WEAK SOLUTIONS AND MAXIMAL REGULARITY FOR ABSTRACT EVOLUTION INEQUALITIES

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Summary. We study the regularity and the approximation of the solution of a parabolic evolution inequality

$$(\star) \qquad u'(t) + A(t)u(t) + \partial\phi(u(t)) \ni f(t), \qquad u(0) = u_0$$

in the framework of a Hilbert triple $\{V, H, V'\}$; here $A(t)$, $t \in]0, T[$, is a family of linear continuous and coercive operators from V to V' and ϕ is a proper convex l.s.c. function defined in V with values in $]-\infty, \infty]$.

We give a weak formulation of $(\star)$ which allows $f \in H^{-1/2+\delta}(0, T; H)$, for some $\delta > 0$ and we prove that the weak solution belongs to $H^{1/2-\epsilon}(0, T; H)$, $\forall \epsilon > 0$, besides the standard $L^2(0, T; V) \cap L^\infty(0, T; H)$.

Under suitable regularity hypotheses on the data f (in particular $f \in H^{1/2+\delta}(0, T; H)$ for some $\delta > 0$ is sufficient) and u_0, the strong solution is in $H^{3/2-\epsilon}(0, T; H)$, $\forall \epsilon > 0$. In this case we prove an optimal error estimate for the backward Euler discretization of $(\star)$ in $L^2(0, T; V) \cap L^\infty(0, T; H)$ and in $H^1(0, T; H)$. We apply these results to the porous medium and the Stefan problem.

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0. - Introduction.

In the usual framework of a real Hilbert triple

$$V \subset H \equiv H' \subset V' \ (^1)$$

we are given a proper convex lower semicontinuous function $\phi : V \mapsto]-\infty, +\infty]$ with domain $D(\phi) = \{v \in V : \ \phi(v) < +\infty\}$; we denote by $\mathcal{K}$ the closure of $D(\phi)$ in H. Let $A(t)$, $t \in]0,T[$, be a family of linear continuous operators from V to V' such that:

$$t \mapsto \big(A(t)u, v\big) \quad \text{is measurable} \qquad \forall\, u, v \in V$$

(0.1) $$\exists\, M > 0 : \quad \|A(t)u\|_* \le M\,\|u\| \quad \text{a.e. in } t, \quad \forall\, u \in V$$

(0.2) $$\exists\, \alpha, \lambda > 0 : \quad \big((A(t)+\lambda)u, u\big) \ge \alpha \|u\|^2 \quad \text{a.e. in } t, \quad \forall\, u \in D(\phi) - D(\phi)$$

We want to study the approximation and the regularity of the solution of the following evolution equation:

Problem (P). *We are given an initial value $u_0 \in \mathcal{K}$ and a function $f(t)$ defined a.e. on the interval $]0,T[$ with values in V'; we ask for $u(t) \in D(\phi)$ such that $u(0) = u_0$ and, for a.e. $t \in]0,T[$*

(0.3) $$\big(u'(t) + A(t)u(t) - f(t), u(t) - v\big) + \phi(u(t)) - \phi(v) \le 0, \qquad \forall\, v \in D(\phi) \ (^2)$$

Problems of this kind have been deeply studied in [6] (in a more general setting, too); at the lowest level of regularity, say $u_0 \in \mathcal{K}$ and $f \in L^2(0,T;V')$ (^{3}) one can give a weak formulation of (0.3) which admits a unique solution

$$u \in L^2(0,T;V) \cap C^0([0,T];H)$$

(^{1}) i.e. H, V are separable real Hilbert spaces with norms $|\cdot| \le \|\cdot\|$ respectively; the above inclusions are continuous and dense, so that we can extend the scalar product $(\cdot,\cdot)$ of H to the duality pairing between V' and V. The induced dual norm of V' will be denoted by $\|\cdot\|_*$.

(^{2}) if $\partial\phi$ denotes the subdifferential mapping of ϕ with respect to the duality V, V', this is equivalent to:

$$u'(t) + A(t)u(t) + \partial\phi(u(t)) \ni f(t), \quad \textit{a.e. in }]0,T[$$

(^{3}) For a generic Hilbert space $\mathcal{H}$, $L^p(0,T;\mathcal{H})$ $(1 \le p \le +\infty)$ is the Banach space of the strongly Lebesgue–measurable functions f a.e. defined on $]0,T[$ with values in $\mathcal{H}$, such that $t \mapsto \|f(t)\|_{\mathcal{H}}$ belongs to $L^p(0,T)$; the L^p-norm of this function gives the norm in $L^p(0,T;\mathcal{H})$.

for any choice of the data.

In order to explain our results, we refer to a particular case of (0.3): if A is constant and ϕ is the indicatrix function of a closed convex set $\mathbf{K}$ of V ([4]) we are dealing with a variational evolution inequality (see [12], [9])

$$\big(u'(t) + Au(t) - f(t), u(t) - v\big) \le 0, \qquad \forall\, v \in \mathbf{K} \tag{0.4}$$

In this case [1] showed that a further time regularity for the weak solution u can be proved if $\mathbf{K}$ is a cone, and this result was extended to a generic convex set in [14] and [15] (where a non decreasing family of convex sets is allowed); more precisely u belongs to the Besov space $B^{1/2}_{2\infty}(0,T;H)$, i.e. the following seminorm will be finite:

$$\big[u\big]^2_{B^{1/2}_{2\infty}(0,T;H)} = \sup_{0<h<T} \int_0^{T-h} \frac{|u(t+h)-u(t)|^2}{h}\,dt < +\infty \tag{0.5}$$

In particular this implies that $u \in H^{1/2-\epsilon}(0,T;H)$, $\forall\,\epsilon > 0$.

We shall show that also the weak solution of (P) belongs to this space (Theorem 1); an analogous improvement holds for the strong solutions $u \in H^1(0,T;V) \cap W^{1,\infty}(0,T;H)$ ([5])of (P), which are obtained assuming suitable smoothness and compatibility of $\{u_0, f, A\}$. Extending the correponding result of [14] for (0.4), we prove that u' belongs to $B^{1/2}_{2\infty}(0,T;H)$, too, and the mapping

$$\{u_0, f\} \in\ H \times L^2(0,T;V') \mapsto u' \in L^2(0,T;H)$$

is 1/2-Hölder continuous on the bounded sets characterized by the regularity assumptions (Theorem 2). Since u' can have jump discontinuities, this is the maximal time regularity one can expect.

As a natural consequence of these properties, we can give weaker smoothness assumptions on f for the existence of a solution of (P), obtaining a corresponding regularity result (Theorem 3.)

([4]) i.e.

$$\phi(v) = \begin{cases} 0 & \text{if } v \in \mathbf{K}; \\ +\infty & \text{otherwise.} \end{cases}$$

([5]) With $W^{1,p}(0,T;\mathcal{H})$, we denote the Banach space of the absolutely continuous functions in $L^p(0,T;\mathcal{H})$, with derivative in the same space; by induction we define also $W^{n,p}(0,T;\mathcal{H})$. As usual, the norms in this spaces are given by:

$$\|f\|^p_{W^{1,p}(0,T;\mathcal{H})} = \|f\|^p_{L^p(0,T;\mathcal{H})} + \|f'\|^p_{L^p(0,T;\mathcal{H})}$$

with the obvious changes when $p = \infty$. When $p = 2$ we are dealing with Hilbert spaces; in this case we use the notation $W^{n,2}(0,T;\mathcal{H}) = H^n(0,T;\mathcal{H})$ (see [7], [10] for more details).

We use a time discretization technique, which is also interesting from the numerical point of view, and we apply the ideas of [1], where the relations between regularity and approximation results are deeply developped.

Let us introduce the backward Euler scheme associated to (P): we fix a time step $k = T/N > 0$, and a subdivision of $[-k, T[$ given by the intervals

$$J_{k,n} = [k(n-1), kn)[, \qquad n = 0, \ldots, N$$

and we set

$$f_n^k = \frac{1}{k}\int_{J_{k,n}} f(t)\,dt; \qquad (A_n^k v, w) = \frac{1}{k}\int_{J_{k,n}} (A(t)v, w)\,dt; \quad n = 1, \ldots, N \tag{0.6}$$

with

$$u_{-1}^k = u_0, \quad f_0^k = f_1^k, \; A_0^k = A_1^k \tag{0.7}$$

For $n = 0$ to N we must recursively solve the variational inequality in the unknown u_n^k:

$$\begin{cases} u_n^k \in D(\phi): \\ (u_n^k - u_{n-1}^k + k\,A_n^k u_n^k - k\,f_n^k, u_n^k - v) + \phi(u_n^k) - \phi(v) \le 0, \quad \forall\, v \in D(\phi) \end{cases} \tag{0.8}$$

Since the linear operators A_n^k satisfy (0.2), (0.8) admits a unique solution if $\lambda\, k \le 1$ (see [9]), so that the sequence $\{u_n^k\}_{n=0,\ldots,N}$ is well defined and it is contained in $\mathcal{D}(\phi)$.

On $]0, T[$ we build the continuous function $\hat{u}_k(t)$, linear on each $J_{k,n}$, such that $\hat{u}_k(nk) = u_n^k$ and we obtain the convergence of $\hat{u}_k$ to u in $L^2(0,T;V) \cap L^\infty(0,T;H)$ and in $B^{1/2}_{2\infty}(0,T;H)$. In the regular case we can estimate the rate of convergence in $L^2(0,T;V) \cap L^\infty(0,T;H)$ and in $H^1(0,T;H)$:

$$\|u - \hat{u}_k\|_{L^2(0,T;V)\cap L^\infty(0,T;H)} = O(k) \tag{0.9}$$

$$\|u' - \hat{u}_k'\|_{L^2(0,T;H)} = O(k^{1/2}) \tag{0.10}$$

(0.9) was proved by [1] for (0.4) and by [13] for evolution equations in the Hilbert space H of the type (see [7]):

$$u' + \partial\phi(u(t)) \ni 0 \tag{0.11}$$

which can be obtained from our general formulation by choosing

$$V \equiv H \equiv V'; \qquad A \equiv 0. \tag{0.12}$$

(0.10) for (0.4) was proved in [14]; in both cases, our extension for Problem (P) allows lower regularity on f (Theorem 4.).

For the applications of these results we refer to [9], [6]; we only mention the Stefan problem and the porous media equation (see [5]).

The outline of the paper is the following: in the next section we detail our theory, then we collect some preliminary results (whose proofs are given in the appendix) on the vector valued function spaces we use and on the singular perturbation problem associated to (0.8). In sect. 3 and 4 we study the weak formulation of (P) with the related stability and convergence properties of $\hat{u}_k$ and in the last two sections we prove the sharp error estimates and the correponding regularity of the solution.

1. - Notation and main results.

First of all we present the weak formulation of Problem (P) (see [6], [1]), if (0.2) holds with $\lambda = 0$; we introduce the integral functional Φ on $L^2(0,T;V)$

$$\Phi(v) = \begin{cases} \int_0^T \phi(v(t))\,dt & \text{if } \phi(v) \in L^1(0,T), \\ +\infty & \text{otherwise} \end{cases}$$

which is proper convex and l.s.c., with the corresponding domain

$$D(\Phi) = \{v \in L^2(0,T;V) : \ \phi(v) \in L^1(0,T)\}. \tag{1.1}$$

We have:

Problem (Pw). *Given $u_0 \in \mathcal{K}$ and $f \in L^2(0,T;V')$ (6), find $u \in L^\infty(0,T;H) \cap D(\Phi)$ such that $\forall\, v \in H^1(0,T;H) \cap D(\Phi)$*

$$\int_0^T \big(v'(t) + A(t)u(t) - f(t), u(t) - v(t)\big)\,dt + \Phi(u) - \Phi(v) \le \frac{1}{2}|u_0 - v(0)|^2. \tag{1.2}$$

1.1 Remark. When $\lambda > 0$ in (0.2) we shall consider a slightly different form of (1.2), which is obtained by writing the integrals with respect to the measure

$$d\mu_\lambda(t) = e^{-2\lambda t}\,dt;$$

(1.2) will be substituted by

$$\int_0^T \Big[\big(v' + (A+\lambda)u - (f + \lambda v), u - v\big)\Big]\,d\mu_\lambda + \Phi_\lambda(u) - \Phi_\lambda(v) \le \frac{1}{2}|u_0 - v(0)|^2, \tag{1.3}$$

where $\Phi_\lambda(v) = \int_0^T \phi(v)\,d\mu_\lambda$. It is important to note that if u is "regular" (say, $u \in H^1(0,T;H)$) then (P) and (Pw) are equivalent (see [6]). $\square$

As we stated in the introduction, there exists a unique solution of this problem (see [6], where more general hypotheses on A and V are made); In fact u belongs to the space (7)

$$B^{1/2}_{2\infty}(0,T;H) = \big(L^2(0,T;H), H^1(0,T;H)\big)_{1/2,\infty}$$

(6) f could be chosen in the bigger space $L^2(0,T;V') + L^1(0,T;H)$ (see [1]); we shall see that the natural space for f must be in duality with the space of the solution.

(7) We shall use the interpolation real functor $(\cdot,\cdot)_{s,p}$, $0 < s < 1$, $1 \le p \le \infty$ (see [8], [4] and [11] for more details); in particular we set:

$$\begin{cases} B^s_{2\,p}(0,T;\mathcal{H}) = \big(L^2(0,T;\mathcal{H}), H^1(0,T;\mathcal{H})\big)_{s,p}; \\ B^{s+1}_{2\,p}(0,T;\mathcal{H}) = \{v \in B^s_{2\,p}(0,T;\mathcal{H}) : \ v' \in B^s_{2\,p}(0,T;\mathcal{H})\} \end{cases}$$

with the usual simplification when $p = 2$: $H^s(0,T;\mathcal{H}) = B^s_{2\,2}(0,T;\mathcal{H})$. We recall that

$$s > r \ \Rightarrow \ B^s_{2\,p}(0,T;\mathcal{H}) \subset B^r_{2\,q}(0,T;\mathcal{H})$$

with continuous inclusion, independently of the choice of p, q.

of the $L^2(0,T;H)$-functions for which (0.5) holds.

Theorem 1. *The solution u of problem (P) belongs to $B^{1/2}_{2\infty}(0,T;H)$ with:*

$$[u]_{B^{1/2}_{2\infty}(0,T;H)} \leq C\Big\{|u_0| + \|f\|_{L^2(0,T;V')}\Big\} \tag{1.4}$$

$$\lim_{h\to 0} \int_0^{T-h} \frac{|u(t+h)-u(t)|^2}{h}\,dt = 0. \tag{1.5}$$

If we want to obtain a strong solution, it is natural to require more regular data; we define

$$\mathbf{W}(u_0,f) = \Big\{w \in H:\ (w + A(0)u_0 - f(0), u_0 - v) + \phi(u_0) - \phi(v) \leq 0, \quad \forall\, v \in D(\phi)\Big\} \tag{1.6}$$

and we assume that:

$$A \in H^1(0,T;\mathcal{L}(V,V')), \qquad \|A'\|_{L^2(0,T;\mathcal{L}(V,V'))} = M' \tag{1.7}$$

$$f \in H^1(0,T;V')(^8) \tag{1.8}$$

$$u_0 \in D(\phi), \qquad w(u_0,f) = \|u_0\| + \inf_{w\in \mathbf{W}(u_0,f)} |w| < +\infty \tag{1.9}$$

For the sake of simplicity, we shall denote by $\mathcal{V}$ the subset of $H \times L^2(0,T;V')$ defined by (1.8) and (1.9) and we shall write $\|(u_0,f)\|_{\mathcal{V}}$ for the quantity $w(u_0,f) + \|f\|_{H^1(0,T;V')}$.

If (u_0,f) belongs to $\mathcal{V}$, then it is well known that

$$u \in H^1(0,T;V) \cap W^{1,\infty}(0,T;H); \qquad \|u\|_{H^1(0,T;V)\cap W^{1,\infty}(0,T;H)} \leq C\,\|(u_0,f)\|_{\mathcal{V}} \tag{1.10}$$

Even in the scalar case (say $V \equiv H \equiv \mathbf{R}$) with C^∞-data, one cannot expect in general that u' would be continuous (^{9}); however, extending the result of [14], we can prove an extra time regularity which is related to the weaker (1.4):

(^{8}) $f \in H^1(0,T;V') + BV(0,T;H)$ is enough for the desired regularity: see note (^{6}). Since we cannot hope that u' belongs to $C^0([0,T];H)$, we allow $f \in BV(0,T;H)$ instead of $W^{1,1}(0,T;H)$: higher regularity of f couldn't give smoother solutions.

(^{9}) For example, let ϕ be the indicatrix function of the interval $[1,+\infty[$ and consider the differential equation:

$$(u'+u)(u-v) \leq 0 \quad \forall\, v \geq 1, \qquad u(0) = 2$$

whose solution is $u(t) = \max\{1, 2\,e^{-t}\}$.

Theorem 2. *If (1.7), (1.8) and (1.9) hold, then u' belongs to $B^{1/2}_{2\infty}(0,T;H)$; moreover the operator $(u_0,f)\in\mathcal{V}\mapsto u'\in L^2(0,T;H)$ is 1/2-Hölder continuous with respect to the $H\times L^2(0,T;V')$ metric on the bounded subsets of $\mathcal{V}$, that is:*

$$\|u'-v'\|^2_{L^2(0,T;H)}\le C\Big(\|(u_0,f)\|_{\mathcal{V}}+\|(v_0,g)\|_{\mathcal{V}}\Big)\Big(|u_0-v_0|+\|f-g\|_{L^2(0,T;V')}\Big)\tag{1.11}$$

where u and v are the solutions of (Pw) relative to (u_0,f) and (v_0,g) respectively.

1.2 Remark. This regularity is optimal, at least in the family:

$$B^{1+s}_{2\,p}(0,T;H)=\big(H^1(0,T;H),H^2(0,T;H)\big)_{s,p},\quad s\in]0,1[,\quad p\in[1,\infty]$$

as the previous note shows. □

1.3 Remark. Taking into account note (7) we have $u\in H^{3/2-\epsilon}(0,T;H)$, $\forall\,\epsilon>0$, so that the regularity results announced in the introduction are justified. □

These results allow us to weaken the hypotheses on f in order to obtain the existence and the regularity theorem. The natural way is to consider a space for the data which forms a duality pair with the solution space, so that the integral in (1.2):

$$\int_0^T\big(f(t),u(t)-v(t)\big)\,dt\tag{1.12}$$

makes sense. We already noticed (see (6)) that $f\in L^1(0,T;H)$ is enough to obtain the existence of a weak solution which belongs to $L^\infty(0,T;H)$. Now we know that $u\in B^{1/2}_{2\infty}(0,T;H)$, too; if we require "a priori" that the solution u have this regularity, we can substitute the integral (1.12) with the duality

$${}_{B^{-1/2}_{2\,1}(0,T;H)}\langle f,u-v\rangle_{\,L^\infty(0,T;H)\cap B^{1/2}_{2\infty}(0,T;H)}$$

where $B^{-1/2}_{2\,1}(0,T;H)=\big(L^2(0,T;H),H^{-1}(0,T;H)\big)_{1/2,1}$, since we shall show that (10)

$$L^\infty(0,T;H)\cap B^{1/2}_{2\infty}(0,T;H)\subset\Big(L^2(0,T;H),H^1_0(0,T;H)\Big)_{1/2,\infty}\equiv\big(B^{-1/2}_{2\,1}(0,T;H)\big)'\tag{1.13}$$

with continuous inclusion. In order to take into account three different kinds of regularity for f, we introduce the "sum" space

$$f\in S(0,T)=L^2(0,T;V')+L^1(0,T;H)+B^{-1/2}_{2\,1}(0,T;H)\ (^{11})\tag{1.14}$$

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(10) In order to avoid interpolation between $L^2(0,T;H)$ and $\big(H^1(0,T;H)\big)'$, which isn't a distribution space; see [10].

(11) If B_1,B_2 are Banach spaces continuously embedded in a Hausdorff topological vector

By (1.13) the scalar product in $L^2(0,T;H)$ can be extended to a duality pairing between $S(0,T)$ and:

$$I(0,T) = L^2(0,T;V) \cap L^\infty(0,T;H) \cap B^{1/2}_{2\infty}(0,T;H) \tag{1.15}$$

and we can identify in this way $I(0,T)$ with the dual of $S(0,T)$:

$$I(0,T) \equiv \big(S(0,T)\big)'. \tag{1.16}$$

We can state our foundamental result (in the case $\lambda = 0$; the modifications in the other case are obvious)

Theorem 3. *For any $f \in S(0,T)$, $u_0 \in \mathcal{K}$, there exists a unique function $u \in I(0,T) \cap D(\Phi)$ such that, $\forall\, v \in H^1(0,T;H) \cap D(\Phi)$:*

$$_{S(0,T)}\langle v' + Au - f, u - v\rangle_{\,I(0,T)} + \Phi(u) - \Phi(v) \le \frac{1}{2}|u_0 - v(0)|^2. \tag{1.17}$$

The solution u belongs to $C^0([0,T];H)$ and satisfies (12)

$$u(0) = u_0; \qquad \|u\|_{I(0,T)} \le C\big[|u_0| + \|f\|_{S(0,T)}\big].$$

Moreover, if

$$f \in S^1(0,T) = H^1(0,T;V') + B^{1/2}_{2\,1}(0,T;H) + BV(0,T;H) \ (^{13}) \tag{1.18}$$

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space $\mathcal{T}$, we set:

$$B_1 + B_2 = \{b \in \mathcal{T}:\ b = b_1 + b_2,\ \text{with } b_1 \in B_1,\ b_2 \in B_2\}$$

with norm

$$\|b\|^2_{B_1+B_2} = \inf_{b_1+b_2=b} \|b_1\|^2_{B_1} + \|b_2\|^2_{B_2}$$

Moreover, if a Hilbert space $\mathcal{H} \equiv \mathcal{H}'$ (in our case $L^2(0,T;H)$) is densely imbedded in B_1 and B_2, we can identify B_1' and B_2' with subspaces of $\mathcal{H}$ and we have

$$\big(B_1 + B_2\big)' \equiv B_1' \cap B_2'$$

(12) We shall denote by C the constants which depend only on $\alpha, \lambda, M, M', T$ and by c the constants depending on T.

(13) $BV(0,T;\mathcal{H})$ is the space of the function $h : [0,T] \mapsto \mathcal{H}$ of bounded variation; the variation of h is:

$$V_0^T(h) = \sup \sum_{m=0}^{n} \|h(t_{m+1}) - h(t_m)\|_{\mathcal{H}}$$

where the supremum is taken over all the subdivisions of the interval $[0,T]$: $0 = t_0 < t_1 < \ldots < t_{n+1} = T$. We shall write $h(0)$ for $\lim_{t\to 0^+} h(t)$ and $\|h\|_{BV(0,T;H)}$ for $\|h\|_{L^1(0,T;H)} + V_0^T(h)$ (see [7] for more details).

and (1.7), (1.9) *hold, then*

$$u \in I^1(0,T) = \{v \in I(0,T) : \, v' \in I(0,T)\} \tag{1.19}$$

with

$$\|u'\|_{I(0,T)} \le C\left\{w(u_0,f) + \|f\|_{S^1(0,T)}\right\} \tag{1.20}$$

where $w(u_0,f)$ *is given by* (1.9).

1.4 Remark. The functions of $B^{1/2}_{2\,1}(0,T;H)$ are continuous (see [15]) and the right limit exists for a $BV(0,T;H)$-function, so that (1.9) makes sense. □

1.5 Remark. In the linear case (that is $\phi \equiv 0$ ([14])) [16], [17] give an isomorphism result under slightly weaker hypotheses on f, which can be chosen in $L^2(0,T;V') + H^{-1/2}(0,T;H)$. With these assumptions, however, we loose the continuity and the boundedness in H of the solution and the initial condition is verified in a particular sense, depending on the injection of V in H, too. On the contrary, the choice of $B^{-1/2}_{2\,1}(0,T;H)$ saves these two properties and is the best possibility in the interpolation family

$$B^{-s}_{2\,p}(0,T;H) = \big(L^2(0,T;H), H^{-1}(0,T;H)\big)_{s,p}.$$

By a simple application of interpolation, from our results we recover this intermediate regularity property (see [17])

$$f \in L^2(0,T;H), \quad u_0 \in \big(D(A(0),H\big)_{1/2,2} \Rightarrow u' \in L^2(0,T;H), \quad Au \in L^2(0,T;H)$$

and, in a different direction (see [2])

$$\left.\begin{array}{l} f \in H^s(0,T;V') + H^{s-1/2}(0,T;H) \\ u_0 \in \big(H,V\big)_{2s,2} \end{array}\right\} \Rightarrow \; u \in H^s(0,T;V) \cap H^{s+1/2}(0,T;H) \tag{1.21}$$

if $0 < s < 1/2$, with the analogous one if $1/2 < s < 1$. □

1.6 Remark. Let us consider the equation in H:

$$u' + \partial\phi(u) \ni f, \qquad u(0) = u_0$$

([14]) (0.3) of problem (P) becomes

$$u'(t) + A(t)u(t) = f(t), \qquad \textit{in } \mathcal{D}'(0,T;V').$$

given by the particular choice (0.12); thanks to the regularizing effect of the subdifferential mapping, it is well known that

$$(1.22) \qquad u_0 \in D(\phi),\ f \in L^2(0,T;H) \ \Rightarrow\ u \in H^1(0,T;H).$$

We can now "complete" this formula at the weakest and strongest level:

$$(1.23) \qquad u_0 \in \mathcal{K},\ f \in B^{-1/2}_{2\,1}(0,T;H) \ \Rightarrow\ u \in B^{1/2}_{2\infty}(0,T;H) \cap C^0([0,T];H)$$

$$(1.24) \qquad u_0 \in D(\partial\phi),\ f \in B^{1/2}_{2\,1}(0.T;H) \ \Rightarrow\ u \in B^{3/2}_{2\infty}(0,T;H) \cap W^{1,\infty}(0,T;H).$$

The "quasi optimal" correspondence between the regularity of f and u can be overcome if we work with intermediate regularity assumptions. In a forthcoming paper we shall prove that

$$(1.25) \qquad f \in H^s(0,T;H) \ \Rightarrow u \in H^{s+1}(0,T;H), \quad \forall\, s \in]-1/2, 1/2[$$

if the initial datum u_0 belongs to a suitable interpolation class between $\mathcal{K}$ and $D(\partial\phi)$ (see [3]). $\square$

We give a simple application of the previous remark:

Stefan problem. Let Ω be a Lipshitz bounded open subset of $\mathbf{R}^d$ and

$$\beta : \mathbf{R} \mapsto \mathbf{R}, \quad \beta(u) = (u-1)^+ - u^- \quad (^{15}), \quad j(u) = \int_0^u \beta(s)\,ds$$

The operator $u \mapsto -\Delta(\beta(u))$ is maximal monotone on $H^{-1}(\Omega)$ with its usual scalar product and it is the subgradient of (see [5]):

$$\phi(u) = \begin{cases} \int_\Omega j(u)\,dx, & \text{if } u, j(u) \in L^1(\Omega); \\ +\infty & \text{otherwise} \end{cases}$$

So, given $u_0 \in L^1(\Omega)$ with $\beta(u_0) \in H^1_0(\Omega)$ and $f \in B^{1/2}_{2\,1}(0,T;H^{-1}(\Omega))$, the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta\beta(u) = f \\ u(0) = u_0, \qquad u_{|\partial\Omega} = 0 \end{cases}$$

admits a unique solution u which satisfies:

$$\begin{cases} u \in B^{3/2}_{2\infty}(0,T;H^{-1}(\Omega)) \cap W^{1,\infty}(0,T;H^{-1}(\Omega)) \\ \beta(u) \in B^{1/2}_{2\infty}(0,T;H^1_0(\Omega)) \cap L^\infty(0,T;H^1_0(\Omega)) \end{cases}$$

(15) The same results hold also for a generic maximal monotone surjective graph of $\mathbf{R}\times\mathbf{R}$; in particular we could consider the multi-phase Stefan problem or the porous media equation.

Moreover, if we compare two different solutions u, v relative to the initial data u_0, v_0 (with $f \equiv 0$ for simplicity) we have:

$$\int_0^T \|\beta(u) - \beta(v)\|^2_{H_0^1(\Omega)}\, dt \le C\, \|u_0 - v_0\|_{H^{-1}(\Omega)} \left(\|\beta(u_0)\|_{H_0^1(\Omega)} + \|\beta(v_0)\|_{H_0^1(\Omega)}\right) \quad \square$$

As we already said in the introduction, the proof of the previous theorems will follow from sharp error estimates for the time discretization of (P) via the backward Euler scheme given by (0.8).

We observe that the definition (0.6) of f_n^k makes sense also for $f \in B^{-1/2}_{2\,1}(0, T; H)$, if we read the integral as the duality between f and the characteristic function of $J_{k,n}$, so that the piecewise linear function $\hat{u}_k$ is well defined. We have:

Theorem 4. *Assume that $f \in S(0,T)$ and $u_0 \in \mathcal{K}$; then the family $\hat{u}_k$ converges to the solution u of (P) in $I(0,T)$ as $k \to 0$. Moreover, if the regularity assumptions* (1.18) *and* (1.9) *hold, then there exist constants $C > 0$ independent of k with*

$$\|\hat{u}_k\|_{I^1(0,T)} \le C \Big\{ \|f\|_{S^1(0,T)} + w(u_0, f) \Big\} \tag{1.26}$$

$$\|u - \hat{u}_k\|_{L^2(0,T;V) \cap L^\infty(0,T;H)} \le C\, k \Big\{ \|f\|_{S^1(0,T)} + w(u_0, f) \Big\} \tag{1.27}$$

$$\|u' - \hat{u}_k'\|_{L^2(0,T;H)} \le C\, k^{1/2} \Big\{ \|f\|_{S^1(0,T)} + w(u_0, f) \Big\} \tag{1.28}$$

where $w = w(u_0, f)$ is given by (1.9).

1.7 Remark. Let us return to the linear situation considered in 1.5; by interpolation from (1.27), assuming $u_0 \in V$, $f \in L^2(0,T;H)$ we get the optimal error bound:

$$\|u - \hat{u}_k\|_{L^2(0,T;V) \cap L^\infty(0,T;H)} \le C\sqrt{k} \Big\{ \|f\|_{L^2(0,T;H)} + \|u_0\| \Big\}$$

without any assumption on the domain of $A(0)$. In a similar way, we could operate in the framework of [2]. $\square$

1.8 Remark. Let $s \in]0,1[$ and $v \in L^2(0,T;\mathcal{H})$; then v belongs to $B^s_{2\infty}(0,T;\mathcal{H})$ iff there exists a sequence $\{v_k\}_{k=T/N,\, N \in \mathbf{N}}$ with:

$$v_k \in H^1(0,T;\mathcal{H}); \quad \|v - v_k\|_{L^2(0,T;H)} \le C\, k^s, \quad \|v_k'\|_{L^2(0,T;\mathcal{H})} \le \frac{C}{k^{1-s}} \tag{1.29}$$

and C independent of k; moreover it holds:

$$[v]_{B^{1/2}_{2\infty}(0,T;H)} \le c \sup_k \Big\{ k^{1-s} \|v_k'\|_{L^2(0,T;\mathcal{H})} + k^{-s} \|v - v_k\|_{L^2(0,T;\mathcal{H})} \Big\} \tag{1.30}$$

(see [4], [11], [1]); this characterization shows, via a regularization of $\hat{u}_k$, that (1.26) and (1.28) imply $u' \in B^{1/2}_{2\infty}(0,T;H)$. $\square$

1.9 Remark. It is not restrictive to assume that

$$\forall\, v \in V \qquad \phi(v) \geq \phi(0) = 0 \tag{1.31}$$

In fact, even if ϕ does not satisfy (1.31), we know that

$$\exists\, \eta \in V'\; \xi \in D(\phi): \quad \phi(v) - \phi(\xi) \geq (\eta, v - \xi) \text{ (}^{16}\text{)}$$

Introducing the new functions

$$\tilde{\phi}(v) = \phi(v + \xi) - (\eta, v) - \phi(\xi), \quad \tilde{f}(t) = f(t) - A(t)\xi - \eta, \quad \tilde{u}_0 = u_0 - \xi$$

the solution $\tilde{u}$ of problem (Pw) with respect to these data is related to the original one by $\tilde{u} = u - \xi$; moreover the existence and regularity hypotheses are not changed by this substitution and now $\tilde{\phi}$ satisfies (1.31). Therefore we shall assume it in oder to semplify the proofs. □

([16]) This is equivalent to $\eta \in D(\partial\phi)$.

2. - Preliminary remarks.

In this section we collect some basic results on the function spaces we introduced and on the approximation of their functions by piecewise constant ones. Our first aim is to prove (1.16) with some related properties.

In order to have shorter notation, we denote by $X(0,T;H)$ the dual of $B^{-1/2}_{2\,1}(0,T;H)$; by standard interpolation results

$$X(0,T;H) = \big(L^2(0,T;H), H^1_0(0,T;H)\big)_{1/2,\infty} \quad (^{17}) \tag{2.1}$$

In order to assign an explicit norm to $X(0,T;H)$ we shall consider the trivial extension p_0 outside $]0,T[$. By interpolation we have

2.1 Lemma. *$X(0,T;H)$ is the space of the $L^2(0,T;H)$-functions f such that $f_0 = p_0 f$ belongs to $B^{1/2}_{2\infty}(\mathbf{R};H)$; we can choose the norm:*

$$\|f\|^2_{X(0,T;H)} = \|f\|^2_{L^2(0,T;H)} + \sup_{h>0} \int_{\mathbf{R}} \frac{|f_0(t+h) - f_0(t)|^2}{h}\, dt \tag{2.2}$$

Proof. Since p_0 is a linear isometry of $L^2(0,T;H)$ and $H^1_0(0,T;H)$ into $L^2(\mathbf{R};H)$ and $H^1(\mathbf{R};H)$ respectively, by interpolation

$$p_0 \in \mathcal{L}(X(0,T;H), B^{1/2}_{2\infty}(\mathbf{R};H)); \qquad \|p_0\| \le 1.$$

On the other side, there exists a linear operator R (see [10]) which maps continuously $L^2(\mathbf{R};H)$ and $H^1(\mathbf{R};H)$ into $L^2(0,T;H)$ and $H^1_0(0,T;H)$ respectively with the property

$$R(f_0) = f, \quad \forall\, f \in L^2(0,T;H).$$

By interpolation R maps $B^{1/2}_{2\infty}(\mathbf{R};H)$ into $X(0,T;H)$, so that $f_0 \in B^{1/2}_{2\infty}(\mathbf{R};H)$ implies $f \in X(0,T;H)$. ∎

2.2 Corollary. *If $f \in L^\infty(0,T;H) \cap B^{1/2}_{2\infty}(0,T;H)$ then f belongs to $X(0,T;H)$ with:*

$$\|f\|^2_{X(0,T;H)} \le \|f\|^2_{B^{1/2}_{2\infty}(0,T;H)} + 2\|f\|^2_{L^\infty(0,T;H)} \tag{2.3}$$

In particular (1.16) *hold.*

Proof. Let f be in $L^\infty(0,T;H) \cap B^{1/2}_{2\infty}(0,T;H)$ and set $f_0 = p_0 f$; we have only to check that

$$x_h = \int_{\mathbf{R}} \frac{|f_0(t+h) - f_0(t)|^2}{h}\, dt \le Ch, \qquad \forall\, h > 0 \tag{2.4}$$

---

(17) $H^1_0(0,T;H)$ is the closure in $H^1(0,T;H)$ of the subspace of the functions with compact support in $]0,T[$. The inclusion $X(0,T;H) \subset B^{1/2}_{2\infty}(0,T;H)$ is strict: consider for example the function $t \mapsto \log|\log t|$ in $]0,1/2[$, $H \equiv \mathbf{R}$.

If $h \geq b-a$ we have:

$$x_h = 2\int_a^b |f(t)|^2\,dt \leq 2(b-a)\|f\|^2_{L^\infty(0,T;H)} \leq 2\,h\|f\|^2_{L^\infty(0,T;H)}$$

If $h < b-a$ we find:

$$x_h = \int_a^{a+h} |f(t)|^2\,dt + \int_{b-h}^b |f(t)|^2\,dt + \int_a^{b-h} |f(t+h)-f(t)|^2\,dt \leq$$
$$2h\|f\|^2_{L^\infty(0,T;H)} + h\big[f\big]^2_{B^{1/2}_{2\infty}(0,T;H)}$$

In both cases (2.4) holds with $C = 2\|f\|^2_{L^\infty(0,T;H)} + \big[f\big]^2_{B^{1/2}_{2\infty}(0,T;H)}$; we deduce (2.3) and

$$L^\infty(0,T;H)\cap B^{1/2}_{2\infty}(0,T;H) \equiv L^\infty(0,T;H)\cap X(0,T;H) \equiv \Big(L^1(0,T;H)+B^{-1/2}_{2\,1}(0,T;H)\Big)'$$

consequently (1.16) is verified, by the density of $L^2(0,T;H)$ in $S(0,T)$. ■

2.3 Remark. $L^2(0,T;H)$ is densely imbedded in $S(0,T)$. □

We are now interested in the property of the truncation operator:

$$f \mapsto r_{a,b}f(t) = \begin{cases} f(t) & \text{if } t \in]a,b[; \\ 0 & \text{otherwise} \end{cases}; \qquad]a,b[\subset]0,T[$$

2.4 Lemma. *The linear operator $r_{a,b}$ is bounded from $I(0,T)$ to $I(0,T)$; moreover it can be extended continuously to $S(0,T)$. In particular we have $\forall\, f \in S(0,T)$*

$$\|f\|_{S(a,b)} \leq \|r_{a,b}f\|_{S(0,T)} \leq c\|f\|_{S(0,T)}; \quad \lim_{b\to a^+} \|r_{a,b}f\|_{S(0,T)} = 0 \tag{2.5}$$

$${}_{S(0,T)}\langle r_{a,b}f, v\rangle_{I(0,T)} = {}_{S(0,T)}\langle r_{a,b}f, r_{a,b}v\rangle_{I(0,T)} = {}_{S(0,T)}\langle f, r_{a,b}v\rangle_{I(0,T)} \tag{2.6}$$

(See the Appendix for the proof)

Let us now consider the approximation of $S(0,T)$-functions by piecewise constant ones. We already observed that definition (0.6) makes sense also for $S(0,T)$-functions f.

2.5 Notation. We denote by $\mathbf{P}_k(\mathcal{H})$ the subspace of $L^2(0,T;\mathcal{H})$ whose functions are constant on each $J_{k,n}$, and by Π_k the projection on $\mathbf{P}_k(\mathcal{H})$; f^k_n are the values of $\Pi_k f$ on $J_{k,n}$, that is:

$$\Pi_k f \equiv \frac{1}{k}\int_{J_{k,n}} f(t)\,dt = f^k_n, \quad \text{on } J_{k,n} \tag{2.7}$$

The following formal properties are straightforward

$$\int_0^T \big(\Pi_k v, w\big)\,dt = \int_0^T \big(v, \Pi_k w\big)\,dt = \int_0^T \big(\Pi_k v, \Pi_k w\big)\,dt \tag{2.8}$$

$$\Pi_k(A\Pi_k v)) = A_k\Pi_k v \quad \square \tag{2.9}$$

We study some properties of Π_k with respect to the $S(0,T)$-norm:

2.6 Lemma. *Let f_k, F_k be the step functions which take the value f_n^k, f_{n-1}^k on $J_{k,n}$ respectively. We have:*

$$\|f_k\|_{S(0,T)} \le c\|f\|_{S(0,T)}, \qquad \lim_{k\to 0}\|f-f_k\|_{S(0,T)} = \lim_{k\to 0}\|f_k - F_k\|_{S(0,T)} = 0 \tag{2.10}$$

$$\lim_{\substack{h\to 0^+, k\to 0^+\\ h=pk, p\in\mathbf{N}}} \|f_k(t+h) - f_k(t)\|_{S(0,T-h)} = 0 \tag{2.11}$$

$$\left.\begin{array}{l}\|f-f_k\|_{S(0,T)}\\ \|f_k - F_k\|_{S(0,T)}\end{array}\right\} \le Ck\|f\|_{S^1(0,T)} \tag{2.12}$$

$\forall\, s \in [0,T]$*:*

$$\int_0^s (f-f_k, v)\, dt \le c\, k^2\|f\|_{S^1(0,T)}\,\|v\|_{I^1(0,T)} + c\, k\|f\|_{S^1(0,T)}\|v\|_{L^2(0,T;V)\cap L^\infty(0,T;H)} \tag{2.13}$$

(The proof is given in the Appendix)

The following properties for A are straightforward:

2.7 Remark. For every function $v \in L^2(0,T;V)$ we have

$$\lim_{k\to 0}\|(A-A_k)v\|_{L^2(0,T;V')} = \lim_{k\to 0} k\|\hat{A}'_k v\|_{L^2(0,T;V')} = 0. \tag{2.14}$$

Moreover if (1.7) holds, we have:

$$\|(A-A_k)v\|_{L^2(0,T;V')} + k\|\hat{A}'_k v\|_{L^2(0,T;V')} \le M'\, k\|v\|_{L^\infty(0,T;V)}. \tag{2.15}$$

Now we consider the singular perturbation problem arising from (0.8), with $n = 0$ and $k\to 0$:

$$\begin{cases}\text{Given } u_0 \in \mathcal{K},\ A_0^k \text{ and } f_0^k \text{ as in (0.6), find } u_0^k \in D(\phi) \text{ such that:}\\ \Big(u_0^k - u_0 + k\, A_0^k u_0^k - k f_0^k, u_0^k - v\Big) + k\phi(u_0^k) - k\phi(v) \le 0, \quad \forall\, v \in D(\phi)\end{cases}. \tag{2.16}$$

Arguing as in [1], we have:

2.8 Proposition. *There exists a constant $C > 0$ such that, for $k\lambda < 1$*

$$|u_0^k|^2 + k\|u_0^k\|^2 + k\phi(u_0^k) \le C\big\{|u_0|^2 + \|f\|^2_{S(0,T)}\big\} \tag{2.17}$$

with

$$\lim_{k\to 0} |u_0^k - u_0| + \sqrt{k}\,\|u_0^k\| + k\phi(u_0^k) = 0. \tag{2.18}$$

Moreover, if (1.18) *and* (1.9) *hold, we have:*

$$|u_0^k - u_0|^2 + k\|u_0^k - u_0\|^2 + k|\phi(u_0^k) - \phi(u_0)| \le C\, k^2\big\{w(u_0,f)^2 + \|f\|^2_{S^1(0,T)}\big\}. \tag{2.19}$$

3. - Stability estimates.

The aim of this section is to prove the following stability result under the weakest regularity assumptions on the data:

3.1 Theorem. *There exists a constant $C > 0$ such that:*

$$\|\hat{u}_k\|^2_{I(0,T)} + k\|\hat{u}'_k\|^2_{L^2(0,T;H)} + \Phi(\hat{u}_k) \leq C\Big\{|u_0|^2 + \|f\|^2_{S(0,T)}\Big\}. \tag{3.1}$$

The *proof* will be given in a few steps, which will contain further useful pieces of information. Now we fix now some notation:

3.2 Notation. We denote by $s(0,T)$ the space $L^2(0,T;V') + L^1(0,T;H)$ and by $i(0,T)$ its dual $L^2(0,T;V) \cap L^\infty(0,T;H)$; observe that:

$$I(0,T) \subset i(0,T) \subset L^2(0,T;H) \subset s(0,T) \subset S(0,T) \tag{3.2}$$

Correspondingly, we set:

$$i^1(0,T) = H^1(0,T;V) \cap W^{1,\infty}(0,T;H) = \big\{v \in i(0,T) : \ v' \in i(0,T)\big\}$$
$$s^1(0,T) = H^1(0,T;V') + BV(0,T;H).$$

3.3 Notation. u_k, f_k, A_k are the step functions which respectively take the values

$$u^k_n, \ f^k_n, \ A^k_n \qquad \text{on } J_{k,n} = [(n-1)k, nk[, \quad n = 0, \dots, N.$$

We denote their translated version by

$$U_k(t) = u_k(t-k), \qquad F_k(t) = f_k(t-k) \quad \text{ defined in } [0, T+k[.$$

As $\hat{u}_k$, $\hat{f}_k$ and $\hat{A}_k$ are the corresponding piecewise linear functions which take the values f^k_n and A^k_n in nk. Finally $\ell_k(t)$ is the $k-$*periodic* function defined as $\ell_k(t) = t/k - (n-1)$ on $J_{k,n}$.

3.4 Remark. Every piecewise constant function is related to the linear one through ([18])

$$\hat{u}_k(t) = (1-\ell_k(t))u_k(t-k) + \ell_k(t)u_k(t) = (1-\ell_k)U_k + \ell_k u_k \tag{3.3}$$

$$\hat{u}'_k(t) = \frac{u_k(t) - u_k(t-k)}{k}; \qquad \hat{u}_k(t) - u_k(t) = k(1-\ell_k(t))\hat{u}'_k(t). \tag{3.4}$$

In many situations it will be easier to get estimates in terms of u_k, U_k than $\hat{u}_k$; $\hat{u}_k$ will be recovered by these formulae. □

We start with a lemma:

([18]) We only write the formula for u_k and $\hat{u}_k$; the behaviour of f_k, $\hat{f}_k$ and A_k, $\hat{A}_k$ is the same.

3.5 Lemma. *There exists a constant $C > 0$ such that:*

$$\left.\begin{array}{l} \sup\limits_{1\le n\le N} |u_n^k|^2 \\ \sum\limits_{n=1}^{N} \left[k\|u_n^k\|^2 + k\phi(u_n^k) + |u_n^k - u_{n-1}^k|^2 \right] \end{array}\right\} \le C\Big\{ |u_0^k|^2 + \sup_{1\le m\le N} k \sum_{n=0}^{m} (f_n^k, u_n^k) \Big\}. \tag{3.5}$$

Proof. Let us choose in (0.8) $v = 0$ (see remark 1.9); we have ([19])

$$|u_n^k|^2 + |u_n^k - u_{n-1}^k|^2 + 2\alpha\, k\|u_n^k\|^2 + 2k\phi(u_n^k) \le |u_{n-1}^k|^2 + 2\lambda\, k|u_n^k|^2 + 2k(f_n^k, u_n^k)$$

Summing up for $n = 1$ to m we get

$$|u_m^k|^2 + \sum_{n=1}^{m} \Big\{ |u_n^k - u_{n-1}^k|^2 + 2\alpha\, k\|u_n^k\|^2 + 2k\phi(u_n^k) \Big\} \le |u_0^k|^2 + 2k \sum_{n=1}^{m} (f_n^k, u_n^k) + \lambda |u_n^k|^2. \tag{3.6}$$

Let us denote by x_m the left hand member and by a_m the non-decreasing sequence:

$$a_m = |u_0^k|^2 + 2 \sup_{1\le n\le m} k \sum_{j=1}^{n} (f_j^k, u_j^k).$$

Formula (3.6) gives

$$x_m \le a_m + 2\lambda\, k \sum_{j=1}^{m} x_j$$

and, by a discrete version of the Gronwall lemma, if $2\lambda\, k < 1$

$$x_m \le \Big[|u_0^k|^2 + 2k \sup_{1\le n\le m} \sum_{j=1}^{n} (f_j^k, u_j^k) \Big] e^{\frac{2\lambda\, km}{1-2\lambda k}}.$$

Since $km \le T$ we get (3.5). ■

3.6 Corollary. *There exists a constant $C > 0$ such that:*

$$\begin{aligned} &\|u_k\|^2_{L^2(0,T;V)\cap L^\infty(0,T;H)} + \Phi(u_k) + k\, \|\hat{u}'_k\|^2_{L^2(0,T;H)} \le \\ &\qquad C\Big\{ |u_0^k|^2 + \sup_{1\le m\le N} \int_0^{mk} \big(f_k(t), u_k(t)\big)\, dt \Big\}. \end{aligned} \tag{3.7}$$

([19]) We shall often use the identity:

$$2(x, x - y) = |x|^2 + |x - y|^2 - |y|^2, \qquad \forall\, x, y \in H$$

Proof. It is an easy consequence of previous result, since

$$\|u_k\|^2_{L^2(0,T;H)} = k\sum_{n=1}^{N}|u_n^k|^2, \quad \|u_k\|_{L^\infty(0,T;H)} = \sup_{1\le n\le N}|u_n^k|$$

$$\|\hat{u}_k'\|^2_{L^2(0,T;H)} = k\sum_{n=1}^{N}\left|\frac{u_n^k - u_{n-1}^k}{k}\right|^2; \qquad \Phi(u_k) = k\sum_{n=1}^{N}\phi(u_n^k) \quad \blacksquare$$

3.7 Remark. If $f \in s(0,T)$ we could deduce the uniform boundedness of u_k in $i(0,T)$ from (3.7), since the last integral would be easily controlled by the Schwarz-Hölder inequality:

$$\sup_{1\le m\le N}\int_0^{mk}\big(f_k(t), u_k(t)\big)\,dt \le \frac{1}{2\epsilon}\|f_k\|^2_{s(0,T)} + \frac{\epsilon}{2}\|u_k\|^2_{i(0,T)}$$

for every $\epsilon > 0$; recalling that $\|f_k\|_{s(0,T)} \le \|f\|_{s(0,T)}$, from (3.7) we get:

$$\|u_k\|^2_{i(0,T)} + \Phi(u_k) + k\,\|\hat{u}_k'\|^2_{L^2(0,T;H)} \le C\left\{|u_0^k|^2 + \|f\|^2_{s(0,T)}\right\}. \tag{3.8}$$

On the contrary, if f contains a proper $B^{-1/2}_{2\,1}(0,T;H)$ term, we only have:

$$\sup_{1\le m\le N}\int_0^{mk}\big(f_k(t), u_k(t)\big)\,dt \le c\,\|f_k\|_{S(0,T)}\,\|u_k\|_{I(0,T)} \tag{3.9}$$

and we need a further estimate on u_k, which is provided by the next lemma. $\square$

3.8 Lemma. *There exists a constant $C > 0$ such that:*

$$\big[u_k\big]^2_{B^{1/2}_{2\infty}(0,T;H)} \le C\Big\{\|f\|_{S(0,T)}\|u_k\|_{I(0,T)} + \Phi(u_k) + \|u_k\|^2_{L^2(0,T;V)}\Big\}. \tag{3.10}$$

Proof. The $B^{1/2}_{2\infty}(0,T;H)$ seminorm for piecewise constant functions is given by (see [14]):

$$\big[u_k\big]^2_{B^{1/2}_{2\infty}(0,T;H)} \le \sup_{1\le p<N}\frac{1}{p}\sum_{n=1}^{N-p}\big|u_{n+p}^k - u_n^k\big|^2;$$

then we have to check that

$$\sum_{n=1}^{N-p}\big|u_{n+p}^k - u_n^k\big|^2 \le C\,p\left\{\|f\|_{S(0,T)}\|u_k\|_{I(0,T)} + \Phi(u_k) + \|u_k\|^2_{L^2(0,T;V)}\right\} \tag{3.11}$$

for some constant $C > 0$ independent of k and p. In order to get this bound, we choose $v = u_m^k$ in (0.8), for $n = m+1$ to $n = m+p$. Summing up with respect to n we get ([20])

$$|u_{m+p}^k - u_m^k|^2 \le 2k \sum_{j=1}^{p} \left(f_{m+j}^k - A_{m+j}^k\, u_{m+j}^k, u_{m+j}^k - u_m^k\right) + \phi(u_m^k) - \phi(u_{m+j}^k).$$

Summing again:
(3.12)

$$\sum_{m=1}^{N-p} |u_{m+p}^k - u_m^k|^2 \le 2 \sum_{j=1}^{p} \sum_{m=1}^{N-p} \Big\{ k\big(f_{m+j}^k, u_{m+j}^k - u_m^k\big) +$$

$$+ M\, k\, \|u_{m+j}^k\|\, \|u_{m+j}^k - u_m^k\| + k\phi(u_m^k) - k\phi(u_{m+j}^k) \Big\} \le$$

$$2 \sum_{j=1}^{p} \int_0^{T-pk} \Big\{ \big(f_k(t+jk), u_k(t+jk) - u_k(t)\big) +$$

$$+ M\|u_k(t+jk)\|\, \|u_k(t+jk) - u_k(t)\| + \phi(u_k(t+jk)) - \phi(u_k(t)) \Big\}\, dt \le$$

$$4\, p \Big\{ c\, \|f_k\|_{S(0,T)} \|u_k\|_{I(0,T)} + M\|u_k\|^2_{L^2(0,T;V)} + \Phi(u_k) \Big\} \quad \blacksquare$$

We can now conclude the proof of theorem 3.1:

3.9 Corollary. *There exists a constant $C > 0$ such that:*

(3.13) $$\|u_k\|^2_{I(0,T)} + k\|\hat{u}_k'\|^2_{L^2(0,T;H)} + \Phi(u_k) \le C\Big\{ |u_0|^2 + \|f\|^2_{S(0,T)} \Big\}$$

Moreover (3.1) *holds and u_k is related to $\hat{u}_k$ by:*

(3.14) $$\lim_{k\to 0^+} \|u_k - \hat{u}_k\|_{L^2(0,T;H)} = 0; \qquad (u_k - \hat{u}_k) \rightharpoonup^* 0 \ \ in \ \ I(0,T)$$

Proof. Combining (3.7) with (3.10) and taking account of (3.9) and (2.17) we find (3.13). Since

(3.15) $$\|\hat{u}_k\|^2_{i(0,T)} \le k\|u_0^k\|^2 + |u_0^k|^2 + \|u_k\|^2_{i(0,T)}, \qquad \Phi(\hat{u}_k) \le k\phi(u_0^k) + \Phi(u_k)$$

and (see the Appendix):

(3.16) $$\big[\hat{u}_k\big]^2_{B^{1/2}_{2\infty}(0,T;H)} \le c\left\{ k\|\hat{u}_k'\|^2_{L^2(0,T;H)} + \big[u_k\big]^2_{B^{1/2}_{2\infty}(0,T;H)} \right\}$$

via (2.17) we obtain (3.1), too. $\blacksquare$

With regard to the last calculations of (3.12), we can obtain a finer estimate, which will be useful later on:

([20]) We apply the identity ([14])

$$2\sum_{j=1}^{p} (v_j - v_{j-1}, v_j - v_0) = |v_p - v_0|^2 + \sum_{j=1}^{p} |v_j - v_{j-1}|^2.$$

3.10 Lemma. *There exists a constant $C > 0$ such that $\frac{1}{p}\sum_{m=1}^{N-p} |u^k_{m+p} - u^k_m|^2$ is bounded by:*

$$C\left\{\begin{array}{l}\displaystyle\int_0^{pk} \phi(u_k)\,dt + \|u_k\|_{I(0,T)}\Big[\|r_{0,pk}f\|_{S(0,T)} + \|r_{T-pk,T}\,f\|_{S(0,T)}\Big] + \\ \|u_k\|_{I(0,T)}\Big[\displaystyle\sup_{1\le j\le p}\big(\|f_k(t+jk) - f_k(t)\|_{S(0,T-jk)} + \|u_k(t+jk) - u_k(t)\|_{L^2(0,T-jk;V)}\big)\Big]\end{array}\right.$$

Proof. Let us start from the second last inequality of (3.12); thanks to corollary 2.3 we have:

$$\int_0^{T-pk} \big(f_k(t+jk), u_k(t+jk) - u_k(t)\big)\,dt = \int_0^{T-pk} \big(f_k(t) - f_k(t+jk), u_k(t)\big)\,dt -$$
$$\int_0^{jk} \big(f_k(t), u_k(t)\big)\,dt + \int_{T-pk}^{T-(p-j)k} \big(f_k(t), u_k(t)\big)\,dt \le$$
$$\Big\{\|f_k(t+jk) - f_k(t)\|_{S(0,T-pk)} + \|r_{0,pk}f\|_{S(0,T)} + \|r_{T-pk,T}\,f\|_|ST\Big\}\|u_k\|_{I(0,T)}$$

and
$$\int_0^{T-pk} \Big(\phi(u_k(t)) - \phi(u_k(t+jk))\Big)\,dt \le \int_0^{pk} \phi(u_k)\,dt \quad ■$$

We can obtain a further stability result, which also involves the piecewise quadratic interpolant $\tilde{u}_k$:

3.11 Notation. $\tilde{u}_k$ is defined by:

$$\tilde{u}_k(t) = \frac{1}{k}\int_t^{t+k} \hat{u}_k(\tau)\,d\tau \qquad (^{21}).$$

Observe that:

$$\tilde{u}'_k(t) = (1-\ell_k)\hat{u}'_k(t) + \ell_k\hat{u}'_k(t+k) \qquad \tilde{u}''_k(t) = \frac{\hat{u}'_k(t+k) - \hat{u}'_k(t)}{k} \quad \Box \tag{3.17}$$

3.12 Lemma. *There exists a constant $C > 0$ such that:*

$$\begin{aligned}&\|\hat{u}'_k\|^2_{i(0,T)} + k\|\tilde{u}''_k\|^2_{L^2(0,T;H)} \le \\ &\qquad C\Big\{\Big|\frac{u^k_0 - u_0}{k}\Big|^2 + \|\hat{A}'_k\,u_k\|^2_{L^2(0,T;V')} + \sup_{1\le m\le N}\int_0^{mk} (\hat{f}'_k, \hat{u}'_k)\,dt\Big\}\end{aligned} \tag{3.18}$$

(^{21}) We define $\hat{u}_k(t) \equiv u^N_k$ for $t \ge T$.

Proof. Let us choose $v = u^k_{n+1}$ in (0.8) and $v = u^k_n$ in the same inequality at the next time step; summing up these relations and dividing by k^2 we find:

$$\left|\frac{u^k_{n+1} - u^k_n}{k}\right|^2 - \left|\frac{u^k_n - u^k_{n-1}}{k}\right|^2 + \left|\frac{u^k_{n+1} - u^k_n - (u^k_n - u^k_{n-1})}{k}\right|^2 + \alpha k \left\|\frac{u^k_{n+1} - u^k_n}{k}\right\|^2 \leq$$
$$2k\left(\frac{f^k_{n+1} - f^k_n}{k}, \frac{u^k_{n+1} - u^k_n}{k}\right) + 2\lambda k \left|\frac{u^k_{n+1} - u^k_n}{k}\right|^2 + \frac{2k}{\alpha}\|u^k_{n+1}\|^2 \left\|\frac{A^k_{n+1} - A^k_n}{k}\right\|^2_{\mathcal{L}(V,V')}$$

Taking the sum for $n = 0$ to $m - 1 < N$ and recalling that $f^0_k = f^1_k$, $A^0_k = A^1_k$ we get

$$\left|\frac{u^k_m - u^k_{m-1}}{k}\right|^2 + \sum_{n=0}^{m-1}\left|\frac{u^k_{n+1} - u^k_n - (u^k_n - u^k_{n-1})}{k}\right|^2 + \frac{\alpha k}{2}\left\|\frac{u^k_{n+1} - u^k_n}{k}\right\|^2 \leq$$
$$\left|\frac{u^k_0 - u_0}{k}\right|^2 +$$
$$\sum_{n=0}^{m-1} 2k\left\{\left(\frac{f^k_{n+1} - f^k_n}{k}, \frac{u^k_{n+1} - u^k_n}{k}\right) + \lambda\left|\frac{u^k_{n+1} - u^k_n}{k}\right|^2 + \frac{\|u^k_{n+1}\|^2}{\alpha}\left\|\frac{A^k_{n+1} - A^k_n}{k}\right\|^2_{\mathcal{L}(V,V')}\right\}$$

A further application of the discrete Gronwall lemma as in 3.1 gives (3.18). ∎

As we already observed in 3.7, standard assumptions on f and u_0 give directly the stability of the derivative of $\hat{u}_k$ in the energy norm:

3.13 Corollary. *Suppose that* (1.7), (1.8) *and* (1.9) *hold true; then there exists a constant* $C > 0$ *(only depending on* $\alpha, \lambda, T, M, M'$*) such that:*

$$\|\hat{u}'_k\|_{i(0,T)} + \sqrt{k}\|\tilde{u}''_k\|_{L^2(0,T;H)} \leq C\Big\{w(u_0, f) + \|f'\|_{s(0,T)}\Big\}. \tag{3.19}$$

4. - Existence and convergence results with low regularity assumptions.

The aim of this chapter is to prove the following

4.1 Theorem. *$\{\hat{u}_k\}_{k>0}$ is a Cauchy's family in $I(0,T)$ as $k\to 0$, and its limit u is the unique solution of (Pw). Moreover, setting:*

(4.1)

$$\mathcal{E}_k^2 = \mathcal{E}_k^2(u_0, f; T) = \begin{cases} |u_0^k - u_0|^2 + k\|u_1^k - u_0^k\| + k^2\|u_0^k\|^2 + \\ \sup\limits_{0<t\leq T} {}_{S(0,t)}\langle f - f_k, u - \hat{u}_k\rangle_{I(0,t)} + k^2 \sup\limits_{0<j\leq N} \int_0^{jk} (\hat{f}_k', \hat{u}_k')\, dt + \\ \|(A - A_k)u_k\|^2_{L^2(0,T;V')} + k^2\|\hat{A}_k' u_k\|^2_{L^2(0,T;V')} \end{cases}$$

we have the estimates

$$\|\hat{u}_k'\|_{L^2(0,T;V)\cap L^\infty(0,T;H)} + \sqrt{k}\|\tilde{u}_k''\|_{L^2(0,T;H)} \leq C\mathcal{E}_k/k \tag{4.2}$$

$$\|u - \hat{u}_k\|_{L^2(0,T;V)\cap L^\infty(0,T;H)} \leq C\,\mathcal{E}_k. \tag{4.3}$$

Observe that (4.2) is a simple reformulation of (3.18); as in the previous section we divide the *proof* of the remaining part of this theorem in some steps. The basic tool consists in writing a "continuous" equation which is satisfied by $\hat{u}_k$ and u_k and in using the stability estimates to pass to the limit in a weak form; then, a finer analysis will give a stronger convergence, the uniqueness of the limit and estimate (4.3).

The time dependence of A ([22]) and the need of optimal estimates are the basic technical difficulties.

Our starting point is the following

([22]) which may be very irregular, at least at the lowest level of our hypotheses; in particular we cannot deduce in general that

$$\|Av_k - A_k v_k\|_{L^2(0,T;V')}\to 0, \qquad k\to 0$$

if v_k is only a bounded family of $L^2(0,T;V)$. Here we have an example: fix $V \equiv \mathbf{R}$, $T = 1$ and $A(t) = \chi_{[0,1/3[}(t)$. Let us consider $n_k = \min\{n : nk > 1/3\}$ and set

$$v_k(t) = \begin{cases} k^{-1/2} & \text{if } t \in J_{k,n_k}, \\ 0 & \text{otherwise;} \end{cases} \qquad \|v_k\|_{L^2(0,1)} = 1.$$

Choosing $k = 10^{-j}$, $j \in \mathbf{N}$ we have:

$$\|A\,v_k - A_k\,v_k\|_{L^2(0,1)} = \frac{2}{3}.$$

4.2 Proposition. *Functions u_k and $\hat{u}_k$ satisfy,* $\forall\, v \in D(\phi)$

$$(4.4) \qquad \big(\hat{u}_k' + A_k u_k - f_k, u_k - v\big) + \phi(u_k) - \phi(v) \le 0, \quad \textit{a.e. in }]0,T[$$

and, $\forall\, v \in H^1(0,T;H) \cap D(\Phi)$

$$(4.5) \qquad \begin{aligned} &\int_0^T \Big\{ \big(v', \hat{u}_k - v\big) + \big((A_k + \lambda)u_k - (f_k + \lambda v), u_k - v\big) \Big\}\, d\mu_\lambda + \Phi_\lambda(u_k) - \Phi_\lambda(v) \le \\ &\qquad \le \frac{1}{2}|u_0^k - v(0)|^2 + k\phi(u_0^k) + 2\lambda \int_0^T (u_k - \hat{u}_k, u_k - v)\, d\mu_\lambda. \end{aligned}$$

Proof. The first inequality is an immediate consequence of (0.6).
Assume now that $v = v(t) \in H^1(0,T;H) \cap D(\Phi)$ in (4.4); since

$$(\hat{u}_k', u_k - \hat{u}_k) = k\big(1 - \ell_k\big)|\hat{u}_k'|^2 \ge 0$$

we have

$$(v', \hat{u}_k - v) + (A_k u_k - f_k, u_k - v) + (\hat{u}_k' - v', \hat{u}_k - v) + \phi(u_k) - \phi(v) \le 0;$$

we multiply by $e^{-2\lambda t}$ and integrate between 0 and T; taking into account

$$\frac{1}{2}\frac{d}{dt}|e^{-\lambda t}(\hat{u}_k(t) - v)|^2 + \lambda e^{-2\lambda t}|\hat{u}_k - v|^2 = e^{-2\lambda t}(\hat{u}_k' - v', \hat{u}_k - v)$$

We obtain

$$\begin{aligned} &\int_0^T \Big\{ (v', \hat{u}_k - v) + (A_k u_k - f_k, u_k - v) \Big\}\, d\mu_\lambda \\ &+ \int_0^T \lambda|\hat{u}_k - v|^2 + \phi(u_k) - \phi(v)\, d\mu_\lambda \le \frac{1}{2}|u_0^k - v(0)|^2. \end{aligned}$$

Since

$$|\hat{u}_k - v|^2 - |u_k - v|^2 = (\hat{u}_k + u_k - 2v, \hat{u}_k - u_k) \ge 2(u_k - v, \hat{u}_k - u_k)$$

we obtain (4.5), too. ∎

We proved in the previous section that the family $\{u_k\}$ is uniformly bounded in $I(0,T)$; since $I(0,T)$ is the dual of $S(0,T)$ we can find a weakly* convergent subsequence; thanks to proposition 4.2 we deduce the existence of a solution of (Pw). The next lemma will solve the main technical difficulty.

4.3 Lemma. *Let $k_n \to 0$ be a decreasing subsequence of $\{T/N\}_{N\in\mathbf{N}}$ such that*

$$(4.6) \qquad u_{k_n} \rightharpoonup^* u \qquad \textit{in } I(0,T), \quad n \to +\infty;$$

then, $\forall\, v \in L^2(0,T;V)$

(4.7) $$\int_0^T \big((A+\lambda)u, u-v\big)\; d\mu_\lambda \le \liminf_{n\to\infty} \int_0^T \big((A_{k_n}+\lambda)u_{k_n}, u_{k_n}-v\big)\; d\mu_\lambda .$$

Proof. Observe that $u_{k_n}, A_{k_n}u_{k_n} \in \mathbf{P}_{k_n}$ so that by (2.8):

$$\int_0^T \Big((A_{k_n}+\lambda)u_{k_n}, u_{k_n}-v\Big)\; d\mu_\lambda = \int_0^T \Big((A_{k_n}+\lambda)u_{k_n}, \Pi_{k_n}\big(u_{k_n}-v\big)e^{-2\lambda t}\big)\Big)\, dt =$$
$$\int_0^T \Big((A+\lambda)u_{k_n}, \Pi_{k_n}\big(u_{k_n}-v\big)e^{-2\lambda t}\big)\Big)\, dt = \int_0^T \Big((A+\lambda)u_{k_n}, u_{k_n}-v\Big)\; d\mu_\lambda +$$
$$\int_0^T \Big((A+\lambda)u_{k_n}, (u_{k_n}-v)e^{-2\lambda t} - \Pi_{k_n}\big(u_{k_n}-v\big)e^{-2\lambda t}\big)\Big)\, dt.$$

Due to the monotonicity of $A+\lambda$ it is a standard fact that

$$\liminf_{n\to\infty} \int_0^T \big((A+\lambda)u_{k_n}, u_{k_n}-v\big)\; d\mu_\lambda \ge \int_0^T \big((A+\lambda)u, u-v\big)\; d\mu_\lambda$$

whereas the second term of the previous formula can be bounded by

$$(M+\lambda)\|u_{k_n}\|_{L^2(0,T;V)} \Big\{2\lambda k_n \|u_{k_n}\|_{L^2(0,T;V)} + \|v\, e^{-2\lambda t} - \Pi_{k_n} v e^{-2\lambda t}\|_{L^2(0,T;V)}\Big\}$$

which tends to 0. ∎

4.4 Proposition. *Let* $k_n \to 0$ *be a decreasing subsequence of* $\{T/N\}_{N\in\mathbf{N}}$ *as in the previous lemma; then* u *is a solution of* (Pw).

Proof. Consider (4.5) and observe that

- by the lower semicontinuity of Φ_λ we have:

$$\Phi_\lambda(u) \le \liminf_{n\to\infty} \Phi_\lambda(u_{k_n});$$

- since $f_{k_n} \to f$ in $S(0,T)$ and $\hat u_{k_n} \rightharpoonup^* u$ in $I(0,T)$ we get

$$\int_0^T \Big\{(v', \hat u_k - v) - (f_{k_n} + \lambda v, u_{k_n} - v)\Big\}\, d\mu_\lambda \to {}_{S(0,T)}\big\langle v' - (f+\lambda v), (u-v)e^{-2\lambda t}\big\rangle_{I(0,T)};$$

- $|u_0^{k_n} - v(0)|^2 \to |u_0 - v(0)|^2$ by (2.18).

Taking into account the previous lemma, we conclude. ∎

4.5 Proposition. *Problem* (Pw) *admits a unique solution* u *and*

$$\lim_{k\to 0} \|u-u_k\|_{L^2(0,T;V)} = 0. \tag{4.8}$$

Proof. Choose $v=u$ in (4.4) and integrate from 0 to T

$$\int_0^T \Big\{ \big(\hat{u}_k', \hat{u}_k - u\big) + \big((A_k+\lambda)u_k - (f_k+\lambda u), u_k - u\big) - \lambda |u_k - u|^2 \Big\}\, d\mu_\lambda + \Phi_\lambda(u_k) - \Phi_\lambda(u) \le 0.$$

Analogously, we choose $v = \hat{u}_k$ in (1.3) ([23])

$$\int_0^T \Big\{ \big(\hat{u}_k', u - \hat{u}_k\big) + \big(A_k+\lambda)u - (f+\lambda u_k), u - u_k\big) \Big\}\, d\mu_\lambda + \Phi_\lambda(u) - \Phi_\lambda(u_k) \le$$

$$\le \int_0^T \Big\{ \big((A_k - A)u - \lambda(u_k - \hat{u}_k), u - u_k\big) + \big((A+\lambda)u - f, \hat{u}_k - u_k\big) \Big\}\, d\mu_\lambda +$$

$$\frac{1}{2}|u_0 - u_0^k|^2 + \Phi_\lambda(\hat{u}_k) - \Phi_\lambda(u_k).$$

Summing up we obtain:

$$\alpha \int_0^T \|u-u_k\|^2 \, d\mu_\lambda \le \Phi_\lambda(\hat{u}_k) - \Phi_\lambda(u_k) + \frac{1}{2}|u_0 - u_0^k|^2 +$$

$$\int_0^T \Big\{ \big((A_k - A)u - \lambda(u_k - \hat{u}_k), u - u_k\big) + \big((A+\lambda)u - f, \hat{u}_k - u_k\big) \Big\}\, d\mu_\lambda.$$

Since $\|A_k u - Au\|_{L^2(0,T;V')}\to 0$ by (2.14) and (2.18), (3.14) hold, the right-hand term of this last inequality tends to 0 if $\Big\{\Phi_\lambda(\hat{u}_k) - \Phi_\lambda(u_k)\Big\}\to 0$; due to the convexity of Φ_λ, we have:

$$\int_0^T \Big\{ \phi(\hat{u}_k) - \phi(u_k) \Big\}\, d\mu_\lambda \le$$

$$\int_0^T \big[(1-\ell_k)\phi(U_k) + \ell_k \phi(u_k) - \phi(u_k) \big]\, d\mu_\lambda =$$

$$\int_0^T (1-\ell_k)[\phi(U_k) - \phi(u_k)]\, d\mu_\lambda =$$

$$\int_0^T e^{-2\lambda(t)}(1-\ell_k)\phi(U_k)\, dt - \int_k^{T+k} e^{-2\lambda(t-k)}(1-\ell_k)\phi(U_k)\, dt \le k\phi(u_0^k) \quad \blacksquare$$

4.6 Remark. Combining (2.18) with (3.15) we get

$$\limsup_{k\to 0} \|\hat{u}_k\|_{L^2(0,T;V)} \le \lim_{k\to 0} \big(\sqrt{k}\|u_0^k\| + \|u_k\|_{L^2(0,T;V)}\big) = \|u\|_{L^2(0,T;V)}$$

so that $\hat{u}_k \to u$ in $L^2(0,T;V)$. $\square$

([23]) For the sake of simplicity, we write the integrals instead of the duality pairing between $S(0,T)$ and $I(0,T)$.

4.7 Corollary.

$$\lim_{k\to 0} \mathcal{E}_k(u_0, f) = 0. \tag{4.9}$$

Proof. Taking into account (2.18), the stability result of Theorem 3.1 and (2.10), it remains to check that:

$$\lim_{k\to 0} \Big\{ \|(A_k - A_k(t-k))u_k\|_{L^2(0,T;V')} + \|(A - A_k)u_k\|_{L^2(0,T;V')} \Big\} = 0.$$

Since u_k and $\hat{u}_k$ converge to u in $L^2(0,T;V)$ and A_k are uniformly bounded in $\mathcal{L}(V, V')$, we reduce it to

$$\lim_{k\to 0} \Big\{ \|(A_k(t) - A_k(t-k))u\|_{L^2(0,T;V')} + \|Au - A_k u\|_{L^2(0,T;V')} \Big\} = 0$$

which is given by (2.14). ∎

In order to obtain stronger estimates, we need a refinement of formula (4.4):

4.8 Proposition. *For any $v \in D(\phi)$ we have:*

$$\big(\hat{u}_k' + A\hat{u}_k - f_k, \hat{u}_k - v\big) + \phi(\hat{u}_k) - \phi(v) \le r_k(u_0, f; v) \tag{4.10}$$

with

$$r_k(u_0, f; v) = \begin{cases} (1-\ell_k)\Big(\hat{u}_k' + A_k u_k - f_k - (\hat{u}_k'(t-k) + A_k(t-k)U_k - F_k), U_k - u_k\Big) + \\ + \big(A\hat{u}_k - A_k u_k, \hat{u}_k - v\big) \end{cases}$$

Proof. Let us start from the left-hand member of (4.10):

(4.11)

$$\begin{aligned}
&\big(\hat{u}_k' + A\hat{u}_k - f_k, \hat{u}_k - v\big) + \phi(\hat{u}_k) - \phi(v) = \\
&\big(\hat{u}_k' + A_k u_k - f_k, \hat{u}_k - v\big) + \phi((1-\ell_k)U_k + \ell_k u_k) - \phi(v) + (A\hat{u}_k - A_k u_k, \hat{u}_k - v) \le \\
&\big(\hat{u}_k' + A_k u_k - f_k, u_k - v\big) + \big(\hat{u}_k' + A_k u_k - f_k, \hat{u}_k - u_k\big) + \\
&\qquad + (1-\ell_k)\phi(U_k) + \ell_k\phi(u_k) - \phi(v) + (A\hat{u}_k - A_k u_k, \hat{u}_k - v) \le \\
&(1-\ell_k)\Big[\big(\hat{u}_k' + A_k u_k - f_k, U_k - u_k\big) + \phi(U_k) - \phi(u_k)\Big] + (A\hat{u}_k - A_k u_k, \hat{u}_k - v)
\end{aligned}$$

thanks to (4.4); we multiply now (4.4) at $t-k$ by $(1-\ell_k)$ and we choose $v = u_k$:

$$(1-\ell_k)\Big[\big(\hat{u}_k'(t-k) + A_k(t-k)u_k(t-k) - f_k(t-k), u_k - U_k\big) + \phi(u_k) - \phi(U_k)\Big] \ge 0.$$

Summing this quantity to the last member of (4.11) we get (4.10). ∎

We must bound the integral of r_k in $]0,T[$:

4.9 Lemma. *For any $v \in D(\Phi)$ let us set*

$$\mathcal{R}_k(u_0, f; v) = \sup_{0 \le s \le T} \int_0^s r_k(u_0, f; v(t))\, dt;$$

then we have

$$\mathcal{R}_k(u_0, f; v) \le (C/\epsilon)\, \mathcal{E}_k^2 + \epsilon\, \|\hat{u}_k - v\|^2_{L^2(0,T;V)}, \qquad \forall\, \epsilon > 0. \tag{4.12}$$

Proof. The first part of the integrand r_k is positive for every $t \in [0,T]$; then it is enough to consider the integral on the whole interval.

We note that for piecewise constant functions g_k the following relation holds:

$$\int_0^T (1 - \ell_k(t)) g_k(t)\, dt = \frac{1}{2} \int_0^T g_k(t)\, dt$$

so that

$$\int_0^T (1-\ell_k)\big(\hat{u}_k' - \hat{u}_k'(t-k), U_k - u_k\big)\, dt = \frac{k}{2} \int_0^T \big(\hat{u}_k'(t-k) - \hat{u}_k', \hat{u}_k'\big)\, dt \le$$
$$\frac{k}{2} \int_0^T \big(|\hat{u}_k'(t-k)|^2 - |\hat{u}_k'|^2\big)\, dt = \frac{k}{2} \int_0^k \left| \frac{u_0^k - u_0}{k} \right|^2 \le \frac{1}{2} |u_0^k - u_0|^2 \le \frac{1}{2} \mathcal{E}_k.$$

For the other terms we get:

$$\int_0^T (1-\ell_k)\big(F_k - f_k + A_k u_k - A_k(t-k) U_k, U_k - u_k\big)\, dt \le$$
$$\frac{k^2}{2} \int_0^T \big(\hat{f}_k' - \hat{A}_k' u_k, \hat{u}_k'\big) + M \|\hat{u}_k'\|^2\, dt \le C \mathcal{E}_k^2$$

and, for every $s \in]0,T[$:

$$\int_0^s (A\hat{u}_k - A_k u_k, \hat{u}_k - v)\, dt \le$$
$$(1/2\epsilon) \|A(\hat{u}_k - u_k) + (A - A_k) u_k\|^2_{L^2(0,T;V')} + (\epsilon/2) \|\hat{u}_k - v\|^2_{L^2(0,T;V)} \le$$
$$(1/2\epsilon) \Big[M\, k^2 \|\hat{u}_k'\|^2_{L^2(0,T;V)} + \|(A - A_k) u_k\|^2_{L^2(0,T;V')} \Big] + (\epsilon/2) \|\hat{u}_k - v\|^2 \le$$
$$(C/\epsilon) \mathcal{E}_k^2 + \epsilon \|\hat{u}_k - v\|^2_{L^2(0,T;V)}$$

thanks to (4.2) and to an obvious application of the Schwarz-Hölder inequality. ∎

4.10 Corollary. *Let (u_0, f), (v_0, g) be given in $\mathcal{K} \times S(0,T)$ and let $\hat{u}_k$, $\hat{v}_h$ be the corresponding approximate solutions relative to the time steps k and h; we have:*

$$\text{(4.13)}\quad \|\hat{u}_k - \hat{v}_h\|^2_{L^2(0,T;V)\cap L^\infty(0,T;H)} \le C \begin{cases} |u_0^k - v_0^h|^2 + \sup\limits_{0\le s\le T} \displaystyle\int_0^s (f_k - g_h, \hat{u}_k - \hat{v}_h)\, dt + \\ \mathcal{E}_k(u_0, f) + \mathcal{E}_h(v_0, g) \end{cases}$$

Proof. We write (4.10) for $\hat{u}_k$ and $\hat{v}_h$, choosing $\hat{v}_h$ and $\hat{u}_k$ as test functions respectively; an integration and an application of the previous result give (4.13), via the Gronwall lemma. ∎

4.11 Corollary. *$\hat{u}_k$ converges to u in $i(0,T)$, too, with (4.3).* ∎

4.12 Corollary. *Let (u_0, f), (v_0, g) be given in $\mathcal{K} \times S(0,T)$ and let u, v be the correponding solutions of (Pw); we have*

$$\text{(4.14)}\qquad \|u - v\|^2_{i(0,T)} \le C\Big\{|u_0 - v_0|^2 + \|f - g\|_{S(0,T)}(\|f\|_{S(0,T)} + \|g\|_{S(0,T)})\Big\}$$

and, if $f, g \in s(0,T)$:

$$\text{(4.15)}\qquad \|u - v\|_{i(0,T)} \le C\Big\{|u_0 - v_0| + \|f - g\|_{s(0,T)}\Big\} \quad ∎$$

It remains to prove the convergence of $\hat{u}_k$ with respect to the $[\,\cdot\,]_{B^{1/2}_{2\infty}(0,T;H)}$ seminorm; we state a preliminary result:

4.13 Lemma. *Let v_k, $k \in \{T/n\}_{n\in\mathbf{N}}$, be a bounded family in $B^{1/2}_{2\infty}(0,T;H)$ which converges to v in $L^2(0,T;H)$ as $k\to 0^+$. If*

$$\text{(4.16)}\qquad \lim_{h\to 0^+, k\to 0^+} \frac{1}{h}\int_0^{T-h} |v_k(t+h) - v_k(t)|^2\, dt = 0,$$

then $v_k \to v$ in $B^{1/2}_{2\infty}(0,T;H)$. (See the Appendix for the proof)

4.14 Proposition. *We have*

$$\text{(4.17)}\qquad \lim_{h\to 0^+, k\to 0^+} \frac{1}{h}\int_0^{T-h} |u_k(t+h) - u_k(t)|^2\, dt = 0$$

and consequently $\hat{u}_k \to u$ in $B^{1/2}_{2\infty}(0,T;H)$.

Proof. We observe that the function (see [14])

$$h \mapsto x_h = \int_0^{T-h} |u_k(t+h) - u_k(t)|^2\, dt$$

is linear on each $J_{k,n}$ so that x_h/h is bounded by the values on the nodes $h = pk$, $p \in \mathbf{N}$; (4.17) is then equivalent to

$$\lim_{\substack{h\to 0, k\to 0\\ h=pk}} \frac{1}{h}\int_0^{T-h} |u_k(t+h) - u_k(t)|^2\, dt = 0.$$

Let us go back to lemma 3.10; there exists a constant $C > 0$ (this time depending also on f, u) such that ($h = pk$, $p \in \mathbf{N}$)

$$x_h/h \le C\Big\{ \int_0^h \phi(u_k)\, dt + \|u - u_k\|_{L^2(0,T;V)} + \mu_{h,k}(f,u)\Big\}, \tag{4.18}$$

with:

$$\mu_{h,k}(f,u) = \begin{cases} \|f_k(t+h) - f_k(t)\|_{S(0,T-h)} + \|u(t+h) - u(t)\|_{L^2(0,T;V)} \\ + \|r_{0,h} f\|_{S(0,T)} + \|r_{T,T-h}\, f\|_{S(0,T)}. \end{cases}$$

By (2.11) and (2.5)

$$\lim_{\substack{h\to 0, k\to 0\\ h=pk}} \mu_{h,k}(f,u) = 0; \tag{4.19}$$

it remains to show that

$$\lim_{\substack{h\to 0, k\to 0\\ h=pk}} \int_0^h \phi(u_k)\, dt = 0.$$

From (3.6) we get, for $h = pk$:

$$\begin{aligned} &\int_0^h \phi(u_k)\, dt \le \frac{1}{2}(|\hat u_k(0)|^2 - |\hat u_k(h)|^2) + \int_0^h \Big[(f_k, u_k) + \lambda |u_k|^2\Big]\, dt \le \\ &C\|u_k\|_{i(0,T)}\{|\hat u_k(h) - \hat u_k(0)| + \|r_{0,h}\, f\|_{S(0,T)} + \|u_k\|_{L^2(0,h;H)}\} \le \\ &C\Big\{\|\hat u_k - u\|_{L^\infty(0,T;H)} + \|u_k - u\|_{L^2(0,T;H)} + \\ &\qquad + \|r_{0,h}\, f\|_{S(0,T)} + |u(h) - u(0)| + \|u\|_{L^2(0,h;H)}\Big\} \end{aligned}$$

where the last constant C depends on u_0, f too.

To show (4.16) for $\hat u_k$, we observe that:

$$\frac{1}{h}\int_0^T |\hat u_k(t+h) - \hat u_k(t)|^2\, dt \le h\|\hat u_k'\|^2_{L^2(0,T;H)} \le k\|\hat u_k'\|^2_{L^2(0,T;H)}$$

if $h \le k$, and $\lim_{k\to 0} k\|\hat u_k'\|^2 = 0$ by the previous calculations; if $h > k$ we can easily reduce it to the estimate for u_k:

$$\frac{1}{h}\int_0^T |\hat u_k(t+h) - \hat u_k(t)|^2\, dt \le \frac{2}{h}\int_0^T |u_k(t+h) - u_k(t)|^2\, dt + \frac{4}{k}\|\hat u_k - u_k\|^2$$

and we conclude. ∎

5. - Error estimates and regularity results: the case $f \in H^1(0,T;V')$.

Throughout all this section we assume (1.7), (1.8), (1.9) and we shall prove Theorem 2 with the related convergence estimates (1.27) and (1.28); as we pointed out in note ([5]), $f \in H^1(0,T;V')$ can be replaced by $f \in s^1(0,T) = H^1(0,T;V') + BV(0,T;H)$; on the other hand, this case will be recovered by the next section.

Thanks to the previous results we can immediately prove the following:

5.1 Theorem. *The solution of* (Pw) *belongs to* $i^1(0,T) = H^1(0,T;V) \cap W^{1,\infty}(0,T;H)$ *with*

$$\mathcal{E}_k \leq C\,k\Big\{w(u_0,f) + \|f\|_{H^1(0,T;V')}\Big\} \tag{5.1}$$

and consequently

$$\|u - \hat{u}_k\|_{L^2(0,T;V)\cap L^\infty(0,T;H)} \leq C\,k\Big\{w(u_0,f) + \|f\|_{H^1(0,T;V')}\Big\}. \tag{5.2}$$

Proof. Let us start from (4.1); as usual we denote by I, II, III the three terms of its right-hand member.

Taking into account (2.19), we have:

$$I \leq C\,k^2\big\{w(u_0,f)^2 + \|f\|^2_{S^1(0,T)}\big\} \tag{5.3}$$

Since

$$\|u_k\|^2_{L^\infty(0,T;V)} \leq \|u_0^k\|^2 + 2\|\hat{u}_k'\|_{L^2(0,T;V)}\,\|u_k\|_{L^2(0,T;V)}$$

via (2.15) we can easily control III by:

$$III \leq 2\,k^2 M'^2\,\|u_k\|^2_{L^\infty(0,T;V)} \leq (C/\epsilon)\Big\{|u_0^k|^2 + \|f\|^2_{S(0,T)}\Big\} + \epsilon\|\hat{u}_k'\|^2_{L^2(0,T;V)}. \tag{5.4}$$

Finally, let us consider term II; we have

$$\sup_{0<j\leq N}\int_0^{jk}(\hat{f}_k', \hat{u}_k')\,dt \leq c\|\hat{f}_k'\|_{s(0,T)}\,\|\hat{u}_k'\|_{i(0,T)} \leq c\|f'\|_{i(0,T)}\|\hat{u}_k'\|_{i(0,T)} \tag{5.5}$$

and

$$\int_0^s (f - f_k, u - \hat{u}_k)\,dt \leq C\,k\|f'\|_{s(0,T)}\|u - \hat{u}_k\|_{i(0,T)}. \tag{5.6}$$

Summing up each contributions and substituting in (4.2) and in (4.3), we get (5.1). ▪

Our aim is now to control the quantity $\|u' - \hat{u}'_k\|_{L^2(0,T;H)}$; we try to substitute $u' - \hat{u}'_k$ in the right-hand member of the duality of (0.3) and of (4.10). This will be possible, if we use suitable approximations of the derivatives.

5.2 Notation. For a given function $v \in L^1(0, T+h; V')$ we set:

$$[v]_h(t) = \int_t^{t+h} v(\tau)\, d\tau, \qquad t \in [0,T]. \tag{5.7}$$

Recall that $\tilde{u}_k = [\hat{u}_k]_k$. $\square$

5.3 Remark. We extend u' and $\hat{u}'_k$ outside $[0,T]$ by 0 and consequently we set:

$$u(t) = u(T),\ \hat{u}_k(t) = \hat{u}_k(T) \text{ for } t > T; \qquad u(t) = u(0),\ \hat{u}_k(t) = \hat{u}_k(0) \text{ for } t < 0.$$

We have the formulae (see [14]):

$$[u]'_k = [u']_k, \quad [\hat{u}_k]'_k = [\hat{u}'_k]_k; \qquad [u]_k(T) = u(T), \quad [\hat{u}_k]_k(T) = \hat{u}_k(T) \tag{5.8}$$

$$\int_0^T \big(\hat{u}'_k, [u']_k\big)\, dt = \int_0^T \big([\hat{u}'_k]_k(t-k), u'\big)\, dt \tag{5.9}$$

$$\int_0^T |\tilde{u}'_k|^2\, dt \le \int_0^T |\hat{u}'_k|^2\, dt; \qquad \int_0^T |[u']_k|^2\, dt \le \int_0^T |u'|^2\, dt \tag{5.10}$$

$$\int_0^T |u' - [u']_k|^2\, dt \le c\, k \|u'\|^2_{I(0,T)}; \qquad \|u - [u]_k\|_{I(0,T)} \le c\, k \|u\|_{I^1(0,T)} \quad \square \tag{5.11}$$

We start from this following well known result (see [6])

5.4 Proposition. *The function $t \mapsto \phi(u(t))$ is absolutely continuous and it satisfies the equality:*

$$(u' + Au - f, u') + [\phi(u)]' = 0 \quad \blacksquare \tag{5.12}$$

It follows the following relation:

5.5 Corollary. *In the previous hypothesis, let v be in $D(\Phi)$; we have for a.e. $t \in]0,T[$*

$$\left(u' + Au - f, u' - [v]'_k + \frac{u-v}{k}\right) + \phi(u)' - [\phi(v)]'_k + \frac{\phi(u)-\phi(v)}{k} \leq 0. \tag{5.13}$$

Proof. We have only to explicit the left-hand member of (5.13):

$$\begin{aligned}
&\left(u' + Au - f, -[v]'_k + \frac{u-v}{k}\right) - [\phi(v)]'_k + \frac{\phi(u)-\phi(v)}{k} = \\
&(1/k)\Big(u' + Au - f, v(t) - v(t+k) + u(t) - v(t)\Big) + \\
&\qquad (1/k)\Big[\phi(v(t)) - \phi(v(t+k)) + \phi(u(t)) - \phi(v(t))\Big] = \\
&(1/k)\Big[\big(u' + Au - f, u(t) - v(t+k)\big) + \phi(u) - \phi(v(t+k))\Big] \leq 0 \quad \blacksquare
\end{aligned}$$

Analogously we have:

5.6 Lemma. *For every $v \in D(\Phi)$ the following inequality holds*

$$\left.\begin{aligned}
&\left(\hat{u}'_k + A\hat{u}_k - f_k, \hat{u}'_k - [v]'_k + \frac{\hat{u}_k - v}{k}\right) \\
&+ [\phi(U_k)]'_k - [\phi(v)]'_k + \frac{\phi(\hat{u}_k) - \phi(v)}{k}
\end{aligned}\right\} \leq \frac{\mathrm{r}_k(u_0, f; v(t+k))}{k} + m_k(u_0, f) \tag{5.14}$$

where r_k is given by proposition 4.8 and $m_k(u_0, f) = (A\hat{u}_k - A_k u_k, \hat{u}'_k)$.

Proof. We have easily

$$\begin{aligned}
&(\hat{u}'_k + A\hat{u}_k - f_k, \hat{u}'_k) + [\phi(U_k)]'_k = \\
&\left(\hat{u}'_k + A\hat{u}_k - f_k, \frac{u_k - U_k}{k}\right) + \frac{\phi(u_k) - \phi(U_k)}{k} \leq (A\hat{u}_k - A_k u_k, \hat{u}'_k) = m_k(u_0, f)
\end{aligned}$$

and

$$\begin{aligned}
&\left(\hat{u}'_k + A\hat{u}_k - f_k, -[v]'_k + \frac{\hat{u}_k - v}{k}\right) - [\phi(v)]'_k + \frac{\phi(\hat{u}_k) - \phi(v)}{k} = \\
&\frac{1}{k}\Big((\hat{u}'_k + A\hat{u}_k - f_k, v(t) - v(t+k) + \hat{u}_k(t) - v(t)\Big) + \\
&\qquad \frac{1}{k}\Big[\phi(v(t)) - \phi(v(t+k)) + \phi(\hat{u}_k(t)) - \phi(v(t))\Big] = \\
&\frac{1}{k}\Big((\hat{u}'_k + A\hat{u}_k - f_k, \hat{u}_k(t) - v(t+k)\Big) + \frac{1}{k}\Big[\phi(\hat{u}_k(t)) - \phi(v(t+k))\Big] \leq \\
&\mathrm{r}_k(u_0, f; v(t+k)) \quad \blacksquare
\end{aligned}$$

5.7 Corollary. *Denoting by* $\mathcal{M}_k(u_0, f) = \int_0^T m_k(u_0, f)\, dt$, *the integral:*

$$\int_0^T |u' - \hat{u}_k'|^2\, dt$$

is bounded by

$$(5.15)\qquad \begin{cases} \displaystyle\int_0^T \Big[(u', \tilde{u}_k' - \hat{u}_k') + (\hat{u}_k', [u']_k - u')\Big]\, dt + \\ \displaystyle\phi(u_0) - \phi([u]_k(0)) + \int_0^T (f - Au, u' - [u']_k)\, dt + \\ \displaystyle\phi(\hat{u}_k(0)) - \phi(\tilde{u}_k(0)) + \int_0^T (f_k - A\hat{u}_k, \hat{u}_k' - \tilde{u}_k')\, dt + \\ \displaystyle\int_0^T \Big[\big(f - Au - (f_k - A\hat{u}_k), [u - \hat{u}_k]_k'\big) + \frac{1}{k}\big(f - f_k + \lambda(u - \hat{u}_k), u - \hat{u}_k\big)\Big]\, dt + \\ \displaystyle\mathcal{M}_k(u_0, f) + \frac{|u_0 - u_0^k|^2}{2k} + \frac{1}{k}\mathcal{R}_k(u_0, f; u(t+k)). \end{cases}$$

Proof. We choose $v = \hat{u}_k$ in (5.13) and $v = u$ in (5.14), then we sum the two inequalities and we integrate from 0 to T, observing that $\phi([u]_k(0)) \geq [\phi(u)]_k(0)$. ∎

We call now $I, II, \ldots, V$ the five terms on the right-hand member of (5.15)and we estimate each of them:

5.8 Lemma. *We have*

$$(5.16)\qquad I \leq \frac{1}{2}\int_0^T |u' - \hat{u}_k'|^2 + \frac{k^2}{3}\int_0^T |\tilde{u}_k''|^2\, dt \leq \frac{1}{2}\int_0^T |u' - \hat{u}_k'|^2 + C\,\mathcal{E}_k^2/k;$$

$$(5.17)\qquad II \leq C\,k\Big(w^2(u_0, f) + \|u\|^2_{i^1(0,T)}\Big) + \int_0^T (f', [u]_k - u)\, dt;$$

$$(5.18)\qquad III \leq C\,\mathcal{E}_k^2/k;$$

$$(5.19)\qquad IV \leq C\left(\mathcal{E}_k^2/k + k\,\|u\|^2_{i^1(0,T)}\right) + \int_0^T \big(f - f_k, [u - \hat{u}_k]_k'\big)\, dt;$$

$$(5.20)\qquad V \leq C\left(\mathcal{E}_k^2/k + k\,\|u\|^2_{i^1(0,T)}\right).$$

Proof.

I - Working as in [14] we obtain:

$$\int_0^T \big(u', \tilde{u}_k' - \hat{u}_k'\big)\, dt \le \int_0^T \left\{ \frac{1}{4}|u' - \hat{u}_k'|^2 + |\tilde{u}_k' - \hat{u}_k'|^2 + \big(\hat{u}_k', \tilde{u}_k' - \hat{u}_k'\big) \right\} dt =$$

$$\frac{1}{4}\int_0^T |u' - \hat{u}_k'|^2\, dt + \frac{1}{2}\int_0^T \Big[|\tilde{u}_k'(t) - \hat{u}_k'(t)|^2 + |\tilde{u}_k'|^2 - |\hat{u}_k'|^2\Big] \le$$

$$\frac{1}{4}\int_0^T |u' - \hat{u}_k'|^2\, dt + \frac{1}{2}\int_0^T {\ell_k}^2 |\hat{u}_k'(t+k) - \hat{u}_k'(t)|^2 \le$$

$$\frac{1}{4}\int_0^T |u' - \hat{u}_k'|^2\, dt + \frac{k^2}{6}\int_0^T |\tilde{u}_k''|^2\, dt$$

by (5.10) and (3.17). By (5.9) we have

$$\int_0^T \big(\hat{u}_k', [u']_k - u'\big)\, dt = \int_0^T \big(\tilde{u}_k'(t-k) - \hat{u}_k', u'\big)\, dt$$

and repeating the same calculations we obtain

$$\int_0^T \big(\hat{u}_k', [u']_k - u'\big)\, dt \le \frac{1}{4}\int_0^T |u' - \hat{u}_k'|^2\, dt + \frac{k^2}{6}\int_0^T |\tilde{u}_k''|^2\, dt.$$

We conclude thanks to lemma 3.12.

II - We integrate by parts, recalling (5.8) and (1.9)

$$\phi(u_0) - \phi([u]_k(0)) + \int_0^T \big(f - Au, u' - [u]_k'\big)\, dt =$$

$$(A(0)u_0 - f(0), u_0 - [u]_k(0)) + \phi(u_0) - \phi([u]_k(0)) - \int_0^T \big((f - Au)', u - [u]_k\big)\, dt \le$$

$$w(u_0, f) \cdot |u_0 - [u]_k(0)| + c\,(M + M')k\|u\|^2_{H^1(0,T;V)} + \int_0^T \big(f', [u]_k - u\big)\, dt \le$$

$$C\,k\Big[w^2(u_0, f) + \|u\|^2_{i^1(0,T)}\Big] + \int_0^T (f', [u]_k - u)\, dt.$$

III - We have

$$\phi(u_0^k) - \phi(\tilde{u}_k(0)) + \int_0^T \big(f_k - A\hat{u}_k, \hat{u}_k' - \tilde{u}_k'\big)\, dt =$$

$$\phi(u_0^k) - \phi(\tilde{u}_k(0)) + \frac{1}{k}\int_0^T \big(f_k - A\hat{u}_k, (U_k(t+k) - \hat{u}_k(t+k)) - (U_k(t) - \hat{u}_k(t))\big)\, dt =$$

$$\phi(u_0^k) - \phi(\tilde{u}_k(0)) + \frac{1}{k}\int_0^k \big(A(0)u_0^k - f_0^k, u_0^k - \hat{u}_k(t)\big)\, dt +$$

$$\frac{1}{k}\int_0^T \big(A(t)\hat{u}_k(t) - A(t-k)\hat{u}_k(t-k) - (f_k(t) - f_k(t-k)), U_k(t) - \hat{u}_k(t)\big)\, dt$$

where we used $U_k \equiv \hat{u}_k \equiv u_N^k$ on $[T, T+k[$ (see remark 5.3). Since $2\tilde{u}_k(0) = u_1^k + u_0^k$, we write the first term as:

$$\phi(u_0^k) - \phi(\tilde{u}_k(0)) + \big(A(0)u_0^k - f_0^k, u_0^k - \tilde{u}_k(0)\big) =$$
$$\phi(u_0^k) - \phi(\tilde{u}_k(0)) + \big(A_0^k u_0^k - f_0^k, u_0^k - \tilde{u}_k(0)\big) + \big((A(0) - A_0^k)u_0^k, u_0^k - \tilde{u}_k(0)\big) \le$$
$$\frac{1}{2k}\big|u_1^k - u_0^k\big|^2 + \frac{kM'^2}{4}\|u_0^k\|^2 + \frac{1}{4}\|u_1^k - u_0^k\|^2.$$

The second is bounded by

$$\|A(t)\hat{u}_k(t) - A(t-k)\hat{u}_k(t-k)\|_{L^2(0,T;V')}\|\hat{u}_k'\|_{L^2(0,T;V)} + \frac{k}{2}\int_0^T \ell_k(t)\big(\hat{f}_k', \hat{u}_k'\big)\,dt$$

so that (5.18) is proved.

IV, V - They follow from the estimates of the previous section and from theorem 5.1. ∎

5.9 Corollary. *If* (1.7), (1.8) *and* (1.9) *hold, we have*

$$\|u' - \hat{u}_k'\|^2_{L^2(0,T;H)} \le C \begin{cases} \mathcal{E}_k^2/k + k\|u\|^2_{i^1(0,T)} + \\ \displaystyle\int_0^T \Big[\big(f', u - [u]_k\big) + \big(f - f_k, [u - \hat{u}_k]_k'\big)\Big]\,dt. \end{cases} \tag{5.21}$$

In particular, $u' \in B^{1/2}_{2\infty}(0,T;H)$ *and*

$$\|u' - \hat{u}_k'\|^2_{L^2(0,T;H)} \le C\,k\Big\{w(u_0, f)^2 + \|f\|^2_{H^1(0,T;V')}\Big\}. \tag{5.22}$$

Proof. (5.21) is the consequence of the previous lemma; recalling theorem 5.1, since by a well known result on approximation:

$$\|u - [u]_k\|_{i(0,T)} \le c\,k\|u\|_{i^1(0,T)}$$
$$\|[u - \hat{u}_k]_k\|_{i^1(0,T)} \le c\{\|u\|_{i^1(0,T)} + \|\hat{u}_k\|_{i^1(0,T)}\}$$

we get (5.22); we could apply characterization 1.7 to obtain $u' \in B^{1/2}_{2\infty}(0,T;H)$ but the following remark is more direct. ∎

5.10 Remark. Let $v \in L^2(0,T;H)$ and assume that there exists a sequence $v_k \in \mathbf{P}_k(H)$, $k \in \{T/N\}_{N\in\mathbf{N}}$ such that

$$\|v - v_k\|_{L^2(0,T;H)} \le C\sqrt{k}; \qquad \|v_k(t+k) - v_k(t)\|_{L^2(0,T-k;H)} \le C\sqrt{k}; \tag{5.23}$$

then $v \in B^{1/2}_{2\infty}(0,T;H)$ with $\big[v\big]_{B^{1/2}_{2\infty}(0,T;H)} \le 5C$. In particular the solution u of (Pw) satisfies:

$$\big[u'\big]_{B^{1/2}_{2\infty}(0,T;H)} \le 5\sup_k\Big\{k^{-1/2}\|u' - \hat{u}_k'\|_{L^2(0,T;H)} + k^{1/2}\|\tilde{u}_k''\|_{L^2(0,T;H)}\Big\} \quad \square \tag{5.24}$$

We conclude this section with the proof of (1.11); the same ideas are developped in [14], so we omit the details.

Starting from corollary 5.5 we find the equivalent form of 5.7, with v instead of $\hat{u}_k$:

5.11 Lemma. *Let u, v be as in theorem 2; we have, for any $h > 0$:*

$$
(5.25)\quad \int_0^T |u'-v'|^2\,dt \le \begin{cases} \displaystyle\int_0^T \Big[(u',[v]_h'-v')+(v',[u]_h'-u')\Big]\,dt+ \\ \displaystyle\phi(u_0)-\phi([u]_h(0))+\int_0^T (f-Au,u'-[u']_h)\,dt+ \\ \displaystyle\phi(v_0)-\phi([v]_h(0))+\int_0^T (g-Av,v'-[v']_h)\,dt+ \\ \displaystyle\int_0^T \Big[\big(f-Au-(g-Av),[u-v]_h'\big)+\frac{1}{h}\big(f-g,u-v\big)\Big]\,dt+ \\ \displaystyle\frac{|u_0-v_0|^2}{2h}+\int_0^T \frac{\lambda}{h}\big|u-v\big|^2\,dt \quad \blacksquare \end{cases}
$$

5.12 Corollary. *With the notation of theorem 2, we have, for any $h > 0$*

$$
(5.26)\quad \int_0^T |u'-v'|^2 \le C \begin{cases} h\Big[\|(u_0,f)\|_{\mathcal{V}}^2+\|(v_0,g)\|_{\mathcal{V}}^2\Big]+ \\ \Big[|u_0-v_0|+\|f-g\|_{H^1(0,T;V')}\Big]\Big[\|(u_0,f)\|_{\mathcal{V}}+\|(v_0,g)\|_{\mathcal{V}}\Big]+ \\ \displaystyle\frac{1}{h}\Big[|u_0-v_0|^2+\|f-g\|_{H^1(0,T;V')}^2\Big]. \end{cases}
$$

Choosing $h = |u_0 - v_0| + \|f - g\|_{H^1(0,T;V')}$ we get (1.11).

Proof. We control only the first term of (5.25): the remaining ones can be treated as in lemma 5.8.

$$
\begin{aligned}
&\int_0^T (u',[v]_h'-v')\,dt = \int_0^T (u'-v',[v]_h'-v')\,dt + \int_0^T (v',[v]_h'-v')\,dt \le \\
&\frac{1}{4}\|u'-v'\|_{L^2(0,T;H)}^2 + \int_0^T ([v]_h',[v]_h'-v')\,dt \le \\
&\frac{1}{4}\|u'-v'\|_{L^2(0,T;H)}^2 + \frac{1}{2}\|[v]_h'-v'\|_{L^2(0,T;H)}^2 \le \\
&\frac{1}{4}\|u'-v'\|_{L^2(0,T;H)}^2 + \frac{h}{2}\|v'\|_{I(0,T)}^2 \le \frac{1}{4}\|u'-v'\|_{L^2(0,T;H)}^2 + \frac{h}{2}\|(v_0,g)\|_{\mathcal{V}}^2
\end{aligned}
$$

where we applied (5.11). $\blacksquare$

6. - Error estimates and regularity results: the case $f \in S^1(0,T)$.

We extend the results of the previous section to data in $S^1(0,T)$; the basic idea is to obtain uniform estimates with respect to the norm of this space and then to pass to the limit by using the following "density" argument:

6.1 Lemma. *For every function $f \in S^1(0,T)$ there exists a sequence $f^j \in H^1(0,T;V')$, $j \in \mathbf{N}$, such that:*

$$\limsup_{j\to\infty} \|f^j\|_{S^1(0,T)} \le \|f\|_{S^1(0,T)}, \qquad f^j(0) = f(0); \qquad \lim_{j\to\infty} \|f^j - f\|_{L^2(0,T;V')} = 0 \quad \blacksquare$$

So, given (u_0, f) such that (1.9) and (1.18) hold, we solve problem (Pw) for the data (u_0, f^j) obtaining functions u^j and $\hat{u}_k^j$; thanks to (4.14) we have

$$\lim_{j\to\infty} u^j = u, \qquad \lim_{j\to\infty} \hat{u}_k^j = \hat{u}_k, \qquad \text{in } \ i(0,T)$$

where u and $\hat{u}_k$ are the exact and approximate solution relative to (u_0, f); since $I(0,T)$ is a dual space and $S(0,T)$ is separable, uniform estimates on $u^j, \hat{u}_k^j$ hold also for $u, \hat{u}_k$. In what follows, we omit index j and we take this approximating procedure for granted.

Let us summarize the foundamental estimates in terms of $\mathcal{E}_k$:

6.2 Proposition. *If* (1.7), (1.8) *and* (1.9) *hold, we have:*

$$\|\hat{u}_k'\|^2_{i(0,T)} \le C\mathcal{E}_k^2/k^2; \tag{6.1}$$

$$\begin{aligned} &k\|\tilde{u}_k''\|^2_{L^2(0,T;H)} + \frac{1}{k}\|u' - \hat{u}_k'\|^2_{L^2(0,T;H)} \le \\ &\qquad C\Big\{\mathcal{E}_k^2/k^2 + \|u\|^2_{i(0,T)} + \|f\|_{S^1(0,T)}\big(\|u'\|_{I(0,T)} + \|\hat{u}_k'\|_{I(0,T)}\big)\Big\} \end{aligned} \tag{6.2}$$

where $\mathcal{E}_k$ is defined by (4.1).

Proof. (6.1) is contained in (4.2); for (6.2), we consider also (5.21), recalling that

$$\|f - f_k\|_{S(0,T)} \le C\,k\|f\|_{S^1(0,T)}; \qquad \|u - [u]_k\|_{I(0,T)} \le C\,k\|u'\|_{I(0,T)} \quad \blacksquare$$

We have now to estimate $\mathcal{E}_k$:

6.3 Lemma. *With the same hypotheses, there exists $C > 0$ such that, for every $\epsilon > 0$*

$$\mathcal{E}_k^2/k^2 \le \begin{cases} (C/\epsilon)\Big[w(u_0,f)^2 + \|f\|^2_{S^1(0,T)}\Big] + \epsilon\|\hat{u}_k'\|_{L^2(0,T;V)} + \\ C\|f\|_{S^1(0,T)}\Big[\|\hat{u}_k'\|_{I(0,T)} + \|u'\|_{I(0,T)} + (1/k)\|u - \hat{u}_k\|_{i(0,T)})\Big]. \end{cases} \tag{6.3}$$

Proof. We repeat the argument of Theorem 5.1; we have only to bound term II:

$$\sup_{0<j\le N}\int_0^{jk}(\hat f_k',\hat u_k')\,dt\le c/k\,\|f_k-F_k\|_{S(0,T)}\,\|\hat u_k'\|_{I(0,T)}\le c\|f\|_{S^1(0,T)}\|\hat u_k'\|_{I(0,T)}$$

by (2.12) and

$$\int_0^s(f-f_k,u-\hat u_k)\,dt\le C\,k\|f\|_{S^1(0,T)}\Big(k\|u'\|_{I(0,T)}+k\|\hat u_k'\|_{I(0,T)}+\|u-\hat u_k\|_{i(0,T)}\Big)\le$$
$$C\,k\|f\|_{S^1(0,T)}\Big(k\|u'\|_{I(0,T)}+k\|\hat u_k'\|_{I(0,T)}+\mathcal{E}_k\Big)$$

by (2.13) and (4.3). Summing up each contribution we get (6.3). ■

We substitute (6.3) in (6.1) and (6.2); with a suitable choice of ϵ we get:

6.4 Corollary. *There exist constant $C>0$ such that:*

$$\|\hat u_k'\|^2_{i(0,T)}\le C\Big\{w(u_0,f)^2+\|f\|^2_{S^1(0,T)}\|f\|_{S^1(0,T)}\big(\|\hat u_k'\|_{I(0,T)}+\|u'\|_{I(0,T)}\big)\Big\}\tag{6.4}$$

$$\begin{aligned}&k\|\tilde u_k''\|^2_{L^2(0,T;H)}+\frac{1}{k}\|u'-\hat u_k'\|^2_{L^2(0,T;H)}\le\\&C\Big\{w(u_0,f)^2+\|f\|^2_{S^1(0,T)}+\|u\|^2_{i(0,T)}+\|f\|_{S^1(0,T)}\big(\|u'\|_{I(0,T)}+\|\hat u_k'\|_{I(0,T)}\big)\Big\}\end{aligned}\tag{6.5}$$ ■

Let us now call P^2 and Q^2 the supremum of the left-hand members of (6.4) and (6.5):

$$P^2=\sup_k\|\hat u_k'\|^2_{i(0,T)};\qquad Q^2=\sup_k\Big\{k\|\tilde u_k''\|^2_{L^2(0,T;H)}+\frac{1}{k}\|u'-\hat u_k'\|^2_{L^2(0,T;H)}\Big\}$$

which are finite, thanks to the regularity hypotheses we made. The following bounds are immediate (see remark 5.10):

$$\|u'\|_{i(0,T)}\le P,\quad \big[u'\big]_{B^{1/2}_{2\infty}(0,T;H)}\le 5Q;\qquad \|u'\|_{I(0,T)}\le 5(P+Q).\tag{6.6}$$

Via P,Q we can control $\big[\hat u_k'\big]_{B^{1/2}_{2\infty}(0,T;H)}$, too:

6.5 Lemma. *We have*

$$\big[\hat u_k'\big]_{B^{1/2}_{2\infty}(0,T;H)}\le 7Q,\qquad \|\hat u_k'\|_{I(0,T)}\le 7(P+Q).\tag{6.7}$$

Proof. We slightly modify the argument of remark 5.10: we have to bound the integral

$$x_h^2=\int_0^{T-h}|\hat u_k'(t+h)-\hat u_k'(t)|^2\,dt.$$

If $h \leq k$ then

$$x_h^2 = (h/k) \int_0^{T-k} |\hat{u}_k'(t+k) - \hat{u}_k'(t)|^2 \, dt = (h/k) k^2 \|\tilde{u}_k''\|_{L^2(0,T;H)} \leq hQ^2$$

whereas if $h > k$ we have

$$x_h \leq 2\|u' - \hat{u}_k'\|_{L^2(0,T;H)} + \sqrt{h}\big[u\big]'_{B^{1/2}_{2\infty}(0,T;H)} \leq 2\sqrt{h}Q + 5\sqrt{h}Q \leq 7\sqrt{h}Q \quad \blacksquare$$

6.6 Corollary. *We have:*

$$\|u'\|_{I(0,T)} + \|\hat{u}_k'\|_{I(0,T)} \leq C\Big\{w(u_0,f) + \|f\|_{S^1(0,T)}\Big\} \tag{6.8}$$

and (1.26)*,* (1.27) *and* (1.28) *hold.*

Proof. From (6.4) and (6.5) we get

$$P^2 \leq C\Big\{w(u_0,f)^2 + \|f\|^2_{S^1(0,T)} + \|f\|_{S^1(0,T)}(P+Q)\Big\} \tag{6.9}$$

and

$$Q^2 \leq C\Big\{w(u_0,f)^2 + \|f\|^2_{S^1(0,T)} + P^2 + \|f\|_{S^1(0,T)}(P+Q)\Big\}. \tag{6.10}$$

Combining these formulae we obtain (6.8). Substituting in (6.3), we find the error estimates. $\blacksquare$

Thanks to the initial remarks, theorem 3 and 4 are now completely proved.

7. - Appendix.

Proof of Lemma 2.4

The boundedness of $r_{a,b}$ in $I(0,T)$ follows by the same argument of 2.2.

Observe that for $L^2(0,T;H)$-functions f we have:

$$\langle r_{a,b}f, g\rangle = \langle f, r_{a,b}g\rangle, \qquad \forall g \in I(0,T) \subset L^2(0,T;H)$$

so that we obtain:

$$\|r_{a,b}f\|_{S(0,T)} \le \|f\|_{S(0,T)} \sup_{\|g\|_{I(0,T)}=1} \|r_{a,b}g\|_{I(0,T)} \le c\|f\|_{S(0,T)}$$

By the density of $L^2(0,T;H)$ in $S(0,T)$ we obtain

$$\lim_{b\to a^+} \|r_{a,b}f\|_{S(0,T)} = 0 \tag{7.1}$$

since $r_{a,b}$ is uniformly bounded and (7.1) holds in $L^2(0,T;H)$. (2.6) follows by density, too. ∎

Now we introduce the operators in $L^2(0,T;H)$:

$$\tau_h[f](t) = \begin{cases} f(t-h) & \text{if } t > h; \\ f(h-t) & \text{otherwise.} \end{cases} \qquad h \in]0,T[\tag{7.2}$$

with the corresponding ones for $h < 0$. By the previous lemma τ_h is uniformly bounded in $I(0,T)$ and in $S(0,T)$. We have the estimate

7.1 Lemma. *There exists a constant $c > 0$ such that:*

$$\|f - \tau_h f\|_{S(0,T)} \le c\,h\|f\|_{S^1(0,T)} \tag{7.3}$$

Proof. We have to prove (7.3) only for $B^{1/2}_{2\,1}(0,T;H)$-functions, i.e.

$$\|f - \tau_h f\|_{B^{-1/2}_{2\,1}(0,T;H)} \le c\,h\|f\|_{B^{1/2}_{2\,1}(0,T;H)} \tag{7.4}$$

since the correponding one for $f \in H^1(0,T;V) + BV(0,T;H)$ can be found in [1].

We recall that, if $f \in H^1(0,T;H)$ then:

$$\|f - \tau_h f\|_{L^2(0,T;H)} \le c\,h\|f\|_{H^1(0,T;H)}$$

so that (7.4) will follow by interpolation by

$$\|f - \tau_h f\|_{H^{-1}(0,T;H)} \le c\,h\|f\|_{L^2(0,T;H)}. \tag{7.5}$$

Consider now $f \in L^2(0,T;H)$ and $g \in H^1_0(0,T;H)$; we have

$$\begin{aligned}
&\int_0^T \big(f - \tau_h f, g\big)\, dt = \\
&= \int_0^{T-h} \big(f(t), g(t) - g(t+h)\big)\, dt - \int_0^h \big(f(h-t), g(t)\big)\, dt + \int_{T-h}^T \big(f(t), g(t)\big)\, dt \le \\
&\le c\,h\|f\|_{L^2(0,T;H)}\, \|g\|_{H^1_0(0,T;H)}
\end{aligned}$$

that is (7.5). ∎

By density we obtain:

7.2 Corollary. *For every $f \in S(0,T)$ we have*

$$\lim_{h\to 0^+} \|f - \tau_h f\|_{S(0,T)} = 0. \tag{7.6}$$

7.3 Lemma. *We have the estimate*

$$\|g - g_k\|_{L^2(0,T;H)} \le c\sqrt{k}\|g\|_{B^{1/2}_{2\infty}(0,T;H)} \tag{7.7}$$

and, for a function $g \in I(0,T)$

$$\|g_k\|_{I(0,T)} \le c\|g\|_{I(0,T)}. \tag{7.8}$$

Proof. (7.7) follows easily by interpolation.
We evaluate the $B^{1/2}_{2\infty}(0,T;H)$-seminorm of g_k; if $h > k$ we have

$$\|g_k(t+h) - g_k(t)\|_{L^2(0,T-h;H)} \le 2\|g_k - g|_{L^2(0,T;H)} + \|g(t+h) - g(t)\|_{L^2(0,T-h;H)} \le$$
$$(2\,c\sqrt{k} + \sqrt{h})\|g\|_{B^{1/2}_{2\infty}(0,T;H)} \le c\sqrt{h}\|g\|_{B^{1/2}_{2\infty}(0,T;H)},$$

whereas if $h \le k$ we have

$$\|g_k(t+h) - g_k(t)\|_{L^2(0,T-h;H)} = \sqrt{h/k}\|g_k(t+k) - g_k(t)\|_{L^2(0,T-k;H)} \le C\sqrt{h}\|g\|_{B^{1/2}_{2\infty}(0,T;H)},$$

so that $\left[g_k\right]_{B^{1/2}_{2\infty}(0,T;H)} \le c\|g\|_{B^{1/2}_{2\infty}(0,T;H)}$; taking account of the correponding bounds with respect to the norm of $L^2(0,T;V)$ and $L^\infty(0,T;H)$ we get (7.8). ∎

Proof of Lemma 2.6

Formula (2.10) follows by a duality argument; assume that $f \in L^2(0,T;H)$: we have

$$\|f_k\|_{S(0,T)} = \sup_{\|g\|_{I(0,T)}=1} \int_0^T (f_k, g)\,dt =$$
$$\sup_{\|g\|_{I(0,T)}=1} \int_0^T (f, g_k)\,dt \le \|f\|_{S(0,T)} \sup_{\|g\|_{I(0,T)}=1} \|g_k\|_{I(0,T)} \le c\,\|f\|_{S(0,T)}$$

by (7.8). By density we complete the proof of (2.10).
In order to prove (2.11), we observe that:

$$\|f_k(t+h) - f_k(t)\|_{S(0,T-h)} \le c\|r_{0,T-h}[\tau_{-h} f_k - f_k]\|_{S(0,T)} \le c\|\tau_{-h} f_k - f_k\|_{S(0,T)};$$

now we write the decomposition

$$\|\tau_{-h}f_k-f_k\|_{S(0,T)} = \big\|\tau_{-h}[f_k-f]+[f-f_k]+[\tau_{-h}f-f]\big\| \leq c\|f-f_k\|_{S(0,T)}+\|\tau_{-h}f-f\|_{S(0,T)}$$

and we use (2.10) and (7.6).

The first formula of (2.12) must be proved only for $f \in B^{1/2}_{2\,1}(0,T;H)$; we use the duality with $X(0,T;H)$ again: we have

$$\begin{aligned}(7.9)\qquad &\int_0^T (f-f_k,g)\,dt = \int_0^T (f-f_k, g-g_k)\,dt \leq \\ &c\,\|f-f_k\|_{L^2(0,T;H)}\|g-g_k\|_{L^2(0,T;H)} \leq c\,k\|f\|_{B^{1/2}_{2\,1}(0,T;H)}\|g\|_{X(0,T;H)}\end{aligned}$$

by (7.7), since

$$B^{1/2}_{2\,1}(0,T;H) \subset B^{1/2}_{2\infty}(0,T;H); \qquad X(0,T;H) \subset B^{1/2}_{2\infty}(0,T;H).$$

The second formula of (2.12) is an immediate consequence of this decomposition

$$f_k - F_k = \Pi_k\big(f-\tau_k f\big);$$

we then apply (2.10) and (7.3).

It remains to study $\int_0^s (f-f_k, u-\hat{u}_k)\,dt$, for $0 < s \leq T$. Assume that $s \in [n_0k, (n_0+1)k[$; we split the integral as

$$\int_0^{n_0k} (f-f_k, u-\hat{u}_k)\,dt + \int_{n_0k}^s (f-f_k, u-\hat{u}_k)\,dt.$$

The first term can be treated as in (7.9); for the second one, we limit ourselves to the case $f \in B^{1/2}_{2\,1}(0,T;H)$, since $f \in s^1(0,T)$ gives immediately (2.13). Since $B^{1/2}_{2\,1}(0,T;H) \subset L^\infty(0,T;H)$ we have for the other term

$$\int_{n_0k}^s (f-f_k,u)\,dt \leq C\,k\|f-f_k\|_{B^{1/2}_{2\,1}(0,T;H)}\|v\|_{L^\infty(0,T;H)} \quad ■$$

Proof of (3.16)

Consider

$$x_h^2 = \int_0^{T-h} |\hat{u}_k(t+h)-\hat{u}_k(t)|^2\,dt$$

as usual we distinguish two cases: if $h \leq k$, then we have

$$x_h \leq h\|\hat{u}_k'\|^2_{L^2(0,T;H)} \leq \sqrt{h}\,\sqrt{k}\|\hat{u}_k'\|^2_{L^2(0,T;H)};$$

if $h > k$ we have

$$x_h \leq 2\|\hat{u}_k - u_k\|_{L^2(0,T;H)} + \|u_k(t+h) - u_k(t)\|_{L^2(0,T-h;H)} \leq$$
$$2\,k\|\hat{u}_k'\|_{L^2(0,T;H)} + \sqrt{h}\big[u_k\big]_{B^{1/2}_{2\infty}(0,T;H)} \leq \sqrt{h}\Big\{2\sqrt{k}\|\hat{u}_k'\|_{L^2(0,T;H)} + \big[u_k\big]_{B^{1/2}_{2\infty}(0,T;H)}\Big\} \quad \blacksquare$$

Proof of 4.13

From (4.16) and the $L^2(0,T;H)$-convergence, we deduce that

$$\lim_{h\to 0^+} \frac{1}{h}\int_0^{T-h} |v(t+h) - v(t)|^2\,dt = 0. \tag{7.10}$$

Let us now fix $\epsilon > 0$; then we can find $h_0, k_0 > 0$ such that, $\forall h < h_0,\ k < k_0$

$$\frac{1}{h}\int_0^{T-h} |v_k(t+h) - v(t+h) - (v_k(t) - v(t))|^2\,dt \leq$$
$$\frac{2}{h}\int_0^{T-h} |v_k(t+h) - v_k(t)|^2\,dt + \frac{2}{h}\int_0^{T-h} |v(t+h) - v(t)|^2\,dt \leq \epsilon^2;$$

by the $L^2(0,T;H)$-convergence we can find k_0' such that, for $h > h_0$, $k < k_0'$

$$\frac{1}{h}\int_0^{T-h} |v_k(t+h) - v(t+h) - (v_k(t) - v(t))|^2\,dt \leq \frac{4}{h_0}\int_0^{T} |v_k(t) - v(t)|^2\,dt \leq \epsilon^2.$$

So we conclude that $\big[v - v_k\big]_{B^{1/2}_{2\infty}(0,T;H)} \leq \epsilon$, for $0 < k < \min\{k_0, k_0'\}$. $\blacksquare$

Proof of Remark 5.10

Let us fix $h > 0$ and k such that $k\frac{T}{T+k} < h \leq k$. Since $k/h < 2$ we have

$$\|v(t+h) - v(t)\|_{L^2(0,T-h;H)} \leq 2\|v - v_k\|_{L^2(0,T;H)} + \|v_k(t+h) - v_k(t)\|_{L^2(0,T-h;H)} \leq$$
$$(2\sqrt{k/h} + 1)\sqrt{h}C \leq 5C\sqrt{h}$$

so that

$$\big[v\big]_{B^{1/2}_{2\infty}(0,T;H)} \leq 5C \quad \blacksquare$$

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