# Nonlinear diffusion equations and curvature conditions in metric measure spaces

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**Abstract**

Aim of this paper is to provide new characterizations of the curvature dimension condition in the context of metric measure spaces $(X, \mathsf{d}, \mathfrak{m})$. On the geometric side, our new approach takes into account suitable weighted action functionals which provide the natural modulus of $K$-convexity when one investigates the convexity properties of $N$-dimensional entropies. On the side of diffusion semigroups and evolution variational inequalities, our new approach uses the nonlinear diffusion semigroup induced by the $N$-dimensional entropy, in place of the heat flow. Under suitable assumptions (most notably the quadraticity of Cheeger's energy relative to the metric measure structure) both approaches are shown to be equivalent to the strong $\mathrm{CD}^*(K, N)$ condition of Bacher-Sturm.

## Contents



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# 1 Introduction

Spaces with Ricci curvature bounded from below play an important role in many probabilistic and analytic investigations, that reveal various deep connections between different fields.

Starting from the celebrated paper by BAKRY-ÉMERY [13], the curvature-dimension condition based on $\Gamma$-calculus and the $\Gamma_2$-criterium in Dirichlet spaces provides crucial tools for proving refined estimates on Markov semigroups and many functional inequalities, of Poincaré, Log-Sobolev, Talagrand, and concentration type (see, e.g. [34, 35, 36, 12, 9, 14]).

In the framework of optimal transport, the importance of curvature bounds has been deeply analyzed in [42, 24, 50]. These and other important results led STURM [48, 49] and LOTT-VILLANI [40] to introduce a new synthetic notion of the curvature-dimension condition, in the general framework of a metric-measure space $(X, \mathsf{d}, \mathfrak{m})$.

In recent years more than one paper has been devoted to the investigation of the relation between the differential and metric structures, particularly in connection with Dirichlet forms, see for instance [33], [32], [47], [6] and [7]. In particular, under a suitable infinitesimally Hilbertian assumption on the metric measure structure (and very mild regularity assumptions), thanks to the results of the last two papers we know that the optimal transportation point of view provided by the LOTT-STURM-VILLANI theory coincides with the point of view provided by BAKRY-ÉMERY when the inequalities do not involve any upper bound on the dimension: both the approaches can thus be equivalently used to characterize the class of RCD$(K,\infty)$ spaces with Riemannian Ricci curvature bounded from below by $K \in \mathbb{R}$. More precisely, the logarithmic entropy functional

$$\mathcal{U}_\infty(\mu) := \int_X \varrho \log \varrho \, \mathrm{d}\mathfrak{m} \quad \text{if } \mu = \varrho\mathfrak{m} \ll \mathfrak{m}, \tag{1.1}$$

satisfies the $K$-convexity inequality along geodesics $(\mu_s)_{s\in[0,1]}$ induced by the transport distance $W_2$ (i.e. with cost equal to the square of the distance)

$$\mathcal{U}_\infty(\mu_s) \le (1-s)\mathcal{U}_\infty(\mu_0) + s\mathcal{U}_\infty(\mu_1) - \frac{K}{2} s(1-s) W_2^2(\mu_0,\mu_1) \tag{1.2}$$

if and only if $\Gamma_2(f) \ge K\,\Gamma(f)$.

A natural and relevant question is then to establish a similar equivalence when upper bounds on the dimension are imposed; more precisely one is interested in the equivalence between the condition

$$\Gamma_2(f) \ge K\,\Gamma(f) + \frac{1}{N}(\mathrm{L}f)^2 \tag{1.3}$$

(where L is the infinitesimal generator of the semigroup associated to the Dirichlet form) and the curvature-dimension conditions based on optimal transport. In the dimensional case, the logarithmic entropy functional (1.1) is replaced by the "$N$-dimensional" Rény entropy

$$\mathcal{U}_N(\mu) := \int_X U_N(\varrho)\,\mathrm{d}\mathfrak{m} = N - N\int_X \varrho^{1-\frac{1}{N}}\,\mathrm{d}\mathfrak{m} \qquad \text{if } \mu = \varrho\mathfrak{m} + \mu^\perp, \quad \mu^\perp \perp \mathfrak{m}. \tag{1.4}$$

Except for the case $K = 0$, which can be formulated by means of a geodesic convexity condition analogous to (1.2), the case $K \neq 0$ involves a much more complicated property [49, 10], that gives raise to difficult technical questions.

Aim of this paper is precisely to provide new characterizations of the curvature dimension condition in the context of metric measure spaces $(X, \mathsf{d}, \mathfrak{m})$. On the geometric side, our new approach takes into account suitable weighted action functionals of the form

$$\mathcal{A}_N^{(t)}(\mu; \mathfrak{m}) = \int_0^1 \int_X \mathsf{g}(s,t)\varrho^{1-1/N}(x,s)\bar{v}^2(x,s)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s, \tag{1.5}$$

where $\mu_s = \varrho_s\mathfrak{m}$, $s \in [0,1]$, is a Wasserstein geodesic, $\mathsf{g}$ is a weight function and $\bar{v}$ is the minimal velocity density of $\mu$, a new concept that extends to general metric spaces the notion of Wasserstein velocity vector field developed for Euclidean spaces [3, Chap. 8]. Functionals like (1.5) provide the natural modulus of $K$-convexity when one investigates the convexity properties of the $N$-dimensional Rény entropy (1.4). On the side of diffusion semigroups and evolution variational inequalities, our new approach uses the nonlinear diffusion semigroup induced by the $N$-dimensional entropy, in place of the heat flow. Under suitable assumptions (most notably the quadraticity of Cheeger's energy relative to the metric measure structure) both approaches are shown to be equivalent to the strong $\mathrm{CD}^*(K, N)$ condition of Bacher-Sturm [10].

Apart from the stated equivalence between the Lott-Sturm-Villani and the Bakry-Émery approaches, our results and techniques can hardly be compared with the recent work [27] of Erbar-Kuwada-Sturm, motivated by the same questions. Instead of the Rény entropies (1.4), in their approach an $N$-dependent modification of the logarithmic entropy (1.1) is considered, namely the logarithmic entropy power

$$\mathcal{S}_N(\mu) := \exp\left(-\frac{1}{N}\mathcal{U}_\infty(\mu)\right),$$

and convexity inequalities as well as evolution variational inequalities are stated in terms of $\mathcal{S}_N$, proving equivalence with the strong $\mathrm{CD}^*(K, N)$ condition. A conceptual and technical advantage of their approach is the use of essentially the same objects (logarithmic entropy, heat flow) of the adimensional theory. On the other hand, since power-like nonlinearities appear in a natural way "inside the integral" in the optimal transport approach to the curvature dimension theory, we believe it is interesting to pursue a different line of thought, using the Wasserstein gradient flow induced by the Rény entropies (in the same spirit of the seminal Otto's paper [41] on convergence to equilibrium for porous medium equations). The only point in common of the two papers is that both provide the equivalence between the differential curvature-dimension condition (1.3) and the so-called strong

$\mathrm{CD}^*(K,N)$ condition; however, this equivalence is estabilished passing through convexity and differential properties which are quite different in the two approaches (for instance some of them do not involve at all the distorsion coefficients) and have, we believe, an independent interest.

Our paper starts with Section 2, where we illustrate in the simple framework of a $d$-dimensional Euclidean space the basic heuristic arguments providing the links between contractivity and convexity. It builds upon the fundamental papers [43] and [25]. The main new ingredient here is that the links are provided in terms of monotonicity of the Hamiltonian, instead of monotonicity of the Lagrangian (see [37] for a related discussion of the role of dual Hamiltonian estimates in terms of the so-called Onsager operator). More precisely, if $\mathsf{S}_t : \mathbb{R}^d \to \mathbb{R}^d$ is the flow generated by a smooth vector field $\mathfrak{f} : \mathbb{R}^d \to \mathbb{R}^d$, and if $\mathcal{C}(x,y)$ is the cost functional relative to a Lagrangian $\mathcal{L}$, then we know that the contractivity property

$$\mathcal{C}(\mathsf{S}_t x, \mathsf{S}_t y) \le \mathcal{C}(x,y) \qquad \text{for all } t \ge 0,$$

is equivalent to the action monotonicity

$$\mathrm{d}\mathcal{L}(x(t), w(t))/\mathrm{d}t \le 0 \tag{1.6}$$

whenever $x(t)$ solves the ODE

$$\mathrm{d}x(t)/\mathrm{d}t = \mathfrak{f}(x(t))$$

and $w$ solves the linearized ODE

$$\mathrm{d}w(t)/\mathrm{d}t = \mathrm{D}\mathfrak{f}(x(t))w(t)$$

(in the applications $w$ arises as the derivative w.r.t. $s$ of a smooth curve of initial data for the ODE). In Section 2 we use duality arguments to prove the same equivalence when the action monotonicity (1.6) is replaced by the Hamiltonian monotonicity

$$\mathrm{d}\mathcal{H}(x(t), \varphi(t))/\mathrm{d}t \ge 0, \tag{1.7}$$

where now $\varphi$ solves the *backward* transposed equation

$$\mathrm{d}\varphi(t) = -\mathrm{D}\mathfrak{f}(x(t))^{\mathsf{T}}\varphi(t), \tag{1.8}$$

see Proposition 2.1. Lemma 2.2 provides, in the case when $\mathfrak{f} = -\nabla U$ and $\mathcal{L}$, $\mathcal{H}$ are quadratic forms, the link between the Hamiltonian monotonicity and another contractivity property involving both $\mathcal{C}$ and $U$, see (2.19); this is known to be equivalent to the convexity of $U$ along the geodesics induced by $\mathcal{C}$.

In the context of optimal transportation (say on a smooth, compact Riemannian manifold $(M,\mathfrak{g})$), the role of the Hamiltonian is played by $\mathcal{H}(\varrho,\varphi) := \frac{1}{2}\int_X |\mathrm{D}\varphi|^2_{\mathfrak{g}}\varrho\,\mathrm{d}\mathfrak{m}$, thanks to Benamou-Brenier formula and the Otto formalism:

$$\mathcal{L}(\varrho, w) := \frac{1}{2}\int_X |\mathrm{D}\varphi|^2_{\mathfrak{g}}\varrho\,\mathrm{d}\mathfrak{m}, \qquad -\mathrm{div}_{\mathfrak{g}}(\varrho\nabla_{\mathfrak{g}}\varphi) = w. \tag{1.9}$$

In other words, the cotangent bundle is associated to the velocity gradient $\nabla_{\mathfrak{g}}\varphi$ and the duality between tangent and cotangent bundle is provided by the possibly degenerate elliptic PDE $-\mathrm{div}_{\mathfrak{g}}(\varrho\nabla_{\mathfrak{g}}\varphi) = w$. With a very short computation we show in Example 2.3 how the Bakry-Émery $\mathrm{BE}(0,\infty)$ condition corresponds precisely to the Hamiltonian monotonicity, when the vector field is (up to the sign) the gradient vector of the logarithmic entropy functional. If the entropy $\mathcal{U}(\varrho\,\mathfrak{m}) = \int U(\varrho)\,\mathrm{d}x$ satisfies the (stronger) McCann's $\mathrm{DC}(N)$ condition, then the same correspondence holds with $\mathrm{BE}(0,N)$, see Example 2.4. In both cases the flow corresponds to the diffusion equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\varrho = \Delta_{\mathfrak{g}} P(\varrho) \tag{1.10}$$

with $P(\varrho) := \varrho U'(\varrho) - U(\varrho)$, which is linear only in the case of the logarithmic entropy (1.1).

The computations made in Examples 2.3 and Example 2.4 involve regularity in time and space of the potentials $\varphi$ in (1.9), whose proof is not straightforward already in the smooth Riemannian context. Another difficulty arises from the degeneracy of the PDE $-\mathrm{div}_{\mathfrak{g}}(\varrho\nabla_{\mathfrak{g}}\varphi) = w$, which forces us to consider weak solutions $\varphi$ in "weighted Sobolev spaces". Keeping in mind these technical difficulties, our goal is then to provide tools to extend the calculations of these examples to a nonsmooth context, following on the one hand the $\Gamma$-calculus formalism, on the other hand the calculus in metric measure spaces $(X,\mathsf{d},\mathfrak{m})$ developed in [5], [6], [28] and in the subsequent papers.

Now we pass to a more detailed description of the three main parts of the paper.

**Part I**

This first part, which consists of Section 3 and Section 4, is written in the context of a Dirichlet form $\mathcal{E}$ on $L^2(X,\mathfrak{m})$, for some measurable space $(X,\mathcal{B})$ endowed with a $\sigma$-finite measure $\mathfrak{m}$. We adopt the notation $\mathbb{H}$ for $\mathbb{L}^2(X,\mathfrak{m})$, $\mathbb{V}$ for the domain of the Dirichlet form, $-\mathrm{L}$ for the linear monotone map from $\mathbb{V}$ to $\mathbb{V}'$ induced by $\mathcal{E}$, $\mathsf{P}_t$ for the semigroup whose infinitesimal generator is L.

We already mentioned the difficulties related to the degeneracy of our PDE; in addition, since we don't want to assume a spectral gap, we need also to take into account the possibility that the kernel $\{f:\ \mathcal{E}(f,f)=0\}$ of the Dirichlet form is not trivial. We then consider the abstract completion $\mathbb{V}_{\mathcal{E}}$ of the quotient space of $\mathbb{V}$ and the realization $\mathbb{V}'_{\mathcal{E}}$ of the dual of $\mathbb{V}_{\mathcal{E}}$ as the finiteness domain of the quadratic form $\mathcal{E}^*:\mathbb{V}'\to[0,\infty]$ defined by

$$\frac{1}{2}\mathcal{E}^*(\ell,\ell) := \sup_{f\in\mathbb{V}}\langle\ell,f\rangle - \frac{1}{2}\mathcal{E}(f,f).$$

Section 3.2 is indeed devoted to basic functional analytic properties relative to the completion of quotient spaces w.r.t. a seminorm (duality, realization of the dual, extensions of the action of L). The spaces $\mathbb{V}$, $\mathbb{V}_{\mathcal{E}}$ and their duals are the basic ingredients for the analysis, in Section 3.3, of the nonlinear diffusion equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\varrho - \mathrm{L}P(\varrho) = 0 \tag{1.11}$$

(which corresponds to (1.10)) in the abstract context, for regular monotone nonlinearities $P$; the basic existence and uniqueness result is given in Theorem 3.4, which provides also the natural apriori estimates and contractivity properties.

Chapter 4 is devoted to the linearizations of the diffusion equation (1.11). We first consider in Theorem 4.1 the (backward) PDE

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi + P'(\varrho)\mathrm{L}\varphi = \psi$$

which is the adjoint to the linearized equation and corresponds, when $\psi = 0$, to the backward transposed ODE (1.8) of the heuristic Section 2. Existence, uniqueness and stability for this equation is provided in the class $W^{1,2}(0,T;\mathbb{D},\mathbb{H})$ of $L^2(0,T;\mathbb{D})$ maps with derivative in $L^2(0,T;\mathbb{H})$, where $\mathbb{D}$ is the space of all $f \in \mathbb{V}$ such that $\mathrm{L}f \in \mathbb{H}$, endowed with the natural norm.
In Theorem 4.4 we consider the linearized PDE

$$\frac{\mathrm{d}}{\mathrm{d}t}w = \mathrm{L}(P'(\varrho)w); \tag{1.12}$$

since (1.12) is in "divergence form" we can use the regularity of $P'(\varrho)$ to provide existence and uniqueness (as well as stability) in the large class $W^{1,2}(0,T;\mathbb{H},\mathbb{D}'_{\mathcal{E}})$ of $L^2(0,T;\mathbb{H})$ maps with derivative in $L^2(0,T;\mathbb{D}'_{\mathcal{E}})$. Here $\mathbb{D}'_{\mathcal{E}}$ is the space of all $\ell \in \mathbb{D}'$ such that, for some constant $C$, $|\langle \ell, f\rangle| \le C\|\mathrm{L}f\|_{\mathbb{H}}$ for all $f \in \mathbb{D}$ (endowed with the natural norm provided by the minimal constant $C$). In Theorem 4.5 we prove that the PDE is indeed the linearization of (1.11) by considering suitable families of initial conditions and their derivative.

## Part II

This part is devoted to the metric side of the theory and builds upon the papers [5], [39], [6], [25] with some new developments that we now illustrate.

Chapter 5 is mostly devoted to the introduction of preliminary and by now well estabilished concepts in metric spaces $(X,\mathsf{d})$, as absolutely continuous curves $\gamma_t$, metric derivative $|\dot\gamma_t|$, $p$-action $\mathcal{A}_p(\gamma) = \int |\dot\gamma|^p\,\mathrm{d}t$, slope $|\mathrm{D}f|$ and its one-sided counterparts $|\mathrm{D}^\pm f|$. In Section 5.2 we recall the metric/differential properties of the map

$$Q_t f(x) := \inf_{y\in X} f(y) + \frac{1}{2t}\mathsf{d}^2(x,y) \qquad x \in X,$$

given by the Hopf-Lax formula (which provides a semigroup if $(X,\mathsf{d})$ is a length space). Section 5.3 and Section 5.4 cover basic material on couplings, $p$-th Wasserstein distance $W_p$, absolutely continuous curves w.r.t. $W_p$ and dynamic plans. Particularly important for us is the 1-1 correspondence between absolutely continuous curves $\mu_t$ in $(\mathscr{P}(X), W_p)$ and time marginals probability measures $\boldsymbol{\pi}$ in $\mathrm{C}([0,1];X)$ with finite $p$-action $\mathscr{A}_p(\boldsymbol{\pi}) := \int \mathcal{A}_p(\gamma)\,\mathrm{d}\boldsymbol{\pi}(\gamma)$, provided in [39]. In general only the inequality $|\dot\mu_t|^p \le \int |\dot\gamma_t|^p\,\mathrm{d}\boldsymbol{\pi}(\gamma)$ holds, and [39] provides existence of a distinguished plan $\boldsymbol{\pi}$ for which equality holds, that we call $p$-tightened to $\mu_t$.

Section 5.5 introduces a key ingredient of the metric theory, the Cheeger energy that we shall denote by $\mathsf{Ch}$ and the relaxed slope $|\mathrm{D}f|_w$, so that $\mathsf{Ch}(f) = \frac{1}{2}\int_X |\mathrm{D}f|_w^2 \,\mathrm{d}\mathfrak{m}$. The energy $\mathsf{Ch}$ is by construction lower semicontinuous in $L^2(X,\mathfrak{m})$; furthermore, under an additional quadraticity assumption it has been shown in [6, 28] that $\mathsf{Ch}$ provides a strongly local Dirichlet form, whose Carré du Champ is given by

$$\Gamma(f,g) = \lim_{\varepsilon\downarrow 0} \frac{|\mathrm{D}(f+\epsilon g)|_w^2 - |\mathrm{D}f|_w^2}{2\epsilon}.$$

Motivated by the necessity to solve the PDE $-\mathrm{div}_{\mathfrak{g}}(\varrho\nabla_{\mathfrak{g}}\varphi) = \ell$, whose abstract counterpart is

$$\int_X \varrho\Gamma(\varphi,f)\,\mathrm{d}\mathfrak{m} = \langle \ell, f\rangle \qquad \forall f\in\mathbb{V}, \tag{1.13}$$

in Section 5.6 we consider natural weighted spaces $\mathbb{V}_\rho$ arising from the completion of the seminorm $\sqrt{\int_X \varrho\Gamma(f)\,\mathrm{d}\mathfrak{m}}$, and the extensions of $\Gamma$ to these spaces, denoted by $\Gamma_\varrho$. In connection with these spaces we investigate several stability properties which play a technical role in our proofs.

Section 6 provides a characterization of $p$-absolutely continuous curves $\mu_s : [0,1] \to \mathscr{P}(X)$ in terms of the following control on the increments (where $|\mathrm{D}^*\varphi|$ is the usc relaxation of the slope $|\mathrm{D}\varphi|$):

$$\left|\int_X \varphi\,\mathrm{d}\mu_s - \int_X \varphi\,\mathrm{d}\mu_t\right| \le \int_s^t \int_X |\mathrm{D}^*\varphi| v\,\mathrm{d}\mu_r\,\mathrm{d}r \qquad \varphi\in\mathrm{Lip}_b(X),\ 0\le s\le t\le 1.$$

Any function $v$ in $L^p(X\times(0,1),\mathfrak{m}\otimes\mathscr{L}^1)$ will be called $p$-velocity density. In Theorem 6.7 we show that for all $p\in(1,\infty)$ a $p$-velocity density exists if and only if $\mu_t\in\mathrm{AC}^p([0,1];\mathscr{P}(X))$ (see also [30] for closely related results). In addition we identify a crucial relation between the unique $p$-velocity density $\bar v$ with minimal $L^p$ norm and any plan $\boldsymbol{\pi}$ $p$-tightened to $\mu$, namely

$$\bar v(\gamma_t,t) = |\dot\gamma_t| \quad \text{for } \boldsymbol{\pi}\text{-a.e. } \gamma, \text{ for } \mathscr{L}^1\text{-a.e. } t\in(0,1). \tag{1.14}$$

Heuristically, this means that even though branching cannot be ruled out, the metric velocity of the curve $\gamma$ in the support of $\boldsymbol{\pi}$ depends only on time and position of the curve, and it is independent of $\boldsymbol{\pi}$.

In Section 7 we use the minimal velocity density $\bar v$ to define, under the additional assumption $\mu_s = \varrho_s\mathfrak{m}$, the weighted energy functionals

$$\mathcal{A}_{\mathfrak{Q}}(\mu;\mathfrak{m}) := \int_0^1\int_X \mathfrak{Q}(s,\varrho_s)\,\bar v^p \varrho_s\,\mathrm{d}\mathfrak{m}\,ds, \tag{1.15}$$

where $\mathfrak{Q}(s,r) : [0,1]\times[0,\infty) \to [0,\infty]$ is a suitable weight function (the typical choice will be $\mathfrak{Q}(s,r) = \omega(s)Q(r)$ with $Q(r) = rP'(r) - P(r)$). Notice that when $\mathfrak{Q}\equiv 1$ we have the usual action $\int_0^1\int_X \bar v^p\,\mathrm{d}\mu_s\,\mathrm{d}s = \mathcal{A}_p(\mu)$, which makes sense even for curves not made of absolutely continuous measures. If $\boldsymbol{\pi}$ is a dynamic plan $p$-tightened to $\mu$ (recall that this means $\mathscr{A}_p(\boldsymbol{\pi}) = \mathcal{A}_p(\mu)$), we can use (1.14) to obtain an equivalent expression in terms of $\boldsymbol{\pi}$:

$$\mathcal{A}_{\mathfrak{Q}}(\mu; \mathfrak{m}) = \int_0^1 \int_X \mathfrak{Q}(s, \varrho_s(\gamma_s))|\dot{\gamma}_s|^p \,\mathrm{d}\boldsymbol{\pi}(\gamma)\,\mathrm{d}s.$$

In Theorem 7.1 we provide, by Young measures techniques, continuity and lower semicontinuity properties of $\mu \mapsto \mathcal{A}_{\mathfrak{Q}}(\mu; \mathfrak{m})$ under the assumption that the $p$-actions are convergent.

In Section 8 we restrict ourselves to the case when $p = 2$ and $\mathsf{Ch}$ is quadratic. For curves $\mu_s = \varrho_s \mathfrak{m}$ having uniformly bounded densities w.r.t. $\mathfrak{m}$ we show in Theorem 8.2 that $(\mu_s)_{s\in[0,1]}$ belongs to $\mathrm{AC}^2([0,1];(\mathscr{P}(X), W_2))$ if and only if there exists $\ell \in L^2(0,1;\mathbb{V}')$ satisfying, for all $f \in \mathbb{V}$,

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_X f\varrho_s\,\mathrm{d}\mathfrak{m} = \ell_s(f) \qquad \text{in } \mathscr{D}'(0,1).$$

In addition $\ell_s \in \mathbb{V}'_{\varrho_s}$ for $\mathscr{L}^1$-a.e. $s \in (0,1)$ and they are linked to the minimal velocity $\bar{v}$ by $\mathcal{E}^*(\ell_s, \ell_s) = \int_X |\bar{v}_s|^2 \varrho_s\,\mathrm{d}s$. Thanks to this result, we can obtain by duality the potentials $\phi_s$ associated to the curve, linked to $\ell_s$ by (1.13).

In Section 9 we enter into the core of the matter, by providing on the one hand a characterization of strong $\mathrm{CD}^*(K,N)$ spaces whose Cheeger energy is quadratic in terms of convexity inequalities involving weighted action functionals and on the other hand a characterization involving evolution variational inequalities. These characterizations extend (1.2), known [6, 2] in the case $N = \infty$: the logarithmic entropy and the Wasserstein distance are now replaced by a nonlinear entropy and weighted action functionals. Section 9.1 provides basic results on weighted convexity inequalities and the distorsion multiplicative coefficients $\sigma^{(t)}_\kappa(\delta)$ (see (9.15)), which appear in the formulation of the $\mathrm{CD}^*(K,N)$ condition. Section 9.2 introduces the basic entropies and their regularizations. In Section 9.3 we recall the basic definitions of $\mathrm{CD}(K,\infty)$ space, of strong $\mathrm{CD}(K,\infty)$ space (involving the $K$-convexity (1.2) of the logarithmic entropy along *all* geodesics) and Proposition 9.8 states their main properties, following [45]. We then pass to the part of the theory involving dimensional bounds, by recalling the Baker-Sturm $\mathrm{CD}^*(K,N)$ condition which involves a convexity inequality along $W_2$-geodesics for the Rény entropies $\mathcal{U}_M$ defined in (1.4) and the distortion coefficients $\sigma^{(t)}_{K/M}(\delta)$, see (9.44), for all $M \geq N$.
Theorem 9.15 is our first main result, providing a characterization of strong $\mathrm{CD}^*(K,N)$ spaces in terms of the convexity inequality

$$\mathcal{U}_N(\mu_t) \leq (1-t)\mathcal{U}_N(\mu_0) + t\,\mathcal{U}_N(\mu_1) - K\mathcal{A}^{(t)}_N(\mu;\mathfrak{m}) \quad \text{for every } t \in [0,1].$$

Here $\mathcal{A}^{(t)}_N(\mu;\mathfrak{m})$ is the $(t,N)$-dependent weighted action functional as in (1.15) given by the choice $\mathfrak{Q}^{(t)}(s,r) := \mathsf{g}(s,t)r^{-1/N}$, where $\mathsf{g}$ is the Green function defined in (9.1), so that

$$\mathcal{A}^{(t)}_N(\mu;\mathfrak{m}) = \int_0^1 \int_X \mathsf{g}(s,t)\varrho^{-1/N}(x,s)\bar{v}^2(x,s)\varrho_s\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s.$$

Comparing with the $\mathrm{CD}^*(K,N)$ definition (9.44), we can say that the distortion due to the lower bound $K$ on the Ricci tensor appears just as a multiplicative factor, and that the distortion coefficients $\sigma^{(t)}_{K/M}(\delta)$ are now replaced by the $(t,N)$-dependent weighted action functional. Hence, $K$ and $N$ have more distinct roles, compared to the original definition. Our second main result is given in Theorem 9.19 and Theorem 9.20. More precisely, in Theorem 9.19 we prove that in strong $\mathrm{CD}^*(K,N)$ spaces whose Cheeger energy is a quadratic form, for any regular entropy $U$ in McCann's class $\mathrm{DC}(N)$ the induced functional $\mathcal{U}$ as in (1.4) satisfies the evolution variational inequality

$$\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t}W_2^2(\mathsf{S}_t\varrho\,\mathfrak{m},\nu)+\mathcal{U}(\mathsf{S}_t\varrho\,\mathfrak{m})\le\mathcal{U}(\nu)-K\mathcal{A}_{\omega Q}(\mu_{\cdot,t};\mathfrak{m}),\tag{1.16}$$

where $\mathsf{S}$ is the nonlinear diffusion semigroup studied in Part I, $\omega(s)=(1-s)$, $Q(r)=P(r)/r=U'(r)-U(r)/r$ and $\{\mu_{s,t}\}_{s\in[0,1]}$ is the unique geodesic connecting $\mu_t=\mathsf{S}_t\varrho\,\mathfrak{m}$ to $\nu$. The proof of this result follows the lines of [6] ($N=\infty$, $\mathfrak{m}(X)<\infty$) and [2] (where the assumption on the finiteness of $\mathfrak{m}$ was removed) and uses the calculus tools developed in [5], in particular in the proof of (9.82). In Theorem 9.20, independently of the quadraticity assumption, we adapt the ideas of [25] to prove that the evolution variational inequality above (for all regular entropies $U\in\mathrm{DC}(N)$) implies the strong $\mathrm{CD}^*(K,N)$ condition. Moreover we can use Lemma 9.13 to get the $\mathrm{CD}(K,\infty)$ condition and then apply the characterization of $\mathrm{RCD}(K,\infty)$ spaces provided in [6] to obtain that $\mathsf{Ch}$ is quadratic. Hence, under the quadraticity assumption on $\mathsf{Ch}$, the strong $\mathrm{CD}^*(K,N)$ condition and the evolution variational inequality are equivalent; without this assumption, as in the case $N=\infty$, the evolution variational inequality is stronger.

## Part III

This last part is really the core of the work, where all the tools developed in Parts I and II are combined to prove the main results. The natural setting is provided by a Polish topological space $(X,\tau)$ endowed with a $\sigma$-finite reference Borel measure $\mathfrak{m}$ and a strongly local symmetric Dirichlet form $\mathcal{E}$ in $L^2(X,\mathfrak{m})$ enjoying a *Carré du Champ* $\Gamma: D(\mathcal{E})\times D(\mathcal{E})\to L^1(X,\mathfrak{m})$ and a $\Gamma$-calculus. All the estimates about the Bakry-Émery condition discussed in Section 10 and the action estimates for nonlinear diffusion equations provided in Section 11 do not really need an underlying compatible metric structure. In any case, in Section 12, they will be applied to the case of the Cheeger energy (thus assumed to be quadratic) of the metric measure space $(X,\mathsf{d},\mathfrak{m})$ in order to prove the main results of the paper. Let us now discuss in more detail the content of Part III.

In Section 10 we recall the basic assumptions related to the Bakry-Émery condition and we prove some important properties related to them; in particular, in the case of a locally compact space, we establish useful local and nonlinear criteria to check this condition. More precisely, we introduce the multilinear form $\mathbf{\Gamma}_2$ given by

$$\mathbf{\Gamma}_2(f,g;\varphi):=\frac{1}{2}\int_X\Big(\Gamma\big(f,g\big)\,\mathrm{L}\varphi-\Gamma\big(f,\mathrm{L}g\big)\varphi-\Gamma\big(g,\mathrm{L}f\big)\varphi\Big)\,\mathrm{d}\mathfrak{m},$$

with $(f,g,\varphi) \in D_{\mathbb{V}}(\mathrm{L}) \times D_{\mathbb{V}}(\mathrm{L}) \times D_{L^\infty}(\mathrm{L})$, where we set $\mathbb{V}_\infty := \mathbb{V} \cap L^\infty(X,\mathfrak{m})$, $\mathbb{D}_\infty := \mathbb{D} \cap L^\infty(X,\mathfrak{m})$,

$$\begin{cases} D_{L^p}(\mathrm{L}) := \big\{f \in \mathbb{D} \cap L^p(X,\mathfrak{m}) :\ \mathrm{L}f \in L^p(X,\mathfrak{m})\big\} & p \in [1,\infty], \\ D_{\mathbb{V}}(\mathrm{L}) = \big\{f \in \mathbb{D} :\ \mathrm{L}f \in \mathbb{V}\big\}. \end{cases}$$

When $f = g$ we also set

$$\mathbf{\Gamma}_2(f;\varphi) := \mathbf{\Gamma}_2(f,f;\varphi) = \int_X \Big(\frac{1}{2}\Gamma\big(f\big)\,\mathrm{L}\varphi - \Gamma\big(f,\mathrm{L}f\big)\varphi\Big)\,\mathrm{d}\mathfrak{m}.$$

The $\mathbf{\Gamma}_2$ form provides a weak version (see Definition 10.1 inspired by [12, 15]) of the Bakry-Émery $\mathrm{BE}(K,N)$ condition [13, 11]

$$\mathbf{\Gamma}_2(f;\varphi) \geq K\int_X \Gamma\big(f\big)\,\varphi\,\mathrm{d}\mathfrak{m} + \frac{1}{N}\int_X (\mathrm{L}f)^2\varphi\,\mathrm{d}\mathfrak{m}, \quad \forall (f,\varphi) \in D_{\mathbb{V}}(\mathrm{L}) \times D_{L^\infty}(\mathrm{L}), \varphi \geq 0. \tag{1.17}$$

We say that a metric measure space $(X,\mathsf{d},\mathfrak{m})$ (see § 5.5) satisfies the *metric* $\mathrm{BE}(K,N)$ condition if the Cheeger energy is quadratic, the associated Dirichlet form $\mathcal{E}$ satisfies $\mathrm{BE}(K,N)$, and any $f \in \mathbb{V}_\infty$ with $\Gamma\big(f\big) \in L^\infty(X,\mathfrak{m})$ has a 1-Lipschitz representative.

In Section 10.1, by an approximation lemma, on the one hand we show that in order to get the full $\mathrm{BE}(K,N)$ it is enough to check the validity of (1.17) just for every $f \in D_{\mathbb{V}}(\mathrm{L}) \cap D_{L^\infty}(\mathrm{L})$ and every nonnegative $\varphi \in D_{L^\infty}(\mathrm{L})$. On the other hand, thanks to the improved integrability of $\Gamma$ given by Theorem 10.6, in Corollary 10.7 we extend the domain of $\mathbf{\Gamma}_2$ to the whole $(\mathbb{D}_\infty)^3$ and we give an equivalent reformulation of the $\mathrm{BE}(K,N)$ condition for functions in this larger space. Local and nonlinear characterizations of the $\mathrm{BE}(K,N)$ condition for locally compact spaces are investigated in Section 10.2: in Theorem 10.10 we show that in order to get the full $\mathrm{BE}(K,N)$ it is enough to check the validity of (1.17) just for every $f \in D_{\mathbb{V}}(\mathrm{L}) \cap D_{L^\infty}(\mathrm{L})$ and $\varphi \in D_{L^\infty}(\mathrm{L})$ *with compact support*, and in Theorem 10.11 we give a new nonlinear characterization of the $\mathrm{BE}(K,N)$ condition in terms of regular entropies, namely

$$\mathbf{\Gamma}_2(f;P(\varphi)) + \int_X R(\varphi)\,(\mathrm{L}f)^2\,\mathrm{d}\mathfrak{m} \geq K\int_X \Gamma\big(f\big)\,P(\varphi)\,\mathrm{d}\mathfrak{m}. \tag{1.18}$$

This last formulation will be very convenient later in the work in order to make a bridge between the curvature of the space and the contraction properties of non linear diffusion semigroups.

Chapter 11 is devoted to action estimates along a nonlinear diffusion semigroup. The aim is to give a rigorous proof of the crucial estimate briefly discussed in the formal calculations of Example 2.4. To this purpose, in Theorem 11.1 we prove that if $\varrho_t$ (resp. $\varphi_t$) is a sufficiently regular solution to the nonlinear diffusion equation $\partial_t\varrho_t - \mathrm{L}P(\rho_t) = 0$ (resp. to the backward linearized equation $\partial_t\varphi_t + P'(\varrho_t)\mathrm{L}\varphi_t = 0$) then the map $t \mapsto \mathcal{E}_{\varrho_t}(\varphi_t) = \int_X \rho_t\Gamma\big(\varphi_t\big)\,\mathrm{d}\mathfrak{m}$ is absolutely continuous and we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_X \rho_t\Gamma\big(\varphi_t\big)\,\mathrm{d}\mathfrak{m} = \mathbf{\Gamma}_2(\varphi_t;P(\rho_t)) + \int_X R(\rho_t)(\mathrm{L}\varphi_t)^2\,\mathrm{d}\mathfrak{m} \quad \mathscr{L}^1\text{-a.e. in } (0,T). \tag{1.19}$$

Notice that this formula is exactly the derivative of the hamiltonian $\frac{1}{2}\int_X \rho_t\Gamma\big(\varphi_t\big)\,\mathrm{d}\mathfrak{m}$ along the nonlinear diffusion semigroup. It is clear from (1.18) and (1.19) that the metric $\mathrm{BE}(K,N)$ condition implies a lower bound on the derivative of the hamiltonian, more precisely in Theorem 11.3 we show that the metric $\mathrm{BE}(K,N)$ condition implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_X \rho_t\Gamma\big(\varphi_t\big)\,\mathrm{d}\mathfrak{m} \geq K\int_X P(\varrho_t)\Gamma\big(\varphi_t\big)\,\mathrm{d}\mathfrak{m} \qquad \mathscr{L}^1\text{-a.e. in } (0,T), \tag{1.20}$$

and its natural counterparts in terms of the potentials $\phi_t$ (introduced in Part II) associated to the curve $\varrho_t\mathfrak{m}$. The inequality (1.20) should be considered as the appropriate nonlinear version of the Bakry-Ledoux inequality (see e.g. [15], [7]) for solutions $\varrho_t, \varphi_{T-t}$ to the *linear* Heat flow

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_X \rho_t\Gamma\big(\varphi_t\big)\,\mathrm{d}\mathfrak{m} \geq K\int_X \varrho_t\Gamma\big(\varphi_t\big)\,\mathrm{d}\mathfrak{m} \qquad \mathscr{L}^1\text{-a.e. in } (0,T), \tag{1.21}$$

which characterizes the $\mathrm{BE}(K,\infty)$ condition. In this case, due to the linearity of the Heat flow and to the self-adjointness of the Laplace operator, the backward evolution $\varphi_t$ can be easily constructed by using the time reversed Heat flow and it is independent of $\varrho$.

In the last Chapter 12 we combine all the estimates and tools in order to prove the equivalence between metric $\mathrm{BE}(K,N)$ and $\mathrm{RCD}^*(K,N)$. To this aim, in Section 12.1 we show some technical lemmas about approximation of $W_2$-geodesics via regular curves and about regularization of entropies. Section 12.2 is devoted to the proof of Theorem 12.8 stating that $\mathrm{BE}(K,N)$ implies $\mathrm{CD}^*(K,N)$. This is achieved by showing that the nonlinear diffusion semigroup associated to a regular entropy provides the unique solution of the Evolution Variational Inequality (1.16) which characterizes $\mathrm{RCD}^*(K,N)$. In the same section we prove the facts of independent interest that $\mathrm{BE}(K,N)$ implies contractivity in $W_2$ of the nonlinear diffusion semigroup induced by a regular entropy (see Theorem 12.5), and that $\mathrm{BE}(K,N)$ implies monotonicity of the action $\mathcal{A}_2$ computed on a curve which is moved by a nonlinear diffusion semigroup (see Theorem 12.6).
The last Section 12.3 is devoted to the proof of the converse implication, namely that if $(X,\mathsf{d},\mathfrak{m})$ is an $\mathrm{RCD}^*(K,N)$ space then the Cheeger energy satisfies $\mathrm{BE}(K,N)$. The rough idea here is to differentiate the 2-action of an arbitrary $W_2$-curve along the nonlinear diffusion semigroup and use the arbitrariness of the curve to show that this yields the nonlinear characterization (1.18) of $\mathrm{BE}(K,N)$ obtained in Theorem 10.11.

## Main notation

| | |
|---|---|
| $\mathcal{E}(f,f)$, $\Gamma\big(f,g\big)$ | Symmetric Dirichlet form $\mathcal{E}$ and its Carré du Champ, Sect. 3.1 |
| $\mathbb{H}$, $\mathbb{V}$ | $L^2(X,\mathfrak{m})$ and the domain of $\mathcal{E}$ |
| $\mathbb{V}_\infty$ | $\mathbb{V}\cap L^\infty(X,\mathfrak{m})$ |
| $\mathsf{P}$ | Markov semigroup induced by $\mathcal{E}$ |
| $\mathrm{L}$ | Infinitesimal generator of $\mathsf{P}$ |
| $\mathbb{D}$ | Domain of L, (5.31) |
| $\mathbb{D}_\infty$ | $\mathbb{D}\cap L^\infty(X,\mathfrak{m})$ |
| $\mathbb{V}_{\mathcal{E}}$ | Homogeneous space associated to $\mathcal{E}$, Sect. 3.2 |
| $\mathcal{Q}^*(\ell,\ell)$ | Dual of a quadratic form $\mathcal{Q}$, (3.10) |
| $\mathcal{E}_\varrho(f,f)$, $\Gamma_\varrho(f,g)$ | Weighted quadratic form and Carré du Champ |
| $\mathbb{V}_\varrho$ | Abstract completion of the domain of $\mathcal{E}_\varrho$ |
| $-A^*_\varrho$ | Riesz isomorphism between $\mathbb{V}'_\varrho$ and $\mathbb{V}_\varrho$, (5.40) |
| $\lvert\dot\gamma\rvert$ | metric velocity, or speed, Sect. 5.1 |
| $\mathrm{AC}^p([a,b];(X,\mathsf{d}))$ | $p$-absolutely continuous paths |
| $\mathcal{A}_p(\gamma)$ | $p$-action of a path $\gamma$, (5.4) |
| $\mathrm{Lip}(X)$, $\mathrm{Lip}_b(X)$ | Lipschitz and bounded Lipschitz functions $f:X\to\mathbb{R}$ |
| $\mathrm{Lip}(f)$ | Lipschitz constant of $f\in\mathrm{Lip}(X)$ |
| $\lvert\mathrm{D}f\rvert$, $\lvert\mathrm{D}^\pm f\rvert$, $\lvert\mathrm{D}^* f\rvert$ | Slopes of $f$, (5.5), (5.7) |
| $\mathsf{Q}_t$ | Hopf-Lax semigroup, (5.8) |
| $\mathscr{B}(X)$, $\mathscr{P}(X)$ | Borel sets and Borel probability measures in $X$ |
| $\mathscr{P}_p(X)$ | Probability measures with finite $p$-moment |
| $\mathscr{P}^{ac}(X,\mathfrak{m})$ | Absolutely continuous probability measures |
| $W_p(\mu,\nu)$ | $p$-Wasserstein extended distance in $\mathscr{P}(X)$ |
| $\mathrm{e}_s$ | Evaluation maps $\gamma\mapsto\gamma_s$ at time $s$ |
| $\mathcal{A}_p(\boldsymbol{\pi})$ | $p$-action of $\boldsymbol{\pi}\in\mathscr{P}(\mathrm{C}([0,1];X))$, (5.16) |
| $\mathrm{GeoOpt}(X)$ | Optimal geodesic plans, (5.20) |
| $\mathsf{Ch}(f)$ | Cheeger relaxed energy, (5.24) |
| $\lvert\mathrm{D}f\rvert_w$ | Minimal weak gradient, (5.25) |
| $\mathcal{A}_{\mathfrak{Q}}(\mu;\mathfrak{m})$ | Weighted energy functional induced by $\mathfrak{Q}$, (7.4) |
| $\mathcal{A}_{\mathfrak{Q}}(\mu_0;\mu_1;\mathfrak{m})$ | Weighted energy functional along a geodesic from $\mu_0$ to $\mu_1$ |
| $\mathsf{g}$ | Green function on $[0,1]$, (9.1) |
| $\sigma^{(t)}_\kappa(\delta)$ | Distorted convexity coefficients, (9.15) |
| $U$, $\mathcal{U}$ | Entropy function and the induced entropy functional, (9.27), (9.28) |
| $P$ | Pressure function induced by $U$, (9.27) |
| $\mathsf{S}$ | Nonlinear diffusion semigroup associated to an entropy $U$ |
| $\mathrm{DC}(N)$ | Entropies satisfying the $N$-dimensional McCann condition, Def. 9.14 |
| $\mathrm{CD}(K,\infty),\mathrm{CD}^*(K,N)$ | Curvature dimension conditions, Sect. 9.3 |
| $\mathrm{RCD}(K,\infty)$ | Riemannian curvature dimension condition, Def. 9.18 |
| $\Gamma_2,\mathrm{BE}(K,N)$ | $\Gamma_2$ tensor and Bakry-Émery curvature dimension condition, Sect. 10.1 |

# 2 Contraction and convexity via Hamiltonian estimates: an heuristic argument

Let us consider a smooth Lagrangian $\mathcal{L} : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$, convex and 2-homogeneous w.r.t. the second variable, which is the Legendre transform of a smooth and convex Hamiltonian $\mathcal{H} : \mathbb{R}^d \times (\mathbb{R}^d)^* \to [0, \infty)$, i.e.

$$\mathcal{L}(x, w) = \sup_{\varphi \in (\mathbb{R}^d)^*} \langle w, \varphi \rangle - \mathcal{H}(x, \varphi), \qquad \mathcal{H}(x, \varphi) = \sup_{w \in \mathbb{R}^d} \langle \varphi, w \rangle - \mathcal{L}(x, w); \tag{2.1}$$

We consider the cost functional

$$\mathcal{C}(x_0, x_1) := \inf \Big\{ \int_0^1 \mathcal{L}(x(s), \dot{x}(s))\, \mathrm{d}s : x \in \mathrm{C}^1([0,1]; \mathbb{R}^d),\ x(i) = x_i,\ i = 0, 1 \Big\} \tag{2.2}$$

and the flow $\mathsf{S}_t : \mathbb{R}^d \to \mathbb{R}^d$ given by a smooth vector field $\mathfrak{f} : \mathbb{R}^d \to \mathbb{R}^d$, i.e. $x(t) = \mathsf{S}_t(\bar{x})$ is the solution of

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = \mathfrak{f}(x(t)), \qquad x(0) = \bar{x}. \tag{2.3}$$

We are interested in necessary and sufficient conditions for the contractivity of the cost $\mathcal{C}$ under the action of the flow $\mathsf{S}_t$.

As a direct approach, for every solution $x$ of the ODE (2.3) one can consider the linearized equation

$$\frac{\mathrm{d}}{\mathrm{d}t} w(t) = \mathrm{D}\mathfrak{f}(x(t)) w(t). \tag{2.4}$$

It is well known that if $s \mapsto \bar{x}(s)$ is a smooth curve of initial data for (2.3) and $x(t, s) := \mathsf{S}_t \bar{x}(s)$ are the corresponding solutions, then $\partial_s x(t, s)$ solves (2.4) for all $s$, i.e.

$$w(t, s) := \frac{\partial}{\partial s} x(t, s) \quad \text{satisfies} \quad \frac{\partial}{\partial t} w(t, s) = \mathrm{D}\mathfrak{f}(x(t, s)) w(t, s), \quad w(0, s) = \dot{\bar{x}}(s). \tag{2.5}$$

It is one of the basic tools of [43] to notice that $\mathsf{S}$ satisfies the contraction property

$$\mathcal{C}(\mathsf{S}_T \bar{x}_0, \mathsf{S}_T \bar{x}_1) \le \mathcal{C}(\bar{x}_0, \bar{x}_1) \quad \text{for every } \bar{x}_0, \bar{x}_1 \in \mathbb{R}^d,\ T \ge 0 \tag{2.6}$$

if and only if for every solution $x$ of (2.3) and every solution $w$ of (2.4) one has

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{L}(x(t), w(t)) \le 0. \tag{2.7}$$

As we will see in the next sections, in some situations it is easier to deal with the Hamiltonian $\mathcal{H}$ instead of the Lagrangian $\mathcal{L}$. In order to get a useful condition, we thus introduce the backward transposed equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \varphi(t) = -\mathrm{D}\mathfrak{f}(x(t))^{\mathsf{T}} \varphi(t). \tag{2.8}$$

It is easy to check that $w'(t) = A(t)w(t)$ and $\varphi'(t) = -A(t)^\mathsf{T}\varphi(t)$ imply that the duality pairing $\langle w(t), \varphi(t)\rangle$ is constant. Hence, choosing $A(t) = \mathrm{D}\mathfrak{f}(x(t))$ gives

$$t \mapsto \langle w(t), \varphi(t)\rangle \text{ is constant, whenever } w \text{ solves (2.4), } \varphi \text{ solves (2.8).} \tag{2.9}$$

In the next proposition we assume a mild coercitivity property on $\mathcal{L}$, namely

$$\mathcal{L}(x, w) \geq \gamma(|x|)|w|^2 \qquad \text{with} \qquad \lim_{R\to\infty} \int_0^R \sqrt{\gamma}(r)\,dr = \infty$$

for some continuous function $\gamma : [0, \infty) \to (0, \infty)$. Under this assumption, by differentiating the function $t \mapsto \int_{|x(0)|}^{|x(t)|} \sqrt{\gamma}(r)\,\mathrm{d}r$, it is easily seen that

$$\sup_n |x_n(0)| + \int_0^1 \mathcal{L}(x_n(s), \dot{x}_n(s))\,\mathrm{d}s < \infty \quad \Longrightarrow \quad \sup_n \max_{[0,1]} |x_n| < \infty. \tag{2.10}$$

**Proposition 2.1 (Contractivity is equivalent to Hamiltonian monotonicity)** *The flow* $(\mathsf{S}_t)_{t\geq 0}$ *satisfies the contraction property* (2.6) *if and only if*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(x(t), \varphi(t)) \geq 0 \quad \textit{whenever } x \textit{ solves } (2.3) \textit{ and } \varphi \textit{ solves } (2.8). \tag{2.11}$$

Notice that the monotonicity condition (2.11) can be equivalently stated in differential form as

$$\langle \mathcal{H}_x(x, \varphi), \mathfrak{f}(x)\rangle - \langle \mathcal{H}_\varphi(x, \varphi), \mathrm{D}\mathfrak{f}(x(t))^\mathsf{T}\varphi\rangle \geq 0 \quad \text{for every } x \in \mathbb{R}^d,\ \varphi \in (\mathbb{R}^d)^*. \tag{2.12}$$

*Proof.* Let us first prove that (2.11) yields (2.6). Let $\bar{x} \in \mathrm{C}^1([0,1];\mathbb{R}^d)$ be a curve connecting $\bar{x}_0$ to $\bar{x}_1$, let $x(t,s) := \mathsf{S}_t\bar{x}(s)$ and $w(t,s) := \partial_s x(t,s)$. The thesis follows if we show that

$$\mathcal{L}(x(T,s), w(T,s)) \leq \mathcal{L}(\bar{x}(s), \dot{\bar{x}}(s)) \quad \text{for every } s \in [0,1],\ T \geq 0, \tag{2.13}$$

since then

$$\mathcal{C}(x(T,0), x(T,1)) \leq \int_0^1 \mathcal{L}(x(T,s), w(T,s))\,\mathrm{d}s \leq \int_0^1 \mathcal{L}(\bar{x}(s), \dot{\bar{x}}(s))\,\mathrm{d}s$$

and it is sufficient to take the infimum of the right hand side w.r.t. all the curves connecting $\bar{x}_0$ to $\bar{x}_1$.

For a fixed $s \in [0,1]$ and $T > 0$ we consider a sequence $\bar{\varphi}_n(s)$ such that

$$\mathcal{L}(x(T,s), w(T,s)) = \lim_{n\to\infty} \langle w(T,s), \bar{\varphi}_n(s)\rangle - \mathcal{H}(x(T,s), \bar{\varphi}_n(s)), \tag{2.14}$$

and we consider the solution $\varphi_n(t,s)$ of the backward differential equation with terminal condition

$$\frac{\partial}{\partial t}\varphi_n(t,s) = -\mathrm{D}\mathfrak{f}(x(t,s))^\mathsf{T}\varphi_n(t,s), \quad 0 \leq t \leq T, \quad \varphi_n(T,s) = \bar{\varphi}_n(s).$$

By (2.5) and (2.9) we get $\langle w(T,s), \bar\varphi_n(s)\rangle = \langle w(0,s), \varphi(0,s)\rangle$. In addition we can use the monotonicity assumption (2.11) to get

$$\mathcal{H}(x(T,s), \bar\varphi_n(s)) \geq \mathcal{H}(\bar x(s), \varphi_n(0,s)).$$

It follows that

$$\begin{aligned} \langle w(T,s), \bar\varphi_n(s)\rangle - \mathcal{H}(x(T,s), \bar\varphi_n(s)) &\leq \langle w(0,s), \varphi_n(0,s)\rangle - \mathcal{H}(\bar x(s), \varphi_n(0,s)) \\ &\leq \mathcal{L}(\bar x(s), w(0,s)) \end{aligned}$$

and passing to the limit as $n \to \infty$, by (2.14) we get (2.13) since $w(0,s) = \dot{\bar x}(s)$.

In order to prove the converse implication, let us first prove the asymptotic formula

$$\lim_{\delta\downarrow 0} \frac{\mathcal{C}(x(0), x(\delta))}{\delta^2} = \mathcal{L}(x(0), \dot x(0)) \tag{2.15}$$

for any curve $s \mapsto x(s)$ right differentiable at 0. Indeed, notice first that the inequality

$$\limsup_{\delta\downarrow 0} \frac{\mathcal{C}(x(0), x(\delta))}{\delta^2} \leq \mathcal{L}(x(0), \dot x(0))$$

immediately follows considering an affine function connecting $x(0)$ and $x(\delta)$. In order to get the lim inf inequality, notice that for any curve $s \mapsto y(s)$ and any vector $\varphi$ one has

$$\begin{aligned} \int_0^1 \mathcal{L}(y(s), \dot y(s))\,\mathrm{d}s &\geq \int_0^1 \Big(\langle \dot y(s), \varphi\rangle - \mathcal{H}(y(s), \varphi)\Big)\,\mathrm{d}s \\ &= \langle y(1) - y(0), \varphi\rangle - \int_0^1 \mathcal{H}(y(s), \varphi)\,\mathrm{d}s, \end{aligned}$$

so that choosing an almost (up to the additive constant $\delta^3$) minimizing curve $y = x_\delta : [0,1] \to \mathbb{R}^d$ connecting $x(0)$ to $x(\delta)$ and replacing $\varphi$ by $\delta\varphi$ with $\delta \in (0,1)$, the 2-homogeneity of $\mathcal{L}$ and $\mathcal{H}$ yield

$$\delta + \frac{\mathcal{C}(x(0), x(\delta))}{\delta^2} \geq \left\langle \frac{x(\delta) - x(0)}{\delta}, \varphi \right\rangle - \int_0^1 \mathcal{H}(x_\delta(s), \varphi)\,\mathrm{d}s.$$

Since (2.10) provides the relative compactness of $x_\delta$ in $\mathrm{C}([0,1];\mathbb{R}^d)$ and since $\gamma > 0$, it is easily seen that $x_\delta$ uniformly converge to the constant $x(0)$ as $\delta \downarrow 0$. Therefore, passing to the limit as $\delta \downarrow 0$ we get

$$\liminf_{\delta\downarrow 0} \frac{\mathcal{C}(x(0), x(\delta))}{\delta^2} \geq \langle \dot x(0), \varphi\rangle - \mathcal{H}(x(0), \varphi)$$

and eventually we can take the supremum w.r.t. $\varphi$ to obtain (2.15).

If (2.6) holds and $x(t)$ and $\varphi(t)$ are solutions to (2.3) and (2.8) respectively, we fix $t_0 \geq 0$ and $w \in \mathbb{R}^d$ such that

$$\mathcal{H}(x(t_0), \varphi(t_0)) = \langle w, \varphi(t_0)\rangle - \mathcal{L}(x(t_0), w). \tag{2.16}$$

We then consider the curve $s \mapsto x(t_0) + sw$ and we set $x(t,s) = \mathsf{S}_{t-t_0}(x(t_0)+sw)$, so that $w(t) = \partial_s x(t,s)\big|_{s=0}$ is a solution of (2.4) with Cauchy condition $w(t_0) = w$. For $t > t_0$ we can use twice (2.15) and (2.9) once more to obtain

$$\begin{aligned}
\mathcal{H}(x(t),\varphi(t)) &\geq \langle \varphi(t), w(t)\rangle - \lim_{\delta\downarrow 0} \frac{\mathcal{C}(x(t,0),x(t,\delta))}{\delta^2} \\
&\overset{(2.6)}{\geq} \langle \varphi(t_0), w\rangle - \lim_{\delta\downarrow 0} \frac{\mathcal{C}(x(t_0,0),x(t_0,\delta))}{\delta^2} \\
&= \langle \varphi(t_0), w\rangle - \mathcal{L}(x(t_0),w) \overset{(2.16)}{=} \mathcal{H}(x(t_0),\varphi(t_0)).
\end{aligned}$$

□

We can refine the previous argument to gain further insights when $\mathcal{H}$, $\mathcal{L}$ are quadratic forms and $\mathfrak{f}$ is the gradient of a potential $U$. More precisely, we will suppose that

$$\mathcal{L}(x,w) = \frac{1}{2}\langle \mathsf{G}(x)w, w\rangle, \quad \mathcal{H}(x,\varphi) = \frac{1}{2}\langle \varphi, \mathsf{H}(x)\varphi\rangle, \quad \mathsf{H}(x) = \mathsf{G}(x)^{-1}, \tag{2.17}$$

and $\mathsf{G}(x)$ are symmetric and positive definite linear maps from $\mathbb{R}^d$ to $(\mathbb{R}^d)^*$, smoothly depending on $x \in \mathbb{R}^d$. The vector field $\mathfrak{f}$ is the (opposite) gradient of $U : \mathbb{R}^d \to \mathbb{R}$ with respect to the metric induced by $G$ if

$$\langle \mathsf{G}(x)\mathfrak{f}(x), w\rangle = \langle -\mathrm{D}U(x), w\rangle \quad \text{for every } w \in \mathbb{R}^d, \quad \text{i.e. } \mathfrak{f}(x) = -\mathsf{H}(x)\mathrm{D}U(x). \tag{2.18}$$

In [25] it is shown that $U$ is geodesically convex along the distance induced by the cost $\mathcal{C}$ if and only if

$$\mathcal{C}(\bar{x}_0, \mathsf{S}_t\bar{x}_1) + t\Big(U(\mathsf{S}_t(\bar{x}_1)) - U(\bar{x}_0)\Big) \leq \mathcal{C}(\bar{x}_0,\bar{x}_1) \quad \text{for every } \bar{x}_0,\ \bar{x}_1 \in \mathbb{R}^d,\ t \geq 0. \tag{2.19}$$

Here is a simple argument to deduce (2.19) from (2.11).

**Lemma 2.2** *Let $\mathcal{L}$, $\mathcal{H}$, $\mathfrak{f}$ be given by* (2.17) *and* (2.18)*. Then* (2.11) *yields* (2.19)*.*

*Proof.* Let us consider a curve $\bar{x}(s)$ connecting $\bar{x}_0$ to $\bar{x}_1$ and let us set

$$y(t,s) := \mathsf{S}_t(\bar{x}(s)), \quad x(t,s) := y(st,s), \quad z(t,s) := \frac{\partial}{\partial s} y(t,s), \quad w(t,s) := \frac{\partial}{\partial s} x(t,s),$$

so that (2.18) gives

$$w(t,s) = z(st,s) + t\mathfrak{f}(x(t,s)) = z(st,s) - t\mathsf{H}(x(t,s))\mathrm{D}U(x(t,s)). \tag{2.20}$$

Clearly for every $t \geq 0$ the curve $s \mapsto x(t,s)$ connects $\bar{x}_0$ to $\mathsf{S}_t\bar{x}_1$ and therefore

$$\mathcal{C}(\bar{x}_0, \mathsf{S}_t\bar{x}_1) \leq \int_0^1 \mathcal{L}(x(t,s),w(t,s))\,\mathrm{d}s, \quad U(\mathsf{S}_t\bar{x}_1) - U(\bar{x}_0) = \int_0^1 \langle \mathrm{D}U(x(t,s)), w(t,s)\rangle\,\mathrm{d}s.$$

It follows that

$$\mathcal{C}(\bar x_0,\mathsf{S}_t\bar x_1)+t\Big(U(\mathsf{S}_t(\bar x_1))-U(\bar x_0)\Big)\le\int_0^1\Big(\mathcal{L}(x(t,s),w(t,s))+t\langle \mathrm{D}U(x(t,s)),w(t,s)\rangle\Big)\,\mathrm{d}s.$$

For a fixed $s\in[0,1]$ the integrand satisfies

$$\begin{aligned}&\mathcal{L}(x(t,s),w(t,s))+t\langle \mathrm{D}U(x(t,s)),w(t,s)\rangle\\ =&\sup_{\psi\in(\mathbb{R}^d)^*}\langle\psi+t\mathrm{D}U(x(t,s)),w(t,s)\rangle-\mathcal{H}(x(t,s),\psi)\\ =&\sup_{\varphi\in(\mathbb{R}^d)^*}\langle\varphi,w(t,s)\rangle-\mathcal{H}(x(t,s),\varphi-t\mathrm{D}U(x(t,s))).\end{aligned}\tag{2.21}$$

Substituting the expression (2.20) and recalling (2.17) we get

$$\begin{aligned}&\langle\varphi,w(t,s)\rangle-\mathcal{H}(x(t,s),\varphi-t\mathrm{D}U(x(t,s)))\\ &\quad=\langle\varphi,z(st,s)\rangle-t\langle\varphi,\mathsf{H}(x(t,s))\mathrm{D}U(x(t,s))\rangle-\mathcal{H}(x(t,s),\varphi-t\mathrm{D}U(x(t,s)))\\ &\quad=\langle\varphi,z(st,s)\rangle-\mathcal{H}(x(t,s),\varphi)-t^2\mathcal{H}(x(t,s),\mathrm{D}U(x(t,s)))\le\langle\varphi,z(st,s)\rangle-\mathcal{H}(x(t,s),\varphi).\end{aligned}$$

Choosing now an arbitrary curve $\varphi(s)$ and solutions $\varphi(\tau,s)$ of

$$\frac{\partial}{\partial\tau}\varphi(\tau,s)=-\mathrm{D}\mathfrak{f}(y(\tau,s))^{\mathsf{T}}\varphi(\tau,s),\quad 0\le\tau\le st,\quad \varphi(st,s)=\varphi(s)$$

we can use the monotonicity assumption and (2.9) to obtain

$$\begin{aligned}\langle\varphi(s),z(st,s)\rangle-\mathcal{H}(x(t,s),\varphi(s))&\le\langle\varphi(0,s),z(0,s)\rangle-\mathcal{H}(x(0,s),\varphi(0,s))\\ &=\langle\varphi(0,s),w(0,s)\rangle-\mathcal{H}(x(0,s),\varphi(0,s))\\ &\le\mathcal{L}(\bar x(s),w(0,s)).\end{aligned}$$

Since $\varphi(s)$ is arbitrary and $w(0,s)=\bar x'(s)$, considering a maximizing sequence $(\varphi_n(s))$ in (2.21) we eventually get

$$\mathcal{C}(\bar x_0,\mathsf{S}_t\bar x_1)+t\Big(U(\mathsf{S}_t(\bar x_1))-U(\bar x_0)\Big)\le\int_0^1\mathcal{L}(\bar x(s),\bar x'(s))\,\mathrm{d}s$$

and taking the infimum w.r.t. the initial curve $\bar x$ we conclude. □

In order to understand how to apply the previous arguments for studying contraction and convexity in Wasserstein space, let us consider two basic examples. For simplicity we will consider the case of a compact Riemannian manifold $(\mathbb{M}^d,\mathsf{d},\mathfrak{m})$ endowed with the distance and measure associated to the Riemannian metric tensor $\mathfrak{g}$.

**Example 2.3 (The Bakry-Émery condition for the linear heat equation)** In the subspace of smooth probability densities (identified with the corresponding measures) the

Wasserstein distance cost $\mathcal{C}(\varrho_0, \varrho_1) = \frac{1}{2} W_2^2(\varrho_0 \mathfrak{m}, \varrho_1 \mathfrak{m})$ is naturally associated to the Hamiltonian

$$\mathcal{H}(\varrho, \varphi) := \frac{1}{2} \int_X |\mathrm{D}\varphi|_{\mathfrak{g}}^2 \varrho \, \mathrm{d}\mathfrak{m} : \tag{2.22}$$

in fact the Otto-Benamou-Brenier interpretation yields

$$\mathcal{C}(\varrho_0, \varrho_1) = \inf \Big\{ \int_0^1 \mathcal{L}(\varrho_s, \dot{\varrho}_s) \, \mathrm{d}s : \ s \mapsto \varrho_s \text{ connects } \varrho_0 \text{ to } \varrho_1 \Big\}, \tag{2.23}$$

where

$$\mathcal{L}(\varrho, w) := \frac{1}{2} \int_X |\mathrm{D}\varphi|_{\mathfrak{g}}^2 \varrho \, \mathrm{d}\mathfrak{m}, \qquad -\mathrm{div}_{\mathfrak{g}}(\varrho \nabla_{\mathfrak{g}} \varphi) = w, \tag{2.24}$$

i.e.

$$\mathcal{L}(\varrho, w) = \sup_{\varphi} \int_X \varphi \, w \, \mathrm{d}\mathfrak{m} - \frac{1}{2} \int_X |\mathrm{D}\varphi|_{\mathfrak{g}}^2 \varrho \, \mathrm{d}\mathfrak{m}. \tag{2.25}$$

In other words, the cotangent bundle is associated to the velocity gradient $\nabla_{\mathfrak{g}} \varphi$ and the duality between tangent and cotangent bundle is provided by the possibly degenerate elliptic PDE

$$-\,\mathrm{div}_{\mathfrak{g}}(\varrho \nabla_{\mathfrak{g}} \varphi) = w. \tag{2.26}$$

If we consider the logarithmic entropy functional $\mathcal{U}_\infty(\varrho) := \int_X \varrho \log \varrho \, \mathrm{d}\mathfrak{m}$, then its Wasserstein gradient flow corresponds to the linear differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \varrho = \Delta_{\mathfrak{g}} \varrho \tag{2.27}$$

and thus the backward equation (2.8) for $\varphi$ corresponds to

$$\frac{\mathrm{d}}{\mathrm{d}t} \varphi = -\Delta_{\mathfrak{g}} \varphi. \tag{2.28}$$

Evaluating the derivative of the Hamiltonian one gets

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{H}(\varrho_t, \varphi_t) &= \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_X \varrho_t |\mathrm{D}\varphi_t|_{\mathfrak{g}}^2 \, \mathrm{d}\mathfrak{m} = \frac{1}{2} \int_X \Delta_{\mathfrak{g}} \varrho_t |\mathrm{D}\varphi_t|_{\mathfrak{g}}^2 \, \mathrm{d}\mathfrak{m} + \int_X \varrho_t \langle \mathrm{D}\varphi_t, \mathrm{D}(-\Delta_{\mathfrak{g}} \varphi_t) \rangle_{\mathfrak{g}} \, \mathrm{d}\mathfrak{m} \\ &= \int_X \varrho_t \Big( \frac{1}{2} \Delta_{\mathfrak{g}} |\mathrm{D}\varphi_t|_{\mathfrak{g}}^2 - \langle \mathrm{D}\varphi_t, \mathrm{D}\Delta_{\mathfrak{g}} \varphi_t \rangle_{\mathfrak{g}} \Big) \, \mathrm{d}\mathfrak{m}. \end{aligned}$$

Since $\varrho \geq 0$ and $\varphi$ are arbitrary, (2.11) corresponds to the Bakry-Émery $\mathrm{BE}(0, \infty)$ condition

$$\Gamma_2(\varphi) := \frac{1}{2} \Delta_{\mathfrak{g}} |\mathrm{D}\varphi_t|_{\mathfrak{g}}^2 - \langle \mathrm{D}\varphi_t, \mathrm{D}\Delta_{\mathfrak{g}} \varphi_t \rangle_{\mathfrak{g}} \geq 0 \qquad \text{for every } \varphi.$$

It is remarkable that the above calculations correspond to the Bakry-Ledoux [15] derivation of the $\Gamma_2$ tensor: if $\mathsf{P}_t$ denotes the heat flow associated to (2.27), it is well known that

$$\Gamma_2 \geq 0 \quad \Longleftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}s} \Big( \mathsf{P}_s |\mathrm{D}\mathsf{P}_{t-s} \varphi|_{\mathfrak{g}}^2 \Big) \geq 0$$

or, in the integrated form,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\int_X \mathsf{P}_s\varrho\,|\mathrm{D}\mathsf{P}_{t-s}\varphi|^2_{\mathfrak{g}}\,\mathrm{d}\mathfrak{m} = \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{H}(\mathsf{P}_s\varrho,\mathsf{P}_{t-s}\varphi) \geq 0 \quad \text{for every } \varphi,\varrho \geq 0.$$

Thus the combination of the forward flow $\mathsf{P}_s\varrho_s$ and the backward flow $\mathsf{P}_{t-s}\varphi$ in the derivation of $\Gamma_2$ tensor corresponds to the Hamiltonian monotonicity (2.11).

**Example 2.4 (The Bakry-Émery condition for nonlinear diffusion)** If we apply the previous argument to the entropy functional $\mathcal{U}(\varrho) = \int_X U(\varrho)\,\mathrm{d}\mathfrak{m}$, we are led to study the nonlinear diffusion equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\varrho = \Delta_{\mathfrak{g}}P(\varrho) \qquad \text{with} \qquad P(\varrho) := \varrho U'(\varrho) - U(\varrho). \tag{2.29}$$

The corresponding linearized backward transposed flow is

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi = -P'(\varrho)\Delta_{\mathfrak{g}}\varphi \tag{2.30}$$

and, setting $R(\varrho) := \varrho P'(\varrho) - P(\varrho)$, we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(\varrho_t,\varphi_t)\\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_X \varrho_t|\mathrm{D}\varphi_t|^2_{\mathfrak{g}}\,\mathrm{d}\mathfrak{m}\\
&= \frac{1}{2}\int_X \Delta_{\mathfrak{g}}P(\varrho_t)|\mathrm{D}\varphi_t|^2_{\mathfrak{g}}\,\mathrm{d}\mathfrak{m} - \int_X \varrho_t\langle \mathrm{D}\varphi_t, \mathrm{D}(P'(\varrho_t)\Delta_{\mathfrak{g}}\varphi_t)\rangle_{\mathfrak{g}}\,\mathrm{d}\mathfrak{m}\\
&= \int_X P(\varrho_t)\Gamma_2(\varphi_t)\,\mathrm{d}\mathfrak{m} + \int_X P(\varrho_t)\langle \mathrm{D}\varphi_\mathrm{t}, \mathrm{D}\Delta_{\mathfrak{g}}\varphi_\mathrm{t}\rangle_{\mathfrak{g}} - \int_\mathrm{X} \varrho_\mathrm{t}\langle \mathrm{D}\varphi_\mathrm{t}, \mathrm{D}(\mathrm{P}'(\varrho_\mathrm{t})\Delta_{\mathfrak{g}}\varphi_\mathrm{t})\rangle_{\mathfrak{g}}\,\mathrm{d}\mathfrak{m}\\
&= \int_X P(\varrho_t)\Gamma_2(\varphi_t)\,\mathrm{d}\mathfrak{m} + \int_X \big(-P(\varrho_t) + \varrho_t P'(\varrho_t)\big)\big(\Delta_{\mathfrak{g}}\varphi_t\big)^2\,\mathrm{d}\mathfrak{m} - \int_X \Delta_{\mathfrak{g}}\varphi_t\langle \mathrm{D}\varphi_t, \mathrm{DP}(\varrho_\mathrm{t})\rangle_{\mathfrak{g}}\,\mathrm{d}\mathfrak{m}\\
&+ \int_X P'(\varrho_t)\Delta_{\mathfrak{g}}\varphi_t\langle \mathrm{D}\varphi_\mathrm{t}, \mathrm{D}\varrho_\mathrm{t}\rangle_{\mathfrak{g}}\,\mathrm{d}\mathfrak{m}\\
&= \int_X P(\varrho_t)\Gamma_2(\varphi_t)\,\mathrm{d}\mathfrak{m} + \int_X R(\varrho_t)\big(\Delta_{\mathfrak{g}}\varphi_t\big)^2\,\mathrm{d}\mathfrak{m}.
\end{aligned}$$

If $U$ satisfies McCann's condition DC($N$), so that $R(\varrho) \geq -\frac{1}{N}P(\varrho)$, and the Bakry-Émery condition BE($0,N$) holds, so that $\Gamma_2(\varphi) \geq \frac{1}{N}(\Delta_{\mathfrak{g}}\varphi)^2$, we still get $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(\varrho,\varphi) \geq 0$.

**Example 2.5 (Nonlinear mobilities)** As a last example, consider [26] the case of an Hamiltonian associated to a nonlinear positive mobility $h$

$$\mathcal{H}(\varrho,\varphi) := \frac{1}{2}\int_X h(\varrho)|\mathrm{D}\varphi|^2_{\mathfrak{g}}\,\mathrm{d}\mathfrak{m} \tag{2.31}$$

under the action of the linear heat flows (2.27) and (2.28): with computations similar to those of the previous examples we get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(\varrho_t,\varphi_t) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_X h(\varrho_t)|\mathrm{D}\varphi_t|^2_{\mathfrak{g}}\,\mathrm{d}\mathfrak{m} \\
&= \frac{1}{2}\int_X h'(\varrho_t)\Delta_{\mathfrak{g}}\varrho_t|\mathrm{D}\varphi_t|^2_{\mathfrak{g}}\,\mathrm{d}\mathfrak{m} - \int_X h(\varrho_t)\langle \mathrm{D}\varphi_t, \mathrm{D}\Delta_{\mathfrak{g}}\varphi_t)\rangle_{\mathfrak{g}}\,\mathrm{d}\mathfrak{m} \\
&= \frac{1}{2}\int_X \Delta_{\mathfrak{g}}(h(\varrho_t))|\mathrm{D}\varphi_t|^2_{\mathfrak{g}}\,\mathrm{d}\mathfrak{m} - \int_X h(\varrho_t)\langle \mathrm{D}\varphi_t, \mathrm{D}\Delta_{\mathfrak{g}}\varphi_t)\rangle_{\mathfrak{g}}\,\mathrm{d}\mathfrak{m} - \frac{1}{2}\int_X h''(\varrho_t)|\mathrm{D}\varrho_t|^2_{\mathfrak{g}}\,|\mathrm{D}\varphi_t|^2_{\mathfrak{g}}\,\mathrm{d}\mathfrak{m} \\
&= \int_X h(\varrho_t)\Gamma_2(\varphi_t)\,\mathrm{d}\mathfrak{m} - \frac{1}{2}\int_X h''(\varrho_t)|\mathrm{D}\varrho_t|^2_{\mathfrak{g}}\,|\mathrm{D}\varphi_t|^2_{\mathfrak{g}}\,\mathrm{d}\mathfrak{m}.
\end{aligned}
$$

If $h$ is concave and the Bakry-Émery condition $\mathrm{BE}(0,\infty)$ holds, it has been proved in [22] that we still have $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(\varrho,\varphi) \geq 0$.

# Part I

# Nonlinear diffusion equations and their linearization in Dirichlet spaces

## 3 Dirichlet forms, homogeneous spaces and nonlinear diffusion

### 3.1 Dirichlet forms

In all this first part we will deal with a measurable space $(X,\mathcal{B})$, which is complete with respect to a $\sigma$-finite measure $\mathfrak{m} : \mathcal{B} \to [0,\infty]$, and with a symmetric Dirichlet form $\mathcal{E} : L^2(X,\mathfrak{m}) \to [0,\infty]$ (see e.g. [16] as a general reference) with proper domain

$$
\mathbb{V} = D(\mathcal{E}) := \big\{ f \in L^2(X,\mathfrak{m}) : \mathcal{E}(f) < \infty \big\}, \quad \text{with} \quad \mathbb{V}_\infty := \mathbb{V} \cap L^\infty(X,\mathfrak{m}). \tag{3.1}
$$

$\mathbb{V}$ is an Hilbert space endowed with the norm

$$
\|f\|^2_{\mathbb{V}} := \|f\|^2_{L^2(X,\mathfrak{m})} + \mathcal{E}(f); \tag{3.2}
$$

the inclusion of $\mathbb{V}$ in $\mathbb{H}$ is always continuous and we will assume that it is also dense (we will write $\mathbb{V} \overset{ds}{\hookrightarrow} \mathbb{H}$); we will still denote by $\mathcal{E}(\cdot,\cdot) : \mathbb{V} \to \mathbb{R}$ the symmetric bilinear form associated to $\mathcal{E}$. Identifying $\mathbb{H}$ with its dual $\mathbb{H}'$, $\mathbb{H}$ is also continuously and densely imbedded in the dual space $\mathbb{V}'$, so that

$$
\mathbb{V} \overset{ds}{\hookrightarrow} \mathbb{H} \equiv \mathbb{H}' \overset{ds}{\hookrightarrow} \mathbb{V}' \quad \text{is a standard Hilbert triple} \tag{3.3}
$$

and we have

$$\langle f, g\rangle = \int_X fg \,\mathrm{d}\mathfrak{m} \quad \text{whenever } f \in \mathbb{H},\, g \in \mathbb{V} \tag{3.4}$$

where $\langle\cdot,\cdot\rangle = {}_{\mathbb{V}'}\langle\cdot,\cdot\rangle_{\mathbb{V}}$ denotes the duality pairing between $\mathbb{V}'$ and $\mathbb{V}$, when there will be no risk of confusion.

The locality of $\mathcal{E}$ and the $\Gamma$-calculus are not needed at this level: they will play a crucial role in the next parts. On the other hand, we will repeatedly use the following properties of Dirichlet form:

$$\mathbb{V}_\infty \text{ is an algebra,} \quad \mathcal{E}^{1/2}(fg) \le \|f\|_\infty \mathcal{E}^{1/2}(g) + \|g\|_\infty \mathcal{E}^{1/2}(f) \quad f,\, g \in \mathbb{V}_\infty, \tag{DF1}$$

$$\begin{gathered}\text{if } P:\mathbb{R}\to\mathbb{R} \text{ is } L\text{-Lipschitz with } P(0)=0 \text{ then the map } f\mapsto P\circ f\\ \text{is well defined and continuous from } \mathbb{V} \text{ to } \mathbb{V} \text{ with } \quad \mathcal{E}(P\circ f) \le L^2\,\mathcal{E}(f),\end{gathered} \tag{DF2}$$

$$\begin{gathered}\text{if } P_i:\mathbb{R}\to\mathbb{R} \text{ are Lipschitz and nondecreasing with } P_i(0)=0,\ i=1,2, \text{ then}\\ \mathcal{E}(P_1\circ f, P_2\circ f) \ge 0 \quad \text{for every } f\in\mathbb{V}.\end{gathered} \tag{DF3}$$

We will denote by $-\mathrm{L}$ the linear monotone operator induced by $\mathcal{E}$,

$$\mathrm{L}:\mathbb{V}\to\mathbb{V}', \quad \langle -\mathrm{L}f, g\rangle := \mathcal{E}(f,g) \quad \text{for every } f,g\in\mathbb{V}, \tag{3.6}$$

satisfying

$$\big|\langle \mathrm{L}f, g\rangle\big|^2 \le \mathcal{E}(f,f)\,\mathcal{E}(g,g) \quad \text{for every } f,\, g\in\mathbb{V}, \tag{3.7}$$

and by $\mathbb{D}$ the Hilbert space

$$\mathbb{D} := \big\{f\in\mathbb{V} : \mathrm{L}f\in\mathbb{H}\big\} \quad \text{endowed with the Hilbert norm} \quad \|f\|_{\mathbb{D}}^2 := \|f\|_{\mathbb{V}}^2 + \|\mathrm{L}f\|_{\mathbb{H}}^2. \tag{3.8}$$

Thanks to the interpolation estimate

$$\|\varrho\|_{\mathbb{V}} \le C\|\varrho\|_{\mathbb{H}}^{1/2}\,\|\varrho\|_{\mathbb{D}}^{1/2} \quad \text{for every } \varrho\in\mathbb{D}, \tag{3.9}$$

which easily follows by the identity $\mathcal{E}(\varrho,\varrho) = -\int_X \varrho\mathrm{L}\varrho\,\mathrm{d}\mathfrak{m}$, the norm of $\mathbb{D}$ is equivalent to the norm $\|f\|_{\mathbb{H}}^2 + \|\mathrm{L}f\|_{\mathbb{H}}^2$.

We also introduce the dual quadratic form $\mathcal{E}^*$ on $\mathbb{V}'$, defined by

$$\frac12\mathcal{E}^*(\ell,\ell) := \sup_{f\in\mathbb{V}} \langle\ell, f\rangle - \frac12\mathcal{E}(f,f). \tag{3.10}$$

It is elementary to check that the right hand side in (3.10) satisfies the parallelogram rule, so our notation $\mathcal{E}^*(\ell,\ell)$ is justified (and actually we will prove that $\mathcal{E}^*$, when restricted to its finiteness domain, is canonically associated to a dual Hilbert norm).

The operator L generates a Markov semigroup $(\mathsf{P}_t)_{t\ge0}$ in each $L^p(X,\mathfrak{m})$, $1\le p\le\infty$: for every $f\in\mathbb{H}$ the curve $f_t := \mathsf{P}_t f$ belongs to $\mathrm{C}^1((0,\infty);\mathbb{H})\cap\mathrm{C}^0((0,\infty);\mathbb{D})$ and it is the unique solution in this class of the Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}t}f_t = \mathrm{L}f_t \quad t>0, \qquad \lim_{t\downarrow0} f_t = f \quad \text{strongly in } \mathbb{H}. \tag{3.11}$$

The curve $(f_t)_{t\ge 0}$ belongs to $\mathrm{C}^1([0,\infty);\mathbb{H})$ if and only if $f\in\mathbb{D}$ and in this case

$$\lim_{t\downarrow 0}\frac{f_t-f}{t}=\mathrm{L}f\quad\text{strongly in }\mathbb{H}. \tag{3.12}$$

$(\mathsf{P}_t)_{t\ge 0}$ is in fact an analytic semigroup of linear contractions in $\mathbb{H}$ and in each $L^p(X,\mathfrak{m})$ space, $p\in(1,\infty)$, satisfying the regularization estimate

$$\frac{1}{2}\|\mathsf{P}_t f\|_{\mathbb{H}}^2+t\mathcal{E}(\mathsf{P}_t f)+t^2\|\mathrm{L}\mathsf{P}_t f\|_{\mathbb{H}}^2\le\frac{1}{2}\|f\|_{\mathbb{H}}^2\quad\text{for every }t>0. \tag{3.13}$$

The semigroup $(\mathsf{P}_t)_{t\ge 0}$ is said to be *mass preserving* if

$$\int_X\mathsf{P}_t f\,\mathrm{d}\mathfrak{m}=\int_X f\,\mathrm{d}\mathfrak{m}\quad\forall t\ge 0\qquad\text{for every }f\in L^1\cap L^2(X,\mathfrak{m}). \tag{3.14}$$

Since $(\mathsf{P}_t)_{t\ge 0}$ is a strongly continuous semigroup of contractions in $L^1(X,\mathfrak{m})$, the mass preserving property is equivalent to

$$\int_X\mathrm{L}f\,\mathrm{d}\mathfrak{m}=0\quad\text{for every }f\in\mathbb{D}\cap L^1(X,\mathfrak{m})\text{ with }\mathrm{L}f\in L^1(X,\mathfrak{m}). \tag{3.15}$$

When $\mathfrak{m}(X)<\infty$ then (3.14) is equivalent to the property $1\in D(\mathcal{E})$ with $\mathcal{E}(1)=0$.

## 3.2 Completion of quotient spaces w.r.t. a seminorm

Here we recall a simple construction that we will often use in the following.

Let $N$ be the kernel of $\mathcal{E}$ and $\mathrm{L}$, namely

$$N:=\Big\{f\in\mathbb{V}:\ \mathcal{E}(f,f)=0\Big\}=\Big\{f\in\mathbb{V}:\ \mathrm{L}f=0\Big\}. \tag{3.16}$$

It is obvious that $N$ is a closed subspace of $\mathbb{V}$ and that it induces the equivalence relation

$$f\sim g\quad\Longleftrightarrow\quad f-g\in N. \tag{3.17}$$

We will denote by $\widetilde{\mathbb{V}}:=\mathbb{V}/\!\sim$ the quotient space and by $\tilde f$ the equivalence class of $f$ (still denoted by $f$ when there is no risk of confusion); it is well known that we can identify the dual of $\widetilde{\mathbb{V}}$ with the closed subspace $N^\perp$ of $\mathbb{V}'$, i.e.

$$\widetilde{\mathbb{V}}'=N^\perp=\big\{\ell\in\mathbb{V}':\ \langle\ell,f\rangle=0\quad\text{for every }f\in N\big\}. \tag{3.18}$$

Since $\mathcal{E}$ is nonnegative, we have $\mathcal{E}(f_1,g_1)=\mathcal{E}(f_0,g_0)$ whenever $f_0\sim f_1$ and $g_0\sim g_1$, so that $\mathcal{E}$ can also be considered a symmetric bilinear form on $\widetilde{\mathbb{V}}$, for which we retain the same notation. The bilinear form $\mathcal{E}$ is in fact a scalar product on $\widetilde{\mathbb{V}}$, so that it can be extended to a scalar product on the abstract completion $\mathbb{V}_{\mathcal{E}}$ of $\widetilde{\mathbb{V}}$, with respect to the norm induced by $\mathcal{E}$. The dual of $\mathbb{V}_{\mathcal{E}}$ will be denoted by $(\mathbb{V}_{\mathcal{E}})'$.

In the next proposition we relate $(\mathbb{V}_{\mathcal{E}})'$ to $\mathbb{V}'$ and to the dual quadratic form $\mathcal{E}^*$ in (3.10).

**Proposition 3.1 (Basic duality properties)** *Let* $\mathbb{V}$, $\mathcal{E}$, $\widetilde{\mathbb{V}}$ *be as above and let* $\mathbb{V}_{\mathcal{E}}$ *be the abstract completion of* $\widetilde{\mathbb{V}}$ *w.r.t. the scalar product* $\mathcal{E}$*. Then the following properties hold:*

*(a)* $(\mathbb{V}_{\mathcal{E}})'$ *can be canonically and isometrically realized as the finiteness domain of* $\mathcal{E}^*$ *in* $\mathbb{V}'$*, endowed with the norm induced by* $\mathcal{E}^*$*, that we will denote as* $\mathbb{V}'_{\mathcal{E}}$*.*

*(b) If* $\ell \in \mathbb{V}'_{\mathcal{E}}$ *and* $(f_n)$ *is a maximizing sequence in* (3.10)*, then the corresponding elements in* $\mathbb{V}_{\mathcal{E}}$ *strongly converge in* $\mathbb{V}_{\mathcal{E}}$ *to* $f \in \mathbb{V}_{\mathcal{E}}$ *satisfying*

$$\frac{1}{2}\mathcal{E}^*(\ell,\ell) = \langle \ell, f\rangle - \frac{1}{2}\mathcal{E}(f,f), \qquad \mathcal{E}^*(\ell,\ell) = \mathcal{E}(f,f). \tag{3.19}$$

*(c) The operator* $\mathrm{L}$ *in* (3.6) *maps* $\mathbb{V}$ *into* $\mathbb{V}'_{\mathcal{E}}$*; it can be extended to a continuous and linear operator* $\mathrm{L}_{\mathcal{E}}$ *from* $\mathbb{V}_{\mathcal{E}}$ *to* $\mathbb{V}'_{\mathcal{E}}$ *and* $-\mathrm{L}_{\mathcal{E}} : \mathbb{V}_{\mathcal{E}} \to \mathbb{V}'_{\mathcal{E}}$ *is the Riesz isomorphism associated to the scalar product* $\mathcal{E}$ *on* $\mathbb{V}_{\mathcal{E}}$*;*

*(d)* $\mathcal{E}^*(\ell, -\mathrm{L}f) = \langle \ell, f\rangle$ *for all* $\ell \in \mathbb{V}'_{\mathcal{E}}$*,* $f \in \mathbb{V}$*.*

*Proof.* (a) The inequality $2|\langle \ell, f\rangle| \le \mathcal{E}(f,f) + \mathcal{E}^*(\ell,\ell)$, by homogeneity, gives $|\langle \ell, f\rangle| \le (\mathcal{E}(f,f))^{1/2}(\mathcal{E}^*(\ell,\ell))^{1/2}$. Hence, any element $\ell$ in the finiteness domain of $\mathcal{E}^*$ induces a continuous linear functional on $\widetilde{\mathbb{V}}$ and therefore an element in $(\mathbb{V}_{\mathcal{E}})'$, with (dual) norm less than $(\mathcal{E}^*(\ell,\ell))^{1/2}$. Conversely, any $\ell \in (\mathbb{V}_{\mathcal{E}})'$ induces a continuous linear functional in $\widetilde{\mathbb{V}}$, and then a continuous linear functional $\ell$ in $\mathbb{V}$, satisfying $|\ell(f)| \le \|L\|_{(\mathbb{V}_{\mathcal{E}})'}(\mathcal{E}(f,f))^{1/2}$. By the continuity of $\mathcal{E}$, $\ell \in \mathbb{V}'$; in addition, the Young inequality gives $\mathcal{E}^*(\ell,\ell) \le \|L\|^2_{(\mathbb{V}_{\mathcal{E}})'}$.

(b) The uniform convexity of $g \mapsto \langle \ell, g\rangle - \frac{1}{2}\mathcal{E}(g,g)$ shows that $\mathcal{E}(f_n - f_m, f_n - f_m) \to 0$ as $n$, $m \to \infty$. By definition of $\mathbb{V}_{\mathcal{E}}$ this means that $(f_n)$ is convergent in $\mathbb{V}_{\mathcal{E}}$. Eventually we use the continuity of $\langle \ell, \cdot\rangle$ in $\mathbb{V}_{\mathcal{E}}$ to conclude.

(c) By (3.7), L can also be seen as an operator from $\widetilde{\mathbb{V}}$ to $\mathbb{V}'_{\mathcal{E}}$, with $\|\mathrm{L}f\|_{\mathbb{V}'_{\mathcal{E}}} \le (\mathcal{E}(f,f))^{1/2}$ for all $f \in \widetilde{\mathbb{V}}$. It extends therefore to the completion $\mathbb{V}_{\mathcal{E}}$ of $\widetilde{\mathbb{V}}$. Denoting by $\mathrm{L}_{\mathcal{E}}$ the extension, let us prove that $-\mathrm{L}_{\mathcal{E}}$ is the Riesz isomorphism.

We first prove that $-\mathrm{L}_{\mathcal{E}}$ is onto; this follows easily proving that, for given $\ell \in \mathbb{V}'_{\mathcal{E}}$, the maximizer $f \in \mathbb{V}_{\mathcal{E}}$ given by (b) satisfies $\ell = -\mathrm{L}_{\mathcal{E}}f$. Since $\widetilde{\mathbb{V}}$ is dense in $\mathbb{V}_{\mathcal{E}}$, we obtain that

$$\Big\{\ell \in \mathbb{V}'_{\mathcal{E}} :\ \ell = -\mathrm{L}f \text{ for some } f \in \widetilde{\mathbb{V}}\Big\} \quad \text{is dense in } \mathbb{V}'_{\mathcal{E}}. \tag{3.20}$$

Computing $\mathcal{E}^*(-\mathrm{L}f, -\mathrm{L}f)$ for $f \in \widetilde{\mathbb{V}}$ and using the definition of $\mathcal{E}$ immediately gives $\mathcal{E}^*(-\mathrm{L}f, -\mathrm{L}f) = \mathcal{E}(f,f)$. By density, this proves that $-\mathrm{L}_{\mathcal{E}}$ is the Riesz isomorphism.

(d) When $\ell = -\mathrm{L}g$ for some $g \in \widetilde{\mathbb{V}}$ it follows by polarization of the identity $\mathcal{E}^*(-\mathrm{L}h, -\mathrm{L}h) = \mathcal{E}(h,h)$, already mentioned in the proof of (c). The general case follows by (3.20). $\square$

We can summarize the realization in (a) by writing

$$\mathbb{V}'_{\mathcal{E}} = D(\mathcal{E}^*) = \big\{\ell \in \mathbb{V}' :\ |\langle \ell, f\rangle| \le C\sqrt{\mathcal{E}(f,f)} \quad \text{for every } f \in \mathbb{V}\big\}. \tag{3.21}$$

According to this representation and the identification $\mathbb{H} = \mathbb{H}'$, $f \in \mathbb{H}$ belongs to $\mathbb{V}'_{\mathcal{E}}$ if and only if there exists a constant $C$ such that

$$\Big|\int_X fg \,\mathrm{d}\mathfrak{m}\Big| \le C\Big(\mathcal{E}(g,g)\Big)^{1/2} \qquad \forall g \in \mathbb{V}.$$

If this is the case, we shall write $f \in \mathbb{H} \cap \mathbb{V}'_{\mathcal{E}}$.

**Remark 3.2 (Identification of Hilbert spaces)** In the usual framework of the variational formulation of parabolic problems, one usually considers a Hilbert triple as in (3.3) $V \subset H \equiv H' \subset V'$ so that the duality pairing $\langle \ell, f\rangle$ between $V'$ and $V$ coincides with the scalar product in $H$ whenever $\ell \in H$. In this way the definition of the domain $\mathbb{D}$ of $\mathrm{L}$ as in (3.19) makes sense. In the case of $\mathbb{V}_{\mathcal{E}}$, $\mathbb{V}'_{\mathcal{E}}$ one has to be careful that $\mathbb{V}_{\mathcal{E}}$ is not generally imbedded in $\mathbb{H}$ and therefore $\mathbb{H}$ is not imbedded in $\mathbb{V}'_{\mathcal{E}}$, unless $\mathcal{E}$ is coercive with respect to the $\mathbb{H}$-norm; it is then possibile to consider the intersection $\mathbb{H} \cap \mathbb{V}'_{\mathcal{E}}$ (which can be better understood as $\mathbb{H}' \cap \mathbb{V}'_{\mathcal{E}}$). Similarly, $\mathbb{V}$ is imbedded in $\mathbb{V}_{\mathcal{E}}$ if and only if $\mathbb{V}'_{\mathcal{E}}$ is dense in $\mathbb{V}'$, and this happens if and only if $N = \{0\}$, i.e. $\mathcal{E}$ is a norm on $\mathbb{V}$.

The following lemma will be useful.

**Lemma 3.3** *The following properties of the spaces $\mathbb{H}$, $\mathbb{V}$ and $\mathbb{V}'_{\mathcal{E}}$ hold.*

*(a) A function $f \in \mathbb{H}$ belongs to $\mathbb{V}$ if and only if*

$$\Big|\int_X f\mathrm{L}g \,\mathrm{d}\mathfrak{m}\Big| \le C\Big(\mathcal{E}(g,g)\Big)^{1/2} \qquad \forall g \in \mathbb{D}. \tag{3.22}$$

*(b) $\{\mathrm{L}f : \ f \in \mathbb{D}\}$ is dense in $\mathbb{V}'_{\mathcal{E}}$ and, in particular, $\mathbb{H} \cap \mathbb{V}'_{\mathcal{E}}$ is dense in $\mathbb{V}'_{\mathcal{E}}$.*

*Proof.* (a) If $f \in \mathbb{V}$ we can choose $g = f$ to obtain $\mathcal{E}(f,f) \le C^2$. In the general case we notice that the property (3.22) is stable under the action of the semigroup $(\mathsf{P}_t)_{t\ge 0}$ and we argue by approximation.

(b) Let us consider an element $\ell \in \mathbb{V}'_{\mathcal{E}}$ such that

$$\mathcal{E}^*(\ell, \mathrm{L}f) = 0 \quad \text{for every } \ f \in \mathbb{D}.$$

Applying Proposition 3.1(d) we get

$$\langle \ell, f\rangle = 0 \quad \text{for every } \ f \in \mathbb{D}.$$

Since $\ell \in \mathbb{V}'$ and $\mathbb{D}$ is dense in $\mathbb{V}$ we conclude that $\ell = 0$. □

## 3.3 Nonlinear diffusion

The aim of this section is to study evolution equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\varrho - \mathrm{L}P(\varrho) = 0, \tag{3.23}$$

where $P : \mathbb{R} \to \mathbb{R}$ is a *regular monotone* nonlinearity satisfying

$$P \in \mathrm{C}^1(\mathbb{R}), \quad P(0) = 0, \quad 0 < \mathsf{a} \le \mathsf{P}'(r) \le \mathsf{a}^{-1} \quad \text{for every } r \ge 0. \tag{3.24}$$

The results are more or less standard application of the abstract theory of monotone operators and variational evolution equations in Hilbert spaces [17, 18, 19, 20], with the only caution described in Remark 3.2 and the use of a general Markov operator instead of a particular realization given by a second order elliptic differential operator.

If $H_0$, $H_1$ are Hilbert spaces continuously imbedded in a common Banach space $B$ and $T > 0$ is a given final time, we introduce the spaces of time-dependent functions

$$W^{1,2}(0,T;H_1,H_0) := \Big\{u \in W^{1,2}(0,T;B) : \ u \in L^2(0,T;H_1), \ \dot u \in L^2(0,T;H_0)\Big\}, \tag{3.25}$$

endowed with the norm

$$\|u\|^2_{W^{1,2}(0,T;H_1,H_0)} := \|u\|^2_{L^2(0,T;H_1)} + \|\dot u\|^2_{L^2(0,T;H_0)}. \tag{3.26}$$

Denoting by $(H_0,H_1)_{\vartheta,2}$, $\vartheta \in (0,1)$, the family of (complex or real, [38, 2.1 and Thm. 15.1], [51, 1.3.2]) Hilbert interpolation spaces, the equivalence with the so-called *trace Interpolation method* [38, Thm. 3.1], [51, 1.8.2], shows that

$$\text{if} \quad H_1 \hookrightarrow H_0 \quad \text{then} \quad W^{1,2}(0,T;H_1,H_0) \hookrightarrow \mathrm{C}([0,T];(H_0,H_1)_{1/2,2}), \tag{3.27}$$

with continuous inclusion.

As a possible example, we will consider $W^{1,2}(0,T;\mathbb{V},\mathbb{V}'_{\mathcal{E}})$ (in this case $\mathbb{V}$ and $\mathbb{V}'_{\mathcal{E}}$ are continuously imbedded in $\mathbb{V}'$) and $W^{1,2}(0,T;\mathbb{D},\mathbb{H})$. Since [38, Prop. 2.1]

$$\mathbb{V}'_{\mathcal{E}} \subset \mathbb{V}', \quad (\mathbb{V},\mathbb{V}')_{1/2,2} = \mathbb{H}, \qquad \text{and} \qquad (\mathbb{D},\mathbb{H})_{1/2,2} = \mathbb{V},$$

we easily get

$$W^{1,2}(0,T;\mathbb{V},\mathbb{V}'_{\mathcal{E}}) \hookrightarrow \mathrm{C}([0,T];\mathbb{H}), \qquad W^{1,2}(0,T;\mathbb{D},\mathbb{H}) \hookrightarrow \mathrm{C}([0,T];\mathbb{V}). \tag{3.28}$$

Let us fix a regular function $P$ according to (3.24): we introduce the set

$$\mathcal{ND}(0,T) := \Big\{\varrho \in W^{1,2}(0,T;\mathbb{H}) \cap \mathrm{C}^1([0,T];\mathbb{V}'_{\mathcal{E}}) : \ P(\varrho) \in L^2(0,T;\mathbb{D})\Big\}. \tag{3.29}$$

Notice that

$$\mathcal{ND}(0,T) \subset \mathrm{C}([0,T];\mathbb{V}). \tag{3.30}$$

Indeed, if $\varrho \in \mathcal{ND}(0,T)$ then by the chain rule $P(\varrho) \in W^{1,2}(0,T;\mathbb{D},\mathbb{H})$, so that $P(\varrho) \in \mathrm{C}([0,T];\mathbb{V})$ thanks to (3.28). Composing with the Lipschitz map $P^{-1}$ provides the continuity of $\varrho$ in $\mathbb{V}$ thanks to (DF2).

**Theorem 3.4 (Nonlinear diffusion)** *Let $P$ be a regular function according to (3.24). For every $T > 0$ and every $\bar\varrho \in \mathbb{H}$ there exists a unique curve $\varrho = \mathsf{S}\bar\varrho \in W^{1,2}(0,T;\mathbb{V},\mathbb{V}'_{\mathcal{E}})$ satisfying*

$$\frac{\mathrm{d}}{\mathrm{d}t}\varrho - \mathrm{L}P(\varrho) = 0 \quad \mathscr{L}^1\text{-a.e. in } (0,T), \text{ with } \varrho_0 = \bar\varrho. \tag{3.31}$$

*Moreover:*

(ND1) *For every $t > 0$ the map $\bar\varrho \mapsto \mathsf{S}_t\bar\varrho$ is a contraction with respect to the norm $\mathbb{V}'_{\mathcal{E}}$, with*

$$\|\mathsf{S}_t\bar\varrho^1 - \mathsf{S}_t\bar\varrho^2\|^2_{\mathbb{V}'_{\mathcal{E}}} + 2\mathsf{a}\int_0^t\int_X |\mathsf{S}_r\bar\varrho^1 - \mathsf{S}_r\bar\varrho^2|^2\,\mathrm{d}\mathfrak{m}\,\mathrm{d}r \le \|\bar\varrho^1 - \bar\varrho^2\|^2_{\mathbb{V}'_{\mathcal{E}}}. \tag{3.32}$$

(ND2) *If $W \in \mathrm{C}^{1,1}(\mathbb{R})$ is a nonnegative convex function with $W(0) = 0$, then*

$$\int_X W(\varrho_t)\,\mathrm{d}\mathfrak{m} + \int_0^t \mathcal{E}(P(\varrho_r), W'(\varrho_r))\,\mathrm{d}r = \int_X W(\bar\varrho)\,\mathrm{d}\mathfrak{m} \quad \forall t \ge 0. \tag{3.33}$$

*Moreover, for every convex and lower semicontinuous function $W : \mathbb{R} \to [0,\infty]$*

$$\int_X W(\varrho_t)\,\mathrm{d}\mathfrak{m} \le \int_X W(\bar\varrho)\,\mathrm{d}\mathfrak{m}. \tag{3.34}$$

*In particular, $\mathsf{S}_t$ is positivity preserving and if $0 \le \bar\varrho \le R$ $\mathfrak{m}$-a.e. in $X$, then $0 \le \varrho_t \le R$ $\mathfrak{m}$-a.e. in $X$ for every $t \ge 0$.*

(ND3) *If $\bar\varrho \in \mathbb{V}$ then $\varrho \in \mathcal{ND}(0,T) \subset \mathrm{C}([0,T];\mathbb{V}) \cap \mathrm{C}^1([0,T];\mathbb{V}'_{\mathcal{E}})$ and*

$$\lim_{h\to 0}\frac{1}{h}\big(\varrho_{t+h} - \varrho_t\big) = \mathrm{L}P(\varrho_t) \quad \text{strongly in } \mathbb{V}'_{\mathcal{E}}, \quad \text{for all } t \ge 0. \tag{3.35}$$

(ND4) *The semigroup $\mathsf{S}_t$ is a contraction in $L^1 \cap L^2(X,\mathfrak{m})$ w.r.t. the $L^1(X,\mathfrak{m})$ norm and more generally for every $\bar\varrho_i \in L^1 \cap L^2(X,\mathfrak{m})$, $i = 1,2$,*

$$\int_X (\mathsf{S}_t\bar\varrho_2 - \mathsf{S}_t\bar\varrho_1)_+\,\mathrm{d}\mathfrak{m} \le \int_X (\varrho_2 - \varrho_1)_+\,\mathrm{d}\mathfrak{m} \quad \text{for every } t \ge 0. \tag{3.36}$$

*In particular $\mathsf{S}$ is order preserving, i.e.*

$$\bar\varrho_1 \le \bar\varrho_2 \quad \Rightarrow \quad \mathsf{S}_t\bar\varrho_1 \le \mathsf{S}_t\bar\varrho_2 \quad \text{for every } t \ge 0. \tag{3.37}$$

*Finally, if $\mathsf{P}_t$ is mass preserving then*

$$\int_X \mathsf{S}_t\bar\varrho\,\mathrm{d}\mathfrak{m} = \int_X \bar\varrho\,\mathrm{d}\mathfrak{m} \quad \text{for every } \bar\varrho \in L^1 \cap L^2(X,\mathfrak{m}), \quad t \ge 0. \tag{3.38}$$

We split the proof of the above theorem in various steps. First of all, we introduce the primitive function of $P$,

$$V(r) := \int_0^r P(z)\,\mathrm{d}z, \tag{3.39}$$

which, thanks to (3.24), satisfies

$$\frac{\mathsf{a}}{2}r^2 \le V(r) \le \frac{1}{2\mathsf{a}}r^2 \qquad \forall r \ge 0. \tag{3.40}$$

We adapt to our setting the approach of [18], showing that the nonlinear equation (3.31) can be viewed as a gradient flow in the dual space $\mathbb{V}'_{\mathcal{E}}$ driven by the integral functional $\mathcal{V} : \mathbb{V}' \to [0,\infty]$ defined by

$$\mathcal{V}(\sigma) := \begin{cases} \displaystyle\int_X V(\sigma)\,\mathrm{d}\mathfrak{m} & \text{if } \sigma \in \mathbb{V}' \cap \mathbb{H}, \\ +\infty & \text{if } \sigma \in \mathbb{V}' \setminus \mathbb{H}, \end{cases} \tag{3.41}$$

associated to $V$.

Since $\mathbb{H}$ is not included in $\mathbb{V}'_{\mathcal{E}}$ in general, if $\varrho$ is a solution of (3.31) with an arbitrary $\bar\varrho \in \mathbb{H}$ only the difference $\sigma_t := \varrho_t - \bar\varrho$, will belong to $\mathbb{V}'_{\mathcal{E}}$; therefore it is useful to introduce the family of shifted functionals $\mathcal{V}_\eta : \mathbb{V}'_{\mathcal{E}} \to [0,\infty]$, $\eta \in \mathbb{H}$, defined by

$$\mathcal{V}_\eta(\sigma) := \mathcal{V}(\eta + \sigma) \qquad \text{for every } \sigma \in \mathbb{V}'_{\mathcal{E}}. \tag{3.42}$$

Notice that, thanks to (3.40), the finiteness domain $D(\mathcal{V}_\eta)$ of $\mathcal{V}_\eta$ coincides with $\mathbb{V}'_{\mathcal{E}} \cap \mathbb{H}$. We shall denote by $\partial\mathcal{V}_\eta$ the $\mathcal{E}^*$-subdifferential of $\mathcal{V}_\eta$, defined at any $\sigma \in D(\mathcal{V}_\eta)$ as the collection of all $\ell \in \mathbb{V}'_{\mathcal{E}}$ satisfying

$$\mathcal{E}^*(\ell, \zeta - \sigma) \le \mathcal{V}_\eta(\zeta) - \mathcal{V}_\eta(\sigma) \qquad \forall \zeta \in D(\mathcal{V}_\eta).$$

In the next lemma we characterize the subdifferentiability and the subdifferential of $\mathcal{V}_\eta$.

**Lemma 3.5 (Subdifferential of $\mathcal{V}_\eta$)** *For every $\eta \in \mathbb{H}$ the functional $\mathcal{V}_\eta : \mathbb{V}'_{\mathcal{E}} \to [0,\infty]$ defined by* (3.42) *is convex and lower semicontinuous. Moreover, for every $\sigma \in D(\mathcal{V}_\eta)$ we have*

$$\ell \in \partial\mathcal{V}_\eta(\sigma) \quad \Longleftrightarrow \quad \sigma \in \mathbb{H}, \quad P(\sigma+\eta) \in \mathbb{V}, \quad \ell = -\mathrm{L}P(\sigma+\eta). \tag{3.43}$$

*In particular $\partial\mathcal{V}_\eta$ is single-valued in its domain and $D(\partial\mathcal{V}_\eta) = \{\sigma \in \mathbb{H} :\ P(\sigma+\eta) \in \mathbb{V}\}$.*

*Proof.* The convexity of $\mathcal{V}_\eta$ is clear. The lower semicontinuity is also easy to prove, since $\mathcal{V}_\eta(\sigma_n) \le C < \infty$ and $\sigma_n \to \sigma$ weakly in $\mathbb{V}'_{\mathcal{E}}$ imply that $\sigma \in \mathbb{H}$ and $\sigma_n$ weakly converge to $\sigma$ in $\mathbb{H}$, by the weak compactness of $(\sigma_n)$ in the weak topology of $\mathbb{H}$.

The left implication $\Leftarrow$ in (3.43) is immediate, since by Proposition 3.1(d) and the fact that $\zeta - \sigma \in \mathbb{H} \cap \mathbb{V}'_{\mathcal{E}}$

$$\begin{aligned} \mathcal{E}^*(-\mathrm{L}P(\sigma+\eta), \zeta-\sigma) &= \int_X P(\sigma+\eta)(\zeta-\sigma)\,\mathrm{d}\mathfrak{m} = \int_X P(\sigma+\eta)\left((\zeta+\eta)-(\sigma+\eta)\right)\mathrm{d}\mathfrak{m} \\ &\le \int_X \Big(V(\zeta+\eta) - V(\sigma+\eta)\Big)\,\mathrm{d}\mathfrak{m} = \mathcal{V}_\eta(\zeta) - \mathcal{V}_\eta(\sigma), \end{aligned}$$

where we used the pointwise property $P(x)(y-x) \le V(y) - V(x)$ for every $x,\, y \in \mathbb{R}$.

In order to prove the converse implication $\Rightarrow$, let us suppose that $\ell \in \partial \mathcal{V}_\eta(\sigma)$; choosing $\zeta = \sigma + \varepsilon\varphi$, with $\varphi \in \mathbb{H} \cap \mathbb{V}'_{\mathcal{E}}$, we get

$$\mathcal{E}^*(\ell, \varphi) \le \varepsilon^{-1}\Big(\mathcal{V}_\eta(\sigma+\varepsilon\varphi) - \mathcal{V}_\eta(\sigma)\Big) \le \int_X P(\sigma+\eta+\varepsilon\varphi)\varphi \,\mathrm{d}\mathfrak{m}.$$

Passing to the limit as $\varepsilon \downarrow 0$ and changing $\varphi$ into $-\varphi$ we get

$$\mathcal{E}^*(\ell, \varphi) = \int_X P(\sigma+\eta)\,\varphi \,\mathrm{d}\mathfrak{m} \qquad \text{for every } \varphi \in \mathbb{H} \cap \mathbb{V}'_{\mathcal{E}}.$$

Choosing now $\varphi = -\mathrm{L}f$ with $f \in \mathbb{D}$ we get

$$-\int_X P(\sigma+\eta)\mathrm{L}f \,\mathrm{d}\mathfrak{m} \le \|\ell\|_{\mathbb{V}'_{\mathcal{E}}}\Big(\mathcal{E}(f,f)\Big)^{1/2},$$

so that Lemma 3.3(a) yields $P(\sigma+\eta) \in \mathbb{V}$. Therefore (using Proposition 3.1(d) once more in the last equality), we get

$$\begin{aligned} \mathcal{E}^*(\ell, -\mathrm{L}f) &= -\int_X P(\sigma+\eta)\mathrm{L}f \,\mathrm{d}\mathfrak{m} = \mathcal{E}(P(\sigma+\eta), f) \\ &= -\langle \mathrm{L}P(\sigma+\eta), f\rangle = \mathcal{E}^*(-\mathrm{L}P(\sigma+\eta), -\mathrm{L}f) \end{aligned}$$

for all $f \in \mathbb{D}$, and this proves that $\ell$ coincides with $-\mathrm{L}P(\sigma+\eta)$. □

*Proof of Theorem 3.4.* Let $\bar\varrho \in \mathbb{H}$ and let $\eta \in \mathbb{H}$ be any element such that $\bar\sigma := \bar\varrho - \eta \in \mathbb{V}'_{\mathcal{E}}$ (in particular we can choose $\eta = \bar\varrho$, so that $\bar\sigma = 0$; as a matter of fact, $\eta$ plays only an auxiliary role in the proof and the solution $\varrho$ will be independent of $\eta$). Setting $\sigma_t := \varrho_t - \eta$, the equation (3.31) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma - \mathrm{L}P(\sigma+\eta) = 0, \quad \text{i.e.} \quad \frac{\mathrm{d}}{\mathrm{d}t}\sigma + \partial\mathcal{V}_\eta(\sigma) \ni 0, \quad \text{with} \quad \sigma_0 = \bar\sigma, \tag{3.44}$$

where $\partial\mathcal{V}_\eta$ is the subdifferential of $\mathcal{V}_\eta$, characterized in (3.43).

*Proof of existence of solutions and (ND1).* Since Lemma 3.3(b) provides the density of the domain of $\mathcal{V}_\eta$ in $\mathbb{V}'_{\mathcal{E}}$, existence of a solution $\sigma \in \mathrm{C}([0,T];\mathbb{V}'_{\mathcal{E}})$ satisfying $\partial\mathcal{V}_\eta(\sigma), \frac{\mathrm{d}}{\mathrm{d}t}\sigma \in L^2(0,T;\mathbb{V}'_{\mathcal{E}})$ (and thus $P(\sigma+\eta) \in L^2(0,T;\mathbb{V})$) follows by the general theory of equations in Hilbert spaces governed by the subdifferential of convex and lower semicontinuous functions [18], so that $\varrho_t := \sigma_t + \eta$ satisfies (3.31). Since $P(\varrho) \in L^2(0,T;\mathbb{V})$ and $P$ satisfies the regularity property (3.24), we also get $\varrho \in L^2(0,T;\mathbb{V})$; since $\frac{\mathrm{d}}{\mathrm{d}t}\varrho \in L^2(0,T;\mathbb{V}'_{\mathcal{E}}) \subset L^2(0,T;\mathbb{V}')$ we deduce $\varrho \in \mathrm{C}([0,T];\mathbb{H})$ by (3.28).

The abstract theory also provides the regularization estimates

$$t\,\|\partial\mathcal{V}_\eta(\sigma_t)\|^2_{\mathbb{V}'_{\mathcal{E}}} = t\,\mathcal{E}(P(\varrho_t), P(\varrho_t)) \le \int_X V(\bar\varrho)\,\mathrm{d}\mathfrak{m} \qquad \text{for every } t > 0, \tag{3.45}$$

$$\lim_{h\downarrow 0}\frac{\varrho_{t+h}-\varrho_t}{h}=\mathrm{L}P(\varrho_t) \text{ in } \mathbb{V}'_{\mathcal{E}} \quad \text{for every } t>0, \tag{3.46}$$

and the fact that the semigroup $\mathsf{S}_t:\bar\rho\mapsto\varrho_t$ is nonexpansive in $\mathbb{V}'_{\mathcal{E}}$. If $\bar\varrho\in\mathbb{V}$ (so that $\bar\sigma\in D(\partial\mathcal{V}_\eta)$) the limit in (3.46) holds also at $t=0$. Since $\partial\mathcal{V}_\eta$ is single-valued, this proves (3.35).

In order to prove (3.32) we simply consider two solutions $\varrho^j_t=\sigma^j_t+\eta$, $j=1,2$ (we can choose the same $\eta$ since $\bar\rho^1-\bar\rho^2\in\mathbb{V}'_{\mathcal{E}}$), and we evaluate the time derivative of $\frac12\mathcal{E}^*(\varrho^1_t-\varrho^2_t)$, obtaining

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac12\mathcal{E}^*(\varrho^1_t-\varrho^2_t)&=\frac{\mathrm{d}}{\mathrm{d}t}\frac12\mathcal{E}^*(\sigma^1_t-\sigma^2_t)=\mathcal{E}^*(\sigma^1_t-\sigma^2_t,-\mathrm{L}P(\varrho^1_t)-\mathrm{L}P(\varrho^2_t))\\&=-\int_X(\varrho^1_t-\varrho^2_t)(P(\varrho^1_t)-P(\varrho^2_t))\,\mathrm{d}\mathfrak{m}\\&\le-\mathsf{a}\|\varrho^1_t-\varrho^2_t\|^2_{L^2(X,\mathfrak{m})},\end{aligned}\tag{3.47}$$

where $\mathsf{a}$ is the constant in (3.24).

*Proof of (ND2).* We can apply Lemma 3.5 to the integral functional $\mathcal{W}_\eta$ defined similarly to $\mathcal{V}$, with $W$ instead of $V$; denoting by $G$ the derivative of $W$, the $\mathbb{V}'_{\mathcal{E}}$-subdifferential $\partial\mathcal{W}_\eta$ can then be represented as $-\mathrm{L}G(\sigma+\eta)$ as in (3.43) and its domain is contained in $D(\partial\mathcal{V}_\eta)$. If $\sigma$ is a solution of (3.44), the chain rule for convex and lower semicontinuous functionals in Hilbert spaces yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_X W(\varrho_t)\,\mathrm{d}\mathfrak{m}&=\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}_\eta(\sigma_t)=-\mathcal{E}^*(\frac{\mathrm{d}}{\mathrm{d}t}\sigma_t,\mathrm{L}G(\sigma_t+\eta))=-\mathcal{E}^*(\mathrm{L}P(\sigma_t+\eta),\mathrm{L}G(\sigma_t+\eta))\\&=-\mathcal{E}(P(\varrho_t),G(\varrho_t)).\end{aligned}$$

The inequality (3.34) follows now by (DF3) and by a standard approximation procedure, e.g. by considering the Moreau-Yosida regularization of $W$. Choosing now $W(r):=(r-R)^2_+$ with $R\ge0$ or $W(r):=(R-r)^2_+$ with $R\le0$, we prove the comparison estimates w.r.t. constants.

*Proof of (ND3).* We already proved (3.35) and that $\varrho\in\mathrm{C}^1([0,T];\mathbb{V}'_{\mathcal{E}})$. It remains to prove that $\frac{\mathrm{d}}{\mathrm{d}t}\varrho\in L^2(0,T;\mathbb{H})$ if $\bar\varrho\in\mathbb{V}$. This property follows easily by (3.32) applied to the couple of solutions $\varrho^1_t:=\varrho_t$ and $\varrho^2_t:=\varrho_{t+h}$, since it yields

$$\frac{2\mathsf{a}}{h^2}\int_0^{T-h}\|\varrho_t-\varrho_{t+h}\|^2_{L^2(X,\mathfrak{m})}\,\mathrm{d}t\le\frac{1}{h^2}\|\varrho_h-\varrho_0\|^2_{\mathbb{V}'_{\mathcal{E}}}\le\mathcal{E}(P(\bar\varrho),P(\bar\varrho))\quad\text{for every } h\in(0,T).$$

The regularity $\frac{\mathrm{d}}{\mathrm{d}t}\varrho\in L^2(0,T;\mathbb{H})$ yields $P(\varrho)\in L^2(0,T;\mathbb{D})$.

*Proof of (ND4).* We introduce the resolvent operators $\mathsf{J}_\tau:\mathbb{V}'_{\mathcal{E}}\to D(\partial\mathcal{V}_\eta)$, $\tau>0$, defined by $\mathsf{J}_\tau:=(I+\tau\partial\mathcal{V}_\eta)^{-1}$ so that

$$\varrho'=\mathsf{J}_\tau(\varrho-\eta)+\eta\quad\Longleftrightarrow\quad\varrho'-\tau\mathrm{L}P(\varrho')=\varrho. \tag{3.48}$$

In view of the exponential formula $\mathsf{S}_t(\bar\sigma)=\lim_{n\to\infty}(\mathsf{J}_{t/n})^n\bar\sigma$ strongly in $\mathbb{V}'_{\mathcal{E}}$, the $L^1$-contraction of $\mathsf{S}$ is a consequence of the $L^1$-contraction of $\mathsf{J}_\tau$. More precisely, we will show that

$$\int_X\big(\mathsf{J}_\tau\sigma_1-\mathsf{J}_\tau\sigma_2\big)_+\,\mathrm{d}\mathfrak{m}\le\int_X\big(\sigma_1-\sigma_2\big)_+\,\mathrm{d}\mathfrak{m}\quad\text{for every } \sigma_1,\,\sigma_2\in\mathbb{H}\cap\mathbb{V}'_{\mathcal{E}}, \tag{3.49}$$

which will yield (3.36), the $L^1$-contraction and the order preserving property (3.37). The monotonicity inequality (3.49) can be proved by introducing an increasing sequence of smooth maps approximating the Heaviside function:

$$f_n \in \mathrm{C}^1(\mathbb{R};[0,1]), \quad f_n \equiv 0 \quad \text{in } (-\infty,0), \quad 0 < f_n'(r) \le n, \quad f_n(r) \uparrow 1 \quad \text{for every } r > 0.$$

If $\varrho_i \in L^1 \cap L^2(X,\mathfrak{m})$, $i = 1,2$, (3.48) yields $\varrho_i' = \mathsf{J}_\tau(\varrho_i - \eta) + \eta \in \mathbb{V}$ with $P(\varrho_i) \in \mathbb{D}$ and therefore $f_n(P(\varrho_1') - P(\varrho_2')) \in L^2 \cap L^\infty(X,\mathfrak{m}) \cap \mathbb{V}$ since $f_n$ is Lipschitz and $f_n(0) = 0$. We thus get by (3.48) and the positivity of $f_n$

$$\begin{aligned}\int_X (\varrho_1' - \varrho_2') f_n\big(P(\varrho_1') - P(\varrho_2')\big)\,\mathrm{d}\mathfrak{m} + \tau\mathcal{E}\Big(f_n\big(P(\varrho_1') - P(\varrho_2')\big), P(\varrho_1') - P(\varrho_2')\Big)\\ = \int_X (\varrho_1 - \varrho_2) f_n\big(P(\varrho_1') - P(\varrho_2')\big)\,\mathrm{d}\mathfrak{m} \le \int_X \big(\varrho_1 - \varrho_2\big)_+\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

By neglecting the positive contribution of the Dirichlet form $\mathcal{E}$ thanks to (DF3), we can pass to the limit as $n \to \infty$ by the monotone convergence theorem observing that $(\varrho_1' - \varrho_2') f_n\big(P(\varrho_1') - P(\varrho_2')\big) \uparrow (\varrho_1' - \varrho_2')_+$ as $n \to \infty$, thus obtaining

$$\int_X (\varrho_1' - \varrho_2')_+\,\mathrm{d}\mathfrak{m} = \int_X \big(\mathsf{J}_\tau\sigma_1 - \mathsf{J}_\tau\sigma_2\big)_+\,\mathrm{d}\mathfrak{m} \le \int_X \big(\varrho_1 - \varrho_2\big)_+\,\mathrm{d}\mathfrak{m} = \int_X \big(\sigma_1 - \sigma_2\big)_+\,\mathrm{d}\mathfrak{m}.$$

In order to check the mass preserving property (3.38) in the case when $\mathsf{P}$ is mass preserving, we observe that the exponential formula also holds in $L^1(X,\mathfrak{m})$ by Crandall-Ligget Theorem. Therefore the mass preserving property (3.38) is still a consequence of

$$\int_X \varrho'\,\mathrm{d}\mathfrak{m} = \int_X \varrho\,\mathrm{d}\mathfrak{m} \quad \text{whenever (3.48) holds.} \tag{3.50}$$

Eventually, (3.50) follows by integrating (3.48) and recalling (3.15). □

We only considered nonlinear diffusion problems associated to regular monotone functions $P$ as in (3.24), since they provide a sufficiently general class of equations for our aims. Nevertheless, starting from Theorem 3.4 and adapting its arguments, it would not be difficult to prove existence and uniqueness results under more general assumptions. The next result is a possible example in this direction: a proof can be obtained by the same strategy (we omit the details, since we need only Theorem 3.4 in the sequel); notice that the fact that $\mathsf{S}_t$ preserves $L^\infty$ bounds allows to modify the behaviour of $P$ for large densities, so that its primitive function $V$ has a quadratic growth and its domain coincides with $L^2(X,\mathfrak{m})$ when $\mathfrak{m}(X) < \infty$.

**Theorem 3.6 (Nonlinear diffusion for general nonlinearities)** *Let $P \in \mathrm{C}^0(\mathbb{R})$ be a nondecreasing function and let us suppose that $\mathfrak{m}(X) < \infty$. For every $\bar\varrho \in L^\infty(X,\mathfrak{m})$ there exists a unique curve $\varrho = \mathsf{S}\bar\varrho \in W^{1,2}(0,T;\mathbb{V}_\mathcal{E}') \cap L^\infty(X \times (0,T))$ with $P(\varrho) \in L^2(0,T;\mathbb{V})$ satisfying* (3.31). *$\mathsf{S}_t$ is a semigroup of contractions in $\mathbb{V}_\mathcal{E}'$ and in $L^1(X,\mathfrak{m})$ and properties* (ND2), (ND4) *still hold.*

# 4 Backward and forward linearizations of nonlinear diffusion

In this section we collect a few results concerning linearization of the nonlinear diffusion equations of the form studied by Theorem 3.4.

The linearized PDE discussed in the next proposition corresponds to (2.8) of the heuristic Section 2, while the evolution semigroup is provided by the nonlinear diffusion equation of Theorem 3.4. Recall the notation $W^{1,2}(0,T;\mathbb{D},\mathbb{H}) = L^2(0,T;\mathbb{D}) \cap W^{1,2}(0,T;\mathbb{H})$, (3.29) for $\mathcal{ND}(0,T)$, and that, according to (3.28) and (3.30),

$$\mathcal{ND}(0,T) \subset \mathrm{C}([0,T];\mathbb{V}), \qquad W^{1,2}(0,T;\mathbb{D},\mathbb{H}) \hookrightarrow \mathrm{C}([0,T];\mathbb{V}). \tag{4.1}$$

**Theorem 4.1 (Backward adjoint linearized equation)** *Let $P$ be a regular monotone nonlinearity as in* (3.24) *and let $\varrho \in L^2(0,T;\mathbb{H})$.*

*For every $\bar\varphi \in \mathbb{V}$, $T > 0$ and $\psi \in L^2(0,T;\mathbb{H})$ there exists a unique strong solution $\varphi \in W^{1,2}(0,T;\mathbb{D},\mathbb{H})$ of*

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi + P'(\varrho)\mathrm{L}\varphi = \psi, \qquad \varphi_T = \bar\varphi. \tag{4.2}$$

(BA1) *For all $r \in [0,T]$, the solution $\varphi$ satisfies*

$$\int_r^T \int_X \frac{1}{P'(\varrho)} |\dot\varphi|^2 \,\mathrm{d}\mathfrak{m}\,\mathrm{d}t + \frac{1}{2}\mathcal{E}(\varphi_r,\varphi_r) = \int_r^T \int_X \frac{1}{P'(\varrho)} \psi\dot\varphi \,\mathrm{d}\mathfrak{m}\,\mathrm{d}t + \frac{1}{2}\mathcal{E}(\bar\varphi,\bar\varphi). \tag{4.3}$$

(BA2) *If $\bar\varphi \in L^\infty(X,\mathfrak{m})$ and $\psi \equiv 0$, then $\varphi_t \in L^\infty(X,\mathfrak{m})$ with $|\varphi_t| \le \|\bar\varphi\|_{L^\infty(X,\mathfrak{m})}$ $\mathfrak{m}$-a.e. in $X$ for every $t \in [0,T]$.*

(BA3) *If $\varrho^n \to \varrho^\infty$, $\psi^n \to \psi^\infty$ in $L^2(0,T;\mathbb{H})$, $\bar\varphi^n \to \bar\varphi^\infty$ in $\mathbb{V}$ and $\varphi^n$, $n \in \mathbb{N}\cup\{\infty\}$, are the corresponding solutions of* (4.2)*, then $\varphi^n \to \varphi^\infty$ strongly in $W^{1,2}(0,T;\mathbb{D},\mathbb{H})$.*

**Remark 4.2 (Forward adjoint linearized equation)** By time reversal, the previous Theorem is equivalent to the analogous result for the forward linearized equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\zeta - P'(\varrho)\mathrm{L}\zeta = \psi \in L^2(0,T;\mathbb{H}), \qquad \zeta_0 = \bar\zeta \in \mathbb{V}, \tag{4.4}$$

that admits a unique solution $\zeta \in W^{1,2}(0,T;\mathbb{D},\mathbb{H})$.

*Proof of Theorem 4.1.* Let us fix the final time $T$ and set $\alpha_t := P'(\varrho_{T-t})$, $g_t := \psi_{T-t}$. We can thus consider the forward equation

$$\frac{\mathrm{d}}{\mathrm{d}t} f_t - \alpha_t \mathrm{L} f_t = g_t \quad \text{in } (0,T), \qquad f_0 = \bar f = \bar\varphi, \tag{4.5}$$

where $\alpha$ is a Borel map satisfying (with $\mathsf{a}$ the positive constant in (3.24))

$$0 < \mathsf{a} \le \alpha \le \frac{1}{\mathsf{a}} \qquad \mathfrak{m} \otimes \mathscr{L}^1\text{-a.e. in } X \times (0,T). \tag{4.6}$$

In order to solve (4.5) we use a piecewise constant (in time) discretization of the coefficients $\alpha_t$: we introduce a uniform partition of the time interval $(0,T]$ of step $\tau := T/N$ given by the intervals $I_k^N := ((k-1)\tau, k\tau]$, $k = 1, \ldots, N$ and we set

$$\alpha_k^N := \frac{1}{\tau} \int_{I_k^N} \alpha_r \,\mathrm{d}r, \qquad \bar{\alpha}_t^N := \alpha_k^N \quad \text{if } t \in I_k^N,$$

so that $\mathsf{a} \le \bar{\alpha}^N \le \mathsf{a}^{-1}$. Applying standard result for evolution equation in Hilbert spaces (in particular we write the PDE as the gradient flow of $\mathcal{E}$ w.r.t. the $L^2(X, 1/\alpha_k^N \mathfrak{m})$ norm when $g \equiv 0$ and in the inhomogeneous case we use Duhamel's principle) we can find recursively strong solutions $f_k^N \in W^{1,2}(I_k^N; \mathbb{D}, \mathbb{H})$ of

$$\frac{1}{\alpha_k^N} \frac{\mathrm{d}}{\mathrm{d}t} f_k^N - \mathrm{L} f_k^N = \frac{1}{\alpha_k^N} g \quad \text{in } I_k^N, \qquad f_k^N((k-1)\tau)) = f_{k-1}^N((k-1)\tau), \tag{4.7}$$

with the convention $f_0^N(0) = \bar{f}$. Defining the function $f^N(t) := f_k^N(t)$ if $t \in I_k^N$, we easily check that $f^N \in W^{1,2}(0,T; \mathbb{D}, \mathbb{H})$, that $f^N$ is a strong solution of the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} f^N - \bar{\alpha}^N \mathrm{L} f^N = g \quad \text{in } (0,T), \tag{4.8}$$

and that it satisfies the apriori energy dissipation identity

$$\int_0^s \int_X \frac{1}{\bar{\alpha}^N} \Big| \frac{\mathrm{d}}{\mathrm{d}t} f^N \Big|^2 \mathrm{d}\mathfrak{m} \,\mathrm{d}t + \frac{1}{2} \mathcal{E}(f_s^N, f_s^N) = \int_0^s \int_X \frac{1}{\bar{\alpha}^N} g \frac{\mathrm{d}}{\mathrm{d}t} f^N \,\mathrm{d}\mathfrak{m} \,\mathrm{d}t + \frac{1}{2} \mathcal{E}(\bar{f}, \bar{f}). \tag{4.9}$$

Since $1/\bar{\alpha}^N \ge 1/\mathsf{a}$ and $\bar{f} \in \mathbb{V}$, this shows in particular that $f^N$ is uniformly bounded in $W^{1,2}(0,T; \mathbb{D}, \mathbb{H})$. Since $\bar{\alpha}^N \to \alpha$ in $L^2(0,T; \mathbb{H})$ we can then easily pass to the limit as $N \to \infty$ (see also the more detailed argument below), obtaining (4.5) Since (4.5) holds in the strong form, we can also write it as

$$\frac{1}{\alpha_t} \frac{\mathrm{d}}{\mathrm{d}t} f_t - \mathrm{L} f_t = \frac{g_t}{\alpha_t} \quad \text{in } (0,T),$$

and then the energy identity corresponding to (4.3) follows by multiplying both sides by $\mathrm{d}f_t/\mathrm{d}t$. This proves (BA1).

When $g \equiv 0$ and $\bar{f} \in L^\infty(X, \mathfrak{m})$ satisfies $|\bar{f}| \le F$ $\mathfrak{m}$-a.e. in $X$, a standard truncation argument based on (4.7) yields the recursive estimate

$$\|f_k^N(k\tau)\|_{L^\infty(X,\mathfrak{m})} \le \|f_k^N(t)\|_{L^\infty(X,\mathfrak{m})} \le \|f_{k-1}^N((k-1)\tau)\|_{L^\infty(X,\mathfrak{m})} \quad \text{for } t \text{ in } I_k^N,$$

and therefore $|f^N(t)| \le F$ $\mathfrak{m}$-a.e. in $X$ for every $t \in [0,T]$; this estimate passes to the limit as $N \to \infty$ providing the statement (BA2).

Let us now prove the last statement (BA3); we thus consider a sequence $\alpha^n$ satisfying the uniform bounds $\mathsf{a} \le \alpha^n \le \mathsf{a}^{-1}$ and the limit $\alpha^n \to \alpha^\infty$ $\mathfrak{m} \otimes \mathscr{L}^1$-a.e. in $X \times (0,T)$, and corresponding solutions $f^n$ of

$$\frac{\mathrm{d}}{\mathrm{d}t} f^n - \alpha^n \mathrm{L} f^n = g^n, \qquad f^n(0) = \bar f^n, \tag{4.10}$$

with $\bar f^n \to \bar f^\infty$ strongly in $\mathbb{V}$ and $g^n \to g^\infty$ strongly in $L^2(0,T;\mathbb{H})$. Using the energy identity (4.3) it is easily seen that $(f_n)$ is bounded in $W^{1,2}(0,T;\mathbb{H})$ and in $\mathrm{C}([0,T];\mathbb{V})$; we can also use the PDE (4.10) to show that $(f_n)$ is bounded in $L^2(0,T;\mathbb{D})$. Hence, possibly extracting a suitable subsequence (still denoted by $f^n$), we can assume that $f^n \rightharpoonup f^\infty$ in $W^{1,2}(0,T;\mathbb{D},\mathbb{H})$, so that $\frac{\mathrm{d}}{\mathrm{d}t} f^n \rightharpoonup \frac{\mathrm{d}}{\mathrm{d}t} f^\infty$ and $\mathrm{L} f^n \rightharpoonup \mathrm{L} f^\infty$ in $L^2(0,T;\mathbb{H})$. Since for every $s \in [0,T]$ the linear operator $f \mapsto f(s)$ is continuous from $W^{1,2}(0,T;\mathbb{D},\mathbb{H})$ to $\mathbb{V}$ thanks to (4.1), we also obtain the weak continuity property $f^n(s) \rightharpoonup f^\infty(s)$ in $\mathbb{V}$ for every $s \in [0,T]$. In particular $f^\infty$ satisfies (4.10) with $n = \infty$.

It follows that

$$\liminf_{n\to\infty} \mathcal{E}(f^n(s), f^n(s)) \ge \mathcal{E}(f^\infty(s), f^\infty(s)),$$

$$\liminf_{n\to\infty} \int_0^s \int_X \frac{1}{\alpha^n} \Big|\frac{\mathrm{d}}{\mathrm{d}t} f^n\Big|^2 \,\mathrm{d}\mathfrak{m}\,\mathrm{d}r \ge \int_0^s \int_X \frac{1}{\alpha^\infty} \Big|\frac{\mathrm{d}}{\mathrm{d}t} f^\infty\Big|^2 \,\mathrm{d}\mathfrak{m}\,\mathrm{d}r$$

and

$$\lim_{n\to\infty} \mathcal{E}(\bar f^n, \bar f^n) = \mathcal{E}(\bar f^\infty, \bar f^\infty), \quad \lim_{n\to\infty} \int_0^s \int_X \frac{g^n}{\alpha^n} \frac{\mathrm{d}}{\mathrm{d}t} f^n \,\mathrm{d}\mathfrak{m}\,\mathrm{d}r = \int_0^s \int_X \frac{g^\infty}{\alpha^\infty} \frac{\mathrm{d}}{\mathrm{d}t} f^\infty \,\mathrm{d}\mathfrak{m}\,\mathrm{d}r,$$

so that by (4.3) we obtain

$$\begin{aligned}\lim_{n\to\infty} \Big( \int_0^s \int_X \frac{1}{\alpha^n} \Big|\frac{\mathrm{d}}{\mathrm{d}t} f^n\Big|^2 \,\mathrm{d}\mathfrak{m} + \frac12 \mathcal{E}(f^n(s), f^n(s)) \Big) &= \int_0^s \int_X \frac{g^\infty}{\alpha^\infty} \frac{\mathrm{d}}{\mathrm{d}t} f^\infty \,\mathrm{d}\mathfrak{m}\,\mathrm{d}r + \frac12 \mathcal{E}(\bar f^\infty, \bar f^\infty) \\ &= \int_0^s \int_X \frac{1}{\alpha^\infty} \Big|\frac{\mathrm{d}}{\mathrm{d}t} f^\infty\Big|^2 \,\mathrm{d}\mathfrak{m}\,\mathrm{d}r + \frac12 \mathcal{E}(f^\infty(s), f^\infty(s)).\end{aligned}$$

We conclude (see Remark 4.3 below) that $\frac{\mathrm{d}}{\mathrm{d}t} f^n \to \frac{\mathrm{d}}{\mathrm{d}t} f^\infty$ strongly in $L^2(0,T;\mathbb{H})$, so that $f^n \to f^\infty$ strongly in $W^{1,2}(0,T;\mathbb{D},\mathbb{H})$. □

**Remark 4.3** We will repeatedly use the following simple property, valid for sequences $(a_n)$, $(b_n)$ of nonnegative real numbers: if

$$\liminf_{n\to\infty} a_n \ge a, \quad \liminf_{n\to\infty} b_n \ge b, \quad \limsup_{n\to\infty} (a_n + b_n) \le (a+b),$$

then

$$\lim_{n\to\infty} a_n = a \qquad \text{and} \qquad \lim_{n\to\infty} b_n = b.$$

The next proposition provides existence and regularity for the linearization of the nonlinear diffusion equation of Theorem 3.4.

In the statement we will make use of the space $\mathbb{D}'$, the dual of $\mathbb{D}$, and

$$\mathbb{D}'_{\mathcal{E}} := \Big\{ \ell \in \mathbb{D}' : \ |\langle \ell, f\rangle| \le C\|\mathrm{L}f\|_{\mathbb{H}} \quad \text{for every } f \in \mathbb{D} \Big\}. \tag{4.11}$$

Since $\mathbb{D} \hookrightarrow^{ds} \mathbb{V}$ we have $\mathbb{H} \hookrightarrow^{ds} \mathbb{V}' \hookrightarrow^{ds} \mathbb{D}'$ with continuous and dense inclusions; the duality pairing between $\mathbb{D}'$ and $\mathbb{D}$ is an extension of the one between $\mathbb{V}'$ and $\mathbb{V}$ and of the scalar product in $\mathbb{H}$, and we will still denote it as $\langle\cdot,\cdot\rangle$ whenever no misunderstanding are possible. Denoting by $\|\ell\|_{\mathbb{D}'_{\mathcal{E}}}$ the least constant $C$ in (4.11), $\mathbb{D}'_{\mathcal{E}}$ is also an Hilbert space, precisely it can be identified with the dual of the pre-Hilbert space one obtains endowing $\mathbb{D}$ with the norm $\|\mathrm{L}f\|_{\mathbb{H}}$, smaller than the canonical norm of $\mathbb{D}$. Arguing as in Section 3.2, we can and will identify $\mathbb{D}'_{\mathcal{E}}$ with the finiteness domain in $\mathbb{D}'$ of the lower semicontinuous functional

$$\frac{1}{2}\|\ell\|^2_{\mathbb{D}'_{\mathcal{E}}} := \sup_{f\in\mathbb{D}} \langle \ell, f\rangle - \frac{1}{2}\int_X |\mathrm{L}f|^2 \,\mathrm{d}\mathfrak{m}. \tag{4.12}$$

By duality, any element $h \in \mathbb{H}$ induces an element $\mathrm{L}h \in \mathbb{D}'_{\mathcal{E}}$, via the relation

$$_{\mathbb{D}'}\langle \mathrm{L}h, f\rangle_{\mathbb{D}} = \int_X h\, \mathrm{L}f \,\mathrm{d}\mathfrak{m}.$$

We shall also make use of the space $W^{1,2}(0,T;\mathbb{H},\mathbb{D}'_{\mathcal{E}})$, fitting in our framework because both $\mathbb{H}$ and $\mathbb{D}'_{\mathcal{E}}$ embed into the space $\mathbb{D}'$. Since $\mathbb{D}'_{\mathcal{E}} \hookrightarrow \mathbb{D}'$ and the duality formula for complex interpolation yields $(\mathbb{H},\mathbb{D}')_{1/2} = \mathbb{V}'$, (3.27) yields

$$W^{1,2}(0,T;\mathbb{H},\mathbb{D}'_{\mathcal{E}}) \hookrightarrow W^{1,2}(0,T;\mathbb{H},\mathbb{D}') \hookrightarrow \mathrm{C}([0,T];\mathbb{V}'). \tag{4.13}$$

**Theorem 4.4 (Forward linearized equation)** *Let $P$ be a regular monotone nonlinearity as in (3.24) and let $\rho \in L^2(0,T;\mathbb{H})$.*

(L1) *For every $\bar w \in \mathbb{V}'_{\mathcal{E}}$, $T>0$ there exists a unique solution $w \in W^{1,2}(0,T;\mathbb{H},\mathbb{D}'_{\mathcal{E}})$ of*

$$\frac{\mathrm{d}}{\mathrm{d}t} w = \mathrm{L}(P'(\varrho)w), \qquad w_0 = \bar w \tag{4.14}$$

*in the weak formulation (recall (4.1) and (4.13))*

$$_{\mathbb{V}'}\langle w_s, \vartheta_s\rangle_{\mathbb{V}} - \int_0^s \int_X \Big(\partial_t\vartheta_t + P'(\varrho_t)\mathrm{L}\vartheta_t\Big) w_t \,\mathrm{d}\mathfrak{m}\,\mathrm{d}t = {}_{\mathbb{V}'}\langle \bar w, \vartheta_0\rangle_{\mathbb{V}} \quad \forall s \in [0,T], \tag{4.15}$$

*for every $\vartheta \in W^{1,2}(0,T;\mathbb{D},\mathbb{H})$. In addition, the function $w$ satisfies*

$$\int_0^t \int_X P'(\varrho_r)|w_r|^2 \,\mathrm{d}\mathfrak{m}\,\mathrm{d}r + \frac{1}{2}\|w_t\|^2_{\mathbb{V}'_{\mathcal{E}}} = \frac{1}{2}\|\bar w\|^2_{\mathbb{V}'_{\mathcal{E}}} \qquad \forall s \in [0,T] \tag{4.16}$$

*and, for every solution $\varphi$ of (4.2) with $\psi \equiv 0$ one has*

$$_{\mathbb{V}'}\langle w_t, \varphi_t\rangle_{\mathbb{V}} = {}_{\mathbb{V}'}\langle \bar w, \varphi_0\rangle_{\mathbb{V}}. \tag{4.17}$$

(L2) *If* $\bar w = \mathrm{L}\bar\zeta$ *for some* $\bar\zeta \in \mathbb{V}$, *then* $w_t = \mathrm{L}\zeta_t$ *for every* $t \in [0,T]$, *where* $\zeta \in W^{1,2}(0,T;\mathbb{D},\mathbb{H})$ *is the solution of* (4.4) *with* $\psi \equiv 0$.

(L3) *If* $\varrho^n \to \varrho^\infty$ *in* $L^2(0,T;\mathbb{H})$, $\bar w^n \to \bar w^\infty$ *in* $\mathbb{V}'_{\mathcal{E}}$ *and* $w^n$, $n \in \mathbb{N}\cup\{\infty\}$, *are the corresponding solutions of* (4.14), *then* $w^n \to w^\infty$ *strongly in* $W^{1,2}(0,T;\mathbb{H},\mathbb{D}'_{\mathcal{E}})$.

*Proof of Theorem 4.4.* Let us first show the second claim: if $w_t = \mathrm{L}\zeta_t \in W^{1,2}(0,T;\mathbb{H};\mathbb{D}'_{\mathcal{E}})$ for the solution $\zeta \in W^{1,2}(0,T;\mathbb{D};\mathbb{H})$ of (4.4) and if $\vartheta$ is any function in $W^{1,2}(0,T;\mathbb{D},\mathbb{H})$ we have

$$ {}_{\mathbb{V}'}\langle w_t, \vartheta_t\rangle_{\mathbb{V}} = {}_{\mathbb{V}'}\langle \mathrm{L}\zeta_t, \vartheta_t\rangle_{\mathbb{V}} = -\mathcal{E}(\zeta_t,\vartheta_t), $$

so that $t \mapsto {}_{\mathbb{V}'}\langle w_t, \vartheta_t\rangle_{\mathbb{V}}$ is absolutely continuous in $[0,T]$ and for $\mathscr{L}^1$-a.e. $t \in (0,T)$ its derivative is given by

$$ -\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\zeta_t,\vartheta_t) = \int_X \Big(\mathrm{L}\zeta_t\,\dot\vartheta_t + \mathrm{L}\vartheta_t\,\dot\zeta_t\Big)\,\mathrm{d}\mathfrak{m} = \int_X w_t\Big(\dot\vartheta_t + P'(\varrho_t)\,\mathrm{L}\vartheta_t\Big)\,\mathrm{d}\mathfrak{m}. $$

A further integration in time yields (4.15). In this case (4.16) is a consequence of (4.3) with $\psi \equiv 0$, by noticing that

$$ \mathcal{E}(\zeta_t,\zeta_t) = \mathcal{E}^*(w_t,w_t) = \|w_t\|^2_{\mathbb{V}'_{\mathcal{E}}}, \qquad \dot\zeta_t = P'(\varrho_t)w_t. $$

The uniqueness of the solution to (4.15) is clear thanks to (4.17).

The general result stated in the first claim for arbitrary $\bar w \in \mathbb{V}'_{\mathcal{E}}$ follows by the linearity of the problem, the estimate (4.16), and the density of the set $\{\mathrm{L}\bar\zeta :\ \bar\zeta \in \mathbb{V}\}$ in $\mathbb{V}'_{\mathcal{E}}$, see Lemma 3.3(b).

The proof of (L3) is completely analogous to the proof of (BA3) in Theorem 4.1: the weak convergence of $w^n$ to $w^\infty$ in $W^{1,2}(0,T;\mathbb{H},\mathbb{D}'_{\mathcal{E}})$ follows by the a priori estimate (4.16), the linearity of the problem w.r.t. $w$ for given $\varrho$ and the uniqueness of its solution. Strong convergence can then be obtained by standard lower semicontinuity arguments and Remark 4.3, by passing to the limit in (4.16). □

**Theorem 4.5 (Perturbation properties)** *Let us suppose that* $P$ *is a regular monotone nonlinearity as in* (3.24). *Let* $\bar\varrho_\varepsilon := \bar\varrho + \varepsilon\bar w_\varepsilon$ *with* $\bar\varrho$, $\bar\varrho_\varepsilon \in L^2\cap L^\infty(X,\mathfrak{m})$, $\bar\varrho_\varepsilon$ *uniformly bounded in* $L^2\cap L^\infty(X,\mathfrak{m})$, *and* $\bar w_\varepsilon \to \bar w$ *strongly in* $\mathbb{V}'_{\mathcal{E}}$ *as* $\varepsilon \downarrow 0$. *Let* $\varrho_{\varepsilon,t}$ *(resp.* $\varrho_t$*) be the solutions provided by Theorem 3.4 with initial datum* $\bar\varrho_\epsilon$ *(resp.* $\bar\varrho$*) and set*

$$ w_{\varepsilon,t} := \frac{\varrho_{\varepsilon,t} - \varrho_t}{\varepsilon}. $$

*Then for every* $t \ge 0$ *there exists the limit* $\lim_{\varepsilon\downarrow 0} w_{\varepsilon,t} = w_t$ *strongly in* $\mathbb{V}'_{\mathcal{E}}$, *the limit function* $w$ *belongs to* $W^{1,2}(0,T;\mathbb{H},\mathbb{D}'_{\mathcal{E}})$ *and satisfies* (4.14).

*Proof.* By the Lipschitz estimate (3.32) of $\mathsf{S} : \mathbb{V}'_{\mathcal{E}} \to L^2(0,T;\mathbb{H}) \cap L^\infty(0,T;\mathbb{V}'_{\mathcal{E}})$ we know that $(w_\varepsilon)$ is bounded in $L^2(0,T;\mathbb{H})$ and in $L^\infty(0,T;\mathbb{V}'_{\mathcal{E}})$, in particular this gives $\varrho_\varepsilon \to \varrho$ in $L^2(0,T;\mathbb{H})$. We can then find a subsequence $\varepsilon_n \downarrow 0$ such that $w_{\varepsilon_n} \to w$ weakly in $L^2(0,T;\mathbb{H})$ and weakly* in $L^\infty(0,T;\mathbb{V}'_{\mathcal{E}})$.

Since $P \in \mathrm{C}^1(\mathbb{R})$ and there exists a constant $R > 0$ such that $|\varrho_\varepsilon| \le R,\ |\varrho| \le R$, we can use the inequalities (depending on the parameter $\delta > 0$ and on the fixed constant $R$)

$$\big|P(\varrho_\varepsilon) - P(\varrho) - P'(\varrho)(\varrho_\varepsilon - \varrho)\big| \le \delta|\varrho_\varepsilon - \varrho| + C_\delta|\varrho_\varepsilon - \varrho|^2, \tag{4.18}$$

and the uniform bound of $\varepsilon^{-1}(\varrho_\varepsilon - \varrho)$ in $L^2(0,T;\mathbb{H})$ to obtain

$$\varepsilon_n^{-1}\big(P(\varrho_{\varepsilon_n}) - P(\varrho)\big) \rightharpoonup P'(\varrho)w \quad \text{weakly in } L^2(0,T;\mathbb{H}). \tag{4.19}$$

In fact, since $P$ is Lipschitz, $\varepsilon^{-1}(P(\varrho_\varepsilon) - P(\varrho))$ is also uniformly bounded in $L^2(0,T;\mathbb{H})$ thus we can use a bounded test function $\zeta \in L^2(0,T;\mathbb{H})$ to characterize the weak limit in (4.19). For such a test function, denoting by $E$ an upper bound of $\varepsilon^{-1}\|\varrho_\varepsilon - \varrho\|_{L^2(0,T;\mathbb{H})}$, we have

$$\Big|\int_0^T \int_X \Big(\frac{P(\varrho_\varepsilon) - P(\varrho)}{\varepsilon} - P'(\varrho)\frac{\varrho_\varepsilon - \varrho}{\varepsilon}\Big)\zeta \,\mathrm{d}\mathfrak{m}\,\mathrm{d}t\Big| \le \delta\, E\,\|\zeta\|_{L^2(0,T;\mathbb{H})} + \varepsilon\, C_\delta\, E^2 \sup|\zeta|$$

thus showing (4.19) as $\delta > 0$ is arbitrary.

Let us now consider for every $t > 0$ and $\bar\varphi \in \mathbb{V}$ the solution $\varphi$ of (4.2) with final condition $\varphi_t = \bar\varphi$ and arbitrary $\psi \in L^2(0,T;\mathbb{H})$, thus satisfying (by the Leibniz rule)

$$\int_X w_{\varepsilon,t}\bar\varphi \,\mathrm{d}\mathfrak{m} = \varepsilon^{-1}\int_0^t \int_X \Big(\big(P(\varrho_{\varepsilon,r}) - P(\varrho_r)\big)\mathrm{L}\varphi_r + (\varrho_{\varepsilon,r} - \varrho_r)\dot\varphi_r\Big)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}r + \int_X \bar w_\varepsilon \varphi_0 \,\mathrm{d}\mathfrak{m}.$$

Since $\dot\varphi$, $\mathrm{L}\varphi \in L^2(0,T;\mathbb{H})$ we obtain that for every $t > 0$ the sequence $(w_{\varepsilon_n,t})$ converges weakly in $\mathbb{V}'$ (and thus in $\mathbb{V}'_{\mathcal{E}}$, since it is uniformly bounded in $\mathbb{V}'_{\mathcal{E}}$) and the limit $\hat w_t$ will satisfy

$$\langle \hat w_t, \bar\varphi\rangle = \int_0^t \int_X w_r\psi_r \,\mathrm{d}\mathfrak{m}\,\mathrm{d}r + \int_X \bar w\varphi_0 \,\mathrm{d}\mathfrak{m}. \tag{4.20}$$

Choosing in particular $\psi \equiv 0$, the previous formula identifies the limit, so that $\hat w_t = w_t$ for $\mathscr{L}^1$-a.e. $t \in (0,T)$ and moreover the limit does not depend on the particular subsequence $(\varepsilon_n)$. Since $\psi$ is arbitrary, we also get that $w$ satisfies (4.14) in the weak sense of (4.15).

In order to prove strong convergence of $w_\varepsilon$ to $w$ in $\mathbb{V}'_{\mathcal{E}}$ for every $t \in [0,T]$ and in $L^2(0,T;\mathbb{H})$, we start from (3.47) written for $\varrho^1 := \varrho$ and $\varrho^2 := \varrho_\varepsilon$. Since

$$\liminf_{\varepsilon\downarrow 0} \frac{1}{\varepsilon^2}\mathcal{E}^*(\varrho(t) - \varrho_\varepsilon(t), \varrho(t) - \varrho_\varepsilon(t)) = \liminf_{\varepsilon\downarrow 0} \mathcal{E}^*(w_\varepsilon(t), w_\varepsilon(t)) \ge \mathcal{E}^*(w(t), w(t)),$$

and the limit $w$ satisfies (4.16), by the argument of Remark 4.3 it is sufficient to prove that

$$\liminf_{\varepsilon\downarrow 0} \frac{1}{\varepsilon^2}\int_0^t \int_X (\varrho - \varrho_\varepsilon)(P(\varrho) - P(\varrho_\varepsilon))\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s \ge \int_0^t \int_X P'(\varrho)|w|^2 \,\mathrm{d}\mathfrak{m}\,\mathrm{d}s. \tag{4.21}$$

Setting

$$Z_\varepsilon := \begin{cases} \dfrac{P(\varrho) - P(\varrho_\varepsilon)}{\varrho - \varrho_\varepsilon} & \text{if } \varrho \neq \varrho_\varepsilon \\ P'(\varrho) & \text{if } \varrho = \varrho_\varepsilon, \end{cases}$$

we obtain a family of nonnegative and uniformly bounded functions that satisfies $Z_{\varepsilon_n} \to P'(\varrho)$ $\mathfrak{m} \otimes \mathscr{L}^1$-a.e. in $X \times (0,T)$ whenever $\varrho_{\varepsilon_n} \to \varrho$ $\mathfrak{m} \otimes \mathscr{L}^1$-a.e. in $X \times (0,T)$. On the other hand

$$\frac{1}{\varepsilon^2}(\varrho - \varrho_\varepsilon)(P(\varrho) - P(\varrho_\varepsilon)) = Z_\varepsilon |w_\varepsilon|^2.$$

We conclude by applying the next Lemma 4.6 to a subsequence $(\varepsilon_n)$ on which the $\liminf$ in (4.21) is attained and convergence $\mathfrak{m} \otimes \mathscr{L}^1$-a.e. in $X \times (0,T)$ holds. □

**Lemma 4.6** *Let $Y$ be a Polish space endowed with a nonnegative $\sigma$-finite Borel measure $\mathfrak{n}$, let $w_n \in L^2(Y, \mathfrak{n})$ and $Z_n \in L^\infty(Y, \mathfrak{n})$, $Z_n \geq 0$. If $w_n \rightharpoonup w$ in $L^2(X, \mathfrak{n})$ and $Z_n \to Z$ pointwise $\mathfrak{n}$-a.e. in $Y$, then*

$$\liminf_{n \to \infty} \int_Y Z_n |w_n|^2 \,\mathrm{d}\mathfrak{n} \geq \int_Y Z|w|^2 \,\mathrm{d}\mathfrak{n}. \tag{4.22}$$

*Proof.* Let us first assume that $\mathfrak{n}(Y) < \infty$; by Egorov's Theorem, for every $\delta > 0$ we can find a $\mathfrak{n}$-measurable set $B_\delta \subset Y$ such that $\mathfrak{n}(Y \setminus B_\delta) \leq \delta$ and $Z_n \to Z$ uniformly on $B_\delta$. Since $\|w_n\|_{L^2(Y,\mathfrak{n})} \leq C$ independent of $n$ we obtain

$$\begin{aligned} \liminf_{n \to \infty} \int_Y Z_n |w_n|^2 \,\mathrm{d}\mathfrak{n} &\geq \liminf_{n \to \infty} \int_{B_\delta} Z_n |w_n|^2 \,\mathrm{d}\mathfrak{n} \\ &\geq -C^2 \limsup_{n \to \infty} \|Z_n - Z\|_{L^\infty(B_\delta, \mathfrak{n})} + \liminf_{n \to \infty} \int_{B_\delta} Z|w_n|^2 \,\mathrm{d}\mathfrak{n} \geq \int_{B_\delta} Z|w|^2 \,\mathrm{d}\mathfrak{n}. \end{aligned}$$

By letting $\delta \downarrow 0$ we obtain (4.22). When $\mathfrak{n}(Y) = \infty$, since $\mathfrak{n}$ is $\sigma$-finite, we can find an increasing sequence $Y_k \uparrow Y$ of Borel sets with $\mathfrak{n}(Y_k) < \infty$. By the previous claim, we get

$$\liminf_{n \to \infty} \int_Y Z_n |w_n|^2 \,\mathrm{d}\mathfrak{n} \geq \liminf_{n \to \infty} \int_{Y_k} Z_n |w_n|^2 \,\mathrm{d}\mathfrak{n} \geq \int_{Y_k} Z|w|^2 \,\mathrm{d}\mathfrak{n}$$

for every $k \in \mathbb{N}$. As $k \to \infty$ we recover (4.22). □

Let $\varrho \in \mathcal{ND}(0,T)$ be the solution provided by Theorem 3.4 with initial datum $\bar\varrho \in \mathbb{V}$. By applying Theorem 4.5 to the difference quotients

$$\frac{1}{\varepsilon}(\varrho_{t+\varepsilon} - \varrho_t)$$

and using the strong differentiability of $t \mapsto \varrho_t$ with respect to $\mathbb{V}'_\mathcal{E}$ (see (3.30)) we obtain the following corollary.

**Corollary 4.7** *Let $\varrho \in \mathcal{ND}(0,T)$ be the solution provided by Theorem 3.4 with initial datum $\bar\varrho \in \mathbb{V} \cap L^\infty(X,\mathfrak{m})$. Then $w := \frac{\mathrm{d}}{\mathrm{d}t}\varrho$ is a solution to (4.15), with initial datum $\bar w = \mathrm{L}P(\bar\varrho)$.*

# Part II
# Continuity equation and curvature conditions in metric measure spaces

## 5 Preliminaries

### 5.1 Absolutely continuous curves, Lipschitz functions and slopes

Let $(X,\mathsf{d})$ be a complete metric space, possibly extended (i.e. the distance $\mathsf{d}$ can take the value $+\infty$). A curve $\gamma : [a,b] \to X$ belongs to $\mathrm{AC}^p([a,b];(X,\mathsf{d}))$, $1 \le p \le \infty$, if there exists $v \in L^p(a,b)$ such that

$$\mathsf{d}(\gamma(s),\gamma(t)) \le \int_s^t v(r)\,\mathrm{d}r \quad \text{for every } a \le s \le t \le b. \tag{5.1}$$

We will often use the shorter notation $\mathrm{AC}^p([a,b];X)$ whenever the choice of the distance $\mathsf{d}$ will be clear from the context. The metric velocity of $\gamma$, defined by

$$|\dot\gamma|(r) := \lim_{h\to 0} \frac{\mathsf{d}(\gamma(r+h),\gamma(r))}{|h|}, \tag{5.2}$$

exists for $\mathscr{L}^1$-a.e. $r \in (a,b)$, belongs to $L^p(a,b)$, and provides the minimal function $v$, up to $\mathscr{L}^1$-negligible sets, such that (5.1) holds. We set

$$\mathcal{A}_p(\gamma) := \begin{cases} \displaystyle\int_a^b |\dot\gamma|^p(r)\,\mathrm{d}r & \text{if } \gamma \in \mathrm{AC}^p([a,b];X), \\ +\infty & \text{otherwise.} \end{cases} \tag{5.3}$$

Notice that $\mathsf{d}^p(\gamma(a),\gamma(b)) \le (b-a)^{p-1}\mathcal{A}_p(\gamma)$.

A continuous function $\gamma : [0,1] \to X$ is a length minimizing constant speed curve if $\mathcal{A}_1(\gamma) = \mathsf{d}(\gamma(0),\gamma(1)) = |\dot\gamma|(t)$ for $\mathscr{L}^1$-a.e. $t \in (0,1)$, or, equivalently, if $\mathcal{A}_p(\gamma) = \mathsf{d}^p(\gamma(0),\gamma(1))$ for some (and thus every) $p > 1$. In the sequel, by geodesic we always mean a length minimizing constant speed curve.

The extended metric space $(X,\mathsf{d})$ is a *length space* if

$$\mathsf{d}(x_0,x_1) = \inf\Big\{\mathcal{A}_1(\gamma) : \gamma \in \mathrm{AC}([0,1];X),\ \gamma(i) = x_i\Big\} \quad \text{for every } x_0,x_1 \in X. \tag{5.4}$$

The collection of all Lipschitz real functions defined in $X$ will be denoted by $\mathrm{Lip}(X)$, while $\mathrm{Lip}_b$ will denote the subspace of bounded Lipschitz functions.

The slopes $|\mathrm{D}^{\pm}\varphi|$, the local Lipschitz constant $|\mathrm{D}\varphi|$ and the asymptotic Lipschitz constant $|\mathrm{D}^*\varphi|$ of $\varphi \in \mathrm{Lip}_b(X)$ are respectively defined by

$$|\mathrm{D}^{\pm}\varphi|(x) := \limsup_{y\to x} \frac{\big(\varphi(y)-\varphi(x)\big)_{\pm}}{\mathsf{d}(y,x)}, \quad |\mathrm{D}\varphi|(x) := \limsup_{y\to x} \frac{|\varphi(y)-\varphi(x)|}{\mathsf{d}(y,x)}, \tag{5.5}$$

$$|\mathrm{D}^*\varphi|(x) := \limsup_{\substack{y,z\to x\\ y\neq z}} \frac{|\varphi(y)-\varphi(z)|}{\mathsf{d}(y,z)} = \lim_{r\downarrow 0} \mathrm{Lip}(f, B_r(x)), \tag{5.6}$$

with the convention that all the above quantities are 0 if $x$ is an isolated point. Notice that $|\mathrm{D}^*\varphi|$ is an u.s.c. function and that, whenever $(X,\mathsf{d})$ is a length space,

$$|\mathrm{D}^*\varphi|(x) = \limsup_{y\to x} |\mathrm{D}\varphi|(y), \qquad \mathrm{Lip}(\varphi) = \sup_{x\in X} |\mathrm{D}\varphi(x)| = \sup_{x\in X} |\mathrm{D}^*\varphi(x)|. \tag{5.7}$$

## 5.2 The Hopf-Lax evolution formula

Let us suppose that $(X,\mathsf{d})$ is a metric space; the Hopf-Lax evolution map $\mathsf{Q}_t : \mathrm{C}_b(X) \to \mathrm{C}_b(X)$, $t \ge 0$, is defined by $\mathsf{Q}_0 f = f$ and

$$\mathsf{Q}_t f(x) := \inf_{y\in X} f(y) + \frac{\mathsf{d}^2(y,x)}{2t} \qquad t > 0. \tag{5.8}$$

We shall need the pointwise properties

$$\inf_X f \le \mathsf{Q}_t f(x) \le \sup_X f \quad \text{for every } x \in X,\ t \ge 0, \tag{5.9}$$

$$-\frac{\mathrm{d}^+}{\mathrm{d}t}\mathsf{Q}_t f(x) \ge \frac{1}{2}|\mathrm{D}^*\mathsf{Q}_t f|^2(x) \quad \text{for every } x \in X,\ t \ge 0 \tag{5.10}$$

(these are proved in Proposition 3.3 and Proposition 3.4 of [4], $\mathrm{d}^+/\mathrm{d}t$ denotes the right derivative).

When $(X,\mathsf{d})$ is a length space $(\mathsf{Q}_t)_{t\ge 0}$ is a semigroup and we have the refined identity [5, Thm. 3.6]

$$-\frac{\mathrm{d}^+}{\mathrm{d}t}\mathsf{Q}_t f(x) = \frac{1}{2}|\mathrm{D}\mathsf{Q}_t f|^2(x) \quad \text{for every } x \in X,\ t > 0. \tag{5.11}$$

Inequality (5.10) and the length property of $X$ yield the a priori bounds

$$\mathrm{Lip}(\mathsf{Q}_t f) \le 2\,\mathrm{Lip}(f) \quad \forall t \ge 0, \qquad \mathrm{Lip}\big(\mathsf{Q}_{\cdot} f(x)\big) \le 2\,\big[\mathrm{Lip}(f)\big]^2 \quad \forall x \in X. \tag{5.12}$$

## 5.3 Measures, couplings, Wasserstein distance

Let $(X, \mathsf{d})$ be a complete and separable metric space. We denote by $\mathscr{B}(X)$ the collection of its Borel sets and by $\mathscr{P}(X)$ the set of all Borel probability measures on $X$ endowed with the weak topology induced by the duality with the class $\mathrm{C}_b(X)$ of bounded and continuous functions in $X$. If $\mathfrak{m}$ is a nonnegative $\sigma$-finite Borel measure of $X$, $\mathscr{P}^{ac}(X, \mathfrak{m})$ denotes the convex subset of the probabiliy measures absolutely continuous w.r.t. $\mathfrak{m}$. $\mathscr{P}_p(X)$ denotes the set of probability measures $\mu \in \mathscr{P}(X)$ with finite $p$-moment, i.e.

$$\int_X \mathsf{d}^p(x, x_0)\,\mathrm{d}\mu(x) < \infty \quad \text{for some (and thus any) } x_0 \in X.$$

If $(Y, \mathsf{d}_Y)$ is another sparable metric space, $\boldsymbol{r} : X \to Y$ is a Borel map and $\mu \in \mathscr{P}(X)$, $\boldsymbol{r}_\sharp\mu$ denotes the push-forward measure in $\mathscr{P}(Y)$ defined by $\boldsymbol{r}_\sharp\mu(B) := \mu(\boldsymbol{r}^{-1}(B))$ for every $B \in \mathscr{B}(Y)$.

For every $p \in [1, \infty)$, the $L^p$-Wasserstein (extended) distance $W_p$ between two measures $\mu_0$, $\mu_1 \in \mathscr{P}(X)$ is defined as

$$W_p^p(\mu_1, \mu_2) := \inf\Big\{ \int_{X\times X} \mathsf{d}^p(x_1, x_2)\,\mathrm{d}\boldsymbol{\mu}(x_1, x_2) : \boldsymbol{\mu} \in \mathscr{P}(X \times X),\ \pi^i_\sharp\boldsymbol{\mu} = \mu_i \Big\}, \tag{5.13}$$

where $\pi^i : X \times X \to X$, $i = 1, 2$, denote the projections $\pi^i(x_1, x_2) = x_i$. A measure $\boldsymbol{\mu}$ with $\pi^i_\sharp\boldsymbol{\mu} = \mu_i$ as in (5.13) is called a coupling between $\mu_1$ and $\mu_2$. If $\mu_1$, $\mu_2 \in \mathscr{P}_p(X)$ then a coupling $\boldsymbol{\mu}$ minimizing (5.13) exists, $W_p(\mu_0, \mu_1) < \infty$, and $(\mathscr{P}_p(X), W_p)$ is a complete and separable metric space; it is also a length space if $X$ is a length space. Notice that if $X$ is unbounded $(\mathscr{P}(X), W_p)$ is an extended metric space, even if $\mathsf{d}$ is a finite distance on $X$.

The dual Kantorovich characterization of $W_p$ provides the useful representation formula (here stated only in the case $p = 2$)

$$\frac{1}{2}W_2^2(\mu_0, \mu_1) = \sup\Big\{ \int_X \mathsf{Q}_1\varphi\,\mathrm{d}\mu_1 - \int_X \varphi\,\mathrm{d}\mu_0 : \varphi \in \mathrm{Lip}_b(X) \Big\}, \tag{5.14}$$

where $(\mathsf{Q}_t)_{t>0}$ is defined in (5.8).

## 5.4 $W_p$-absolutely continuous curves and dynamic plans

A dynamic plan $\boldsymbol{\pi}$ is a Borel probability measure on $\mathrm{C}([0, 1]; X)$. For each dynamic plan $\boldsymbol{\pi}$ one can consider the (weakly) continuous curve $\mu = (\mu_s)_{s\in[0,1]} \subset \mathscr{P}(X)$ defined by $\mu(s) := (\mathrm{e}_s)_\sharp\boldsymbol{\pi}$, $s \in [0, 1]$ (we will often write $\mu_s$ instead of $\mu(s)$ and we will also use an analogous notation for "time dependent" densities or functions); here

$$\mathrm{e}_s : \mathrm{C}([0, 1]; X) \to X, \qquad \mathrm{e}_s(\gamma) := \gamma(s) \tag{5.15}$$

is the evaluation map at time $s \in [0, 1]$.

We say that $\boldsymbol{\pi}$ has finite $p$-energy, $p \in [1, \infty)$, if

$$\mathscr{A}_p(\boldsymbol{\pi}) := \int \mathcal{A}_p(\gamma)\,\mathrm{d}\boldsymbol{\pi}(\gamma) < \infty, \tag{5.16}$$

a condition that in particular yields $\gamma \in \mathrm{AC}^p([0,1];X)$ for $\boldsymbol{\pi}$-almost every $\gamma$. If for some $p > 1$ the dynamic plan $\boldsymbol{\pi}$ has finite $p$-energy, it is not hard to show that the induced curve $\mu$ belongs to $\mathrm{AC}^p([0,1];(\mathscr{P}(X),W_p))$ and that

$$|\dot\mu_s|^p \le \int |\dot\gamma_s|^p \,\mathrm{d}\boldsymbol{\pi}(\gamma) \text{ for } \mathscr{L}^1\text{-a.e. } s \in (0,1), \quad \text{so that} \quad \int_0^1 |\dot\mu_s|^p \,\mathrm{d}s \le \mathscr{A}_p(\boldsymbol{\pi}), \tag{5.17}$$

where $|\dot\mu_s|$ denotes the metric derivative of the curve $\mu$ in $(\mathscr{P}(X),W_p)$. Notice that the second inequality in (5.17) can also be written as $\mathcal{A}_p(\mu) \le \mathscr{A}_p(\boldsymbol{\pi})$. The converse inequalities, which involve a special choice of $\boldsymbol{\pi}$, provide a metric version of the so-called superposition principle, and their proof is less elementary.

**Theorem 5.1 ([39])** *For any $\mu \in \mathrm{AC}^p([0,1];(\mathscr{P}(X),W_p))$ there exists a dynamic plan $\boldsymbol{\pi}$ with finite p-energy such that*

$$\mu_t = (\mathrm{e}_t)_\sharp \boldsymbol{\pi} \quad \textit{for every } t \in [0,1], \qquad \int_0^1 |\dot\mu_t|^p \,\mathrm{d}t = \mathscr{A}_p(\boldsymbol{\pi}). \tag{5.18}$$

We say that the dynamic plan $\boldsymbol{\pi}$ is *$p$-tightened to* $\mu$ if (5.18) holds. For this class of plans equality holds in (5.17), namely

$$|\dot\mu_s|^p = \int |\dot\gamma_s|^p \,\mathrm{d}\boldsymbol{\pi}(\gamma) \text{ for } \mathscr{L}^1\text{-a.e. } s \in (0,1). \tag{5.19}$$

Focusing now on the case $p = 2$, the distinguished class of optimal geodesic plans $\mathrm{GeoOpt}(X)$ consists by those dynamic plans whose 2-action coincides with the squared $L^2$-Wasserstein distance between the marginals at the end points:

$$\boldsymbol{\pi} \in \mathrm{GeoOpt}(X) \quad \text{if} \quad \mathscr{A}_2(\boldsymbol{\pi}) = W_2^2(\mu_0,\mu_1), \quad \mu_i = (\mathrm{e}_i)_\sharp \boldsymbol{\pi}. \tag{5.20}$$

It is not difficult to check that (5.20) is equivalent to

$$\boldsymbol{\pi}\text{-a.e. } \gamma \text{ is a geodesic and } (\mathrm{e}_0,\mathrm{e}_1)_\sharp \boldsymbol{\pi} \text{ is an optimal coupling between } \mu_0,\ \mu_1. \tag{5.21}$$

It follows that a curve $\mu \in \mathrm{Lip}([0,1];(\mathscr{P}_2(X),W_2))$ is a geodesic if and only if there exists an optimal geodesic plan $\boldsymbol{\pi} \in \mathrm{OptGeo}(X)$ 2-tightened to $\mu$.

Finally, when a reference $\sigma$-finite and nonnegative Borel measure $\mathfrak{m}$ is fixed, we say that $\boldsymbol{\pi} \in \mathscr{P}(\mathrm{C}([0,1];\mathscr{P}(X)))$ is a *test plan* if it has finite 2-energy and there exists a constant $R > 0$ such that

$$\mu_t := (\mathrm{e}_t)_\sharp \boldsymbol{\pi} = \varrho_t \mathfrak{m} \ll \mathfrak{m}, \quad \varrho_t \le R \quad \mathfrak{m}\text{-a.e. in } X \text{ for every } t \in [0,1]. \tag{5.22}$$

## 5.5 Metric measure spaces and the Cheeger energy

In this paper a *metric measure space* $(X, \mathsf{d}, \mathfrak{m})$ will always consist of:

- a complete and separable metric space $(X, \mathsf{d})$;
- a nonnegative Borel measure $\mathfrak{m}$ having full support and satisfying the growth condition

$$\mathfrak{m}(B_r(x_0)) \le A\mathrm{e}^{Br^2} \quad \text{for some constants } A,\ B \ge 0, \text{ and some } x_0 \in X. \tag{5.23}$$

The Cheeger energy of a function $f \in L^2(X, \mathfrak{m})$ is defined as

$$\mathsf{Ch}(f) := \inf\Big\{ \liminf_{n\to\infty} \frac{1}{2}\int_X |\mathrm{D}f_n|^2 \,\mathrm{d}\mathfrak{m} : f_n \in \mathrm{Lip}_b(X), \quad f_n \to f \text{ in } L^2(X, \mathfrak{m})\Big\}. \tag{5.24}$$

If $f \in L^2(X, \mathfrak{m})$ with $\mathsf{Ch}(f) < \infty$, then there exists a unique function $|\mathrm{D}f|_w \in L^2(X, \mathfrak{m})$, called *minimal weak gradient of $f$*, satisfying the two conditions

$$\begin{gathered}\mathrm{Lip}_b(X) \cap L^2(X, \mathfrak{m}) \ni f_n \rightharpoonup f,\ |\mathrm{D}f_n| \rightharpoonup G \quad \text{in } L^2(X, \mathfrak{m}) \quad \Rightarrow \quad |\mathrm{D}f|_w \le G \ \mathfrak{m}\text{-a.e.}\\ \mathsf{Ch}(f) = \frac{1}{2}\int_X |\mathrm{D}f|_w^2 \,\mathrm{d}\mathfrak{m}.\end{gathered} \tag{5.25}$$

In (5.24) we can also replace $|\mathrm{D}f|$ with $|\mathrm{D}^*f|$ since a further approximation result of [4, §8.3] (see [1] for a detailed proof) yields for every $f \in L^2(X, \mathfrak{m})$ with $\mathsf{Ch}(f) < \infty$

$$\exists\, f_n \in \mathrm{Lip}_b(X) \cap L^2(X, \mathfrak{m}) : \quad f_n \to f, \quad |\mathrm{D}^*f_n| \to |\mathrm{D}f|_w \quad \text{strongly in } L^2(X, \mathfrak{m}). \tag{5.26}$$

We will denote by $W^{1,2}(X, \mathfrak{m})$ the vector space of the $L^2(X, \mathfrak{m})$ functions with finite Cheeger energy endowed with the canonical norm

$$\|f\|^2_{W^{1,2}(X,\mathfrak{m})} := \|f\|^2_{L^2(X,\mathfrak{m})} + 2\mathsf{Ch}(f) \tag{5.27}$$

that induces on $W^{1,2}(X, \mathfrak{m})$ a Banach space structure. We say that $\mathsf{Ch}$ is a quadratic form if it satisfies the parallelogram identity

$$\mathsf{Ch}(f + g) + \mathsf{Ch}(f - g) = 2\mathsf{Ch}(f) + 2\mathsf{Ch}(g) \quad \text{for every } f, g \in W^{1,2}(X, \mathfrak{m}). \tag{5.28}$$

In this case we will denote by $\mathcal{E}$ the associated bilinear Dirichlet form, so that $\mathsf{Ch}(f) = \frac{1}{2}\mathcal{E}(f, f)$; if $\mathcal{B}$ is the $\mathfrak{m}$-completion of the collection of Borel sets in $X$, we are in the setting of Section 3.1; keeping that notation, $\mathbb{H} = L^2(X, \mathfrak{m})$ and $\mathbb{V}$ is the separable Hilbert space $W^{1,2}(X, \mathfrak{m})$ endowed with the norm (5.27). Under the quadraticity assumption on $\mathsf{Ch}$ it is possible to prove [6, Thm. 4.18] that (5.28) can be localized, namely

$$|\mathrm{D}(f + g)|_w^2 + |\mathrm{D}(f - g)|_w^2 = 2|\mathrm{D}f|_w^2 + 2|\mathrm{D}g|_w^2 \quad \mathfrak{m}\text{-a.e. in } X. \tag{5.29}$$

It follows that

$$(f,g) \mapsto \Gamma\big(f,g\big) := \frac{1}{4}|\mathrm{D}(f+g)|_w^2 - \frac{1}{4}|\mathrm{D}(f-g)|_w^2 = \lim_{\varepsilon\downarrow 0} \frac{|\mathrm{D}(f+\varepsilon g)|_w^2 - |\mathrm{D}f|_w^2}{2\varepsilon} \tag{5.30}$$

is a strongly continuous bilinear map from $\mathbb{V}$ to $L^1(X,\mathfrak{m})$, with $\Gamma\big(f\big) = |\mathrm{D}f|_w^2$. The operator $\Gamma$ is the *Carré du Champ* associated to $\mathcal{E}$ and $\mathcal{E}$ is a strongly local Dirichlet form enjoying useful $\Gamma$-calculus properties, see e.g. [16, 7, 46], and the mass preserving property (3.14) (thanks to (5.23)). In the measure-metric setting we will use the Laplace operator symbol $-\Delta : \mathbb{V} \to \mathbb{V}'$ to denote the linear operator L associated to $\mathcal{E}$, with

$$\mathbb{D} := \big\{ f \in \mathbb{V} : \ \Delta f \in \mathbb{H} \big\}, \tag{5.31}$$

the domain of $\Delta$ as unbounded selfadjoint operator in $\mathbb{H}$, endowed with the Hilbertian norm $\|f\|_{\mathbb{D}}^2 := \|f\|_{\mathbb{V}}^2 + \|\Delta f\|_{\mathbb{H}}^2$. The operator $-\Delta$ generates a measure preserving Markov semigroup $(\mathsf{P}_t)_{t\geq 0}$ in each $L^p(X,\mathfrak{m})$, $1 \leq p \leq \infty$.

## 5.6 Weighted Cheeger energies

In the metric-measure setting of Section 5.5, consider a nonnegative function $\varrho \in L^\infty(X,\mathfrak{m})$ and the symmetric and continuous bilinear form in $\mathbb{V}\times\mathbb{V}$

$$\mathcal{E}_\varrho(f,g) := \int_X \varrho\,\Gamma\big(f,g\big)\,\mathrm{d}\mathfrak{m} \qquad f,\, g \in \mathbb{V}, \tag{5.32}$$

which induces a seminorm: we will denote by $\mathbb{V}_\varrho = \mathbb{V}_{\mathcal{E}_\varrho}$ the abstract Hilbert spaces constructed from $\mathcal{E}_\varrho$ as in Section 3.2, namely the completion of the quotient space of $\mathbb{V}$ induced by the equivalence relation $f \sim g$ if $\mathcal{E}_\varrho(f-g,f-g) = 0$, with respect to the norm induced by the quotient scalar product. If $\varphi \in \mathbb{V}$ then its equivalence class in $\mathbb{V}_\varrho$ will be denoted by $\varphi_\varrho$ (or still by $\varphi$ when there is no risk of confusion), whereas we will still use the symbol $\mathcal{E}_\varrho$ to denote the scalar product in $\mathbb{V}_\varrho$. Notice that the quadratic form $\frac{1}{2}\mathcal{E}_\varrho$ is always larger than the Cheeger energy induced by the measure $\varrho\mathfrak{m}$. When $\varrho \equiv 1$, $\mathbb{V}_1$ corresponds to the homogeneous space $\mathbb{V}_{\mathcal{E}}$ associated to $\mathcal{E}$ already introduced in Section 5.5.

The following two simple results provide useful tools to deal with the abstract spaces $\mathbb{V}_\varrho$.

**Lemma 5.2 (Extension of $\Gamma$ to the weighted spaces $\mathbb{V}_\varrho$)** *Let $\varrho \in L^\infty_+(X,\mathfrak{m})$, and let $(\varphi_n) \subset \mathbb{V}$ be a Cauchy sequence with respect to the seminorm of $\mathbb{V}_\varrho$, thus converging to $\phi \in \mathbb{V}_\varrho$. Then $\Gamma\big(\varphi_n\big)$ is strongly converging in $L^1(X,\varrho\mathfrak{m})$ to a limit that depends only on $\varrho$ and $\phi$ and that we will denote by $\Gamma_\varrho(\phi)$. When $\phi = \varphi_\varrho$ for some $\varphi \in \mathbb{V}$ then $\Gamma_\varrho(\phi) = \Gamma\big(\varphi\big)$ $\varrho\mathfrak{m}$-a.e. in $X$. The map*

$$\Gamma_\varrho(\phi,\psi) := \frac{1}{4}\Gamma_\varrho(\phi+\psi) - \frac{1}{4}\Gamma_\varrho(\phi-\psi) \tag{5.33}$$

*is a continuous bilinear map from $\mathbb{V}_\varrho$ to $L^1(X,\varrho\mathfrak{m})$ and (5.32) extends to $\mathbb{V}_\varrho$ as follows:*

$$\mathcal{E}_\varrho(\phi,\psi) = \int_X \varrho\,\Gamma_\varrho(\phi,\psi)\,\mathrm{d}\mathfrak{m} \qquad \phi,\, \psi \in \mathbb{V}_\varrho. \tag{5.34}$$

*Proof.* The convergence of $\Gamma\big(\varphi_n\big)$ in $L^1(X,\varrho\mathfrak{m})$ and the independence of the limit follow from the obvious inequality

$$\int_X \Big|\Gamma\big(\psi_1\big)-\Gamma\big(\psi_2\big)\Big|\,\varrho\,\mathrm{d}\mathfrak{m}=\int_X \Gamma\big(\psi_1-\psi_2\big)^{1/2}\Gamma\big(\psi_1+\psi_2\big)^{1/2}\,\varrho\,\mathrm{d}\mathfrak{m}\le \|\psi_1-\psi_2\|_{\mathbb{V}_\varrho}\|\psi_1+\psi_2\|_{\mathbb{V}_\varrho},$$

for every $\psi_1,\,\psi_2\in\mathbb{V}$. When $\phi=\varphi_\varrho$ then we can choose the constant sequence $\varphi_n\equiv\varphi$, thus showing that $\Gamma_\varrho(\phi)=\Gamma\big(\varphi\big)$ $\varrho\mathfrak{m}$-a.e. in $X$. It is immediate to check that $\Gamma_\varrho(\cdot)$ satisfies the parallelogram rule, so that (5.33) and (5.34) follow from the corresponding properties of $\Gamma$ and $\mathcal{E}_\varrho$ in $\mathbb{V}$. □

**Lemma 5.3 (Stability)** *Let $\varrho_t\in L^\infty_+(X,\mathfrak{m})$, $t\in[0,1]$, be a uniformly bounded family, continuous with respect to the convergence in $\mathfrak{m}$-measure, let $\varrho\in L^\infty_+(X,\mathfrak{m})$ and let $B_t:\mathbb{V}\to\mathbb{V}$ be a family of linear operators satisfying*

$$\int_X \varrho_t\Gamma\big(B_t\varphi\big)\,\mathrm{d}\mathfrak{m}\le C\int_X \varrho\Gamma\big(\varphi\big)\,\mathrm{d}\mathfrak{m}\quad \textit{for every } t\in[0,1],\ \varphi\in\mathbb{V}, \tag{5.35}$$

$$t\mapsto B_t\varphi\in \mathrm{C}([0,1];\mathbb{V})\quad \textit{for every } \varphi\in\mathbb{V}. \tag{5.36}$$

*Then $B_t$ can be extended by continuity to a family of uniformly bounded linear operators from $\mathbb{V}_\varrho$ to $\mathbb{V}_{\varrho_t}$ such that*

$$\mathcal{E}_{\varrho_t}(B_t\phi)\le\mathcal{E}_\varrho(\phi),\quad \textit{for every } t\in[0,1],\ \phi\in\mathbb{V}_\varrho, \tag{5.37}$$

$$t\mapsto \varrho_t\Gamma_{\varrho_t}(B_t\phi)\in\mathrm{C}([0,1];L^1(X,\mathfrak{m}))\quad \textit{for every } \phi\in\mathbb{V}_\varrho. \tag{5.38}$$

*Proof.* Assumption (5.35) shows that for every $t\in[0,1]$ the operator $B_t$ is compatible with the equivalence relations associated to $\mathbb{V}_\varrho$ and $\mathbb{V}_{\varrho_t}$, so that it can be extended by continuity to a linear map between the two spaces, still denoted $B_t$ and satisfying (5.37). Given any $\varphi\in\mathbb{V}_\varrho$, choosing $(\varphi_n)\subset\mathbb{V}$ such that the corresponding elements $(\tilde\varphi_n)_\varrho$ converge to $\phi$ in $\mathbb{V}_\varrho$, the estimate (5.35) shows that $\varrho_t\Gamma_{\varrho_t}(B_t\varphi_n)$ converges uniformly in time to $\varrho_t\Gamma_{\varrho_t}(B_t\phi)$ in $L^1(X,\mathfrak{m})$, so that the continuity property (5.38) follows from the continuity of each curve $t\mapsto\varrho_t\Gamma_{\varrho_t}(B_t\varphi_n)$. □

Finally, we discuss dual spaces, following the general scheme described in Section 3.2, see in particular Proposition 3.1. The space $\mathbb{V}_\varrho'$ is the realization of the dual of $\mathbb{V}_\varrho$ in $\mathbb{V}'$. It can be seen as the finiteness domain of the quadratic form

$$\frac12\mathcal{E}^*_\varrho(\ell,\ell):=\sup_{\varphi\in\mathbb{V}}\langle\ell,\varphi\rangle-\frac12\mathcal{E}_\varrho(\varphi,\varphi),\qquad w\in\mathbb{V}'. \tag{5.39}$$

We shall denote by $\mathcal{E}^*_\varrho(\cdot,\cdot)$ the quadratic form on $\mathbb{V}_\varrho'$ induced by $\mathcal{E}^*_\varrho$. We denote by $-A_\varrho$ the Riesz isomorphism between $\mathbb{V}_\varrho$ and $\mathbb{V}_\varrho'$, and by $-A^*_\varrho$ its inverse. It is characterized by

$$\phi=-A^*_\varrho\ell\quad\Longleftrightarrow\quad\mathcal{E}_\varrho(\phi,\psi)=\langle\ell,\psi\rangle\quad\text{for every }\psi\in\mathbb{V}_\varrho. \tag{5.40}$$

Notice that it is equivalent in (5.40) to require the validity of the equality for all $\psi \in \mathbb{V}$; in this sense, (5.40) corresponds in our abstract framework to the weak formulation of the PDE $-\mathrm{div}(\varrho\nabla\phi) = \ell$ in (2.26), and $-A^*_\varrho$ is the solution operator. Since $-A_\varrho$ is the Riesz isomorphism, we get

$$\mathcal{E}^*_\varrho(\ell, \ell) = \mathcal{E}_\varrho(A^*_\varrho\ell, A^*_\varrho\ell). \tag{5.41}$$

Correspondingly we set

$$\Gamma^*_\varrho(\ell) := \Gamma_\varrho(A^*_\varrho\ell) \quad \text{whenever } \ell \in \mathbb{V}'_\varrho. \tag{5.42}$$

It is clear that $\Gamma^*_\varrho : \mathbb{V}'_\varrho \mapsto L^1(X, \varrho\mathfrak{m})$ is a nonnegative quadratic map.

**Lemma 5.4 (Dual characterization of $\Gamma^*_\varrho$)** *For every $\ell \in \mathbb{V}'$ and $\varrho \in L^\infty_+(X, \mathfrak{m})$ let us consider the (possibily empty) closed convex subset of $L^2(X, \varrho\mathfrak{m})$ defined by*

$$G(\varrho, \ell) := \Big\{ g \in L^2(X, \varrho\mathfrak{m}) : \big|\langle \ell, \varphi\rangle\big| \le \int_X g\sqrt{\Gamma(\varphi)}\,\varrho\,\mathrm{d}\mathfrak{m} \quad \text{for every} \ \ \varphi \in \mathbb{V}\Big\}. \tag{5.43}$$

*Then $\ell \in \mathbb{V}'_\varrho$ if and only if $G(\varrho, \ell)$ is not empty; if $\ell \in \mathbb{V}'_\varrho$ then $\sqrt{\Gamma^*_\varrho(\ell)}$ is the element of minimal $L^2(X, \varrho\mathfrak{m})$-norm in $G(\varrho, \ell)$.*

*Proof.* If $g \in G(\varrho, \ell)$ then

$$\langle \ell, \varphi\rangle \le \|g\|_{L^2(X,\varrho\mathfrak{m})}\Big(\mathcal{E}_\varrho(\varphi, \varphi)\Big)^{1/2} \quad \text{for every } \varphi \in \mathbb{V},$$

so that $\ell \in \mathbb{V}'_\varrho$ and

$$\int_X \Gamma^*_\varrho(\ell)\,\varrho\,\mathrm{d}\mathfrak{m} = \mathcal{E}^*_\varrho(\ell, \ell) \le \int_X g^2\,\varrho\,\mathrm{d}\mathfrak{m}. \tag{5.44}$$

Conversely, let us suppose that $\ell \in \mathbb{V}'_\varrho$ and let $\phi = -A^*_\varrho\ell$; (5.40) yields

$$|\langle \ell, \psi\rangle| \le \int_X \Gamma_\varrho(\phi, \psi)\,\varrho\,\mathrm{d}\mathfrak{m} \le \int_X \sqrt{\Gamma_\varrho(\phi)}\,\sqrt{\Gamma_\varrho(\psi)}\,\varrho\,\mathrm{d}\mathfrak{m} \quad \text{for every } \psi \in \mathbb{V}$$

so that $\sqrt{\Gamma^*_\varrho(\ell)} = \sqrt{\Gamma_\varrho(\phi)} \in G(\varrho, \ell)$. Combining with (5.44) we conclude that $\sqrt{\Gamma^*_\varrho(\ell)}$ is the element of minimal norm in $G(\varrho, \ell)$. □

The following lower semicontinuity lemma with respect to the weak topology of $\mathbb{V}'$ will also be useful.

**Lemma 5.5** *Let $\varrho_n \overset{*}{\rightharpoonup} \varrho$ in $L^\infty(X, \mathfrak{m})$ be nonnegative and assume that $\ell_n \rightharpoonup \ell$ in $\mathbb{V}'$. Then*

$$\liminf_{n\to\infty} \mathcal{E}^*_{\varrho_n}(\ell_n, \ell_n) \ge \mathcal{E}^*_\varrho(\ell, \ell). \tag{5.45}$$

*If moreover $\varrho_n \to \varrho$ also in the strong topology of $L^1(X, \mathfrak{m})$ and*

$$\limsup_{n\to\infty} \mathcal{E}^*_{\varrho_n}(\ell_n, \ell_n) \le \mathcal{E}^*_{\varrho}(\ell, \ell) < \infty, \tag{5.46}$$

*then for every continuous and bounded function $Q : [0, \infty) \to [0, \infty)$ we have*

$$\lim_{n\to\infty} \int_X Q(\varrho_n)\Gamma^*_{\varrho_n}(\ell_n)\varrho_n \,\mathrm{d}\mathfrak{m} = \int_X Q(\varrho)\Gamma^*_{\varrho}(\ell)\,\varrho \,\mathrm{d}\mathfrak{m}. \tag{5.47}$$

*Proof.* Concerning (5.45), for every $\varphi \in \mathbb{V}$, we have

$$\langle \ell, \varphi\rangle - \frac{1}{2}\int_X \varrho\Gamma\big(\varphi\big)\,\mathrm{d}\mathfrak{m} = \lim_{n\to\infty}\Big(\langle \ell_n, \varphi\rangle - \frac{1}{2}\int_X \varrho_n\Gamma\big(\varphi\big)\,\mathrm{d}\mathfrak{m}\Big) \le \liminf_{n\to\infty} \mathcal{E}^*_{\varrho_n}(\ell_n, \ell_n).$$

Taking the supremum with respect to $\varphi \in \mathbb{V}$ we get (5.45).

Let us consider the second part of the statement and let us set $g_n = \sqrt{\Gamma^*_{\varrho_n}(\ell_n)}$, $h_n = g_n\varrho_n$. Since $h_n$ is uniformly bounded in $L^2(X, \mathfrak{m})$, possibly extracting a suitable subsequence we can assume that $h_n$ weakly converge in $L^2(X, \mathfrak{m})$ to $h$. Since the measures $h_n\mathfrak{m}$, $\varrho_n\mathfrak{m}$ weakly converge respectively to $h\mathfrak{m}$ and $\varrho\mathfrak{m}$ and the densities $g_n$ of $h_n\mathfrak{m}$ w.r.t. $\varrho_n\mathfrak{m}$ satisfy $\sup_n \|g_n\|_{L^2(X,\varrho_n\mathfrak{m})} < \infty$ we can apply a standard joint lower semicontinuity lemma (see, for instance [3, Lemma 9.4.3]) to write $h = g\varrho$ for some $g \in L^2(X, \varrho\mathfrak{m})$, with

$$\int_X g^2\,\varrho \,\mathrm{d}\mathfrak{m} \le \liminf_{n\to\infty} \int_X g_n^2\,\varrho_n \,\mathrm{d}\mathfrak{m}. \tag{5.48}$$

Passing to the limit in the inequalities

$$|\langle \ell_n, \psi\rangle| \le \int_X \sqrt{\Gamma^*_{\varrho_n}(\ell_n)}\,\sqrt{\Gamma\big(\psi\big)}\,\varrho_n \,\mathrm{d}\mathfrak{m} = \int_X h_n\,\sqrt{\Gamma\big(\psi\big)}\,\mathrm{d}\mathfrak{m} \quad \text{for every } \psi \in \mathbb{V},$$

we get

$$|\langle \ell, \psi\rangle| \le \int_X h\sqrt{\Gamma\big(\psi\big)}\,\mathrm{d}\mathfrak{m} = \int_X g\sqrt{\Gamma\big(\psi\big)}\,\varrho \,\mathrm{d}\mathfrak{m} \quad \text{for every } \psi \in \mathbb{V},$$

which shows that $g \in G(\varrho, \ell)$. On the other hand, (5.46) and (5.48) yield

$$\int_X g^2\,\varrho \,\mathrm{d}\mathfrak{m} \le \liminf_{n\to\infty} \int_X g_n^2\,\varrho_n \,\mathrm{d}\mathfrak{m} = \liminf_{n\to\infty} \mathcal{E}^*_{\varrho_n}(\ell_n, \ell_n) \le \mathcal{E}^*_{\varrho}(\ell, \ell) = \int_X \Gamma^*_{\varrho}(\ell)\varrho \,\mathrm{d}\mathfrak{m},$$

so that Lemma 5.4 gives $g = \Gamma^*_{\varrho}(\ell)$. An application of [3, Thm. 5.4.4] yields (5.47). □

# 6 Absolutely continuous curves in Wasserstein spaces and continuity inequalities in a metric setting

In this section we extend to general metric spaces some aspects of the results of [3, Chap. 8]. Even if we will use only the case $p = 2$, we state some results in the general case for possible future reference.

Let $(X, \mathsf{d})$ be a complete and separable metric space; we set $\tilde{X} := X \times [0,1]$ and define $\tilde{\mathrm{e}} : \mathrm{C}([0,1], X) \times [0,1] \to \tilde{X}$ by $\tilde{\mathrm{e}}(\gamma, t) := (\gamma(t), t)$. For every dynamic plan $\boldsymbol{\pi}$ we consider the measures

$$\lambda := \mathscr{L}^1|_{[0,1]}, \qquad \tilde{\boldsymbol{\pi}} := \boldsymbol{\pi} \otimes \lambda, \qquad \tilde{\mu} := \tilde{\mathrm{e}}_\sharp\big(\tilde{\boldsymbol{\pi}}\big) \in \mathscr{P}(X \times [0,1]). \tag{6.1}$$

Notice that the disintegration of $\tilde{\mu}$ with respect to time is exactly $((\mathrm{e}_t)_\sharp \boldsymbol{\pi})_{t \in [0,1]}$, i.e. $\tilde{\mu}$ admits the representation

$$\tilde{\mu} = \int_0^1 \mu_t \, \mathrm{d}\lambda(t) \quad \text{with} \quad \mu_t := (\mathrm{e}_t)_\sharp \boldsymbol{\pi}. \tag{6.2}$$

If $\boldsymbol{\pi}$ has finite $p$-energy for some $p \in (1, \infty)$, the Borel map $(\gamma, t) \mapsto \tilde{\mathrm{v}}(\gamma, t) := |\dot{\gamma}|(t)$ (defined where the metric derivative exists) belongs to $L^p(\mathrm{C}([0,1]; X) \times [0,1], \tilde{\boldsymbol{\pi}})$, so that the mean velocity $v$ of $\boldsymbol{\pi}$ can be defined by

$$\tilde{\mathrm{e}}_\sharp(\tilde{\mathrm{v}}\, \tilde{\boldsymbol{\pi}}) = v \tilde{\mu} \quad \text{with} \quad v \in L^p(\tilde{X}, \tilde{\mu}), \quad v(x,t) = \int |\dot{\gamma}_t| \, \mathrm{d}\tilde{\boldsymbol{\pi}}_{x,t}(\gamma) \tag{6.3}$$

(here $(\tilde{\boldsymbol{\pi}}_{x,t})_{(x,t) \in \tilde{X}} \subset \mathscr{P}(\mathrm{C}([0,1]; X))$ is the disintegration of $\tilde{\boldsymbol{\pi}}$ w.r.t. its image $\tilde{\mu}$).

In the next definition we make precise the concept of a square integrable velocity density for a curve of probability measures: differently from [3], here we can consider only the "modulus" of the velocity field, but this already provides an interesting information in many situations.

**Definition 6.1 (Velocity density)** *Let $\mu \in \mathrm{C}([0,1]; \mathscr{P}(X))$, $\tilde{\mu} := \int \mu_t \, \mathrm{d}\lambda \in \mathscr{P}(\tilde{X})$. We say that $v \in L^1(\tilde{X}, \tilde{\mu})$ is a velocity density for $\mu$ if for every $\varphi \in \mathrm{Lip}_b(X)$ one has*

$$\Big| \int_X \varphi \, \mathrm{d}\mu_t - \int_X \varphi \, \mathrm{d}\mu_s \Big| \le \int_{X \times (s,t)} |\mathrm{D}^* \varphi| \, v \, \mathrm{d}\tilde{\mu} \quad \textit{for every} \ \ 0 \le s < t \le 1. \tag{6.4}$$

*The set of velocity densities is a closed convex set in $L^1(\tilde{X}, \tilde{\mu})$. We say that $\bar{v} \in L^p(\tilde{X}, \tilde{\mu})$ is* the minimal $p$-velocity density *if $\bar{v}$ is the element of minimal $L^p(\tilde{X}, \tilde{\mu})$-norm among all the velocity densities.*

**Remark 6.2 (Lipschitz test functions with bounded support)** We obtain an equivalent definition by asking that (6.4) holds for every test function $\varphi \in \mathrm{Lip}_b(X)$ *with bounded support*: in fact, fixing $x_0 \in X$ and the family of cut-off functions

$$\psi_R(x) = \eta(\mathsf{d}(x, x_0)/R) \quad \text{where} \quad \eta(y) = (1 - (y-1)_+)_+, \tag{6.5}$$

every $\varphi \in \mathrm{Lip}_b(X)$ can be approximated by the sequence $\varphi_n := \varphi \cdot \psi_n$; if $v \in L^1(\tilde{X}, \tilde{\mu})$ satisfies (6.4) for every Lipschitz function with bounded support, we can use the dominated convergence theorem to pass to the limit as $n \to \infty$ in

$$\Big| \int_X \varphi_n \, \mathrm{d}\mu_t - \int_X \varphi_n \, \mathrm{d}\mu_s \Big| \le \int_{X \times (s,t)} |\mathrm{D}^* \varphi| \, v \, \mathrm{d}\tilde{\mu} + \sup |\varphi| \int_{(\overline{B}_{2n}(x_0) \setminus B_n(x_0)) \times (s,t)} v \, \mathrm{d}\tilde{\mu},$$

since

$$|\mathrm{D}^* \varphi_n|(x) \le |\mathrm{D}^* \varphi|(x) \, \psi_n(x) + \sup |\varphi| \chi_{\overline{B}_{2n}(x_0) \setminus B_n(x_0)}(x).$$

For $p \in (1,\infty)$, we are going to show that the minimal $p$-velocity density exists for curves $\mu \in \mathrm{AC}^p([0,1];(\mathscr{P}(X),W_p))$ and that it is provided exactly by (6.3), for every dynamic plan with finite $p$-energy $\boldsymbol{\pi}$ tightened to $\mu$. Heuristically, this means that for a tightened plan $\boldsymbol{\pi}$ associated to $\mu$, while branching may occur, the speed of curves at a given point at a given time is independent of the curve and given by the minimal $p$-velocity. The starting point of our investigation is provided by the following simple result.

**Lemma 6.3 (The mean velocity is a velocity density)** *Let $\boldsymbol{\pi}$ be a dynamic plan with finite $p$-energy and let $\mu$, $\tilde\mu$, $v$ be defined as in* (6.1), (6.2), (6.3)*. Then $v \in L^p(\tilde X,\tilde\mu)$ is a velocity density for $\mu$.*

*Proof.* Immediate, since the upper gradient property of $|\mathrm{D}^*\varphi|$ yields

$$\int_X \varphi\,\mathrm{d}\mu_t - \int_X \varphi\,\mathrm{d}\mu_s = \int \Big(\varphi(\gamma(t)) - \varphi(\gamma(s))\Big)\,\mathrm{d}\boldsymbol{\pi}(\gamma) \le \int\int_s^t |\mathrm{D}^*\varphi|(\gamma(r))|\dot\gamma|(r)\,\mathrm{d}r\,\mathrm{d}\boldsymbol{\pi}(\gamma)$$
$$= \int_{\mathrm{C}([0,1];X)\times(s,t)} |\mathrm{D}^*\varphi|(\gamma(r))\tilde{\mathrm{v}}(\gamma,r)\,\mathrm{d}\tilde{\boldsymbol{\pi}}(\gamma,r) = \int_{X\times(s,t)} |\mathrm{D}^*\varphi| v\,\mathrm{d}\tilde\mu. \qquad \square$$

Choosing now plans $\boldsymbol{\pi}$ tightened to $\mu$, applying (5.19) and Jensen's inequality we immediately get:

**Corollary 6.4** *For every $\mu \in \mathrm{AC}^p([0,1];(\mathscr{P}(X),W_p))$ and every dynamic plan $\boldsymbol{\pi}$ tightened to $\mu$ the velocity density $v \in L^p(\tilde X,\tilde\mu)$ defined in* (6.3) *satisfies*

$$\int_{\tilde X} v^p\,\mathrm{d}\tilde\mu \le \mathcal{A}_p(\boldsymbol{\pi}). \tag{6.6}$$

The next Lemma shows that we can use a velocity density even with time-dependent test functions.

**Lemma 6.5** *Let $\mu \in \mathrm{C}([0,1];\mathscr{P}(X))$, $\tilde\mu := \int \mu_t\,\mathrm{d}\lambda \in \mathscr{P}(\tilde X)$ and let $v \in L^1(\tilde X,\tilde\mu)$ be a velocity density for $\mu$. Then $\mu \in \mathrm{AC}([0,1];(\mathscr{P}_1(X),W_1))$ and for every $\varphi \in \mathrm{Lip}_b(\tilde X)$ one has*

$$\int_X \varphi_t\,\mathrm{d}\mu_t - \int_X \varphi_s\,\mathrm{d}\mu_s \le \int_{X\times(s,t)} \big(\partial_r^+\varphi_r + |\mathrm{D}^*\varphi_r|\,v\big)\,\mathrm{d}\tilde\mu \quad \textit{for every } 0 \le s < t \le 1, \tag{6.7}$$

*where*

$$\partial_r^+\varphi_r(x) = \limsup_{h\downarrow 0} \frac{1}{h}\big(\varphi_{r+h}(x) - \varphi_r(x)\big). \tag{6.8}$$

*Proof.* If $\varphi$ is 1-Lipschitz then $|\mathrm{D}^*\varphi| \le 1$, so that from (6.4) and the dual chacracterization of $W_1$ we easily get

$$W_1(\mu_s,\mu_t) = \sup_{\varphi\in\mathrm{Lip}_b(X),\ \mathrm{Lip}(\varphi)\le 1} \Big|\int_X \varphi\,\mathrm{d}\mu_t - \int_X \varphi\,\mathrm{d}\mu_s\Big| \le \int_s^t m(r)\,\mathrm{d}r, \quad \text{where}$$
$$m(r) := \int_X v\,\mathrm{d}\mu_r, \quad \text{so that } m \in L^1(0,1).$$

If we consider the map $\eta(s,t) := \int_X \varphi_s \, \mathrm{d}\mu_t$ and we call $L$ the Lipschitz constant of $\varphi$, we easily get for every $0 \le s \le s' \le 1, \; 0 \le t \le t' \le 1$

$$|\eta(s',t) - \eta(s,t)| \le L|s'-s|, \qquad |\eta(s,t') - \eta(s,t)| \le L \int_t^{t'} m(r) \, \mathrm{d}r,$$

so that we can apply [3, Lemma 4.3.4] to get the absolute continuity of $t \mapsto \eta(t,t)$ with

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,t) \le \limsup_{h \downarrow 0} \frac{1}{h} \int_X \varphi_t \, \mathrm{d}(\mu_t - \mu_{t-h}) + \limsup_{h \downarrow 0} \frac{1}{h} \int_X (\varphi_{t+h} - \varphi_t) \, \mathrm{d}\mu_t. \tag{6.9}$$

Choosing a Lebesgue point of $t \mapsto \int_X |\mathrm{D}^* \varphi| v \, \mathrm{d}\mu_t$ and applying Fatou's Lemma we conclude that we can estimate from above the derivative of $t \mapsto \eta(t,t)$ by

$$\int_X \partial_t^+ \varphi_t \, \mathrm{d}\mu_t + \int_X |\mathrm{D}^* \varphi_t| v_t \, \mathrm{d}\mu_t.$$

Since $t \mapsto \eta(t,t)$ is absolutely continuous, by integration we get the result. $\square$

**Theorem 6.6 (The metric derivative can be estimated with any velocity density)** *Let $\mu \in \mathrm{C}([0,1]; \mathscr{P}(X))$, $\tilde{\mu} := \int \mu_t \, \mathrm{d}\lambda \in \mathscr{P}(\tilde{X})$ and let $v \in L^p(\tilde{X}, \tilde{\mu})$ be a $p$-velocity density for $\mu$, for some $p \in (1, \infty)$. Then $\mu \in \mathrm{AC}^p([0,1]; (\mathscr{P}(X), W_p))$ and*

$$|\dot{\mu}_t|^p \le \int_X v_t^p \, \mathrm{d}\mu_t \quad \textit{for } \lambda\textit{-a.e. } t \in (0,1). \tag{6.10}$$

*Proof.* We give the proof in the case $p = 2$, the general case is completely analogous. With the notation of Kuwada's Lemma [5, Lemma 6.1], denoting by $\mathsf{Q}_t \varphi$ the Hopf-Lax evolution map given by (5.8), one has

$$\frac{1}{2} W_2^2(\mu_s, \mu_t) = \sup_{\varphi \in \mathrm{Lip}_b(X)} \int_X \mathsf{Q}_1 \varphi \, \mathrm{d}\mu_t - \int_X \varphi \, \mathrm{d}\mu_s \qquad 0 \le s \le t \le 1.$$

Setting $\ell = t - s$ and recalling that (5.10) gives

$$\partial_r^+ \mathsf{Q}_{r/\ell} \varphi \le - \frac{|\mathrm{D}^* \mathsf{Q}_{r/\ell} \varphi|^2}{2\ell} \quad \text{in } X \times [0, \ell],$$

the inequality (6.7) yields

$$\begin{aligned} \int_X \mathsf{Q}_1 \varphi \, \mathrm{d}\mu_t - \int_X \varphi \, \mathrm{d}\mu_s &\le \int_0^\ell \int_X \Big( - \frac{|\mathrm{D}^* \mathsf{Q}_{r/\ell} \varphi|^2}{2\ell} + |\mathrm{D}^* \mathsf{Q}_{r/\ell} \varphi| \, v_{s+r} \Big) \, \mathrm{d}\mu_{s+r} \, \mathrm{d}r \\ &\le \frac{\ell}{2} \int_0^\ell \int_X v_{s+r}^2 \, \mathrm{d}\mu_{s+r} \, \mathrm{d}r, \end{aligned}$$

where we used that $2|\mathrm{D}^* \mathsf{Q}_{r/\ell} \varphi| \, v_{s+r} \le |\mathrm{D}^* \mathsf{Q}_{r/\ell} \varphi|^2/\ell + \ell v_{s+r}^2$. We conclude that

$$\frac{1}{2} W_2^2(\mu_s, \mu_t) \le \frac{1}{2}(t-s) \int_s^t \Big( \int_X v_r^2 \, \mathrm{d}\mu_r \Big) \, \mathrm{d}r,$$

that yields first the 2-absolute continuity of the curve $t \mapsto \mu_t$ in $(\mathscr{P}_2(X), W_2)$. Also inequality (6.10) follows, becaust we have

$$|\dot\mu_t|^2 = \lim_{h\to 0} \frac{W_2^2(\mu_{t+h}, \mu_t)}{h^2} \le \lim_{h\to 0} \frac{1}{h} \int_t^{t+h} \left( \int_X v_r^2 \, \mathrm{d}\mu_r \right) \mathrm{d}r = \int_X v_t^2 \, \mathrm{d}\mu_t$$

for $\lambda$-a.e. $t \in (0,1)$. □

**Theorem 6.7 (Existence and characterization of the metric velocity density)**

[M.1] *A curve $\mu \in \mathrm{C}([0,1]; \mathscr{P}(X))$ belongs to $\mathrm{AC}^p([0,1]; (\mathscr{P}(X), W_p))$, $p \in (1,\infty)$, if and only if $\mu$ admits a velocity density in $L^p(\tilde X, \tilde\mu)$. In this case there exists a unique (up to $\tilde\mu$-negligible sets) minimal $p$-velocity density $\bar v \in L^p(\tilde X, \tilde \mu)$ and*

$$|\dot\mu_t|^p = \int_X \bar v^p \, \mathrm{d}\mu_t \quad \textit{for } \lambda\textit{-a.e. } t \in (0,1). \tag{6.11}$$

[M.2] *If $\boldsymbol{\pi}$ is a dynamical plan $p$-tightened to $\mu$ and the mean velocity $v$ of $\boldsymbol{\pi}$ is defined as in (6.3), then $\bar v = v$ $\tilde\mu$-a.e. in $\tilde X$ and*

$$\bar v(\gamma(t), t) = |\dot\gamma|(t) \quad \textit{for } \tilde{\boldsymbol{\pi}}\textit{-a.e. } (\gamma, t). \tag{6.12}$$

*In particular, the velocity of curves depends $\tilde{\boldsymbol{\pi}}$-a.e. only on $(\gamma(t), t)$ and it is independent of the choice of $\tilde{\boldsymbol{\pi}}$.*

*Proof.* The characterization of $\mathrm{AC}^p([0,1]; (\mathscr{P}_p(X), W_p))$ in terms of the existence of a velocity density in $L^p(\tilde X, \tilde\mu)$ follows by Corollary 6.4 and Theorem 6.6. The existence and the uniqueness of the minimal $p$-velocity density is a consequence of the strict convexity of the $L^p$-norm.

If $\boldsymbol{\pi}$ is a dynamic plan $p$-tightened to $\mu$ and $v$ is defined in terms of (6.3), we can combine Corollary 6.4 and Theorem 6.6 (which provides the sharp lower bound on the $L^p$ norm of velocity densities) to obtain that $v$ is the minimal $p$-velocity density and that $|\dot\mu_t|^p = \int_X v_t^p \, \mathrm{d}\mu_t$ for $\lambda$-a.e. $t \in (0,1)$, so (6.11) follows. Combining this information with (5.19) yields

$$\int_X v_t^p \, \mathrm{d}\mu_t = \int_X |\dot\gamma_t|^p \, d\boldsymbol{\pi}(\gamma) \qquad \text{for } \lambda\text{-a.e. } t \in (0,1),$$

so that, recalling (6.3), we get

$$\int_{\tilde X} \left( \int |\dot\gamma_t| \, \mathrm{d}\tilde{\boldsymbol{\pi}}_{x,t}(\gamma) \right)^p \mathrm{d}\tilde\mu(x,t) = \int_{\tilde X} \int |\dot\gamma_t|^p \, \mathrm{d}\tilde{\boldsymbol{\pi}}_{x,t}(\gamma) \, \mathrm{d}\tilde\mu(x,t).$$

It follows that, for $\tilde\mu$-a.e. $(x,t)$, $|\dot\gamma_t|$ is $\tilde{\boldsymbol{\pi}}_{x,t}$-equivalent to a constant. By the definition of $v$, this gives (6.12) with $v$ in place of $\bar v$. Using the coincidence of $v$ and $\bar v$ we conclude. □

# 7 Weighted energy functionals along absolutely continuous curves

Let $\mathfrak{m}$ be a reference measure in $X$ such that $(X, \mathsf{d}, \mathfrak{m})$ is a metric measure space according to Section 5.5, and set $\tilde{\mathfrak{m}} := \mathfrak{m} \otimes \lambda$, with $\lambda = \mathscr{L}^1|_{[0,1]}$. Let $\mathfrak{Q} : [0,1] \times [0,\infty) \to [0,\infty]$ be a lower semicontinuous function satisfying

$$\lim_{r \downarrow 0} r\, \mathfrak{Q}(s,r) = 0 \quad \text{for every } s \in [0,1]. \tag{7.1}$$

Our typical example will be of the form

$$\mathfrak{Q}(s,r) = \omega(s) Q(r). \tag{7.2}$$

Let us fix an exponent $p \in (1,\infty)$ and let us consider a curve $\mu \in \mathrm{AC}^p([0,1]; (\mathscr{P}(X), W_p))$. We denote $\tilde{\mu} = \int_0^1 \mu_s \, \mathrm{d}\lambda(s) \in \mathscr{P}(\tilde{X})$ and by $v \in L^p(\tilde{X}, \tilde{\mu})$ the minimal $p$-velocity density of $\mu$. We suppose that

$$\tilde{\mu} = \varrho \tilde{\mathfrak{m}} \ll \tilde{\mathfrak{m}}, \quad \text{so that} \quad \varrho(s,\cdot) = \varrho_s(\cdot) = \frac{\mathrm{d}\mu_s}{\mathrm{d}\mathfrak{m}} \quad \text{for } \lambda\text{-a.e. } s \in (0,1), \tag{7.3}$$

where $\mathrm{d}\nu/\mathrm{d}\mathfrak{m}$ denotes the Radon-Nikodym density of the measure $\nu$ with respect to $\mathfrak{m}$. Then we introduce the functional

$$\mathcal{A}_{\mathfrak{Q}}(\mu; \mathfrak{m}) := \int_{\tilde{X}} \mathfrak{Q}(s, \varrho_s)\, v^p \, \mathrm{d}\tilde{\mu} = \int_{\tilde{X}} \varrho\, \mathfrak{Q}(s, \varrho_s)\, v^p \, \mathrm{d}\tilde{\mathfrak{m}}. \tag{7.4}$$

We omit to indicate the dependence on $p$ in the notation of the functional $\mathcal{A}_{\mathfrak{Q}}$, since $p$ will be fixed throughout this section. Notice that when $\mathfrak{Q}(s,r) = 1$ we have the usual action $\int_{\tilde{X}} v^p \, \mathrm{d}\tilde{\mu} = \mathcal{A}_p(\mu)$, the functional is independent of $\mathfrak{m}$ and it makes sense even for curves not contained in $\mathscr{P}^{ac}(X, \mathfrak{m})$.

If $\boldsymbol{\pi}$ is a dynamic plan $p$-tightened to $\mu$ (recall that this means $\mathscr{A}_p(\boldsymbol{\pi}) = \mathcal{A}_p(\mu)$), thanks to (6.12) we have the equivalent expression

$$\mathcal{A}_{\mathfrak{Q}}(\mu; \mathfrak{m}) = \iint_0^1 \mathfrak{Q}(s, \varrho_s(\gamma(s)))|\dot{\gamma}|^p(s) \, \mathrm{d}s \, \mathrm{d}\boldsymbol{\pi}(\gamma). \tag{7.5}$$

**Theorem 7.1 (Stability)** *Let $(\mu_n)$ be a sequence in $\mathrm{AC}^p([0,1]; (\mathscr{P}(X), W_p))$ with $\tilde{\mu}_n = \varrho_n \tilde{\mathfrak{m}} \ll \tilde{\mathfrak{m}}$, such that*

$$\lim_{n \to \infty} \varrho_n = \varrho_\infty \quad \textit{strongly in } L^1(\tilde{X}, \tilde{\mathfrak{m}}) \tag{7.6}$$

*and, writing $\mu_\infty := \varrho_\infty \tilde{\mathfrak{m}}$,*

$$\limsup_{n \to \infty} \mathcal{A}_p(\mu_n) \le \mathcal{A}_p(\mu_\infty) < \infty. \tag{7.7}$$

*Then*

$$\liminf_{n\to\infty} \mathcal{A}_{\mathfrak{Q}}(\mu_n;\mathfrak{m}) \geq \mathcal{A}_{\mathfrak{Q}}(\mu_\infty;\mathfrak{m}), \tag{7.8}$$

*and, whenever* $\mathfrak{Q}$ *is continuous and bounded,*

$$\lim_{n\to\infty} \mathcal{A}_{\mathfrak{Q}}(\mu_n;\mathfrak{m}) = \mathcal{A}_{\mathfrak{Q}}(\mu_\infty;\mathfrak{m}). \tag{7.9}$$

*Proof.* We assume, possibly extracting a subsequence, that the lim inf in (7.8) is a finite limit. Still up to extraction of a subsequence, (7.6) and equi-continuity in the weak topology yield

$$\mu_{n,s} \to \mu_{\infty,s} \text{ weakly in } \mathscr{P}(X) \text{ for every } s \in [0,1]. \tag{7.10}$$

We can apply [3, Thm. 5.4.4] first to the sequence $(\tilde\mu_n, v_n)$, with $v_n$ equal to the velocity densities of $\mu_n$. We find that the family of plans $\boldsymbol{\nu}_n := (\boldsymbol{i} \times v_n)_\sharp \tilde\mu_n$ has a limit point $\boldsymbol{\nu}_\infty$ in $\mathscr{P}(\tilde X \times [0,\infty))$ whose first marginal is $\tilde\mu_\infty$ and satisfies

$$\int_{\tilde X\times[0,\infty)} |y|^p \, d\boldsymbol{\nu}_\infty(x,s,y) \leq \liminf_{n\to\infty} \int_{\tilde X\times[0,\infty)} |y|^p \, d\boldsymbol{\nu}_n(x,s,y) = \mathcal{A}_p(\mu_\infty).$$

If $(\nu_{x,s})_{(x,s)\in\tilde X} \subset \mathscr{P}(\tilde X)$ is the disintegration of $\boldsymbol{\nu}_\infty$ w.r.t. $\tilde\mu_\infty$, setting

$$v_\infty(x,s) := \int_0^\infty y \, \mathrm{d}\nu_{x,s}(y),$$

we obtain from the previous inequality and Jensen's inequality that $v_\infty$ belongs to $L^p(\tilde X, \tilde\mu_\infty)$, with $\|v_\infty\|^p_{L^p(\tilde X,\tilde\mu_\infty)} \leq \mathcal{A}_p(\mu_\infty)$. For every $\varphi \in \mathrm{Lip}_b(X)$ and $0 \leq r < s \leq 1$, we can use the upper semicontinuity of $|\mathrm{D}^*\varphi|$ to pass to the limit in the family of inequalities corresponding to (6.4)

$$\Big|\int_X \varphi \, \mathrm{d}\mu_{n,s} - \int_X \varphi \, \mathrm{d}\mu_{n,r}\Big| \leq \int_{X\times(r,s)} |\mathrm{D}^*\varphi| \, v_n \, \mathrm{d}\tilde\mu_n = \int_{\tilde X\times(r,s)} |\mathrm{D}^*\varphi|(x) y \, d\boldsymbol{\nu}_n(x,s,y),$$

obtaining

$$\Big|\int_X \varphi \, \mathrm{d}\mu_{\infty,s} - \int_X \varphi \, \mathrm{d}\mu_{\infty,r}\Big| \leq \int_{X\times(r,s)} |\mathrm{D}^*\varphi| \, v_\infty \, \mathrm{d}\tilde\mu_\infty.$$

It follows that $v_\infty$ is a velocity density for the curve $\mu_\infty$, and therefore (by minimality) is the $p$-velocity density, so that $\|v_\infty\|^p_{L^p(\tilde X,\tilde\mu_\infty)} = \mathcal{A}_p(\mu_\infty)$. From (7.7) and Jensen's inequality we get

$$\limsup_{n\to\infty} \int_{\tilde X\times[0,\infty)} |y|^p \, d\boldsymbol{\nu}_n = \limsup_{n\to\infty} \mathcal{A}_p(\mu_n) \leq \mathcal{A}_p(\mu_\infty) = \|v_\infty\|^p_{L^p(\tilde X,\tilde\mu_\infty)} \leq \int_{\tilde X\times[0,\infty)} |y|^p \, d\boldsymbol{\nu}_\infty,$$

so that [3, Thm. 5.4.4] yields that $\boldsymbol{\nu}_\infty = (\boldsymbol{i} \times v_\infty)_\sharp \mu_\infty$.

Denoting now by $(\tilde{x}, v, r) = (x, s, v, r)$ the coordinates in $\tilde{X} \times \mathbb{R} \times \mathbb{R}$, let us now consider the plans $\boldsymbol{\sigma}_n := (\tilde{x}, v, \varrho_n(\tilde{x}))_\sharp \boldsymbol{\nu}_n = (\tilde{x}, v_n(\tilde{x}), \varrho_n(\tilde{x}))_\sharp \tilde{\mu}_n \in \mathscr{P}(\tilde{X} \times [0, \infty) \times [0, \infty))$. From (7.6) we obtain the existence of $\zeta : [0, \infty) \to [0, \infty)$ with $\zeta(r) \to +\infty$ as $r \to \infty$ such that

$$Z \geq \int_{\tilde{X}} \varrho_n \zeta(\varrho_n) \, \mathrm{d}\tilde{\mathfrak{m}} = \int \zeta(r) \boldsymbol{\sigma}_n(\tilde{x}, v, r),$$

so that $\boldsymbol{\sigma}_n$ are tight (the marginals of $\boldsymbol{\sigma}_n$ with respect to the block of variables $(\tilde{x}, v)$ are $\boldsymbol{\nu}_n$, thus are tight). We can then extract a subsequence (still denoted by $\boldsymbol{\sigma}_n$) weakly converging to $\boldsymbol{\sigma}_\infty$, whose marginal w.r.t. $(x, v)$ is $\boldsymbol{\nu}_\infty$.

The strong $L^1$ convergence in (7.6) also shows that for every $\zeta \in \mathrm{C}_b(\tilde{X} \times \mathbb{R})$ one has

$$\begin{aligned}\int \zeta(\tilde{x}, r) \, \mathrm{d}\boldsymbol{\sigma}_\infty(\tilde{x}, v, r) &= \lim_{n \to \infty} \int \zeta(\tilde{x}, r) \, \mathrm{d}\boldsymbol{\sigma}_n(\tilde{x}, v, r) = \lim_{n \to \infty} \int_{\tilde{X}} \zeta(\tilde{x}, \varrho_n(\tilde{x})) \varrho_n(\tilde{x}) \, \mathrm{d}\tilde{\mathfrak{m}}(\tilde{x}) \\ &= \int_{\tilde{X}} \zeta(\tilde{x}, \varrho_\infty(\tilde{x})) \varrho_\infty(\tilde{x}) \, \mathrm{d}\tilde{\mathfrak{m}}(\tilde{x}) = \int_{\tilde{X}} \zeta(\tilde{x}, \varrho_\infty(\tilde{x})) \, \mathrm{d}\tilde{\mu}_\infty(\tilde{x}),\end{aligned}$$

so that $(\boldsymbol{i} \times \varrho_\infty)_\sharp \tilde{\mu}_\infty$ is the marginal w.r.t. $(\tilde{x}, r)$ of $\boldsymbol{\sigma}_\infty$.

Hence, disintegrating $\boldsymbol{\sigma}_\infty$ with respect to $\tilde{\mu}_\infty$ we obtain $\boldsymbol{\sigma}_\infty = (\tilde{x}, v_\infty(\tilde{x}), \varrho_\infty(\tilde{x}))_\sharp \mu_\infty$. Since the map

$$\tilde{X} \times [0, \infty) \times [0, \infty] \ni (x, s, v, r) \mapsto \mathfrak{Q}(s, r) v^p$$

is lower semicontinuous and nonnegative we get (7.8), since

$$\begin{aligned}\liminf_{n \to \infty} \mathcal{A}_{\mathfrak{Q}}(\mu_n; \mathfrak{m}) &= \liminf_{n \to \infty} \int \mathfrak{Q}(s, r) v^p \, \mathrm{d}\boldsymbol{\sigma}_n(x, s, v, r) \geq \int \mathfrak{Q}(s, r) v^p \, \mathrm{d}\boldsymbol{\sigma}_\infty(x, s, v, r) \\ &= \int_{\tilde{X}} \mathfrak{Q}(s, \varrho_\infty) v_\infty^p \, \mathrm{d}\tilde{\mu}_\infty = \mathcal{A}_{\mathfrak{Q}}(\mu_\infty; \mathfrak{m}).\end{aligned}$$

When $\mathfrak{Q}$ is continuous and bounded, say by a constant $M > 0$, then the thesis follows by (7.7), that yields the limit

$$\lim_{n \to \infty} \int M \, v^p \, \mathrm{d}\boldsymbol{\sigma}_n(\tilde{x}, v, r) = \int M \, v^p \, \mathrm{d}\boldsymbol{\sigma}_\infty(\tilde{x}, v, r),$$

by the lower semicontinuity property (7.8) applied to the functionals $\mathcal{A}_{\mathfrak{Q}}$ and $\mathcal{A}_{M - \mathfrak{Q}}$, and by Remark 4.3. □

# 8 Dynamic Kantorovich potentials, continuity equation and dual weighted Cheeger energies

In this section we will still consider a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ according to the definition of Section 5.5 and we will focus on the particular case when

$$p = 2 \text{ and the Cheeger energy } \mathsf{Ch} \text{ is quadratic (see (5.28))}, \tag{8.1}$$

so that $\mathbb{V} = W^{1,2}(X, \mathfrak{m})$ is a separable Hilbert space; we will also consider continuous curves $(\mu_s)_{s\in[0,1]} \subset \mathscr{P}(X)$ with uniformly bounded densities w.r.t. $\mathfrak{m}$, i.e.

$$\mu \in \mathrm{C}([0,1];\mathscr{P}(X)), \qquad \mu_s = \varrho_s\mathfrak{m}, \qquad R := \sup_s \|\varrho_s\|_{L^\infty(X,\mathfrak{m})} < \infty. \tag{8.2}$$

Our main aim is to show that the weighted energies $\mathcal{E}_{\varrho_s}$ (or better, their dual forms $\mathcal{E}^*_{\varrho_s}$) provide a useful characterization of the minimal 2-velocity of absolutely continuous curves $\mu$ in $(\mathscr{P}_2(X), W_2)$, now not only in the form of inequality as in (6.4), but in the form of equality, see (8.5).

**Lemma 8.1 (Absolute continuity w.r.t. $\mathbb{V}'_{\mathcal{E}}$)** *Let $\mu$ as in* (8.2) *and let $v \in L^2(\tilde X, \tilde\mu)$ be a velocity density for $\mu$, i.e. satisfying* (6.4)*. Then for every $\varphi \in \mathbb{V}$ one has*

$$\Big|\int_X \varphi\,(\varrho_t - \varrho_s)\,\mathrm{d}\mathfrak{m}\Big| \le \int_{X\times(s,t)} |\mathrm{D}\varphi|_w\, v\,\mathrm{d}\tilde\mu \quad \textit{for every} \ \ 0 \le s < t \le 1. \tag{8.3}$$

*In addition $\varrho : [0,1] \to L^1_+ \cap L^\infty_+(X,\mathfrak{m})$ has finite 2-energy with respect to the $\mathbb{V}'_{\mathcal{E}}$ norm, more precisely*

$$\|\varrho_s - \varrho_t\|^2_{\mathbb{V}'_{\mathcal{E}}} \le R(t-s)\int_{X\times(s,t)} v^2\,\mathrm{d}\tilde\mu \quad \textit{for every } 0 \le s < t \le 1. \tag{8.4}$$

*Proof.* In order to prove (8.3) we simply approximate $\varphi$ with a sequence of Lipschitz functions $\varphi_n$ strongly converging to $\varphi$ in $L^2(X,\mathfrak{m})$ such that $|\mathrm{D}^*\varphi_n| \to |\mathrm{D}\varphi|_w$ in $L^2(X,\mathfrak{m})$ and we pass to the limit in (6.4), using the fact that $\mu_t = \varrho_t\mathfrak{m}$ with uniformly bounded densities.

By (8.3) it follows that

$$\begin{aligned}\Big|\int_X (\varrho_t - \varrho_s)\varphi\,\mathrm{d}\mathfrak{m}\Big| &\le \int_s^t \Big(\int_X |\mathrm{D}\varphi|^2_w\,\varrho_r\mathrm{d}\mathfrak{m}\Big)^{1/2}\Big(\int_X v_r^2\,\varrho_r\,\mathrm{d}\mathfrak{m}\Big)^{1/2}\mathrm{d}r\\ &\le R^{1/2}\Big(\int_X |\mathrm{D}\varphi|^2_w\,\mathrm{d}\mathfrak{m}\Big)^{1/2}\int_s^t\Big(\int_X v_r^2\,\varrho_r\,\mathrm{d}\mathfrak{m}\Big)^{1/2}\mathrm{d}r,\end{aligned}$$

and since $\varphi$ is arbitrary we obtain (8.4). □

**Theorem 8.2 (Dual Kantorovich potentials and links with the minimal velocity)** *Let us assume that* $\mathsf{Ch}$ *is a quadratic form and let $\mu$ as in* (8.2)*. Then $\mu$ belongs to* $\mathrm{AC}^2([0,1];(\mathscr{P}(X),W_2))$ *if and only if there exists $\ell \in L^2(0,1;\mathbb{V}')$ such that*

$$\int_X \varphi\,(\varrho_t - \varrho_s)\,\mathrm{d}\mathfrak{m} = \int_s^t \langle \ell(r), \varphi\rangle\,\mathrm{d}r \quad \textit{for every } \varphi \in \mathbb{V}, \tag{8.5}$$

*and, recalling the definition* (5.39) *of $\mathcal{E}^*_{\varrho_r}$,*

$$\int_0^1 \mathcal{E}^*_{\varrho_r}(\ell(r),\ell(r))\,\mathrm{d}r < \infty. \tag{8.6}$$

*In particular $\ell(r) \in \mathbb{V}'_{\varrho_r}$ for $\mathscr{L}^1$-a.e. $r \in (0,1)$ and, moreover, it is linked to the minimal velocity density $v$ of $\mu$ by*

$$\int_X v_r^2\, \varrho_r \,\mathrm{d}\mathfrak{m} = \mathcal{E}^*_{\varrho_r}(\ell(r), \ell(r)) \quad \text{for } \mathscr{L}^1\text{-a.e. } r \in (0,1), \tag{8.7}$$

$$v_r^2 = \Gamma_{\varrho_r}(\phi_r) \quad \mu_r\text{-a.e. in } X, \qquad \text{for } \mathscr{L}^1\text{-a.e. } r \in (0,1), \tag{8.8}$$

*where $\phi_r = -A^*_\varrho \ell(r) \in \mathbb{V}_{\varrho_r}$ is the solution of (5.40) with $\ell = \ell(r)$.*

*Proof.* If $\mu \in \mathrm{AC}^2([0,1]; (\mathscr{P}(X), W_2))$ then the existence of $\ell$ and (8.5) follow immediately by Lemma 8.1, Theorem 6.7 and the fact that $\mathbb{V}'$ is a separable Hilbert space. Differentiating (8.3) with $v$ equal to the minimal velocity density in a Lebesgue point for $s \mapsto \int_X |\mathrm{D}\varphi|_w v_s \varrho_s \,\mathrm{d}\mathfrak{m}$ and for a countable dense set of test functions $\varphi$ in $\mathbb{V}$ we get

$$\mathcal{E}^*_{\varrho_r}(\ell(r), \ell(r)) \le \int_X v_r^2\, \varrho_r \,\mathrm{d}\mathfrak{m} \quad \text{for } \mathscr{L}^1\text{-a.e. } r \in (0,1), \tag{8.9}$$

which in particular yields (8.6).

In order to prove the converse implication (and that equality holds in (8.7), as well as (8.8)), let us start from $\mu$ as in (8.2), satisfying (8.5) and (8.6) for some $\ell \in L^2(0,1;\mathbb{V}')$. Let us consider $\psi \in \mathrm{Lip}(X)$ with bounded support, the solution $\phi_r = -A^*_\varrho \ell(r) \in \mathbb{V}_{\varrho_r}$ of (5.40) with $\ell = \ell(r)$ and $\psi_{\varrho_r}$ the equivalence class associated to $\psi$ in $\mathbb{V}_{\varrho_r}$, so that

$$\langle \ell(r), \psi \rangle = \int \varrho_r \Gamma_{\varrho_r}(\phi_r, \psi_{\varrho_r}) \,\mathrm{d}\mathfrak{m}.$$

Now observe that (8.5) and (8.6) yield for every $0 \le s < t \le 1$

$$\Big| \int_X \psi \,\mathrm{d}\mu_t - \int_X \psi \,\mathrm{d}\mu_s \Big| \le \int_s^t \Big| \int_X \varrho_r \Gamma_{\varrho_r}(\phi_r, \psi_{\varrho_r}) \,\mathrm{d}\mathfrak{m} \Big| \,\mathrm{d}r \le \int_s^t \int_X \varrho_r \big(\Gamma_{\varrho_r}(\phi_r)\big)^{1/2} |\mathrm{D}^*\psi| \,\mathrm{d}\mathfrak{m} \,\mathrm{d}r$$

since for $\psi \in \mathrm{Lip}_b(X)$

$$\Gamma_{\varrho_r}(\psi_{\varrho_r}) \le \Gamma\big(\psi\big) = |\mathrm{D}\psi|_w^2 \le |\mathrm{D}^*\psi|^2 \quad \varrho_r\mathfrak{m}\text{-a.e. in } X.$$

In view of Remark 6.2, we see that $\hat{v}_r = \big(\Gamma_{\varrho_r}(\phi_r)\big)^{1/2}$ is a velocity density for the curve $\mu$. Applying Theorem 6.6 and (5.41) we get $\mu \in \mathrm{AC}^2([0,1]; (\mathscr{P}(X), W_2))$. In addition, since

$$\int_X \hat{v}_r^2\, \varrho_r \mathrm{d}\mathfrak{m} = \int_X \Gamma_{\varrho_r}(\phi_r) \varrho_r \,\mathrm{d}\mathfrak{m} = \mathcal{E}_{\varrho_r}(\phi_r, \phi_r) = \mathcal{E}^*_{\varrho_r}(\ell(r), \ell(r)) \quad \text{for } \mathscr{L}^1\text{-a.e. } r \in (0,1),$$

comparing with (8.9) we obtain that $\hat{v}$ is the minimal velocity density $v$, thus obtaining (8.7) and (8.8). □

# 9 The $\mathrm{RCD}^*(K,N)$ condition and its characterizations through weighted convexity and evolution variational inequalities

## 9.1 Green functions on intervals

We define the function $\mathsf{g} : [0,1] \times [0,1] \to [0,1]$ by

$$\mathsf{g}(s,t) := \begin{cases} (1-t)s & \text{if } s \in [0,t], \\ t(1-s) & \text{if } s \in [t,1], \end{cases} \tag{9.1}$$

so that for all $t \in (0,1)$ one has

$$-\frac{\partial^2}{\partial s^2}\mathsf{g}(s,t) = \delta_t \quad \text{in } \mathscr{D}'(0,1), \qquad \mathsf{g}(0,t) = \mathsf{g}(1,t) = 0. \tag{9.2}$$

It is not difficult to check that (see e.g. [52, Chap. 16]) the condition $u'' \geq f$ can be characterized in terms of an integral inequality involving $\mathsf{g}$.

**Lemma 9.1 (Integral formulation of $u'' \geq f$)** *Let $u \in \mathrm{C}([0,1])$ and $f \in L^1(0,1)$. Then*

$$u'' \geq f \quad \text{in } \mathscr{D}'(0,1), \tag{9.3}$$

*if and only if for every $0 \leq r_0 \leq r_1 \leq 1$ and $t \in [0,1]$ one has*

$$u((1-t)r_0 + tr_1) \leq (1-t)u(r_0) + tu(r_1) - (r_1 - r_0)^2 \int_0^1 f((1-s)r_0 + sr_1)\mathsf{g}(s,t)\,\mathrm{d}s. \tag{9.4}$$

*Proof.* In order to prove the implication from (9.3) to (9.4) it is not restrictive to assume $u \in \mathrm{C}^2([0,1])$ and $f \in \mathrm{C}([0,1])$. The proof of (9.4) follows easily from the elementary identity

$$u((1-t)r_0 + tr_1) = (1-t)u(r_0) + tu(r_1) - \int_0^1 u''((1-s)r_0 + sr_1)\mathsf{g}(s,t)\,\mathrm{d}s.$$

Concerning the converse implication, we choose $r_1 := r + h$, $r_0 = r - h$ and $t = \frac{1}{2}$ obtaining

$$\frac{1}{2}u(r+h) + \frac{1}{2}u(r-h) - u(r) \geq 4h^2 \int_0^1 f(r - h + 2hs)\mathsf{g}(s, 1/2)\,\mathrm{d}s.$$

Multiplying by $2h^{-2}$ and by a nonnegative test function $\zeta \in \mathrm{C}_c^\infty(0,1)$ we get after an integration

$$\frac{1}{h^2}\int_0^1 u(r)\big(\zeta(r+h) + \zeta(r-h) - 2\zeta(r)\big)\,\mathrm{d}r \geq 8\int_0^1 \mathsf{g}(s,1/2)\Big(\int_0^1 f(r-h+2hs)\zeta(r)\,\mathrm{d}r\Big)\,\mathrm{d}s.$$

Passing to the limit as $h \downarrow 0$ we obtain

$$\int_0^1 u\,\zeta''\,\mathrm{d}r \geq 8\int_0^1 \mathsf{g}(s,1/2)\,\mathrm{d}s \int_0^1 f\,\zeta\,\mathrm{d}r = \int_0^1 f\,\zeta\,\mathrm{d}r. \qquad \square$$

In the next lemma we show that functions satisfying the weighted convexity condition (9.4) are locally Lipschitz, this will allow us to apply Lemma 9.1.

**Lemma 9.2** *Let $\mathfrak{D} \subset \mathbb{R}$, $\mathfrak{D} \neq \{0\}$, be a $\mathbb{Q}$-vector space and let $u : (0,1) \cap \mathfrak{D} \to \mathbb{R}$ satisfy* (9.4) *for some $f \in L^1_{\mathrm{loc}}(0,1)$, for every $r_0$, $r_1 \in (0,1) \cap \mathfrak{D}$, $t \in [0,1]$ such that $(1-t)r_0 + tr_1 \in \mathfrak{D}$. Then $u$ is locally Lipschitz in $(0,1)$, more precisely for every closed subinterval $[a,b] \subset (0,1)$ there exists $C \geq 0$ such that*

$$|u(x) - u(y)| \leq C|x - y| \qquad \forall x,\, y \in (a,b) \cap \mathfrak{D}. \tag{9.5}$$

*Proof.* Since the statement is local and $\mathfrak{D}$ is dense, we can assume with no loss of generality that $f \in L^1(0,1)$, that $0$, $1 \in \mathfrak{D}$ and that (9.4) holds $r_0$, $r_t$, $r_1 \in [0,1] \cap \mathfrak{D}$, with $r_t := (1-t)r_0 + tr_1$. First of all note that (9.4) is equivalent to the following control on the incremental ratios: for every $r_0$, $r_t$, $r_1 \in [0,1] \cap \mathfrak{D}$ one has

$$\frac{u(r_t) - u(r_0)}{r_t - r_0} \leq \frac{u(r_1) - u(r_t)}{r_1 - r_t} - \frac{r_1 - r_0}{t(1-t)} \int_0^1 f(r_s)\mathsf{g}(s,t)\,\mathrm{d}s. \tag{9.6}$$

Observing that $0 \leq \mathsf{g}(s,t) \leq t(1-t)$, we can easily estimate the remainder in the last inequality by

$$\left|\frac{r_1 - r_0}{t(1-t)} \int_0^1 f(r_s)\mathsf{g}(s,t)\,\mathrm{d}s\right| \leq \int_{r_0}^{r_1} |f(r)|\,\mathrm{d}r = \|f\|_{L^1(r_0,r_1)}. \tag{9.7}$$

Given $a < b \in (0,1) \cap \mathfrak{D}$, for every $x$, $y \in \mathfrak{D} \cap (a,b)$, $x < y$, we want to use (9.6) iteratively in order to estimate the difference quotient $|u(x) - u(y)|/|x - y|$.
Applying (9.6) with $r_0 = 0$, $r_1 = x$, $r_t = a$ we obtain

$$\frac{u(a) - u(0)}{a} \leq \frac{u(x) - u(a)}{x - a} + \|f\|_{L^1(0,x)}. \tag{9.8}$$

Analogously, choosing $r_0 = a$, $r_1 = y$, $r_t = x$ in (9.6) yields

$$\frac{u(x) - u(a)}{x - a} \leq \frac{u(y) - u(x)}{y - x} + \|f\|_{L^1(a,y)}. \tag{9.9}$$

Putting together (9.8) and (9.9) we obtain the desired lower bound

$$\frac{u(y) - u(x)}{y - x} \geq \frac{u(a) - u(0)}{a} - 2\|f\|_{L^1(0,1)}.$$

Along the same lines one gets also the upper bound

$$\frac{u(y) - u(x)}{y - x} \leq \frac{u(1) - u(b)}{1 - b} + 2\|f\|_{L^1(0,1)}.$$

Since the last two estimates hold for every $x$, $y \in \mathfrak{D} \cap (a,b)$ with $x \neq y$, the proof is complete. □

The next lemma provides a subdifferential inequality, in a quantitative form involving $f$.

**Lemma 9.3** *Suppose that $u \in \mathrm{C}([0,1])$ satisfies $u'' \geq f$ in $\mathscr{D}'(0,1)$ for some $f \in L^1(0,1)$. Then, setting $u'(0_+) := \limsup_{t\downarrow 0}(u(t)-u(0))/t$, we get*

$$u(1) - u(0) - u'(0_+) \geq \int_0^1 f(s)(1-s)\,\mathrm{d}s. \tag{9.10}$$

*Proof.* Notice that by (9.4)

$$u(t) - u(0) \leq t(u(1)-u(0)) - \int_0^1 f(s)\mathsf{g}(s,t)\,\mathrm{d}s.$$

Dividing by $t$ and passing to the limit as $t \downarrow 0$, since $\lim_{t\downarrow 0} t^{-1}\mathsf{g}(s,t) = 1-s$ pointwise in $(0,1]$ and $0 \leq t^{-1}\mathsf{g}(s,t) \leq (1-s)$, we get (9.10). □

A similar result holds for the solutions $u$ of the differential inequality

$$u \in \mathrm{C}([0,1]), \quad u'' + \kappa\, u \leq 0 \quad \text{in } \mathscr{D}'(0,1), \qquad \kappa \in \mathbb{R}. \tag{9.11}$$

In this case, choosing $[r_0, r_1] \subset [0,1]$ with $\delta = r_1 - r_0 \in (0,1]$, we can compare the function $t \mapsto u((1-t)r_0 + tr_1)$, which solves $w'' + \kappa\delta^2 w \leq 0$ in $\mathscr{D}'(0,1)$, with the solution of the Dirichlet problem

$$v'' + \kappa\,\delta^2\, v = 0 \quad \text{in } (0,1), \quad v(0) = u(r_0),\ v(1) = u(r_1), \tag{9.12}$$

given by

$$v(t) = u(r_0)\frac{\sin(\omega(1-t))}{\sin(\omega)} + u(r_1)\frac{\sin(\omega t)}{\sin(\omega)} \qquad \text{if } \kappa\,\delta^2 = \omega^2 \in (0,\pi^2), \tag{9.13}$$

and by

$$v(t) = u(r_0)\frac{\sinh(\omega(1-t))}{\sinh(\omega)} + u(r_1)\frac{\sinh(\omega t)}{\sinh(\omega)} \qquad \text{if } \kappa\,\delta^2 = -\omega^2 < 0, \tag{9.14}$$

observing that the comparison principle gives $u((1-t)r_0 + tr_1) \geq v(t)$ for every $t \in [0,1]$.

By introducing the factors

$$\sigma_\kappa^{(t)}(\delta) := \begin{cases} +\infty & \text{if } \kappa\,\delta^2 \geq \pi^2, \\ \dfrac{\sin(\omega t)}{\sin(\omega)} & \text{if } \kappa\,\delta^2 = \omega^2 \in (0,\pi^2), \\ t & \text{if } \kappa = 0, \\ \dfrac{\sinh(\omega t)}{\sinh(\omega)} & \text{if } \kappa\,\delta^2 = -\omega^2 < 0, \end{cases} \tag{9.15}$$

the solution $v$ of (9.12), thanks to (9.13) and (9.14), can be expressed in the form

$$v(t) = \sigma_\kappa^{(1-t)}(r_1 - r_0)u(r_0) + \sigma_\kappa^{(t)}(r_1 - r_0)u(r_1)$$

and the following result holds (see for instance [52, Thm. 14.28]):

**Lemma 9.4** *Let $u \in \mathrm{C}([0,1])$ nonnegative and $\kappa \in \mathbb{R}$. Then $u'' + \kappa u \le 0$ in $\mathscr{D}'(0,1)$ if and only if for every $t \in [0,1]$ and for every $0 \le r_0 < r_1 \le 1$ with $\kappa(r_1 - r_0)^2 < \pi^2$ one has*

$$u((1-t)r_0 + tr_1) \ge \sigma_\kappa^{(1-t)}(r_1 - r_0)\, u(r_0) + \sigma_\kappa^{(t)}(r_1 - r_0)\, u(r_1). \tag{9.16}$$

In the same spirit of Lemma 9.2, where we proved that "weighted convex" functions are locally Lipschitz, next we show that functions satisfying the concavity condition (9.16) have the same regularity; this will allow to apply Lemma 9.4.

**Lemma 9.5** *Let $\mathfrak{D} \subset \mathbb{R}$ be a $\mathbb{Q}$-vector space with $\mathfrak{D} \neq \{0\}$. Let $\kappa \in \mathbb{R}$ and let $u : [0,1] \cap \mathfrak{D} \to \mathbb{R}$ satisfy (9.16) for every $r_0$, $r_1 \in [0,1] \cap \mathfrak{D}$ with $\kappa(r_1 - r_0)^2 < \pi^2$ and $t \in [0,1]$ such that $(1-t)r_0 + tr_1 \in \mathfrak{D}$. Then the following hold.*

*(a) There exists $\varepsilon_0 = \varepsilon_0(\kappa) > 0$ with the following property: if*

$$\sup_{n \in \mathbb{N}, n \le \lfloor \frac{1}{\varepsilon} \rfloor} u(n\varepsilon) < \infty, \tag{9.17}$$

*for some $\varepsilon \in (0, \varepsilon_0) \cap \mathfrak{D}$, then $\sup_{r \in \mathfrak{D} \cap [0,1]} u(r) < \infty$.*

*(b) There exists $\varepsilon_0 = \varepsilon_0(\kappa) > 0$ with the following property: if*

$$\sup_{n \in \mathbb{N}, n \le \lfloor \frac{1}{\varepsilon} \rfloor} |u(n\varepsilon)| < \infty, \tag{9.18}$$

*for some $\varepsilon \in (0, \varepsilon_0) \cap \mathfrak{D}$, then $\sup_{r \in \mathfrak{D} \cap [0,1]} |u(r)| < \infty$.*

*(c) If in addition $u : [0,1] \cap \mathfrak{D} \to \mathbb{R}$ is locally bounded then $u$ is locally Lipschitz in $(0,1)$, i.e. for every $r \in (0,1) \cap \mathfrak{D}$ there exist $\varepsilon$, $C > 0$ such that $[r - \varepsilon, r + \varepsilon] \subset [0,1]$ and*

$$|u(x) - u(y)| \le C|x - y| \quad \forall x,\, y \in [r - \varepsilon, r + \varepsilon] \cap \mathfrak{D}. \tag{9.19}$$

*Proof.* For simplicity of notation we can assume $\mathbb{Q} \subset \mathfrak{D}$.

(a) Assume by contradiction the existence of a sequence $(s_n) \subset (0,1) \cap \mathfrak{D}$ such that $u(s_n) \to +\infty$. Clearly there exists $\bar{s} \in [0,1]$ such that, up to subsequences, $s_n \to \bar{s}$; let us start by assuming $\bar{s} = 0$, without loss of generality we can also assume that $s_n \in [0, \varepsilon/4]$ for every $n \in \mathbb{N}$ ($\varepsilon_0$ will be chosen later just depending on $\kappa$). Applying (9.16) with $r_0 = s_n$, $r_t = \varepsilon$ and $r_1 = 2\varepsilon$ we get

$$u(\varepsilon) \ge \sigma_\kappa^{(1-t_n)}(2\varepsilon - s_n)\, u(s_n) + \sigma_\kappa^{(t_n)}(2\varepsilon - s_n)\, u(2\varepsilon), \tag{9.20}$$

where $t_n = \frac{\varepsilon - s_n}{2\varepsilon - s_n} \to \frac{1}{2}$ as $n \to \infty$.
By a Taylor expansion at 0 of the function $r \to \sigma_\kappa^{(1-t_n)}(r)$ it is easy to see that

$$\sigma_\kappa^{(1-t_n)}(2\varepsilon - s_n) = (1 - t_n) + o_\kappa(\varepsilon_0) \ge \frac{1}{4}, \tag{9.21}$$

provided $\varepsilon_0 = \varepsilon_0(\kappa) > 0$ is chosen small enough. But then, observing that $\inf_n \sigma_\kappa^{(t_n)}(2\varepsilon - s_n)\, u(2\varepsilon) > -\infty$, combining (9.20) and (9.21) we get

$$u(\varepsilon) \geq \frac{1}{4} u(s_n) + \sigma_\kappa^{(t_n)}(2\varepsilon - s_n)\, u(2\varepsilon) \to +\infty \quad \text{ as } n \to +\infty,$$

contradicting (9.17). If instead $s_n \to 1$, applying (9.16) with $r_0 = 1 - 2\varepsilon$, $r_t = 1 - \varepsilon$ and $r_1 = s_n$, with analogous arguments we get

$$u(1-\varepsilon) \geq \sigma_\kappa^{(1-t_n)}(s_n - (1-2\varepsilon))\, u(1-2\varepsilon) + \frac{1}{4} u(s_n) \to +\infty \quad \text{ as } n \to +\infty,$$

contradicting (9.17). Finally, if $\lim_n s_n = \bar{s} \in (0,1)$ we can repeat the first argument with 0 replaced by $\bar{s}$ thus reaching a contradiction. The proof of the first statement is then complete.

(b) Let $\varepsilon_0 = \varepsilon_0(\kappa) > 0$ be as above. Since by the first statement we already know that $u$ is uniformly bounded above, here it is enough to prove a uniform bound from below. Applying (9.16) to $r_0 = n\varepsilon$ and $r_1 = (n+1)\varepsilon$ for every $n \in \mathbb{N} \cap [0, \lfloor \frac{1}{\varepsilon} \rfloor]$, we get that

$$u(r) \geq \sigma_\kappa^{(1-t_r)}(\varepsilon)\, u(n\varepsilon) + \sigma_\kappa^{(t_r)}(\varepsilon)\, u((n+1)\varepsilon) \geq -C \sup_{n \in \mathbb{N}, n \leq \lfloor \frac{1}{\varepsilon} \rfloor} |u(n\varepsilon)| > -\infty\ ,$$

for every $r \in [n\varepsilon, (n+1)\varepsilon] \cap \mathfrak{D}$, for some $C > 0$ independent of $n$. Applying the same argument to $r_0 = \lfloor \frac{1}{\varepsilon} \rfloor \varepsilon$, $r_1 = 1$ we also obtain a uniform lower bound on $[\lfloor \frac{1}{\varepsilon} \rfloor, 1] \cap \mathfrak{D}$ and the conclusion follows.

(c) Since the statement is local and $\mathfrak{D}$ is dense, we can assume with no loss of generality that $u : [0,1] \cap \mathfrak{D} \to \mathbb{R}$ is bounded, that $0, 1 \in \mathfrak{D}$ and that (9.16) holds for every $r_0, r_1 \in [0,1] \cap \mathfrak{D}$ with $\kappa(r_1 - r_0)^2 < \pi^2$ and $t \in [0,1] \cap \mathbb{Q}$. First of all note that (9.16) is equivalent to the following control on distorted incremental ratios: for every $r_0, r_1 \in [0,1] \cap \mathfrak{D},\ t \in [0,1] \cap \mathbb{Q}$ with $\kappa(r_1 - r_0)^2 < \pi^2$ it holds

$$\frac{u(r_t) - \frac{1}{1-t}\sigma_\kappa^{(1-t)}(r_1 - r_0)\ u(r_0)}{r_t - r_0} \geq \frac{\frac{1}{t}\sigma_\kappa^{(t)}(r_1 - r_0)\ u(r_1) - u(r_t)}{r_1 - r_t}, \tag{9.22}$$

where $r_t := (1-t)r_0 + t r_1 \in [0,1] \cap \mathfrak{D}$.

Given $r \in (0,1) \cap \mathfrak{D}$, $\varepsilon > 0$ with $\varepsilon < \min\{r, 1-r\}$ and $4\kappa\varepsilon^2 \leq \pi^2$, and $x, y \in \mathfrak{D} \cap [r-\varepsilon, r+\varepsilon]$, $x < y$, we want to use (9.22) iteratively in order to estimate the difference quotient $|u(x) - u(y)|/|x-y|$. We will prove that this is possible provided $\varepsilon$ is sufficiently small.

At first apply (9.22) with $r_0 = 0$, $r_1 = x$, $r_t = r - \varepsilon$. Noting that $1 - t = \frac{x - (r-\varepsilon)}{x} \leq C_r \varepsilon$, with a first order Taylor expansion at $t = 1$ of the explicit expression (9.15) of $\frac{1}{1-t}\sigma_\kappa^{(1-t)}(x)$ one checks the existence of $C_r > 0$, $\varepsilon_r > 0$ satisfying (with the above choice of $t = t(x,r)$)

$$\left| \frac{1}{1-t}\sigma_\kappa^{(1-t)}(x) \right| \leq C_r \quad \text{for all } x \in [r-\varepsilon, r+\varepsilon], \text{ for every } \varepsilon \in (0, \varepsilon_r). \tag{9.23}$$

Analogously, possibly enlarging $C_r$ and reducing $\varepsilon_r$ we can also achieve

$$\left|\frac{1}{t}\sigma_\kappa^{(t)}(x) - 1\right| \le C_r(x - (r-\varepsilon)) \text{ for every } \varepsilon \in (0, \varepsilon_r). \tag{9.24}$$

The combination of (9.22), (9.23) and (9.24) gives

$$\frac{u(r-\varepsilon) + C_r|u(0)|}{r-\varepsilon} \ge \frac{u(x) - u(r-\varepsilon)}{x - (r-\varepsilon)} - C_r \quad \text{for every } \varepsilon \in (0, \varepsilon_r) \cap \mathfrak{D}. \tag{9.25}$$

Observing that $|\frac{1}{t}\sigma_\kappa^{(t)}(y - (r-\varepsilon)) - 1| \le C_r t(y - (r-\varepsilon))$, applying (9.22) with $r_0 = r - \varepsilon$, $r_1 = y$, $r_t = x$, yields

$$\frac{u(x) - u(r-\varepsilon)}{x - (r-\varepsilon)} + C_r \ge \frac{u(y) - u(x)}{y - x}\ , \text{ for every } \varepsilon \in (0, \varepsilon_r) \cap \mathfrak{D}. \tag{9.26}$$

Putting together (9.25) and (9.26) we obtain the desired upper bound

$$\frac{u(y) - u(x)}{y - x} \le \frac{u(r-\varepsilon) + C_r|u(0)|}{r - \varepsilon} + 2C_r.$$

Along the same lines one gets also the lower bound

$$\frac{u(y) - u(x)}{y - x} \ge \frac{u(r+\varepsilon) - u(y)}{(r+\varepsilon) - y} - C_r \ge \frac{-C_r|u(1)| - u(r+\varepsilon)}{1 - (r+\varepsilon)} - 2C_r.$$

Since the last two estimates hold for every $x$, $y \in \mathfrak{D} \cap [r-\varepsilon, r+\varepsilon]$ with $x \neq y$, the proof is complete. □

## 9.2 Entropies and their regularizations

Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space as in Section 5.5. We consider continuous and convex entropy functions $U : [0,\infty) \to \mathbb{R}$ with locally Lipschitz derivative in $(0,\infty)$ and $U(0) = 0$. We set

$$P(r) := rU'(r) - U(r), \quad Q(r) := r^{-1}P(r) \in \mathrm{Lip}_{\mathrm{loc}}(0,\infty), \quad R(r) := rP'(r) - P(r). \tag{9.27}$$

The induced entropy functional is defined by

$$\mathcal{U}(\mu) := \int_X U(\varrho)\,\mathrm{d}\mathfrak{m} + U'(\infty)\mu^\perp(X) \quad \text{if } \mu = \varrho\mathfrak{m} + \mu^\perp, \quad \mu^\perp \perp \mathfrak{m}, \tag{9.28}$$

where $U'(\infty) = \lim_{r\to\infty} U'(r)$. Since $U(0) = 0$ and the negative part of $U$ grows at most linearly, $\mathcal{U}$ is well defined and with values in $(-\infty, +\infty]$ if $\mu$ has bounded support.

We say that $P$ is *regular* if, for some constant $\mathsf{a} = \mathsf{a}(P) > 0$, one has

$$P \in \mathrm{C}^1([0,\infty)), \quad P(0) = 0, \quad 0 < \mathsf{a} \le P'(r) \le \mathsf{a}^{-1} \quad \text{for every } r \ge 0. \tag{9.29}$$

Notice that in this case $Q$, extended at 0 with the value $P'(0)$, is continuous in $[0,\infty)$ and it satisfies the analogous bounds

$$\mathsf{a} \le Q(r) \le \mathsf{a}^{-1} \quad \text{for every } r \ge 0. \tag{9.30}$$

When $P$ is regular, we still denote by $P : \mathbb{R} \to \mathbb{R}$ its odd extension, namely $P(-r) := -P(r)$ for every $r \ge 0$.

Once a regular function $P$ is assigned, a corresponding entropy function $U$ can be determined up to a linear term by the formula

$$U(r) = r \int_1^r \frac{P(s)}{s^2}\,\mathrm{d}s, \tag{9.31}$$

so that (9.29) yields

$$\mathsf{a}\big|r\log r\big| \le |U(r)| \le \mathsf{a}^{-1}\big|r\log r\big| \qquad \text{for every } r \ge 0. \tag{9.32}$$

Motivated by (9.31), we call the entropies $U$ satisfying $U(1) = 0$, *normalized*. Notice that $P$ uniquely determines the normalized entropy $U$. Thus in the case of regular $P$, the asymptotic behaviour of $U$ near $r = 0$ or $r = \infty$ is controlled by the one of the logarithmic entropy functional $\mathcal{U}_\infty$ associated to $U_\infty$, namely

$$U_\infty(r) := r\log r, \quad P_\infty(r) = r, \quad Q_\infty(r) = 1, \quad R_\infty(r) = 0. \tag{9.33}$$

In particular, using (5.23) one can prove that $\mathcal{U}(\mu)$ is always well defined, with values in $(-\infty,+\infty]$, if $\mu \in \mathscr{P}_2(X)$, see [5, §7,1]. The choice of the base point 1 in the integral formula (9.31) provides, thanks to Jensen's inequality, the lower bound $\mathcal{U}(\mu) \ge 0$ whenever $\mathfrak{m} \in \mathscr{P}(X)$.

**Remark 9.6 (Regularized entropies)** Let $P \in \mathrm{C}^1((0,\infty))$ with $P'(r) > 0$ for every $r > 0$ and $0 = P(0) = \lim_{r\downarrow 0} P(r)$. It is easy to approximate $P$ by regular functions: we set for $0 < \varepsilon < M < \infty$

$$P_\varepsilon(r) := P(r+\varepsilon) - P(\varepsilon), \qquad P_{\varepsilon,M}(r) := \begin{cases} P_\varepsilon(r) & \text{if } 0 \le r \le M, \\ P_\varepsilon(M) + (r-M)P'_\varepsilon(M) & \text{if } r > M. \end{cases} \tag{9.34}$$

Notice that

$$rP'_\varepsilon(r) - P_\varepsilon(r) = R(r+\varepsilon) - R(\varepsilon) + \varepsilon(P'(\varepsilon) - P'(r+\varepsilon)). \tag{9.35}$$

Besides (9.33), our main example is provided by the family depending on $N \in (1,\infty)$

$$\begin{aligned} &U_N(r) := Nr(1-r^{-1/N}), \quad P_N(r) = r^{1-1/N}, \quad Q_N(r) := r^{-1/N}, \\ &R_N(r) = -\frac{r^{1-1/N}}{N} = -\frac{1}{N}P_N(r) \end{aligned} \tag{9.36}$$

together with the regularized functions $P_{N,\varepsilon}$ and $P_{N,\varepsilon,M}$ as in (9.34).

Notice that a simple computation provides:

$$R_{N,\varepsilon}(r) = -\frac{1}{N} P_{N,\varepsilon}(r) + \varepsilon(P'_{N,\varepsilon}(0) - P'_{N,\varepsilon}(r)) \quad \text{for every } r \in [0, \infty), \tag{9.37}$$

so that the concavity and the monotonicity of $P_{N,\varepsilon}$ give

$$-\frac{1}{N} P_{N,\varepsilon}(r) + (1 - \frac{1}{N})\varepsilon^{1-1/N} \geq R_{N,\varepsilon}(r) \geq -\frac{1}{N} P_{N,\varepsilon}(r) \quad \text{for every } r \in [0, \infty). \tag{9.38}$$

The entropies corresponding to $U_N$ will be denoted with $\mathcal{U}_N$:

$$\mathcal{U}_N(\mu) := \int_X U_N(\varrho)\, \mathrm{d}\mathfrak{m} = N - N \int_X \varrho^{1-\frac{1}{N}}\, \mathrm{d}\mathfrak{m} \qquad \text{if } \mu = \varrho\mathfrak{m} + \mu^\perp, \quad \mu^\perp \perp \mathfrak{m}. \tag{9.39}$$

## 9.3 The $\mathrm{CD}^*(K, N)$ condition and its characterization via weighted action convexity

In this section we start by recalling what does it mean for a metric measure space to have "Ricci tensor bounded below by $K \in \mathbb{R}$ and dimension bounded above by $N \in (1, \infty]$", this corresponds to the the so-called curvature dimension conditions $\mathrm{CD}(K, N)$ or to the reduced curvature dimension conditions $\mathrm{CD}^*(K, N)$. First, let us recall the notion of $\mathrm{CD}(K, \infty)$ space introduced independently by Lott-Villani [40] and Sturm [48] (see also [52] for a comprehensive treatment).

**Definition 9.7 ($\mathrm{CD}(K, \infty)$ condition)** *Let $K \in \mathbb{R}$. We say that $(X, \mathsf{d}, \mathfrak{m})$ satisfies the $\mathrm{CD}(K, \infty)$ condition if for every $\mu_i = \varrho_i\mathfrak{m} \in D(\mathcal{U}_\infty) \cap \mathscr{P}_2(X)$, $i = 0, 1$, there exists a $W_2$-geodesic $(\mu_s)_{s\in[0,1]}$ connecting $\mu_0$ to $\mu_1$ such that*

$$\mathcal{U}_\infty(\mu_s) \leq (1 - s)\mathcal{U}_\infty(\mu_0) + s\mathcal{U}_\infty(\mu_1) - \frac{K}{2} s(1 - s)W_2^2(\mu_0, \mu_1) \qquad \forall s \in (0, 1). \tag{9.40}$$

*If morever* (9.40) *is satisfied along* any *geodesic $\mu_s$ connecting $\mu_0$ to $\mu_1$, we say that $(X, \mathsf{d}, \mathfrak{m})$ is a* strong $\mathrm{CD}(K, \infty)$ *space.*

It is well known that smooth Riemannian manifolds with Ricci curvature bounded below by $K$ are $\mathrm{CD}(K, \infty)$-spaces; one reason of the geometric relevance of such spaces is that they form a class which is stable under measured Gromov-Hausdorff convergence (for proper spaces see [40], for normalized spaces with finite total volume, see [48], for the general case without any finiteness or local compactness assumption see [31]).

In strong $\mathrm{CD}(K, \infty)$ spaces $(X, \mathsf{d}, \mathfrak{m})$, quite stronger metric properties have been proved in [45]; we list them in the next proposition.

**Proposition 9.8 (Properties of strong $\mathrm{CD}(K, \infty)$ spaces)** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a strong $\mathrm{CD}(K, \infty)$ space. Then:*

[RS1] *For every* $\mu_0 = \varrho_0\mathfrak{m}$, $\mu_1 = \varrho_1\mathfrak{m} \in \mathscr{P}^{ac}(X,\mathfrak{m}) \cap \mathscr{P}_2(X)$ *there exists only one optimal geodesic plan* $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu_0,\mu_1)$ *(and thus only one geodesic connecting* $\mu_0$, $\mu_1$*);*

[RS2] $\boldsymbol{\pi}$ *is concentrated on a set of nonbranching geodesics and it is induced by a map;*

[RS3] *all the interpolated measures* $\mu_s = (\mathrm{e}_s)_\sharp\boldsymbol{\pi}$ *are absolutely continuous w.r.t.* $\mathfrak{m}$*; if moreover* $\mu_0, \mu_1 \in D(\mathcal{U}_\infty)$*, then* $\mu_s$ *have uniformly bounded logarithmic entropies* $\mathcal{U}_\infty(\mu_s)$*.*

[RS4] *if* $\varrho_0$, $\varrho_1$ *are* $\mathfrak{m}$*-essentially bounded and have bounded supports, then the interpolated measures* $\mu_s = \varrho_s\mathfrak{m} = (\mathrm{e}_s)_\sharp\boldsymbol{\pi}$ *have uniformly bounded densities. More precisely the following estimate holds:*

$$\|\varrho_s\|_{L^\infty(X,\mathfrak{m})} \le e^{K^-D^2/12}\max\{\|\varrho_0\|_{L^\infty(X,\mathfrak{m})}, \|\varrho_1\|_{L^\infty(X,\mathfrak{m})}\}, \tag{9.41}$$

*where* $D := \mathrm{diam}(\mathrm{supp}\,\varrho_0 \cup \mathrm{supp}\,\varrho_1)$ *and* $K^- := \max\{0,-K\}$.

As bibliographical remark let us mention also [29] about existence of optimal maps in non branching spaces; also note that [RS4] is well known [52, Thm. 30.32, (30.51)], [6, §3] as soon as the branching phenomenon is ruled out. Remarkably, this property holds even without the non-branching assumption [44].

**Remark 9.9** Notice also the following general fact, holding regardless of curvature assumptions: if $\mu_0$, $\mu_1 \in \mathscr{P}(X)$ have bounded support, then there exists a bounded subset $E$ of $X$ containing all the traces of the geodesics from a point of $\mathrm{supp}\,\mu_0$ to a point of $\mathrm{supp}\,\mu_1$; in particular we have that $\mathrm{supp}[(\mathrm{e}_s)_\sharp\boldsymbol{\pi}] \subset E$ for every $s \in [0,1]$ for every $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu_0,\mu_1)$.

**Lemma 9.10** $(X,\mathsf{d},\mathfrak{m})$ *is a strong* $\mathrm{CD}(K,\infty)$ *space if and only if every couple of measures* $\mu_0$, $\mu_1 \in D(\mathcal{U}_\infty) \cap \mathscr{P}_2(X)$ *with bounded support can be connected by a* $W_2$*-geodesic and* (9.40) *is satisfied along any geodesic connecting* $\mu_0$ *to* $\mu_1$*.*

*Proof.* Let us first prove that every couple $\mu_0$, $\mu_1 \in D(\mathcal{U}_\infty) \cap \mathscr{P}_2(X)$ can be connected by a $W_2$-geodesic. For $\bar{x} \in X$ fixed and $N$ sufficiently big, we can define the measures $\mu_i^N := \frac{1}{c_N}\mu_i \llcorner B_N(\bar{x}) \in \mathscr{P}_2(X)$.

By choosing a constant $C > B$ (recall (5.23)), we can introduce the normalized probability measure $\bar{\mathfrak{m}} \in \mathscr{P}_2(X)$

$$\mathrm{z} := \int_X \mathrm{e}^{-C\mathsf{d}^2(x,\bar{x})}\,\mathrm{d}\mathfrak{m}(x), \quad \bar{\mathfrak{m}} := \frac{1}{\mathrm{z}}\mathrm{e}^{-C\mathsf{d}^2(x,\bar{x})}\mathfrak{m},$$

and the corresponding relative entropy functional $\tilde{\mathcal{U}}_\infty$, satisfying the identity

$$\mathcal{U}_\infty(\mu) = \tilde{\mathcal{U}}_\infty(\mu) - C\int_X \mathsf{d}^2(x,\bar{x})\,\mathrm{d}\mu - \log \mathrm{z}. \tag{9.42}$$

Let us denote by $\tilde\varrho_i, \tilde\varrho_i^N$ the densities of $\mu_i, \mu_i^N$ w.r.t. $\tilde{\mathfrak m}$. From $c_N \uparrow 1$ it is easy to check that $W_2(\mu_i^N, \mu_i) \to 0$ and $\|\tilde\varrho_i^N - \tilde\varrho_i\|_{L^1(X,\tilde{\mathfrak m})} \to 0$ as $N \uparrow \infty$. Since $\tilde\varrho_i^N \le c_N^{-1}\tilde\varrho_i$, the uniform bound

$$\mathrm{e}^{-1} \le \tilde\varrho_i^N \log(\tilde\varrho_i^N) \le \tilde\varrho_i^N(\log \varrho_i - \log c_N) \le c_N\varrho_i(\log\varrho_i)_+ - c_N \log c_N \varrho_i$$

and the fact that $\tilde{\mathfrak m}(X)$ is finite yields $\tilde{\mathcal U}_\infty(\mu_i^N) \to \tilde{\mathcal U}_\infty(\mu_i)$ as $N \uparrow \infty$ and therefore, by (9.42), $\mathcal U_\infty(\mu_i^N) \to \mathcal U_\infty(\mu_i)$ as $N \to \infty$.

Since $\mu_i^N$ have bounded support we can find a $W_2$-geodesic $(\mu_s^N)_{s\in[0,1]}$ connecting them and satisfying the corresponding uniform bound

$$\mathcal U_\infty(\mu_s^N) \le (1-s)\mathcal U_\infty(\mu_0^N) + s\mathcal U_\infty(\mu_1^N) - \frac{K}{2}s(1-s)W_2^2(\mu_0^N, \mu_1^N) \qquad \forall s \in (0,1), \quad (9.43)$$

which in particular shows that $\tilde{\mathcal U}_\infty(\mu_s^N) \le S < \infty$ for every $s \in [0,1]$ and $N$ sufficiently big. Since the sublevels of $\tilde{\mathcal U}_\infty$ are relatively compact in $\mathscr P(X)$ and the curves $[0,1] \ni s \mapsto \mu_s^N$ are equi-Lipschitz with respect to $W_2$, we can extract (see e.g. [3, Prop. 3.3.1]) an increasing subsequence $h \mapsto N(h)$ and a limit geodesic $(\mu_s)_{s\in[0,1]}$ such that $\mu_s^{N(h)} \to \mu_s$ weakly in $\mathscr P(X)$ as $h \to \infty$. In particular $(\mu_s)_{s\in[0,1]}$ is a geodesic connecting $\mu_0$ to $\mu_1$.

Let us now prove that (9.40) holds along any geodesic connecting $\mu_0, \mu_1 \in D(\mathcal U_\infty) \cap \mathscr P_2(X)$. Let $\mu_s = (\mathrm e_s)_\sharp \boldsymbol\pi$ be a geodesic induced by $\boldsymbol\pi \in \mathrm{GeoOpt}(\mu_0, \mu_1)$; we consider

$$\Gamma_R := \left\{\gamma \in \mathrm C^0([0,T];X) : \ \gamma([0,1]) \subset \overline B_R(\bar x)\right\}, \qquad c_R := \boldsymbol\pi(\Gamma_R), \quad \boldsymbol\pi^R := \frac{1}{c_R}\boldsymbol\pi \llcorner \Gamma_R$$

and $\mu_s^R := (\mathrm e_s)_\sharp \boldsymbol\pi^R$; since $\boldsymbol\pi^R \in \mathrm{GeoOpt}(\mu_0^R, \mu_1^R)$, $(\mu_s^R)_{s\in[0,1]}$ is a geodesic and the measures $\mu_s^R$ have bounded support in $\overline B_R(\bar x)$. Thus for every $R > 0$ one has that $\mu_0^R, \mu_s^R, \mu_1^R$ satisfy (9.40); arguing as in the previous step, we can pass to the limit as $R \to \infty$ using the facts that $W_2(\mu_s^R, \mu_s) \to 0$ for every $s \in [0,1]$, $\mathcal U_\infty(\mu_i^R) \to \mathcal U_\infty(\mu_i)$ if $i = 0$, $1$, and $\liminf_{R\to\infty} \mathcal U_\infty(\mu_s^R) \ge \mathcal U_\infty(\mu_s)$ and we obtain the corresponding inequality for $\mu_0, \mu_s, \mu_1$. □

Next, let us recall the definition of reduced curvature dimension condition $\mathrm{CD}^*(K,N)$ introduced by Bacher-Sturm [10].

**Definition 9.11 ($\mathrm{CD}^*(K,N)$ condition)** *We say that $(X, \mathsf d, \mathfrak m)$ satisfies the $\mathrm{CD}^*(K,N)$ condition, $N \in [1,\infty)$, if for every $\mu_i = \varrho_i \mathfrak m \in \mathscr P^{ac}(X,\mathfrak m)$, $i = 0, 1$, with bounded support there exists $\boldsymbol\pi \in \mathrm{GeoOpt}(\mu_0,\mu_1)$ such that*

$$\mathcal U_M(\mu_s) \le M - M\int\Big(\sigma_{K/M}^{(1-s)}(\mathsf d(\gamma_0,\gamma_1))\varrho_0(\gamma_0)^{-1/M} + \sigma_{K/M}^{(s)}(\mathsf d(\gamma_0,\gamma_1))\varrho_1(\gamma_1)^{-1/M}\Big)\,\mathrm d\boldsymbol\pi(\gamma) \tag{9.44}$$

*for every $s \in [0,1]$ and $M \ge N$, where $\mu_s = (\mathrm e_s)_\sharp \boldsymbol\pi$, the coefficients $\sigma$ are defined in (9.15) and $\mathcal U_M$ is defined in (9.39).*

*If moreover (9.44) is satisfied along* any $\boldsymbol\pi \in \mathrm{GeoOpt}(\mu_0,\mu_1)$, *we say that $(X, \mathsf d, \mathfrak m)$ satisfies the* strong $\mathrm{CD}^*(K,N)$ *condition.*

**Remark 9.12** Definition 9.11 coincides with the original definition of $\mathrm{CD}^*(K,N)$ spaces given in [10]. Note that the additional terms in the right hand side of (9.44) are due to our definition of entropy as $\mathcal{U}_M(\varrho\mathfrak{m}) := M\int_X \varrho(1-\varrho^{-\frac{1}{M}})\,\mathrm{d}\mathfrak{m} = M - M\int_X \varrho^{1-\frac{1}{M}}\,\mathrm{d}\mathfrak{m}$, while the one adopted in [10] was $-\int_X \varrho^{1-\frac{1}{M}}\,\mathrm{d}\mathfrak{m}$ (for absolutely continuous measures). This convention will be convenient in our work in order to use regularized entropies and analyze the corresponding non linear diffusion semigroups.

It can be proved that a strong $\mathrm{CD}^*(K,N)$-space is also a strong $\mathrm{CD}(K,\infty)$ space, and thus properties [RS1-4] hold, see Lemma 9.13 below, whose proof included for completeness follows the lines of [49, Prop. 1.6]. Conversely, any $\mathrm{CD}^*(K,N)$ space satisfying [RS1-4] is clearly strong. Therefore a $\mathrm{CD}^*(K,N)$ space is strong if and only if [RS1-4] hold.

**Lemma 9.13** *If $(X,\mathsf{d},\mathfrak{m})$ satisfies the (strong) $\mathrm{CD}^*(K,N)$ condition for some $K\in\mathbb{R}$, $N\in[1,\infty)$, then $(X,\mathsf{d},\mathfrak{m})$ is a strong $\mathrm{CD}(K,\infty)$ space.*

*Proof.* By Lemma 9.10 it is sufficient to prove (9.40) along every $W_2$-geodesic $(\mu_s)_{s\in[0,1]}$ induced by $\boldsymbol{\pi}\in\mathrm{GeoOpt}(\mu_0,\mu_1)$ with $\mu_s$ supported in a bounded set and $\mu_i\in D(\mathcal{U}_\infty)$, $i=0,1$.

Let us first notice that for every $r\geq 0$

$$\lim_{N\to\infty} U_N(r) = U_\infty(r),\quad N\mapsto U_N(r)\text{ is increasing if } r>1\text{, decreasing if } 0<r<1.$$

If $\mu_s=\varrho_s\mathfrak{m}$, since $\mu_s$ is supported in a bounded set with finite $\mathfrak{m}$-measure, it is then not difficult to prove that

$$\lim_{N\to\infty}\mathcal{U}_N(\mu_s)=\mathcal{U}_\infty(\mu_s)\quad\text{for every } s\in[0,1]. \tag{9.45}$$

The second important property concerns the coefficients $\sigma_\kappa^{(s)}(\delta)$: if $\delta>0$

$$M\Big(s-\sigma^{(s)}_{K/M}(\delta)\Big) = K\frac{s\sin(\sqrt{K/M}\delta)-\sin(\sqrt{K/M}s\delta)}{(K/M)\sin(\sqrt{K/M}\delta)} = \frac{\delta^2}{6}K\big(s^3-s\big)+o(1) \tag{9.46}$$

as $M\uparrow\infty$, and a similar property holds when $K<0$.

We thus get

$$\begin{aligned}&\mathcal{U}_M(\mu_s)-(1-s)\mathcal{U}_M(\mu_0)-s\mathcal{U}_M(\mu_1)\\ &\leq M\int\Big(\big(s-\sigma^{(s)}_{K/M}(\mathsf{d}(x_0,x_1))\big)\varrho_0^{-1/M}(\gamma_0)+\big(1-s-\sigma^{(1-s)}_{K/M}(\mathsf{d}(x_0,x_1))\big)\varrho_1^{-1/M}(\gamma_1)\Big)\,\mathrm{d}\boldsymbol{\pi}(\gamma)\end{aligned}$$

and passing to the limit as $M\uparrow\infty$ by applying (9.45) and (9.46) we obtain

$$\begin{aligned}\mathcal{U}_\infty(\mu_s)-(1-s)\mathcal{U}_\infty(\mu_0)+s\mathcal{U}_\infty(\mu_1) &\leq -\frac{K}{2}s(1-s)\int\mathsf{d}^2(x_0,x_1)\,\mathrm{d}\boldsymbol{\pi}(\gamma)\\ &= -\frac{K}{2}s(1-s)W_2^2(\mu_0,\mu_1).\end{aligned}$$

□

Let us also introduce a more general class of natural entropy functionals, used for instance in Lott-Villani's approach of CD spaces [40, 52]. They will play a crucial role in the next chapters.

**Definition 9.14 ([52, Def. 17.1])** *We say that the entropy density $U$ belongs to McCann's class* $\mathrm{DC}(N)$, $N \in [1,\infty]$, *if the corresponding pressure function* $P = rU' - U$ *satisfies* $P(0) = \lim_{r\downarrow 0} P(r) = 0$ *and* $r \mapsto r^{\frac{1}{N}-1}P(r)$ *is nondecreasing, i.e.*

$$R(r) = rP'(r) - P(r) \geq -\frac{1}{N}P(r) \quad \text{for } \mathscr{L}^1\text{-a.e. } r > 0. \tag{9.47}$$

*We say that $U$ is* regular *and write* $U \in \mathrm{DC}_{reg}(N)$ *if, in addition, $U$ is normalized and $P$ is regular according to* (9.29).

If $P$ is regular, we can also write $P \in \mathrm{DC}(N)$ (resp. $P \in \mathrm{DC}_{reg}(N)$) if the corresponding normalized entropy $U$ belongs to $\mathrm{DC}(N)$ (resp. $\mathrm{DC}_{reg}(N)$). Directly from (9.47) it is immediate to see that

$$U \in \mathrm{DC}(N) \quad \Rightarrow \quad P \text{ is nonnegative} \tag{9.48}$$

moreover, the function $V : (0,\infty) \to \mathbb{R}^+$ defined by

$$V(r) := r^N U(r^{-N}) \quad \text{is convex and nonincreasing.} \tag{9.49}$$

The last condition is actually equivalent to $U \in \mathrm{DC}(N)$.

Before stating the next result, we introduce a family of weighted energy functionals taylored to a pressure function $P$ as in § 9.2: if $Q(r) := P(r)/r$, we consider the weight $\mathfrak{Q}^{(t)}(s,r) := \mathsf{g}(s,t)Q(r)$, where $\mathsf{g}$ is the Green function defined in (9.1).

We adopt the notation of Section 6. If $\mu \in \mathrm{AC}^2([0,1];(\mathscr{P}(X),W_2))$ with $\tilde\mu = \varrho\tilde{\mathfrak{m}} \ll \mathfrak{m}$ and $v$ is its minimal 2-velocity density, we set

$$\mathcal{A}^{(t)}_Q(\mu;\mathfrak{m}) := \mathcal{A}_{\mathfrak{Q}^{(t)}}(\mu;\mathfrak{m}) = \int_{\tilde X} \mathsf{g}(s,t)Q(\varrho(x,s))v^2(x,s)\,\mathrm{d}\tilde\mu(x,s). \tag{9.50}$$

In the following theorem we relate the $\mathrm{CD}^*(K,N)$ condition, defined in terms of the distortion coefficients $\sigma_{K/N}$, to a modulus of convexity along Wasserstein geodesics of the entropies induced by maps $U \in \mathrm{DC}(N)$, very much like in the case $N = \infty$. The main difference is that the modulus of convexity is not the squared Wasserstein distance, but the action $\mathcal{A}^{(t)}_Q(\mu;\mathfrak{m})$ of (9.50).

**Theorem 9.15** *Let us assume that* [RS1-4] *hold. The following properties are equivalent:*

[CD1] $(X,\mathsf{d},\mathfrak{m})$ *is a strong* $\mathrm{CD}^*(K,N)$ *space, for some* $K \in \mathbb{R}$ *and* $N \in [1,\infty)$.

[CD2] *For every* $\mu_0 = \varrho_0\mathfrak{m}$, $\mu_1 = \varrho_1\mathfrak{m} \in \mathscr{P}^{ac}(X,\mathfrak{m})$ *with densities* $\varrho_i$ $\mathfrak{m}$*-essentially bounded with bounded support, the geodesic* $(\mu_t)_{t\in[0,1]}$ *connecting* $\mu_0$ *to* $\mu_1$ *satisfies (with* $Q_N(r) = r^{-1/N}$ *as in* (9.36)*)*

$$\mathcal{U}_N(\mu_t) \leq (1-t)\mathcal{U}_N(\mu_0) + t\,\mathcal{U}_N(\mu_1) - K\mathcal{A}^{(t)}_{Q_N}(\mu;\mathfrak{m}) \quad \text{for every } t \in [0,1]. \tag{9.51}$$

[CD3] *For every* $\mu_0 = \varrho_0\mathfrak{m}$, $\mu_1 = \varrho_1\mathfrak{m} \in \mathscr{P}^{ac}(X,\mathfrak{m}) \cap \mathscr{P}_2(X)$, *the geodesic* $(\mu_t)_{t\in[0,1]}$ *connecting* $\mu_0$ *to* $\mu_1$ *satisfies* (9.51).

[CD4] *For every* $\mu_0 = \varrho_0\mathfrak{m}$, $\mu_1 = \varrho_1\mathfrak{m} \in \mathscr{P}^{ac}(X,\mathfrak{m}) \cap \mathscr{P}_2(X)$ *and every* $U \in \mathrm{DC}(N)$, *the geodesic* $(\mu_t)_{t\in[0,1]}$ *connecting* $\mu_0$ *to* $\mu_1$ *satisfies*

$$\mathcal{U}(\mu_t) \le (1-t)\mathcal{U}(\mu_0) + t\mathcal{U}(\mu_1) - K\mathcal{A}_Q^{(t)}(\mu;\mathfrak{m}) \quad \text{for every } t \in [0,1]. \tag{9.52}$$

[CD5] *For every* $\mu_0$, $\mu_1 \in \mathscr{P}^{ac}(X,\mathfrak{m}) \cap \mathscr{P}_2(X)$ *and every regular* $U \in \mathrm{DC}_{reg}(N)$ *the geodesic* $(\mu_t)_{t\in[0,1]}$ *connecting* $\mu_0$ *to* $\mu_1$ *satisfies* (9.52).

*Proof.* The implications [CD4] ⇒ [CD3] ⇒ [CD2] and [CD4] ⇒ [CD5] are trivial. We will prove [CD1] ⇒ [CD2] ⇒ [CD3] ⇒ [CD4] ⇒ [CD1] and [CD5] ⇒ [CD2].

[CD1] ⇒ [CD2]. Let $\mu_0 = \varrho_0\mathfrak{m}$, $\mu_1 = \varrho_1\mathfrak{m} \in \mathscr{P}^{ac}(X,\mathfrak{m})$ with densities $\varrho_i$ $\mathfrak{m}$-essentially bounded with bounded support. By [RS1-4] there exists a unique geodesic $\mu_t = (\mathrm{e_t})_\sharp\boldsymbol{\pi}$ connecting $\mu_0$ to $\mu_1$, it is made of absolutely continuous measures with bounded densities and it is given by optimal maps: $\varrho_t\mathfrak{m} = \mu_t = (T_t)_\sharp\mu_0$. Since $\boldsymbol{\pi}$ is concentrated on non-branching geodesics, we can apply [10, Proposition 2.8 (iii)] to infer that for every $t \in (0,1)$ there exists a Borel subset $E_t \subset \operatorname{supp}\mu_0$ with $\mu_0(X \setminus E_t) = 0$ such that

$$\mathfrak{d}_t(x) \ge \sigma_{K/N}^{(1-t)}(\mathsf{d}(x,T_1(x)))\,\mathfrak{d}_0(x) + \sigma_{K/N}^{(t)}(\mathsf{d}(x,T_1(x)))\,\mathfrak{d}_1(x), \quad \forall x \in E_t, \tag{9.53}$$

where $\mathfrak{d}_t(x) := (\varrho_t^{-1/N} \circ T_t)(x)$. Moreover, by [44, Theorem 1.2], the convexity property (9.44) holds for all intermediate times (note that in this case one could argue directly by knowing that the $W_2$-geodesic is unique). It follows that, for any fixed countable $\mathbb{Q}$-vector space $\mathfrak{D} \subset \mathbb{R}$, there exists a Borel subset $E_\mathfrak{D} \subset \operatorname{supp}\mu_0$ with $\mu_0(X \setminus E_\mathfrak{D}) = 0$ such that for every $x \in E_\mathfrak{D}$, every $r_0, r_1 \in [0,1] \cap \mathfrak{D}$ and $t \in [0,1] \cap \mathbb{Q}$ it holds

$$\mathfrak{d}_{r_t}(x) \ge \sigma_{K/N}^{(1-t)}(\mathsf{d}(T_{r_0}(x),T_{r_1}(x)))\,\mathfrak{d}_{r_0}(x) + \sigma_{K/N}^{(t)}(\mathsf{d}(T_{r_0}(x),T_{r_1}(x)))\,\mathfrak{d}_{r_1}(x), \tag{9.54}$$

where $r_t := (1-t)r_0 + tr_1$. For the moment simply choose $\mathfrak{D} = \mathbb{Q}$ and, fixed some $n \in \mathbb{N}$, define

$$\mathfrak{F} := \{m/n \,:\, m \in \mathbb{N}, m \le n\}.$$

By Lemma 9.17 below we have that the map $\mathfrak{F} \in r \mapsto \mathfrak{d}_r(x)$ is uniformly bounded in $[0,1]$ for for every $x \in E$, where $E \subset E_\mathfrak{D}$ satisfies $\mu_0(E) = 1$. Observe also that $\mathsf{d}(T_{r_0}(x),T_{r_1}(x)) = (r_1-r_0)\mathsf{d}(x,T_1(x))$ is uniformly bounded since we are assuming $\mu_i$ to have bounded support, $i = 0$, 1. Choosing $n \in \mathbb{N}$ large enough in the definition of $\mathfrak{F}$, by Lemma 9.5(b) we infer that the map $\mathbb{Q} \in r \mapsto \mathfrak{d}_r(x)$ is uniformly bounded for every $x \in E$. But then part (c) of Lemma 9.5 applied to the function $[0,1] \cap \mathbb{Q} \ni r \mapsto \mathfrak{d}_r(x) \in \mathbb{R}^+$ gives that such a map is locally Lipschitz, so it admits a unique continuous extension $[0,1] \ni t \mapsto \bar{\mathfrak{d}}_t(x) \in \mathbb{R}^+$.

Observing now that

$$\sigma_{K/N}^{(1-s)}(\mathsf{d}(T_{r_0}(x),T_{r_1}(x))) = \sigma_{K/N}^{(1-s)}((r_1-r_0)\mathsf{d}(x,T_1(x))) = \sigma_{\frac{K}{N}\mathsf{d}(x,T_1(x))}^{(1-s)}(r_1-r_0),$$

Lemma 9.4 implies that the continuous map $t \mapsto \bar{\mathfrak{d}}_t(x)$ satisfies the differential inequality

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bar{\mathfrak{d}}_t(x) \le -\frac{K}{N}\mathsf{d}^2(x, T_1(x))\,\bar{\mathfrak{d}}_t(x) \quad \text{in } \mathscr{D}'(0,1), \text{ for every } x \in E. \tag{9.55}$$

But then, Lemma 9.1 gives

$$\bar{\mathfrak{d}}_t(x) \ge (1-t)\bar{\mathfrak{d}}_0(x) + t\bar{\mathfrak{d}}_1(x) + \frac{K}{N}\int_0^1 \bar{\mathfrak{d}}_s(x)\mathsf{d}^2(x, T_1(x))\,\mathsf{g}(s,t)\,\mathrm{d}s \quad \forall t \in [0,1],\ \forall x \in E. \tag{9.56}$$

We now claim that for every $t \in [0,1]$ it holds $\mathfrak{d}_t = \bar{\mathfrak{d}}_t$ $\mu_0$-a.e. If it is not the case then there exists $\bar{t} \in [0,1]$ and a subset $F_{\bar{t}} \subset \operatorname{supp}\mu_0$ with $\mu_0(F_{\bar{t}}) > 0$ such that

$$\mathfrak{d}_{\bar{t}}(x) \neq \bar{\mathfrak{d}}_{\bar{t}}(x) = \lim_{n\to\infty}\mathfrak{d}_{t_n}(x) \quad \forall x \in F_{\bar{t}},\ t_n \in \mathbb{Q}\cap[0,1] \text{ with } t_n \to \bar{t}. \tag{9.57}$$

But choosing $\mathfrak{D} = \{q_1 + q_2\bar{t} \,:\, q_1, q_2 \in \mathbb{Q}\}$, we get that there exists a subset $E'_{\mathfrak{D}} \subset \operatorname{supp}\mu_0$ with $\mu_0(X \setminus E'_{\mathfrak{D}}) = 0$ such that the inequality (9.54) holds for every $x \in E'_{\mathfrak{D}}$; therefore, repeating the arguments above, Lemma 9.5 yields that the function $[0,1]\cap\mathfrak{D} \ni r \mapsto \mathfrak{d}_r(x) \in \mathbb{R}^+$ is locally Lipschitz for every $x \in E'_{\mathfrak{D}}$. This is in contradiction with the discontinuity (9.57) at $\bar{t}$, since $\mu_0(F_{\bar{t}}) > 0$ and $\mu_0(X \setminus E'_{\mathfrak{D}}) = 0$.

Integrating now (9.56) in $\mathrm{d}\mu_0(x)$, since by construction $\mu_0(X \setminus E) = 0$ and $\mathfrak{d}_t = \bar{\mathfrak{d}}_t$ $\mu_0$-a.e, we get (9.51). Indeed

$$\int_E \bar{\mathfrak{d}}_t\,\mathrm{d}\mu_0 = \int_E \mathfrak{d}_t\,\mathrm{d}\mu_0 = \int_X \varrho_t^{-1/N}\circ T_t\,\mathrm{d}\mu_0 = \int_X \varrho_t^{1-\frac{1}{N}}\,\mathrm{d}\mathfrak{m} = 1 - \frac{1}{N}\mathcal{U}_N(\mu_t);$$

and

$$\begin{aligned}
\int_0^1 \left[\int_X \bar{\mathfrak{d}}_s(x)\mathsf{d}^2(x,T_1(x))\mathrm{d}\mu_0(x)\right]\mathsf{g}(s,t)\,\mathrm{d}s \quad &= \quad \int_0^1 \left[\int_X \mathfrak{d}_s(x)\mathsf{d}^2(x,T_1(x))\mathrm{d}\mu_0(x)\right]\mathsf{g}(s,t)\,\mathrm{d}s \\
&= \quad \int_0^1 \left[\int_X \varrho_s^{-\frac{1}{N}}(x)\,v^2(x)\mathrm{d}\mu_s(x)\right]\mathsf{g}(s,t)\mathrm{d}s \\
&= \quad \int_{\tilde{X}} \mathsf{g}(s,t)Q_N(\varrho(x,s))v^2(x)\mathrm{d}\tilde{\mu}(x,s) \\
&= \quad \mathcal{A}^{(t)}_{Q_N}(\mu;\mathfrak{m}),
\end{aligned}$$

where we used the fact that, since the plan $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu_0,\mu_1)$ is concentrated on constant speed geodesics and recalling (6.12), the minimal 2-velocity $v$ is constant in time and given by $v(x) = \mathsf{d}(x, T_1(x))$.

[CD2] $\Rightarrow$ [CD3]. Let us start by assuming that $\mu_0 = \varrho_0\mathfrak{m}$, $\mu_1 = \varrho_1\mathfrak{m}$, $\varrho_t\mathfrak{m} = \mu_t = (T_t)_\sharp\mu_0 = (\mathrm{e_t})_\sharp\boldsymbol{\pi}$ are as in the above implication, i.e. $\varrho_i$, $i = 0,1$, are $\mathfrak{m}$-essentially bounded with bounded support, so that by hypothesis we know that $\mu_t$ satisfies (9.51). For any Borel subset $A \subset \operatorname{supp}\mu_0$ with $\mu_0(A) > 0$, consider the localized and normalized measure $\mu_0^A :=$

$\frac{1}{\mu_0(A)}\ \mu_0 \llcorner A = \frac{1}{\mu_0(A)}\chi_A\ \mu_0$ and its push forwards $\mu_t^A := (T_t)_\sharp \mu_0^A$. By cyclical monotonicity of the measure-theoretic support, it is well known that $\frac{1}{\mu_0(A)}(\chi_A \circ \mathrm{e}_0)\boldsymbol{\pi} \in \mathrm{GeoOpt}(\mu_0^{\mathrm{A}}, \mu_1^{\mathrm{A}})$ so that $\mu_t^A$ is the $W_2$-geodesic from $\mu_0^A$ to $\mu_1^A$. Moreover, since the map $T_t$ is $\mu_0$-essentially injective, we have

$$\mu_t^A = \frac{1}{\mu_0(A)}(T_t)_\sharp(\mu_0 \llcorner A) = \frac{1}{\mu_0(A)}\mu_t \llcorner T_t(A) = \frac{1}{\mu_0(A)}(\chi_{T_t(A)} \cdot \varrho_t)\, \mathfrak{m}.$$

Applying (9.51) to the geodesic $\mu_t^A$ gives the localized convexity inequality

$$\int_A \mathfrak{d}_t \,\mathrm{d}\mu_0 \geq (1-t)\int_A \mathfrak{d}_0 \,\mathrm{d}\mu_0 + t\int_A \mathfrak{d}_1 \,\mathrm{d}\mu_0 + \frac{K}{N}\int_A \int_0^1 \mathfrak{d}_s(x)\mathsf{d}^2(x, T_1(x))\, \mathsf{g}(s,t)\,\mathrm{d}s\,\mathrm{d}\mu_0(x),$$

for every $t \in [0,1]$, where $\mathfrak{d}_t(x) := (\varrho_t^{-1/N} \circ T_t)(x)$ as before. The arbitrariness of the Borel set $A$ implies that for all $t \in [0,1]$ one has

$$\mathfrak{d}_t(x) \geq (1-t)\mathfrak{d}_0(x) + t\mathfrak{d}_1(x) + \frac{K}{N}\int_0^1 \mathfrak{d}_s(x)\mathsf{d}^2(x, T_1(x))\, \mathsf{g}(s,t)\,\mathrm{d}s, \quad \text{for } \mu_0\text{-a.e. } x. \tag{9.58}$$

Now let instead $\mu_i = \varrho_i \mathfrak{m} \in \mathscr{P}^{ac}(X, \mathfrak{m}) \cap \mathscr{P}_2(X)$, $i = 0, 1$, and $\varrho_t \mathfrak{m} = \mu_t = (T_t)_\sharp \mu_0 = (\mathrm{e_t})_\sharp \boldsymbol{\pi}$ be the unique $W_2$-geodesic joining them. Consider the approximating geodesic $\mu_t^k = \varrho_t^k \mathfrak{m}$ given by Lemma 9.16 below. Since $\varrho_i^k$ are $\mathfrak{m}$-essentially bounded with bounded support, (9.58) holds for $\mu_t^k$ by assumption. It follows that there exists $E_{k,t} \subset \operatorname{supp} \mu_0^k \subset \operatorname{supp} \mu_0$ with $\mu_0^k(X \setminus E_{k,t}) = 0$ such that

$$\mathfrak{d}_t^k(x) \geq (1-t)\mathfrak{d}_0^k(x) + t\mathfrak{d}_1^k(x) + \frac{K}{N}\int_0^1 \mathfrak{d}_s^k(x)\mathsf{d}^2(x, T_1(x))\, \mathsf{g}(s,t)\,\mathrm{d}s, \tag{9.59}$$

for every $x \in E_{k,t}$, where $\mathfrak{d}_t^k(x) = (\varrho_t^k \circ T_t)^{-1/N}(x)$. Defining $E_t := \bigcap_{k\in\mathbb{N}} E_{k,t}$, by using Lemma 9.16(4), we get that $E_t \subset \operatorname{supp}\mu_0$ and $\mu_0(X \setminus E_t) = 0$. Moreover, observe that (9.59) is still true for the renormalized measures $c_k\, \varrho_t^k$, since the constants just simplify from both sides thanks to the homogeneity of the entropy. But then, Lemma 9.16(3) implies that for $\mu_0$-a.e. $x \in E$ one has $\mathfrak{d}_t^k(x) = \mathfrak{d}_t(x)$, provided $k$ is large enough. Passing to the limit for $k \to \infty$, we conclude that (9.58) holds and the thesis follows by integration in $\mathrm{d}\mu_0(x)$ as in the implication [CD1] $\Rightarrow$ [CD2].

[CD3] $\Rightarrow$ [CD4] Let $\mu_i = \varrho_i \mathfrak{m} \in \mathscr{P}^{ac}(X, \mathfrak{m}) \cap \mathscr{P}_2(X)$, $i = 0, 1$, and $\varrho_t \mathfrak{m} = \mu_t = (T_t)_\sharp \mu_0 = (\mathrm{e_t})_\sharp \boldsymbol{\pi}$ be the unique $W_2$-geodesic joining them. Observing that the restriction to a subinterval $[r_0, r_1] \subset [0,1]$ of a geodesic is still a geodesic, the localization argument of the implication [CD2] $\Rightarrow$ [CD3] ensures that for every $r_0, r_1, t \in [0,1]$ one has

$$\mathfrak{d}_{r_t}(x) \geq (1-t)\mathfrak{d}_{r_0}(x) + t\mathfrak{d}_1(x) + \frac{K}{N}(r_1 - r_0)^2 \int_0^1 \mathfrak{d}_{r_s}(x)\mathsf{d}^2(x, T_1(x))\, \mathsf{g}(s,t)\,\mathrm{d}s \quad \mu_0\text{-a.e. } x, \tag{9.60}$$

where $r_t := (1-t)r_0 + tr_1$ as before. It follows that, for any fixed countable $\mathbb{Q}$-vector space $\mathfrak{D} \subset \mathbb{R}$, there exists a Borel subset $E_{\mathfrak{D}} \subset \operatorname{supp}\mu_0$ with $\mu_0(X \setminus E_{\mathfrak{D}}) = 0$ such that (9.60) holds for every $x \in E_{\mathfrak{D}}$, every $r_0, r_1 \in [0,1] \cap \mathfrak{D}$ and $t \in [0,1] \cap \mathbb{Q}$. Choosing simply $\mathfrak{D} = \mathbb{Q}$, for every fixed $x \in E := E_{\mathbb{Q}}$, we can apply Lemma 9.2 to the function $[0,1] \cap \mathbb{Q} \ni r \mapsto \mathfrak{d}_r(x) \in \mathbb{R}^+$ and infer that such a map is locally Lipschitz, so it admits a unique continuous extension $[0,1] \ni t \mapsto \bar{\mathfrak{d}}_t(x) \in \mathbb{R}^+$ satisfying (9.56). Lemma 9.1 gives then

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bar{\mathfrak{d}}_t(x) \le -\frac{K}{N}\mathsf{d}^2(x, T_1(x))\,\bar{\mathfrak{d}}_t(x) \quad \text{in } \mathscr{D}'(0,1), \text{ for every } x \in E\,. \tag{9.61}$$

Given now $U \in \mathrm{DC}(N)$, recalling (9.49) and taking (9.61) into account, we get the following chain of inequalities in distributional sense

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}V(\bar{\mathfrak{d}}_t(x)) &= V''(\bar{\mathfrak{d}}_t(x))\left(\frac{\mathrm{d}}{\mathrm{d}t}\bar{\mathfrak{d}}_t(x)\right)^2 + V'(\bar{\mathfrak{d}}_t(x))\,\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bar{\mathfrak{d}}_t(x) \\
&\ge -\frac{K}{N}\,\bar{\mathfrak{d}}_t(x)\,\mathsf{d}^2(x, T_1(x))\,V'(\bar{\mathfrak{d}}_t(x)) \quad \text{in } \mathscr{D}'(0,1), \text{ for every } x \in E.
\end{aligned}$$

Applying again Lemma 9.1, this time with $u(t) := V(\bar{\mathfrak{d}}_t(x))$, we obtain

$$V(\bar{\mathfrak{d}}_t(x)) \le (1-t)V(\bar{\mathfrak{d}}_0(x)) + tV(\bar{\mathfrak{d}}_1(x)) + \frac{K}{N}\int_0^1 \bar{\mathfrak{d}}_s(x)\,\mathsf{d}^2(x, T_1(x))\,V'(\bar{\mathfrak{d}}_s(x))\,\mathsf{g}(s,t)\,\mathrm{d}s, \tag{9.62}$$

for every $x \in E$. With the same argument as in the proof of [CD1] $\Rightarrow$ [CD2], we have that for every $t \in [0,1]$ it holds $\bar{\mathfrak{d}}_t(x) = \mathfrak{d}_t(x) = (\varrho_t^{-1/N} \circ T_t)(x)$ for $\mu_0$-a.e. $x$. The desired inequality (9.52) follows then by integrating (9.62) in $\mathrm{d}\mu_0(x)$, since by construction $\mu_0(X \setminus E) = 0$. Indeed, recalling that $V(\mathfrak{d}_t(x)) = V(\varrho_t^{-1/N} \circ T_t(x)) = \frac{U(\varrho_t \circ T_t(x))}{\varrho_t \circ T_t(x)}$, we have

$$\begin{aligned}
\int_E V(\bar{\mathfrak{d}}_t)\,\mathrm{d}\mu_0 &= \int_X V(\mathfrak{d}_t)\,\mathrm{d}\mu_0 = \int_X \frac{U(\varrho_t \circ T_t)}{\varrho_t \circ T_t}\,\mathrm{d}\mu_0 = \int_X \frac{U(\varrho_t)}{\varrho_t}\,\mathrm{d}((T_t)_\sharp\mu_0) \\
&= \int_X \frac{U(\varrho_t)}{\varrho_t}\,\mathrm{d}(\varrho_t\mathfrak{m}) = \int_X U(\varrho_t)\,\mathrm{d}\mathfrak{m} = \mathcal{U}(\mu_t) \quad .
\end{aligned}$$

For the action term in (9.52) observe that, since the plan $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu_0, \mu_1)$ is concentrated on constant speed geodesics and recalling (6.12), the minimal 2-velocity $v$ is constant in time and given by $v(x) = \mathsf{d}(x, T_1(x))$. Therefore, noting that $Q(r) = -\frac{1}{N}r^{-\frac{1}{N}}V'(r^{-\frac{1}{N}})$ for $\mathscr{L}^1$-a.e. $r \in (0,1)$, we obtain

$$\begin{aligned}
&\frac{K}{N}\int_E \left[\int_0^1 \bar{\mathfrak{d}}_s(x)\,\mathsf{d}^2(x, T_1(x))\,V'(\bar{\mathfrak{d}}_s(x))\,\mathsf{g}(s,t)\,\mathrm{d}s\right]\mathrm{d}\mu_0(x) \\
&\qquad = K\int_0^1 \left[\int_X v^2(x)\frac{1}{N}\mathfrak{d}_s(x)\,V'(\mathfrak{d}_s(x))\,\mathrm{d}\mu_0(x)\right]\,\mathsf{g}(s,t)\,\mathrm{d}s \\
&\qquad = -K\int_0^1 \left[\int_X v^2(x)Q(\varrho_s(x))\,\mathrm{d}\mu_s(x)\right]\,\mathsf{g}(s,t)\,\mathrm{d}s = -K\mathcal{A}_Q^{(t)}(\mu;\mathfrak{m}).
\end{aligned}$$

[CD4] $\Rightarrow$ [CD1]. Let $\mu_0 = \varrho_0\mathfrak{m}$, $\mu_1 = \varrho_1\mathfrak{m} \in \mathscr{P}^{ac}(X,\mathfrak{m})$ with densities $\varrho_i$ having bounded support, so in particular $\mu_i \in \mathscr{P}_2(X)$. By [RS1-4] there exists a unique $W_2$-geodesic $\mu_t = (\mathrm{e_t})_\sharp\boldsymbol{\pi}$ from $\mu_0$ to $\mu_1$, it is made of absolutely continuous measures and it is given by optimal maps: $\varrho_t\mathfrak{m} = \mu_t = (T_t)_\sharp\mu_0$. Choosing $U = U_N$, we get that $\mu_t$ satisfies (9.51) by assumption. Localizing in space and time as above, we obtain (9.60), namely

$$\mathfrak{d}_{r_t}(x) \geq (1-t)\mathfrak{d}_{r_0}(x) + t\mathfrak{d}_1(x) + \frac{K}{N}(r_1-r_0)^2\int_0^1 \mathfrak{d}_{r_s}(x)\mathsf{d}^2(x,T_1(x))\,\mathsf{g}(s,t)\,\mathrm{d}s \quad \mu_0\text{-a.e. } x.$$

It follows that, for any fixed countable $\mathbb{Q}$-vector space $\mathfrak{D} \subset \mathbb{R}$, there exists a Borel subset $E_\mathfrak{D} \subset \operatorname{supp}\mu_0$ with $\mu_0(X \setminus E_\mathfrak{D}) = 0$ such that (9.60) holds for every $x \in E_\mathfrak{D}$, every $r_0, r_1 \in [0,1]\cap\mathfrak{D}$ and $t \in [0,1]\cap\mathbb{Q}$. Choosing simply $\mathfrak{D} = \mathbb{Q}$, for every fixed $x \in E := E_\mathbb{Q}$, we can apply Lemma 9.2 to the function $[0,1]\cap\mathbb{Q} \ni r \mapsto \mathfrak{d}_r(x) \in \mathbb{R}^+$ and infer that such a map is locally Lipschitz, so it admits a unique continuous extension $[0,1] \ni t \mapsto \bar{\mathfrak{d}}_t(x) \in \mathbb{R}^+$ satisfying (9.56). Lemma 9.1 gives then (9.55) and Lemma 9.4 yields

$$\bar{\mathfrak{d}}_t(x) \geq \sigma^{(1-t)}_{K/N}(\mathsf{d}(x,T_1(x)))\,\bar{\mathfrak{d}}_0(x) + \sigma^{(t)}_{K/N}(\mathsf{d}(x,T_1(x)))\,\bar{\mathfrak{d}}_1(x), \forall x \in E, \forall t \in \mathbb{Q}\cap[0,1]. \quad (9.63)$$

Arguing as in the implication [CD1] $\Rightarrow$ [CD2], we get that for every $t \in [0,1]$ one has $\bar{\mathfrak{d}}_t = \mathfrak{d}_t$, $\mu_0$-a.e. in $X$. Integrating (9.63) in $\mathrm{d}\mu_0(x)$ gives (9.44) for $\mathcal{U}_N$; since for every $M > N$ one has $U_M \in \mathrm{DC}(N)$, the argument for any other $M > N$ is completely analogous: just replace $N$ with $M$ in the formulas above.

[CD5] $\Rightarrow$ [CD2]. Let $\mu_i = \varrho_i\mathfrak{m} \in \mathscr{P}^{ac}(X,\mathfrak{m})$ with $\varrho_i$ $\mathfrak{m}$-essentially bounded having bounded supports, $i = 0,1$, and $\varrho_t\mathfrak{m} = \mu_t = (T_t)_\sharp\mu_0 = (\mathrm{e_t})_\sharp\boldsymbol{\pi}$ be the unique $W_2$-geodesic joining them. Under our working assumptions we have that $\varrho_t$, $t \in [0,1]$, are uniformly $\mathfrak{m}$-essentially bounded with uniformly bounded supports. Given the $N$-dimensional entropy $U(r) := Nr(1-r^{-1/N})$ with associated pressure $P(r) := r^{1-1/N}$, for every $k \in \mathbb{N}$ call $P_k$ the regularized pressure $P_k := P_{1/k,k}$ where $P_{1/k,k}$ was defined in (9.34). Called $U_k$ the regularized and normalized entropy associated to $P_k$ as in (9.31), observe that

$$P_k \to P \text{ and } U_k \to U \text{ uniformly on } [0,R] \text{ for every } R \in \mathbb{R}^+. \quad (9.64)$$

Since $U_k \in \mathrm{DC}_{reg}(N)$, by assumption for every $k \in \mathbb{N}$ we have

$$\begin{aligned}\int_{\operatorname{supp}\mu_t} U_k(\varrho_t)\,\mathrm{d}\mathfrak{m} \;\; &\leq \;\; (1-t)\int_{\operatorname{supp}\mu_0} U_k(\varrho_0)\,\mathrm{d}\mathfrak{m} + t\int_{\operatorname{supp}\mu_1} U_k(\varrho_1)\,\mathrm{d}\mathfrak{m} \\ &\quad -K\int_0^1\left[\int_{\operatorname{supp}\mu_s} P_k(\varrho_s)\,\mathsf{d}^2(x,T_1(x))\,\mathrm{d}\mathfrak{m}(x)\right]\mathsf{g}(s,t)\,\mathrm{d}s\;, \quad (9.65)\end{aligned}$$

where we used that $Q(r) = P(r)/r$ by definition. Recalling that $\varrho_t$ are uniformly $\mathfrak{m}$-essentially bounded with uniformly bounded supports, we infer that $\mathfrak{m}\left(\bigcup_t \operatorname{supp}(\mu_t)\right) < \infty$ and the uniform convergence (9.64) allows to pass to the limit in (9.65), obtaining (9.51). $\square$

**Lemma 9.16** *Let $(X,\mathsf{d},\mathfrak{m})$ be a strong* $\mathrm{CD}(K,\infty)$ *space, so that* [RS1-4] *hold. Consider $\mu_i = \varrho_i\mathfrak{m} \in \mathscr{P}^{ac}(X,\mathfrak{m}) \cap \mathscr{P}_2(X)$, $i = 0,1$, and let $\boldsymbol{\pi} \in \mathrm{GeoOpt}(\mu_0,\mu_1)$ be the plan representing the $W_2$-geodesic $\varrho_t\mathfrak{m} = \mu_t = (\mathrm{e}_t)_\sharp\boldsymbol{\pi} = (T_t)_\sharp(\mu_0)$ from $\mu_0$ to $\mu_1$.*

*Then there exist sequences of measures $\mu_0^k = \varrho_0^k\mathfrak{m} \in \mathscr{P}^{ac}(X,\mathfrak{m})$ and constants $c_k \uparrow 1$ such that the curve $\varrho_t^k\mathfrak{m} := \mu_t^k := (T_t)_\sharp(\mu_0^k)$ is the $W_2$-geodesic from $\mu_0^k$ to $\mu_1^k$ and it satisfies the following:*

*(1) $\varrho_i^k$ are $\mathfrak{m}$-essentially bounded and with bounded support, $i = 0,1$;*

*(2) $c_k\,\varrho_t^k \le \varrho_t$ $\mathfrak{m}$-a.e.in $X$ for every $t \in [0,1]$;*

*(3) for every $t \in [0,1]$ it holds $c_k\,\rho_t^k(x) = \rho_t(x)$ for $\mathfrak{m}$-a.e. $x \in X$, for $k$ large enough possibly depending on $x$;*

*(4) $\mu_0^k = c_k^{-1}\sigma_k\mu_0$ with $\sigma_k \uparrow 1$, $\mu_0$-a.e. on $X$.*

*Proof.* Fix a base point $\bar{x} \in \operatorname{supp}\mu_0$, call $B_k := B_k(\bar{x})$ the ball of center $\bar{x}$ and radius $k \in \mathbb{N}$. For every $k \in \mathbb{N}$ consider first the densities $\bar{\varrho}_0^k := \chi_{B_k}\min\{k,\varrho_0\}$ and the push forward measures $\bar{\mu}_1^k := (T_1)_\sharp(\bar{\varrho}_0^k\mathfrak{m})$. Since clearly $\bar{\varrho}_0^k \le \varrho_0$ and $\bar{\varrho}_0^k = \varrho_0$ on $\{x \in B_k : \varrho_0(x) \le k\}$, and since by assumption $T_1$ is $\mu_0$-essentially injective, we have

$$\bar{\varrho}_1^k\mathfrak{m} := \bar{\mu}_1^k \le \mu_1 \quad \text{and} \quad \bar{\varrho}_1^k = \varrho_1 \quad \text{on } T_1(\{x \in B_k : \varrho_0(x) \le k\}). \tag{9.66}$$

Consider now $\tilde{\varrho}_1^k := \chi_{B_k}\min\{k,\bar{\varrho}_1^k\}$. Using again the $\mu_0$-essential injectivity of $T_1$ and observing that $\tilde{\varrho}_1^k \le \bar{\varrho}_1^k \le \varrho_1$, we can define $\tilde{\mu}_0^k := (T_1^{-1})_\sharp(\tilde{\varrho}_1^k\mathfrak{m})$. By construction we have

$$\tilde{\varrho}_0^k\mathfrak{m} := \tilde{\mu}_0^k \le \bar{\varrho}_0^k\mathfrak{m} \le \mu_0 \ \text{ and } \ \tilde{\varrho}_0^k = \varrho_0 \text{ on } T_1^{-1}\big(T_1\big(B_k \cap \big\{\max\{\varrho_0,\bar{\varrho}_1^k\} \le k\big\}\big)\big); \tag{9.67}$$

in particular we have that $\tilde{\varrho}_i^k \le k$ and $\operatorname{supp}\tilde{\varrho}_i^k \subset B_k$, $i = 0,1$. Moreover, for $\mathfrak{m}$-a.e. $x$ we have $\tilde{\varrho}_0^k(x) = \varrho_0(x)$ for $k$ large enough (possibly depending on $x$).
Setting $c_k := \tilde{\mu}_0^k(X)$, $\mu_0^k := c_k^{-1}\tilde{\mu}_0^k$ and $\mu_t^k := (T_t)_\sharp(\mu_0^k)$ we get the thesis. □

**Lemma 9.17** *Let $\mu_0 \in \mathscr{P}(X)$ and let $T : \operatorname{supp}\mu_0 \to X$ be a $\mu_0$-essentially injective map such that $\varrho_1\mathfrak{m} := \mu_1 := T_\sharp(\mu_0) \in \mathscr{P}^{ac}(X,\mathfrak{m})$. Then*

$$\mu_0(\{x \in \operatorname{supp}\mu_0 : \varrho_1(T(x)) = 0\}) = 0. \tag{9.68}$$

*In particular, given $\mu_t = \varrho_t\mathfrak{m} = (T_t)_\sharp(\mu_0)$ a $W_2$-geodesic as in [RS1-3], for any finite subset $\mathfrak{F} \subset [0,1]$ we have*

$$\mu_0\big(\{x \in \operatorname{supp}\mu_0 : \min_{r\in\mathfrak{F}}\varrho_r(T_r(x)) > 0\}\big) = 1. \tag{9.69}$$

*Proof.* Let us consider the set $A := \{x \in \operatorname{supp}\mu_0 : \varrho_1(T(x)) = 0\}$. Since by assumption $\mu_1 = T_\sharp(\mu_0)$ and $T$ is $\mu_0$-essentially injective, we have that $T$ is $\mu_1$-a.e. invertible and $\mu_0 = (T^{-1})_\sharp(\mu_1)$. It follows that

$$\mu_0(A) = \mu_0\big(T^{-1}(T(A))\big) = \mu_1(T(A)) = \int_{T(A)}\varrho_1\,\mathrm{d}\mathfrak{m} = 0,$$

since, by definition of $A$, we have $\rho_1 \equiv 0$ on $T(A)$. This proves the first statement. The second one is an easy consequence of the finiteness of $\mathfrak{F}$; indeed, called

$$A_r := \{x \in \operatorname{supp}\mu_0 \,:\, \varrho_r(T_r(x)) = 0\},$$

by the first part of the lemma we have that $\mu_0(A_r) = 0$ for every $r \in \mathfrak{F}$. Denoted with

$$C_n := \left\{x \in \operatorname{supp}\mu_0 \,:\, \varrho_r(T_r(x)) \geq \frac{1}{n} \text{ for every } r \in \mathfrak{F}\right\},$$

using the finiteness of $\mathfrak{F}$ we have

$$\bigcup_{n\in\mathbb{N}} C_n = X \setminus \bigcup_{r\in\mathfrak{F}} A_r\,.$$

We conclude that $\bigcup_{n\in\mathbb{N}} C_n$ is of full $\mu_0$-measure and the proof is complete. □

## 9.4 $\mathrm{RCD}(K,\infty)$ spaces and a criterium for $\mathrm{CD}^*(K,N)$ via EVI

Let us first recall the definition of $\mathrm{RCD}(K,\infty)$ spaces, introduced and characterized in [6] (see also [2] for the present simplified axiomatization and extension to $\sigma$-finite measures); in the statements involving the so-called evolution variational inequalities, characterized by differential inequalities involving the squared distance, the entropy and suitable action functionals, we will use the notation

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\zeta(t) := \limsup_{h\downarrow 0} \frac{\zeta(t+h)-\zeta(t)}{h} \tag{9.70}$$

for the upper right Dini derivative.

**Definition 9.18 ($\mathrm{RCD}(K,\infty)$ metric measure spaces)** *A metric measure space $(X,\mathsf{d},\mathfrak{m})$ is an $\mathrm{RCD}(K,\infty)$ space if it satisfies one of the following equivalent conditions:*

(RCD1) *$(X,\mathsf{d},\mathfrak{m})$ satisfies the $\mathrm{CD}(K,\infty)$ condition and the Cheeger energy is quadratic.*

(RCD2) *For every $\mu \in D(\mathcal{U}_\infty)\cap\mathscr{P}_2(X)$ there exists a curve $\mu_t = \mathsf{H}_t\mu$, $t \geq 0$, such that*

$$\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t}W_2^2(\mu_t,\nu) + \mathcal{U}_\infty(\mu) \leq \mathcal{U}_\infty(\nu) - \frac{K}{2}W_2^2(\mu_t,\nu) \quad \textit{for every } t\geq 0,\ \nu \in D(\mathcal{U}_\infty). \tag{9.71}$$

Among the important consequences of the above property, we recall that:

1. $\mathrm{RCD}(K,\infty)$ spaces are *strong* $\mathrm{CD}(K,\infty)$ spaces and thus satisfy properties [RS1-4].
2. The map $(\mathsf{H}_t)_{t\geq 0}$ is uniquely characterized by (9.71), it is a $K$-contraction in $\mathscr{P}_2(X)$ and it coincides with the heat flow $\mathsf{P}_t$, i.e.

$$\mathsf{H}_t(\varrho\mathfrak{m}) = (\mathsf{P}_t\varrho)\mathfrak{m} \quad \text{for every } \varrho\mathfrak{m} \in D(\mathcal{U}_\infty)\cap\mathscr{P}_2(X). \tag{9.72}$$

3. Lipschitz functions essentially coincide with functions $f \in \mathbb{V}$ with $|\mathrm{D}f|_w \in L^\infty(X,\mathfrak{m})$, more precisely (recall that, according to (3.1), $\mathbb{V}_\infty$ stands for $\mathbb{V} \cap L^\infty(X,\mathfrak{m})$):

$$\text{every } f \in \mathbb{V}_\infty \text{ with } |\mathrm{D}f|_w \le 1 \ \mathfrak{m}\text{-a.e. in } X \text{ admits a 1-Lipschitz representative.} \tag{9.73}$$

4. The Cheeger energy satisfies the Bakry-Émery $\mathrm{BE}(K,\infty)$ condition: we will discuss this aspect in the next Section 10.

We will show that a similar characterization holds for strong $\mathrm{CD}^*(K,N)$ spaces.

In order to deal with a general class of entropy functionals $\mathcal{U}$ with entropy density satisfying the McCann condition $\mathrm{DC}(N)$ and arbitrary curvature bounds $K \in \mathbb{R}$, for every $\mu \in \mathrm{AC}^2([0,1];(\mathscr{P}_2(X),W_2))$ with $\mu_s \ll \mathfrak{m}$ for $\mathscr{L}^1$-a.e. $s \in (0,1)$ we consider the weighted action functional associated to $\mathfrak{Q}(s,r) = \omega(s)Q(r)$ as in (7.4), with $\omega(s) := 1-s$:

$$\mathcal{A}_{\mathfrak{Q}}(\mu;\mathfrak{m}) := \mathcal{A}_{\omega Q}(\mu;\mathfrak{m}) = \int_{\tilde{X}} (1-s)Q(\varrho(x,s))v^2(x,s)\,\mathrm{d}\tilde{\mu}(x,s). \tag{9.74}$$

If $(X,\mathsf{d},\mathfrak{m})$ is a strong $\mathrm{CD}(K,\infty)$ space then for every $\mu_0,\ \mu_1 \in \mathscr{P}^{ac}(X,\mathfrak{m})$ we can also set

$$\mathcal{A}_{\omega Q}(\mu_0,\mu_1;\mathfrak{m}) := \mathcal{A}_{\omega Q}(\mu;\mathfrak{m}), \quad \text{with } \mu \text{ the unique geodesic connecting } \mu_0 \text{ to } \mu_1. \tag{9.75}$$

Since $\omega(1-s)+\omega(s)=1$, we obtain the useful identity

$$\mathcal{A}_Q(\mu_0,\mu_1;\mathfrak{m}) = \mathcal{A}_{\omega Q}(\mu_0,\mu_1;\mathfrak{m}) + \mathcal{A}_{\omega Q}(\mu_1,\mu_0;\mathfrak{m}). \tag{9.76}$$

**Theorem 9.19** *Let $(X,\mathsf{d},\mathfrak{m})$ be a strong $\mathrm{CD}^*(K,N)$ space and let us suppose that the Cheeger energy $\mathsf{Ch}$ is quadratic as in (5.28). Let $U \in \mathrm{DC}_{reg}(N)$, $P$, $Q$ as in (9.27), $\Lambda := \inf_{r\ge0} KQ(r)$ and let $(\mathsf{S}_t)_{t\ge0}$ be the flow defined by Theorem 3.4.*
*Then $\mathsf{S}_t$ induces a $\Lambda$-contraction in $(\mathscr{P}_2(X),W_2)$ and for every $\mu = \varrho\mathfrak{m} \in D(\mathcal{U}) \cap \mathscr{P}_2(X)$ the curve $\mu_t := (\mathsf{S}_t\varrho)\mathfrak{m}$ satisfies*

$$\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t}W_2^2(\mu_t,\nu) + \mathcal{U}(\mu_t) \le \mathcal{U}(\nu) - K\mathcal{A}_{\mathfrak{Q}}(\mu_t,\nu;\mathfrak{m}) \quad \textit{for every } \nu \in D(\mathcal{U}) \cap \mathscr{P}_2(X),\ t \ge 0. \tag{9.77}$$

*Proof.* The proof of (9.77) follows the lines of [2] (where the case $\mathcal{U} = \mathcal{U}_\infty$ was considered), which extends to the $\sigma$-finite case the analogous result proved with finite reference measures $\mathfrak{m}$ in [6]. Specifically, first the proof is reduced to the case of measures $\mu = \varrho\mathfrak{m}$ and $\nu$ with $\varrho \in L^\infty(X,\mathfrak{m})$ and $\nu$ with bounded support. Then, using the dual formulation (5.14) of the optimal transport problem, one can show that for $\mathscr{L}^1$-a.e. $t > 0$ one has (see [2, Thm. 6.3])

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t,\nu) = \int_X \Gamma_{\varrho_t}(\varphi_t, P(\varrho_t))\,\mathrm{d}\mathfrak{m} \tag{9.78}$$

for *any* optimal Kantorovich potential $\varphi_t$ from $\mu_t$ to $\nu$. On the other hand, one can also use the calculus tools developed in [5, 6] to estimate (see [2, Thm. 6.5])

$$\mathcal{U}(\nu) - \mathcal{U}(\mu_t) - K\mathcal{A}_{\mathfrak{Q}}(\mu_t, \nu; \mathfrak{m}) \geq \int_X \Gamma_{\varrho_t}(\varphi_t, U'(\varrho_t))\varrho_t \,\mathrm{d}\mathfrak{m} \tag{9.79}$$

for *some* optimal Kantorovich potential $\varphi_t$ from $\mu_t$ to $\nu$. Since $P'(z) = zU''(z)$, the combination of (9.78) and (9.79) gives (9.77).

In turn, the proof of (9.79) goes as follows. First of all one notices that

$$\lim_{s\downarrow 0} \frac{1}{s}\mathcal{A}_Q^{(s)}(\mu_{\cdot,t}; \mathfrak{m}) = \mathcal{A}_{\mathfrak{Q}}(\mu_t, \nu; \mathfrak{m}), \tag{9.80}$$

where $s \mapsto \mu_{s,t}$ is any constant speed geodesic joining $\mu_t$ to $\nu$. Indeed, setting $\mu_{s,t} = \varrho_{s,t}\mathfrak{m}$ and denoting by $v_{s,t}(x)$ the minimal velocity density of $\mu_{\cdot,t}$, we can use the expression (9.1) of $\mathsf{g}$ to write

$$\mathcal{A}_Q^{(s)}(\mu_{\cdot,t}; \mathfrak{m}) = \int_0^s (1-s)r \int_X Q(\varrho_{r,t})v_{r,t}^2\varrho_{r,t}\,\mathrm{d}\mathfrak{m}\,\mathrm{d}r + \int_s^1 (1-r)s \int_X Q(\varrho_{r,t})v_{r,t}^2\varrho_{r,t}\,\mathrm{d}\mathfrak{m}\,\mathrm{d}r.$$

Since the first term in the right hand side is $o(s)$ (recall that $Q$ is a bounded function), by monotone convergence we obtain (9.80).

Now, by the convexity inequality (9.52) one has

$$\mathcal{U}(\nu) - \mathcal{U}(\mu_t) - \liminf_{s\downarrow 0} \frac{1}{s}K\mathcal{A}_Q^{(s)}(\mu_{\cdot,t}; \mathfrak{m}) \geq \limsup_{s\downarrow 0} \frac{\mathcal{U}(\mu_{s,t}) - \mathcal{U}(\mu_t)}{s}. \tag{9.81}$$

In addition, if $\varrho_t$ decays sufficiently fast at infinity, one can estimate the directional derivative of $\mathcal{U}$ as follows:

$$\limsup_{s\downarrow 0} \frac{\mathcal{U}(\mu_{s,t}) - \mathcal{U}(\mu_t)}{s} \geq \limsup_{s\downarrow 0} \int_X U'(\varrho_t)\frac{\varrho_{s,t} - \varrho_t}{s}\,\mathrm{d}\mathfrak{m} \geq \int_X \Gamma_{\varrho_t}(\varphi_t, U'(\varrho_t))\varrho_t\,\mathrm{d}\mathfrak{m}. \tag{9.82}$$

The combination of (9.81) and (9.82) gives (9.79), taking (9.80) into account. Then the decay assumption on $\varrho_t$ is removed by an approximation argument, recovering (9.79) in the general case. This concludes the proof of (9.77).

Since geodesics have constant speed, from (6.11) we obtain the identity

$$\int_0^1 (1-r)\int_X v_{r,t}^2\varrho_{r,t}\,\mathrm{d}\mathfrak{m}\,\mathrm{d}r = \frac{1}{2}W_2^2(\mu_t, \nu).$$

Hence, from (9.77) we get the standard $\mathsf{EVI}$ condition (9.71) with $\mathcal{U}_\infty$ replaced by $\mathcal{U}$ and $K$ replaced by $\Lambda$, and it is well-known (see for instance [3, Cor. 4.3.3]) that this leads to $\Lambda$-contractivity. $\square$

Conversely, we can now prove adapting the proof of [25] that the infinitesimal version of (9.77) leads to the strong $\mathrm{CD}^*(K, N)$ condition.

**Theorem 9.20** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a strong $\mathrm{CD}(K, \infty)$ metric measure space. Suppose that for every $U \in \mathrm{DC}_{reg}(N)$ and every $\bar\mu = \varrho\mathfrak{m} \in \mathscr{P}_2(X)$ with $\varrho \in L^\infty(X, \mathfrak{m})$ there exists a curve $\mu_t = \mathsf{S}_t\bar\mu \in \mathscr{P}_2(X)$, $t \geq 0$, such that*

$$\limsup_{h\downarrow 0} \frac{W_2^2(\mu_h, \nu) - W_2^2(\bar\mu, \nu)}{2h} + \mathcal{U}(\bar\mu) \leq \mathcal{U}(\bar\nu) - K\mathcal{A}_{\mathfrak{Q}}(\bar\mu, \bar\nu; \mathfrak{m}) \tag{9.83}$$

*for every $\bar\nu \in D(\mathcal{U}) \cap \mathscr{P}_2(X)$. Then $(X, \mathsf{d}, \mathfrak{m})$ satisfies the strong $\mathrm{CD}^*(K, N)$ condition and the Cheeger energy is quadratic.*

*Proof.* We prove the validity of [CD2] of Theorem 9.15. So, let us fix $\mu_0$, $\mu_1 \in D(\mathcal{U})$ with bounded densities and support and let $(\mu_s)_{s\in[0,1]}$ be the geodesic connecting $\mu_0$ to $\mu_1$. For a given $s \in (0, 1)$, let $\mu_{s,t} := \mathsf{S}_t\mu_s$ be the curve starting from $\bar\mu = \mu_s$ and satisfying (9.83).

Choosing $\nu := \mu_0$ and taking the right upper derivative at $t = 0$ (still denoted for simplicity by $\mathrm{d}^+/\mathrm{d}t$) we get

$$\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t} W_2^2(\mu_{s,t}, \mu_0)_{|t=0} + \mathcal{U}(\mu_s) - \mathcal{U}(\mu_0) \leq -K\mathcal{A}_{\mathfrak{Q}}(\mu_s, \mu_0; \mathfrak{m}).$$

Similarly, choosing $\nu := \mu_1$, we get

$$\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t} W_2^2(\mu_{s,t}, \mu_1)_{|t=0} + \mathcal{U}(\mu_s) - \mathcal{U}(\mu_1) \leq -K\,\mathcal{A}_{\mathfrak{Q}}(\mu_s, \mu_1; \mathfrak{m}).$$

Let us observe, as in [25], that

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\Big((1-s)W_2^2(\mu_{s,t}, \mu_0) + sW_2^2(\mu_{s,t}, \mu_1)\Big)_{|t=0_+} \geq 0$$

since the inequality $(a + b)^2 \leq a^2/s + b^2/(1 - s)$ gives

$$(1-s)W_2^2(\mu_{s,t}, \mu_0) + sW_2^2(\mu_{s,t}, \mu_1) \geq s(1-s)W_2^2(\mu_0, \mu_1) = (1-s)W_2^2(\mu_s, \mu_0) + sW_2^2(\mu_s, \mu_1).$$

Hence, taking a convex combination of the two inequalities with weights $(1 - s)$ and $s$ respectively, we obtain

$$(1-s)\mathcal{U}(\mu_0) + s\mathcal{U}(\mu_1) - \mathcal{U}(\mu_s) \geq (1-s)K\,\mathcal{A}_{\mathfrak{Q}}(\mu_s, \mu_0; \mathfrak{m}) + sK\,\mathcal{A}_{\mathfrak{Q}}(\mu_s, \mu_1; \mathfrak{m}).$$

Now observe that (for $\Theta_r = \int Q(\varrho_r)v_r^2\,\mathrm{d}\mu_r$, $s(1-\xi) = r$)

$$\mathcal{A}_{\mathfrak{Q}}(\mu_s, \mu_0; \mathfrak{m}) = s^2\int_0^1 \Theta_{s(1-\xi)}(1-\xi)\,\mathrm{d}\xi = \int_0^s \Theta_r\, r\,\mathrm{d}r$$

and, analogously, that (for $\Theta_r$ as above, $s + (1-s)\xi = r$)

$$\mathcal{A}_{\mathfrak{Q}}(\mu_s, \mu_1; \mathfrak{m}) = (1-s)^2\int_0^1 \Theta_{s+(1-s)\xi}(1-\xi)\,\mathrm{d}\xi = \int_s^1 \Theta_r(1-r)\,\mathrm{d}r,$$

so that the definition (9.1) of $\mathsf{g}$ gives

$$(1-s)\mathcal{A}_{\mathfrak{Q}}(\mu_s,\mu_0;\mathfrak{m}) + s\mathcal{A}_{\mathfrak{Q}}(\mu_s,\mu_1;\mathfrak{m}) = \int_0^s \Theta_r\,(1-s)r\,\mathrm{d}r + \int_s^1 \Theta_r s(1-r)\,\mathrm{d}r$$
$$= \int_0^1 \Theta_r \mathsf{g}(r,s)\,\mathrm{d}r = \mathcal{A}_Q^{(s)}(\mu;\mathfrak{m}).$$

This proves that (9.52) of Theorem 9.15 holds and therefore $(X,\mathsf{d},\mathfrak{m})$ is a strong $\mathrm{CD}^*(K,N)$ space. It remains to show that the Cheeger energy is quadratic; by applying the characterization of $\mathrm{RCD}(K,\infty)$ spaces recalled in Definition 9.18 it is sufficient to check that (9.83) yields (9.71) as a particular case. In fact, we can choose the regular entropy $U_\infty(r) := r\log r \in \mathrm{DC}_{reg}(N)$ with $Q_\infty \equiv 1$, and observe that the associated weighted action on constant speed geodesics is nothing but half of the standard 2-action:

$$\mathcal{A}_{\omega Q_\infty}(\mu_0,\mu_1;\mathfrak{m}) = \int_0^1\int_X (1-s)\,v^2(x,s)\,\mathrm{d}\mu_s\,\mathrm{d}s = \int_0^1 (1-s)|\dot\mu_s|^2\,\mathrm{d}s = \frac{1}{2}W_2^2(\mu_0,\mu_1)^2,$$

where in the second equality we recalled (6.11) and in the last one we used that $(\mu_s)_{s\in[0,1]}$ is a constant speed geodesic. $\square$

# Part III
# Bakry-Émery condition and nonlinear diffusion

## 10 The Bakry-Émery condition

In this section we will recall the basic assumptions related to the Bakry-Émery condition and we will prove some important properties related to them. In the case of a locally compact space we will also establish a useful local criterium to check this condition.

### 10.1 The Bakry-Émery condition for local Dirichlet forms and interpolation estimates

The natural setting is provided by a Polish topological space $(X,\tau)$ endowed with a $\sigma$-finite reference Borel measure $\mathfrak{m}$ and a strongly local symmetric Dirichlet form $\mathcal{E}$ in $L^2(X,\mathfrak{m})$ enjoying a *Carré du Champ* $\Gamma: D(\mathcal{E})\times D(\mathcal{E}) \to L^1(X,\mathfrak{m})$ and a $\Gamma$-calculus (see e.g. [7, § 2]). All the estimates we are discussing in this section and in the next one, Section 11, devoted to action estimates for nonlinear diffusion equations do not really need an underlying compatible metric structure, as the one discussed in [7, § 3]. We refer to [7, §2] for the basic notation and assumptions; in any case, we will apply all the results to the case of the

Cheeger energy (thus assumed to be quadratic) of the metric measure space $(X, \mathsf{d}, \mathfrak{m})$ and we keep the same notation of the previous Section 5.5, just using the calculus properties of the Dirichlet form that are related to the $\Gamma$-formalism.

In the following we set $\mathbb{V}_\infty := \mathbb{V} \cap L^\infty(X, \mathfrak{m})$, $\mathbb{D}_\infty := \mathbb{D} \cap L^\infty(X, \mathfrak{m})$,

$$\begin{cases} D_{L^p}(\mathrm{L}) := \big\{f \in \mathbb{D} \cap L^p(X, \mathfrak{m}) : \ \mathrm{L}f \in L^p(X, \mathfrak{m})\big\} & p \in [1, \infty], \\ D_{\mathbb{V}}(\mathrm{L}) = \big\{f \in \mathbb{D} : \ \mathrm{L}f \in \mathbb{V}\big\}, \end{cases} \tag{10.1}$$

endowed with the norms

$$\|f\|_{D_{L^p}} := \|f\|_{\mathbb{V}} + \|f - \mathrm{L}f\|_{L^2 \cap L^p(X,\mathfrak{m})}, \quad \|f\|^2_{D_{\mathbb{V}}} := \|f\|^2_{L^2(X,\mathfrak{m})} + \|\mathrm{L}f\|^2_{\mathbb{V}}, \tag{10.2}$$

and we introduce the multilinear form $\mathbf{\Gamma}_2$ given by

$$\mathbf{\Gamma}_2(f, g; \varphi) := \frac{1}{2} \int_X \Big(\Gamma\big(f, g\big) \, \mathrm{L}\varphi - \Gamma\big(f, \mathrm{L}g\big)\varphi - \Gamma\big(g, \mathrm{L}f\big)\varphi\Big) \, \mathrm{d}\mathfrak{m} \qquad (f, g, \varphi) \in D(\mathbf{\Gamma}_2), \tag{10.3}$$

where $D(\mathbf{\Gamma}_2) := D_{\mathbb{V}}(\mathrm{L}) \times D_{\mathbb{V}}(\mathrm{L}) \times D_{L^\infty}(\mathrm{L})$. When $f = g$ we also set

$$\mathbf{\Gamma}_2(f; \varphi) := \mathbf{\Gamma}_2(f, f; \varphi) = \int_X \Big(\frac{1}{2}\Gamma\big(f\big) \, \mathrm{L}\varphi - \Gamma\big(f, \mathrm{L}f\big)\varphi\Big) \, \mathrm{d}\mathfrak{m}, \tag{10.4}$$

so that

$$\mathbf{\Gamma}_2(f, g; \varphi) = \frac{1}{4}\mathbf{\Gamma}_2(f + g; \varphi) - \frac{1}{4}\mathbf{\Gamma}_2(f - g; \varphi). \tag{10.5}$$

$\mathbf{\Gamma}_2$ provides a weak version (inspired by [12, 15]) of the Bakry-Émery condition [13, 11].

**Definition 10.1 (Bakry-Émery conditions)** *Let $K \in \mathbb{R}$, $N \in [1, \infty]$. We say that the strongly local Dirichlet form $\mathcal{E}$ satisfies the* BE$(K, N)$ *condition, if it admits a Carré du Champ $\Gamma$ and for every $(f, \varphi) \in D_{\mathbb{V}}(\mathrm{L}) \times D_{L^\infty}(\mathrm{L})$ with $\varphi \geq 0$ one has*

$$\mathbf{\Gamma}_2(f; \varphi) \geq K \int_X \Gamma\big(f\big) \, \varphi \, \mathrm{d}\mathfrak{m} + \frac{1}{N} \int_X (\mathrm{L}f)^2 \varphi \, \mathrm{d}\mathfrak{m}. \tag{10.6}$$

*We say that a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ (see § 5.5) satisfies the* metric BE$(K, N)$ *condition if the Cheeger energy is quadratic, the associated Dirichlet form $\mathcal{E}$ satisfies* BE$(K, N)$*, and*

$$\textit{any} \quad f \in \mathbb{V}_\infty \quad \textit{with} \quad \Gamma\big(f\big) \in L^\infty(X, \mathfrak{m}) \quad \textit{has a 1-Lipschitz representative.} \tag{10.7}$$

**Remark 10.2 (Pointwise gradient estimates for** BE$(K, \infty)$**)** When $N = \infty$, the inequality (10.6) is in fact equivalent (see [7, Cor. 2.3] for a proof in the abstract setup of this section) to either of the following pointwise gradient estimates

$$\Gamma\big(\mathsf{P}_t f\big) \leq \mathrm{e}^{-2Kt} \, \mathsf{P}_t\big(\Gamma\big(f\big)\big) \quad \mathfrak{m}\text{-a.e. in } X, \text{ for every } f \in \mathbb{V}, \tag{10.8}$$

$$2\mathrm{I}_{2K}(t)\Gamma\big(\mathsf{P}_t f\big) \le \mathsf{P}_t f^2 - \big(\mathsf{P}_t f\big)^2 \quad \mathfrak{m}\text{-a.e. in } X, \quad \text{for every } t > 0,\ f \in L^2(X,\mathfrak{m}), \tag{10.9}$$

where $\mathrm{I}_K$ denotes the real function

$$\mathrm{I}_K(t) := \int_0^t \mathrm{e}^{Kr}\,\mathrm{d}r = \begin{cases} \dfrac{1}{K}(\mathrm{e}^{Kt} - 1) & \text{if } K \neq 0, \\ t & \text{if } K = 0. \end{cases}$$

It will be useful to have different expressions for $\mathbf{\Gamma}_2(f;\varphi)$, that make sense under weaker condition on $f$, $\varphi$. Typically their equivalence will be proved by regularization arguments, which will be based on the following approximation result.

**Lemma 10.3 (Density of $D_{\mathbb{V}}(\mathrm{L}) \cap D_{L^\infty}(\mathrm{L})$)** *The vector space $D_{\mathbb{V}}(\mathrm{L}) \cap D_{L^\infty}(\mathrm{L})$ is dense in $D_{\mathbb{V}}(\mathrm{L})$. In addition, if $f \in D_{L^p}(\mathrm{L})$, $p \in [1,\infty]$ satisfies the uniform bounds $c_0 \le f \le c_1$ $\mathfrak{m}$-a.e. in $X$ for some real constants $c_0$, $c_1$, then we can find an approximating sequence $(f_n) \subset D_{\mathbb{V}}(\mathrm{L}) \cap D_{L^\infty}(\mathrm{L})$ converging to $f$ in $D_{L^p}(\mathrm{L})$ with $f_n \to f$ in $\mathbb{V}$ and $\mathrm{L}f_n \to \mathrm{L}f$ in $L^2 \cap L^p$ if $p < \infty$ (in the weak$^*$ sense when $p = \infty$), as $n \to \infty$ and satisfying the same bounds $c_0 \le f_n \le c_1$ $\mathfrak{m}$-a.e. in $X$.*

*Proof.* The proof of the density of $D_{\mathbb{V}}(\mathrm{L}) \cap D_{L^\infty}(\mathrm{L})$ in $D_{\mathbb{V}}(\mathrm{L})$ has been given in [8, Lemma 4.2]. In order to prove the second approximation result, we introduce the mollified heat flow

$$\mathfrak{H}^\varepsilon f := \frac{1}{\varepsilon}\int_0^\infty \mathsf{P}_r f\,\kappa(r/\varepsilon)\,\mathrm{d}r, \tag{10.10}$$

where $\kappa \in \mathrm{C}_c^\infty(0,\infty)$ is a nonnegative regularization kernel with $\int_0^\infty \kappa(r)\,\mathrm{d}r = 1$.

Setting $f_n := \mathfrak{H}^{1/n} f$, since $f \in L^2 \cap L^\infty(X,\mathfrak{m})$ it is not difficult to check that $f_n \in D_{\mathbb{V}}(\mathrm{L}) \cap D_{L^\infty}(\mathrm{L})$. In addition, $c_0 \le f_n \le c_1$, since the heat flow preserves global lower or upper bounds by constants.

We then use the fact L is the generator of a strongly continuous semigroup in each $L^p(X,\mathfrak{m})$ if $p < \infty$ (and of a weak$^*$-continuous semigroup in $L^\infty(X,\mathfrak{m})$). □

**Corollary 10.4** *If (10.6) holds for every $f \in D_{\mathbb{V}}(\mathrm{L}) \cap D_{L^\infty}(\mathrm{L})$ and every nonnegative $\varphi \in D_{L^\infty}(\mathrm{L})$, then the $\mathrm{BE}(K,N)$ condition holds.*

A first representation of $\mathbf{\Gamma}_2$ is provided by the following lemma, whose proof is an easy consequence of the Lebniz rule for $\Gamma$, see [8, Lemma 4.1].

**Lemma 10.5** *If $f \in D_{\mathbb{V}}(\mathrm{L}) \cap D_{L^\infty}(\mathrm{L})$ and $\varphi \in D_{L^\infty}(\mathrm{L})$ then*

$$\mathbf{\Gamma}_2(f;\varphi) = \int_X \Big(\frac{1}{2}\Gamma\big(f\big)\mathrm{L}\varphi + \mathrm{L}f\,\Gamma\big(f,\varphi\big) + \varphi(\mathrm{L}f)^2\Big)\,\mathrm{d}\mathfrak{m}. \tag{10.11}$$

Recalling (10.5) we also get

$$\mathbf{\Gamma}_2(f,g;\varphi) = \frac{1}{2}\int_X \Big(\Gamma\big(f,g\big)\mathrm{L}\varphi + \mathrm{L}f\,\Gamma\big(g,\varphi\big) + \mathrm{L}g\,\Gamma\big(f,\varphi\big) + 2\varphi\,\mathrm{L}f\,\mathrm{L}g\Big)\,\mathrm{d}\mathfrak{m}. \tag{10.12}$$

Notice that (10.11) makes sense even if $f$, $\varphi \in \mathbb{D}_\infty$, provided $\Gamma\big(f\big)$ and $\Gamma\big(f,\varphi\big)$ belong to $L^2(X,\mathfrak{m})$. This extra integrability of $\Gamma$ is a general consequence of the $\mathrm{BE}(K,\infty)$ condition.

**Theorem 10.6 (Gradient interpolation, [8, Thm. 3.1])** *Assume that* $\mathrm{BE}(K,\infty)$ *holds, let* $\lambda \geq K_-$, $p \in \{2,\infty\}$, $f \in L^2 \cap L^\infty(X,\mathfrak{m})$ *with* $\mathrm{L}f \in L^p(X,\mathfrak{m})$. *Then* $\Gamma(f) \in L^p(X,\mathfrak{m})$ *and*

$$\|\Gamma(f)\|_{L^p(X,\mathfrak{m})} \leq c\|f\|_{L^\infty(X,\mathfrak{m})}\,\|\lambda f - \mathrm{L}f\|_{L^p(X,\mathfrak{m})} \tag{10.13}$$

*for a universal constant c independent of* $\lambda, X, \mathfrak{m}, f$ *(*$c = \sqrt{2\pi}$ *when* $p = \infty$*).*

*Moreover, if* $f_n \in \mathbb{D}_\infty$ *with* $\sup_n \|f_n\|_{L^\infty(X,\mathfrak{m})} < \infty$ *and* $f_n \to f$ *strongly in* $\mathbb{D}$*, then* $\Gamma(f_n) \to \Gamma(f)$ *and* $\Gamma(f_n - f) \to 0$ *strongly in* $L^2(X,\mathfrak{m})$*.*

An important consequence of Theorem 10.6 is that $\mathbb{D}_\infty$ is an algebra, also preserved by left composition with functions $h \in \mathrm{C}^2(\mathbb{R})$ vanishing at 0: this can be easily checked by the formula

$$\mathrm{L}(fg) = f\mathrm{L}g + g\mathrm{L}f + 2\Gamma(f,g), \qquad \mathrm{L}(h(f)) = h'(f)\mathrm{L}f + h''(f)\Gamma(f) \tag{10.14}$$

using the fact that $\Gamma(f)$, $\Gamma(f,g) \in L^2(X,\mathfrak{m})$ whenever $f$, $g \in \mathbb{D}_\infty$.

Thanks to the improved integrability of $\Gamma$ given by Theorem 10.6 and to the previous approximation result, we can now extend the domain of $\mathbf{\Gamma}_2$ to the whole of $(\mathbb{D}_\infty)^3$.

**Corollary 10.7 (Extension of $\mathbf{\Gamma}_2$)** *If* $\mathrm{BE}(K,\infty)$ *holds then* $\mathbf{\Gamma}_2$ *can be extended to a continuous multilinear form in* $\mathbb{D}_\infty \times \mathbb{D}_\infty \times \mathbb{D}_\infty$ *by* (10.12) *and* $\mathrm{BE}(K,N)$ *holds if and only if*

$$\int_X \Big(\frac{1}{2}\Gamma(f)\mathrm{L}\varphi + \mathrm{L}f\,\Gamma(f,\varphi) + (1-\frac{1}{N})\varphi(\mathrm{L}f)^2\Big)\,\mathrm{d}\mathfrak{m} \geq K\int_X \Gamma(f)\,\varphi\,\mathrm{d}\mathfrak{m}. \tag{10.15}$$

*is satisfied by every choice of* $f$, $\varphi \in \mathbb{D}_\infty$ *with* $\varphi \geq 0$*.*

*Proof.* Notice that (10.12) makes sense if $f$, $g$, $\varphi \in \mathbb{D}_\infty$ since $\Gamma(f)$, $\Gamma(g)$, $\Gamma(\varphi) \in L^2(X,\mathfrak{m})$ by Theorem 10.6 and that it provides an extension of $\mathbf{\Gamma}_2$ by Lemma 10.5.

In order to check (10.15) under the $\mathrm{BE}(K,N)$ assumption whenever $f$, $\varphi \in \mathbb{D}_\infty$, $\varphi \geq 0$, we first approximate $f, \varphi$ in $\mathbb{D}_\infty$ with elements in $D_{\mathbb{V}}(\mathrm{L})$ via the Heat flow, and then we apply Lemma 10.3 with a diagonal argument to find $f_n$, $\varphi_n \in D_{\mathbb{V}}(\mathrm{L}) \cap D_{L^\infty}(\mathrm{L})$ with $\varphi_n \geq 0$ such that (10.6) and (10.11) yield

$$\int_X \Big(\frac{1}{2}\Gamma(f_n)\mathrm{L}\varphi_n + \mathrm{L}f_n\,\Gamma(f_n,\varphi_n) + (1-\frac{1}{N})\varphi_n(\mathrm{L}f_n)^2\Big)\,\mathrm{d}\mathfrak{m} \geq K\int_X \Gamma(f_n)\,\varphi_n\,\mathrm{d}\mathfrak{m}.$$

Since

$$f_n \to f,\ \varphi_n \to \varphi \quad \text{strongly in } \mathbb{D}, \quad \|f_n\|_{L^\infty(X,\mathfrak{m})} \leq \|f\|_{L^\infty(X,\mathfrak{m})}, \quad \|\varphi_n\|_{L^\infty(X,\mathfrak{m})} \leq \|\varphi\|_{L^\infty(X,\mathfrak{m})}$$

we can apply the estimates stated in Theorem 10.6 to pass to the limit in the previous inequality as $n \to \infty$.

Conversely, if (10.15) holds for every $f$, $\varphi \in \mathbb{D}_\infty$ with $\varphi \geq 0$, it clearly holds for every $f \in D_{\mathbb{V}}(\mathrm{L}) \cap D_{L^\infty}(\mathrm{L})$ and nonnegative $\varphi \in D_{L^\infty}(\mathrm{L})$, thus with the expression of $\mathbf{\Gamma}_2$ given by (10.4), thanks to Lemma 10.5. We can then apply Corollary 10.4. □

## 10.2 Local and "nonlinear" characterization of the metric $\mathrm{BE}(K,N)$ condition in locally compact spaces

When $(X,\mathsf{d},\mathfrak{m})$ is a locally compact space satisfying the metric $\mathrm{BE}(K,\infty)$ condition, the $\mathbf{\Gamma}_2$ form enjoys a few localization properties, that will turn to be useful in the following.

Let us first recall that if $(X,\mathsf{d},\mathfrak{m})$ satisfies the metric $\mathrm{BE}(K,\infty)$ condition, then $(X,\mathsf{d})$ is a length space and the Dirichlet form $\mathcal{E}$ associated to the Cheeger energy is quasi-regular [46, Thm. 4.1].

In the locally compact case, the length condition also yields that $(X,\mathsf{d})$ is proper (i.e. every closed bounded subset of $X$ is compact) and thus geodesic (every couple of points can be joined by a minimal geodesic), see, e.g., [21, Prop. 2.5.22].

A further important property (see e.g. [8, Remark 6.3]) is that $\mathcal{E}$ is regular, i.e. $\mathbb{V}\cap\mathrm{C}_c(X)$ is dense both in $\mathbb{V}$ (w.r.t. the $\mathbb{V}$ norm) and in $\mathrm{C}_c(X)$ (w.r.t. the uniform norm). In particular, by Fukushima's theory (see e.g. [23, 16]), every $\varphi\in\mathbb{V}$ admits a $\mathcal{E}$-quasi continuous representative $\tilde\varphi$ uniquely determined up to $\mathcal{E}$-polar sets and every linear functional $\ell:\mathbb{V}\to\mathbb{R}$ which is nonnegative (i.e. such that $\langle\ell,\varphi\rangle\ge 0$ for every nonnegative $\varphi\in\mathbb{V}$) can be uniquely represented by a $\sigma$-finite Borel measure $\mu_\ell$ which does not charge $\mathcal{E}$-polar sets, so that $\langle\ell,\varphi\rangle=\int_X\tilde\varphi\,\mathrm{d}\mu_\ell$ for every $\varphi\in\mathbb{V}$. We refer to [8, Sect. 5] for more details. We will often identify $\varphi$ with $\tilde\varphi$, when there is no risk of confusion.

Before stating our locality results, we recall two useful facts, obtained in [8] and slightly improving earlier results in [46]. See Corollary 5.7 for statement (i), and Lemma 6.7 of [8] for statement (ii) (more precisely, the statement of [8, Lemma 6.7] deals with a Lipschitz cut off function $\chi$ with $\mathrm{L}\chi\in L^\infty(X,\mathfrak{m})$ and $\Gamma\big(\chi\big)\in\mathbb{V}_\infty$, but since $\chi$ is built of the form $\eta\circ f$ with $\eta$ constant near 0, from Lemma 10.8(i) below and (10.14) one can get also $\mathrm{L}\chi\in\mathbb{V}_\infty$.)

**Lemma 10.8** *Let us suppose that* $(X,\mathsf{d},\mathfrak{m})$ *satisfies the* metric $\mathrm{BE}(K,\infty)$ *condition for some* $K\in\mathbb{R}$.

*(i) For every* $f,\,g\in D_{L^4}(\mathrm{L})$ *we have* $\Gamma\big(f,g\big)\in\mathbb{V}$ *and the bounded linear functional*

$$\mathbb{V}\ni\varphi\mapsto\int_X\Big(-\frac12\Gamma\big(\Gamma\big(f\big),\varphi\big)+\mathrm{L}f\,\Gamma\big(f,\varphi\big)+\big((\mathrm{L}f)^2-K\Gamma\big(f\big)\big)\varphi\Big)\,\mathrm{d}\mathfrak{m}\tag{10.16}$$

*can be represented by a finite nonnegative Borel measure denoted by* $\Gamma^*_{2,K}[f]$, *satisfying*

$$\mathbf{\Gamma}_2(f;\varphi)-K\int_X\Gamma\big(f\big)\varphi\,\mathrm{d}\mathfrak{m}=\int_X\varphi\,\mathrm{d}\Gamma^*_{2,K}[f]\tag{10.17}$$

*for every* $f\in D_{L^4}(\mathrm{L})\cap L^\infty(X,\mathfrak{m})$ *and* $\varphi\in\mathbb{D}_\infty$, *where in* (10.17) *we use the extension of* $\mathbf{\Gamma}_2(f;\varphi)$ *provided by Corollary 10.7.*

*(ii) If* $(X,\mathsf{d})$ *is locally compact, then for every compact set* $E$ *and every open neighborhood* $U\supset E$ *there exists a Lipschitz cutoff function* $\chi:X\to[0,1]$ *such that* $\mathrm{supp}(\chi)\subset U$, $\chi\equiv 1$ *in a neighborhood of* $E$, $\mathrm{L}\chi\in\mathbb{V}_\infty$ *and* $\Gamma\big(\chi\big)\in\mathbb{V}_\infty$.

**Corollary 10.9 (Locality w.r.t. $\varphi$)** *Let $K \in \mathbb{R}$ and $N < \infty$. Let us suppose that $(X, \mathsf{d})$ is locally compact and that $(X, \mathsf{d}, \mathfrak{m})$ satisfies the* metric $\mathrm{BE}(K, \infty)$ *condition. If* (10.6) *holds for every $f \in D_{\mathbb{V}}(\Delta) \cap D_{L^\infty}(\mathrm{L})$ and every nonnegative $\varphi \in D_{L^\infty}(\mathrm{L})$ with compact support, then $(X, \mathsf{d}, \mathfrak{m})$ satisfies the metric $\mathrm{BE}(K, N)$ condition.*

*Proof.* We argue by contradiction: if $\mathrm{BE}(K, N)$ does not hold, by Corollary 10.4 we can find $f \in D_{\mathbb{V}}(\mathrm{L}) \cap D_{L^\infty}(\mathrm{L})$ and a nonnegative $\varphi \in D_{L^\infty}(\mathrm{L})$ such that

$$\mathbf{\Gamma}_2(f; \varphi) - K \int_X \Gamma\big(f\big)\varphi \,\mathrm{d}\mathfrak{m} - \frac{1}{N}\int_X (\mathrm{L}f)^2 \varphi \,\mathrm{d}\mathfrak{m} < 0.$$

Since $D_{\mathbb{V}}(\mathrm{L}) \cap D_{L^\infty}(\mathrm{L}) \subset D_{L^4}(\mathrm{L}) \cap L^\infty(X, \mathfrak{m})$ we can apply the representation result (10.17), thus obtaining that the measure

$$\mu := \varphi \Gamma^*_{2,K}[f] - \frac{\varphi 1}{N}(\Delta f)^2 \mathfrak{m}$$

has a nontrivial negative part. Since $X$ is Polish, we can find a compact set $E$ such that $\mu(E) < 0$; approximating $E$ by a sequence of open set $U_n \downarrow E$, Lemma 10.8(ii) provides a corresponding sequence of nonnegative test functions $\chi_n \in D_{L^\infty}(\Delta)$ such that

$$\lim_{n \to \infty} \int_X \chi_n \,\mathrm{d}\mu = \mu(E) < 0.$$

Choosing $n$ sufficiently large, since $\varphi \chi_n$ has compact support and belongs to $D_{L^\infty}(\mathrm{L})$, this contradicts the assumptions of the Corollary. □

**Theorem 10.10 (Local characterization of $\mathrm{BE}(K, N)$)** *Let us suppose that $(X, \mathsf{d}, \mathfrak{m})$ satisfies the* metric $\mathrm{BE}(K, \infty)$ *condition for some $K \in \mathbb{R}$, and that $(X, \mathsf{d})$ is locally compact. If* (10.6) *with $N < \infty$ holds for every $f \in D_{L^\infty}(\mathrm{L}) \cap D_{\mathbb{V}}(\mathrm{L})$ with compact support and for every nonnegative $\varphi \in D_{L^\infty}(\mathrm{L})$ with compact support and with $\inf_{\mathrm{supp}\, f} \varphi > 0$, then $(X, \mathsf{d}, \mathfrak{m})$ satisfies the metric $\mathrm{BE}(K, N)$ condition.*

*Proof.* By the previous Corollary, we have to check that (10.6) holds if $f \in D_{\mathbb{V}}(\mathrm{L}) \cap D_{L^\infty}(\mathrm{L})$ and $\varphi \in D_{L^\infty}(\mathrm{L})$ nonnegative with compact support. Choosing a cutoff function $\chi \in D_{L^\infty}(\mathrm{L}) \cap D_{\mathbb{V}}(\mathrm{L})$ with compact support, values in $[0, 1]$ and such that $\chi \equiv 1$ on a neighborhood of $\mathrm{supp}(\varphi)$ as in Lemma 10.8(ii), it is easy to check, using Theorem 10.6, the locality properties of $\Gamma$, $\mathrm{L}$ as well as the computation rules

$$\chi f \in \mathbb{D}_\infty, \quad \mathrm{L}(\chi f) = \chi \mathrm{L} f + 2\Gamma\big(\chi, f\big) + f \mathrm{L}\chi, \quad \mathrm{L}(\chi f) = \chi \mathrm{L} f \quad \text{on } \mathrm{supp}(\varphi),$$

that $\chi f \in D_{L^\infty}(\mathrm{L}) \cap D_{\mathbb{V}}(\mathrm{L}) \subset D_{L^4}(\mathrm{L}) \cap L^\infty(X, \mathfrak{m})$ and that

$$\begin{aligned}
\mathbf{\Gamma}_2(f; \varphi) - K \int_X \Gamma(f)\varphi \,\mathrm{d}\mathfrak{m} &= \mathbf{\Gamma}_2(f; \chi\varphi) - K \int_X \Gamma(f)\chi\varphi \,\mathrm{d}\mathfrak{m} = \\
&= \int_X \Big( -\frac{1}{2}\Gamma\big(\Gamma\big(f\big), \chi\varphi\big) + \mathrm{L}f\, \Gamma\big(f, \chi\varphi\big) + (\mathrm{L}f)^2 \chi\varphi - K\Gamma\big(f\big)\chi\varphi \Big) \,\mathrm{d}\mathfrak{m} \\
&= \int_X \Big( -\frac{1}{2}\Gamma\big(\Gamma\big(\chi f\big), \varphi\big) + \mathrm{L}(\chi f)\, \Gamma\big(\chi f, \varphi\big) + (\mathrm{L}(\chi f))^2 \varphi - K\Gamma\big(\chi f\big)\varphi \Big) \,\mathrm{d}\mathfrak{m} \\
&= \mathbf{\Gamma}^*_{2,K}(\chi f; \varphi) = \lim_{\varepsilon \downarrow 0} \mathbf{\Gamma}^*_{2,K}(\chi f; \psi_\varepsilon),
\end{aligned}$$

where $\psi_\varepsilon = \varphi + \varepsilon\hat{\chi}$ and $\hat{\chi} \in D_{L^\infty}(\mathrm{L})$ is another nonnegative cutoff function with compact support such that $\hat{\chi} \equiv 1$ in an open neighborhood of $\mathrm{supp}(\chi f)$. Since by assumption $\mathbf{\Gamma}^*_{2,K}(\chi f; \psi_\varepsilon) \geq 0$ we conclude. □

**Theorem 10.11 (A nonlinear version of the $\mathrm{BE}(K,N)$ condition)** *If the* $\mathrm{BE}(K,N)$ *condition holds and* $P \in \mathrm{DC}(N)$ *is regular with* $R(r) = rP'(r) - P(r)$*, then for every* $f \in \mathbb{D}_\infty$ *and every nonnegative function* $\varphi \in \mathbb{V}_\infty$ *with* $P(\varphi) \in \mathbb{D}_\infty$ *we have*

$$\mathbf{\Gamma}_2(f; P(\varphi)) + \int_X R(\varphi)\,(\mathrm{L}f)^2\,\mathrm{d}\mathfrak{m} \geq K \int_X \Gamma\big(f\big)\,P(\varphi)\,\mathrm{d}\mathfrak{m}. \tag{10.18}$$

*Conversely, let us assume that* $(X, \mathsf{d}, \mathfrak{m})$ *is locally compact and satisfies the metric* $\mathrm{BE}(K,\infty)$*-condition. If* (10.18) *holds for every function* $P = P_{N,\varepsilon,M}$*,* $\varepsilon$*,* $M > 0$ *as in* (9.34) *and* (9.36)*, every* $f \in D_{\mathbb{V}}(\mathrm{L}) \cap D_{L^\infty}(\mathrm{L})$ *with compact support and every nonnegative* $\varphi \in D_{L^\infty}(\mathrm{L})$ *with compact support and* $\inf_{\mathrm{supp}\, f} \varphi > 0$*, then* $(X, \mathsf{d}, \mathfrak{m})$ *satisfies the metric* $\mathrm{BE}(K,N)$ *condition.*

*Proof.* The inequality (10.18) is an obvious consequence of $\mathrm{BE}(K,N)$ (in the form of Corollary 10.7) since $P \in \mathrm{DC}(N)$ yields $R(r) \geq -\frac{1}{N}P(r)$ .

In order to prove the second part of the statement, we apply the previous Theorem 10.10: we fix $f \in D_{\mathbb{V}}(\mathrm{L}) \cap D_{L^\infty}(\mathrm{L})$ and $\varphi \in D_{L^\infty}(\mathrm{L})$ nonnegative, both with compact support and satisfying $\inf\{\varphi(x) : x \in \mathrm{supp}(f)\} > 0$; with this choice of $f$ and $\varphi$ we need to prove (10.6).

We fix $\varepsilon > 0$ and we set $\tilde{\varphi} = P^{-1}_{N,\varepsilon}(\varphi)$; since $\varphi$ is bounded, $\tilde{\varphi} \in D_{L^\infty}(\mathrm{L})$ and therefore we can choose $M > 0$ sufficiently large such that $\tilde{\varphi} \leq M$ and consequently $\varphi = P_{N,\varepsilon,M}(\tilde{\varphi})$. Applying (10.18) with this choice of $f$ and $\tilde{\varphi}$ and recalling the inequality (9.38) we get

$$\mathbf{\Gamma}_2(f; \varphi) - \frac{1}{N}\int_X \varphi\,(\mathrm{L}f)^2\,\mathrm{d}\mathfrak{m} + \varepsilon^{1-1/N}\int_X (\mathrm{L}f)^2\,\mathrm{d}\mathfrak{m} \geq K\int_X \Gamma\big(f\big)\,\varphi\,\mathrm{d}\mathfrak{m}.$$

Passing to the limit as $\varepsilon \downarrow 0$ we get (10.6). □

# 11 Nonlinear diffusion equations and action estimates

In this section we give a rigorous proof of the crucial estimate we briefly discussed in the formal calculations of Example 2.4. The estimate requires extra continuity and summability properties on $\Gamma\big(\varrho\big)$ and $\Gamma\big(\varphi\big)$, that will be provided by the interpolation estimates of Theorem 10.6.

We will assume that $P$ is regular according to (9.29), we introduce the functions $R(z) = zP'(z) - P(z)$ and $Q(r) := P(r)/r$, and we recall the definition of the $\Gamma_2$ multilinear form

$$\mathbf{\Gamma}_2(\varphi; \varrho) = \int_X \Big(\frac{1}{2}\mathrm{L}\varrho\,\Gamma\big(\varphi\big)\,\mathrm{d}\mathfrak{m} + \varrho\big(\mathrm{L}\varphi\big)^2 + \Gamma\big(\varrho, \varphi\big)\mathrm{L}\varphi\Big)\,\mathrm{d}\mathfrak{m}$$

whenever $\varrho, \varphi \in \mathbb{D}_\infty$ with $\Gamma(\varrho), \Gamma(\varphi) \in \mathbb{H}$. Recall that, under the $\mathrm{BE}(K,\infty)$ assumption, $f \in \mathbb{D}_\infty$ implies $\Gamma(f) \in \mathbb{H}$. Notice also that $P(\varrho) \in L^2(0,T;\mathbb{D})$ and $\varrho$ bounded imply $\Gamma\big(P(\varrho)\big) \in L^2(0,T;\mathbb{H})$, so that the regularity of $P$ and the chain rule yield $\Gamma(\varrho) \in L^2(0,T;\mathbb{H})$.

**Theorem 11.1 (Derivative of the Hamiltonian)** *Let us assume that $\varrho \in \mathcal{ND}(0,T)$, $\varphi \in W^{1,2}(0,T;\mathbb{D},\mathbb{H})$ are bounded solutions, respectively, of*

$$\partial_t\varrho - \mathrm{L}P(\varrho) = 0 \tag{11.1a}$$

$$\partial_t\varphi + P'(\varrho)\mathrm{L}\varphi = 0 \tag{11.1b}$$

*with $\Gamma(\varrho), \Gamma(\varphi) \in L^2(0,T;\mathbb{H})$. Then the map $t \mapsto \mathcal{E}_{\varrho_t}(\varphi_t) = \int_X \rho_t\Gamma(\varphi_t)\,\mathrm{d}\mathfrak{m}$ is absolutely continuous in $[0,T]$ and we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_X \rho_t\Gamma\big(\varphi_t\big)\,\mathrm{d}\mathfrak{m} = \mathbf{\Gamma}_2(\varphi_t;P(\rho_t)) + \int_X R(\rho_t)(\mathrm{L}\varphi_t)^2\,\mathrm{d}\mathfrak{m} \quad \mathscr{L}^1\text{-a.e. in } (0,T). \tag{11.2}$$

The *proof* is based on the following Lemma:

**Lemma 11.2** *Let us assume that $\varrho \in \mathcal{ND}(0,T)$, $\varphi \in W^{1,2}(0,T;\mathbb{D},\mathbb{H})$ are bounded with $\Gamma(\varrho), \Gamma(\varphi) \in L^2(0,T;\mathbb{H})$. Then, for every $\eta \in \mathrm{C}^\infty_\mathrm{c}(0,T)$ we have*

$$\frac{1}{2}\int_0^T\int_X \frac{\mathrm{d}}{\mathrm{d}t}(\varrho_t\eta_t)\Gamma\big(\varphi_t\big)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}t = \int_0^T \eta_t\int_X\Big(\varrho_t\mathrm{L}\varphi_t\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t + \Gamma\big(\varrho_t,\varphi_t\big)\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t\Big)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}t. \tag{11.3}$$

*Proof.* Let us consider the functions $\varphi^\varepsilon_t := \varepsilon^{-1}\int_0^\varepsilon \varphi_{t+r}\,\mathrm{d}r$: $t \mapsto \varphi^\varepsilon_t$ are differentiable in $\mathbb{V}$ with $\frac{\mathrm{d}}{\mathrm{d}t}\varphi^\varepsilon_t = \varepsilon^{-1}(\varphi_{t+\varepsilon} - \varphi_t)$, so that

$$\frac{1}{2}\int_X \varrho_t\frac{\mathrm{d}}{\mathrm{d}t}\Big(\Gamma\big(\varphi^\varepsilon_t\big)\Big)\,\mathrm{d}\mathfrak{m} = \int_X \varrho_t\Gamma\big(\frac{\mathrm{d}}{\mathrm{d}t}\varphi^\varepsilon_t,\varphi^\varepsilon_t\big)\,\mathrm{d}\mathfrak{m} = -\int_X \varrho_t\mathrm{L}\varphi^\varepsilon_t\frac{\mathrm{d}}{\mathrm{d}t}\varphi^\varepsilon_t\,\mathrm{d}\mathfrak{m} - \int_X \Gamma\big(\varrho_t,\varphi^\varepsilon_t\big)\frac{\mathrm{d}}{\mathrm{d}t}\varphi^\varepsilon_t\,\mathrm{d}\mathfrak{m}.$$

For every $\eta \in \mathrm{C}^\infty_\mathrm{c}(0,T)$ we thus have

$$\frac{1}{2}\int_0^T\int_X \frac{\mathrm{d}}{\mathrm{d}t}(\varrho_t\eta_t)\Gamma\big(\varphi^\varepsilon_t\big)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}t = \int_0^T \eta_t\Big(\int_X \varrho_t\mathrm{L}\varphi^\varepsilon_t\frac{\mathrm{d}}{\mathrm{d}t}\varphi^\varepsilon_t\,\mathrm{d}\mathfrak{m} + \int_X \Gamma\big(\varrho_t,\varphi^\varepsilon_t\big)\frac{\mathrm{d}}{\mathrm{d}t}\varphi^\varepsilon_t\,\mathrm{d}\mathfrak{m}\Big)\,\mathrm{d}t.$$

In order to pass to the limit as $\varepsilon \downarrow 0$ in the last identity, we observe that $\frac{\mathrm{d}}{\mathrm{d}t}\varphi^\varepsilon_t \to \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t$ and that $\mathrm{L}\varphi^\varepsilon \to \mathrm{L}\varphi$ strongly in $L^2(0,T;\mathbb{H})$. Moreover, it is easy to check that the convexity of $\zeta \mapsto \sqrt{\Gamma\big(\zeta\big)}$ yields

$$\Gamma\big(\varphi^\varepsilon_t\big) \le \frac{1}{\varepsilon}\int_0^\varepsilon \Gamma\big(\varphi_{t+r}\big)\,\mathrm{d}r \quad \mathfrak{m}\text{-a.e. in } X, \quad \text{for every } t \in [0,T-\varepsilon], \tag{11.4}$$

so that the convolution inequality $\int_0^{T-\varepsilon}\varepsilon^{-1}\int_0^\varepsilon \psi(t+r)\,\mathrm{d}r\,\mathrm{d}t \le \int_0^T \psi(t)\,\mathrm{d}t$ gives

$$\int_0^{T-\varepsilon}\int_X \big(\Gamma\big(\varphi^\varepsilon_t\big)\big)^2\,\mathrm{d}\mathfrak{m}\,\mathrm{d}t \le \frac{1}{\varepsilon}\int_0^{T-\varepsilon}\int_0^\varepsilon\int_X \big(\Gamma\big(\varphi_{t+r}\big)\big)^2\,\mathrm{d}\mathfrak{m}\,\mathrm{d}r\,\mathrm{d}t \le \int_0^T\int_X \big(\Gamma\big(\varphi_t\big)\big)^2\,\mathrm{d}\mathfrak{m}\,\mathrm{d}t. \tag{11.5}$$

Since $\varphi_t^\varepsilon \to \varphi_t$ strongly in $\mathbb{V}$ as $\varepsilon \downarrow 0$ for every $t \in [0,T]$, we have $\Gamma\big(\varphi_t^\varepsilon\big) \to \Gamma\big(\varphi_t\big)$ pointwise in $\mathbb{H}$, so that (11.5) yields the strong convergence of $\Gamma\big(\varphi^\varepsilon\big)$ to $\Gamma\big(\varphi\big)$ in $L^2(0,T;\mathbb{H})$. The above mentioned convergences are then sufficient to get (11.3). $\square$

*Proof of Theorem 11.1.* The map $t \mapsto \mathcal{E}_{\varrho_t}(\varphi_t)$ is continuous since $t \mapsto \varrho_t$ is weakly* continuous in $L^\infty(X,\mathfrak{m})$ and $t \mapsto \Gamma\big(\varphi_t\big)$ is strongly continuous in $L^1(X,\mathfrak{m})$ (thanks to Theorem 10.6). For every $\eta \in \mathrm{C}_\mathrm{c}^\infty(0,T)$, using the differentiability of $\varrho$ in $L^2(0,T;\mathbb{H})$ we have

$$
\begin{aligned}
-\int_0^T \mathcal{E}_{\varrho_t}(\varphi_t)\frac{\mathrm{d}}{\mathrm{d}t}\eta_t\,\mathrm{d}t &= -\frac{1}{2}\int_0^T\int_X \varrho_t\Gamma\big(\varphi_t\big)\,\mathrm{d}\mathfrak{m}\,\frac{\mathrm{d}}{\mathrm{d}t}\eta_t\mathrm{d}t\\
&= -\frac{1}{2}\int_0^T\int_X \frac{\mathrm{d}}{\mathrm{d}t}(\varrho_t\eta_t)\Gamma\big(\varphi_t\big)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}t + \frac{1}{2}\int_0^T\int_X (\frac{\mathrm{d}}{\mathrm{d}t}\varrho_t)\eta_t\Gamma\big(\varphi_t\big)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}t\\
&= -\frac{1}{2}\int_0^T\int_X \frac{\mathrm{d}}{\mathrm{d}t}(\varrho_t\eta_t)\Gamma\big(\varphi_t\big)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}t + \frac{1}{2}\int_0^T \eta_t\Big(\int_X \mathrm{L}P(\varrho_t)\Gamma\big(\varphi_t\big)\,\mathrm{d}\mathfrak{m}\Big)\,\mathrm{d}t.
\end{aligned}
$$

On the other hand, (11.3) yields

$$
\begin{aligned}
-\frac{1}{2}\int_0^T\int_X \frac{\mathrm{d}}{\mathrm{d}t}(\varrho_t\eta_t)\Gamma\big(\varphi_t\big)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}t &= \int_0^T \eta_t\int_X \Big(\varrho_t P'(\varrho_t)\big(\mathrm{L}\varphi_t\big)^2 + \Gamma\big(\varrho_t,\varphi_t\big)\,P'(\varrho_t)\mathrm{L}\varphi_t\Big)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}t\\
= \int_0^T \eta_t\int_X \Big(P(\varrho_t)\big(\mathrm{L}\varphi_t\big)^2 + \Gamma\big(P(\varrho_t),\varphi_t\big)\,\mathrm{L}\varphi_t\Big)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}t &+ \int_0^T \eta_t\int_X R(\varrho_t)\big(\mathrm{L}\varphi_t\big)^2\,\mathrm{d}\mathfrak{m}\,\mathrm{d}t.
\end{aligned}
$$

Combining the two formulas, we get (11.2). $\square$

**Theorem 11.3 (Action and dual action monotonicity)** *Let us assume that the* $\mathrm{BE}(K,N)$ *condition holds, and that* $P \in \mathrm{DC}_{reg}(N)$.

(i) *If* $\varrho \in \mathcal{ND}(0,T), \varphi \in W^{1,2}(0,T;\mathbb{D},\mathbb{H})$ *are bounded solutions of* (11.1a,b) *then the map* $t \mapsto \int_X \varrho_t\Gamma\big(\varphi_t\big)\,\mathrm{d}\mathfrak{m}$ *is absolutely continuous in* $[0,T]$ *and we have*

$$
\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_X \rho_t\Gamma\big(\varphi_t\big)\,\mathrm{d}\mathfrak{m} \geq K\int_X P(\varrho_t)\Gamma\big(\varphi_t\big)\,\mathrm{d}\mathfrak{m} \qquad \mathscr{L}^1\text{-a.e. in } (0,T). \tag{11.6}
$$

(ii) *Setting*

$$
\Lambda := \inf_{r>0} KQ(r) > -\infty, \tag{11.7}
$$

*if* $w \in W^{1,2}(0,T;\mathbb{H},\mathbb{D}'_\mathcal{E})$ *is a solution of* (4.14) *with* $\bar{w} \in \mathbb{V}'_{\bar\varrho} \subset \mathbb{V}'_\mathcal{E}$, *then* $w_t \in \mathbb{V}'_{\varrho_t}$ *for all* $t \in [0,T]$, *with*

$$
\mathcal{E}^*_{\varrho_s}(w_s,w_s) \leq \mathrm{e}^{-2\Lambda(s-t)}\mathcal{E}^*_{\varrho_t}(w_t,w_t) \quad \textit{for every } 0 \leq t < s \leq T. \tag{11.8}
$$

(iii) *If moreover* $\phi_t = -A^*_{\varrho_t}(w_t) \in \mathbb{V}_{\varrho_t}$ *is the potential associated to* $w_t$ *according to* (5.40), *i.e.*

$$\mathcal{E}_{\varrho_t}(\phi_t, \zeta) = \langle w_t, \zeta \rangle \quad \textit{for every } \zeta \in \mathbb{V}_{\varrho_t}, \tag{11.9}$$

*we have*

$$\limsup_{h \downarrow 0} \frac{1}{2h}\Big(\mathcal{E}^*_{\varrho_t}(w_t, w_t) - \mathcal{E}^*_{\varrho_{t-h}}(w_{t-h}, w_{t-h})\Big) \le -K \int_X Q(\varrho_t)\varrho_t \Gamma_{\varrho_t}(\phi_t)\,\mathrm{d}\mathfrak{m}. \tag{11.10}$$

*Proof.* Since $\mathrm{BE}(K, N)$ holds (and thus in particular $\mathrm{BE}(K, \infty)$) we can apply (11.2) of Theorem 11.1, since the interpolation estimate (10.13) and the regularity properties of $\varrho$ and $\varphi$ yield $\Gamma\big(\varrho\big), \Gamma\big(\varphi\big) \in L^2(0, T; \mathbb{H})$. The estimate (11.6) follows then by the combination of (11.2) with Theorem 10.11.

For $0 \le t < s \le T$, let us now call $B_{s,t} : \mathbb{V}_\infty \to \mathbb{V}_\infty$ the linear map that to each function $\bar\varphi \in \mathbb{V}_\infty$ associates the value at time $t$ of the unique solution $\varphi$ of (11.1b) with final condition $\varphi_s = \bar\varphi$, given by Theorem 4.1. If (11.6) holds and $\Lambda$ is defined as in (11.7) we have

$$\int_X \varrho_t \Gamma\big(\varphi_t\big)\,\mathrm{d}\mathfrak{m} \le \mathrm{e}^{-2\Lambda(s-t)} \int_X \varrho_s \Gamma\big(\bar\varphi\big)\,\mathrm{d}\mathfrak{m}, \tag{11.11}$$

so that

$$\begin{aligned}
& \langle \mathrm{e}^{\Lambda(s-t)}\, w_s, \mathrm{e}^{-\Lambda(s-t)}\bar\varphi \rangle - \frac{1}{2}\mathcal{E}_{\varrho_s}(\mathrm{e}^{-\Lambda(s-t)}\bar\varphi, \mathrm{e}^{-\Lambda(s-t)}\bar\varphi) \\
= \;& \langle w_s, \bar\varphi \rangle - \mathrm{e}^{-2\Lambda(s-t)}\,\frac{1}{2}\mathcal{E}_{\varrho_s}(\bar\varphi, \bar\varphi) \overset{(4.17)}{=} \langle w_t, B_{s,t}\bar\varphi \rangle - \mathrm{e}^{-2\Lambda(s-t)}\frac{1}{2}\mathcal{E}_{\varrho_s}(\bar\varphi, \bar\varphi) \\
\overset{(11.11)}{\le} \;& \langle w_t, B_{s,t}\bar\varphi \rangle - \frac{1}{2}\mathcal{E}_{\varrho_t}(B_{s,t}\bar\varphi, B_{s,t}\bar\varphi) \overset{(5.39)}{\le} \frac{1}{2}\mathcal{E}^*_{\varrho_t}(w_t, w_t).
\end{aligned}$$

Taking the supremum with respect to $\bar\varphi \in \mathbb{V}_\infty$ we get (11.8).

Similarly, we can choose a maximizing sequence $(\varphi_n) \subset \mathbb{V}_\infty$ in

$$\frac{1}{2}\mathcal{E}^*_{\varrho_t}(w_t, w_t) = \sup_{\varphi \in \mathbb{V}_\infty} \langle w_t, \varphi \rangle - \frac{1}{2}\mathcal{E}_{\varrho_t}(\varphi, \varphi),$$

so that $\varphi_n$ converge in $\mathbb{V}_{\varrho_t}$ to the potential $\phi_t = -A^*_{\varrho_t}(w_t)$. Recalling (11.6) and (4.17) we have

$$\begin{aligned}
\langle w_t, \varphi_n \rangle - \frac{1}{2}\mathcal{E}_{\varrho_t}(\varphi_n, \varphi_n) \le \;& \langle w_{t-h}, B_{t,t-h}\varphi_n \rangle - \frac{1}{2}\mathcal{E}_{\varrho_{t-h}}(B_{t,t-h}\varphi_n, B_{t,t-h}\varphi_n) \\
& - K \int_{t-h}^t \int_X Q(\varrho_r)\varrho_r \Gamma\big(B_{t,r}\varphi_n\big)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}r.
\end{aligned}$$

Passing to the limit as $n \to \infty$ and recalling Lemma 5.3 we get

$$\begin{aligned}
\frac{1}{2}\mathcal{E}^*_{\varrho_t}(w_t, w_t) \le \;& \langle w_{t-h}, B_{t,t-h}\phi_t \rangle - \frac{1}{2}\mathcal{E}_{\varrho_{t-h}}(B_{t,t-h}\phi_t, B_{t,t-h}\phi_t) \\
& - K \int_{t-h}^t \int_X Q(\varrho_r)\varrho_r \Gamma_{\varrho_r}(B_{t,r}\phi_t)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}r \\
\le \;& \frac{1}{2}\mathcal{E}^*_{\varrho_{t-h}}(w_{t-h}, w_{t-h}) - K \int_{t-h}^t \int_X Q(\varrho_r)\varrho_r \Gamma_{\varrho_r}(B_{t,r}\phi_t)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}r.
\end{aligned}$$

Dividing by $h$ and passing to the limit as $h \downarrow 0$, a further application of Lemma 5.3 yields (11.10). $\square$

**Corollary 11.4** *Let us assume that the* $\mathrm{BE}(K,N)$ *holds, and that* $P \in \mathrm{DC}_{reg}(N)$*. If* $\varrho \in \mathcal{ND}(0,T)$ *is a nonnegative bounded solution of* (3.31) *with* $\sqrt{\bar\varrho} \in \mathbb{V}$ *then* $w_t := \frac{\mathrm{d}}{\mathrm{d}t}\varrho_t$ *satisfies*

$$\mathcal{E}^*_{\varrho_t}(w_t, w_t) \le \mathrm{e}^{-2\Lambda t}\mathcal{E}^*_{\bar\varrho}(w_0, w_0) \le 4\mathsf{a}^{-2}\, e^{2\Lambda^- T}\mathcal{E}(\sqrt{\bar\varrho}, \sqrt{\bar\varrho}) < \infty \tag{11.12}$$

*with* $\mathsf{a}$ *given by* (3.24).

*Proof.* Since $w_0 = \Delta P(\bar\varrho)$, we have for every $\varphi \in \mathbb{V}$

$$\begin{aligned}
-\langle w_0, \varphi\rangle &= \mathcal{E}(P(\bar\varrho), \varphi) = \int_X P'(\bar\varrho)\Gamma\big(\bar\varrho, \varphi\big)\,\mathrm{d}\mathfrak{m} \\
&= 2\int_X P'(\bar\varrho)\sqrt{\bar\varrho}\,\Gamma\big(\sqrt{\bar\varrho}, \varphi\big)\,\mathrm{d}\mathfrak{m} \\
&\le 2\mathsf{a}^{-2}\mathcal{E}(\sqrt{\bar\varrho}, \sqrt{\bar\varrho}) + \frac{1}{2}\mathcal{E}_{\bar\varrho}(\varphi, \varphi),
\end{aligned}$$

which yields $\mathcal{E}^*_{\bar\varrho}(w_0, w_0) \le 4\mathsf{a}^{-2}\mathcal{E}(\sqrt{\bar\varrho}, \sqrt{\bar\varrho})$. Since, thanks to Corollary 4.7, $w$ solves (4.14), we can apply (11.8) to obtain (11.12). $\square$

# 12 The equivalence between $\mathrm{BE}(K,N)$ and $\mathrm{RCD}^*(K,N)$

## 12.1 Regular curves and regularized entropies

Let us first recall the notion (adapted from [7, Def. 4.10]) of regular curve.

**Definition 12.1 (Regular curves)** *Let* $\mu_s = \varrho_s\mathfrak{m} \in \mathscr{P}_2(X)$, $s \in [0,1]$*. We say that* $\mu$ *is a regular curve if:*

(a) *There exists a constant* $R > 0$ *such that* $\varrho_s \le R$ $\mathfrak{m}$*-a.e. in* $X$ *for every* $s \in [0,1]$*.*

(b) $\mu \in \mathrm{Lip}([0,1]; \mathscr{P}_2(X))$ *and in particular* (8.3) *yields* $\varrho \in \mathrm{Lip}([0,1]; \mathbb{V}'_{\mathcal{E}})$*.*

(c) $g_s := \sqrt{\varrho_s} \in \mathbb{V}$ *and there exists a constant* $E > 0$ *such that* $\mathcal{E}(g_s) \le E$ *for every* $s \in [0,1]$ *(in combination with (a), this yields that* $\varrho_s \in \mathbb{V}$ *and also* $\mathcal{E}(\varrho_s) \le 4RE$ *are uniformly bounded).*

The next approximation result is an improvement of [7, Prop. 4.11], since we are able to approximate with curves having uniformly bounded densities (while in the original version only a uniform bound on entropies was imposed). This improvement is possible thanks to [44, Thm. 1.3].

**Lemma 12.2 (Approximation by regular curves)** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, \infty)$ space and $\mu_0$, $\mu_1 \in \mathscr{P}_2(X)$. Then there exist a geodesic $(\mu_s)_{s\in[0,1]}$ connecting $\mu_0$ to $\mu_1$ in $\mathscr{P}_2(X)$ and regular curves $(\mu_s^n)_{s\in[0,1]}$, $n \in \mathbb{N}$, such that*

$$\lim_{n\to\infty} W_2(\mu_s^n, \mu_s) = 0 \quad \textit{for every } s \in [0,1], \qquad \limsup_{n\to\infty} \int_0^1 |\dot\mu_s^n|^2 \,\mathrm{d}s \le W_2^2(\mu_0, \mu_1). \tag{12.1}$$

*Proof.* First of all we approximate $\mu_i$, $i = 0, 1$, in $\mathscr{P}_2(X)$ by two sequences $\nu_i^n = \sigma_i^n \mathfrak{m}$ with bounded support and bounded densities $\sigma_i^n \in L^\infty(X, \mathfrak{m})$. Applying [44, Thm. 1.3] we can find geodesics $(\nu_s^n)_{s\in[0,1]}$ in $\mathscr{P}_2(X)$ connecting $\nu_0^n$ to $\nu_1^n$ with uniformly bounded entropies and densities $\sigma_s^n$ satisfying $\sup_{s\in[0,1]} \|\sigma_s^n\|_{L^\infty(X,\mathfrak{m})} < \infty$ for every $n \in \mathbb{N}$. Since $\nu^n \in \mathrm{AC}^2(0,1;\mathbb{V}_{\mathcal{E}}')$, we can then apply the same argument of [7, Prop. 4.11] (precisely, an averaging procedure w.r.t. $s$ and a short time action of the semigroup, to gain $\mathbb{V}$ regularity) to construct regular curves $\nu^{n,k}$, $k \in \mathbb{N}$, in the sense of Definition 12.1 approximating $\nu^n$ in energy and Wasserstein distance as $k \to \infty$. A standard diagonal argument yields a subsequence $\mu_s^n := \nu_s^{n,k_n}$ satisfying the properties stated in the Lemma. □

Given $U : [0, \infty) \to \mathbb{R}$ continuous, with $U(0) = U(1) = 0$ and $U'$ locally Lipschitz in $(0, \infty)$, with $P(r) = rU'(r) - U(r)$ regular, we now introduce the regularized convex entropies $U_\varepsilon \in \mathrm{C}^2([0, \infty))$, $\varepsilon > 0$, defined by

$$\begin{aligned} U_\varepsilon(r) \;&:=\; (r+\varepsilon)\int_0^r \frac{P(s)}{(s+\varepsilon)^2}\,\mathrm{d}s - r\int_0^1 \frac{P(s)}{(s+\varepsilon)^2}\,\mathrm{d}s \\ &=\; r\int_1^r \frac{P(s)}{(s+\varepsilon)^2}\,\mathrm{d}s + \varepsilon\int_0^r \frac{P(s)}{(s+\varepsilon)^2}\,\mathrm{d}s, \\ &=\; U(r) + \varepsilon\int_0^r \frac{P(s)}{(s+\varepsilon)^2}\,\mathrm{d}s, \end{aligned} \tag{12.2}$$

that satisfy (since $P(0) = 0$)

$$U_\varepsilon(0) = 0, \qquad U_\varepsilon'(0) = -\int_0^1 \frac{P(s)}{(s+\varepsilon)^2}\,\mathrm{d}s, \qquad U_\varepsilon''(r) = \frac{P'(r)}{r+\varepsilon}. \tag{12.3}$$

Notice that, since $U$ is normalized, for every $R > 0$ there exists a constant $C_R$ such that

$$\min\{U(r), 0\} \le U_\varepsilon(r) \quad \forall r \in [0,1], \qquad 0 \le U_\varepsilon(r) \le C_R r \quad \forall r \in [1, R], \tag{12.4}$$

moreover one has the convergence property

$$\lim_{\varepsilon\downarrow 0} U_\varepsilon(r) = U(r). \tag{12.5}$$

We also set

$$Z(r) := \int_0^r \frac{P'(s)}{\sqrt{s}}\,\mathrm{d}s, \tag{12.6}$$

so that (3.24) gives

$$2\mathsf{a}\sqrt{r} \le Z(r) \le 2\mathsf{a}^{-1}\sqrt{r}. \tag{12.7}$$

**Lemma 12.3 (Derivative of the regularized Entropy)** *Let $(\varrho_s)_{s\in[0,1]}$ be uniformly bounded densities in $W^{1,2}(0,1;\mathbb{V},\mathbb{V}'_{\mathcal{E}})$. Then the map $s\mapsto\int_X U_\varepsilon(\varrho_s)\,\mathrm{d}\mathfrak{m}$ is absolutely continuous in $[0,1]$ and*

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_X U_\varepsilon(\varrho_s)\,\mathrm{d}\mathfrak{m} = {}_{\mathbb{V}'}\langle\frac{\mathrm{d}}{\mathrm{d}s}\varrho_s, U'_\varepsilon(\varrho_s)\rangle_{\mathbb{V}} \qquad \textit{for } \mathscr{L}^1\textit{-a.e. } s\in(0,1). \tag{12.8}$$

*Proof.* The convexity of $U_\varepsilon$ yields

$$\begin{aligned}\int_X U_\varepsilon(\varrho_r)\,\mathrm{d}\mathfrak{m} - \int_X U_\varepsilon(\varrho_s)\,\mathrm{d}\mathfrak{m} &\le \int_X U'_\varepsilon(\varrho_s)(\varrho_s-\varrho_r)\,\mathrm{d}\mathfrak{m} \le \mathcal{E}(U'_\varepsilon(\varrho_s))^{1/2}\mathcal{E}^*(\varrho_s-\varrho_r)^{1/2}\\ &\le \sup|U''_\varepsilon|\,\mathcal{E}(\varrho_s)^{1/2}\mathcal{E}^*(\varrho_s-\varrho_r)^{1/2},\end{aligned}$$

so that (3.24) and the last identity in (12.3) give

$$\Big|\int_X U_\varepsilon(\varrho_r)\,\mathrm{d}\mathfrak{m} - \int_X U_\varepsilon(\varrho_s)\,\mathrm{d}\mathfrak{m}\Big| \le \frac{1}{\mathsf{a}\varepsilon}\max\Big(\mathcal{E}(\varrho_s)^{1/2},\mathcal{E}(\varrho_r)^{1/2}\Big)\mathcal{E}^*(\varrho_s-\varrho_r)^{1/2}. \tag{12.9}$$

This shows the absolute continuity (see e.g. [3, Thm. 1.2.5]). The derivation of (12.8) is then standard. □

**Lemma 12.4** *Let $\varrho\in\mathbb{V}_\infty$ be nonnegative.*

(i) *$\sqrt{\varrho}\in\mathbb{V}$ if and only if $Z(\varrho)\in\mathbb{V}$ if and only if $\int_{\{\varrho>0\}}\varrho^{-1}\Gamma\big(P(\varrho)\big)\,\mathrm{d}\mathfrak{m}<\infty$. In this case*

$$\mathcal{E}(Z(\varrho),Z(\varrho)) = \int_{\{\varrho>0\}}\frac{\Gamma\big(P(\varrho)\big)}{\varrho}\,\mathrm{d}\mathfrak{m} = \lim_{\varepsilon\downarrow0}\int_X \varrho\Gamma\big(U'_\varepsilon(\varrho)\big)\,\mathrm{d}\mathfrak{m}. \tag{12.10}$$

(ii) *If $Z(\varrho)\in\mathbb{V}$ then $\Delta P(\varrho)\in\mathbb{V}'_\varrho$, $U'_\varepsilon(\varrho)\to A^*_\varrho(\Delta P(\varrho))$ in $\mathbb{V}_\varrho$ as $\varepsilon\downarrow0$ and*

$$\lim_{\varepsilon\downarrow0}\int_X\varrho\Gamma\big(U'_\varepsilon(\varrho)\big)\,\mathrm{d}\mathfrak{m} = \int_X\Gamma\big(Z(\varrho)\big)\,\mathrm{d}\mathfrak{m} = \mathcal{E}^*(\Delta P(\varrho),\Delta P(\varrho)). \tag{12.11}$$

*Motivated by this, we will call $U'(\varrho)\in\mathbb{V}_\varrho$ the limit $A^*_\varrho(\Delta P(\varrho))$ of $U'_\varepsilon(\varrho)$ in $\mathbb{V}_\varrho$.*

(iii) *If $\mu_s=\varrho_s\mathfrak{m}$, $s\in[0,1]$, is a regular curve, then $s\mapsto\mathcal{U}(\mu_s)$ is absolutely continuous and*

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{U}(\mu_s) = {}_{\mathbb{V}'_{\varrho_s}}\langle\frac{\mathrm{d}}{\mathrm{d}s}\varrho_s, U'(\varrho_s)\rangle_{\mathbb{V}_{\varrho_s}} \quad \textit{for } \mathscr{L}^1\textit{-a.e. } s\in(0,1). \tag{12.12}$$

(iv) *If $Z(\varrho)\in\mathbb{V}$, $\varrho_t=\mathsf{S}_t\varrho$, $\mathrm{BE}(K,N)$ holds and $\Lambda$ is defined as in (11.7), then*

$$Z(\varrho_t)\in\mathbb{V},\quad \mathcal{E}(Z(\varrho_t),Z(\varrho_t))\le \mathrm{e}^{-2\Lambda t}\mathcal{E}(Z(\varrho),Z(\varrho)) \qquad \forall t\ge0. \tag{12.13}$$

*Proof.* The proof of the first claim is standard, see e.g. [5, Lemma 4.10].

In order to prove (ii), let us first notice that

$$\mathcal{E}^*_\varrho(\Delta P(\varrho), \Delta P(\varrho)) \le \mathcal{E}(Z(\varrho), Z(\varrho)). \tag{12.14}$$

In fact for every $\varphi \in \mathbb{V}$ there holds

$$-\langle \Delta P(\varrho), \varphi\rangle = \int_X P'(\varrho)\Gamma\big(\varrho, \varphi\big)\,\mathrm{d}\mathfrak{m} = \int_X \sqrt{\varrho}\,\Gamma\big(Z(\varrho), \varphi\big)\,\mathrm{d}\mathfrak{m} \le \mathcal{E}(Z(\varrho), Z(\varrho))^{1/2}\mathcal{E}_\varrho(\varphi, \varphi)^{1/2}.$$

On the other hand, choosing as test functions $\varphi_\varepsilon := -U'_\varepsilon(\varrho)$, taking the last identity in (12.3) into account we get

$$\begin{aligned}
\mathcal{E}_\varrho(\varphi_\varepsilon, \varphi_\varepsilon) &= \int_X \varrho\Gamma\big(U'_\varepsilon(\varrho)\big)\,\mathrm{d}\mathfrak{m} \le \int_X (\varrho+\varepsilon)(U''_\varepsilon(\varrho))^2\Gamma\big(\varrho\big)\,\mathrm{d}\mathfrak{m} \le \mathcal{E}(Z(\varrho), Z(\varrho)),\\
\langle \Delta P(\varrho), \varphi_\varepsilon\rangle &= \int_X \Gamma\big(P(\varrho), U'_\varepsilon(\varrho)\big)\,\mathrm{d}\mathfrak{m} = \int_X \frac{1}{\varrho+\varepsilon}\Gamma\big(P(\varrho)\big)\,\mathrm{d}\mathfrak{m} \uparrow \mathcal{E}(Z(\varrho), Z(\varrho)) \quad \text{as } \varepsilon \downarrow 0.
\end{aligned}$$

This shows that $\{\varphi_\varepsilon\}_{\varepsilon>0}$ is an optimal family as $\varepsilon \to 0$, thus we can apply Proposition 3.1(b) to obtain that $\varphi_\varepsilon$ converge in $\mathbb{V}_\varrho$ to a $-A^*_\varrho(\Delta P(\varrho))$, and that (12.11) holds.

In order to prove (12.12) we pass to the limit as $\varepsilon \downarrow 0$ in the identity obtained integrating (12.8)

$$\int_X U_\varepsilon(\varrho_t)\,\mathrm{d}\mathfrak{m} - \int_X U_\varepsilon(\varrho_s)\,\mathrm{d}\mathfrak{m} = \int_s^t {}_{\mathbb{V}'_{\varrho_r}}\langle \frac{\mathrm{d}}{\mathrm{d}r}\varrho_r, U'_\varepsilon(\varrho_r)\rangle_{\mathbb{V}_{\varrho_r}}\,\mathrm{d}r \quad \text{for every } 0 \le s \le t \le 1. \tag{12.15}$$

Indeed, in the left hand side it is sufficient to apply the dominated convergence theorem, thanks to the uniform bounds of (12.4) and (9.32). Since the curve $\mu$ is regular, the modulus of the integrand in the right hand side is bounded from above by

$$\frac{1}{2}\mathcal{E}(Z(\varrho_r), Z(\varrho_r)) + \frac{1}{2}\mathcal{E}^*_{\varrho_r}(\frac{\mathrm{d}}{\mathrm{d}r}\varrho_r, \frac{\mathrm{d}}{\mathrm{d}r}\varrho_r) \le C \quad \text{for every } r \in [0,1],$$

so that we can pass to the limit thanks to (ii).

The inequality (12.13) follows by (11.12), the fact that $\frac{\mathrm{d}}{\mathrm{d}t}\varrho_t = \Delta P(\varrho_t)$ and (12.11). □

## 12.2 $\mathrm{BE}(K,N)$ yields $\mathrm{EVI}$ for regular entropy functionals in $\mathrm{DC}(N)$

**Theorem 12.5 ($\mathrm{BE}(K,N)$ implies contractivity)** *Let us assume that metric $\mathrm{BE}(K,N)$ holds and that $P \in \mathrm{DC}_{reg}(N)$. If $\Lambda$ is defined as in (11.7), then the nonlinear diffusion semigroup $\mathsf{S}$ is $\Lambda$-contractive in $\mathscr{P}_2(X)$, i.e.*

$$W_2(\mu_t, \nu_t) \le \mathrm{e}^{-\Lambda t}\,W_2(\mu_0, \nu_0) \quad \textit{with} \quad \mu_t = (\mathsf{S}_t\varrho)\mathfrak{m},\ \nu_t = (\mathsf{S}_t\sigma)\mathfrak{m}. \tag{12.16}$$

*Proof.* We assume first that $\varrho$ and $\sigma$ are the extreme points of a regular curve $\bar\mu_s = \bar\varrho_s\mathfrak{m}$. We set $\mu_{s,t} = \varrho_{s,t}\mathfrak{m}$, with $\varrho_{s,t} = \mathsf{S}_t\bar\varrho_s$. Since $\bar\mu_s$ is Lipschitz with respect to $W_2$ and $\bar\varrho_s$ are uniformly bounded, $s \mapsto \bar\varrho_s$ is also Lipschitz and weakly differentiable with respect to $\mathbb{V}'_{\mathcal{E}}$: we set $w_{s,t} := \partial_s\varrho_{s,t}$.

By Kantorovich duality,

$$\frac{1}{2}W_2^2(\mu_{0,t}, \mu_{1,t}) = \sup\Big\{\int_X \mathsf{Q}_1\varphi\,\mathrm{d}\mu_{1,t} - \int_X \varphi\,\mathrm{d}\mu_{0,t}\Big\} \tag{12.17}$$

where $\varphi$ runs among all Lipschitz and bounded functions with bounded support. If $\varphi$ is such a function with Lipschitz constant $L$, setting $\varphi_s := Q_s\varphi$, the map $\eta(s,r) := \int_X \varphi_s\,\mathrm{d}\mu_{r,t}$ is Lipschitz: in fact, recalling that

$$\mathrm{Lip}(\varphi_s) \le 2L, \qquad \sup_{x\in X}|\varphi_s(x) - \varphi_r(x)| \le 2L^2|s-r|$$

we easily have

$$|\eta(s,r) - \eta(s',r)| \le 2L^2|s-s'|, \qquad |\eta(s,r) - \eta(s,r')| \le 2L\sqrt{\mathfrak{m}(S)}\,\|\varrho_{s,r} - \varrho_{s,r'}\|_{\mathbb{V}'_{\mathcal{E}}},$$

where $S$ is a bounded set containing all the supports of $\varphi_s$, $s \in [0,1]$. From (5.11) we eventually find

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_X \varphi_s\,\mathrm{d}\mu_{s,t} \le -\frac{1}{2}\int_X |\mathrm{D}\varphi_s|^2\,\mathrm{d}\mu_{s,t} + \langle w_{s,t}, \varphi_s\rangle.$$

Denoting now by $r \mapsto \varphi_{s,r}$ the solution of the backward linearized equation (4.2) (corresponding to (11.1b)) in the interval $[0,t]$ with final condition $\varphi_{s,t} := \varphi_s$, recalling Corollary 4.7 we get by (4.17) of Theorem 4.4 and (11.6) of Theorem 11.3

$$\langle w_{s,t}, \varphi_s\rangle = \langle w_{s,0}, \varphi_{s,0}\rangle = \int_X \varphi_{s,0}\,\partial_s\varrho_s\,\mathrm{d}\mathfrak{m}, \qquad \int_X |\mathrm{D}\varphi_s|_w^2\,\mathrm{d}\mu_{s,t} \ge \mathrm{e}^{2\Lambda t}\int_X |\mathrm{D}\varphi_{s,0}|_w^2\,\mathrm{d}\mu_s,$$

and therefore the relations (6.11) and (8.7) between minimal 2-velocity and metric derivative, together with Lemma 8.1, give

$$\begin{aligned}\int_X \varphi_1\,\mathrm{d}\mu_{1,t} - \int_X \varphi_0\,\mathrm{d}\mu_{0,t} &\le \int_0^1 \Big(-\frac{1}{2}\int_X |\mathrm{D}\varphi_s|_w^2\,\mathrm{d}\mu_{s,t} + \langle w_{s,t}, \varphi_s\rangle\Big)\,\mathrm{d}s\\ &\le \int_0^1 \Big(-\frac{1}{2}\mathrm{e}^{2\Lambda t}\int_X |\mathrm{D}\varphi_{s,0}|_w^2\,\mathrm{d}\mu_s + \int_X (\partial_s\varrho_s)\,\varphi_{s,0}\,\mathrm{d}\mathfrak{m}\Big)\,\mathrm{d}s\\ &\le \frac{1}{2}\mathrm{e}^{-2\Lambda t}\int_0^1 |\dot\mu_s|^2\,\mathrm{d}s.\end{aligned}$$

Taking now the supremum with respect to $\varphi$ we get $W_2^2((\mathsf{S}_t\varrho)\mathfrak{m}, (\mathsf{S}_t\sigma)\mathfrak{m}) \le e^{-2\Lambda t}\int_0^1 |\dot\mu_s|^2\,\mathrm{d}s$. Using Lemma 12.2 we obtain the same bound for an arbitrary couple of initial measures. $\square$

Let us recall the notation (see (7.4))

$$\mathcal{A}_Q(\mu;\mathfrak{m}) = \int_0^1 \int_X Q(\varrho_s) v_s^2 \rho_s \,\mathrm{d}\mathfrak{m}\mathrm{d}s$$

for the weighted action of a curve $\mu_s = \varrho_s \mathfrak{m}$ w.r.t. $\mathfrak{m}$, where $v_s$ is the velocity density of the curve.

**Theorem 12.6 (Action monotonicity)** *Let us assume that metric* $\mathrm{BE}(K,N)$ *holds, and that* $P \in \mathrm{DC}_{reg}(N)$*. Let* $\mu_s = \varrho_s\mathfrak{m}$*,* $s \in [0,1]$*, be a regular curve and let* $\mu_{s,t} := (\mathsf{S}_t\varrho_s)\mathfrak{m}$*. Denoting by* $\mu_{\cdot,t}$ *the curve* $s \mapsto \mu_{s,t}$*, we have*

$$\frac{1}{2}\mathcal{A}_2(\mu_{\cdot,t_1}) + K\int_{t_0}^{t_1} \mathcal{A}_Q(\mu_{\cdot,t};\mathfrak{m})\,\mathrm{d}t \le \frac{1}{2}\mathcal{A}_2(\mu_{\cdot,t_0}) \qquad 0 \le t_0 \le t_1 \le 1. \tag{12.18}$$

*Proof.* It is sufficient to prove that for every $t > 0$

$$\limsup_{h\downarrow 0} \frac{1}{2h}\Big(\mathcal{A}_2(\mu_{\cdot,t}) - \mathcal{A}_2(\mu_{\cdot,t-h})\Big) \le -K\mathcal{A}_Q(\mu_{\cdot,t};\mathfrak{m}). \tag{12.19}$$

Let us fix $t > h > 0$; thanks to Theorem 3.4 and Theorem 4.5, the curves $\varrho_{\cdot,t}$ and $\varrho_{\cdot,t-h}$ are $\mathscr{L}^1$-a.e. in $(0,1)$ differentiable in $\mathbb{V}'$, with derivatives $w_{s,t} \in \mathbb{V}'_{\varrho_{s,t}}$, $w_{s,t-h} \in \mathbb{V}'_{\varrho_{s,t-h}}$.

Recall also the relations (8.7) and (8.8) of Theorem 8.2, linking the minimal velocity density of a regular curve $\nu_s = \varrho_s\mathfrak{m}$, its $\mathbb{V}'$ derivative $\ell_s$ and the potential $\phi_r = -A^*_{\varrho_s}(\ell_s)$.

By (8.7) we get

$$\frac{1}{2h}\Big(\mathcal{A}_2(\mu_{\cdot,t}) - \mathcal{A}_2(\mu_{\cdot,t-h})\Big) = \frac{1}{2h}\int_0^1 \Big(\mathcal{E}^*_{\varrho_{s,t}}(w_{s,t},w_{s,t}) - \mathcal{E}^*_{\varrho_{s,t-h}}(w_{s,t-h},w_{s,t-h})\Big)\,\mathrm{d}s.$$

Recalling (11.8) and the definition (11.7) of $\Lambda$, one has

$$\mathcal{E}^*_{\varrho_{s,t}}(w_{s,t},w_{s,t}) - \mathcal{E}^*_{\varrho_{s,t-h}}(w_{s,t-h},w_{s,t-h}) \le \big(e^{-2\Lambda h} - 1\big)\mathcal{E}^*_{\varrho_{s,t-h}}(w_{s,t-h},w_{s,t-h})$$

which is uniformly bounded (using (11.8) once more) by $C(t)h$, if $h < t/2$. Hence, applying (11.10), (8.7) and Fatou's Lemma we get

$$\limsup_{h\downarrow 0} \frac{1}{2h}\Big(\mathcal{A}_2(\mu_{\cdot,t}) - \mathcal{A}_2(\mu_{\cdot,t-h})\Big) \le -K\int_0^1\int_X Q(\varrho_{s,t})\varrho_{s,t}v_{s,t}^2\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s,$$

where $v_{\cdot,t}$ is the minimal velocity density of $\mu_{\cdot,t}$. □

Let us now refine the previous argument. In this refinement we shall use the weighted action

$$\mathcal{A}_{\omega Q}(\mu;\mathfrak{m}) = \int_0^1\int_X (1-s)Q(\varrho_s)v_s^2\rho_s\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s,$$

where $\omega(s) = (1-s)$. Notice that this is the same weighted action appearing in the EVI property (9.77).

**Theorem 12.7 (Action and energy monotonocity)** *Let us assume that metric* $\mathrm{BE}(K,N)$ *holds and that* $P \in \mathrm{DC}_{reg}(N)$. *Let* $\mu_s = \varrho_s\mathfrak{m}$, $s \in [0,1]$, *be a regular curve and let* $\mu_{s,t} := (\mathsf{S}_{st}\varrho_s)\mathfrak{m}$. *Denoting by* $\mu_{\cdot,t}$ *the curve* $s \mapsto \mu_{s,t}$, *we have*

$$\frac{1}{2}\mathcal{A}_2(\mu_{\cdot,t}) + t\mathcal{U}(\mu_{1,t}) + K\int_0^t \mathcal{A}_{\omega Q}(\mu_{\cdot,r};\mathfrak{m})\,\mathrm{d}r \le \frac{1}{2}\mathcal{A}_2(\mu_{\cdot,0}) + t\mathcal{U}(\mu_{0,0}). \tag{12.20}$$

*Proof.* Since by assumption $U$ is continuous and convex, by (3.34) we already know that the map $t \mapsto \mathcal{U}(\mu_{1,t})$ is nonincreasing; thus it is sufficient to prove that

$$\limsup_{h\downarrow 0} \frac{1}{2h}\Big(\mathcal{A}_2(\mu_{\cdot,t}) - \mathcal{A}_2(\mu_{\cdot,t-h})\Big) \le \mathcal{U}(\mu_{0,0}) - \mathcal{U}(\mu_{1,t}) - K\mathcal{A}_{\omega Q}(\mu_{\cdot,t};\mathfrak{m}). \tag{12.21}$$

We thus fix $0 < h < t$ and for $\tau \ge 0$ we set $\varrho_{s,t} := \mathsf{S}_{st}\varrho_s$ and

$$w_{s,t} := \partial_s\varrho_{s,t}, \quad \varsigma_{s,\tau} := \varrho_{s,t-h+\tau}, \quad \nu_{s,\tau} := \varsigma_{s,\tau}\mathfrak{m} = \mu_{s,t-h+\tau}, \quad z_{s,\tau} = w_{s,t-h+\tau}, \tag{12.22}$$

so that $\nu_{s,0} = \mu_{s,t-h}$, $\nu_{s,h} = \mu_{s,t}$, $\varsigma_{s,\tau} = \mathsf{S}_{s\tau}\varsigma_{s,0}$, $z_{s,\tau} = \partial_s\varsigma_{s,\tau}$. Therefore, recalling Theorem 6.7 and Theorem 8.2, we have

$$\mathcal{A}_2(\mu_{\cdot,t}) = \int_0^1 \mathcal{E}^*_{\varsigma_{s,h}}(z_{s,h})\,\mathrm{d}s, \qquad \mathcal{A}_2(\mu_{\cdot,t-h}) = \int_0^1 \mathcal{E}^*_{\varsigma_{s,0}}(z_{s,0})\,\mathrm{d}s. \tag{12.23}$$

It is easy to check that for every $\tau > 0$ the curve $s \mapsto \nu_{s,\tau}$ is Lipschitz in $\mathscr{P}_2(X)$ and $s \mapsto \varsigma_{s,\tau}$ is Lipschitz in $\mathbb{V}'_{\mathcal{E}}$, since for every $0 \le s_1 < s_2 \le 1$

$$\|\varsigma_{s_1,\tau} - \varsigma_{s_2,\tau}\|_{\mathbb{V}'_{\mathcal{E}}} \le \|\varsigma_{s_1,\tau} - \mathsf{S}_{s_1\tau}\varsigma_{s_2,0}\|_{\mathbb{V}'_{\mathcal{E}}} + \|\mathsf{S}_{s_1\tau}\varsigma_{s_2,0} - \varsigma_{s_2,\tau}\|_{\mathbb{V}'_{\mathcal{E}}} \le \|\varsigma_{s_1,0} - \varsigma_{s_2,0}\|_{\mathbb{V}'_{\mathcal{E}}} + C\,\tau(s_2 - s_1),$$

for some constant $C$ independent of $s$ and $\tau$, where in the last inequality we used the contractivity (3.32) of $\mathsf{S}$ in $\mathbb{V}'_{\mathcal{E}}$ and Theorem 3.4 (ND3); the above Lipschitz estimate ensures a uniform bound on $s \mapsto \|\varsigma_{s,\tau}\|_{\mathbb{V}'_{\mathcal{E}}}$ which yields $W_2$-Lipschitz continuity of the curve $s \mapsto \nu_{s,\tau}$ thanks to Theorem 6.7 and Theorem 8.2.

For every $r \in [0,1]$ also the curves $s \mapsto \varsigma^h_{s,r} = \mathsf{S}_{hr}\varsigma_{s,0}$; are regular: we set $z^h_{s,r} := \partial_s\varsigma^h_{s,r}$. Since we have

$$\lim_{k\to 0}\frac{\varsigma^h_{s,r+k} - \varsigma^h_{s,r}}{k} = h\Delta P(\varsigma^h_{s,r}) \quad \text{for every } h \ge 0, \quad s,\, r \in [0,1].$$

It follows that the derivative of $s \mapsto \varsigma_{s,h}$ in $\mathbb{V}'_{\mathcal{E}}$ is

$$\partial_s\varsigma_{s,h} = z_{s,h} = z^h_{s,s} + h\Delta P(\varsigma_{s,h}).$$

Applying Lemma 12.4(ii,iii) we get

$$\frac{\partial}{\partial s}\int_X U(\varsigma_{s,h})\,\mathrm{d}\mathfrak{m} = {}_{\mathbb{V}'_{\varsigma_{s,h}}}\langle z^h_{s,s}, U'(\varsigma_{s,h})\rangle_{\mathbb{V}_{\varsigma_{s,h}}} - h\int_X \varsigma_{s,h}\Gamma\big(U'(\varsigma_{s,h})\big)\,\mathrm{d}\mathfrak{m} \quad \mathscr{L}^1\text{-a.e. in } (0,1).$$

For every $s \in [0,1]$, let $\varphi_s^n \in \mathbb{V}$ be an optimal sequence for $z_{s,h}$, thus satisfying

$$\frac{1}{2}\mathcal{E}^*_{\varsigma_{s,h}}(z_{s,h}, z_{s,h}) = \lim_{n\to\infty} {}_{\mathbb{V}'_{\varsigma_{s,h}}}\langle z^h_{s,s} + h\Delta P(\varsigma_{s,h}), \varphi^n_s\rangle_{\mathbb{V}_{\varsigma_{s,h}}} - \frac{1}{2}\int_X \varsigma_{s,h}\Gamma\big(\varphi^n_s\big)\,\mathrm{d}\mathfrak{m}.$$

Let $\upsilon_{s,h} := -U'(\varsigma_{s,h}) \in \mathbb{V}_{\varsigma_{s,h}}$ and $\psi^n_{s,h} := \varphi^n_s - h\upsilon_{s,h}$. We get

$$\begin{aligned}
&{}_{\mathbb{V}'_{\varsigma_{s,h}}}\langle z^h_{s,s} + h\Delta P(\varsigma_{s,h}), \varphi^n_s\rangle_{\mathbb{V}_{\varsigma_{s,h}}} - \frac{1}{2}\int_X \varsigma_{s,h}\Gamma\big(\varphi^n_s\big)\,\mathrm{d}\mathfrak{m}\\
&= {}_{\mathbb{V}'_{\varsigma_{s,h}}}\langle z^h_{s,s} + h\Delta P(\varsigma_{s,h}), \psi^n_{s,h} + h\upsilon_{s,h}\rangle_{\mathbb{V}_{\varsigma_{s,h}}} - \frac{1}{2}\int_X \varsigma_{s,h}\Gamma_{\varsigma_{s,h}}(\psi^n_{s,h} + h\upsilon_{s,h})\,\mathrm{d}\mathfrak{m}\\
&= -h\frac{\partial}{\partial s}\int_X U(\varsigma_{s,h})\,\mathrm{d}\mathfrak{m} + {}_{\mathbb{V}'_{\varsigma_{s,h}}}\langle z^h_{s,s}, \psi^n_{s,h}\rangle_{\mathbb{V}_{\varsigma_{s,h}}} - \frac{1}{2}\mathcal{E}_{\varsigma_{s,h}}(\psi^n_{s,h}, \psi^n_{s,h}) - h\ {}_{\mathbb{V}'_{\varsigma_{s,h}}}\langle \Delta P(\varsigma_{s,h}), \psi^n_{s,h}\rangle_{\mathbb{V}_{\varsigma_{s,h}}}\\
&\qquad - h\int_X \varsigma_{s,h}\Gamma_{\varsigma_{s,h}}(\psi^n_{s,h}, \upsilon_{s,h})\,\mathrm{d}\mathfrak{m} - \frac{h^2}{2}\int_X \varsigma_{s,h}\Gamma\big(\upsilon_{s,h}\big)\,\mathrm{d}\mathfrak{m}\\
&\le -h\frac{\partial}{\partial s}\int_X U(\varsigma_{s,h})\,\mathrm{d}\mathfrak{m} + {}_{\mathbb{V}'_{\varsigma_{s,h}}}\langle z^h_{s,s}, \psi^n_{s,h}\rangle_{\mathbb{V}_{\varsigma_{s,h}}} - \frac{1}{2}\mathcal{E}_{\varsigma_{s,h}}(\psi^n_{s,h}, \psi^n_{s,h}),
\end{aligned}$$

where we used Lemma 12.4 (ii) to get the second equality, and to simplify the third and second to last terms in order to obtain the last inequality. We observe that $\psi^n_{s,h}$ is an optimal sequence for $z^h_{s,s}$: we will denote by $\psi_{s,h}$ its limit in $\mathbb{V}'_{\varsigma_{\mathrm{s,h}}}$ and by $\phi_s$ the limit of $\varphi^n_s$. They are related by

$$\phi_s = \psi_{s,h} + h\upsilon_{s,h}. \tag{12.24}$$

Passing to the limit in the previous inequality we obtain

$$\frac{1}{2}\mathcal{E}^*_{\varsigma_{s,h}}(z_{s,h}, z_{s,h}) \le -h\frac{\partial}{\partial s}\int_X U(\varsigma_{s,h})\,\mathrm{d}\mathfrak{m} + \frac{1}{2}\mathcal{E}^*_{\varsigma_{s,h}}(z^h_{s,s}, z^h_{s,s}). \tag{12.25}$$

Let now $B^t_{s,r}$ be the operator, given by Theorem 4.1, mapping $\zeta \in \mathbb{V}$ into the solution $\zeta_{s,r}$ of

$$\frac{\mathrm{d}}{\mathrm{d}r}\zeta_{s,r} = -(1-s)P'(\varrho_{s,r})\Delta\zeta_{s,r} \qquad r \in [0,t], \quad \zeta_{s,t} := \zeta, \tag{12.26}$$

Theorem 11.3, (12.22) and the fact that $z^0_{s,s} = z_{s,0}$ yield

$$\frac{1}{2}\Big[\mathcal{E}^*_{\varsigma_{s,h}}(z^h_{s,s}, z^h_{s,s}) - \mathcal{E}^*_{\varsigma_{s,0}}(z_{s,0}, z_{s,0})\Big] \le -K(1-s)\int_{t-h}^t\int_X Q(\varrho_{s,r})\varrho_{s,r}\Gamma_{\varrho_{s,r}}(B^t_{s,r}(\psi_{s,h}))\,\mathrm{d}\mathfrak{m}\,\mathrm{d}r.$$

Using the estimate

$$\Gamma_{\varrho_{s,r}}(B^t_{s,r}(\psi_{s,h})) \le (1+\delta)\Gamma_{\varrho_{s,r}}(B^t_{s,r}(\phi_s)) + h^2\Big(1+\frac{1}{\delta}\Big)\Gamma_{\varrho_{s,r}}(B^t_{s,r}(\upsilon_{s,h}))$$

and the uniform bound

$$\int_X \varrho_{s,r}\Gamma_{\varrho_{s,r}}(B^t_{s,r}(\upsilon_{s,h}))\,\mathrm{d}\mathfrak{m} \le C\int_X \varrho_{s,t}\Gamma_{\varrho_{s,t}}(\upsilon_{s,h})\,\mathrm{d}\mathfrak{m} \le C',$$

Lemma 5.3 eventually yields

$$\limsup_{h\downarrow 0}\frac{1}{2h}\Big(\mathcal{E}^*_{\varsigma_{s,h}}(z^h_{s,s},z^h_{s,s})-\mathcal{E}^*_{\varsigma_{s,0}}(z_{s,0},z_{s,0})\Big)\le -K(1-s)\int_X Q(\varrho_{s,t})\varrho_{s,t}\Gamma_{\varrho_{s,t}}(\phi_s)\,\mathrm{d}\mathfrak{m}.$$

Combining this estimate with (12.25) integrated w.r.t. $s$ in $(0,1)$, (12.23) and Theorem 8.2, we obtain (12.21). □

**Theorem 12.8 (**$\mathrm{BE}(K,N)$ **implies** $\mathrm{CD}^*(K,N)$**)** *Let us suppose that* $(X,\mathsf{d},\mathfrak{m})$ *is a metric measure space satisfying the metric* $\mathrm{BE}(K,N)$ *condition. Then for every entropy function* $U$ *in* $\mathrm{DC}_{reg}(N)$ *and every* $\mu=\varrho\mathfrak{m}\in D(\mathcal{U})\cap\mathscr{P}_2(X)$ *the curve* $\mu_t=(\mathsf{S}_t\varrho)\mathfrak{m}$ *is the unique solution of the Evolution Variational Inequality* (9.77)*. In particular* $(X,\mathsf{d},\mathfrak{m})$ *is a strong* $\mathrm{CD}^*(K,N)$ *space.*

*Proof.* Under the above conditions, one can apply [7, Cor. 4.18] to obtain that $(X,\mathsf{d},\mathfrak{m})$ is an $\mathrm{RCD}(K,\infty)$ space, in particular the assumptions of Lemma 12.2 are satisfied. Now let $\mu_t=(\mathsf{S}_t\varrho)\mathfrak{m}$ be the solution of the nonlinear diffusion equation given by Theorem 3.4 and let $\nu=\sigma\mathfrak{m}\in D(\mathcal{U})\cap\mathscr{P}_2(X)$; we consider a family of regular curves $\mu^{(n)}_s=\varrho^{(n)}_s\mathfrak{m}$ approximating a geodesic $\mu_s$ from $\nu$ to $\mu$ in the sense of Lemma 12.2 and we set $\mu^n_{s,t}=(\mathsf{S}_{st}\varrho^{(n)}_s)\mathfrak{m}$. Applying (12.20) of Theorem 12.7 we get

$$\frac{1}{2}\mathcal{A}_2(\mu^{(n)}_{\cdot,t})+t\mathcal{U}(\mu^{(n)}_{1,t})+K\int_0^t\mathcal{A}_{\omega Q}(\mu^{(n)}_{\cdot,r};\mathfrak{m})\,\mathrm{d}r\le\frac{1}{2}\mathcal{A}_2(\mu^{(n)}_{\cdot,0})+t\mathcal{U}(\mu^{(n)}_{0,0}).\tag{12.27}$$

We can then pass to the limit as $n\to\infty$ applying Theorem 7.1, obtaining

$$\frac{1}{2}W_2^2(\mu_t,\nu)+t\mathcal{U}(\mu_t)+K\int_0^t\mathcal{A}_{\omega Q}(\mu_{\cdot,r};\mathfrak{m})\,\mathrm{d}r\le\frac{1}{2}W_2^2(\mu_0,\nu)+t\mathcal{U}(\nu),\tag{12.28}$$

where $\mu_{s,t}:=(\mathsf{S}_{st}\varrho_s)\mathfrak{m}$. A further application of Theorem 7.1 yields

$$\limsup_{t\downarrow 0}\frac{1}{2t}\Big(W_2^2(\mu_t,\nu)-W_2^2(\mu_0,\nu)\Big)+\mathcal{U}(\mu_0)+K\mathcal{A}_{\omega Q}(\nu,\mu_0;\mathfrak{m})\le\mathcal{U}(\nu).\tag{12.29}$$

This proves (9.77) at $t=0$. By the semigroup property we get (9.77) at every time $t\ge 0$.

The last statement, relative to the strong $\mathrm{CD}^*(K,N)$ property, is an immediate consequence of Theorem 9.20. □

## 12.3 $\mathrm{RCD}^*(K,N)$ implies $\mathrm{BE}(K,N)$

In this section we will assume that $(X,\mathsf{d},\mathfrak{m})$ is an $\mathrm{RCD}^*(K,N)$ space and we will show that the Cheeger energy satisfies $\mathrm{BE}(K,N)$. By [7] we already know that $\mathrm{BE}(K,\infty)$ holds.

In the following, we consider an entropy density function $U=U_{N,\varepsilon,M}\in\mathrm{DC}_{reg}(N)$ of the form given by (9.36) through the regularization (9.34) and we will denote by $(\mathsf{S}_t)_{t\ge 0}$ the nonlinear diffusion flow provided by Theorem 3.4 and satisfying the EVI property (9.77) by Theorem 9.19.

**Lemma 12.9** *Let $\mu_s = \varrho_s \mathfrak{m}$ be a Lipschitz curve in $\mathscr{P}_2(X)$ with uniformly bounded supports, $s \mapsto \varrho_s$ continuous w.r.t. the $L^1(X, \mathfrak{m})$ topology and $\sup_s \|\varrho_s\|_{L^\infty(X,\mathfrak{m})} < \infty$. For a given integer $J$, consider the uniform partition $0 = s_0 < s_1 < \cdots < s_J = 1$ of the time interval $[0,1]$ of size $\sigma := J^{-1}$ and the piecewise geodesic $\mu_s^J = \varrho_s^J \mathfrak{m}$, $s \in [0,1]$, obtained by glueing all the geodesics connecting $\mu_{s_{j-1}}$ to $\mu_{s_j}$. Then*

1. *$\varrho_s^J \to \varrho_s$ in $L^p(X, \mathfrak{m})$ as $J \to \infty$ for all $s \in [0,1]$ and for all $p \in [1, \infty)$,*
2. *$\sup_{s,J} \|\varrho_s^J\|_{L^\infty(X,\mathfrak{m})} < \infty$, and the supports of $\{\varrho_s^J\}_{s\in[0,1],J\in\mathbb{N}}$ are uniformly bounded,*
3. *$\varrho_{(\cdot)}^J \to \varrho_{(\cdot)}$ in $L^1(X \times [0,1], \mathfrak{m} \otimes \mathscr{L}^1)$.*

*Proof.* First of all, since $\mu_{(\cdot)}$ is a Lipschitz curve in $\mathscr{P}_2(X)$, it is clear that the geodesic interpolation converges, i.e. $\mu_{(\cdot)}^J \to \mu_{(\cdot)}$ in $C^0([0,1], \mathscr{P}_2(X))$. Therefore for every $s \in [0,1]$ we have $\mu_s^J \to \mu_s$ weakly and thus (see for instance [3, Lemma 9.4.3])

$$\mathrm{Ent}_{\mathfrak{m}}(\mu_s) \le \liminf_{J\to\infty} \mathrm{Ent}_{\mathfrak{m}}(\mu_s^J), \quad \forall s \in [0,1]. \tag{12.30}$$

On the other hand it is not difficult to prove also the converse inequality

$$\mathrm{Ent}_{\mathfrak{m}}(\mu_s) \ge \limsup_{J\to\infty} \mathrm{Ent}_{\mathfrak{m}}(\mu_s^J), \quad \forall s \in [0,1]. \tag{12.31}$$

Indeed, the $K$-geodesic convexity of the entropy along geodesics ensured by $\mathrm{RCD}(K,\infty)$ yields

$$\mathrm{Ent}_{\mathfrak{m}}(\mu^J_{(1-t)s_j+ts_{j+1}}) \le (1-t)\mathrm{Ent}_{\mathfrak{m}}(\mu_{s_j}) + t\mathrm{Ent}_{\mathfrak{m}}(\mu_{s_{j+1}}) - K\frac{t(1-t)}{2} W_2^2(\mu_{s_j}, \mu_{s_{j+1}}), \tag{12.32}$$

for all $t \in [0,1]$. Since the maps $s \mapsto \mathrm{Ent}_{\mathfrak{m}}(\mu_s) \in \mathbb{R}^+$ and $s \mapsto \mu_s \in \mathscr{P}_2(X)$ are continuous, we get (12.31) by passing to the limit as $J \to \infty$ in (12.32).
Now observe that the supports of the measures $\mu_s^J$ are uniformly bounded for $s \in [0,1]$ and $J \in \mathbb{N}$, since $\mu_s^J$ are concentrated on geodesics with initial and final point in a fixed bounded set; by (9.41) we then infer that

$$\sup_{s,J} \|\varrho_s^J\|_{L^\infty(X,\mathfrak{m})} < \infty. \tag{12.33}$$

By combining the informations above, for every fixed $s \in [0,1]$, we get that $\mu_s^J$ have uniformly bounded supports, $\varrho_s^J \to \varrho_s$ in the $L^\infty$-weak* sense, and $\mathrm{Ent}_{\mathfrak{m}}(\mu_s^J) \to \mathrm{Ent}_{\mathfrak{m}}(\mu_s)$. Since the function $t \mapsto t \log t$ is strictly convex in $[0, \infty)$, it is well known that the above properties imply $\rho_s^J \to \rho_s$ in measure. The thesis follows then by the Dominated Convergence Theorem once recalled (12.33) and the uniform bound on the supports. □

**Lemma 12.10 (Density of measures with bounded support)** *Let $\mu_s = \varrho_s \mathfrak{m}$ be a Lipschitz curve in $\mathscr{P}_2(X)$ with $s \mapsto \varrho_s$ continuous w.r.t. the $L^1(X, \mathfrak{m})$ topology and*

$\sup_s \|\varrho_s\|_{L^\infty(X,\mathfrak{m})} < \infty$. *Then for all* $\bar{x} \in X$ *there exist curves* $\mu_s^R = \varrho_s^R \mathfrak{m}$ *with the same properties, with* $\varrho_s^R = 0$ $\mathfrak{m}$*-a.e. on* $X \setminus \overline{B}_R(\bar{x})$, $\sup_s \|\varrho_s^R\|_{L^\infty(X,\mathfrak{m})} \le 2 \sup_s \|\varrho_s\|_{L^\infty(X,\mathfrak{m})}$ *and*

$$\lim_{R\to\infty} \|\varrho_s^R - \varrho_s\|_{L^1(X,\mathfrak{m})} = 0 \qquad \textit{uniformly in } s \in [0,1], \tag{12.34}$$

$$\mathcal{A}_2(\mu^R_{(\cdot)}) \to \mathcal{A}_2(\mu_{(\cdot)}) \quad \textit{as } R \to \infty. \tag{12.35}$$

*Proof.* By using reparameterizations very close to the identity and a diagonal argument we can assume with no loss of generality that the curve $\varrho$ to be approximated is constant in $[0, \varepsilon]$ and in $[\varepsilon, 1]$ for some $\varepsilon > 0$. Let $\boldsymbol{\pi}$ be a 2-plan tightened to $\mu$ and, for $\bar{x} \in X$ fixed, define

$$\Gamma_R := \left\{\gamma \in \mathrm{C}^0([0,T];X) :\ \gamma([0,1]) \subset \overline{B}_R(\bar{x})\right\}, \qquad c_R := \boldsymbol{\pi}(\Gamma_R).$$

We first approximate $\varrho_s$ by the densities $\rho_s^R$ of the measures

$$\mu_s^R := (\mathrm{e}_s)_\sharp \frac{1}{c_R} \boldsymbol{\pi} \llcorner \Gamma_R$$

where $R \ge R_0$, with $R_0 > 1/\varepsilon$ satisfying $\boldsymbol{\pi}(\Gamma_{R_0}) \ge 1/2$. It is immediately seen that $\rho_s^R$ have support in $\overline{B}_R(\bar{x})$ and, since $\int |\dot\gamma(t)|^2 \, \mathrm{d}\boldsymbol{\pi}(\gamma) \in L^\infty(0,1)$, that $s \mapsto \mu_s^R$ are Lipschitz (even uniformly w.r.t. $R$). Since $c_R \ge 1/2$, one has also that $\rho_s^R$ are uniformly bounded w.r.t. $s$ and $R$ by $2 \sup_s \|\varrho_s\|_{L^\infty(X,\mathfrak{m})}$.

In connection with the convergence properties, from $c_R \uparrow 1$ and $\Gamma_R \uparrow \mathrm{C}^0([0,T];X)$, we obtain

$$\lim_{R\to\infty} \sup_{s\in[0,1]} \|\rho_s^R - \varrho_s\|_{L^1(X,\mathfrak{m})} = 0$$

and (12.35).

By a convolution of $\rho_s^R$ with scale $1/R < \varepsilon$ with respect to the parameter $s$ these properties are preserved (as well as the Lipschitz property w.r.t. $W_2$, thanks also to the convexity of $W_2^2$) and continuity with respect to the $L^1(X,\mathfrak{m})$ is obtained, thus proving the lemma with the mollified family $\varrho_s^R$. □

**Lemma 12.11** *Let* $\mu_s = \varrho_s \mathfrak{m}$ *be a Lipschitz curve in* $\mathscr{P}_2(X)$ *with* $s \mapsto \varrho_s$ *continuous w.r.t. the* $L^1(X,\mathfrak{m})$ *topology and* $\sup_s \|\varrho_s\|_{L^\infty(X,\mathfrak{m})} < \infty$. *Then, defining* $\mu_{s,t} = (\mathsf{S}_t \varrho_s)\mathfrak{m}$, *one has*

$$\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t} \mathcal{A}_2(\mu_{\cdot,t}) \le -K \mathcal{A}_Q(\mu_{\cdot,t};\mathfrak{m}) \quad \textit{for every } t \ge 0. \tag{12.36}$$

*Proof.* By Theorem 3.4 we know that $(s,t) \mapsto \varrho_{s,t}$ is continuous w.r.t. the $L^1(X,\mathfrak{m})$ topology, the $L^\infty(X,\mathfrak{m})$ norms of $\varrho_{s,t}$ are uniformly bounded, and $s \mapsto \mu_{s,t} = \varrho_{s,t}\mathfrak{m}$ is a Lipschitz curve in $\mathscr{P}_2(X)$.

For $t$ fixed, let then $\varrho_{s,t,R}$ be given by Lemma 12.10; we shall use the convergence properties provided by the Lemma and

$$\mathcal{A}_Q(\mu_{\cdot,t,R};\mathfrak{m}) \to \mathcal{A}_Q(\mu_{\cdot,t};\mathfrak{m}) \quad \text{as } R \to \infty, \text{ for every } t > 0, \tag{12.37}$$

provided by Lemma 7.1.

For a fixed integer $J$ we consider the uniform partition $0 = s_0 < s_1 < \cdots < s_J = 1$ of the time interval $[0,1]$ of size $\sigma := J^{-1}$, and the corresponding piecewise geodesic approximation $\mu^J_{s,t,R}$ of $\mu_{s,t,R}$.

Summing up the Evolution Variational Inequality (9.77) for $\mu_{s_{j-1},t,R}$ and test measure $\mu_{s_j,t,R}$ and the corresponding one for $\mu_{s_j,t,R}$ and test measure $\mu_{s_{j-1},t,R}$ we use the Leibniz rule [3, Lemma 4.3.4] to get that $t \mapsto W_2^2(\mu_{s_{j-1},t}, \mu_{s_j,t})$ is locally absolutely continuous in $[0,\infty)$, and that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}W_2^2(\mu_{s_{j-1},t,R}, \mu_{s_j,t,R}) \leq -K\left(\mathcal{A}_{\omega Q}(\mu_{s_{j-1},t,R}, \mu_{s_j,t,R}; \mathfrak{m}) + \mathcal{A}_{\omega Q}(\mu_{s_j,t,R}, \mu_{s_{j-1},t,R}; \mathfrak{m})\right)$$

for $j = 1, \ldots, J$ and $\mathscr{L}^1$-a.e. $t > 0$. Denoting by $\mu^J_{\cdot,t,R}$ the piecewise geodesic curve as in the previous lemma, we obviously have

$$\mathcal{A}_2(\mu^J_{\cdot,t,R}) = \sigma \sum_{j=1}^{J} W_2^2(\mu_{s_{j-1},t,R}, \mu_{s_j,t,R}),$$

while (9.76) gives

$$\mathcal{A}_Q(\mu^J_{\cdot,t,R}; \mathfrak{m}) = \sigma \sum_{j=1}^{J}\left(\mathcal{A}_{\omega Q}(\mu_{s_{j-1},t,R}, \mu_{s_j,t,R}; \mathfrak{m}) + \mathcal{A}_{\omega Q}(\mu_{s_j,t,R}, \mu_{s_{j-1},t,R}; \mathfrak{m})\right).$$

We end up with

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}_2(\mu^J_{\cdot,t,R}) \leq -K\mathcal{A}_Q(\mu^J_{\cdot,t,R}; \mathfrak{m}) \qquad \text{for } \mathscr{L}^1\text{-a.e. } t > 0, \tag{12.38}$$

or, in the equivalent integral form,

$$\frac{1}{2}\mathcal{A}_2(\mu^J_{\cdot,t_2,R}) - \frac{1}{2}\mathcal{A}_2(\mu^J_{\cdot,t_1,R}) \leq -K\int_{t_1}^{t_2} \mathcal{A}_Q(\mu^J_{\cdot,t,R}; \mathfrak{m})\,\mathrm{d}t \qquad 0 \leq t_1 < t_2. \tag{12.39}$$

By Lemma 12.9, we know that the curves $\mu^J_{\cdot,t,R}$ converge to the curves $\mu_{\cdot,t,R}$ in $L^1(X \times [0,1], \mathfrak{m} \otimes \mathscr{L}^1)$ as $J \to \infty$. This enables us to apply Lemma 7.1 so that we can pass to the limit as $J \uparrow \infty$ in (12.39) and use (7.9) to get

$$\frac{1}{2}\mathcal{A}_2(\mu_{\cdot,t_2,R}) - \frac{1}{2}\mathcal{A}_2(\mu_{\cdot,t_1,R}) \leq -K\int_{t_1}^{t_2} \mathcal{A}_Q(\mu_{\cdot,t,R}; \mathfrak{m})\,\mathrm{d}t \qquad \text{for all } 0 \leq t_1 < t_2 \leq T,\ R \geq \bar{R}. \tag{12.40}$$

By using (12.35) and (12.37) we can then pass to the limit in (12.40) as $R \to \infty$ and get the thesis. □

**Corollary 12.12** *Under the same assumptions and notation of the previous Lemma 12.11, if $\Lambda$ is defined as in (11.7) then*

$$\mathcal{A}_2(\mu_{\cdot,t}) \leq \mathrm{e}^{-2\Lambda t}\mathcal{A}_2(\mu_{\cdot,0}) \quad \textit{for every } t \geq 0. \tag{12.41}$$

*In particular, if $L$ is the Lipschitz constant of the initial curve $(\mu_s)_{s\in[0,1]}$ in $(\mathscr{P}_2(X), W_2)$ and $s \mapsto \varrho_{s,t} \in \mathrm{C}^1([0,1];\mathbb{V}')$ then*

$$\mathcal{E}^*_{\varrho_{s,t}}(\partial_s\varrho_{s,t}, \partial_s\varrho_{s,t}) \le \mathrm{e}^{-2\Lambda t} L^2 \qquad \forall\, s \in [0,1],\ \forall t \ge 0. \tag{12.42}$$

*Proof.* The action estimate (12.41) follows easily by (12.36) and the fact that the definition of $\Lambda$ gives $-K\mathcal{A}_Q(\mu_{\cdot,t};\mathfrak{m}) \le -\Lambda\mathcal{A}_2(\mu_{\cdot,t})$.

By repeating the estimate above to every subinterval of $[0,1]$, the identity (8.7) of Theorem 8.2 and the equality (6.11) between minimal velocity and metric derivative yield

$$\mathcal{E}^*_{\varrho_{s,t}}(\partial_s\varrho_{s,t}, \partial_s\varrho_{s,t}) \le \mathrm{e}^{-2\Lambda t} L^2 \qquad \mathscr{L}^1\text{-a.e. } s \in [0,1],\ \forall t \ge 0.$$

The thesis (12.42) then follows by the lower semicontinuity of the map $s \mapsto \mathcal{E}^*_{\varrho_{s,t}}(\partial_s\varrho_{s,t}, \partial_s\varrho_{s,t})$ ensured by Lemma 5.5, since the maps $s \mapsto \partial_s\varrho_{s,t}$, $s \mapsto \varrho_{s,t}$ are continuos in $\mathbb{V}'$ and weak*-$L^\infty(X,\mathfrak{m})$ respectively. □

**Theorem 12.13** *If $(X, \mathsf{d}, \mathfrak{m})$ satisfies $\mathrm{RCD}^*(K,N)$ then the metric $\mathrm{BE}(K,N)$ condition holds.*

*Proof.* Let us first remark that $(X, \mathsf{d}, \mathfrak{m})$ satisfies the metric $\mathrm{BE}(K,\infty)$ condition and that $(X, \mathsf{d})$ is locally compact; in order to check $\mathrm{BE}(K,N)$ we can thus apply Theorem 10.11.

We fix $f \in D_{\mathbb{V}}(\mathrm{L}) \cap D_{L^\infty}(\mathrm{L})$ with compact support and $\mu = \varrho\mathfrak{m} \in \mathscr{P}(X)$ with compactly supported density $\varrho \in D_{L^\infty}(\Delta)$ satisfying $0 < r_0 \le \varrho$ $\mathfrak{m}$-a.e. on the support of $f$. With these choices, our goal is to prove the inequality

$$\mathbf{\Gamma}_2(f; P(\varrho)) + \int_X R(\varrho)\, (\mathrm{L}f)^2 \,\mathrm{d}\mathfrak{m} \ge K \int_X \Gamma\big(f\big)\, P(\varrho)\,\mathrm{d}\mathfrak{m}. \tag{12.43}$$

We define

$$\psi := -\varrho\mathrm{L}f - \Gamma\big(\varrho, f\big).$$

Since $\varrho$ and $f$ are Lipschitz in $X$, recalling Theorem 10.6 and Lemma 10.8 one has $\psi \in \mathbb{V}_\infty$ and

$$|\psi| \le a\varrho \quad \text{for some constant } a > 0. \tag{12.44}$$

In addition, $\psi$ has compact support and

$$\int_X \psi\zeta \,\mathrm{d}\mathfrak{m} = \int_X \varrho\,\Gamma\big(f,\zeta\big)\,\mathrm{d}\mathfrak{m} \quad \forall \zeta \in \mathbb{V}, \qquad \int_X \psi \,\mathrm{d}\mathfrak{m} = 0, \tag{12.45}$$

$$\frac{1}{2}\mathcal{E}^*_\varrho(\psi,\psi) = \frac{1}{2}\mathcal{E}_\varrho(f,f) = \langle \psi, f\rangle - \frac{1}{2}\int_X \varrho\Gamma\big(f\big)\,\mathrm{d}\mathfrak{m}. \tag{12.46}$$

We then set $\varrho_s := \varrho + s\psi$, so that $\partial_s\varrho_s \equiv \psi$, and we observe that (12.44) gives

$$(1-as)\varrho \le \varrho_s \le (1+as)\varrho. \tag{12.47}$$

This, together with (12.45), implies that $\varrho_s\mathfrak{m} \in \mathscr{P}(X)$ for $s \in [0, 1/a]$; moreover, (12.47) also gives $(1 - as)\mathcal{E}_\varrho(\varphi) \le \mathcal{E}_{\varrho_s}(\varphi) \le (1 + as)\mathcal{E}_\varrho(\varphi)$ for all $\varphi \in \mathbb{V}$, so that by duality we get

$$(1 + as)^{-1}\mathcal{E}^*_\varrho(\psi, \psi) \le \mathcal{E}^*_{\varrho_s}(\psi, \psi) \le (1 - as)^{-1}\mathcal{E}^*_\varrho(\psi, \psi). \tag{12.48}$$

It follows that $\varrho_s$ is Lipschitz in $\mathscr{P}_2(X)$ by Theorem 8.2 and

$$\lim_{s\downarrow 0} \mathcal{E}^*_{\varrho_s}(\psi, \psi) = \mathcal{E}^*_\varrho(\psi, \psi) = \mathcal{E}_\varrho(f, f). \tag{12.49}$$

We set $\varrho^t_s := \mathsf{S}_t\varrho_s$, $w^t_s := \partial_s\varrho^t_s$, $\varrho^t = \mathsf{S}_t\varrho$. Recall that, thanks to Corollary 4.7, $t \mapsto w^t_s$ belong to $W^{1,2}(0, T; \mathbb{H}, \mathbb{D}'_{\mathcal{E}}) \subset C([0, T]; \mathbb{V}')$ and solve the PDE $\partial_t w = \Delta(P'(\rho^t_s)w)$ of Theorem 4.4 with the initial condition $\bar w = \partial_s\varrho_s = \psi$. The contraction property of $\mathsf{S}$ in $L^1(X, \mathfrak{m})$ and the integrability of $\psi$ yield

$$\|\varrho^t_s - \varrho^t\|_{L^1(X,\mathfrak{m})} \le \|\varrho_s - \varrho\|_{L^1(X,\mathfrak{m})} = s\|\psi\|_{L^1(X,\mathfrak{m})} \quad \forall s \in (0, 1/a),\ \forall t \in [0, T]. \tag{12.50}$$

Combining Theorem 4.4, the estimate (12.42) and (12.48) we also get

$$\sup_{0\le s\le S} \mathcal{E}^*_{\varrho^t_s}(w^t_s, w^t_s) \le \frac{\mathrm{e}^{-2\Lambda t}}{1 - aS}\mathcal{E}^*_\varrho(\psi, \psi) \quad \forall t \ge 0,\ \forall S \in (0, 1/a). \tag{12.51}$$

Theorem 4.4(L3) in combination with (12.47) and (12.50) also shows that

$$\lim_{s\downarrow 0} \sup_{0\le t\le T} \|w^t_s - w^t_0\|_{\mathbb{V}'_{\mathcal{E}}} = 0 \quad \text{for every } T > 0 \quad \text{and} \quad \lim_{s,t\downarrow 0} \|w^t_s - \psi\|_{\mathbb{V}'_{\mathcal{E}}} = 0. \tag{12.52}$$

Combining the lower semicontinuity property (5.45) with (12.51), (12.52) and recalling (12.46), we get

$$\lim_{s,t\downarrow 0} \mathcal{E}^*_{\varrho^t_s}(w^t_s, w^t_s) = \mathcal{E}^*_\varrho(\psi, \psi) = \mathcal{E}_\varrho(f, f); \tag{12.53}$$

we are then in position to apply Lemma 5.5 and infer that

$$\lim_{s,t\downarrow 0} \int_X Q(\varrho^t_s)\varrho^t_s\Gamma^*_{\varrho^t_s}(w^t_s)\,\mathrm{d}\mathfrak{m} = \int_X Q(\varrho)\varrho\Gamma\big(f\big)\,\mathrm{d}\mathfrak{m}. \tag{12.54}$$

Moreover, by (12.50) and (12.52) we can find a nondecreasing function $(0, 1) \ni t \mapsto S(t) > 0$ with $S(t) \le t^2$, such that

$$\lim_{t\downarrow 0} \sup_{0<s<S(t)} t^{-1}\|w^t_s - w^t_0\|_{\mathbb{V}'_{\mathcal{E}}} = 0, \qquad \lim_{t\downarrow 0} \sup_{0<s<S(t)} t^{-1}\|\varrho^t_s - \varrho^t\|_{L^1(X,\mathfrak{m})} = 0,$$

so that

$$\lim_{t\downarrow 0} \fint_0^{S(t)} \frac{1}{t}\langle w^t_s - \psi, f\rangle\,\mathrm{d}s = \lim_{t\downarrow 0} \frac{1}{t}\langle w^t_0 - \psi, f\rangle \tag{12.55}$$

and

$$\lim_{t\downarrow 0} \fint_0^{S(t)} \int_X \frac{1}{t}(\varrho^t_s - \varrho)\Gamma\big(f\big)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s = \lim_{t\downarrow 0} \int_X \frac{1}{t}(\varrho^t - \varrho)\Gamma\big(f\big)\,\mathrm{d}\mathfrak{m}, \tag{12.56}$$

provided the limits in the right hand sides exist. Eventually, (12.46), (12.48) and $1-as \geq \frac{1}{2}$ yield

$$\frac{1}{2}\mathcal{E}^*_{\varrho_s}(\psi,\psi) \leq \frac{1}{2}(1+2as)\mathcal{E}^*_{\varrho}(\psi,\psi) = \langle \psi, f\rangle - \frac{1}{2}\int_X \varrho\Gamma\big(f\big)\,\mathrm{d}\mathfrak{m} + as\,\mathcal{E}^*_{\varrho}(\psi,\psi)$$

so that the bound $S(t) \leq t^2$ yields

$$\frac{1}{2}\fint_0^{S(t)} \mathcal{E}^*_{\varrho_s}(\psi,\psi)\,\mathrm{d}s \leq \langle \psi, f\rangle - \frac{1}{2}\int_X \varrho\Gamma\big(f\big)\,\mathrm{d}\mathfrak{m} + \frac{1}{2}at^2\mathcal{E}_{\varrho}(f,f). \tag{12.57}$$

Combining Theorem 8.2 and Lemma 12.11 (applied to the rescaled curves in the interval $(0,S(t))$) we get

$$\frac{1}{2}\fint_0^{S(t)} \mathcal{E}^*_{\varrho^t_s}(w^t_s,w^t_s)\,\mathrm{d}s + K\int_0^t \fint_0^{S(t)}\int_X Q(\varrho^r_s)\varrho^r_s\Gamma^*_{\varrho^r_s}(w^r_s)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s\,\mathrm{d}r \leq \frac{1}{2}\fint_0^{S(t)} \mathcal{E}^*_{\varrho_s}(\psi,\psi)\,\mathrm{d}s, \tag{12.58}$$

so that (12.57) and the very definition of $\mathcal{E}^*_{\varrho^t_s}$ yield

$$\begin{aligned}\fint_0^{S(t)}\Big(\langle w^t_s, f\rangle - \frac{1}{2}\int_X \varrho^t_s\Gamma\big(f\big)\,\mathrm{d}\mathfrak{m}\Big)\,\mathrm{d}s + K\int_0^t \fint_0^{S(t)}\int_X Q(\varrho^r_s)\varrho^r_s\Gamma^*_{\varrho^r_s}(w^r_s)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s\,\mathrm{d}r\\ \leq \langle \psi, f\rangle - \frac{1}{2}\int_X \varrho\Gamma\big(f\big)\,\mathrm{d}\mathfrak{m} + \frac{1}{2}at^2\mathcal{E}_{\varrho}(f,f),\end{aligned}$$

and, dividing by $t>0$,

$$\begin{aligned}\fint_0^{S(t)}\Big(\frac{1}{t}\langle w^t_s-\psi, f\rangle - \frac{1}{2}\int_X \frac{1}{t}(\varrho^t_s-\varrho)\Gamma\big(f\big)\,\mathrm{d}\mathfrak{m}\Big)\,\mathrm{d}s + K\fint_0^t \fint_0^{S(t)}\int_X Q(\varrho^r_s)\varrho^r_s\Gamma^*_{\varrho^r_s}(w^r_s)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s\,\mathrm{d}r\\ \leq \frac{1}{2}ta\mathcal{E}_{\varrho}(f,f).\end{aligned}$$

Passing to the limit as $t \downarrow 0$ and recalling (12.54), (12.55) and (12.56) we eventually get

$$\lim_{t\downarrow 0}\langle \frac{w^t_0-\psi}{t}, f\rangle - \frac{1}{2}\lim_{t\downarrow 0}\int_X \frac{\varrho^t-\varrho}{t}\Gamma\big(f\big)\,\mathrm{d}\mathfrak{m} + K\int_X Q(\varrho)\varrho\Gamma\big(f\big)\,\mathrm{d}\mathfrak{m} \leq 0. \tag{12.59}$$

Observe now that

$$\frac{1}{t}\langle w^t_0-\psi, f\rangle = \fint_0^t \langle \mathrm{L}(P'(\varrho^r_0)w^r_0), f\rangle\,\mathrm{d}r = \fint_0^t \langle P'(\varrho^r_0)w^r_0, \mathrm{L}f\rangle\,\mathrm{d}r = \fint_0^t \langle w^r_0, P'(\varrho^r_0)\mathrm{L}f\rangle\,\mathrm{d}r.$$

We can then pass to the limit since $w^r_0 \to \psi$ in $\mathbb{V}'_{\mathcal{E}}$, $P'(\varrho^r_0) \to P'(\varrho)$ in $\mathbb{V}$ (thanks to (3.30)) with uniform $L^\infty$ bound and $\mathrm{L}f \in \mathbb{V}_\infty$. We get, by the definition of $\psi$, that

$$\lim_{t\downarrow 0}\frac{1}{t}\langle w^t_0-\psi, f\rangle = \langle \psi, P'(\varrho)\mathrm{L}f\rangle = -\int_X \Big(\varrho P'(\varrho)(\mathrm{L}f)^2 + \Gamma\big(P(\varrho),f\big)\,\mathrm{L}f\Big)\,\mathrm{d}\mathfrak{m}. \tag{12.60}$$

Similarly, since $\frac{1}{t}(\varrho^t - \varrho) \to \mathrm{L}P(\varrho)$ in $\mathbb{V}'$, $\Gamma\big(f\big) \in \mathbb{V}$ and $P(\varrho) \in \mathbb{D}_\infty$, we obtain

$$\lim_{t\downarrow 0} \int_X \frac{\varrho^t - \varrho}{t} \Gamma\big(f\big)\,\mathrm{d}\mathfrak{m} = \int_X \mathrm{L}P(\varrho)\Gamma\big(f\big)\,\mathrm{d}\mathfrak{m}. \tag{12.61}$$

Combining (12.59) with (12.60) and (12.61) we obtain

$$-\int_X P'(\varrho)(\varrho(\mathrm{L}f)^2 + \Gamma\big(P(\varrho), f\big))\mathrm{L}f + \frac{1}{2}\mathrm{L}P(\varrho)\Gamma\big(f\big)\,\mathrm{d}\mathfrak{m} \le -K \int_X P(\varrho)\Gamma\big(f\big)\,\mathrm{d}\mathfrak{m} \tag{12.62}$$

and finally (12.43) is achieved. By applying Theorem 10.11 we then get $\mathrm{BE}(K, N)$. □

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