Regularity Results for Elliptic Equations in Lipschitz Domains

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Abstract

We develop a simple variational argument based on the usual Nirenberg's difference quotient technique to deal with the regularity of the solutions of Dirichlet and Neumann problems for some linear and quasilinear elliptic equation in Lipschitz domains. We obtain optimal regularity results in the natural family of Sobolev spaces associated with the variational structure of the equations. In the linear case, we find in a completely different way some of the results of D. Jerison & C.E. Kenig about the Laplace equation.

1 Introduction.

In a bounded Lipschitz open set $\Omega \subset \mathbb{R}^N$ let us consider the homogeneous Dirichlet problem of elliptic type

$$\begin{cases} -\operatorname{div} A(x)\nabla u(x) &= f(x) \quad \text{in } \Omega, \\ u(x) &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $A(x)$ are symmetric matrices with measurable coefficients satisfying the uniform ellipticity assumption

$$\exists\, \alpha, \mu > 0 : \quad \alpha|\boldsymbol{\xi}|^2 \le A(x)\boldsymbol{\xi} \cdot \boldsymbol{\xi} \le \mu|\boldsymbol{\xi}|^2, \quad \forall\, \boldsymbol{\xi} \in \mathbb{R}^N, \quad \text{for a.e. } x \in \Omega. \tag{2}$$

It is well known that for every choice of $f \in H^{-1}(\Omega)$ the usual variational formulation of (1) admits a unique solution $u \in H^1_0(\Omega)$ and the correspondence between f and u establishes a linear isomorphism between these two Hilbert spaces.

If f is more regular, say $f \in L^2(\Omega)$, and A is Lipschitz, i.e.

$$\exists L > 0 : \quad |A(x) - A(y)| \leq L|x - y|, \quad \forall\, x, y \in \Omega, \tag{3}$$

then u belongs to $H^2_{loc}(\Omega)$ and this regularity holds up to the boundary, i.e. $u \in H^2(\Omega)$, if $\partial\Omega$ is of class $C^{1,1}$ or Ω is convex (see e.g. [11], Theorems 2.2.2.3 and 3.2.1.2).

A possible way to prove the local result is to use the difference quotient technique of L. Nirenberg (see [21, 19, 18]); by means of a suitable change of coordinates the global result can also be achieved, provided that the boundary of Ω is regular. Unfortunately, when $\partial\Omega$ is only Lipschitz continuous, this transformation "destroys" the regularity of the coefficients of $A(x)$ and the above method does not work.

This is not only a technical difficulty, since the solution of (1) may have a singular behavior near the irregular points of $\partial\Omega$, even in the simplest case of the Laplace equation, corresponding to $A(x) \equiv I$; in this case, if Ω is a non-convex polygon in $\mathbb{R}^2$, it is well known (cf. [11]) that $u \notin H^2(\Omega)$ in general, even if $f \in C^\infty(\overline{\Omega})$. More precisely, for every $\varepsilon > 0$, there exists a polygon $\Omega := \Omega_\varepsilon$ and a smooth function $f := f_\varepsilon$ such that the corresponding solution $u := u_\varepsilon$ of (1) does not belong to $H^{3/2+\varepsilon}(\Omega_\varepsilon)$.

In order to give an insight into this phenomenon, the regularity analysis can be carried out in weighted function spaces related to the geometry of $\partial\Omega$ and it can be shown that some compatibility conditions between the data, the elliptic operator and the boundary have to be imposed in order to recover smoother solutions (see [11, 5, 12] and the references therein). However, when Ω is not a polygon or A is not constant, it could be difficult to make these conditions explicit; furthermore, it could be interesting to know what is the maximal (Sobolev) regularity of the solution with respect to the data without assuming any compatibility on them or any particular structure on $\partial\Omega$, except for the Lipschitz property.

In the constant coefficients case and in the Hilbertian framework, a first answer to this question follows from estimates of D. Jerison and C.E. Kenig [13] for the homogeneous Dirichlet problem; thanks to the usual technique of reducing the inhomogeneous problem to the homogeneous one and to real interpolation (as detailed in [15]), those estimates in particular imply

$$u \in H^{3/2}(\Omega) \quad \text{if} \quad f \in H^{-1/2+\varepsilon}(\Omega), \quad \text{for some } \varepsilon > 0, \tag{4}$$

and (see also [14])

$$f \in H^{-1+s}(\Omega) \quad \Rightarrow \quad u \in H^{1+s}_0(\Omega), \quad \forall\, s \in]-1/2, 1/2[. \tag{5}$$

By deeply using the powerful tools of Harmonic Analysis (cf. [16]), these optimal regularity results have been recently generalized to L^p-Sobolev spaces by [15], to which we refer for a complete list of results and counterexamples [1].

In our paper we come back to the original difference quotient technique and, developing an idea of [22], we propose a simple variant of it which can be applied to Lipschitz open sets and which gives (5) also for the equations with variable coefficients (1). More generally, we apply the method to Dirichlet or Neumann problem for quasi-linear elliptic equations of the type

$$-\operatorname{div} \boldsymbol{a}(x, \nabla u(x)) = f(x), \quad \text{in } \Omega, \tag{6}$$

where $\boldsymbol{a} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a "regular and coercive" vector field which is the gradient of a scalar convex function $F : \Omega \times \mathbb{R}^N \to \mathbb{R}$, satisfying some usual growth conditions. In other words, we assume that (6) is the Euler-Lagrange equation related to the integral functional

$$\mathcal{F}_f(u) := \int_\Omega F(x, \nabla u(x))\, dx - \int_\Omega f(x)\, u(x)\, dx, \tag{7}$$

in a suitable Sobolev space $W^{1,p}(\Omega)$. This natural generalization, which in particular covers the case of the p-Laplacian, does not require more effort (in this context, of course) than the linear case and it clarifies the simple variational argument behind the proof.

Applications are also given to problems of transmission type through a Lipschitz interface (where the coefficients of A have a jump discontinuity, destroying (3)) and to other boundary value problems for the biharmonic operator and the linear Stokes equation in two dimensions; in a forthcoming paper we will apply an interpolation estimate obtained for (1) to handle linear parabolic equations in non-cylindric Lipschitz domains via the abstract framework of [9, 23]; in particular we will be able to give a more refined answer to a problem proposed by E. De Giorgi in [6].

The plan of the paper is the following: in the next section we point out, in an abstract setting, the elementary variational principle to be used in the following. Section 3 contains the basic local estimates to deal with (6). After a brief recall of some basic properties of the intermediate Sobolev and Besov spaces, the other sections are devoted to the various applications.

[1] In particular, it is showed that (4) does not hold for $\varepsilon = 0$; see also [4] for further developments.

2 The abstract estimate

In a Banach space V let us consider a convex function $\mathcal{F} : V \to \mathbb{R}$ which is Gâteaux–differentiable at every point of a convex set $K \subset V$. Let us denote by $\mathcal{A} : K \to V'$ its differential, which is a monotone operator on K, and by $[\cdot]$ a given seminorm of V.

Theorem 1 *Let us assume that $\mathcal{A}$ is p-coercive on K w.r.t. $[\cdot]$ for a given $p \in [2, +\infty[$, i.e.*

$$\exists \alpha > 0 : \langle \mathcal{A}u - \mathcal{A}v, u - v\rangle \geq \alpha [u - v]^p, \quad \forall\, u, v \in K. \tag{8}$$

Then, if u realizes the minimum of $\mathcal{F}$ on K we have

$$\frac{\alpha}{p}\,[u - v]^p \leq \mathcal{F}(u) - \mathcal{F}(v), \quad \forall\, v \in K. \tag{9}$$

Proof. Since u satisfies the variational inequality

$$\langle \mathcal{A}u, v - u\rangle \geq 0, \quad \forall\, v \in K, \tag{10}$$

(9) will be a consequence of the following general inequality:

$$\mathcal{F}(u) - \mathcal{F}(v) - \langle \mathcal{A}u, v - u\rangle \geq \frac{\alpha}{p}[v - u]^p, \quad \forall\, u, v \in K. \tag{11}$$

To prove (11), let u, v be a couple of vectors of K and let us define

$$g(t) := \mathcal{F}(u + t(v - u)), \quad t \in [0, 1].$$

g is a convex real function of class C^1 in the closed interval $[0, 1]$ [17, Ch. 2, Prop. 1.1], with

$$g'(t) = \langle \mathcal{A}(u + t(v - u)), v - u\rangle$$

We get

$$\begin{aligned}
\mathcal{F}(v) - \mathcal{F}(u) - \langle \mathcal{A}u, v - u\rangle &= g(1) - g(0) - g'(0) = \int_0^1 [g'(t) - g'(0)]\, dt \\
&= \int_0^1 \langle \mathcal{A}(u + t(v - u)) - \mathcal{A}u, t(v - u)\rangle \,\frac{dt}{t} \\
&\geq \alpha \int_0^1 t^p [v - u]^p \,\frac{dt}{t} = \frac{\alpha}{p}[v - u]^p \quad \blacksquare
\end{aligned}$$

Remark 2.1 *Let us make a few comments about the assumptions of the previous theorem.*

- First of all, we could consider weaker differentiability properties of $\mathcal{F}$: we chose this formulation for simplicity, since the statement of Theorem 1 is enough for our purposes.
- (11) is in fact equivalent to the p-coercivity of $\mathcal{A}$.
- Even if the previous calculations hold also for $1 \leq p < 2$, it is easy to see that no gradient operator $\mathcal{A}$ satisfies (8) w.r.t. a non-trivial seminorm: otherwise for every couple of vectors u, v such that $[u-v] > 0$ the function $g'(t)$ previously defined would be non decreasing but nowhere differentiable in $[0,1]$. □

Here is a possible way to apply Theorem 1: we will assign a family of maps

$$T_h : K \to K, \quad h \text{ varying in a given subset } D \subset \mathbb{R}^N, \tag{12}$$

and we will check that our functional $\mathcal{F}$ is (T,D)*-regular on* K in the sense that

$$\forall\, u \in K, \quad \omega(u) = \omega(u; \mathcal{F}, T, D) := \sup_{h \in D \setminus \{0\}} \frac{\mathcal{F}(T_h u) - \mathcal{F}(u)}{|h|} < +\infty. \tag{13}$$

If this will be the case, we will immediately deduce the following estimate.

Corollary 1 *Let us assume that* $\mathcal{A}$ *is p-coercive* (8) *and* $\mathcal{F}$ *is* (T,D)*-regular on* K (13)*. Then a minimum point* u *of* $\mathcal{F}$ *on* K *satisfies*

$$\alpha\, [u - T_h u]^p \leq p\, \omega(u)\, |h|, \quad \forall\, h \in D \setminus \{0\} \quad \blacksquare \tag{14}$$

Remark 2.2 We note that $\omega(\ldots)$ is subadditive w.r.t. its $\mathcal{F}$-argument, i.e.

$$\omega(u; \mathcal{F}' + \mathcal{F}'', T, D) \leq \omega(u; \mathcal{F}', T, D) + \omega(u; \mathcal{F}'', T, D).$$

Since we will be concerned with linear perturbations of a given functional $\mathcal{F}_0$, we will extensively use this property by studying separately the regularity of the linear and the nonlinear part of the functionals. □

In the following section we will make precise what kind of maps T_h we will consider; let us now list some basic examples of functionals $\mathcal{F}$ which satisfy the assumptions of Theorem 1.

Basic examples

In all the following examples we deal with Sobolev spaces V of functions defined in a connected *bounded Lipschitz* open set $\Omega \subset \mathbb{R}^N$ and, according to the previous remark, we will set

$$\mathcal{F}_f := \mathcal{F}_0 - \mathcal{L}_f, \quad \mathcal{L}_f(v) := \int_\Omega f(x)v(x)\,dx. \tag{15}$$

E 1. Let $V := H^1(\Omega)$, $K := H^1_0(\Omega)$, $[v] := \|\nabla v\|_{L^2(\Omega;\mathbb{R}^N)}$, $p := 2$, $f \in L^2(\Omega)$, and

$$\mathcal{F}_0(v) := \frac{1}{2}\int_\Omega A(x)\nabla v(x)\cdot\nabla v(x)\,dx, \tag{16}$$

where the matrices $A(x)$ satisfy (2); then $\mathcal{F}_f$ admits a unique minimum point u on K, which solves the Dirichlet problem (1).

E 2. As in the previous example, but with $K \equiv V \equiv H^1(\Omega)$; if the integral of f vanishes, then the minimum points of $\mathcal{F}_f$ (determined up to an additive constant) satisfy the elliptic equation of (1) with homogeneous Neumann boundary conditions.

E 3. Let $p \in]1,\infty[$, p' the conjugate exponent of p, $V := W^{1,p}(\Omega)$, $K := W^{1,p}_0(\Omega)$, $[v] := \|\nabla v\|_{L^p(\Omega;\mathbb{R}^N)}$, $f \in L^{p'}(\Omega)$, and

$$\mathcal{F}_0(v) := \frac{1}{p}\int_\Omega |\nabla v(x)|^p\,dx. \tag{17}$$

Then $\mathcal{F}_f$ admits a unique minimum point $u \in W^{1,p}_0(\Omega)$ [17, Ch. 2, 2.3.1], which satisfies the equation

$$-\operatorname{div}\left(|\nabla u(x)|^{p-2}\nabla u(x)\right) = f(x), \quad \text{in } \Omega. \tag{18}$$

More generally, let us define

$$\mathcal{F}_0(v) := \int_\Omega F(x,\nabla v(x))\,dx, \tag{19}$$

where $F(x,\boldsymbol{\xi}) : \overline{\Omega}\times\mathbb{R}^N \to \mathbb{R}$ is a continuous function which is convex and differentiable w.r.t. $\boldsymbol{\xi} \in \mathbb{R}^N$; we set $\boldsymbol{a} := \nabla_\xi F$. If F and $\boldsymbol{a}$ satisfy the usual p-growth conditions

$$\exists\,\mu > 0 : \quad |F(x,\boldsymbol{\xi})| \le \mu(1+|\boldsymbol{\xi}|^p), \quad |\boldsymbol{a}(x,\boldsymbol{\xi})| \le \mu(1+|\boldsymbol{\xi}|^{p-1}), \tag{20}$$

then $\mathcal{F}_0$ is well defined and its Gâteaux differential $\mathcal{A}_0 : V \to V'$ is given by (see e.g. [10], Chap. I, Thm. 5.1)

$$\langle \mathcal{A}_0 u, v\rangle := \int_\Omega \boldsymbol{a}(x, \nabla u(x)) \cdot \nabla v(x)\, dx, \quad \forall\, u, v \in W^{1,p}(\Omega). \tag{21}$$

We distinguish two cases:

(i) If $p \geq 2$ and, for every choice of $x \in \Omega$ and $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^N$, $\boldsymbol{a}$ satisfies

$$\exists\, \alpha > 0 : \quad (\boldsymbol{a}(x, \boldsymbol{\xi}) - \boldsymbol{a}(x, \boldsymbol{\eta})) \cdot (\boldsymbol{\xi} - \boldsymbol{\eta}) \geq \alpha\, |\boldsymbol{\xi} - \boldsymbol{\eta}|^p, \tag{22}$$

then $\mathcal{A}_0$ is also p-coercive on V (see (8)).

(ii) If $p < 2$ and $\boldsymbol{a}$ satisfies

$$\exists\, \alpha > 0 : \quad (\boldsymbol{a}(x, \boldsymbol{\xi}) - \boldsymbol{a}(x, \boldsymbol{\eta})) \cdot (\boldsymbol{\xi} - \boldsymbol{\eta}) \geq \alpha\, \frac{|\boldsymbol{\xi} - \boldsymbol{\eta}|^2}{(|\boldsymbol{\xi}| + |\boldsymbol{\eta}|)^{2-p}} \tag{23}$$

then $\mathcal{A}_0$ is 2-coercive on the *bounded* sets of $W^{1,p}(\Omega)$; actually, for every couple of vector field $\boldsymbol{\xi}(x), \boldsymbol{\eta}(x)$ in $L^p(\Omega; \mathbb{R}^N)$ we deduce from (23) and from Hölder inequality

$$\begin{aligned}
&\alpha^{p/2} \int_\Omega |\boldsymbol{\xi} - \boldsymbol{\eta}|^p\, dx \leq \\
&\leq \int_\Omega \Big[(\boldsymbol{a}(x, \boldsymbol{\xi}) - \boldsymbol{a}(x, \boldsymbol{\eta})) \cdot (\boldsymbol{\xi} - \boldsymbol{\eta})\Big]^{p/2} \Big[|\boldsymbol{\xi}| + |\boldsymbol{\eta}|\Big]^{p(2-p)/2} dx \leq \\
&\leq \left(\int_\Omega \Big((\boldsymbol{a}(x, \boldsymbol{\xi}) - \boldsymbol{a}(x, \boldsymbol{\eta})) \cdot (\boldsymbol{\xi} - \boldsymbol{\eta})\Big)\, dx\right)^{\frac{p}{2}} \left(\int_\Omega \Big(|\boldsymbol{\xi}| + |\boldsymbol{\eta}|\Big)^p\, dx\right)^{\frac{2-p}{2}}.
\end{aligned}$$

In both cases $\mathcal{F}_f$ admits a unique minimum point u on $W^{1,p}_0(\Omega)$, satisfying (6) in the sense of distribution.

We recall that for the particular choice (17) we have

$$\boldsymbol{a}(x, \boldsymbol{\xi}) = |\boldsymbol{\xi}|^{p-2}\boldsymbol{\xi},$$

which satisfies conditions *(i)* or *(ii)* above: see e.g. [26, p. 487], [24, (2.2)], or [8, Ch. I, 4-(iii)]

E 4. Let $V := H^2_0(\Omega)$, $f \in L^2(\Omega)$, and

$$\mathcal{F}_0(v) = \frac{1}{2}[v]^2 := \frac{1}{2} \int_\Omega \sum_{i,j=1}^N \left|\frac{\partial^2 v}{\partial x_i \partial x_j}\right|^2 dx. \tag{24}$$

The minimum point of $\mathcal{F}_f$ on $H^2_0(\Omega)$ solves the 4^{th}-order elliptic problem (ν being the exterior unit normal to $\partial\Omega$)

$$\begin{cases} \Delta^2 u &= f(x) \quad \text{in } \Omega, \\ u(x),\ \partial_\nu u(x) &= 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{25}$$

As in the previous example, we could consider more complicated functionals and equations involving higher order derivatives: since we do not claim any completeness, we preferred to explain the basic ideas of our method in simpler situations.

3 Local estimates

The aim of this section is to apply Corollary 1 in the framework of the difference-quotient technique. Roughly speaking, if u is a function belonging to one of the Sobolev spaces V on Ω previously quoted, we would like to choose $T_h u(x) \approx u(x+h)$, for suitable choices of $h \in D \subset \mathbb{R}^N$, in formulae (12,13,14): the behavior of $[u - T_h u]$ as h goes to 0 given by (14) would then tell us some further informations about the regularity of u. Of course, if Ω is not globally invariant[2] with respect to the translations of D, this choice is not allowed, so that we have to use a suitable localization procedure. First of all we fix some notation.

Notation 3.1 For every function $v : \Omega \to \mathbb{R}^M$, we will denote by v_* its trivial extension to 0 outside Ω and we set

$$v_h(x) := v_*(x+h), \quad \forall\, x, h \in \mathbb{R}^N.$$

$\rho \in]0,1]$ will indicate a given radius and x_0 a given point in $\mathbb{R}^N$; $\Omega_\rho(x_0)$ is the intersection $\Omega \cap B_\rho(x_0)$.
For every angle $\theta \in]0,\pi]$ and unitary vector $\boldsymbol{n} \in \mathbb{S}^{N-1}$ we will consider the cone with vertex at 0, height ρ, opening θ, and the axis pointing towards $\boldsymbol{n}$:

$$\mathcal{C}_\rho(\boldsymbol{n},\theta) := \Big\{ h \in \mathbb{R}^N : |h| \le \rho,\ h \cdot \boldsymbol{n} \ge |h| \cos\theta \Big\}. \tag{26}$$

We choose a C^∞ "cut-off" function $\phi = \phi_{x_0,\rho}$[3] centered at x_0 with support contained in $B_{2\rho}(x_0)$:

$$0 \le \phi(x) \le 1, \quad |D^k \phi(x)| \le C_k \rho^{-k}; \quad \phi(x) \equiv 1, \text{ on } B_\rho(x_0), \tag{27}$$

[2] i.e., if Ω does not satisfies $\Omega + h \subset \Omega$ (for Neumann-type boundary conditions) or $\Omega + h \supset \Omega$ (for homogeneous Dirichlet problems) for every $h \in D$.

[3] we will omit to indicate the explicit dependence on x_0, ρ, when no misunderstanding is possible.

Finally, once x_0, ρ, and ϕ are fixed, for every function $v : \Omega \to \mathbb{R}^M$ and every vector $h \in \mathbb{R}^N$, we define

$$T_h v := \phi v_h + (1-\phi)v. \quad \square \tag{28}$$

We try to apply Corollary 1 to each example of the previous section with this choice of T. First of all we have to determine a set $D := D_\rho(x_0) \subset \mathbb{R}^N$ ensuring that

$$v \in K, \quad h \in D_\rho(x_0) \quad \Rightarrow \quad T_h(v) \in K,$$

for every concrete choice of $K \subset V$.

In order to point out the local geometric properties of Ω related to this question, we introduce the following definitions:

Definition 3.2 *For every $x_0 \in \mathbb{R}^N$ and $\rho \in]0,1]$, $\mathcal{I}_\rho(x_0)$ is the set of the admissible inward vectors of magnitude less than ρ "near" x_0*

$$\mathcal{I}_\rho(x_0) := \Big\{ h \in \mathbb{R}^N : |h| \le \rho, \quad (B_{3\rho}(x_0) \cap \Omega) + h \subset \Omega \Big\},^4 \tag{29}$$

and $\mathcal{O}_\rho(x_0)$ is the set of the admissible outward vectors

$$\mathcal{O}_\rho(x_0) := \Big\{ h \in \mathbb{R}^N : |h| \le \rho, \quad (B_{3\rho}(x_0) \setminus \Omega) + h \subset \mathbb{R}^N \setminus \Omega \Big\}. \tag{30}$$

They are related by

$$\kappa \mathcal{I}_{\rho/\kappa}(x_0) \subset -\mathcal{O}_\rho(x_0) \subset \kappa^{-1} \mathcal{I}_{\kappa\rho}(x_0), \quad \forall \kappa \in]0, 3/4]. \quad \square \tag{31}$$

It is easy to see that, for every integer k

$$\begin{cases} u \in W^{k,p}(\Omega), & h \in \mathcal{I}_\rho(x_0) \quad \Rightarrow \quad T_h u \in W^{k,p}(\Omega), \\ u \in W^{k,p}_0(\Omega), & h \in \mathcal{O}_\rho(x_0) \quad \Rightarrow \quad T_h u \in W^{k,p}_0(\Omega). \end{cases} \tag{32}$$

When Ω is a bounded *Lipschitz* open set it is well known that Ω satisfies the *uniform cone property* (see [11], def. 1.2.2.1 and Thm. I.2.2.2); with our notation, this is equivalent to the following statement.

Proposition 3.3 *Assume Ω is a bounded Lipschitz open set; then there exist*

$$\bar\rho := \bar\rho(\Omega) \in]0,1], \quad \bar\theta := \bar\theta(\Omega) \in]0,\pi], \quad \text{and a map } \bar{\boldsymbol{n}} : \mathbb{R}^N \to \mathbb{S}^{N-1} \tag{33}$$

such that for every $x \in \mathbb{R}^N$

$$\mathcal{O}_{\bar\rho}(x) \supset \mathcal{C}_{\bar\rho}(\bar{\boldsymbol{n}}(x), \bar\theta), \quad \mathcal{I}_{\bar\rho}(x) \supset \mathcal{C}_{\bar\rho}(-\bar{\boldsymbol{n}}(x), \bar\theta). \quad \blacksquare \tag{34}$$

[4] When $B_{4\rho}(x_0) \subset \Omega$ or $B_{3\rho}(x_0) \cap \Omega = \emptyset$, then $\mathcal{I}_\rho(x_0) \equiv B_\rho(0)$. Of course, this notion is more expressive when $x_0 \in \partial\Omega$; however many further expression can be treated in a simpler unified way if we allow every $x_0 \in \mathbb{R}^N$ in this definition.

The main calculations are collected in the following proposition.

Proposition 3.4 *Let F be given as in the previous example* **E** 3*, and let us assume that it is Lipschitz with respect to x, with*

$$\exists\, L > 0 : \quad |F(x,\boldsymbol{\xi}) - F(y,\boldsymbol{\xi})| \le L\,|x-y|\,(1+|\boldsymbol{\xi}|^p), \quad \forall\, x,y \in \Omega, \quad \forall\, \boldsymbol{\xi} \in \mathbb{R}^N. \tag{35}$$

Then, for every $x_0 \in \mathbb{R}^N$ and $\rho \in]0,1]$, the corresponding functional $\mathcal{F}_0$ given by (19) *is $(T,\mathcal{O}_\rho(x_0))$-regular on $W^{1,p}_0(\Omega)$ and there exists a constant C depending only on p, L, μ, ρ, such that (recall* (13)*)*

$$\omega(u;\mathcal{F}_0,T,\mathcal{O}_\rho(x_0)) \le C\Bigg(1 + \int\limits_{\Omega_{3\rho}(x_0)} |\nabla u|^p\,dx\Bigg), \quad \forall\, u \in W^{1,p}_0(\Omega), \tag{36}$$

Proof. For simplicity we omit to indicate the dependence on x_0, taking it fixed, and we denote by the same letter C different constants depending only on p, L, μ, ρ. We split the calculations in three steps.

- For every $\boldsymbol{\eta}, \boldsymbol{\zeta} \in L^p(\Omega;\mathbb{R}^N)$, $\mathrm{supp}(\boldsymbol{\zeta}) \subset B_{2\rho}$, we have

$$\int_\Omega \Big[F(x,\boldsymbol{\eta}(x)+\boldsymbol{\zeta}(x)) - F(x,\boldsymbol{\eta}(x))\Big]\,dx \le$$
$$C\|\boldsymbol{\zeta}\|_{L^p(\Omega_{2\rho};\mathbb{R}^N)}\Big(1+\|\boldsymbol{\zeta}\|_{L^p(\Omega_{2\rho};\mathbb{R}^N)}+\|\boldsymbol{\eta}\|_{L^p(\Omega_{2\rho};\mathbb{R}^N)}\Big)^{p-1} \tag{37}$$

  To show (37) we use (20) obtaining

$$F(\boldsymbol{\eta}(x)+\boldsymbol{\zeta}(x)) - F(x,\boldsymbol{\eta}(x)) \le \mu\,|\boldsymbol{\zeta}(x)|\,(1+|\boldsymbol{\eta}(x)|+|\boldsymbol{\zeta}(x)|)^{p-1}$$

  and then we integrate on Ω applying Hölder inequality.

- For every vector field $\boldsymbol{\xi} \in L^p(\Omega;\mathbb{R}^N)$ and every $h \in \mathcal{O}_\rho(x_0)$ we get

$$\int_\Omega \Big[F(x,T_h\boldsymbol{\xi}(x)) - F(x,\boldsymbol{\xi}(x))\Big]\,dx \le C|h|\,\Big(1+\|\boldsymbol{\xi}\|^p_{L^p(\Omega_{3\rho};\mathbb{R}^N)}\Big). \tag{38}$$

  By the convexity of F and (28) we have, for almost every $x \in \Omega$,

$$\begin{aligned} &F(x,T_h\boldsymbol{\xi}(x)) - F(x,\boldsymbol{\xi}(x)) \le \\ &\quad\le (1-\phi(x))F(x,\boldsymbol{\xi}(x)) + \phi(x)F(x,\boldsymbol{\xi}_h(x)) - F(x,\boldsymbol{\xi}(x)) = \\ &\quad= \phi(x)\Big[F(x,\boldsymbol{\xi}_h(x)) - F(x,\boldsymbol{\xi}(x))\Big]. \end{aligned}$$

Let us denote again by F an extension of the function $\Omega \ni x \mapsto F(x,0)$ to the whole $\mathbb{R}^N$: we can assume that the Lipschitz constant of this extension does not exceed L [3, 3.1.1]. Since in $B_{3\rho} \setminus \Omega$ we have $\boldsymbol{\xi}_h \equiv \boldsymbol{\xi}_* \equiv 0$ by (30), recalling the support property of ϕ and integrating in Ω we get

$$
\begin{aligned}
&\int_\Omega [F(x,T_h\boldsymbol{\xi}) - F(x,\boldsymbol{\xi})]\,dx \le \int_{B_{2\rho}} \phi[F(x,\boldsymbol{\xi}_h) - F(x,\boldsymbol{\xi}_*)]\,dx = \\
&= \int_{B_{2\rho}+h} \phi(x-h)F(x-h,\boldsymbol{\xi}_*)\,dx - \int_{B_{2\rho}} \phi(x)F(x,\boldsymbol{\xi}_*)\,dx = \\
&= \int_{B_{3\rho}} \Big[\phi(x-h)F(x-h,\boldsymbol{\xi}_*) - \phi(x)F(x-h,\boldsymbol{\xi}_*)\Big]\,dx \\
&\qquad + \int_{B_{2\rho}} \phi(x)\Big[F(x-h,\boldsymbol{\xi}_*) - F(x,\boldsymbol{\xi}_*)\Big]\,dx \le \qquad (39)\\
&\le C|h|\Big(1 + \|\boldsymbol{\xi}\|^p_{L^p(\Omega_{3\rho};\mathbb{R}^N)}\Big). \qquad (40)
\end{aligned}
$$

- Now we deduce (36). Since the gradient of $T_h u$ is

$$
\nabla[T_h u] = \phi\nabla u_h + (1-\phi)\nabla u + \nabla\phi(u_h - u) = T_h\nabla u + \nabla\phi(u_h - u), \quad (41)
$$

  we have

$$
\begin{aligned}
&\mathcal{F}_0(T_h u) - \mathcal{F}_0(u) \le \\
&\qquad \int_\Omega \Big[F(x, T_h\nabla u + \nabla\phi(u_h - u)) - F(x, T_h\nabla u)\Big]\,dx + \\
&\qquad\qquad + \int_\Omega \Big[F(x, T_h\nabla u) - F(x, \nabla u)\Big]\,dx. \qquad (42)
\end{aligned}
$$

  The first integral can be estimated from above by (37), choosing $\boldsymbol{\eta} := T_h\nabla u$ and $\boldsymbol{\zeta} := \nabla\phi(u_h - u)$, and recalling that

$$
\|u_h - u\|_{L^p(\Omega_{2\rho})} \le |h|\,\|\nabla u\|_{L^p(\Omega_{3\rho};\mathbb{R}^N)} \qquad (43)
$$

  as $u_* \in W^{1,p}(\mathbb{R}^N)$.
  The second integral of (42) can be estimated by (38), choosing $\boldsymbol{\xi} := \nabla u$. ■

Remark 3.5 If F is non negative, the previous result holds even if we replace $W^{1,p}_0(\Omega)$ and $\mathcal{O}_\rho(x_0)$ with $W^{1,p}(\Omega)$ and $\mathcal{I}_\rho(x_0)$ respectively; the proof of this fact is based on almost the same calculations and on (32). The only changes occur in (40), which can be modified as follows:

$$\begin{aligned}
&\int_\Omega [F(x,T_h\boldsymbol{\xi})-F(x,\boldsymbol{\xi})]\,dx \le \int_{\Omega_{2\rho}} \phi[F(x,\boldsymbol{\xi}_h)-F(x,\boldsymbol{\xi})]\,dx = \\
&= \int_{\Omega_{2\rho}+h} \phi(x-h)F(x-h,\boldsymbol{\xi})\,dx - \int_{\Omega_{2\rho}} \phi(x)F(x,\boldsymbol{\xi})\,dx \le \\
&= \int_{\Omega_{2\rho}+h} \Big[\phi(x-h)F(x-h,\boldsymbol{\xi}) - \phi(x)F(x-h,\boldsymbol{\xi})\Big]\,dx \\
&\qquad + \int_{\Omega_{2\rho}+h} \phi(x)\Big[F(x-h,\boldsymbol{\xi})-F(x,\boldsymbol{\xi})\Big]\,dx \le \qquad (44)\\
&\le C|h|\Big(1+\|\boldsymbol{\xi}\|^p_{L^p(\Omega_{3\rho};\mathbb{R}^N)}\Big), \qquad (45)
\end{aligned}$$

since F is non negative and ϕ vanishes in $(\Omega_{2\rho}+h)\setminus\Omega_{2\rho}$. □

Remark 3.6 If we examine the proof of Proposition 3.4, we can check that the only occurrence of the Lipschitz assumption (35) on F is in (39) (in (44), for the Neumann boundary conditions) and these calculations still remain valid if we assume only a one side control on F along the direction h. More precisely, we suppose for the sake of simplicity that $F(x,0)\equiv 0$, [5] and we denote by $\mathcal{R}_{\rho,L}(x_0)$ the set of "regular directions"

$$\begin{aligned}
\mathcal{R}_{\rho,L}(x_0) \;:=\; \Big\{ & h\in B_\rho(0) : F(x-h,\boldsymbol{\xi})-F(x,\boldsymbol{\xi}) \le L|h|\,(1+|\boldsymbol{\xi}|^p), \\
& \forall\,\boldsymbol{\xi}\in\mathbb{R}^N, \quad \text{for a.e. } x\in\Omega_{2\rho}(x_0)\cap(\Omega+h)\Big\}. \qquad (46)
\end{aligned}$$

If we substitute $\mathcal{O}_\rho(x_0)$ ($\mathcal{I}_\rho(x_0)$) with $\mathcal{O}_\rho(x_0)\cap\mathcal{R}_{\rho,L}(x_0)$ (resp. $\mathcal{I}_\rho(x_0)\cap\mathcal{R}_{\rho,L}(x_0)$) then[6] (40) (resp. (45)) and consequently Proposition 3.4 (resp. Remark 3.5) hold without other changes. This extension will be useful in the discussion of the transmission problems in the fifth section. □

Remark 3.7 In the case of example **E** 1, (35) reduces to (3). □

To deal with example **E** 4, we only have to take account of a slightly more complicated situation.

[5] so that we can trivially extend $x\mapsto F(x,0)$ outside Ω, without assuming a global Lipschitz property

[6] (46) is enough to bound the integrand of (39): in fact, if $x\in B_{2\rho}\setminus\Omega$ then $\boldsymbol{\xi}_*$ vanishes together to the difference $F(x-h,\boldsymbol{\xi}_*)-F(x,\boldsymbol{\xi}_*)$, whereas $x\in\Omega_{2\rho}$ implies $x\in\Omega+h$ since $h\in\mathcal{O}_\rho(x_0)$.

Proposition 3.8 *The functional $\mathcal{F}_0$ defined by (24) is $(T, \mathcal{O}_\rho(x_0))$-regular and there exists a constant $C > 0$ depending only on ρ such that*

$$\omega(u; \mathcal{F}_0, T, \mathcal{O}_\rho(x_0)) \leq C \int_{\Omega_{3\rho}(x_0)} \Big[|\nabla u|^2 + |Hu|^2 \Big]\, dx, \tag{47}$$

where Hu denotes the Hessian-matrix of the function u.

Proof. We follow the same structure of the previous argument. We have

$$\begin{aligned} H(T_h v) = \ & \Big((v_h - v) H\phi + \nabla(v_h - v)\, \nabla^T \phi + \nabla\phi \nabla^T (v_h - v) \Big) \\ & + \phi H v_h + (1-\phi) H v = Z_h + T_h H v \end{aligned} \tag{48}$$

where $Z_h(x)$ is supported in $B_{2\rho}(x_0)$ and

$$|Z_h| \leq C(|v_h - v| + |\nabla v_h - \nabla v|). \tag{49}$$

As in (42) we perform the splitting

$$\begin{aligned} \mathcal{F}_0(T_h v) - \mathcal{F}_0(v) \ &= \ \frac{1}{2} \int_\Omega \Big[\, |T_h H v + Z_h|^2 - |T_h H v|^2 \,\Big]\, dx + \\ & \quad + \frac{1}{2} \int_\Omega \Big[\, |T_h H v|^2 - |Hv|^2 \,\Big]\, dx, \end{aligned} \tag{50}$$

and we estimate separately these last two integrals.
By (49) we have

$$\begin{aligned} & |T_h H v + Z_h|^2 - |T_h H v|^2 \leq \\ & \leq C\Big(|v_h - v| + |\nabla(v_h - v)| \Big) \Big(|v_h - v| + |\nabla(v_h - v)| + |Hv| + |Hv_h| \Big), \end{aligned} \tag{51}$$

and integrating on Ω

$$\int_\Omega \Big[\, |T_h H v + Z_h|^2 - |T_h H v|^2 \,\Big]\, dx \leq C|h| \int_{\Omega_{3\rho}(x_0)} \Big[|\nabla u|^2 + |Hu|^2 \Big], \tag{52}$$

where we used the well known estimates for a function $v \in H^2_0(\Omega)$

$$\|v_h - v_*\|_{L^2(B_{2\rho}(x_0))} \ \leq \ |h|\, \|\nabla v_*\|_{L^2(B_{3\rho}(x_0);\mathbb{R}^N)}, \tag{53}$$

$$\|\nabla v_h - \nabla v_*\|_{L^2(B_{2\rho}(x_0);\mathbb{R}^N)} \ \leq \ |h|\, \|H v_*\|_{L^2(B_{3\rho}(x_0);\mathbb{M}^{N\times N})}. \tag{54}$$

Finally, by the convexity of the norm and the support property of ϕ, the last integral of (50) satisfies

$$\begin{aligned} \int_\Omega \Big[\, |T_h H v|^2 - |Hv|^2 \,\Big]\, dx &\leq \int_{B_{2\rho}(x_0)} \phi(x) \Big(\, |Hv_h|^2 - |Hv_*|^2 \Big)\, dx = \\ &= \int_{B_{3\rho}(x_0)} \Big(\phi(x-h) - \phi(x) \Big) |Hv_*|^2 \, dx \leq C|h| \int_{B_{3\rho}(x_0)} |Hv_*|^2 \, dx \quad \blacksquare \end{aligned}$$

4 Application: linear and quasilinear elliptic equation of second order.

Let us briefly recall the definition and the basic properties of the intermediate Sobolev-Besov spaces we need (for a complete treatment of the relative theory, we refer to [1, 20, 28]). We shall extensively use the real interpolation functor $(\cdot,\cdot)_{s,q}$ [2, 1, 28].

Intermediate Sobolev-Besov spaces.

B 1. *Definition.* Let $s \in]0,1[,\ p,q \in [1,\infty]$; we define

$$B^s_{p,q}(\Omega) := (L^p(\Omega), W^{1,p}(\Omega))_{s,q}, \quad B^{-s}_{p,q}(\Omega) := (L^p(\Omega), W^{-1,p}(\Omega))_{s,q}$$
$$B^{1+s}_{p,q}(\Omega) := (W^{1,p}(\Omega), W^{2,p}(\Omega))_{s,q} = \left\{ u \in W^{1,p}(\Omega) : \nabla u \in B^s_{p,q}(\Omega;\mathbb{R}^N) \right\}$$

with the well known particular cases

$$W^{s,p}(\Omega) := B^s_{p,p}(\Omega), \quad H^s(\Omega) := W^{s,2}(\Omega) = B^s_{2,2}(\Omega).$$

B 2. *Difference quotients.* Let us denote by Ω_λ, $\lambda > 0$, the set of points $x \in \Omega$ whose distance from $\partial\Omega$ is greater than λ and let D be a set generating $\mathbb{R}^N$ and star-shaped with respect to 0. For $s \in]0,1[,\ p \in [1,+\infty]$, we consider the seminorm

$$[u]^p_{s,p;\Omega} := \sup_{h\in D\setminus\{0\}} \int_{\Omega_{|h|}} \left| \frac{u(x+h)-u(x)}{|h|^s} \right|^p dx \tag{55}$$

which characterizes $B^s_{p,\infty}(\Omega)$ in the sense that [7].

$$u \in B^s_{p,\infty}(\Omega) \ \Leftrightarrow\ u \in L^p(\Omega) \text{ and } [u]_{s,p;\Omega} < +\infty. \tag{56}$$

Moreover, there exist positive constants C_0, C_1 depending only on s, p, Ω, and D such that

$$C_0\|u\|_{B^s_{p,\infty}(\Omega)} \le \|u\|_{L^p(\Omega)} + [u]_{s,p;\Omega} \le C_1\|u\|_{B^s_{p,\infty}(\Omega)}$$

If $\Omega := B_\rho(x_0)$ then these constants do not change if we substitute D with $D' := QD$, Q being an orthogonal matrix. In particular they are independent on the choice of $\boldsymbol{n}$, when D is a cone $\mathcal{C}_\rho(\theta,\boldsymbol{n})$.

[7] When $\Omega = \mathbb{R}^N$ the equivalence between these and other characterizations follows from general interpolation results: see e.g. [1, Ch. 6]; in the general case, we employ the extension results of [20, def. 4.3.4 and Th.. 1, p. 381] or of [7].

B 3. *Duality.* For $s \in]0,1[$, $p \in [1,+\infty[$, and $q \in [1,+\infty]$, we define

$$\dot{B}^s_{p,q}(\Omega) := \Big\{ u \in B^s_{p,q}(\Omega) : u_* \in B^s_{p,q}(\mathbb{R}^N) \Big\} = (L^p(\Omega), W^{1,p}_0(\Omega))_{s,q}, \tag{57}$$

which satisfies, when $p, q > 1$

$$\dot{B}^s_{p,q}(\Omega) = \Big(B^{-s}_{p',q'}(\Omega) \Big)'. \tag{58}$$

B 4. *Localization.* If $\{U_j\}_{j=1,\ldots,m}$ is a finite collection of open balls covering Ω, then a function v belongs to $B^s_{p,q}(\Omega)$ if and only if $v_{|\Omega \cap U_j} \in B^s_{p,q}(\Omega \cap U_j)$ for every $j = 1, \ldots, m$, and there exist constants C_0, C_1 depending only on m such that

$$C_0 \|v\|^p_{B^s_{p,q}(\Omega)} \le \sum_{j=1}^m \|v\|^p_{B^s_{p,q}(\Omega \cap U_j)} \le C_1 \|v\|^p_{B^s_{p,q}(\Omega)}. \tag{59}$$

B 5. *Reiteration.* For every $s \in]0,1]$, $p, q \in [1,+\infty]$, and $\lambda \in]0,1[$ we have

$$\begin{gathered}(W^{1,p}(\Omega), B^{1+s}_{p,q}(\Omega))_{\lambda,p} = W^{1+\lambda s,p}(\Omega), \\ (W^{-1,p}(\Omega), B^{-1+s}_{p,q}(\Omega))_{\lambda,p} = W^{-1+\lambda s,p}(\Omega). \quad \square\end{gathered}$$

We conclude by recalling two useful properties: the first one is a particular case of the nonlinear interpolation results of Tartar [26, Theorem 1].

Proposition 4.1 (Tartar, [26]) *Let us given two couples of Banach spaces $E_0 \subset E_1$, $F_0 \subset F_1$ (the inclusions are continuous) and an open subset U of E_1. Let $\mathcal{T} : U \to F_1$ be an operator mapping $E_0 \cap U$ into F_0 and let us assume that there exist $p \in [2,\infty[$ and positive constants c_0, c_1 such that*

$$\begin{aligned} &u \in U \cap E_0 &&\Rightarrow \quad \|\mathcal{T}u\|^p_{F_0} \le c_0 \Big(1 + \|u\|^{p'}_{E_0} \Big), \\ &u, v \in U &&\Rightarrow \quad \|\mathcal{T}u - \mathcal{T}v\|^p_{F_1} \le c_1 \|u - v\|^{p'}_{E_1}; \end{aligned} \tag{60}$$

then for every $\sigma \in]0,1[$

$$\mathcal{T} \text{ maps } U \cap (E_0, E_1)_{\sigma,p'} \text{ into } (F_0, F_1)_{\sigma,p}. \quad \blacksquare \tag{61}$$

The second property we are recalling follows by the same arguments of [1, 3.5(b)].

Proposition 4.2 *Suppose that $E_0 \subset E_1$ is a couple of Banach spaces, the inclusion being continuous, and suppose that $\mathcal{T}$ is a linear bounded operator mapping E_0 into a Banach space F and there exist $C > 0$ and $\sigma \in]0,1[$ such that*

$$\|\mathcal{T}e\|_F \leq C\|e\|_{E_0}^{1-\sigma}\,\|e\|_{E_1}^{\sigma}, \quad \forall\, e \in E_0. \tag{62}$$

Then $\mathcal{T}$ can be continuously extended to a bounded linear operator between $(E_0, E_1)_{\sigma,1}$ and F. ■

Regularity results.

Now we have all the elements to state our main results.

Theorem 2 *Let Ω be a Lipschitz bounded open set, $p \in [2, +\infty[$, and $u \in W_0^{1,p}(\Omega)$ be the solution of*

$$-\operatorname{div} \boldsymbol{a}(x, \nabla u) = f \in W^{-1,p'}(\Omega), \tag{63}$$

under the assumptions (20), (22), *and* (35). *If f belongs to $L^{p'}(\Omega)$ we have $u \in B^{1+1/p}_{p,\infty}(\Omega)$ and the same regularity holds even if $f \in B^{-1+1/p'}_{p',1}(\Omega)$. Moreover, we have*

$$f \in W^{-1+\lambda/p',p'}(\Omega) \;\Rightarrow\; u \in W^{1+\lambda/p,p}(\Omega), \quad \forall\,\lambda \in [0,1[\,. \tag{64}$$

Proof. First of all we note that if $f \in L^{p'}(\Omega)$ then $\mathcal{L}_f$ is $(T, \mathcal{O}_\rho(x_0))$ regular for every x_0 and ρ, since by (43) we get

$$\begin{aligned}\langle \mathcal{L}_f, T_h u - u\rangle &= \int_\Omega \phi\, f\,(u_h - u)\,dx \leq \|f\|_{L^{p'}(\Omega_{2\rho}(x_0))}\|u_h - u_*\|_{L^p(B_{2\rho}(x_0))}\\ &\leq |h|\,\|f\|_{L^{p'}(\Omega_{2\rho}(x_0))}\|\nabla u_*\|_{L^p(B_{3\rho}(x_0);\mathbb{R}^N)}.\end{aligned} \tag{65}$$

Taking Proposition 3.4, (34), Remark 2.2, and Corollary 1 into account, we deduce the estimate

$$\begin{aligned}&\int_{B_{\bar\rho}(x_0)} \Big|\nabla u_h - \nabla u_*\Big|^p dx \leq\\ &\leq C\,|h|\Big\{1 + \|\nabla u\|^p_{L^p(\Omega_{3\bar\rho}(x_0);\mathbb{R}^N)} + \|f\|_{L^{p'}(\Omega_{3\bar\rho}(x_0))}\|\nabla u\|_{L^p(\Omega_{3\bar\rho}(x_0);\mathbb{R}^N)}\Big\},\end{aligned} \tag{66}$$

for every $x_0 \in \mathbb{R}^N$ and $h \in \mathcal{C}_{\bar\rho}(\bar\theta, \boldsymbol{n}(x_0))$. By the characterization **B** 2, choosing $D := \mathcal{C}_{\bar\rho}(\bar\theta, \boldsymbol{n}(x_0))$ we deduce

$$[\nabla u_*]^p_{1/p,p;B_{\bar\rho}(x_0)} \leq C\Big\{1 + \|\nabla u\|^p_{L^p(\Omega_{3\bar\rho}(x_0))} + \|f\|_{L^{p'}(\Omega_{3\bar\rho}(x_0))}\|\nabla u\|_{L^p(\Omega_{3\bar\rho}(x_0))}\Big\}, \tag{67}$$

and, covering Ω by a finite number of balls of radius $\bar\rho$, by the localization property **B** 4 we infer $\nabla u \in \dot B^{1/p}_{p,\infty}(\Omega)$ and the global bound

$$\|\nabla u\|^p_{\dot B^{1/p}_{p,\infty}(\Omega)} \le C\Big\{1 + \|f\|^{p'}_{W^{-1,p'}(\Omega)} + \|f\|_{L^{p'}(\Omega)}\|f\|^{p'-1}_{W^{-1,p'}(\Omega)}\Big\}. \qquad (68)$$

Here the constant C only depends on $\bar\rho(\Omega), \bar\theta(\Omega), \mathrm{diam}(\Omega)$, and we used the boundedness estimate for (63)

$$\|\nabla u\|^p_{L^p(\Omega)} \le C\Big(1 + \|f\|^{p'}_{W^{-1,p'}(\Omega)}\Big). \qquad (69)$$

To prove the second part of the statement, we refine (65); first of all, **B** 3 implies

$$\langle \mathcal{L}_f, T_h u - u\rangle \le \|f\|_{B^{-1/p}_{p',1}(\Omega)}\|T_h u - u\|_{\dot B^{1/p}_{p,\infty}(\Omega)}. \qquad (70)$$

Since for every $h \in \mathcal{O}_\rho(x_0)$ we have

$$\|T_h u - u\|_{L^p(\Omega)} \le C|h|\,\|u\|_{W^{1,p}_0(\Omega)}, \quad \|T_h u - u\|_{W^{1,p}_0(\Omega)} \le C|h|\,\|u\|_{W^{2,p}_0(\Omega)}, \qquad (71)$$

by interpolation we get

$$\|T_h u - u\|_{\dot B^{1/p}_{p,\infty}(\Omega)} \le C|h|\,\|u\|_{\dot B^{1+1/p}_{p,\infty}(\Omega)}, \quad \forall\, h \in \mathcal{O}_\rho(x_0). \qquad (72)$$

Repeating the previous arguments, we easily find that the last term in the right-hand side of (66,68) can be replaced by the product

$$\|f\|_{B^{-1/p}_{p',1}(\Omega)}\,\|u\|_{\dot B^{1+1/p}_{p,\infty}(\Omega)}$$

and, if $f \in L^{p'}(\Omega)$, we obtain the estimate

$$\|u\|^p_{\dot B^{1+1/p}_{p,\infty}(\Omega)} \le C\Big\{1 + \|f\|^{p'}_{B^{-1+1/p'}_{p',1}(\Omega)}\Big\}. \qquad (73)$$

Since $L^{p'}$ is dense in $B^{-1+1/p'}_{p',1}(\Omega)$, $\dot B^{1+1/p}_{p,\infty}(\Omega)$ is a dual Banach space by (58), and the map

$$\mathcal{T} : f \in W^{-1,p'}(\Omega) \to u \in W^{1,p}_0(\Omega), \quad u \text{ is the solution of (63)} \qquad (74)$$

is continuous, we conclude by a standard approximation argument.[8]

[8] In the linear case, estimate (73) follows directly by (68) applying Proposition 4.2.

Finally, (64) follows by applying Proposition 4.1 to the operator (74), and by the reiteration property **B** 5. The employed estimates are (73) and the well known Hölder property of $\mathcal{T}$,

$$\|\mathcal{T}f-\mathcal{T}g\|^p_{W^{1,p}_0(\Omega)} \le c\|f-g\|^{p'}_{W^{-1,p'}(\Omega)}, \quad \forall\, f,g \in W^{-1,p'}(\Omega), \tag{75}$$

which is a direct consequence of (22). ■

We state the analogous version for $p \le 2$:

Theorem 2' *Let* Ω *be a Lipschitz bounded open set,* $p \in]1,2]$*, and* $u \in W^{1,p}_0(\Omega)$ *be the solution of*

$$-\operatorname{div} \boldsymbol{a}(x,\nabla u) = f \in W^{-1,p'}(\Omega), \tag{76}$$

under the assumptions (20)*,* (23)*, and* (35)*. If* f *belongs to* $L^{p'}(\Omega)$ *we have* $u \in B^{1+1/2}_{p,\infty}(\Omega)$ *and the same regularity holds even if* $f \in B^{-1+1/2}_{p',1}(\Omega)$*. Moreover, we have*

$$f \in W^{-1+s,p'}(\Omega) \;\Rightarrow\; u \in W^{1+s,p}(\Omega), \quad \forall\, s \in [0,1/2[\,. \tag{77}$$

Proof. We adapt the previous arguments to this case, without repeating the details of the proof. Taking account of **E** 3 *(ii),* we know that the differential operator (21) is 2-coercive on the bounded sets of $W^{1,p}_0(\Omega)$; moreover, the maps T_h defined by (28) are *uniformly* bounded in $W^{1,p}_0(\Omega)$. Setting

$$K_R := \{v \in W^{1,p}_0(\Omega) : \|\nabla v\|_{L^p(\Omega;\mathbb{R}^N)} < R\}, \tag{78}$$

we know that for every $R>0$ there exists $R'>0$ such that $\|f\|_{W^{-1,p'}(\Omega)} < R'$ implies $u \in K_R$, and $T_h u \in K_R$, for every choice of $x_0 \in \mathbb{R}^N$ and $h \in \mathcal{O}_{\bar\rho}(x_0)$. Of course, if $f \in L^{p'}(\Omega)$, u is a minimum point for $\mathcal{F}_f$ on K_R; arguing as in the previous proof, we can deduce $u \in B^{3/2}_{p,\infty}(\Omega)$ and the formula, analogous to (73),

$$\|u\|^2_{\dot B^{1+1/2}_{p,\infty}(\Omega)} \le C_R\Big\{1+\|f\|^2_{B^{-1/2}_{p',1}(\Omega)}\Big\}, \tag{79}$$

where, of course, the constant C_R depends also on the omitted quantities $\bar\rho(\Omega), \bar\theta(\Omega)$, and $\operatorname{diam}(\Omega)$, besides R.

Since now the operator $\mathcal{T}$ is locally Lipschitz in $W^{-1,p'}(\Omega)$, i.e. for every $f,g \in W^{-1,p'}(\Omega)$ with norm less than R',

$$\|\mathcal{T}f-\mathcal{T}g\|_{W^{1,p}_0(\Omega)} \le c_{R'}\|f-g\|_{W^{-1,p'}(\Omega)}, \tag{80}$$

(77) follows from (79) and (80) by applying Proposition 4.1. ■

Remark 4.3 The regularity results (64, 77) are optimal, also for smooth open sets: cf. [24]. □

In the linear case we can easily deduce further informations.

Theorem 3 *Let Ω be a bounded Lipschitz open set and let us assume that* (2) *and* (3) *hold. For every $s \in]-1/2, 1/2[$, if*

$$f \in H^{-1+s}(\Omega), \quad g \in H^{1/2+s}(\partial\Omega), \tag{81}$$

the non-homogeneous Dirichlet problem

$$\begin{cases} -\operatorname{div} A(x)\nabla u(x) &= f(x) \quad \textit{in } \Omega, \\ u(x) &= g(x) \quad \textit{on } \partial\Omega, \end{cases} \tag{82}$$

admits a unique solution $u \in H^{1+s}(\Omega)$.

Proof. When $g \equiv 0$ and $s > 0$ the thesis is a particular case of the previous results; when $s < 0$ it follows by a standard transposition technique (see e.g. [18], Chap. 2, 6.1).

The general case $g \neq 0$ is an immediate consequence of the trace results [11, Thoerem 1.5.1.1]. ■

Remark 4.4 When $g \equiv 0$ and $f \in B^{-1/2}_{2,1}(\Omega) \supset L^2(\Omega)$, then u_* belongs to $B^{3/2}_{2,\infty}(\mathbb{R}^N)$ and there exists a constant C such that

$$\|\nabla u_*\|^2_{B^{1/2}_{2,\infty}(\mathbb{R}^N)} \leq C\|f\|_{B^{-1/2}_{2,1}(\Omega)}. \tag{83}$$

Moreover, if f belongs to $L^2(\Omega)$, we get the interpolation estimate

$$\|\nabla u_*\|^2_{B^{1/2}_{2,\infty}(\mathbb{R}^N)} \leq C\|\nabla u\|_{L^2(\Omega)}\|f\|_{L^2(\Omega)}, \tag{84}$$

which could be useful in the study of parabolic problems in non cylindrical Lipschitz domains (cf. [9] and [23]). It is important to note that in both cases the constant C depends only on $\bar{\theta}(\Omega), \bar{\rho}(\Omega)$, and on the diameter of Ω. □

The analogous regularity results for the equations with boundary conditions of Neumann type follow by the same arguments, thanks to Remark 3.5. Here we only consider the linear case.

Theorem 4 *Let Ω be a Lipschitz bounded open set, let ν be the exterior unit normal to its boundary, and let us assume that* (2) *and* (3) *hold; then, for every $s \in]-1/2, 1/2[$, if*

$$f \in L^2(\Omega), \quad g \in H^{-1/2+s}(\partial\Omega), \tag{85}$$

the non-homogeneous Neumann problem (with $\lambda > 0$ and $\nu_A := A\nu$)

$$\begin{cases} -\operatorname{div} A(x)\nabla u(x) + \lambda\, u &= f(x) \quad \text{in } \Omega \\ \partial_{\nu_A} u(x) &= g(x) \quad \text{on } \partial\Omega, \end{cases} \tag{86}$$

admits a unique solution $u \in H^{1+s}(\Omega)$.

Proof. Let us first assume $g \equiv 0$; thanks to remark (3.5), the same argument of Theorem 2 shows that

$$f \in L^2(\Omega) \;\Rightarrow\; u \in B^{3/2}_{2,\infty}(\Omega), \quad \text{with} \quad \|\nabla u\|^2_{B^{1/2}_{2,\infty}(\Omega);\mathbb{R}^N} \le C\|f\|_{L^2(\Omega)}\|u\|_{H^1(\Omega)}. \tag{87}$$

As usual, let us identify $L^2(\Omega)$ with its dual, so that it can be densely injected into the dual space of $H^1(\Omega)$: in this way, $H^1(\Omega) \subset L^2(\Omega) \subset (H^1(\Omega))'$ becomes a Hilbert triplet and we can consider the linear map

$$\mathcal{G} : \ell \in (H^1(\Omega))' \mapsto u \in H^1(\Omega)$$

defined by the variational equation

$$\int_\Omega \Big[A(x)\nabla u(x)\cdot\nabla v(x) + \lambda u(x)v(x)\Big]\,dx = {}_{H^1(\Omega)'}\langle \ell, v\rangle_{H^1(\Omega)} \quad \forall\, v \in H^1(\Omega). \tag{88}$$

Then (87) says that

$$\ell \in L^2(\Omega) \;\Rightarrow\; \mathcal{G}\ell \in B^{3/2}_{2,\infty}(\Omega), \quad \|\mathcal{G}\ell\|_{B^{3/2}_{2,\infty}(\Omega)} \le C\|\ell\|_{L^2(\Omega)}\|\ell\|_{(H^1(\Omega))'} \tag{89}$$

and by Proposition 4.2,

$$\ell \in \Big((H^1(\Omega))', L^2(\Omega)\Big)_{1/2,1} \;\Rightarrow\; \mathcal{G}\ell \in B^{3/2}_{2,\infty}(\Omega). \tag{90}$$

Applying the Reiteration and Duality theorems (cf. [1] 3.5 and 3.7) we deduce that, for every $s \in]0, 1/2[$

$$\ell \in \Big((H^1(\Omega))', L^2(\Omega)\Big)_{s,2} = \Big(H^{1-s}(\Omega)\Big)' \;\Rightarrow\; \mathcal{G}\ell \in H^{1+s}(\Omega). \tag{91}$$

Finally, choosing ℓ of the type

$$\langle \ell, v\rangle := \int_\Omega f v\, dx + {}_{H^{-1/2}(\partial\Omega)}\langle g, v\rangle_{H^{1/2}(\partial\Omega)}\,, \quad \forall\, v \in H^1(\Omega), \tag{92}$$

with $f \in L^2(\Omega)$ and $g \in H^{-1/2+s}(\partial\Omega)$, we have $\ell \in (H^{1-s}(\Omega))'$ and by (91) we prove the theorem for $s > 0$. The remaining cases follow by transposition again. $\blacksquare$

Remark 4.5 The regularity assumption on f of (85) is surely not optimal; $f \in H^{-1/2}(\Omega)$ would have been sufficient[9]. Another possible choice is the family of spaces $\Xi^{-1+s}(\Omega)$ introduced by [18, Ch. 2, 6.3]. $\square$

5 Application: transmission problems.

In this section we show that in some cases the global Lipschitz assumption with respect to x on the integrand $F(x, \boldsymbol{\xi})$ can be weakened, in order to consider problems of transmission type through a Lipschitz interface, as in the following simple model.

Let us suppose that the Lipschitz domain Ω is the disjoint union of two Lipschitz bounded open sets Ω_1, Ω_2 and their common interface $\Gamma := \partial\Omega_1 \cap \partial\Omega_2 \cap \Omega$, oriented from Ω_1 to Ω_2 by the unit normal ν. We assign two positive constant $0 < \alpha_1 < \alpha_2$, two functions $f_i \in L^2(\Omega_i)$, $i = 1, 2$, and we look for a couple u_1, u_2 solving

$$\begin{cases} -\alpha_i \Delta u_i(x) &= f_i(x) & \text{in } \Omega_i, \\ u_1(x) &= u_2(x) & \text{on } \Gamma, \\ \alpha_1 \partial_\nu u_1 &= \alpha_2 \partial_\nu u_2 & \text{on } \Gamma, \\ u_i(x) &= 0 & \text{on } \partial\Omega_i \cap \partial\Omega. \end{cases} \tag{93}$$

Setting

$$F(x, \boldsymbol{\xi}) := \frac{1}{2}\sum_{i=1}^2 \alpha_i \chi_{\Omega_i}(x)|\boldsymbol{\xi}|^2, \quad f(x) := \sum_{i=1}^2 f_i(x)\chi_{\Omega_i}(x), \tag{94}$$

the variational formulation of (93) has the same form of the examples $\mathbf{E}\,1, 3$, i.e. the global solution $u(x) := \sum_{i=1}^2 u_i(x)\chi_{\Omega_i}(x)$ is the minimum point in

[9] This regularity is also sufficient to give a "meaning" to the conormal derivative, following the same ideas of [18, Ch. 2, Rem. 6.2], and [11, Thm. 1.53.10]. I wish to thank G. Gilardi for suggesting me this remark.

$H^1_0(\Omega)$ of

$$\begin{aligned}\mathcal{F}_f(v) &:= \int_\Omega \Big\{ F(x, \nabla v(x)) - f(x)\, v(x) \Big\}\, dx \\ &= \sum_{i=1}^2 \int_{\Omega_i} \Big\{ \frac{\alpha_i}{2} |\nabla v(x)|^2 - f_i(x) v(x) \Big\}\, dx \end{aligned} \tag{95}$$

but, of course, F does not satisfy (35) since $\alpha_1 < \alpha_2$.

In this case u does not belong to $H^2(\Omega)$, due to the jump of its normal derivative across Γ: in the family of the real interpolation spaces between $H^1(\Omega)$ and $H^2(\Omega)$, $B^{3/2}_{2,\infty}(\Omega)$ is the maximal regularity which is compatible with this kind of discontinuity. When Γ is of class $C^{1,1}$ this regularity is a consequence of the results of G. Stampacchia [25] (cf. also the bibliographycal notes of [18], I Ch. 2, 10.3), at least far from the junction points $\partial'\Gamma := \bar{\Gamma} \cap \partial\Omega$: by the same methods we discussed in the Introduction, he proved that

$$u_i \in H^2(\Omega_i \cap \Omega'), \text{ for every open set } \Omega' \subset \Omega \text{ with } \overline{\Omega'} \cap \partial'\Gamma = \emptyset. \tag{96}$$

We shall see how the local estimates of section 2 can be employed to prove the optimal Besov-type regularity even if Γ is only Lipschitz and therefore (96) is no longer true in general; if a suitable geometric compatibility condition between Ω and Ω_1 is satisfied (see also Remark 5.1) the regularity holds up to $\partial\Omega$.

Theorem 5 *Let $\Omega, \Omega_1, \Omega_2$, and Γ be given as described before, and let $u \in H^1_0(\Omega)$ be the solution of (93) with $f_i \in L^2(\Omega_i)$. Then for every open set $\Omega' \subset \Omega$ such that $\overline{\Omega'}$ does not intersect $\partial'\Gamma$ we have $u \in B^{3/2}_{2,\infty}(\Omega')$ and*

$$f \in H^{-1+s}(\Omega) \;\Rightarrow\; u \in H^{1+s}(\Omega'), \quad \forall\, s \in]0, 1/2[\,. \tag{97}$$

Moreover, these results hold globally (i.e. we can replace Ω' with the whole Ω) if for every $x_0 \in \partial'\Gamma$ there exists a cone $\mathcal{C} := \mathcal{C}_\rho(\theta, \boldsymbol{n})$ with $\rho, \theta > 0$, such that

$$\mathcal{C} \subset \mathcal{O}_\rho(x_0) \cap \mathcal{O}^1_\rho(x_0), \tag{98}$$

where $\mathcal{O}_\rho(x_0)$ and $\mathcal{O}^1_\rho(x_0)$, are the sets of the admissible outward directions defined by (30) with respect to Ω and Ω_1 respectively.

Proof. By repeating the same arguments of the proof of Theorem 2 and taking account of Remark 3.6, the thesis follows if for every point $x_0 \in \overline{\Omega'}$

we can find a radius $\rho > 0$ such that the set of "regular outward directions" $\mathcal{O}_\rho(x_0) \cap \mathcal{R}_{\rho,L}(x_0)$ (cf. (46)) contains a non-degenerate cone; a standard covering technique allows then to piece together all the local estimates. Of course, only the points x_0 of $\overline{\Gamma}$ have to be checked, since F is locally constant outside.

Let x be a point of $B_{2\rho}(x_0) \cap \Omega$ and h a vector of $\mathcal{O}^1_\rho(x_0)$; if $x - h \in \Omega_2$ then $x \in \Omega_2$, too, by the definition (30) of $\mathcal{O}^1_\rho(x_0)$: in particular, the difference $F(x-h, \boldsymbol{\xi}) - F(x, \boldsymbol{\xi})$ of (46) vanishes, whereas it is surely non positive if $x - h \in \Omega_1$, since $\alpha_1 < \alpha_2$. This elementary fact implies that $\mathcal{R}_{\rho,0}(x_0)$ contains $\mathcal{O}^1_\rho(x_0)$, for every $x_0 \in \overline{\Gamma}$.

Now we distinguish two cases. If $x_0 \in \Gamma \subset \Omega$ then there exists a ball $B_{3\rho}(x_0)$ completely contained in Ω and consequently

$$\mathcal{O}_\rho(x_0) \cap \mathcal{R}_{\rho,0}(x_0) = \mathcal{R}_{\rho,0}(x_0) \supset \mathcal{O}^1_\rho(x_0).$$

Since Ω_1 is Lipschitz, by 3.3 we are able to find a non degenerate cone $\mathcal{C} \subset \mathcal{O}_\rho(x_0) \cap \mathcal{R}_{\rho,0}(x_0)$: as we said just before, this fact guarantees the local estimates and (97).

Finally, if $x_0 \in \partial'\Gamma$ then we invoke (98) to obtain the same conclusion. ■

Remark 5.1 It is not difficult to see that (98) always holds if Ω is locally of class C^1 or convex near the points of $\partial'\Gamma$. In the first case, we observe that if x_0 is a regular point of $\partial\Omega$ and ν is the outward unitary normal to $\partial\Omega$ at x_0, then $\mathcal{O}_\rho(x_0)$ contains a cone $\mathcal{C}_\rho(\theta, \nu)$ whose opening θ tends to $\pi/2$ as ρ goes to 0. On the other hand, $\mathcal{O}^1_\rho(x_0)$ is surely contained in the half-space $\{h \in \mathbb{R}^N : \ h \cdot \nu \geq 0\}$ so that (98) is satisfied for a suitable small $\rho > 0$.
In the convex case, we choose a cone $\mathcal{C}_\rho(\theta, \boldsymbol{n})$ contained in $\mathcal{I}^1_\rho(x_0)$ and we observe that

$$x_0 + \mathcal{C}_\rho(\theta, \boldsymbol{n}) \subset \overline{\Omega}_1 \subset \overline{\Omega}.$$

By the (local) convexity of Ω there exists a couple of smaller $\theta', \rho' > 0$ such that $\mathcal{C}_{\rho'}(\theta', \boldsymbol{n}) \subset \mathcal{I}_\rho(x_0)$; recalling (31), we conclude. □

Remark 5.2 We could restate the previous theorem in the more general framework of example **E** 3, by assigning two functions $F_i : \Omega_i \times \mathbb{R}^N \to \mathbb{R}$ satisfying in their proper domains (20), (22) (or (23) according to p), (35), and $F_i(x, 0) \equiv 0$, and by setting

$$F(x, \boldsymbol{\xi}) := \sum_{i=1}^{2} \chi_{\Omega_i}(x) F_i(x, \boldsymbol{\xi}). \tag{99}$$

The crucial assumption which allows to repeat the previous proof is a sort of *compatibility* of F_1 and F_2 on $\overline{\Gamma}$: for every point x_0 of $\overline{\Gamma}$ there exists a neighborhood U of x_0 such that

$$F(x,\boldsymbol{\xi}) \leq F(y,\boldsymbol{\xi}), \quad \forall\, \boldsymbol{\xi} \in \mathbb{R}^N, \quad \forall\, x \in U \cap \Omega_1, \quad \forall\, y \in U \cap \Omega_2. \tag{100}$$

Then a local (and global, if (98) holds) result in the spirit of Theorems 2 and 2' holds. □

6 Application: biharmonic and Stokes operator.

Theorem 6 *Let Ω be a bounded Lipschitz open set and $u \in H^2_0(\Omega)$ be the unique variational solution of the equation*

$$\Delta^2 u = f \in H^{-2}(\Omega). \tag{101}$$

For every $s \in]0,1/2[$, if $f \in H^{-2+s}(\Omega)$ then u belongs to $H_0^{2+s}(\Omega)$. Moreover, the linear operator mapping f into u can be extended by continuity to a continuous linear operator between $H^{-2-s}(\Omega)$ into $H_0^{2-s}(\Omega)$.

Proof. We are in the framework of example **E** 4 and Proposition 3.8. As before we have to check the $(T, \mathcal{O}_\rho(x_0))$-regularity of $\mathcal{L}_f$, where now we choose $f \in H^{-1}(\Omega)$.

By (71) we deduce

$$\langle \mathcal{L}_f, T_h u - u\rangle \leq C|h| \|f\|_{H^{-1}(\Omega)} \|u\|_{H^2_0(\Omega)}. \tag{102}$$

and arguing as in the proof of Theorem 2, we get

$$\|u\|_{\dot{B}^{1/2}_{2,\infty}(\Omega)} \leq C\, \|f\|_{H^{-1}(\Omega)} \|f\|_{H^{-2}(\Omega)}. \tag{103}$$

We conclude applying Proposition 4.2 as in the previous Theorem 4, and the standard transposition technique. ■

Theorem 7 *Let us assume that Ω is a bounded Lipschitz domain of $\mathbb{R}^2$, and let s be in the interval $]-1/2,1/2[$. For every set of data*

$$\boldsymbol{f} \in H^{-1+s}(\Omega;\mathbb{R}^2), \quad g \in H^s(\Omega), \quad \boldsymbol{\phi} \in H^{1/2+s}(\Omega;\mathbb{R}^2), \tag{104}$$

satisfying the compatibility condition

$$\int_\Omega g\, dx = \int_{\partial\Omega} \boldsymbol{\phi} \cdot \nu \, d\mathcal{H}^1, \tag{105}$$

there exist a unique $\boldsymbol{u} \in H^{1+s}(\Omega; \mathbb{R}^2)$ *and a* $p \in H^s(\Omega)$ *uniquely determined up to the addition of a constant, which are solutions of the non homogeneous Stokes problem*

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = g & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{\phi} & \text{on } \partial\Omega. \end{cases} \tag{106}$$

Proof. Arguing as in [27, Ch. I, 2.4], it is possible to reduce (106) to the homogeneous case $g = 0, \boldsymbol{\phi} = 0$. Since we are in dimension 2 it is possible to reduce the study of the regularity of the Stokes equation to the previous biharmonic problem, as detailed in [27, Ch. I, Prop. 2.3]. Therefore we can apply Theorem 6. ■

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