ABSTRACT EVOLUTION EQUATIONS ON VARIABLE DOMAINS: AN APPROACH BY MINIMIZING MOVEMENTS

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Abstract. Starting from a problem on parabolic equations in non-cylindrical domains suggested by E. De Giorgi in his recent work on Minimizing Movements, we study the applications of this theory to general abstract evolution equations in variable domains.

Key words: Minimizing Movements, non-cilindrical domains, evolution equations.

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0. - Introduction.

In a recent work (see [15]) E. De Giorgi has proposed a very general method, the so-called *Minimizing Movements method*, which provides a unifying framework for different problems relative to Variational Calculus, Partial Differential Equations and Geometric Measure Theory.

Here we consider a problem suggested in [15] concerning a parabolic equation on a non-cilindrical domain. The Minimizing Movement's tool leads to study a time discretization of an associated penalized equation in a fixed domain, the discretization step and the penalizing term being related by themselves.

We study this problem in the framework of abstract evolution equations in Hilbert spaces (see [23], [24], [3], [7], [8], [21]) so that De Giorgi's problem will be recovered as a special case; the same abstract setting can be applied to study parabolic equation on a fixed domain but with mixed (and varying) lateral boundary conditions and parabolic variational inequalities on variable convex sets.

We study the convergence properties of the approximation procedure under general assumptions on the data and on the interplay between discretization and penalization, proving weak and strong convergence results depending on the regularity of the solution of the continuous problem. New regularity results for this solution are also given, with sharp error estimates in the "energy norm".

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The plan of the work is the following: in Section 1 we introduce the notation and state the main results with the related applications; in Section 2 we prove the basic existence and convergence results; refinement of the regularity properties of the continuous solution are given in the next Section and in the last one we prove the stronger convergence and continuity results with the error estimates.

1. - Notation and Main Results.

We begin with De Giorgi's definition (see [15]) of general Minimizing Movements ([1])

1.1 Definition. *Let us consider a topological space S, a functional*

$$F :]0,1[\times \mathbf{N} \times S \times S \to \mathbf{R} \cup \{-\infty, \infty\}$$

and an initial datum $u^0 \in S$; we say that $u : [0,\infty[\mapsto S$ is a Minimizing Movement in S associated to F and u^0 *and we write $u \in MM(F, u^0; S)$ if there exists a family of sequences $\{u_\tau^k\}_{k\in\mathbf{N}}$ depending on $\tau \in]0,1[$ such that*

$$\begin{cases} u_\tau^0 = u^0 \\ F(\tau, k, u_\tau^{k+1}, u_\tau^k) = \min\limits_{s\in S} F(\tau, k, s, u_\tau^k), \qquad \forall\, k \in \mathbf{N},\ \tau \in]0,1[\end{cases} \tag{1.1}$$

and u is the pointwise limit in S, as τ goes to 0, of the step functions $u_\tau : [0,\infty[\mapsto S$ defined as

$$u_\tau(t) = u_\tau^k, \qquad \textit{if} \quad t \in I_\tau^k = [k\tau, (k+1)\tau[,$$

that is

$$\lim_{\tau\to 0^+} u_\tau(t) = u(t), \qquad \forall\, t \in [0,\infty[\quad \square \tag{1.2}$$

Let us now choose $S = H^1(\mathbf{R}^n)$ and a measurable function

$$f : \mathbf{R}^n \times [0,\infty[\mapsto \mathbf{R}$$

with an open set $E \subset \mathbf{R}^n \times [0,\infty[$, whose sections at fixed $t \in [0,\infty[$ we call

$$E_t = \{x \in \mathbf{R}^n : \ (x,t) \in E\};$$

De Giorgi suggested the following

Problem 1. *Let*

$$F(\tau, k, v, w) = \begin{cases} \dfrac{1}{\tau} \displaystyle\int_{I_\tau^k} dt \left(\int_{\mathbf{R}^n} \left\{ \frac{|v(x) - w(x)|^2}{\tau} + |\nabla v(x)|^2 - 2f(x,t)\, v(x) \right\} dx + \right. \\ \left. \qquad (-\log\tau) \displaystyle\int_{\mathbf{R}^n \setminus E_t} |v(x)|^2 dx \right). \end{cases} \tag{1.3}$$

([1]) The original definition of [15] is slightly different and can be obtained by the change of parameters $\lambda = \tau^{-1}$; following [1], we also made explicit the initial datum u^0.

Find conditions on f, u^0 and E in order to obtain $MM(F,u^0;S) \neq \emptyset$. $\square$

De Giorgi himself observed that, setting $u(x,t) = u(t)(x)$ for $u \in MM(F,u^0;S)$, the term

$$-\log\tau \int_{\mathbf{R}^n \setminus E_t} |v(x)|^2\,dx$$

in the previous problem leads in the regular cases (for example $E = \{(x,t) :\ |x|^2 < t+1\}$) to the parabolic boundary value problem in a non-cylindrical domain:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_x u = f & \text{in } E, \\ u(x,t) = 0 & \text{on } \partial E_t, \quad t > 0, \\ u(x,0) = u^0 & \text{in } E_0. \end{cases} \tag{1.4}$$

Therefore, besides the general question stated in Problem 1 above, it is also interesting to characterize the elements of $MM(F,u^0;S)$ as the solutions, in a suitable sense, of (1.4). In this case, the choice of $H^1(\mathbf{R}^n)$ for S (instead of the weaker $L^2(\mathbf{R}^n)$) requires stronger assumptions to obtain the convergence of the approximating family u_τ but allows uniqueness and better regularity properties for $u \in MM(F,u^0;S)$. We say in advance that we can give a satisfactory answer to these questions when $\{E_t\}_{t\geq 0}$ is a non decreasing family of open sets.

1.2 Remark. It is obvious that different topologies on the same set S give rise to different classes of Minimizing Movements; therefore we shall distinguish between the weak and the strong topology of $H^1(\mathbf{R}^n)$. Strong convergence will be achieved when E is sufficiently smooth. $\square$

Before stating our results, let us point out the particular structure of the functional F which is common to more general situations; we set

$$a(v) = \int_{\mathbf{R}^n} |\nabla v(x)|^2\,dx, \qquad \forall\, v \in H^1(R^n) \tag{1.5}$$

$$b(t;v) = \int_{\mathbf{R}^n \setminus E_t} |v(x)|^2\,dx, \qquad \forall\, t \geq 0, \quad \forall\, v \in H^1(R^n) \tag{1.6}$$

In order to semplify the integrals defining F, we also introduce:

$$\forall\, v \in H^1(\mathbf{R}^n) \qquad b_\tau^k(v) = \frac{1}{\tau}\int_{k\tau}^{(k+1)\tau} b(t;v)\,dt \tag{1.7}$$

and

$$f_\tau^k(x) = \frac{1}{\tau}\int_{k\tau}^{(k+1)\tau} f(x,t)\,dt, \qquad \text{for a.e. } x \in \mathbf{R}^n. \tag{1.8}$$

In this way F becomes:

$$(1.9) \qquad F(\tau,k,v,w) = \frac{1}{\tau}\|v-w\|^2_{L^2(\mathbf{R}^n)} + a(v) - \log\tau\, b^k_\tau(v) - 2\big(f^\tau_k, v\big)_{L^2(\mathbf{R}^n)}$$

and it is easy to see that F admits a natural generalization in the usual framework of every Hilbert triple.

More precisely, let $V \subset H$ be a couple of real (separable) Hilbert spaces, the inclusion being continuous and dense; the norms on V and H and the scalar product on H are denoted respectively by $\|\cdot\|$, $|\cdot|$ and $(\cdot,\cdot)$. We identify H with its dual H', so that the dual space V' is the completion of H with respect to the dual norm and the relations

$$(1.10) \qquad V \subset H \equiv H' \subset V'$$

hold with continuous and dense imbeddings; moreover $(\cdot,\cdot)$ can also be used for the duality pairing between V and V'.

Let us consider a symmetric bilinear form

$$a(\cdot,\cdot) : V\times V \to \mathbf{R}, \qquad a(v) = a(v,v)$$

which we assume continuous and (weakly) coercive

$$(H1) \qquad \begin{cases} \exists\, M_a > 0: & \forall\, v,w\in V, \quad a(v,w) \le M_a\,\|v\|\,\|w\| \\ \forall\epsilon>0 \ \ \exists\alpha_\epsilon > 0: & \forall\, v\in V, \quad a(v,v) + \epsilon|v|^2 \ge \alpha_\epsilon\|v\|^2. \end{cases}$$

We also consider a (weakly) measurable ([2]) family of lower semicontinuous convex functions

$$b(t;\cdot):\ [0,\infty[\times V \to \mathbf{R}\cup\{+\infty\}$$

with

$$(H2) \qquad \forall\, t\ge 0,\ \forall\, v\in V, \quad b(t;v) \ge 0$$

and define

$$(1.11) \qquad \forall\, v\in V, \qquad b^k_\tau(v) = \frac{1}{\tau}\int_{k\tau}^{(k+1)\tau} b(t;v)\,dt.$$

The term $-\log\tau$ will be replaced by a penalty coefficient $\varepsilon_\tau^{-1} > 0$, for $\tau \in]0,1[$, such that

$$(H3) \qquad \lim_{\tau\to 0} \varepsilon_\tau = 0.$$

([2]) that is, $\forall\, v\in V$ the map $t\mapsto b(t;v)$ is measurable.

Let a function $f \in L^1_{loc}([0,+\infty[;H)$ (3) be given and set, as in (1.8):

$$f_k^\tau = \frac{1}{\tau}\int_{k\tau}^{(k+1)\tau} f(t)\,dt \in H.$$

Now Problem 1 is a particular case of the following

Problem 2. *Let*

$$F(\tau,k,v,w) = \frac{|v-w|^2}{\tau} + a(v) + \frac{1}{\varepsilon_\tau} b_\tau^k(v) - 2(f_\tau^k, v). \tag{1.12}$$

Find conditions on a, b, u^0, f such that there exists a function $u : [0,\infty[\mapsto V$ with $u \in MM(F,u^0;V)$.

1.3 Remark. Let us see another example which can be formulated in our abstract framework (see [24], [2], [7]). Let us fix a smooth open subset $\Omega \subset \mathbf{R}^n$ and consider $G \subset \partial\Omega \times [0,\infty[$, with $G_t = G \cap (\partial\Omega \times \{t\})$. Choosing $V = H^1(\Omega)$, $H = L^2(\Omega)$,

$$a(v,w) = \int_\Omega (\nabla v, \nabla w)\,dx, \qquad b(t;v) = \int_{\partial\Omega\setminus G_t} |v|^2\,d\mathcal{H}^{n-1}$$

the Minimizing Movements procedure of Problem 2 leads (at least formally) to the mixed boundary value problem:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} - \Delta u(x,t) = f(x,t) & \text{in } \Omega, \\ \dfrac{\partial u(x,t)}{\partial n} = 0 & \text{on } G, \\ u(x,t) = 0 & \text{on } \partial\Omega \times [0,\infty[\setminus G, \\ u(x,0) = u^0 & \text{in } \Omega. \end{cases} \qquad \square$$

To state our result we assume that, $\forall\, v \in V$

$(H4)$ the family $b(t;v)$ is non-increasing in time: $t_1 \le t_2 \Rightarrow b(t_1;v) \ge b(t_2;v)$

(3) For a generic Hilbert space $\mathcal{H}$, $L^p(0,T;\mathcal{H})$ will be the Banach space of the strongly measurable (classes of) functions $v :]0,T[\mapsto \mathcal{H}$ such that the map $t \mapsto \|v(t)\|_{\mathcal{H}}$ is in the usual $L^p(0,T)$ space, for $p \in [1,\infty]$, $T > 0$; the correponding norm will be (for $p < \infty$)

$$\|v\|_{L^p(0,T;\mathcal{H})} = \left\{ \int_0^T \|v(t)\|_{\mathcal{H}}^p\,dt \right\}^{1/p}$$

with the obvious changes in the $p = \infty$-case. Analogously, $v : [0,\infty[\mapsto \mathcal{H}$ belongs to $L^p_{loc}([0,\infty[;\mathcal{H})$ if its restriction to any interval $]0,T[$ is in $L^p(0,T;\mathcal{H})$.

and we choose

$$(H5) \qquad f \in L^2_{loc}([0,\infty[;H); \qquad u^0 \in V: \quad b(0,u^0) = 0 \ (^4).$$

We have the following results:

Theorem 1. *Let us assume that $(H1-H5)$ hold; then there exists a unique element u of $MM(F,u^0;V_w)$, where V_w is the topological vector space V endowed with its weak topology.*

In order to characterize the Minimizing Movement u, it will be useful to introduce the family of closed convex sets

$$(1.13) \qquad N_t = \{u \in V: \ b(t;u) = 0\}$$

which, by $(H4)$, are nondecreasing with respect to t. As usual we denote by $A: V \mapsto V'$ the linear continuous operator induced by $a(\cdot,\cdot)$ [5]. We have:

Theorem 2. *The Minimizing Movement u belongs to $H^1(0,T;H) \cap L^\infty(0,T;V)$ [6] for any $T > 0$ and it satisfies the inequality:*

$$(1.14) \qquad \begin{cases} u(t) \in N_t & \text{for } t > 0, \\ \big(u'(t) + Au(t) - f(t), u(t) - v\big) \le 0 & \forall\, v \in N_t, \quad \text{for a.e. } t > 0, \\ u(0) = u^0. \end{cases}$$

1.4 Remark. If N_t are subspaces [7] then the second of (1.14) becomes:

$$\big(u'(t) + Au(t) - f(t), v\big) = 0 \qquad \forall\, v \in N_t, \quad \text{for a.e. } t > 0 \quad \square$$

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[4] We could also choose $f \in BV_{loc}([0,\infty[;V')$ or in a "sum space" as in [3], [4], [29], obtaining analogous results; we limit ourselves to the L^2 setting in order to simplify the proofs. The condition on u^0 can be replaced by the slightly weaker

$$b(t,u^0) = 0, \ \forall\, t > 0.$$

[5] which is defined as:

$$\big(Av, w\big) = a(v,w), \qquad \forall\, v, w \in V$$

[6] $H^1(0,T;\mathcal{H})$ is the Hilbert space of the absolutely continuous functions $w:]0,T[\mapsto \mathcal{H}$ such that $w' \in L^2(0,T;\mathcal{H})$ (see [24]).

[7] for example if $b(t;\cdot)$ is $p-$homogeneous (with $p \ge 1$), that is

$$b(t;\lambda v) = |\lambda|^p b(t;v), \qquad \forall \lambda \in \mathbf{R}, \ \forall\, v \in V.$$

1.5 Remark. From now on we choose the usual Lebesgue representative for u, that is we assume that

$$u(t) = \lim_{\sigma\to 0^+} \frac{1}{2\sigma}\int_{t-\sigma}^{t+\sigma} u(\xi)\,d\xi \tag{1.15}$$

we easily find that u is continuous in H and in V_w, too. $\square$

1.6 Remark (*Problem 1***).** We can now give a precise meaning to (1.4); in fact if E is an open subset of $R^n\times\,]0,\infty[$ with

$$t_0 < t_1 \;\Rightarrow\; E_{t_0} \subset E_{t_1}$$

and

$$f \in L^2_{loc}([0,\infty[; L^2(\mathbf{R}^n)) \qquad u^0 \in H^1(\mathbf{R}^n) \text{ with } u^0(x) = 0 \text{ for a.e. } x \notin E_0 \tag{1.16}$$

then $u \in MM(F, u^0; H^1_w(\mathbf{R}^n))$ belongs to $H^1_{loc}([0,\infty[; L^2(\mathbf{R}^n)) \cap L^\infty_{loc}([0,\infty[; H^1(\mathbf{R}^n))$ with

$$\operatorname{supp}(u(\cdot;t)) \subset \overline{E_t}, \quad \forall\, t > 0 \text{ (}^8\text{)}.$$

Moreover, if we denote with u again the restriction of u to $E^T = E \cap \mathbf{R}^n\times]0,T[$, we have

$$u_t,\; \Delta u \in L^2(E^T); \quad u_t - \Delta u = f \in L^2(E^T)$$

so that the heat equation is surely satisfied in the sense of distributions on E. $\square$

We investigate about further regularity properties of u:

Theorem 3. *The solution u is right-continuous with respect to the strong topology of V and the set of its discontinuities is (at most) countable; moreover it belongs to the Besov space $B^{1/2}_{2\,\infty}(0,T;V)$, $\forall\, T > 0$, that is*

$$\exists C = C(T) > 0 : \qquad \int_0^T \|u(t+h) - u(t)\|^2\,dt \le C\,h, \qquad \forall\, h > 0. \tag{1.17}$$

1.7 Remark. Let us recall that in the framework of evolution equations on a constant domain (that is, if $N_t \equiv V$, $\forall\, t$) this result is quite easy, since we read from the equation

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(^{8}) in this sense the lateral boundary condition is satisfied; this relation holds for *any* t thanks to the weak continuity in $H^1(\mathbf{R}^n)$ and the easy property

$$E_t = \cup_{s<t} E_s, \qquad \forall\, t > 0.$$

Observe that if E_t has a continuous boundary, then the restriction of $u(\cdot;t)$ to E_t belongs to $H^1_0(E_t)$ (see [20]) that $Au \in L^2_{loc}([0,\infty[;H)$ and use the well known interpolation results of [28] ([9]). When $N_t \equiv \mathbf{K}$ is a proper convex subset of V, to estimate Au in H some compatibility condition (see [11], [13]) are required; nevertheless the symmetry of A ensures the continuity in V (see [11]) and the Besov's intermediate regularity (see [29]). In our case, on the contrary, we have neither a similar information on Au (which is false, in general; see [7]) nor a fixed convex set, and we must follow a different procedure to obtain the previous theorem. Let us recall that analogous interpolation estimates, at a lower level of regularity, can be found in [30]. □

We can give some more information about the convergence of u_τ, which show that a u in $MM(F,u^0;V_w)$ is "almost" in $MM(F,u^0;V)$

Theorem 4. *The family u_τ converges in H to u uniformly on every compact interval $[0,T]$ and strongly in $L^p(0,T;V)$, $\forall\, p<\infty$. Morevoer, if u is continuous at t, we have:*

$$\lim_{\tau\to 0^+} u_\tau(t) = u(t) \quad \textit{strongly in } V. \tag{1.18}$$

In particular (1.18) *holds except for an (at most) countable subset of* $]0,\infty[$ *and u belongs to $MM(F,u^0;V)$ if it is continuous with values in V.*

Thanks to this last result, a "weak" Minimizing Movement is also "strong" if it is strongly continuous in V; therefore it is interesting to find general conditions ensuring this continuity.

A natural way in the framework of *inequalities* is to introduce a compatibility assumption of the type (see [13]):

$$\exists\, g\in L^2_{loc}([0,\infty[;H): \quad v+\lambda(A-g(t))v\in N_t \Rightarrow v\in N_t, \qquad \text{for a.e. } t>0 \tag{1.19}$$

That yields an estimate of Au in $L^2_{loc}([0,\infty[;H)$ and the desired continuity, as briefly sketched in the previous remark.

However, since Problem 1 doesn't satisfy (1.19), we have to consider a different setting; for the sake of simplicity we assume that

$$N_t \text{ are subspaces} \tag{$H6$}$$

and we consider the related family of Hilbert spaces:

$$D_t = \{v\in N_t:\ \exists\, C>0,\ a(v,w)\le C|w|, \quad \forall\, w\in N_t\} \tag{1.20}$$

with the seminorm ([10])

$$[v]_t = \sup_{w\in N_t,|w|\le 1} a(v,w) \tag{1.21}$$

and the norm $\|v\|^2_{D_t} = \|v\|^2 + [v]^2_t$. We have:

([9]) In fact u would be in $C^0([0,T];V)\cap H^{1/2}(0,T;V)$, $\forall\, T>0$; see also [27]
([10]) If $a(\cdot)$ is coercive on N_t this is a norm.

Theorem 5. *Assume that for any $T>0$ there exist two constants $\delta_T, C_T>0$ such that, $\forall\, t, t+h \in [T-\delta_T, T]$*

$$(H7) \qquad a(u,u-v) \le C_T \|u\|_{D_t}\Big\{|u-v| + h\|v\|_{D_{t+h}}\Big\}, \qquad \forall u \in N_t, \quad \forall\, v \in N_{t+h}.$$

Then u is strongly continuous in V and belongs to $MM(F,u^0;V)$.

1.8 Remark. $(H7)$ can be substituted by intermediate conditions of the type:

$$(H7') \qquad \begin{aligned} a(u,u-v) \le & C_T \left\{ \|u\|_{D_t}|u-v| + d(t;h)\left[\|u\|_{D_t}\,\|v\|_{D_{t+h}}\right]^{1-\theta}\left[\|u\|\|v\|\right]^{\theta}\right\} \\ & \forall u \in N_t, \quad \forall\, v \in N_{t+h}, \qquad \forall\, t, t+h \in\, [T-\delta_T, T] \end{aligned}$$

where $d : [T-\delta_T, T]\times]0,\delta_T[\,\mapsto [0,\infty[$ is a positive function such that

$$(H7'') \qquad d(t,h) \le \int_t^{t+h} \rho(\lambda)\, d\lambda, \qquad \rho \in L^{1/\theta}(T-\delta_T, T)$$

for some $\theta \in]0,1[$. For "$\theta = 0$" we find $(H7)$ again; observe that a larger θ requires a stronger $(H7')$ but a weaker $(H7'')$. □

Application. In the context of Problem 1, with (1.16), let us assume that ([11])

$$(1.22) \qquad \begin{cases} E_t \text{ is a nondecreasing family either of bounded convex sets} \\ \text{or of bounded uniform } C^{1,1} \text{ regular sets} \end{cases}$$

Then we will show that $(H7')$ holds if $\theta = 1/2$ and $d(t;h) = \mathrm{dist}(\partial E_t, \partial E_{t+h})$, where dist is the usual Hausdorff distance between closed sets ([12]). Consequently, if $\exists\, \rho \in L^2(T-\delta_T, T)$ such that:

$$(1.23) \qquad \mathrm{dist}(\partial E_t, \partial E_{t+h}) \le \int_t^{t+h} \rho(\lambda)\, d\lambda, \qquad \forall\, t, t+h \in [T-\delta_T, T] \ ([13])$$

([11]) see [20]; in both cases, it is easy to see that

$$D_t = \left\{ v \in H^1(\mathbf{R}^n) : \ v_{|_{E_t}} \in H_0^1(E_t) \cap H^2(E_t)\right\}$$

.

([12]) We recall that given two bounded closed sets $B_1, B_2 \subset \mathbf{R}^N$ we define

$$\mathrm{dist}(B_1, B_2) = \sup_{b_1 \in B_1} d(b_1, B_2) + \sup_{b_2 \in B_2} d(b_2, B_1).$$

([13]) Let us remark that $(H7)$ requires the more readable:

$$\mathrm{dist}(\partial E_t, \partial E_{t+h}) \le C_T\, h, \qquad \forall\, t, t+h \in [T-\delta_T, T]$$

then Problem 1 has a unique solution; let us point out that (1.22), (1.23) allow a dicrete set of $t \geq 0$ such that:

$$E_t \neq \bigcap_{s>t} E_s$$

and also "tangential points" as in

$$E = \{(x,t) \in \mathbf{R}^n \times [0,\infty[: \quad |x| < 2 + \frac{t-1}{|t-1|^\eta}\}, \qquad \eta < 1/2 \quad \square$$

Finally, we want to study some error estimates between u_τ and u in the "energy norm" of $L^\infty(0,T;H) \cap L^2(0,T;V)$.

Let us recall that in the simplest case when b is the indicatrix function of a (fixed) closed convex set $\mathbf{K} \subset V$, that is

$$b(t;u) = I_{\mathbf{K}}(u) = \begin{cases} 0 & \text{if } u \in \mathbf{K}, \\ +\infty & \text{otherwise,} \end{cases} \quad (^{14})$$

we know that (see [31] for the linear case and [29] for the nonlinear one) the optimal estimate is

$$\|u - u_\tau\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} = O(\sqrt{\tau}). \tag{1.24}$$

In our case we must take into account the penalty term and we want to highlight some simple parameters which the order of convergence will depend on.

Since N_t is the null set of $b(t;\cdot)$ we shall assume that this penalty function measures the "distance" from N_t with respect to some intermediate norm between V and H; correspondingly we suppose a sort of compatibility between D_t, V and A, which we shall make precise by the tool of interpolation theory (see [21], [25] for a different type of assumptions and applications).

Following [14], [6], we denote by $(\mathcal{H}_0, \mathcal{H}_1)_{\sigma,p}$, $\sigma \in]0,1[$, $p \in [1,\infty]$, the family of the real interpolation spaces between $\mathcal{H}_0, \mathcal{H}_1$. We have

1.9 Definition. *We say that the couple $a(\cdot), b(t;\cdot)$ is (uniformly) of class γ, for $\gamma \in]0,\infty[$ if there exists $\theta \in]0,1[$ and $p \in [1,\infty]$ such that* (^{15})

$$N_t \text{ is continuously imbedded in } (V, D(A))_{\theta,p} \tag{1.25}$$

with a uniform (with respect to t) bound of the embedding norm, and for any $M > 0$ there exists a constant $C = C_M > 0$ such that:

$$\inf_{w\in N_t} \|v - w\|^\gamma_{(V,H)_{\theta,p'}} \leq C_M b(t;v), \qquad \forall\, v \in V,\ \|v\| \leq M \tag{1.26}$$

with $1/p + 1/p' = 1$. $\square$

(^{14}) which obviously implies $N_t \equiv \mathbf{K}$.

(^{15}) We recall that

$$D(A) = \{v \in V : \ Av \in H\} \quad \text{equipped by the norm} \quad \|v\|^2_{D(A)} = \|v\|^2 + |Av|^2$$

1.10 Remark. Let us quickly consider the limiting cases "$\theta = 0, 1$" we excluded in the previous definition. When "$\theta = 0$" (1.25) is always satisfied and (1.26) says that $b(t; v)$ penalizes the distance from N_t (at the power γ) with respect to the (strongest) V-norm. When "$\theta = 1$" we are penalizing the (weakest) H-distance from N_t but (1.25) requires $D_t \subset D(A)$. In both these cases $(H6)$ is unnecessary and (1.25) for $\theta = 1$ can be replaced by (1.19). ∎

When the function $b(t; \cdot)$ is related to the $H-$distance from N_t, in order to check the previous definition it could be useful the following

1.11 Remark. Assume that (1.25) holds with $p = \infty$ and a proper θ; moreover suppose there exists a family of operators $\{\mathcal{P}_t\}_{t\in[0,\infty[}$ with $\mathcal{P}_t : V \mapsto N_t$ such that:

$$\forall u \in V \quad \|\mathcal{P}_t u\| \le C\,\|u\|; \qquad |u - \mathcal{P}_t u|^\beta \le cb(t; u). \tag{1.27}$$

Then the couple $a(\cdot), b(\cdot; \cdot)$ is of class β/θ. In fact, by the usual interpolation inequalities, we have:

$$\inf_{w\in N_t} \|v-w\|^\gamma_{(V,H)_{\theta,p'}} \le \|v-\mathcal{P}_t v\|_{(V,H)_{\theta,p'}} \le \|v-\mathcal{P}_t v\|^{1-\theta}|v-\mathcal{P}_t v|^\theta \le (1+C)M^{1-\theta}cb(t; v)^{\theta/\beta}$$

In particular we will show that for Problem 1 we can choose $\theta = 1/2$ and $\beta = 2$, so that the couple a, b is of class 4. □

Theorem 6. *Assume that*

$$a(\cdot), b(\cdot; \cdot) \text{ are of class } \gamma; \tag{$H8$}$$

then we have the estimate

$$\|u - u_\tau\|^2_{L^\infty(0,T;H)} + \|u - u_\tau\|^2_{L^2(0,T;V)} \le C(f, u^0; T)\Big[\tau + \varepsilon^\sigma_\tau\Big]$$

where we set:

$$\sigma = \begin{cases} \frac{1}{\gamma-1} & \textit{if } \gamma \ge 2, \\ \frac{2}{\gamma} & \textit{if } \gamma \in [1, 2[. \end{cases} \tag{1.28}$$

In particular, choosing

$$\varepsilon_\tau = O(\tau^{\frac{1}{\sigma}})$$

we obtain the optimal order of convergence $O(\sqrt{\tau})$.

1.12 Remark. In the framework of previous remark with $\beta = 2$ we obtain $\sigma = \theta/(2-\theta)$, so that for the solution of Problem 1 we obtain

$$\left.\begin{array}{r} \displaystyle\sup_{0\le s\le T} \int_{\mathbf{R}^N} |u(x, s) - u_\tau(x, s)|^2\,dx \\ \displaystyle\int_0^T \int_{\mathbf{R}^n} |\nabla u(x, t) - \nabla u_\tau(x, t)|^2\,dx\,dt \end{array}\right\} \le C(f, u^0, T)\Big[\tau + \varepsilon^{1/3}_\tau\Big]$$

2. - Proof of Theorem 1 and 2.

In this Section we consider the functional F as in the formulation of Problem 2 and we assume that $(H1-5)$ hold true. First of all we fix $\tau \in]0,1[$ and look for u_τ^k given by the recursive formula (1.1); by standard results on convex functions (see [25], [17]) it is easy to see that:

2.1 Proposition. *For every $\tau \in]0,1[$ there exists a unique sequence $\{u_\tau^k\}_{k\in\mathbf{N}}$ which satisfies (1.1); for each $k \in \mathbf{N}$, u_τ^{k+1} solves the variational inequality*

$$(2.1)\quad \Big(\frac{u_\tau^{k+1}-u_\tau^k}{\tau} + Au_\tau^{k+1} - f_\tau^k, u_\tau^{k+1} - w\Big) + \frac{1}{2\varepsilon_\tau} b_\tau^k(u_\tau^{k+1}) \le \frac{1}{2\varepsilon_\tau} b_\tau^k(w) \qquad \forall\, w \in V \quad \blacksquare$$

In the previous Section we have already defined $u_\tau(t)$ as the piecewise constant function whose value in $I_\tau^k = [k\tau, (k+1)\tau)[$ is u_τ^k. We set

$$U_\tau(t) = u_\tau(t+\tau), \quad \text{so that} \quad U_\tau(t) = u_\tau^{k+1} \quad \text{if } t \in I_\tau^k,$$

and we also use the piecewise linear interpolant $\hat{u}_\tau(t)$ which satisfies:

$$\hat{u}_\tau(k\tau) = u_\tau^k, \qquad \hat{u}_\tau(t) = (t/\tau - k)u_\tau^{k+1} + (k+1-t/\tau)u_\tau^k \quad \text{on } I_\tau^k.$$

Analogously, we call $b_\tau(t;\cdot)$ the piecewise constant family of convex l.s.c. functions such that:

$$(2.2)\qquad \forall\, v \in V \qquad b_\tau(t;v) = b_\tau^k(v), \quad \text{if } t \in I_\tau^k.$$

The next proposition gives the basic stability estimates on $u_\tau(t)$ and $\hat{u}_\tau(t)$ in some suitable function spaces.

2.2 Proposition. *Assume that $(H1-5)$ hold; with the previous notation, we have:*

$$(2.3)\qquad \left.\begin{array}{c} \|\hat{u}'_\tau\|^2_{L^2(0,T;H)} \\ \sup\limits_{t\in[0,T]} a(u_\tau(t)) \\ \dfrac{1}{\varepsilon_\tau}\, b_\tau(u_\tau(t+\tau)) \end{array}\right\} \le \Big[a(u^0) + \|f\|^2_{L^2(0,T+\tau;H)}\Big], \qquad \forall\, T > 0$$

and there exists a constant $C = C(T) > 0$ such that (16)

$$(2.4)\qquad \|u_\tau\|_{L^\infty(0,T;V)} \le \|\hat{u}_\tau\|_{L^\infty(0,T;V)} \le C(T)\Big\{\|u^0\| + \|f\|_{L^2(0,T+\tau;H)}\Big\}.$$

(16) If a is strongly coercive on V we can choose C independent of T.

Proof. If we choose $w = u_\tau^k$ in (2.1) we obtain ([17])

$$
(2.5)\qquad \begin{aligned} 2\tau \left|\frac{u_\tau^{k+1}-u_\tau^k}{\tau}\right|^2 + a(u_\tau^{k+1}) + a(u_\tau^{k+1}-u_\tau^k) + \frac{1}{\varepsilon_\tau} b_\tau^k(u_\tau^{k+1}) \leq \\ \leq a(u_\tau^k) + \frac{1}{\varepsilon_\tau}\, b_\tau^{k-1}(u_\tau^k) + 2(f_\tau^k, u_\tau^{k+1}-u_\tau^k) \end{aligned}
$$

where we took into account that

$$
b_\tau^k(u_\tau^k) \leq b_\tau^{k-1}(u_\tau^k).
$$

and we set $u_\tau^{-1} = u^0, b_\tau^{-1}(\cdot) = b_\tau^0(\cdot)$. If we define

$$
\delta_\tau^k = \frac{u_\tau^{k+1}-u_\tau^k}{\tau}
$$

and add up for $k = 0, \ldots, m \leq T/\tau$ we obtain

$$
(2.6)\qquad \begin{aligned} \tau \sum_{k=0}^m |\delta_\tau^k|^2 + a(u_\tau^{m+1}) + \tau^2 \sum_{k=0}^m a(\delta_\tau^k) + \frac{1}{\varepsilon_\tau} b_\tau^m(u_\tau^{m+1}) \leq \\ \leq a(u^0) + \frac{1}{\varepsilon_\tau}\, b_\tau^0(u^0) + \tau \sum_{k=0}^m |f_\tau^k|^2 \end{aligned}
$$

By $(H4)$ we have $b_\tau^0(u^0) = 0$ and by construction we have

$$
|f_\tau^k|^2 \leq \frac{1}{\tau} \int_{k\tau}^{(k+1)\tau} |f(t)|^2\, dt, \qquad \forall\, k \geq 0
$$

so that the sum on the right hand member is bounded by $\int_0^{T+\tau} |f(t)|^2\, dt$; analogously, since

$$
\hat{u}_\tau'(t) = \delta_\tau^k, \quad \text{if } t \in [k\tau, (k+1)\tau[
$$

we obtain (2.3); Finally, (2.4) follows from (2.3), and the (weak) coercivity assumption (H1). ■

Let us now fix a $T > 0$; we denote by $\mathcal{N}(0,T)$ the closed convex subset of $L^2(0,T;V)$

$$
(2.7)\qquad \mathcal{N}(0,T) = \left\{ v \in L^2(0,T;V) : \quad v(t) \in N_t, \quad \text{for a.e. } t \in]0,T[\right\}
$$

([17]) We use the simple identity:

$$
2(Au, u-v) = a(u,u) - a(v,v) + a(u-v, u-v), \qquad \forall\, u, v \in V
$$

which holds for every symmetric bilinear form on a vector space.

which can also be viewed as the "kernel" of the lower semicontinuous functional (see [12]):

$$v \in L^2(0,T;V) \mapsto \int_0^T b(t;v(t))\,dt.$$

Analogously we set:

$$\mathcal{N}_\tau(0,T) = \big\{ v \in L^2(0,T;V) : \quad b_\tau(t;v(t)) = 0, \quad \text{for a.e. } t \in]0,T[\big\}. \tag{2.8}$$

This simple lemma shows the relation between $\mathcal{N}$ and $\mathcal{N}_\tau$:

2.3 Lemma. *For each $\tau > 0$ $\mathcal{N}_\tau(0,T)$ is a closed convex subset of $\mathcal{N}(0,T)$; we have*

$$\mathcal{N}(0,T) = \overline{\bigcup_\tau \mathcal{N}_\tau(0,T)}^{L^2(0,T;V)} = \overline{\bigcup_n \mathcal{N}_{\tau_n}(0,T)}^{L^2(0,T;V)} \tag{2.9}$$

for every sequence $\{\tau_n\}_{n\in\mathbf{N}} \subset]0,1[$ with $\lim_{n\to\infty} \tau_n = 0$.

Proof. We observe that

$$v \in \mathcal{N}_\tau(0,T) \;\Leftrightarrow\; b_\tau^k(v(t)) = 0, \quad \text{for a.e. } t \in [k\tau,(k+1)\tau[$$

and

$$\big\{ v \in V : \; b_\tau^k(v) = 0 \big\} = \bigcap_{t\in I_\tau^k} N_t$$

so that $\mathcal{N}_\tau(0,T) \subset \mathcal{N}(0,T)$.

On the other hand, it easy to see that if $v \in \mathcal{N}(0,T)$ then the function:

$${}_\tau v(t) = \begin{cases} v(t-\tau) & \text{if } t \in]\tau,T[, \\ 0 & \text{if } t \in]0,\tau] \end{cases} \tag{2.10}$$

belongs to $\mathcal{N}_\tau(0,T)$; since

$$\lim_{\tau\to 0^+} \|{}_\tau v - v\|_{L^2(0,T;V)} = 0$$

we are done. ∎

2.4 Theorem. *The family $\hat{u}_\tau$ weakly* converges in $H^1(0,T;H) \cap L^\infty(0,T;V)$ to the unique solution u of*

$$\begin{cases} u \in \mathcal{N}(0,T) \cap H^1(0,T;H) \\ \displaystyle\int_0^T (u'(t) + Au(t) - f(t), u(t) - v(t))\,dt \le 0 \quad \forall v \in \mathcal{N}(0,T) \\ u(0) = u^0. \end{cases} \tag{2.11}$$

Proof. Taking account of (2.1), we have that $\hat{u}_\tau$ and U_τ satisfy

$$(2.12)\quad \big(\hat{u}'_\tau + AU_\tau - f_\tau, U_\tau - v\big) + \frac{1}{2\varepsilon_\tau}\, b_\tau(t;U_\tau) \le \frac{1}{2\varepsilon_\tau}\, b_\tau(t;v) \qquad \forall v\in V, \quad \text{a.e. in}\,]0,T[.$$

If we choose $v = v(t) \in \mathcal{N}_\tau(0,T)$ and we integrate from 0 to T, by (2.8) and $(H2)$ we obtain

$$(2.13)\quad \int_0^T \Big[(\hat{u}'_\tau, U_\tau) + a(U_\tau,U_\tau) + (\hat{u}'_\tau + AU_\tau, -v) - (f_\tau, U_\tau - v)\Big]\,dt \le 0, \qquad \forall\, v\in\mathcal{N}_\tau(0,T).$$

In order to pass to the limit in previous formula, we observe that ([18])

$$\int_0^T (\hat{u}'_\tau, U_\tau)\,dt = \int_0^T \Big[(\hat{u}'_\tau, U_\tau - \hat{u}_\tau) + (\hat{u}'_\tau, \hat{u}_\tau)\Big]\,dt \ge \frac12|\hat{u}_\tau(T)|^2 - \frac12|u^0|^2$$

and we get $\forall\, v\in\mathcal{N}_\tau(0,T)$

$$(2.14)\qquad \frac12|\hat{u}_\tau(T)|^2 + \int_0^T a(U_\tau)\,dt + \int_0^T \Big[(\hat{u}'_\tau + AU_\tau, -v) - (f_\tau, U_\tau - v)\Big]\,dt \le \frac12|u^0|^2.$$

We now choose a decreasing sequence $j \mapsto \tau_j \in]0,1[$ such that $\hat{u}_{\tau_j}$ weakly* converges to a function u in $H^1(0,T;H)\cap L^\infty(0,T;V)$; u is surely the weak* limit for $\{U_{\tau_j}\}_{j\in\mathbf{N}}$ in $L^\infty(0,T;V)$ too, since the first formula of (2.3) and the previous note imply

$$(2.15)\qquad \|\hat{u}_\tau - U_\tau\|_{L^\infty(0,T;H)} \le C(f,u^0;T)\sqrt{\tau};$$

in particular we have

$$(2.16)\qquad |u(T)|^2 \le \liminf_{j\to\infty}|\hat{u}_{\tau_j}(T)|^2; \qquad \int_0^T a(u(t))\,dt \le \liminf_{j\to\infty}\int_0^T a(U_{\tau_j}(t))\,dt.$$

Let us now fix a function $v\in\mathcal{N}(0,T)$ and set $v_j = {}_{\tau_j}v$ as in (2.10), so that

$$v_j \in \mathcal{N}_{\tau_j}(0,T); \qquad \lim_{j\to\infty}\|v_j - v\|_{L^2(0,T;V)} = 0.$$

Substituting τ with τ_j and v with v_j in (2.14), and passing to the limit as $j\to\infty$ we get

$$(2.17)\qquad \frac12|u(T)|^2 + \int_0^T a(u(t))\,dt + \int_0^T \Big[(u' + Au, -v) - (f, u - v)\Big]\,dt \le \frac12|u^0|^2,$$

([18]) Recall that

$$U_\tau(t) - \hat{u}_\tau(t) = (k+1-t/\tau)[u_\tau^{k+1} - u_\tau^k] = \tau(k+1-t/\tau)\hat{u}'_\tau(t), \quad \text{if } t\in I_\tau^k$$

and $0 \le k+1-t/\tau \le 1$ for $t\in I_\tau^k$; consequently, $(\hat{u}'_\tau, U_\tau - \hat{u}_\tau) \ge 0$.

and consequently

$$\int_0^T (u' + Au - f, u - v)\, dt \leq 0 \qquad \forall v \in \mathcal{N}(0,T).$$

Let us check that $u \in \mathcal{N}(0,T)$, too; from (2.3) we have:

$$\frac{1}{\varepsilon_\tau} \int_0^T b_\tau(t; U_\tau(t))\, dt \leq a(u^0) + \|f\|^2_{L^2(0,T+\tau;H)}.$$

On the other hand, on any interval I_τ^k the integral of $b_\tau(t; U_\tau(t))$ is the same as the integral of $b(t; U_\tau(t))$, $U_\tau(t)$ being constant. Therefore

$$\int_0^T b(t; U_\tau(t))\, dt \leq C\varepsilon_\tau.$$

As $\varepsilon_\tau \to 0$ we obtain

$$\lim_{\tau \to 0^+} \int_0^T b(t; U_\tau(t))\, dt = 0$$

and since b is positive and lower semicontinuous we can conclude that

$$\int_0^T b(t; u(t))\, dt \leq \liminf_{\tau \to 0^+} \int_0^T b(t; U_\tau(t))\, dt = 0$$

that is $u \in \mathcal{N}(0,T)$.

At this level of regularity the uniqueness of the solution of (2.11) follows by standard arguments (see [11], [24]); consequently we obtain the (weak*) convergence of the whole family $\hat{u}_\tau$ to u. ∎

2.5 Remark. The pointwise formulation of Theorem 2 is a straightforward consequence of the integral one (see [11]): if we choose in (2.11)

$$v(t) = \begin{cases} u(t) \text{ if } t < t_0 \\ \underline{v} \text{ if } t \in (t_0, t_0 + \sigma) \\ u(t) \text{ if } t > t_0 + \sigma \end{cases}$$

and $\underline{v} \in N_{t_0}$ it is clear that we obtain

$$\frac{1}{\sigma} \int_{t_0}^{t_0+\sigma} (u'(t) + Au(t) - f(t), u(t) - \underline{v})\, dt \leq 0.$$

If we now pass to the limit, for a.e. t_0 we have

$$(u'(t_0) + Au(t_0) - f(t_0), u(t_0) - \underline{v}) \leq 0 \qquad \forall\, \underline{v} \in N_{t_0} \quad \square$$

2.6 Corollary. *With the notation of the previous theorem, we have $u \in MM(F, u^0; V_w)$.*

Proof. By the weak convergence of $\hat{u}_\tau$ to u in $H^1(0,T;H)$ and by (2.15) we have

$$u_\tau(t) \rightharpoonup u(t) \quad \text{in } H, \qquad \forall\, t \geq 0$$

Being $\{u_\tau(t)\}_{\tau \in]0,1[}$ bounded in V, the weak convergence in V follows immediately. ∎

3. - Proof of theorems 3 and 4.

3.1 Proposition. *The solution* u *of* (1.14) *belongs to* $B^{1/2}_{2\,\infty}(0,T;V)$, $\forall\, T>0$.

Proof. Since $u \in \mathcal{N}(0,T)$ implies that $u(t-\sigma) \in \mathcal{N}(0,T)$, $\forall\,\sigma > 0$ ([19]), from (2.11) we obtain

$$\int_0^T (u'(t) + Au(t) - f(t), u(t) - u(t-\sigma))\, dt \le 0$$

We then have

$$\int_0^T \Big[\frac{1}{2}a(u(t)) - \frac{1}{2}a(u(t-\sigma)) + \frac{1}{2}a(u(t) - u(t-\sigma))\Big]\, dt \le$$
$$\le \int_0^T \big(f(t) - u'(t), u(t) - u(t-\sigma)\big)\, dt$$

and also

$$\int_{T-\sigma}^T a(u(t))\, dt + \int_0^T a(u(t) - u(t-\sigma))\, dt \le$$
$$\le \sigma\, a(u^0) + 2\|f - u'\|_{L^2(0,T;H)}\, \|u(t) - u(t-\sigma)\|_{L^2(0,T;H)}.$$

Since

$$\|u(t) - u(t-\sigma)\|_{L^2(0,T;H)} \le \sigma \|u\|_{H^1(0,T;H)} \le C\sigma \Big\{ \|u^0\| + \|f\|_{L^2(0,T;H)} \Big\}$$

using the weak coerciveness of $a(\cdot)$ we obtain (1.17). ∎

By the same technique, we prove:

3.2 Proposition. *The solution* u *of* (1.14) *satisfies:*

$$a(u(t_1)) \le a(u(t_0)) + 2\int_{t_0}^{t_1} \big(f(t) - u'(t), u'(t)\big)\, dt, \qquad \forall\, t_0 < t_1 \tag{3.1}$$ ∎

Proof. Let us start from the pointwise inequality (1.14) and choose

$$v(t) = \begin{cases} u(t-\sigma) & \text{if } t \in [t_0+\sigma, t_1]; \\ u(t_0) & \text{if } t \in [t_0, t_0+\sigma[. \end{cases}$$

Integrating between t_0 and t_1 and repeating the previous calculations, we get

$$\frac{1}{\sigma}\int_{t_1-\sigma}^{t_1} a(u(t))\, dt \le a(u(t_0)) + 2\int_{t_0}^{t_1} \big(f(t) - u'(t), \frac{u(t) - u(t-\sigma)}{\sigma}\big)\, dt$$

Passing to the limit as $\sigma \to 0^+$ and using the weak continuity of u in V, we get (3.1). ∎

([19]) we put $u(t) = u^0$ when $t \in [-\sigma, 0[$.

3.3 Corollary. *The function u is right continuous in (the strong topology of) V.*

Proof. We already know that u is weakly continuous; from (3.1) we deduce that:

$$\limsup_{t\to t_0^+} a(u(t)) \le a(u(t_0)), \qquad \forall\, t_0 \in [0,\infty[$$

and by note ([16])

$$\lim_{t\to t_0^+} a(u(t)-u(t_0)) = 0$$

that implies the strong limit in the V-norm by the weak coercivity of a. ∎

3.4 Corollary. *The discontinuity set of u (with respect to the strong topology of V) is at most countable.*

Proof. By the previous argument we find that:

$$\lim_{t\to t_0} \|u(t)-u(t_0)\| = 0 \;\Leftrightarrow\; \lim_{t\to t_0} a(u(t)) = a(u(t_0))$$

From (3.1) we deduce that the map

$$t \mapsto a(u(t)) - 2\int_0^t \big(f(s)-u'(s), u'(s)\big)\, ds$$

is non increasing, so that it has an (at most) countable discontinuity set.

Since $t \mapsto 2\int_0^t \big(f(s)-u'(s),u'(s)\big)\,ds$ is absolutely continuous, we conclude. ∎

3.5 Theorem. *$\hat u_\tau$ converges to u in $L^\infty(0,T;H)\cap L^2(0,T;V)$, $\forall\, T>0$.*

Proof. In order to semplify our formulas we call

$$_\tau u(t) = \begin{cases} u(t-\tau) & \text{if } t \ge \tau; \\ u^0 & \text{if } t \in [0,\tau[. \end{cases}$$

Starting from (2.12), we choose $v = {_\tau u(t)}$ obtaining:

$$\begin{aligned}\big(\hat u_\tau', \hat u_\tau - {_\tau u}\big) + a(U_\tau - {_\tau u}) + \frac{1}{2\varepsilon_\tau}\, b_\tau(\cdot;U_\tau) \le \\ \le \big(f_\tau, U_\tau - {_\tau u}\big) + \big(\hat u_\tau', \hat u_\tau - U_\tau\big) + a({_\tau u}, {_\tau u} - U_\tau)\end{aligned}$$

since $b_\tau(t; {_\tau u(t)}) = 0$. Recalling that (see ([17]))

$$\big(\hat u_\tau'(t), \hat u_\tau(t) - U_\tau(t)\big) \le 0, \qquad \text{for a.e. } t>0$$

we have:

$$\begin{aligned}\big(\hat u_\tau' - {_\tau u'}, \hat u_\tau - {_\tau u}\big) + a(U_\tau - {_\tau u}) + \frac{1}{2\varepsilon_\tau}\, b_\tau(\cdot;U_\tau) \le \\ \le \big(f_\tau, U_\tau - {_\tau u}\big) + \big({_\tau u'}, {_\tau u} - \hat u_\tau\big) + a({_\tau u}, {_\tau u} - U_\tau)\end{aligned}$$

and integrating from 0 to $t \le T$ we get:
(3.2)
$$\frac{1}{2}|\hat{u}_\tau(t) - {}_\tau u(t)|^2 + \int_0^t \Big[a(U_\tau(s) - {}_\tau u(s)) + \frac{1}{2\varepsilon_\tau}\, b_\tau(s; U_\tau(s))\Big]\, ds \le$$
$$\le \int_0^t \Big[\big(f_\tau(s), U_\tau(s) - {}_\tau u(s)\big) + \big({}_\tau u'(s), {}_\tau u(s) - \hat{u}_\tau(s)\big) + a({}_\tau u(s), {}_\tau u(s) - U_\tau(s))\Big]\, ds .$$

By the previous weak convergence results we deduce that the right hand member goes to 0 as $\tau \to 0$, so that:

$$\hat{u}_\tau(t) \to u(t) \quad \text{in } H, \quad \forall\, t \ge 0; \qquad \lim_{\tau\to 0} \int_0^T a(U_\tau - u(t))\, dt = 0. \tag{3.3}$$

By (2.15), u_τ and U_τ pointwise converge to u in H, too; the uniform boundedness implies the convergence in $L^2(0,T;H)$ and the convergence in $L^2(0,T;V)$ follows now from (3.3).

Finally, from (3.2) we find

$$\|\hat{u}_\tau - {}_\tau u\|_{L^\infty(0,T;H)} \le C(f, u^0; T)\Big\{\|{}_\tau u - \hat{u}_\tau\|_{L^2(0,T;H)} + \|U_\tau - {}_\tau u\|_{L^2(0,T;V)}\Big\}$$

and we obtain the uniform convergence in H. ▪

3.6 Remark. The convergence in $L^p(0,T;V)$, $\forall p < \infty$, follows from the above result and the uniform boundedness in $L^\infty(0,T;V)$. □

From (3.2), we easily find

3.7 Corollary. *For any $T > 0$ we have:*

$$\lim_{\tau\to 0^+} \frac{1}{\varepsilon_\tau} \int_0^T b_\tau(t; U_\tau(t))\, dt = 0 \quad ▪$$

3.8 Theorem. *Assume that for a fixed $t > 0$ it holds*

$$\limsup_{s\to t^-} a(u(s)) \le a(u(t));$$

then we have:

$$\lim_{\tau\to 0} \|u_\tau(t) - u(t)\| = 0. \tag{3.4}$$

Proof. We shall show that, under the previous assumption, from every decreasing sequence $\{\tau_n\}_{n\in\mathbf{N}}$ with $\lim_{n\to\infty} \tau_n = 0$ we can extract a subsequence τ_{n_j} such that

$$\limsup_{j\to\infty} a(u_{\tau_{n_j}}(t)) \le a(u(t)) \tag{3.5}$$

We choose τ_j ([20]) in such a way that for a.e. $s > 0$

$$\lim_{j\to\infty} \|u_{\tau_j}(s) - u(s)\| = 0; \qquad \lim_{j\to\infty} \frac{1}{\varepsilon_{\tau_j}} b_{\tau_j}(s; U_{\tau_j}(s)) = 0 \tag{3.6}$$

which is always possible, thanks to the integral convergence of the previous results.

Let Z be the subset of $[0, \infty[$ where (3.6) holds; in particular t is an accumulation point of Z, since its complement has empty interior.

Let us fix $s \in Z$ with $s < t$ and choose

$$s_j, t_j \in \{k\tau_j\}_{k\in\mathbf{N}} : \qquad s_j \le s < s_j + \tau_j, \quad t_j \le t < t_j + \tau_j$$

with:

$$m_j = s_j/\tau_j, \qquad M_j = t_j/\tau_j \text{ ([21])}.$$

Repeating the argument of Proposition 2.2, we obtain:

$$a(u_{\tau_j}^{M_j}) \le a(u_{\tau_j}^{m_j}) + \frac{1}{\varepsilon_{\tau_j}} b_{\tau_j}^{m_j}(u_{\tau_j}^{m_j+1}) + \tau_j \sum_{k=m_j+1}^{M_j-1} |f_{\tau_j}^k|^2$$

that is:

$$a(u_{\tau_j}(t)) \le a(u_{\tau_j}(s)) + \frac{1}{\varepsilon_{\tau_j}} b_{\tau_j}(s; U_{\tau_j}(s)) + \int_{s_j}^{t_j} |f(\lambda)|^2 \, d\lambda.$$

Passing to the limit as $j \to \infty$:

$$\limsup_{j\to\infty} a(u_{\tau_j}(t)) \le a(u(s)) + \int_s^t |f(\lambda)|^2 \, d\lambda, \qquad \forall\, s \in Z,\ s < t$$

and as $s \to t$:

$$\limsup_{j\to\infty} a(u_{\tau_j}(t)) \le \limsup_{s\to t^-} a(u(s)) \le a(u(t)) \quad \blacksquare$$

([20]) For the sake of simplcity, we write τ_j for τ_{n_j}

([21]) In this way we have $u_{\tau_j}^{M_j} = u_{\tau_j}(t)$ and $u_{\tau_j}^{m_j+1} = U_{\tau_j}(s)$.

4. - Proof of theorem 5 and 6.

We assume now $(H6-7',7'')$ and we prove theorem 5; we fix $T>0$ and we have to show that

$$\limsup_{s\to T^-} a(u(s)) \le a(u(T)). \tag{4.1}$$

Since we are interested to the behaviour of $u(t)$ for $t\le T$, it is not restrictive to assume that

$$N_t \equiv N_T, \qquad \text{for } t\ge T \tag{4.2}$$

since the solution of (1.14) relative to this new family of subspaces coincide with u in the interval $[0,T]$.

Now we choose $h\in]0,\delta_T[$, $s\in [T-\delta_T, T-h[$, and recall that

$$\frac{1}{2}\int_s^{s+h} a(u(t))\,dt \le \frac{1}{2}\int_T^{T+h} a(u(t))\,dt + \int_s^T a(u(t),u(t)-u(t+h))\,dt \tag{4.3}$$

Our aim is to estimate the last integral.

We observe that from (1.14) we obtain

$$\int_0^{T+\delta_T} \|u(t)\|^2_{D_t}\,dt \le C(f,u^0;T), \tag{4.4}$$

so that

$$\begin{aligned}
&\int_s^T a(u(t),u(t)-u(t+h))\,dt \le \\
&\qquad \le\ C_T\int_s^T \|u(t)\|_{D_t}|u(t)-u(t+h)|\,dt + \\
&\qquad + C_T\int_0^h d\lambda\int_s^T \rho(t+\lambda)\Big[\|u(t)\|_{D_t}\|u(t+h)\|_{D_{t+h}}\Big]^{1-\theta}\Big[\|u(t)\|\,\|u(t+h)\|\Big]^{\theta}\,dt
\end{aligned}$$

where we extended $\rho(\lambda)$ to 0 outside $[T-\delta_T,T]$. The first integral is bounded by

$$h\int_s^{T+h} \|u(t)\|^2_{D_t} + |u'(t)|^2\,dt$$

while the second one is controlled via

$$h\|\rho\|^{\theta}_{L^{1/\theta}(T-\delta_T,T)}\|u\|^{2\theta}_{L^\infty(0,T+h;V)}\left\{\int_s^{T+h}\|u(t)\|^2_{D_t}\,dt\right\}^{1-\theta}.$$

Recalling (4.3) we have:

$$\frac{1}{2h}\int_s^{s+h} a(u(t))\,dt \le \frac{1}{2h}\int_T^{T+h} a(u(t))\,dt+$$
$$+\,C(f,u^0,T,\rho)\left\{\int_s^{T+h} \|u\|_{D_t}^2 + |u'|^2\,dt + \left[\int_s^{T+h} \|u(t)\|_{D_t}^2\,dt\right]^{1-\theta}\right\}.$$

As $h \to 0^+$, by the right continuity of u with respect to the V-norm we get

$$a(u(s)) \le a(u(T)) + C\int_s^T \left[\|u(t)\|_{D_t}^2 + |u'(t)|^2\right] dt + C\left[\int_s^T \|u(t)\|_{D_t}^2\,dt\right]^{1-\theta}$$

and finally

$$\limsup_{s\to T^-} a(u(s)) \le a(u(t)) \quad \blacksquare$$

4.1 Remark. We already observed that $(H7)$ is a particular case of these more general assumption; the $L^{1/\theta}$-norm of ρ becomes the L^∞-one, and the calculations are the same. $\square$

Let us now check the validity of the application given in Section 1. First of all we recall some basic estimates on functions of Sobolev spaces and their traces at the boundary.

4.2 Lemma ([30]). *Let Ω be a (strongly) Lipschitz open subset of $\mathbf{R}^n$; then there exists a constant $C > 0$, depending only on the Lipschitz bound of the boundary, such that:*

$$\int_{\partial\Omega} |v|^2\,d\mathcal{H}^{n-1} \le C\|v\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega;\mathbf{R}^n)}, \qquad \forall\, v \in H^1(\Omega) \tag{4.5}$$

4.3 Lemma ([30]). *Let $\Omega_0 \subset \Omega$, be (strongly) Lipschitz open subsets of $\mathbf{R}^n$; then there exists a constant $C > 0$, depending only on the Lipschitz bound of their boundaries, such that:*

$$\int_{\Omega\setminus\Omega_0} |v(x)|^2\,dx \le C\,\mathrm{dist}(\partial\Omega_0,\partial\Omega)\|v\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega;\mathbf{R}^n)}, \qquad \forall\, v \in H^1(\Omega) \tag{4.6}$$

4.4 Corollary. *In the same hypotheses of the previous lemma, we have:*

$$\int_{\Omega\setminus\Omega_0} |v(x)|^2\,dx \le C\,\mathrm{dist}^2(\partial\Omega_0,\partial\Omega)\|\nabla v\|_{L^2(\Omega;\mathbf{R}^n)}^2, \qquad \forall\, v \in H_0^1(\Omega) \tag{4.7}$$

Proof. It is sufficient to apply (4.6) to the new open sets $\mathbf{R}^n \setminus \overline{\Omega} \subset \mathbf{R}^n \setminus \overline{\Omega_0}$ and to the trivial extension of u outside Ω. $\blacksquare$

4.5 Lemma ([20]). *Let $\{E_t\}_{t\geq 0}$ be a non decreasing family of convex open bounded sets and $T>0$ such that*

$$\lim_{t\to T^-} \operatorname{dist}(\partial E_t, \partial E_T) = 0. \tag{4.8}$$

Then there exists a $\delta_T > 0$ such that E_t are uniformly Lipschitz for $t \in [T-\delta_T, T]$.

We have now all the elements to show $(H7', H7'')$. First we note that $\forall\, t \geq 0$:

$$D_t = \left\{ v \in H^1(\mathbf{R}^n) :\ v_{|_{E_t}} \in H^1_0(E_t) \cap H^2(E_t) \right\}$$

with (see [20])

$$[v]_t^2 = \int_{E_t} |\Delta v(x)|^2\, dx \geq \sum_{i,j=1}^n \int_{E_t} \left| \frac{\partial^2 v}{\partial x_i \partial x_j} \right|^2 dx. \tag{4.9}$$

If $u \in D_t$, $v \in D_{t+h}$, standard Green formula gives:

$$\begin{aligned} a(u, u-v) = &\int_{\mathbf{R}^n} \big(\nabla u(x), \nabla u(x) - \nabla v(x)\big)\, dx = \int_{E_t} \big(\nabla u(x), \nabla u(x) - \nabla v(x)\big)\, dx = \\ &- \int_{E_t} \Delta u(x)(u(x) - v(x))\, dx - \int_{\partial E_t} \frac{\partial u(x)}{\partial n} v(x)\, d\mathcal{H}^{n-1} \leq \\ &\|u\|_{D_t} \|u-v\|_{L^2(\mathbf{R}^n)} + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\partial E_t)} \|v_{|_{\partial E_t}}\|_{L^2(\partial E_t)} \end{aligned}$$

(1.23) implies that (4.8) is satisfied; applying (4.5) in the left neighborhood of $T>0$ given by the previous lemma, we get $\forall\, t, t+h \in [T-\delta_T, T]$

(4.10)

$$\|v\|^2_{L^2(\partial E_t)} \leq C_T \|v\|_{L^2(\mathbf{R}^n \setminus E_t)} \|\nabla v\|_{L^2(\mathbf{R}^n \setminus E_t; R^n)} = C_T \|v\|_{L^2(E_{t+h} \setminus E_t)} \|\nabla v\|_{L^2(E_{t+h} \setminus E_t; R^n)}$$

and

$$\left\| \frac{\partial u}{\partial n} \right\|^2_{L^2(\partial E_t)} \leq C_T \|\nabla u\|_{L^2(E_t; \mathbf{R}^n)} [u]_t. \tag{4.11}$$

Since $v \in H^1_0(E_{t+h})$, by (4.7) we get

$$\|v\|_{L^2(E_{t+h} \setminus E_t)} \leq C_T \operatorname{dist}(\partial E_t, \partial E_{t+h}) \|\nabla v\|_{L^2(E_{t+h} \setminus E_t; R^n)}.$$

Applying (4.5) to $\nabla v \in H^1(E_{t+h}; \mathbf{R}^n)$ we get, thanks to (4.9):

$$\|\nabla v\|^2_{L^2(E_{t+h} \setminus E_t; R^n)} \leq C_T \operatorname{dist}(\partial E_t, \partial E_{t+h}) \|\nabla v\|_{L^2(E_{t+h}; \mathbf{R}^n)} [v]_{t+h}.$$

Combining all these estimates we get:

$$\begin{aligned} a(u, u-v) \leq &\|u\|_{D_t} \|u-v\|_{L^2(\mathbf{R}^n)} + \\ &C_T \operatorname{dist}(\partial E_t, \partial E_{t+h}) \Big(\|\nabla u\|_{L^2(E_t; \mathbf{R}^n)} [u]_t \|\nabla v\|_{L^2(E_{t+h}; \mathbf{R}^n)} [v]_{t+h} \Big)^{1/2} \end{aligned}$$

and by (1.23) we conclude. ∎

4.6 Remark. It is now easy to check that uniform $C^{1,1}$ regularity for E_t allows analogous bounds. $\square$

Now we assume that $(H1-6,8)$ and prove the last theorem of Section 1; we shall denote by c all the constants independent of the data and by C those that depend only on f, u^0 and T. We start with a simple lemma:

4.7 Lemma. *The bilinear form* $a(\cdot,\cdot)$ *can be continuously extended to* $D_t \times (V,H)_{\theta,p'}$ *with*

$$a(v,w) \leq c\|v\|_{D_t}\,\|w\|_{(V,H)_{\theta,p'}}, \qquad \forall\, v \in D_t \;\forall\, w \in (V,H)_{\theta,p'} \tag{4.12}$$

Proof. Observe that the real bilinear form

$$(v,w) \mapsto a(v,w)$$

is continuous in the product spaces $V \times V$ and $D(A) \times H$. By standard results on interpolation (see [28], [6]) it is also continuous in

$$(V, D(A))_{\theta,p} \times (V,H)_{\theta,p'}, \qquad \theta \in]0,1[$$

Since $D_t \subset (V, D(A))_{\theta,p}$ we conclude. ∎

4.8 Proposition. *Let* u *be the solution of* (1.14), *and* $\hat{u}_\tau$ *be the usual piecewise linear Minimizing Movement; then there exists a constant* $C = C(f, u^0; T)$ *such that*

$$\begin{aligned} &\int_0^T \Big[(u', u-\hat{u}_\tau) + a(u, u-U_\tau)\Big]\, dt \leq \\ &\leq \int_0^T \Big[(f, u-U_\tau) + \frac{1}{2\varepsilon_\tau} b(t; U_\tau)\Big]\, dt + C\Big[\tau + \varepsilon_\tau^\sigma\Big] \end{aligned} \tag{4.13}$$

Proof. Let us start from the lefthand side of (4.13), and choose $w \in N_t$; we easily get

$$\begin{aligned} (u',&u-\hat{u}_\tau) + a(u, u-U_\tau) = \\ &= (u' + Au - f, u - U_\tau) + (u', U_\tau - \hat{u}_\tau) + (f, u-U_\tau) \leq \\ &\leq (u' + Au - f, w - U_\tau) + (u', U_\tau - \hat{u}_\tau) + (f, u-U_\tau) \leq \\ &\leq |u' - f|\,|w - U_\tau| + c\|u\|_{D_t}\|U_\tau - w\|_{(V,H)_{\theta,p'}} + (f, u-U_\tau) + \tau|u'|\,|\hat{u}_\tau'| \end{aligned}$$

Since w is arbitrary and $(V,H)_{\theta,p'} \subset H$ we obtain

$$\begin{aligned} (u',&u-\hat{u}_\tau) + a(u, u-U_\tau) \leq \\ &\leq (f, u-U_\tau) + \tau|u'|\,|\hat{u}_\tau'| + \Big[|u'-f| + c\|u\|_{D_t}\Big] \inf_{w\in N_t} \|U_\tau - w\|_{(V,H)_{\theta,p'}}. \end{aligned} \tag{4.14}$$

Choosing in (1.26)

$$M=\sup_{\tau\in]0,1[}\|U_\tau\|_{L^\infty(0,T;V)}$$

we can estimate the last addendum of the righthand member; from Proposition 2.2 we would get

$$\inf_{w\in N_t}\|U_\tau-w\|_{(V,H)_{\theta,p'}}\le[C_M\,b(t,U_\tau)]^{\frac{1}{\gamma}}\le C\varepsilon_\tau^{\frac{1}{\gamma}} \tag{4.15}$$

but we can obtain a better exponent; (4.14) is bounded by (22)

$$C\varepsilon_\tau^{\frac{1}{\gamma-1}}\Big[|u'-f|+c\|u\|_{D_t}\Big]^{\frac{\gamma}{\gamma-1}}+\frac{1}{2\varepsilon_\tau}b(t;U_\tau),\qquad \text{if }\gamma\ge2$$

and

$$\left.\begin{aligned}&C\inf_{w\in N_t}\|U_\tau-w\|^{1-\gamma/2}_{(V,H)_{\theta,p'}}\Big[|u'-f|+\|u\|_{D_t}\Big]b(t;U_\tau)^{1/2}\le\\ &C\varepsilon_\tau^{2/\gamma}\Big[|u'-f|^2+\|u\|^2_{D_t}\Big]+\frac{1}{2\varepsilon_\tau}b(t;U_\tau).\end{aligned}\right.\qquad\text{if }1\le\gamma\le2$$

Integrating on $(0,T)$ we obtain (4.13). ∎

4.9 Proposition. *With the same hypotheses of the previous proposition, we have:*

$$\begin{aligned}&\int_0^T\big(\hat u'_\tau,\hat u_\tau-u\big)+a(U_\tau,U_\tau-u)+\frac{1}{2\varepsilon_\tau}b(t;U_\tau)\,dt\le\\ &\le\int_0^T\big(f_\tau,U_\tau-u\big)+\frac12a(u-U_\tau)\,dt+C\tau.\end{aligned}\tag{4.16}$$

Proof. Again we have:

$$\begin{aligned}&\big(\hat u'_\tau,\hat u_\tau-u\big)+a(U_\tau,U_\tau-u)+\frac{1}{2\varepsilon_\tau}b(t;U_\tau)=\\ &=\big(\hat u'_\tau,\hat u_\tau-{}_\tau u\big)+a(U_\tau,U_\tau-{}_\tau u)+\frac{1}{2\varepsilon_\tau}b(t;U_\tau-{}_\tau u)+\big(\hat u'_\tau+AU_\tau,{}_\tau u-u\big)\le\\ &\le(f_\tau,U_\tau-{}_\tau u)+(\hat u'_\tau,\hat u_\tau-U_\tau)+\big(\hat u'_\tau+AU_\tau,{}_\tau u-u\big)\le\\ &\le(f_\tau,U_\tau-{}_\tau u)+|\hat u'_\tau|\,|{}_\tau u-u|+a(U_\tau-u,{}_\tau u-u)+a(u,{}_\tau u-u)\le\\ &\le(f_\tau,U_\tau-{}_\tau u)+|\hat u'_\tau|\,|{}_\tau u-u|+\frac12a(U_\tau-u)+\frac12a({}_\tau u)-\frac12a(u)\end{aligned}$$

Integrating, we conclude. ∎

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(22) We use the standard inequality ($p,q\in]1,\infty[$)

$$xy\le\frac{\alpha^p}{p}x^p+\frac{1}{q\alpha^q}y^q,\qquad\forall\,x,y,\alpha\ge0,\qquad 1/p+1/q=1.$$

4.10 Corollary. *In the usual hypotheses we obtain:*

$$\|u - \hat{u}_\tau\|^2_{L^\infty(0,T;H)} + \|u - U_\tau\|^2_{L^2(0,T;V)} \leq C\Big[\tau + \varepsilon_\tau^\sigma\Big]$$

where σ is given by (1.28).

Proof. Summing up (4.13) and (4.16) we obtain

$$\frac{1}{2}|u(T) - \hat{u}_\tau(T)|^2 + \frac{1}{2}\int_0^T a(u - U_\tau)\,dt \leq \int_0^T (f - f_\tau, u - U_\tau)\,dt + C\Big[\tau + \varepsilon_\tau^\sigma\Big]$$

and we only have to control that

$$\int_0^T (f - f_\tau, u - U_\tau)\,dt \leq C\,\tau\,\|f\|_{L^2(0,T+\tau;H)}\|u\|_{H^1(0,T+\tau;H)}.$$

We refer to [29]. ■

Finally we control that Problem 1 belongs to the class 4, following Remark 1.11.

From [30] we have:

$$N_t \subset \big(H^2(\mathbf{R}^N), H^1(\mathbf{R}^N)\big)_{1/2,\infty}$$

uniformly with respect to t, so that (1.25) is satisfied with $\theta = 1/2$; we conclude explaining how to construct $\mathcal{P}_t$.

We know that for every Lipschitz open set Ω there exists a bounded linear extension operator $\mathcal{T} : L^2(\Omega) \mapsto L^2(\mathbf{R}^n)$ such that

$$\big(\mathcal{T}v\big)_{|\Omega} = v$$

and

$$v \in H^1(\Omega) \Rightarrow \mathcal{T}v \in H^1(\mathbf{R}^n), \qquad \|\mathcal{T}v\|_{H^1(\mathbf{R}^n)} \leq C_{\mathcal{T}}\|v\|_{H^1(\Omega)}.$$

Moreover the norms of $\mathcal{T}$ as linear operator in L^2 and H^1 depend only on the Lipschitz bound of Ω.

Therefore we consider the extension operator $\mathcal{T}_t$ relative to $\mathbf{R}^n \setminus \overline{E}_t$ and we set

$$\mathcal{P}_t v = v - \mathcal{T}_t\big(v_{|\mathbf{R}^n \setminus \bar{E}_t}\big).$$

Since the restriction operator is bounded from $H^1(\mathbf{R}^n)$ to $H^1(\mathbf{R}^n \setminus \bar{E}_t)$ we easily check the first bound of (1.27); moreover

$$\|v - \mathcal{P}_t v\|_{L^2(\mathbf{R}^n)} = \|\mathcal{T}_t\big(v_{|\mathbf{R}^n \setminus \bar{E}_t}\big)\|_{L^2(\mathbf{R}^n)} \leq C\|v\|_{L^2(\mathbf{R}^n \setminus \bar{E}_t)} \leq Cb(t;v)^{1/2} \quad ■$$

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