mu_t(x) \, dt < +\infty, \tag{2.15}$$

and we suppose that (2.14) holds in the sense of distributions, i.e.

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^d} \Big(\partial_t \varphi(x,t) + \langle \boldsymbol{v}_t(x), \nabla_x \varphi(x,t) \rangle \Big) \, d\mu_t(x) \, dt = 0, \\ \forall \, \varphi \in C_c^\infty(\mathbb{R}^d \times (0,T)). \end{aligned} \tag{2.16}$$

Remark 2.5 (More general test functions) By a simple regularization argument via convolution, it is easy to show that (2.16) holds if $\varphi \in C^1_c\left(\mathbb{R}^d \times (0,T)\right)$ as well. Moreover, under condition (2.15), we can also consider bounded test functions φ, with bounded gradient, whose support has a compact projection in $(0,T)$ (that is, the support in x need not be compact): it suffices to approximate φ by $\varphi\chi_R$ where $\chi_R \in C^\infty_c(\mathbb{R}^d)$, $0 \le \chi_R \le 1$, $|\nabla \chi_R| \le 2$ and $\chi_R = 1$ on $B_R(0)$.

First of all we recall some technical preliminaries.

Lemma 2.6 (Continuous representative) *Let μ_t be a Borel family of probability measures satisfying* (2.16) *for a Borel vector field $\boldsymbol{v}_t$ satisfying* (2.15)*. Then there exists a* narrowly continuous *curve $t \in [0,T] \mapsto \tilde\mu_t \in \mathscr{P}(\mathbb{R}^d)$ such that $\mu_t = \tilde\mu_t$ for $\mathscr{L}^1$-a.e. $t \in (0,T)$. Moreover, if $\varphi \in C^1_c(\mathbb{R}^d \times [0,T])$ and $t_1 \le t_2 \in [0,T]$ we have*

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x,t_2) \, d\tilde\mu_{t_2}(x) - \int_{\mathbb{R}^d} \varphi(x,t_1) \, d\tilde\mu_{t_1}(x) \\ = \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \Big(\partial_t \varphi + \langle \nabla \varphi, \boldsymbol{v}_t \rangle \Big) \, d\mu_t(x) \, dt. \end{aligned} \tag{2.17}$$

Proof. Let us take $\varphi(x,t) = \eta(t)\zeta(x)$, $\eta \in C^\infty_c(0,T)$ and $\zeta \in C^\infty_c(\mathbb{R}^d)$; we have

$$-\int_0^T \eta'(t) \Big(\int_{\mathbb{R}^d} \zeta(x) \, d\mu_t(x) \Big) \, dt = \int_0^T \eta(t) \Big(\int_{\mathbb{R}^d} \langle \nabla \zeta(x), \boldsymbol{v}_t(x) \rangle \, d\mu_t(x) \Big) \, dt,$$

so that the map

$$t \mapsto \mu_t(\zeta) = \int_{\mathbb{R}^d} \zeta(x) \, d\mu_t(x)$$

belongs to $W^{1,1}(0,T)$ with distributional derivative

$$\dot\mu_t(\zeta) = \int_{\mathbb{R}^d} \langle \nabla\zeta(x), \boldsymbol{v}_t(x)\rangle \, d\mu_t(x) \quad \text{for } \mathscr{L}^1\text{-a.e. } t \in (0,T) \tag{2.18}$$

with

$$|\dot\mu_t(\zeta)| \le V(t) \sup_{\mathbb{R}^d} |\nabla\zeta|, \quad V(t) := \int_{\mathbb{R}^d} |\boldsymbol{v}_t(x)| \, d\mu_t(x), \quad V \in L^1(0,T). \tag{2.19}$$

If L_ζ is the set of its Lebesgue points, we know that $\mathscr{L}^1((0,T)\setminus L_\zeta) = 0$. Let us now take a countable set Z which is dense in $C^1_c(\mathbb{R}^d)$ with respect the usual C^1 norm $\|\zeta\|_{C^1} = \sup_{\mathbb{R}^d}(|\zeta|, |\nabla\zeta|)$ and let us set $L_Z := \cap_{\zeta\in Z} L_\zeta$. The restriction of the curve μ to L_Z provides a uniformly continuous family of bounded functionals on $C^1_c(\mathbb{R}^d)$, since (2.19) shows

$$|\mu_t(\zeta) - \mu_s(\zeta)| \le \|\zeta\|_{C^1} \int_s^t V(\lambda)\, d\lambda \quad \forall\, s,\, t \in L_Z.$$

Therefore, it can be extended in a unique way to a continuous curve $\{\tilde\mu_t\}_{t\in[0,T]}$ in $[C^1_c(\mathbb{R}^d)]'$. If we show that $\{\mu_t\}_{t\in L_Z}$ is also tight, the extension provides a continuous curve in $\mathscr{P}(\mathbb{R}^d)$.

For, let us consider nonnegative, smooth functions $\zeta_k : \mathbb{R}^d \to [0,1]$, $k \in \mathbb{N}$, such that

$$\zeta_k(x) = 1 \quad \text{if } |x| \le k, \quad \zeta_k(x) = 0 \quad \text{if } |x| \ge k+1, \quad |\nabla\zeta_k(x)| \le 2.$$

It is not restrictive to suppose that $\zeta_k \in Z$. Applying the previous formula (2.18), for t, $s \in L_Z$ we have

$$|\mu_t(\zeta_k) - \mu_s(\zeta_k)| \le a_k := 2\int_0^T \int_{k<|x|<k+1} |\boldsymbol{v}_\lambda(x)| \, d\mu_\lambda(x)\, d\lambda,$$

with $\sum_{k=1}^{+\infty} a_k < +\infty$. For a fixed $s \in L_Z$ and $\varepsilon > 0$, being μ_s tight, we can find $k \in \mathbb{N}$ such that $\mu_s(\zeta_k) > 1 - \varepsilon/2$ and $a_k < \varepsilon/2$. It follows that

$$\mu_t(\overline{B_{k+1}(0)}) \ge \mu_t(\zeta_k) \ge 1 - \varepsilon \quad \forall\, t \in L_Z.$$

Now we show (2.17). Let us choose $\varphi \in C^1_c(\mathbb{R}^d \times [0,T])$ and set $\varphi_\varepsilon(x,t) = \eta_\varepsilon(t)\varphi(x,t)$, where $\eta_\varepsilon \in C^\infty_c(t_1,t_2)$ such that

$$0 \le \eta_\varepsilon(t) \le 1, \quad \lim_{\varepsilon\downarrow 0} \eta_\varepsilon(t) = \chi_{(t_1,t_2)}(t)\ \forall\, t \in [0,T], \quad \lim_{\varepsilon\downarrow 0} \eta'_\varepsilon = \delta_{t_1} - \delta_{t_2}$$

in the duality with continuous functions in $[0,T]$. We get

$$\begin{aligned} 0 &= \int_0^T \int_{\mathbb{R}^d} \Big(\partial_t(\eta_\varepsilon\varphi) + \langle \nabla_x(\eta_\varepsilon\varphi), \boldsymbol{v}_t\rangle \Big)\, d\mu_t(x)\, dt \\ &= \int_0^T \eta_\varepsilon(t) \int_{\mathbb{R}^d} \Big(\partial_t\varphi(x,t) + \langle \boldsymbol{v}_t(x), \nabla_x\varphi(x,t)\rangle \Big)\, d\mu_t(x)\, dt \\ &\quad + \int_0^T \eta'_\varepsilon(t) \int_{\mathbb{R}^d} \varphi(x,t)\, d\tilde\mu_t(x)\, dt. \end{aligned}$$

Passing to the limit as ε vanishes and invoking the continuity of $\tilde{\mu}_t$, we get (2.17). $\square$

Lemma 2.7 (Time rescaling) *Let* $\mathsf{t} : s \in [0, T'] \to \mathsf{t}(s) \in [0, T]$ *be a strictly increasing absolutely continuous map with absolutely continuous inverse* $\mathsf{s} := \mathsf{t}^{-1}$*. Then* $(\mu_t, \boldsymbol{v}_t)$ *is a distributional solution of* (2.14) *if and only if*

$$\hat{\mu} := \mu \circ \mathsf{t},\ \hat{\boldsymbol{v}} := \mathsf{t}' \boldsymbol{v} \circ \mathsf{t}, \qquad \text{is a distributional solution of (2.14) on } (0, T').$$

Proof. By an elementary smoothing argument we can assume that s is continuously differentiable and $\mathsf{s}' > 0$. We choose $\hat{\varphi} \in C^1_c(\mathbb{R}^d \times (0, T'))$ and let us set $\varphi(x, t) := \hat{\varphi}(x, \mathsf{s}(t))$; since $\varphi \in C^1_c(\mathbb{R}^d \times (0, T))$ we have

$$\begin{aligned}
0 &= \int_0^T \int_{\mathbb{R}^d} \big(\mathsf{s}'(t) \partial_s \hat{\varphi}(x, \mathsf{s}(t)) + \langle \nabla \hat{\varphi}(x, \mathsf{s}(t)), \hat{\boldsymbol{v}}_t(x) \rangle\big)\, d\mu_t(x)\, dt \\
&= \int_0^T \mathsf{s}'(t) \int_{\mathbb{R}^d} \Big(\partial_s \hat{\varphi}(x, \mathsf{s}(t)) + \langle \nabla_x \hat{\varphi}(x, \mathsf{s}(t)), \frac{\boldsymbol{v}_t(x)}{\mathsf{s}'(t)} \rangle\Big)\, d\mu_t(x)\, dt \\
&= \int_0^{T'} \int_{\mathbb{R}^d} \Big(\partial_s \hat{\varphi}(x, s) + \langle \nabla_x \hat{\varphi}(x, s), \mathsf{t}'(s) \boldsymbol{v}_{\mathsf{t}(s)}(x) \rangle\Big)\, d\hat{\mu}_s(x)\, ds.
\end{aligned}$$

$\square$

When the velocity field $\boldsymbol{v}_t$ is more regular, the classical method of characteristics provides an explicit solution of (2.14). First we recall an elementary result of the theory of ordinary differential equations.

Lemma 2.8 (The characteristic system of ODE) *Let* $\boldsymbol{v}_t$ *be a Borel vector field such that for every compact set* $B \subset \mathbb{R}^d$

$$\int_0^T \Big(\sup_B |\boldsymbol{v}_t| + \mathrm{Lip}(\boldsymbol{v}_t, B)\Big)\, dt < +\infty. \tag{2.20}$$

Then, for every $x \in \mathbb{R}^d$ *and* $s \in [0, T]$ *the ODE*

$$X_s(x, s) = x, \quad \frac{d}{dt} X_t(x, s) = \boldsymbol{v}_t(X_t(x, s)), \tag{2.21}$$

admits a unique maximal solution defined in an interval $I(x, s)$ *relatively open in* $[0, T]$ *and containing* s *as (relatively) internal point.*
Furthermore, if $t \mapsto |X_t(x, s)|$ *is bounded in the interior of* $I(x, s)$ *then* $I(x, s) = [0, T]$*; finally, if* $\boldsymbol{v}$ *satisfies the global bounds analogous to* (2.20)

$$S := \int_0^T \Big(\sup_{\mathbb{R}^d} |\boldsymbol{v}_t| + \mathrm{Lip}(\boldsymbol{v}_t, \mathbb{R}^d)\Big)\, dt < +\infty, \tag{2.22}$$

then the flow map X *satisfies*

$$\int_0^T \sup_{x \in \mathbb{R}^d} |\partial_t X_t(x, s)|\, dt \le S, \qquad \sup_{t, s \in [0, T]} \mathrm{Lip}(X_t(\cdot, s), \mathbb{R}^d) \le e^S. \tag{2.23}$$

For simplicity, we set $X_t(x) := X_t(x,0)$ in the particular case $s=0$ and we denote by $\tau(x) := \sup I(x,0)$ the length of the maximal time domain of the characteristics leaving from x at $t=0$.

Remark 2.9 (The characteristics method for first order linear PDE's)

Characteristics provide a useful representation formula for classical solutions of the backward equation (formally adjoint to (2.14))

$$\partial_t\varphi + \langle \boldsymbol{v}_t, \nabla\varphi\rangle = \psi \quad \text{in } \mathbb{R}^d \times (0,T), \quad \varphi(x,T) = \varphi_T(x) \quad x \in \mathbb{R}^d, \tag{2.24}$$

when, e.g., $\psi \in C^1_b(\mathbb{R}^d \times (0,T)), \varphi_T \in C^1_b(\mathbb{R}^d)$ and $\boldsymbol{v}$ satisfies the global bounds (2.22), so that maximal solutions are always defined in $[0,T]$. A direct calculation shows that

$$\varphi(x,t) := \varphi_T(X_T(x,t)) - \int_t^T \psi(X_s(x,t),s)\,ds \tag{2.25}$$

solve (2.24). For $X_s(X_t(x,0),t) = X_s(x,0)$ yields

$$\varphi(X_t(x,0),t) = \varphi_T(X_T(x,0)) - \int_t^T \psi(X_s(x,0),s)\,ds,$$

and differentiating both sides with respect to t we obtain

$$\left[\frac{\partial\varphi}{\partial t} + \langle \boldsymbol{v}_t, \nabla\varphi\rangle\right](X_t(x,0),t) = \psi(X_t(x,0),t).$$

Since x (and then $X_t(x,0)$) is arbitrary we conclude that (2.31) is fulfilled.

Now we use characteristics to prove the existence, the uniqueness, and a representation formula of the solution of the continuity equation, under suitable assumption on v.

Lemma 2.10 *Let $\boldsymbol{v}_t$ be a Borel velocity field satisfying* (2.20), (2.15)*, let $\mu_0 \in \mathscr{P}(\mathbb{R}^d)$, and let X_t be the maximal solution of the ODE* (2.21) *(corresponding to $s=0$). Suppose that for some $\bar t \in (0,T]$*

$$\tau(x) > \bar t \quad \textit{for } \mu_0\textit{-a.e. } x \in \mathbb{R}^d. \tag{2.26}$$

Then $t \mapsto \mu_t := (X_t)_\# \mu_0$ is a continuous solution of (2.14) *in $[0,\bar t]$.*

Proof. The continuity of μ_t follows easily since $\lim_{s\to t} X_s(x) = X_t(x)$ for μ_0-a.e. $x \in \mathbb{R}^d$: thus for every continuous and bounded function $\zeta : \mathbb{R}^d \to \mathbb{R}$ the dominated convergence theorem yields

$$\lim_{s\to t}\int_{\mathbb{R}^d} \zeta\,d\mu_s = \lim_{s\to t}\int_{\mathbb{R}^d} \zeta(X_s(x))\,d\mu_0(x) = \int_{\mathbb{R}^d} \zeta(X_t(x))\,d\mu_0(x) = \int_{\mathbb{R}^d} \zeta\,d\mu_t.$$

For any $\varphi \in C^\infty_c(\mathbb{R}^d \times (0,\bar t))$ and for μ_0-a.e. $x \in \mathbb{R}^d$ the maps $t \mapsto \varphi_t(x) := \varphi(X_t(x),t)$ are absolutely continuous in $(0,\bar t)$, with

$$\dot\varphi_t(x) = \partial_t\varphi(X_t(x),t) + \langle \nabla\varphi(X_t(x),t), \boldsymbol{v}_t(X_t(x))\rangle = \Lambda(\cdot,t)\circ X_t,$$

where $\Lambda(x,t) := \partial_t\varphi(x,t) + \langle\nabla\varphi(x,t), v_t(x)\rangle$. We thus have

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} |\dot\varphi_t(x)|\,d\mu_0(x)\,dt &= \int_0^T \int_{\mathbb{R}^d} |\Lambda(X_t(x),t)|\,d\mu_0(x)\,dt \\ &= \int_0^T \int_{\mathbb{R}^d} |\Lambda(x,t)|\,d\mu_t(x)\,dt \\ &\le \mathrm{Lip}(\varphi)\Big(T + \int_0^T \int_{\mathbb{R}^d} |\boldsymbol{v}_t(x)|\,d\mu_t(x)\,dt\Big) < +\infty\end{aligned}$$

and therefore

$$\begin{aligned}0 &= \int_{\mathbb{R}^d} \varphi(x,\bar t)\,d\mu_{\bar t}(x) - \int_{\mathbb{R}^d} \varphi(x,0)\,d\mu_0(x) = \int_{\mathbb{R}^d} \Big(\varphi(X_{\bar t}(x),\bar t) - \varphi(x,0)\Big)\,d\mu_0(x) \\ &= \int_{\mathbb{R}^d} \Big(\int_0^{\bar t} \dot\varphi_t(x)\,dt\Big)\,d\mu_0(x) = \int_0^{\bar t}\int_{\mathbb{R}^d} \big(\partial_t\varphi + \langle\nabla\varphi, \boldsymbol{v}_t\rangle\big)\,d\mu_t\,dt,\end{aligned}$$

by a simple application of Fubini's theorem. □

We want to prove that, under reasonable assumptions, in fact *any* solution of (2.14) can be represented as in Lemma 2.10. The first step is a uniqueness theorem for the continuity equation under minimal regularity assumptions on the velocity field. Notice that the only global information on v_t is (2.27). The proof is based on a classical duality argument (see for instance [41, 7, 19]).

Proposition 2.11 (Uniqueness and comparison for the continuity equation) *Let σ_t be a narrowly continuous family of signed measures solving*

$$\partial_t\sigma_t + \nabla\cdot(\boldsymbol{v}_t\sigma_t) = 0 \quad \textit{in } \mathbb{R}^d \times (0,T),$$

with $\sigma_0 \le 0$,

$$\int_0^T \int_{\mathbb{R}^d} |\boldsymbol{v}_t|\,d|\sigma_t|dt < +\infty, \tag{2.27}$$

and

$$\int_0^T \Big(|\sigma_t|(B) + \sup_B |\boldsymbol{v}_t| + \mathrm{Lip}(\boldsymbol{v}_t, B)\Big)\,dt < +\infty$$

for any bounded closed set $B \subset \mathbb{R}^d$. Then $\sigma_t \le 0$ for any $t \in [0,T]$.

Proof. Fix $\psi \in C_c^\infty(\mathbb{R}^d \times (0,T))$ with $0 \le \psi \le 1$, $R > 0$, and a smooth cut-off function

$$\begin{gathered}\chi_R(\cdot) = \chi(\cdot/R) \in C_c^\infty(\mathbb{R}^d) \quad \text{such that } 0 \le \chi_R \le 1,\ |\nabla\chi_R| \le 2/R, \\ \chi_R \equiv 1 \text{ on } B_R(0), \text{ and } \chi_R \equiv 0 \text{ on } \mathbb{R}^d \setminus B_{2R}(0).\end{gathered} \tag{2.28}$$

We define $\boldsymbol{w}_t$ so that $\boldsymbol{w}_t = \boldsymbol{v}_t$ on $B_{2R}(0) \times (0,T)$, $\boldsymbol{w}_t = 0$ if $t \notin [0,T]$ and

$$\sup_{\mathbb{R}^d} |\boldsymbol{w}_t| + \mathrm{Lip}(\boldsymbol{w}_t, \mathbb{R}^d) \le \sup_{B_{2R}(0)} |\boldsymbol{v}_t| + \mathrm{Lip}(\boldsymbol{v}_t, B_{2R}(0)) \quad \forall\, t \in [0,T]. \tag{2.29}$$

Let $\boldsymbol{w}_t^\varepsilon$ be obtained from $\boldsymbol{w}_t$ by a double mollification with respect to the space and time variables: notice that $\boldsymbol{w}_t^\varepsilon$ satisfy

$$\sup_{\varepsilon\in(0,1)}\int_0^T \Big(\sup_{\mathbb{R}^d}|\boldsymbol{w}_t^\varepsilon| + \mathrm{Lip}(\boldsymbol{w}_t^\varepsilon,\mathbb{R}^d)\Big)\,dt < +\infty. \tag{2.30}$$

We now build, by the method of characteristics described in Remark 2.9, a smooth solution $\varphi^\varepsilon : \mathbb{R}^d \times [0,T] \to \mathbb{R}$ of the PDE

$$\frac{\partial \varphi^\varepsilon}{\partial t} + \langle \boldsymbol{w}_t^\varepsilon, \nabla\varphi^\varepsilon\rangle = \psi \quad \text{in } \mathbb{R}^d\times(0,T), \quad \varphi^\varepsilon(x,T) = 0 \quad x\in\mathbb{R}^d. \tag{2.31}$$

Combining the representation formula (2.25), the uniform bound (2.30), and the estimate (2.23), it is easy to check that $0 \geq \varphi^\varepsilon \geq -T$ and $|\nabla\varphi^\varepsilon|$ is uniformly bounded with respect to ε, t and x.

We insert now the test function $\varphi^\varepsilon\chi_R$ in the continuity equation and take into account that $\sigma_0 \leq 0$ and $\varphi^\varepsilon \leq 0$ to obtain

$$\begin{aligned}
0 \quad &\geq \quad -\int_{\mathbb{R}^d}\varphi^\varepsilon\chi_R\,d\sigma_0 = \int_0^T\int_{\mathbb{R}^d}\chi_R\frac{\partial\varphi^\varepsilon}{\partial t} + \langle \boldsymbol{v}_t, \chi_R\nabla\varphi^\varepsilon + \varphi^\varepsilon\nabla\chi_R\rangle\,d\sigma_t dt \\
&= \quad \int_0^T\int_{\mathbb{R}^d}\chi_R(\psi + \langle \boldsymbol{v}_t - \boldsymbol{w}_t^\varepsilon, \nabla\varphi^\varepsilon\rangle)\,d\sigma_t dt + \int_0^T\int_{\mathbb{R}^d}\varphi^\varepsilon\langle\nabla\chi_R, \boldsymbol{v}_t\rangle\,d\sigma_t dt \\
&\geq \quad \int_0^T\int_{\mathbb{R}^d}\chi_R(\psi + \langle \boldsymbol{v}_t - \boldsymbol{w}_t^\varepsilon, \nabla\varphi^\varepsilon\rangle)\,d\sigma_t dt - \int_0^T\int_{\mathbb{R}^d}|\nabla\chi_R||\boldsymbol{v}_t|\,d|\sigma_t|\,dt.
\end{aligned}$$

Letting $\varepsilon\downarrow 0$ and using the uniform bound on $|\nabla\varphi^\varepsilon|$ and the fact that $\boldsymbol{w}_t = \boldsymbol{v}_t$ on $\operatorname{supp}\chi_R\times[0,T]$, we get

$$\int_0^T\int_{\mathbb{R}^d}\chi_R\psi\,d\sigma_t\,dt \leq \int_0^T\int_{\mathbb{R}^d}|\nabla\chi_R||\boldsymbol{v}_t|\,d|\sigma_t|\,dt \leq \frac{2}{R}\int_0^T\int_{R\leq|x|\leq 2R}|\boldsymbol{v}_t|\,d|\sigma_t|\,dt.$$

Eventually letting $R\to\infty$ we obtain that $\int_0^T\int_{\mathbb{R}^d}\psi\,d\sigma_t dt \leq 0$. Since ψ is arbitrary the proof is achieved. □

Proposition 2.12 (Representation formula for the continuity equation) *Let μ_t, $t\in[0,T]$, be a narrowly continuous family of Borel probability measures solving the continuity equation* (2.14) *w.r.t. a Borel vector field $\boldsymbol{v}_t$ satisfying* (2.20) *and* (2.15). *Then for μ_0-a.e. $x\in\mathbb{R}^d$ the characteristic system* (2.21) *admits a globally defined solution $X_t(x)$ in $[0,T]$ and*

$$\mu_t = (X_t)_\#\mu_0 \quad \forall\, t\in[0,T]. \tag{2.32}$$

Moreover, if

$$\int_0^T\int_{\mathbb{R}^d}|\boldsymbol{v}_t(x)|^2\,d\mu_t(x)\,dt < +\infty \tag{2.33}$$

then the velocity field $\boldsymbol{v}_t$ is the time derivative of X_t in the L^2-sense

$$\lim_{h\downarrow 0}\int_0^{T-h}\int_{\mathbb{R}^d}\left|\frac{X_{t+h}(x)-X_t(x)}{h}-\boldsymbol{v}_t(X_t(x))\right|^2 d\mu_0(x)\,dt=0, \tag{2.34}$$

$$\lim_{h\to 0}\frac{X_{t+h}(x,t)-x}{h}=\boldsymbol{v}_t(x) \quad \textit{in } L^2(\mu_t;\mathbb{R}^d) \quad \textit{for } \mathscr{L}^1\textit{-a.e. } t\in(0,T). \tag{2.35}$$

Proof. Let $E_s=\{\tau>s\}$ and let us use the fact that, proved in Lemma 2.10, that $t\mapsto X_{t\#}(\chi_{E_s}\mu_0)$ is the solution of (2.14) in $[0,s]$. By Proposition 2.11 we get also

$$X_{t\#}(\chi_{E_s}\mu_0)\le\mu_t \qquad \text{whenever } 0\le t\le s.$$

Using the previous inequality with $s=t$ we can estimate:

$$\begin{aligned}
\int_{\mathbb{R}^d}\sup_{(0,\tau(x))}|X_t(x)-x|\,d\mu_0(x) &\le \int_{\mathbb{R}^d}\int_0^{\tau(x)}|\dot X_t(x)|\,d\mu_0(x)\\
&= \int_{\mathbb{R}^d}\int_0^{\tau(x)}|\boldsymbol{v}_t(X_t(x))|\,d\mu_0(x)\\
&= \int_0^T\int_{E_t}|\boldsymbol{v}_t(X_t(x))|\,d\mu_0(x)\,dt\\
&\le \int_0^T\int_{\mathbb{R}^d}|\boldsymbol{v}_t|\,d\mu_t\,dt.
\end{aligned}$$

It follows that $X_t(x)$ is bounded on $(0,\tau(x))$ for μ_0-a.e. $x\in\mathbb{R}^d$ and therefore X_t is globally defined in $[0,T]$ for μ_0-a.e. in $\mathbb{R}^d$. Applying Lemma 2.10 and Proposition 2.11 we obtain (2.32).

Now we observe that the differential quotient $D_h(x,t):=h^{-1}(X_{t+h}(x)-X_t(x))$ can be bounded in $L^2(\mu_0\times\mathscr{L}^1)$ by

$$\begin{aligned}
&\int_0^{T-h}\int_{\mathbb{R}^d}\left|\frac{X_{t+h}(x)-X_t(x)}{h}\right|^2 d\mu_0(x)\,dt\\
=&\int_0^{T-h}\int_{\mathbb{R}^d}\left|\frac{1}{h}\int_0^h\boldsymbol{v}_{t+s}(X_{t+s}(x))\,ds\right|^2 d\mu_0(x)\,dt\\
\le&\int_0^{T-h}\int_{\mathbb{R}^d}\frac{1}{h}\int_0^h|\boldsymbol{v}_{t+s}(X_{t+s}(x))|^2\,ds\,d\mu_0(x)\,dt\\
\le&\int_0^T\int_{\mathbb{R}^d}|\boldsymbol{v}_t(X_t(x))|^2\,d\mu_0(x)\,dt<+\infty.
\end{aligned}$$

Since we already know that D_h is pointwise converging to $\boldsymbol{v}_t\circ X_t$ $\mu_0\times\mathscr{L}^1$-a.e. in $\mathbb{R}^d\times(0,T)$, we obtain the strong convergence in $L^2(\mu_0\times\mathscr{L}^1)$, i.e. (2.34).

Finally, we can consider $t\mapsto X_t(\cdot)$ and $t\mapsto\boldsymbol{v}_t(X_t(\cdot)$ as maps from $(0,T)$ to $L^2(\mu_0;\mathbb{R}^d)$; (2.34) is then equivalent to

$$\lim_{h\downarrow 0}\int_0^{T-h}\left\|\frac{X_{t+h}-X_t}{h}-\boldsymbol{v}_t(X_t)\right\|^2_{L^2(\mu_0;\mathbb{R}^d)}dt=0,$$

and it shows that $t \mapsto X_t(\cdot)$ belongs to $AC^2(0,T;L^2(\mu_0;\mathbb{R}^d))$. General results for absolutely continuous maps with values in Hilbert spaces yield that X_t is differentiable $\mathscr{L}^1$-a.e. in $(0,T)$, so that

$$\lim_{h\to 0}\int_{\mathbb{R}^d}\left|\frac{X_{t+h}(x)-X_t(x)}{h}-\boldsymbol{v}_t(X_t(x))\right|^2 d\mu_0(x)=0 \quad \text{for } \mathscr{L}^1\text{-a.e. } t\in(0,T).$$

Since $X_{t+h}(x)=X_h(X_t(x),t)$, we obtain (2.35). □

Now we state an approximation result for general solution of (2.14) with more regular ones, satisfying the conditions of the previous Proposition 2.12.

Lemma 2.13 (Approximation by regular curves) *Let μ_t be a time-continuous solution of* (2.14) *w.r.t. a velocity field satisfying the integrability condition*

$$\int_0^T\int_{\mathbb{R}^d}|\boldsymbol{v}_t(x)|^2\,d\mu_t(x)\,dt<+\infty. \tag{2.36}$$

Let $(\rho_\varepsilon)\subset C^\infty(\mathbb{R}^d)$ be a family of strictly positive *mollifiers in the x variable, (e.g. $\rho_\varepsilon(x)=(2\pi\varepsilon)^{-d/2}\exp(-|x|^2/2\varepsilon)$), and set*

$$\mu_t^\varepsilon:=\mu_t*\rho_\varepsilon,\quad E_t^\varepsilon:=(\boldsymbol{v}_t\mu_t)*\rho_\varepsilon,\quad v_t^\varepsilon:=\frac{E_t^\varepsilon}{\mu_t^\varepsilon}. \tag{2.37}$$

Then μ_t^ε is a continuous solution of (2.14) *w.r.t. v_t^ε, which satisfies the local regularity assumptions* (2.20) *and the uniform integrability bounds*

$$\int_{\mathbb{R}^d}|v_t^\varepsilon(x)|^2\,d\mu_t^\varepsilon(x)\le\int_{\mathbb{R}^d}|\boldsymbol{v}_t(x)|^2\,d\mu_t(x)\quad\forall\, t\in(0,T). \tag{2.38}$$

Moreover, $E_t^\varepsilon\to\boldsymbol{v}_t\mu_t$ narrowly and

$$\lim_{\varepsilon\downarrow 0}\|v_t^\varepsilon\|_{L^2(\mu_t^\varepsilon;\mathbb{R}^d)}=\|\boldsymbol{v}_t\|_{L^2(\mu_t;\mathbb{R}^d)}\qquad\forall t\in(0,T). \tag{2.39}$$

Proof. With a slight abuse of notation, we are denoting the measure μ_t^ε and its density w.r.t. $\mathscr{L}^d$ by the same symbol. Notice first that $|E^\varepsilon|(t,\cdot)$ and its spatial gradient are uniformly bounded in space by the product of $\|\boldsymbol{v}_t\|_{L^1(\mu_t)}$ with a constant depending on ε, and the first quantity is integrable in time. Analogously, $|\mu_t^\varepsilon|(t,\cdot)$ and its spatial gradient are uniformly bounded in space by a constant depending on ε. Therefore, as $v_t^\varepsilon=E_t^\varepsilon/\mu_t^\varepsilon$, the local regularity assumptions (2.20) is fulfilled if

$$\inf_{|x|\le R,\,t\in[0,T]}\mu_t^\varepsilon(x)>0\qquad\text{for any }\varepsilon>0,\ R>0.$$

This property is immediate, since μ_t^ε are continuous w.r.t. t and equi-continuous w.r.t. x, and therefore continuous in both variables.

Lemma 2.14 below shows that (2.38) holds. Notice also that μ_t^ε solve the continuity equation

$$\partial_t\mu_t^\varepsilon+\nabla\cdot(v_t^\varepsilon\mu_t^\varepsilon)=0\qquad\text{in }\mathbb{R}^d\times(0,T), \tag{2.40}$$

because, by construction, $\nabla \cdot (v_t^\varepsilon \mu_t^\varepsilon) = \nabla \cdot ((\boldsymbol{v}_t \mu_t) * \rho_\varepsilon) = (\nabla \cdot (\boldsymbol{v}_t \mu_t)) * \rho_\varepsilon$. Finally, general lower semicontinuity results on integral functionals defined on measures of the form

$$(E, \mu) \mapsto \int_{\mathbb{R}^d} \left| \frac{E}{\mu} \right|^2 d\mu$$

(see for instance Theorem 2.34 and Example 2.36 in [8]) provide (2.39). $\square$

Lemma 2.14 *Let $\mu \in \mathscr{P}(\mathbb{R}^d)$ and let E be a $\mathbb{R}^m$-valued measure in $\mathbb{R}^d$ with finite total variation and absolutely continuous with respect to μ. Then*

$$\int_{\mathbb{R}^d} \left| \frac{E * \rho}{\mu * \rho} \right|^2 \mu * \rho \, dx \le \int_{\mathbb{R}^d} \left| \frac{E}{\mu} \right|^2 d\mu$$

for any convolution kernel ρ.

Proof. We use Jensen inequality in the following form: if $\Phi : \mathbb{R}^{m+1} \to [0, +\infty]$ is convex, l.s.c. and positively 1-homogeneous, then

$$\Phi \left(\int_{\mathbb{R}^d} \psi(x) \, d\theta(x) \right) \le \int_{\mathbb{R}^d} \Phi(\psi(x)) \, d\theta(x)$$

for any Borel map $\psi : \mathbb{R}^d \to \mathbb{R}^{m+1}$ and any positive and finite measure θ in $\mathbb{R}^d$ (by rescaling θ to be a probability measure and looking at the image measure $\psi_\# \theta$ the formula reduces to the standard Jensen inequality). Fix $x \in \mathbb{R}^d$ and apply the inequality above with $\psi := (E/\mu, 1)$, $\theta := \rho(x - \cdot)\mu$ and

$$\Phi(z, t) := \begin{cases} \dfrac{|z|^2}{t} & \text{if } t > 0 \\ 0 & \text{if } (z, t) = (0, 0) \\ +\infty & \text{if either } t < 0 \text{ or } t = 0, z \ne 0, \end{cases}$$

to obtain

$$\begin{aligned} \left| \frac{E * \rho(x)}{\mu * \rho(x)} \right|^2 \mu * \rho(x) &= \Phi \left(\int_{\mathbb{R}^d} \frac{E}{\mu}(y) \rho(x - y) \, d\mu(y), \int \rho(x - y) d\mu(y) \right) \\ &\le \int_{\mathbb{R}^d} \Phi(\frac{E}{\mu}(y), 1) \rho(x - y) \, d\mu(y) \\ &= \int_{\mathbb{R}^d} \left| \frac{E}{\mu} \right|^2 (y) \rho(x - y) \, d\mu(y). \end{aligned}$$

An integration with respect to x leads to the desired inequality. $\square$

2.6 The tangent bundle to the Wasserstein space

In this section we endow $\mathscr{P}_2(\mathbb{R}^d)$ with a kind of differential structure, consistent with the metric structure introduced in Section 2.2. Our starting point is the analysis of absolutely continuous curves $\mu_t : (a,b) \to \mathscr{P}_2(\mathbb{R}^d)$: recall that this concept depends only on the metric structure of $\mathscr{P}_2(\mathbb{R}^d)$, by Definition 2.1. We show in Theorem 2.15 that this class of curves coincides with (distributional) solutions of the continuity equation

$$\frac{\partial}{\partial t}\mu_t + \nabla \cdot (\boldsymbol{v}_t \mu_t) = 0 \qquad \text{in } \mathbb{R}^d \times (a,b).$$

More precisely, given an absolutely continuous curve μ_t, one can find a Borel time-dependent velocity field $\boldsymbol{v}_t : \mathbb{R}^d \to \mathbb{R}^d$ such that $\|\boldsymbol{v}_t\|_{L^2(\mu_t)} \le |\mu'|(t)$ for $\mathscr{L}^1$-a.e. $t \in (a,b)$ and the continuity equation holds. Here $|\mu'|(t)$ is the metric derivative of μ_t, defined in (2.2). Conversely, if μ_t solve the continuity equation for some Borel velocity field $\boldsymbol{w}_t$ with $\int_a^b \|\boldsymbol{w}_t\|_{L^2(\mu_t)}\,dt < +\infty$, then μ_t is an absolutely continuous curve and $\|\boldsymbol{w}_t\|_{L^2(\mu_t)} \ge |\mu'|(t)$ for $\mathscr{L}^1$-a.e. $t \in (a,b)$.

As a consequence of Theorem 2.15 we see that among all velocity fields $\boldsymbol{w}_t$ which produce the same flow μ_t, there is a unique optimal one with smallest $L^2(\mu_t;\mathbb{R}^d)$-norm, equal to the metric derivative of μ_t; we view this optimal field as the "tangent" vector field to the curve μ_t. To make this statement more precise, one can show that the minimality of the L^2 norm of $\boldsymbol{w}_t$ is characterized by the property

$$\boldsymbol{w}_t \in \overline{\{\nabla\varphi :\ \varphi \in C_c^\infty(\mathbb{R}^d)\}}^{L^2(\mu_t;\mathbb{R}^d)} \qquad \text{for } \mathscr{L}^1\text{-a.e. } t \in (a,b). \tag{2.41}$$

The characterization (2.41) of tangent vectors strongly suggests to consider the following tangent bundle to $\mathscr{P}_2(\mathbb{R}^d)$

$$\mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d) := \overline{\{\nabla\varphi :\ \varphi \in C_c^\infty(\mathbb{R}^d)\}}^{L^2(\mu;\mathbb{R}^d)} \qquad \forall \mu \in \mathscr{P}_2(\mathbb{R}^d), \tag{2.42}$$

endowed with the natural L^2 metric. Moreover, as a consequence of the characterization of absolutely continuous curves in $\mathscr{P}_2(\mathbb{R}^d)$, we recover the Benamou–Brenier (see [15], where the formula was introduced for numerical purposes) formula for the Wasserstein distance:

$$W_2^2(\mu_0,\mu_1) = \min\left\{\int_0^1 \|\boldsymbol{w}_t\|^2_{L^2(\mu_t;\mathbb{R}^d)}\,dt :\ \frac{d}{dt}\mu_t + \nabla\cdot(\boldsymbol{w}_t\mu_t) = 0\right\}. \tag{2.43}$$

Indeed, for any admissible curve we use the inequality between L^2 norm of $\boldsymbol{w}_t$ and metric derivative to obtain:

$$\int_0^1 \|\boldsymbol{w}_t\|^2_{L^2(\mu_t;\mathbb{R}^d)}\,dt \ge \int_0^1 |\mu'|^2(t)\,dt \ge W_2^2(\mu_0,\mu_1).$$

Conversely, since we know that $\mathscr{P}_2(\mathbb{R}^d)$ is a length space, we can use a geodesic μ_t and its tangent vector field $\boldsymbol{v}_t$ to obtain equality in (2.43). We also show that optimal transport maps belong to $\mathrm{Tan}_\mu\mathscr{P}_2(\mathbb{R}^d)$ under quite general conditions.

In this way we recover in a more general framework the *Riemannian interpretation* of the Wasserstein distance developed by Otto in [74] (see also [73],

[57]) and used to study the long time behavior of the porous medium equation. In the original paper [74], (2.43) is derived using formally the concept of Riemannian submersion and the family of maps $\phi \mapsto \phi_{\#}\mu$ (indexed by $\mu \ll \mathscr{L}^d$) from ARNOLD's space of diffeomorphisms into the Wasserstein space. In OTTO's formalism tangent vectors are rather thought as $s = \frac{d}{dt}\mu_t$ and these vectors are identified, via the continuity equation, with $-D \cdot (\boldsymbol{v}_s \mu_t)$. Moreover $\boldsymbol{v}_s$ is chosen to be the gradient of a function ψ_s, so that $D \cdot (\nabla\psi_s \mu_t) = -s$. Then the metric tensor is induced by the identification $s \mapsto \nabla\phi_s$ as follows:

$$\langle s, s' \rangle_{\mu_t} := \int_{\mathbb{R}^d} \langle \nabla\psi_s, \nabla\psi_{s'} \rangle \, d\mu_t.$$

As noticed in [74], both the identification between tangent vectors and gradients and the scalar product depend on μ_t, and these facts lead to a non trivial geometry of the Wasserstein space. We prefer instead to consider directly $\boldsymbol{v}_t$ as the tangent vectors, allowing them to be not necessarily gradients: this leads to (2.42).

Another consequence of the characterization of absolutely continuous curves is a result, given in Proposition 2.20, concerning the infinitesimal behavior of the Wasserstein distance along absolutely continuous curves μ_t: given the tangent vector field $\boldsymbol{v}_t$ to the curve, we show that

$$\lim_{h\to 0} \frac{W_2(\mu_{t+h}, (\boldsymbol{i} + h\boldsymbol{v}_t)_{\#}\mu_t)}{|h|} = 0 \qquad \text{for } \mathscr{L}^1\text{-a.e. } t \in (a,b).$$

Moreover the rescaled optimal transport maps between μ_t and μ_{t+h} converge to the transport plan $(\boldsymbol{i} \times \boldsymbol{v}_t)_{\#}\mu_t$ associated to $\boldsymbol{v}_t$ (see (2.56)). As a consequence, we will obtain in Theorem 2.21 a key formula for the derivative of the map $t \mapsto W_2^2(\mu_t, \nu)$.

Theorem 2.15 (Absolutely continuous curves in $\mathscr{P}_2(\mathbb{R}^d)$) *Let I be an open interval in $\mathbb{R}$, let $\mu_t : I \to \mathscr{P}_2(\mathbb{R}^d)$ be an absolutely continuous curve and let $|\mu'| \in L^1(I)$ be its metric derivative, given by (2.2). Then there exists a Borel vector field $\boldsymbol{v} : (x,t) \mapsto \boldsymbol{v}_t(x)$ such that*

$$\boldsymbol{v}_t \in L^2(\mu_t; \mathbb{R}^d), \qquad \|\boldsymbol{v}_t\|_{L^2(\mu_t;\mathbb{R}^d)} \le |\mu'|(t) \qquad \textit{for } \mathscr{L}^1\textit{-a.e. } t \in I, \tag{2.44}$$

and the continuity equation

$$\partial_t \mu_t + \nabla \cdot (\boldsymbol{v}_t \mu_t) = 0 \qquad \textit{in } \mathbb{R}^d \times I \tag{2.45}$$

holds in the sense of distributions, i.e.

$$\int_I \int_{\mathbb{R}^d} \Big(\partial_t \varphi(x,t) + \langle \boldsymbol{v}_t(x), \nabla_x \varphi(x,t) \rangle \Big) \, d\mu_t(x) \, dt = 0 \quad \forall\, \varphi \in C_c^\infty(\mathbb{R}^d \times I). \tag{2.46}$$

Moreover, for $\mathscr{L}^1$-a.e. $t \in I$ $\boldsymbol{v}_t$ belongs to the closure in $L^2(\mu_t, \mathbb{R}^d)$ of the subspace generated by the gradients $\nabla\varphi$ with $\varphi \in C_c^\infty(\mathbb{R}^d)$.

Conversely, if a narrowly continuous curve $\mu_t : I \to \mathscr{P}_2(\mathbb{R}^d)$ satisfies the continuity equation for some Borel velocity field $\boldsymbol{w}_t$ with $\|\boldsymbol{w}_t\|_{L^2(\mu_t;\mathbb{R}^d)} \in L^1(I)$ then $\mu_t : I \to \mathscr{P}_2(\mathbb{R}^d)$ is absolutely continuous and $|\mu'|(t) \le \|\boldsymbol{w}_t\|_{L^2(\mu_t;\mathbb{R}^d)}$ for $\mathscr{L}^1$-a.e. $t \in I$.
In particular equality holds in (2.44).

Proof. Taking into account that any absolutely continuous curve can be reparametrized by arc length (see for instance [11]) and Lemma 2.7, we will assume with no loss of generality that $|\mu'| \in L^\infty(I)$ in the proof of the first statement. To fix the ideas, we also assume that $I = (0,1)$.

First of all we show that for every $\varphi \in C_c^\infty(\mathbb{R}^d)$ the function $t \mapsto \mu_t(\varphi)$ is absolutely continuous, and its derivative can be estimated with the metric derivative of μ_t. Indeed, for $s, t \in I$ and $\boldsymbol{\mu}_{st} \in \Gamma_o(\mu_s, \mu_t)$ we have, using the Hölder inequality,

$$|\mu_t(\varphi) - \mu_s(\varphi)| = \left| \int_{\mathbb{R}^d} \big(\varphi(y) - \varphi(x)\big)\, d\boldsymbol{\mu}_{st} \right| \le \mathrm{Lip}(\varphi) W_2(\mu_s, \mu_t),$$

whence the absolute continuity follows. In order to estimate more precisely the derivative of $\mu_t(\varphi)$ we introduce the upper semicontinuous and bounded map

$$H(x,y) := \begin{cases} |\nabla\varphi(x)| & \text{if } x = y, \\[2ex] \dfrac{|\varphi(x) - \varphi(y)|}{|x-y|} & \text{if } x \ne y, \end{cases}$$

and notice that, setting $\boldsymbol{\mu}_h = \boldsymbol{\mu}_{(s+h)s}$, we have

$$\begin{aligned} \frac{|\mu_{s+h}(\varphi) - \mu_s(\varphi)|}{|h|} &\le \frac{1}{|h|} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y| H(x,y)\, d\boldsymbol{\mu}_h \\ &\le \frac{W_2(\mu_{s+h}, \mu_s)}{|h|} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} H^2(x,y)\, d\boldsymbol{\mu}_h \right)^{1/2}. \end{aligned}$$

If t is a point where $s \mapsto \mu_s$ is metrically differentiable, using the fact that $\boldsymbol{\mu}_h \to (x,x)_\# \mu_t$ narrowly (because their marginals are narrowly converging, any limit point belongs to $\Gamma_o(\mu_t,\mu_t)$ and is concentrated on the diagonal of $\mathbb{R}^d \times \mathbb{R}^d$) we obtain

$$\limsup_{h \to 0} \frac{|\mu_{t+h}(\varphi) - \mu_t(\varphi)|}{|h|} \le |\mu'|(t) \left(\int_{\mathbb{R}^d} H^2(x,x)\, d\mu_t \right)^{1/2} = |\mu'|(t) \|\nabla\varphi\|_{L^2(\mu_t;\mathbb{R}^d)}. \tag{2.47}$$

Set $Q = \mathbb{R}^d \times I$ and let $\mu = \int \mu_t\, dt \in \mathscr{P}(Q)$ be the measure whose disintegration is $\{\mu_t\}_{t \in I}$. For any $\varphi \in C_c^\infty(Q)$ we have

$$\begin{aligned} \int_Q \partial_s \varphi(x,s)\, d\mu(x,s) &= \lim_{h \downarrow 0} \int_Q \frac{\varphi(x,s) - \varphi(x,s-h)}{h}\, d\mu(x,s) \\ &= \lim_{h \downarrow 0} \int_I \frac{1}{h} \Big(\int_{\mathbb{R}^d} \varphi(x,s)\, d\mu_s(x) - \int_{\mathbb{R}^d} \varphi(x,s)\, d\mu_{s+h}(x) \Big)\, ds. \end{aligned}$$

Taking into account (2.47), Fatou's Lemma yields

$$\begin{aligned}\left|\int_Q \partial_s\varphi(x,s)\,d\mu(x,s)\right| &\le \int_J |\mu'|(s)\Big(\int_{\mathbb{R}^d} |\nabla\varphi(x,s)|^q\,d\mu_s(x)\Big)^{1/q}\,ds\\ &\le \Big(\int_J |\mu'|^p(s)\,ds\Big)^{1/p}\Big(\int_Q |\nabla\varphi(x,s)|^q\,d\mu(x,s)\Big)^{1/q},\end{aligned} \tag{2.48}$$

where $J\subset I$ is any interval such that $\operatorname{supp}\varphi\subset J\times\mathbb{R}^d$. If $\mathscr{V}$ denotes the closure in $L^2(\mu;\mathbb{R}^d)$ of the subspace $V:=\Big\{\nabla\varphi,\quad \varphi\in C^\infty_c(Q)\Big\}$, the previous formula says that the linear functional $L: V\to\mathbb{R}$ defined by

$$L(\nabla\varphi) := -\int_Q \partial_s\varphi(x,s)\,d\mu(x,s)$$

can be uniquely extended to a bounded functional on $\mathscr{V}$. Therefore the minimum problem

$$\min\left\{\frac{1}{2}\int_Q |w(x,s)|^2\,d\mu(x,s) - L(w):\ w\in\mathscr{V}\right\} \tag{2.49}$$

admits a unique solution $\boldsymbol{v}$ satisfying

$$\int_Q \langle \boldsymbol{v}(x,s),\nabla\varphi(x,s)\rangle\,d\mu(x,s) = \langle L,\nabla\varphi\rangle \qquad \forall\varphi\in C^\infty_c(Q). \tag{2.50}$$

Setting $\boldsymbol{v}_t(x)=\boldsymbol{v}(x,t)$ and using the definition of L we obtain (2.46). Moreover, choosing a sequence $(\nabla\varphi_n)\subset V$ converging to w in $L^2(\mu;\mathbb{R}^d)$, it is easy to show that for $\mathscr{L}^1$-a.e. $t\in I$ there exists a subsequence $n(i)$ (possibly depending on t) such that $\nabla\varphi_{n(i)}(\cdot,t)\in C^\infty_c(\mathbb{R}^d)$ converge in $L^2(\mu_t;\mathbb{R}^d)$ to $\boldsymbol{v}(\cdot,t)$).

Finally, choosing an interval $J\subset I$ and $\eta\in C^\infty_c(J)$ with $0\le\eta\le 1$, (2.50) and (2.48) yield

$$\begin{aligned}&\int_Q \eta(s)|v(x,s)|^2\,d\mu(x,s) = \int_Q \eta\langle v,w\rangle\,d\mu = \lim_{n\to\infty}\int_Q \eta\langle v,\nabla\varphi_n\rangle\,d\mu\\ &= \lim_{n\to\infty}\langle L,\nabla(\eta\varphi_n)\rangle \le \||\mu'|\|_{L^2(J)}\lim_{n\to\infty}\Big(\int_{\mathbb{R}^d\times J}|\nabla\varphi_n|^2\,d\mu\Big)^{1/2}\\ &= \||\mu'|\|_{L^2(J)}\Big(\int_{\mathbb{R}^d\times J}|\boldsymbol{v}|^2\,d\mu\Big)^{1/2}.\end{aligned}$$

Taking a sequence of smooth approximations of the characteristic function of J we obtain

$$\int_J\int_{\mathbb{R}^d}|\boldsymbol{v}_s(x)|^2\,d\mu_s(x)\,ds \le \int_J |\mu'|^2(s)\,ds, \tag{2.51}$$

and therefore

$$\|\boldsymbol{v}_t\|_{L^2(\mu_t,\mathbb{R}^d)} \le |\mu'|(t) \quad \text{for } \mathscr{L}^1\text{-a.e. } t\in I.$$

Now we show the converse implication. We apply the regularization Lemma 2.13, finding approximations μ^ε_t, $\boldsymbol{w}^\varepsilon_t$ satisfying the continuity equation, the uniform integrability condition (2.15) and the local regularity assumptions (2.20). Therefore, we can apply Proposition 2.12, obtaining the representation formula $\mu_t^\varepsilon = (T_t^\varepsilon)_\# \mu_0^\varepsilon$, where T_t^ε is the maximal solution of the ODE $\dot T_t^\varepsilon = \boldsymbol{w}_t^\varepsilon(T_t^\varepsilon)$ with the initial condition $T_0^\varepsilon = x$ (see Lemma 2.8).

Now, taking into account Lemma 2.14, we estimate

$$\begin{aligned}\int_{\mathbb{R}^d} |T_{t_2}^\varepsilon(x) - T_{t_1}^\varepsilon(x)|^2 \, d\mu_0^\varepsilon &\le (t_2 - t_1) \int_{\mathbb{R}^d} \int_{t_1}^{t_2} |\dot T_t^\varepsilon(x)|^2 \, dt \, d\mu_0^\varepsilon \qquad (2.52)\\ &= (t_2 - t_1) \int_{t_1}^{t_2} \int_{\mathbb{R}^d} |\boldsymbol{w}_t^\varepsilon(x)|^2 \, d\mu_t^\varepsilon \, dt\\ &\le (t_2 - t_1) \int_{t_1}^{t_2} \int_{\mathbb{R}^d} |\boldsymbol{w}_t|^2 \, d\mu_t dt,\end{aligned}$$

therefore the transport plan $\boldsymbol{\gamma}^\varepsilon := (T_{t_1}^\varepsilon \times T_{t_2}^\varepsilon)_\# \mu_0^\varepsilon \in \Gamma(\mu_{t_1}^\varepsilon, \mu_{t_2}^\varepsilon)$ satisfies

$$W_2^2(\mu_{t_1}^\varepsilon, \mu_{t_2}^\varepsilon) \le \int_{\mathbb{R}^{2d}} |x-y|^2 \, d\boldsymbol{\gamma}^\varepsilon \le (t_2 - t_1) \int_{t_1}^{t_2} \int_{\mathbb{R}^d} |\boldsymbol{w}_t|^2 \, d\mu_t \, dt.$$

Since for every $t \in I$ μ_t^ε converges narrowly to μ_t as $\varepsilon \to 0$, a compactness argument (see Lemma 5.2.2 or Proposition 7.1.3 of [9]) gives

$$W_2^2(\mu_{t_1}, \mu_{t_2}) \le \int_{\mathbb{R}^{2d}} |x-y|^2 \, d\boldsymbol{\gamma} \le (t_2 - t_1) \int_{t_1}^{t_2} \int_{\mathbb{R}^d} |\boldsymbol{w}_t|^2 \, d\mu_t dt$$

for some optimal transport plan γ between μ_{t_1} and μ_{t_2}. Since t_1 and t_2 are arbitrary this implies that μ_t is absolutely continuous and that its metric derivative is less than $\|\boldsymbol{v}_t\|_{L^2(\mu_t;\mathbb{R}^d)}$ for $\mathscr{L}^1$-a.e. $t \in I$. □

Notice that the continuity equation (2.45) involves only the action of $\boldsymbol{w}_t$ on $\nabla\varphi$ with $\varphi \in C_c^\infty(\mathbb{R}^d)$. Moreover, Theorem 2.15 shows that the minimal norm among all possible velocity fields $\boldsymbol{w}_t$ is the metric derivative and that $\boldsymbol{v}_t$ belongs to the L^2 closure of gradients of functions in $C_c^\infty(\mathbb{R}^d)$. These facts suggest a "canonical" choice of $\boldsymbol{v}_t$ and the following definition of tangent bundle to $\mathscr{P}_2(\mathbb{R}^d)$.

Definition 2.16 (Tangent bundle) *Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$. We define*

$$\mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d) := \overline{\{\nabla\varphi : \ \varphi \in C_c^\infty(\mathbb{R}^d)\}}^{L^2(\mu;\mathbb{R}^d)}.$$

This definition is motivated by the following variational selection principle:

Lemma 2.17 (Variational selection of the tangent vectors) *A vector $\boldsymbol{v} \in L^2(\mu;\mathbb{R}^d)$ belongs to the tangent space $\mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$ iff*

$$\|\boldsymbol{v}+\boldsymbol{w}\|_{L^2(\mu;\mathbb{R}^d)} \ge \|\boldsymbol{v}\|_{L^2(\mu;\mathbb{R}^d)} \quad \forall\, \boldsymbol{w} \in L^2(\mu;\mathbb{R}^d) \text{ such that } \nabla\cdot(\boldsymbol{w}\mu) = 0. \qquad (2.53)$$

In particular, for every $\boldsymbol{v} \in L^2(\mu;\mathbb{R}^d)$, denoting by $\Pi(\boldsymbol{v})$ its orthogonal projection on $\mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$, we have $\nabla \cdot ((\boldsymbol{v} - \Pi(\boldsymbol{v}))\mu) = 0$.

Proof. By the convexity of the L^2 norm, (2.53) holds iff

$$\int_{\mathbb{R}^d} \langle \boldsymbol{v}, \boldsymbol{w} \rangle \, d\mu = 0 \quad \text{for any } \boldsymbol{w} \in L^2(\mu; \mathbb{R}^d) \text{ such that } \nabla \cdot (\boldsymbol{w}\mu) = 0. \tag{2.54}$$

As the space of w such that $\nabla \cdot (\boldsymbol{w}\mu) = 0$ is the orthogonal space to gradients of $C_c^\infty(\mathbb{R}^d)$ functions (in the duality induced by the scalar product of $L^2(\mu; \mathbb{R}^d)$), standard Hilbert duality gives that (2.54) holds iff $\boldsymbol{v}$ belongs to the L^2 closure of $\left\{ \nabla\phi : \ \phi \in C_c^\infty(\mathbb{R}^d) \right\}$. Therefore (2.53) holds iff $\boldsymbol{v}$ belongs to $\mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$. $\square$

The remarks above lead also to the following characterization of divergence-free vector fields (we skip the elementary proof of this statement):

Proposition 2.18 *Let $\boldsymbol{w} \in L^2(\mu; \mathbb{R}^d)$. Then $\nabla \cdot (\boldsymbol{w}\mu) = 0$ iff*

$$\|\boldsymbol{v} - \boldsymbol{w}\|_{L^2(\mu;\mathbb{R}^d)} \geq \|\boldsymbol{v}\|_{L^2(\mu;\mathbb{R}^d)} \qquad \forall \boldsymbol{v} \in \mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d).$$

Moreover equality holds for some $\boldsymbol{v}$ iff $\boldsymbol{w} = 0$.

By the characterization (2.54) of $\mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$ we obtain also

$$\mathrm{Tan}_\mu^\perp \mathscr{P}_2(\mathbb{R}^d) = \left\{ \boldsymbol{v} \in L^2(\mu, \mathbb{R}^d) : \ \nabla \cdot (\boldsymbol{v}\mu) = 0 \right\}. \tag{2.55}$$

The following two propositions show that the notion of tangent space is consistent with the metric structure, with the continuity equation, and with optimal transport maps (if any).

Proposition 2.19 (Tangent vector to a.c. curves) *Let $\mu_t : I \to \mathscr{P}_2(\mathbb{R}^d)$ be an absolutely continuous curve and let $\boldsymbol{v}_t \in L^2(\mu_t; \mathbb{R}^d)$ be such that* (2.45) *holds. Then $\boldsymbol{v}_t$ satisfies* (2.44) *as well if and only if $\boldsymbol{v}_t \in \mathrm{Tan}_{\mu_t} \mathscr{P}_2(\mathbb{R}^d)$ for $\mathscr{L}^1$-a.e. $t \in I$. The vector $\boldsymbol{v}_t$ is uniquely determined $\mathscr{L}^1$-a.e. in I by* (2.44) *and* (2.45).

Proof. The uniqueness of $\boldsymbol{v}_t$ is a straightforward consequence of the linearity with respect to the velocity field of the continuity equation and of the strict convexity of the L^2 norm.

In the proof of Theorem 2.15 we built vector fields $\boldsymbol{v}_t \in \mathrm{Tan}_{\mu_t} \mathscr{P}_2(\mathbb{R}^d)$ satisfying (2.44) and (2.45). By uniqueness, it follows that conditions (2.44) and (2.45) imply $\boldsymbol{v}_t \in \mathrm{Tan}_{\mu_t} \mathscr{P}_2(\mathbb{R}^d)$ for $\mathscr{L}^1$-a.e. t. $\square$

In the following proposition we recover the tangent vector field to a curve $(\mu_t) \subset \mathscr{P}_2^a(\mathbb{R}^d)$ through the infinitesimal behavior of optimal transport maps along the curve. See Proposition 8.4.6 of [9] for a more general result in the case of curves $(\mu_t) \subset \mathscr{P}_2(\mathbb{R}^d)$.

Proposition 2.20 (Optimal plans along a.c. curves) *Let $\mu_t : I \to \mathscr{P}_2^a(\mathbb{R}^d)$ be an absolutely continuous curve and let $\boldsymbol{v}_t \in \mathrm{Tan}_{\mu_t} \mathscr{P}_2(\mathbb{R}^d)$ be characterized by Proposition 2.19. Then, for $\mathscr{L}^1$-a.e. $t \in I$ the following properties hold:*

$$\lim_{h \to 0} \frac{1}{h} (\boldsymbol{t}_{\mu_t}^{\mu_{t+h}} - \boldsymbol{i}) = \boldsymbol{v}_t \qquad \text{in } L^2(\mu_t; \mathbb{R}^d), \tag{2.56}$$

where $\boldsymbol{t}_{\mu_t}^{\mu_{t+h}}$ is the unique optimal transport map between μ_t and μ_{t+h}, and

$$\lim_{h\to 0} \frac{W_2(\mu_{t+h},(\boldsymbol{i}+h\boldsymbol{v}_t)_\#\mu_t)}{|h|} = 0. \tag{2.57}$$

Proof. Let $\mathscr{D} \subset C_c^\infty(\mathbb{R}^d)$ be a countable set with the following property: for any integer $R > 0$ and any $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp}\varphi \subset B_R$ there exist $(\varphi_n) \subset \mathscr{D}$ with $\operatorname{supp}\varphi_n \subset B_R$ and $\varphi_n \to \varphi$ in $C^1(\mathbb{R}^d)$.

We fix $t \in I$ such that $W_2(\mu_{t+h},\mu_t)/|h| \to |\mu'|(t) = \|\boldsymbol{v}_t\|_{L^2(\mu_t)}$ and

$$\lim_{h\to 0} \frac{\mu_{t+h}(\varphi)-\mu_t(\varphi)}{h} = \int_{\mathbb{R}^d} \langle \nabla\varphi, \boldsymbol{v}_t\rangle \, d\mu_t \qquad \forall \varphi \in \mathscr{D}. \tag{2.58}$$

Since $\mathscr{D}$ is countable, the metric differentiation theorem implies that both conditions are fulfilled for $\mathscr{L}^1$-a.e. $t \in I$. Set

$$\boldsymbol{s}_h := \frac{\boldsymbol{t}_{\mu_t}^{\mu_{t+h}} - \boldsymbol{i}}{h}$$

and fix $\varphi \in \mathscr{D}$ and a weak limit point $\boldsymbol{s}_0$ of $\boldsymbol{s}_h$ as $h \to 0$. We use the identity

$$\begin{aligned} \frac{\mu_{t+h}(\varphi)-\mu_t(\varphi)}{h} &= \frac{1}{h}\int_{\mathbb{R}^d} \varphi(\boldsymbol{t}_{\mu_t}^{\mu_{t+h}}(x)) - \varphi(x)\, d\mu_t \\ = \quad \frac{1}{h}\int_{\mathbb{R}^d} \varphi(x+h\boldsymbol{s}_h(x)) - \varphi(x)\, d\mu_h &= h\int_{\mathbb{R}^d} \langle \nabla\varphi(x), \boldsymbol{s}_h(x)\rangle + \omega_x(h)\, d\mu_h \end{aligned}$$

with $\omega_x(h)$ bounded and infinitesimal as $h \to 0$, to obtain

$$\int_{\mathbb{R}^d} \langle \nabla\varphi, \boldsymbol{v}_t\rangle \, d\mu_t = \int_{\mathbb{R}^d} \langle \nabla\varphi, \boldsymbol{s}_0\rangle \, d\mu_t(x).$$

By the density of $\mathscr{D}$ it follows that

$$\nabla \cdot ((\boldsymbol{s}_0 - \boldsymbol{v}_t)\mu_t) = 0. \tag{2.59}$$

We now claim that

$$\int_{\mathbb{R}^d} |\boldsymbol{s}_0|^2 \, d\mu_t(x) \le \left[|\mu'|(t)\right]^2. \tag{2.60}$$

Indeed

$$\begin{aligned} \int_{\mathbb{R}^d} |\boldsymbol{s}_0|^2 \, d\mu_t(x) &\le \liminf_{h\to 0} \int_{\mathbb{R}^d} |\boldsymbol{s}_h|^2 \, d\mu_t \\ &= \liminf_{h\to 0} \frac{1}{h^2}\int_{\mathbb{R}^d} |\boldsymbol{t}_{\mu_t}^{\mu_{t+h}}(x) - x|^2 d\mu_t \\ &= \liminf_{h\to 0} \frac{W_2(\mu_{t+h},\mu_t)}{h^2} = |\mu'|^2(t). \end{aligned}$$

From (2.60) we obtain that $\|\boldsymbol{s}_0\|_{L^2(\mu_t;\mathbb{R}^d)} \le [|\mu'|(t)] = \|\boldsymbol{v}_t\|_{L^2(\mu_t;\mathbb{R}^d)}$. Therefore Proposition 2.18 entails that $\boldsymbol{s}_0 = \boldsymbol{v}_t$. Moreover, the first inequality above is strict if $\boldsymbol{s}_h$ converge weakly, but not strongly, to $\boldsymbol{s}_0$. Therefore (2.56) holds.

Now we show (2.57). By (2.8) we can estimate the distance between μ_{t+h} and $(\boldsymbol{i}+h\boldsymbol{v}_t)_\#\mu_t$ with $\|\boldsymbol{i}+h\boldsymbol{v}_t - \boldsymbol{t}_{\mu_t}^{\mu_{t+h}}\|_{L^2(\mu_t;\mathbb{R}^d)}$, and because of (2.56) this norm tends to 0 faster than h. ∎

As an application of (2.57) we are now able to show the $\mathscr{L}^1$-a.e. differentiability of $t \mapsto W_2(\mu_t, \sigma)$ along absolutely continuous curves μ_t, with $\mu_t \in \mathscr{P}_2^a(\mathbb{R}^d)$.

Theorem 2.21 (Generic differentiability of $W_2(\mu_t, \sigma)$) Let $\mu_t : I \to \mathscr{P}_2^a(\mathbb{R}^d)$ be an absolutely continuous curve, let $\sigma \in \mathscr{P}_2(\mathbb{R}^d)$ and let $\boldsymbol{v}_t \in \mathrm{Tan}_{\mu_t}\mathscr{P}_2(\mathbb{R}^d)$ be its tangent vector field, characterized by Proposition 2.19. Then

$$\frac{d}{dt}W_2^2(\mu_t, \sigma) = 2\int_{\mathbb{R}^d} \langle x - \boldsymbol{t}_{\mu_t}^{\sigma}(x), \boldsymbol{v}_t(x)\rangle \, d\mu_t(x) \qquad \text{for } \mathscr{L}^1\text{-a.e. } t \in I. \tag{2.61}$$

Proof. We show that the stated property is true at any $\bar{t}$ where (2.57) holds and the derivative of $t \mapsto W_2(\mu_t, \sigma)$ exists (recall that this map is absolutely continuous). Due to (2.57), we know that the limit

$$L := \lim_{h\to 0} \frac{W_2^2((\boldsymbol{i} + h\boldsymbol{v}_{\bar{t}})_{\#}\mu_{\bar{t}}, \sigma) - W_2^2(\mu_{\bar{t}}, \sigma)}{h}$$

exists and coincides with $\frac{d}{dt}W_2^2(\mu_t, \sigma)$ evaluated at $t = \bar{t}$, and we have to show that it is equal to the left hand side in (2.61).

Using the transport map $\boldsymbol{s} := (\boldsymbol{i} + h\boldsymbol{v}_{\bar{t}}) \circ \boldsymbol{t}_{\sigma}^{\mu_{\bar{t}}}$ to estimate from above $W_2((\boldsymbol{i} + h\boldsymbol{v}_{\bar{t}})_{\#}\mu_{\bar{t}}, \sigma)$, we get

$$\begin{aligned} W_2^2((\boldsymbol{i} + h\boldsymbol{v}_{\bar{t}})_{\#}\mu_{\bar{t}}, \sigma) &\leq \int_{\mathbb{R}^d} |(\boldsymbol{i} + h\boldsymbol{v}_{\bar{t}}) \circ \boldsymbol{t}_{\sigma}^{\mu_{\bar{t}}} - \boldsymbol{i}|^2 \, d\sigma \\ &= \int_{\mathbb{R}^d} |\boldsymbol{i} + h\boldsymbol{v}_{\bar{t}} - \boldsymbol{t}_{\mu_{\bar{t}}}^{\sigma}|^2 \, d\mu_{\bar{t}} = 2\int_{\mathbb{R}^d} \langle \boldsymbol{i} - \boldsymbol{t}_{\mu_{\bar{t}}}, \boldsymbol{v}_{\bar{t}}\rangle \, d\mu_{\bar{t}} + o(h). \end{aligned}$$

Dividing both sides by h and taking limits as $h \downarrow 0$ or $h \uparrow 0$ we obtain

$$L \leq 2\int_{\mathbb{R}^d} \langle x - \boldsymbol{t}_{\mu_t}^{\sigma}(x), \boldsymbol{v}_t(x)\rangle \, d\mu_t(x) \leq L. \quad \Box$$

The argument in the previous proof leads to the so-called super-differentiability property of the Wasserstein distance, a theme used in many papers on this subject (see in particular [66] and Chapter 10 of [9]). Finally, we compare the tangent space arising from the closure of gradients of smooth compactly supported function with the tangent space built using optimal maps. Proposition 2.20 suggests indeed another possible definition of tangent cone to a measure $\mu \in \mathscr{P}_2^a(\mathbb{R}^d)$: we define

$$\mathrm{Tan}_{\mu}^{r}\mathscr{P}_2(\mathbb{R}^d) := \overline{\left\{\lambda(\boldsymbol{t}_{\mu}^{\nu} - \boldsymbol{i}) : \nu \in \mathscr{P}_2(\mathbb{R}^d),\ \lambda > 0\right\}}^{L^2(\mu;\mathbb{R}^d)}. \tag{2.62}$$

As a matter of fact, the two concepts coincide (see also §8.5 of [9] for a more general statement).

Theorem 2.22 *For any $\mu \in \mathscr{P}_2^a(\mathbb{R}^d)$ we have* $\mathrm{Tan}_{\mu}\mathscr{P}_2(\mathbb{R}^d) = \mathrm{Tan}_{\mu}^{r}\mathscr{P}_2(\mathbb{R}^d)$.

Proof. We show first that optimal transport maps $\boldsymbol{t} = \boldsymbol{t}_{\mu_t}^{\sigma}$ belong to $\mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$. Assume that $\operatorname{supp}\sigma$ is contained in $\overline{B}_R(0)$ for some $R > 0$. We know that we can represent $\boldsymbol{t} = \nabla\varphi$, where φ is a Lipschitz convex function. We consider now the mollified functions φ_ε. A truncation argument enabling an approximation by gradients with compact support gives that $\nabla\varphi_\varepsilon$ belong to $\mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$. Due to the absolute continuity of μ it is immediate to check using the dominated convergence theorem that $\nabla\varphi_\varepsilon$ converge to $\nabla\varphi$ in $L^2(\mu;\mathbb{R}^d)$, therefore $\nabla\varphi \in \mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$ as well. In the case when the support of σ is not bounded we approximate σ in $\mathscr{P}_2(\mathbb{R}^d)$ by measures with compact support (details are worked out in Lemma 8.5.3 of [9]).

Now we show the opposite inclusion: if $\varphi \in C_c^\infty(\mathbb{R}^d)$ it is always possible to choose $\lambda > 0$ such that $x \mapsto \frac{1}{2}|x|^2 + \lambda^{-1}\phi(x)$ is convex. Therefore $\boldsymbol{r} := \boldsymbol{i} + \lambda^{-1}\nabla\varphi$ is the optimal map between μ and $\nu := \boldsymbol{r}_{\#}\mu$; by (2.62) we obtain that $\nabla\phi = \lambda(\boldsymbol{r} - \boldsymbol{i})$ belongs to $\mathrm{Tan}_\mu^r \mathscr{P}_2(\mathbb{R}^d)$. $\square$

3 Convex functionals in $\mathscr{P}_2(\mathbb{R}^d)$

The importance of geodesically convex functionals in Wasserstein spaces was firstly pointed out by McCann [67], who introduced the three basic examples we will discuss in detail in 3.4, 3.6, 3.8. His original motivation was to prove the uniqueness of the minimizer of an energy functional which results from the sum of the above three contributions.

Applications of this idea have been given to (im)prove many deep functional (Brunn-Minkowski, Gaussian, (logarithmic) Sobolev, Isoperimetric, etc.) inequalities: we refer to Villani's book [86, Chap. 6] (see also the survey [49]) for a detailed account on this topic. Connections with evolution equations have also been exploited [70, 74, 75, 1, 29], mainly to study the asymptotic decay of the solution to the equilibrium.

From our point of view, convexity is a crucial tool to study the well posedness and the basic regularity properties of gradient flows. Thus in this section we discuss the basic notions and properties related to this concept: the first part of Section 3.1 is devoted to fixing the notion of convexity along geodesics in $\mathscr{P}_2(\mathbb{R}^d)$.

Section 3.2 discusses in great generality the main examples of geodesically convex functionals: potential, interaction and internal energy. We consider also the convexity properties of the map $\mu \mapsto -W_2^2(\mu,\nu)$ and its geometric implications.

In the last section we give a closer look to the convexity properties of general Relative Entropy functionals, showing that they are strictly related to the log-concavity of the reference measures.

3.1 λ-geodesically convex functionals in $\mathscr{P}_2(\mathbb{R}^d)$

In McCann's approach, a functional $\phi : \mathscr{P}_2^a(\mathbb{R}^d) \to (-\infty,+\infty]$ is *displacement convex* if

$$\begin{aligned} &\text{setting}\quad \mu_t^{1\to 2} := \big(\boldsymbol{i} + t(\boldsymbol{t}-\boldsymbol{i})\big)_\# \mu^1, \text{ with } \boldsymbol{t} = \boldsymbol{t}_{\mu^1}^{\mu^2},\\ &\text{the map}\quad t\in[0,1]\mapsto \phi\big(\mu_t^{1\to 2}\big)\quad \text{is convex},\quad \forall\, \mu^1,\,\mu^2\in \mathscr{P}_2^a(\mathbb{R}^d). \end{aligned} \tag{3.1}$$

We have seen that the curve $\mu_t^{1\to 2}$ is the unique constant speed geodesic connecting μ^1 to μ^2; therefore the following definition seems natural, when we consider functionals whose domain contains general probability measures.

Definition 3.1 (λ-convexity along geodesics) *Let $\phi : \mathscr{P}_2(\mathbb{R}^d) \to (-\infty,+\infty]$. Given $\lambda\in\mathbb{R}$, we say that ϕ is* λ-geodesically convex *in $\mathscr{P}_2(\mathbb{R}^d)$ if for every couple μ^1, $\mu^2 \in \mathscr{P}_2(\mathbb{R}^d)$* there exists *$\boldsymbol{\mu}\in\Gamma_o(\mu^1,\mu^2)$ such that*

$$\phi(\mu_t^{1\to 2}) \le (1-t)\phi(\mu^1) + t\phi(\mu^2) - \frac{\lambda}{2} t(1-t) W_2^2(\mu^1,\mu^2)\quad \forall\, t\in[0,1], \tag{3.2}$$

where $\mu_t^{1\to 2} = \big((1-t)\pi^1 + t\pi^2\big)_\# \boldsymbol{\mu}$, π^1, π^2 being the projections onto the first and the second coordinate in $\mathbb{R}^d\times\mathbb{R}^d$, respectively.

Remark 3.2 (The map $t \mapsto \phi(\mu_t^{1\to 2})$ is λ-convex) The standard definition of λ-convexity for a map $\varphi : \mathbb{R}^n \to \mathbb{R}$ requires

$$\varphi(tx+(1-t)y) \leq t\varphi(x)+(1-t)\varphi(y)-\frac{\lambda}{2}t(1-t)|x-y|^2 \qquad \forall t \in [0,1],\ x,\, y \in \mathbb{R}^n \tag{3.3}$$

(equivalently, if φ is continuous, one might ask that $D^2\varphi \geq \lambda I$ in the sense of distributions). The definition of λ-convexity expressed through (3.2) implies that

$$\text{the map } t \in [0,1] \mapsto \phi(\mu_t^{1\to 2}) \text{ is } \lambda\text{-convex,} \tag{3.4}$$

thus recovering an (apparently) stronger and more traditional form. This equivalence follows easily by the fact that for $t_1 < t_2$ in $[0,1]$ with $\{t_1, t_2\} \neq \{0,1\}$ the plan $\Big(\big((1-t_1)\pi^1 + t_1\pi^2\big) \times \big((1-t_2)\pi^1 + t_2\pi^2\big)\Big)_\#\boldsymbol{\mu}$ is the *unique* element of $\Gamma_o(\mu_{t_1}^{1\to 2}, \mu_{t_2}^{1\to 2})$.

Let us discuss now the convexity properties of the squared Wasserstein distance. In the 1-dimensional case it can be easily shown (see Theorem 6.0.2 of [9]) that $\mathscr{P}_2(\mathbb{R})$ is isometrically isomorphic to a closed convex subset of an Hilbert space: precisely the space of nondecreasing functions in $(0,1)$ (the inverses of distribution functions), viewed as a subset of $L^2(0,1)$. Thus the 2-Wasserstein distance in $\mathbb{R}$ satisfies the generalized parallelogram rule

$$\begin{gathered} W_2^2(\mu^1, \mu_t^{2\to 3}) = (1-t)W_2^2(\mu^1, \mu^2) + tW_2^2(\mu^1, \mu^3) - t(1-t)W_2^2(\mu^2, \mu^3) \\ \forall\, t \in [0,1], \quad \mu^1, \mu^2, \mu^3 \in \mathscr{P}_2(\mathbb{R}). \end{gathered} \tag{3.5}$$

On the other hand, if the ambient space has dimension ≥ 2 the following example shows that there is no constant λ such that $W_2^2(\cdot, \mu^1)$ is λ-convex along geodesics.

Example 3.3 (The squared distance function is not λ-convex) Let $d = 2$ and

$$\mu^2 := \frac{1}{2}\left(\delta_{(0,0)} + \delta_{(2,1)}\right), \qquad \mu^3 := \frac{1}{2}\left(\delta_{(0,0)} + \delta_{(-2,1)}\right).$$

It is easy to check that the unique optimal map $\boldsymbol{r}$ pushing μ^2 to μ^3 maps $(0,0)$ in $(-2,1)$ and $(2,1)$ in $(0,0)$, therefore there is a unique constant speed geodesic joining the two measures, given by

$$\mu_t^{2\to 3} := \frac{1}{2}\left(\delta_{(-2t,t)} + \delta_{(2-2t,1-t)}\right) \qquad t \in [0,1].$$

Choosing $\mu^1 := \frac{1}{2}\left(\delta_{(0,0)} + \delta_{(0,-2)}\right)$, there are two maps $\boldsymbol{r}_t$, $\boldsymbol{s}_t$ pushing μ^1 to $\mu_t^{2\to 3}$, given by

$$\begin{aligned} \boldsymbol{r}_t(0,0) &= (-2t,t), \quad \boldsymbol{r}_t(0,-2) = (2-2t, 1-t), \\ \boldsymbol{s}_t(0,0) &= (2-2t,1-t), \quad \boldsymbol{s}_t(0,-2) = (-2t,t). \end{aligned}$$

Therefore

$$W_2^2(\mu_t^{2\to 3}, \mu^1) = \min\left\{5t^2 - 7t + \frac{13}{2}, 5t^2 - 3t + \frac{9}{2}\right\}$$

has a concave cusp at $t = 1/2$ and therefore is not λ-convex along the geodesic $\mu_t^{2\to 3}$ for any $\lambda \in \mathbb{R}$.

3.2 Examples of convex functionals in $\mathscr{P}_2(\mathbb{R}^d)$

In this section we introduce the main classes of geodesically convex functionals.

Example 3.4 (Potential energy) Let $V : \mathbb{R}^d \to (-\infty, +\infty]$ be a proper, lower semicontinuous function whose negative part has a quadratic growth, i.e.

$$V(x) \geq -A - B|x|^2 \quad \forall\, x \in \mathbb{R}^d \quad \text{for some} \quad A,\, B \in \mathbb{R}^+. \tag{3.6}$$

In $\mathscr{P}_2(\mathbb{R}^d)$ we define

$$\mathcal{V}(\mu) := \int_{\mathbb{R}^d} V(x)\, d\mu(x). \tag{3.7}$$

Evaluating $\mathcal{V}$ on Dirac's masses we check that $\mathcal{V}$ is proper; since V^- has at most quadratic growth Lemma 1.2 gives that $\mathcal{V}$ is lower semicontinuous in $\mathscr{P}_2(\mathbb{R}^d)$. If V is bounded from below we have even lower semicontinuity w.r.t. narrow convergence.

The following simple proposition shows that $\mathcal{V}$ is convex along all interpolating curves induced by admissible plans; choosing optimal plans one obtains in particular that $\mathcal{V}$ is convex along geodesics.

Proposition 3.5 (Convexity of $\mathcal{V}$) *If V is λ-convex then for every μ^1, $\mu^2 \in D(\mathcal{V})$ and $\boldsymbol{\mu} \in \Gamma(\mu^1, \mu^2)$ we have*

$$\mathcal{V}(\mu_t^{1\to 2}) \leq (1-t)\mathcal{V}(\mu^1) + t\mathcal{V}(\mu^2) - \frac{\lambda}{2} t(1-t) \int_{\mathbb{R}^d \times \mathbb{R}^d} |x_1 - x_2|^2\, d\boldsymbol{\mu}(x_1, x_2). \tag{3.8}$$

In particular $\mathcal{V}$ is λ-convex along geodesics.

Proof. Since V is bounded from below either by a continuous affine functional (if $\lambda \geq 0$) or by a quadratic function (if $\lambda < 0$) its negative part satisfies (3.6); therefore the definition (3.7) makes sense.

Integrating (3.3) along any admissible transport plan $\boldsymbol{\mu} \in \Gamma(\mu^1, \mu^2)$ with μ^1, $\mu^2 \in D(\mathcal{V})$ we obtain (3.8), since

$$\begin{aligned}
\mathcal{V}(\mu_t^{1\to 2}) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} V((1-t)x_1 + tx_2)\, d\boldsymbol{\mu}(x_1, x_2) \\
&\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \Big((1-t)V(x_1) + tV(x_2) - \frac{\lambda}{2} t(1-t)|x_1 - x_2|^2 \Big)\, d\boldsymbol{\mu}(x_1, x_2) \\
&= (1-t)\mathcal{V}(\mu^1) + t\mathcal{V}(\mu^2) - \frac{\lambda}{2} t(1-t) \int_{\mathbb{R}^d \times \mathbb{R}^d} |x_1 - x_2|^2\, d\boldsymbol{\mu}(x_1, x_2).
\end{aligned}$$

Since $\mathcal{V}(\delta_x) = V(x)$, it is easy to check that the conditions on V are also necessary for the validity of the previous proposition.

Example 3.6 (Interaction energy) Let us fix an integer $k > 1$ and let us consider a lower semicontinuous function $W : \mathbb{R}^{kd} \to (-\infty, +\infty]$, whose negative part satisfies the usual quadratic growth condition. Denoting by $\mu^{\times k}$ the measure $\mu \times \mu \times \cdots \times \mu$ on $\mathbb{R}^{kd}$, we set

$$\mathcal{W}_k(\mu) := \int_{\mathbb{R}^{kd}} W(x_1, x_2, \ldots, x_k)\, d\mu^{\times k}(x_1, x_2, \ldots, x_k). \tag{3.9}$$

If
$$\exists\, x \in \mathbb{R}^d : W(x, x, \ldots, x) < +\infty, \tag{3.10}$$
then $\mathcal{W}_k$ is proper; its lower semicontinuity follows from the fact that
$$\mu_n \to \mu \quad \text{in } \mathscr{P}_2(\mathbb{R}^d) \quad \Longrightarrow \quad \mu_n^{\times k} \to \mu^{\times k} \quad \text{in } \mathscr{P}_2(\mathbb{R}^{kd}). \tag{3.11}$$
Here the typical example is $k = 2$ and $W(x_1, x_2) := \tilde{W}(x_1 - x_2)$ for some $\tilde{W} : \mathbb{R}^d \to (-\infty, +\infty]$ with $\tilde{W}(0) < +\infty$.

Proposition 3.7 (Convexity of $\mathcal{W}$) *If W is convex then the functional $\mathcal{W}_k$ is convex along the interpolating curve $\mu_t^{1\to2}$ induced by any $\boldsymbol{\mu} \in \Gamma(\mu^1, \mu^2)$, in $\mathscr{P}_2(\mathbb{R}^d)$.*

Proof. Observe that $\mathcal{W}_k$ is the restriction to the subset
$$\mathscr{P}_2^{\times}(\mathbb{R}^{kd}) := \Big\{ \mu^{\times k} : \mu \in \mathscr{P}_2(\mathbb{R}^d) \Big\}$$
of the potential energy functional $\mathcal{W}$ on $\mathscr{P}_2(\mathbb{R}^{kd})$ given by
$$\mathcal{W}(\boldsymbol{\mu}) := \int_{R^{kd}} W(x_1, \ldots, x_k)\, d\boldsymbol{\mu}(x_1, \ldots, x_k).$$
We consider the linear permutation of coordinates $P : (\mathbb{R}^{2d})^k \to (\mathbb{R}^{kd})^2$ defined by
$$P\Big((x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k) \Big) := \Big((x_1, \ldots x_k), (y_1, \ldots y_k) \Big).$$
If $\boldsymbol{\mu} \in \Gamma(\mu_1, \mu_2)$ then it is easy to check that $P_\# \boldsymbol{\mu}^{\times k} \in \Gamma(\mu_1^{\times k}, \mu_2^{\times k}) \subset \mathscr{P}((\mathbb{R}^{kd})^2)$ and
$$(\pi_t^{1\to2})_\# P_\# (\boldsymbol{\mu}^{\times k}) = P_\# \Big((\pi_t^{1\to2})_\# \boldsymbol{\mu} \Big)^{\times k}.$$
Therefore all the convexity properties of $\mathcal{W}_k$ follow from the corresponding ones of $\mathcal{W}$. □

Example 3.8 (Internal energy) Let $F : [0, +\infty) \to (-\infty, +\infty]$ be a proper, lower semicontinuous convex function such that
$$F(0) = 0, \quad \liminf_{s \downarrow 0} \frac{F(s)}{s^\alpha} > -\infty \quad \text{for some } \alpha > \frac{d}{d+2}. \tag{3.12}$$
We consider the functional $\mathcal{F} : \mathscr{P}_2(\mathbb{R}^d) \to (-\infty, +\infty]$ defined by
$$\mathcal{F}(\mu) := \begin{cases} \int_{\mathbb{R}^d} F(u(x))\, d\mathscr{L}^d(x) & \text{if } \mu = u \cdot \mathscr{L}^d \in \mathscr{P}_2^a(\mathbb{R}^d), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.13}$$

Remark 3.9 (The meaning of condition (3.12)) Condition (3.12) simply guarantees that the negative part of $F(\mu)$ is integrable in $\mathbb{R}^d$. For, let us observe that there exist nonnegative constants c_1, c_2 such that the negative part of F satisfies

$$F^-(s) \le c_1 s + c_2 s^\alpha \quad \forall\, s \in [0,+\infty),$$

and it is not restrictive to suppose $\alpha \le 1$. Since $\mu = u\,\mathscr{L}^d \in \mathscr{P}_2(\mathbb{R}^d)$ and $\frac{2\alpha}{1-\alpha} > d$ we have

$$\begin{aligned}&\int_{\mathbb{R}^d} u^\alpha(x)\, d\mathscr{L}^d(x) = \int_{\mathbb{R}^d} u^\alpha(x)(1+|x|)^{2\alpha}(1+|x|)^{-2\alpha}\, d\mathscr{L}^d(x)\\ &\le \Big(\int_{\mathbb{R}^d} u(x)(1+|x|)^2\, d\mathscr{L}^d(x)\Big)^\alpha \Big(\int_{\mathbb{R}^d} (1+|x|)^{-2\alpha/(1-\alpha)}\, d\mathscr{L}^d(x)\Big)^{1-\alpha} < +\infty\end{aligned}$$

and therefore $F^-(u) \in L^1(\mathbb{R}^d)$.

Remark 3.10 (Lower semicontinuity of $\mathcal{F}$) General results on integral functionals (see for instance [8]) show that $\mathcal{F}$ is narrowly lower semicontinuous if F has a superlinear growth at infinity. Indeed, under this assumption sequences $\mu_n = u_n \mathscr{L}^d$ on which $\mathcal{F}$ is bounded have the property that (u_n) is sequentially weakly relatively compact in $L^1(\mathbb{R}^d)$, and the convexity of $\mathcal{F}$ together with the lower semicontinuity of F ensure the sequential lower semicontinuity with respect to the weak L^1 topology.

In the next proposition we prove the geodesic convexity of the internal energy functional (3.13) by using the change of variable formula (1.24). This was first shown by McCann [67] with a different argument.

Proposition 3.11 (Convexity of $\mathcal{F}$) *If F has a superlinear growth at infinity and*

$$\textit{the map} \quad s \mapsto s^d F(s^{-d}) \quad \textit{is convex and non increasing in } (0,+\infty), \tag{3.14}$$

then the functional $\mathcal{F}$ is convex along geodesics in $\mathscr{P}_2(\mathbb{R}^d)$.

Proof. We consider two measures $\mu^i = u^i \mathscr{L}^d \in D(\mathcal{F})$, $i = 1, 2$ and the optimal transport map $\boldsymbol{r}$ such that $\boldsymbol{r}_\# \mu^1 = \mu^2$. Setting $\boldsymbol{r}_t := (1-t)\boldsymbol{i} + t\boldsymbol{r}$, by the characterization of constant speed geodesics we know that $\boldsymbol{r}_t$ is the optimal transport map between μ^1 and $\mu_t := \boldsymbol{r}_{t\#}\mu^1$ for any $t \in [0,1]$, and $\mu_t = u_t \mathscr{L}^d \in \mathscr{P}_2^a(\mathbb{R}^d)$, with

$$u_t(\boldsymbol{r}_t(x)) = \frac{u^1(x)}{\det \nabla \boldsymbol{r}_t(x)} \quad \text{for } \mu^1\text{-a.e. } x \in \mathbb{R}^d.$$

By (1.24) it follows that

$$\mathcal{F}(\mu_t) = \int_{\mathbb{R}^d} F(u_t(y))\, dy = \int_{\mathbb{R}^d} F\Big(\frac{u(x)}{\det \nabla \boldsymbol{r}_t(x)}\Big) \det \nabla \boldsymbol{r}_t(x)\, dx.$$

Since for a diagonalizable map D with nonnegative eigenvalues

$$t \mapsto \det((1-t)I + tD)^{1/d} \quad \text{is concave in } [0,1], \tag{3.15}$$

the integrand above may be seen as the composition of the convex and nonincreasing map $s \mapsto s^d F(u(x)/s^d)$ and of the concave map in (3.15), so that the resulting map is convex in $[0,1]$ for μ^1-a.e. $x \in \mathbb{R}^d$. Thus we have

$$F\Big(\frac{u^1(x)}{\det \nabla \boldsymbol{r}_t(x)}\Big) \det \nabla \boldsymbol{r}_t(x) \le (1-t)F(u^1(x)) + tF(u^2(x))$$

and the thesis follows by integrating this inequality in $\mathbb{R}^d$.

In order to express (3.14) in a different way, we introduce the function

$$L_F(z) := zF'(z) - F(z) \quad \text{which satisfies } -L_F(e^{-z})e^z = \frac{d}{dz}F(e^{-z})e^z; \tag{3.16}$$

denoting by $\hat F$ the modified function $F(e^{-z})e^z$ we have the simple relation

$$\begin{gathered} \hat L_F(z) = -\frac{d}{dz}\hat F(z), \quad \widehat{L^2_F}(z) = -\frac{d}{dz}\hat L_F(z) = \frac{d^2}{dz^2}\hat F(z), \quad \text{where} \\ L^2_F(z) := L_{L_F}(z) = zL_F'(z) - L_F(z). \end{gathered} \tag{3.17}$$

The nonincreasing part of condition (3.14) is equivalent to say that

$$L_F(z) \ge 0 \quad \forall\, z \in (0,+\infty), \tag{3.18}$$

and it is in fact implied by the convexity of F. A simple computation in the case $F \in C^2(0,+\infty)$ shows

$$\frac{d^2}{ds^2}F(s^{-d})s^d = \frac{d^2}{ds^2}\hat F(d\cdot \log s) = \hat L^2_F(d\cdot\log s)\frac{d^2}{s^2} + \hat L_F(d\cdot \log s)\frac{d}{s^2},$$

and therefore

$$\text{(3.14) is equivalent to } L^2_F(z) \ge -\frac{1}{d}L_F(z) \quad \forall\, z\in(0,+\infty), \tag{3.19}$$

i.e.

$$zL_F'(z) \ge \big(1-\frac{1}{d}\big)L_F(z), \quad \text{the map } z \mapsto z^{1/d-1}L_F(z) \text{ is non increasing.} \tag{3.20}$$

Observe that the bigger is the dimension d, the stronger are the above conditions, which always imply the convexity of F.

Remark 3.12 (A "dimension free" condition) The weakest condition on F yielding the geodesic convexity of $\mathcal{F}$ in *any dimension* is therefore

$$L^2_F(z) = zL_F'(z) - L_F(z) \ge 0 \quad \forall\, z \in (0,+\infty). \tag{3.21}$$

Taking into account (3.17), this is also equivalent to ask that

$$\textit{the map} \quad s \mapsto F(e^{-s})e^s \quad \textit{is convex and nonincreasing in } (0,+\infty). \tag{3.22}$$

Among the functionals F satisfying (3.14) we quote:

$$\text{the entropy functional:} \quad F(s) = s \log s, \tag{3.23}$$

$$\text{the power functional:} \quad F(s) = \frac{1}{m-1} s^m \quad \text{for } m \geq 1 - \frac{1}{d}. \tag{3.24}$$

Observe that the entropy functional and the power functional with $m > 1$ have a superlinear growth. In order to deal with the power functional with $m \leq 1$, due to the failure of the lower semicontinuity property one has to introduce a suitable relaxation $\mathcal{F}^*$ of it, defined by [55, 24]

$$\mathcal{F}^*(\mu) := \frac{1}{m-1} \int_{\mathbb{R}^d} F(u(x))\, d\mathscr{L}^d(x) \quad \text{with } \mu = u \cdot \mathscr{L}^d + \mu_s,\ \mu_s \perp \mathscr{L}^d. \tag{3.25}$$

In this case the functional takes only account of the density of the absolutely continuous part of μ w.r.t. $\mathscr{L}^d$ and the domain of $\mathcal{F}^*$ is the whole $\mathscr{P}_2(\mathbb{R}^d)$. The functional $\mathcal{F}^*$ retains the convexity properties of $\mathcal{F}$, see [9].

Example 3.13 (The opposite Wasserstein distance) Let us fix a base measure $\mu^1 \in \mathscr{P}_2(\mathbb{R}^d)$ and let us consider the functional

$$\phi(\mu) := -\frac{1}{2} W_2^2(\mu^1, \mu). \tag{3.26}$$

Proposition 3.14 *For each couple μ^2, $\mu^3 \in \mathscr{P}_2(\mathbb{R}^d)$ and each transfer plan $\boldsymbol{\mu}^{2\,3} \in \Gamma(\mu^2, \mu^3)$ we have*

$$\begin{aligned} W_2^2(\mu^1, \mu_t^{2\to 3}) \geq{}& (1-t) W_2^2(\mu^1, \mu^2) + t W_2^2(\mu^1, \mu^3) \\ & - t(1-t) \int_{\mathbb{R}^d \times \mathbb{R}^d} |x_2 - x_3|^2 \, d\boldsymbol{\mu}^{2\,3}(x_2, x_3) \quad \forall\, t \in [0,1]. \end{aligned} \tag{3.27}$$

In particular the map $\phi : \mu \mapsto -\frac{1}{2} W_2^2(\mu^1, \mu)$ is (-1)-convex along geodesics.

Proof. For $\boldsymbol{\mu}^{2\,3} \in \Gamma(\mu^2, \mu^3)$, we can find (see Proposition 7.3.1 of [9]) $\boldsymbol{\mu} \in \mathscr{P}(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$ whose projection on the second and third variable is $\boldsymbol{\mu}^{2\,3}$ and such that

$$(\pi^1, (1-t)\pi^2 + t\pi^3)_\# \boldsymbol{\mu} \in \Gamma_o(\mu^1, \mu_t^{2\to 3}), \tag{3.28}$$

with $\mu_t^{2\to 3} := ((1-t)\pi^2 + t\pi^3)_\# \boldsymbol{\mu}^{2\,3}$. Therefore

$$\begin{aligned} W_2^2(\mu^1, \mu_t^{2\to 3}) &= \int_{\mathbb{R}^{3d}} |(1-t)x_2 + t x_3 - x_1|^2 \, d\boldsymbol{\mu}(x_1, x_2, x_3) \\ &= \int_{\mathbb{R}^{3d}} \Big((1-t)|x_2 - x_1|^2 + t|x_3 - x_1|^2 - t(1-t)|x_2 - x_3|^2 \Big) \, d\boldsymbol{\mu}(x_1, x_2, x_3) \\ &\geq (1-t) W_2^2(\mu^1, \mu^2) + t W_2^2(\mu^1, \mu^3) - t(1-t) \int_{\mathbb{R}^{2d}} |x_2 - x_3|^2 \, d\boldsymbol{\mu}^{2\,3}(x_2, x_3). \end{aligned}$$

□

In particular, choosing optimal plans in (3.27), we obtain the *semiconcavity inequality* of the Wasserstein distance from a fixed measure μ^3 along the constant speed geodesics $\mu_t^{1\to 2}$ connecting μ^1 to μ^2:

$$W_2^2(\mu_t^{1\to 2}, \mu^3) \geq (1-t)W_2^2(\mu^1, \mu^3) + tW_2^2(\mu^2, \mu^3) - t(1-t)W_2^2(\mu^1, \mu^2). \quad (3.29)$$

According to Aleksandrov's metric notion of curvature (see [5], [58]), this inequality can be interpreted by saying that the Wasserstein space is a positively curved metric space (in short, a PC-space). This was already pointed out by a formal computation in [74], showing also that generically the inequality is strict. An example where strict inequality occurs can be obtained as follows: let $d = 2$ and

$$\mu^1 := \frac{1}{2}\left(\delta_{(1,1)} + \delta_{(5,3)}\right), \quad \mu^2 := \frac{1}{2}\left(\delta_{(-1,1)} + \delta_{(-5,3)}\right), \quad \mu^3 := \frac{1}{2}\left(\delta_{(0,0)} + \delta_{(0,-4)}\right).$$

Then, it is immediate to check that $W_2^2(\mu^1, \mu^2) = 40$, $W_2^2(\mu^1, \mu^3) = 30$, and $W_2^2(\mu^2, \mu^3) = 30$. On the other hand, the unique constant speed geodesic joining μ^1 to μ^2 is given by

$$\mu_t := \frac{1}{2}\left(\delta_{(1-6t,1+2t)} + \delta_{(5-6t,3-2t)}\right)$$

and a simple computation gives

$$24 = W_2^2(\mu_{1/2}, \mu^3) > \frac{30}{2} + \frac{30}{2} - \frac{40}{4}.$$

3.3 Relative entropy and convex functionals of measures

In this section we study in detail the relative entropy functional; although we confine the discussion to a finite-dimensional situation, the formalism used in this section is well adapted to the extension to an infinite-dimensional context, see [9].

Definition 3.15 (Relative entropy) *Let γ, μ be Borel probability measures on $\mathbb{R}^d$; the* relative entropy *of μ w.r.t. γ is*

$$\mathcal{H}(\mu|\gamma) := \begin{cases} \displaystyle\int_{\mathbb{R}^d} \frac{d\mu}{d\gamma} \log\left(\frac{d\mu}{d\gamma}\right) d\gamma & \text{if } \mu \ll \gamma, \\ +\infty & \text{otherwise.} \end{cases} \quad (3.30)$$

As in Example 3.8 we introduce the nonnegative, l.s.c. and convex function

$$H(s) := \begin{cases} s(\log s - 1) + 1 & \text{if } s > 0, \\ 1 & \text{if } s = 0, \\ +\infty & \text{if } s < 0, \end{cases} \quad (3.31)$$

and we observe that, whenever $\mu \ll \gamma$, we have

$$\mathcal{H}(\mu|\gamma) = \int_{\mathbb{R}^d} H\Big(\frac{d\mu}{d\gamma}\Big)\, d\gamma \geq 0; \quad \mathcal{H}(\mu|\gamma) = 0 \quad \Leftrightarrow \quad \mu = \gamma. \quad (3.32)$$

Remark 3.16 (Changing γ) Let γ be a Borel measure on $\mathbb{R}^d$ and let $V : \mathbb{R}^d \to (-\infty, +\infty]$ a Borel map such that

$$V^+ \quad \text{has at most quadratic growth,} \quad \tilde\gamma := e^{-V} \cdot \gamma \quad \text{is a probability measure.} \tag{3.33}$$

Then for measures in $\mathscr{P}_2(\mathbb{R}^d)$ the relative entropy w.r.t. γ is well defined by the formula

$$\mathcal{H}(\mu|\gamma) := \mathcal{H}(\mu|\tilde\gamma) - \int_{\mathbb{R}^d} V(x)\, d\mu(x) \in (-\infty, +\infty] \qquad \forall\, \mu \in \mathscr{P}_2(\mathbb{R}^d). \tag{3.34}$$

In particular, when γ is the d-dimensional Lebesgue measure, we find the standard entropy functional introduced in (3.23).

More generally, we can consider a

$$\begin{gathered} \text{proper, l.s.c., convex function } F : [0, +\infty) \to [0, +\infty] \\ \text{with superlinear growth} \end{gathered} \tag{3.35}$$

and the related functional

$$\mathcal{F}(\mu|\gamma) := \begin{cases} \displaystyle\int_{\mathbb{R}^d} F\Big(\frac{d\mu}{d\gamma}\Big)\, d\gamma & \text{if } \mu \ll \gamma, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.36}$$

Lemma 3.17 (Joint lower semicontinuity) *Let (γ^n), $(\mu^n) \subset \mathscr{P}(\mathbb{R}^d)$ be two sequences narrowly converging to γ, μ in $\mathscr{P}(\mathbb{R}^d)$. Then*

$$\liminf_{n\to\infty} \mathcal{H}(\mu^n|\gamma^n) \ge \mathcal{H}(\mu|\gamma), \qquad \liminf_{n\to\infty} \mathcal{F}(\mu^n|\gamma^n) \ge \mathcal{F}(\mu|\gamma). \tag{3.37}$$

The proof of this lemma follows easily from the next representation formula; before stating it, we need to introduce the conjugate function of F

$$F^*(s^*) := \sup_{s\ge 0} s \cdot s^* - F(s) < +\infty \quad \forall\, s^* \in \mathbb{R}, \tag{3.38}$$

so that

$$F(s) = \sup_{s^*\in\mathbb{R}} s^* \cdot s - F^*(s^*); \tag{3.39}$$

if $s_0 \ge 0$ is a minimizer of F then

$$F^*(s^*) \ge s^* s_0 - F(s_0), \quad s \ge s_0 \quad \Rightarrow \quad F(s) = \sup_{s^*\ge 0} s^* \cdot s - F^*(s^*). \tag{3.40}$$

In the case of the entropy functional, we have $H^*(s^*) = e^{s^*} - 1$. Now we recall a classical duality formula for functionals defined on measures; we recall its proof for the reader's convenience.

Lemma 3.18 (Duality formula) *For any γ, $\mu \in \mathscr{P}(\mathbb{R}^d)$ we have*

$$\mathcal{F}(\mu|\gamma) = \sup\Big\{ \int_{\mathbb{R}^d} S^*(x)\, d\mu(x) - \int_{\mathbb{R}^d} F^*(S^*(x))\, d\gamma(x) : S^* \in C^0_b(\mathbb{R}^d) \Big\}. \tag{3.41}$$

Proof. Up to an addition of a constant, we can always assume $F^*(0) = -\min_{s\geq 0} F(s) = -F(s_0) = 0$. Let us denote by $\mathcal{F}'(\mu|\gamma)$ the right hand side of (3.41). It is obvious that $\mathcal{F}'(\mu|\gamma) \leq \mathcal{H}(\mu|\gamma)$, so that we have to prove only the converse inequality.

First of all we show that $\mathcal{F}'(\mu|\gamma) < +\infty$ yields that $\mu \ll \gamma$. For, let us fix s^*, $\varepsilon > 0$ and a Borel set A with $\gamma(A) \leq \varepsilon/2$. Since μ, γ are finite measures we can find a compact set $K \subset A$, an open set $G \supset A$ and a continuous function $\zeta : \mathbb{R}^d \to [0, s^*]$ such that

$$\mu(G \setminus K) \leq \varepsilon, \quad \gamma(G) \leq \varepsilon, \quad \zeta(x) = s^* \quad \text{on } K, \quad \zeta(x) = 0 \quad \text{on } \mathbb{R}^d \setminus G.$$

Since F^* is increasing (by the definition (3.38)) and $F^*(0) = 0$, we have

$$\begin{aligned} s^*\mu(K) - F^*(s^*)\varepsilon &\leq \int_K \zeta(x)\, d\mu(x) - \int_G F^*(\zeta(x))\, d\gamma(x) \\ &\leq \int_{\mathbb{R}^d} \zeta(x)\, d\mu(x) - \int_{\mathbb{R}^d} F^*(\zeta(x))\, d\gamma(x) \leq \mathcal{F}'(\mu|\gamma) \end{aligned}$$

Taking the supremum w.r.t. $K \subset A$ and $s^* \geq 0$, and using (3.40) we get

$$\varepsilon F\big(\mu(A)/\varepsilon\big) \leq \mathcal{F}'(\mu|\gamma) \quad \text{if } \mu(A) \geq \varepsilon s_0.$$

Since $F(s)$ has a superlinear growth as $s \to +\infty$, we conclude that $\mu(A) \to 0$ as $\varepsilon \downarrow 0$.

Now we can suppose that $\mu = \rho \cdot \gamma$ for some Borel function $\rho \in L^1(\gamma)$, so that

$$\mathcal{F}'(\mu|\gamma) = \sup\Big\{ \int_{\mathbb{R}^d} \big(S^*(x)\rho(x) - F^*(S^*(x))\big)\, d\gamma(x) : S^* \in C^0_b(\mathbb{R}^d)\Big\}$$

and, for a suitable dense countable set $C = \{s^*_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$

$$\begin{aligned} \mathcal{F}(\mu|\gamma) &= \int_{\mathbb{R}^d} \sup_{s^*\in C} \big(s^*\rho(x) - F^*(s^*)\big)\, d\gamma(x) \\ &= \lim_{k\to\infty} \int_{\mathbb{R}^d} \sup_{s^*\in C_k} \big(s^*\rho(x) - F^*(s^*)\big)\, d\gamma(x) \end{aligned}$$

where $C_k = \{s^*_1, \cdots, s^*_k\}$. Our thesis follows if we show that for every k

$$\int_{\mathbb{R}^d} \max_{s^*\in C_k} \big(s^*\rho(x) - F^*(s^*)\big)\, d\gamma(x) \leq \mathcal{F}'(\mu|\gamma). \tag{3.42}$$

For we call

$$A_j = \Big\{x \in \mathbb{R}^d : s^*_j\rho(x) - F^*(s^*_j) \geq s^*_i\rho(x) - F^*(s^*_i) \quad \forall\, i \in \{1, \dots, k\}\Big\},$$

and

$$A'_1 = A_1, \quad A'_{j+1} = A_{j+1} \setminus \Big(\bigcup_{i=1}^j A_i\Big).$$

We find compact sets $K_j \subset A'_j$, open sets $G_j \supset A_j$ with $G_j \cap K_i = \varnothing$ if $i \neq j$, and continuous functions ζ_j such that

$$\sum_{j=1}^{k} \gamma(G_j \setminus K_j) + \mu(G_j \setminus K_j) \le \varepsilon, \quad \zeta_j \equiv s^*_j \text{ on } K_j, \quad \zeta_j \equiv 0 \text{ on } \mathbb{R}^d \setminus G_j.$$

Denoting by $\zeta := \sum_{j=1}^k \zeta_j$, $M := \sum_{j=1}^k |s^*_j|$, since the negative part of $F^*(s^*)$ is bounded above by $|s^*|s_0$ we have

$$\begin{aligned}
\int_{\mathbb{R}^d} \max_{s^*\in C_k} \big(s^*\rho(x) - F^*(s^*)\big)\, d\gamma(x) &= \sum_{j=1}^{k} \int_{A'_j} \big(s^*_j\rho(x) - F^*(s^*_j)\big)\, d\gamma(x) \\
&\le \sum_{j=1}^{k} \int_{K_j} \big(s^*_j\rho(x) - F^*(s^*_j)\big)\, d\gamma(x) + \varepsilon(M + Ms_0) \\
&= \sum_{j=1}^{k} \int_{K_j} \big(\zeta(x)\rho(x) - F^*(\zeta(x))\big)\, d\gamma(x) + \varepsilon(M + Ms_0) \\
&\le \int_{\mathbb{R}^d} \big(\zeta(x)\rho(x) - F^*(\zeta(x))\big)\, d\gamma(x) + \varepsilon(M + Ms_0 + M + F^*(M)).
\end{aligned}$$

Passing to the limit as $\varepsilon \downarrow 0$ we get (3.42). $\square$

3.4 Log-concavity and displacement convexity

We want to characterize the probability measures γ inducing a geodesically convex relative entropy functional $\mathcal{H}(\cdot|\gamma)$ in $\mathscr{P}_2(\mathbb{R}^d)$. The following lemma provides the first crucial property; the argument is strictly related to the proof of the Brunn-Minkowski inequality for the Lebesgue measure, obtained via optimal transportation inequalities [86]. See also [18] for the link between log-concavity and representation formulae like (3.50).

Lemma 3.19 (γ is log-concave if $\mathcal{H}(\cdot|\gamma)$ is displacement convex) *Suppose that for each couple of probability measures μ^1, $\mu^2 \in \mathscr{P}(\mathbb{R}^d)$ with bounded support there exists $\boldsymbol{\mu} \in \Gamma(\mu^1, \mu^2)$ such that $\mathcal{H}(\cdot|\gamma)$ is convex along the interpolating curve $\mu_t^{1\to 2} = \big((1-t)\pi^1 + t\pi^2\big)_{\#}\boldsymbol{\mu}$, $t \in [0,1]$. Then for each couple of open sets A, $B \subset \mathbb{R}^d$ and $t \in [0,1]$ we have*

$$\log \gamma((1-t)A + tB) \ge (1-t)\log\gamma(A) + t\log\gamma(B). \tag{3.43}$$

Proof. We can obviously assume that $\gamma(A) > 0$, $\gamma(B) > 0$ in (3.43); we consider

$$\mu^1 := \gamma(\cdot|A) = \frac{1}{\gamma(A)}\chi_A \cdot \gamma, \quad \mu^2 := \gamma(\cdot|B) = \frac{1}{\gamma(B)}\chi_B \cdot \gamma,$$

observing that

$$\mathcal{H}(\mu^1|\gamma) = -\log\gamma(A), \quad \mathcal{H}(\mu^2|\gamma) = -\log\gamma(B). \tag{3.44}$$

If $\mu_t^{1\to 2}$ is induced by a transfer plan $\boldsymbol{\mu} \in \Gamma(\mu^1, \mu^2)$ along which the relative entropy is displacement convex, we have

$$\mathcal{H}(\mu_t^{1\to 2}|\gamma) \le (1-t)\mathcal{H}(\mu^1|\gamma) + t\mathcal{H}(\mu^2|\gamma) = -(1-t)\log\gamma(A) - t\log\gamma(B).$$

On the other hand the measure $\mu_t^{1\to 2}$ is concentrated on $(1-t)A + tB = \pi_t^{1\to 2}(A\times B)$ and the next lemma shows that

$$-\log\gamma((1-t)A + tB) \le \mathcal{H}(\mu_t^{1\to 2}|\gamma). \quad \square$$

Lemma 3.20 (Relative entropy of concentrated measures) Let γ, $\mu \in \mathscr{P}(\mathbb{R}^d)$; if μ is concentrated on a Borel set A, i.e. $\mu(\mathbb{R}^d \setminus A) = 0$, then

$$\mathcal{H}(\mu|\gamma) \ge -\log\gamma(A). \tag{3.45}$$

Proof. It is not restrictive to assume $\mu \ll \gamma$ and $\gamma(A) > 0$; denoting by γ_A the probability measure $\gamma(\cdot|A) := \gamma(A)^{-1}\chi_A \cdot \gamma$, we have

$$\begin{aligned}
\mathcal{H}(\mu|\gamma) &= \int_{\mathbb{R}^d} \log\Big(\frac{d\mu}{d\gamma}\Big)\,d\mu = \int_A \log\Big(\frac{d\mu}{d\gamma_A}\cdot\frac{1}{\gamma(A)}\Big)\,d\mu \\
&= \int_A \log\Big(\frac{d\mu}{d\gamma_A}\Big)\,d\mu - \int_A \log\big(\gamma(A)\big)\,d\mu = \mathcal{H}(\mu|\gamma_A) - \log\big(\gamma(A)\big) \\
&\ge -\log\big(\gamma(A)\big). \quad \square
\end{aligned}$$

The previous results justifies the following definition:

Definition 3.21 (log-concavity of a measure) *We say that a Borel probability measure $\gamma \in \mathscr{P}(\mathbb{R}^d)$ is* log-concave *if for every couple of open sets A, $B \subset \mathbb{R}^d$ we have*

$$\log\gamma((1-t)A + tB) \ge (1-t)\log\gamma(A) + t\log\gamma(B). \tag{3.46}$$

In Definition 3.21 and also in the previous theorem we confined ourselves to pairs of open sets, to avoid the non trivial issue of the measurability of $(1-t)A + tB$ when A and B are only Borel (in fact, it is an open set whenever A and B are open). Observe that a log-concave measure γ in particular satisfies

$$\log\gamma(B_r((1-t)x_0 + tx_1)) \ge (1-t)\log\gamma(B_r(x_0)) + t\log\gamma(B_r(x_1)), \tag{3.47}$$

for every couple of points x_0, $x_1 \in \mathbb{R}^d$, $r > 0$, $t \in [0,1]$.

We want to show that in fact log concavity is equivalent to the geodesic convexity of the Relative Entropy functional $\mathcal{H}(\cdot|\gamma)$.

Let us first recall some elementary properties of convex sets in $\mathbb{R}^d$. Let $C \subset \mathbb{R}^d$ be a convex set; the *affine dimension* $\dim C$ of C is the linear dimension of its affine envelope

$$\operatorname{aff} C = \Big\{(1-t)x_0 + tx_1 : x_0, x_1 \in C,\ t \in \mathbb{R}\Big\}, \tag{3.48}$$

which is an affine subspace of $\mathbb{R}^d$. We denote by $\operatorname{int} C$ the relative interior of C as a subset of $\operatorname{aff} C$: it is possible to show that

$$\operatorname{int} C \neq \varnothing, \quad \overline{\operatorname{int} C} = \overline{C}, \quad \mathscr{H}^k(\overline{C} \setminus \operatorname{int} C) = 0 \quad \text{if } k = \dim C, \tag{3.49}$$

where $\mathscr{H}^k$ is the k-dimensional Hausdorff measure in $\mathbb{R}^d$. The previous theorem shows that log-concavity of γ is equivalent to the convexity of $\mathcal{H}(\mu|\gamma)$ along geodesics of the Wasserstein space $\mathscr{P}_2(\mathbb{R}^d)$: the link between these two concepts is provided by the representation formula (3.50).

Theorem 3.22 *Let us suppose that $\gamma \in \mathscr{P}(\mathbb{R}^d)$ satisfies the* log-concavity *assumptions on balls* (3.47). *Then $\operatorname{supp}\gamma$ is convex and there exists a convex l.s.c. function $V : \mathbb{R}^d \to (\infty, +\infty]$ such that*

$$\gamma = e^{-V} \cdot \mathscr{H} k|_{\operatorname{aff}(\operatorname{supp}\gamma)}, \quad \textit{where } k = \dim(\operatorname{supp}\gamma). \tag{3.50}$$

Conversely, if γ admits the representation (3.50) *then γ is* log-concave *and the relative entropy functional $\mathcal{H}(\cdot|\gamma)$ is convex along any geodesic of $\mathscr{P}_2(\mathbb{R}^d)$.*

Proof. Let us suppose that γ satisfies the log-concave inequality on balls and let k be the dimension of $\operatorname{aff}(\operatorname{supp}\gamma)$. Observe that the measure γ satisfies the same inequality (3.47) for the balls of $\operatorname{aff}(\operatorname{supp}\gamma)$: up to an isometric change of coordinates it is not restrictive to assume that $k = d$ and $\operatorname{aff}(\operatorname{supp}\gamma) = \mathbb{R}^d$.

Let us now introduce the set

$$D := \Big\{x \in \mathbb{R}^d : \liminf_{r\downarrow 0} \frac{\gamma(B_r(x))}{r^d} > 0\Big\}. \tag{3.51}$$

Since (3.47) yields

$$\frac{\gamma(B_r(x_t))}{r^k} \geq \left(\frac{\gamma(B_r(x_0))}{r^k}\right)^{1-t} \left(\frac{\gamma(B_r(x_1))}{r^k}\right)^t \quad t \in (0,1), \tag{3.52}$$

it is immediate to check that D is a convex subset of $\mathbb{R}^d$ with $D \subset \operatorname{supp}\gamma$.

General results on derivation of Radon measures in $\mathbb{R}^d$ (see for instance Theorem 2.56 in [8]) show that

$$\limsup_{r\downarrow 0} \frac{\gamma(B_r(x))}{r^d} < +\infty \quad \text{for } \mathscr{L}^d\text{-a.e. } x \in \mathbb{R}^d \tag{3.53}$$

and

$$\limsup_{r\downarrow 0} \frac{r^d}{\gamma(B_r(x))} < +\infty \quad \text{for } \gamma\text{-a.e. } x \in \mathbb{R}^d. \tag{3.54}$$

Using (3.54) we see that actually γ is concentrated on D (so that $\operatorname{supp}\gamma \subset \overline{D}$) and therefore, being d the dimension of $\operatorname{aff}(\operatorname{supp}\gamma)$, it follows that d is also the dimension of $\operatorname{aff}(D)$.

If a point $\bar{x} \in \mathbb{R}^d$ exists such that

$$\limsup_{r\downarrow 0} \frac{\gamma(B_r(\bar{x}))}{r^d} = +\infty,$$

then (3.52) forces every point of $\mathrm{int}(D)$ to verify the same property, but this would be in contradiction with (3.53), since we know that $\mathrm{int}(D)$ has strictly positive $\mathscr{L}^d$-measure. Therefore

$$\limsup_{r\downarrow 0} \frac{\gamma(B_r(x))}{r^d} < +\infty \quad \text{for all } x \in \mathbb{R}^d \tag{3.55}$$

and we obtain that $\gamma \ll \mathscr{L}^d$, again by the theory of derivation of Radon measures in $\mathbb{R}^d$. In the sequel we denote by g the density of γ w.r.t. $\mathscr{L}^d$ and notice that by Lebesgue differentiation theorem $g > 0$ $\mathscr{L}^d$-a.e. in D and $g = 0$ $\mathscr{L}^d$-a.e. in $\mathbb{R}^d \setminus D$.

By (3.47) the maps

$$V_r(x) = -\log\Big(\frac{\gamma(B_r(x))}{\omega_d r^d}\Big)$$

are convex on $\mathbb{R}^d$, and (3.55) gives that the family $V_r(x)$ is bounded as $r \downarrow 0$ for any $x \in D$. Using the pointwise boundedness of V_r on D and the convexity of V_r it is easy to show that V_r are locally equi-bounded (hence locally equi-continuous) on $\mathrm{int}(D)$ as $r \downarrow 0$. Let W be a limit point of V_r, with respect to the local uniform convergence, as $r \downarrow 0$: W is convex on $\mathrm{int}(D)$ and Lebesgue differentiation theorem shows that

$$\exists \lim_{r\downarrow 0} V_r(x) = -\log g(x) = W(x) \quad \text{for } \mathscr{L}^d\text{-a.e.} \quad x \in \mathrm{int}(D), \tag{3.56}$$

so that $\gamma = g\mathscr{L}^d = e^{-W}\chi_{\mathrm{int}(D)}\mathscr{L}^d$. In order to get a globally defined convex and l.s.c function V we extend W with the $+\infty$ value out of $\mathrm{int}(D)$ and define V to be its convex and l.s.c. envelope. It turns out that V coincides with W on $\mathrm{int}(D)$, so that still the representation $\gamma = e^{-V}\mathscr{L}^d$ holds.

Conversely, let us suppose that γ admits the representation (3.50) for a given convex l.s.c. function V and let μ^1, $\mu^2 \in \mathscr{P}_2(\mathbb{R}^d)$; if their relative entropies are finite then they are absolutely continuous w.r.t. γ and therefore their supports are contained in $\mathrm{aff}(\mathrm{supp}\,\gamma)$. It follows that the support of any optimal plan $\boldsymbol{\mu} \in \Gamma_o(\mu^1, \mu^2)$ in $\mathscr{P}_2(\mathbb{R}^d)$ is contained in $\mathrm{aff}(\mathrm{supp}\,\gamma) \times \mathrm{aff}(\mathrm{supp}\,\gamma)$: up to a linear isometric change of coordinates, it is not restrictive to suppose $\mathrm{aff}(\mathrm{supp}\,\gamma) = \mathbb{R}^d$, μ^1, $\mu^2 \in \mathscr{P}_2^a(\mathbb{R}^d)$, $\gamma = e^{-V} \cdot \mathscr{L}^d \in \mathscr{P}(\mathbb{R}^d)$.

In this case we introduce the densities u^i of μ^i w.r.t. $\mathscr{L}^d$, observing that

$$\frac{d\mu^i}{d\gamma} = u^i e^V \quad i = 1, 2,$$

where we adopted the convention $0 \cdot (+\infty) = 0$ (recall that $u^i(x) = 0$ for $\mathscr{L}^d$-a.e. $x \in \mathbb{R}^d \setminus D(V)$). Therefore the entropy functional can be written as

$$\mathcal{H}(\mu^i|\gamma) = \int_{\mathbb{R}^d} u^i(x) \log u^i(x)\, dx + \int_{\mathbb{R}^d} V(x)\, d\mu^i(x), \tag{3.57}$$

i.e. the sum of two geodesically convex functionals, as we proved discussing Examples 3.4 and Examples 3.8. Lemma 3.19 yields the log-concavity of γ.

□

If γ is log-concave and F satisfies (3.22), then all the integral functionals $\mathcal{F}(\cdot|\gamma)$ introduced in (3.36) are geodesically convex in $\mathscr{P}_2(\mathbb{R}^d)$.

Theorem 3.23 (Geodesic convexity for relative integral functionals) Suppose that γ is log-concave and $F : [0, +\infty) \to [0, +\infty]$ satisfies conditions (3.35) and (3.22). Then the integral functional $\mathcal{F}(\cdot|\gamma)$ is geodesically convex in $\mathscr{P}_2(\mathbb{R}^d)$.

Proof. Arguing as in the final part of the proof of Theorem 3.22 we can assume that $\gamma := e^{-V}\mathscr{L}^d$ for a convex l.s.c. function $V : \mathbb{R}^d \to (-\infty, +\infty]$ whose domain has not empty interior. For every couple of measures μ^1, $\mu^2 \in D(\mathcal{F}(\cdot|\gamma))$ we have

$$\mu^i = u^i e^V \cdot \gamma, \quad \mathcal{F}(\mu^i|\gamma) = \int_{\mathbb{R}^d} F(u^i(x)e^{V(x)})e^{-V(x)}\,dx, \quad i = 1, 2. \tag{3.58}$$

We denote by $\boldsymbol{r}$ the optimal transport map for the Wasserstein distance pushing μ^1 to μ^2 and we set $\boldsymbol{r}^t := (1-t)\boldsymbol{i} + t\boldsymbol{r}$, $\mu_t := (\boldsymbol{r}^t)_\#\mu^1$; arguing as in Proposition 3.11, we get

$$\mathcal{F}(\mu_t|\gamma) = \int_{\mathbb{R}^d} F\Big(\frac{u(x)e^{V(\boldsymbol{r}_t(x))}}{\det\nabla\boldsymbol{r}^t(x)}\Big) \det\nabla\boldsymbol{r}^t(x)e^{-V(\boldsymbol{r}_t(x))}\,dx, \tag{3.59}$$

and the integrand above may be seen as the composition of the convex and nonincreasing map $s \mapsto F(u(x)e^{-s})e^s$ with the concave curve

$$t \mapsto -V(\boldsymbol{r}_t(x)) + \log(\det\nabla\boldsymbol{r}_t(x)),$$

since $D(x) := \nabla\boldsymbol{r}(x)$ is a diagonalizable map with nonnegative eigenvalues and

$$t \mapsto \log\det\big((1-t)I + tD(x)\big) \quad \text{is concave in } [0,1].$$

$\square$

4 Subdifferential calculus in $\mathscr{P}_2(\mathbb{R}^d)$

Let X be an Hilbert space. In the classical theory of subdifferential calculus (see e.g. [22]) lower semicontinuous functionals $\phi : X \to (-\infty, +\infty]$ with

$$\text{proper domain} \quad D(\phi) := \Big\{ v \in X : \phi(v) < +\infty \Big\} \neq \emptyset, \tag{4.1}$$

the *Fréchet Subdifferential* $\partial\phi : X \to 2^X$ of ϕ is a multivalued operator defined as

$$\xi \in \partial\phi(v) \quad \iff \quad v \in D(\phi), \quad \liminf_{w \to v} \frac{\phi(w) - \phi(v) - \langle \xi, w - v \rangle}{|w - v|} \geq 0, \tag{4.2}$$

which we will also write in the equivalent form for $v \in D(\phi)$

$$\xi \in \partial\phi(v) \quad \iff \quad \phi(w) \geq \phi(v) + \langle \xi, w - v \rangle + o\big(|w - v|\big) \quad \text{as } w \to v. \tag{4.3}$$

As usual in multivalued analysis, the proper domain $D(\partial\phi) \subset D(\phi)$ is defined as the set of all $v \in X$ such that $\partial\phi(v) \neq \varnothing$; we will use this convention for all the multivalued operators we will introduce.

The metric counterpart of the Fréchet Subdifferential is represented by the *metric slope* of ϕ, which for every $v \in D(\phi)$ is defined by

$$|\partial\phi|(v) = \limsup_{w \to v} \frac{(\phi(v) - \phi(w))^+}{|w - v|}, \tag{4.4}$$

and can also be characterized by an asymptotic expansion similar to (4.3) for $s \geq 0$

$$s \geq |\partial\phi|(v) \quad \iff \quad \phi(w) \geq \phi(v) - s|w - v| + o\big(|w - v|\big) \quad \text{as } w \to v. \tag{4.5}$$

It is then immediate to check that

$$\xi \in \partial\phi(v) \quad \Longrightarrow \quad |\partial\phi|(v) \leq |\xi|. \tag{4.6}$$

The Fréchet subdifferential and the metric slope occur quite naturally in the Euler equations for minima of (smooth perturbation of) ϕ:

A. Euler equation for quadratic perturbations. If v_τ is a minimizer of

$$w \mapsto \Phi(\tau, v; w) := \phi(w) + \frac{1}{2\tau}|w - v|^2 \quad \text{for some} \quad \tau > 0, \ v \in X \tag{4.7}$$

then

$$v_\tau \in D(\partial\phi) \quad \text{and} \quad -\frac{v_\tau - v}{\tau} \in \partial\phi(v_\tau); \tag{4.8}$$

concerning the slope we easily get

$$v_\tau \in D(|\partial\phi|) \quad \text{and} \quad |\partial\phi|(v) \leq \frac{|v - v_\tau|}{\tau}. \tag{4.9}$$

For λ-***convex functionals*** the Fréchet subdifferential enjoys at least two other simple but fundamental properties, which play a crucial role in the corresponding variational theory of evolution equations:

B. Characterization by variational inequalities and monotonicity. If ϕ is λ-convex, then

$$\xi \in \partial\phi(v) \quad \Longleftrightarrow \quad \phi(w) \geq \phi(v) + \langle \xi, w - v\rangle + \frac{\lambda}{2}|w-v|^2 \quad \forall\, w \in D(\phi); \tag{4.10}$$

in particular,

$$\xi_i \in \partial\phi(v_i) \quad \Longrightarrow \quad \langle \xi_1 - \xi_2, v_1 - v_2\rangle \geq \lambda |v_1 - v_2|^2 \quad \forall\, v_1,\, v_2 \in D(\partial\phi). \tag{4.11}$$

As in (4.10), the slope of a λ-convex functional can also be characterized by a system of inequalities for $s \geq 0$

$$s \geq |\partial\phi|(v) \quad \Longleftrightarrow \quad \phi(w) \geq \phi(v) - s|w-v| + \frac{\lambda}{2}|w-v|^2 \quad \forall\, w \in D(\phi), \tag{4.12}$$

which can equivalently reformulated as

$$|\partial\phi|(v) = \sup_{w \neq v} \left(\frac{\phi(v) - \phi(w)}{|v-w|} + \frac{\lambda}{2}|v-w| \right)^+. \tag{4.13}$$

C. Convexity and strong-weak closure. [22, Chap. II, Ex. 2.3.4, Prop. 2.5] If ϕ is λ-convex, then $\partial\phi(v)$ is closed and convex, and for every sequences $(v_n) \subset X$, $(\xi_n) \subset X$ we have

$$\xi_n \in \partial\phi(v_n),\ v_n \to v,\ \xi_n \rightharpoonup \xi \quad \Longrightarrow \quad \xi \in \partial\phi(v), \quad \phi(v_n) \to \phi(v). \tag{4.14}$$

The slope is l.s.c.

$$v_n \to v \quad \Longrightarrow \quad \liminf_{n\to\infty} |\partial\phi|(v_n) \geq |\partial\phi|(v). \tag{4.15}$$

Modeled on the last property **C**, and following a terminology introduced by F.H. Clarke, see e.g. [80, Chap. 8], we say that a functional ϕ is *regular* if

$$\left.\begin{array}{c} \xi_n \in \partial\phi(v_n), \quad \varphi_n = \phi(v_n) \\ v_n \to v, \quad \xi_n \rightharpoonup \xi, \quad \varphi_n \to \varphi \end{array}\right\} \quad \Longrightarrow \quad \xi \in \partial\phi(v), \quad \varphi = \phi(v). \tag{4.16}$$

D. Minimal selection and slope. If ϕ is regular (in particular if ϕ is λ-convex) $|\partial\phi|(v)$ *is finite* if and only if $\partial\phi(v) \neq \emptyset$ and

$$|\partial\phi|(v) = \min\Big\{ |\xi| : \xi \in \partial\phi(v) \Big\}. \tag{4.17}$$

The inequality $\leq$ in (4.17) follows directly from (4.6). The other one is simple to check, using the Hahn-Banach theorem, in the λ-convex case. In the more general case when ϕ is regular, one can use the existence (proved even in a general metric setting in Lemma 3.1.5 of [9]) of an infinitesimal sequence $(\tau_n) \subset (0, +\infty)$ and minimizers v_n of $w \mapsto \phi(w) + |w - v|^2/2\tau_n$ such that $\phi(v_n) \to \phi(v)$ and

$$\lim_{n\to\infty} \frac{|v - v_n|}{\tau_n} = |\partial\phi|(v).$$

As $(v - v_n)/\tau_n \in \partial\phi(v_n)$ we can use the regularity property and a weak compactness argument to obtain $\xi \in \partial\phi(v)$ with $|\xi| \le |\partial\phi|(v)$.

E. Chain rule. If $v : (a, b) \to D(\phi)$ is a curve in X then

$$\frac{d}{dt}\phi(v(t)) = \langle \xi, v'(t) \rangle \quad \forall\, \xi \in \partial\phi(v(t)), \tag{4.18}$$

at each point t where v and $\phi \circ v$ are differentiable and $\partial\phi(v(t)) \neq \emptyset$. In particular (see [22, Chap. III, Lemma 3.3] and Corollary 2.4.10 in [9]) if ϕ is also λ-convex, $v \in AC(a, b; X)$, and

$$\int_a^b |\partial\phi|(v(t))|v'(t)|\, dt < +\infty, \tag{4.19}$$

then $\phi \circ v$ is absolutely continuous in (a, b) and (4.18) holds for $\mathscr{L}^1$-a.e. $t \in (a, b)$.

The aim of this section is to extend the notion of Fréchet subdifferentiability and these properties to the Wasserstein framework (see also [29] for related results).

4.1 Definition of the subdifferential for a.c. measures

In this section we focus our attention to functionals ϕ defined on $\mathscr{P}_2(\mathbb{R}^d)$. The formal mechanism for translating statements from the euclidean framework to the Wasserstein formalism is simple: if $\mu \leftrightarrow v$ is the reference point, scalar products $\langle \cdot, \cdot \rangle$ have to be intended in the reference Hilbert space $L^2(\mu; \mathbb{R}^d)$ (which contains the tangent space $\mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$) and displacement vectors $w - v$ corresponds to transport maps $\boldsymbol{t}_\mu^\nu - \boldsymbol{i}$, which is well defined if $\mu \in \mathscr{P}_2^a(\mathbb{R}^d)$. According to these two natural rules, the transposition of (4.2) yields:

Definition 4.1 (Fréchet subdifferential and metric slope) *Let us consider a functional $\phi : \mathscr{P}_2(\mathbb{R}^d) \to (-\infty, +\infty]$ and a measure $\mu \in D(\phi) \cap \mathscr{P}_2^a(\mathbb{R}^d)$. We say that $\boldsymbol{\xi} \in L^2(\mu; \mathbb{R}^d)$ belongs to the Fréchet subdifferential $\partial\phi(\mu)$ if*

$$\phi(\nu) - \phi(\mu) \ge \int_{\mathbb{R}^d} \langle \boldsymbol{\xi}(x), \boldsymbol{t}_\mu^\nu(x) - x \rangle\, d\mu(x) + o\big(W_2(\mu, \nu)\big). \tag{4.20}$$

When $\boldsymbol{\xi} \in \partial\phi(\mu)$ also satisfies

$$\phi(\boldsymbol{t}_\# \mu) - \phi(\mu) \ge \int_{\mathbb{R}^d} \langle \boldsymbol{\xi}(x), \boldsymbol{t}(x) - x \rangle\, d\mu(x) + o\big(\|\boldsymbol{t} - \boldsymbol{i}\|_{L^2(\mu;\mathbb{R}^d)}\big), \tag{4.21}$$

then we will say that $\boldsymbol{\xi}$ is a strong *subdifferential.*

It is obvious that $\partial\phi(\mu)$ is a closed convex subset of $L^2(\mu;\mathbb{R}^d)$; in fact, we could also impose that it is contained in the tangent space $\mathrm{Tan}_\mu\mathscr{P}_2(\mathbb{R}^d)$, since the vector $\boldsymbol{\xi}$ in (4.20) acts only on tangent vectors (see Theorem 2.22): for, if Π denotes the orthogonal projection onto $\mathrm{Tan}_\mu\mathscr{P}_2(\mathbb{R}^d)$ in $L^2(\mu;\mathbb{R}^d)$,

$$\boldsymbol{\xi} \in \partial\phi(\mu) \quad \Longrightarrow \quad \Pi\boldsymbol{\xi} \in \partial\phi(\mu). \tag{4.22}$$

It is interesting to note that elements in $\partial\phi(\mu) \cap \mathrm{Tan}_\mu\mathscr{P}_2(\mathbb{R}^d)$ are in fact *strong* subdifferentials.

Proposition 4.2 (Subdifferentials in $\mathrm{Tan}_\mu\mathscr{P}_2(\mathbb{R}^d)$ are strong) *Let $\mu \in D(\phi) \cap \mathscr{P}_2^a(\mathbb{R}^d)$ and let $\boldsymbol{\xi} \in \partial\phi(\mu) \cap \mathrm{Tan}_\mu\mathscr{P}_2(\mathbb{R}^d)$. Then $\boldsymbol{\xi}$ is a* strong *subdifferential.*

Proof. We argue by contradiction, and we assume that a constant $\delta > 0$ and a sequence $(\boldsymbol{s}_n) \subset L^2(\mu;\mathbb{R}^d)$ with $\varepsilon_n := \|\boldsymbol{s}_n - \boldsymbol{i}\|_{L^2(\mu;\mathbb{R}^d)} \to 0$ as $n \to \infty$ exist such that

$$\phi(\mu_n) - \phi(\mu) - \int_{\mathbb{R}^d} \langle \boldsymbol{\xi}, \boldsymbol{s}_n - \boldsymbol{i}\rangle \, d\mu \le -\delta\,\varepsilon_n, \quad \mu_n := (\boldsymbol{s}_n)_\#\mu. \tag{4.23}$$

Let us denote by $\boldsymbol{t}_n$ the optimal transport pushing μ onto μ_n: we know that

$$\|\boldsymbol{t}_n - \boldsymbol{i}\|_{L^2(\mu;\mathbb{R}^d)} = W_2(\mu,\mu_n) \le \varepsilon_n \to 0. \tag{4.24}$$

By the definition of subdifferential, there exists $n_0 \in \mathbb{N}$ such that for every $n \ge n_0$

$$\phi(\mu_n) - \phi(\mu) \ge \int_{\mathbb{R}^d} \langle \boldsymbol{\xi}, \boldsymbol{t}_n - \boldsymbol{i}\rangle \, d\mu - \frac{\delta}{2}\varepsilon_n;$$

combining with (4.23) we obtain

$$\int_{\mathbb{R}^d} \langle \boldsymbol{\xi}, \boldsymbol{t}_n - \boldsymbol{s}_n\rangle \, d\mu \le -\frac{\delta}{2}\varepsilon_n \qquad \forall\, n \ge n_0. \tag{4.25}$$

Up to an extraction of a suitable subsequence, we can assume that

$$\frac{\boldsymbol{s}_n - \boldsymbol{i}}{\varepsilon_n} \rightharpoonup \tilde{\boldsymbol{s}}, \quad \frac{\boldsymbol{t}_n - \boldsymbol{i}}{\varepsilon_n} \rightharpoonup \tilde{\boldsymbol{t}} \quad \text{weakly in } L^2(\mu;\mathbb{R}^d) \quad \text{as } n \to \infty; \tag{4.26}$$

by (4.25) we get

$$\int_{\mathbb{R}^d} \langle \boldsymbol{\xi}, \tilde{\boldsymbol{t}} - \tilde{\boldsymbol{s}}\rangle \, d\mu \le -\frac{\delta}{2} < 0. \tag{4.27}$$

On the other hand, for every function $\zeta \in C_c^\infty(\mathbb{R}^d)$, the global estimates

$$\zeta(y) - \zeta(x) \le \langle D\zeta(x), y - x\rangle + C|y-x|^2, \quad \zeta(x) - \zeta(y) \le \langle D\zeta(x), x - y\rangle + C|y-x|^2$$

for some constant $C \ge 0$ yield

$$\begin{aligned}
0 &= \int_{\mathbb{R}^d} \Big(\zeta(\boldsymbol{t}_n(x)) - \zeta(\boldsymbol{s}_n(x))\Big)\, d\mu(x) \le \int_{\mathbb{R}^d} \langle D\zeta(x), \boldsymbol{t}_n(x) - \boldsymbol{s}_n(x)\rangle \, d\mu(x) \\
&\quad + C\int_{\mathbb{R}^d} \Big(|\boldsymbol{s}_n(x) - x|^2 + |\boldsymbol{t}_n(x) - x|^2\Big)\, d\mu(x) \\
&\ge \int_{\mathbb{R}^d} \langle D\zeta(x), \boldsymbol{t}_n(x) - \boldsymbol{s}_n(x)\rangle \, d\mu(x) + 2C\varepsilon_n^2.
\end{aligned}$$

Dividing by ε_n and passing to the limit as $n \to \infty$ we get

$$\int_{\mathbb{R}^d} \langle D\zeta, \tilde{\boldsymbol{t}} - \tilde{\boldsymbol{s}} \rangle \, d\mu \geq 0 \qquad \forall \, \zeta \in C_c^\infty(\mathbb{R}^d). \tag{4.28}$$

Since the gradients of $C_c^\infty(\mathbb{R}^d)$ functions are dense in $\mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$, (4.28) contradicts (4.27). $\square$

The DeGiorgi's definition of the metric slope of ϕ is in fact common to functionals defined in arbitrary metric spaces [37].

Definition 4.3 (Metric slope) *Let us consider a functional $\phi : \mathscr{P}_2(\mathbb{R}^d) \to (-\infty, +\infty]$ and a measure $\mu \in D(\phi)$. The metric slope of ϕ at μ is defined by*

$$|\partial\phi|(\mu) = \limsup_{\nu \to \mu} \frac{(\phi(\nu) - \phi(\mu))^+}{W_2(\nu, \mu)}, \tag{4.29}$$

or, equivalently, by

$$\begin{aligned} |\partial\phi|(\mu) := \inf \Big\{ s \geq 0 : \ \phi(\nu) \geq \phi(\mu) - sW_2(\nu, \mu) + o\big(W_2(\nu, \mu)\big) \\ \text{as } W_2(\nu, \mu) \to 0 \Big\}. \end{aligned} \tag{4.30}$$

4.2 Subdifferential calculus in $\mathscr{P}_2^a(\mathbb{R}^d)$

We now try to reproduce in the Wasserstein framework the calculus properties for the subdifferential, we briefly discussed at the beginning of the present section.

In order to simplify some technical point, we are supposing that

$$\begin{gathered} \phi : \mathscr{P}_2(\mathbb{R}^d) \to (-\infty, +\infty] \text{ is proper and lower semicontinuous,} \\ \text{with } D(|\partial\phi|) \subset \mathscr{P}_2^a(\mathbb{R}^d), \end{gathered} \tag{4.31a}$$

and that for some $\tau_* > 0$ the functional

$$\begin{gathered} \nu \mapsto \Phi(\tau, \mu; \nu) = \frac{1}{2\tau} W_2^2(\mu, \nu) + \phi(\nu) \quad \text{admits at least} \\ \text{a minimum point } \mu_\tau, \text{ for all } \tau \in (0, \tau_*) \text{ and } \mu \in \mathscr{P}_2(\mathbb{R}^d). \end{gathered} \tag{4.31b}$$

Notice that $D(\phi) \subset \mathscr{P}_2^a(\mathbb{R}^d)$ is a sufficient but not necessary condition for (4.31a): the internal energy functionals induced by a class of sublinear functions F satisfy (4.31a), but have a domain strictly larger than $\mathscr{P}_2^a(\mathbb{R}^d)$ (see Theorem 10.4.8 of [9]).

A. Euler equation for quadratic perturbations. When we want to minimize the perturbed functional (4.31b) we get a result completely analogous to the euclidean one:

Lemma 4.4 *Let ϕ be satisfying* (4.31a,b)*. Each minimizer μ_τ of* (4.31b) *belongs to $\mu_\tau \in D(|\partial\phi|)$ and*

$$\frac{1}{\tau}\big(\boldsymbol{t}_{\mu_\tau}^{\mu} - \boldsymbol{i}\big) \in \partial\phi(\mu_\tau) \quad \textit{is a strong subdifferential.} \tag{4.32}$$

Proof. The minimality of μ_τ gives for every $\nu \in \mathscr{P}_2(\mathbb{R}^d)$

$$\begin{aligned}\phi(\nu) - \phi(\mu_\tau) &= \Phi(\tau,\mu;\nu) - \Phi(\tau,\mu;\mu_\tau) + \frac{1}{2\tau}\Big(W_2^2(\mu_\tau,\mu) - W_2^2(\nu,\mu)\Big)\\ &\geq \frac{1}{2\tau}\Big(W_2^2(\mu_\tau,\mu) - W_2^2(\nu,\mu)\Big) \qquad (4.33)\\ &\geq -\frac{1}{2\tau}W_2(\mu_\tau,\nu)\Big(W_2(\mu_\tau,\mu) + W_2(\nu,\mu)\Big). \qquad (4.34)\end{aligned}$$

Letting ν converge to μ_τ, (4.34) yields

$$|\partial\phi|(\mu_\tau) \leq \frac{W_2(\mu_\tau,\nu)}{\tau}. \tag{4.35}$$

By (4.31a) we get $\mu_\tau \in \mathscr{P}_2^a(\mathbb{R}^d)$; if $\nu = \boldsymbol{t}_\#\mu_\tau$ we have

$$W_2^2(\mu_\tau,\mu) = \int_{\mathbb{R}^d} |\boldsymbol{t}_{\mu_\tau}^{\mu}(x) - x|^2\,d\mu_\tau(x), \quad W_2^2(\nu,\mu) \quad \leq \int_{\mathbb{R}^d} |\boldsymbol{t}(x) - \boldsymbol{t}_{\mu_\tau}^{\mu}(x)|^2\,d\mu_\tau(x),$$

and therefore the elementary identity $\frac{1}{2}|a|^2 - \frac{1}{2}|b|^2 = \langle a, a-b\rangle - \frac{1}{2}|a-b|^2$ and (4.33) yield

$$\begin{aligned}\phi(\nu) - \phi(\mu_\tau) &\geq \frac{1}{2\tau}\int_{\mathbb{R}^d}\Big(|\boldsymbol{t}_{\mu_\tau}^{\mu}(x) - x|^2 - |\boldsymbol{t}_{\mu_\tau}^{\mu}(x) - \boldsymbol{t}(x)|^2\Big)\,d\mu_\tau(x)\\ &= \int_{\mathbb{R}^d}\Big(\frac{1}{\tau}\langle \boldsymbol{t}_{\mu_\tau}^{\mu}(x) - x, \boldsymbol{t}(x) - x\rangle - \frac{1}{2\tau}|\boldsymbol{t}(x) - x|^2\Big)\,d\mu_\tau(x)\\ &= \int_{\mathbb{R}^d}\frac{1}{\tau}\langle \boldsymbol{t}_{\mu_\tau}^{\mu}(x) - x, \boldsymbol{t}(x) - x\rangle\,d\mu_\tau(x) - \frac{1}{2\tau}\|\boldsymbol{t} - \boldsymbol{i}\|^2_{L^2(\mu_\tau;\mathbb{R}^d)}.\end{aligned}$$

We deduce $\frac{1}{\tau}\big(\boldsymbol{t}_{\mu_\tau}^{\mu} - \boldsymbol{i}\big) \in \partial\phi(\mu_\tau)$ and the strong subdifferentiability condition. □

The above result, though simple, is very useful and usually provides the first crucial information when one looks for the properties of solutions of the variational problem (4.31b). The nice argument which combines the minimality of μ_τ and the possibility to use any "test" transport map $\boldsymbol{t}$ to estimate $W_2^2(\boldsymbol{t}_\#\nu,\mu)$ was originally introduced by F. Otto.

4.3 The case of λ-convex functionals along geodesics

Let us now focus our attention to the case of a λ-convex functional:

$$\phi \quad \text{is } \lambda\text{-convex on geodesics, according to Definition 3.1.} \tag{4.36}$$

B. Characterization by Variational inequalities and monotonicity. *Suppose that ϕ satisfies* (4.31a,b) *and* (4.36). *Then a vector $\boldsymbol{\xi} \in L^2(\mu;\mathbb{R}^d)$ belongs to the Fréchet subdifferential of ϕ at μ iff*

$$\phi(\nu) - \phi(\mu) \geq \int_{\mathbb{R}^d} \langle \boldsymbol{\xi}(x), \boldsymbol{t}_\mu^\nu(x) - x\rangle \, d\mu(x) + \frac{\lambda}{2} W_2^2(\mu,\nu) \quad \forall\, \nu \in D(\phi). \tag{4.37}$$

In particular if $\boldsymbol{\xi}_i \in \partial\phi(\mu_i)$, $i = 1, 2$, and $\boldsymbol{t} = \boldsymbol{t}_{\mu_1}^{\mu_2}$ is the optimal transport map, then

$$\int_{\mathbb{R}^d} \langle \boldsymbol{\xi}_2(\boldsymbol{t}(x)) - \boldsymbol{\xi}_1(x), \boldsymbol{t}(x) - x\rangle \, d\mu_1(x) \geq \lambda W_2^2(\mu_1,\mu_2). \tag{4.38}$$

Concerning the slope of ϕ we have for every $s \geq 0$

$$s \geq |\partial\phi|(\mu) \quad \Longleftrightarrow \quad \phi(\nu) \geq \phi(\mu) - sW_2(\nu,\mu) + \frac{\lambda}{2} W_2^2(\nu,\mu) \quad \forall\, \nu \in D(\phi), \tag{4.39}$$

or, equivalently,

$$|\partial\phi|(\mu) = \sup_{\nu\neq\mu} \left(\frac{\phi(\mu)-\phi(\nu)}{W_2(\mu,\nu)} + \frac{\lambda}{2} W_2(\mu,\nu) \right)^+. \tag{4.40}$$

Proof. One implication of (4.37) and of (4.39) is trivial. To prove the other one, in the case of (4.37) suppose that $\boldsymbol{\xi} \in \partial\phi(\mu)$ and $\nu \in D(\phi)$; for $t \in [0,1]$ we set $\mu_t := (\boldsymbol{i} + t(\boldsymbol{t}_\mu^\nu - \boldsymbol{i}))_{\#}\mu$ and we recall that the λ-convexity yields

$$\frac{\phi(\mu_t) - \phi(\mu)}{t} \leq \phi(\nu) - \phi(\mu) - \frac{\lambda}{2}(1-t) W_2^2(\mu,\nu). \tag{4.41}$$

On the other hand, since $W_2(\mu,\mu_t) = tW_2(\mu,\nu)$, Fréchet differentiability yields

$$\begin{aligned}\liminf_{t\downarrow 0} \frac{\phi(\mu_t)-\phi(\mu)}{t} &\geq \liminf_{t\to 0^+} \frac{1}{t} \int_{\mathbb{R}^d} \langle \boldsymbol{\xi}(x), \boldsymbol{t}_\mu^{\mu_t}(x) - x\rangle \, d\mu(x) \\ &\geq \int_{\mathbb{R}^d} \langle \boldsymbol{\xi}(x), \boldsymbol{t}_\mu^\nu(x) - x\rangle \, d\mu(x),\end{aligned}$$

since $\boldsymbol{t}_\mu^{\mu_t}(x) = x + t(\boldsymbol{t}_\mu^\nu(x) - x)$.

In the case of the slope (4.39), (4.41) and the fact that

$$\liminf_{t\downarrow 0} \frac{\phi(\mu_t)-\phi(\mu)}{t} \geq -|\partial\phi|(\mu)\, W_2(\mu,\nu) \tag{4.42}$$

yield (4.39). □

A simple consequence of (4.40) is the lower semicontinuity of the slope:

$$\mu_n \to \mu \quad \text{in } \mathscr{P}_2(\mathbb{R}^d) \quad \Longrightarrow \quad \liminf_{n\to\infty} |\partial\phi|(\mu_n) \geq |\partial\phi|(\mu). \tag{4.43}$$

Indeed, if $\nu \neq \mu$ then $\nu \neq \mu_n$ for n large enough, hence

$$\liminf_{n\to\infty} \frac{\phi(\mu_n)-\phi(\nu)}{W_2(\mu_n,\nu)} + \lambda W_2(\mu_n,\nu) \geq \frac{\phi(\mu)-\phi(\nu)}{W_2(\mu,\nu)} + \lambda W_2(\mu,\nu).$$

By estimating the left hand side with $\liminf_n |\partial\phi|(\mu_n)$ and taking the supremum w.r.t. ν we obtain (4.43).

C. Convexity and strong-weak closure. The next step is to show the closure of the graph of $\partial\phi$: here one has to be careful in the meaning of the convergence of vectors $\boldsymbol{\xi}_n \in L^2(\mu_n;\mathbb{R}^m)$, which belongs to different L^2-spaces, and we will adopt the following natural one.

Definition 4.5 *Let $(\mu_n) \subset \mathscr{P}(\mathbb{R}^d)$ be narrowly converging to μ in $\mathscr{P}(\mathbb{R}^d)$ and let $\boldsymbol{v}_n \in L^1(\mu_n;\mathbb{R}^m)$. We say that $\boldsymbol{v}_n$ weakly converge to $\boldsymbol{v} \in L^1(\mu;\mathbb{R}^m)$ if*

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \zeta(x)\boldsymbol{v}_n(x)\,d\mu_n(x) = \int_{\mathbb{R}^d} \zeta(x)\boldsymbol{v}(x)\,d\mu(x) \qquad \forall \zeta \in C_c^\infty(\mathbb{R}^d). \tag{4.44}$$

Clearly, if $\|\boldsymbol{v}_n\|_{L^1(\mu_n;\mathbb{R}^m)}$ is bounded, a density argument shows that the convergence above is equivalent to the narrow convergence (i.e. in the duality with $C_b(\mathbb{R}^d)$) of the vector-valued measures $\boldsymbol{v}_n\mu_n$ to $\boldsymbol{v}\mu$. We now state (see [9, Theorem 5.4.4] for a more general statement) some basic properties of this convergence.

Theorem 4.6 *Let $(\mu_n) \subset \mathscr{P}_2(\mathbb{R}^d)$ be converging to μ in $\mathscr{P}_2(\mathbb{R}^d)$ and let $\boldsymbol{v}_n \in L^2(\mu_n;\mathbb{R}^m)$ be such that*

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d} |\boldsymbol{v}_n(x)|^2\,d\mu_n(x) < +\infty. \tag{4.45}$$

Then the sequence $(\boldsymbol{v}_n)$ has weak limit points as $n \to \infty$, and if $\boldsymbol{v}$ is any limit point, along some subsequence $n(k)$, we have

$$\int_{\mathbb{R}^d} |\boldsymbol{v}(x)|^2\,d\mu(x) \le \liminf_{k\to\infty} \int_{\mathbb{R}^d} |\boldsymbol{v}_{n(k)}(x)|^2\,d\mu_{n(k)}, \tag{4.46}$$

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} \langle \boldsymbol{v}_{n(k)}, \varphi\rangle\,d\mu_{n(k)}(x) = \int_{\mathbb{R}^d} \langle \boldsymbol{v}(x), \varphi\rangle\,d\mu(x), \tag{4.47}$$

for every continuous function $\varphi : \mathbb{R}^d \to \mathbb{R}^m$ with at most linear growth.

Proof. The first statement is a direct consequence of the lower semicontinuity of the relative entropy functional (3.36), in the case when $F(z) = z^2$, see Lemma 3.17 (here actually only the narrow convergence of the μ_n is needed). The convergence property (4.47) follows by a simple truncation argument, taking into account that, $|x|^2$ is uniformly integrable w.r.t. $\{\mu_n\}_{n\in\mathbb{N}}$. □

Lemma 4.7 (Closure of the subdifferential) *Let ϕ be a λ-convex functional satisfying (4.31a), let (μ_n) be converging to $\mu \in D(\phi)$ in $\mathscr{P}_2(\mathbb{R}^d)$, let $\boldsymbol{\xi}_n \in \partial\phi(\mu_n)$ be satisfying*

$$\sup_n \int_{\mathbb{R}^d} |\boldsymbol{\xi}_n(x)|^2\,d\mu_n(x) < +\infty, \tag{4.48}$$

and converging to $\boldsymbol{\xi}$ according to Definition 4.5. Then $\boldsymbol{\xi} \in \partial\phi(\mu)$.

Proof. Let $\nu \in D(\phi)$ and let C be the constant in (4.48). We have to pass to the limit as $n \to \infty$ in the subdifferential inequality

$$\phi(\nu) - \phi(\mu_n) \geq \int_{\mathbb{R}^d} \langle \boldsymbol{\xi}_n(x), \boldsymbol{t}_{\mu_n}^{\nu}(x) - x\rangle \, d\mu_n(x) + \frac{\lambda}{2} W_2^2(\mu_n, \nu). \tag{4.49}$$

By the lower semicontinuity of ϕ the upper limit of $\phi(\nu) - \phi(\mu_n)$ is less than $\phi(\nu) - \phi(\mu)$. Passing to the right hand side, given $\varepsilon > 0$ we choose $\bar{\boldsymbol{t}} \in C_b^0(\mathbb{R}^d; \mathbb{R}^d)$ such that $\|\boldsymbol{t}_\mu^\nu - \bar{\boldsymbol{t}}\|_{L^2(\mu;\mathbb{R}^d)} < \varepsilon^2$ and split the integrals as

$$\int_{\mathbb{R}^d} \langle \boldsymbol{\xi}_n(x), \boldsymbol{t}_{\mu_n}^{\nu}(x) - \bar{\boldsymbol{t}}(x)\rangle \, d\mu_n(x) + \int_{\mathbb{R}^d} \langle \boldsymbol{\xi}_n(x), \bar{\boldsymbol{t}}(x) - x\rangle \, d\mu_n(x). \tag{4.50}$$

By the Young inequality, the first integrals can be estimated with

$$\frac{C\varepsilon}{2} + \frac{1}{2\varepsilon} \limsup_{n\to\infty} \int_{\mathbb{R}^d} |\boldsymbol{t}_{\mu_n}^{\nu} - \bar{\boldsymbol{t}}|^2 \, d\mu_n = \frac{C\varepsilon}{2} + \frac{1}{2\varepsilon} \limsup_{n\to\infty} \int_{\mathbb{R}^d\times\mathbb{R}^d} |y - \bar{\boldsymbol{t}}(x)|^2 \, d\boldsymbol{\gamma}_n,$$

where $\boldsymbol{\gamma}_n = (\boldsymbol{i} \times \boldsymbol{t}_{\mu_n}^{\nu})_{\#}\mu_n$ are the optimal plans induced by $\boldsymbol{t}_{\mu_n}^{\nu}$. Now, by Proposition 7.1.3 of [9] (showing that optimal plans are stable under narrow convergence), we know that $\boldsymbol{\gamma}_n$ narrowly converge to the plan $\boldsymbol{\gamma} = (\boldsymbol{i} \times \boldsymbol{t}_\mu^\nu)_{\#}\mu$ induced by $\boldsymbol{t}_\mu^\nu$; moreover, as $|y|^2$ is uniformly integrable with respect to $\{\boldsymbol{\gamma}_n\}$ (because the second marginal of $\boldsymbol{\gamma}_n$ is constant), Lemma 1.2 gives that the upper limits above are less than

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} |y - \bar{\boldsymbol{t}}(x)|^2 \, d\boldsymbol{\gamma} = \int_{\mathbb{R}^d\times\mathbb{R}^d} |\boldsymbol{t}_\mu^\nu - \bar{\boldsymbol{t}}|^2 \, d\mu \leq \varepsilon^2.$$

Summing up, we proved that the limsup of the first integrals in (4.50) is less than $(C+1)\varepsilon/2$. The convergence of the second integrals in (4.50) to

$$\int_{\mathbb{R}^d} \langle \xi(x), \bar{\boldsymbol{t}}(x) - x\rangle \, d\mu(x)$$

follows directly from Theorem 4.6(i). As a consequence

$$\begin{aligned}\liminf_{n\to\infty} \int_{\mathbb{R}^d} \langle \boldsymbol{\xi}_n(x), \boldsymbol{t}_{\mu_n}^{\nu}(x) - x\rangle \, d\mu_n(x) &\geq \int_{\mathbb{R}^d} \langle \boldsymbol{\xi}(x), \boldsymbol{t}_\mu^\nu(x) - x\rangle \, d\mu(x) \\ &\quad - \frac{\varepsilon}{2}(C+1) - \int_{\mathbb{R}^d} |\xi(x)| \cdot |\bar{\boldsymbol{t}}(x) - \boldsymbol{t}_\mu^\nu| \, d\mu(x).\end{aligned}$$

As ε is arbitrary, the variational inequality (4.49) passes to the limit. □

4.4 Regular functionals

Definition 4.8 *A functional $\phi : \mathscr{P}_2(\mathbb{R}^d) \to (-\infty, +\infty]$ satisfying* (4.31a) *is* regular *if, whenever the* strong *subdifferentials $\boldsymbol{\xi}_n \in \partial\phi(\mu_n)$, $\varphi_n = \phi(\mu_n)$ satisfy*

$$\begin{cases} \mu_n \to \mu \text{ in } \mathscr{P}_2(\mathbb{R}^d), \quad \varphi_n \to \varphi, \quad \sup_n \|\boldsymbol{\xi}_n\|_{L^2(\mu_n;\mathbb{R}^d)} < +\infty \\ \boldsymbol{\xi}_n \to \boldsymbol{\xi} \text{ weakly, according to Definition 4.5,} \end{cases} \tag{4.51}$$

then $\boldsymbol{\xi} \in \partial\phi(\mu)$ and $\varphi = \phi(\mu)$.

We just proved that λ-convex functionals are indeed regular.

In the "differential" proof of the convergence of the implicit Euler scheme for gradient flows we will use the following time-dependent variant of Lemma 4.7 whose proof uses the same approximation arguments.

Remark 4.9 Let $\mu_t^n : [0,T] \to \mathscr{P}_2^a(\mathbb{R}^d)$ be uniformly bounded and pointwise converging in $[0,T]$ to $\mu_t : [0,T] \to \mathscr{P}_2^a(\mathbb{R}^d)$ as $n \to \infty$. Let $\boldsymbol{\xi}_n$, $\boldsymbol{\xi} : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be such that

$$\sup_n \int_0^T \int_{\mathbb{R}^d} |\boldsymbol{\xi}_n|^2 \, d\mu_t^n dt < +\infty$$

and

$$\lim_{n\to\infty} \int_0^T \int_{\mathbb{R}^d} \boldsymbol{\xi}_n \varphi \, d\mu_t^n dt = \int_0^T \int_{\mathbb{R}^d} \boldsymbol{\xi} \varphi \, d\mu_t dt \qquad \forall \varphi \in C_c^\infty\left((0,T) \times \mathbb{R}^d\right).$$

Then, for all $\nu \in \mathscr{P}_2(\mathbb{R}^d)$ we have

$$\lim_{n\to\infty} \int_0^T \int_{\mathbb{R}^d} \langle \boldsymbol{t}_{\mu_t^n}^\nu - \boldsymbol{i}, \boldsymbol{\xi}^n \rangle \, d\mu_t^n dt = \int_0^T \int_{\mathbb{R}^d} \langle \boldsymbol{t}_{\mu_t}^\nu - \boldsymbol{i}, \boldsymbol{\xi} \rangle \, d\mu_t dt.$$

D. Minimal selection and slope.

Lemma 4.10 *Let ϕ be a regular functional satisfying* (4.31a,b). *$\mu \in D(|\partial\phi|)$ if and only if $\partial\phi(\mu)$ is not empty and*

$$|\partial\phi|(\mu) = \min\Big\{ \|\boldsymbol{\xi}\|_{L^2(\mu;\mathbb{R}^d)} : \boldsymbol{\xi} \in \partial\phi(\mu) \Big\}, \tag{4.52}$$

where the metric slope $|\partial\phi|(\mu)$ is defined in (4.4).
By the convexity of $\partial\phi(\mu)$ there exists a unique vector $\boldsymbol{\xi} \in \partial\phi(\mu)$ which attains the minimum in (4.52)*: we will denote it by $\partial^\circ\phi(\mu)$, it belongs to* $\mathrm{Tan}_\mu\mathscr{P}_2(\mathbb{R}^d)$ *and it is also a* strong *subdifferential.*

Proof. It is clear from the very definition of Fréchet subdifferential that

$$|\partial\phi|(\mu) \le \|\boldsymbol{\xi}\|_{L^2(\mu;\mathbb{R}^d)} \quad \forall \boldsymbol{\xi} \in \partial\phi(\mu);$$

thus we should prove that if $|\partial\phi|(\mu) < +\infty$ there exists $\boldsymbol{\xi} \in \partial\phi(\mu)$ such that $\|\boldsymbol{\xi}\|_{L^2(\mu;\mathbb{R}^d)} \le |\partial\phi|(\mu)$. We argue by approximation: for $\mu \in D(|\partial\phi|)$ and $\tau \in (0,\tau_*)$, let μ_τ be a minimizer of (4.31b); by Lemma 4.4 we know that

$$\boldsymbol{\xi}_\tau = \frac{1}{\tau}\left(\boldsymbol{t}_{\mu_\tau}^\mu - \boldsymbol{i}\right) \in \partial\phi(\mu_\tau), \qquad \int_{\mathbb{R}^d} |\boldsymbol{\xi}_\tau(x)|^2 \, d\mu_\tau(x) = \frac{W_2^2(\mu,\mu_\tau)}{\tau^2},$$

and $\boldsymbol{\xi}_\tau$ is a strong subdifferential. Furthermore, it is proved in Lemma 3.1.5 of [9] (in a general metric space setting) that there exists a sequence $(\tau_n) \downarrow 0$ such that

$$\lim_{n\to\infty} \frac{W_2^2(\mu_{\tau_n},\mu)}{\tau^2} = |\partial\phi|^2(\mu). \tag{4.53}$$

By Theorem 4.6(i) we know that $\boldsymbol{\xi}_\tau$ has some limit point $\boldsymbol{\xi} \in L^2(\mu;\mathbb{R}^d)$ as $\tau \downarrow 0$, according to Definition 4.5. By (4.51) we get $\boldsymbol{\xi} \in \partial\phi(\mu)$ with $\|\boldsymbol{\xi}\|_{L^2(\mu;\mathbb{R}^d)} \le |\partial\phi|(\mu)$, so that $\boldsymbol{\xi}$ is the (unique) element of minimal norm in $\partial\phi(\mu)$.

By (4.22) we also deduce that $\boldsymbol{\xi} \in \mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$ and Proposition 4.2 shows that $\boldsymbol{\xi}$ is a strong subdifferential. □

Remark 4.11 (The λ-convex case) When ϕ satisfies the λ-convexity assumption (4.36), the proof of Property (4.53) is considerably easier, since μ_τ satisfies the a priori bound [9, Thm. 3.1.6]

$$(1+\lambda\tau)\frac{W_2(\mu_\tau,\mu)}{\tau} \le |\partial\phi|(\mu). \tag{4.54}$$

For, we choose $\mu_t := (\boldsymbol{i}+t(\boldsymbol{t}_\mu^{\mu_\tau}-\boldsymbol{i}))_\#\mu$ and we recall that λ-convexity of ϕ yields

$$\begin{aligned}\frac{1}{2\tau}W_2^2(\mu,\mu_\tau)+\phi(\mu_\tau) &\le \frac{1}{2\tau}W_2^2(\mu,\mu_t)+\phi(\mu_t)\\ &\le \frac{t}{2\tau}\Big(t-\lambda\tau\,(1-t)\Big)W_2^2(\mu,\mu_\tau)+(1-t)\phi(\mu)+t\phi(\mu_\tau).\end{aligned}$$

Since the right hand quadratic function has a minimum for $t=1$, taking the left derivative we obtain

$$\Big(\frac{\lambda}{2}+\frac{1}{\tau}\Big)W_2^2(\mu,\mu_\tau)+\phi(\mu_\tau)-\phi(\mu)\le 0,$$

and therefore, by (4.40)

$$\begin{aligned}\frac{1}{2}(1+\lambda\tau)\frac{W_2^2(\mu,\mu_\tau)}{\tau^2} &\le \frac{\phi(\mu)-\phi(\mu_\tau)}{\tau}-\frac{W_2^2(\mu,\mu_\tau)}{2\tau^2}\\ &\le |\partial\phi|(\mu)\frac{W_2(\mu_\tau,\mu)}{\tau}-(1+\lambda\tau)\frac{W_2^2(\mu_\tau,\mu)}{2\tau^2}\\ &\le \frac{1}{2(1+\lambda\tau)}|\partial\phi|^2(\mu),\end{aligned}$$

which yields (4.54).

E. Chain rule. *Let $\phi : \mathscr{P}_2(\mathbb{R}^d) \to (-\infty,+\infty]$ be a regular functional satisfying* (4.31a,b)*, and let $\mu : (a,b) \mapsto \mu_t \in D(\phi) \subset \mathscr{P}_2(\mathbb{R}^d)$ be an absolutely continuous curve with tangent velocity vector $\boldsymbol{v}_t$. Let $\Lambda \subset (a,b)$ be the set of points $t \in (a,b)$ such that*

- *(a)* $|\partial\phi|(\mu_t) < +\infty$;
- *(b)* $\phi\circ\mu$ *is differentiable at* t;
- *(c) condition* (2.56) *of Proposition 2.20 holds.*

Then

$$\frac{d}{dt}\phi(\mu_t) = \int_{\mathbb{R}^d} \langle \boldsymbol{\xi}_t(x), \boldsymbol{v}_t(x)\rangle \, d\mu_t(x) \quad \forall\, \boldsymbol{\xi}_t \in \partial\phi(\mu_t), \quad \forall t \in \Lambda. \tag{4.55}$$

Moreover, if ϕ is λ-convex along geodesics and

$$\int_a^b |\partial\phi|(\mu_t)|\mu'|(t)\, dt < +\infty, \tag{4.56}$$

then the map $t \mapsto \phi(\mu_t)$ is absolutely continuous, and $(a,b)\setminus\Lambda$ is $\mathscr{L}^1$-negligible.

Proof. Let $\bar t \in \Lambda$; observing that

$$\boldsymbol{v}_h := \frac{1}{h}\big(\boldsymbol{t}_{\mu_{\bar t}}^{\mu_{\bar t+h}} - \boldsymbol{i}\big) \to v_{\bar t} \quad \text{in } L^2(\mu_{\bar t};\mathbb{R}^d), \tag{4.57}$$

we have

$$\phi(\mu_{\bar t+h}) - \phi(\mu_{\bar t}) \geq h \int_{\mathbb{R}^d} \langle \boldsymbol{v}_h(x), \boldsymbol{\xi}_{\bar t}(x)\rangle \, d\mu_{\bar t}(x) + o(h). \tag{4.58}$$

Dividing by h and taking the right and left limits as $h \to 0$ we obtain that the left and right derivatives $d/dt_{\pm}\phi(\mu_t)$ satisfy

$$\begin{aligned}
\frac{d}{dt_+}\phi(\mu_t)\big|_{t=\bar t} &\geq \int_{\mathbb{R}^d} \langle v_{\bar t}(x), \boldsymbol{\xi}_{\bar t}(x)\rangle \, d\mu_{\bar t}(x),\\
\frac{d}{dt_-}\phi(\mu_t)\big|_{t=\bar t} &\leq \int_{\mathbb{R}^d} \langle v_{\bar t}(x), \boldsymbol{\xi}_{\bar t}(x)\rangle \, d\mu_{\bar t}(x)
\end{aligned}$$

and therefore we find (4.55).

In the λ-convex case, using (4.40) it can be shown (see Corollary 2.4.10 in [9]) that (4.56) implies that $t \mapsto \phi(\mu_t)$ is absolutely continuous in (a,b) and thus conditions (a,b,c) hold $\mathscr{L}^1$-a.e. in (a,b). $\square$

4.5 Examples of subdifferentials

In this section we consider in the detail the subdifferential of the convex functionals presented in Section 3.2 (potential energy, interaction energy, internal energy, negative Wasserstein distance), with a particular attention to the characterization of the elements with minimal norm.

We start by considering a general, but *smooth*, situation.

4.5.1 Variational integrals: the smooth case

In order to clarify the underlying structure of many examples and the link between the notion of Wasserstein subdifferential and the standard variational calculus for integral functionals, we first consider the case of a variational integral of the type

$$\mathscr{F}(\mu) := \begin{cases} \displaystyle\int_{\mathbb{R}^d} F(x, u(x), \nabla u(x))\, dx & \text{if } \mu = u\cdot\mathscr{L}^d \text{ with } u \in C^1(\mathbb{R}^d)\\ +\infty & \text{otherwise.}\end{cases} \tag{4.59}$$

Since we are not claiming any generality and we are only interested in the form of the subdifferential, we will assume enough regularity to justify all the computations; therefore, we suppose that $F : \mathbb{R}^d \times [0, +\infty) \times \mathbb{R}^d \to [0, +\infty)$ is a C^2 function with $F(x, 0, p) = 0$ for every x, $p \in \mathbb{R}^d$ and we consider the case of a smooth and strictly positive density u: as usual, we denote by $(x, z, p) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$ the variables of F and by $\delta\mathscr{F}/\delta u$ the first variation density

$$\frac{\delta\mathscr{F}}{\delta u}(x) := F_z(x, u(x), \nabla u(x)) - \nabla \cdot F_p(x, u(x), \nabla u(x)). \tag{4.60}$$

Lemma 4.12 *If $\mu = u \cdot \mathscr{L}^d \in \mathscr{P}_2^a(\mathbb{R}^d)$ with $u \in C^1(\mathbb{R}^d)$ satisfies $\mathscr{F}(\mu) < +\infty$ and $\boldsymbol{w} \in L^2(\mu; \mathbb{R}^d)$ belongs to the strong subdifferential of $\mathscr{F}$ at μ (in particular, by Proposition 4.2, if $\boldsymbol{w} \in \partial\phi(\mu) \cap \mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$), then*

$$\boldsymbol{w}(x) = \nabla \frac{\delta\mathscr{F}}{\delta u}(x) \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d, \tag{4.61}$$

and for every vector field $\boldsymbol{\xi} \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} \boldsymbol{w}(x) \cdot \boldsymbol{\xi}(x)\, d\mu(x) = -\int_{\mathbb{R}^d} \frac{\delta\mathscr{F}}{\delta u}(x)\, \nabla \cdot \big(u(x)\boldsymbol{\xi}(x)\big)\, dx. \tag{4.62}$$

Proof. We take a smooth vector field $\boldsymbol{\xi} \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^d)$ and we set for $\varepsilon \in \mathbb{R}$ sufficiently small $\mu_\varepsilon := (\boldsymbol{i} + \varepsilon\boldsymbol{\xi})_{\#}\mu$. If $\boldsymbol{w}$ is a strong subdifferential, we know that

$$\limsup_{\varepsilon \uparrow 0} \frac{\mathscr{F}(\mu_\varepsilon) - \mathscr{F}(\mu)}{\varepsilon} \le \int_{\mathbb{R}^d} \boldsymbol{w}(x) \cdot \boldsymbol{\xi}(x)\, d\mu(x) \le \liminf_{\varepsilon \downarrow 0} \frac{\mathscr{F}(\mu_\varepsilon) - \mathscr{F}(\mu)}{\varepsilon}; \tag{4.63}$$

on the other hand, by the change of variables formula we know that $\mu_\varepsilon = u_\varepsilon \mathscr{L}^d$ with

$$u_\varepsilon(y) = \frac{u}{\det(I + \varepsilon\nabla\boldsymbol{\xi})} \circ \big(\boldsymbol{i} + \varepsilon\boldsymbol{\xi}\big)^{-1}(y) \quad \forall\, y \in \mathbb{R}^d. \tag{4.64}$$

The map $(x, \varepsilon) \mapsto u_\varepsilon(x)$ is of class C^2 with $u_\varepsilon(x) = u(x)$ outside a compact set and

$$u_\varepsilon(x)|_{\varepsilon=0} = u(x), \qquad \frac{\partial u_\varepsilon(x)}{\partial \varepsilon}|_{\varepsilon=0} = -\nabla \cdot \big(u(x)\boldsymbol{\xi}(x)\big). \tag{4.65}$$

Standard variational formulae (see e.g. [53, Vol. I, 1.2.1]) yield

$$\lim_{\varepsilon \to 0} \frac{\mathscr{F}(\mu_\varepsilon) - \mathscr{F}(\mu)}{\varepsilon} = -\int_{\mathbb{R}^d} \frac{\delta\mathscr{F}}{\delta u}(x)\, \nabla \cdot \big(u(x)\boldsymbol{\xi}(x)\big)\, dx, \tag{4.66}$$

which shows (4.62).

Let us now suppose that $\boldsymbol{w} \in \partial\mathscr{F}(\mu) \cap \mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$; then (4.66) holds whenever $\boldsymbol{i} + \varepsilon\boldsymbol{\xi}$ is, an optimal transport map for $|\varepsilon|$ small enough, and in particular for gradient vector fields $\boldsymbol{\xi} = \nabla\zeta$ with $\zeta \in C_c^\infty(\mathbb{R}^d)$. Since $\mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$ is the closure in $L^2(\mu; \mathbb{R}^d)$ of the space of such gradients, we have

$$\int_{\mathbb{R}^d} \boldsymbol{w}(x) \cdot \boldsymbol{\xi}(x)\, d\mu(x) = -\int_{\mathbb{R}^d} \nabla \frac{\delta\mathscr{F}}{\delta u}(x) \cdot \boldsymbol{\xi}(x)\, d\mu(x) \quad \forall\, \boldsymbol{\xi} \in \mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d). \tag{4.67}$$

We obtain (4.61) noticing that $\delta\mathscr{F}/\delta u \in \mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$, by the assumption that $u \in C_c^2(\mathbb{R}^d)$. $\square$

4.5.2 The potential energy

Let $V : \mathbb{R}^d \to (-\infty, +\infty]$ be a proper, l.s.c. and λ-convex functional and let $\mathcal{V}(\mu) = \int_{\mathbb{R}^d} V\, d\mu$ be defined on $\mathscr{P}_2(\mathbb{R}^d)$. We denote by $\operatorname{graph} \partial V$ the graph of the Fréchet subdifferential of V in $\mathbb{R}^d \times \mathbb{R}^d$, i.e. the subset of the couples $(x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d$ satisfying

$$V(x_3) \geq V(x_1) + \langle x_2, x_3 - x_1 \rangle + \frac{\lambda}{2}|x_1 - x_2|^2 \quad \forall\, x_3 \in \mathbb{R}^d. \tag{4.68}$$

As usual, $\partial^o V(x)$ denotes the element of minimal norm in $\partial V(x)$.

Notice that the potential energy functional (as well as the interaction energy functional) fails to satisfy (4.31a), and for this reason it would be more appropriate to consider a more general notion of subdifferential, involving plans and not only maps as elements of the subdifferential, and, at the same time, takes onto account transport plans and not only transport maps (see §10.3 of [9]).

In the present case, we choose an intermediate generalization, and say that $\boldsymbol{\xi} \in L^2(\mu; \mathbb{R}^d)$ belongs to the Fréchet subdifferential $\partial \mathcal{V}(\mu)$ at $\mu \in D(\mathcal{V})$ if

$$\mathcal{V}(\nu) - \mathcal{V}(\mu) \geq \inf_{\gamma \in \Gamma_o(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle \boldsymbol{\xi}(x), y - x \rangle \, d\gamma(x) + o\big(W_2(\mu, \nu)\big). \tag{4.69}$$

The following characterization of $\partial \mathcal{V}$ and of its minimal selection is proved in Proposition 10.4.2 of [9].

Proposition 4.13 *Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ and $\xi \in L^2(\mu; \mathbb{R}^d)$. Then*

(i) ξ is a subdifferential of $\mathcal{V}$ at μ iff $\xi(x) \in \partial V(x)$ for μ-a.e. x.

(ii) $\partial^\circ \mathcal{V}(\mu) = \partial^\circ V(x)$ for μ-a.e. $x \in \mathbb{R}^d$.

4.5.3 The internal energy

Let $\mathcal{F}$ be the functional

$$\mathcal{F}(\mu) := \begin{cases} \displaystyle\int_{\mathbb{R}^d} F(u(x))\, d\mathscr{L}^d(x) & \text{if } \mu = u \cdot \mathscr{L}^d \in \mathscr{P}_2^a(\mathbb{R}^d), \\ +\infty & \text{otherwise,} \end{cases} \tag{4.70}$$

for a convex differentiable function satisfying

$$F(0) = 0, \quad \liminf_{s \downarrow 0} \frac{F(s)}{s^\alpha} > -\infty \quad \text{for some } \alpha > \frac{d}{d+2} \tag{4.71}$$

as in Example 3.8. Recall that if F has superlinear growth at infinity then the functional $\mathcal{F}$ is l.s.c. with respect to the narrow convergence (indeed, under this growth condition the lower semicontinuity can be checked w.r.t. to the stronger weak L^1 convergence, by Dunford-Pettis theorem, and lower semicontinuity w.r.t. weak L^1 convergence is a direct consequence of the convexity of $\mathcal{F}$).

We confine our discussion to the case when F has a more than linear growth at infinity, i.e.

$$\lim_{z\to+\infty}\frac{F(z)}{z}=+\infty, \tag{4.72}$$

see Theorem 10.4.6 and Theorem 10.4.8 of [9] for a discussion of the (sub)linear case.

We set $L_F(z)=zF'(z)-F(z):[0,+\infty)\to[0,+\infty)$ and we observe that L_F is strictly related to the *convex* function

$$G(z,s):=sF(z/s),\quad z\in[0,+\infty),\ s\in(0,+\infty), \tag{4.73}$$

since

$$\frac{\partial}{\partial s}G(z,s)=-\frac{z}{s}F'(z/s)+F(z/s)=-L_F(z/s). \tag{4.74}$$

In particular (recall that $F(0)=0$, by (4.71))

$$G(z,s)\le F(z)\quad\text{for } s\ge 1,\qquad \frac{F(z)-G(z,s)}{s-1}\uparrow L_F(z)\quad\text{as } s\downarrow 1. \tag{4.75}$$

We will also suppose that F satisfies the condition

$$\text{the map}\quad s\mapsto s^dF(s^{-d})\quad\text{is convex and non increasing in }(0,+\infty), \tag{4.76}$$

yielding the geodesic convexity of $\mathcal{F}$.

The following lemma shows the existence of the directional derivative of $\mathcal{F}$ along a suitable class of directions including all optimal transport maps.

Lemma 4.14 (Directional derivative of $\mathcal{F}$) *Suppose that $F:[0,+\infty)\to\mathbb{R}$ is a convex differentiable function satisfying* (4.71), (4.72) *and* (4.76)*. Let $\mu=u\mathscr{L}^d\in D(\mathcal{F})$, $\boldsymbol{r}\in L^2(\mu;\mathbb{R}^d)$ and $\bar t>0$ be such that*

(i) $\boldsymbol{r}$ is differentiable $u\mathscr{L}^d$-a.e. and $\boldsymbol{r}_t:=(1-t)\boldsymbol{i}+t\boldsymbol{r}$ is $u\mathscr{L}^d$-injective with $|\det\nabla\boldsymbol{r}_t(x)|>0$ $u\mathscr{L}^d$-a.e., for any $t\in[0,\bar t]$;

(ii) $\nabla\boldsymbol{r}_{\bar t}$ is diagonalizable with positive eigenvalues;

(iii) $\mathcal{F}((\boldsymbol{r}_{\bar t})_\#\mu)<+\infty$.

Then the map $t\mapsto t^{-1}\big(\mathcal{F}((\boldsymbol{r}_t)_\#\mu)-\mathcal{F}^*(\mu)\big)$ is nondecreasing in $[0,\bar t]$ and

$$+\infty>\lim_{t\downarrow 0}\frac{\mathcal{F}((\boldsymbol{r}_t)_\#\mu)-\mathcal{F}(\mu)}{t}=-\int_{\mathbb{R}^d}L_F(u)\mathrm{tr}\,\tilde\nabla(\boldsymbol{r}-\boldsymbol{i})\,dx. \tag{4.77}$$

The identity above still holds when assumptions (ii) on $\boldsymbol{r}$ is replaced by

(ii') $\|\tilde\nabla(\boldsymbol{r}-\boldsymbol{i})\|_{L^\infty(u\mathscr{L}^d;\mathbb{R}^{d\times d})}<+\infty$ (in particular if $\boldsymbol{r}-\boldsymbol{i}\in C^\infty_c(\mathbb{R}^d;\mathbb{R}^d)$),

and F satisfies in addition the "doubling" condition

$$\exists\, C>0:\quad F(z+w)\le C\big(1+F(z)+F(w)\big)\quad\forall\, z,\,w. \tag{4.78}$$

Proof. By assumptions (i) and (ii), taking into account Lemma 1.3 we have

$$\begin{aligned}\mathcal{F}((\boldsymbol{r}_t)_\#\mu) - \mathcal{F}(\mu) &= \int_{\mathbb{R}^d} F\left(\frac{u(x)}{\det \nabla \boldsymbol{r}_t(x)}\right) \det \nabla \boldsymbol{r}_t(x)\,dx - \int_{\mathbb{R}^d} F(u(x))\,dx \\ &= \int_{\mathbb{R}^d} \Big(G(u(x), \det \nabla \boldsymbol{r}_t(x)) - F(u(x))\Big)\,dx\end{aligned}$$

for any $t \in (0, \bar{t}]$. Assumption (4.76), together with the concavity of the map $t \mapsto [\det((1-t)I + t\nabla \boldsymbol{r})]^{1/d}$, implies that the function

$$\frac{G(u(x), \det \nabla \boldsymbol{r}_t) - F(u(x))}{t} \qquad t \in (0, \bar{t}] \tag{4.79}$$

is nondecreasing w.r.t. t and bounded above by an integrable function (take $t = \bar{t}$ and apply (iii)). Therefore the monotone convergence theorem gives

$$\lim_{t \downarrow 0} \frac{\mathcal{F}((\boldsymbol{r}_t)_\#\mu) - \mathcal{F}(\mu)}{t} = \int_{\mathbb{R}^d} \frac{d}{dt} G(u(x), \det \nabla \boldsymbol{r}_t(x))\big|_{t=0}\,dx$$

and the expansion $\det \nabla \boldsymbol{r}_t = 1 + t \operatorname{tr} \nabla(\boldsymbol{r} - \boldsymbol{i}) + o(t)$ together with (4.74) give the result.

In the case when (ii') holds, the argument is analogous but, since condition (ii) fails, we cannot rely anymore on the monotonicity of the function in (4.79). However, using the inequalities

$$F(w) - F(0) \leq wF'(w) \leq F(2w) - F(w)$$

and the doubling condition we easily see that the derivative w.r.t. s of the function $G(z, s)$ can be bounded by $C(1 + F^+(z))$ for $|s - 1| \leq 1/2$. Therefore we can use the dominated convergence theorem instead of the monotone convergence theorem to pass to the limit. □

The next technical lemma shows that we can "integrate by parts" in (4.77) preserving the inequality, if $L_F(u)$ is locally in $W^{1,1}$.

Lemma 4.15 (A "weak" integration by parts formula) *Under the same assumptions of Lemma 4.14, let us suppose that*

(i) $\operatorname{supp} \mu \subset \overline{\Omega}$, *$\Omega$ being a convex open subset of $\mathbb{R}^d$ (not necessarily bounded);*

(ii) $L_F(u) \in W^{1,1}_{\mathrm{loc}}(\Omega)$*;*

(iii) $\operatorname{supp}((\boldsymbol{r}_{\bar{t}})_\#\mu)$ *is a compact subset of Ω for some $\bar{t} \in [0, 1]$;*

(iv) $\boldsymbol{r} \in BV_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^d)$ *and* $D \cdot \boldsymbol{r} \geq 0$.

Then we can find an increasing family of nonnegative Lipschitz functions $\chi_k : \mathbb{R}^d \to [0, 1]$ with compact support in Ω such that $\chi_k \uparrow \chi_\Omega$ and

$$-\int_{\mathbb{R}^d} L_F(u(x)) \operatorname{tr} \nabla(\boldsymbol{r} - \boldsymbol{i})\,dx \geq \limsup_{k \to \infty} \int_{\mathbb{R}^d} \langle \nabla L_F(u), \boldsymbol{r} - \boldsymbol{i} \rangle \chi_k\,dx. \tag{4.80}$$

Proof. Possibly replacing $\boldsymbol{r}$ by $\boldsymbol{r}_{\bar{t}}$, we can assume that $\bar{t} = 1$ in (iii). Let us first recall that by Calderon-Zygmund theorem (see for instance [8]) the pointwise divergence $\operatorname{tr}(\nabla \boldsymbol{r})$ is the absolutely continuous part of the distributional divergence $D \cdot \boldsymbol{r}$; therefore we have

$$\int_{\mathbb{R}^d} v \operatorname{tr}(\nabla \boldsymbol{r})\, dx \le -\int_{\mathbb{R}^d} \langle \nabla v, \boldsymbol{r} \rangle\, dx, \tag{4.81}$$

provided $v \in C_c^\infty(\mathbb{R}^d)$ is nonnegative. As $\boldsymbol{r}$ is bounded, by approximation the same inequality remains true for every nonnegative function $v \in W^{1,1}(\mathbb{R}^d)$. For every Lipschitz function $\eta : \mathbb{R}^d \to [0,1]$ with compact support in Ω, choosing $v := \eta L_F(u) \in W^{1,1}(\mathbb{R}^d)$ we get

$$\int_{\mathbb{R}^d} \big(\eta L_F(u)\big) \operatorname{tr}(\nabla \boldsymbol{r})\, dx \le -\int_{\mathbb{R}^d} \langle \nabla\big(\eta L_F(u)\big), \boldsymbol{r} \rangle\, dx. \tag{4.82}$$

On the other hand, a standard integration by parts yields

$$\int_{\Omega} \big(\eta L_F(u)\big) \operatorname{tr}(\nabla \boldsymbol{i})\, dx = -\int_{\Omega} \langle \nabla\big(\eta L_F(u)\big), \boldsymbol{i} \rangle\, dx; \tag{4.83}$$

summing up with (4.82) and inverting the sign we find

$$-\int_{\mathbb{R}^d} \big(\eta L_F(u)\big) \operatorname{tr}(\nabla(\boldsymbol{r} - \boldsymbol{i}))\, dx \ge \int_{\mathbb{R}^d} \langle \nabla\big(\eta L_F(u)\big), \boldsymbol{r} - \boldsymbol{i} \rangle\, dx. \tag{4.84}$$

Now we choose carefully the test function η. We consider an increasing family bounded open convex sets Ω_k such that

$$\overline{\Omega_k} \subset\subset \Omega, \quad \Omega = \bigcup_{k=1}^{\infty} \Omega_k$$

and for each convex set Ω_k we consider the function

$$\chi_k(x) := k\, d(x, \mathbb{R}^d \setminus \Omega_k) \wedge 1. \tag{4.85}$$

χ_k is an increasing family of nonnegative Lipschitz functions which take their values in $[0,1]$ and satisfy $\chi_k(x) \equiv 1$ if $d(x, \mathbb{R}^d \setminus \Omega_k) \ge \frac{1}{k}$; in particular, $\chi_k \equiv 1$ in K for k sufficiently large. Moreover χ_k is concave in Ω_k, since the distance function $d(\cdot, \mathbb{R}^d \setminus \Omega_k)$ is concave. Choosing $\eta := \chi_k$ in (4.84) we get

$$\begin{aligned} -\int_{\mathbb{R}^d} \big(\chi_k L_F(u)\big) \operatorname{tr}(\nabla(\boldsymbol{r} - \boldsymbol{i}))\, dx &\ge \int_{\mathbb{R}^d} \langle \nabla L_F(u), \boldsymbol{r} - \boldsymbol{i} \rangle\, \chi_k\, dx \\ &\quad + \int_{\Omega_k} \langle \nabla \chi_k, \boldsymbol{r} - \boldsymbol{i} \rangle\, L_F(u)\, dx \qquad (4.86) \\ &\ge \int_{\mathbb{R}^d} \langle \nabla L_F(u), \boldsymbol{r} - \boldsymbol{i} \rangle\, \chi_k\, dx \end{aligned}$$

since the integrand of (4.86) is nonnegative: in fact, for $\mathscr{L}^d$-a.e. $x \in \Omega_k$ where $L_F(u(x))$ is strictly positive, the concavity of χ_k and $\boldsymbol{r}(x) \in K$ yields

$$\langle \nabla \chi_k(x), \boldsymbol{r}(x) - \boldsymbol{i}(x) \rangle \geq \chi_k(\boldsymbol{r}(x)) - \chi_k(x) = 1 - \chi_k(x) \geq 0.$$

Passing to the limit as $k \to \infty$ in the previous integral inequality, we obtain (4.80) (recall that the function in the left hand side of (4.80) is semiintegrable by (4.77)). $\square$

In the following theorem we characterize the minimal selection in the subdifferential of $\mathcal{F}$ and give, under the doubling condition, a formula for the slope of the functional.

Theorem 4.16 (Slope and subdifferential of $\mathcal{F}$) *Let $F : [0, +\infty) \to \mathbb{R}$ be a convex differentiable function satisfying (4.71), (4.72), (4.76) and (4.78). Assume that $\mathcal{F}$ has finite slope at $\mu = u\mathscr{L}^d \in \mathscr{P}_2^a(\mathbb{R}^d)$. Then $L_F(u) \in W^{1,1}(\mathbb{R}^d)$, $\nabla L_F(u) = \boldsymbol{w}u$ for some function $\boldsymbol{w} \in L^2(u\mathscr{L}^d; \mathbb{R}^d)$ and*

$$\Big(\int_{\mathbb{R}^d} |\boldsymbol{w}(x)|^2 u(x)\, dx \Big)^{1/2} = |\partial \mathcal{F}|(\mu) < +\infty. \tag{4.87}$$

Conversely, if $\nabla L_F(u) \in W^{1,1}_{\rm loc}(\mathbb{R}^d)$ and $\nabla L_F(u) = \boldsymbol{w}u$ for some $\boldsymbol{w} \in L^2(\mu; \mathbb{R}^d)$, then $\mathcal{F}$ has a finite slope at $\mu = u\mathscr{L}^d$ and $\boldsymbol{w} = \partial^\circ \mathcal{F}(\mu)$.

Proof. (a) We apply first (4.77) with $\boldsymbol{r} = 0$ $u\mathscr{L}^d$-a.e. and $\boldsymbol{r} = \boldsymbol{i}$ and take into account that

$$W_2(\mu, ((1-t)\boldsymbol{i} + t\boldsymbol{r})_\# \mu) \leq t \|\boldsymbol{i}\|_{L^2(u\mathscr{L}^d; \mathbb{R}^d)}$$

to obtain

$$d \int_{\mathbb{R}^d} L_F(u)\, dx \leq |\partial \mathcal{F}|(\mu) \|\boldsymbol{i}\|_{L^2(u\mathscr{L}^d; \mathbb{R}^d)},$$

so that $L_F(u) \in L^1(\mathbb{R}^d)$. Next, we apply (4.77) with $\boldsymbol{r} - \boldsymbol{i}$ equal to a $C_c^\infty(\mathbb{R}^d; \mathbb{R}^d)$ function $\boldsymbol{t}$ (notice that condition (i) holds with $\bar{t} < \sup |\nabla \boldsymbol{t}|$) and use again the inequality $W_2(\mu, ((1-t)\boldsymbol{i} + t\boldsymbol{r})_\# \mu) \leq t \|\boldsymbol{r} - \boldsymbol{i}\|_{L^2(u\mathscr{L}^d)}$ to obtain

$$\int_{\mathbb{R}^d} L_F(u) \mathrm{tr}\, (\nabla \boldsymbol{t})\, dx \leq |\partial \mathcal{F}^*|(\mu) \|\boldsymbol{t}\|_{L^p(u\mathscr{L}^d)} \leq |\partial \mathcal{F}^*|(\mu) \sup_{\mathbb{R}^d} |\boldsymbol{t}|.$$

As $\boldsymbol{t}$ is arbitrary, Riesz theorem gives that $L_F(u)$ is a function of bounded variation (i.e. its distributional derivative $DL_F(u)$ is a finite $\mathbb{R}^d$-valued measure in $\mathbb{R}^d$), so that we can rewrite the inequality as

$$\left| \sum_{i=1}^d \int_{\mathbb{R}^d} \boldsymbol{t}_i\, dD_i L_F(u) \right| \leq |\partial \mathcal{F}|(\mu) \|\boldsymbol{t}\|_{L^2(u\mathscr{L}^d; \mathbb{R}^d)}.$$

By L^2 duality theory there exists $\boldsymbol{w} \in L^2(u\mathscr{L}^d; \mathbb{R}^d)$ with $\|\boldsymbol{w}\|_2 \leq |\partial \mathcal{F}|(\mu)$ such that

$$\sum_{i=1}^d \int_{\mathbb{R}^d} \boldsymbol{t}_i\, dD_i L_F(u) = \int_{\mathbb{R}^d} \langle \boldsymbol{w}, \boldsymbol{t} \rangle\, du\mathscr{L}^d \qquad \forall \boldsymbol{t} \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d).$$

Therefore $L_F(u) \in W^{1,1}(\mathbb{R}^d)$ and $\nabla L_F(u) = \boldsymbol{w}u$. This leads to the inequality $\le$ in (4.87).

In order to show that equality holds in (4.87) we will prove that $(\boldsymbol{i} \times \boldsymbol{w})_{\#}\mu$ belongs to $\boldsymbol{\partial}\mathcal{F}(\mu)$. We have to show that (4.37) holds for any $\nu \in D(\mathcal{F})$. Using the doubling condition it is also easy to find a sequence of measures ν_h with compact support converging to ν in $\mathscr{P}_2(\mathbb{R}^d)$ and such that $\mathcal{F}(\nu_h)$ converges to $\mathcal{F}(\nu)$, hence we can also assume that $\operatorname{supp}\nu$ is compact.

As $\boldsymbol{t}_\mu^\nu$ is induced by the gradient of a Lipschitz and convex map φ, we know that all the conditions of Lemma 4.14 are fulfilled with $\boldsymbol{r} = \nabla\varphi$, and also Lemma 4.15 holds; therefore, by applying (4.77), the geodesic convexity of $\mathcal{F}$, and (4.80) we obtain

$$\begin{aligned}\mathcal{F}(\nu) - \mathcal{F}(\mu) &\ge \limsup_{h\to\infty} \int_{\mathbb{R}^d} \langle \nabla L_F(u), (\boldsymbol{r} - \boldsymbol{i})\rangle \chi_h \, dx \\ &= \limsup_{h\to\infty} \int_{\mathbb{R}^d} \langle \boldsymbol{w}, (\boldsymbol{r} - \boldsymbol{i})\rangle \chi_h u \, dx = \int_{\mathbb{R}^d} \langle \boldsymbol{w}, \boldsymbol{r} - \boldsymbol{i}\rangle \, d\mu,\end{aligned}$$

proving that $\boldsymbol{w} \in \partial\mathcal{F}(\mu)$.

Finally, we notice that our proof that $\boldsymbol{w} = \nabla L_F(u)/u \in \partial\mathcal{F}(\mu)$ does not use the finiteness of slope, but only the assumption $\boldsymbol{w} \in L^2(\mu;\mathbb{R}^d)$, therefore these conditions imply that the subdifferential is not empty and that the slope is finite. $\square$

4.5.4 The relative internal energy

In this section we briefly discuss the modifications which should be apported to the previous results, when one consider a relative energy functional as in Section 3.3.

We thus consider a log-concave probability measure $\gamma = e^{-V} \cdot \mathscr{L}^d \in \mathscr{P}(\mathbb{R}^d)$ induced by a convex l.s.c. potential

$$V : \mathbb{R}^d \to (-\infty, +\infty], \quad \text{with} \quad \Omega = \operatorname{int} D(V) \neq \emptyset. \tag{4.88}$$

We are also assuming that the energy density

$$\begin{gathered} F : [0,+\infty) \to [0,+\infty] \quad \text{is convex and l.s.c.,} \\ \text{it satisfies the doubling property (4.78),} \\ \text{and the geodesic convexity condition (3.22),} \end{gathered} \tag{4.89}$$

which yield that the map $s \mapsto \hat{F}(s) := F(e^{-s})e^s$ is convex and non increasing in $\mathbb{R}$. The functional

$$\mathcal{F}(\mu|\gamma) := \int_{\mathbb{R}^d} F(\rho)\, d\gamma = \int_\Omega F\Big(u/e^{-V}\Big)e^{-V}\, dx, \qquad \mu = \rho\cdot\gamma = u\mathscr{L}^d \tag{4.90}$$

is therefore geodesically convex in $\mathscr{P}_2(\mathbb{R}^d)$, by Theorem 3.23. It is easy to check that whenever $\hat{F}$ is not constant (case which corresponds to a linear F and a constant functional $\mathcal{F}$), F has a superlinear growth and therefore $\mathcal{F}$ is lower semicontinuous in $\mathscr{P}_2(\mathbb{R}^d)$.

Theorem 4.17 (Subdifferential of $\mathcal{F}(\cdot|\gamma)$) *The functional $\mathcal{F}(\cdot|\gamma)$ has finite slope at $\mu = \rho \cdot \gamma \in D(\mathcal{F})$ if and only if $L_F(\rho) \in W^{1,1}_{\rm loc}(\Omega)$ and $\nabla L_F(\rho) = \rho \boldsymbol{w}$ for some function $\boldsymbol{w} \in L^2(\mu; \mathbb{R}^d)$. In this case*

$$\Big(\int_{\mathbb{R}^d} |\boldsymbol{w}(x)|^2 \, d\mu(x) \Big)^{1/2} = |\partial \mathcal{F}|(\mu), \tag{4.91}$$

and $\boldsymbol{w} = \partial^\circ \mathcal{F}(\mu)$.

Proof. We argue as in Theorem 4.16: in the present case the directional derivative formula (4.77) becomes

$$\begin{aligned} +\infty > \lim_{t \downarrow 0} & \frac{\mathcal{F}((\boldsymbol{r}_t)_\# \mu|\gamma) - \mathcal{F}(\mu|\gamma)}{t} \\ &= -\int_{\mathbb{R}^d} L_F(u/e^{-V}) \Big(e^{-V} \operatorname{tr} \tilde{\nabla}(\boldsymbol{r} - \boldsymbol{i}) - e^{-V} \langle \nabla V, \boldsymbol{r} - \boldsymbol{i} \rangle \Big) \, dx \\ &= -\int_{\mathbb{R}^d} L_F(\rho) \operatorname{tr} \tilde{\nabla} \Big(e^{-V} (\boldsymbol{r} - \boldsymbol{i}) \Big) \, dx \end{aligned} \tag{4.92}$$

for every vector field $\boldsymbol{r}$ satisfying the assumptions of Lemma 4.14 and $\mathcal{F}(\boldsymbol{r}_\# \mu|\gamma)$ is finite. Choosing as before $\boldsymbol{r} = \boldsymbol{i} + e^V \boldsymbol{t}$, $\boldsymbol{t} \in C^\infty_c(\Omega; \mathbb{R}^d)$, since V is bounded in each compact subset of Ω, we get

$$\int_\Omega L_F(\rho) \operatorname{tr} \nabla \boldsymbol{t} \, dx \le |\partial \mathcal{F}|(\mu) \sup_{\mathbb{R}^d} |e^V \boldsymbol{t}|,$$

so that $L_F(\rho) \in BV_{\rm loc}(\Omega)$. Choosing now $\boldsymbol{r} = \boldsymbol{i} + \boldsymbol{t}$ with $\boldsymbol{t} \in C^\infty_c(\Omega; \mathbb{R}^d)$ we get

$$\left| \sum_{i=1}^d \int_\Omega \boldsymbol{t}_i \, dD_i L_F(\rho) \, d\gamma \right| \le |\partial \mathcal{F}|(\mu) \|\boldsymbol{t}\|_{L^2(\mu; \mathbb{R}^d)}$$

so that there exists $\boldsymbol{w} \in L^2(\mu; \mathbb{R}^d)$ such that

$$\sum_{i=1}^d \int_\Omega \boldsymbol{t}_i \, dD_i L_F(\rho) \, d\gamma = \int_{\mathbb{R}^d} \langle \boldsymbol{w}, \boldsymbol{t} \rangle \, d\mu = \int_{\mathbb{R}^d} \langle u \boldsymbol{w}, \boldsymbol{t} \rangle e^{-V} \, dx \quad \forall \boldsymbol{t} \in C^\infty_c(\Omega; \mathbb{R}^d),$$

thus showing that $L_F(\rho) \in W^{1,1}_{\rm loc}(\Omega)$ and $\nabla L_F(\rho) = u e^{-V} \boldsymbol{w} = \rho \boldsymbol{w}$.

Conversely, if $L_F(\rho) \in W^{1,1}_{\rm loc}(\Omega)$ with $\nabla L_F(\rho) = \rho \boldsymbol{w}$ and $\boldsymbol{w} \in L^2(\mu; \mathbb{R}^d)$, arguing as in Lemma 4.15 we have for every measure $\nu = \boldsymbol{r}_\# \mu$ with compact support in Ω

$$\begin{aligned} \mathcal{F}(\nu|\gamma) - \mathcal{F}(\mu|\gamma) &\ge \limsup_{k \to \infty} -\int_\Omega L_F(\rho) \operatorname{tr} \tilde{\nabla} \big(e^{-V} (\boldsymbol{r} - \boldsymbol{i}) \big) \chi_k \, dx \\ &\ge \limsup_{k \to \infty} \int_\Omega \langle \chi_k \nabla L_F(\rho) + L_F(\rho) \nabla \chi_k, \boldsymbol{r} - \boldsymbol{i} \rangle \, d\gamma \\ &\ge \limsup_{k \to \infty} \int_\Omega \langle \nabla L_F(\rho), \boldsymbol{r} - \boldsymbol{i} \rangle \chi_k \, d\gamma \\ &\ge \limsup_{k \to \infty} \int_\Omega \langle \boldsymbol{w}, \boldsymbol{r} - \boldsymbol{i} \rangle \chi_k \, d\mu = \int_\Omega \langle \boldsymbol{w}, \boldsymbol{r} - \boldsymbol{i} \rangle \, d\mu, \end{aligned}$$

which shows, through a density argument, that $\boldsymbol{w} \in \partial \mathcal{F}(\mu)$. $\square$

4.5.5 The interaction energy

In this section we consider the interaction energy functional $\mathcal{W} : \mathscr{P}_2(\mathbb{R}^d) \to [0, +\infty]$ defined by

$$\mathcal{W}(\mu) := \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} W(x-y)\, d\mu \times \mu(x,y).$$

Without loss of generality we shall assume that $W : \mathbb{R}^d \to [0, +\infty)$ is an even function; our main assumption, besides the convexity of $\mathbb{R}^d$, is the doubling condition

$$\exists\, C_W > 0: \quad W(x+y) \le C_W\big(1 + W(x) + W(y)\big) \quad \forall\, x,\, y \in \mathbb{R}^d. \tag{4.93}$$

Let us first state a preliminary result: we are denoting by $\bar\mu$ the barycenter of the measure μ:

$$\bar\mu := \int_{\mathbb{R}^d} x\, d\mu(x). \tag{4.94}$$

Lemma 4.18 *Assume that $W : \mathbb{R}^d \to [0, +\infty)$ is convex, Gateaux differentiable, even, and satisfies the doubling condition* (4.93)*. Then for any $\mu \in D(\mathcal{W})$ we have*

$$\int_{\mathbb{R}^d} W(x)\, d\mu(x) \le C_W\big(1 + \mathcal{W}(\mu) + W(\bar\mu)\big) < +\infty, \tag{4.95}$$

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |\nabla W(x-y)|\, d\mu \times \mu(x,y) \le C_W\big(1 + S_W + \mathcal{W}(\mu)\big) < +\infty, \tag{4.96}$$

where $S_W := \sup_{|y| \le 1} W(y)$. In particular $\boldsymbol{w} := (\nabla W) * \mu$ is well defined for μ-a.e. $x \in \mathbb{R}^d$, it belongs to $L^1(\mu; \mathbb{R}^d)$, and it satisfies

$$\begin{aligned} \int_{\mathbb{R}^{2d} \times \mathbb{R}^d} &\langle \nabla W(x_1 - x_2), y_1 - x_1 \rangle\, d\boldsymbol{\gamma}(x_1, y_1)\, d\mu(x_2) \\ &= \int_{\mathbb{R}^{2d}} \langle \boldsymbol{w}(x_1), y_1 - x_1 \rangle\, d\boldsymbol{\gamma}(x_1, y_1), \end{aligned} \tag{4.97}$$

for every $\boldsymbol{\gamma} \in \Gamma(\mu, \nu)$ with $\nu \in D(\mathcal{W})$. In particular, choosing $\boldsymbol{\gamma} := (\boldsymbol{i} \times \boldsymbol{r})_\# \mu$, we have

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \langle \nabla W(x-y), \boldsymbol{r}(x) \rangle\, d\mu \times \mu(x,y) = \int_{\mathbb{R}^d} \langle \boldsymbol{w}(x), \boldsymbol{r}(x) \rangle\, d\mu(x) \tag{4.98}$$

for every vector field $\boldsymbol{r} \in L^\infty(\mu; \mathbb{R}^d)$ and for $\boldsymbol{r} := \lambda \boldsymbol{i}$, $\lambda \in \mathbb{R}$.

Proof. By Jensen inequality we have

$$W(x - \bar\mu) \le \int_{\mathbb{R}^d} W(x-y)\, d\mu(y) \quad \forall\, x \in \mathbb{R}^d, \tag{4.99}$$

so that a further integration yields

$$\int_{\mathbb{R}^d} W(x - \bar\mu)\, d\mu(x) \le \mathcal{W}(\mu); \tag{4.100}$$

(4.95) follows directly from (4.100) and the doubling condition (4.93), since $W(x) \le C_W\big(1 + W(x-\bar\mu) + W(\bar\mu)\big)$.

Combining the doubling condition and the convexity of W we also get

$$\begin{aligned}|\nabla W(x)| = \sup_{|y|\le 1} \langle \nabla W(x), y\rangle &\le \sup_{|y|\le 1} W(x+y) - W(x)\\ &\le C_W\big(1 + W(x) + \sup_{|y|\le 1} W(y)\big),\end{aligned} \tag{4.101}$$

which yields (4.96).

If now $\nu \in D(\mathcal{W})$ and $\boldsymbol\gamma \in \Gamma(\mu,\nu)$, then the positive part of the map $(x_1,y_1,x_2) \mapsto \langle \nabla W(x_1-x_2), y_1 - x_1\rangle$ belongs to $L^1(\boldsymbol\gamma\times\mu)$ since convexity yields

$$\langle \nabla W(x_1-x_2), y_1-x_1\rangle \le W(y_1-x_2) - W(x_1-x_2),$$

and the right hand side of this inequality is integrable:

$$\int_{\mathbb{R}^{3d}} W(y_1-x_2)\, d\boldsymbol\gamma\times\mu = \int_{\mathbb{R}^{2d}} W(y_1-x_2)\, d\nu\times\mu \le C\big(1+\mathcal{W}(\nu)+\mathcal{W}(\mu)+W(\bar\nu-\bar\mu)\big),$$

$$\int_{\mathbb{R}^{3d}} W(x_1-x_2)\, d\boldsymbol\gamma\times\mu = \int_{\mathbb{R}^{2d}} W(x_1-x_2)\, d\mu\times\mu = \mathcal{W}(\mu).$$

Therefore we can apply Fubini-Tonelli theorem to obtain

$$\begin{aligned}&\int_{\mathbb{R}^{3d}} \langle \nabla W(x_1-x_2), y_1-x_1\rangle\, d\boldsymbol\gamma\times\mu(x_1,y_1,x_2)\\ &= \int_{\mathbb{R}^{2d}} \Big(\int_X \langle \nabla W(x_1-x_2), y_1-x_1\rangle\, d\mu(x_2)\Big)\, d\boldsymbol\gamma(x_1,y_1)\\ &= \int_{\mathbb{R}^{2d}} \langle\Big(\int_X \nabla W(x_1-x_2)\, d\mu(x_2)\Big), y_1-x_1\rangle\, d\boldsymbol\gamma(x_1,y_1)\\ &= \int_{\mathbb{R}^{2d}} \langle \boldsymbol{w}(x_1), y_1-x_1\rangle\, d\boldsymbol\gamma(x_1,y_1),\end{aligned}$$

which yields (4.97). □

As the interaction energy fails to satisfy (4.31a), as we did for the potential energy functional we say that $\boldsymbol\xi \in L^2(\mu;\mathbb{R}^d)$ belongs to the Fréchet subdifferential $\partial\mathcal{W}(\mu)$ at $\mu\in D(\mathcal{W})$ if

$$\mathcal{W}(\nu) - \mathcal{W}(\mu) \ge \inf_{\gamma\in\Gamma_o(\mu,\nu)} \int_{\mathbb{R}^d\times\mathbb{R}^d} \langle \boldsymbol\xi(x), y-x\rangle\, d\gamma(x) + o\big(W_2(\mu,\nu)\big). \tag{4.102}$$

Theorem 4.19 (Minimal subdifferential of $\mathcal{W}$) *Assume that $W:\mathbb{R}^d\to[0,+\infty)$ is convex, Gateaux differentiable, even, and satisfies the doubling condition (4.93). Then $\mu\in\mathscr{P}_2(\mathbb{R}^d)$ belongs to $D(|\partial\mathcal{W}|)$ if and only if $\boldsymbol{w} = (\nabla W)*u \in L^2(\mu;\mathbb{R}^d)$. In this case $\boldsymbol{w} = \partial^\circ\mathcal{W}(\mu)$.*

Proof. As we did for the internal energy functional, we start by computing the directional derivative of $\mathcal{W}$ along a direction induced by a transport map $\boldsymbol{r} = \boldsymbol{i} + \boldsymbol{t}$, with $\boldsymbol{t}$ bounded and with a compact support (by the growth condition on W, this ensures that $\mathcal{W}(\boldsymbol{r}_{\#}\mu) < +\infty$). Since the map

$$t \mapsto \frac{W\left((x-y) + t(\boldsymbol{t}(x) - \boldsymbol{t}(y))\right) - W(x-y)}{t}$$

is nondecreasing w.r.t. t, the monotone convergence theorem and (4.98) give (taking into account that ∇W is an odd function)

$$\begin{aligned} +\infty \quad > \quad & \lim_{t \downarrow 0} \frac{\mathcal{W}((\boldsymbol{i} + t\boldsymbol{t})_{\#}\mu - \mathcal{W}(\mu)}{t} \\ = \quad & \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle \nabla W(x-y), (\boldsymbol{t}(x) - \boldsymbol{t}(y)) \rangle \, d\mu \times \mu = \int_{\mathbb{R}^d} \langle \boldsymbol{w}, \boldsymbol{t} \rangle \, d\mu. \end{aligned}$$

On the other hand, since $|\partial \mathcal{W}|(\mu) < +\infty$, using the inequality $W_2((\boldsymbol{i}+t\boldsymbol{t})_{\#}\mu, \mu) \leq \|\boldsymbol{t}\|_{L^2(\mu;\mathbb{R}^d)}$ we get

$$\int_{\mathbb{R}^d} \langle \boldsymbol{w}, \boldsymbol{t} \rangle \, d\mu \geq -|\partial \mathcal{W}|(\mu) \|\boldsymbol{t}\|_{L^2(\mu;\mathbb{R}^d)};$$

changing the sign of $\boldsymbol{t}$ we obtain

$$\left| \int_{\mathbb{R}^d} \langle \boldsymbol{w}, \boldsymbol{t} \rangle \, d\mu \right| \leq |\partial \mathcal{W}|(\mu) \|\boldsymbol{t}\|_{L^2(\mu;\mathbb{R}^d)},$$

and this proves that $\boldsymbol{w} \in L^2(\mu; \mathbb{R}^d)$ and that $\|\boldsymbol{w}\|_{L^2} \leq |\partial \mathcal{W}|(\mu)$.

Now we prove that if $\boldsymbol{w} = (\nabla W) * \mu \in L^2(\mu; \mathbb{R}^d)$, then it belongs to $\partial \mathcal{W}(\mu)$. Let us consider a test measure $\nu \in D(\mathcal{W})$, a plan $\boldsymbol{\gamma} \in \Gamma(\mu, \nu)$, and the directional derivative of $\mathcal{W}$ along the direction induced by $\boldsymbol{\gamma}$. Since the map

$$t \mapsto \frac{W\left((1-t)(x_1 - x_2) + t(y_1 - y_2)\right) - W(x_1 - x_2)}{t}$$

is nondecreasing w.r.t. t, the monotone convergence theorem, the fact that ∇W is an odd function, and (4.98) give

$$\begin{aligned} \mathcal{W}(\nu) - \mathcal{W}(\mu) \quad \geq \quad & \lim_{t \downarrow 0} \frac{\mathcal{W}(((1-t)\pi^1 + t\pi^2)_{\#}\boldsymbol{\gamma} - \mathcal{W}(\mu)}{t} \\ = \quad & \frac{1}{2} \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} \langle \nabla W(x_1 - x_2), (y_1 - x_1) - (y_2 - x_2) \rangle \, d\boldsymbol{\gamma} \times \boldsymbol{\gamma} \\ = \quad & \int_{\mathbb{R}^{2d}} \langle \boldsymbol{w}(x_1), y_1 - x_1 \rangle \, d\boldsymbol{\gamma}(x_1, y_1), \end{aligned}$$

and this proves that $(\boldsymbol{i} \times \boldsymbol{w})_{\#}\mu \in \boldsymbol{\partial}\mathcal{W}(\mu)$. $\square$

4.5.6 The opposite Wasserstein distance

In this section we compute the (metric) slope of the function $\psi(\cdot) := -\frac{1}{2}W_2^2(\cdot,\mu^2)$, i.e. the limit

$$\frac{1}{2}\limsup_{\nu\to\mu}\frac{W_2^2(\nu,\mu^2)-W_2^2(\mu,\mu^2)}{W_2(\nu,\mu)} = |\partial\psi|(\mu); \tag{4.103}$$

observe that the triangle inequality shows that the "lim sup" above is always less than $W_2(\mu,\mu^2)$; however this inequality is always strict when optimal plans are not induced by transports, as the following theorem shows ([9, Theorem 10.4.12]); the right formula for the slope involves the minimal L^2 norm of the *barycentric projection* of the optimal plans and gives that the minimal selection is always induced by a map. We recall that, given $\boldsymbol{\gamma}\in\Gamma(\mu,\nu)$, the barycentric projection $\bar{\boldsymbol{\gamma}}$ is the map in $L^2(\mu)$ characterized by $\pi^1_\#(y\boldsymbol{\gamma})=\bar{\boldsymbol{\gamma}}\mu$, or equivalently by

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} y\varphi(x)\,d\boldsymbol{\gamma} = \int_{\mathbb{R}^d}\bar{\boldsymbol{\gamma}}(x)\varphi(x)\,d\mu(x) \qquad \forall\varphi\in C_c^\infty(\mathbb{R}^d).$$

Theorem 4.20 (Minimal subdifferential of $-\frac{1}{2}W_2^2(\cdot,\nu)$)
Let $\psi(\mu) = -\frac{1}{2}W_2^2(\mu,\mu^2)$. Then

$$\partial\psi(\mu) = \{\bar{\boldsymbol{\gamma}}-\boldsymbol{i}:\ \boldsymbol{\gamma}\in\Gamma_o(\mu,\nu)\} \qquad \forall\mu\in\mathscr{P}_2(\mathbb{R}^d).$$

In particular

$$|\partial\psi|^2(\mu) = \min\left\{\int_{\mathbb{R}^d}|\bar{\boldsymbol{\gamma}}-\boldsymbol{i}|^2\,d\mu:\ \boldsymbol{\gamma}\in\Gamma_o(\mu,\mu^2)\right\} \qquad \forall\mu\in\mathscr{P}_2(\mathbb{R}^d), \tag{4.104}$$

and $\partial^\circ\psi(\mu)=\bar{\boldsymbol{\gamma}}-\boldsymbol{i}$ is a strong *subdifferential, where $\boldsymbol{\gamma}$ is the unique minimizing plan above.*
Finally $\mu\mapsto|\partial\psi|(\mu)$ is lower semicontinuous with respect to narrow convergence in $\mathscr{P}(\mathbb{R}^d)$, along sequences bounded in $\mathscr{P}_2(\mathbb{R}^d)$.

4.5.7 The sum of internal, potential and interaction energy

In this section we consider, as in [29], the functional $\phi:\mathscr{P}_2(\mathbb{R}^d)\to(-\infty,+\infty]$ given by the sum of internal, potential and interaction energy:

$$\phi(\mu) := \int_{\mathbb{R}^d}F(u)\,dx + \int_{\mathbb{R}^d}V\,d\mu + \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d}W\,d\mu\times\mu \quad \text{if } \mu = u\mathscr{L}^d, \tag{4.105}$$

setting $\phi(\mu)=+\infty$ if $\mu\in\mathscr{P}_2(\mathbb{R}^d)\setminus\mathscr{P}_2^a(\mathbb{R}^d)$. Recalling the "doubling condition" stated in (4.78), we make the following assumptions on F, V and W:

- **(F)** $F:[0,+\infty)\to\mathbb{R}$ is a doubling, convex differentiable function with superlinear growth satisfying (4.71) (i.e. the bounds on F^-) and (4.76) (yielding the geodesic convexity of the internal energy).
- **(V)** $V:\mathbb{R}^d\to(-\infty,+\infty]$ is a l.s.c. λ-convex function with proper domain $D(V)$ with nonempty interior $\Omega\subset\mathbb{R}^d$.

(W) $W : \mathbb{R}^d \to [0, +\infty)$ is a convex, differentiable, even function satisfying the doubling condition (4.93).

The finiteness of ϕ yields

$$\operatorname{supp}\mu \subset \overline{\Omega} = \overline{D(V)}, \quad \mu(\partial\Omega) = 0, \tag{4.106}$$

so that its density u w.r.t. $\mathscr{L}^d$ can be considered as a function of $L^1(\Omega)$.

The same monotonicity argument used in the proof of Lemma 4.14 gives

$$+\infty > \lim_{t\downarrow 0} \frac{\int_{\mathbb{R}^d} V\, d((1-t)\boldsymbol{i} + t\boldsymbol{r}))_{\#}\mu - \int_{\mathbb{R}^d} V\, d\mu}{t} = \int_{\mathbb{R}^d} \langle \nabla V, \boldsymbol{r} - \boldsymbol{i}\rangle\, d\mu, \tag{4.107}$$

whenever both $\int_{\mathbb{R}^d} V\, d\mu < +\infty$ and $\int_{\mathbb{R}^d} V\, d\boldsymbol{r}_{\#}\mu < +\infty$.

Analogously, denoting by $\mathcal{W}$ the interaction energy functional induced by $W/2$, arguing as in the first part of Theorem 4.19 we have

$$+\infty > \lim_{t\downarrow 0} \frac{\mathcal{W}(((1-t)\boldsymbol{i} + t\boldsymbol{r})_{\#}\mu) - \mathcal{W}(\mu)}{t} = \int_{\mathbb{R}^d} \langle (\nabla W) * \mu), \boldsymbol{r} - \boldsymbol{i}\rangle\, d\mu, \tag{4.108}$$

whenever $\mathcal{W}(\mu) + \mathcal{W}(\boldsymbol{r}_{\#}\mu) < +\infty$. The growth condition on W ensures that $\mu \in D(\mathcal{W})$ implies $\boldsymbol{r}_{\#}\mu \in D(\mathcal{W})$ if either $\boldsymbol{r} - \boldsymbol{i}$ is bounded or $\boldsymbol{r} = 2\boldsymbol{i}$ (here we use the doubling condition).

We have the following characterization of the minimal selection in the subdifferential $\partial^\circ\phi(\mu)$:

Theorem 4.21 (Minimal subdifferential of ϕ) *A measure $\mu = u\mathscr{L}^d \in D(\phi) \subset \mathscr{P}_2(\mathbb{R}^d)$ belongs to $D(|\partial\phi|)$ if and only if $L_F(u) \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and*

$$u\boldsymbol{w} = \nabla L_F(u) + u\nabla V + u(\nabla W) * u \quad \text{for some} \quad \boldsymbol{w} \in L^2(\mu; \mathbb{R}^d). \tag{4.109}$$

In this case the vector $\boldsymbol{w}$ defined μ-a.e. by (4.109) is the minimal selection in $\partial\phi(\mu)$, i.e. $\boldsymbol{w} = \partial^\circ\phi(\mu)$.

Proof. We argue exactly as in the proof of Theorem 4.16, computing the Gateaux derivative of ϕ in several directions $\boldsymbol{r}$, using Lemma 4.14 for the internal energy and (4.107), (4.108) respectively for the potential and interaction energy.

Choosing $\boldsymbol{r} = \boldsymbol{i} + \boldsymbol{t}$, with $\boldsymbol{t} \in C^\infty_c(\Omega; \mathbb{R}^d)$, we obtain

$$-\int_{\mathbb{R}^d} L_F(u)\nabla\cdot\boldsymbol{t}\, dx + \int_{\mathbb{R}^d} \langle \nabla V, \boldsymbol{t}\rangle\, d\mu + \int_{\mathbb{R}^d} \langle (\nabla W) * u, \boldsymbol{t}\rangle\, d\mu \geq -|\partial\phi|(\mu)\|\boldsymbol{t}\|_{L^2(\mu)}. \tag{4.110}$$

Since V is locally Lipschitz in Ω and $\nabla W * u$ is locally bounded, following the same argument of Theorem 4.16, we obtain from (4.110) first that $L_F(u) \in BV_{\mathrm{loc}}(\mathbb{R}^d)$ and then that $L_F(u) \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$, with

$$\nabla L_F(u) + u\nabla V + u(\nabla W) * u = \boldsymbol{w}u \quad \text{for some } \boldsymbol{w} \in L^2(\mu; \mathbb{R}^d) \tag{4.111}$$

with $\|\boldsymbol{w}\|_{L^2} \leq |\partial\phi|(\mu)$.

In order to show that the vector $\boldsymbol{w}$ is in the subdifferential (and then, by the previous estimate, it is the minimal selection) we choose eventually a test measure $\nu \in D(\phi)$ with compact support contained in Ω and the associated optimal transport map $\boldsymbol{r} = \boldsymbol{t}_\mu^\nu$; Lemma 4.14, (4.107), (4.108), and Lemma 4.15 yield

$$
\begin{aligned}
\phi(\nu) - \phi(\mu) &\geq \frac{d}{dt}\phi\left(((1-t)\boldsymbol{i} + t\boldsymbol{r})_{\#}\mu\right)\Big|_{t=0^+} \\
&= -\int_\Omega L_F(u)\nabla\cdot(\boldsymbol{r}-\boldsymbol{i})\,dx + \int_\Omega \langle\nabla V, \boldsymbol{r}-\boldsymbol{i}\rangle\,d\mu + \int_\Omega \langle(\nabla W)*u, \boldsymbol{r}-\boldsymbol{i}\rangle\,d\mu \\
&\geq \limsup_{h\to\infty}\int_\Omega \langle\nabla L_F(u), \boldsymbol{r}-\boldsymbol{i}\rangle\chi_h\,dx + \int_\Omega \langle\nabla V + (\nabla W)*u, \boldsymbol{r}-\boldsymbol{i}\rangle\,d\mu \\
&= \limsup_{h\to\infty}\int_\Omega \langle\nabla L_F(u) + u\nabla V + u(\nabla W)*u, \boldsymbol{r}-\boldsymbol{i}\rangle\chi_h\,dx \\
&= \int_\Omega \langle u\boldsymbol{w}, \boldsymbol{r}-\boldsymbol{i}\rangle\,dx = \int_\Omega \langle\boldsymbol{w}, \boldsymbol{r}-\boldsymbol{i}\rangle\,d\mu.
\end{aligned}
$$

Finally, we notice that the proof that $\boldsymbol{w}$ belongs to the subdifferential did not use the finiteness of slope, but only the assumption (previously derived by the finiteness of slope) that $L_F(u) \in W^{1,1}_{\mathrm{loc}}(\Omega)$, (4.109), and $\phi(\mu) < +\infty$; therefore these conditions imply that the subdifferential is not empty, hence the slope is finite and the vector $\boldsymbol{w}$ is the minimal selection in $\partial\phi(\mu)$. □

An interesting particular case of the above result is provided by the relative entropy functional: let us choose $W \equiv 0$ and

$$
F(s) := s\log s, \quad \gamma := \frac{1}{Z}e^{-V}\cdot\mathscr{L}^d = e^{-(V(x)+\log Z)}\cdot\mathscr{L}^d,
$$

with $Z > 0$ chosen so that $\gamma(\mathbb{R}^d) = 1$. Recalling Remark 3.16, the functional ϕ can also be written as

$$
\phi(\mu) = \mathcal{H}(\mu|\gamma) - \log Z. \tag{4.112}
$$

Since in this case $L_F(u) = u$, a vector $\boldsymbol{w} \in L^2(\mu;\mathbb{R}^d)$ is the minimal selection $\partial^\circ\phi(\mu)$ if and only if

$$
-\int_{\mathbb{R}^d} \nabla\cdot\boldsymbol{\zeta}(x)\,d\mu(x) = \int_{\mathbb{R}^d} \langle\boldsymbol{w}(x), \boldsymbol{\zeta}(x)\rangle\,d\mu(x) - \int_{\mathbb{R}^d} \langle\nabla V(x), \boldsymbol{\zeta}(x)\rangle\,d\mu(x), \tag{4.113}
$$

for every test function $\boldsymbol{\zeta} \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$; (4.113) can also be written in terms of $\rho = \frac{d\mu}{d\gamma}$ as

$$
-\int_{\mathbb{R}^d} \rho\nabla\cdot(e^{-V(x)}\boldsymbol{\zeta}(x))\,dx = \int_{\mathbb{R}^d} \langle\rho\boldsymbol{w}(x), e^{-V(x)}\boldsymbol{\zeta}(x)\rangle\,dx, \tag{4.114}
$$

which shows that $\rho\boldsymbol{w} = \nabla\rho$.

5 Gradient flows of λ-geodesically convex functionals in $\mathscr{P}_2(\mathbb{R}^d)$

In this chapter we state some structural results, concerning existence, uniqueness, approximation, and qualitative properties of gradient flows in $\mathscr{P}_2(\mathbb{R}^d)$ generated by

$$\text{a proper and l.s.c. functional} \quad \phi : \mathscr{P}_2(\mathbb{R}^d) \to (-\infty, +\infty]. \tag{5.1a}$$

We will also assume that

$$\phi \quad \text{is } \lambda\text{-geodesically convex, according to Definition 3.1.} \tag{5.1b}$$

Since we are mostly concerned with absolutely continuous measures, some technical details will be simpler assuming that

$$D(|\partial\phi|) \subset \mathscr{P}_2^a(\mathbb{R}^d); \tag{5.1c}$$

finally, the (simplified) existence theory we are presenting here will also require that for some $\tau_* > 0$

$$\begin{gathered}\text{the map} \quad \nu \mapsto \Phi(\tau, \mu; \nu) = \frac{1}{2\tau} W_2^2(\mu, \nu) + \phi(\nu) \quad \text{admits at least} \\ \text{a minimum point } \mu_\tau, \text{ for all } \tau \in (0, \tau_*) \text{ and } \mu \in \mathscr{P}_2(X).\end{gathered} \tag{5.1d}$$

Notice that (5.1b) gives that any minimizer μ_τ in (5.1d) belongs to $\mathscr{P}_2^a(\mathbb{R}^d)$, due to Lemma 4.4.

Remark 5.1 (5.1d) is slightly more restrictive than lower semicontinuity in $\mathscr{P}_2(\mathbb{R}^d)$; by the standard direct method in Calculus of Variations, it surely holds if ϕ satisfies the following coerciveness-l.s.c. conditions:

$$\inf_{\mu \in \mathscr{P}_2(\mathbb{R}^d)} \phi(\mu) + \frac{1}{2\tau_*} \mathsf{m}_2^2(\mu) > -\infty, \tag{5.2a}$$

$$\left.\begin{aligned} \mu_n \to \mu \quad &\textit{narrowly} \text{ in } \mathscr{P}(\mathbb{R}^d) \\ \sup_n \mathsf{m}_2(\mu_n) &< +\infty \end{aligned}\right\} \quad \Longrightarrow \quad \liminf_{n\to\infty} \phi(\mu_n) \ge \phi(\mu). \tag{5.2b}$$

Another sufficient condition yielding (5.1d) and satisfied by our main examples is (5.61): it will be introduced in the "existence" Theorem 5.8.

The inclusion (5.1c) is a simplifying assumption, which ensures that the flows stay inside the absolutely continuous measures, thus avoiding more complicated notions of subdifferentials (see Chapter 11 of [9], where this restriction is completely removed).

Definition 5.2 (Gradient flows) *We say that* $\mu_t \in AC^2_{\rm loc}\big((0, +\infty); \mathscr{P}_2(\mathbb{R}^d)\big)$ *is a solution of the gradient flow equation*

$$\boldsymbol{v}_t \in -\partial\phi(\mu_t) \quad t > 0, \tag{5.3}$$

if, for $\mathscr{L}^1$-a.e. $t>0$, $\mu_t \in \mathscr{P}_2^a(\mathbb{R}^d)$ and its velocity vector field $\boldsymbol{v}_t \in \mathrm{Tan}_{\mu_t}\mathscr{P}_2(\mathbb{R}^d)$ belongs to the subdifferential (4.20) *of ϕ at μ_t.*

Recalling the characterization of the tangent velocity field to an absolutely continuous curve, the above definition is equivalent to the requirement that there exists a Borel vector field $\boldsymbol{v}_t$ such that

$$\boldsymbol{v}_t \in \mathrm{Tan}_{\mu_t}\mathscr{P}_2(\mathbb{R}^d) \text{ for } \mathscr{L}^1\text{-a.e. } t>0, \quad \|\boldsymbol{v}_t\|_{L^2(\mu_t;\mathbb{R}^d)} \in L^2_{\mathrm{loc}}(0,+\infty), \tag{5.4a}$$

the continuity equation

$$\partial_t \mu_t + \nabla\cdot\big(\boldsymbol{v}_t\mu_t\big) = 0 \quad \text{in } \mathbb{R}^d\times(0,+\infty) \tag{5.4b}$$

holds in the sense of distributions according to (2.46), and finally

$$-\boldsymbol{v}_t \in \partial\phi(\mu_t) \quad \text{for } \mathscr{L}^1\text{-a.e. } t>0. \tag{5.4c}$$

Before studying the question of existence of solutions to (5.3), which we will postpone to the next sections, we want to discuss some preliminary issues.

5.1 Characterizations of gradient flows, uniqueness and contractivity

Theorem 5.3 (Gradient flows, E.V.I., and curves of Maximal Slope) *Let $\phi : \mathscr{P}_2(\mathbb{R}^d) \to (-\infty,+\infty]$ be as in* (5.1a,b). *An absolutely continuous curve $\mu \in AC^2_{\mathrm{loc}}\big((0,+\infty);\mathscr{P}_2(\mathbb{R}^d)\big)$ with $\mu_t \in \mathscr{P}_2^a(\mathbb{R}^d)$ for $\mathscr{L}^1$-a.e. $t\in(0,+\infty)$ is a gradient flow of ϕ according to Definition* 5.2 *if and only if it satisfies one of the following equivalent characterizations:*

i) There exists a Borel vector field $\tilde{\boldsymbol{v}}_t$ with $\|\tilde{\boldsymbol{v}}_t\|_{L^2(\mu_t;\mathbb{R}^d)}$ in $L^2_{\mathrm{loc}}(0,+\infty)$ such that

$$\partial_t \mu_t + \nabla\cdot(\tilde{\boldsymbol{v}}_t\mu_t) = 0 \quad \textit{in } \mathbb{R}^d\times(0,+\infty), \tag{5.5a}$$

in the sense of distributions, and

$$-\int_{\mathbb{R}^d} \langle \tilde{\boldsymbol{v}}_t, \boldsymbol{t}_{\mu_t}^\sigma - \boldsymbol{i}\rangle\, d\mu_t \le \phi(\sigma) - \phi(\mu_t) - \frac{\lambda}{2}W_2^2(\sigma,\mu_t) \quad \forall\sigma\in D(\phi), \tag{5.5b}$$

$\mathscr{L}^1$-a.e. in $(0,+\infty)$.

ii) Every Borel vector field $\tilde{\boldsymbol{v}}_t$ with $\|\tilde{\boldsymbol{v}}_t\|_{L^2(\mu_t;\mathbb{R}^d)}$ in $L^2_{\mathrm{loc}}(0,+\infty)$ (in particular the velocity vector field $\boldsymbol{v}_t \in \mathrm{Tan}_{\mu_t}\mathscr{P}_2(\mathbb{R}^d)$) satisfying the continuity equation

$$\partial_t \mu_t + \nabla\cdot(\tilde{\boldsymbol{v}}_t\mu_t) = 0 \quad \textit{in } \mathbb{R}^d\times(0,+\infty), \tag{5.6a}$$

in the sense of distributions, satisfies the variational inequality

$$-\int_{\mathbb{R}^d} \langle \tilde{\boldsymbol{v}}_t, \boldsymbol{t}_{\mu_t}^\sigma - \boldsymbol{i}\rangle\, d\mu_t \le \phi(\sigma) - \phi(\mu_t) - \frac{\lambda}{2}W_2^2(\sigma,\mu_t) \quad \forall\sigma\in D(\phi), \tag{5.6b}$$

for $t\in(0,+\infty)\setminus\mathscr{N}$, $\mathscr{N}$ being a $\mathscr{L}^1$-negligible set.

iii) The metric Evolution Variational Inequalities

$$\frac{1}{2}\frac{d}{dt}W_2^2(\mu_t,\sigma)+\frac{\lambda}{2}W_2^2(\mu_t,\sigma)\le\phi(\sigma)-\phi(\mu_t)\quad \textit{for } \mathscr{L}^1\textit{-a.e. } t>0 \tag{5.7}$$

hold for every $\sigma\in D(\phi)$.

iv) The map $t\mapsto\phi(\mu_t)$ *is locally absolutely continuous in* $(0,+\infty)$ *and*

$$-\frac{d}{dt}\phi(\mu_t)\ge\frac{1}{2}\|\boldsymbol{v}_t\|^2_{L^2(\mu_t;\mathbb{R}^d)}+\frac{1}{2}|\partial\phi|^2(\mu_t)\quad \mathscr{L}^1\textit{-a.e. in } (0,+\infty). \tag{5.8}$$

v) The map $t\mapsto\phi(\mu_t)$ *is locally absolutely continuous in* $(0,+\infty)$ *and*

$$-\frac{d}{dt}\phi(\mu_t)=\|\boldsymbol{v}_t\|^2_{L^2(\mu_t;\mathbb{R}^d)}=|\partial\phi|^2(\mu_t)\quad \mathscr{L}^1\textit{-a.e. in } (0,+\infty). \tag{5.9}$$

In particular, (5.3) *and v) yield*

$$-\boldsymbol{v}_t=\partial^\circ\phi(\mu_t)\quad \textit{for } \mathscr{L}^1\textit{-a.e. } t>0. \tag{5.10}$$

Proof. **i)** If μ_t is a gradient flow according to Definition 5.2, recalling the property of the subdifferential (4.37), it is immediate that μ_t and its velocity vector field $\boldsymbol{v}_t$ satisfy (5.5a,b).

Conversely, suppose that $\tilde{\boldsymbol{v}}_t$ satisfies (5.5a,b) and let us denote by $\boldsymbol{v}_t\in\mathrm{Tan}_{\mu_t}\mathscr{P}_2(\mathbb{R}^d)$ the tangent velocity vector of μ_t. Since, by (2.55), for $\mathscr{L}^1$-a.e. $t>0$ $\boldsymbol{v}_t$ is the orthogonal projection of $\tilde{\boldsymbol{v}}_t$ on $\mathrm{Tan}_{\mu_t}\mathscr{P}_2(\mathbb{R}^d)$, the difference $\tilde{\boldsymbol{v}}_t-\boldsymbol{v}_t$ is orthogonal to the tangent space, and therefore by Theorem 2.22 we have

$$\int_{\mathbb{R}^d}\langle\tilde{\boldsymbol{v}}_t-\boldsymbol{v}_t,\boldsymbol{t}_{\mu_t}^\sigma-\boldsymbol{i}\rangle\,d\mu_t=0\quad\forall\,\sigma\in\mathscr{P}_2(\mathbb{R}^d),\quad \text{for } \mathscr{L}^1\text{-a.e. } t>0. \tag{5.11}$$

As a consequence, $\boldsymbol{v}_t$ fulfills (5.5b) for $\mathscr{L}^1$-a.e. t, and this property characterizes the elements of the subdifferential.

ii) follows by the same argument, thanks to (5.11).

iii) Assume that (5.7) holds for all $\sigma\in D(\phi)$. For any $\sigma\in D(\phi)$ fixed, the differentiability of W_2^2 stated in Lemma 2.21 gives

$$\frac{1}{2}\frac{d}{dt}W_2^2(\mu_t,\sigma)=\int_{\mathbb{R}^d}\langle\boldsymbol{v}_t,\boldsymbol{i}-\boldsymbol{t}_{\mu_t}^\sigma\rangle\,d\mu_t\quad \text{for } \mathscr{L}^1\text{-a.e. } t\in(0,+\infty).$$

Therefore we can find, for any countable set $\mathcal{D}\subset D(\phi)$, a $\mathscr{L}^1$-negligible set of times $\mathcal{N}$ such that

$$-\int_{\mathbb{R}^d}\langle\boldsymbol{v}_t,\boldsymbol{t}_{\mu_t}^\sigma-\boldsymbol{i}\rangle\,d\mu_t\le\phi(\sigma)-\phi(\mu_t)-\frac{\lambda}{2}W_2^2(\sigma,\mu_t) \tag{5.12}$$

holds for all $t\in(0,+\infty)\setminus\mathcal{N}$ and all $\phi\in\mathcal{D}$. Choosing $\mathcal{D}$ to be dense relative to the distance $W_2(\mu,\nu)+|\phi(\mu)-\phi(\nu)|$ in $D(\phi)$, we obtain that (5.5b) holds for all $t\in(0,+\infty)\setminus\mathcal{N}$. The converse implication is analogous.

iv) If μ_t is a gradient flow in the sense of (5.3), taking into account that $|\mu_t'| = \|\boldsymbol{v}_t\|_{L^2(\mu_t;\mathbb{R}^d)}$ and that $|\partial\phi(\mu_t)| \le \|\boldsymbol{v}_t\|_{L^2(\mu_t;\mathbb{R}^d)}$ (by (4.52)) we obtain

$$|\partial\phi(\mu_t)||\mu_t'| \in L^1_{\rm loc}(0,+\infty)\,.$$

Thanks to the λ-convexity and the lower semicontinuity of ϕ, this implies (see Corollary 2.4.10 in [9]) that $t \mapsto \phi(\mu_t)$ is locally absolutely continuous in $(0,+\infty)$. Then, the chain rule (4.55) easily yields

$$-\frac{d}{dt}\phi(\mu_t) = \int_{\mathbb{R}^d} |\boldsymbol{v}_t|^2\, d\mu_t \ge |\partial\phi|^2(\mu_t) \tag{5.13}$$

for $\mathscr{L}^1$-a.e. $t > 0$, and therefore (5.8).

Conversely, if $t \mapsto \phi(\mu_t)$ is locally absolutely continuous and μ_t satisfies (5.8), we know that $\partial\phi(\mu_t) \neq \emptyset$ for $\mathscr{L}^1$-a.e. $t > 0$; thus the chain rule (4.55) shows that

$$\frac{d}{dt}\varphi(t) = \int_{\mathbb{R}^d} \langle \boldsymbol{\xi}, \boldsymbol{v}_t\rangle\, d\mu_t \quad \forall\, \boldsymbol{\xi} \in \partial\phi(\mu_t), \quad \text{for } \mathscr{L}^1\text{-a.e. } t > 0. \tag{5.14}$$

Choosing in particular $\boldsymbol{\xi}_t = \partial^\circ\phi(\mu_t)$, for $\mathscr{L}^1$-a.e. $t > 0$ we get

$$\int_{\mathbb{R}^d} \Big(\frac{1}{2}|\boldsymbol{v}_t|^2 + \frac{1}{2}|\boldsymbol{\xi}_t|^2 + \langle \boldsymbol{\xi}_t, \boldsymbol{v}_t\rangle\Big)\, d\mu_t \le 0. \tag{5.15}$$

It follows that

$$\boldsymbol{\xi}_t(x) = -\boldsymbol{v}_t(x) \quad \text{for } \mu_t\text{-a.e. } x \in \mathbb{R}^d,$$

i.e. $\boldsymbol{v}_t = -\partial^\circ\phi(\mu_t)$.

v) is equivalent to **iv)** by the previous argument. □

Remark 5.4 The "purely metric" formulations (5.7) or (5.8) do not require that μ_t is an absolutely continuous measure at $\mathscr{L}^1$-a.e. $t \in (0,+\infty)$ and do not depend on an explicit expression of the subdifferential of ϕ, as only the metric slope is involved; therefore they can be used to define the gradient flow of ϕ under more general assumptions: again, we refer to [9] for a complete development of this approach. Different points of view have been considered in [29, 76].

Theorem 5.5 (Uniqueness and contractivity of gradient flows) *If $\mu_t^i : (0,+\infty) \to \mathscr{P}_2(\mathbb{R}^d)$, $i = 1,2$, are gradient flows satisfying $\mu_t^i \to \mu^i \in \mathscr{P}_2(\mathbb{R}^d)$ as $t \downarrow 0$ in $\mathscr{P}_2(\mathbb{R}^d)$, then*

$$W_2(\mu_t^1, \mu_t^2) \le e^{-\lambda t} W_2(\mu^1, \mu^2) \quad \forall\, t > 0. \tag{5.16}$$

In particular, for any $\mu_0 \in \mathscr{P}_2(\mathbb{R}^d)$ there is at most one gradient flow μ_t satisfying the initial Cauchy condition $\mu_t \to \mu_0$ as $t \downarrow 0$.

Proof. If μ_t^1, μ_t^2 are two gradient flows satisfying the initial Cauchy condition $\mu_t^i \to \mu^i$ as $t \downarrow 0$, $i = 1,2$, by the E.V.I. formulation (5.7) we can apply the next Lemma 5.6 with the choices $d(s,t) := W_2^2(\mu_s^1, \mu_t^2)$, $\delta(t) := d(t,t)$, thus obtaining $\delta' \le -2\lambda\delta$. Since $\delta(0_+) = W_2^2(\mu^1, \mu^2)$ we obtain (5.16). □

Lemma 5.6 *Let $d(s,t) : (a,b)^2 \to \mathbb{R}$ be a map satisfying*

$$|d(s,t) - d(s',t)| \le |v(s) - v(s')|, \quad |d(s,t) - d(s,t')| \le |v(t) - v(t')|$$

for any s, t, s', $t' \in (a,b)$, for some locally absolutely continuous map $v : (a,b) \to \mathbb{R}$ and let $\delta(t) := d(t,t)$. Then δ is locally absolutely continuous in (a,b) and

$$\frac{d}{dt}\delta(t) \le \limsup_{h\downarrow 0} \frac{d(t,t) - d(t-h,t)}{h} + \limsup_{h\downarrow 0} \frac{d(t,t+h) - d(t,t)}{h} \quad \mathscr{L}^1\text{-a.e. in } (a,b).$$

Proof. Since $|\delta(s) - \delta(t)| \le 2|v(s) - v(t)|$ the function δ is locally absolutely continuous. We fix a nonnegative function $\zeta \in C_c^\infty(a,b)$ and $h > 0$ such that $\pm h + \operatorname{supp}\zeta \subset (a,b)$. We have then

$$-\int_a^b \delta(t)\frac{\zeta(t+h) - \zeta(t)}{h}\,dt = \int_a^b \zeta(t)\frac{d(t,t) - d(t-h,t-h)}{h}\,dt$$
$$= \int_a^b \zeta(t)\frac{d(t,t) - d(t-h,t)}{h}\,dt + \int_a^b \zeta(t+h)\frac{d(t,t+h) - d(t,t)}{h}\,dt,$$

where the last equality follows by adding and subtracting $d(t-h,t)$ and then making a change of variables in the last integral. Since

$$h^{-1}\,|d(t,t) - d(t-h,t)| \le h^{-1}\,|v(t) - v(t-h)| \to |v'(t)| \quad \text{in } L^1_{\mathrm{loc}}(a,b) \text{ as } h \downarrow 0$$

and an analogous inequality holds for the other difference quotient, we can apply (an extended version of) Fatou's Lemma and pass to the upper limit in the integrals as $h \downarrow 0$ (recall that Fatou's lemma with the limsup holds even for sequences bounded above by a sequence strongly convergent in L^1); denoting by α and β the two upper derivatives in the statement of the Lemma we get $-\int \delta\zeta'\,dt \le \int(\alpha+\beta)\zeta\,dt$, whence the inequality between distributions follows. □

5.2 Main properties of Gradient Flows

In this section we collect the main properties of the gradient flow generated by a functional $\phi : \mathscr{P}_2(\mathbb{R}^d) \to (-\infty, +\infty]$ satisfying the assumptions (5.1a,b,c). We limit this exposition to functionals ϕ whose modulus of (geodesic) convexity is quadratic (λ-convexity according to Definition 3.1); more general assumptions could also be considered as in [29].

Theorem 5.7 (Main properties of gradient flows) *Let us suppose that $\phi : \mathscr{P}_2(\mathbb{R}^d) \to (-\infty,+\infty]$ satisfies* (5.1a,b,c) *and let us suppose that its gradient flow μ_t exists for every initial value $\mu_0 \in D$, D being a dense subset of $D(\phi)$.*

λ-contractive semigroup. *For every $\mu_0 \in \overline{D(\phi)}$ there exists a unique solution $\mu := S[\mu_0]$ of the Cauchy problem associated to* (5.3) *with $\lim_{t\downarrow 0}\mu_t = \mu_0$. The map $\mu_0 \mapsto S_t[\mu_0]$ is a λ-contracting semigroup on $\overline{D(\phi)}$, i.e.*

$$W_2(S[\mu_0](t), S[\nu_0](t)) \le e^{-\lambda t} W_2(\mu_0,\nu_0) \qquad \forall\, \mu_0,\, \nu_0 \in \overline{D(\phi)}. \qquad (5.17)$$

Regularizing effect. *S_t maps $\overline{D(\phi)}$ into $D(\partial\phi) \subset D(\phi)$* for every $t > 0$,

$$\textit{the map } t \mapsto e^{\lambda t}|\partial\phi|(\mu_t) \textit{ is non increasing,} \tag{5.18}$$

and each solution $\mu_t = S_t[\mu_0]$ satisfies the following regularization estimates:

$$\begin{cases} \phi(\mu_t) \leq \dfrac{1}{2t} W_2^2(\mu_0, \sigma) + \phi(\nu) & \textit{if } \lambda = 0 \\ \phi(\mu_t) \leq \dfrac{\lambda}{2(e^{\lambda t} - 1)} W_2^2(\mu_0, \sigma) + \phi(\nu) & \textit{if } \lambda \neq 0 \end{cases} \tag{5.19}$$

$$\begin{aligned} e^{-2\lambda^- t}|\partial\phi|^2(\mu_t) \quad &\leq \quad |\partial\phi|^2(\nu) - \frac{\lambda}{2t} W_2^2(\mu_t, \sigma) + \frac{1}{t^2} W_2^2(\mu_0, \nu) \\ &- \quad \frac{\lambda}{t^2} \int_0^t W_2^2(\mu_s, \nu)\, ds \end{aligned} \tag{5.20}$$

for every $\sigma \in D(\partial\phi)$.

Energy identity. *If $\boldsymbol{v}_t \in \mathrm{Tan}_{\mu_t}\mathscr{P}_2(\mathbb{R}^d)$ is the tangent velocity field of a gradient flow $\mu_t = S_t[\mu_0]$, then the energy identity holds:*

$$\int_a^b \int_{\mathbb{R}^d} |\boldsymbol{v}_t(x)|^2 \, d\mu_t(x)\, dt + \phi(\mu_b) = \phi(\mu_a) \qquad \forall\, 0 \leq a < b < +\infty. \tag{5.21}$$

Asymptotic behavior. *If $\lambda > 0$, then ϕ admits a unique minimum point $\overline{\mu}$ and for $t \geq t_0$ we have*

$$\frac{\lambda}{2} W_2^2(\mu_t, \bar{\mu}) \leq \phi(\mu_t) - \phi(\bar{\mu}) \leq \frac{1}{2\lambda}|\partial\phi|^2(\mu_t) \quad \forall\, t \geq 0, \tag{5.22a}$$

$$W_2(\mu_t, \overline{\mu}) \leq W_2(\mu_{t_0}), \overline{\mu}) e^{-\lambda(t - t_0)}, \tag{5.22b}$$

$$\phi(\mu_t) - \phi(\overline{\mu}) \leq \Big(\phi(\mu_{t_0}) - \phi(\overline{\mu})\Big) e^{-2\lambda(t - t_0)}, \tag{5.22c}$$

$$|\partial\phi|(\mu_t) \leq |\partial\phi|(\mu_{t_0}) e^{-\lambda(t - t_0)}. \tag{5.22d}$$

If $\lambda = 0$ and $\overline{\mu}$ is any minimum point of ϕ then we have

$$\begin{gathered} |\partial\phi|(\mu_t) \leq \frac{W_2(\mu_0, \overline{\mu})}{t}, \quad \phi(\mu_t) - \phi(\overline{\mu}) \leq \frac{W_2^2(\mu_0, \overline{\mu})}{2t}, \\ \textit{the map} \quad t \mapsto W_2(\mu_t, \overline{\mu}) \quad \textit{is not increasing.} \end{gathered} \tag{5.23}$$

Right and left limits, precise pointwise formulation of the equation. *For every $t > 0$ the* right limit

$$\boldsymbol{v}_{t+} := \lim_{h \downarrow 0} \frac{\boldsymbol{t}_{\mu_t}^{\mu_{t+h}} - \boldsymbol{i}}{h} \quad \textit{exists in } L^2(\mu_t; \mathbb{R}^d) \tag{5.24}$$

and satisfies

$$-\boldsymbol{v}_{t+} = \partial^\circ\phi(\mu_t) \quad \forall\, t > 0, \tag{5.25}$$

$$\frac{d}{dt_+}\phi(\mu_t) = -\int_{\mathbb{R}^d} |\boldsymbol{v}_{t+}|^2 \, d\mu_t = -|\partial\phi|^2(\mu_t) \quad \forall\, t > 0. \tag{5.26}$$

(5.24), (5.25), and (5.26) hold at $t = 0$ iff $\mu_0 \in D(\partial\phi) = D(|\partial\phi|)$. Moreover, there exists an at most countable set $\mathcal{C} \subset (0, +\infty)$ such that the analogous identities for the left *limits hold for every $t \in (0, +\infty) \setminus \mathcal{C}$:*

$$\begin{cases} \boldsymbol{v}_{t-} = \lim\limits_{h\downarrow 0} \dfrac{\boldsymbol{t}_{\mu_t}^{\mu_{t-h}} - \boldsymbol{i}}{h} = -\partial^\circ\phi(\mu_t), \\ \dfrac{d}{dt_-}\phi(\mu_t) = -|\partial\phi|^2(\mu_t). \end{cases} \quad \forall\, t \in (0, +\infty) \setminus \mathcal{C}. \tag{5.27}$$

Proof.

Regularizing effect. We first observe that for every $h > 0$ the map $t \mapsto \mu_{t+h}$ is still a gradient flow, and therefore estimate (5.16) yields

$$W_2(\mu_{t+h}, \mu_t) \le e^{-\lambda(t-t_0)} W_2(\mu_{t_0+h}, \mu_{t_0}) \quad \forall\, 0 \le t_0 < t < +\infty. \tag{5.28}$$

Setting

$$\delta(t) := \limsup_{h\downarrow 0} \frac{W_2(\mu_{t+h}, \mu_t)}{h} \quad t \ge 0, \tag{5.29}$$

(5.28) yields

$$\text{the map} \quad t \mapsto e^{\lambda t}\delta(t) \quad \text{is nonincreasing.} \tag{5.30}$$

We denote by $\mathscr{N}$ the subset of $(0, +\infty)$ whose points t_0 satisfies $\mu_{t_0} \in D(\partial\phi) \subset \mathscr{P}_2^a(\mathbb{R}^d)$, the metric derivative of μ_t coincides with $\|\boldsymbol{v}_t\|_{L^2(\mu_t;\mathbb{R}^d)}$ and $-\boldsymbol{v}_{t_0} = \partial^\circ\phi(\mu_{t_0})$: by the definition of gradient flow, Theorem 2.15, and point *v)* of Theorem 5.3, $\mathscr{L}^1\big((0, +\infty) \setminus \mathscr{N}\big) = 0$ and

$$\delta(t) = \|\boldsymbol{v}_t\|_{L^2(\mu_{t_0};\mathbb{R}^d)} = |\partial\phi|(\mu_t) < +\infty \quad \forall\, t \in (0, +\infty) \setminus \mathscr{N}; \tag{5.31}$$

in particular, (5.30) yields $\delta(t) < +\infty$ for every $t > 0$.

We want to show now that

$$\delta(t) = |\partial\phi|(\mu_t) \quad \forall\, t \ge 0. \tag{5.32}$$

Integrating the E.V.I. (5.7) in the interval $(t, t+h)$ and dividing by h we get for every $\sigma \in D(\phi)$

$$\begin{aligned} \frac{1}{h}\int_0^h \Big(\phi(\mu_{t+s}) + \frac{\lambda}{2} W_2^2(\mu_{t+s}, \sigma)\Big)\, ds - \phi(\sigma) \\ \le & \frac{1}{2h} W_2^2(\mu_t, \sigma) - \frac{1}{2h} W_2^2(\mu_{t+h}, \sigma) \\ \le & \frac{W_2(\mu_{t+h}, \mu_t)}{2h}\Big(W_2(\mu_t, \sigma) + W_2(\mu_{t+h}, \sigma)\Big). \end{aligned} \tag{5.33}$$

Passing to the limit as $h \downarrow 0$ and recalling that the map $t \mapsto \phi(\mu_t)$ is (absolutely) continuous, we obtain

$$\phi(\mu_t) - \phi(\sigma) + \frac{\lambda}{2} W_2^2(\mu_t, \sigma) \le \delta(t) W_2(\mu_t, \sigma), \tag{5.34}$$

which yields

$$|\partial\phi|(\mu_t) \le \delta(t) \quad \forall\, t \ge 0. \tag{5.35}$$

Choosing $\sigma := \mu_t$ in (5.33), and rescaling the integrand, we can use (4.40) to obtain

$$\begin{aligned}
\frac{1}{2h^2} W_2^2(\mu_{t+h}, \mu_t) &\le \frac{1}{h} \int_0^1 \Big(\phi(\mu_t) - \phi(\mu_{t+hs}) - \frac{\lambda}{2} W_2^2(\mu_{t+hs}, \mu_t) \Big)\, ds \\
&\le |\partial\phi|(\mu_t) \int_0^1 \frac{W_2(\mu_{t+hs}, \mu_t)}{hs} s\, ds - \lambda \int_0^1 \frac{W_2^2(\mu_{t+hs}, \mu_t)}{h}\, ds.
\end{aligned}$$

Passing to the limit as $h \downarrow 0$ we obtain

$$\frac{1}{2}\delta^2(t) \le |\partial\phi|(\mu_t) \int_0^1 \delta(t) s\, ds = \frac{1}{2} |\partial\phi|(\mu_t) \delta(t), \tag{5.36}$$

which yields (5.32) and in particular (5.18).

The estimates (5.19) follow easily by integrating in the interval $(0, t)$ the following form of (5.7)

$$\frac{d}{ds} \frac{e^{\lambda s}}{2} W_2^2(\mu_s, \sigma) + e^{\lambda s} \phi(\mu_s) \le e^{\lambda s} \phi(\sigma) \tag{5.37}$$

and recalling that $t \mapsto \phi(\mu_t)$ is nonincreasing; when $\lambda \ne 0$ we get

$$\begin{aligned}
\frac{e^{\lambda t} - 1}{\lambda} \phi(\mu_t) &\le \int_0^t \frac{d}{ds} \frac{e^{\lambda s}}{2} W_2^2(\mu_s, \sigma)\, ds + \int_0^t e^{\lambda s} \phi(\sigma)\, ds \\
&\le \frac{1}{2} W_2^2(\mu_0, \sigma) + \frac{e^{\lambda t} - 1}{\lambda} \phi(\sigma).
\end{aligned}$$

In order to show (5.20) we apply (5.18), the fact that $-\frac{d}{dt}\phi(\mu_t) = |\partial\phi|^2(\mu_t)$ and finally the E.V.I. to obtain

$$\begin{aligned}
\frac{e^{-2\lambda^- t} t^2}{2} |\partial\phi|^2(\mu_t) &\le \int_0^t s e^{-2\lambda^- s} |\partial\phi|^2(\mu_s)\, ds \le - \int_0^t s \big(\phi(\mu_s)\big)'\, ds \\
&= \int_0^t \phi(\mu_s)\, ds - t\phi(\mu_t) \\
&\le t(\phi(\sigma) - \phi(\mu_t)) + \frac{1}{2} W_2^2(\mu_0, \sigma) - \frac{1}{2} W_2^2(\mu_t, \sigma) - \frac{\lambda}{2} \int_0^t W_2^2(\mu_s, \sigma)\, ds.
\end{aligned}$$

If $\sigma \in D(\partial\phi)$, using (4.40) we can bound the right hand side by

$$\begin{aligned}
t|\partial\phi|(\sigma) W_2(\mu_t, \sigma) &- \frac{1}{2}(t\lambda + 1) W_2^2(\mu_t, \sigma) + \frac{1}{2} W_2^2(\mu_0, \sigma) - \frac{\lambda}{2} \int_0^t W_2^2(\mu_s, \sigma)\, ds \\
&\le \frac{t^2}{2} |\partial\phi|^2(\sigma) - \frac{t\lambda}{2} W_2^2(\mu_t, \sigma) + \frac{1}{2} W_2^2(\mu_0, \sigma) - \frac{\lambda}{2} \int_0^t W_2^2(\mu_s, \sigma)\, ds,
\end{aligned}$$

which yields (5.20).

λ-contractive semigroup. Thanks to the λ-contraction estimate of Theorem 5.5, it is now easy to extend the semigroup S defined on D to its closure, which coincides with $\overline{D(\phi)}$. Observe that each trajectory μ_t of the extended semigroup still satisfies the E.V.I. formulation (5.7); moreover, the previous regularization estimates show that $t \mapsto \mu_t$ is locally Lipschitz and $\mu_t \in D(|\partial\phi|)$ for every $t > 0$, in particular $\mu_t \in \mathscr{P}_2^a(\mathbb{R}^d)$ for every $t > 0$. Theorem 5.3 then shows that μ_t is a gradient flow for ϕ.

Energy identity. It is an immediate consequence of (5.9).

Asymptotic behavior. When $\lambda > 0$ (5.28) shows that for every gradient flow μ_t the sequence $k \mapsto \mu_k$ satisfies the Cauchy condition in $\mathscr{P}_2(\mathbb{R}^d)$, since

$$W_2(\mu_{k+1}, \mu_k) \le e^{-\lambda} W_2(\mu_k, \mu_{k-1}). \tag{5.38}$$

Therefore it is convergent to some limit $\bar\mu$; (5.19) and the lower semicontinuity of ϕ show that $\bar\mu$ is a minimum point for ϕ; in particular, the constant curve $t \mapsto \bar\mu$ is a gradient flow. (5.22b) is a particular case of the λ-contraction property (5.17) and in particular it shows that the minimum point $\bar\mu$ is unique, when $\lambda > 0$.

The inequality (5.22d) is simply (5.30), while (5.22a) is a general property of λ-geodesically convex functions (even in metric spaces, see Theorem 2.4.14 of [9]): for, if $\mu \in D(\partial\phi)$, property (4.40) of the slope and Young inequality yield

$$\phi(\mu) - \phi(\bar\mu) \le |\partial\phi|(\mu) W_2(\mu, \bar\mu) - \frac{\lambda}{2} W_2^2(\mu, \bar\mu) \le \frac{1}{2\lambda} |\partial\phi|^2(\mu). \tag{5.39}$$

For the opposite inequality, being $0 \in \partial\phi(\bar\mu)$, from (4.37) we easily get

$$\phi(\mu) - \phi(\bar\mu) \ge \frac{\lambda}{2} W_2^2(\mu, \bar\mu). \tag{5.40}$$

The estimate (5.22c) now follows by observing that (5.39) yields

$$\frac{d}{dt}\big(\phi(\mu_t) - \phi(\bar\mu)\big) = -|\partial\phi|^2(\mu_t) \le -2\lambda\big(\phi(\mu_t) - \phi(\bar\mu)\big). \tag{5.41}$$

Right and left limits, precise pointwise formulation of the equation. Here, for the sake of simplicity, we are assuming that $\lambda \ge 0$.

We already know that $\partial\phi(\mu_t)$ is not empty for $t > 0$: we set $\boldsymbol{\xi}_t = \partial^\circ\phi(\mu_t)$; since the slope $|\partial\phi|$ is lower semicontinuous (see (4.43)) and the map $t \mapsto |\partial\phi|(\mu_t)$ is nonincreasing, we obtain

$$|\partial\phi|(\mu_t) = \lim_{h\downarrow 0} |\partial\phi|(\mu_{t+h}). \tag{5.42}$$

Moreover, the map $t \mapsto \phi(\mu_t)$ is absolutely continuous, nonincreasing, and its time derivative coincides $\mathscr{L}^1$-a.e. with the nondecreasing map $-|\partial\phi|^2(\mu_t)$; it follows that $t \mapsto \phi(\mu_t)$ is continuous and convex, so that

$$\exists \frac{d}{dt_+}\phi(\mu_t) = \lim_{h\downarrow 0} \frac{\phi(\mu_{t+h}) - \phi(\mu_t)}{h} = -|\partial\phi|^2(\mu_t) = -\delta^2(t) \quad \forall\, t > 0. \tag{5.43}$$

Let now fix $t > 0$ and an infinitesimal sequence h_n such that

$$\frac{\boldsymbol{t}_{\mu_t}^{\mu_{t+h_n}} - \boldsymbol{i}}{h_n} \rightharpoonup \tilde{\boldsymbol{v}}_t \quad \text{weakly in } L^2(\mu_t; \mathbb{R}^d). \tag{5.44}$$

By the definition of subdifferential, it is immediate to check that

$$-|\partial\phi|^2(\mu_t) = -\|\boldsymbol{\xi}_t\|^2_{L^2(\mu_t;\mathbb{R}^d)} = \frac{d}{dt+}\phi(\mu_t) \geq \int_{\mathbb{R}^d} \langle \boldsymbol{\xi}_t, \tilde{\boldsymbol{v}}_t\rangle \, d\mu_t. \tag{5.45}$$

On the other hand

$$\|\tilde{\boldsymbol{v}}_t\|_{L^2(\mu_t;\mathbb{R}^d)} \leq \delta(t) = \|\boldsymbol{\xi}_t\|_{L^2(\mu_t;\mathbb{R}^d)}. \tag{5.46}$$

It follows that $\tilde{\boldsymbol{v}}_t = -\boldsymbol{\xi}_t$; since the limit is uniquely determined independently of the subsequence h_n, we obtain that

$$\lim_{h\downarrow 0} \frac{\boldsymbol{t}_{\mu_t}^{\mu_{t+h}} - \boldsymbol{i}}{h} = -\boldsymbol{\xi}_t \quad \text{weakly in } L^2(\mu_t; \mathbb{R}^d). \tag{5.47}$$

On the other hand

$$\limsup_{h\downarrow 0} \left\| \frac{\boldsymbol{t}_{\mu_t}^{\mu_{t+h}} - \boldsymbol{i}}{h} \right\|_{L^2(\mu_t;\mathbb{R}^d)} = \limsup_{h\downarrow 0} \frac{W_2(\mu_t, \mu_{t+h})}{h} = \delta(t) = \|\boldsymbol{\xi}_t\|_{L^2(\mu_t;\mathbb{R}^d)}$$

and therefore the limit in (5.47) is also *strong* in $L^2(\mu_t; \mathbb{R}^d)$.

The same argument can be applied for the left limit at each continuity point of the map $t \mapsto |\partial\phi|(\mu_t)$ (whose complement $\mathcal{C}$ in $(0, +\infty)$ is at most countable) i.e. for every t such that

$$\lim_{h\downarrow 0} |\partial\phi|(\mu_{t-h}) = |\partial\phi|(\mu_t), \tag{5.48}$$

observing that in this case

$$\exists \frac{d}{dt}\phi(\mu_t) = -|\partial\phi|^2(\mu_t) \tag{5.49}$$

and (by the $\mathscr{L}^1$-a.e. equality of $|\mu_t|'$ and $|\partial\phi(\mu_t)|$ and the monotonicity of $|\partial\phi(\mu_t)|$)

$$\frac{W_2(\mu_{t-h}, \mu_t)}{h} = \frac{1}{h}\int_{t-h}^t |\mu_s'| \, ds = \frac{1}{h}\int_{t-h}^t |\partial\phi|(\mu_s) \, ds \leq |\partial\phi|(\mu_{t-h}), \tag{5.50}$$

and therefore for any $t \in (0, +\infty) \setminus \mathcal{C}$ we have

$$\limsup_{h\downarrow 0} \left\| \frac{\boldsymbol{t}_{\mu_t}^{\mu_{t-h}} - \boldsymbol{i}}{h} \right\|_{L^2(\mu_t;\mathbb{R}^d)} = \limsup_{h\downarrow 0} \frac{W_2(\mu_{t-h}, \mu_t)}{h} \leq |\partial\phi|(\mu_t). \tag{5.51}$$

$\square$

5.3 Existence of Gradient Flows by convergence of the "Minimizing Movement" scheme

The existence of solutions to the Cauchy problem for (5.3) will be obtained as limit of a variational approximation scheme (the "Minimizing Movement" scheme, in De Giorgi's terminology [36]), which we will briefly recall.

The variational approximation scheme. Let us introduce a uniform partition $\mathcal{P}_\tau$ of $(0,+\infty)$ by intervals I_τ^n of size $\tau > 0$

$$\mathcal{P}_\tau := \left\{ 0 < t_\tau^1 = \tau < t_\tau^2 = 2\tau < \cdots < t_\tau^n = n\tau < \cdots \right\}, \quad I_\tau^n := ((n-1)\tau, n\tau],$$

and a given family of "discrete" values M_τ^0 approximating the initial value $\mu_0 \in \overline{D(\phi)}$ so that

$$M_\tau^0 \to \mu_0 \quad \text{in } \mathscr{P}_2(\mathbb{R}^d), \quad \phi(M_\tau^0) \to \phi(\mu_0) \quad \text{as } \tau \downarrow 0. \tag{5.52}$$

If (5.1c) and (5.1d) are satisfied, for every $\tau \in (0, \tau_*)$ we can find sequences $(M_\tau^n)_{n\in\mathbb{N}} \subset \mathscr{P}_2^a(\mathbb{R}^d)$ recursively defined by solving the variational problem

$$M_\tau^n \quad \text{minimizes} \quad \mu \mapsto \Phi(\tau, M_\tau^{n-1}; \mu) = \frac{1}{2\tau} W_2^2(\mu, M_\tau^{n-1}) + \phi(\mu). \tag{5.53}$$

We call "discrete solution" the piecewise constant interpolant

$$\overline{M}_\tau(t) := M_\tau^n \quad \text{if } t \in ((n-1)\tau, n\tau], \tag{5.54}$$

and we say that a curve μ_t is a **Minimizing Movement** of Φ starting from μ_0, writing $\mu_t \in MM(\Phi; \mu_0)$, if there exists a family of discrete solutions $\overline{M}_\tau$ such that

$$\overline{M}_\tau(t) \to \mu_t \quad \text{in } \mathscr{P}_2(\mathbb{R}^d) \text{ for every } t > 0, \quad \text{as } \tau \downarrow 0. \tag{5.55}$$

In order to clarify why this variational scheme provides an approximation of the Gradient Flow equation (5.3), we introduce the optimal transport maps $\boldsymbol{t}_\tau^n = \boldsymbol{t}_{M_\tau^n}^{M_\tau^{n-1}}$ pushing M_τ^n to M_τ^{n-1}, and we define the discrete velocity vector $\boldsymbol{V}_\tau^n$ as $(\boldsymbol{i} - \boldsymbol{t}_\tau^n)/\tau$. By Lemma 4.4 and Theorem 4.20

$$-\boldsymbol{V}_\tau^n = \frac{\boldsymbol{t}_\tau^n - \boldsymbol{i}}{\tau} \in \partial\phi(M_\tau^n), \tag{5.56}$$

which can be considered as an Euler implicit discretization of (5.3). By introducing the piecewise constant interpolant

$$\overline{\boldsymbol{V}}_\tau(t) := \boldsymbol{V}_\tau^n \quad \text{if } t \in ((n-1)\tau, n\tau], \tag{5.57}$$

the identity (5.56) reads

$$-\overline{\boldsymbol{V}}_\tau(t) \in \partial\phi(\overline{M}_\tau(t)) \quad \text{for } t > 0. \tag{5.58}$$

By general compactness arguments, it is not difficult to show that, up to subsequences, $\overline{\boldsymbol{V}}_\tau \overline{M}_\tau \rightharpoonup \boldsymbol{v}\mu$ in the distribution sense in $\mathbb{R}^d \times (0,+\infty)$, for some vector field $\boldsymbol{v}(t,x) = \boldsymbol{v}_t(x)$ satisfying

$$\partial_t \mu_t + \nabla \cdot (\boldsymbol{v}_t \mu_t) = 0 \quad \text{in } \mathbb{R}^d \times (0,+\infty), \quad \|\boldsymbol{v}_t\|_{L^2(\mu_t;\mathbb{R}^d)} \in L^2_{\rm loc}(0,+\infty). \tag{5.59}$$

The main difficulty is to show that the nonlinear equation (5.58) is preserved in the limit.

Here we present two proofs of this fact based on two qualitatively different assumptions: the first one is a *coercivity assumption*: for every $C > 0$ the sublevels

$$\Big\{\mu \in \mathscr{P}_2(\mathbb{R}^d) : \phi(\mu) \le C,\ \mathsf{m}_2(\mu) \le C\Big\} \quad \text{are } \textit{compact} \text{ in } \mathscr{P}_2(\mathbb{R}^d). \tag{5.60}$$

The second one is a *strong convexity assumption*: for every $\mu \in D(|\partial\phi|)$ and $\sigma_0,\, \sigma_1 \in D(\phi)$

$$\text{the map} \quad \begin{cases} s \mapsto \phi\big(\sigma_s\big) - \dfrac{\lambda}{2} W_2^2(\sigma_0,\sigma_1)s^2 \\ \sigma_s := \big((1-s)\boldsymbol{t}_\mu^{\sigma_0} + s\boldsymbol{t}_\mu^{\sigma_1}\big)_{\#}\,\mu \end{cases} \quad \text{is convex in } [0,1]. \tag{5.61}$$

The first assumption is typically satisfied when the domain of ϕ consists of measures supported in a bounded domain (as in this case convergence in $\mathscr{P}_2(\mathbb{R}^d)$ reduces to the narrow convergence). The second assumption is slightly stronger than λ-convexity along geodesics (corresponding to the case when either $\mu = \sigma_0$ or $\mu = \sigma_1$), but it happens that the conditions imposed on the internal, potential and interaction energy functionals to ensure convexity along geodesics, ensure (5.61) as well. The same phenomenon occurs for $-W_2^2(\cdot,\nu)$, that turns out to satisfy (5.61) with $\lambda = -1$.

In [9] (see in particular Theorem 11.3.2 therein) one can find more general results where one imposes only compactness with respect to the narrow topology of $\mathscr{P}(\mathbb{R}^d)$ and convexity along geodesics: in this case one has to impose that both ϕ and $|\partial\phi|$ are lower semicontinuous with respect to the narrow convergence, an assumption that is fulfilled in many cases of interest. However, the proof of these convergence results is much harder, compared to the one presented here, and it involves a deep *variational* interpolation argument due to De Giorgi.

Theorem 5.8 (Existence and approximation of Gradient Flows) *Let us assume that $\phi : \mathscr{P}_2(\mathbb{R}^d) \to (-\infty,+\infty]$ satisfy* (5.1a,b,c) *and at least one of the conditions* (5.60), (5.61) *hold. Then for every $\mu_0 \in \overline{D(\phi)}$ there exists a unique solution μ_t of the gradient flow (according to Definition 5.2) satisfying the Cauchy condition*

$$\lim_{t\downarrow 0} \mu_t = \mu_0 \quad \textit{in } \mathscr{P}_2(\mathbb{R}^d). \tag{5.62}$$

Moreover, for every choice of the discrete initial values M_τ^0 satisfying (5.52), *the discrete solutions $\overline{M}_\tau(t)$ converge to μ_t in $\mathscr{P}_2(\mathbb{R}^d)$, uniformly in each bounded time interval.*

Finally, if condition (5.61) *holds with* $\lambda \ge 0$ *and* $M_\tau^0 = \mu_0 \in D(\phi)$*, for every* $t = k\tau \in \mathcal{P}_\tau$ *we have the* a priori *error estimate*

$$W_2^2(\mu_t, \overline{M}_\tau(t)) \le \tau\big(\phi(\mu_0) - \phi_\tau(\mu_0)\big) \le \frac{\tau^2}{2}|\partial\phi|^2(\mu_0), \tag{5.63}$$

where we set

$$\phi_\tau(\mu) := \inf_{\nu \in \mathscr{P}_2(\mathbb{R}^d)} \phi(\nu) + \frac{1}{2\tau}W_2^2(\mu,\nu) = \inf_{\nu \in \mathscr{P}_2(\mathbb{R}^d)} \Phi(\tau,\mu;\nu). \tag{5.64}$$

We give two separate proofs of this result, in the coercive case and in the strongly convex case. For the sake of simplicity, we also assume that $\phi \ge 0$; the *a priori* estimates needed in the more general coercive case can be found in [9].

Proof of Theorem 5.8 in the coercive case.

A priori estimates. We easily have

$$\frac{\tau}{2}\frac{W_2^2(M_\tau^n, M_\tau^{n-1})}{\tau^2} + \phi(M_\tau^n) \le \phi(M_\tau^{n-1}), \tag{5.65}$$

which yields

$$\phi(M_\tau^n) \le \phi(M_\tau^0) \quad \forall\, n \in \mathbb{N}, \qquad \sum_{n=1}^{+\infty} \frac{W_2^2(M_\tau^n, M_\tau^{n-1})}{\tau^2} \le \frac{2\phi(M_\tau^0)}{\tau}. \tag{5.66}$$

In terms of $\overline{M}_\tau$, this means that

$$\sup_{t \ge 0} \phi(\overline{M}_\tau(t)) \le \phi(M_\tau^0) \qquad \forall\, \tau > 0. \tag{5.67}$$

From the last inequality of (5.66) we get for $0 \le m \le n$

$$\begin{aligned} W_2(M_\tau^n, M_\tau^m) &\le \tau \sum_{k=m+1}^{n} \frac{W_2(M_\tau^k, M_\tau^{k-1})}{\tau} \\ &\le \tau \left(\sum_{k=1}^{n} \frac{W_2^2(M_\tau^k, M_\tau^{k-1})}{\tau^2} \right)^{1/2} \Big((n-m)\Big)^{1/2} \\ &\le \Big(2\phi(M_\tau^0)\Big)^{1/2} \Big((m-n)\tau\Big)^{1/2}. \end{aligned} \tag{5.68}$$

Compactness and limit trajectory μ_t***.*** (5.68) and (5.52) show that in each bounded interval $(0,T)$ the values $\{\phi(\overline{M}_\tau(t))\}_{\tau>0}$ are bounded and $\{\overline{M}_\tau(t)\}_{\tau>0}$ are bounded in $\mathscr{P}_2(\mathbb{R}^d)$, thus belong to a fixed compact set of $\mathscr{P}_2(\mathbb{R}^d)$ thanks to the coercivity assumption (5.60).

By connecting every pair of consecutive discrete values M_τ^{n-1}, M_τ^n with a constant speed geodesic parametrized in the interval $[t_\tau^{n-1}, t_\tau^n]$, we obtain by (5.68) a family of Lipschitz curves $\hat{M}_\tau$ satisfying

$$\begin{aligned} W_2(\hat{M}_\tau(t), \hat{M}_\tau(s)) \le C(t-s)^{1/2}, \\ W_2(\hat{M}_\tau(t), \overline{M}_\tau(t)) \le C\sqrt{\tau} \qquad \forall\, t,\, s \in [0,T], \end{aligned} \tag{5.69}$$

where C is a constant independent of τ. Since the curves $\hat{M}_\tau$ are uniformly equicontinuous w.r.t. W_2, Ascoli-Arzelà Theorem yields the relative compactness of the family $\{\hat{M}_{\tau_h}\}_{h\in\mathbb{N}}$ in $C^0([0,T];\mathscr{P}_2(\mathbb{R}^d))$ for each bounded interval $[0,T]$; we can therefore extract a vanishing sequence (τ_h) such that $\overline{M}_{\tau_h}(t) \to \mu_t$ in $\mathscr{P}_2(\mathbb{R}^d)$ for any $t \in [0,+\infty)$.

Space-time measures and construction of $\boldsymbol{v}$. Recall that $\boldsymbol{t}_\tau^n$ is the optimal transport map pushing M_τ^n to M_τ^{n-1}, and that the discrete velocity vector $\boldsymbol{V}_\tau^n$ is defined by $(\boldsymbol{i} - \boldsymbol{t}_\tau^n)/\tau$. Let us introduce the piecewise constant interpolants

$$\overline{\boldsymbol{t}}_\tau(t) := \boldsymbol{t}_\tau^n \quad \text{if } t \in ((n-1)\tau, n\tau]. \tag{5.70}$$

For every bounded time interval $I_T := (0,T]$, denoting by $X_T := \mathbb{R}^d \times I_T$, we can canonically identify $T^{-1}\overline{M}_\tau$ and $T^{-1}\mu$ to elements of $\mathscr{P}_2(X_T)$ simply by integrating with respect to the (normalized) Lebesgue measure $T^{-1}\mathscr{L}^1$ in I_T. Therefore $\overline{\boldsymbol{V}}_\tau$ is a vector field in $L^2(\overline{M}_\tau;\mathbb{R}^d)$ and (5.66) yields

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^d} |\overline{\boldsymbol{V}}_\tau(x,t)|^2 \, d\overline{M}_\tau(x)\, dt &= \int_{X_T} |\overline{\boldsymbol{V}}_\tau(x,t)|^2 \, d\overline{M}_\tau(x,t) \\ &= \tau \sum_{n=1}^{+\infty} \frac{\|\boldsymbol{V}_\tau^n\|^2_{L^2(M_\tau^n)}}{\tau^2} \le 2\phi(\mu_0). \end{aligned} \tag{5.71}$$

Hence, by Theorem 4.6, and taking into account the convergence in $\mathscr{P}_2(X_T)$ of $T^{-1}\overline{M}_\tau$ to $T^{-1}\mu$, the family $\overline{\boldsymbol{V}}_\tau$ has limit points as $\tau \downarrow 0$. We denote by $\boldsymbol{v}$ the limit (up to the extraction of a further subsequence, not relabeled) of $\boldsymbol{V}_{\tau_h}$.

Then, (4.46) and (5.71) give

$$\int_{X_T} |\boldsymbol{v}(x,t)|^2 \, d\mu(x,t) \le \liminf_{h\to\infty} \int_{X_T} |\overline{\boldsymbol{V}}_{\tau_h}(x,t)|^2 \, d\overline{M}_{\tau_h}(x,t) \le 2\phi(\mu_0). \tag{5.72}$$

***The limits μ, $\boldsymbol{v}$ satisfy the continuity equation* (5.4b).** The following argument was introduced in [57]. Let us first observe that for every $\psi \in C_c^\infty(\mathbb{R}^d)$

we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \psi(x)\, d\overline{M}_\tau(t)(x) - \int_{\mathbb{R}^d} \psi(x)\, d\overline{M}_\tau(t-\tau)(x) \\
&\quad = \int_{\mathbb{R}^d} \big(\psi(x) - \psi(\bar{\boldsymbol{t}}_\tau(x,t))\big)\, d\overline{M}_\tau(t)(x) \\
&\quad = \int_{\mathbb{R}^d} \langle \nabla\psi(x), x - \bar{\boldsymbol{t}}_\tau(x,t)\rangle\, d\overline{M}_\tau(t)(x) + \varepsilon(\tau,\psi,t) \\
&\quad = \tau \int_{\mathbb{R}^d} \langle \nabla\psi(x), \overline{\boldsymbol{V}}_\tau(x)\rangle\, d\overline{M}_\tau(t)(x) + \varepsilon(\tau,\psi,t),
\end{aligned}$$

where, for a suitable constant C_ψ depending only on the second derivatives of ψ

$$\begin{aligned}
|\varepsilon(\tau,\psi,t)| &= \left| \int_{\mathbb{R}^d} \Big(\psi(x) - \psi(\bar{\boldsymbol{t}}_\tau(x,t)) - \nabla\psi(x)\cdot(x - \bar{\boldsymbol{t}}_\tau(x,t))\Big)\, d\overline{M}_\tau(t)(x) \right| \\
&\le C_\psi \int_{\mathbb{R}^d} |x - \bar{\boldsymbol{t}}_\tau(x,t)|^2\, d\overline{M}_\tau(t)(x) = C_\psi \tau^2 \int_{\mathbb{R}^d} \left|\overline{\boldsymbol{V}}_\tau(x,t)\right|^2\, d\overline{M}_\tau(t)(x).
\end{aligned}$$

Choosing now $\varphi \in C_c^\infty((0,T)\times\mathbb{R}^d)$, applying the estimate above with $\psi(\cdot) = \varphi(t,\cdot)$ and taking into account (5.71), we have

$$\begin{aligned}
&-\int_{X_T} \partial_t\varphi(x,t)\, d\mu(x,t) = \lim_{h\to\infty} -\int_{X_T} \partial_t\varphi(x,t)\, d\overline{M}_{\tau_h}(x,t) = \\
&\quad = \lim_{h\to\infty} -\tau_h^{-1} \int_{X_T} \big(\varphi(x,t+\tau_h) - \varphi(x,t)\big)\, d\overline{M}_{\tau_h}(x,t) \\
&\quad = \lim_{h\to\infty} \int_{X_T} \langle \nabla\varphi(t,x), \overline{\boldsymbol{V}}_{\tau_h}\rangle\, d\overline{M}_{\tau_h}(x,t) + \tau_h^{-1} \int_0^T \varepsilon(\tau_h, \phi(t,\cdot), t)\, dt \\
&\quad = \int_{X_T} \langle \nabla\varphi(t,x), \boldsymbol{v}\rangle\, d\mu(x,t).
\end{aligned}$$

The limits μ, $\boldsymbol{v}$ satisfy the equation $-\boldsymbol{v}_t \in \partial\phi(\mu_t)$. For $\sigma \in \mathscr{P}_2(\mathbb{R}^d)$ fixed we can use the variational characterization of the subdifferential (4.37) and (5.58) to obtain

$$\phi(\sigma) \ge \phi(\overline{M}_\tau(t)) - \int_{\mathbb{R}^d} \langle \bar{\boldsymbol{t}}_\tau(t) - \boldsymbol{i}, \overline{\boldsymbol{V}}_\tau(t)\rangle\, d\overline{M}_\tau(t) + \lambda W_2^2(\sigma, \overline{M}_\tau(t))$$

for all $\tau > 0$, $t > 0$. Then, we choose a nonnegative $\eta \in C_c^\infty((0,T))$ with $\int \eta\, dt = 1$ and integrate in time the previous inequality multiplied by $\eta(t)$ to find

$$\begin{aligned}
\phi(\sigma) \ge &\int_0^T \phi(M_\tau(t))\eta(t)\, dt - \int_{X_T} \langle \bar{\boldsymbol{s}}_\tau(t) - \boldsymbol{i}, \overline{\boldsymbol{V}}_\tau(t)\rangle\, d\overline{M}_\tau(t)\eta(t)\, dt \qquad (5.73) \\
&+ \lambda \int_0^T W_2^2(\sigma, \overline{M}_\tau(t))\eta(t)\, dt,
\end{aligned}$$

where $\overline{\boldsymbol{s}}_\tau(t)$ is the optimal transport map between $\overline{M}_\tau(t)$ and σ. Next, we set $\tau = \tau_h$ in (5.73) and pass to the limit as $h \to \infty$. By the lower semicontinuity of ϕ and the convergence of $\overline{M}_{\tau_h}(t)$ to μ_t, the convergence of the first and third integrals in the right hand side is trivial. Concerning the second integrals, their passage to the limit is ensured by the time-dependent version of Lemma 4.7, see Remark 4.9. Therefore we obtain

$$\phi(\sigma) \geq \int_0^T \phi(\mu_t)\eta(t)\,dt - \int_{X_T} \langle \boldsymbol{t}_{\mu_t}^\sigma - \boldsymbol{i}, \boldsymbol{v}_t\rangle\,d\mu_t\eta(t)\,dt + \lambda\int_0^T W_2^2(\sigma,\mu_t)\eta(t)\,dt.$$

If $\bar{t} \in (0,T)$ is a Lebesgue point for the map

$$t \mapsto \int_{\mathbb{R}^d} \langle \boldsymbol{t}_{\mu_t}^\sigma - \boldsymbol{i}, \boldsymbol{v}_t\rangle\,d\mu_t,$$

choosing a family η_i converging to $\delta_{\bar{t}}$ in the inequality above we get

$$\phi(\sigma) \geq \phi(\mu_{\bar{t}}) - \int_{\mathbb{R}^d} \langle \boldsymbol{t}_{\mu_{\bar{t}}}^\sigma - \boldsymbol{i}, \boldsymbol{v}_{\bar{t}}\rangle\,d\mu_{\bar{t}}.$$

As σ is arbitrary, (4.37) again gives that $-\boldsymbol{v}_{\bar{t}} \in \partial\phi(\mu_{\bar{t}})$.

In conclusion, the uniqueness of gradient flows gives that μ, $\boldsymbol{v}$ do not depend on the chosen subsequence, and so there is full convergence as $\tau \downarrow 0$. Finally, a simple compactness argument based on the equi-continuity of $\overline{M}_\tau$ gives the local uniform convergence in $[0,+\infty)$.

Proof of Theorem 5.8 in the strongly convex case.

We shall only give a brief sketch of the proof (showing a rough error estimate, still sufficient to prove convergence) in a simplified setting, by assuming that the strong convexity assumption (5.61) holds for $\lambda \geq 0$, ϕ is nonnegative, and μ_0, $M_\tau^0 \in D(\phi)$.

As a preliminary remark, let us observe that if σ_s is defined as in (5.61) we have

$$\begin{aligned} W_2^2(\mu,\sigma_s) &= \int_{\mathbb{R}^d} \Big|(1-s)\boldsymbol{t}_\mu^{\sigma_0} + s\boldsymbol{t}_\mu^{\sigma_1} - \boldsymbol{i}\Big|^2\,d\mu \\ &= \int_{\mathbb{R}^d} \left((1-s)\Big|\boldsymbol{t}_\mu^{\sigma_0} - \boldsymbol{i}\Big|^2 + s\Big|\boldsymbol{t}_\mu^{\sigma_1} - \boldsymbol{i}\Big| - s(1-s)\Big|\boldsymbol{t}_\mu^{\sigma_0} - \boldsymbol{t}_\mu^{\sigma_1}\Big|^2\right)d\mu \\ &= (1-s)W_2^2(\mu,\sigma_0) + sW_2^2(\mu,\sigma_1) - s(1-s)\int_{\mathbb{R}^d}\Big|\boldsymbol{t}_\mu^{\sigma_0} - \boldsymbol{t}_\mu^{\sigma_1}\Big|^2\,d\mu \\ &\leq (1-s)W_2^2(\mu,\sigma_0) + sW_2^2(\mu,\sigma_1) - s(1-s)W_2^2(\sigma_0,\sigma_1). \qquad (5.74) \end{aligned}$$

This inequality reflects a nice convexity property of the functional Φ defined in (5.53) and provides the starting point of our estimates.

A "metric variational inequality" for M^n_τ. The first step consists in writing a variational inequality for the discrete solution, analogous to (5.7): here we will use in a crucial way (5.61) and (5.74). In fact, it is easy to see that they yield the following strong convexity property for the functionals $s \mapsto \Phi(\tau, \mu; \sigma_s)$

$$\Phi(\tau, \mu; \sigma_s) \leq (1-s)\Phi(\tau, \mu; \sigma_0) + s\Phi(\tau, \mu; \sigma_1) - \frac{1}{2\tau} s(1-s) W_2^2(\sigma_0, \sigma_1). \tag{5.75}$$

Starting from the minimum property (5.53) and applying (5.75) with $\mu := M^{n-1}_\tau$, $\sigma_0 := M^n_\tau$, $\sigma := \sigma_1 \in D(\phi)$, we get

$$\begin{aligned} &\Phi(\tau, M^{n-1}_\tau; M^n_\tau) \leq \Phi(\tau, M^{n-1}_\tau; \sigma_s) \\ \leq &(1-s)\Phi(\tau, M^{n-1}_\tau; M^n_\tau) + s\Phi(\tau, M^{n-1}_\tau; \sigma) - \frac{1}{2\tau} s(1-s) W_2^2(M^n_\tau, \sigma). \end{aligned}$$

The minimum condition says that the right derivative at $s = 0$ of the right hand side is nonnegative; thus we find

$$\Phi(\tau, M^{n-1}_\tau; \sigma) - \Phi(\tau, M^{n-1}_\tau; M^n_\tau) - \frac{1}{2\tau} W_2^2(M^n_\tau, \sigma) \geq 0 \quad \forall \sigma \in D(\phi), \tag{5.76}$$

which can also be written as

$$\begin{aligned} \frac{1}{\tau}\Big(\frac{1}{2} W_2^2(M^n_\tau, \sigma) - \frac{1}{2} W_2^2(M^{n-1}_\tau, \sigma)\Big) \leq \phi(\sigma) - \phi(M^n_\tau) \\ - \frac{1}{2\tau} W_2^2(M^n_\tau, M^{n-1}_\tau). \end{aligned} \tag{5.77}$$

***A continuous formulation of* (5.77).** We want to write (5.77) as a true differential evolution inequality for the discrete solution $\overline{M}_\tau$, in order to compare two discrete solutions corresponding to different time steps τ, $\eta > 0$, and to try to reproduce the same comparison argument which we used in Theorem 5.5. Therefore, we set

$$\phi_\tau(t) := \text{"the linear interpolant of } \phi(M^{n-1}_\tau) \text{ and } \phi(M^n_\tau)\text{" if } t \in (t^{n-1}_\tau, t^n_\tau],$$

i.e.

$$\phi_\tau(t) := \frac{t^n_\tau - t}{\tau} \phi(M^{n-1}_\tau) + \frac{t - t^{n-1}_\tau}{\tau} \phi(M^n_\tau) \qquad t \in (t^{n-1}_\tau, t^n_\tau]. \tag{5.78}$$

Analogously, for any $\sigma \in D(\phi)$ we set

$$W_\tau^2(t; \sigma) := \frac{t^n_\tau - t}{\tau} W_2^2(M^{n-1}_\tau, \sigma) + \frac{t - t^{n-1}_\tau}{\tau} W_2^2(M^n_\tau, \sigma) \qquad t \in (t^{n-1}_\tau, t^n_\tau]. \tag{5.79}$$

Since

$$\frac{d}{dt} W_\tau^2(t; \sigma) = \frac{1}{\tau}\Big(W_2^2(M^n_\tau, \sigma) - W_2^2(M^{n-1}_\tau, \sigma)\Big) \qquad t \in (t^{n-1}_\tau, t^n_\tau],$$

neglecting the last negative term, (5.77) becomes

$$\frac{d}{dt}\frac{1}{2}W_\tau^2(t;\sigma) \le \phi(\sigma) - \phi_\tau(t) + \frac{1}{2}\mathscr{R}_\tau(t) \qquad \forall\, t \in (0,T)\setminus \mathcal{P}_\tau, \tag{5.80}$$

where we set, for $t \in (t_\tau^{n-1}, t_\tau^n]$,

$$\frac{1}{2}\mathscr{R}_\tau(t) := \phi_\tau(t) - \phi(M_\tau^n) = \frac{t_\tau^n - t}{\tau}\Big(\phi(M_\tau^{n-1}) - \phi(M_\tau^n)\Big) \ge 0. \tag{5.81}$$

The comparison argument. We consider now another time step $\eta > 0$ inducing the partition $\mathcal{P}_\eta$, a corresponding discrete solution (M_η^k), and the piecewise linear interpolating functions

$$W_{\tau,\eta}^2(t,s) := \frac{t_\eta^k - s}{\eta} W_\tau^2(t, M_\eta^k) + \frac{s - t_\eta^{k-1}}{\eta} W_\tau^2(t, M_\eta^{k-1}) \quad s \in (t_\eta^{k-1}, t_\eta^k], \tag{5.82}$$

observing that

$$W_{\tau,\eta}^2(t,s) = W_{\eta,\tau}^2(s,t) \quad \forall\, s,\, t \ge 0, \qquad W_{\tau,\eta}^2(t_\tau^n, s_\eta^k) = W_2^2(M_\tau^n, M_\eta^k). \tag{5.83}$$

Taking a convex combination w.r.t. the variable $s \in I_\tau^k$ of (5.80) written for $\sigma := M_\eta^{k-1}$ and $\sigma := M_\eta^k$, we easily get

$$\frac{\partial}{\partial t}\frac{1}{2}W_{\tau,\eta}^2(t,s) \le \phi_\eta(s) - \phi_\tau(t) + \frac{1}{2}\mathscr{R}_\tau(t) \qquad t \in (0,+\infty)\setminus \mathcal{P}_\tau,\ s > 0. \tag{5.84}$$

Reversing the rôles of η and τ, and recalling (5.83), we also find

$$\frac{\partial}{\partial s}\frac{1}{2}W_{\tau,\eta}^2(t,s) \le \phi_\tau(t) - \phi_\eta(s) + \frac{1}{2}\mathscr{R}_\eta(s) \qquad t > 0,\ s \in (0,+\infty)\setminus \mathcal{P}_\eta. \tag{5.85}$$

Summing (5.84) and (5.85) we end up with

$$\frac{\partial}{\partial t}W_{\tau,\eta}^2(t,s) + \frac{\partial}{\partial s}W_{\tau,\eta}^2(s,t) \le \mathscr{R}_\tau(t) + \mathscr{R}_\eta(s) \quad t \in (0,+\infty)\setminus\mathcal{P}_\tau,\ s \in (0,+\infty)\setminus\mathcal{P}_\eta. \tag{5.86}$$

Choosing $s = t$ we eventually find

$$\frac{d}{dt}W_{\tau,\eta}^2(t,t) \le \mathscr{R}_\tau(t) + \mathscr{R}_\eta(t) \qquad t \in (0,\infty)\setminus(\mathcal{P}_\tau \cup \mathcal{P}_\eta), \tag{5.87}$$

and therefore, being $t \mapsto W_{\tau,\eta}^2(t,t)$ continuous,

$$W_{\tau,\eta}^2(T,T) \le W_{\tau,\eta}^2(0,0) + \int_0^T \Big(\mathscr{R}_\tau(t) + \mathscr{R}_\eta(t)\Big)\,dt \qquad \forall\, T > 0. \tag{5.88}$$

Observe now that

$$\begin{aligned}\int_0^{+\infty} \mathscr{R}_\tau(t)\,dt = \sum_{j=1}^{+\infty}\int_{t_\tau^{j-1}}^{t_\tau^j} \mathscr{R}_\tau(t)\,dt &= \sum_{j=1}^{+\infty}\tau\big(\phi(M_\tau^{j-1}) - \phi(M_\tau^j)\big) \\ &\le \tau\phi(M_\tau^0),\end{aligned} \tag{5.89}$$

so that (5.88) yields

$$W_{\tau,\eta}^2(T,T) \le W_2^2(M_\tau^0, M_\eta^0) + \tau\phi(M_\tau^0) + \eta\phi(M_\eta^0) \qquad \forall\, T > 0. \tag{5.90}$$

Convergence and rough error estimates. Recalling that

$$W_2^2(M_\tau^n, M_\tau^{n-1}) \le \tau\phi(M_\tau^0), \qquad W_2^2(M_\eta^k, M_\eta^{k-1}) \le \eta\phi(M_\eta^0),$$

and that, for $t \in I_\tau^n \cap I_\eta^k$,

$$W_2^2(\overline{M}_\tau(t), \overline{M}_\eta(t)) \le 3\Big(W_{\tau,\eta}^2(t,t) + W_2^2(M_\tau^n, M_\tau^{n-1}) + W_2^2(M_\eta^k, M_\eta^{k-1})\Big),$$

we get

$$\sup_{t\ge 0} W_2^2(\overline{M}_\tau(t), \overline{M}_\eta(t)) \le 3\big(W_2^2(M_\tau^0, M_\eta^0) + 2\tau\phi(M_\tau^0) + 2\eta\phi(M_\eta^0)\big), \tag{5.91}$$

thus showing that $\tau \mapsto \overline{M}_\tau(t)$ is a Cauchy sequence in $\mathscr{P}_2(\mathbb{R}^d)$ for every $t \ge 0$. Denoting by μ_t its limit, we can pass to the limit in (5.90) as $\eta \downarrow 0$ by taking τ fixed and choosing $t \in \mathcal{P}_\tau$, thus obtaining the error estimate

$$\sup_{t\in\mathcal{P}_\tau} W_2^2(\overline{M}_\tau(t), \mu_t) \le W_2^2(M_\tau^0, \mu_0) + \tau\phi(M_\tau^0). \tag{5.92}$$

μ_t ***is the gradient flow.*** To this aim, it suffices to check that μ_t satisfies the metric Evolution Variational Inequality (5.7) with $\lambda = 0$ for every $\sigma \in D(\phi)$. Starting from the integrated form of (5.80) and recalling (5.89), we get for every $0 < a < b < +\infty$

$$\frac{1}{2}W_\tau^2(b,\sigma) - \frac{1}{2}W_\tau^2(a,\sigma) + \int_a^b \phi_\tau(t)\,dt \le (b-a)\phi(\sigma) + \tau\phi(M_\tau^0). \tag{5.93}$$

Since

$$\lim_{\tau\downarrow 0} W_\tau^2(t,\sigma) = W_2^2(\mu_t,\sigma), \quad \liminf_{\tau\downarrow 0} \phi_\tau(t) \ge \phi(\mu_t), \quad \lim_{\tau\downarrow 0} \phi(M_\tau^0) = \phi(\mu_0) < +\infty,$$

we easily get

$$\frac{1}{2}W_2^2(\mu_b,\sigma) - \frac{1}{2}W_2^2(\mu_a,\sigma) + \int_a^b \phi(\mu_t)\,dt \le (b-a)\phi(\sigma) \qquad \forall\,\sigma \in D(\phi), \tag{5.94}$$

which yields (5.7). The regularization estimates of Theorem 5.7 (which depend only on the metric E.V.I. formulation), together with (5.1c), show then that $\mu_t \in \mathscr{P}_2^a(\mathbb{R}^d)$ for $t > 0$. □

5.4 Bibliographical notes

The notion of gradient flows. There are at least four possible approaches to gradient flows which can be adapted to the framework of Wasserstein spaces:

1. **The "Minimizing Movement" approximation.** We can simply consider any limit curve of the variational approximation scheme we introduced in

Section 5.3, a "Generalized minimizing movement" in the terminology suggested by E. De Giorgi in [36]. In the context of $\mathscr{P}_2(\mathbb{R}^d)$ this procedure has been first used in [57, 71, 72, 70, 73] and subsequently it has been applied in many different contexts, e.g. by [56, 68, 74, 50, 51, 54, 46, 26, 27, 1, 52, 43, 10]. It has the advantage to allow for the greatest generality of functionals ϕ, it provides a simple constructive method for proving existence of gradient flows, and it can be applied to arbitrary metric spaces, in particular to $\mathscr{P}_p(\mathbb{R}^d)$, the space of probability measures endowed with the p-Wasserstein distance.

2. **Curves of Maximal Slope.** We can look for absolutely continuous curves $\mu_t \in AC^2_{\rm loc}((0,+\infty);\mathscr{P}_2(\mathbb{R}^d))$ which satisfy the differential form of the Energy inequality

$$\frac{d}{dt}\phi(\mu_t) \le -\frac{1}{2}|\mu'|^2(t) - \frac{1}{2}|\partial\phi|^2(\mu_t) \le -|\partial\phi|(\mu_t)\cdot|\mu'|(t) \tag{5.95}$$

for $\mathscr{L}^1$-a.e. $t \in (0,+\infty)$. This definition, introduced in a slightly different form in [37] and further developed in [38, 64, 9], it is still purely metric and it provides a general strategy to deduce differential properties satisfied by the limit curves of the Minimizing Movement scheme.

3. **The pointwise differential formulation.** It is the notion we adopted in Definition 5.2 and which requires the richest structure: since we have at our disposal a notion of tangent space and the related concepts of velocity vector field $\boldsymbol{v}_t$ and (sub)differential $\partial\phi(\mu_t)$, we can reproduce the simple definition of gradient flow modeled on smooth Riemannian manifold, i.e.

$$\boldsymbol{v}_t \in -\partial\phi(\mu_t). \tag{5.96}$$

The *a priori* assumption that $\mu_t \in \mathscr{P}^a_2(\mathbb{R}^d)$ avoids subtle technical complications arising from the introduction of "plan-" (or measure valued-) subdifferentials instead of the simpler vector fields. The general theory, which also covers the case of an underlying separable Hilbert space of infinite dimension, has been presented in [9]. A different approach has been developed in [29].

4. **Systems of Evolution Variational Inequalities (E.V.I.).** In the case of λ-convex functionals along geodesics in $\mathscr{P}_2(\mathbb{R}^d)$, one can try to find solutions of the family of "metric" variational inequalities

$$\frac{1}{2}\frac{d}{dt}W_2^2(\mu_t,\nu) \le \phi(\nu) - \phi(\mu_t) - \frac{\lambda}{2}W_2^2(\mu_t,\nu) \quad \forall\,\nu \in D(\phi). \tag{5.97}$$

This formulation can be considered as a "metric" version of Bénilan [16] notion of integral solutions of contraction semigroups in Banach spaces generated by m-accretive operators; it provides the best kind of solutions, for which in particular one can prove not only uniqueness, but also various regularization effects and nice asymptotic behavior. These results are in fact completely analogous to the corresponding ones of the Hilbertian theory, thus showing that they do not strictly depend on the linearity of the underlying space.

Of course, the fact that such strong formulation always admits a solution involves the (geodesic) convexity of the functional ϕ and a crucial "curvature" property of the distance (3.27). In [9] we discussed the role of these properties and presented new existence results in general metric spaces, extending the previous theory of [65].

Convergence of the variational approximation scheme. The variational approximation scheme is one of the basic tools for proving existence of gradient flows.

a) At the highest level of generality, when the functional ϕ does not satisfy any convexity or regularity assumption, one can only hope to prove the existence of a limit curve which will satisfy a sort of "relaxed" differential equation. In this case the proof relies on compactness arguments: passing to the limit in the discrete equation satisfied at each step by the approximating sequence M_τ^n, one tries to write a relaxed form of the limit differential equation, assuming only narrow convergences of weak type. A possible formalization of this point of view has been discussed in [9, Thm. 11.1.6] and an application to fourth order evolution equations is presented in [52] (see also [81] in the simpler framework of the Hilbert theory).

   It may happen that under suitable closure and convexity assumptions on the sections of the subdifferential, which should be checked in each particular situation, this relaxed version coincides with the stronger one, and therefore one gets an effective solution to (5.3). Here we outlined the main points of this argument in the first proof of Theorem 5.8: in this case a final relaxation of the limit differential inclusion can be avoided, thanks to the (geodesic) convexity of the functional.

   In general, this direct approach could be considered as a first basic step, which should be common to each attempt to apply the Wasserstein formalism for studying a gradient flow.

b) A second approach involves the *regularity of the functional* according to Definition 4.8, and still works with general distances and functionals. In this case the metric formulation of gradient flows as curves of maximal slope (see (5.95) and (5.8)) plays a crucial role.

   The key ingredient, which allows to pass to the limit, is a refined discrete energy estimate (related to De Giorgi's variational interpolation) and the lower semicontinuity of the slope, which follows from the regularity of the functional. We presented a detailed analysis of this point of view in [9].

c) A third approach, presented in the second proof of Theorem 5.8, can be performed only if the distance of the metric space, as in the case of $\mathscr{P}_2(\mathbb{R}^d)$, satisfies strong "'curvature-like" bounds (related to Example 3.13): moreover, the functional should satisfies a strong λ-convexity condition.

It extends to the Wasserstein framework previous results: the celebrated Crandall-Liggett [32] generation theorem for nonlinear contraction semigroups in Banach spaces, the optimal error estimates of [14, 82, 69] for gradient flows in Hilbert spaces, the convergence results of [65] in *non positively curved metric spaces* (we refer to [9] for a more detailed discussion).

Despite the strong convexity requirements on ϕ, which are nevertheless satisfied by all the examples of Section 4.5 in $\mathscr{P}_2(\mathbb{R}^d)$, this approach has interesting features:

- it does not require compactness assumptions of the sublevels of ϕ in $\mathscr{P}_2(\mathbb{R}^d)$: the convergence of the "Minimizing movement" scheme is proved by a Cauchy-type estimate.
- It provides an explicit bound for the error between a discrete approximation and the continuous solution.
- it is well suited to study the stability of the gradient flow with respect to Γ-convergence of the generating functionals (see [9, Theorem 11.2.1]).

6 Applications to Evolution PDE's

In this section we present some applications of the theory developed in the previous section to some relevant PDE's. Since many approaches are obviously possible, let us briefly mention some advantages of the "Wasserstein" one:

a) The gradient flow formulation (5.3) suggests a general variational scheme (the Minimizing Movement approach, which we discussed in the first part of this paper) to approximate the solution of (6.4a,b,c): proving its convergence is interesting both from the theoretical (cf. the papers quoted at the end of the previous section) and the numerical point of view [59].

b) The variational scheme exhibits solutions which are *a priori* nonnegative, even if the equation does not satisfies any maximum principle as in the fourth order case [72, 52].

c) Working in Wasserstein spaces allows for weak assumptions on the data: initial values which are general measures (as for fundamental solutions, in the linear cases) fit quite naturally in this framework.

d) The gradient flow structure suggests new contraction and energy estimates, which may be useful to study the asymptotic behavior of solutions to (6.4a,b,c) [74, 13, 25, 29, 2, 83, 42], or to prove uniqueness under weak assumptions on the data.

e) The interplay with the theory of Optimal Transportation provides a novel point of view to get new functional inequalities with sharp constants [75, 85, 3, 31, 12, 39, 62, 84].

f) The variational structure provides an important tool in the study of the dependence of solutions from perturbation of the functional.

g) The setting in space of measures is particularly well suited when one considers evolution equations in infinite dimensions and tries to "pass to the limit" as the dimension d goes to ∞.

First of all we mention the basic (but formal, at this level) example, which provides one of the main motivations to study this kind of gradient flows.

6.1 Gradient flows and evolutionary PDE's of diffusion type

In the space-time open cylinder $\mathbb{R}^d \times (0, +\infty)$ we look for *nonnegative solutions* $u : \mathbb{R}^d \times (0, +\infty)$ of a parabolic equation of the type

$$\partial_t u - \nabla \cdot \Big(\nabla\big(\frac{\delta \mathscr{F}}{\delta u}\big)u\Big) = 0 \quad \text{in } \mathbb{R}^d \times (0, +\infty), \tag{6.1}$$

where

$$\frac{\delta \mathscr{F}(u)}{\delta u} = -\nabla \cdot F_p(x, u, \nabla u) + F_z(x, u, \nabla u). \tag{6.2}$$

This is the first variation of a typical integral functional as in (4.59a,b)

$$\mathscr{F}(u) = \int_{\mathbb{R}^d} F(x, u(x), \nabla u(x))\, dx \tag{6.3}$$

associated to a (smooth) Lagrangian $F = F(x, z, p) : \mathbb{R}^d \times [0, +\infty) \times \mathbb{R}^d \to \mathbb{R}$.

Observe that (6.1) has the following structure:

$$\partial_t u + \nabla \cdot (\boldsymbol{v} u) = 0 \qquad \textit{(continuity equation)}, \tag{6.4a}$$

$$\boldsymbol{v} u = u \nabla \psi \qquad \textit{(gradient condition)}, \tag{6.4b}$$

$$\psi = -\frac{\delta \mathscr{F}(u)}{\delta u} \qquad \textit{(nonlinear relation)}. \tag{6.4c}$$

Observe that in the case when F depends only on $z = u$ then we have

$$\frac{\delta \mathscr{F}(u)}{\delta u} = F_z(u), \quad u \nabla F_z(x, u) = \nabla L_F(u), \quad L_F(z) := zF'(z) - F(z). \tag{6.5}$$

Since we look for *nonnegative solutions* having (constant, by (6.4a), normalized) *finite mass*

$$u(x, t) \geq 0, \quad \int_{\mathbb{R}^d} u(x, t)\, dx = 1 \qquad \forall t \geq 0, \tag{6.6}$$

and *finite quadratic momentum*

$$\int_{\mathbb{R}^d} |x|^2 u(x, t)\, dx < +\infty \quad \forall\, t \geq 0. \tag{6.7}$$

Recalling Example 4.5.1, we can

$$\text{identify } u \text{ with the measures } \mu_t := u(\cdot, t) \mathscr{L}^d, \tag{6.8}$$

and we consider $\mathscr{F}$ as a functional defined in $\mathscr{P}_2(\mathbb{R}^d)$. Then any smooth positive function u is a solution of the system (6.4a,b,c) if and only if μ is a solution in $\mathscr{P}_2(\mathbb{R}^d)$ of the Gradient Flow equation (5.3) for the functional $\mathscr{F}$.

Observe that (6.4a) coincides with (5.4b), the gradient constraint (6.4b) corresponds to the tangent condition $\boldsymbol{v}_t \in \mathrm{Tan}_{\mu_t} \mathscr{P}_2(\mathbb{R}^d)$ of (5.4c), and the nonlinear coupling $\psi = -\delta \mathscr{F}(u)/\delta u$ is equivalent to the differential inclusion $\boldsymbol{v}_t \in -\partial \mathscr{F}(\mu_t)$ of (5.4c).

At this level of generality the equivalence between the system (6.4a,b,c) and the evolution equation (5.3) is known only for smooth solution (which, by the way, may not exist); nevertheless, the point of view of gradient flow in the Wasserstein spaces, which was introduced by F. Otto in a series of pioneering and enlightening papers [71, 57, 73, 74], still presents some interesting features, whose role should be discussed in each concrete case.

6.1.1 Changing the reference measure

In many situations the choice of the Lebesgue measure $\mathscr{L}^d$ as a reference measure, thus inducing the identification (6.8), looks quite natural; nevertheless there are some interesting cases where a different measure γ plays a crucial role (see e.g. the example of Section 4.5.4 and the next Section 6.3) and it may happen that an evolution PDE takes a simpler form by an appropriate choice of γ.

From the Wasserstein point of view, an integral functional ϕ inducing the gradient flow is defined on measures μ, but its *explicit* form *depends* on the reference γ, so that different PDE's involving the density of μ w.r.t. γ could arise from the same functional.

Let us suppose, e.g., that ϕ takes the integral form

$$\phi(\mu) = \mathscr{F}_\gamma(\rho) = \int_{\mathbb{R}^d} \tilde{F}(x, \rho(x), \nabla\rho(x))\, d\gamma(x) \quad \text{if } \mu = \rho\gamma, \tag{6.9}$$

where γ is a probability measure induced by the (smooth) potential V, i.e.

$$\gamma := e^{-V}\mathscr{L}^d \in \mathscr{P}_2(\mathbb{R}^d). \tag{6.10}$$

Since

$$u = \frac{d\mu}{d\mathscr{L}^d} = e^{-V}\rho \quad \text{and} \quad \nabla\rho = e^V(u\nabla V + \nabla u), \tag{6.11}$$

the integrand $\tilde{F}(x, \tilde{z}, \tilde{p})$ of (6.9) is related to the integrand F of the representation (6.3) by the relation

$$\begin{gathered}\tilde{z} := e^{V(x)}z, \quad \tilde{p} = e^{V(x)}(z\nabla V(x) + p)\\ F(x, z, p) = e^{-V(x)}\tilde{F}(x, \tilde{z}, \tilde{p}) = e^{-V(x)}\tilde{F}\big(x, e^{V(x)}z, e^{V(x)}(z\nabla V(x) + p)\big).\end{gathered} \tag{6.12}$$

In this case it could be better to write the solution of the gradient flow μ_t generated by ϕ in terms of the density

$$\rho_t := \frac{d\mu_t}{d\gamma} = e^V\frac{d\mu_t}{d\mathscr{L}^d}, \tag{6.13}$$

and to use the differential operators associated with γ

$$\nabla_\gamma\rho := e^V\nabla\big(e^{-V}\rho\big) = \nabla\rho - \rho\nabla V, \tag{6.14a}$$

$$\nabla_\gamma\cdot\boldsymbol{\xi} := e^V\nabla\cdot(e^{-V}\boldsymbol{\xi}) = \nabla\cdot\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla V, \tag{6.14b}$$

which satisfy the "integration by parts formulae" with respect to the measure γ

$$\int_{\mathbb{R}^d}\boldsymbol{\xi}\cdot\nabla\zeta\, d\gamma = -\int_{\mathbb{R}^d}\zeta\nabla_\gamma\cdot\boldsymbol{\xi}\, d\gamma, \quad \int_{\mathbb{R}^d}\nabla\cdot\boldsymbol{\xi}\,\zeta\, d\gamma = -\int_{\mathbb{R}^d}\nabla_\gamma\zeta\cdot\boldsymbol{\xi}\, d\gamma, \tag{6.15}$$

when $\zeta \in C_c^\infty(\mathbb{R}^d)$, $\boldsymbol{\xi} \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$. The system (6.4a,b,c) preserves the same structure and takes the form

$$\partial_t\rho + \nabla_\gamma\cdot(\boldsymbol{v}\rho) = 0 \qquad \textit{(continuity equation)}, \tag{6.16a}$$

$$\boldsymbol{v}\rho = \rho\nabla\psi \qquad \textit{(gradient condition)}, \tag{6.16b}$$

$$\psi = -\frac{\delta\mathscr{F}_\gamma(\rho)}{\delta\rho} \qquad \textit{(nonlinear relation)}, \tag{6.16c}$$

where
$$\frac{\delta \mathscr{F}_\gamma(\rho)}{\delta \rho} := -\nabla_\gamma \cdot \tilde F_{\tilde p}(x,\rho,\nabla\rho) + \tilde F_{\tilde z}(x,\rho,\nabla\rho). \tag{6.17}$$
For, (6.16a) (resp. (6.16b)) can be transformed into (6.4a) (resp. (6.4b)), simply by multiplying the equation by e^{-V} and recalling (6.14b). The equivalence of (6.16c) and (6.4c) follows by a direct computation starting from (6.12): by (6.11) we get (with the obvious convention to evaluate F in $(x,u,\nabla u)$ and $\tilde F$ in $(x,\rho,\nabla\rho)$)
$$\begin{aligned}\frac{\delta \mathscr{F}(u)}{\delta u} &= F_z - \nabla\cdot F_p = \tilde F_{\tilde z} + \nabla V\cdot \tilde F_{\tilde p} - \nabla\cdot \tilde F_{\tilde p}\\ &= \tilde F_{\tilde z} - \nabla_\gamma\cdot \tilde F_{\tilde p} = \frac{\delta \mathscr{F}_\gamma(\rho)}{\delta\rho}.\end{aligned}$$

Remark 6.1 (Equations in bounded sets and Neumann B.C.) The possibility to change the reference measure is also useful to study evolution equations in a bounded open set $\Omega\subset\mathbb{R}^d$: they correspond to a measure γ whose support is included in $\overline{\Omega}$, e.g.
$$\gamma := \mathscr{L}^d|_\Omega$$
Observe that in any case the family of time-dependent measures $\mu_t = u_t\mathscr{L}^d|_\Omega$, which solves of the gradient flow equation according to Definition 5.2, still satisfies the continuity equation (5.4b) in $\mathbb{R}^d\times(0,+\infty)$. This can be seen as a weak formulation of the continuity equation for u_t in $\Omega\times(0,+\infty)$ with Neumann boundary conditions on $\partial\Omega\times(0,+\infty)$:
$$\partial_t u_t + \nabla\cdot(\boldsymbol{v}_t u_t) = 0 \quad \text{in } \Omega\times(0,+\infty), \qquad u_t\boldsymbol{v}_t\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega\times(0,+\infty). \tag{6.18}$$

6.2 The linear transport equation for λ-convex potentials

Let $V:\mathbb{R}^d\to(-\infty,+\infty]$ be a proper, l.s.c. and λ-convex potential. We are looking for curves $t\mapsto\mu_t\in\mathscr{P}_2(\mathbb{R}^d)$ which solve the evolution equation
$$\frac{\partial}{\partial t}\mu_t + \nabla\cdot(\boldsymbol{v}_t\mu_t) = 0, \quad \text{with } -\boldsymbol{v}_t(x)\in\partial V(x) \text{ for } \mu_t\text{-a.e. } x\in\mathbb{R}^d, \tag{6.19}$$
which is the gradient flow in $\mathscr{P}_2(\mathbb{R}^d)$ of the potential energy functional discussed in Example 3.4:
$$\mathcal{V}(\mu) := \int_{\mathbb{R}^d} V(x)\,d\mu(x). \tag{6.20}$$
If V is differentiable, (6.19) can also be written as
$$\frac{\partial}{\partial t}\mu_t = \nabla\cdot(\nabla V\mu_t) \quad \text{in the distribution sense.} \tag{6.21}$$
In the statement of the following theorem we denote by T the λ-contractive semigroup on $\overline{D(V)}\subset\mathbb{R}^d$ induced by the differential inclusion
$$\frac{d}{dt}T_t(x)\in -\partial V(T_t(x)), \quad T_0(x) = x \quad \forall\, x\in\overline{D(V)}. \tag{6.22}$$

Recall also that, according to Brezis theorem, $\frac{d}{dt}T_t(x)$ equals $-\partial^\circ V(T_t(x))$ at each point $t > 0$ of differentiability.

Theorem 6.2 *For every $\mu_0 \in \mathscr{P}_2(\mathbb{R}^d)$ with $\operatorname{supp}\mu_0 \subset \overline{D(V)}$ there exists a unique solution $(\mu_t, \boldsymbol{v})$ of (6.19) satisfying*

$$\lim_{t\downarrow 0} \mu_t = \mu_0, \qquad \int_{\mathbb{R}^d} |\boldsymbol{v}_t(x)|^2 \, d\mu_t(x) \in L^1_{\mathrm{loc}}(0, +\infty); \tag{6.23}$$

this solution is the gradient flow of $\mathcal{V}$ in the sense of the E.V.I. formulation (6.19) and of the Energy Identity (5.9) of Theorem 5.3. In particular it induces a λ-contractive semigroup on $\{\mu \in \mathscr{P}_2(\mathbb{R}^d) : \operatorname{supp}(\mu) \subset \overline{D(V)}\}$ and it exhibits the regularizing effect and the asymptotic behavior as in Theorem 5.7.
Moreover, for every $t > 0$ we have the representation formulas:

$$\mu_t = (T_t)_\# \mu_0, \qquad \boldsymbol{v}_t(x) = -\partial^\circ V(x) \quad \text{for } \mu_t\text{-a.e. } x \in \mathbb{R}^d. \tag{6.24}$$

Proof. Proposition 3.5 shows that the functional $\mathcal{V}$ satisfies (5.1a), (5.1b), (5.1d); it is also easy to check that (5.61) holds. On the other hand, $\mathcal{V}$ does not satisfy (5.1c), thus our simplified existence results can not be directly applied. Nevertheless, the more general theory of [9] covers also this case and yields the present result.

In any case, the solution to (6.19) can also be directly constructed by the representation formula (6.24). For, it is immediate to check directly that if we choose μ_0 of the type

$$\mu_0 := \sum_{k=1}^K \alpha_k \delta_{x_k}, \quad \alpha_k \geq 0, \quad \sum_{k=1}^K \alpha_k = 1, \quad x_k \in D(V), \tag{6.25}$$

then

$$\mu_t = \sum_{k=1}^K \alpha_k \delta_{T_t(x_k)} = (T_t)_\# \mu_0 \tag{6.26}$$

solves (6.19) (see also Section 2.5, where the connection between characteristics and solutions of the continuity equation is studied in detail), whereas (6.23) follows by the energy identity

$$\int_a^b |\partial^\circ V(T_t(x))|^2 \, dt + \phi(T_b(x)) = \phi(T_a(x)) \quad \forall\, x \in \overline{D(V)}.$$

Arguing as in the proof of Theorem 2.21 we also get for every $\sigma \in D(\mathcal{V})$ and every $\boldsymbol{\gamma} \in \Gamma_o(\mu_t, \sigma)$

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} W_2^2(\mu_t, \sigma) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle \boldsymbol{v}_t(x), x - y\rangle \, d\boldsymbol{\gamma}(x, y) \\ &\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \Big(V(y) - V(x) - \frac{\lambda}{2}|x - y|^2 \Big) \, d\boldsymbol{\gamma}(x, y) \\ &= \mathcal{V}(\sigma) - \mathcal{V}(\mu_t) - \frac{\lambda}{2} W_2^2(\mu_t, \sigma) \end{aligned} \tag{6.27}$$

at any t where $s \mapsto W_2(\mu_s, \sigma)$ and all $s \mapsto T_s(x_i)$ are differentiable. The measures $\mu_t = (T_t)_\# \mu_0$ thus solves the E.V.I. formulation (5.7) of the gradient flow for every initial datum μ_0 which is a convex combination of Dirac masses in $D(V)$. A standard approximation argument via (5.17) and Theorem 5.7 yields the same result for $\mu_t = (T_t)_\# \mu_0$ and every admissible initial measure $\mu_0 \in \overline{D(\mathcal{V})}$: for, being $\operatorname{supp} \mu_0 \subset \overline{D(V)}$, we can find a sequence $(\nu_n) \subset D(\mathcal{V})$ of convex combination of Dirac masses

$$\nu_n := \sum_{k=1}^{K_n} \alpha_{n,k} \delta_{x_{n,k}}, \quad \alpha_{n,k} \geq 0, \quad \sum_{k=1}^{K_n} \alpha_{n,k} = 1, \quad x_{n,k} \in D(V), \tag{6.28}$$

such that $\nu_n \to \mu_0$ in $\mathscr{P}_2(\mathbb{R}^d)$. $\square$

6.3 Kolmogorov-Fokker-Planck equation

The aim of this section is to present a systematic study of the "Wasserstein" approach to Kolmogorov-Fokker-Planck (KFP in the following) equation, which was firstly proposed by Jordan, Kinderlehrer, and Otto [57].

From this point of view, this equation is the gradient flow of the Relative Entropy functional discussed in 3.3; when the involved potential V is λ-convex, we have at our disposal all the tools to develop a self-contained variational theory for the generation of a λ-contracting semigroup in $\mathscr{P}_2(\mathbb{R}^d)$ with nice regularizing properties, independently of the growth of V (for other kind of estimates we refer to [34] and the references therein).

The particular "linear" structure of the subdifferential of the Entropy yields the linearity of the semigroup. Under quite general assumptions, which can be be applied to more general situations, the construction of a family of kernels and of general representation formulae is particularly easy in the Wasserstein framework, as well as the extension of the semigroup to L^p-spaces with respect to the invariant measure $\gamma := e^{-V} \mathscr{L}^d$. The λ-contractivity in $\mathscr{P}_2(\mathbb{R}^d)$ and the regularizing effect of the Wasserstein construction are also crucial to derive the Feller property for the KFP semigroup. We also show the equivalence with the more usual approach by Dirichlet forms in $L^2(\gamma)$.

Even if the theory presented here is finite-dimensional, we tried to develop sufficiently general arguments which could be extended to an infinite dimensional setting, taking also account of the more general theory available in [9]. It would be interesting to compare this point of view with other well established approaches (see e.g. [35, 18]).

6.3.1 Relative Entropy and Fisher Information

Let us consider

$$\begin{gathered} \text{a l.s.c. } \lambda\text{-convex potential} \quad V : \mathbb{R}^d \to (-\infty, +\infty] \\ \text{with } \Omega := \operatorname{Int}\big(D(V)\big) \neq \emptyset; \end{gathered} \tag{6.29}$$

for the sake of simplicity, we assume that the reference measure induced by the potential V is a probability measure with finite quadratic moment, i.e.

$$\gamma := e^{-V}\mathscr{L}^d \in \mathscr{P}_2(\mathbb{R}^d). \tag{6.30}$$

This condition, up to a renormalization, is always satisfied if, e.g., $\lambda > 0$. Observe that the density e^{-V} of γ with respect to $\mathscr{L}^d$ is 0 outside $\overline{\Omega} = \overline{D(V)}$. We adopt the convention to write a measure $\mu \in \mathscr{P}_2^a(\mathbb{R}^d)$ supported in $\overline{\Omega}$ as

$$\mu = u\mathscr{L}^d|_\Omega = \rho\gamma, \quad u = e^{-V}\rho; \tag{6.31}$$

the *Relative Entropy* (see Section 3.3) of μ w.r.t. γ is defined as

$$\mathcal{H}(\mu|\gamma) = \int_\Omega \rho \log \rho \, d\gamma = \int_\Omega u\Big(\log u + V\Big)\, dx, \tag{6.32}$$

whereas the *Relative Fisher Information* is defined as

$$\mathcal{I}(\mu|\gamma) := \int_\Omega \left|\frac{\nabla\rho}{\rho}\right|^2 d\mu = \int_\Omega \frac{\big|\nabla\rho\big|^2}{\rho}\, d\gamma = \int_\Omega \frac{\big|\nabla u + u\nabla V\big|^2}{u}\, dx \tag{6.33}$$

whenever $u, \rho \in W^{1,1}_{\text{loc}}(\Omega)$ (recall that V is locally Lipschitz in Ω); as usual, we set $\mathcal{H}(\mu|\gamma) = +\infty$ if μ is not absolutely continuous, and $\mathcal{I}(\mu|\gamma) = +\infty$ if $\rho \notin W^{1,1}_{\text{loc}}(\Omega)$.

Let us collect in the following proposition the main properties of these two functionals, we already discussed in Sections 3 and 4 and 5.

Proposition 6.3 (Entropy and Fisher information) *Let V, γ be as in* (6.29) *and* (6.30).

i) ***λ-convexity of the Relative Entropy.*** *The functional $\mu \mapsto \mathcal{H}(\mu|\gamma)$ is* λ-displacement convex *and it also satisfies the strong convexity assumption* (5.61).

ii) ***Subdifferential and slope of the Entropy:*** *A measure $\mu = \rho\gamma = u\mathscr{L}^d|_\Omega$ belongs to $D(\partial\mathcal{H}) = D(|\partial\mathcal{H}|)$ iff $\mathcal{I}(\mu|\gamma) < +\infty$, i.e.*

$$\rho, u \in W^{1,1}_{\text{loc}}(\Omega) \quad \textit{and} \quad \frac{\nabla\rho}{\rho} = \frac{\nabla u}{u} + \nabla V \in L^2(\mu;\mathbb{R}^d); \tag{6.34}$$

in this case

$$\boldsymbol{\xi} = \partial^\circ\mathcal{H}(\mu|\gamma) \quad \Longleftrightarrow \quad \xi = \frac{\nabla\rho}{\rho} \in L^2(\mu;\mathbb{R}^d), \tag{6.35}$$

so that

$$\mathcal{I}(\mu|\gamma) = \int_\Omega \big|\boldsymbol{\xi}\big|^2\, d\mu = |\partial\mathcal{H}|^2(\mu). \tag{6.36}$$

iii) ***Variational inequality for the logarithmic gradient.*** *If $\mathcal{I}(\mu|\gamma) < +\infty$, the logarithmic gradient $\boldsymbol{\xi} = \nabla\rho/\rho$ satisfies*

$$\int_\Omega \Big(\big(\boldsymbol{t}_\mu^\sigma - x\big)\cdot\boldsymbol{\xi} + \frac{\lambda}{2}|\boldsymbol{t}_\mu^\sigma - x|^2\Big)\,d\mu \le \mathcal{H}(\sigma|\gamma) - \mathcal{H}(\mu|\gamma) \quad \forall\,\sigma \in \mathscr{P}_2^a(\mathbb{R}^d). \tag{6.37}$$

iv) ***Log-Sobolev inequality.*** *If $\lambda > 0$ then*

$$\mathcal{H}(\mu|\gamma) \le \frac{1}{2\lambda}\mathcal{I}(\mu|\gamma) \quad \forall\,\mu \in \mathscr{P}_2^a(\mathbb{R}^d). \tag{6.38}$$

v) ***Derivative of the Entropy along curves.*** *Let $\mu : t \in [0,T] \mapsto \mu_t = \rho_t\,\gamma \in \mathscr{P}_2(\mathbb{R}^d)$ be a continuous family of measures satisfying the continuity equation*

$$\partial_t\mu + \nabla\cdot\big(\boldsymbol{v}\,\mu\big) = 0 \quad \textit{in } \mathcal{D}'(\mathbb{R}^d \times (0,T)) \tag{6.39}$$

for a Borel vector field $\boldsymbol{v}$ with

$$\int_0^T\int_\Omega \big|\boldsymbol{v}_t\big|^2\,d\mu_t\,dt < +\infty, \qquad \int_0^T \mathcal{I}(\mu_t|\gamma)\,dt < +\infty. \tag{6.40}$$

Then the map $t \mapsto \mathcal{H}(\mu_t|\gamma)$ is absolutely continuous in $[0,T]$ and for $\mathscr{L}^1$-a.e. $t \in (0,T)$ its derivative is

$$\frac{d}{dt}\mathcal{H}(\gamma|\mu_t) = \int_\Omega \boldsymbol{v}_t\cdot\frac{\nabla\rho_t}{\rho_t}\,d\mu_t = \int_\Omega \boldsymbol{v}_t\cdot\nabla\rho_t\,d\gamma. \tag{6.41}$$

Proof. *i)* follows from Propositions 3.5 and 3.11. The generalized convexity property (5.61) follows by analogous arguments (see [9, Prop. 9.3.9]).
ii) and *iii)* have been proved in Theorem 4.21 and (4.37).
iv) follows from (5.22a).
v) follows from the general Chain rule (4.55). □

6.3.2 Wasserstein formulation of the Kolmogorov-Fokker-Planck equation

Under the same assumption (6.29), (6.30) of the previous section, and recalling the differential operators of (6.14a,b), let us introduce the Laplacian operator Δ_γ induced by γ:

$$\Delta_\gamma\rho := \nabla_\gamma\cdot(\nabla\rho) = e^V\nabla\cdot\big(e^{-V}\nabla\rho\big) = \Delta\rho - \nabla\rho\cdot\nabla V, \tag{6.42}$$

and its formal adjoint (with respect to the Lebesgue measure) Fokker-Planck operator

$$\Delta_\gamma^* u := e^{-V}\Delta_\gamma\big(e^V u\big) = \nabla\cdot\big(\nabla u + u\nabla V\big). \tag{6.43}$$

Indeed, we formally have

$$\begin{aligned} e^{-V}\Delta_\gamma(e^V u) &= e^{-V}\left[\Delta(e^V u) - \nabla(e^V u)\cdot\nabla V\right] \\ &= e^{-V}\left[\nabla\cdot(e^V\nabla u + e^V u\nabla V) - \nabla(e^V u)\cdot\nabla V\right] \\ &= \Delta u + \nabla u\cdot\nabla V + u\Delta V = \nabla\cdot(\nabla u + u\nabla V). \end{aligned}$$

For smooth functions with compact support in Ω they satisfy

$$-\int_\Omega \Delta_\gamma \rho\, \zeta\, d\gamma = \int_\Omega \nabla\rho\cdot\nabla\zeta\, d\gamma = -\int_\Omega \rho\, \Delta_\gamma \zeta\, d\gamma, \tag{6.44}$$

$$-\int_\Omega \Delta^*_\gamma u\, \zeta\, dx = -\int_\Omega u\, \Delta_\gamma \zeta\, dx. \tag{6.45}$$

In the case of the centered Gaussian measure with variance λ^{-1} we have

$$V(x) = \frac{1}{2}\big(\lambda|x|^2 - \lambda\log(\lambda/2\pi)\big), \quad \gamma = \frac{1}{(2\pi/\lambda)^{d/2}} e^{-\frac{\lambda}{2}|x|^2}\mathscr{L}^d \tag{6.46}$$

Δ_γ is the *Ornstein-Uhlenbeck* operator $\Delta - \lambda x\cdot\nabla$.

The general definition of gradient flow, when particularized to the Relative Entropy functional, reads as follows:

Definition 6.4 ("Wasserstein" solutions of K.F.P. equations) *A continuous family $\mu_t = \rho_t\gamma = u_t\mathscr{L}^d|_\Omega \in C^0((0,+\infty);\mathscr{P}_2(\mathbb{R}^d))$ is a Wasserstein solution of the Kolmogorov-Fokker-Plank equation if $t\mapsto \mathcal{I}(\mu_t|\gamma)$ belongs to $L^2_{\rm loc}(0,+\infty)$ so that for $\mathscr{L}^1$-a.e. $t\in(0,+\infty)$*

$$\rho_t, u_t \in W^{1,1}_{\rm loc}(\Omega), \qquad \boldsymbol{\xi}_t = \frac{\nabla\rho_t}{\rho_t} = \frac{\nabla u_t}{u_t} + \nabla V \in L^2(\mu_t;\mathbb{R}^d), \tag{6.47}$$

and

$$\partial_t\mu_t - \nabla\cdot\big(\mu_t\frac{\nabla\rho_t}{\rho_t}\big) = 0 \quad \textit{in } \mathscr{D}'(\mathbb{R}^d\times(0+\infty)). \tag{6.48}$$

In terms of test functions (6.48) means

$$\int_0^{+\infty}\int_\Omega \Big(-\partial_t\zeta + \frac{\nabla\rho_t}{\rho_t}\cdot\nabla\zeta\Big)\, d\mu_t\, dt = 0 \quad \forall\,\zeta\in C^\infty_c(\mathbb{R}^d\times(0,+\infty)), \tag{6.49}$$

so that ρ_t satisfy the weak formulation

$$\int_0^{+\infty}\int_\Omega \Big(-\rho_t\,\partial_t\zeta + \nabla\rho_t\cdot\nabla\zeta\Big)\, d\gamma\, dt = 0 \quad \forall\,\zeta\in C^\infty_c(\mathbb{R}^d\times(0,+\infty)) \tag{6.50}$$

of

$$\partial_t\rho_t - \Delta_\gamma\rho_t = 0 \quad \text{in } \Omega\times(0,+\infty), \quad e^{-V}\partial_{\boldsymbol{n}}\rho_t = 0 \quad \text{on } \partial\Omega\times(0,+\infty) \tag{6.51}$$

Remark 6.5 In terms of the Lebesgue density u_t, (6.48) reads

$$\int_0^{+\infty}\int_\Omega \Big(-u\,\partial_t\zeta + \big(\nabla u + u\nabla V\big)\cdot\nabla\zeta\Big)\, dx\, dt = 0 \quad \forall\,\zeta\in C^\infty_c(\mathbb{R}^d\times(0,+\infty)), \tag{6.52}$$

corresponding to the Fokker-Planck equation

$$\partial_t u - \Delta^*_\gamma u = \partial_t u - \nabla\cdot\big(\nabla u + u\nabla V\big) = 0 \quad \text{in } \Omega\times(0,+\infty), \tag{6.53}$$

with homogeneous boundary conditions $\big(\nabla u + u\nabla V\big)\cdot\boldsymbol{n} = 0$ on $\partial\Omega\times(0,+\infty)$.

We introduce the narrowly closed and convex (both in the metric and linear sense) subset of $\mathscr{P}_2(\mathbb{R}^d)$

$$\mathscr{P}_2(\overline{\Omega}) := \Big\{\mu \in \mathscr{P}_2(\mathbb{R}^d) : \mathrm{supp}(\mu) \subset \overline{\Omega}\Big\}. \tag{6.54}$$

Theorem 6.6 *For every $\mu_0 \in \mathscr{P}_2(\overline{\Omega})$ there exists a unique Wasserstein solution $\mu_t = \rho_t\gamma = u_t\mathscr{L}^d|_{\Omega}$ of the Kolmogorov-Fokker-Planck equation (6.48) satisfying $\mu_t \to \mu_0$ in $\mathscr{P}_2(\mathbb{R}^d)$ as $t \downarrow 0$ and it coincides with the Wasserstein gradient flow generated by the functional $\phi(\mu) := \mathcal{H}(\mu|\gamma)$.*
The maps $S_t : \mu_0 \mapsto \mu_t$, $t \geq 0$, define a continuous λ-contractive semigroup in $\mathscr{P}_2(\overline{\Omega})$ which can be characterized by the system of E.V.I.

$$\frac{1}{2}\frac{d}{dt}W_2^2(\mu_t,\sigma) + \frac{\lambda}{2}W_2^2(\mu_t,\sigma) \leq \mathcal{H}(\sigma|\gamma) - \mathcal{H}(\mu_t|\gamma) \quad \forall\, \sigma \in \mathscr{P}_2(\overline{\Omega}). \tag{6.55}$$

It exhibits the ***regularizing effect***

$$\mathcal{H}(\mu_t|\gamma) < +\infty, \quad \mathcal{I}(\mu_t|\gamma) < +\infty \quad \forall\, t > 0, \tag{6.56}$$

with, for $\lambda \geq 0$,

$$\mathcal{H}(\mu_t|\gamma) \leq \frac{1}{2t}W_2^2(\mu_t,\gamma), \quad \mathcal{I}(\mu_t|\gamma) \leq \frac{1}{t^2}W_2^2(\mu_t,\gamma). \tag{6.57}$$

The map $t \mapsto e^{2\lambda t}\mathcal{I}(\mu_t|\gamma)$ is non increasing and it satisfies the ***Energy Identity***

$$\mathcal{H}(\mu_b|\gamma) + \int_a^b \mathcal{I}(\mu_t|\gamma)\, dt = \mathcal{H}(\mu_a|\gamma) \quad \forall\, 0 \leq a \leq b \leq +\infty. \tag{6.58}$$

When $\lambda > 0$ the ***asymptotic behavior*** *of μ_t as $t_0 \leq t \to +\infty$ is governed by*

$$\begin{gathered} W_2(\mu_t,\gamma) \leq e^{-\lambda(t-t_0)}W_2(\mu_{t_0},\gamma), \quad \mathcal{H}(\mu_t|\gamma) \leq e^{-2\lambda(t-t_0)}\mathcal{H}(\mu_{t_0}|\gamma), \\ \mathcal{I}(\mu_t|\gamma) \leq e^{-2\lambda(t-t_0)}\mathcal{I}(\mu_{t_0}|\gamma). \end{gathered} \tag{6.59}$$

Moreover, for every $t > 0$ (and also for $t = 0$, provided $\mathcal{I}(\mu_0|\gamma) < +\infty$)

$$\begin{aligned} \exists \lim_{h\downarrow 0} \frac{\boldsymbol{t}_{\mu_t}^{\mu_{t+h}} - \boldsymbol{i}}{h} &= \frac{\nabla \rho_t}{\rho_t} \quad \textit{in } L^2(\mu_t;\mathbb{R}^d), \\ \exists \lim_{h\downarrow 0} \frac{\mathcal{H}(\mu_{t+h}|\gamma) - \mathcal{H}(\mu_t|\gamma)}{h} &= \mathcal{I}(\mu_t|\gamma). \end{aligned} \tag{6.60}$$

Proof. Is is not difficult to check that

$$\overline{D(\phi)} = \mathscr{P}_2(\overline{\Omega}). \tag{6.61}$$

For, $D(\phi)$ contains all the measures of the type

$$\mu_{x_0,\rho} := \frac{1}{\gamma(B_\rho(x_0))}\chi_{B_\rho(x_0)} \cdot \gamma \quad \text{with} \quad B_\rho(x_0) \subset\subset \Omega,$$

and their convex combinations, so that

$$\sum_i \alpha_i \delta_{x_i} \in \overline{D(\phi)} \quad \text{if } x_i \in \Omega, \quad \alpha_i \geq 0, \quad \sum_i \alpha_i = 1.$$

Since the subset of all the finite convex combinations of δ-measures concentrated in Ω is dense in $\mathscr{P}_2(\overline{\Omega})$, we get (6.61).

By Proposition 6.3 the Relative Entropy functional $\mu \mapsto \mathcal{H}(\mu|\gamma)$ satisfies all the assumptions of Theorem 5.3, Theorem 5.7, and Theorem 5.8 (in the strongly convex case). Theorem 6.6 is a simple transposition of the results of Section 4, taking also into account the particular form of the subdifferential of $\mathcal{H}$ expressed by (6.35) and the fact that γ is the unique minimum of $\mathcal{H}$ with $\mathcal{H}(\gamma|\gamma) = 0$. □

We conclude this section by briefly discussing some further properties of the semigroup constructed by Theorem 6.6. We first introduce the "transition probabilities" $\nu_{x,t} = \vartheta_{x,t}\gamma$

$$\nu_{x,t} := S_t[\delta_x] \quad \text{with densities} \quad \vartheta_{x,t} := \frac{d\nu_{x,t}}{d\gamma} \in L^1(\gamma) \quad \forall\, x \in \overline{\Omega},\ t > 0. \tag{6.62}$$

Besicovitch differentiation theorem and the narrow continuity of $x \mapsto S_t[\delta_x]$ give that the explicit formula

$$\vartheta_{x,t}(y) := \limsup_{r \downarrow 0} \frac{\nu_{x,t}(B_r(y))}{\gamma(B_r(y))}$$

provides us with a pointwise definition of the densities $\vartheta_{x,t}$ satisfying

$$\text{for every } t > 0 \text{ the map } (x, y) \in \overline{\Omega} \times \overline{\Omega} \to \vartheta_{x,t}(y) \text{ is Borel.} \tag{6.63}$$

Theorem 6.7 (The associated Markovian semigroup) *Let $(S_t)_{t \geq 0}$ be the semigroup constructed in the previous Theorem 6.6 and let us consider the set of densities*

$$B_\gamma := \Big\{ \rho \in L^1(\gamma) : \rho\gamma \in \mathscr{P}_2(\mathbb{R}^d) \Big\}. \tag{6.64}$$

Extension to a contraction semigroup in $L^p(\gamma)$. *There exists a unique strongly continuous semigroup of linear contraction operators $(\mathscr{S}_t)_{t \geq 0}$ in $L^1(\gamma)$ such that*

$$(\mathscr{S}_t[\rho_0] = \rho_t \quad \Longleftrightarrow \quad S_t[\rho_0\gamma] = \rho_t\gamma) \qquad \forall\, \rho_0 \in B_\gamma. \tag{6.65}$$

For every $p \in [1, +\infty]$ $\mathscr{S}_t$ is a continuous (only weakly continuous, if $p = +\infty$) contraction semigroup in $L^p(\gamma)$*

$$\big\|\mathscr{S}_t[\rho]\big\|_{L^p(\gamma)} \leq \big\|\rho\big\|_{L^p(\gamma)} \quad \forall\, \rho \in L^p(\gamma), \tag{6.66}$$

it is order preserving

$$\rho_0 \leq \rho_1 \quad \Longrightarrow \quad \mathscr{S}_t[\rho_0] \leq \mathscr{S}_t[\rho_1], \tag{6.67}$$

and regularizing, since

$$\mathscr{S}_t(L^\infty(\gamma)) \subset C_b(\overline{\Omega}) \quad \forall\, t > 0. \tag{6.68}$$

Moreover

$$\begin{gathered}\mathscr{S}_t\big(\mathrm{Lip}(\Omega)\big) \subset \mathrm{Lip}(\Omega) \quad \forall\, t > 0,\\ \mathrm{Lip}(\mathscr{S}_t[\rho];\Omega) \le e^{-\lambda t}\,\mathrm{Lip}(\rho,\Omega) \quad \forall\, \rho \in \mathrm{Lip}(\Omega).\end{gathered} \tag{6.69}$$

Representation formula. *The semigroups S_t, $\mathscr{S}_t$ admit the representation formulas*

$$S_t[\mu] = \rho_t\gamma \qquad \textit{with} \quad \rho_t(x) = \int_{\mathbb{R}^d} \vartheta_{y,t}(x)\,d\mu(y) \quad \gamma\textit{-a.e.} \tag{6.70}$$

$$\mathscr{S}_t[\rho_0] = \rho_t \qquad \textit{with} \quad \rho_t(x) = \int_{\Omega} \vartheta_{y,t}(x)\rho_0(y)\,d\gamma(y) \quad \gamma\textit{-a.e.} \tag{6.71}$$

Dirichlet form. *$\mathscr{S}_t$ coincides in $L^2(\gamma)$ with the (analytic) semigroup $\tilde{\mathscr{S}_t}$ associated to the symmetric Dirichlet form with domain*

$$W^{1,2}_\gamma(\Omega) := \Big\{\rho \in W^{1,2}_{\mathrm{loc}}(\Omega) : \rho \in L^2(\gamma),\ \nabla\rho \in L^2(\gamma;\mathbb{R}^d)\Big\} \subset L^2(\gamma), \tag{6.72}$$

$$a_\gamma(\rho,\eta) := \int_\Omega \nabla\rho\cdot\nabla\eta\,d\gamma \qquad \forall\, \rho,\,\eta \in W^{1,2}_\gamma(\Omega). \tag{6.73}$$

In particular, if $\rho_0 \in L^2(\gamma)$ then the solution $\rho_t = \mathscr{S}_t[\rho_0]$ satisfies

$$\rho \in L^2_{\mathrm{loc}}([0,+\infty);W^{1,2}_\gamma(\Omega)) \cap C^0([0,+\infty);L^2(\gamma)) \tag{6.74}$$

and

$$\int_0^{+\infty} \Big(-(\rho,\partial_t\eta)_{L^2(\gamma)} + a_\gamma(\rho,\eta)\Big)\,dt = 0 \qquad \forall\, \eta \in C^1_c((0,+\infty);W^{1,2}_\gamma(\Omega)). \tag{6.75}$$

Symmetry of the transition densities. *For every $t > 0$ the transition densities $\vartheta_{x,t}$ satisfy*

$$\vartheta_{x,t}(y) = \vartheta_{y,t}(x) \quad \textit{for } \gamma\times\gamma\textit{-a.e. } (x,y) \in \Omega\times\Omega, \tag{6.76}$$

so that the "adjoint" representation formula holds

$$\mathscr{S}_t[\rho_0] = \rho_t \quad \textit{with} \quad \rho_t(x) = \int_\Omega \vartheta_{x,t}(y)\rho_0(y)\,d\gamma(y), \tag{6.77}$$

which provides the continuous representative of ρ_t when $\rho_0 \in L^\infty(\gamma)$.

Proof. Most of the results stated in the theorem are a direct consequence of the "linearity" of the semigroup $\mathscr{S}_t$ and of its regularizing effect; therefore, we postpone their proof to the next section, where we will discuss from a general point of view the construction of a Markov semigroup starting from a "linear" Wasserstein semigroup.

Here we only consider the last two properties, establishing the link with the "Dirichlet form" approach. Let us first observe that $W^{1,2}_\gamma(\Omega)$ is dense in $L^2_\gamma(\Omega)$ and it is an Hilbert space with the norm

$$\|\rho\|^2_{W^{1,2}_\gamma(\Omega)} := \|\rho\|^2_{L^2(\gamma)} + a_\gamma(\rho,\rho) = \int_\Omega \Big(|\rho|^2 + |\nabla\rho|^2\Big)\,d\gamma. \tag{6.78}$$

In fact, this is equivalent to the lower semicontinuity property of a_γ with respect to convergence in L^2_γ

$$\left.\begin{matrix}\rho_n \in W^{1,2}_\gamma(\Omega), \quad \rho_n \to \rho \quad \text{in } L^2(\gamma)\\ \sup_n a_\gamma(\rho_n,\rho_n) \le C\end{matrix}\right\} \quad \Rightarrow \quad \rho \in W^{1,2}_\gamma(\Omega), \quad a_\gamma(\rho,\rho) \le C. \tag{6.79}$$

Formulation (6.75) is stronger than the Wasserstein one as ρ is supposed to be in $L^2_{\rm loc}([0,+\infty);W^{1,2}_\gamma(\Omega))$; whenever this extra regularity holds, then more general test functions in $W^{1,2}_\gamma(\Omega)$ are allowed in (6.50), since it is not difficult to check that $C^\infty_c(\mathbb{R}^d)$ functions are dense in $W^{1,2}_\gamma(\Omega)$; it is then possible to recover (6.75) directly from (6.50).

The main idea is then to prove that a Wasserstein solution starting from $\mu_0 := \rho_0\gamma$ with $\rho_0 \in L^2(\gamma)$ satisfies the energy estimate (in fact an identity)

$$2\int_0^T \int_\Omega |\nabla\rho_t|^2\,d\gamma\,dt + \int_\Omega |\rho_T|^2\,d\gamma \le \int_\Omega |\rho_0|^2\,d\gamma \quad \forall\, T>0, \tag{6.80}$$

by evaluating the time derivative of the $L^2(\gamma)$-norm of ρ along the solution of the gradient flow.

For, we need a preliminary regularization and we consider the family of real convex superlinear functions $F_k : [0,+\infty) \to [0,+\infty)$ (depending on $k>0$)

$$F_k(\rho) := \begin{cases}\rho^2 & \text{if } \rho \le k,\\ k\rho(1-\log k + \log\rho) & \text{if } \rho \ge k,\end{cases} \tag{6.81}$$

which satisfy

$$0 \le F_k(\rho) \le c_k + k\rho\log\rho, \quad F_k(\rho) \uparrow \rho^2 \quad \text{as } k \uparrow +\infty \quad \forall\,\rho \ge 0. \tag{6.82}$$

F_k induces the relative energy functional

$$\mathcal{F}_k(\mu|\gamma) := \int_\Omega F_k(\rho)\,d\gamma. \tag{6.83}$$

A direct calculations shows that F_k satisfies (3.22) and

$$L_{F_k}(\rho) = \begin{cases} \rho^2 & \text{if } \rho \le k, \\ k\rho & \text{if } \rho \ge k, \end{cases} \tag{6.84}$$

so that for a measure $\mu = \rho\gamma$

$$\boldsymbol{\xi} = \partial^\circ \mathcal{F}_k(\mu|\gamma) \quad \Leftrightarrow \quad \rho \in W^{1,1}_\gamma(\Omega), \quad \boldsymbol{\xi} = \frac{\nabla L_{F_k}(\rho)}{\rho} \in L^2(\gamma; \mathbb{R}^d). \tag{6.85}$$

Being L_F Lipschitz, the Chain rule for Sobolev functions $\rho \in W^{1,1}_\gamma(\Omega)$ yields

$$\nabla L_{F_k}(\rho) = \begin{cases} 2\rho\nabla\rho & \text{in } \Omega \cap \{x : \rho(x) \le k\}, \\ k\nabla\rho & \text{in } \Omega \cap \{x : \rho(x) > k\}. \end{cases} \tag{6.86}$$

if $\mathcal{I}(\mu|\gamma) < +\infty$ then $\mu \in D(\partial\mathcal{F}_k)$ since

$$\int_\Omega \left| \frac{\nabla L_{F_k}(\rho)}{\rho} \right|^2 \rho \, d\gamma \le 4k^2 \mathcal{I}(\mu|\gamma) < +\infty. \tag{6.87}$$

If $\mu_0 = \rho_0\gamma,\ \rho_0 \in L^2(\gamma)$ then $\mathcal{F}_k(\mu_0|\gamma) < +\infty, \mathcal{H}(\mu_0|\gamma) < +\infty$, and the chain rule (4.55) yields

$$\mathcal{F}_k(\mu_t|\gamma) + \int_0^T \int_\Omega \frac{\nabla L_{F_k}(\rho_t) \cdot \nabla\rho_t}{\rho_t} \, d\gamma \, dt = \mathcal{F}_k(\rho_0) \le \int_\Omega |\rho_0|^2 \, d\gamma < +\infty. \tag{6.88}$$

By (6.86)

$$\int_\Omega \frac{\nabla F_k(\rho_t) \cdot \nabla\rho_t}{\rho_t} \, d\gamma \ge 2 \int_{\Omega \cap \{\rho_t \le k\}} |\nabla\rho_t|^2 \, d\gamma,$$

so that the Monotone Convergence Theorem yields (6.80).

Let us now check the last statement of Theorem 6.7. (6.75) and the regularity (6.74) yields that for every $\eta \in W^{1,2}_\gamma(\Omega)$ the map

$$\begin{gathered} t \mapsto \int_\Omega \mathscr{S}_t[\rho] \, \eta \, d\gamma \quad \text{is absolutely continuous, with} \\ \frac{d}{dt} \int_\Omega \mathscr{S}_t[\rho] \, \eta \, d\gamma + a_\gamma(\mathscr{S}_t[\rho], \eta) = 0. \end{gathered} \tag{6.89}$$

By integrating (6.89) and choosing initial data $\rho, \eta \in W^{1,2}_\gamma$, being a_γ a symmetric form it is immediate to check that $\mathscr{S}_t$ is self-adjoint in $L^2(\gamma)$ and we have

$$\int_\Omega \rho \, \mathscr{S}_t[\eta] \, d\gamma = \int_\Omega \mathscr{S}_t[\rho] \, \eta \, d\gamma \qquad \forall \, \rho, \, \eta \in L^2(\gamma). \tag{6.90}$$

For every bounded nonnegative ρ, $\eta \in L^\infty(\gamma)$, (6.71) yields

$$\int_\Omega \rho(x) \Big(\int_\Omega \vartheta_{y,t}(x) \eta(y) \, d\gamma(y) \Big) \, d\gamma(x) = \int_\Omega \Big(\int_\Omega \vartheta_{x,t}(y) \rho(x) \, d\gamma(x) \Big) \eta(y) \, d\gamma(y). \tag{6.91}$$

By (6.63) and Fubini's Theorem, we get (6.76).

Finally, the fact that $\mathscr{S}$ is a continuous semigroup in $L^1(\gamma)$ follows directly from the estimate (6.80): being $\mathscr{S}_t$ non expansive, it is sufficient to check that $\mathscr{S}_t[\rho] \to \rho$ strongly in $L^1(\gamma)$ as $t \downarrow 0$ on the dense subset $L^2(\gamma)$. The uniform bound of (6.80) provides both the weak and the strong convergence of $\mathscr{S}_t[\rho_0]$ to ρ_0 in $L^2(\gamma)$ as $t \downarrow 0$. □

Remark 6.8 (Dirichlet forms and analytic Markovian semigroups) Since the variational solution of (6.75) is unique (by J.L. Lions' Theorem on variational evolution equations in a Hilbert triplet, see e.g. [23]), in the proof of Theorem 6.7 we do not really need the converse implication showing that solutions of (6.75) are Wasserstein solutions of (6.50) with

$$\int_a^b \mathcal{I}(\rho_t\gamma|\gamma)\, dt < +\infty \quad \forall\, 0 < a < b < +\infty. \tag{6.92}$$

Nevertheless, we briefly mention how one can pass from (6.75) to the Wasserstein formulation; the main point is to show that the relative Fisher information is locally integrable in $(0, +\infty)$.

Let us first recall that for every $\rho_0 \in L^2(\gamma)$ Lions' Theorem provides a unique solution

$$\rho \in L^2_{\rm loc}([0,+\infty); W^{1,2}_\gamma(\Omega)) \cap H^1_{\rm loc}([0,+\infty); (W^{1,2}_\gamma(\Omega))') \subset C^0([0,+\infty); L^2(\gamma))$$

solving (6.75) or, equivalently,

$$\frac{d}{dt}\int_\Omega \rho\,\eta\, d\gamma + \int_\Omega \nabla\rho\cdot\nabla\eta\, d\gamma = 0 \qquad \forall\, \eta \in W^{1,2}_\gamma(\Omega), \quad \mathscr{L}^1\text{-a.e. in } (0,+\infty), \tag{6.93}$$

and such that $\lim_{t\downarrow 0}\rho_t = \rho_0$ strongly in $L^2(\gamma)$. Moreover ρ_t satisfies the energy identity

$$\int_0^T \int_\Omega |\nabla\rho_t|^2\, d\gamma\, dt + \frac{1}{2}\int_\Omega |\rho_T|^2\, d\gamma = \frac{1}{2}\int_\Omega |\rho_0|^2\, d\gamma, \tag{6.94}$$

and since a_γ is symmetric the map $\mathscr{S}_t : \rho_0 \mapsto \rho_t$ is a contraction *analytic* semigroup in $L^2(\gamma)$. In particular, ρ enjoys the nicer property $\rho \in C^\infty((0,+\infty); W^{1,2}_\gamma(\Omega))$.

Moreover, a standard truncation argument in Sobolev space yields

$$a_\gamma(\rho^+\wedge 1, \rho^+\wedge 1) \le a_\gamma(\rho,\rho) \quad \forall\, \rho \in W^{1,2}_\gamma(\Omega), \tag{6.95}$$

so that a_γ is a *closed and symmetric Dirichlet form* in $L^2(\gamma)$ (see e.g. [63]); in particular

$$\mathscr{S}_t(c) = c \quad \forall\, c \in \mathbb{R}; \qquad \rho_0 \le \rho_1 \quad \Rightarrow \quad \mathscr{S}_t(\rho_0) \le \mathscr{S}_t(\rho_1). \tag{6.96}$$

In order to check the equivalence with the Wasserstein formulation, we observe that for every initial datum $\rho_0 \in L^\infty(\gamma)$ with $\rho_0(x) \ge r > 0$ for γ-a.e. $x \in \Omega$, the unique solution ρ_t of (6.93) still satisfies the lower bound $\rho_t \geq r$ by (6.96); moreover, (6.94) yields

$$\int_0^{+\infty} \mathcal{I}(\rho_t\gamma|\gamma)\,dt \leq r^{-1}\int_0^{+\infty}\int_\Omega |\nabla\rho_t|^2\,d\gamma\,dt < +\infty, \tag{6.97}$$

so that, by Theorem 5.3, the measures $\mu_t = \rho_t\gamma$ provide the unique Wasserstein solution of (6.50) (since γ is a finite measure, $C_c^\infty(\mathbb{R}^d)$ is a subset of $W^{1,2}_\gamma(\Omega)$). Therefore the semigroups $\mathscr{S}$ and $\tilde{\mathscr{S}}$ coincide on $L^2(\gamma)$-densities bounded away from 0: a simple density argument shows that they coincide on $L^2(\gamma)$.

Remark 6.9 The measures $(\nu_{x,t})_{t\geq 0}$ are a Markovian semigroup of kernels associated with $(\mathcal{S}_t)_{t\geq 0}$ [63, II-4]

6.3.3 The construction of the Markovian semigroup

Among general λ-contracting semigroups in $\mathscr{P}_2(\mathbb{R}^d)$, the Kolmogorov-Fokker-Planck equation enjoys several other interesting features, due to its linearity. As we will see in the next Lemma, this is a direct consequence of the following "linearity condition"

$$\left.\begin{aligned}\boldsymbol{\xi}_i = \partial^\circ\phi(\mu_i),\ \alpha_i \geq 0,\ \alpha_1+\alpha_2 = 1\\ \boldsymbol{\xi}(\alpha_1\mu_1+\alpha_2\mu_2) = \alpha_1\boldsymbol{\xi}_1\mu_1+\alpha_2\boldsymbol{\xi}_2\mu_2\end{aligned}\right\} \quad\Longrightarrow\quad \boldsymbol{\xi}\in\partial\phi(\alpha_1\mu_1+\alpha_1\mu_2) \tag{6.98}$$

satisfied by the Wasserstein subdifferential of $\phi(\mu) := \mathcal{H}(\mu|\gamma)$.

The aim of this section is to show how easily one can deduce contraction and regularizing estimates starting from a "linear" Wasserstein semigroup; in particular, the construction of the fundamental solutions is particularly simple. It should not be too difficult to extend the following results to infinite dimensional underlying spaces, taking into account that the existence and the uniqueness of the gradient flow of the Relative Entropy functional extend to this context (see [9]).

Lemma 6.10 (Linearity of the gradient flow) *Let $\phi : \mathscr{P}_2(\mathbb{R}^d) \to (-\infty, +\infty]$ be a functional satisfying* (5.1a,b,c,d) *and let S_t be the λ-contractive semigroup generated by its gradient flow on $\overline{D(\phi)}$ as in Theorem 5.7. If ϕ satisfies* (6.98)*, then the semigroup S_t satisfies the "linearity" property*

$$S_t[\alpha_1\mu_1+\alpha_2\mu_2] = \alpha_1 S_t[\mu_1]+\alpha_2 S_t[\mu_2] \quad \forall\,\mu_1,\mu_2\in\overline{D(\phi)},\ \alpha_1,\alpha_2\geq 0,\ \alpha_1+\alpha_2 = 1. \tag{6.99}$$

Proof. Take two initial data $\mu_1, \mu_2 \in \overline{D(\phi)}$ and set $\mu_{i,t} := S_t[\mu_i]$, $\boldsymbol{v}_{i,t} = -\partial^\circ\phi(\mu_{i,t})$ their velocity vector fields, $\mu_t = \alpha_1\mu_{1,t}+\alpha_2\mu_{2,t}$, and define the vector field $\boldsymbol{v}_t$ so that

$$\boldsymbol{v}_t\mu_t := \alpha_1\boldsymbol{v}_{1,t}\mu_{1,t}+\alpha_2\boldsymbol{v}_{2,t}\mu_{2,t}. \tag{6.100}$$

Assuming $\alpha_i > 0$ and introducing the densities

$$\rho_{i,t} := \frac{d\mu_{i,t}}{d\mu_t}, \quad \text{so that} \quad \boldsymbol{v}_t = \alpha_1\rho_{1,t}\boldsymbol{v}_{1,t} + \alpha_2\rho_{2,t}\boldsymbol{v}_{2,t}, \quad \alpha_1\rho_{1,t} + \alpha_2\rho_{2,t} = 1,$$

it is easy to check that for every $t > 0$

$$\begin{aligned}
\int_{\mathbb{R}^d} |\boldsymbol{v}_t|^2 \, d\mu_t &= \int_{\mathbb{R}^d} |\alpha_1\rho_{1,t}\boldsymbol{v}_{1,t} + \alpha_2\rho_{2,t}\boldsymbol{v}_{2,t}|^2 \, d\mu_t \\
&\le \alpha_1 \int_{\mathbb{R}^d} |\boldsymbol{v}_{1,t}|^2 \rho_{1,t} \, d\mu_t + \alpha_2 \int_{\mathbb{R}^d} |\boldsymbol{v}_{2,t}|^2 \rho_{2,t} \, d\mu_t \\
&= \alpha_1 \int_{\mathbb{R}^d} |\boldsymbol{v}_{1,t}|^2 \, d\mu_{1,t} + \alpha_2 \int_{\mathbb{R}^d} |\boldsymbol{v}_{2,t}|^2 \, d\mu_{2,t}. \qquad (6.101)
\end{aligned}$$

It follows that the map $t \mapsto \|\boldsymbol{v}_t\|_{L^2(\mu_t;\mathbb{R}^d)}$ belongs to $L^2_{\mathrm{loc}}(0,+\infty)$ and, by linearity, μ_t satisfies the continuity equation

$$\partial_t \mu_t + \nabla \cdot (\boldsymbol{v}_t \, \mu_t) = 0 \quad \text{in } \mathbb{R}^d \times (0,+\infty). \qquad (6.102)$$

Since $\boldsymbol{v}_t \in \partial\phi(\mu_t)$ by (6.98), μ_t is the unique gradient flow with initial datum $\alpha_1\mu_1 + \alpha_2\mu_2$, that is $\mu_t = S_t(\alpha_1\mu_1 + \alpha_2\mu_2)$. □

Let γ be a nonnegative Borel measure on $\mathbb{R}^d$, with support D, and let B_γ be defined as in (6.64);

Theorem 6.11 *For $t \ge 0$, let $S_t : \mathscr{P}_2(D) \to \mathscr{P}_2(D)$ be satisfying the following assumptions:*

$$S_t \text{ is a continuous } \lambda\text{-contracting semigroup.} \qquad (6.103a)$$

$$S_t[\mu] \ll \gamma \quad \forall\, \mu \in \mathscr{P}_2(D),\ t > 0. \qquad (6.103b)$$

$$S_t[\alpha\mu+\beta\nu] = \alpha S_t[\mu] + \beta S_t[\nu] \quad \forall\, \mu,\, \nu \in \mathscr{P}_2(D),\ \alpha,\, \beta \ge 0,\ \alpha+\beta = 1. \qquad (6.103c)$$

The the following properties hold:

Extension to $L^1(\gamma)$. *There exists a unique narrowly continuous semigroup (denoted by $\mathscr{S}_t$) of bounded linear operators on $L^1(\gamma)$ such that*

$$S_t[\rho\gamma] = \mathscr{S}_t[\rho]\gamma \quad \forall\, \rho \in B_\gamma. \qquad (6.104)$$

Contraction and order preserving properties. *$\mathscr{S}_t$ is in fact a contraction and order preserving semigroup, i.e.*

$$\big\|\mathscr{S}_t[\rho]\big\|_{L^1(\gamma)} \le \big\|\rho\big\|_{L^1(\gamma)}, \quad \rho_1 \le \rho_2 \quad \Longrightarrow \quad \mathscr{S}_t[\rho_1] \le \mathscr{S}_t[\rho_2]. \qquad (6.105)$$

Representation formula. *Denoting by $\nu_{t,x} = \theta_{t,x}\gamma$ the "transition probabilities"*

$$\nu_{x,t} := S_t[\delta_x], \quad \textit{with densities} \quad \vartheta_{x,t} := \frac{d\nu_{x,t}}{d\gamma} \in L^1(\gamma) \quad \forall\, x \in D,\ t > 0, \qquad (6.106)$$

the semigroup S_t admits the representation formula

$$S_t[\mu] = \rho_t\gamma \quad \textit{with} \quad \rho_t(x) = \int_{\mathbb{R}^d} \vartheta_{y,t}(x)\, d\mu(y) \quad \textit{for } \gamma\textit{-a.e. } x \in D. \tag{6.107}$$

Invariant measure and Markov property. *If*

$$\gamma \in \mathscr{P}_2(\mathbb{R}^d) \quad \textit{is an invariant measure, i.e.} \quad S_t[\gamma] = \gamma \quad \forall\, t \geq 0, \tag{6.108}$$

then

$$\mathscr{S}_t\big(L^p(\gamma)\big) \subset L^p(\gamma) \qquad \forall p \in [1, +\infty] \tag{6.109}$$

and the restriction of $\mathscr{S}_t$ to $L^p(\gamma)$ is a continuous (weakly continuous if $p = \infty$) contraction semigroup.*

Proof. Let us first extend S by homogeneity to the cone $\mathscr{M}_2(D)$ of nonnegative finite measures with finite second moment

$$\mathscr{M}_2(D) := \Big\{\lambda\mu : \mu \in \mathscr{P}_2(D), \quad \lambda \geq 0\Big\} \tag{6.110}$$

simply by setting

$$S_t[\lambda\mu] = \lambda S_t[\mu] \quad \forall\, \mu \in \mathscr{P}_2(D),\ \lambda \geq 0. \tag{6.111}$$

It is easy to check that this extension preserves properties (6.103a,b) and, moreover, (6.103c) holds for every couple of nonnegative coefficients α, β:

$$S_t[\alpha\mu + \beta\nu] = \alpha S_t[\mu] + \beta S_t[\nu] \quad \forall\, \mu,\, \nu \in \mathscr{M}_2(D),\ \alpha, \beta \geq 0. \tag{6.112}$$

The uniqueness of $\mathscr{S}_t$ is then immediate: if $\rho\gamma \in \mathscr{M}_2(D)$ then by (6.104) and (6.103b)

$$\mathscr{S}_t[\rho] = \frac{S_t[\rho\gamma]}{\gamma}. \tag{6.113}$$

Being $\mathscr{S}_t$ continuous, it is sufficient to determine it on the set

$$C_\gamma := \Big\{\rho \in L^1(\gamma) : \int |x|^2 |\rho(x)|\, d\gamma(x) < +\infty\Big\}, \tag{6.114}$$

which is clearly dense in $L^1(\gamma)$; since each $\rho \in C_\gamma$ can be decomposed as

$$\rho = \rho_+ - \rho_- \quad \text{where} \quad \rho_+\gamma,\, \rho_-\gamma \in \mathscr{M}_2(D) \tag{6.115}$$

$\mathscr{S}_t[\rho]$ should be equal to the difference between $\mathscr{S}_t[\rho_+]$ and $\mathscr{S}_t[\rho_-]$. Let us check that this representation is independent of the particular decomposition: if ρ'_+, ρ'_- is another admissible couple as in (6.115), then $\rho_+ + \rho'_- = \rho'_+ + \rho_-$ and therefore

$$\mathscr{S}_t[\rho_+] + \mathscr{S}_t[\rho'_-] = \mathscr{S}_t[\rho_+ + \rho'_-] = \mathscr{S}_t[\rho'_+ + \rho_-] = \mathscr{S}_t[\rho'_+] + \mathscr{S}_t[\rho_-],$$

showing that

$$\mathscr{S}_t[\rho_+]-\mathscr{S}_t[\rho_-]=\mathscr{S}_t[\rho'_+]-\mathscr{S}_t[\rho'_-].$$

Choosing in particular $\rho_+:=\max[\rho,0]$ and $\rho_-:=-\min[\rho,0]$ we get the bound

$$\begin{aligned}\big\|\mathscr{S}_t[\rho]\big\|_{L^1(\gamma)} &\le \big\|\mathscr{S}_t[\rho_+]\big\|_{L^1(\gamma)}+\big\|\mathscr{S}_t[\rho_-]\big\|_{L^1(\gamma)} \\ &= \big\|\rho_+\big\|_{L^1(\gamma)}+\big\|\rho_-\big\|_{L^1(\gamma)}=\big\|\rho\big\|_{L^1(\gamma)},\end{aligned} \tag{6.116}$$

which shows that $\mathscr{S}_t$ is nonexpansive. Therefore, it can also be uniquely extended to a nonexpansive linear operator on $L^1(\gamma)$.

From the narrow continuity of $x\mapsto S_t[\delta_x]$ we also get

$$\text{the map } x\mapsto\int_D\varphi(y)\vartheta_{x,t}(y)\,d\gamma(y)\quad\text{is continuous}\quad\forall\,\varphi\in C^0_b(\mathbb{R}^d). \tag{6.117}$$

In order to prove the representation formula (6.107) we observe that for every initial measure $\nu=\sum_i\alpha_i\delta_{x_i}\in\mathscr{P}_2(D)$ and every $\varphi\in C^0_b(D)$, $\nu_t=S_t[\nu]$ satisfies

$$\begin{aligned}\int_D\varphi(y)\,d\nu_t(y)&=\sum_i\alpha_i\int_D\varphi(y)\vartheta_{x_i,t}(y)\,d\gamma(y)=\\ &=\int_D\Big(\int_D\varphi(y)\vartheta_{x,t}(y)\,d\gamma(y)\Big)\,d\nu(x).\end{aligned} \tag{6.118}$$

Therefore, by approximating in $\mathscr{P}_2(\mathbb{R}^d)$ an arbitrary measure $\mu\in\mathscr{P}_2(D)$ by a sequence of concentrated measures $\nu^k=\sum_i\alpha_i^k\delta_{x_i^k}$, since $S_t[\nu^k]\to S_t[\mu]=\mu_t=\rho_t\,\gamma$ in $\mathscr{P}_2(\mathbb{R}^d)$, (6.117) yields

$$\int_D\varphi(y)\,\rho_t(y)\,d\gamma(y)=\int_D\Big(\int_D\varphi(y)\vartheta_{x,t}(y)\,d\gamma(y)\Big)\,d\mu(x), \tag{6.119}$$

and therefore (6.107) follows by Fubini's Theorem.

Finally, if γ is an invariant measure, then $\mathscr{S}_t[1]=1$; the order preserving property shows that $\|\mathscr{S}_t[\rho]\|_{L^\infty(\gamma)}\le\|\rho\|_{L^\infty(\gamma)}$. By interpolation, the same property holds for every space $L^p(\gamma)$. $\square$

In order to study the adjoint semigroup $\mathscr{S}^*$ of $\mathscr{S}$ we further suppose that

$$\sup_{y\in D\cap B_r(x_0)}\int_D\varphi(\vartheta_{y,t}(x))\,d\gamma(x)<+\infty\qquad\forall\,x_0\in D,\ t,r>0 \tag{6.120a}$$

for some continuous convex function $\varphi:[0,+\infty)\to[0,+\infty)$ with more than linear growth at infinity, and

$$\limsup_{t\downarrow0}\int_D\varphi(\mathscr{S}_t[\rho](x))\,d\gamma(x)<+\infty\quad\forall\,\rho\in B_\gamma\cap L^\infty(\gamma). \tag{6.120b}$$

In the case of the KFP semigroup we have seen that these properties hold with $\varphi(z)=z\ln z$. For every function $\zeta\in L^\infty(\gamma)$ we can thus define

$$\zeta_t(x)=\mathscr{S}^*_t[\zeta](x):=\int_{\mathbb{R}^d}\vartheta_{x,t}(y)\zeta(y)\,d\gamma(y). \tag{6.121}$$

The next result show that $\mathscr{S}_t^*$ is the adjoint semigroup of $\mathscr{S}_t$ and it exhibits the Feller regularizing property.

Theorem 6.12 (The adjoint semigroup) *Under the same assumption of the previous theorem and* (6.120a,b)*, $\mathscr{S}_t$ is a strongly continuous semigroup in $L^1(\gamma)$ and the maps $\mathscr{S}_t^*$ defined by* (6.121) *are the weakly*-continuous, nonexpansive, adjoint semigroup on $L^\infty(\gamma)$ induced by $\mathscr{S}_t$, i.e. they satisfy*

$$\int_{\mathbb{R}^d} \mathscr{S}_t^*[\zeta]\rho\,d\gamma = \int_{\mathbb{R}^d} \zeta\mathscr{S}_t[\rho]\,d\gamma \qquad \forall\,\rho\in L^1(\gamma),\,\zeta\in L^\infty(\gamma). \tag{6.122}$$

Moreover, for every $t>0$

$$\mathscr{S}_t^*\big(L^\infty(\gamma)\big)\subset C_b^0(D),\quad \mathscr{S}_t^*\big(\mathrm{Lip}(D)\big)\subset \mathrm{Lip}(D), \tag{6.123}$$

$$\mathrm{Lip}(\mathscr{S}_t[\rho];D)\le e^{-\lambda t}\,\mathrm{Lip}(\rho,D)\quad \forall\rho\in\mathrm{Lip}(D). \tag{6.124}$$

Proof. We already know that $\mathscr{S}_t$ is a narrowly continuous contraction semigroup in $L^1(\gamma)$. For linear semigroups, strong continuity is equivalent to weak continuity [77]; therefore, being $B_\gamma\cap L^\infty(\gamma)$ a dense subset in (the positive cone of) $L^1(\gamma)$, it is sufficient to check that

$$\mathscr{S}_t[\rho_0]\rightharpoonup\rho\quad\text{weakly in } L^1(\gamma)\quad\forall\,\rho_0\in B_\gamma\cap L^\infty(\gamma). \tag{6.125}$$

(6.125) follows then directly from the narrow continuity of the map $t\mapsto\mathscr{S}_t[\rho_0]$ and its weak compactness in $L^1(\gamma)$ given by the uniform bound (6.120b).

Let us denote by $\tilde{\mathscr{S}}_t^*$ the adjoint semigroup, defined as in (6.122), and by ζ_t the image of $\zeta\in L^\infty(\gamma)$ by $\tilde{\mathscr{S}}_t^*$; we introduce the measures

$$\gamma^r_{x_0} := \frac{1}{\gamma(B_r(x_0))}\chi_{B_r(x_0)}\cdot\gamma\in\mathscr{P}_2(\mathbb{R}^d),\quad \forall\,x_0\in D=\mathrm{supp}(\gamma),\ r>0, \tag{6.126}$$

satisfying

$$\gamma^r_{x_0}\to\delta_{x_0}\quad\text{in }\mathscr{P}_2(\mathbb{R}^d)\quad\text{as } r\downarrow 0,\quad\forall\,x_0\in D. \tag{6.127}$$

Let us check that the functions

$$\vartheta^r_{x_0,t} := \mathscr{S}_t\Big[\frac{\chi_{B_r(x_0)}}{\gamma(B_r(x_0))}\Big] = \frac{dS_t[\gamma^r_{x_0}]}{d\gamma},\quad \vartheta^r_{x_0,t}(x)=\int_D\vartheta_{y,t}(x)\,d\gamma^r_{x_0}(y)$$

satisfy

$$\vartheta^r_{x_0,t}\rightharpoonup\vartheta_{x_0,t}\quad\text{weakly in } L^1(\gamma)\quad\text{as } r\downarrow 0. \tag{6.128}$$

For, narrow convergence is provided by (6.127) and the continuity of S_t in $\mathscr{P}_2(\mathbb{R}^d)$, whereas narrow compactness (when $r\in(0,r_0]$) is provided by (6.107), Jensen inequality, and (6.120a), since

$$\begin{aligned}
\int_D\varphi\big(\vartheta^r_{x_0,t}(x)\big)\,d\gamma(x) &= \int_D\varphi\Big(\int_D\vartheta_{y,t}(x)\,d\gamma^r_{x_0}(y)\Big)\,d\gamma(x)\\
&\le \int_D\Big(\int_D\Big(\varphi\big(\vartheta_{y,t}(x)\big)\,d\gamma^r_{x_0}(y)\Big)\,d\gamma(x) = \int_D\Big(\int_D\varphi\big(\vartheta_{y,t}(x)\big)\,d\gamma(x)\Big)\,d\gamma^r_{x_0}(y)\\
&\le \sup_{y\in D\cap B_{r_0}(x_0)}\int_D\varphi\big(\vartheta_{y,t}(x)\big)\,d\gamma(x)<+\infty.
\end{aligned}$$

It follows that for every $\zeta \in L^\infty(\gamma)$ and every $x_0 \in D, t > 0$ the limit

$$\tilde{\zeta}_t(x_0) := \lim_{r\downarrow 0} \frac{1}{\gamma(B_r(x_0))} \int_{B_r(x_0)} \zeta_t(x)\, d\gamma(x) \tag{6.129}$$

exists since

$$\begin{aligned}\frac{1}{\gamma(B_r(x_0))} \int_{B_r(x_0)} \zeta_t(x)\, d\gamma(x) &= \int_{\mathbb{R}^d} \mathscr{S}_t^*[\zeta] \frac{\chi_{B_r(x_0)}}{\gamma(B_r(x_0))}\, d\gamma \\ &= \int_{\mathbb{R}^d} \zeta \mathscr{S}_t\Big[\frac{\chi_{B_r(x_0)}}{\gamma(B_r(x_0))}\Big]\, d\gamma = \int_{\mathbb{R}^d} \zeta\, \vartheta^r_{x_0,t}\, d\gamma,\end{aligned}$$

and therefore

$$\tilde{\zeta}_t(x_0) = \lim_{r\downarrow 0} \int_{\mathbb{R}^d} \zeta\, \vartheta^r_{x_0,t}\, d\gamma = \int_{\mathbb{R}^d} \zeta\, \vartheta_{x_0,t}\, d\gamma = \mathscr{S}_t^*[\zeta](x_0). \tag{6.130}$$

Then, Lebesgue differentiation theorem yields $\zeta_t(x) = \mathscr{S}_t^*[\zeta](x)$ for γ-a.e. $x \in D$, thus showing that $\tilde{\mathscr{S}}^* = \mathscr{S}^*$.

From (6.120a) (providing compactness with respect to the weak L^1 topology) and the narrow continuity of $x \mapsto S_t[\delta_x]$ we obtain

$$\theta_{x,t} \rightharpoonup \theta_{x_0,t} \quad \text{weakly in } L^1(\gamma) \text{ as } x \to x_0 \quad \forall\, t > 0, \tag{6.131}$$

and therefore $\tilde{\zeta}_t$ is the continuous representative of ζ_t; this also shows the first inclusion of (6.123).

The second inclusion of (6.52) follows easily, since for each $\zeta \in \mathrm{Lip}(D)$, setting $\zeta_t = \mathscr{S}_t^*(\zeta)$, for each couple of points $x, y \in D$ we have

$$\begin{aligned}|\zeta_t(x) - \zeta_t(y)| &= \Big| \int_{\mathbb{R}^d} \zeta_t\, dS_t[\delta_x] - \int_{\mathbb{R}^d} \zeta_t\, dS_t[\delta_y] \Big| \le \mathrm{Lip}(\zeta; D) W_2(S_t[\delta_x], S_t[\delta_y]) \\ &\le \mathrm{Lip}(\zeta; D) e^{-\lambda t} W_2(\delta_x, \delta_y) = e^{-\lambda t}\, \mathrm{Lip}(\zeta; D)|x - y|.\end{aligned}$$

□

6.4 Nonlinear diffusion equations

In this section we consider the case of nonlinear diffusion equations in $\mathbb{R}^d$.

Let us consider a convex differentiable function $F : [0, +\infty) \to \mathbb{R}$ which satisfies (4.71), (4.76) and (4.78): F is the density of the *internal energy functional* $\mathcal{F}$ defined in (4.70).

Setting $L_F(z) := zF'(z) - F(z)$, we are looking for *nonnegative* solution of the evolution equation

$$\partial_t u_t - \Delta(L_F(u_t)) = 0 \quad \text{in } \mathbb{R}^d \times (0, +\infty), \tag{6.132a}$$

satisfying the (normalized) mass conservation

$$u_t \in L^1(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} u_t(x)\, dx = 1 \quad \forall\, t > 0, \tag{6.132b}$$

the finiteness of the quadratic moment

$$\int_{\mathbb{R}^d} |x|^2 u_t(x)\,dx < +\infty \quad \forall\, t > 0, \tag{6.132c}$$

the integrability condition $L_F(u) \in L^1_{\rm loc}(\mathbb{R}^d \times (0,+\infty))$, and the initial Cauchy condition

$$\lim_{t\downarrow 0} u_t \cdot \mathscr{L}^d = \mu_0 \text{ in } \mathscr{P}_2(\mathbb{R}^d). \tag{6.132d}$$

Therefore (6.132a) has the usual distributional meaning

$$\int_0^{+\infty} \int_{\mathbb{R}^d} \Big(- u_t\,\partial_t\zeta - L_F(u_t)\Delta\zeta\Big)\,dx\,dt = 0 \qquad \forall\,\zeta \in C^\infty_c\left(\mathbb{R}^d \times (0,+\infty)\right).$$

We can always assume possibly redefining u_t in a $\mathscr{L}^1$-negligible set of times, that $t \mapsto u_t\mathscr{L}^d$ is narrowly continuous in $[0,+\infty)$.

Theorem 6.13 *Suppose that F has a superlinear growth as in* (4.72)*. Then for every $\mu_0 \in \mathscr{P}_2(\mathbb{R}^d)$ there exists a unique solution*

$$u \in AC^2_{\rm loc}((0,+\infty); \mathscr{P}_2(\mathbb{R}^d))$$

of (6.132a,b,c,d) *among those satisfying*

$$L_F(u) \in L^1_{\rm loc}((0,+\infty); W^{1,1}_{\rm loc}(\mathbb{R}^d)), \quad \int_{\mathbb{R}^d} \frac{|\nabla L_F(u)|^2}{u}\,dx \in L^1_{\rm loc}(0,+\infty). \tag{6.133}$$

The map $t \mapsto S_t[\mu_0] = \mu_t = u_t\mathscr{L}^d$ is the unique gradient flow in $\mathscr{P}_2(\mathbb{R}^d)$ of the functional $\mathcal{F}$ defined in (4.70)*, which is geodesically convex (and also satisfies* (5.61) *with $\lambda = 0$).*
The gradient flow satisfies all properties of Theorem 5.7 for $\lambda = 0$. In particular, it is characterized by the system of E.V.I.

$$\frac{1}{2}\frac{d}{dt} W_2^2(\mu_t,\sigma) \le \mathcal{F}(\sigma) - \mathcal{F}(\mu_t) \quad \mathscr{L}^1\text{-a.e.}, \qquad \forall\,\sigma \in D(\mathcal{F}), \tag{6.134}$$

it is non expansive

$$W_2(S_t[\mu_0], S_t[\nu_0]) \le W_2(\mu_0,\nu_0) \quad \forall\,\mu_0,\,\nu_0 \in \mathscr{P}_2(\mathbb{R}^d) \tag{6.135}$$

and regularizing

$$\begin{gathered}
\sup_{0<t\le 1} t \int_{\mathbb{R}^d} F(u_t(x))\,dx < +\infty,\\
\sup_{0<t\le 1} t^2 \int_{\mathbb{R}^d} \frac{|\nabla L_F(u_t(x))|^2}{u_t(x)}\,dx < +\infty,\\
t \mapsto \int_{\mathbb{R}^d} \frac{|\nabla L_F(u_t(x))|^2}{u_t(x)}\,dx \quad \textit{is nonincreasing},
\end{gathered} \tag{6.136}$$

and for every $t > 0$

$$\exists \lim_{h\downarrow 0} \frac{\boldsymbol{t}_{\mu_t}^{\mu_{t+h}} - \boldsymbol{i}}{h} = \frac{\nabla L_F(u_t)}{u_t} \quad \text{in } L^2(\mu_t;\mathbb{R}^d),$$
$$\exists \lim_{h\downarrow 0} \frac{\mathcal{F}(\mu_{t+h}) - \mathcal{F}(\mu_t)}{h} = \int_{\mathbb{R}^d} \frac{|\nabla L_F(u_t(x))|^2}{u_t(x)}\,dx. \tag{6.137}$$

Proof. The proof is a simple combination of Theorem 5.3, Theorem 5.7 and Theorem 5.8 (in the strongly convex case, see also [9, Proposition 9.3.9]), and of the results of Section 4.5.3 for the functional $\mathcal{F}$, noticing that the domain of $\mathcal{F}$ is dense in $\mathscr{P}_2(\mathbb{R}^d)$. □

Remark 6.14 When F has a sublinear growth and satisfies

$$\lim_{z\to+\infty} \frac{F(z)}{z} = 0, \quad \lim_{z\to+\infty} \frac{F(z)}{z^{1-1/d}} = -\infty, \tag{6.138}$$

then it is possible to prove [9, Thm. 10.4.8] that $\mathcal{F}$ still satisfies (5.1c) and the Wasserstein semigroup generated by $\mathcal{F}$ provides the unique solution of (6.132a) in the above precise meaning: for, even if μ_0 is not regular (e.g. a Dirac mass), the regularizing effect of the Wasserstein semigroup shows that $\mu_t := S[\mu_0](t)$ is absolutely continuous w.r.t. the Lebesgue measure $\mathscr{L}^d$ for all $t > 0$: its density u_t w.r.t. $\mathscr{L}^d$ is therefore well defined and solves (6.132a).

Remark 6.15 Equation (6.132a) is a very classical problem: it has been studied by many authors from different points of view, which is impossible to recall in detail here.
We only mention that in the case of homogeneous Dirichlet boundary conditions in a bounded domain, H. Brézis showed that the equation is the gradient flow (see [22]) of the convex functional (since L_F is monotone)

$$\psi(u) := \int_{\mathbb{R}^d} G_F(u)\,dx, \quad \text{where} \quad G_F(u) := \int_0^u L_F(r)\,dr,$$

in the space $H^{-1}(\Omega)$. We refer to the paper of Otto [74] for a detailed comparison of the two notions of solutions and for a physical justification of the interest of the Wasserstein approach.

It is also possible to prove that the differential operator $-\Delta(L_F(u))$ is m-accretive in $L^1(\mathbb{R}^d)$ and therefore it induces a (nonlinear) contraction semigroup in $L^1(\mathbb{R}^d)$. Notice that here we allow for more general initial data (an arbitrary probability measure), whereas in the H^{-1} (or L^1) formulation Dirac masses are not allowed (but see [78, 30] in the fast diffusion case).

6.5 Drift diffusion equations with non local terms

Let us consider, as in [28, 29], a functional ϕ which is the sum of internal, potential, and interaction energy:

$$\phi(\mu) := \int_{\mathbb{R}^d} F(u)\,dx + \int_{\mathbb{R}^d} V\,d\mu + \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} W\,d\mu\times\mu \quad \text{if } \mu = u\mathscr{L}^d.$$

Here F, V, W satisfy the assumptions considered in Section 4.5.7; as usual we set $\phi(\mu) = +\infty$ if $\mu \in \mathscr{P}_2(\mathbb{R}^d) \setminus \mathscr{P}_2^a(\mathbb{R}^d)$. The gradient flow of ϕ in $\mathscr{P}_2(\mathbb{R}^d)$ leads to the equation

$$\partial_t u_t - \nabla \cdot \Big(\nabla L_F(u_t) + u_t \nabla V + u_t (\nabla W) \star u_t\Big) = 0, \tag{6.139}$$

coupled with conditions (6.132b), (6.132c), (6.132d).

Theorem 6.16 *For every $\mu_0 \in \mathscr{P}_2(\mathbb{R}^d)$ there exists a unique distributional solution u_t of* (6.139) *among those satisfying $u_t \mathscr{L}^d \to \mu_0$ in $\mathscr{P}_2(\mathbb{R}^d)$ as $t \downarrow 0$, $L_F(u_t) \in L^1_{\rm loc}((0,+\infty); W^{1,1}_{\rm loc}(\Omega))$, and*

$$\left\| \frac{\nabla L_F(u_t)}{u_t} + \nabla V + (\nabla W) \star u_t \right\|_{L^2(\mu_t;\mathbb{R}^d)} \in L^2_{\rm loc}(0,+\infty). \tag{6.140}$$

Furthermore, this solution is the unique gradient flow in $\mathscr{P}_2(\mathbb{R}^d)$ of the functional ϕ, which is λ-geodesically convex, and therefore satisfies all the properties stated in Theorem 5.7. In particular, when $\lambda > 0$ there exists a unique minimizer $\overline{\mu}$ of ϕ and the gradient flow generates a λ-contracting and regularizing semigroup which exhibits the asymptotic behavior of (5.22a,b,c,d).

Proof. The existence of u_t follows by Theorem 5.8 (besides ((5.1)) the function ϕ satisfies the strong convexity assumption (5.61), see [9, Theorem 9.3.5]) and by the characterization, given in Section 4.5.7, of the (minimal) subdifferential of ϕ. The same characterization proves that any u_t as in the statement of the theorem is a gradient flow; therefore the uniqueness Theorem 5.5 can be applied. $\square$

In the limiting case F, $V = 0$, the generated semigroup loses its regularizing effect and its existence and main properties follow from the more general theory of [9]. In this way it is possibile to study a model equation for the evolution of granular flows (see e.g. [25]).

Notice that, as we did in Section 6.3, we can also consider evolution equations in convex (bounded or unbounded) domains $\Omega \subset \mathbb{R}^d$ with homogeneous Neumann boundary conditions, simply by setting $V(x) \equiv +\infty$ for $x \in \mathbb{R}^d \setminus \overline{\Omega}$.

6.6 Gradient flow of $-W^2/2$ and geodesics

For a fixed reference measure $\sigma \in \mathscr{P}_2(\mathbb{R}^d)$ let us now consider the functional $\phi(\mu) := -\frac{1}{2} W_2^2(\mu, \sigma)$, as in Theorem 4.20. Being ϕ (-1)-convex along generalized geodesics, we can apply Theorem 5.7 to show that ϕ generates an evolution semigroup on $\mathscr{P}_2(\mathbb{R}^d)$. The following result ([9, Theorem 11.2.10]) shows that this evolution semigroup coincides with the (unique) extension of the geodesic between σ and μ_0 as long as this extension is still a minimizing geodesic.

Theorem 6.17 *Let be given two measures $\sigma, \mu_0 \in \mathscr{P}_2(\mathbb{R}^d)$ and suppose that $\boldsymbol{\gamma} \in \Gamma_o(\sigma, \mu_0)$ satisfies the following property: the constant speed geodesic*

$$\gamma(s) := \big((1-s)\pi^1 + s\pi^2\big)_\# \boldsymbol{\gamma}$$

can be extended to an interval $[0, T]$, *with* $T > 1$. *Then the formula*

$$t \to \mu(t) := \gamma(e^t), \qquad \text{for } 0 \leq t \leq \log(T), \tag{6.141}$$

gives the gradient flow of $\mu \mapsto -\frac{1}{2}W_2^2(\mu, \sigma)$ *starting from* μ_0.

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