ON THE REGULARITY OF THE POSITIVE PART OF FUNCTIONS

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0. - Introduction and main results.

The aim of this paper is to study the regularity properties of the truncation operator ([1])

$$u \mapsto u^+ = \max(u,0) = \begin{cases} u & \text{if } u > 0; \\ 0 & \text{otherwise} \end{cases} \tag{0.1}$$

in various Banach spaces of functions defined on a Lipschitz open set $\Omega \subset \mathbf{R}^N$.
It is a well known result (see [1] for example), that (0.1) defines a contraction in $L^p(\Omega)$ which is bounded in $W^{1,p}(\Omega)$, $\forall\, p \in [1,+\infty]$, that is:

$$\|u^+ - v^+\|_{L^p(\Omega)} \le \|u - v\|_{L^p(\Omega)}, \qquad \forall u, v \in L^p(\Omega) \tag{0.2}$$

$$\|u^+\|_{W^{1,p}(\Omega)} \le \|u\|_{W^{1,p}(\Omega)}, \qquad \forall\, u \in W^{1,p}(\Omega) \tag{0.3}$$

On the other hand, $u \in W^{2,p}(\Omega)$ does not imply the same regularity for u^+, since jump discontinuities of the gradient along manifolds of dimension $N-1$ ([2]) may occour.

([1]) In order to fix our ideas, we only consider the positive part; obviously, all our statements hold for the negative one, and can be extended to $u \mapsto |u|$ and so on.

([2]) For $W^{1,p}(\Omega)$-functions, we have the formula ([1]):

$$\nabla(u^+) = H(u)\cdot\nabla u, \quad \text{a.e. in } \Omega \quad \text{where } H(u) = \begin{cases} 1 & \text{if } u > 0 \\ 0 & \text{otherwise} \end{cases} \text{ is the Heaviside function.}$$

We recall that further generalizations are given by [2] (where the positive part is replaced by any Lipschitz continuous function) and by [3] (where a general chain rule is established for BV-functions u).

One can ask if $u \in W^{s,p}(\Omega) \Rightarrow u^+ \in W^{s,p}(\Omega)$ for some intermediate $s \in]1,2[$ ([3]). The same remark on the gradient discontinuities carries to exclude the case $s \geq 1/p$. In [9] it is proved that the range $s \in [1, 1+1/p[$ can be allowed, that is (0.1) maps $W^{s,p}(\Omega)$ into itself, $\forall\, s \in [1, 1+1/p[$, $1 \leq p < \infty$, with

$$\|u^+\|_{W^{s,p}(\Omega)} \leq C\|u\|_{W^{s,p}(\Omega)} \tag{0.4}$$

for a suitable constant $C > 0$. By (nonlinear) interpolation the same result holds for the Besov spaces $B^s_{p\,q}(\Omega)$, with $q \in [1, \infty]$ and $1 \leq s < 1 + 1/p$.

0.1 Remark. In the hilbertian framework of the family $W^{s,2}(\Omega) = H^s(\Omega)$ (see [10]) we have

$$u \in H^s(\Omega) \;\Rightarrow\; u^+ \in H^s(\Omega), \qquad \forall\, s < 3/2 \quad \square$$

The limiting case $p = 1$ has an interesting development which is not covered by the previous result; instead of $W^{2,1}(\Omega)$ we can consider the space (see [11]):

$$BH(\Omega) = \big\{u \in W^{1,1}(\Omega) : D^2u \text{ is a matrix-valued Radon measure}\big\}$$

or equivalently $\nabla u \in BV(\Omega; \mathbf{R}^N)$, that is the total variation

$$|D^2u|_T = \sup\left\{\sum_{i=1}^{N} \int_\Omega \frac{\partial u}{\partial x_i} \mathrm{div}\phi_i(x)\, dx, \quad \phi_i \in C_0^\infty(\Omega; \mathbf{R}^N), \quad \sum_i |\phi_i(x)|^2 \leq 1 \text{ in } \Omega\right\}$$

is finite ([4]). We prove:

([3]) For the definitions, see [4], [5], [6]; we shall use the interpolation real functor $(\cdot,\cdot)_{\theta,p}$, $0 < \theta < 1$, $1 \leq p \leq \infty$ (see [7], [8] and section 2 for more details) and we set

$$B^s_{p\,q}(\Omega) = \big(L^p(\Omega), W^{2,p}(\Omega)\big)_{s/2,q}; \qquad 0 < s < 2, \quad p, q \in [1, +\infty]$$

with

$$W^{s,p}(\Omega) = B^s_{p\,p}(\Omega) \quad \text{if } \; 0 < s < 2, \; s \neq 1$$

([4]) $BH(\Omega)$ is a (dual) Banach space, equipped by the norm:

$$\|u\|_{BH(\Omega)} = \|u\|_{W^{1,1}(\Omega)} + |D^2u|_T.$$

Observe that $W^{2,1}(\Omega) \subset BH(\Omega)$ and for a $W^{2,1}(\Omega)$ function u the total variation of the Hessian becomes:

$$|D^2u|_T = \int_\Omega \big|D^2u\big|\, dx.$$

Theorem 1. *If $u \in BH(\Omega)$ then $u^+ \in BH(\Omega)$ and we can choose an equivalent norm in $BH(\Omega)$ such that:*

$$\|u^+\|_{BH(\Omega)} \leq \|u\|_{BH(\Omega)}$$

0.2 Remark. $BH(\Omega)$ has been thoroughly studied by F.Demengel in [11] as natural domain of minimizing hessian dependent functionals with linear growth at infinity. This space allows creasing whithout fracture and it is not surprising that the positive part doesn't modify this regularity. Another interesting function space related to this kind of problem is $SBH(\Omega)$, which has been studied in [12]. $\square$

0.3 Remark. We can also characterize $BH(\Omega)$ as the Gagliardo completion of $W^{2,1}(\Omega)$ relative to $L^1(\Omega)$ (see [13] and sect.4); in particular, $BH(\Omega)$ is an exact interpolation space between the other two. This result allows us to extend some properties proved in [11] and [12] for C^2 domains to any Lipschitz open subset of $\mathbf{R}^N$. We only mention the extension theorem, the Poincaré type inequality and the continuous inclusion

$$BH(\Omega) \subset C^0(\bar{\Omega}), \qquad N = 2 \quad \square$$

In the one dimensional case we can be more precise; if v is a locally integrable real function defined on the open (and possibly unbounded) interval (a, b), we define the *essential variation* of v as:

$$\text{ess-}V_a^b(v) = \sup\left\{\int_a^b v(x)\phi'(x)\,dx, \quad \phi \in C_0^\infty(a,b),\ |\phi(x)| \leq 1 \text{ in } (a,b)\right\} = |v'|_T \quad (0.5)$$

and we say that v is of essential bounded variation ($v \in BV(a,b)$), if ess-$V_a^b(v) < +\infty$ ([5]). For this kind of function it is possible to define two linear trace operators which we call for simplicity:

$$v \mapsto v_+(a), \qquad v \mapsto v_-(b)$$

We have:

([5]) We know (see [14], [15]) that we can find a real function $w : (a,b) \mapsto \mathbf{R}$ such that $w = v$ a.e. on (a,b) and

$$V_a^b(w) = \sup\Big\{\sum_{j=1}^m \big|w(t_j) - w(t_{j-1})\big|, \qquad a < t_0 < t_1 < \ldots < t_m < b\Big\} = \text{ess-}V_a^b(v)$$

In sect. 1 we use the analogous definition of ess-$V_a^b(v)$ by using the "approximate" notions of limit and continuity.

Theorem 2. *Let v be a locally absolutely continuous function defined on (a,b) and let u be the positive part of v; if* ess-$V_a^b(v') < +\infty$ *then we have* ess-$V_a^b(u') < +\infty$ *with:*

$$\text{ess-}V_a^b(u') + |u'_+(a)| + |u'_-(b)| \leq \text{ess-}V_a^b(v') + |v'_+(a)| + |v'_-(b)| \tag{0.6}$$

In particular, if $v'_+(a) = v'_-(b) = 0$ or $v(a), v(b) > 0$ then:

$$\text{ess-}V_a^b(u') \leq \text{ess-}V_a^b(v') \tag{0.7}$$

0.4 Remark. In general we cannot control ess-$V_a^b(u')$ just by ess-$V_a^b(v')$: consider for example the function

$$v(x) = x, \qquad x \in (-1,1)$$

we have ess-$V_{-1}^1(u') = 1$ and ess-$V_{-1}^1(v') = 0$. However, (0.7) holds if $v'_+(a) \leq 0$ or $v(a) > 0$, and $v'_-(b) \geq 0$ or $v(b) > 0$. When $(a,b) = \mathbf{R}$ and $v \in L^p(\mathbf{R})$ or $v' \in L^p(\mathbf{R})$ for some $p < \infty$, (0.7) is always verified. □

0.5 Remark. We denoted by $BV(a,b)$ the space of functions of essential bounded variation; $BV(a,b)$ becomes a Banach space with the norm:

$$\|u\|_{BV(a,b)} = |u_+(a)| + |u_-(b)| + \text{ess-}V_a^b(u)$$

and we easily control that $BV(a,b) \subset L^\infty(a,b)$, with continuous inclusion. Observe that the term $|u_+(a)| + |u_-(b)|$ is usually substituted by the $L^1(a,b)$-norm; if (a,b) is bounded these norms are equivalent whereas in the unbounded cases the L^1-norm raises a smaller space. Our choice allows a slightly bigger generality and is justified by (0.6). □

Let us now consider the two extreme cases with respect to the summability power p and to the corresponding regularity, that is $u \in W^{1,\infty}(a,b)$ and $u \in BH(a,b)$; since $u \mapsto u^+$ is bounded in both spaces, it would be interesting to know if it is bounded in the family of real interpolation spaces obtained from them. In other words, let $(\cdot,\cdot)_{\theta,p}$ be the real interpolation functor introduced by the $K-$method of J. Peetre (see [8], [7] and sect.2) and let us define (following [16]), for $0 < \theta < 1$, $1 \leq p \leq \infty$

$$\begin{aligned} Z^{\theta,p}(a,b) &= \big(L^\infty(a,b), BV(a,b)\big)_{\theta,p} \\ Z^{1+\theta,p}(a,b) &= \Big\{w \in W^{1,\infty}(a,b) : w' \in Z^{\theta,p}(a,b)\Big\} \end{aligned} \tag{0.8}$$

Non-linear intrpolation theory (see [17]) can not be applied, as it requires a Lipschitz or Hölder hypothesis for the mapping on some of the spaces; however we can prove:

Theorem 3. *Let $v \in AC_{loc}(a,b)$ with $v' \in Z^{\theta,p}(a,b)$; if $u = v^+$ then we have:*

$$u' \in Z^{\theta,p}(a,b), \qquad \|u'\|_{Z^{\theta,p}(a,b)} \leq \|v'\|_{Z^{\theta,p}(a,b)} \tag{0.9}$$

In particular, $v \in Z^{1+\theta,p}(a,b) \Rightarrow v^+ \in Z^{1+\theta,p}(a,b)$.

0.6 Remark. The family $Z^{\theta,p}$ has been introduced in [16] to study the regularity properties of the solutions of scalar conservation laws. In this framework the good function spaces have to contain discontinuous functions and their behaviour with respect to truncations plays an important role. $\square$

At this point it is natural to ask if the spaces $Z^{\theta,p}(a,b)$ are related to those of the more familiar Besov class; we can prove a conjecture stated in [16]

Theorem 4. *Choosing in* (0.8) $p = 1/\theta$, *the space*

$$Z^{1+\theta}(a,b) = W^{1,p}(a,b) \cap Z^{1+\theta,p}(a,b), \qquad 0 < \theta < 1$$

is of class $(1+\theta)/2$ *between* $L^p(a,b)$ *and* $W^{2,p}(a,b)$, *that is*

$$B^{1+\theta}_{p\,1}(a,b) \subset Z^{1+\theta}(a,b) \subset B^{1+\theta}_{p\,\infty}(a,b).$$

This last property says that $Z^{1+\theta}(a,b)$, for $0 < \theta < 1$, is an "upper" limiting bound for the family $B^s_{p\,q}(a,b)$, $1 \leq p,q \leq +\infty$, $1 \leq s < 1 + 1/p$; then it is not surprisingly that the result of [9] can be recovered by (nonlinear) interpolation. In this way we can also give a further information on the constant C of (0.4):

Theorem 5. (0.1) *maps* $B^s_{p\,q}(\Omega)$ *into itself,* $\forall\, s \in [1, 1+1/p[$, $1 \leq p,q \leq \infty$, *and we can choose an equivalent norm on these spaces such that:*

$$\|u^+\|_{B^s_{p\,q}(\Omega)} \leq \|u\|_{B^s_{p\,q}(\Omega)} \tag{0.10}$$

0.7 Remark. In [9] is given an example of a $B^{1+1/p}_{p\,\infty}(\mathbf{R})$-function v whose positive part doesn't belong to the same space; theorem 3 and 4 show, however, that this regularity can be recovered as soon as v is a little more regular, that is $v \in B^{1+1/p}_{p\,1}(\mathbf{R})$. For $p = 2$ this fact has a rather surprising counterpart in the results of [18] for evolution variational inequalities. On the other side our technique can also be applied to the evolution problems of the type sutdied in [19], extending the results of [20] and [18]. In a forthcoming article we exploit this feature. $\square$

The outline of the paper is the following: in the next two sections we deal with $BH(a,b)$ and $Z^{1+\theta,p}(a,b)$ respectively; we prove theorem 2 and 3 for piecewise linear functions and we extend them to the whole spaces by an approximation technique. The last two theorems will be proved in section 3; the proof of theorem 1 with remark 0.3 will be given in the last section, but it depends only on the results of section 1.

1. - Proof of theorem 2.

We begin by recalling the main properties of functions of bounded variation in (a, b). In order to work with a.e. defined functions we must consider the "approximate" notions of limit and continuity (see [14], [15], [21]); for a measurable set E, $|E|$ is its Lebesgue measure.

1.1 Definition. *Let $u : (a, b) \mapsto \mathbf{R}$ be a Lebesgue measurable function and $x_0 \in [a, b]$. We say that*

$$\operatorname{ap}\lim_{x\to x_0} u(x) = l$$

if

$$\lim_{r\to 0^+} \frac{\big|\{x \in \mathbf{R} : |u(x) - l| > \varepsilon\} \cap B_r(x_0)\big|}{2r} = 0, \qquad \forall\, \varepsilon > 0$$

Consequently, x_0 is a point of approximate continuity for u if

$$\operatorname{ap}\lim_{x\to x_0} u(x) = u(x_0)$$

Left and right approximate limits are analogously defined (6)*. We denote by $\mathcal{C}(u)$ the set of approximate continuity points of u; recall that $|(a, b) \setminus \mathcal{C}(u)| = 0$.* □

It is possible to give an equivalent characterization of BV functions

1.2 Theorem ([14], [15], [21])**.** *For a function $u \in L^1_{loc}(a, b)$ we have:*

$$\text{ess-}V_a^b(u) = \sup\Big\{ \sum_{j=1}^{n} |u(t_j) - u(t_{j-1})|, \qquad a < t_0 < t_1 < \ldots < t_n < b, \quad t_j \in \mathcal{C}(u) \Big\}$$

Moreover, if $u \in BV(a, b)$ then, $\forall\, x_0 \in (a, b)$

$$\exists \operatorname{ap}\lim_{x\to x_0^-} u(x) = \lim_{\substack{x\to x_0^- \\ x\in\mathcal{C}(u)}} u(x) = u_-(x_0); \qquad \exists \operatorname{ap}\lim_{x\to x_0^+} u(x) = \lim_{\substack{x\to x_0^+ \\ x\in\mathcal{C}(u)}} u(x) = u_+(x_0)$$

and these values are equal outside a countable set. Analogously we have:

$$\exists \operatorname{ap}\lim_{x\to a^+} u(x) = \lim_{\substack{x\to a^+ \\ x\in\mathcal{C}(u)}} u(x) = u_+(a); \qquad \exists \operatorname{ap}\lim_{x\to b^-} u(x) = \lim_{\substack{x\to b^- \\ x\in\mathcal{C}(u)}} u(x) = u_-(b) \quad \blacksquare$$

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(6) *if $x_0 = +\infty$ we set:*

$$\operatorname{ap}\lim_{x\to+\infty} u(x) = l \iff \operatorname{ap}\lim_{x\to 0^+} u(1/x) = l$$

with the obvious changes if $x_0 = -\infty$.

1.3 Remark. Thanks to (0.5) the ess-V_a^b functional is lower semicontinuous with respect to the $\mathcal{D}'(a,b)$-topology. Since $BV(a,b)$ is a dual space, also its norm is lower semicontinuous. $\square$

1.4 Remark. We can study the ess-V_α^β functional with respect to the extrema $\alpha, \beta \in [a,b]$; setting:

$$[u](x_0) = |u_+(x_0) - u_-(x_0)|, \qquad \forall\, x_0 \in (a,b)$$

the following formula hold for each function of $BV(a,b)$ and for $a < \alpha < \beta < b$

$$\text{ess-}V_a^b(u) = \text{ess-}V_a^\alpha(u) + \text{ess-}V_\alpha^\beta(u) + \text{ess-}V_\beta^b(u) + [u](\alpha) + [u](\beta)$$

$$\lim_{\beta\to b^-} \text{ess-}V_a^\beta(u) = \text{ess-}V_a^b(u); \qquad \lim_{\beta\to a^+} \text{ess-}V_a^\beta(u) = 0 \quad \square$$

Now we want to study some approximation properties in $BV(a,b)$. We consider a finite subdivision $a < t_0 < t_1 < \ldots < t_m < b$ and we call $\mathcal{S}$ the associated family of intervals

$$(a,t_0),\ (t_0,t_1), \ldots, (t_{m-1},t_m),\ (t_m,b) \tag{1.1}$$

We denote by $\mathbf{P}^0(\mathcal{S})$ the vector space of the piecewise constant function u such that:

$$u_{|_I} \equiv c_I, \qquad \forall I \in \mathcal{S}$$

$\mathbf{P}^1(\mathcal{S})$ will be the space of continuous piecewise linear functions u such that $u' \in \mathbf{P}^0(\mathcal{S})$. Given a function $u \in BV(a,b)$ we set (7)

$$\bar{u} = \Pi^0_{\mathcal{S}} u \in \mathbf{P}^0(\mathcal{S}): \quad \bar{u}(x) = \begin{cases} \frac{1}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} u(x)\,dx & \text{if } x \in (t_{j-1},t_j),\ j = 1,\ldots,m; \\ u_+(a) & \text{if } x \in (a,t_0); \\ u_-(b) & \text{if } x \in (t_m,b). \end{cases}$$

If $u \in AC_{loc}(a,b)$ and $u' \in BV(a,b)$ we set:

$$\hat{u} = \Pi^1_{\mathcal{S}} u \in \mathbf{P}^1(\mathcal{S}): \quad \hat{u}(t_j) = u(t_j),\ j = 0,1,\ldots,m; \quad u'(x) = \begin{cases} u'_+(a) & \text{if } x \in (a,t_0), \\ u'_-(b) & \text{if } x \in (t_m,b). \end{cases}$$

The following theorem summarizes the results we shall use:

1.5 Theorem. *Let $\mathcal{S}$ be a subdivision of (a,b) as in* (1.1)*, and $u \in BV(a,b)$; we have:*

$$\|\bar{u}\|_{L^\infty(a,b)} \le \|u\|_{L^\infty(a,b)} \tag{1.2}$$

$$\text{ess-}V_a^b(\bar{u}) \le \text{ess-}V_a^b(u), \quad \|\bar{u}\|_{BV(a,b)} \le \|u\|_{BV(a,b)} \tag{1.3}$$

(7) We cannot choose the mean value on the intervals (a,t_0) and (t_m,b) since they may be unbounded.

and, if $u \in AC_{loc}(a,b)$ *and* $u' \in BV(a,b)$*, we have:*

$$\|\hat{u}\|_{L^\infty(t_0,t_m)} \leq \|u\|_{L^\infty(t_0,t_m)} \tag{1.4}$$

$$(\Pi^1_{\mathcal{S}} u)' = \Pi^0_{\mathcal{S}} u'. \tag{1.5}$$

Moreover, there exists a sequence $\mathcal{S}_n$ *of subdivision of* (a,b) *such that, setting* $\bar{u}_n, \hat{u}_n$ *the corresponding projection of a function* u*:*

$$u \in BV(a,b) \Rightarrow \lim_{n\to\infty} \|u - \bar{u}_n\|_{L^p(\alpha,\beta)} = 0, \qquad \forall\, a < \alpha < \beta < b, \quad \forall\, p \in [1,+\infty) \tag{1.6}$$

$$u \in AC_{loc}(a,b), u' \in BV(a,b) \Rightarrow \ \|u - \hat{u}_n\|_{L^\infty(\alpha,\beta)} \to 0, \qquad \forall\, a < \alpha < \beta < b \tag{1.7}$$

$$\text{ess-}V^b_a(u) = \lim_{n\to\infty} \text{ess-}V^b_a(\bar{u}_n) \qquad \|u\|_{BV(a,b)} = \lim_{n\to\infty} \|\bar{u}_n\|_{BV(a,b)}. \tag{1.8}$$

Proof. The first three formulae are well known and (1.5) follows by the above definitions.
A good choice of $\mathcal{S}_n$ requires that:

$$\lim_{n\to\infty} L_n = a, \quad \lim_{n\to\infty} R_n = b; \qquad \lim_{n\to\infty} \Delta_n = 0 \tag{1.9}$$

where L_n, R_n are the left and the right extremum of the internal intervals of $\mathcal{S}_n$ respectively and Δ_n is the maximum amplitude of the internal intervals of $\mathcal{S}_n$.
In this case (1.6) reduces to standard approximation results on piecewise constant functions and (1.7) is a consequence of the uniform continuity of u on every bounded interval.
Finally (1.8) follows from (1.3) and from the l.s.c. of the ess-V^b_a functional and of the BV-norm. ▪
We begin now the proof of theorem 2 by considering the special case of piecewise linear functions. If v is such a function, so is $u = v^+$ and we can find a common subdivision $\mathcal{S}$ such that $u, v \in \mathbf{P}^1(\mathcal{S})$. Moreover, u is related to v by:

$$u(x) \geq 0; \qquad u'(x) = v'(x)\, H(u(x))(^8), \qquad \forall\, x \in (a,b) \setminus \{t_0, \dots t_n\} \tag{1.10}$$

We focus our attention on this relation and we fix the following

1.6 Definition. *For a given function* $w \in \mathbf{P}^0(\mathcal{S})$ *we denote by* $\mathbf{K}[w]$ *the family of* $u \in \mathbf{P}^1(\mathcal{S})$ *such that*

$$u \geq 0; \qquad u'(x) = w(x)\, H(u(x)), \qquad \forall\, x \in (a,b) \setminus \{t_0, \dots t_n\} \quad \square \tag{1.11}$$

By (1.10) we have:

$$u = v^+ \quad \Rightarrow u \in \mathbf{K}[v']$$

(8) Observe that $H(u(x)) = H(v(x))$. All our results hold even if H is replaced by the characteristic function of any open interval of $\mathbf{R}$.

and we shall see that this is sufficient to obtain the bound:

$$\|u'\|_{BV(a,b)} \leq \|v'\|_{BV(a,b)}.$$

The idea is the following: we divide the zero-set of u into the disjoint union of maximal intervals $I_1, \ldots, I_k$ and we observe that u' can be obtained by applying to v' the functions

$$\mathcal{N}_j(x) = \begin{cases} 0 & \text{if } x \in I_j \\ x & \text{if } x \notin I_j \end{cases}, \qquad j = 1, \ldots, k.$$

Our thesis follows if we shall show that each step does not increase the BV-norm of the function.

1.7 Definition. *We say that an interval $I = (\alpha, \beta) \subset (a, b)$ belongs to $\mathcal{Z}(u)$, the family of the maximal zero intervals of $u \in \mathbf{P}^1(\mathcal{S})$, if*

$$u_{|I} \equiv 0; \quad I \subset I' = (\alpha', \beta'),\ u_{|I'} \equiv 0 \ \Rightarrow I = I' \tag{1.12}$$

For an interval $I = (\alpha, \beta)$, $\mathcal{N}_I(x)$ will be the function which takes the value 0 on I and x outside:

$$\mathcal{N}_I(x) = x\big(1 - \chi_I(x)\big) = \begin{cases} 0 & \text{if } x \in I; \\ x & \text{if } x \notin I. \end{cases} \quad \square$$

We can state now our first result:

1.8 Theorem. *If $u \in \mathbf{K}[w]$ then:*

$$\|u'\|_{BV(a,b)} \leq \|w\|_{BV(a,b)}$$

Proof. We observe that, if $I \in \mathcal{Z}(u)$ is a maximal zero interval of u and $u \in \mathbf{K}[w]$ then also u and $\mathcal{N}_I(w)$ satisfy (1.11):

$$u'(x) = \Big[\mathcal{N}_I\big(w(x)\big)\Big]\, H\big(u(x)\big), \quad u \in \mathbf{K}[\mathcal{N}_I(w)] \tag{1.13}$$

We call $I_1, I_2, \ldots, I_k$ the disjoint intervals of $\mathcal{Z}(u)$ (which is obviously finite) and we define

$$w_0(x) = w(x); \qquad w_j(x) = \mathcal{N}_{I_j}\big(w_{j-1}(x)\big), \qquad j = 1, \ldots, k$$

By induction, we have:

$$u \in \mathbf{K}[w_j], \qquad j = 0, 1, \ldots, k$$

and

$$u'(x) = w_k(x)$$

since the zero of u outside $I_1, \ldots, I_k$ are isolated. So, our thesis follws by induction from the basic lemma:

1.9 Lemma. *Let $u \in \mathbf{K}[w]$ be as in the statement of theorem 1.8, and let $I = (\alpha, \beta) \in \mathcal{Z}(u)$ be a maximal zero interval of u; then*

$$\|\mathcal{N}_I(w)\|_{BV(a,b)} \leq \|w\|_{BV(a,b)} \tag{1.14}$$

Proof. Call $\tilde{w} = \mathcal{N}_I(w)$; since u is piecewise linear and non negative, I has the following nice property:

$$\alpha \neq a \ \Rightarrow u'_-(\alpha) < 0; \qquad \beta \neq b \ \Rightarrow u'_+(\beta) > 0 \tag{1.15}$$

which holds for $w, \tilde{w}$ too, by (1.11):

$$u'_-(\alpha) = w_-(\alpha) = \tilde{w}_-(\alpha); \qquad u'_+(\beta) = w_+(\beta) = \tilde{w}_+(\beta) \tag{1.16}$$

Except the trivial case $I \equiv (a,b)$, we distinguish three other cases:

I. $\alpha = a$ and $\beta < b$:
as $\tilde{w}_+(a) = 0$ and $\tilde{w}_-(b) = w_-(b)$, we have to prove that:

$$\text{ess-}V_a^b(\tilde{w}) \leq \text{ess-}V_a^b(w) + |w_+(a)|$$

By remark 1.4, we have:

$$\text{ess-}V_a^b(\tilde{w}) = \text{ess-}V_a^\beta(\tilde{w}) + \text{ess-}V_\beta^b(\tilde{w}) + [\tilde{w}](\beta) = \text{ess-}V_\beta^b(w) + \tilde{w}_+(\beta) = \text{ess-}V_\beta^b(w) + w_+(\beta)$$

since

$$\tilde{w}|_{(a,\beta)} \equiv 0, \qquad \tilde{w}|_{(\beta,b)} \equiv w$$

But

$$w_+(\beta) = w_+(\beta) - w_+(a) + w_+(a) \leq [w](\beta) + \text{ess-}V_a^\beta(w) + w_+(a)$$

so that, by 1.4, we conclude.

II. $a < \alpha$ and $\beta = b$:
we use the same argument.

III. $a < \alpha < \beta < b$:
In this case we have:

$$\begin{aligned}\text{ess-}V_a^b(\tilde{w}) =& \text{ess-}V_a^\alpha(\tilde{w}) + \text{ess-}V_\beta^b(\tilde{w}) + [\tilde{w}](\alpha) + [\tilde{w}](\beta) = \\ =& \text{ess-}V_\alpha^b(w) + \text{ess-}V_\beta^b(w) + |\tilde{w}_-(\alpha)| + |\tilde{w}_+(\beta)| = \\ =& \text{ess-}V_\alpha^b(w) + \text{ess-}V_\beta^b(w) + w_+(\beta) - w_-(\alpha)\end{aligned}$$

thanks to (1.15) and (1.16). As in previous calculations we obtain:

$$w_+(\beta) - w_-(\alpha) \leq [w](\beta) + \text{ess-}V_\alpha^\beta(w) + [w](\alpha)$$

Summing up we conclude, as $\tilde{w}_+(a) = w_+(a)$ and $\tilde{w}_-(b) = w_-(b)$. ∎

Before concluding the proof of theorem 2, let us note that the previous proof can be adapted to a more general situation.
Let $\mathcal{T}$ be an operator defined on $AC_{loc}(a,b)$ with the following properties (which are obviously satisfied by $u \mapsto u^+$):

$$\begin{cases} \exists\, p \in [1,\infty[\colon u_n \to u \text{ in } W^{1,p}_{(loc)}(a,b) \ \Rightarrow \mathcal{T}[u_n] \to \mathcal{T}[u] \text{ in } \mathcal{D}'(a,b) \\ \ \mathcal{T} \text{ maps piecewise linear functions into piecewise linear ones} \\ \mathcal{T}[u] \in \mathbf{K}[u'], \qquad \forall\, u \in \mathbf{P}^1(\mathcal{S}),\ \forall \mathcal{S} \end{cases} \tag{1.17}$$

We have:

1.10 Theorem. *Let $v \in AC_{loc}(a,b), v' \in BV(a,b)$ and $u = \mathcal{T}[v]$; then $u' \in BV(a,b)$ and:*

$$\|u'\|_{BV(a,b)} \leq \|v'\|_{BV(a,b)} \tag{1.18}$$

Proof. Let $v_n \in \mathbf{P}^1(\mathcal{S}_n)$ be a sequence of piecewise linear functions which approximate v as in theorem 1.5 and set $u_n = \mathcal{T}[v_n]$.
From (1.17) we deduce that:

$$\|u_n'\|_{BV(a,b)} \leq \|v_n'\|_{BV(a,b)} \qquad \lim_{n\to\infty} \|v_n'\|_{BV(a,b)} = \|v'\|_{BV(a,b)}$$

and by thorem 1.5:

$$v_n \to v \text{ in } W^{1,p}_{(loc)}(a,b)$$

with p given by (1.17); it follows that $u_n' \to u'$ in $D'(a,b)$ and by the lower semicontinuity of the $BV-$norm we obtain:

$$\|u'\|_{BV(a,b)} \leq \liminf_{n\to\infty} \|u_n'\|_{BV(a,b)} \leq \liminf_{n\to\infty} \|v_n'\|_{BV(a,b)} = \|v'\|_{BV(a,b)} \quad \blacksquare$$

2. - Proof of theorem 3.

We first recall the definition of interpolation spaces by the $K-$method of J. Peetre (see [7], [8])

2.1 Definition. *Let $E_0 \supset E_1$ be two Banach spaces (the inclusion being continuous) and for $v \in E_0$ let us define*

$$K(t,v) = \inf_{x \in E_1} \Big[\|x - v\|_{E_0} + t\|x\|_{E_1} \Big] \tag{2.1}$$

Then for $0 < \theta < 1$ and $1 \le p \le \infty$, $\big(E_0, E_1\big)_{\theta,p}$ denotes the Banach spaces of $v \in E_0$ such that

$$t^{-\theta} K(t,v) \in L^p_*(0,\infty) = L^p(0,\infty; dt/t) \ (^9)$$

with norm:

$$\|v\|_{(E_0,E_1)_{\theta,p}} = \|t^{-\theta}K(t,v)\|_{L^p_*(0,\infty)} \quad \square$$

Our purpose is to extend theorem 1.10 to the interpolation family obtained from the Banach spaces $BH(a,b)$ and $W^{1,\infty}(a,b)$.
We have:

2.2 Theorem. *Let $v \in AC_{loc}(a,b)$ with $v' \in Z^{\theta,p}(a,b)$; if $u = \mathcal{T}[v]$, with $\mathcal{T}$ satisfying* (1.17)*, we have $u' \in Z^{\theta,p}(a,b)$ with*

$$\|u'\|_{Z^{\theta,p}(a,b)} \le \|v'\|_{Z^{\theta,p}(a,b)} \tag{2.2}$$

We divide the *proof* in some steps; first we study the approximation by $\mathbf{P}^0$-functions:

2.3 Lemma. *Let $\mathcal{S}$ be a subdivision of (a,b); the associated linear map*

$$\Pi^0_{\mathcal{S}} : u \in BV(a,b) \mapsto \bar{u} \in \mathbf{P}^0(\mathcal{S})$$

is a contraction with respect to the $L^\infty(a,b)$-norm and it can be extended to a linear contraction on each $Z^{\theta,p}(a,b)$.

Proof. It is an easy consequence of (1.2) and (1.3). Let us denote by $X(a,b)$ the closure of $BV(a,b)$ in $L^\infty(a,b)$; by (1.2) $\Pi^0_{\mathcal{S}}$ can be extended by density to a linear contraction of $X(a,b)$. On the other hand (1.3) says that $\Pi^0_{\mathcal{S}}$ is a contraction also of $BV(a,b)$ and, by interpolation, of $\big(X(a,b), BV(a,b)\big)_{\theta,p}$ for any admissible choice of θ and p. But we know that (see [8])

$$\big(E_0, E_1\big)_{\theta,p} = \big(\overline{E_1}^{E_0}, E_1\big)_{\theta,p}, \qquad \text{with equal norms}$$

so that

$$Z^{\theta,p}(a,b) = \big(X(a,b), BV(a,b)\big)_{\theta,p} \quad \blacksquare$$

We study now the semicontinuity properties of the interpolated norms:

(^{9}) *that is the L^p-space on $(0,\infty)$ with respect to the Haar measure dt/t; when $p = \infty$ we are dealing with the usual one.*

2.4 Proposition. *Assume that E_0 is the dual of a separable space; then the intepolated norms $\|\cdot\|_{(E_0,E_1)_{\theta,p}}$ are sequentially l.s.c. with respect to the weak* topology of E_0.*

Proof. Let $u_n \in \big(E_0, E_1\big)_{\theta,p}$ with:

$$u_n \rightharpoonup^* u \text{ in } E_0$$

Since $K(t,\cdot)$ is an equivalent norm for E_0 (see [7]) we have, for all $t>0$:

$$K(t,u) \le \liminf_{n\to\infty} K(t,u_n)$$

and by Fatou's lemma ([10]):

$$\|u\|^p_{(E_0,E_1)_{\theta,p}} = \int_0^\infty K^p(t,u)\frac{dt}{t^{1-\theta p}} \le \liminf_{n\to\infty}\int_0^\infty K^p(t,u_n)\frac{dt}{t^{1-\theta p}} = \liminf_{n\to\infty}\|u_n\|^p_{((E_0,E_1)_{\theta,p}} \quad \blacksquare$$

2.5 Corollary. *Let $\mathcal{S}_n$ be a sequence of subdivisions satisfying* (1.9) *and set $\bar u_n = \Pi^0_{\mathcal{S}_n} u$, $\hat u_n = \Pi^1_{\mathcal{S}_n} u$; we have:*

$$\lim_{n\to\infty}\|\bar u_n\|_{Z^{\theta,p}(a,b)} = \|u\|_{Z^{\theta,p}(a,b)}$$

for any $u\in Z^{\theta,p}(a,b)$; if $u \in AC_{loc}(a,b)$ and $u' \in Z^{\theta,p}(a,b)$, then

$$\lim_{n\to\infty}\|\hat u_n'\|_{Z^{\theta,p}(a,b)} = \|u'\|_{Z^{\theta,p}(a,b)}.$$

Now we study the K-functional of a piecewise constant function:

2.6 Lemma. *Let $\mathcal{S}$ be a subdivision of (a,b) and $w\in \mathbf{P}^0(\mathcal{S})$; we have:*

$$K(t,w) = \inf_{x\in\mathbf{P}^0(\mathcal{S})}\Big[\|x-w\|_{L^\infty(a,b)} + t\|x\|_{BV(a,b)}\Big] \tag{2.3}$$

Proof. We shall show that, for any $x\in BV(a,b)$, $\bar x \in \mathbf{P}^0(\mathcal{S})$ realizes a better choice for the minimizing functional, i.e.:

$$\|\bar x - w\|_{L^\infty(a,b)} + t\|\bar x\|_{BV(a,b)} \le \|x-w\|_{L^\infty(a,b)} + t\|x\|_{BV(a,b)}$$

In fact, by theorem 1.5, we know that:

$$\|\bar x\|_{BV(a,b)} \le \|x\|_{BV(a,b)};$$

from the other hand, we have $\bar w \equiv w$, w being in $\mathbf{P}^0(\mathcal{S})$, so that:

$$\|\bar x - w\|_{L^\infty(a,b)} = \|\Pi^0_{\mathcal{S}}(x-w)\|_{L^\infty(a,b)} \le \|x-w\|_{L^\infty(a,b)} \quad \blacksquare$$

2.7 Corollary. *For any $w\in\mathbf{P}^0(\mathcal{S})$ there exists a family $w^t \in \mathbf{P}^0(\mathcal{S})$ such that:*

$$K(t,w) = \|w-w^t\|_{L^\infty(a,b)} + t\|w^t\|_{BV(a,b)} \tag{2.4}$$

Proof. Immediate, since $\mathbf{P}^0(\mathcal{S})$ is finite dimensional. $\blacksquare$

([10]) The case $p=\infty$ is immediate, too.

The second step is to consider the situation of lemma 1.9 and to bound $K(t,\tilde{w})$ by $K(t,w)$:

2.8 Lemma. *Let* $w \in \mathbf{P}^0(\mathcal{S})$ *and* $u' \in \mathbf{P}^1(\mathcal{S})$*, with* $u \in \mathbf{K}[w]$*, as in* (1.11)*. If* $I = (\alpha,\beta) \in \mathcal{Z}(u)$ *is a maximal zero interval of* u *and* $\tilde{w} = \mathcal{N}_I(w)$ *(see definition* 1.7*), then:*

$$K(t,\tilde{w}) \le K(t,w), \qquad \forall\, t > 0. \tag{2.5}$$

Proof. We choose a family $w^t \in \mathbf{P}^0(\mathcal{S})$ as in previous corollary and we want to exhibit a correponding $\tilde{w}^t$ such that

$$\|\tilde{w} - \tilde{w}^t\|_{L^\infty(a,b)} \le \|w - w^t\|_{L^\infty(a,b)}, \tag{2.6}$$

$$\|\tilde{w}^t\|_{BV(a,b)} \le \|w^t\|_{BV(a,b)}. \tag{2.7}$$

The cases $\alpha = a$ or $\beta = b$ are easy to check if we choose $\tilde{w}^t = \mathcal{N}_I(w^t)$; (2.6) is immediate since

$$\|\tilde{w} - \tilde{w}^t\|_{L^\infty(a,b)} = \|\mathcal{N}_I(w - w^t)\|_{L^\infty(a,b)} \le \|w - w^t\|_{L^\infty(a,b)}$$

and for the other one we have (in the case $\alpha = a$):

$$\begin{aligned}\|\tilde{w}^t\|_{BV(a,b)} =& |w^t_+(\beta)| + \text{ess-}V^b_a(w^t) + |w^t_-(b)| \le \\ \le& |w^t_+(a)| + |w^t_+(\beta) - w^t_+(a)| + \text{ess-}V^b_a(w^t) + |w^t_-(b)| \le \|w^t\|_{BV(a,b)}\end{aligned}.$$

The case $a < \alpha$, $\beta < b$ is more delicate. We denote by I_- the interval of $\mathcal{S}$ preceding I and by I_+ the following one; we know that

$$u'|_{I_-} = \tilde{w}|_{I_-} = w|_{I_-} = w_-(\alpha) < 0, \qquad u'|_{I_+} = \tilde{w}|_{I_+} = w|_{I_+} = w_+(\beta) > 0. \tag{2.8}$$

We distinguish two situations again, according to the sign of w^t on I_- and I_+.

I. $w^t|_{I_-} \cdot w^t|_{I_+} \le 0$
we set $\tilde{w}^t = \mathcal{N}_I(w^t)$ and we immediately have (2.6); on the other hand we have:

$$\begin{aligned}\text{ess-}V^b_a(\tilde{w}^t) =& \text{ess-}V^\alpha_a(\tilde{w}^t) + |\tilde{w}^t_-(\alpha)| + |\tilde{w}^t_+(\beta)| + \text{ess-}V^b_\beta(\tilde{w}^t) = \\ =& \text{ess-}V^\alpha_a(w^t) + |w^t_-(\alpha)| + |w^t_+(\beta)| + \text{ess-}V^b_\beta(w^t) = \\ =& \text{ess-}V^\alpha_a(w^t) + |w^t_-(\alpha) - w^t_+(\beta)| + \text{ess-}V^b_\beta(w^t) \le \text{ess-}V^b_a(w^t)\end{aligned}$$

since we assumed that

$$|w^t_-(\alpha)| + |w^t_+(\beta)| = |w^t_-(\alpha) - w^t_+(\beta)|.$$

Consequently (2.7) holds, too.

II. $w^t|_{I_-} \cdot w^t|_{I_+} > 0$

in order to fix ideas, suppose that they are both positive; then we set:

$$\tilde{w}^t(x) = \begin{cases} w^t_-(\alpha) & \text{if } x \in I; \\ w^t & \text{otherwise.} \end{cases}$$

$\tilde{w}^t - \tilde{w}$ differs from $w^t - w$ only on I, where we have

$$|\tilde{w}^t - \tilde{w}| = \tilde{w}^t = w^t_-(\alpha) \leq w^t|_{I_-} - w|_{I_-} \leq \|w_t - w\|_{L^\infty(a,b)}$$

thanks to (2.8). The variation of $\tilde{w}^t$ can be bounded by

$$\begin{aligned}\text{ess-}V^b_a(\tilde{w}^t) =& \text{ess-}V^\alpha_a(\tilde{w}^t) + [\tilde{w}^t](\beta) + \text{ess-}V^b_\beta(\tilde{w}^t) = \\ =& \text{ess-}V^\alpha_a(w^t) + |w^t_+(\beta) - w^t_-(\alpha)| + \text{ess-}V^b_\beta(w^t) \leq \text{ess-}V^b_a(w^t)\end{aligned}$$

obtaining (2.7) again.
If $w^t|_{I_-}$ and $w^t|_{I_+}$ are both negative, we set analogously:

$$\tilde{w}^t(x) = \begin{cases} w^t_+(\beta) & \text{if } x \in I; \\ w^t & \text{otherwise.} \end{cases} \quad \blacksquare$$

As we argued in theorem 1.8, we obtain

2.9 Corollary. *Let $u \in \mathbf{K}[w]$ as in theorem 1.8. Then*

$$\|u'\|_{Z^{\theta,p}(a,b)} \leq \|w\|_{Z^{\theta,p}(a,b)} \quad \blacksquare$$

By our usual approximation argument, we conclude the proof of theorem 2.2.

3. - Proof of theorems 4 and 5.

We start with a

3.1 Definition. *For* $0 < \theta < 1$, $Z^\theta(a,b)$ *will be the space* (11) $L^p(a,b) \cap Z^{\theta,p}(a,b)$, *with* $p = 1/\theta$ *and* $Z^{1+\theta}(a,b) = \{u \in W^{1,p}(a,b) : u' \in Z^\theta(a,b)\}$ *with:*

$$\|u\|_{Z^{1+\theta}(a,b)} = \|u\|_{W^{1,p}(a,b)} + \|u'\|_{Z^{\theta,p}(a,b)} \quad \square$$

The particular choice of θ, p of this definition is justified by the following

3.2 Theorem. $Z^\theta(a,b)$ *is of interpolation class* θ *between* $L^p(a,b)$ *and* $W^{1,p}(a,b)$, *that is:*

$$B^\theta_{p\,1}(a,b) = \big(L^p(a,b), W^{1,p}(a,b)\big)_{\theta,1} \subset Z^\theta(a,b) \subset \big(L^p(a,b), W^{1,p}(a,b)\big)_{\theta,\infty} = B^\theta_{p\,\infty}(a,b)$$

Proof. For the sake of simplicity we suppose $(a,b) = \mathbf{R}$. The right inclusion is showed in [16]; in order to prove the other one, it is sufficient to obtain the estimate

$$\|u\|_{Z^{\theta,p}(a,b)} \leq C\|u'\|^\theta_{L^p(a,b)}\,\|u\|^{1-\theta}_{L^p(a,b)}, \qquad p = 1/\theta$$

for some constant $C > 0$ and every nonnegative function $u \in W^{1,p}(\mathbf{R})$.
We use a consequence of the "power theorem" (see [7]) which is related to the original representation of $\big(E_0, E_1\big)_{\theta,p}$ as "espaces de moyennes" given by J.L. Lions and J. Peetre in [22]: for a couple $E_0 \supset E_1$ of Banach spaces let v be a given element of E_0 and $v^t \in E_1$, $t > 0$ be a strongly measurable family such that

$$\|t^{1-\theta} v^t\|_{L^1_*(E_1)} = \int_0^\infty t^{1-\theta}\|v^t\|_{E_1}\,\frac{dt}{t} < +\infty$$

and

$$\|t^{-\theta}(v - v^t)\|_{L^\infty_*(E_0)} = \operatorname*{ess-sup}_{t>0} t^{-\theta}\|v - v^t\|_{E_0} < +\infty.$$

Then $v \in \big(E_0, E_1\big)_{\theta,p}$ with $p = 1/\theta$ and there exists a constant $c > 0$ independent of v such that

$$\|v\|_{(E_0,E_1)_{\theta,p}} \leq c\|t^{1-\theta}v^t\|^\theta_{L^1_*(E_1)} \cdot \|t^{-\theta}(v - v^t)\|^{1-\theta}_{L^\infty_*(E_0)} \tag{3.1}$$

In our case we choose

$$u^t(x) = \big(u(x) - t^\theta\big)^+$$

so that

$$t^{-\theta}\|u - u^t\|_{L^\infty(\mathbf{R})} \leq 1, \qquad \forall t > 0$$

---

(11) *When* (a,b) *is bounded,* $Z^\theta = Z^{\theta,p}$, $p = 1/\theta$.

and $u^t \in BV(\mathbf{R})$ as $u^t \in W^{1,p}(\mathbf{R})$ and is 0 outside a set of finite measure. We have $u^t_+(-\infty) = u^t_-(+\infty) = 0$ and

$$\begin{aligned}\int_0^\infty \Big(t^{1-\theta}\text{ess-}V(u^t)\Big)\frac{dt}{t} &= \int_0^\infty t^{1-\theta}\Big(\int_{\mathbf{R}} |u'(x)|\,\chi_{[u>t^\theta]}(x)\,dx\Big)\frac{dt}{t} = \\ &= \int_{\mathbf{R}} |u'(x)|\Big(\int_0^{u^p(x)} t^{-\theta}\,dt\Big)\,dx \le \\ &\le \frac{1}{1-\theta}\int_{\mathbf{R}} |u'(x)|\,\big(u(x)\big)^{p(1-\theta)}\,dx \le \\ &\le \frac{1}{1-\theta}\|u'\|_{L^p(\mathbf{R})}\,\|u\|^{p(1-\theta)}_{L^p(\mathbf{R})}\end{aligned}$$

By applying (3.1) and using $p = 1/\theta$ we conclude. ∎

By a simple application of the reiteration theorem (see [22], [7]) we find

3.3 Corollary. *For any $s \in (0,1)$ and $q \in [1,+\infty]$ we have*

$$\big(L^p(a,b), Z^{1+\theta}(a,b)\big)_{s,q} = B^{(1+\theta)s}_{p\,q}(a,b), \qquad p = 1/\theta$$

with equivalent norms. ∎

By the results of the previous section we have

$$u \in Z^{1+\theta}(a,b) \;\Rightarrow\; u^+ \in Z^{1+\theta}(a,b), \qquad \|u^+\|_{Z^{1+\theta}(a,b)} \le \|u\|_{Z^{1+\theta}(a,b)}. \tag{3.2}$$

Since $u \mapsto u^+$ is a contraction of $L^p(\mathbf{R})$, by nonlinear interpolation (see [17]) we deduce

$$u \in \big(L^p(a,b), Z^{1+\theta}(a,b)\big)_{s,q} \;\Rightarrow\; u^+ \in \big(L^p(a,b), Z^{1+\theta}(a,b)\big)_{s,q}$$

with

$$\|u^+\|_{\left(L^p(a,b),Z^{1+\theta}(a,b)\right)_{s,q}} \le \|u\|_{\left(L^p(a,b),Z^{1+\theta}(a,b)\right)_{s,q}}$$

and we conclude the proof of theorem 5 in the 1-dimensional case.

Now we study the multidimensional case, by using a slicing argument. Let ν be a unitary vector of $\mathbf{R}^N$; we set:

$$\begin{aligned}&\pi_\nu = \big\{x \in \mathbf{R}^N : (x,\nu) = 0\big\} \\ &u_\nu(x;t) = u(x+t\nu), \qquad \forall\, x \in \pi_\nu,\ t \in \mathbf{R}.\end{aligned} \tag{3.3}$$

We denote by $\mathcal{H}^{N-1}$ the $(N-1)$-dimensional Hausdorff measure and by $\mathbf{e}_1, \ldots, \mathbf{e}_N$ the canonical basis of $\mathbf{R}^N$.

Fubini's theorem ensures that $u \in L^p(\mathbf{R}^N)$ if and only if

$$\begin{cases} u_{\mathbf{e}_j}(x;\cdot) \in L^p(\mathbf{R}) \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in \pi_{\mathbf{e}_j}, \qquad j = 1, \ldots, N \\ \displaystyle\int_{\pi_{\mathbf{e}_j}} \|u_{\mathbf{e}_j}(x;\cdot)\|^p_{L^p(\mathbf{R})}\,d\mathcal{H}^{N-1}(x) = \|u\|^p_{L^p(\mathbf{R}^N)} < +\infty. \end{cases}$$

An analogous characterization holds for $W^{s,p}(\mathbf{R}^N)$ (see [5], thm. 5.6.2):

$$u \in W^{s,p}(\mathbf{R}^N) \;\Leftrightarrow \begin{cases} u_{\mathbf{e}_j}(x;\cdot) \in W^{s,p}(\mathbf{R}) \text{ for } \mathcal{H}^{N-1}\text{–a.e. } x \in \pi_{\mathbf{e}_j}, \\ \displaystyle\int_{\pi_{\mathbf{e}_j}} \|u_j(x;\cdot)\|^p_{W^{s,p}(\mathbf{R})}\, d\mathcal{H}^{N-1}(x) < +\infty, \quad j = 1,\ldots,N \end{cases}$$

and we can choose as norm of $W^{s,p}(\mathbf{R}^N)$ the following:

$$\|u\|^p_{W^{s,p}(\mathbf{R}^N)} = \sum_{j=1}^{N} \int_{\pi_{\mathbf{e}_j}} \|u_{\mathbf{e}_j}(x;\cdot)\|^p_{W^{s,p}(\mathbf{R})}\, d\mathcal{H}^{N-1}(x)$$

With the aid of this powerful result, the proof of the corollary for $\mathbf{R}^N$ is immediately given, since for a fixed $u \in W^{s,p}(\mathbf{R}^N)$, $s < 1 + 1/p$, we have

$$\left(u^+\right)_{\mathbf{e}_j}(x;t) = \left(u_{\mathbf{e}_j}\right)^+(x;t)$$

and we already know the one dimensional result. The case of a general Besov space follows now easily by interpolation.
Finally, if Ω is a given Lipschitz subset of $\mathbf{R}^N$, we know that every $B^s_{p\,q}(\Omega)$ function admits an extension to $\mathbf{R}^N$ with uniformly bounded norm; so we can choose as norm in $B^s_{p\,q}(\Omega)$

$$\|u\|_{B^s_{p\,q}(\Omega)} = \inf\left\{\|v\|_{B^s_{p\,q}(\mathbf{R}^N)},\; v_{|\Omega} \equiv u\right\}$$

Since

$$v_{|\Omega} \equiv u \;\Rightarrow\; v^+{}_{|\Omega} \equiv u^+$$

we conclude. ∎

4. - Proof of theorem 1.

We begin to consider the case $\Omega = \mathbf{R}^N$; for $BH(\Omega)$ an analogous characterization of previous section doesn't hold, so we have to use a different argument.
With the vectors $\mathbf{e}_j$, $j = 1, \dots, N$, of the canonical basis of $\mathbf{R}^N$ we define

$$\mathbf{e}_{ii} = \mathbf{e}_i, \qquad \mathbf{e}_{ij} = \frac{1}{\sqrt{2}}(\mathbf{e}_i + \mathbf{e}_j),\ i \neq j.$$

For $C_0^\infty(\mathbf{R}^N)$ functions we have $(i \neq j)$

$$\frac{\partial^2 f}{\partial \mathbf{e}_i \partial \mathbf{e}_j} = \frac{\partial^2 f}{\partial \mathbf{e}_{ij}^2} - \frac{1}{2}\Big(\frac{\partial^2 f}{\partial \mathbf{e}_i^2} + \frac{\partial^2 f}{\partial \mathbf{e}_j^2}\Big),$$

so that

$$\|f\|_{W^{1,1}(\mathbf{R}^N)} + \sum_{i,j} \int_{\mathbf{R}^N} \Big|\frac{\partial^2 f}{\partial \mathbf{e}_{ij}^2}\Big|\, dx \tag{4.1}$$

is an equivalent norm for $W^{2,1}(\mathbf{R}^N)$, thanks the density of $C_0^\infty(\mathbf{R}^N)$-functions.
We want to extend (4.1) to $BH(\mathbf{R}^N)$ by using the essential variation along a fixed direction.
For a unitary vector ν of $\mathbf{R}^N$ and a function $u \in L^1(\mathbf{R}^N)$ such that (see (3.3))

$$t \mapsto u_\nu(x; t) \in BV(\mathbf{R}), \qquad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \pi_\nu,$$

we set

$$\text{ess-}V_\nu(u) = \int_{\pi_\nu} \text{ess-}V(u_\nu(x; \cdot))\, d\mathcal{H}^{N-1}(x)$$

and we know that

$$u \in BV(\mathbf{R}^N) \ \Leftrightarrow\ \text{ess-}V_\nu(u) < \infty \qquad \forall\, \nu \in \mathbf{R}^N,\ |\nu| = 1. \tag{4.2}$$

Moreover, if $u \in W^{1,1}(\mathbf{R}^N)$ then

$$\text{ess-}V_\nu(u) = \int_{\mathbf{R}^N} \Big|\frac{\partial u}{\partial \nu}\Big|\, dx \tag{4.3}$$

It is not surprising that a good substitute of (4.1) can be obtained by taking the variation of any partial derivative along the same direction:

4.1 Proposition. *A function $u \in W^{1,1}(\mathbf{R}^N)$ belongs to $BH(\mathbf{R}^N)$ iff*

$$[u]_{BH(\mathbf{R}^N)} = \sum_{i,j} \text{ess-}V_{\mathbf{e}_{ij}}\Big(\frac{\partial u}{\partial \mathbf{e}_{ij}}\Big) < +\infty \tag{4.4}$$

Moreover $\|u\|_{W^{1,1}(\mathbf{R}^N)} + [u]_{BH(\mathbf{R}^N)}$ is an equivalent norm for $BH(\mathbf{R}^N)$.
Proof. (4.4) is well defined and continuous on $BH(\mathbf{R}^N)$ thanks to (4.2).

Let u be a $W^{1,1}(\mathbf{R}^N)$ function such that $[u]_{BH(\mathbf{R}^N)}$ is finite; we shall prove that $u \in BH(\mathbf{R}^N)$ by showing that there exists a constant $c > 0$ independent of u and ε such that

$$\|u^\varepsilon\|_{BH(\mathbf{R}^N)} \leq c\Big\{\|u\|_{W^{1,1}(\mathbf{R}^N)} + [u]_{BH(\mathbf{R}^N)}\Big\}, \tag{4.5}$$

where u^ε is the usual approximation of u by a convolution with a symmetric mollifier η ([12]):

$$u^\varepsilon = u * \eta^\varepsilon; \qquad \eta^\varepsilon(x) = \varepsilon^{-N}\eta(x/\varepsilon).$$

It is well known that

$$\|u^\varepsilon\|_{W^{1,1}(\mathbf{R}^N)} \leq \|u\|_{W^{1,1}(\mathbf{R}^N)}$$

and we want to check the same bound for the seminorm $[\cdot]_{BH(\mathbf{R}^N)}$.
Denoting by $v_{ij} = \frac{\partial u}{\partial \mathbf{e}_{ij}}$ we have

$$\int_{\mathbf{R}^n} |v_{ij}(x + h\mathbf{e}_{ij}) - v_{ij}(x)|\, dx \leq \text{ess-}V_{\mathbf{e}_{ij}}(v_{ij}) \cdot h;$$

By the usual properties of the mollifiers

$$v_{ij}^\varepsilon = \frac{\partial u^\varepsilon}{\partial \mathbf{e}_{ij}} = \frac{\partial u}{\partial \mathbf{e}_{ij}} * \eta^\varepsilon$$

so that

$$\int_{\mathbf{R}^N} \Big|v_{ij}^\varepsilon(x + h\mathbf{e}_{ij}) - v_{ij}^\varepsilon(x)\Big|\, dx \leq \int_{\mathbf{R}^N} \Big|[v_{ij}(x + h\mathbf{e}_{ij}) - v_{ij}(x)] * \eta^\varepsilon\Big|\, dx \leq$$
$$\int_{\mathbf{R}^n} |v_{ij}(x + h\mathbf{e}_{ij}) - v_{ij}(x)|\, dx \leq \text{ess-}V_{\mathbf{e}_{ij}}(v_{ij}) \cdot h.$$

Since v_{ij}^ε belong to C^∞ we deduce that

$$\left\|\frac{\partial v_{ij}^\varepsilon}{\partial \mathbf{e}_{ij}}\right\|_{L^1(\mathbf{R}^N)} \leq \text{ess-}V_{\mathbf{e}_{ij}}(v_{\mathbf{e}_{ij}})$$

and $[u^\varepsilon]_{BH(\mathbf{R}^N)} \leq [u]_{BH(\mathbf{R}^N)}$. Since we already pointed out that for C^∞-functions we have

$$\|u^\varepsilon\|_{BH(\mathbf{R}^N)} = \|u^\varepsilon\|_{W^{2,1}(\mathbf{R}^N)} \leq c\Big[\|u^\varepsilon\|_{W^{1,1}(\mathbf{R}^N)} + [u^\varepsilon]_{BH(\mathbf{R}^N)}\Big]$$

with $c > 0$ independent of u^ε, we conclude our proof. ∎

([12]) For example see [15]; (4.5) is equivalent to our thesis by the lower semicontinuity of the BH-norm.

4.2 Corollary. *If $u \in BH(\mathbf{R}^N)$ then also $u^+ \in BH(\mathbf{R}^N)$ with*

$$[u^+]_{BH(\mathbf{R}^N)} \le [u]_{BH(\mathbf{R}^N)}. \tag{4.6}$$

Proof. For any unitary vector ν we have

$$(u^+)_\nu = (u_\nu)^+; \qquad \left(\frac{\partial u}{\partial \nu}\right)_\nu (x;t) = \frac{\partial}{\partial t} u_\nu(x;t)$$

so that, for H^{N-1} a.e. $x \in \pi_\nu$ we have

$$\text{ess-}V_\nu\Big((\frac{\partial u^+}{\partial \nu})_\nu \Big) \le \text{ess-}V_\nu\Big((\frac{\partial u}{\partial \nu})_\nu \Big).$$

Choosing $\nu = \mathbf{e}_{ij}$ and summing up we obtain (4.6). ∎

In order to prove the same result for any Lipschitz open set Ω, we have to refine the extension theorem of [11] for BH functions, which requires a boundary of class C^2. First we show that $BH(\Omega)$ is an exact interpolation space with respect to $L^1(\Omega)$ and $W^{2,1}(\Omega)$:

4.3 Theorem. *Let Ω be an open subset of $\mathbf{R}^N$; $BH(\Omega)$ is the Gagliardo completion of $W^{2,1}(\Omega)$ relative to $L^1(\Omega)$; in particular it is an exact interpolation space between $W^{2,1}(\Omega)$ and $L^1(\Omega)$.*

Proof. Let us recall (see [13]) that for a given couple $E_0 \supset E_1$ of Banach spaces, the Gagliardo completion of E_1 relative to E_0 is defined by

$$E_1^{E_0,c} = \Big\{ v \in E_0 :\ \exists v_n \in E_1 \text{ with } \|v_n\|_{E_1} \text{ bounded and } \lim_{n\to\infty} \|v_n - v\|_{E_0} = 0 \Big\} \tag{4.7}$$

with

$$\|v\|_{E_1^{E_0,c}} = \inf \big\{ \sup_n \|v_n\|_{E_1}, \qquad v_n \text{ satisfyng (4.7)} \big\}$$

and it is an interpolation space between E_0 and E_1.
In our case let us denote by X the Gagliardo completion of $W^{2,1}(\Omega)$ in $L^1(\Omega)$; by the lower semiconinuity of the norm of $BH(\Omega)$ with respect to the $L^1(\Omega)$ convergence, we have easily

$$X \subset BH(\Omega); \qquad \|u\|_{BH(\Omega)} \le \|u\|_X, \qquad \forall\, u \in X. \tag{4.8}$$

On the other hand, given a function $u \in BH(\Omega)$ we know (see [11], [23] and the remark of [12]) that there exists a sequence u_n of $C^\infty(\Omega)$ functions such that:

$$\lim_{n\to\infty} \|u - u_n\|_{L^1(\Omega)} = 0; \qquad \lim_{n\to\infty} \|u_n\|_{BH(\Omega)} = \|u\|_{BH(\Omega)}$$

Since for $C^\infty(\Omega)$ functions the norms of $W^{2,1}(\Omega)$ and $BH(\Omega)$ coincide, we obtain the reverse inclusion and inequality of (4.8). ∎

4.4 Corollary. *If Ω is a strongly Lipschitz open subset of $\mathbf{R}^N$, there exist a constant $M = M(\Omega) > 0$ and a linear continuous operator $\mathcal{P} : BH(\Omega) \mapsto BH(\mathbf{R}^N)$ such that*

$$\big(\mathcal{P}u\big)_{|\Omega} \equiv u, \qquad \|\mathcal{P}u\|_{BH(\mathbf{R}^N)} \leq M\|u\|_{BH(\Omega)} \qquad \forall\, u \in BH(\Omega),$$

$$\mathcal{P}\big(W^{2,1}(\Omega)\big) \subset W^{2,1}(\mathbf{R}^N).$$

Proof. Let $\mathcal{P}$ be the "universal" linear extension operator of [24] (chap.VI, thm. 5); in particular we know that:

$$\mathcal{P} : L^1(\Omega) \mapsto L^1(\mathbf{R}^N) \ \text{ and } \ W^{2,1}(\Omega) \mapsto W^{2,1}(\mathbf{R}^N) \tag{4.9}$$

with bounded norm. Thanks to the previous result we conclude by interpolation. ▪

As we argued in the previous section,

$$\|u\|_{BH(\Omega)} = \|u\|_{W^{1,1}(\Omega)} + \inf\Big\{[v]_{BH(\mathbf{R}^N)}, \qquad v_{|\Omega} \equiv u\Big\}$$

is an equivalent norm on $BH(\Omega)$ for which theorem 1 holds.

REFERENCES

1. **G. Stampacchia**, Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus, *Ann. Inst. Fourier (Grenoble)*, 15, 189-258 (1965).

2. **M. Marcus, V.J. Mizel**, Absolute continuity on tracks and mappings of Sobolev spaces, *Arch. Rational Mech. Anal.*, 45, 294-320 (1972).

3. **L.Ambrosio, G.Dal Maso**, A general chain rule for distributional derivatives, *Proc. A.M.S.*, 108, 691-702 (1990).

4. **R. Adams**, *Sobolev Spaces*, Academic Press, New York (1975).

5. **S.M. Nikolśkii**, *Approximation of Functions of Several Variables and Imbedding Theorems*, Springer-Verlag, Berlin (1975).

6. **H. Triebel**, *Theory of Function Spaces II*, Birkäuser, Boston (1992).

7. **J. Bergh, J. Löfström**, *Interpolation Spaces*, Springer-Verlag, Berlin (1976).

8. **P.L. Butzer, H. Berens**, *Semi-Groups of Operators and Approximation*, Springer-Verlag, Berlin (1967).

9. **G. Bourdaud, Y. Meyer**, Fonctions qui opèrent sur les espaces de Sobolev, *Journal of functional analysis*, 97, 351-360 (1991).

10. **J.L. Lions, E. Magenes**, *Non homogeneous boundary value problems and applications I*, Springer Verlag, Berlin (1972).

11. **F. Demengel**, Fonctions a Hessian Borné, *Ann. Inst. Fourier*, 34, 155-190 (1984).

12. **M. Carriero, A. Leaci, F. Tomarelli**, Special bounded hessian and elastic-plastic plate, *Rend. Accad. Naz. Sci. dei XL, Mem. Mat.*, XVI, 223-258 (1992).

13. **E.Gagliardo**, A unified structure in various families of function spaces. Compactness and closure theorems, *Proc. Internat. Symp. Linear Spaces*, Jerusalem, 237-241 (1960).

14. **H. Federer**, *Geometric Measure Theory*, Springer, Berlin (1969).

15. **E. Giusti**, *Minimal Surfaces and Functions of Bounded Variation*, Birkhäuser, Boston (1984).

16. **L. Tartar**, Remarks on some interpolation spaces, *in "Boundary value problems for PDEs and applications"; C. Baiocchi - J.L. Lions editors*, Masson, 229-252 (1993).

17. **L. Tartar**, Interpolation non linéaire et régularité, *J. Funct. Analsis*, 9, 469-489 (1972).

18. **G. Savaré**, Weak solutions and maximal regularity for abstract evolution inequalities, *Adv. Math. Sci. Appl.* (to appear).

19. **A.Visintin**, Stefan problem with phase relaxation, *IMA J. Appl. Math.*, 34, 225-245 (1985).

20. **G. Savaré**, Approximation and regularity of evolution variational inequalities, *Rnd. Acc. Naz. Sci. dei XL, Memorie di Matematica*, XVII, 83-111 (1993).

21. **W.P. Ziemer**, *Weakly Differentiable Functions*, Springer-Berlag, New York (1988).

22. **J.L. Lions, J. Peetre**, Sur une classe d'espaces d'interpolation, *Inst. des Hautes Études Scientifiques Publ. Math.*, 19, 5-68 (1964).

23. **R. Temam**, *Problèmes mathématiques en plasticité*, Dunod, Paris (1983).

24. **E.M. Stein**, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, N.J. (1970).