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Self-improvement of the Bakry-Émery condition and Wasserstein contraction of the heat flow in $\mathrm{RCD}(K,\infty)$ metric measure spaces

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Abstract

We prove that the linear "heat" flow in a $\mathrm{RCD}(K,\infty)$ metric measure space $(X,\mathsf{d},\mathfrak{m})$ satisfies a contraction property with respect to every L^p-Kantorovich-Rubinstein-Wasserstein distance, $p\in[1,\infty]$. In particular, we obtain a precise estimate for the optimal W_∞-coupling between two fundamental solutions in terms of the distance of the initial points.

The result is a consequence of the equivalence between the $\mathrm{RCD}(K,\infty)$ lower Ricci bound and the corresponding Bakry-Émery condition for the canonical Cheeger-Dirichlet form in $(X,\mathsf{d},\mathfrak{m})$. The crucial tool is the extension to the non-smooth metric measure setting of the Bakry's argument, that allows to improve the commutation estimates between the Markov semigroup and the *Carré du Champ* Γ associated to the Dirichlet form.

This extension is based on a new a priori estimate and a capacitary argument for regular and tight Dirichlet forms that are of independent interest.

Contents

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1 Introduction

The investigation of the deep connections between lower Ricci curvature bounds (also in the broader sense of the Bakry-Émery curvature-dimension condition $\mathrm{BE}(K,N)$ [10]) and optimal transport in Riemannian geometry started with the pioneering papers [31, 20]. Since then a big effort have been made to develop a synthetic theory of curvature-dimension bounds for a general metric-measure space $(X,\mathsf{d},\mathfrak{m})$ in absence of a smooth differential structure.

Lott-Sturm-Villani $\mathrm{CD}(K,\infty)$ spaces

In the approach developed by Sturm [34, 35] and Lott-Villani [27] (see also [36]), optimal transport provides a very useful and far-reaching point of view, in particular to obtain a stable notion with respect to measured Gromov-Hausdorff (or Gromov-Prokhorov) convergence that includes all possibile Gromov-Hausdorff limits of Riemannian manifolds under uniform dimension and lower curvature bounds [16, 17, 18].

According to Lott-Sturm-Villani, a complete and separable metric space (X,d) endowed with a Borel probability measure $\mathfrak{m}\in\mathscr{P}(X)$ (here we assume $\mathfrak{m}(X)=1$ for simplicity, see § 4.1 for a more general condition) satisfies the $\mathrm{CD}(K,\infty)$ curvature bound if the relative entropy functional $\mathrm{Ent}_{\mathfrak{m}}:\mathscr{P}(X)\to[0,\infty]$ induced by $\mathfrak{m}$ is displacement K-convex in the Wasserstein space $\mathscr{P}_2(X)$. The latter is the space of Borel probability measures with finite quadratic moment endowed with the L^2 Kantorovich-Rubinstein-Wasserstein distance W_2, see § 4.1.

A question that naturally arises in this metric setting concerns the relationships between the optimal transport and the Bakry-Émery's approaches. Since the latter makes sense only in the framework of a Dirichlet form $\mathcal{E}$ generating a linear Markov semigroup $(\mathsf{P}_t)_{t\ge0}$ in $L^2(X,\mathfrak{m})$, one has first to understand how to construct a diffusion semigroup and an energy functional in a $\mathrm{CD}(K,\infty)$ space.

Since the $\mathrm{CD}(K,\infty)$ condition involves the geodesic K-convexity of the entropy functional in the Wasserstein space, it is quite natural to consider the metric gradient flow $(\mathsf{H}_t)_{t\ge0}$ [2] of $\mathrm{Ent}_{\mathfrak{m}}$ in $(\mathscr{P}_2(X),W_2)$ (see [22, 3]). As showed initially by [25] in $\mathbb{R}^n$ and then extended to many different situations by [21, 36, 6, 30, 23], it turns out [4] that $(\mathsf{H}_t)_{t\ge0}$ essentially coincides with the L^2-gradient flow $(\mathsf{P}_t)_{t\ge0}$ of the convex and lower semicontinuous *Cheeger energy*

$$\mathsf{Ch}(f):=\inf\Big\{\liminf_{n\to\infty}\frac12\int_X|\mathrm{D}f_n|^2\,\mathrm{d}\mathfrak{m}:f_n\in\mathrm{Lip}_b(X),\quad f_n\to f\text{ in }L^2(X,\mathfrak{m})\Big\},\tag{1.1}$$

where the metric slope $|\mathrm{D}f|$ of a Lipschitz function $f:X\to\mathbb{R}$ is defined by $|\mathrm{D}f|(x):=\limsup_{y\to x}|f(y)-f(x)|/\mathsf{d}(x,y)$.

$(\mathsf{P}_t)_{t\ge0}$ thus defines a (possibly nonlinear) semigroup of contractions in $L^2(X,\mathfrak{m})$ and, in fact, in any $L^p(X,\mathfrak{m})$. Since it is also positivity preserving, it is a Markov semigroup if and only if it is linear, or, equivalently, if Ch is a quadratic form in $L^2(X,\mathfrak{m})$, thus satisfying the parallelogram rule

$$\mathsf{Ch}(f+g)+\mathsf{Ch}(f-g)=2\mathsf{Ch}(f)+2\mathsf{Ch}(g)\quad\text{for every }f,g\in D(\mathsf{Ch}).\tag{Q-Ch}$$

$\mathrm{RCD}(K,\infty)$-metric measure spaces and the Bakry-Émery $\mathrm{BE}(K,\infty)$ condition

Spaces satisfying Lott-Sturm-Villani $\mathrm{CD}(K,\infty)$ conditions and (Q-Ch) have been introduced in [4] as metric measure spaces with *Riemannian Ricci curvature* bounded from below, RCD(K,∞) spaces in short. This more restrictive class of spaces can also be characterised in terms of the *Evolution variational inequality* formulation of $(\mathsf{H}_t)_{t\geq 0}$, see (4.15), that provides the W_2 contraction property

$$W_2(\mathsf{H}_t\mu, \mathsf{H}_t\nu) \leq \mathrm{e}^{-Kt} W_2(\mu,\nu) \quad \text{for every } \mu,\nu \in \mathscr{P}_2(X). \tag{1.2}$$

The RCD(K,∞) condition is still stable with respect to measured Gromov-Hausdorff convergence [4, 24] and thus includes all possibile measured Gromov-Hausdorff limits of Riemannian manifolds under uniform lower curvature bounds.

In RCD(K,∞) spaces $\mathcal{E} := 2\mathsf{Ch}$ is a strongly local Dirichlet form admitting a *Carré du champ* $\Gamma(f)$ that coincides with the squared minimal weak upper gradient $|\mathrm{D}f|_w^2$ associated to (1.1), see (4.8) and (4.9). In terms of the generator $\mathsf{L} : D(\mathsf{L}) \subset L^2(X,\mathfrak{m}) \to L^2(X,\mathfrak{m})$ of $(\mathsf{P}_t)_{t\geq 0}$ this provides useful the Leibnitz and composition rules

$$2\Gamma\big(f,g\big) = \mathsf{L}(fg) - f\mathsf{L}g - g\mathsf{L}f, \quad \mathsf{L}(\Phi(f)) = \Phi'(f)\mathsf{L}f + \Phi''(f)\Gamma\big(f\big)$$

at least for a suitable class of functions in $D(\mathsf{L})$, see § 2.2.

Distance and energy are intimately correlated by the explicit formula (1.1) (that involves the metric slope of Lipschitz functions) and by the somehow dual property that expresses d as the canonical distance [12] associated to $\mathcal{E}$:

$$\text{every bounded function } f \in D(\mathcal{E}) \text{ with } \Gamma\big(f\big) \leq 1 \text{ has a continuous representative } \tilde{f}, \tag{1.3a}$$

$$\mathsf{d}(x,y) := \sup\Big\{\psi(x) - \psi(y) : \psi \in D(\mathcal{E}) \cap \mathrm{C}_b(X), \quad \Gamma\big(f\big) \leq 1\Big\}. \tag{1.3b}$$

Having a Carré du champ at disposal, it is then possibile to consider a weak version (see (3.1)) of the *Carré du champ itéré*

$$2\Gamma_2(f,g) := \mathsf{L}\Gamma\big(f,g\big) - \Gamma\big(f,\mathsf{L}g\big) - \Gamma\big(g,\mathsf{L}f\big), \tag{1.4}$$

and to prove a weak BE(K,∞) condition of the type

$$\Gamma_2(f) \geq K\Gamma\big(f\big), \quad \text{where} \quad \Gamma\big(f\big) := \Gamma\big(f,f\big), \quad \Gamma_2(f) := \Gamma_2(f,f), \tag{1.5}$$

in a suitable weaker integral form (Definition 3.1), but still sufficient to get the crucial pointwise gradient bound

$$\Gamma\big(\mathsf{P}_t f\big) \leq |\mathrm{D}\mathsf{P}_t f|^2 \leq \mathrm{e}^{-2Kt}\mathsf{P}_t\Gamma\big(f\big) \quad \text{for every } f \in \mathrm{Lip}_b(X). \tag{1.6}$$

It turns out that the implication RCD$(K,\infty) \Rightarrow$ BE(K,∞) can also be inverted and the two points of view are eventually equivalent. This has been shown by [5]: starting from a Polish topological space (X,τ) endowed with a local Dirichlet form $\mathcal{E}$ with the associated *Carré du champ* Γ and the intrinsic distance d satisfying (1.3a,b) and inducing the topology τ, if BE(K,∞) holds, then $(X,\mathsf{d},\mathfrak{m})$ is a RCD(K,∞) metric measure space.

Applications of BE(K,∞): refined gradient estimates and Wasserstein contraction

The identification between RCD(K,∞) and BE(K,∞) lead to the possibility to apply a large numbers of the results and techniques originally proved for smoother spaces satisfying the Bakry-Émery condition. Performing this project is not always simple, since proofs often use extra regularity or algebraic assumptions (see e.g. [8, Page 24]) that prevent a direct application to the non smooth context.

Among the most useful properties, Bakry [7, 8] showed that the Γ_2 condition expressed through the pointwise bounds (1.6) is potentially self-improving, since it leads to the stronger commutation inequality

$$\Big(\Gamma\big(\mathsf{P}_t f\big)\Big)^\alpha \le \mathrm{e}^{-2\alpha K t}\mathsf{P}_t\Big(\Gamma\big(f\big)^\alpha\Big) \quad \text{for every } \alpha\in[1/2,2]. \tag{1.7}$$

(1.7) is in fact a consequence of the crucial estimate

$$\Gamma\big(\Gamma\big(f\big)\big) \le 4\Big(\Gamma_2(f)-K\Gamma\big(f\big)\Big)\Gamma\big(f\big), \tag{1.8}$$

a formula whose meaning can be better understood recalling that in a Riemannian manifold $(\mathbb{M}^d,\mathsf{g})$ endowed with the canonical Riemannian volume $\mathfrak{m}=\mathrm{Vol}_\mathsf{g}$, we have

$$\Gamma\big(f\big)=|\mathrm{D}f|^2_\mathsf{g},\quad \Gamma_2(f)-K\Gamma\big(f\big)\ge|\mathrm{D}^2f|^2_\mathsf{g},\quad \Gamma\big(\Gamma\big(f\big)\big)=\Big|\mathrm{D}\big|\mathrm{D}f\big|^2_\mathsf{g}\Big|^2_\mathsf{g}\le 4|\mathrm{D}^2f|^2_\mathsf{g}\,|\mathrm{D}f|^2_\mathsf{g}. \tag{1.9}$$

(1.8) can be derived by applying the Γ_2 inequality (1.5) to polynomials of two or more functions $f_1,f_2,\cdots$. However the Bakry's clever strategy of [7, 8] requires a multivariate differential formula for the Γ_2 operator, that typically involves further smoothness assumptions.

The aim of the present paper is twofold: from one side, we want to show how to obtain the estimate (1.8) in a very general setting, starting from the weak integral formulation of $\mathrm{BE}(K,\infty)$.

This result is independent of the theory of metric measure spaces, and it is obtained for general Dirichlet forms in Polish spaces satisfying standard regularity and tightness assumptions. It relies on a simple estimate showing that $\Gamma\big(f\big)\in D(\mathcal{E})$ if f belongs to the space $\mathbb{D}_\infty$, whose elements f are characterised by $f\in D(\mathsf{L})$ with $\Gamma\big(f\big)\in L^\infty(X,\mathfrak{m})$, $\mathsf{L}f\in D(\mathcal{E})$. Tightness and regularity of $\mathcal{E}$ are then sufficient to give a measure-theoretic sense to $\mathsf{L}\Gamma\big(f\big)$, to $\Gamma_2(f)$ and to multivariate calculus for $\Phi\circ f$ thanks to capacitary arguments. The main point here is that $\Gamma_2(f)$ may be singular with respect to $\mathfrak{m}$, but its singular part is nonnegative; moreover, the multiplication of the measure $\Gamma_2(f)$ with functions in $D(\mathcal{E})$ still makes sense since the latter admit a quasi continuous representative and polar sets are negligible w.r.t. the measure $\Gamma_2(f)$.

The derivation of (1.7) from (2.17) follows then the ideas of [11, 9, 37], suitably adapted to the weak integral version of (1.5).

Finally, the application of (1.5) to contraction estimates for the heat flow $(\mathsf{H}_t)_{t\ge0}$ in Wasserstein spaces follows the Kuwada's duality approach [26], thanks to (1.3a), (1.3b) and the refined argument developed in [5]. We can then prove the optimal contraction estimate for every L^p-Wasserstein distance

$$W_p(\mathsf{H}_t\mu,\mathsf{H}_t\nu)\le\mathrm{e}^{-Kt}W_p(\mu,\nu)\quad\text{for every }\mu,\nu\in\mathscr{P}(X),\quad p\in[1,\infty], \tag{1.10}$$

and, when $K\ge0$, for any transport cost depending on the distance d in an increasing way, see (4.1) (see [29] for a similar estimate in $\mathbb{R}^n$).

Plan of the paper

We will recall in Section 2 a few basic results concerning Dirichlet forms, *Carré du champ*, multivariate differential calculus and capacities. A simple but important estimate is proved in Lemma 2.6.

After a brief review of the weak formulation of the $\mathrm{BE}(K,\infty)$ condition, Section 3 contains the main properties for the measure theoretic interpretation of the *Carré du champ itéré* Γ_2 and the corresponding multivariate calculus rules. The main estimates are then proved in Theorem 3.4 and its Corollary 3.5.

Applications to $\mathrm{RCD}(K,\infty)$ spaces and to Wasserstein contraction of the heat flow are eventually discussed in the last section 4.

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2 Preliminaries

2.1 Notation, Dirichlet forms and Carré du Champ

Let (X,τ) be a Polish topological space. We will denote by $\mathscr{B}(X)$ the collection of its Borel sets and by $\mathscr{M}(X)$ the space of Borel signed measures with finite total variation, i.e. σ-additive maps $\mu:\mathscr{B}(X)\to\mathbb{R}$. $\mathscr{M}(X)$ is endowed with the weak convergence with respect to the duality with the continuous and bounded functions of $\mathrm{C}_b(X)$. $\mathscr{M}_+(X)$ and $\mathscr{P}(X)$ will denote the convex subsets of nonnegative finite measures and of probabilities measures in X, respectively.

We will consider a σ-finite Borel measure $\mathfrak{m}\in\mathscr{M}_+(X)$ with full support $\mathrm{supp}(\mathfrak{m})=X$ and a strongly local, symmetric Dirichlet form $\mathcal{E}:L^2(X,\mathfrak{m})\to[0,\infty]$ with proper domain $\mathbb{V}:=\big\{f\in L^2(X,\mathfrak{m}):\mathcal{E}(f)<\infty\big\}$ dense in $L^2(X,\mathfrak{m})$. $\mathcal{E}$ generates a mass preserving Markov semigroup $(\mathsf{P}_t)_{t\ge0}$ in $L^2(X,\mathfrak{m})$ with generator L and domain $D(\mathsf{L})$ dense in $\mathbb{V}$.

We will still use the symbol $\mathcal{E}$ to denote the associated bilinear form in $\mathbb{V}$. $\mathbb{V}$ is an Hilbert space with the graph norm induced by $\mathcal{E}$:

$$\|f\|_{\mathbb{V}}^2:=\|f\|^2_{L^2(X,\mathfrak{m})}+\mathcal{E}(f,f).$$

We will assume that $\mathcal{E}$ admits a *Carré du Champ* $\Gamma\big(\cdot,\cdot\big)$: it is a symmetric, bilinear and continuous map $\Gamma:\mathbb{V}\times\mathbb{V}\to L^1(X,\mathfrak{m})$, which is uniquely characterised in the algebra $\mathbb{V}\cap L^\infty(X,\mathfrak{m})$ by

$$2\int_X\Gamma\big(f,g\big)\varphi\,\mathrm{d}\mathfrak{m}=\mathcal{E}(f,g\varphi)+\mathcal{E}(g,f\varphi)-\mathcal{E}(fg,\varphi)\quad\text{for every }\varphi\in\mathbb{V}\cap L^\infty(X,\mathfrak{m}).$$

In the following we set

$$\mathbb{V}_\infty:=\mathbb{V}\cap L^\infty(X,\mathfrak{m}),\quad \mathbb{G}_\infty:=\{f\in\mathbb{V}_\infty:\Gamma\big(f\big)\in L^\infty(X,\mathfrak{m})\}.\tag{2.1}$$

2.2 Leibnitz rule and multivariate calculus

We recall now a few useful calculus rules. We will consider smooth functions $\Phi, \Psi : \mathbb{R}^n \to \mathbb{R}$ with $\Phi(0) = \Psi(0) = 0$, we set $\Phi_i := \partial_i \Phi$, $\Phi_{ij} := \partial_{ij}\Phi$, $i, j = 1, \cdots, n$, and similarly for Ψ. We will denote by $\boldsymbol{f} := (f_i)_{i=1}^n$ a n-uple of real measurable functions defined on X and by $\Phi(\boldsymbol{f}) = \Phi(f_1, \cdots, f_n)$ the corresponding composed function.

For a proof of the following properties, we refer to [14, Ch. I, §6]: notice that we do not assume any bounds on the derivatives of Φ and Ψ since they will be composed with (essentially) bounded functions.

⟨L.1⟩ $\mathbb{V}_\infty$ and $\mathbb{G}_\infty$ are closed with respect to pointwise multiplication (see [14, Ch. I, Cor. 3.3.2] and the next Leibnitz rule (2.2)).

⟨L.2⟩ If $f \in \mathbb{V}$ and $g \in \mathbb{G}_\infty$ then $fg \in \mathbb{V}$.

⟨L.3⟩ If $f, g \in \mathbb{V}_\infty$ (or $f \in \mathbb{V}$ and $g \in \mathbb{G}_\infty$) and $h \in \mathbb{V}$ then [14, Ch. I, Cor. 6.1.3]

$$\Gamma\big(fg, h\big) = f\Gamma\big(g, h\big) + g\Gamma\big(f, h\big), \quad \Gamma\big(fg\big) = f^2\Gamma\big(g\big) + g^2\Gamma\big(f\big) + 2fg\Gamma\big(f, g\big). \tag{2.2}$$

⟨L.4⟩ If $(f_i)_{i=1}^n \in (\mathbb{V}_\infty)^n$ the functions $\Phi(\boldsymbol{f})$, $\Psi(\boldsymbol{f})$ belong to $\mathbb{V}_\infty$ and [14, Ch. I, Cor. 6.1.3]

$$\Gamma\big(\Phi(\boldsymbol{f}), \Psi(\boldsymbol{f})\big) = \sum_{i,j} \Phi_i(\boldsymbol{f})\Psi_j(\boldsymbol{f})\Gamma\big(f_i, f_j\big). \tag{2.3}$$

⟨L.5⟩ If $f_i \in D(\mathsf{L}) \cap \mathbb{G}_\infty$ then $\Phi(\boldsymbol{f}) \in D(\mathsf{L}) \cap \mathbb{G}_\infty$ with [14, Ch. I, Cor. 6.1.4]

$$\mathsf{L}(\Phi(\boldsymbol{f})) = \sum_i \Phi_i(\boldsymbol{f})\mathsf{L}f_i + \sum_{i,j} \Phi_{ij}(\boldsymbol{f})\Gamma\big(f_i, f_j\big). \tag{2.4}$$

⟨L.6⟩ $D(\mathsf{L}) \cap \mathbb{G}_\infty$ is closed with respect to pointwise multiplication: if $f_i \in D(\mathsf{L}) \cap \mathbb{G}_\infty$ then

$$\mathsf{L}(f_1 f_2) = f_1 \mathsf{L}f_2 + f_2 \mathsf{L}f_1 + 2\Gamma\big(f_1, f_2\big). \tag{2.5}$$

2.3 Quasi-regular Dirichlet forms, capacity and measures with finite energy.

We follow here the approach developed by Ma and Röckner, see [28, III.2, III.3, IV.3] (covering the general case of a possibly non-symmetric Dirichlet form) and [19, 1.3]. If F is a closed subset of X we set

$$\mathbb{V}_F := \Big\{f \in \mathbb{V} : f(x) = 0 \text{ for } \mathfrak{m}\text{-a.e. } x \in X \setminus F\Big\}.$$

Definition 2.1 (Nests, polar sets, and quasi continuity [28, III.2.1], [19, 1.2.12]) *An $\mathcal{E}$-nest is an increasing sequence $(F_k)_{k\in\mathbb{N}}$ of closed subsets of X such that $\cup_{k\in\mathbb{N}}\mathbb{V}_{F_k}$ is dense in $\mathbb{V}$.*
A set $N \subset X$ is $\mathcal{E}$-polar if there is an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ such that $N \subset X \setminus \cup_{k\in\mathbb{N}}F_k$. If a property of points in X holds in a complement of an $\mathcal{E}$-polar set we say that it holds $\mathcal{E}$-quasi-everywhere ($\mathcal{E}$-q.e.).
A function $f : X \to \mathbb{R}$ is said to be $\mathcal{E}$-quasi-continuous if there exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ such that every restriction $f_{|_{F_k}}$ is continuous on F_k.

$\mathcal{E}$-nests and $\mathcal{E}$-polar sets can also be characterized in terms of capacities; we recall here a version that we will be useful later on. The capacity Cap (it corresponds to $\mathrm{Cap}_{h,1}$ with $h \equiv 1$ in the notation of [19]) of an open set $A \subset X$ is defined by

$$\mathrm{Cap}(A) := \inf \Big\{ \|u\|_{\mathbb{V}}^2 : \ u \geq 1 \ \mathfrak{m}\text{-a.e. in } A \Big\},$$

and it can be extended to arbitrary sets $B \subset X$ by

$$\mathrm{Cap}(B) := \inf \Big\{ \mathrm{Cap}(A) : B \subset A, A \text{ open} \Big\}.$$

Notice also that $\mathrm{Cap}(A) \geq \mathfrak{m}(A)$.

Theorem 2.2 ([19, 1.2.14]) *Let us suppose that there exists a nondecreasing sequence $(X_n)_{n\in\mathbb{N}}$ of open subsets of X such that*

$$\mathrm{Cap}(X_n) < \infty, \quad \overline{X}_n \subset X_{n+1}, \quad (\overline{X}_n)_{n\in\mathbb{N}} \textit{ is an } \mathcal{E}\textit{-nest.} \tag{2.6}$$

(*i*) *A nondecreasing sequence of closed subsets $F_k \subset X$ is an $\mathcal{E}$-nest if and only if* $\lim_{k\to\infty} \mathrm{Cap}(X_n \setminus F_k) = 0$ *for every $n \in \mathbb{N}$.*

(*ii*) *$N \subset X$ is an $\mathcal{E}$-polar set if and only if* $\mathrm{Cap}(N) = 0$.

When $\mathfrak{m}(X) < \infty$ then $\mathrm{Cap}(X) = \mathfrak{m}(X) < \infty$, so that (2.6) is always satisfied by choosing $X_n \equiv X$. In this case a function $f : X \to \mathbb{R}$ is $\mathcal{E}$-quasi-continuous if for every $\varepsilon > 0$ there exists a closed set $C_\varepsilon \subset X$ such that $f_{|C_\varepsilon}$ is continuous and $\mathrm{Cap}(X \setminus C_\varepsilon) < \varepsilon$.

Definition 2.3 (Quasi-regular Dirichlet forms) *The Dirichlet form $\mathcal{E}$ is quasi-regular if*

$\langle$QR.1$\rangle$ *There exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ consisting of compact sets.*

$\langle$QR.2$\rangle$ *There exists a dense subset of $\mathbb{V}$ whose elements have $\mathcal{E}$-quasi-continuous representatives.*

$\langle$QR.3$\rangle$ *There exists an $\mathcal{E}$-polar set $N \subset X$ and a countable collection of $\mathcal{E}$-quasi-continuous functions $(f_k)_{k\in\mathbb{N}} \subset \mathbb{V}$ separating the points of $X \setminus N$.*

If $\mathcal{E}$ is quasi-regular, then [19, Remark 1.3.9(ii)]

$$\text{every function } f \in \mathbb{V} \text{ admits an } \mathcal{E}\text{-quasi-continuous representative } \tilde{f}, \tag{2.7}$$

$\tilde{f}$ is unique up to q.e. equality. Notice that

$$\text{if } f \in \mathbb{V}_\infty \text{ with } |f| \leq M \ \mathfrak{m} \text{ a.e. in } X, \text{ then } |\tilde{f}| \leq M \text{ q.e.} \tag{2.8}$$

When $\mathfrak{m}(X) < \infty$ so that $\mathrm{Cap}(X) < \infty$, Theorem 2.2(*i*) shows that $\langle$QR.1$\rangle$ is equivalent to the tightness condition

$$\text{there exist compact sets } K_n \subset X,\ n \geq 1, \text{ such that } \lim_{n\to\infty} \mathrm{Cap}(X \setminus K_n) = 0. \tag{2.9}$$

In the general case of a σ-finite measure $\mathfrak{m}$ satisfying (2.6), we have the following simple criterium of quasi-regularity, where (with a slight abuse of notation) we will denote by $\mathbb{V} \cap \mathrm{C}(X)$ the subspace of $\mathbb{V}$ consisting of those functions which admits a continuous representative.

Lemma 2.4 (A criterium for quasi-regularity) *Let us assume that there exists a nondecreasing sequence $(X_n)_{n\in\mathbb{N}}$ of open subsets of X satisfying (2.6) and let us suppose that*

$\langle$QR.1$'\rangle$ *For every $n, m \in \mathbb{N}$ there exists a compact set $K_{n,m} \subset X$ such that $\mathrm{Cap}(X_n \setminus K_{n,m}) \le 1/m$.*

$\langle$QR.2$'\rangle$ *$\mathbb{V} \cap \mathrm{C}(X)$ is dense in $\mathbb{V}$ and it separates the points of X.*

Then $\mathcal{E}$ is quasi-regular.

Proof. Let us set $F_k := \bigcup_{j=1}^k K_{j,j}$. $(F_k)_{k\in\mathbb{N}}$ is a nondecreasing sequence of compact sets and whenever $k \ge n$ we get

$$\mathrm{Cap}(X_n \setminus F_k) \le \mathrm{Cap}(X_k \setminus F_k) \le \mathrm{Cap}(X_k \setminus K_{k,k}) \le 1/k,$$

so that $\lim_{k\to\infty} \mathrm{Cap}(X_n \setminus F_k) = 0$. Applying Theorem 2.2$(i)$ we obtain that $(F_k)_{k\in\mathbb{N}}$ is an $\mathcal{E}$-nest, so that $\langle$QR.1$\rangle$ holds.

$\langle$QR.2$\rangle$ is a trivial consequence of $\langle$QR.2$'\rangle$; $\langle$QR.3$\rangle$ still follows by $\langle$QR.2$'\rangle$ thanks to [33, Ch. II, Prop. 4]. □

We introduce the convex set $\mathbb{V}_+ := \{\phi \in \mathbb{V} : \phi \ge 0 \ \mathfrak{m}\text{-a.e. in } X\}$; $\mathbb{V}'_+$ denotes the set of linear functionals $\ell \in \mathbb{V}'$ such that $\langle \ell, \phi\rangle \ge 0$ for all $\phi \in \mathbb{V}_+$; we also set $\mathbb{V}'_\pm := \mathbb{V}'_+ - \mathbb{V}'_+$.

By Lax-Milgram Lemma, for every $\ell \in \mathbb{V}'_+$ there exists a unique $u_\ell \in \mathbb{V}$ representing ℓ in the sense that

$$\langle \ell, \varphi\rangle = \int_X u_\ell \varphi \,\mathrm{d}\mathfrak{m} + \mathcal{E}(u_\ell, \varphi) \quad \text{for every } \varphi \in \mathbb{V}. \tag{2.10}$$

u_ℓ is 1-excessive according to [19, Def. 1.2.1, Lemma 1.2.4] (in particular u_ℓ is non negative). The proof of the next result can be found as a consequence of the so-called "transfer method" of [28, Ch. VI, Prop. 2.1] (see also [14, Ch. I, § 9.2] in the case of a finite measure $\mathfrak{m}(X) < \infty$), applied to the representation of ℓ through the 1-excessive function u_ℓ of (2.10).

Proposition 2.5 *Let us assume that $\mathcal{E}$ is quasi-regular. Then for every $\ell \in \mathbb{V}'_+$ there exists a (unique) σ-finite and nonnegative Borel measure μ in X*
such that every $\mathcal{E}$-polar set is μ-negligible and

$$\forall f \in \mathbb{V} \quad \textit{the } \mathcal{E}\textit{-q.c. representative } \tilde f \in L^1(X,\mu), \quad \langle \ell, f\rangle = \int_X \tilde f \,\mathrm{d}\mu. \tag{2.11}$$

If moreover

$$\langle \ell, \varphi\rangle \le M \quad \textit{for every } \varphi \in \mathbb{V}_+, \quad \varphi \le 1 \ \mathfrak{m}\textit{-a.e. in } X, \tag{2.12}$$

then μ is a finite measure and $\mu(X) \le M$.

We will identify ℓ with μ. Notice that if $\mu \in \mathbb{V}'_+$ and $0 \le \nu \le c\mu$, then also $\nu \in \mathbb{V}'_+$ since

$$\Big|\int_X \tilde\varphi \,\mathrm{d}\nu\Big| \le \int_X |\tilde\varphi| \,\mathrm{d}\nu \le c\int_X |\tilde\varphi|\,\mathrm{d}\mu \le c\|\mu\|_{\mathbb{V}'}\,\|\varphi\|_{\mathbb{V}}.$$

The next Lemma provides a simple but important application of the previous Proposition to the case of a function u with measure-valued $\mathsf{L}u$. We first recall a well known approximation procedure (see e.g. [32, Proof of Thm. 2.7]), that will turn to be useful in the sequel. For $f \in L^2(X, \mathfrak{m})$ let us set

$$\mathfrak{P}_\varepsilon f := \frac{1}{\varepsilon}\int_0^\infty \mathsf{P}_r f\,\kappa(r/\varepsilon)\,\mathrm{d}r = \int_0^\infty \mathsf{P}_{\varepsilon s} f\,\kappa(s)\,\mathrm{d}s, \quad \varepsilon > 0, \quad \text{where}$$
$$\kappa \in \mathrm{C}_c^\infty(0,\infty) \text{ is a nonnegative kernel with } \int_0^\infty \kappa(r)\,\mathrm{d}r = 1. \tag{2.13}$$

$\mathfrak{P}_\varepsilon$ is positivity preserving and it is not difficult to check that for $\varepsilon > 0$ $\mathfrak{P}_\varepsilon f \in D(\mathsf{L})$ and for every $f \in L^p(X, \mathfrak{m})$, $p \in [1, \infty]$, we have

$$\mathsf{L}f = -\frac{1}{\varepsilon^2}\int_0^\infty \mathsf{P}_r f\,\kappa'(r/\varepsilon)\,\mathrm{d}r \in L^p(X, \mathfrak{m}). \tag{2.14}$$

Lemma 2.6 *Let us assume that the strongly local Dirichlet form $\mathcal{E}$ is quasi-regular, according to Definition 2.3. Let $u \in L^1 \cap L^\infty(X, \mathfrak{m})$ be nonnegative and let $g \in L^1 \cap L^2(X, \mathfrak{m})$ such that*

$$\int_X u\mathsf{L}\varphi\,\mathrm{d}\mathfrak{m} \geq -\int_X g\varphi\,\mathrm{d}\mathfrak{m} \quad \textit{for any nonnegative } \varphi \in D(\mathsf{L}) \cap L^\infty(X, \mathfrak{m}) \textit{ with } \mathsf{L}\varphi \in L^\infty(X, \mathfrak{m}). \tag{2.15}$$

Then

$$u \in \mathbb{V}, \quad \mathcal{E}(u) \leq \int_X u\,g\,\mathrm{d}\mathfrak{m}, \quad \int_X g\,\mathrm{d}\mathfrak{m} \geq 0, \tag{2.16}$$

and there exists a unique finite Borel measure $\mu := \mu_+ - g\,\mathfrak{m}$ with $\mu_+ \geq 0$, $\mu_+(X) \leq \int_X g\,\mathrm{d}\mathfrak{m}$ such that every $\mathcal{E}$-polar set is $|\mu|$-negligible, the q.c. representative of any function in $\mathbb{V}$ belongs to $L^1(X, |\mu|)$, and

$$-\mathcal{E}(u, \varphi) = -\int_X \Gamma\big(u, \varphi\big)\,\mathrm{d}\mathfrak{m} = \int_X \tilde{\varphi}\,\mathrm{d}\mu \quad \textit{for every } \varphi \in \mathbb{V}. \tag{2.17}$$

Proof. Let $u_\varepsilon := \mathfrak{P}_\varepsilon u$, $\varepsilon \geq 0$, and notice that by the regularisation properties of $(\mathfrak{P}_\varepsilon)_{\varepsilon>0}$ $u_\varepsilon \in D(\mathsf{L})$ with $\mathsf{L}u_\varepsilon \in L^1 \cap L^\infty(X, \mathfrak{m})$. It follows that for every $\varphi \in L^2 \cap L^\infty(X, \mathfrak{m})$ nonnegative

$$\int_X \mathsf{L}u_\varepsilon \varphi\,\mathrm{d}\mathfrak{m} = \int_X u\,\mathsf{L}\mathfrak{P}_\varepsilon\varphi\,\mathrm{d}\mathfrak{m} \geq -\int_X g\mathfrak{P}_\varepsilon\varphi\,\mathrm{d}\mathfrak{m} \geq -\int_X g_+\,\mathfrak{P}_\varepsilon\varphi\,\mathrm{d}\mathfrak{m}, \tag{2.18}$$

which in particular yields $\mathsf{L}u_\varepsilon + \mathfrak{P}_\varepsilon g \geq 0$. Choosing $\varphi := u_\varepsilon$ in (2.18) and inverting the sign of the inequality we obtain

$$\mathcal{E}(u_\varepsilon) = -\int_X \mathsf{L}u_\varepsilon\,u_\varepsilon\,\mathrm{d}\mathfrak{m} \leq \int_X u_\varepsilon\,\mathfrak{P}_\varepsilon g\,\mathrm{d}\mathfrak{m}$$

We can then pass to the limit as $\varepsilon \downarrow 0$ obtaining (2.16).

Moreover, taking nonnegative functions $\phi, \psi \in L^2 \cap L^\infty(X, \mathfrak{m})$ with $0 \leq \varphi(x) \leq 1$ and $\psi(x) > 0$ for $\mathfrak{m}$-a.e. $x \in X$ (such a function exists since $\mathfrak{m}$ is σ-finite) and setting $\varphi_n(x) := 1 \wedge (\varphi(x) + n\psi(x))$, (2.18) applied to the differences $\varphi_{n+1} - \varphi_n \geq 0$ (notice that $\varphi \equiv \varphi_0$), yields that for every $n \geq 0$

$$0 \leq \int_X (\mathsf{L}u_\varepsilon + \mathfrak{P}_\varepsilon g)\varphi\,\mathrm{d}\mathfrak{m} \leq \int_X (\mathsf{L}u_\varepsilon + \mathfrak{P}_\varepsilon g)\varphi_n\,\mathrm{d}\mathfrak{m} \leq \int_X (\mathsf{L}u_\varepsilon + \mathfrak{P}_\varepsilon g)\varphi_{n+1}\,\mathrm{d}\mathfrak{m}.$$

Passing to the limit as $n \to \infty$, since $\varphi_n \uparrow 1$ $\mathfrak{m}$-a.e. we obtain

$$0 \le \int_X (\mathsf{L} u_\varepsilon + \mathfrak{P}_\varepsilon g)\varphi \,\mathrm{d}\mathfrak{m} \le \int_X (\mathsf{L} u_\varepsilon + \mathfrak{P}_\varepsilon g)\,\mathrm{d}\mathfrak{m} = \int_X \mathfrak{P}_\varepsilon g \,\mathrm{d}\mathfrak{m} = \int_X g \,\mathrm{d}\mathfrak{m} \tag{2.19}$$

since $(\mathsf{P}_t)_{t\ge0}$ is mass preserving and thus $\int_X \mathsf{L} u_\varepsilon \,\mathrm{d}\mathfrak{m} = 0$. Let us now denote by ℓ the linear functional in $\mathbb{V}'$

$$\langle \ell, \varphi\rangle := -\mathcal{E}(u,\varphi) + \int_X g\,\varphi\,\mathrm{d}\mathfrak{m}$$

Choosing a nonnegative $\varphi \in \mathbb{V}_\infty$ in (2.18) and passing to the limit $\varepsilon \downarrow 0$ we easily find that $\ell \in \mathbb{V}'_+$; if moreover $\varphi \le 1$ then (2.19) yields

$$\langle \ell, \varphi\rangle = \lim_{\varepsilon\downarrow0}\Big(-\mathcal{E}(u_\varepsilon,\varphi) + \int_X \mathfrak{P}_\varepsilon g\,\varphi\,\mathrm{d}\mathfrak{m}\Big) \le \int_X g\,\mathrm{d}\mathfrak{m}.$$

Applying the previous Proposition 2.5 we conclude. □

We denote by $\mathbb{M}_\infty$ the space of $u \in \mathbb{V}_\infty$ such that there exist $\mu = \mu_+ - \mu_-$ with $\mu_\pm \in \mathbb{V}'_+$ such that

$$-\mathcal{E}(u,\varphi) = \int_X \tilde\varphi\,\mathrm{d}\mu \quad \text{for every } \varphi \in \mathbb{V}, \qquad \text{and we will write } \mathsf{L}^\star u = \mu. \tag{2.20}$$

For functions u with measure-valued $\mathsf{L}^\star u$ we can extend the calculus rule (2.5):

Corollary 2.7 *Under the same assumptions of Proposition 2.5, for every $u \in \mathbb{M}_\infty$ and $f \in D(\mathsf{L}) \cap \mathbb{G}_\infty$ we have $fu \in \mathbb{M}_\infty$ with*

$$\mathsf{L}^\star(f\,u) = \tilde f \mathsf{L}^\star u + u\mathsf{L} f\,\mathfrak{m} + 2\Gamma\big(u,f\big)\mathfrak{m}. \tag{2.21}$$

Proof. By (2.8) $\tilde f$ belongs to $L^\infty(X,|\mu|)$ where $\mu = \mathsf{L}^\star u$ and coincide with f up to a $\mathfrak{m}$-negligible set; we have for every $\zeta \in \mathbb{V}_\infty$

$$\begin{aligned} -\mathcal{E}(fu,\zeta) &\overset{(2.2)}{=} -\int \Big(f\Gamma\big(u,\zeta\big) + u\Gamma\big(f,\zeta\big)\Big)\,\mathrm{d}\mathfrak{m} \overset{(2.2)}{=} -\int \Big(\Gamma\big(u,f\zeta\big) + \Gamma\big(f,u\zeta\big) - 2\zeta\,\Gamma\big(f,u\big)\Big)\,\mathrm{d}\mathfrak{m} \\ &\overset{(2.20)}{=} \int_X \tilde f\tilde\zeta\,\mathrm{d}\big(\mathsf{L}^\star u\big) + \int_X \big(u\mathsf{L} f + 2\langle\Gamma\big(f,u\big)\big)\zeta\,\mathrm{d}\mathfrak{m}. \end{aligned}$$

By a standard approximation argument by truncation we extend the previous identity to arbitrary $\zeta \in \mathbb{V}$ (notice that $\tilde f$ is essentially bounded and $\tilde\zeta \in L^1(X,|\mu|)$). □

3 The Bakry-Émery condition and the measure-valued operator Γ_2

3.1 The Bakry-Émery condition

Let us assume that the Dirichlet form $\mathcal{E}$ admits a *Carré du champ* Γ and let us introduce the multilinear form $\mathbf{\Gamma}_2$

$$\mathbf{\Gamma}_2[f,g;\varphi] := \frac12 \int_X \Big(\Gamma\big(f,g\big)\,\mathsf{L}\varphi - \big(\Gamma\big(f,\mathsf{L} g\big) + \Gamma\big(g,\mathsf{L} f\big)\big)\varphi\Big)\,\mathrm{d}\mathfrak{m} \quad (f,g,\varphi) \in D(\mathbf{\Gamma}_2), \tag{3.1}$$

where $D(\mathbf{\Gamma}_2) := D_{\mathbb{V}}(\mathsf{L}) \times D_{\mathbb{V}}(\mathsf{L}) \times D_{L^\infty}(\mathsf{L})$, and

$$D_{\mathbb{V}}(\mathsf{L}) = \big\{f \in D(\mathsf{L}) : \mathsf{L}f \in \mathbb{V}\big\}, \quad D_{L^\infty}(\mathsf{L}) := \big\{\varphi \in D(\mathsf{L}) \cap L^\infty(X,\mathfrak{m}) : \mathsf{L}\varphi \in L^\infty(X,\mathfrak{m})\big\}.$$

When $f = g$ we also set

$$\mathbf{\Gamma}_2[f;\varphi] := \mathbf{\Gamma}_2[f,f;\varphi] = \int_X \Big(\frac{1}{2}\Gamma\big(f\big)\,\mathsf{L}\varphi - \Gamma\big(f,\mathsf{L}f\big)\varphi\Big)\,\mathrm{d}\mathfrak{m},$$

so that

$$\mathbf{\Gamma}_2[f,g;\varphi] = \frac{1}{4}\mathbf{\Gamma}_2[f+g;\varphi] - \frac{1}{4}\mathbf{\Gamma}_2[f-g;\varphi].$$

$\mathbf{\Gamma}_2$ provides a weak version (inspired by [9, 11]) of the Bakry-Émery condition [10, 8].

Definition 3.1 (Bakry-Émery condition) *We say that the strongly local Dirichlet form $\mathcal{E}$ satisfies the* $\mathrm{BE}(K,\infty)$ *condition,* $K \in \mathbb{R}$*, if it admits a Carré du Champ* Γ *and*

$$\mathbf{\Gamma}_2[f;\varphi] \geq K \int_X \Gamma\big(f\big)\,\varphi\,\mathrm{d}\mathfrak{m} \quad \textit{for every } (f,\varphi) \in D(\mathbf{\Gamma}_2),\ \varphi \geq 0. \tag{BE(K,∞)}$$

(BE(K,∞)) is in fact equivalent [5, Corollary 2.3] to the properties

$$\Gamma\big(\mathsf{P}_t f\big) \leq \mathrm{e}^{-2Kt}\,\mathsf{P}_t\Gamma\big(f\big) \quad \mathfrak{m}\text{-a.e. in } X, \quad \text{for every } t \geq 0,\ f \in \mathbb{V}, \tag{3.2}$$

and

$$2\mathrm{I}_{2K}(t)\Gamma\big(\mathsf{P}_t f\big) \leq \mathsf{P}_t f^2 - \big(\mathsf{P}_t f\big)^2 \quad \mathfrak{m}\text{-a.e. in } X, \quad \text{for every } t > 0,\ f \in L^2(X,\mathfrak{m}), \tag{3.3}$$

where $\mathrm{I}_{2K}(t) = \int_0^t \mathrm{e}^{2Kt}\,\mathrm{d}t$.

3.2 An estimate for $\Gamma\big(f\big)$ and multivariate calculus for Γ_2

Let us introduce the space

$$\mathbb{D}_\infty := \big\{f \in D(\mathsf{L}) \cap \mathbb{G}_\infty : \mathsf{L}f \in \mathbb{V}\big\}. \tag{3.4}$$

The following Lemma provides a further crucial regularity property for $\Gamma\big(f\big)$ when $f \in \mathbb{D}_\infty$ and shows how to define a measure-valued $\Gamma_2^\star(f)$ operator.

Lemma 3.2 *Let $\mathcal{E}$ be a strongly local and quasi-regular Dirichlet form. If* $\mathrm{BE}(K,\infty)$ *holds then for every $f \in \mathbb{D}_\infty$ we have $\Gamma\big(f\big) \in \mathbb{M}_\infty$ with*

$$\mathcal{E}(\Gamma\big(f\big)) \leq -\int_X \Big(2K\Gamma\big(f\big)^2 + \Gamma\big(f\big)\Gamma\big(f,\mathsf{L}f\big)\Big)\,\mathrm{d}\mathfrak{m} \tag{3.5}$$

and

$$\frac{1}{2}\mathsf{L}^\star\Gamma\big(f\big) - \Gamma\big(f,\mathsf{L}f\big)\mathfrak{m} \geq K\Gamma\big(f\big)\mathfrak{m}. \tag{3.6}$$

Moreover, $\mathbb{D}_\infty$ is an algebra (closed w.r.t. pointwise multiplication) and if $\boldsymbol{f} = (f_i)_{i=1}^n \in (\mathbb{D}_\infty)^n$ then $\Phi(\boldsymbol{f}) \in \mathbb{D}_\infty$ for every smooth function $\Phi : \mathbb{R}^n \to \mathbb{R}$ with $\Phi(0) = 0$.

Proof. Let us first notice that for every $f \in \mathbb{G}_\infty$ we have $\Gamma\big(f\big) \in L^1(X,\mathfrak{m}) \cap L^\infty(X,\mathfrak{m}) \subset L^p(X,\mathfrak{m})$ for every $p \in [1,\infty]$.

If $f \in \mathbb{D}_\infty$ then $\mathrm{BE}(K,\infty)$ and Lemma 2.6 with $-g := \Gamma\big(f, \mathsf{L}f\big) + K\Gamma\big(f\big)$ and $u := \Gamma\big(f\big)$ yield $\Gamma\big(f\big) \in \mathbb{V}$. (3.5) then follows from (2.16).

Since every function $\varphi \in \mathbb{V}_+$ can be (strongly) approximated by nonnegative functions in $D_{L^\infty}(\mathsf{L})$ by means of the regularization operators (2.13), (3.6) is a direct consequence of $\mathrm{BE}(K,\infty)$, (2.17), (2.20),

We already observed in $\langle$L.6$\rangle$, §2.2, that if $f_1, f_2 \in \mathbb{D}_\infty$ than $f_1 f_2 \in D(\mathsf{L}) \cap \mathbb{G}_\infty$; (2.5) and $\langle$L.2$\rangle$ also show that $\mathsf{L}(f_1 f_2) \in \mathbb{V}$. A similar argument, based on $\langle$L.5$\rangle$, shows that $\Phi(\boldsymbol{f}) \in \mathbb{D}_\infty$ whenever $\boldsymbol{f} \in (\mathbb{D}_\infty)^n$. $\square$

For every $f \in \mathbb{D}_\infty$ we denote by $\Gamma_2^\star(f)$ the finite Borel measure

$$\Gamma_2^\star(f) := \frac{1}{2}\mathsf{L}^\star\,\Gamma\big(f\big) - \Gamma\big(f, \mathsf{L}f\big)\mathfrak{m}. \tag{3.7}$$

By Lemma 2.6, $\Gamma_2^\star(f)$ has finite total variation, since

$$\Gamma_2^\star(f) = K\Gamma\big(f\big)\mathfrak{m} + \mu_+, \quad \text{with} \quad \mu_+ \geq 0, \quad \mu_+(X) \leq -\int_X \Big(\Gamma\big(f, \mathsf{L}f\big) + K\Gamma\big(f\big)\Big)\,\mathrm{d}\mathfrak{m}. \tag{3.8}$$

The measure $\Gamma_2^\star(u)$ vanishes on sets of 0 capacity. We denote by $\gamma_2(u) \in L^1(X,\mathfrak{m})$ its density with respect to $\mathfrak{m}$:

$$\Gamma_2^\star(f) = \gamma_2(f)\mathfrak{m} + \Gamma_2^\perp(f), \quad \Gamma_2^\perp(f) \perp \mathfrak{m}, \quad \gamma_2(f) \geq K\Gamma\big(f\big) \quad \mathfrak{m}\text{-a.e. in } X, \quad \Gamma_2^\perp(f) \geq 0. \tag{3.9}$$

The main point is that $\Gamma_2^\star(\cdot)$ can have a singular part $\Gamma_2^\perp(\cdot)$ w.r.t. $\mathfrak{m}$, but this is nonnegative and it does not affect many crucial inequalities.

According to (3.1) we also set for $f, g \in \mathbb{D}_\infty$

$$\Gamma_2^\star(f,g) := \frac{1}{4}\Gamma_2^\star(f+g) - \frac{1}{4}\Gamma_2^\star(f-g) = \frac{1}{2}\Big(\mathsf{L}^\star\,\Gamma\big(f,g\big) - \Gamma\big(f,\mathsf{L}g\big)\mathfrak{m} - \Gamma\big(g,\mathsf{L}f\big)\mathfrak{m}\Big), \tag{3.10}$$

and similarly

$$\gamma_2(f,g) := \frac{1}{4}\gamma_2(f+g) - \frac{1}{4}\gamma_2(f-g), \quad \Gamma_2^\star(f,g) = \gamma_2(f,g)\mathfrak{m} + \Gamma_2^\perp(f,g). \tag{3.11}$$

The next lemma extends to the present nonsmooth setting the multivariate calculus for Γ_2 of [7, 8].

Lemma 3.3 (The fundamental identity) *Under the same assumptions of the previous Lemma 3.2, let $\boldsymbol{f} = (f^i)_{i=1}^n \in \mathbb{D}_\infty^n$ and let $\Phi \in \mathrm{C}^3(\mathbb{R}^n)$ with $\Phi(0) = 0$. Then $\Phi(\boldsymbol{f}) \in \mathbb{D}_\infty$ and*

$$\begin{aligned}\Gamma_2^\star(\Phi(\boldsymbol{f})) = &\sum_{i,j} \Phi_i(\tilde{\boldsymbol{f}})\,\Phi_j(\tilde{\boldsymbol{f}})\,\Gamma_2^\star(f^i,f^j) \\ &+ \Big(2\sum_{i,j,k}\Phi_i(\boldsymbol{f})\Phi_{jk}(\boldsymbol{f})\mathrm{H}[f^i](f^j,f^k) + \sum_{i,j,k,h}\Phi_{ik}(\boldsymbol{f})\Phi_{jh}(\boldsymbol{f})\Gamma\big(f^i,f^j\big)\Gamma\big(f^k,f^h\big)\Big)\mathfrak{m},\end{aligned} \tag{3.12}$$

where for $f, g, h \in \mathbb{D}_\infty$

$$\mathrm{H}[f](g,h) = \frac{1}{2}\Big(\Gamma\big(g,\Gamma(f,h)\big) + \Gamma\big(h,\Gamma(f,g)\big) - \Gamma\big(f,\Gamma(g,h)\big)\Big). \tag{3.13}$$

Similarly

$$\begin{aligned}\gamma_2(\Phi(\boldsymbol{f})) &= \sum_{i,j} \Phi_i(\boldsymbol{f})\,\Phi_j(\boldsymbol{f})\,\gamma_2(f^i,f^j) \\ &\quad + 2\sum_{i,j,k} \Phi_i(\boldsymbol{f})\Phi_{jk}(\boldsymbol{f})\mathrm{H}[f^i](f^j,f^k) + \sum_{i,j,k,h} \Phi_{ik}(\boldsymbol{f})\Phi_{jh}(\boldsymbol{f})\Gamma\big(f^i,f^j\big)\Gamma\big(f^k,f^h\big).\end{aligned} \tag{3.14}$$

Proof. The fact that $\Phi(\boldsymbol{f}) \in \mathbb{D}_\infty$ has been proved in Lemma 3.2.

In the following we will assume that the indices i,j,h,k run from 1 to n and we will use Einstein summation convention.

We set $g^{ij} := \Gamma\big(f^i,f^j\big) \in \mathbb{M}_\infty$, $\ell^i := \mathsf{L} f^i \in \mathbb{V}$, $\phi_i := \Phi_i(\tilde{\boldsymbol{f}})$, $\phi_{ij} := \Phi_{ij}(\tilde{\boldsymbol{f}})$, $\phi_{ijk} := \Phi_{ijk}(\tilde{\boldsymbol{f}})$ in $\mathbb{D}_\infty$; we will also consider the quasi-continuous representative.

By (2.3) and Lemma 3.2 we have

$$\Gamma\big(\Phi(\boldsymbol{f})\big) = g^{ij}\phi_i\phi_j \in \mathbb{M}_\infty$$

Since $\phi_i\phi_j \in \mathbb{D}_\infty$ by $\langle$L.6$\rangle$ and $g^{ij} \in \mathbb{M}_\infty$ by Lemma 3.2, we can apply (2.21) obtaining

$$\frac12 \mathsf{L}^\star \Gamma\big(\Phi(\boldsymbol{f})\big) = \frac12 \phi_i\phi_j \mathsf{L}^\star g^{ij} + \Big(\frac12 g^{ij}\mathsf{L}(\phi_i\phi_j) + \Gamma\big(\phi_i\phi_j, g^{ij}\big)\Big)\mathfrak{m} = I + \big(II + III\big)\mathfrak{m}.$$

$$\begin{aligned} II &\overset{(2.5)}{=} \frac12 g^{ij}\Big(\phi_i\mathsf{L}\phi_j + \phi_j\mathsf{L}\phi_i + 2\Gamma\big(\phi_i,\phi_j\big)\Big) = g^{ij}\Big(\phi_i\mathsf{L}\phi_j + \Gamma\big(\phi_i,\phi_j\big)\Big) \\ &\overset{(2.4)}{=} g^{ij}\Big[\phi_i\Big(\phi_{jk}\ell^k + \phi_{jkh}\,g^{kh}\Big) \overset{(2.3)}{+} \phi_{ik}\,\phi_{jh}\,g^{kh}\Big]\end{aligned}$$

where we used $g^{ij} = g^{ji}$,

$$III \overset{(2.3)}{=} \Big(\phi_{ik}\,\phi_j + \phi_{jk}\phi_i\Big)\Gamma\big(f^k, g^{ij}\big) = \phi_i\phi_{jk}\Big(\Gamma\big(f^k,g^{ij}\big) + \Gamma\big(f^j, g^{ik}\big)\Big)$$

where we used the identity $\phi_{ik}\,\phi_j\Gamma\big(f^k,g^{ij}\big) = \phi_i\phi_{jk}\Gamma\big(f^j,g^{ik}\big)$ obtained by performing a cyclic permutation $i \to k \to j \to i$.

On the other hand

$$\begin{aligned}\Gamma\big(\Phi(\boldsymbol{f}), \mathsf{L}\Phi(\boldsymbol{f})\big) &\overset{(2.3)}{=} \phi_i\Gamma\big(f^i, \mathsf{L}\Phi(\boldsymbol{f})\big) \overset{(2.4)}{=} \phi_i\Gamma\big(f^i, \phi_k\ell^k + \phi_{kh}\,g^{kh}\big) \\ &= \phi_i\phi_k\Gamma\big(f^i,\ell^k\big) + \phi_i\ell^k\,\phi_{kj}\,g^{ij} + \phi_i\,g^{kh}\,\phi_{khj}\,g^{ij} + \phi_i\phi_{kh}\Gamma\big(f^i,g^{kh}\big) \\ &= \phi_i\phi_j\Gamma\big(f^i,\ell^j\big) + \phi_i\ell^k\,\phi_{kj}\,g^{ij} + \phi_i\,g^{kh}\,\phi_{khj}\,g^{ij} + \phi_i\phi_{jk}\Gamma\big(f^i,g^{jk}\big),\end{aligned}$$

where we changed k with j in the first term and h with j in the last one. We end up with

$$\begin{aligned}\Gamma_2^\star(\Phi(\boldsymbol{f})) &= \frac12\phi_i\phi_j\mathsf{L}^\star g_{ij} - \phi_i\phi_j\Gamma\big(f_i,\ell_j\big)\mathfrak{m} \\ &\quad + \phi_{ik}\,\phi_{jh}\,g^{ij}\,g^{kh}\,\mathfrak{m} \\ &\quad + \phi_i\phi_{jk}\Big(\Gamma\big(f^k,g^{ij}\big) + \Gamma\big(f^j,g^{ik}\big) - \Gamma\big(f^i,g^{jk}\big)\Big)\mathfrak{m}\end{aligned}$$

that gives (3.12). □

It could be useful to remember that in the smooth context of a Riemannian manifold $(\mathbb{M}^n, \mathsf{g})$ as for (1.9) we have [8, Page 96]

$$\mathrm{H}[f](g,h) = \langle \mathrm{D}^2 f\,\mathrm{D}g, \mathrm{D}h\rangle_{\mathsf{g}}.$$

3.3 A pointwise estimate for $\Gamma(\Gamma(f))$

Applying the previous results and adapting the ideas of [7] we can now state our first fundamental estimates.

Theorem 3.4 *Let $\mathcal{E}$ be a strongly local and quasi-regular Dirichlet form. If* (BE(K,∞)) *holds then for every $f,g,h \in \mathbb{D}_\infty$ (so that $\Gamma(f),\Gamma(g),\Gamma(h) \in \mathbb{V}_\infty$) we have (all the inequalities are to be intended $\mathfrak{m}$-a.e. in X)*

$$\Big|\mathrm{H}[f](g,h)\Big|^2 \le \big(\gamma_2(f) - K\Gamma(f)\big)\Gamma(g)\Gamma(h), \tag{3.15}$$

$$\sqrt{\Gamma\big(\Gamma(f,g)\big)} \le \sqrt{\gamma_2(f) - K\Gamma(f)}\,\sqrt{\Gamma(g)} + \sqrt{\gamma_2(g) - K\Gamma(g)}\,\sqrt{\Gamma(f)}, \tag{3.16}$$

$$\Gamma\big(\Gamma(f)\big) \le 4\Big(\gamma_2(f) - K\Gamma(f)\Big)\Gamma(f). \tag{3.17}$$

Proof. Lemma 3.2 shows that $\Gamma(f) \in \mathbb{V}_\infty$.

We choose the polynomial $\Phi : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$\Phi(\boldsymbol{f}) := \lambda f^1 + (f^2 - a)(f^3 - b) - ab, \quad \lambda, a, b \in \mathbb{R}; \tag{3.18}$$

keeping the same notation of Lemma 3.3 we have

$$\Phi_1(\boldsymbol{f}) = \lambda, \quad \Phi_2(\boldsymbol{f}) = f^3 - b, \quad \Phi_3(\boldsymbol{f}) = f^2 - a$$
$$\Phi_{23}(\boldsymbol{f}) = \Phi_{32}(\boldsymbol{f}) = 1, \quad \Phi_{ij}(\boldsymbol{f}) = 0 \quad \text{if } (i,j) \notin \{(2,3),(3,2)\}.$$

If $\boldsymbol{f} \in \mathbb{D}_\infty$ Lemma 3.2 yields $\Phi(\boldsymbol{f}) \in \mathbb{D}_\infty$ and we can then apply the inequality (3.9) obtaining

$$\gamma_2(\Phi(\boldsymbol{f})) \ge K\,\Gamma\big(\Phi(\boldsymbol{f})\big) \quad \mathfrak{m}\text{-a.e. in } X, \tag{3.19}$$

where both sides of the inequality depend on $\lambda, a, b \in \mathbb{R}$. Evaluating $\gamma_2(\Phi(\boldsymbol{f}))$ by (3.14), and choosing a countable dense set Q of the parameters (λ, a, b) in $\mathbb{R}^3$, for $\mathfrak{m}$-almost every $x \in X$ the previous inequality holds for every $(\lambda, a, b) \in Q$. Since the dependence of the left and right side of the inequality w.r.t. λ, a, b is continuous, we conclude that for $\mathfrak{m}$-almost every $x \in X$ the inequality holds for every $(\lambda, a, b) \in \mathbb{R}^3$. Apart from a $\mathfrak{m}$-negligible set, for every x we can then choose $a := f^2(x)$, $b = f^3(x)$ so that $\Phi_2(\boldsymbol{f})(x) = \Phi_3(\boldsymbol{f})(x) = 0$ obtaining

$$\lambda^2\gamma_2(f^1) + 4\lambda\mathrm{H}[f^1](f^2,f^3) + 2\Big(\Gamma(f^2)\Gamma(f^3) + \Gamma(f^2,f^3)^2\Big) \ge K\lambda^2\Gamma(f^1).$$

Since λ is arbitrary and

$$\Gamma(f^2)\Gamma(f^3) + \Gamma(f^2,f^3)^2 \le 2\Gamma(f^2)\Gamma(f^3),$$

we eventually obtain

$$\Big(\mathrm{H}[f^1](f^2,f^3)\Big)^2 \le \Big(\gamma_2(f^1) - K\Gamma(f^1)\Big)\Gamma(f^2)\Gamma(f^3) \tag{3.20}$$

that provides (3.15). (3.16) then follows by first noticing that

$$\mathrm{H}[f](g,h) + \mathrm{H}[g](f,h) = \Gamma\big(\Gamma(f,g),h\big), \tag{3.21}$$

so that

$$\left|\Gamma\big(\Gamma(f,g),h\big)\right| \le \left[\sqrt{\gamma_2(f)-K\Gamma(f)}\sqrt{\Gamma(g)}+\sqrt{\gamma_2(g)-K\Gamma(g)}\sqrt{\Gamma(f)}\right]\sqrt{\Gamma(h)}. \tag{3.22}$$

We argue now by approximation, fixing $f,g\in\mathbb{D}_\infty$ and approximating an arbitrary $h\in\mathbb{V}_\infty$ with a sequence $h_n\in\mathbb{D}_\infty$ (e.g. by (2.13)) converging to h in energy with

$$\Gamma\big(h_n\big)\to\Gamma\big(h\big),\quad \Gamma\big(h_n,\Gamma(f,g)\big)\to\Gamma\big(h,\Gamma(f,g)\big)$$

pointwise and in $L^1(X,\mathfrak{m})$, thanks to (3.2) (see also Remark 2.5 and (4.5) of [5]): (3.22) thus hold for arbitrary $h\in\mathbb{V}_\infty$ and we can then choose $h:=\Gamma\big(f,g\big)$ obtaining (3.16). (3.17) then follows by choosing $g:=f$ in (3.16). □

Corollary 3.5 *Under the same assumption of Theorem 3.4, for every $f\in\mathbb{V}$ and $\alpha\in[1/2,1]$ we have*

$$\Gamma\big(\mathsf{P}_t f\big)^\alpha \le \mathrm{e}^{-2\alpha Kt}\mathsf{P}_t\big(\Gamma\big(f\big)^\alpha\big). \tag{3.23}$$

Proof. We adapt here the strategy of [37]. Since the case $\alpha=1$ has been already covered by (3.2), we can also assume $1/2\le\alpha<1$.

We consider the concave and smooth function $\eta_\varepsilon(r):=\big(\varepsilon+r\big)^\alpha-\varepsilon^\alpha$, $\varepsilon>0$, $r\ge 0$, and for a time $t>0$, a nonnegative $\zeta\in\mathbb{V}_\infty$, and an arbitrary $f\in\mathbb{D}_\infty$ we define the curves

$$f_\tau:=\mathsf{P}_\tau f,\ \zeta_s:=\mathsf{P}_s\zeta,\ u_\tau:=\Gamma\big(f_\tau\big),\quad G_\varepsilon(s):=\int_X\eta_\varepsilon(u_{t-s})\,\zeta_s\,\mathrm{d}\mathfrak{m},\quad \tau,s\in[0,t]. \tag{3.24}$$

Notice that η_ε is smooth and Lipschitz; a direct computation yields

$$\eta_\varepsilon(r)\le r^\alpha,\quad r\,\eta_\varepsilon'(r)\ge\alpha\eta_\varepsilon,\quad 2\eta_\varepsilon'(r)+4r\,\eta_\varepsilon''(r)\ge 0. \tag{3.25}$$

Moreover, for every $s\in[0,t]$ $f_{t-s}\in\mathbb{D}_\infty$ so that $u_{t-s}\in\mathbb{V}\cap L^1\cap L^\infty(X,\mathfrak{m})$,

$$\frac{\mathrm{d}}{\mathrm{d}s}u_{t-s}=-2\Gamma\big(f_{t-s},\mathsf{L}f_{t-s}\big),\quad \frac{\mathrm{d}}{\mathrm{d}s}\eta_\varepsilon(u_{t-s})=-2\eta_\varepsilon'(u_{t-s})\Gamma\big(f_{t-s},\mathsf{L}f_{t-s}\big)\quad\text{in } L^1\cap L^2(X,\mathfrak{m}).$$

Differentiating with respect to $s\in(0,t)$ we get

$$\begin{aligned}
G'(s)&=\int_X\Big(\eta_\varepsilon(u_{t-s})\,\mathsf{L}\zeta_s-2\eta_\varepsilon'(u_{t-s})\Gamma\big(f_{t-s},\mathsf{L}f_{t-s}\big)\zeta_s\Big)\,\mathrm{d}\mathfrak{m}\\
&=-\int_X\eta_\varepsilon'(u_{t-s})\Gamma\big(\Gamma\big(f_{t-s}\big),\zeta_s\big)+2\Gamma\big(f_{t-s},\mathsf{L}f_{t-s}\big)\eta_\varepsilon'(u_{t-s})\zeta_s\Big)\,\mathrm{d}\mathfrak{m}\\
&=-\int_X\Big(\Gamma\big(\Gamma\big(f_{t-s}\big),\eta_\varepsilon'(u_{t-s})\zeta_s\big)-\Gamma\big(\Gamma\big(f_{t-s}\big)\big)\eta_\varepsilon''(u_{t-s})\zeta_s+2\Gamma\big(f_{t-s},\mathsf{L}f_{t-s}\big)\eta_\varepsilon'(u_{t-s})\zeta_s\Big)\,\mathrm{d}\mathfrak{m}\\
&=2\int_X\eta_\varepsilon'(\tilde u_{t-s})\tilde\zeta_s\,\mathrm{d}\Gamma_2^\star(f_{t-s})+\int\Gamma\big(\Gamma\big(f_{t-s}\big)\big)\eta_\varepsilon''(u_{t-s})\zeta_s\,\mathrm{d}\mathfrak{m}\\
&\ge 2\int_X\eta_\varepsilon'(u_{t-s})\zeta_s\,\gamma_2(f_{t-s})\,\mathrm{d}\mathfrak{m}+4\int_X\eta_\varepsilon''(u_{t-s})\big(\gamma_2(f_{t-s})-Ku_{t-s}\big)u_{t-s}\,\zeta_s\,\mathrm{d}\mathfrak{m}\\
&=\int_X\Big(2\eta_\varepsilon'(u_{t-s})+4\eta_\varepsilon''(u_{t-s})u_{t-s}\Big)\Big(\gamma_2(f_{t-s})-Ku_{t-s}\Big)\zeta_s\,\mathrm{d}\mathfrak{m}+2K\int_X\eta_\varepsilon'(u_{t-s})u_{t-s}\zeta_s\,\mathrm{d}\mathfrak{m}\\
&\ge 2K\int_X\eta_\varepsilon'(u_{t-s})u_{t-s}\zeta_s\,\mathrm{d}\mathfrak{m}\overset{(3.25)}{\ge}2\alpha\,K\int_X\eta_\varepsilon(u_{t-s})\zeta_s\,\mathrm{d}\mathfrak{m}=2\alpha\,KG_\varepsilon(s).
\end{aligned}$$

thanks to (3.17).

Since G is continuous, we obtain $G_\varepsilon(0)\mathrm{e}^{2\alpha Kt} \le G_\varepsilon(t)$ which yields, after passing to the limit as $\varepsilon \downarrow 0$

$$\mathrm{e}^{2\alpha Kt}\int_X \Gamma\big(\mathsf{P}_t f\big)^\alpha\,\zeta\,\mathrm{d}\mathfrak{m} \le \int_X \Gamma\big(f\big)^\alpha\,\mathsf{P}_t\zeta\,\mathrm{d}\mathfrak{m} = \int_X \mathsf{P}_t\big(\Gamma\big(f\big)^\alpha\big)\,\zeta\,\mathrm{d}\mathfrak{m}. \tag{3.26}$$

Since $\mathbb{D}_\infty$ is dense in $\mathbb{V}$ we can extend (3.26) to arbitrary $f \in \mathbb{V}$ and then obtain (3.23), since ζ is arbitrary. □

4 $\mathrm{RCD}(K,\infty)$-metric measure spaces

In this section we will apply the previous result to prove new contraction properties w.r.t. transport costs (in particular W_p Wasserstein distance) for the heat flow in $\mathrm{RCD}(K,\infty)$ metric measure spaces.

4.1 Basic notions

Metric measure spaces, transport and Wasserstein distances, entropy

We will quickly recall a few basic facts concerning optimal transport of probability measures, also to fix notation; we refer to [2, 36] for more details.

Let (X,d) be a complete and separable metric space endowed with a Borel measure $\mathfrak{m}$ satisfying

$$\mathrm{supp}(\mathfrak{m}) = X, \qquad \mathfrak{m}(B_r(\bar x)) \le \mathsf{c}_1\exp(\mathsf{c}_2 r^2) \quad \text{for every } r>0, \tag{$\mathfrak{m}$-exp}$$

for some constants $\mathsf{c}_1,\mathsf{c}_2 \ge 0$ and a point $\bar x \in X$.

Recall that for every Borel probability measure $\mu \in \mathscr{P}(Y)$ in a separable metric space Y and every Borel map $\boldsymbol{r}: Y \to X$, the push-forward $\boldsymbol{r}_\sharp\mu \in \mathscr{P}(X)$ is defined by $\boldsymbol{r}_\sharp\mu(B) = \mu(\boldsymbol{r}^{-1}(B))$ for every $B \in \mathscr{B}(X)$. If $\mu_i \in \mathscr{P}(X)$, $i=1,2$, we denote by $\Pi(\mu_1,\mu_2)$ the collection of all couplings $\boldsymbol{\mu}$ between μ_1 and μ_2, i.e. measures in $\mathscr{P}(X\times X)$ whose marginals $\pi^i_\sharp\boldsymbol{\mu}$ coincide with μ_i (here $\pi^i(x_1,x_2)=x_i$). Given a nondecreasing continuous function $h:[0,\infty)\to[0,\infty)$, we consider the transport cost

$$\mathcal{C}_h(\mu_1,\mu_2) := \min\Big\{\int_{X\times X} h(\mathsf{d}(x,y))\,\mathrm{d}\boldsymbol{\mu}(x,y) : \boldsymbol{\mu}\in\Pi(\mu_1,\mu_2)\Big\}, \tag{4.1}$$

where we implicitly assume that the minimum is $+\infty$ if couplings with finite cost do not exist. In the particular case $h(r) := r^p$ we set

$$W_p(\mu_1,\mu_2) := \big(\mathcal{C}_h(\mu_1,\mu_2)\big)^{1/p}, \quad h(r) := r^p, \tag{4.2}$$

and we also set

$$W_\infty(\mu_1,\mu_2) = \min\Big\{\|\mathsf{d}\|_{L^\infty(X\times X,\boldsymbol{\mu})} : \boldsymbol{\mu}\in\Pi(\mu_1,\mu_2)\Big\} = \lim_{p\uparrow\infty} W_p(\mu_1,\mu_2). \tag{4.3}$$

Denoting by $\mathscr{P}_p(X)$ the space of Borel probability measures with finite p-th moment, i.e.

$$\mu\in\mathscr{P}_p(X) \quad\Longleftrightarrow\quad \int_X \mathsf{d}^p(x,\bar x)\,\mathrm{d}\mu(x) < \infty \quad \text{for some (and thus any) } \bar x \in X, \tag{4.4}$$

$(\mathscr{P}_p(X), W_p)$ is a complete and separable metric space.

The relative entropy of a measure $\mu \in \mathscr{P}_2(X)$ is defined as

$$\mathrm{Ent}_{\mathfrak{m}}(\mu) := \begin{cases} \int_X \rho \log \rho \, \mathrm{d}\mathfrak{m} & \text{if } \mu = \rho\mathfrak{m} \ll \mathfrak{m}, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.5}$$

The entropy functional is well defined and lower semicontinuous w.r.t. W_2 convergence (see e.g. [3, §7.1]

The Cheeger energy and its L^2-gradient flow

We first recall that the metric slope of a Lipschitz function $f : X \to \mathbb{R}$ is defined by

$$|\mathrm{D}f|(x) := \limsup_{y \to x} \frac{|f(y) - f(x)|}{\mathsf{d}(x, y)}. \tag{4.6}$$

The Cheeger energy [15, 3] is obtained as the L^2-lower semicontinuous envelope of the functional $f \mapsto \frac{1}{2} \int_X |\mathrm{D}f|^2 \, \mathrm{d}\mathfrak{m}$:

$$\mathsf{Ch}(f) := \inf \Big\{ \liminf_{n \to \infty} \frac{1}{2} \int_X |\mathrm{D}f_n|^2 \, \mathrm{d}\mathfrak{m} : f_n \in \mathrm{Lip}_b(X), \quad f_n \to f \text{ in } L^2(X, \mathfrak{m}) \Big\}. \tag{4.7}$$

If $\mathsf{Ch}(f) < \infty$ it is possibile to show that the collection

$$S(f) := \Big\{ G \in L^2(X, \mathfrak{m}) : \exists f_n \in \mathrm{Lip}_b(X),\ f_n \to f,\ |\mathrm{D}f_n| \rightharpoonup G \text{ in } L^2(X, \mathfrak{m}) \Big\}$$

admits a unique element of minimal norm, *the minimal weak upper gradient* $|\mathrm{D}f|_w$, that it is also minimal with respect to the order structure [3, §4], i.e.

$$|\mathrm{D}f|_w \in S(f), \quad |\mathrm{D}f|_w \le G \quad \mathfrak{m}\text{-a.e.} \quad \text{for every } G \in S(f). \tag{4.8}$$

By $|\mathrm{D}f|_w$ we can also represent $\mathsf{Ch}(f)$ as

$$\mathsf{Ch}(f) = \frac{1}{2} \int_X |\mathrm{D}f|_w^2 \, \mathrm{d}\mathfrak{m}. \tag{4.9}$$

It turns out that Ch is a 2-homogeneous, l.s.c., convex functional in $L^2(X, \mathfrak{m})$, whose proper domain $D(\mathsf{Ch}) := \{f \in L^2(X, \mathfrak{m}) : \mathsf{Ch}(f) < \infty\}$ is a dense linear subspace of $L^2(X, \mathfrak{m})$.

Its L^2-gradient flow is a continuous semigroup of contractions $(\mathsf{h}_t)_{t \ge 0}$ in $L^2(X, \mathfrak{m})$, whose continuous trajectories $f_t = \mathsf{h}_t f$, $t \ge 0$ and $f \in L^2(X, \mathfrak{m})$, are locally Lipschitz curves in $(0, \infty)$ with values in $L^2(X, \mathfrak{m})$ characterised by the differential inclusion

$$\frac{\mathrm{d}}{\mathrm{d}t} f_t + \partial \mathsf{Ch}(f_t) \ni 0 \quad \text{a.e. in } (0, \infty). \tag{4.10}$$

4.2 $\mathrm{RCD}(K, \infty)$-spaces

In order to state the main equivalent definitions of $\mathrm{RCD}(K, \infty)$ spaces, let us first recall a list of relevant properties:

[Q-Ch]: The Cheeger energy is quadratic, i.e.

$$\mathsf{Ch}(f+g)+\mathsf{Ch}(f-g)=2\mathsf{Ch}(f)+2\mathsf{Ch}(g)\quad \text{for every } f,g\in D(\mathsf{Ch}). \tag{4.11}$$

[CD(K,∞)]: The entropy functional is displacement K-convex in $\mathscr{P}_2(X)$ [34, 27], i.e. for every $\mu_0,\mu_1\in D(\mathrm{Ent}_{\mathfrak{m}})\subset\mathscr{P}_2(X)$ and $t\in[0,1]$ there exists $\mu_t\in\mathscr{P}_2(X)$ such that

$$\begin{gathered} W_2(\mu_0,\mu_t)=tW_2(\mu_0,\mu_1),\quad W_2(\mu_t,\mu_1)=(1-t)W_2(\mu_0,\mu_1),\\ \mathrm{Ent}_{\mathfrak{m}}(\mu_t)\le(1-t)\mathrm{Ent}_{\mathfrak{m}}(\mu_0)+t\mathrm{Ent}_{\mathfrak{m}}(\mu_1)-\frac{K}{2}t(1-t)W_2(\mu_0,\mu_1). \end{gathered} \tag{4.12}$$

[Length]: (X,d) is a length space, i.e. for every $x_0,x_1\in X$ and $\varepsilon>0$ there exists an ε-middle point $x_\varepsilon\in X$ such that

$$\mathsf{d}(x_0,x_\varepsilon)<\frac12\mathsf{d}(x_0,x_1)+\varepsilon,\quad \mathsf{d}(x_1,x_\varepsilon)<\frac12\mathsf{d}(x_0,x_1)+\varepsilon. \tag{4.13}$$

[Cont]: Every bounded function $f\in D(\mathsf{Ch})$ with $|\mathrm{D}f|_w\le 1$ admits a continuous representative.

[W_2-cont]: For every $f_0,f_1\in L^2(X,\mathfrak{m})$ with $f_i\mathfrak{m}\in\mathscr{P}_2(X)$ we have

$$W_2(\mathsf{h}_tf_0\,\mathfrak{m},\mathsf{h}_tf_1\,\mathfrak{m})\le \mathrm{e}^{-Kt}W_2(f_0\mathfrak{m},f_1\mathfrak{m})\quad t\ge0. \tag{4.14}$$

[BE(K,∞)]: Assuming that the Cheeger energy is quadratic, then the Dirichlet form $\mathcal{E}:=2\mathsf{Ch}$ satisfies the Bakry-Émery condition according to Definition 3.1.

[EVI$_K$]: For every $\bar\mu\in\mathscr{P}_2(X)$ there exists a curve $(\mu_t)_{t\ge0}\subset D(\mathrm{Ent}_{\mathfrak{m}})$ such that $\lim_{t\downarrow0}\mu_t=\bar\mu$ and

$$\frac{\mathrm{d}}{\mathrm{d}t_+}\frac12W_2^2(\mu_t,\nu)+\frac{K}{2}W_2^2(\mu_t,\nu)\le\mathrm{Ent}_{\mathfrak{m}}(\nu)-\mathrm{Ent}_{\mathfrak{m}}(\mu_t)\quad t>0. \tag{4.15}$$

Let us now recall the main equivalence results:

Theorem 4.1 *Let $(X,\mathsf{d},\mathfrak{m})$ be a complete, length, and separable metric measure space satisfying condition* ($\mathfrak{m}$-exp) *and let $K\in\mathbb{R}$. The following set of conditions for $(X,\mathsf{d},\mathfrak{m})$ are equivalent:*

(I) **[Q-Ch]** *and* **[CD(K,∞)]***;*

(II) **[Q-Ch]**, **[Cont]**, *and* **[W_2-cont]***;*

(III) **[Q-Ch]**, **[Cont]**, *and* **[BE(K,∞)]***;*

(IV) **[EVI$_K$]**.

Moreover, if one of the above conditions hold then $\mathcal{E}:=2\mathsf{Ch}$ is a strongly local and quasi-regular (see Definition 2.3) Dirichlet form, it admits the Carré du Champ

$$\Gamma(f)=|\mathrm{D}f|_w^2\quad \textit{for every } f\in D(\mathsf{Ch}), \tag{4.16}$$

the subdifferential $\partial\mathsf{Ch}$ is single-valued and coincides with the linear generator L, $(\mathsf{h}_t)_{t\ge0}=(\mathsf{P}_t)_{t\ge0}$, and for every $\bar\mu=f\mathfrak{m}\in\mathscr{P}_2(X)$ with $f\in L^2(X,\mathfrak{m})$ the curve $\mu_t=\mathsf{h}_tf\mathfrak{m}$ is the unique solution of (4.15)*. Eventually, any essentially bounded function $f\in D(\mathsf{Ch})$ with $|\mathrm{D}f|_w\le L$ admits a L-Lipschitz representative $\tilde f$, and for every $f\in\mathrm{Lip}_b(X), g\in\mathrm{C}_b(X)$ we have*

$$|\mathrm{D}f|_w\le|\mathrm{D}f|;\quad |\mathrm{D}f|_w\le g\quad\Longrightarrow\quad |\mathrm{D}f|\le g. \tag{4.17}$$

Proof. The implication (I)⇔(IV) has been proved in [4, Thm. 5.1] in the case when $\mathfrak{m} \in \mathscr{P}_2(X)$ and extended to the general case by [1]. (IV)⇒(II),(III) has been proved in [4, Thm. 6.2, Thm. 6.10] and the relations (II)⇔(III)⇒(IV) have been proved in [5, Thm. 3.17, Cor. 3.18, Cor. 4.18]. (4.16) follows from [4, Thm. 4.18]; see [5, Prop. 3.11] for (4.17).

Let us eventually check that $\mathcal{E}$ is quasi-regular, by applying Lemma 2.4. $\langle$QR.2$'\rangle$ is always true for a Cheeger energy, since Lipschitz functions are dense in $\mathbb{V}$ by (4.7).

When $\mathfrak{m}(X) < \infty$ we can always choose $X_n := X$ and $\langle$QR.1$'\rangle$ reduces to the tightness property (2.9), that has been proved in [4, Lemma 6.7], following an argument of [28, Proposition IV.4.2]. In the general case we can adapt the same argument: we recall here the various steps for the easy of the reader.

Let us fix a point $\bar{x} \in X$ and let us set $X_n := B_n(\bar{x})$. In order to prove that $(\overline{X}_n)_{n\in\mathbb{N}}$ is an $\mathcal{E}$-nest, we introduce the 1-Lipschitz cut-off functions $\psi_n : X \to [0,1]$

$$\psi_n(x) := 0 \vee (n - \mathsf{d}(x,\bar{x})) \wedge 1, \quad \text{so that} \quad \psi_n(x) = \begin{cases} 1 & \text{if } x \in \overline{X}_{n-1}, \\ 0 & \text{if } x \in X \setminus X_n, \end{cases} \quad \psi_n(x) \uparrow 1 \text{ as } n \to \infty.$$

For every $f \in \mathbb{V}$ we can consider the approximations $f_n := \psi_n f$ in $\mathbb{V}_{\bar{X}_n}$. The Lebesgue's Dominated Convergence Theorem shows that $f_n \to f$ strongly in $L^2(X,\mathfrak{m})$ as $n \to \infty$. The Leibniz rule yields

$$|\mathrm{D}(f - f_n)|_w \le \big(|\mathrm{D}f|_w + |f|\big)\chi_{X\setminus B_{n-1}(\bar{x})}$$

so that $\lim_{n\to\infty} \mathcal{E}(f - f_n) = 0$ as well. This shows that $f_n \to f$ strongly in $\mathbb{V}$ and $(\overline{X}_n)_{m\in\mathbb{N}}$ is an $\mathcal{E}$-nest.

In order to prove $\langle$QR.1$'\rangle$, we fix $n \in \mathbb{N}$, we consider a dense sequence $(x_j)_{j\in\mathbb{N}}$ in X_{n+1}, and we define the functions $w_k : X \to [0,1]$

$$w_k(x) := \psi_{n+1}(x) \wedge \min_{1\le j\le k} \mathsf{d}(x, x_j) \quad x \in X.$$

It is easy to check that w_k are 1-Lipschitz and pointwise nonincreasing, they satisfy $0 \le w_k \le \psi_{n+1} \le 1$ and the pointwise limit $w_k \downarrow 0$ as $k \to \infty$, so that $w_k \to w$ strongly in $L^2(X,\mathfrak{m})$ since $\mathrm{supp}(w_k) \subset \overline{X}_{n+1}$ and $\mathfrak{m}(\overline{X}_{n+1}) < \infty$. The finiteness of $\mathfrak{m}(\overline{X}_{n+1})$ also yields that $(w_k)_{k\in\mathbb{N}}$ is bounded in $\mathbb{V}$, so that $w_k \rightharpoonup 0$ weakly in $\mathbb{V}$ as $k \to \infty$.

The Banach-Saks theorem ensures the existence of an increasing subsequence $(k_h)_{h\in\mathbb{N}}$ such that the Cesaro means $v_h := \frac{1}{h}\sum_{i=1}^h w_{k_i}$ converge to 0 strongly in $\mathbb{V}$. This implies [19, Thm. 1.3.3] that a subsequence $(v_{h(l)})$ of (v_h) converges to 0 quasi-uniformly, i.e. for all integers $m \ge 1$ there exists a closed set $G_m \subset X$ such that $\mathrm{Cap}(X_{n+1} \setminus G_m) < 1/m$ and $v_{h(l)} \to 0$ uniformly on G_m. As $w_{k_{h(l)}} \le v_{h(l)}$, if we set $F_m = \cup_{i\le m} G_i$, we have that $w_{k_{h(l)}} \to 0$ as $l \to \infty$ uniformly on F_m for all m and $\mathrm{Cap}(X_{n+1} \setminus F_m) \le 1/m$.

Therefore, for every $\delta > 0$ we can find an integer $p \in \mathbb{N}$ such that $w_p < \delta$ on F_m; since $\psi_{n+1}(x) \equiv 1$ when $x \in \overline{X}_n$, the definition of w_p implies

$$\forall\, x \in \overline{X}_n \cap F_m \quad \exists j \in \mathbb{N},\ j \le p:\ \mathsf{d}(x, x_j) < \delta, \quad \text{i.e.} \quad \overline{X}_n \cap F_m \subset \bigcup_{j=1}^p B(x_i, \delta).$$

Since δ is arbitrary this proves that $K_{n,m} := \overline{X}_n \cap F_m$ is totally bounded, hence compact and $\mathrm{Cap}(X_n \setminus K_{n,m}) \le \mathrm{Cap}(X_{n+1} \setminus F_m) \le 1/m$. □

Definition 4.2 *We say that* $(X, \mathsf{d}, \mathfrak{m})$ *is* $\mathrm{RCD}(K, \infty)$*-metric measure space if it is complete, separable and length,* $\mathfrak{m}$ *satisfies* ($\mathfrak{m}$-exp)*, and at least one of the (equivalent) properties* (I)–(IV) *holds.*

By Corollary 3.5 we thus obtain:

Corollary 4.3 *If* $(X, \mathsf{d}, \mathfrak{m})$ *is a* $\mathrm{RCD}(K, \infty)$ *metric measure space, then for every* $t > 0$, $\beta \in [1, 2]$ *and* $f \in \mathbb{V}_\infty$

$$|\mathsf{D}\mathsf{P}_t f|^\beta \le \mathrm{e}^{-\beta K t}\mathsf{P}_t(|\mathrm{D}f|_w^\beta). \tag{4.18}$$

We call $\mathsf{H}_t\bar{\mu}$ the unique solution to (4.15): by [5, Prop. 3.2] $(\mathsf{H})_{t\ge 0}$ can be extended in a unique way to a semigroup of weakly continuous operators in $\mathscr{P}(X)$ satisfying

$$\lim_{t\downarrow 0} \mathsf{H}_t\mu = \mu \text{ in } \mathscr{P}(X), \quad W_2(\mathsf{H}_t\mu_0, \mathsf{H}_t\mu_1) \le \mathrm{e}^{-Kt} W_2(\mu_0, \mu_1) \text{ for every } \mu_0, \mu_1 \in \mathscr{P}(X). \tag{4.19}$$

In particular we can consider the fundamental solutions

$$\varrho_{t,x} := \mathsf{H}_t\delta_x \in \mathscr{P}_2(X). \tag{4.20}$$

4.3 New contraction properties for the heat flow $(\mathsf{H}_t)_{t\ge 0}$

Let us fix a parameter $K \in \mathbb{R}$ and for every nondecreasing cost function $h : [0, \infty) \to [0, \infty)$ let us consider the perturbed cost functions

$$h_{Kt}(r) := h(\mathrm{e}^{Kt}\, r), \tag{4.21}$$

and the associated transportation costs $\mathcal{C}_h$, $\mathcal{C}_{h_{Kt}}$.

Theorem 4.4 *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a* $\mathrm{RCD}(K, \infty)$ *metric measure space. Then*

i) *For every* $x, y \in X$ *the fundamental solutions* $\varrho_{t,x}$, $\varrho_{t,y}$ *defined by* (4.20) *satisfy*

$$W_\infty(\varrho_{t,x}, \varrho_{t,y}) \le \mathrm{e}^{-Kt}\mathsf{d}(x, y). \tag{4.22}$$

ii) *For every* $\mu, \nu \in \mathscr{P}(X)$

$$\mathcal{C}_{h_{Kt}}(\mathsf{H}_t\mu, \mathsf{H}_t\nu) \le \mathcal{C}_h(\mu, \nu). \tag{4.23}$$

iii) *For every* $\mu, \nu \in \mathscr{P}(X)$ *and every* $p \in [1, \infty]$

$$W_p(\mathsf{H}_t\mu, \mathsf{H}_t\nu) \le \mathrm{e}^{-Kt} W_p(\mu, \nu). \tag{4.24}$$

Proof. **i)** follows from (4.18) by the Kuwada's duality argument [26, Prop. 3.7] as developed by [5, Lemma 3.4, Theorem 3.5].

ii) Let μ, ν with $\mathcal{C}_h(\mu, \nu) < \infty$ and let $\boldsymbol{\gamma} \in \Pi(\mu, \nu)$ be an optimal plan for $\mathcal{C}_h$. We may use a measurable selection theorem (see for instance [13, Theorem 6.9.2] to select in a $\boldsymbol{\gamma}$-measurable way optimal plans $\boldsymbol{\gamma}_{x,y}$ for W_∞ between $\varrho_{t,x}$ and $\varrho_{t,y}$. Then, we define

$$\boldsymbol{\sigma} := \int_{X\times X} \boldsymbol{\gamma}_{x,y}\, \mathrm{d}\boldsymbol{\gamma}(x, y).$$

Notice that $\boldsymbol{\sigma} \in \Gamma(\mathsf{H}_t\mu, \mathsf{H}_t\nu)$ since e.g. for every $\varphi \in \mathrm{C}_b(X)$ we have

$$\begin{aligned}\int_{X\times X} \varphi(x)\,\mathrm{d}\boldsymbol{\sigma}(x,y) &= \int_{X\times X}\int_{X\times X} \varphi(x)\,\mathrm{d}\gamma_{u,v}(x,y)\,\mathrm{d}\boldsymbol{\gamma}(u,v) = \int_{X\times X}\int_X \varphi(x)\,\mathrm{d}\varrho_{t,u}(x)\,\mathrm{d}\boldsymbol{\gamma}(u,v)\\ &= \int_X\int_X \varphi(x)\,\mathrm{d}\varrho_{t,u}(x)\,\mathrm{d}\mu(u) = \int_X \varphi(x)\,\mathrm{d}\mathsf{H}_t\mu(x),\end{aligned}$$

and a similar computation holds integrating functions depending only on y. Therefore, since (4.22) yields

$$\mathsf{d}(x,y) \le \mathrm{e}^{-Kt}\mathsf{d}(u,v) \quad \text{for } \boldsymbol{\gamma}_{u,v}\text{-a.e. } (x,y) \in X\times X, \tag{4.25}$$

$$\begin{aligned}\mathcal{C}_{h_{Kt}}(\mathsf{H}_t\mu, \mathsf{H}_t\nu) &\le \int_{X\times X} h_{Kt}\big(\mathsf{d}(x,y)\big)\,\mathrm{d}\boldsymbol{\sigma}(x,y)\\ &= \int_{X\times X}\int_{X\times X} h\big(\mathrm{e}^{Kt}\mathsf{d}(x,y)\big)\,\mathrm{d}\gamma_{u,v}(x,y)\,\mathrm{d}\boldsymbol{\gamma}(u,v)\\ &\overset{(4.25)}{\le} \int_{X\times X}\int_{X\times X} h\big(\mathsf{d}(u,v)\big)\,\mathrm{d}\gamma_{u,v}(x,y)\,\mathrm{d}\boldsymbol{\gamma}(u,v) = \int_{X\times X} h\big(\mathsf{d}(u,v)\big)\,\mathrm{d}\boldsymbol{\gamma}(u,v)\\ &= \mathcal{C}_h(\mu,\nu).\end{aligned}$$

iii) follows immediately by (4.23) by choosing $h(r) := r^p$ so that $h_{Kt}(r) = \mathrm{e}^{pKt}r^p$. □

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