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A WASSERSTEIN APPROACH TO THE ONE-DIMENSIONAL STICKY PARTICLE SYSTEM*

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Abstract. We present a simple approach for studying the one-dimensional pressureless Euler system via adhesion dynamics in the Wasserstein space $\mathcal{P}_2(\mathbb{R})$ of probability measures with finite quadratic moments. Starting from a discrete system of a finite number of "sticky" particles, we obtain new explicit estimates of the solution in terms of the initial mass and momentum, and we are able to construct an evolution semigroup in a measure-theoretic phase space, allowing mass distributions in $\mathcal{P}_2(\mathbb{R})$ and the corresponding L^2-velocity fields. We investigate various interesting properties of this semigroup, in particular its link with the gradient flow of the (opposite) squared Wasserstein distance. Our arguments rely on an equivalent formulation of the evolution as a gradient flow in the convex cone of nondecreasing functions in the Hilbert space $L^2(0,1)$, which corresponds to the Lagrangian system of coordinates given by the canonical monotone rearrangement of the measures.

Key words. pressureless Euler equation, sticky particles, Wasserstein distance, monotone rearrangement, gradient flows

AMS subject classifications. 35L65, 76N15, 82C22

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1. Introduction.

In recent years considerable attention has been devoted to the one-dimensional pressureless Euler system

$$\begin{cases} \partial_t\rho + \partial_x(\rho\, v) = 0 \\ \partial_t(\rho\, v) + \partial_x(\rho\, v^2) = 0 \end{cases} \quad \text{in } \mathbb{R}\times(0,+\infty), \quad \rho|_{t=0} = \rho_0, \quad v|_{t=0} = v_0 \tag{1.1}$$

in connection with the Zel′dovich model [32] for the evolution of a sticky particle system (SPS) via adhesion dynamics. This model describes the behavior of a finite collection of particles, freely moving in the absence of forces and sticking under collision. They can be mathematically represented by a time-dependent discrete measure $\rho_t^N := \sum_{i=1}^n m_i\delta_{x_i(t)}$ concentrated in a finite set of N particles $P_i(t) := (m_i, x_i(t), v_i(t))$, $i = 1,\dots,N$ with positive mass m_i, ordered positions $x_1(t) \le x_2(t) \le \cdots \le x_{N-1}(t) \le x_N(t)$, and velocities $v_i(t)$.

Denoting by $J_i(t) := \{j : x_j(t) = x_i(t)\}$ the collection of (the indices of) the particles $P_j(t)$ coinciding with $P_i(t)$ at time t, the adhesion dynamic imposes that the sets $J_i(t)$ are nondecreasing in time, so that $v_j(t+) = v_i(t+)$ for every $j \in J_i(t)$. We can thus order in a finite and monotone sequence $0 < t_1 < t_2 < \cdots$ the collection of times when the cardinality of some $J_i(t)$ has a discontinuity (corresponding to some collision). In each open interval $[t_k, t_{k+1})$ the (right-continuous) velocities $v_i(t) = \dot{x}_i(t)$ are thus supposed to be constant and, at each collision time t_k, the conservation of mass and momentum yields the updated equation for the velocities

$$v_i(t_k+) = \frac{\sum_{j\in J_i(t_k)} m_j v_j(t_k-)}{\sum_{j\in J_i(t_k)} m_j}, \qquad i = 1,\dots N. \tag{1.2}$$

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It is not difficult to check that the measures ρ^N and $(\rho v)^N_t := \sum_{i=1}^N m_i v_i(t)\, \delta_{x_i(t)}$ solve (1.1). Starting from the discrete SPS, the existence of measure-valued solutions to (1.1) with general initial data and satisfying suitable entropy conditions [5] has been proved by Grenier [17] and E, Rykov, and Sinai [15] (see also the contribution of Martin and Piasecki [19]) as limits (in the sense of weak convergence of measures) of the discrete particle evolutions ρ^N_t as $N \uparrow +\infty$. Here we also cite the different approaches of Bouchut and James [6], Poupaud and Rascle [22], and Sever [27] in the multidimensional case; viscous regularizations of (1.1) have been studied by Sobolevskiĭ [29] and Boudin [7], and a different model, starting from particles of finite size, has been considered by Wolansky [31].

The convergence result has been extended further and refined by Brenier and Grenier [11], Huang and Wang [18], and Nguyen and Tudorascu [21]. (By a different probabilistic approach, Moutsinga [20] has recently been able to consider initial velocities with nonpositive jumps at each point of the support of ρ_0.) The basic assumption is that the discrete initial velocity v_i is the value in x_i of a given continuous function v with at most quadratic growth and (the total mass being normalized to 1) the sequence ρ^N_0 converges to ρ_0 with respect to the L^2-Wasserstein distance in the space $\mathcal{P}_2(\mathbb{R})$ of probability measures with finite quadratic moment. This includes the case (considered in [11]) of a sequence ρ^N_0 with uniformly bounded support and weakly converging to ρ_0 in the duality with continuous real functions.

All these results depend on a remarkable characterization of the solution ρ found by Brenier and Grenier [11]: by introducing the cumulative distribution function M_ρ associated to a probability measure $\rho \in \mathcal{P}(\mathbb{R})$

$$M_\rho(x) := \rho((-\infty, x]) \quad \forall x \in \mathbb{R}, \quad \text{so that } \rho = \partial_x M_\rho \quad \text{in } \mathscr{D}'(\mathbb{R}), \tag{1.3}$$

they prove that the function $M(t,\cdot) := M_{\rho_t}(\cdot)$ is the unique entropy solution of the scalar conservation law

$$\partial_t M + \partial_x A(M) = 0 \quad \text{in } \mathbb{R} \times (0, +\infty), \tag{1.4}$$

where $A : [0,1] \to \mathbb{R}$ is a continuous flux function depending only on ρ_0 and v_0 (see Theorem 6.1 for a precise statement).

It can also be shown [21] that this solution satisfies the Oleinik entropy condition

$$v_t(x_2) - v_t(x_1) \le \frac{1}{t}(x_2 - x_1) \quad \text{for } \rho_t\text{-a.e. } x_1, x_2 \in \mathbb{R},\ x_1 \le x_2. \tag{1.5}$$

In the present paper we discuss various refinements of the Brenier–Grenier result by a different approach. Our starting point (Theorem 2.2) is an explicit Lipschitz estimate (in the L^p-Wasserstein distance W_p for every $p \ge 1$—see (2.1)) of the dependence of ρ_t with respect to the initial data $\rho_0, (\rho v)_0$: for $p = 2$ it shows that $(\rho^N_t)_{N\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{P}_2(\mathbb{R})$ and, in particular, yields the convergence results of [15, 11, 21], allowing general initial measures in $\mathcal{P}_2(\mathbb{R})$ and (possibly discontinuous) velocity fields $v_0 \in L^2(\rho_0)$. We also show that a suitable L^2-like integral distance between the momentum ρv of two solutions can be controlled in terms of the initial data, and we prove further precise representation properties of the solution and its velocity field (Theorem 2.3).

This leads to the construction of a semigroup $\mathscr{S}_t$ associated to the evolution of the SPS, which exhibits interesting links with another semigroup (recently studied by Ambrosio, Gigli, and Savaré [1]), obtained as the gradient flow in $\mathcal{P}_2(\mathbb{R})$ of the (opposite) squared Wasserstein distance from a fixed reference measure.

This link (which at first glance may be unexpected) can be better understood in the simpler case when the initial velocity field v satisfies a one-sided monotonicity condition (see section 5.4.2 of Villani's book [30] for more details). Still, considering the simpler discrete case, if

$$-\delta^{-1} := \min_{x_i \neq x_j} \frac{v(x_i) - v(x_j)}{x_i - x_j} < 0, \quad v(x_i) := v_i \tag{1.6}$$

for $t \in [0, \delta)$, then the map $\mathsf{x}_0^t(x) := x + tv(x)$ is nondecreasing on the support of ρ_0 (the finite set $\{x_i : i = 1, \dots N\}$), so that the first collision occurs at $t := \delta$ and, in the interval $[0, \delta)$, one has the freely moving measures

$$\rho_t := (\mathsf{x}_0^t)_\# \rho_0 = \sum_{i=1}^N m_i \delta_{x_i + tv_i}, \quad (\rho v)_t = \sum_{i=1}^N m_i v_i \delta_{x + tv_i}, \ t \in [0, \delta), \tag{1.7}$$

solving the pressureless Euler system (1.1). On the other hand, the curve $t \mapsto \rho_t$, $t \in [0, \delta]$ is a constant speed minimal geodesic in $\mathcal{P}_2(\mathbb{R})$ connecting ρ_0 with $\eta := \rho_\delta$; as in any Riemannian manifold, it coincides (up to a suitable rescaling; see [1, Theorem 11.2.10]) with the gradient flow in $\mathcal{P}_2(\mathbb{R})$ of the functional $\phi^{\rho_0}(\rho) := -\frac{1}{2} W_2^2(\rho, \rho^0)$. After the collision at time $t = \delta$ the trajectory of the gradient flow no longer coincides with the free motion (1.7), since its velocity has a jump which can be described exactly by (1.2) [1, Theorem 10.4.12]. At a later time, the velocity field induced by the (rescaled) Wasserstein gradient flow can be characterized by the formula

$$v_i(t+) = t^{-1} \left(x_i(t) - \frac{\sum_{j \in J_i(t)} m_j x_j(0)}{\sum_{j \in J_i(t)} m_j} \right), \qquad i = 1, \dots N. \tag{1.8}$$

There is an interesting property (stated in Theorems 2.4 and 2.5) that the two different laws (1.2) and (1.8) give rise to the same evolution, even for arbitrary initial data.

In order to obtain these results, we adopt the viewpoint of one-dimensional optimal transportation and represent each probability measure $\rho \in \mathcal{P}_2(\mathbb{R})$ by its monotone rearrangement X_ρ, which is the pseudoinverse of the distribution function M_ρ of (1.3):

$$X_\rho(w) := \inf \{x : M_\rho(x) > w\} = \inf \{x : \rho((-\infty, x]) > w\}, \quad w \in (0, 1). \tag{1.9}$$

(A similar approach, in a probabilistic framework, was used in [20]; see also [16] for other applications.) The map $\rho \mapsto X_\rho$ is an isometry between $\mathcal{P}_2(\mathbb{R})$ (endowed with the L^2-Wasserstein distance) and the convex cone $\mathcal{K}$ of nondecreasing functions in the Hilbert space $L^2(0, 1)$. Through this isometry, any gradient flow with respect to W_2 in $\mathcal{P}_2(\mathbb{R})$ can be rephrased as a gradient flow in $\mathcal{K}$ with respect to the $L^2(0, 1)$-distance and one can use the powerful tools of the classical theory of variational evolution inequalities in Hilbert spaces (we refer the reader to the book by Brézis [12]). It turns out (see Theorem 2.6) that in this Lagrangian formulation the solution X_{ρ_t} admits three simple characterizations—in terms of the $L^2(0, 1)$-projection $\mathsf{P}_\mathcal{K}$ onto $\mathcal{K}$:

$$X_{\rho_t} = \mathsf{P}_\mathcal{K}(X_{\rho_0} + tV_0), \quad V_0 = v_0 \circ X_{\rho_0}, \tag{1.10}$$

and in terms of the differential inclusions:

$$\frac{\mathrm{d}}{\mathrm{d}t} X_{\rho_t} + \partial I_\mathcal{K}(X_{\rho_t}) \ni V_0, \qquad t \frac{\mathrm{d}}{\mathrm{d}t} X_{\rho_t} + \partial I_\mathcal{K}(X_{\rho_t}) \ni X_{\rho_t} - X_{\rho_0}, \tag{1.11}$$

where $I_{\mathcal{K}}$ is the indicator function associated to $\mathcal{K}$ (see (2.28)). Equations (1.10) and (1.11) encode all the qualitative information on the measure-valued solution ρ_t, and their proof in the case of the discrete SPS constitutes the core of our argument. The proof relies on an elementary but careful description of the $L^2(0,1)$-projection operator $\mathsf{P}_{\mathcal{K}}$ and on the subdifferential of $I_{\mathcal{K}}$, which we discuss in section 3. Once ρ_t has been determined, its velocity $v_t \in L^2_{\rho_t}(\mathbb{R})$ can be recovered from the right derivative $V(t) := \frac{\mathrm{d}^+}{\mathrm{d}t} X_{\rho_t} \in L^2(0,1)$. In fact, as a byproduct of the second differential inclusion of (1.11), $V(t)$ is a function of $X(t)$ and, therefore, one obtains

$$V(t) = v_t \circ X_{\rho_t}. \tag{1.12}$$

The projection formula (1.10) (which was introduced by Shnirel′man [28] (see also [2]) in a slightly different form; see Remark 2.9) lies more or less explicitly at the core of the formulations by [15] and [11]. As was nicely explained by Andrievsky, Gurbatov, and Sobolevsky [2], elaborating on the contribution of [28], (1.10) is equivalent to the *generalized variational principle* of [15], which can be expressed through the convex envelope of the primitive function of the map $X_{\rho_0} + tV_0$. As stated in full generality by Theorem 3.1, this convexification characterizes the L^2-projection on $\mathcal{K}$. On the other hand, a convexification is also involved in the second Hopf formula for the solutions of the Hamilton–Jacobi equation associated to (1.4), as has already been observed in [11, sect. 4]. We will detail this point in Theorem 6.1.

The link between the formulation based on the scalar conservation law (1.4) and the Hilbertian theory of gradient flows such as (1.11) is not at all surprising after the illuminating paper by Brenier [10] (whose ideas, in particular concerning the SPS, could be traced back to [8, 9]). Wasserstein contraction properties of solutions of one-dimensional scalar conservation laws have also been recently obtained by Bolley, Brenier, and Loeper [4] (see also the further contribution by Carrillo, Di Francesco, and Lattanzio [13]). So it would be possible, in principle, to approach the SPS starting from (1.4) and try to apply the techniques developed there. Note, however, that two solutions originating from different initial distributions of position and velocity give rise to two scalar conservation laws differing not only by the initial data but also by the flux functions, so that their comparison does not look immediate. Moreover, the present self-contained approach is very simple, since it relies on elementary tools of convex analysis and direct computations on the discrete case; the simultaneous characterization of the evolution by (1.10) and (1.11) provides a more refined description of the solution and, as a byproduct, a new direct proof of the Brenier–Grenier theorem.

Plan of the paper. In the next section we recall some basic definition and notation and we state our main results. Section 3 collects the main properties related to the convex cone $\mathcal{K}$ in $L^2(0,1)$ (projection, polar cone, subdifferential of the indicator function)—they provide simple but crucial tools for the analysis of the discrete SPS presented in section 4, which contains all the basic calculations. Section 5 deals with the existence, stability, and uniqueness of the solution in the Lagrangian formulation. The final steps of the proofs (mainly concerning the various limit processes) will be detailed in the last section, where we also show a new derivation of the Brenier–Grenier theorem [11] from the Lagrangian representation of the SPS.

2. Main results.

Couplings, Wasserstein distance, and monotone rearrangements. For $p \in [1, +\infty)$, let us denote by $\mathcal{P}_p(\mathbb{R})$ the space of Borel probability measures ρ with finite p-moment $\int_{\mathbb{R}} |x|^p \,\mathrm{d}\rho(x) < +\infty$. The L^p Kantorovich–Rubinstein–Wasserstein distance $W_p(\rho^1, \rho^2)$ between two measures $\rho^1, \rho^2 \in \mathcal{P}_p(\mathbb{R})$ can be defined in terms of couplings, i.e., probability measures $\boldsymbol{\rho} \in \mathcal{P}(\mathbb{R} \times \mathbb{R})$ such that $\pi^i_\# \boldsymbol{\rho} = \rho^i$, $i = 1, 2$, by the formula

$$W_p^p(\rho^1, \rho^2) := \min \left\{ \int_{\mathbb{R}\times\mathbb{R}} |x - y|^p \,\mathrm{d}\boldsymbol{\rho}(x, y) : \boldsymbol{\rho} \in \mathcal{P}(\mathbb{R} \times \mathbb{R}),\ \pi^i_\# \boldsymbol{\rho} = \rho^i \right\}. \tag{2.1}$$

Here $\pi^i(x_1, x_2) = x_i$ is the usual projection on the ith coordinate and, for a general Borel map $\mathsf{T} : \mathbb{R}^m \to \mathbb{R}^n$ and a Borel measure $\mu \in \mathcal{P}(\mathbb{R}^m)$, the *push-forward* $\nu = \mathsf{T}_\# \mu$ is the measure defined by $\nu(A) = \mu(\mathsf{T}^{-1}(A))$ for every Borel set $A \subset \mathbb{R}^n$. We will repeatedly use the change-of-variable formula

$$\int_{\mathbb{R}^n} \zeta(y) \,\mathrm{d}(\mathsf{T}_\# \mu)(y) = \int_{\mathbb{R}^m} \zeta(\mathsf{T}(x)) \,\mathrm{d}\mu(x) \quad \text{for every Borel } \zeta : \mathbb{R}^n \to [0, +\infty]. \tag{2.2}$$

More generally, given a convex, even, and lower semicontinuous function $\psi : \mathbb{R} \to [0, +\infty]$, we can consider the cost $c_\psi(x, y) := \psi(x - y)$, $x, y \in \mathbb{R}$, and the associated optimal mass transportation problem

$$\mathcal{C}_\psi(\rho^1, \rho^2) := \inf \left\{ \int_{\mathbb{R}\times\mathbb{R}} \psi(x - y) \,\mathrm{d}\boldsymbol{\rho}(x, y) : \boldsymbol{\rho} \in \mathcal{P}(\mathbb{R} \times \mathbb{R}),\ \pi^i_\# \boldsymbol{\rho} = \rho^i \right\}. \tag{2.3}$$

In the present one-dimensional case, there exists a unique optimal coupling $\boldsymbol{\rho} = \Gamma_o(\rho^1, \rho^2)$ realizing the minimum of (2.1) and (2.3) (at least when the cost is finite): it can be explicitly characterized by inverting the distribution functions of ρ^1, ρ^2. More precisely, for every $\rho \in \mathcal{P}(\mathbb{R})$ we consider its monotone rearrangement X_ρ (see (1.9)), a right-continuous and nondecreasing function satisfying

$$(X_\rho)_\# \lambda = \rho, \quad \lambda := \mathscr{L}^1|_{(0,1)}, \qquad \int_{\mathbb{R}} \zeta(x) \,\mathrm{d}\rho(x) = \int_0^1 \zeta(X_\rho(w)) \,\mathrm{d}w \tag{2.4}$$

for every nonnegative Borel map $\zeta : \mathbb{R} \to [0, +\infty]$. In particular, $\rho \in \mathcal{P}_p(\mathbb{R})$ if and only if $X_\rho \in L^p(0, 1)$. Moreover, thanks to the Hoeffding–Fréchet theorem [23, sect. 3.1], the joint map $X_{\rho^1,\rho^2}(w) := (X_{\rho^1}(w), X_{\rho^2}(w))$, $w \in (0, 1)$ characterizes the optimal coupling $\boldsymbol{\rho} = \Gamma_o(\rho^1, \rho^2)$ by the formula

$$\boldsymbol{\rho} = \left(X_{\rho^1,\rho^2}\right)_\# \lambda \tag{2.5}$$

so that [14, 23, 30]

$$\begin{aligned} W_p^p(\rho^1, \rho^2) &= \int_0^1 \left| X_{\rho^1}(w) - X_{\rho^2}(w) \right|^p \mathrm{d}w, \\ \mathcal{C}_\psi(\rho^1, \rho^2) &= \int_0^1 \psi\left(X_{\rho^1}(w) - X_{\rho^2}(w) \right) \mathrm{d}w, \end{aligned} \tag{2.6}$$

and the map $\rho \in \mathcal{P}(\mathbb{R}) \longmapsto X_\rho$ is an isometry between $\mathcal{P}_2(\mathbb{R})$ and the convex subset $\mathcal{K}$ of $L^2(0, 1)$ of (essentially) nondecreasing functions (which can be identified with their right-continuous representatives).

An explicit estimate through Wasserstein distance. We introduce the set

$$\mathcal{V}_p(\mathbb{R}) := \left\{\mu = (\rho, \rho v) \in \mathcal{P}_p(\mathbb{R}) \times \mathcal{M}(\mathbb{R}) : v \in L^p_\rho(\mathbb{R})\right\}, \quad p \in [1, +\infty), \tag{2.7}$$

where $\mathcal{M}(\mathbb{R})$ is the set of all signed Borel measures with finite total variation, the semidistances (here $\mu^i = (\rho^i, \rho^i v^i)$)

$$U_p^p(\mu^1, \mu^2) := \int_{\mathbb{R}\times\mathbb{R}} |v^1(x) - v^2(y)|^p \,\mathrm{d}\boldsymbol{\rho}(x, y) \quad \boldsymbol{\rho} = \Gamma_o(\rho^1, \rho^2) \tag{2.8}$$

$$= \int_0^1 |v^1(X_{\rho^1}(w)) - v^2(X_{\rho^2}(w))|^p \,\mathrm{d}w, \tag{2.9}$$

and the distances

$$D_p^p(\mu^1, \mu^2) := W_p^p(\rho^1, \rho^2) + U_p^p(\mu^1, \mu^2). \tag{2.10}$$

We also set

$$[\mu]_p^p := \int_{\mathbb{R}} (|x|^p + |v(x)|^p)\,\mathrm{d}\rho(x) = D_p^p\left(\mu, (\delta_0, 0)\right). \tag{2.11}$$

PROPOSITION 2.1. *D_p is a distance in $\mathcal{V}_p(\mathbb{R})$ and $(\mathcal{V}_p(\mathbb{R}), D_p)$ is a metric (but not complete) space whose topology is stronger than the one induced by the weak convergence of measures. The collection of discrete measures*

$$\hat{\mathcal{V}}(\mathbb{R}) := \left\{\mu = \left(\sum_{i=1}^N m_i \delta_{x_i}, \sum_{i=1}^N m_i v_i \delta_{x_i}\right) : m_i > 0,\ \sum_{i=1}^N m_i = 1,\ x_i, v_i \in \mathbb{R}\right\} \tag{2.12}$$

is a dense subset of $\mathcal{V}_p(\mathbb{R})$. A sequence $\mu_n = (\rho_n, \rho_n v_n)$, $n \in \mathbb{N}$ converges to $\mu = (\rho, \rho v)$ in $\mathcal{V}_p(\mathbb{R})$, $p > 1$ if and only if (see [1, Def. 5.4.3]*)*

$$W_p(\rho_n, \rho) \to 0, \quad \rho_n v_n \rightharpoonup \rho v \quad \textit{weakly in } \mathcal{M}(\mathbb{R}), \quad \int_{\mathbb{R}} |v_n|^p \,d\rho_n \to \int_{\mathbb{R}} |v|^p \,\mathrm{d}\rho. \tag{2.13}$$

Let us denote by $\mathscr{S}_t : \hat{\mathcal{V}}(\mathbb{R}) \to \hat{\mathcal{V}}(\mathbb{R})$ the map associating to any discrete initial datum $(\rho_0, \rho_0 v_0)$ the solution $(\rho_t, \rho_t v_t)$ of the (discrete) SPS. $\mathscr{S}_t$ is a semigroup in $\hat{\mathcal{V}}(\mathbb{R})$.

THEOREM 2.2 (stability with respect to the initial data). *Let $\mu_t^\ell = (\rho_t^\ell, \rho_t^\ell v_t^\ell) = \mathscr{S}_t[\mu_0^\ell]$, $\ell = 1, 2$ be the solutions of the (discrete) SPS with initial data $\mu_0^\ell \in \hat{\mathcal{V}}(\mathbb{R})$. Then, for every convex cost* (2.3) *and every $p \geq 1$,*

$$\mathcal{C}_\psi(\rho_t^1, \rho_t^2) \leq \int_{\mathbb{R}\times\mathbb{R}} \psi\left(x + tv^1(x) - (y + tv^2(y)\right) \mathrm{d}\boldsymbol{\rho}(x, y), \quad \boldsymbol{\rho} = \Gamma_o(\rho^1, \rho^2), \tag{2.14a}$$

$$W_p(\rho_t^1, \rho_t^2) \leq W_p(\rho_0^1, \rho_0^2) + tU_p(\mu_0^1, \mu_0^2), \tag{2.14b}$$

$$\int_0^t U_2^2(\mu_r^1, \mu_r^2)\,\mathrm{d}r \leq C(1 + t)\left([\mu^1]_2 + [\mu^2]_2\right)\left(W_2(\rho_0^1, \rho_0^2) + U_2(\mu_0^1, \mu_0^2)\right) \tag{2.14c}$$

for a suitable "universal" constant C independent of t and the data.

We say that a map $\mathscr{S} : \mathcal{V}_p(\mathbb{R}) \to \mathcal{V}_p(\mathbb{R})$ is *strongly-weakly continuous* if, for every $\mu^n, \mu \in \mathcal{V}_p(\mathbb{R})$ with $\mathscr{S}[\mu^n] = (\tilde{\rho}^n, \tilde{\rho}^n \tilde{v}^n)$, $\mathscr{S}[\mu] = (\tilde{\rho}, \tilde{\rho}\tilde{v}) \in \mathcal{V}_p(\mathbb{R})$,

$$\lim_{n\uparrow+\infty} D_p(\mu_n, \mu) = 0 \quad \Longrightarrow \quad \lim_{n\uparrow+\infty} W_p(\tilde{\rho}^n, \tilde{\rho}) = 0, \quad \tilde{\rho}_n \tilde{v}_n \rightharpoonup \tilde{\rho}\tilde{v} \quad \text{weakly in } \mathcal{M}(\mathbb{R}). \tag{2.15}$$

THEOREM 2.3 (the evolution semigroup in $\mathcal{V}_p(\mathbb{R})$).

(a) *The semigroup $\mathscr{S}_t$ can be uniquely extended by density to a right-continuous semigroup (still denoted $\mathscr{S}_t$) of strongly-weakly continuous transformations in $\mathcal{V}_p(\mathbb{R})$, $p \geq 2$, thus satisfying*

$$\mathscr{S}_{s+t}[\mu] = \mathscr{S}_s[\mathscr{S}_t[\mu]] \quad \forall\, s,t \geq 0, \qquad \lim_{t\downarrow 0} D_p(\mathscr{S}_t[\mu],\mu) = 0 \quad \forall\, \mu \in \mathcal{V}_p(\mathbb{R}). \tag{2.16}$$

$\mathscr{S}_t$ complies with the same estimates (2.14a)–(2.14c) *of Theorem* 2.2.

(b) $(\rho_t, \rho_t v_t) = \mathscr{S}_t[\mu]$, $\mu \in \mathcal{V}_2(\mathbb{R})$ *is a distributional solution of* (1.1) *satisfying Oleinik entropy condition* (1.5).

(c) *If $\psi : \mathbb{R} \to \mathbb{R}$ is a convex function such that $\psi(v_0) \in L^1_{\rho_0}(\mathbb{R})$ and $(\rho_t, \rho_t v_t) = \mathscr{S}_t[\mu_0]$, then the map*

$$t \mapsto \int_{\mathbb{R}} \psi(v_t)\,\mathrm{d}\rho_t(x) \tag{2.17}$$

is nonincreasing in $[0,+\infty)$, and its jump set is contained in an at most countable set $\mathcal{T} = \mathcal{T}(\mu)$ independent of ψ.

(d) *If $\mu \in \mathcal{V}_p(\mathbb{R})$ and $\mu_t = (\rho_t, \rho_t v_t) = \mathscr{S}_t[\mu]$, $t \in [0,+\infty)$, the curve $t \mapsto \rho_t$ is Lipschitz in $\mathcal{P}_p(\mathbb{R})$ with respect to W_p, and the curve $t \mapsto \rho_t v_t$ is continuous with respect to the weak topology in $\mathcal{M}(\mathbb{R})$, right-continuous in $[0,+\infty)$ with respect to the (semi-) distance U_p, and left-continuous at each $t \in (0,+\infty) \setminus \mathcal{T}$, where $\mathcal{T}$ is the at most countable jump set of* (2.17).

(e) *Let $\mu^n_t = (\rho^n_t, \rho^n_t v^n_t) = \mathscr{S}_t[\mu^n]$ and $\mu_t = (\rho_t, \rho_t v_t) = \mathscr{S}_t[\mu]$; if μ^n converges to μ in $\mathcal{V}_p(\mathbb{R})$ as $n \uparrow +\infty$, then for every $t \in [0,+\infty)$ ρ^n_t converges to ρ_t in $\mathcal{P}_p(\mathbb{R})$ and $\rho^n_t v^n_t$ weakly converges to $\rho_t v_t$ in $\mathcal{M}(\mathbb{R})$; moreover, μ^n_t converges to $(\rho_t, \rho_t v_t) = \mathscr{S}_t[\mu]$ in $\mathcal{V}_p(\mathbb{R})$ for every $t \in [0,+\infty) \setminus \mathcal{T}(\mu)$.*

(f) *For every $0 \leq s < t$, there exists a ρ_s-essentially unique monotone map $\mathsf{x}^t_s \in L^2_{\rho_s}(\mathbb{R})$ such that*

$$\rho_t = (\mathsf{x}^t_s)_{\#}\rho_s, \quad \lim_{h\downarrow 0} \frac{\mathsf{x}^{s+h}_s - \mathsf{i}}{h} = v_s \quad \textit{in } L^2_{\rho_s}(\mathbb{R}), \quad \mathsf{i}(x) \equiv x, \tag{2.18}$$

$$v_t(y) = \int_{\mathbb{R}} v_s(x)\,\mathrm{d}\rho^{s\to t}_y(x) = (t-s)^{-1}\left(y - \int_{\mathbb{R}} \mathsf{x}^t_s(x)\,\mathrm{d}\rho^{s\to t}_y(x)\right) \quad \textit{for } \rho_t\textit{-a.e. } y \in \mathbb{R}, \tag{2.19}$$

where $\rho^{s\to t}_y$ is the disintegration of ρ_s with respect to x^t_s.

Let us recall that the disintegration $\rho^{s\to t}_y$ of ρ_s with respect to the Borel (monotone) map x^t_s is a Borel family of parametrized measures uniquely determined for ρ_t-a.e. $y \in \mathbb{R}$ such that $\rho_s = \int_{\mathbb{R}} \rho^{s\to t}_y\,\mathrm{d}\rho_t(y)$ with $\rho^{s\to t}_y((\mathsf{x}^t_s)^{-1}(y)) = 1$ (see, e.g., [1, Thm. 5.3.1]).

Note that for a fixed t the map $\mathscr{S}_t : \mathcal{V}_p(\mathbb{R}) \to \mathcal{V}_p(\mathbb{R})$ may fail to be continuous with respect to the distance D_p, at least in the momentum component ρv.

The gradient flow of the (opposite) squared Wasserstein distance. Equations (2.18) and (2.19) show an interesting connection between the semigroup $\mathscr{S}_t$ in $\mathcal{V}_2(\mathbb{R})$ and the gradient flow $\mathscr{G}^\sigma_t$ in $\mathcal{P}_2(\mathbb{R})$ of the (opposite) squared distance functional

$$\phi^\sigma(\rho) := -\frac{1}{2}W_2^2(\rho,\sigma) \quad \forall\, \rho,\sigma \in \mathcal{P}_2(\mathbb{R}). \tag{2.20}$$

Let us recall [1] that for every choice of a reference measure $\sigma \in \mathcal{P}_2(\mathbb{R})$ it is possible to define a unique continuous and 1-expansive semigroup $\mathscr{G}^\sigma_\tau : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$, $\tau \geq 0$ whose Lipschitz trajectories $\hat\rho_\tau := \mathscr{G}^\sigma_\tau(\rho)$ can be uniquely characterized by the evolution variational inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}W_2^2(\hat\rho_\tau,\eta) - \frac{1}{2}W_2^2(\hat\rho_\tau,\eta) \leq \phi^\sigma(\eta) - \phi^\sigma(\hat\rho_\tau) \quad \forall\, \eta \in \mathcal{P}_2(\mathbb{R}). \tag{2.21}$$

The next result shows that $\mathscr{S}_t$ and $\mathscr{G}^{\rho_0}_\tau$ basically coincide up to the rescaling

$$\tau = \log t, \quad t = e^\tau, \quad \hat\rho_\tau = \rho_{e^\tau}. \tag{2.22}$$

Theorem 2.4 (gradient flow of the Wasserstein distance and SPS). *Let* $(\rho_t, \rho_t v_t) = \mathscr{S}_t(\rho_0, \rho_0 v_0) \in \mathcal{V}_2(\mathbb{R})$ *be the semigroup solution of the SPS. The Lipschitz curve* $(\rho_t)_{t\geq 0}$ *in* $\mathcal{P}_2(\mathbb{R})$ *solves the evolution variational inequality*

$$\frac{t}{2}\frac{\mathrm{d}}{\mathrm{d}t}W_2^2(\rho_t,\eta) - \frac{1}{2}W_2^2(\rho_t,\eta) \leq \phi^{\rho_0}(\eta) - \phi^{\rho_0}(\rho_t) \quad \textit{a.e. in } (0,+\infty) \tag{2.23}$$

for every $\eta \in \mathcal{P}_2(\mathbb{R})$. *Equivalently, the reparametrized solutions* $\hat\rho_\tau = \rho_{e^\tau}$ *satisfy* (2.21) *with* $\sigma := \rho_0$ *and we thus get the representation formula*

$$\hat\rho_\tau = \mathscr{G}^{\rho_0}_{\tau-\delta}\hat\rho_\delta \quad \textit{or, equivalently,} \quad \rho_t = \mathscr{G}^{\rho_0}_{\log(t/\varepsilon)}\rho_\varepsilon \quad \forall\, \tau = \log t \geq \delta = \log\varepsilon. \tag{2.24}$$

Conversely, if $t \mapsto \rho_t$ *is a Lipschitz curve in* $\mathcal{P}_2(\mathbb{R})$ *satisfying* (2.23) *and the initial velocity condition*

$$\lim_{t\downarrow 0} t^{-2}\int_{\mathbb{R}} |x + tv_0(x) - y|^2 \,\mathrm{d}\boldsymbol{\rho}_t(x,y) = 0 \quad \boldsymbol{\rho}_t = \Gamma_o(\rho_0,\rho_t), \tag{2.25}$$

then there exists a unique Borel velocity vector field $v_t \in L^2_{\rho_t}(\mathbb{R})$ *such that* $(\rho_t, \rho_t v_t) = \mathscr{S}_t(\rho_0, \rho_0 v_0)$. v_t *is the Wasserstein velocity field of* ρ_t [1, Thm. 8.4.5].

Note that (2.25) corresponds to (2.18) for $s = 0$ in the case (which a posteriori is always verified) $\boldsymbol{\rho}_t = (\mathsf{i} \times \mathsf{x}^t_0)_\# \rho_0$.

We can use (2.24) to exhibit the solution ρ_t of the SPS by a simple limit procedure.

Theorem 2.5. *Let* $(\rho_t, \rho_t v_t) = \mathscr{S}_t(\rho_0, \rho_0 v_0) \in \mathcal{V}_2(\mathbb{R})$ *be the solution of the SPS, and let* $\tilde\rho_\varepsilon := (i + \varepsilon v_0)_\# \rho_0$, $\varepsilon > 0$. *Then*

$$\rho_t = \lim_{\varepsilon\downarrow 0} \mathscr{G}^{\rho_0}_{\log(t/\varepsilon)}(\tilde\rho_\varepsilon) \quad \textit{in } \mathcal{P}_2(\mathbb{R}). \tag{2.26}$$

Moreover, if for some $\varepsilon_0 > 0$ *the map* $i + \varepsilon_0 v_0$ *is* ρ_0*-essentially nondecreasing, then*

$$\rho_\varepsilon = \tilde\rho_\varepsilon, \quad \rho_t = \mathscr{G}^{\rho_0}_{\log(t/\varepsilon)}(\tilde\rho_\varepsilon) \quad \forall\, \varepsilon \in (0,\varepsilon_0],\ t \geq \varepsilon. \tag{2.27}$$

The evolution in Lagrangian coordinates. We conclude this section with an even more explicit formula for the evolution of the monotone rearrangement function $X(t) = X_{\rho_t}$. We denote by $I_{\mathcal{K}}$ the indicator (convex, lower semicontinuous) function of $\mathcal{K}$ in $L^2(0,1)$

$$I_{\mathcal{K}}(X) = \begin{cases} 0 & \text{if } X \in \mathcal{K}, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.28}$$

with (multivalued) subdifferential $\partial I_{\mathcal{K}} : L^2(0,1) \to 2^{L^2(0,1)}$. We also introduce the closed subspace $\mathcal{H}_X \subset L^2(0,1)$, $X \in \mathcal{K}$, whose functions $Y \in L^2(0,1)$ are essentially constant in each open interval $(a,b) \subset (0,1)$, where X is constant. It is not difficult to check that for every $X \in \mathcal{K}$ and $Y \in L^2(0,1)$

$$Y \in \mathcal{H}_X \quad \text{iff} \quad Y = y \circ X \text{ for some Borel map } y \in L^2_\rho(\mathbb{R}),\ \rho = X_\# \lambda. \tag{2.29}$$

THEOREM 2.6 (Lagrangian evolution). *A curve $(\rho_t, \rho_t v_t) \in \mathcal{V}_2(\mathbb{R})$, $t \ge 0$ is the semigroup solution $\mathscr{S}_t(\rho_0, \rho_0 v_0)$ of the SPS as in Theorem 2.3 if and only if its monotone rearrangement $X(t) = X_{\rho_t} \in \mathcal{K} \subset L^2(0,1)$ satisfies one of the following three (equivalent) characterizations in terms of the couple $X_0 := X_{\rho_0}$ and $V_0 := v_0(X_0) \in \mathcal{H}_{X_0}$:*

I. *X is the unique strong (i.e., absolutely continuous) solution of the Cauchy problem for the subdifferential inclusion*

$$\frac{\mathrm{d}}{\mathrm{d}t} X \in -\partial I_{\mathcal{K}}(X) + V_0, \quad X(0) = X_0, \tag{L.I}$$

which is the gradient flow in $L^2(0,1)$ of the convex functional

$$X \mapsto I_{\mathcal{K}}(X) - (V_0|X), \quad X \in L^2(0,1). \tag{2.30}$$

II. *X admits the representation formula*

$$X(t) = \mathsf{P}_{\mathcal{K}}(X_0 + tV_0), \tag{L.II}$$

where $\mathsf{P}_{\mathcal{K}}$ is the L^2-projection on the convex cone $\mathcal{K} \subset L^2(0,1)$.

III. *X is the unique strong solution of the rescaled gradient flow*

$$\begin{aligned} &t \frac{\mathrm{d}}{\mathrm{d}t} X(t) \in -\partial I_{\mathcal{K}}(X(t)) + X(t) - X_0, \\ &\lim_{t \downarrow 0} t^{-1}(X(t) - X_0) = V_0 \quad \text{in } L^2(0,1). \end{aligned} \tag{L.III}$$

In each of these cases the curve $t \mapsto X(t)$ is Lipschitz continuous in $L^2(0,1)$ and right-differentiable at each time t; the velocity field v_t can be recovered by the formula

$$V(t) = \frac{\mathrm{d}^+}{\mathrm{d}t} X(t) = v_t \circ X(t) = \mathsf{P}_{\mathcal{H}_{X(t)}}(V_0) \in \mathcal{H}_{X(t)} \quad \forall\, t \ge 0, \tag{L.a}$$

where $\mathsf{P}_{\mathcal{H}_X}$ denotes the L^2-orthogonal projection on the closed subspace $\mathcal{H}_X \subset L^2(0,1)$. The closed subspaces $\mathcal{H}_{X(t)}$ are nonincreasing:

$$\mathcal{H}_{X(t)} \subset \mathcal{H}_{X(s)} \quad \text{if } 0 \le s \le t, \tag{L.b}$$

and X, V satisfy the semigroup identities

$$X(t) = \mathsf{P}_{\mathcal{K}}(X(s) + (t-s)V(s)), \quad V(t) = \mathsf{P}_{\mathcal{H}_{X(t)}}(V(s)) \quad \forall\, 0 \le s \le t. \tag{L.c}$$

This result shows that the natural evolution space for the Lagrangian sticky particles flow is

$$\mathcal{X}_p(0,1) := \{(X,V) \in L^p(0,1) \times L^p(0,1) : X \in \mathcal{K},\ V = v \circ X \in \mathcal{H}_X\}, \quad p \ge 2, \tag{2.31}$$

endowed with the product distance in $L^p(0,1) \times L^p(0,1)$. The bijective map

$$(\rho, \rho v) \in \mathcal{V}_p(\mathbb{R}) \longleftrightarrow (X,V) \in \mathcal{X}_p(0,1), \quad X = X_\rho,\ V = v \circ X_\rho \tag{2.32}$$

is, in fact, an isometry with respect to D_p of (2.10).

COROLLARY 2.7 (Lagrangian semigroup). *For every $p \ge 2$, the time-dependent transformations $\mathsf{S}_t : \mathcal{X}_p(0,1) \to \mathcal{X}_p(0,1)$, $t \ge 0$, which map a couple $(X_0, V_0) \in \mathcal{X}_p(0,1)$ into the couple $(X(t), V(t)) = \mathsf{S}_t(X_0, V_0) \in \mathcal{X}_p(0,1)$, where X is the solution of (one of the equivalent)* (L.I), (L.II), (L.III) *and $V = \frac{\mathrm{d}^+}{\mathrm{d}t} X$ as in* (L.a)*, define a right-continuous semigroup in $\mathcal{X}_p(0,1)$, satisfying*

$$\begin{aligned} (X(t), V(t)) = \mathsf{S}_t(X_0, V_0) \quad &\Longleftrightarrow \quad (\rho_t, \rho_t v_t) = \mathscr{S}_t(\rho_0, \rho_0 v_0), \\ \textit{where} \qquad \rho_t = (X(t))_{\#}\,\lambda, &\quad V(t) = v_t \circ X(t). \end{aligned} \tag{2.33}$$

Remark 2.8 (rescaling). Up to the rescaling $\tau = \log t$, $\hat{X}(\tau) = X(e^\tau)$, (L.III) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \hat{X}(t) \in -\partial I_{\mathcal{K}}(\hat{X}(t)) + \hat{X}(t) - X_0. \tag{2.34}$$

We shall show (see Theorem 3.1) that $\mathsf{P}_{\mathcal{K}}$ is a contraction in every $L^p(0,1)$ so that (L.II) provides a simple and sharp way to estimate $X(t)$ in terms of the initial data corresponding to (2.14b). Applying a general result of [25, 26], one can obtain (2.14c) from the representation (L.I).

Let us finally remark that the Wasserstein gradient flow of Theorem 2.4 is equivalent to (L.III)–(2.34). It is sufficient to introduce the functional Φ^σ

$$\Phi^\sigma(X) := -\frac{1}{2}\|X - X_\sigma\|_{L^2(0,1)} + I_{\mathcal{K}}(X), \quad X \in L^2(0,1), \tag{2.35}$$

which is related to ϕ^σ by

$$\phi^\sigma(\rho) = \Phi^\sigma(X_\rho) \quad \forall\, \rho \in \mathcal{P}_2(\mathbb{R}) \tag{2.36}$$

and is a smooth quadratic perturbation of the convex and lower semicontinuous indicator functional $I_{\mathcal{K}}$; since

$$\partial \Phi^\sigma(X) = \partial I_{\mathcal{K}}(X) - (X - X_\sigma), \tag{2.37}$$

(2.34) is the subdifferential formulation in $L^2(0,1)$ of the gradient flow of Φ^{ρ_0}, whose metric characterization [1] yields (2.21) thanks to the isometry $\rho \leftrightarrow X_\rho$ between $\mathcal{P}_2(\mathbb{R})$ and $\mathcal{K}$.

Remark 2.9 (minimal Lagrangian description). One can use (as in [28, 2]) the initial measure $\rho_0 \in \mathcal{P}(\mathbb{R})$ as a reference for the Lagrangian evolution, thus representing ρ_t as $\mathsf{x}(t)_{\#}\rho_0$ for the optimal monotone map $\mathsf{x}(t) = \mathsf{x}_0^t \in L^2_{\rho_0}(\mathbb{R})$ according to Theorem 2.3(f). We can therefore introduce the convex set $\mathcal{K}(\rho_0)$ of essentially nonincreasing Borel maps in the Hilbert space $L^2_{\rho_0}(\mathbb{R})$ and we have the corresponding formulae for the evolution in $L^2_{\rho_0}(\mathbb{R})$ ($\mathsf{i} : \mathbb{R} \to \mathbb{R}$ denotes the identity map):

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{x}(t) \in -\partial I_{\mathcal{K}(\rho_0)}(\mathsf{x}(t)) + v_0, \quad \mathsf{x}(0) = \mathsf{i}, \tag{L.I$'$}$$

$$\mathsf{x}(t) = \mathsf{P}_{\mathcal{K}(\rho_0)}(\mathsf{i} + t v_0), \quad \mathsf{i}(x) = x, \tag{L.II$'$}$$

$$t\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{x}(t) \in -\partial I_{\mathcal{K}(\rho_0)}(\mathsf{x}(t)) + \mathsf{x}(t) - \mathsf{i}, \tag{L.III$'$}$$

to be completed with the expression for the velocity $\frac{\mathrm{d}^+}{\mathrm{d}t}\mathsf{x}(t) = \mathsf{v}(t) = v_t \circ \mathsf{x}(t)$. All these relations could be easily deduced by Theorem 2.6, since the correspondence

$\mathsf{x} \leftrightarrow X = \mathsf{x} \circ X_0$ is an isometry between $L^2_{\rho_0}(\mathbb{R})$ and the closed subspace $\mathcal{H}_{X_0}$ of $L^2(0,1)$. On the other hand, it is easier to deal with the convex set $\mathcal{K}$ in the space $L^2(0,1)$ with the uniform Lebesgue measure as a reference. The description provided by Theorem 2.6 is more general, since it allows us to compare solutions arising from different initial data.

3. Main properties of $\mathcal{K}$. In this section we will study the properties of the convex set $\mathcal{K}$ of nondecreasing functions in $L^2(0,1)$, in particular the $L^2(0,1)$-projection operator $\mathsf{P}_\mathcal{K}$ and the subdifferential of the indicator function $I_\mathcal{K}$ (2.28). Denoting by $(\cdot|\cdot)$ (resp., $\|\cdot\|$) the usual scalar product (resp., the induced norm) in $L^2(0,1)$, since $\mathcal{K}$ is a convex cone, $\mathsf{P}_\mathcal{K}$ can be characterized by

$$g = \mathsf{P}_\mathcal{K}(f) \iff g \in \mathcal{K}, \quad (f-g|z-g) \le 0 \quad \forall\, z \in \mathcal{K}, \tag{3.1}$$

$$\iff g \in \mathcal{K}, \quad (f-g|z) \le 0 \quad \forall\, z \in \mathcal{K}, \qquad (f-g|g) = 0. \tag{3.2}$$

The next result provides a useful characterization of $\mathsf{P}_\mathcal{K}(f)$ in terms of the convex envelope of the primitive of f. Recall that the convex envelope of a given continuous function $F:[0,1]\to\mathbb{R}$ is defined as

$$F^{**}(w) := \sup\left\{a + bw : a, b \in \mathbb{R},\ a + bv \le F(v) \quad \forall\, v \in [0,1]\right\}, \quad w \in [0,1], \tag{3.3}$$

and it is the greatest bounded, (lower semi-) continuous, and convex function G satisfying $G \le F$ in $[0,1]$; it is therefore right- and left-differentiable at every point $t \in (0,1)$ and its right derivative $g := \frac{\mathrm{d}^+}{\mathrm{d}w} F^{**}$ is nondecreasing and right-continuous.

THEOREM 3.1 (projection on $\mathcal{K}$). *Let $f \in L^2(0,1)$ and let $F(w) = \int_0^w f(s)ds$ be its primitive. Then*

$$\mathsf{P}_\mathcal{K}(f) = g = \frac{\mathrm{d}^+}{\mathrm{d}w} F^{**},$$

where F^{**} is the convex envelope of F defined by (3.3)*. Moreover, for every convex lower semicontinuous function $\psi : \mathbb{R} \to (-\infty, +\infty]$ and every $f, h \in L^2(0,1)$, we have*

$$\int_\mathbb{R} \psi\left(\mathsf{P}_\mathcal{K}(f)\right) \mathrm{d}w \le \int_\mathbb{R} \psi(f)\,\mathrm{d}w, \quad \int_\mathbb{R} \psi\left(\mathsf{P}_\mathcal{K}(f) - \mathsf{P}_\mathcal{K}(h)\right) \mathrm{d}w \le \int_\mathbb{R} \psi(f-h)\,\mathrm{d}w. \tag{3.4}$$

In particular, $\mathsf{P}_\mathcal{K}$ is a contraction in every space $L^p(0,1)$, $p \in [1,+\infty]$:

$$\|\mathsf{P}_\mathcal{K}(f) - \mathsf{P}_\mathcal{K}(h)\|_{L^p(0,1)} \le \|f-g\|_{L^p(0,1)} \quad \forall\, f, h \in L^p(0,1). \tag{3.5}$$

We split the proof into several steps. Here is a preliminary lemma.

LEMMA 3.2. *For every $f \in L^2(0,1)$, F^{**} is continuous in $[0,1]$, locally Lipschitz in $(0,1)$, and coincides with F at $w=0$ and $w=1$. If $f \in L^\infty(0,1)$, then F and F^{**} are Lipschitz continuous in the closed interval $[0,1]$.*

Proof. Let us first assume $f \in L^\infty(0,1)$, and let L be the Lipschitz constant of F; then

$$F(0) - Lw \le F(w), \quad F(1) + L(w-1) \le F(w) \quad \forall\, w \in [0,1]$$

so that $F^{**}(0) = F(0)$, $F^{**}(1) = F(1)$, and

$$F(0) - Lw \le F^{**}(w), \quad F(1) + L(w-1) \le F^{**}(w) \quad \forall\, w \in [0,1]. \tag{3.6}$$

Therefore the right derivative g of F^{**} satisfies $-L \le g(0) \le g(w) \le \frac{\mathrm{d}^-}{\mathrm{d}w} F^{**}(1) \le L$ so that F^{**} is a Lipschitz function.

In the general case when $f \in L^2(0,1)$, we can approximate its (absolutely continuous) primitive F by an increasing sequence of Lipschitz functions F_n uniformly converging to F, e.g., by setting

$$F_n(w) = \inf_{v \in [0,1]} F(v) + n|v-w|.$$

Thus F_n^{**} is an increasing sequence of Lipschitz functions satisfying $F_n^{**}(w) = F_n(w)$ at $w = 0, 1$ and pointwise converging to some lower semicontinuous convex function G as $n \uparrow +\infty$ with

$$G(w) \le F^{**}(w) \le F(w) \quad \forall\, w \in [0,1]. \tag{3.7}$$

On the other hand, for $w = 0, 1$ we have $G(w) = \lim_{n\uparrow+\infty} F_n(w) = F(w)$ so that $F^{**}(w) = F(w)$. $\square$

Let us now consider the set

$$\Lambda = \left\{ w \in [0,1] : (F - F^{**})(w) > 0 \right\}. \tag{3.8}$$

Since Λ is open and does not contain 0 and 1, it is the disjoint union of an (at most countable) collection $\mathcal{O}$ of open intervals.

LEMMA 3.3. *If $(a,b) \in \mathcal{O}$ is a connected component of Λ, then for every $w = (1-\theta)a + \theta b$, $\theta \in [0,1]$*

$$\begin{gathered} F^{**}((1-\theta)a + \theta b) = (1-\theta)F(a) + \theta F(b) \quad \forall\, \theta \in [0,1], \\ F(a) = F^{**}(a),\ F(b) = F^{**}(b). \end{gathered} \tag{3.9}$$

Proof. Since $a, b \notin \Lambda$, one has $F(a) = F^{**}(a)$ and $F(b) = F^{**}(b)$. Let $\bar{w} \in [a,b]$ be a minimizer of the continuous function

$$w \mapsto F(w) - L(w), \quad L(w) := F(a) + (w-a)\frac{F(b) - F(a)}{b-a}$$

so that $F(w) \ge F(\bar{w}) + L(w) - L(\bar{w})$ for every $w \in [a,b]$. The continuous function

$$G(w) := \begin{cases} F^{**}(w) & \text{if } w \notin [a,b], \\ \max\left(F^{**}(w), F(\bar{w}) + L(w) - L(\bar{w})\right) & \text{if } w \in [a,b] \end{cases} \tag{3.10}$$

provides a convex lower bound of F and therefore $G(w) \le F^{**}(w)$ for every $w \in [0,1]$. Since $G(\bar{w}) = F(\bar{w})$, we deduce that $\bar{w} \notin \Lambda$ and therefore $\bar{w}$ coincides with a or b and the inequality $G(w) \le F^{**}(w)$ yields $F^{**}((1-\theta)a + \theta b) \ge (1-\theta)F(a) + \theta F(b)$. The opposite inequality is a consequence of the convexity of F^{**}. $\square$

The next lemma contains the crucial inequality we need to characterize $\mathsf{P}_{\mathcal{K}}$.

LEMMA 3.4. *Let $\psi \in C^1(\mathbb{R})$ be a convex function. For every $f \in L^2(0,1)$ and $z \in \mathcal{K}$ with $g := (F^{**})'$, if $(f-g)\psi'(z-g) \in L^1(0,1)$, we have*

$$\int_0^1 \left(f(w) - g(w)\right)\psi'\left(z(w) - g(w)\right)\mathrm{d}w \le 0 \le \int_0^1 \left(f(w) - g(w)\right)\psi'\left(g(w) - z(w)\right)\mathrm{d}w. \tag{3.11}$$

Proof. We decompose $[0,1]$ in the disjoint union of the open intervals $(a,b) \in \mathcal{O}$ covering Λ (see (3.8)) and of $[0,1] \setminus \Lambda$; note that $F(w) = F^{**}(w)$ on $[0,1] \setminus \Lambda$, and so

$f(w) = g(w)$ a.e. in $[0,1] \setminus \Lambda$ (recall that F^{**} is locally Lipschitz). In each $(a,b) \in \mathcal{O}$, F^{**} is linear, g is constant, and the function $w \mapsto \psi'(z(w) - g)$ is bounded and nondecreasing; thus, its distributional derivative is a nonnegative finite measure $\gamma_{a,b}$. Since $F = F^{**}$ in $\{a,b\}$, we have

$$\int_0^1 (f-g)\,\psi'(z-g)\,\mathrm{d}w = \int_\Lambda (f-g)\psi'(z-g)\,\mathrm{d}w + \int_{[0,1]\setminus\Lambda} (f-g)\psi'(z-g)\,\mathrm{d}w$$
$$= \sum_{(a,b)\in\mathcal{O}} \int_a^b (f-g)\psi'(z-g)\,\mathrm{d}w = -\sum_{(a,b)\in\mathcal{O}} \int_a^b (F(w) - F^{**}(w))\,d\gamma_{a,b}(w) \le 0.$$

The second inequality of (3.11) can be obtained simply by considering the convex function $\tilde{\psi}(r) := \psi(-r)$. ◻

End of the proof of Theorem 3.1. Concerning the projection in $L^2(0,1)$, (3.1) follows (3.11) by choosing $\psi(r) := \frac{1}{2}r^2$.

In order to prove (3.4), a standard approximation of ψ by the increasing sequence of its Moreau–Yosida approximations $\psi_n(r) := \min_{s\in\mathbb{R}} \psi(s) + \frac{n}{2}|s-r|^2$ shows that it is not restrictive to assume ψ convex, C^1, and at most quadratically growing as $|r| \to \infty$. We can then apply the standard convexity inequality $\psi(s) - \psi(r) \ge \psi'(r)(s-r)$ and Lemma 3.4, obtaining

$$\int_\mathbb{R} \left(\psi\left(f-h\right) - \psi\left(\mathsf{P}_\mathcal{K}(f) - \mathsf{P}_\mathcal{K}(h)\right)\right)\,\mathrm{d}w$$
$$\ge \int_\mathbb{R} \psi'\left(\mathsf{P}_\mathcal{K}(f) - \mathsf{P}_\mathcal{K}(h)\right)\left((f - \mathsf{P}_\mathcal{K}(f)) - (h - \mathsf{P}_\mathcal{K}(h))\right)\,\mathrm{d}w \overset{(3.11)}{\ge} 0.$$

The first inequality of (3.4) is a particular case of the second one, with $h = \mathsf{P}_\mathcal{K}(h) = 0$. ◻

The following result is a simple consequence of Theorem 3.1. Let us first introduce for a given $f \in L^2(0,1)$ the open set $\Omega_f \subset (0,1)$, where f is locally constant

$$\Omega_f := \{w \in (0,1) : f \text{ is essentially constant in a neighborhood of } w\}. \tag{3.12}$$

Equivalently Ω_f is the complement of the support of the distributional derivative of f.

COROLLARY 3.5. *Let $f \in L^2(0,1)$ and $g = \mathsf{P}_\mathcal{K}(f)$. Then*

$$\Omega_f \subset \Omega_g. \tag{3.13}$$

Proof. Note that $\Lambda \subset \Omega_g$ (Λ has been defined by (3.8)); if $w \in \Omega_f \setminus \Lambda$, then $F(w) = F^{**}(w)$, so that any linear part of the graph of F in an open interval containing w should locally coincide with F^{**}; it follows that $F^{**} = F$ in a neighborhood of w so that $w \in \Omega_g$. ◻

DEFINITION 3.6 (the polar cone and the subdifferential of the indicator function $I_\mathcal{K}$). *We denote by $\mathcal{K}^\circ$ the polar cone of $\mathcal{K}$, defined by*

$$f \in \mathcal{K}^\circ \iff (f|z) \le 0 \quad \forall\, z \in \mathcal{K} \iff \mathsf{P}_\mathcal{K}(f) = 0. \tag{3.14}$$

The subdifferential $\partial I_\mathcal{K}(g)$ of the indicator function of $\mathcal{K}$ (see (2.35)*) at some function $g \in \mathcal{K}$ is the subset of $L^2(0,1)$ characterized by*

$$\xi \in \partial I_\mathcal{K}(g) \iff (\xi|z-g) \le 0 \quad \forall\, z \in \mathcal{K}. \tag{3.15}$$

Remark 3.7. $\mathcal{K}^\circ$ and $\partial I_{\mathcal{K}}$ are clearly linked by $\mathcal{K}^\circ = \partial I_{\mathcal{K}}(0)$ and

$$\xi \in \partial I_{\mathcal{K}}(g) \quad \Longleftrightarrow \quad \xi \in \mathcal{K}^\circ, \quad (\xi|g) = 0. \tag{3.16}$$

$\mathcal{K}^\circ$ provides an equivalent reformulation of (3.1), since

$$g = \mathsf{P}_{\mathcal{K}}(f) \iff g \in \mathcal{K},\ \ f - g \in \mathcal{K}^\circ,\ (f-g|g) = 0 \iff f - g \in \partial I_{\mathcal{K}}(g). \tag{3.17}$$

If Ω is an open subset of $(0,1)$, we denote by $\mathcal{N}_\Omega$ the convex cone

$$\mathcal{N}_\Omega := \left\{ F \in C^0([0,1]) : F \geq 0 \text{ in } [0,1], \quad F = 0 \text{ in } [0,1] \setminus \Omega \right\}. \tag{3.18}$$

We can give a useful characterization of $\mathcal{K}^\circ$ in terms of the cone $\mathcal{N} := \mathcal{N}_{(0,1)}$.

PROPOSITION 3.8 (a characterization of the polar cone $\mathcal{K}^\circ$). *A function f belongs to the polar cone $\mathcal{K}^\circ$ if and only if its primitive $F(w) := \int_0^w f(s)\,\mathrm{d}s$ belongs to $\mathcal{N}$.*

Proof. If $F \in \mathcal{N}$, then one easily gets for every $z \in \mathcal{K} \cap C^1([0,1])$

$$(f|z) = \int_0^1 F'(w)\, z(w)\,\mathrm{d}w = -\int_0^1 F(w)\, z'(w)\,\mathrm{d}w \leq 0, \tag{3.19}$$

since $F, z' \geq 0$, $F(0) = F(1) = 0$.

Let us now assume that $f \in \mathcal{K}^\circ$; for every continuous and nonnegative function $z \geq 0$ and $c \in \mathbb{R}$ we replace $Z(w) = \int_0^w z(s)\,\mathrm{d}s - c$. Since $Z \in \mathcal{K}$ we have

$$0 \geq (f|Z) = \int_0^1 f(w)\, Z(w)\,\mathrm{d}w = -\int_0^1 F(w)\, z(w)\,\mathrm{d}w + F(1)(Z(1) - c).$$

Since c, z are arbitrary, we conclude that $F \in \mathcal{N}$. □

The last result of this section concerns a precise characterization of $\partial I_{\mathcal{K}}$. Let us first define for $f \in L^2(0,1)$ the closed subspace $\mathcal{H}_f \subset L^2(0,1)$, defined as

$$\mathcal{H}_f := \left\{ h \in L^2(0,1) : h \text{ is essentially constant in each connected component of } \Omega_f \right\}. \tag{3.20}$$

We denote by $\mathsf{P}_{\mathcal{H}_f}$ the orthogonal L^2-projection on $\mathcal{H}_f$. It is easy to check that

$$\begin{aligned} &\mathsf{P}_{\mathcal{H}_g}(f) = f \text{ a.e. in } (0,1) \setminus \Omega_g, \\ &\mathsf{P}_{\mathcal{H}_g}(f) \equiv \fint_\alpha^\beta f(w)\,\mathrm{d}w \text{ a.e. in every connected component } (\alpha,\beta) \subset \Omega_g. \end{aligned} \tag{3.21}$$

Moreover, denoting by F the primitive function of f,

$$\text{if } F \in \mathcal{N}_{\Omega_g}, \text{ then } f \text{ is orthogonal to } \mathcal{H}_g, \tag{3.22}$$

since $f = F'$ vanishes a.e. outside Ω_g and for every connected component (α, β) of Ω_g we have $\int_\alpha^\beta f(w)\,\mathrm{d}w = F(\beta) - F(\alpha) = 0$.

THEOREM 3.9 (the subdifferential of $I_{\mathcal{K}}$). *Let $g \in \mathcal{K}$, $\xi \in L^2(0,1)$, and $\Xi(w) := \int_0^w \xi(s)\,\mathrm{d}s$. Then we have*

$$\xi \in \partial I_{\mathcal{K}}(g) \quad \Longleftrightarrow \quad \Xi \in \mathcal{N}_{\Omega_g}. \tag{3.23}$$

In particular,

$$\text{if } \xi \in \partial I_{\mathcal{K}}(g), \text{ then } \quad \begin{cases} \xi = 0 \ \text{a.e. in } [0,1] \setminus \Omega_g, \\ \displaystyle\int_\alpha^\beta \xi(w)\,\mathrm{d}w = 0 \quad \text{for every connected component } (\alpha,\beta) \text{ of } \Omega_g \end{cases} \tag{3.24}$$

so that ξ *is orthogonal to* $\mathcal{H}_g$ *and we have by* (3.17) *and* (3.13)

$$g = \mathsf{P}_{\mathcal{K}}(f) \quad \Rightarrow \quad g = \mathsf{P}_{\mathcal{H}_g}(f), \quad \mathcal{H}_g \subset \mathcal{H}_f. \tag{3.25}$$

Proof. The left implication in (3.23) is immediate, since $\Xi \in \mathcal{N}_{\Omega_g}$ implies $\Xi \in \mathcal{N}$ and therefore $\xi \in \mathcal{K}^\circ$ by Proposition 3.8; moreover, ξ is orthogonal to $\mathcal{H}_g$ by (3.22) and therefore it is also orthogonal to $g \in \mathcal{H}_g$, so that $\xi \in \partial I_{\mathcal{K}}(g)$ by (3.16).

Conversely, if $\xi \in \partial I_{\mathcal{K}}(g)$, then $\Xi \in \mathcal{N}$ by (3.16) and Proposition 3.8. Moreover, denoting by $\gamma = g'$ the nonnegative Radon measure associated to the distributional derivative of g in $(0,1)$, the next Lemma 3.10 yields

$$0 \overset{(3.16)}{=} \int_0^1 \xi(w) g(w)\,\mathrm{d}w \overset{(3.27)}{=} -\int_0^1 \Xi(w)\,\mathrm{d}\gamma(w), \tag{3.26}$$

which shows that $\Xi(w) = 0$ on the support of γ and yields $\Xi \in \mathcal{N}_{\Omega_g}$. ∎

LEMMA 3.10. *Let* $g \in \mathcal{K}$ *and* $\xi \in \mathcal{K}^\circ$ *with (nonnegative) primitive* $\Xi \in \mathcal{N}$*. If* $\gamma = g'$ *is the nonnegative Radon measure associated to the distributional derivative of* g *in* $(0,1)$*, then* $\Xi \in L^1(\gamma)$ *and*

$$\int_0^1 g(w)\xi(w)\,\mathrm{d}w = -\int_0^1 \Xi(w)\,\mathrm{d}\gamma(w). \tag{3.27}$$

Proof. Since γ is a nonnegative Radon measure in $(0,1)$ but is not necessarily finite, we need an approximation argument to justify (3.27). Let $\varphi_n \in C_0^\infty(0,1)$ be an increasing sequence of nonnegative functions such that $\lim_{n\uparrow+\infty}\varphi_n(w) = 1$, $|\varphi_n'| \le 2n$, and $\varphi_n(w) \equiv 1$ for $1/n \le w \le 1-1/n$. We have

$$\int_0^1 g\xi\varphi_n\,\mathrm{d}w = -\int_0^1 \Xi\varphi_n\,\mathrm{d}\gamma - \int_0^1 \Xi g\varphi_n'\,\mathrm{d}w \tag{3.28a}$$

$$= -\int_0^1 \Xi\varphi_n\,\mathrm{d}\gamma - \int_0^{1/n} \Xi g\varphi_n'\,\mathrm{d}w - \int_{1-1/n}^1 \Xi g\varphi_n'\,\mathrm{d}w. \tag{3.28b}$$

Applying Hardy's inequality, we get

$$\left|\int_0^{1/n} \Xi g\varphi_n'\,\mathrm{d}w\right| \le 2n\|w^{-1}\Xi\|_{L^2(0,1/n)}\|wg\|_{L^2(0,1/n)} \le 2C\|\xi\|_{L^2(0,1)}\|g\|_{L^2(0,1/n)}$$

so that the integral vanishes as $n\uparrow+\infty$. A similar argument holds for the last integral of (3.28b). Passing to the limit in (3.28a), (3.28b) as $n\uparrow+\infty$ and using the Lebesgue dominated (since $g\xi \in L^1(0,1)$) or monotone (since $\Xi \ge 0$ and φ_n is increasing) convergence theorem, we conclude. ∎

The last lemma of this section provides a useful example concerning a class of elements in $\partial I_{\mathcal{K}}(g)$.

LEMMA 3.11 (an example of minimal selection in $\partial I_{\mathcal{K}}$). *If* $g, h \in \mathcal{K}$*, then*

$$\xi_h := \mathsf{P}_{\mathcal{H}_g}(h) - h \in \partial I_{\mathcal{K}}(g). \tag{3.29}$$

Moreover,

$$\|z - h - \xi_h\|_{L^2(0,1)} \le \|z - h - \xi\|_{L^2(0,1)} \quad \forall\,\xi \in \partial I_{\mathcal{K}}(g), \quad z \in \mathcal{H}_g. \tag{3.30}$$

In particular,

$$\text{(3.31)} \qquad \textit{if } z \in \mathcal{H}_g, \textit{ then} \quad \|z\|_{L^2(0,1)} \le \|z-\xi\|_{L^2(0,1)} \quad \forall\, \xi \in \partial I_K(g).$$

Proof. Since $h - \mathsf{P}_{\mathcal{H}_g}(h)$ is orthogonal to $\mathcal{H}_g$ (thus, in particular, to g), by (3.16) we have to check that $\xi_h \in \mathcal{K}^\circ$ by applying Proposition 3.8. By (3.21), $\xi_h = 0$ a.e. in $(0,1)\setminus\Omega_g$, so that the primitive Ξ_h of ξ_h satisfies

$$\Xi_h(w) = \int_{\Omega_g \cap (0,w)} \xi_h(s)\,\mathrm{d}s.$$

The thesis then follows if we show that for every connected component (α,β) of Ω_g we have $\Xi_h(\alpha) = \Xi_h(\beta) = 0 = \min_{[\alpha,\beta]} \Xi_h$. Since the characteristic function $\chi_{(0,\alpha)}$ of $(0,\alpha)$ belongs to $\mathcal{H}_g$, we have

$$\Xi_h(\alpha) = \int_0^\alpha \xi_h(w)\,\mathrm{d}w = (h - \mathsf{P}_{\mathcal{H}_g}(h)|\chi_{(0,\alpha)}) = 0.$$

A similar argument shows that $\Xi_h(\beta) = 0$. Moreover, for $w \in (\alpha,\beta)$ we have

$$\Xi_h(w) = \int_\alpha^w \xi_h(s)\,\mathrm{d}s \stackrel{(3.21)}{=} (w-\alpha) \fint_\alpha^\beta h(w)\,\mathrm{d}w - \int_\alpha^w h(w)\,\mathrm{d}w,$$

which shows that Ξ_h is concave and therefore nonnegative in (α,β).

Equation (3.30) follows immediately by observing that $\xi, \xi_h \in (\mathcal{H}_g)^\perp$ and $z-h-\xi_h$ belongs to $\mathcal{H}_g$ and, therefore, it is the orthogonal projection of $z-h$ and of $z-h-\xi$ onto $\mathcal{H}_g$. ∎

4. The Lagrangian formulation of the discrete sticky particle system. In this section we shall show that the discrete SPS satisfies the three characterizations of Theorem 2.6 and we prove Theorem 2.2.

Notation 4.1. Let us recapitulate our basic notation and definitions.

1. $P_i(t) = (m_i, x_i(t), v_i(t))$, $i \in I = \{1,\dots,N\}$, $t \ge 0$ is a solution of the discrete SPS.
2. The positions of the particles are ordered: $x_1(t) \le x_2(t) \le \cdots \le x_N(t)$.
3. The sets $J_i(t) := \{j \in I : x_j(t) = x_i(t)\}$ are nondecreasing with respect to time. They correspond to a single particle of mass $\sum_{j\in J_i(t)} m_j$.
4. At each time t we pick up the collection of minimal indices
$$I(t) := \{\min J_i(t) : i = 1,\dots,N\} = \big\{i_1(t) < \cdots < i_{N(t)}(t)\big\} \subset I$$
so that each $J_i(t)$ is of the form $\{j \in I : i_k(t) \le j < i_{k+1}(t)\}$ for some k and $(J_i(t))_{i\in I(t)}$ is a partition of I.
5. We denote by $0 < t_1 < t_2 < \cdots t_h < \cdots < t_{H-1}$ the (finite) sequence of times at which the cardinality of some $J_i(t)$ has an increasing jump; setting $t_0 = 0$ and $t_H = +\infty$, $\{[t_h, t_{h+1})\}_{h=0}^H$ is the associate partition of the positive real line with step sizes $\delta_h := t_h - t_{h-1}$.
6. The functions x_i are globally continuous and linear on each interval $[t_h, t_{h+1})$, with piecewise constant, right-continuous derivatives $v_i(t)$ satisfying (1.2). Each set $J_i(t)$ and $I(t)$ is also constant in each interval $[t_h, t_{h+1})$.

Let $\rho_t = \sum_{i\in I} m_i \delta_{x_i(t)}$ be the measure induced by the discrete SPS. In order to explicitly write the function $X(t) := X_{\rho_t}$ we consider the subdivision of $[0,1]$ given by

$$\text{(4.1)} \qquad w_0 = 0 < w_1 < \cdots < w_N = 1, \quad w_i = w_{i-1} + m_i = \sum_{j=1}^i m_j, \qquad i \in I.$$

We also set

$$W_i := [w_{i-1}, w_i), \quad W_i(t) = \bigcup_{j\in J_i(t)} W_j, \qquad i \in I, \tag{4.2}$$

and we note that

$$X(t) = \sum_{i=1}^N x_i(t)\mathbb{1}_{W_i}, \quad \frac{\mathrm{d}^+}{\mathrm{d}t} X(t) = V(t) = \sum_{i=1}^N v_i(t)\mathbb{1}_{W_i}. \tag{4.3}$$

The main result of this section is contained in the following theorem.

THEOREM 4.2 (Lagrangian formulation of the discrete SPS). *The couple* (X, V), *defined by* (4.3), *satisfies the equations* (L.I)–(L.III) *and the properties* (L.a)–(L.c) *of Theorem* 2.6. *In particular, it defines a semigroup* S_t *in the discrete subspace*

$$\begin{aligned}\hat{\mathcal{X}} := \Big\{ &(X, V) \in \mathcal{X}_p(0,1) : X = \sum_{i=1}^N x_i \mathbb{1}_{W_i} \\ &\textit{for a finite interval partition } (W_i)_{i=1}^N \textit{ of } [0,1) \Big\}.\end{aligned} \tag{4.4}$$

Proof. We split the proof into various steps.

The collection $(W_i(t))_{i\in I(t)}$ is a partition of $[0,1)$. In $L^2(0,1)$ we introduce the decreasing family of finite-dimensional spaces $\mathcal{H}(t)$ whose elements are piecewise constant on each interval $W_i(t)$, $i \in I(t)$. Note that, by the very definitions of $\Omega_{X(t)}$ and $\mathcal{H}_{X(t)}$, (3.12) and (3.20),

$$\Omega_{X(t)} = (0,1) \setminus \{w_i : i \in I(t)\}, \quad \mathcal{H}(t) = \mathcal{H}_{X(t)}. \tag{4.5}$$

Besides (4.3), the crucial features describing the evolution of $X(t)$ are

$$\begin{aligned}&X(t) \in \mathcal{K} \cap \mathcal{H}(t), \quad V(t) \in \mathcal{H}(t), \\ &\mathcal{H}(t), V(t) \text{ are constant in each time interval } [t_h, t_{h+1})\end{aligned} \tag{4.6}$$

(we set $\mathcal{H}_h := \mathcal{H}(t)$, $V_h := V(t)$ if $t \in [t_h, t_{h+1})$) and the update rule for the velocity (1.2): $V(t_h)$ is constant in each interval $W_i(t_h) = \cup_{j\in J_i(t_h)} W_j$ and its value is given by

$$V(t_h+)|_{W_i(t_h)} = \frac{\sum_{j\in J_i(t_h)} m_j v_j(t_{h-1})}{\sum_{j\in J_i(t_h)} m_i} = \big(\mathscr{L}^1(W_i(t_h))\big)^{-1} \int_{W_i(t_h)} V(t_{h-1})\,\mathrm{d}w,$$

so that by (3.21)

$$V_h = \mathsf{P}_{\mathcal{H}_h}(V_{h-1}) = \mathsf{P}_{\mathcal{H}_h}(V_0) \quad \text{since} \quad \mathcal{H}_0 \supset \mathcal{H}_1 \supset \mathcal{H}_2 \supset \cdots \mathcal{H}_h, \tag{4.7}$$

which yields (L.a) and (L.b). The next lemma shows (L.III).

LEMMA 4.3. *Let* $\tilde{X}(t) := X_0 + tV_0$ *be associated to the free system* $\tilde{P}_i = (m_i, \tilde{x}_i, \tilde{v}_i)$ *given by* $\tilde{x}_i(t) = x_i(0) + tv_i(0)$, $\tilde{v}_i(t) \equiv \tilde{v}_i = v_i(0)$. *Then*

$$X(t) = \mathsf{P}_{\mathcal{H}(t)}(\tilde{X}(t)) = \mathsf{P}_{\mathcal{H}(t)}(X_0 + tV_0), \tag{4.8}$$

$$t\frac{\mathrm{d}^+}{\mathrm{d}t} X(t) = tV(t) = X(t) - X_0 - \Xi(t) \quad \textit{for} \quad \Xi(t) := -X_0 + \mathsf{P}_{\mathcal{H}(t)}(X_0) \in \partial I_{\mathcal{K}}(X(t)).$$

Proof. Suppose that $t \in [t_h, t_{h+1})$; since $X(t) \in \mathcal{H}(t) = \mathcal{H}_h \subset \mathcal{H}(r)$ and $V(r) = \mathsf{P}_{\mathcal{H}(r)}(V_0)$ for $0 \le r \le t$ by (4.7), we have by the linearity of $\mathsf{P}_{\mathcal{H}(r)}$

$$\begin{aligned} X(t) = X_0 + \int_0^t V(r)\,\mathrm{d}r &\overset{(4.6)}{=} \mathsf{P}_{\mathcal{H}(t)}\left(X_0 + \int_0^t V(r)\,\mathrm{d}r\right) \\ &\overset{(4.7)}{=} \mathsf{P}_{\mathcal{H}(t)}(X_0) + \int_0^t \mathsf{P}_{\mathcal{H}(t)}(\mathsf{P}_{\mathcal{H}(r)}(V_0))\,\mathrm{d}r = \mathsf{P}_{\mathcal{H}(t)}(X_0) + \int_0^t \mathsf{P}_{\mathcal{H}(t)}(V_0)\,\mathrm{d}r \\ &= \mathsf{P}_{\mathcal{H}(t)}\left(X_0 + \int_0^t V_0\,\mathrm{d}r\right) = \mathsf{P}_{\mathcal{H}(t)}(\tilde{X}(t)). \end{aligned}$$

From (4.8) we have

$$t\frac{\mathrm{d}^+}{\mathrm{d}t}X(t) = tV(t) \overset{(4.7)}{=} \mathsf{P}_{\mathcal{H}(t)}(tV_0) \overset{(4.8)}{=} X(t) - \mathsf{P}_{\mathcal{H}(t)}(X_0) = X(t) - X_0 - \Xi(t),$$

where $\Xi(t) = \mathsf{P}_{\mathcal{H}(t)}(X_0) - X_0$; since $X_0 \in \mathcal{K}$ and $\mathcal{H}(t) = \mathcal{H}_{X(t)}$, by Lemma 3.11 we conclude that $\Xi(t) \in \partial I_{\mathcal{K}}(X(t))$. ∎

We now conclude the proof of (L.I) and (L.II); note that (L.c) follows directly from (L.II) and (L.a) via the semigroup property of $\mathscr{S}_t$ in $\hat{\mathcal{V}}(\mathbb{R})$.

LEMMA 4.4. *Under the same notation and assumptions as before, we have*

$$U(t) := V_0 - V(t) = V_0 - \mathsf{P}_{\mathcal{H}(t)}(V_0) \in \partial I_{\mathcal{K}}(X(t)), \quad X(t) = \mathsf{P}_{\mathcal{K}}(X_0 + tV_0). \tag{4.9}$$

Proof. Since $(U(t)|X(t)) = 0$ $(X(t) \in \mathcal{H}(t))$, the first inclusion of (4.9) is equivalent to

$$U(t) = V_0 - V(t) \in \mathcal{K}^\circ \quad \forall\, t \ge 0 \tag{4.10}$$

by (3.16). It is not restrictive to assume that $t = t_h$ and $V(t) = V_h$ for some $h \in \{1, \dots, H-1\}$. Since $\mathcal{K}^\circ$ is a cone and $V_0 - V_h$ can be decomposed into the sum

$$V_0 - V_h = \sum_{k=0}^{h-1}(V_k - V_{k+1}), \tag{4.11}$$

it is sufficient to prove that $V_k - V_{k+1} \in \mathcal{K}^\circ$ or, equivalently, that $\delta_{k+1}(V_k - V_{k+1}) \in \mathcal{K}^\circ$. Since $V_{k+1} = \mathsf{P}_{\mathcal{H}_{k+1}}(V_k)$ we obtain

$$\begin{aligned} \delta_{k+1}(V_k - V_{k+1}) &= \delta_{k+1}V_k - \mathsf{P}_{\mathcal{H}_{k+1}}(\delta_{k+1}V_k) = (X_{k+1} - X_k) - \mathsf{P}_{\mathcal{H}_{k+1}}(X_{k+1} - X_k) \\ &= X_{k+1} - \mathsf{P}_{\mathcal{H}_{k+1}}(X_{k+1}) + \mathsf{P}_{\mathcal{H}_{k+1}}(X_k) - X_k = \mathsf{P}_{\mathcal{H}_{k+1}}(X_k) - X_k \in \mathcal{K}^\circ \end{aligned}$$

by Lemma 3.11 and (3.16).

The second identity of (4.9) follows now by a similar argument by checking the conditions of (3.17). Since $X(t) \in \mathcal{K}$ and $(\tilde{X}(t) - X(t)|X(t)) = 0$ by (4.8), it is sufficient to show that $\tilde{X}(t) - X(t) \in \mathcal{K}^\circ$. On the other hand,

$$\tilde{X}(t) - X(t) = tV_0 - \int_0^t V(r)\,\mathrm{d}r = \int_0^t (V_0 - V(r))\,\mathrm{d}r = \int_0^t U(r)\,\mathrm{d}r$$

and (4.10) shows that $U(r) \in \mathcal{K}^\circ$ for every $r \ge 0$. Since $\mathcal{K}^\circ$ is a cone, we conclude. ∎

Proof of Theorem 2.2. Let us now consider two discrete Lagrangian solutions $(X^\ell(t), V^\ell(t)) = \mathsf{S}_t(X_0^\ell, V_0^\ell) \in \hat{\mathcal{X}}$, $\ell = 1, 2$. Equations (3.4), (3.5), and (L.II) immediately yield the estimates

$$\int_0^1 \psi\left(X^1(t) - X^2(t)\right) \mathrm{d}w \le \int_0^1 \psi\left(X_0^1 - X_0^2 + t(V_0^1 - V_0^2)\right) \mathrm{d}w, \tag{4.12}$$

$$\|X^1(t) - X^2(t)\|_{L^p(0,1)} \le \|X_0^1 - X_0^2\|_{L^p(0,1)} + t\|V_0^1 - V_0^2\|_{L^p(0,1)}, \tag{4.13}$$

which are equivalent to (2.14a) and (2.14b). Equation (L.I) yields [25, Thm. 3], [26, Thm. 1.2]

$$\int_0^t \|V^1 - V^2\|^2 \,\mathrm{d}r \le C(1+t)\left(\sum_{\ell=1,2} \|X_0^\ell\| + \|V_0^\ell\|\right)\left(\|X_0^1 - X_0^2\| + \|V_0^1 - V_0^2\|\right) \tag{4.14}$$

(here $\|\cdot\|$ denotes the norm of $L^2(0,1)$), which is equivalent to (2.14c). ◻

5. Stability and uniqueness of Lagrangian solutions. Our first result concerns the stability of Lagrangian solutions to (L.I)–(L.III) of Theorem 2.6 (in particular, it applies to those obtained by the discrete SPS in $\hat{\mathcal{X}}$).

LEMMA 5.1. *Let $X^n, V^n := \frac{\mathrm{d}^+}{\mathrm{d}t} X^n$ be curves satisfying the equations* (L.I)–(L.III) *and the properties* (L.a)–(L.c) *stated in Theorem* 2.6 *with respect to initial data $X_0^n, V_0^n = v_0^n(X_0^n)$ converging to $X_0, V_0 = v_0(X_0)$ in $L^p(0,1)$, $p \ge 2$. Then we have the following.*

(a) *$X^n(t)$ converges to $X(t)$ in $L^p(0,1)$ uniformly in each compact interval; X is Lipschitz continuous with values in $L^p(0,1)$.*

(b) *The Lipschitz curve X is right-differentiable at each point t, with right-continuous derivative $V(t)$, and it satisfies* (L.I)–(L.III) *and* (L.a)–(L.c) *of Theorem* 2.6.

(c) *V^n strongly converges to V in $L^2(0,T; L^2(0,1))$ for every $T > 0$.*

(d) *The curve X is differentiable in $L^p(0,1)$ and V is continuous at each point of $(0,+\infty) \setminus \mathcal{T}$, where $\mathcal{T}$ is the jump set of the nonincreasing map $t \mapsto \|V(t)\|_{L^2(0,1)}$.*

(e) *If $\bar{V}$ is any weak accumulation point of $V^n(t)$ in $L^p(0,1)$, then $\mathsf{P}_{\mathcal{H}_{X(t)}}(\bar{V}) = V(t)$.*

(f) *$V^n(t) \to V(t)$ in $L^p(0,1)$ for every $t \in [0,+\infty) \setminus \mathcal{T}$.*

Proof. (a) The proof of (a) is an immediate consequence of (L.II) and (3.5), which also shows that X^n is uniformly Lipschitz continuous with values in $L^p(0,1)$ and Lipschitz constant bounded by $\|V_0^n\|_{L^p(0,1)}$. The convergence is therefore uniform in each compact interval and the limit function X satisfies the same Lipschitz bound with constant $\|V_0\|_{L^p(0,1)}$.

(b), (c) Standard stability results for gradient flows in Hilbert spaces [12] show that X solves (L.I) and (L.III); in particular X is right-differentiable in $L^2(0,1)$ at each $t \ge 0$, with $L^2(0,1)$ right derivative $V(t)$ which is right-continuous. Equation (4.14) shows that V is the limit of V^n in $L^2(0,T; L^2(0,1))$ for every $T > 0$ (this proves point (c)): in particular, up to the extraction of a suitable subsequence n_k, we can find an $\mathscr{L}^1$-negligible set $N \subset (0,+\infty)$ such that $V^{n_k}(t) \to V(t)$ in $L^2(0,1)$ for every $t \in [0,+\infty) \setminus N$ as $k \uparrow +\infty$. Passing to the limit in (L.c) and in (L.I), we obtain that

$$X(t) = \mathsf{P}_{\mathcal{K}}(X(s) + (t-s)V(s)), \qquad \frac{\mathrm{d}^+}{\mathrm{d}t} X(t) = V(t) \in -\partial I_{\mathcal{K}}(X(t)) + V(s) \tag{5.1}$$

for every $s \in [0, +\infty) \setminus N$ and $t \geq s$. Since V is right-continuous, (5.1) eventually holds for every $0 \leq s \leq t$.

The projection formula of (5.1) shows that for every $h \geq 0$

$$\|X(t+h) - X(t)\|_{L^p(0,1)} \leq h\|V(t)\|_{L^p(0,1)} \leq h\|V(s)\|_{L^p(0,1)} \quad \forall\, 0 \leq s \leq t, \tag{5.2}$$

and, more generally,

$$\int_0^1 \psi\left(h^{-1}(X(t+h) - X(t))\right) \mathrm{d}w \leq \int_0^1 \psi(V_s)\, \mathrm{d}w \leq \int_0^1 \psi(V_0)\, \mathrm{d}w \quad \forall\, 0 \leq s \leq t \tag{5.3}$$

for every convex nonnegative function $\psi : \mathbb{R} \to \mathbb{R}$. Equation (5.2) and the right-differentiability of X in $L^2(0,1)$ yields that $V(t)$ is also the right derivative of X in $L^p(0,1)$, its L^p-norm is not increasing, and, by (5.3), the family V_s is uniformly p-integrable (by the Dunford–Pettis criterion, it is sufficient to choose a convex function ψ with $\psi(r)/|r|^p \to +\infty$ as $|r| \to +\infty$ and $\psi \circ V_0 \in L^1(0,1)$; see, e.g., [24, Lem. 3.7]).

From (L.III) we deduce that $tV(t) = X(t) - X_0 - \Xi(t)$, where $\Xi(t)$ is characterized by

$$\Xi(t) \in \partial I_{\mathcal{K}}(X(t)), \qquad \|X(t) - X_0 - \Xi(t)\| \leq \|X(t) - X_0 - \xi\| \quad \forall\, \xi \in \partial I_{\mathcal{K}}(X(t)). \tag{5.4}$$

Applying Lemma 3.11 with $g := X(t)$ and $h := X_0$, we obtain $\Xi(t) = \mathsf{P}_{\mathcal{H}_{X(t)}}(X_0) - X_0$ and, therefore,

$$tV(t) = X(t) - \mathsf{P}_{\mathcal{H}_{X(t)}}(X_0), \quad V(t) \in \mathcal{H}_{X(t)}. \tag{5.5}$$

It follows by (3.25) and (5.1) that $\mathcal{H}_{X(s)} \supset \mathcal{H}_{X(t)}$ if $0 \leq s \leq t$; moreover, by (2.29), there exists a Borel map $v_t \in L^p_{\rho_t}(\mathbb{R})$ such that

$$V(t) = v_t \circ X(t), \quad V(t) = \mathsf{P}_{\mathcal{H}_{X(t)}}(V(s)) \quad \forall\, 0 \leq s \leq t, \tag{5.6}$$

where the last identity follows from the second formula of (5.1) and the fact that $V(t)$ belongs to $\mathcal{H}_{X(t)}$, whereas $\partial I_{\mathcal{K}}(X(t))$ is orthogonal to $\mathcal{H}_{X(t)}$.

(d) Let $\mathcal{T}$ be the jump set of the L^2-norm of $V(t)$; we show that V is left-continuous at every $\bar{t} \in (0, +\infty) \setminus \mathcal{T}$ (this also yields the left-differentiability of X at $\bar{t}$). Equation (L.I) provides the minimal selection characterization of V at every time $t \geq 0$:

$$V(t) \in V_0 - \partial I_{\mathcal{K}}(X(t)), \quad \|V(t)\|_{L^2(0,1)} \leq \|V_0 - \xi\|_{L^2(0,1)} \quad \forall\, \xi \in \partial I_{\mathcal{K}}(X(t)). \tag{5.7}$$

Take an arbitrary increasing sequence $t_n \uparrow \bar{t}$ such that $V(t_n) \rightharpoonup \bar{V}$ in $L^p(0,1)$. Since the graph of $\partial I_{\mathcal{K}}$ is strongly-weakly closed in $L^2(0,1)$, we have $\bar{V} \in V_0 - \partial I_K(X(\bar{t}))$. Passing to the limit in (5.7) we obtain

$$\|\bar{V}\|_{L^2(0,1)} \leq \lim_{n\to\infty} \|V(t_n)\|_{L^2(0,1)} = \|V(\bar{t})\|_{L^2(0,1)} \leq \|V_0 - \bar{\xi}\|_{L^2(0,1)} \quad \forall\, \bar{\xi} \in \partial I_{\mathcal{K}}(X(\bar{t})).$$

Since $\partial I_K(X(\bar{t}))$ is a closed convex set, it follows that $\bar{V} = V(\bar{t})$ and that the convergence is strong in $L^2(0,1)$ and therefore also in $L^p(0,1)$, since $V(t_n)$ is uniformly p-integrable.

(e) Let n_k be an arbitrary subsequence such that $V^{n_k}(t) \rightharpoonup \bar V$ in $L^p(0,1)$. Passing to the limit in the inclusion $V^n(t) \in V_0^n - \partial I_{\mathcal K}(X^n(t))$, we obtain $\bar V \in V_0 - \partial I_{\mathcal K}(X(t))$. By Theorem 3.9 any element in $\partial I_{\mathcal K}(X(t))$ is orthogonal to $\mathcal H_{X(t)}$ so that

$$\mathsf P_{\mathcal H_{X(t)}}(\bar V) = \mathsf P_{\mathcal H_{X(t)}}(V_0) \overset{\text{(L.a)}}{=} V(t).$$

(f) Now let $t \in (0,+\infty) \setminus \mathcal T$, and let $n_k, \bar V$ be as in the previous point (e). Up to the extraction of a further subsequence (still denoted by n_k), there exists a dense set $S \subset (0,+\infty)$ such that $V^{n_k}(s) \to V(s)$ for every $s \in S$ so that $\forall\, s \in S,\ s < t$

$$\|\bar V\|_{L^2(0,1)} \le \limsup_{k\uparrow+\infty} \|V^{n_k}(t)\|_{L^2(0,1)} \le \limsup_{k\uparrow+\infty} \|V^{n_k}(s)\|_{L^2(0,1)} = \|V(s)\|_{L^2(0,1)}.$$

Since t is a continuity point for V, we obtain by (5.7)

$$\|\bar V\|_{L^2(0,1)} \le \|V(t)\|_{L^2(0,1)} \le \|V_0 - \xi\|_{L^2(0,1)} \quad \forall\, \xi \in \partial I_{\mathcal K}(X(t)), \tag{5.8}$$

which yields $\bar V = V(t)$, $\limsup_{k\uparrow+\infty} \|V^{n_k}(t)\|_{L^2(0,1)} \le \|V(t)\|_{L^2(0,1)}$ and the strong convergence of $V^n(t)$ to $V(t)$ in $L^2(0,1)$. The strong convergence in $L^p(0,1)$ follows by the uniform p-integrability estimate (5.3). ◻

Corollary 5.2 (existence of the Lagrangian semigroup). *For every initial data* $(X_0,V_0) \in \mathcal X_2(0,1)$*, there exists a unique Lipschitz curve* X *in* $L^2(0,1)$ *satisfying the equations* (L.I)–(L.III) *and the properties* (L.a)–(L.c) *stated in Theorem* 2.6*. Setting* $V(t) := \frac{\mathrm d^+}{\mathrm dt} X(t)$*, the map* $\mathsf S_t : (X_0,V_0) \mapsto (X(t),V(t))$ *defines a right-continuous semigroup in each space* $\mathcal X_p(0,1)$*,* $p \ge 2$*.*

Proof. It is sufficient to approximate $(X_0,V_0) \in \mathcal X_p(0,1)$ by a sequence $(X_0^n,V_0^n) \in \hat{\mathcal X}$ of initial data arising from finite discrete distributions of space and velocities in $\hat{\mathcal V}(\mathbb R)$ and to apply the previous lemma. ◻

Corollary 5.3 (equivalent characterizations). *Let* $(X_0,V_0) \in \mathcal X_2(0,1)$ *be given initial data. If* X *is a solution of one of the equations* (L.I)*,* (L.II)*,* (L.III)*, then it satisfies all the formulations* (L.I)–(L.III) *and the properties* (L.a)–(L.c) *stated in Theorem* 2.6*.*

Proof. The thesis is obvious in the case of (L.I) and (L.II), whose solution is unique, and should coincide with the Lagrangian evolution provided by Corollary 5.2.

Let us now assume that X is a Lipschitz curve solving (L.III), let $\tilde X$ be the Lagrangian solution given by the previous Corollary 5.2 with initial data X_0, V_0, and let us set $V_0^n := n(X(n^{-1}) - X_0)$, $X^n(t) := \mathsf P_{\mathcal K}(X_0 + tV_0^n)$. $X^n(t)$ is thus a Lagrangian flow satisfying (L.I)–(L.III) with respect to the initial data X_0, V_0^n; in particular,

$$t\frac{\mathrm d}{\mathrm dt}X^n(t) \in -\partial I_{\mathcal K}(X^n(t)) + X^n(t) - X_0, \quad X^n(n^{-1}) = X(n^{-1}) \tag{5.9}$$

so that $X^n(t) = X(t)$ for $t \ge n^{-1}$. On the other hand, the stability Lemma 5.1 yields

$$\|X^n(t) - \tilde X(t)\| \le t\|V_0^n - V_0\| = t\|n(X(n^{-1}) - X_0) - V_0\| \overset{\text{(L.III)}}{\to} 0 \quad \text{as } n\uparrow+\infty \tag{5.10}$$

so that $X = \tilde X$. ◻

6. The continuous sticky particle system in Eulerian coordinates. In this section we conclude the proofs of the various theorems of section 2.

Proof of Proposition 2.1. Starting from (2.9) it is immediate to check that D_p is a metric on $\mathcal V_p(\mathbb R)$. Let us check the equivalence characterization (2.13): assuming first that $D_p(\mu_n,\mu)\to 0$, we obviously have $W_p(\rho_n,\rho)\to 0$; since $X_n = X_{\rho_n}\to X = X_\rho$ and $v_n(X_n)\to v(X)$ in $L^p(0,1)$ as $n\uparrow+\infty$, for a continuous and bounded test function $\zeta:\mathbb{R}\to\mathbb{R}$ we easily get

$$\begin{aligned}\lim_{n\uparrow+\infty}\int_{\mathbb{R}}\zeta(x)v_n(x)\,\mathrm{d}\rho_n(x) &= \lim_{n\uparrow+\infty}\int_0^1\zeta(X_n(w))v_n(X_n(w))\,\mathrm{d}w\\ &= \int_0^1\zeta(X(w))v(X(w))\,\mathrm{d}w = \int_{\mathbb{R}}\zeta(x)v(x)\,\mathrm{d}\rho(x),\end{aligned}\tag{6.1}$$

showing that $\rho_n v_n \rightharpoonup \rho v$ and

$$\begin{aligned}\lim_{n\uparrow+\infty}\int_{\mathbb{R}}|v_n(x)|^p\,\mathrm{d}\rho_n(x) &= \lim_{n\uparrow+\infty}\int_0^1|v_n(X_n(w))|^p\,\mathrm{d}w\\ &= \int_0^1|v(X(w))|^p\,\mathrm{d}w = \int_{\mathbb{R}}|v(x)|^p\,\mathrm{d}\rho(x).\end{aligned}$$

The converse implication is a particular case of [1, Thm. 5.4.4]; here is a simplified argument. If (2.13) holds, then one gets the strong convergence of X_n to X in $L^p(0,1)$; since $V_n := v_n\circ X_n$ is bounded in $L^p(0,1)$ up to the extraction of a suitable subsequence, one has $V_n\rightharpoonup V$ in $L^p(0,1)$ and, arguing as in (6.1),

$$\int_0^1\zeta(X(w))V(w)\,\mathrm{d}w = \int_0^1\zeta(X(w))v(X(w))\,\mathrm{d}w \quad \forall\,\zeta\in C_b(\mathbb{R}).\tag{6.2}$$

Note that a function in $L^p(0,1)$ of the form $b\circ X$ for some Borel map $b:\mathbb{R}\to\mathbb{R}$ belongs to $\mathcal{H}_X$; a simple approximation argument shows that the set $\{\zeta\circ X:\zeta\in C_b(\mathbb{R})\}$ is dense in $\mathcal{H}_X$ so that (6.2) yields

$$v\circ X = \mathsf{P}_{\mathcal{H}_X}V.\tag{6.3}$$

On the other hand, the last limit property stated in (2.13) yields

$$\|V\|_{L^p(0,1)} \le \lim_{n\uparrow+\infty}\|V_n\|_{L^p(0,1)} = \|v\circ X\|_{L^p(0,1)} = \|\mathsf{P}_{\mathcal{H}_X}(V)\|_{L^p(0,1)} \le \|V\|_{L^p(0,1)}\tag{6.4}$$

so that $v\circ X$ should coincide with V, which is also the strong limit of V_n in $L^p(0,1)$.

Let us finally consider the density of $\hat{\mathcal{V}}$: if $(\rho,\rho v)\in\mathcal{V}_p(\mathbb{R})$, we can first approximate v in $L^p_\rho(\mathbb{R})$ by a sequence of bounded and continuous functions $v_n\in C_b(\mathbb{R})$. We can then find a sequence $\rho^N = \sum_{j=1}^N m_{j,N}\delta_{x_{j,N}}$, $N\in\mathbb{N}$ such that $\rho^N\to\rho$ in $\mathcal{P}_p(\mathbb{R})$. It is then easy to check that $v_n\rho^N\rightharpoonup v_n\rho$ as $N\uparrow+\infty$ according to (2.13). ∎

Proof of Theorem 2.3. (a) The extension of the semigroup $\mathscr{S}$ is not difficult, using the estimates of Theorem 2.2 and the density of $\hat{\mathcal{V}}(\mathbb{R})$ in $\mathcal{V}_p(\mathbb{R})$, but it is not completely trivial since the space $\mathcal{V}_p(\mathbb{R})$ is not complete and (2.14c) and (4.14) do not provide a pointwise continuous dependence of the velocity on the initial data. Therefore, we will use the equivalence stated in Theorem 2.6 (which we already proved at the level of discrete data in Theorem 4.2) and the Lagrangian stability result of Lemma 5.1. It is clear that the only possible extension of $\mathscr{S}_t$ to $\mathcal{V}_p(\mathbb{R})$ is given by formula (2.33). Since S_t is a semigroup in $\mathcal{X}_p(0,1)$ satisfying $\lim_{t\downarrow 0}\mathsf{S}_t(X_0,V_0) = (X_0,V_0)$ strongly in $L^p(0,1)^2$, $\mathscr{S}_t$ satisfies (2.16).

In order to check that $\mathscr{S}_t$ is strongly-weakly continuous, we take a sequence $\mu^n_t = (\rho^n_t,\rho^n_t v^n_t) = \mathscr{S}_t[\mu^n_0]\in\hat{\mathcal{V}}$, with μ^n_0 converging to $\mu = (\rho,\rho v)\in\mathcal{V}_p(\mathbb{R})$ with respect to D_p, and we consider the associated maps $(X^n(t), V^n(t)) = \mathsf{S}_t(X_0^n, V_0^n)$. By Lemma 5.1(f), for every weakly converging sequence $V^{n_k} \rightharpoonup \bar V$ in $L^p(0,1)$ and every test function $\zeta \in C_b^0(\mathbb{R})$ we have

$$\int_{\mathbb{R}} \zeta\, v_t^{n_k}\,\mathrm{d}\rho_t^{n_k} = \int_0^1 \zeta(X^{n_k}(t)) v_t^{n_k}(X^{n_k}(t))\,\mathrm{d}w \overset{\text{(L.a)}}{=} \int_0^1 \zeta(X_t^{n_k}) V^{n_k}(t)\,\mathrm{d}w$$
$$\xrightarrow{k\uparrow+\infty} \int_0^1 \zeta(X(t))\bar V\,\mathrm{d}w \overset{\text{Lemma 5.1(e)}}{=} \int_0^1 \zeta(X(t))V(t)\,\mathrm{d}w \overset{\text{(L.a)}}{=} \int_{\mathbb{R}} \zeta\, v_t\,\mathrm{d}\rho_t,$$

where we used the fact that $\zeta(X^n(t)) \to \zeta(X(t))$ strongly in $L^p(0,1)$.

(b) It is immediate to check that $(\rho, \rho v) = \mathscr{S}(\rho_0, \rho_0 v_0)$ is a distributional solution of (1.1), since in Lagrangian coordinates the continuity equation reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \zeta(x)\,\mathrm{d}\rho_t(x) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \zeta(X(t))\,\mathrm{d}w \overset{\text{(L.a)}}{=} \int_0^1 \zeta'(X(t))V(t)\,\mathrm{d}w$$
$$\overset{\text{(L.a)}}{=} \int_0^1 \zeta'(X(t))v_t(X(t))\,\mathrm{d}w = \int_{\mathbb{R}} \zeta'(x)v_t(x)\,\mathrm{d}\rho_t(x),$$

and the momentum equation similarly becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \zeta(x)v_t(x)\,\mathrm{d}\rho_t(x) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \zeta(X(t))V(t)\,\mathrm{d}w \overset{\text{(2.29) (L.a)}}{=} \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \zeta(X(t))V_0\,\mathrm{d}w$$
$$\overset{\text{(L.a)}}{=} \int_0^1 \zeta'(X(t))V(t)V_0\,\mathrm{d}w = \int_0^1 \zeta'(X(t))v_t(X(t))V_0\,\mathrm{d}w$$
$$\overset{\text{(2.29) (L.a)}}{=} \int_0^1 \zeta'(X(t))v_t^2(X(t))\,\mathrm{d}w = \int_0^1 \zeta'(x)v_t^2(x)\,\mathrm{d}\rho_t(x).$$

Oleinik entropy condition (1.5) follows easily by (5.5), by observing that $\mathsf{P}_{\mathcal{H}_{X(t)}}(X_0)$ is a nondecreasing map, $V(t) = v_t(X(t))$, and $\rho_t = (X(t))_\#\lambda$.

(c) The proof of (c) follows from (5.3).

(d) The proof is equivalent to point (d) of Lemma 5.1; concerning the left continuity of $\rho_t v_t$ in the weak topology, we fix an arbitrary bounded Lipschitz test function $\zeta : \mathbb{R} \to \mathbb{R}$, and we observe that

$$\lim_{s\uparrow t}\int_{\mathbb{R}} \zeta(x)v_s(x)\,\mathrm{d}\rho_s(x) = \lim_{s\uparrow t}\int_0^1 \zeta(X(s))V(s)\,\mathrm{d}w = \lim_{s\uparrow t}\int_0^1 \zeta(X(t))V(s)\,\mathrm{d}w$$

since $X(s) \to X(t)$ in $L^2(0,1)$ as $s \uparrow t$. On the other hand, since $\zeta \circ X(t) \in \mathcal{H}_{X(t)}$, we have

$$\int_0^1 \zeta(X(t))V(s)\,\mathrm{d}w = \int_0^1 \zeta(X(t))V(t)\,\mathrm{d}w$$
$$= \int_0^1 \zeta(X(t))v_t(X(t))\,\mathrm{d}w = \int_{\mathbb{R}} \zeta(x)v_t(x)\,\mathrm{d}\rho_t(x).$$

(e) The proof of (e) has already been discussed in point (a), except for the convergence at $t \in (0,+\infty) \setminus \mathcal{T}$, which follows from Lemma 5.1(f).

(f) Equation (2.18) follows by the projection representation (5.1) and Corollary 3.5. The limit in (2.18) can be obtained in the Lagrangian coordinate:

$$\lim_{h\downarrow 0}\int_{\mathbb{R}} \left|h^{-1}(\mathsf{x}_s^{s+h} - \mathsf{i}) - v_s\right|^2 \mathrm{d}\rho_s = \lim_{h\downarrow 0}\int_0^1 \left|h^{-1}(X(s+h) - X(s)) - V(s)\right|^2 \mathrm{d}w = 0,$$

since $t \mapsto X(t)$ is right-differentiable. Equation (2.19) is an immediate consequence of (5.5), which yields

$$(t-s)V(t) = X(t) - \mathsf{P}_{\mathcal{H}_{X(t)}}(X(s)) \quad \forall\, 0 \le s < t. \qquad \square$$

Proof of Theorem 2.6. The proof now follows by applying Lemma 5.1 and its Corollaries 5.2 and 5.3. $\square$

Proof of Theorem 2.4. Equation (2.23) follows from a simple calculation starting from (L.III): we introduce Z, the monotone rearrangement of the measure $\eta \in \mathcal{P}_2(\mathbb{R})$ and we observe that $W_2^2(\rho_t, \eta) = \|X(t) - Z\|^2$ (we refer to the usual notation for (X, V) and we denote by $\|\cdot\|$ the norm in $L^2(0,1)$). We get for some $\Xi(t) \in \partial I_{\mathcal{K}}(X(t))$

$$\begin{aligned}\frac{t}{2}\frac{\mathrm{d}^+}{\mathrm{d}t} W_2^2(\rho_t, \eta) &= \frac{t}{2}\frac{\mathrm{d}^+}{\mathrm{d}t}\|X(t) - Z\|^2 = t(\dot{X}(t)|X(t) - Z) \\ &\overset{\text{(L.III)}}{=} (X(t) - X_0 - \Xi(t)|X(t) - Z) \overset{(3.15)}{\le} (X(t) - X_0|X(t) - Z) \\ &= \frac{1}{2}\|X(t) - Z\|^2 - \frac{1}{2}\|Z - X_0\|^2 + \frac{1}{2}\|X(t) - X_0\|^2 \\ &= \frac{1}{2}W_2^2(\rho_t, \eta) - \phi^{\rho_0}(\rho_t) + \phi^{\rho_0}(\eta).\end{aligned}$$

Let us now consider the converse implication: if ρ_t satisfies (2.23), then $X(t) = X_{\rho_t}$ satisfies (see (2.35))

$$\text{(6.5)} \qquad \frac{t}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|X(t) - Z\|^2 - \frac{1}{2}\|X(t) - Z\|^2 \le \Phi^{\rho_0}(Z) - \Phi^{\rho_0}(X(t)) \quad \forall\, Z \in \mathcal{K},$$

which is the equivalent metric formulation [1] of the differential inclusion (L.III).

Since $\boldsymbol{\rho}_t = (X_0, X(t))_{\#}\lambda$, (2.25) yields

$$\text{(6.6)} \qquad \lim_{t \downarrow 0} t^{-2} \int_0^1 |X_0 + tV_0 - X(t)|^2 \,\mathrm{d}w = 0,$$

i.e., $X(t)$ also satisfies the initial limit condition of (L.III). Therefore, setting $V := \frac{\mathrm{d}}{\mathrm{d}t}X = v \circ X$, by Corollary 5.3 the couple $(X(t), V(t))$ coincides with the Lagrangian flow $\mathsf{S}_t(X_0, V_0)$ so that $(\rho_t, \rho_t v_t) = \mathscr{S}_t(\rho_0, \rho_0 v_0)$. $\square$

Proof of Theorem 2.5. Let us first note that when $\mathsf{i} + \varepsilon_0 v_0$ is ρ_0-essentially nondecreasing, (2.27) follows directly from (2.24), since the collision-free motion $\rho_t = (\mathsf{i} + tv_0)_{\#}\rho_0$ for $t \in [0, \varepsilon_0)$ is a solution of the SPS.

Let us now consider the general case, setting $\tilde{\rho}_{\varepsilon,t} := \mathscr{G}^{\rho_0}_{\log(t/\varepsilon)}(\tilde{\rho}_\varepsilon)$. For every $\varepsilon > 0$ let us consider the convex set of bounded Lipschitz functions

$$BL(\varepsilon) := \left\{ u \in C^{0,1}(\mathbb{R}) : \sup|u| \le \varepsilon^{-1}, \mathrm{Lip}(u) \le (2\varepsilon)^{-1} \right\},$$

and let $u_\varepsilon \in BL(\varepsilon)$ be a minimizer of

$$\text{(6.7)} \qquad m_\varepsilon = \min_{u \in BL(\varepsilon)} \|v_0 - u\| = \|v_0 - u_\varepsilon\|.$$

By standard approximation results, $\lim_{\varepsilon \downarrow 0} m_\varepsilon = 0$ so that u_ε converges to v_0.

By the definition of $BL(\varepsilon)$ the map $\mathsf{i} + \varepsilon u_\varepsilon$ is monotone and, therefore, it is the optimal map pushing ρ to $\hat{\rho}_\varepsilon = (\mathsf{i} + \varepsilon u_\varepsilon)_{\#}\rho_0$. The sticky particle solution $(\hat{\rho}_{\varepsilon,t}, \hat{\rho}_{\varepsilon,t}\hat{v}_{\varepsilon,t}) := \mathscr{S}_t(\hat{\rho}_0, \hat{\rho}_0 u_\varepsilon)$ admits the representation (see (2.24))

$$\hat{\rho}_{\varepsilon,t} = \mathscr{G}^{\rho_0}_{\log(t/\varepsilon)}(\hat{\rho}_\varepsilon)$$

so that, by the exponential rate of expansion of $\mathscr{G}$ we get

(6.8)

$$W_2(\hat\rho_{\varepsilon,t},\tilde\rho_{\varepsilon,t}) \le \exp\left(\log(t/\varepsilon)\right) W_2(\hat\rho_\varepsilon,\tilde\rho_\varepsilon) = \frac{t}{\varepsilon} W_2(\hat\rho_\varepsilon,\tilde\rho_\varepsilon) \le t\|v_0-u_\varepsilon\|_{L^2_{\rho_0}(\mathbb{R})} \stackrel{(6.7)}{=} tm_\varepsilon.$$

On the other hand, if $(\rho_t,\rho_t v_t) = \mathscr{S}_t(\rho_0,\rho_0 v_0)$, (2.14b) yields

$$W_2(\hat\rho_{\varepsilon,t},\rho_t) \le t\|v_0-u_\varepsilon\|_{L^2_{\rho_0}(\mathbb{R})} = tm_\varepsilon \quad \text{so that} \quad W_2(\rho_t,\tilde\rho_{\varepsilon,t}) \le 2m_\varepsilon t, \tag{6.9}$$

and concludes the proof of (2.26). ◻

We conclude this section by showing that the representation convergence theorem of Brenier and Grenier [11] can be easily deduced by our result, in particular by formula (L.II) of Theorem 2.6.

THEOREM 6.1 (Brenier–Grenier theorem). *Let $v_0 \in C^0(\mathbb{R})$, let ρ_0^N, $N\in\mathbb{N}$ be a sequence of discrete probability measures supported in a fixed compact interval $[-R,R]$ and weakly converging to ρ_0 in $\mathcal{P}(\mathbb{R})$, and let ρ_t^N be the solution of the discrete SPS with initial data $(\rho_0^N, v_0\rho_0^N)$. For every $t\ge 0$, ρ_t^N weakly converge to a probability measure ρ_t, whose distribution function $M_t(x) := \rho_t((-\infty,x])$, $t\ge 0$ is the unique entropy solution of*

$$\partial_t M + \partial_x(A(M)) = 0, \quad M(0) = M_0, \tag{6.10}$$

where the flux function $A:[0,1]\to\mathbb{R}$ is defined by

$$A(w) := \int_0^w V_0(r)\,\mathrm{d}r, \quad \textit{where} \quad V_0 := v_0\circ X_0, \quad X_0 := X_{\rho_0}. \tag{6.11}$$

Proof. The convergence part follows by Theorem 2.3 and we can represent $X_t := X_{\rho_t}$ by the formula $X_t = \mathsf{P}_{\mathcal{K}}(X_0 + tV_0)$ of Theorem 2.6. Introducing the convex primitive functions $F_t(w) := \int_0^w X_t(r)\,\mathrm{d}r$, Theorem 3.1 yields

$$F_t = (F_0 + tA)^{**} \quad \text{so that} \quad (F_t)^* = (F_0+tA)^*. \tag{6.12}$$

On the other hand, since the derivative X_t of F_t is the pseudoinverse of M_t (1.9), a standard duality result shows that $(F_t)^* = G_t$, where $G_t(x) = \int_{-\infty}^x M_t(y)\,\mathrm{d}y$, so that

$$G_t = (F_0+tA)^* = (G_0^* + tA)^*. \tag{6.13}$$

It was already observed in [11, sect. 4] that (6.13) provides the second Hopf formula [3] for the viscosity solution of the Hamilton–Jacobi equation

$$\partial_t G + A(\partial_x G) = 0 \quad \text{in } \mathbb{R}\times(0,+\infty), \tag{6.14}$$

and therefore the derivative $M_t = \partial_x G_t$ is the entropy solution of (6.10). ◻

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