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MEASURE VALUED SOLUTIONS OF SUB-LINEAR DIFFUSION EQUATIONS WITH A DRIFT TERM

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Abstract. In this paper we study nonnegative, measure-valued solutions of the initial value problem for one-dimensional drift-diffusion equations when the nonlinear diffusion is governed by a strictly increasing C^1 function β with $\lim_{r\to+\infty}\beta(r)<+\infty$. By using tools of optimal transport, we will show that this kind of problems is well posed in the class of nonnegative Borel measures with finite mass $\mathfrak{m}$ and finite quadratic momentum and it is the gradient flow of a suitable entropy functional with respect to the so called L^2-Wasserstein distance.

Due to the degeneracy of diffusion for large densities, concentration of masses can occur, whose support is transported by the drift. We shall show that the large-time behavior of solutions depends on a critical mass $\mathfrak{m}_c$, which can be explicitly characterized in terms of β and of the drift term. If the initial mass is less then $\mathfrak{m}_c$, the entropy has a unique minimizer which is absolutely continuous with respect to the Lebesgue measure. Conversely, when the total mass $\mathfrak{m}$ of the solutions is greater than the critical one, the stationary solution has a singular part in which the exceeding mass $\mathfrak{m}-\mathfrak{m}_c$ is accumulated.

1. **Introduction.**

In this paper we study nonnegative, measure-valued solutions of the Cauchy problem for a one-dimensional drift-diffusion equation

$$\partial_t\rho-\partial_x\big(\partial_x(\beta(\rho))+V'\rho\big)=0\quad\text{in }\mathbb{R}\times(0,+\infty),\qquad \rho(0,\cdot)=\rho_0\quad\text{in }\mathbb{R}.\qquad(1.\mathrm{DDE})\ \boxed{\texttt{eq:rho}}$$

Here we assume that

$$\beta\in C^1([0,+\infty))\text{ is strictly increasing},\quad \beta(0)=0,\quad \beta_\infty:=\lim_{r\to+\infty}\beta(r)<+\infty,\qquad(1.\beta)\ \boxed{\texttt{hp:beta}}$$

and $V:\mathbb{R}\to\mathbb{R}$ is a C^2 driving potential, satisfying the conditions

$$V''(x)\ge\lambda\qquad\text{for all }x\in\mathbb{R};\qquad \liminf_{|x|\to+\infty}\frac{V(x)}{|x|^2}\ge 0.\qquad(1.V)\ \boxed{\texttt{crescita-quadratica}}$$

We will look for solutions $t\mapsto\rho_t$ in the space $\mathscr{M}_2(\mathbb{R},\mathfrak{m})$ of nonnegative Borel measures with finite mass $\mathfrak{m}=\rho(\mathbb{R})$ and finite quadratic momentum

$$\mathfrak{m}_2(\rho):=\int_{\mathbb{R}}|x|^2\,\mathrm{d}\rho(x)<+\infty.$$

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Conditions (1.β) describe the physical situation in which the diffusion operator is very weak and possibly unable to smooth out the solution if initially point masses are present. This behavior is rather different from that described by both classical fast and slow diffusions, in which the source–type solution is immediately smoothed out in time [17].

This peculiarity is reflected by the natural entropy functional $\mathcal{F}$ which generates equations like (1.DDE) as gradient flow in $\mathscr{M}_2(\mathbb{R},\mathfrak{m})$ and in particular decays along the solutions of (1.DDE),

$$
\begin{gathered}
\mathcal{F}(\rho) := \mathcal{E}(\rho) + \mathcal{V}(\rho),\\
\mathcal{E}(\rho) := \int_{\mathbb{R}} E(u(x))\,\mathrm{d}x \quad \text{if } \rho = u\mathscr{L}^1 + \rho^\perp, \quad \mathcal{V}(\rho) = \int_{\mathbb{R}} V(x)\,\mathrm{d}\rho(x),
\end{gathered}
\qquad (1.1)\ \boxed{\texttt{eq:29}}
$$

where the strictly convex energy density function $E : [0,+\infty) \to \mathbb{R}$ is defined as

$$
E(r) := -\beta(r) - r\int_r^{+\infty} \frac{\beta'(s)}{s}\,\mathrm{d}s \quad \text{so that} \quad \beta'(r) = rE''(r), \quad E(0) = 0, \qquad (1.E)\ \boxed{\texttt{eq:26}}
$$

and satisfies

$$
\lim_{r\to+\infty} E(r) = -\beta_\infty \quad \text{and therefore} \quad \lim_{r\to+\infty} \frac{E(r)}{r} = 0, \qquad (1.2)\ \boxed{\texttt{eq:32}}
$$

so that the (lower semicontinuous) integral functional $\mathcal{E}$ defined by (1.1) depends only on the regular part of a Borel measure (see for instance [9]).

Even worse, the energy density E does not satisfy the regularizing condition [2, Thm. 10.4.8] $\lim_{r\to+\infty} E(r) = -\infty$, which prevents a singular part for measures with finite energy dissipation along (1.DDE), thus in particular for any solution ρ_t at positive time $t > 0$.

Sub-linear diffusions and Bose-Einstein distribution. In order to better clarify the physical meaning of condition (1.β), let us briefly describe a situation in $\mathbb{R}^d$ in which the stationary solution of the drift-diffusion equation is explicitly computable. To this aim, for $x \in \mathbb{R}^d$, $d \geq 1$, let us fix $V(x) = |x|^2/2$, while, for a fixed constant $\alpha > 0$, the diffusion function $\beta(r)$ is defined by

$$
\beta(0) = 0, \qquad \beta'(r) = \frac{1}{1+r^\alpha}. \qquad (1.3)\ \boxed{\texttt{alpha}}
$$

Then, since in this case the drift-diffusion equation

$$
\partial_t\rho - \nabla_x \cdot (\nabla_x\beta(\rho) + x\rho) = 0, \qquad x \in \mathbb{R}^d
$$

can be rewritten as

$$
\partial_t\rho - \nabla_x \cdot \left(\rho\nabla_x\left(\frac{1}{\alpha}\ln\frac{\rho^\alpha}{1+\rho^\alpha} + \frac{|x|^2}{2}\right)\right) = 0, \qquad (1.4)\ \boxed{\texttt{gen}}
$$

the stationary solutions of (1.4) are given by

$$
\rho_\infty(x) = \left[e^{\alpha|x|^2/2+\eta} - 1\right]^{-1/\alpha}, \qquad \eta \geq 0. \qquad (1.5)\ \boxed{\texttt{steady-gen}}
$$

The (nonnegative) constant η in (1.5) identifies the mass of the stationary solution

$$
\mathfrak{m}_\eta = \int_{\mathbb{R}^d} \left[e^{\alpha|x|^2/2+\eta} - 1\right]^{-1/\alpha}\,\mathrm{d}x.
$$

Since the mass $\mathfrak{m}_\eta$ is decreasing as soon as η increases, the maximum value of $\mathfrak{m}_\eta$ is attained at $\eta = 0$. Note that, if B_d denotes the measure of the unit sphere in $\mathbb{R}^d$, the value

$$\mathfrak{m}_0 = \int_{\mathbb{R}^d} \left[e^{\alpha|x|^2/2} - 1\right]^{-1/\alpha} \mathrm{d}x = B_d \int_0^{+\infty} r^{d-1} \left[e^{\alpha r^2/2} - 1\right]^{-1/\alpha} \mathrm{d}r$$

is bounded as soon as $\alpha > 2/d$. Whenever the constant α is chosen in this range, the value

$$\mathfrak{m}_c = \mathfrak{m}_0 = B_d \int_0^{+\infty} r^{d-1} \left[e^{\alpha r^2/2} - 1\right]^{-1/\alpha} \mathrm{d}r < +\infty$$

defines the so-called *critical mass* of the problem, namely the maximal mass that can be achieved by a regular stationary solution. It is interesting to remark that, in view of the lower bound on α which implies the existence of a critical mass, since in dimension one $\alpha > 2$, the function β in (1.3) satisfies conditions (1.β), in particular

$$\lim_{r\to+\infty} \beta(r) < +\infty.$$

This condition clearly can fail in higher dimensions.

The most relevant physical example of such type of functions is furnished by the three-dimensional Bose-Einstein distribution [7]

$$u_\infty(x) = \left[e^{|x|^2/2+\eta} - 1\right]^{-1} \tag{1.6}$$ BEsteady

that is the stationary solution of equation (1.4) corresponding to $\alpha = 1$. In this case the function β is explicitly computable to give $\beta(\rho) = \ln(1+\rho)$. Since $\alpha = 1$, if the dimension $d \geq 3$, the Bose-Einstein distribution exhibits a critical mass. We remark that in this case the energy functional $E(u)$ is the Bose-Einstein entropy

$$E(u) = u \ln u - (1+u)\ln(1+u).$$

One of the fundamental problems related to evolution equations that relax towards a stationary state characterized by the existence of a critical mass, is to show how, starting from an initial distribution with a supercritical mass $\mathfrak{m} > \mathfrak{m}_c$, the solution eventually develops a singular part (the condensate), and, as soon as the singular part is present, to be able to follow its evolution. We remark that in general the condensation phenomenon is heavily dependent of the dimension of the physical space. In dimension $d \leq 2$, in fact, the maximal mass $\mathfrak{m}_0$ of the Bose-Einstein distribution (1.6) is unbounded, and the eventual formation of a condensate is lost.

In order to simplify the mathematical difficulties, while maintaining the physical picture in which the stationary solution has a critical mass, in [4] the one-dimensional case corresponding to a stationary solution of the form (1.5), with $\alpha > 2$ has been considered. The analysis of [4] refers to a linear diffusion with a super-linear drift

$$\partial_t \rho = \partial_x \left(\partial_x \rho + x\rho(1+\rho^\alpha)\right), \tag{1.7}$$ BGT

that is reminiscent of the Kaniadakis-Quarati model of Bose-Einstein particles [12]

$$\partial_t \rho = \nabla \cdot \left(\nabla \rho + x\rho(1+\rho)\right).$$

While the stationary solutions of equation (1.7) coincide with (1.5), the evolution problem is considerably different, due to the presence of the non-linear drift operator.

A measure-theoretic setting for diffusion equations. In the present paper we deal with an almost complete description of the time-evolution of the solution of problem (1.DDE) with a Borel measure as initial datum. While the mathematical study of drift-diffusion kinetic equations with the Bose-Einstein density as stationary state has been considered before (cfr. [6, 10] and the references therein), to our knowledge, drift-diffusion equations of type (1.DDE) at present have never been studied systematically. It is worth mentioning that the analysis of [6, 10] refers to Kompaneets equation [13], namely to a kinetic equation which describes the evolution of the radiation distribution in a homogeneous plasma when radiation interacts with matter via Compton scattering. The Kompaneets equation, in analogy with the Kaniadakis-Quarati model [12], is characterized by the presence of the non-linearity in the drift operator.

Motivated by the previous remarks and by the degeneracy of the entropy functional $\mathcal{F}$ introduced in (1.1), whose minimizers could exhibit concentration effect, we address the study of (1.DDE) by the measure-theoretic point of view recently developed in the framework of optimal transport [2]. This approach, started by the pioneering papers of Jordan-Kinderlehrer-Otto [11] and Otto [15], provides a sufficiently general setting for measure-valued solutions to (1.DDE).

$\mathscr{M}_2(\mathbb{R}, \mathfrak{m})$ endowed with the so called L^2-Wasserstein distance is the natural ambient space for carrying on our analysis. A first important fact is that the entropy functional $\mathcal{F}$ (1.1) turns out to be displacement λ-convex, a crucial property which holds only in the one-dimensional case, since the possibility of entropies satisfying (1.2) is prevented by McCann's condition [14] in higher dimensions.

Moreover, we are able to extend the results of [2] (which for sublinear entropies covers the case when $\lim_{r \to +\infty} E(r) = -\infty$) providing an explicit characterization of the dissipation of $\mathcal{F}$, which is strictly related to the "Wasserstein differential" of $\mathcal{F}$. As a crucial byproduct of this analysis, we will find the right condition that measure-valued solutions have to satisfy in order to enjoy nice uniqueness and stability results. It is worth mentioning here that the distributional formulation of (1.DDE) does not provide enough information to characterize the solutions, when a concentration on a Lebesgue negligible set occurs.

Applying the general theory of gradient flows of displacement λ-convex functionals in Wasserstein spaces, we can thus obtain a precise characterization of measure valued solutions to (1.DDE) and we can prove their existence, uniqueness, and stability.

Further justifications showing that the notion of Wasserstein solutions is well adapted to (1.DDE) come from natural regularization/approximation results: we will show that our solutions are both the limit of the simplest vanishing viscosity approximation of (1.DDE) and of smooth solutions generated by regularization of the initial data.

We complete our analysis by studying the propagation of the singularities, the structure of minimizers of $\mathcal{F}$ and of stationary solutions, and the asymptotic behavior of the solutions, showing general convergence results to the minimizer of $\mathcal{F}$.

Plan of the paper. In the next section we will make precise our definition of measure-valued solutions to (1.DDE) (§2.1) and we will present our main results concerning existence, uniqueness, stability, and approximation of Wasserstein solutions (§2.2). The equation governing the propagation of their singularities is considered in §2.3. §2.4 is devoted to a precise characterization of minimizers of $\mathcal{F}$

and of the critical mass; stationary solutions are studied in §2.5 and §2.6 collects some results concerning the asymptotic behaviour of Wasserstein solutions.

Section 3 briefly recalls some definitions and tools of (one-dimensional) optimal transport, Wasserstein distance, and the related (sub)differentiability properties of displacement λ-convex functionals. Theorems 3.4 and 3.6 lie at the core of our further developments. A last paragraph devoted to a simple regularization of $\mathcal{F}$ by Γ-convergence concludes the section.

The last section contains the proofs of all our main results: the connection with the general theory is discussed in §4.1 and §4.2 is devoted to the propagation of the singularities; the study of the minimizers of $\mathcal{F}$ and of the related asymptotic behavior of the solutions to (1.DDE) is performed in the last part.

2. **Definitions and main results.**

In this section we collect the main definitions and results we shall prove in the rest of the paper.

⟨subsec:def⟩ 2.1. **Wasserstein solutions.**

The case of bounded initial densities and Lipschitz drifts. When (the Lebesgue density of) $\rho_0 \in L^\infty(\mathbb{R})$ and the potential V is such that

$$V''(x) \le \mathsf{c} \quad \text{for every } x \in \mathbb{R}, \tag{2.1}$$ eq:8

it is not difficult (see [17] and next Corollary 4.2) to show that a smooth solution ρ_t of (1.DDE) satisfies the *a priori* estimate

$$\sup_{t\in[0,T]} \|\rho_t\|_{L^\infty(\mathbb{R})} \le R_T := \|\rho_0\|_{L^\infty(\mathbb{R})} \mathrm{e}^{\mathsf{c}T} \quad \text{for every } T > 0, \tag{2.2}$$ eq:9

so that it is uniformly bounded in every bounded time interval $[0,T]$. We can infer from (2.2) that the behavior of $\beta(r)$ as $r \uparrow +\infty$ does not play any role, and a solution in $[0,T]$ could be easily obtained by solving (1.DDE) with respect to a nonlinearity $\tilde\beta$ defined for instance by

$$\tilde\beta(r) := \begin{cases} \beta(r) & \text{if } r \le 2R_T, \\ \beta(2R_T) + \beta'(2R_T)(r - 2R_T) & \text{if } r > 2R_T. \end{cases}$$

Denoting by $\mathsf{S}_t(\rho_0)$ the solution ρ_t generated by a bounded initial datum ρ_0, it is possible to check that S_t satisfies the L^1 contraction property

$$\big\|\mathsf{S}_t(\rho) - \mathsf{S}_t(\eta)\big\|_{L^1(\mathbb{R})} \le \|\rho - \eta\|_{L^1(\mathbb{R})} \quad \text{for every } \rho, \eta \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}).$$

Consequently S_t can be extended in a canonical way to a contraction semigroup in the cone $L^1_+(\mathbb{R})$ of nonnegative integrable densities.

Measure-valued solutions. In case the Lebesgue density of ρ_0 is not bounded or V does not satisfy (2.1), the presence of a singular part in the solution ρ of (1.DDE) has to be taken into account, since the boundedness of β is responsible of the (possible) presence of a critical mass. We shall see an example of a solution ρ_t exhibiting a singular part for every $t \ge 0$ in the next Remark 2.23.

In the following we will denote by $\mathscr{M}_+(\mathbb{R})$ (resp. $\mathscr{M}_+(\mathbb{R}, \mathfrak{m})$) the space of non-negative Borel measures in $\mathbb{R}$ with finite mass (resp. with prescribed mass $\mathfrak{m} > 0$) and by $\mathscr{M}_2(\mathbb{R})$ (resp. $\mathscr{M}_2(\mathbb{R}, \mathfrak{m})$) the collection of measures in $\mathscr{M}_+(\mathbb{R})$ (resp. in $\mathscr{M}_+(\mathbb{R}, \mathfrak{m})$) with finite quadratic momentum. In order to enucleate a precise notion of measure-valued solution, for every $\rho \in \mathscr{M}_+(\mathbb{R})$ we consider the classical Lebesgue decomposition

$$\rho = \rho^a + \rho^\perp, \quad \rho^a = u\,\mathscr{L}^1,$$

where $u \in L^1(\mathbb{R})$ is the Lebesgue density of the absolutely continuous part ρ^a of ρ and $\rho^\perp$ is the singular part of ρ, concentrated on a set of Lebesgue measure 0.

It is then natural to substitute the term $\beta(\rho)$ in (1.DDE) by $\beta(u)$ and then interpret (1.DDE) in the sense of distributions. If we want to obtain a good notion of solution, we should add some further requirements to the density u. The first one is of qualitative type, and relies in considering u as a continuous function on $\mathbb{R}$ with values in the extended set $[0, +\infty]$, endowed with the usual topology.

⟨def:right_measures⟩ **Definition 2.1** (Measures with continuous densities)**.** We say that a measure $\rho = \rho^a + \rho^\perp \in \mathscr{M}_+(\mathbb{R})$ has a generalized continuous density $u \in C^0(\mathbb{R}; [0, +\infty])$ with proper domain $\mathsf{D}(u) := \big\{x \in \mathbb{R} : u(x) < +\infty\big\}$ if

$$\rho^\perp\big(\mathsf{D}(u)\big) = 0, \quad \mathscr{L}^1\big(\mathbb{R} \setminus \mathsf{D}(u)\big) = 0, \quad \text{and} \quad \rho^a = u\,\mathscr{L}^1|_{\mathsf{D}(u)}.$$

We set $\mathsf{D}_+(u) := \big\{x \in \mathsf{D}(u) : u(x) > 0\big\}$. We denote by $\mathscr{M}_+^{\mathrm{c}}(\mathbb{R})$ the collection of all measures with generalized continuous density and we set $\mathscr{M}_2^{\mathrm{c}}(\mathbb{R}) := \mathscr{M}_+^{\mathrm{c}}(\mathbb{R}) \cap \mathscr{M}_2(\mathbb{R})$, $\mathscr{M}_2^{\mathrm{c}}(\mathbb{R}, \mathfrak{m}) := \mathscr{M}_+^{\mathrm{c}}(\mathbb{R}) \cap \mathscr{M}_2(\mathbb{R}, \mathfrak{m})$.

Notice that $\mathsf{D}(u)$ is a *dense open* subset of $\mathbb{R}$, $\rho^\perp = \rho|_{\mathbb{R} \setminus \mathsf{D}(u)}$, and

$$\lim_{x \to \bar{x}} u(x) = +\infty \quad \text{for every } \bar{x} \in \partial\mathsf{D}(u) = \mathbb{R} \setminus \mathsf{D}(u).$$

In particular, $\mathscr{M}_+^{\mathrm{c}}(\mathbb{R})$ does not contain any purely singular measure: if $\rho^a = 0$ then also $\rho^\perp$ vanishes.

If $\rho \in \mathscr{M}_+^{\mathrm{c}}(\mathbb{R})$ then we will always identify its Lebesgue density $\mathrm{d}\rho/\mathrm{d}\mathscr{L}^1$ with the (unique) continuous precise representative $u \in C^0(\mathbb{R}; [0, +\infty])$ given by Definition 2.1. By (1.β) we can consider β as a continuous function defined on the extended set $[0, +\infty]$ and therefore the composition $\beta \circ u$ is a well defined real continuous function on $\mathbb{R}$.

The second requirement is a quantitative estimate concerning the "generalized Fisher" dissipation functional.

Definition 2.2 (Generalized Fisher dissipation)**.** If ρ belongs to $\mathscr{M}_+^{\mathrm{c}}(\mathbb{R})$ with continuous density u we set

$$\mathcal{I}(\rho) := \int_{\mathsf{D}_+(u)} \Big|\frac{\partial_x \beta(u)}{u} + V'\Big|^2 u \,\mathrm{d}x + \int_{\mathbb{R}} |V'|^2 \,\mathrm{d}\rho^\perp \quad \text{if} \quad \beta \circ u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}). \tag{2.3}$$ eq:5

When $\beta \circ u \notin W^{1,1}_{\mathrm{loc}}(\mathbb{R})$ or $\rho \notin \mathscr{M}_+^{\mathrm{c}}(\mathbb{R})$, we simply set $\mathcal{I}(\rho) := +\infty$.

It turns out that $\mathcal{I}$ is a lower semicontinuous functional with respect to weak convergence of measures in $\mathscr{M}_+(\mathbb{R})$ (see Theorem 3.5).

⟨def:measuresolution⟩ **Definition 2.3** (Wasserstein solutions to (1.DDE))**.** We say that ρ is a Wasserstein solution of problem (1.DDE) if

$$\rho \in C^0([0, +\infty); \mathscr{M}_2(\mathbb{R}, \mathfrak{m})) \tag{2.4a}$$ cont

and, denoting by ρ_t the measure ρ at the time t,

$$\rho_t \in \mathscr{M}_+^{\mathrm{c}}(\mathbb{R}) \text{ for } \mathscr{L}^1\text{-a.e. } t > 0, \tag{2.4b}$$ eq:3

$$\int_{T_0}^{T_1} \mathcal{I}(\rho_t)\,\mathrm{d}t < +\infty \quad \text{for every } 0 < T_0 < T_1 < +\infty, \tag{2.4c}$$ eq:3bis

and

$$\int_0^{+\infty}\int_{\mathbb{R}}\Big(-\partial_t\varphi+\partial_x\varphi V'\Big)\,\mathrm{d}\rho_t\,\mathrm{d}t+\int_0^{+\infty}\int_{\mathbb{R}}\partial_x\varphi\,\partial_x\beta(u_t)\,\mathrm{d}x\,\mathrm{d}t=0 \quad \text{for every } \varphi\in C^\infty_{\mathrm{c}}(\mathbb{R}\times(0,+\infty)), \qquad \text{(2.4d)}\ \boxed{\texttt{eq-debole}}$$

where u_t is the generalized continuous density of ρ_t for $\mathscr{L}^1$-a.e. $t>0$.

Remark 2.4. In order to clarify the continuity condition (2.4a), we recall that $\mathscr{M}_2(\mathbb{R},\mathfrak{m})$ is a complete metric space endowed with the so called L^2-Wasserstein distance $W_2(\cdot,\cdot)$. More details on this distance will be given in the next section; let us just recall that a sequence ρ_n converges to ρ in $\mathscr{M}_2(\mathbb{R},\mathfrak{m})$ as $n\uparrow+\infty$ if and only if

$$\lim_{n\uparrow+\infty}\int_{\mathbb{R}}\varphi(x)\,\mathrm{d}\rho_n(x)=\int_{\mathbb{R}}\varphi(x)\,\mathrm{d}\rho(x) \quad \text{for every } \varphi\in C^0(\mathbb{R}) \text{ with } \sup_x\frac{|\varphi(x)|}{1+x^2}<+\infty. \qquad \text{(2.5)}\ \boxed{\texttt{eq:14}}$$

Remark 2.5 (The role of the generalized continuous density)**.** By neglecting condition (2.4b) one can easily construct evolutions of purely singular measures which solve (2.4d) and are not influenced at all by the diffusion term. We take a finite number of C^1 curves $x_j:[0,+\infty)\to\mathbb{R}$, $i=1,\cdots,N$, which solve the differential equation $\dot x_j(t)=-V'(x_j(t))$ in $[0,+\infty)$, and we set

$$\rho_t:=\sum_{j=1}^N\alpha_j\delta_{x_j(t)},\qquad \alpha_j\ge 0. \qquad \text{(2.6)}\ \boxed{\texttt{eq:6}}$$

In this case $\rho^a_t\equiv 0$ for every $t\ge 0$, which implies $\beta(u_t)\equiv 0$ and (2.4d) contains just the pure transport contribution given by the first integral. On the other hand, by taking a smooth approximating family $\rho^\varepsilon\to\rho_0$ in $\mathscr{M}_2(\mathbb{R},\mathfrak{m})$, we can see that (2.6) is not the limit of the corresponding solution ρ^ε_t as $\varepsilon\downarrow 0$ (see Theorem 2.6).

Energy functional and Fisher dissipation. In order to understand both the role of the generalized Fisher dissipation and the consequences of (2.4c), let us recall the definition (1.E) of the so-called internal energy density $E:[0,+\infty)\to\mathbb{R}$ by the relation

$$E(r):=-\beta_\infty+\int_r^{+\infty}\big(1-\frac{r}{s}\big)\beta'(s)\,\mathrm{d}s=-\beta(r)-r\int_r^{+\infty}\frac{\beta'(s)}{s}\,\mathrm{d}s.$$

It is simple to check that E is a strictly convex nonpositive function satisfying

$$E\in C^2(0,+\infty),\quad E(0)=0, \qquad \lim_{r\to 0^+}\frac{E(r)}{r\log r}=\beta'(0)\in[0,+\infty),\qquad E_\infty=\lim_{r\uparrow+\infty}E(r)=-\beta_\infty, \qquad \text{(2.7)}\ \boxed{\texttt{propertiesE}}$$

and

$$\beta'(r)=rE''(r),\quad E'(r)=-\int_r^{+\infty}\frac{\beta'(s)}{s}\,\mathrm{d}s<0,\qquad \forall r\in(0,+\infty).$$

We associate the integral functional

$$\mathcal{E}(\rho):=\int_{\mathbb{R}}E(u(x))\,\mathrm{d}x\quad\text{whenever}\quad \rho=u\mathscr{L}^1+\rho^\perp\in\mathscr{M}_+(\mathbb{R}),$$

to the energy density E, the potential energy

$$\mathcal{V}(\rho) := \int_{\mathbb{R}} V(x)\,\mathrm{d}\rho(x)$$

to the potential V, and the energy functional $\mathcal{F} : \mathscr{M}_2(\mathbb{R}, \mathfrak{m}) \to (-\infty, +\infty]$

$$\mathcal{F}(\rho) := \mathcal{E}(\rho) + \mathcal{V}(\rho).$$

Formal computations show that $\mathcal{F}$ and $\mathcal{I}$ satisfy the energy dissipation identity along solutions to (1.DDE)

$$\mathcal{F}(\rho_{t_1}) + \int_{t_0}^{t_1} \mathcal{I}(\rho_t)\,\mathrm{d}t = \mathcal{F}(\rho_{t_0}) \qquad 0 \le t_0 < t_1 < +\infty. \tag{2.8}$$ eq:13

⟨subsec:main⟩ 2.2. **Existence, stability, and approximation results.** Recall that $\lambda \in \mathbb{R}$ is a lower bound for the second derivative of V, see (1.V). Let us set

$$\mathsf{E}_\lambda(t) := \int_0^t \mathrm{e}^{\lambda s}\,\mathrm{d}s = \begin{cases} \frac{\mathrm{e}^{\lambda t}-1}{\lambda} & \text{if } \lambda \neq 0, \\ t & \text{if } \lambda = 0. \end{cases}$$

⟨thm:main1⟩ **Theorem 2.6** (Existence, uniqueness, stability, and comparison)**.** *For every $\rho_0 \in \mathscr{M}_2(\mathbb{R}, \mathfrak{m})$ there exists a unique Wasserstein solution ρ_t to (1.DDE) according to Definition 2.3. This solution satisfies the regularization estimate*

$$\mathcal{F}(\rho_t) + \frac{\mathsf{E}_\lambda(t)}{2}\mathcal{I}(\rho_t) \le \mathfrak{m}V(0) + \frac{1}{2\mathsf{E}_\lambda(t)}\mathfrak{m}_2(\rho_0) \quad \textit{for every } t > 0, \tag{2.9}$$ eq:18

the energy dissipation identity (2.8)*, and the dissipation inequality*

$$\mathcal{I}(\rho_t) \le \mathcal{I}(\rho_{t_0})\mathrm{e}^{-2\lambda(t-t_0)}, \qquad \forall\, t \ge t_0 \ge 0. \tag{2.10}$$ eq:19

The map $\mathsf{S}_t : \mathscr{M}_2(\mathbb{R}, \mathfrak{m}) \to \mathscr{M}_2(\mathbb{R}, \mathfrak{m})$ defined by $\mathsf{S}_t(\rho_0) = \rho_t$ is a semigroup of continuous maps in $\mathscr{M}_2(\mathbb{R}, \mathfrak{m})$ satisfying the stability property

$$W_2(\mathsf{S}_t(\rho_0), \mathsf{S}_t(\eta_0)) \le \mathrm{e}^{-\lambda t} W_2(\rho_0, \eta_0). \tag{2.11}$$ eq:15

If moreover $\rho_0 \le \eta_0$ then $\mathsf{S}_t(\rho_0) \le \mathsf{S}_t(\eta_0)$ for every $t \ge 0$.

⟨rem:singular⟩ **Remark 2.7** (Singularities)**.** Recalling the definition (2.3) of $\mathcal{I}$, the regularization estimate (2.9) shows that the solution given by Theorem 2.6 satisfies $\rho_t \in \mathscr{M}_2^c(\mathbb{R}, \mathfrak{m})$ for every $t > 0$.

In the case when V' is Lipschitz, the stability property (2.11) and a simple regularization of the initial datum show that Wasserstein solutions are the limit of locally bounded solutions satisfying (2.2). Another way to see that Definition 2.3 provides the right notion of solution involves a classical viscous regularization of (1.DDE) combined with a suitable regularization of the potential V. Given a small parameter $\varepsilon > 0$ let us consider the perturbed nonlinear functions

$$\beta^\varepsilon(r) := \beta(r) + \varepsilon r, \quad r \in [0, +\infty), \tag{2.12}$$ eq:16

and a family V^ε of smooth and Lipschitz potentials such that

$$V^\varepsilon(x) \le V(x) + A|x|^2 \quad \lambda \le (V^\varepsilon)''(x) \le \sup_{\mathbb{R}} V'' \qquad \text{for every } x \in \mathbb{R}, \tag{2.13a}$$ eq:82

$$(V^\varepsilon)^{(h)} \to V^{(h)} \quad \text{as } \varepsilon \downarrow 0 \quad \text{uniformly on compact sets of } \mathbb{R}, \quad h = 0, 1, 2, \tag{2.13b}$$ eq:85

$$\liminf_{|x|\to\infty} \frac{V^\varepsilon(x)}{|x|^2} \ge 0 \quad \text{uniformly with respect to } \varepsilon. \tag{2.13c}$$ eq:86

For every $\rho_0^\varepsilon \in \mathscr{M}_2(\mathbb{R}, \mathfrak{m})$ we consider the problem

$$\partial_t \rho^\varepsilon - \partial_x\big(\partial_x \beta^\varepsilon(u^\varepsilon) + (V^\varepsilon)'\rho^\varepsilon\big) = 0, \quad \text{in } (0,+\infty)\times\mathbb{R}; \quad \rho^\varepsilon(0,\cdot) = \rho_0^\varepsilon, \quad \text{in } \mathbb{R}, \tag{2.14}$$ eq:17

the associated energy functional

$$\mathcal{F}^\varepsilon := \mathcal{E}^\varepsilon + \mathcal{V}^\varepsilon,$$

$$\mathcal{E}^\varepsilon(\rho) := \begin{cases} \mathcal{E}(\rho) + \varepsilon \displaystyle\int_{\mathbb{R}} u \log u \,\mathrm{d}x & \text{if } \rho = u\mathscr{L}^1 \ll \mathscr{L}^1 \\ +\infty & \text{if } \rho^\perp \neq 0 \end{cases}, \quad \mathcal{V}^\varepsilon(\rho) := \int_{\mathbb{R}} V^\varepsilon(x)\,\mathrm{d}\rho,$$

and the corresponding Fisher-dissipation

$$\mathcal{I}^\varepsilon(\rho) := \int_{\mathbb{R}} \Big| \frac{\partial_x \beta^\varepsilon(u)}{u} + (V^\varepsilon)' \Big|^2 u \,\mathrm{d}x \qquad \text{if } \rho = u\mathscr{L}^1,\ \beta^\varepsilon(u) \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}).$$

As usual $\mathcal{I}^\varepsilon(\rho) = +\infty$ if $u \notin W^{1,1}_{\mathrm{loc}}(\mathbb{R})$ or $\rho \not\ll \mathscr{L}^1$.

⟨thm:main2⟩ **Theorem 2.8** (Convergence of viscous regularizations)**.** *For every $\rho_0^\varepsilon = u_0^\varepsilon \mathscr{L}^1 \in \mathscr{M}_2(\mathbb{R}, \mathfrak{m})$ with $u_0^\varepsilon \in C^1_c(\mathbb{R})$, there exists a unique smooth solution $\rho^\varepsilon = u^\varepsilon \mathscr{L}^1 \in C^0([0,+\infty); \mathscr{M}_2(\mathbb{R}, \mathfrak{m}))$ of problem (2.14) satisfying $\mathcal{I}^\varepsilon(\rho^\varepsilon) \in L^1_{loc}(0,+\infty)$. Moreover (2.9), (2.10), and (2.8) hold with $\mathcal{F}, \mathcal{I}$ replaced by $\mathcal{F}^\varepsilon, \mathcal{I}^\varepsilon$, respectively.*

If $\rho_0^\varepsilon \to \rho_0$ in $\mathscr{M}_2(\mathbb{R}, \mathfrak{m})$ and $\sup_\varepsilon \mathcal{F}^\varepsilon(\rho^\varepsilon) < +\infty$, then ρ_t^ε converges in $\mathscr{M}_2(\mathbb{R}, \mathfrak{m})$ to the unique Wasserstein solution ρ_t of problem (1.DDE) as $\varepsilon \downarrow 0$ for every $t > 0$. Moreover $u_t^\varepsilon \to u_t$ uniformly on compact sets of $\mathsf{D}(u_t)$ for every $t > 0$.

The proofs of Theorems 2.6 and 2.8 take advantage of the theory of gradient flows of convex functionals with respect to the Wasserstein distance [2] and will be given in Section 4.1.

?⟨rem:nonsmooth⟩? **Remark 2.9** (Non smooth potentials)**.** Theorems 2.6 and 2.8 are still true in the case when V is a general λ-convex function, i.e. the condition (1.V) on the lower bound on V'' (which we assumed for the sake of simplicity) is replaced by

$$x \mapsto V(x) - \frac{\lambda}{2}x^2 \quad \text{is convex in } \mathbb{R}. \tag{2.15}$$ eq:79

Condition (2.15) implies that V is differentiable $\mathscr{L}^1$-almost everywhere, so that the first occurence of V' in the definition (2.3) of $\mathcal{I}$ still makes sense as it is integrated with respect to $\mathscr{L}^1$. The second integral term in (2.3) should be replaced by

$$\int_{\mathbb{R}} |\partial^\circ V(x)|^2 \,\mathrm{d}\rho^\perp(x)$$

where $\partial^\circ V(x)$ denotes the element of minimal norm in the (non empty) Frechet subdifferential ∂V of V.

⟨subsec:propagation⟩ 2.3. **Propagation of singularities.** In this section we want to study the evolution of the singular part $\rho_t^\perp$ of the Wasserstein solution ρ_t to (1.DDE). By Remark 2.7 we know that $\rho_t = u_t\mathscr{L}^1 + \rho_t^\perp \in \mathscr{M}_2^c(\mathbb{R})$ for every $t > 0$, so that the support of $\rho_t^\perp$ coincides with the set where the (continuous representative of the) density u_t takes the value $+\infty$. We thus call

$\mathsf{J}(u_t) := \mathbb{R} \setminus \mathsf{D}(u_t) = \big\{x \in \mathbb{R} : u_t(x) = +\infty\big\}$ and we will show that the evolution of $\mathsf{J}(u_t)$ follows the flow generated by $-V'$.

Let us first introduce the evolution semigroup X on $\mathbb{R}$ generated by $-V'$, thus satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{X}_t(x) = -V'(\mathsf{X}_t(x)), \quad \mathsf{X}_0(x) = x \quad \text{for every } x \in \mathbb{R}.$$

Since V' is of class C^1 and, by (1.V),

$$\big(V'(x) - V'(y)\big)(x-y) \ge \lambda|x-y|^2 \quad \text{for every } x, y \in \mathbb{R},$$

X_t is a family of diffeomorphisms mapping $\mathbb{R}$ onto the open set $\mathsf{R}_t := \mathsf{X}_t(\mathbb{R})$. We set

$$\mathsf{J}_t := \mathsf{X}_t\big(\mathsf{J}(u_0)\big), \quad \mathsf{D}_t := \mathsf{X}_t\big(\mathsf{D}(u_0)\big), \quad t \ge 0,$$

and we notice that $\mathsf{J}_t = \mathsf{R}_t \setminus \mathsf{D}_t$ is a closed subset of R_t, since D_t is open.

If $\sigma \in \mathscr{M}_+(\mathbb{R})$, the push-forward $(\mathsf{X}_t)_\#\sigma$ through X_t is the Borel measure defined by

$$(\mathsf{X}_t)_\#\sigma(A) := \sigma\big(\mathsf{X}_t^{-1}(A)\big) \quad \text{for each Borel set } A \subset \mathbb{R}.$$

⟨thm:main3⟩ **Theorem 2.10** (Propagation of singularities). *If $\rho_0 \in \mathscr{M}_2^{\mathrm{c}}(\mathbb{R})$ and $\rho_t = u_t\mathscr{L}^1 + \rho_t^\perp \in \mathscr{M}_2^{\mathrm{c}}(\mathbb{R})$ is the unique Wasserstein solution of (1.DDE), then*

$$\begin{aligned}&\partial_t\rho_t^\perp - \partial_x\big(\rho_t^\perp V'\big) \le 0 \quad \textit{in the sense of distributions,}\\ &\lim_{t\downarrow 0}\int_{\mathbb{R}} \varphi(x)\mathrm{d}\rho_t^\perp(x) \le \int_{\mathbb{R}}\varphi(x)\mathrm{d}\rho_0^\perp(x) \quad \textit{for every } \varphi \in C^0_{\mathrm{c}}(\mathbb{R};[0,+\infty)).\end{aligned} \tag{2.16}$$

eq:68

In particular

$$\mathsf{J}(u_t) \subset \mathsf{J}_t, \quad \rho_t^\perp \le (\mathsf{X}_t)_\#\rho_0^\perp, \quad \textit{for every } t \ge 0,$$

so that for every Borel set $A \subset \mathbb{R}$

$$\rho_t^\perp(A) \le \rho_0^\perp\big(\mathsf{X}_t^{-1}(A)\big).$$

In particular $\rho_t^\perp$ is concentrated in $\mathsf{X}_t(\mathsf{J}(u_0))$ and u_t is finite in $\mathsf{X}_t(\mathsf{D}(u_0))$.

The proof of Theorem 2.10 will be carried out in Section 4.2.

The case when $\rho_0^\perp = \sum_{j=1}^N \alpha_j\delta_{x_j}$ with $x_1 < x_2 < \cdots < x_N$ and $\alpha_j > 0$ is of particular interest. In this case, from Theorem 2.10 we deduce that $\rho_t = u_t\mathscr{L}^1 + \rho_t^\perp$ with

$$\rho_t^\perp = \sum_{j=1}^N \alpha_j(t)\delta_{x_j(t)}, \quad x_j(t) = \mathsf{X}_t(x_j),$$

where $\alpha_j : [0,+\infty) \to [0,+\infty)$ is nonincreasing.

Theorem 2.10 can be equivalently formulated in terms of the density u_t of the regular part of ρ_t:

Corollary 2.11 (The regular part is a supersolution). *If $\rho_t = u_t\mathscr{L}^1 + \rho_t^\perp \in \mathscr{M}_2^{\mathrm{c}}(\mathbb{R})$ is a Wasserstein solution to (1.DDE) then u_t is a supersolution of (1.DDE), i.e.*

$$\partial_t u - \partial_x\big(\partial_x\beta(u) + V'\,u\big) \ge 0 \quad \textit{in the sense of distributions in } (0,+\infty)\times\mathbb{R}.$$

⟨subsec:minimizers⟩ 2.4. **Minimizers of the energy functional and critical mass.** In this section we consider the functional $\mathcal{F}$ in the space $\mathscr{M}_+(\mathbb{R},\mathfrak{m})$ and we study its minimizers, which are particular stationary solutions of equation (1.DDE). We will assume that the potential V satisfies the coercivity condition

$$\lim_{|x|\to+\infty} V(x) = +\infty, \tag{2.coer}$$

eq:95

and we set

$$V_{\min} := \min_{\mathbb{R}} V, \qquad Q := \big\{x \in \mathbb{R} : V(x) = V_{\min}\big\}.$$

The structure of the minimizers of $\mathcal{F}$ is governed by two critical constants and two functions, with their inverses. The first function is $r \mapsto -E'(r)$: it is a decreasing homeomorphism between $(0,+\infty)$ and the interval $(0,\mathfrak{d})$, which can be characterized by the constant

$$\mathfrak{d} := -\lim_{x\to 0^+} E'(x) = \int_0^{+\infty} \frac{\beta'(s)}{s}\,\mathrm{d}s \in (0,+\infty].$$

Notice that $\mathfrak{d}$ is finite if and only if $s\mapsto \beta'(s)/s$ is integrable in a right neighborhood of 0. We can thus consider the pseudo-inverse function $H:(0,+\infty)\to[0,+\infty)$ defined by

$$H(v) = \begin{cases} (E')^{-1}(-v) & \text{if } v\in(0,\mathfrak{d})\\ 0 & \text{if } \mathfrak{d}<+\infty \text{ and } v\in[\mathfrak{d},+\infty)\end{cases}$$

which is strictly decreasing in the interval $(0,\mathfrak{d})$.

The second function is

$$M_{\mathbb{R}}(v) := \int_{\mathbb{R}} H(V(x)-v)\,\mathrm{d}x, \quad v\le V_{\min}.$$

In order to avoid a degenerate situation, we will assume the integrability condition (depending on E and V)

$$\int_{\mathbb{R}\setminus\tilde{Q}} |x|^\alpha H(V(x)-V_{\min})\,\mathrm{d}x < +\infty, \tag{2.int}$$ eq:96

for some $\alpha>0$ and some bounded open neighborhood $\tilde{Q}$ of Q.

This assumption and (2.coer) ensure that the functional $\mathcal{F}$ is bounded from below in $\mathscr{M}_+(\mathbb{R},\mathfrak{m})$, as we will prove in Theorem 2.12. Moreover (2.int) implies $\int_{\mathbb{R}\setminus\tilde{Q}} H(V(x)-V_{\min})\,\mathrm{d}x < +\infty$ and then $M_{\mathbb{R}}(v) < +\infty$ for every $v<V_{\min}$ so that $M_{\mathbb{R}}$ is an increasing homeomorphism between $(V_{\min}-\mathfrak{d},V_{\min})$ and the interval $(0,\mathfrak{m}_{\mathrm{c}})$, where the critical mass $\mathfrak{m}_{\mathrm{c}}$ is defined by

$$\mathfrak{m}_{\mathrm{c}} := \lim_{v\uparrow V_{\min}} M_{\mathbb{R}}(v) = \int_{\mathbb{R}} H(V(x)-V_{\min})\,\mathrm{d}x \in (0,+\infty]. \tag{2.17}$$ eq:28

If $M_{\mathbb{R}}^{-1}:(0,\mathfrak{m}_{\mathrm{c}})\to(V_{\min}-\mathfrak{d},V_{\min})$ denotes the inverse map of $M_{\mathbb{R}}$, we eventually set

$$\mathfrak{v} := \begin{cases} M_{\mathbb{R}}^{-1}(\mathfrak{m}) & \text{if } \mathfrak{m}<\mathfrak{m}_{\mathrm{c}}\\ V_{\min} & \text{if } \mathfrak{m}\ge\mathfrak{m}_{\mathrm{c}}.\end{cases}$$

⟨thm:main4⟩ **Theorem 2.12** (Characterization of minimizers). *If (2.int) holds and V satisfies (2.coer) then $\mathcal{F}$ is bounded from below in $\mathscr{M}_+(\mathbb{R},\mathfrak{m})$ and attains its minimum. A measure $\rho\in\mathscr{M}_+(\mathbb{R},\mathfrak{m})$ is a minimizer of $\mathcal{F}$ if and only if it belongs to $\mathscr{M}_+^{\mathrm{c}}(\mathbb{R},\mathfrak{m})$ and its decomposition $\rho_{\min} = u_{\min}\mathscr{L}^1 + \rho_{\min}^\perp$ satisfies*

$$u_{\min}(x) = H(V(x)-\mathfrak{v}), \quad \rho^\perp_{\min}(\mathbb{R}\setminus Q) = 0, \quad \rho^\perp_{\min}(Q) = (\mathfrak{m}-\mathfrak{m}_{\mathrm{c}})^+. \tag{2.18}$$ eq:30

If (2.int) holds for $\alpha=2$ then $\rho_{\min}$ belongs to $\mathscr{M}_2(\mathbb{R},\mathfrak{m})$.

⟨rem:examples⟩ **Remark 2.13.**

- In the case when $\mathfrak{m}\le\mathfrak{m}_{\mathrm{c}}$, the minimizer $\rho_{\min} = u_{\min}\mathscr{L}^1$ is unique and $\rho^\perp_{\min} = 0$. If $\mathfrak{m}<\mathfrak{m}_{\mathrm{c}}$, $u_{\min}$ is bounded, whereas if $\mathfrak{m}=\mathfrak{m}_{\mathrm{c}}$, $u_{\min}(x)=+\infty$ for every $x\in Q$. Last, if $\mathfrak{m}>\mathfrak{m}_{\mathrm{c}}$ the minimizer has a nontrivial singular part and it is unique only when Q is a singleton.
- As already pointed out, the existence of the critical mass $\mathfrak{m}_{\mathrm{c}}<+\infty$ depends on the behavior of the $\beta(r)$ for large values of r and on the local behaviour of V near Q.

- If $\mathfrak{d} < +\infty$ then the support of $\rho_{\min}$ is compact and it is contained in the sublevel of V $\{x \in \mathbb{R} : V(x) \le \mathfrak{v} + \mathfrak{d}\}$.
- If Q is an interval (in particular if V is convex) then the minimizer of $\mathcal{F}$ is unique. This property is always true when $\mathfrak{m}_{\mathrm{c}} = +\infty$; when $\mathfrak{m}_{\mathrm{c}} < +\infty$, the fact that Q is a closed interval and (2.17) show that Q is a singleton.

⟨subsec:stationary⟩ 2.5. **Stationary solutions.** In this section we will study the stationary Wasserstein solutions of (1.DDE), i.e. constant measures $\rho \in \mathscr{M}_2(\mathbb{R})$ which solve (1.DDE). As a starting point, we observe that stationary solutions can be characterized as measures with vanishing Fisher dissipation.

⟨thm:Fisher=0⟩ **Theorem 2.14.** *A measure $\rho \in \mathscr{M}_2(\mathbb{R}, \mathfrak{m})$ is a stationary Wasserstein solution of (1.DDE) if and only if $\rho \in \mathscr{M}_2^{\mathrm{c}}(\mathbb{R}, \mathfrak{m})$ and $\mathcal{I}(\rho) = 0$.*

Of course, any minimizer ρ of $\mathcal{F}$ in $\mathscr{M}_2(\mathbb{R}, \mathfrak{m})$ satisfies $\mathcal{I}(\rho) = 0$ and it is a stationary solution, but in general one can expect that other stationary solutions exist. Their structure depends in a crucial way on $\mathfrak{d}$; the simplest case is when $\mathfrak{d} = +\infty$.

⟨thm:statI⟩ **Theorem 2.15** (Characterization of stationary measures I)**.** *Let us suppose that (2.int) holds and that V satisfies (2.coer). If $\mathfrak{d} = +\infty$ then for every $\rho \in \mathscr{M}_+(\mathbb{R}, \mathfrak{m})$*

$$\mathcal{I}(\rho) = 0 \quad \Leftrightarrow \quad \rho \text{ is a minimizer for } \mathcal{F} \text{ in } \mathscr{M}_+(\mathbb{R}, \mathfrak{m}). \tag{2.19}$$ nulldissipation

In particular, a measure $\rho \in \mathscr{M}_2(\mathbb{R}, \mathfrak{m})$ is a stationary solution if and only if it is a minimizer of $\mathcal{F}$.

The case when $\mathfrak{d} < +\infty$ is more complicated and requires some preliminary definition.

⟨def:adm_interval⟩ **Definition 2.16.** Let us suppose that $\mathfrak{d} < +\infty$. We say that a bounded open interval $I = (a, b) \subset \mathbb{R}$ is an admissible local sublevel of V if

$$V(a) = V(b), \quad \mathfrak{v}_I := V(a) - \mathfrak{d} \le V(x) < V(a) \quad \text{for every } x \in (a, b), \tag{2.20}$$ eq:98

and

$$M_I := \int_a^b H\big(V(x) - \mathfrak{v}_I\big)\,\mathrm{d}x < +\infty. \tag{2.21}$$ eq:99

We set $Q_I := \big\{x \in I : V(x) = \min_I V\big\}$.

Notice that Q_I is not empty if and only if

$$\mathfrak{v}_I = V(a) - \mathfrak{d} = \min_I V.$$

If Q_I is empty, i.e. $\mathfrak{v}_I < \min_I V$, then condition (2.21) is always satisfied.

If $u : \mathbb{R} \to [0, +\infty]$ is a continuous map, we set

$$\Omega_+(u) := \big\{x \in \mathbb{R} : u(x) > 0\big\},$$
$$\mathscr{I}(u) := \text{the collection of all the connected components of } \Omega_+(u).$$

⟨thm:main_stationary⟩ **Theorem 2.17** (Characterization of stationary measures II)**.** *Let us suppose that (2.int) holds and that V satisfies (2.coer). If $\mathfrak{d} < +\infty$ a measure $\rho = u\mathscr{L}^1 + \rho^\perp \in \mathscr{M}_+^{\mathrm{c}}(\mathbb{R}, \mathfrak{m})$ satisfies $\mathcal{I}(\rho) = 0$ if and only if it satisfies the following three conditions:*

1. *All the connected components in $\mathscr{I}(u)$ of the open set $\Omega_+(u)$ are admissible local sublevels of V according to Definition 2.16.*
2. $$u_{|_I} = H(V(x) - \mathfrak{v}_I) \quad \text{for every } I \in \mathscr{I}(u). \tag{2.22}$$ eq:103

3. If $Q(u) := \bigcup_{I\in\mathscr{I}(u)} Q_I$

$$\rho^\perp \text{ is concentrated on } Q(u), \text{ and } \quad \mathfrak{m} = \sum_{I\in\mathscr{I}(u)} M_I + \rho^\perp(\mathbb{R}). \tag{2.23}$$ eq:84

⟨cor:obvious⟩ **Corollary 2.18.** *If (2.int) holds with $\alpha = 2$, V satisfies (2.coer) and*

$$\begin{aligned} &\text{the set } Q \text{ is an interval } [q_-, q_+],\\ &V' \geq 0 \text{ in } (q_+, +\infty), \text{ and } V' \leq 0 \text{ in } (-\infty, q_-), \end{aligned} \tag{2.24}$$ eq:110

(assumption (2.24) is always satisfied if V is convex), then (2.19) holds and there exists a unique stationary measure in $\mathscr{M}_2(\mathbb{R},\mathfrak{m})$ which coincides with the unique minimizer of $\mathcal{F}$ in $\mathscr{M}_2(\mathbb{R},\mathfrak{m})$.

?⟨rem:converse_obvious⟩? **Remark 2.19.** It is possible to prove a converse form of Corollary 2.18: if $\mathfrak{d} < +\infty$ and for every value of $\mathfrak{m} > 0$ there exists a unique stationary solution in $\mathscr{M}_2(\mathbb{R},\mathfrak{m})$ then V satisfies (2.24).

?⟨ex:1⟩? **Example 2.20.** Let us choose $\beta(r) = \arctan r$, so that $E(r) = r\log\left(\frac{r}{\sqrt{1+r^2}}\right) - \arctan r$ and $E'(r) = \log\left(\frac{r}{\sqrt{1+r^2}}\right)$. Notice that $\mathfrak{d} = +\infty$. One can compute explicitly $H(v) = \frac{\mathrm{e}^{-v}}{\sqrt{1-\mathrm{e}^{-2v}}}$, for $v > 0$. If the potential is $V(x) = |x|^\alpha$, with $\alpha > 1$, the critical mass is defined by $\mathfrak{m}_\mathrm{c} = \int_\mathbb{R} \frac{\mathrm{e}^{-|x|^\alpha}}{\sqrt{1-\mathrm{e}^{-2|x|^\alpha}}}\,\mathrm{d}x$. It follows that $\mathfrak{m}_\mathrm{c} < +\infty$ if and only if $\alpha < 2$.

We find that

$$u_{\min}(x) = \frac{\mathrm{e}^{-|x|^\alpha+\mathfrak{v}}}{\sqrt{1-\mathrm{e}^{-2|x|^\alpha+2\mathfrak{v}}}}.$$

If $\alpha \geq 2$, for every value of the mass $\mathfrak{m}$, the unique minimum point, which is also the unique stationary solution, can not have a singular part, and it is bounded and positive. The same situation occurs when $\alpha < 2$ and $\mathfrak{m} < \mathfrak{m}_\mathrm{c}$. If $\alpha < 2$ and $\mathfrak{m} = \mathfrak{m}_\mathrm{c}$, then the unique stationary state is infinite at $x = 0$ but without a singular part, whereas for $\mathfrak{m} > \mathfrak{m}_\mathrm{c}$ the singular part is $\rho^\perp_{\min} = (\mathfrak{m} - \mathfrak{m}_\mathrm{c})\delta_0$.

?⟨ex:2⟩? **Example 2.21.** Let us choose $\beta(r) = \frac{r^2}{1+r^2}$. Then $E(r) = -r\arctan(1/r)$ and $E'(r) = \frac{r}{1+r^2} - \arctan(1/r)$. In this case $\mathfrak{d} = \frac{\pi}{2}$.

Let us observe that $E'(r)$ has the same behavior of $r \mapsto -1/r^3$ as $r \to +\infty$. Therefore $H(v)$ has the same behavior of $v \mapsto v^{-1/3}$ for $v \to 0^+$. Considering again the potential $V(x) = |x|^\alpha$, with $\alpha > 1$, it follows that $\mathfrak{m}_\mathrm{c} < +\infty$ if and only if $\alpha < 3$.

The support of the unique stationary state is $\{x \in \mathbb{R} : |x| \leq (\mathfrak{v} + \pi/2)^{1/\alpha}\}$ and it is compact for every value of $\mathfrak{m}$ and α.

Finally we show a measure $\rho \in \mathscr{M}_2^c(\mathbb{R},\mathfrak{m})$ satisfying $\mathcal{I}(\rho) = 0$ which is not of the form (2.18).

To this aim we consider the double well potential $V(x) = \pi(x-1)^2(x+1)^2$. Let $\mathfrak{m} > \mathfrak{m}_\mathrm{c}$. Defining $u(x) = H(V(x))$ for $x > 0$ and $u(x) = 0$ for $x \leq 0$, we observe that u is continuous on $\mathbb{R}$ with values in $[0, +\infty]$ and $\int_\mathbb{R} u(x)\,\mathrm{d}x = \mathfrak{m}_\mathrm{c}/2$. Consequently, the measure $\rho = u\mathscr{L}^1 + (\mathfrak{m} - \mathfrak{m}_\mathrm{c}/2)\delta_1$ belongs to $\mathscr{M}_2^c(\mathbb{R},\mathfrak{m})$, satisfies $\mathcal{I}(\rho) = 0$ but is not of the form (2.18).

We can construct a similar example when V has a local minimum greater than $V_{\min}$. For instance we can consider a potential V defined by $V(x) = 2\pi(x+1)^2 + 1$

for $x < -1/2$, $V(x) = 2\pi(x-1)^2$ for $x > 1/2$ and suitably defined in $[-1/2, 1/2]$ in order to satisfy the condition $V(x) > \pi/2$ and the λ-convexity assumption. Then the support of $\rho_{\min}$ is contained in $[-3/2,-1/2] \cup [1/2, 3/2]$. Let us define $u(x) = H(V(x)-1)$ for $x < 0$ and $u(x) = 0$ for $x \geq 0$, and $\rho = u\mathscr{L}^1 + (\mathfrak{m} - \tilde{\mathfrak{m}}_c)\delta_{-1}$, where $\tilde{\mathfrak{m}}_c := \mathfrak{m}_c - \int_{-\infty}^0 H(V(x))\,\mathrm{d}x$. Then $\rho \in \mathscr{M}_2^c(\mathbb{R}, \mathfrak{m})$, $\mathcal{I}(\rho) = 0$ but ρ is not of the form (2.18) and it is not a minimizer of $\mathcal{F}$.

Remark 2.22. We point out that the case $\mathfrak{d} = +\infty$ reveals some analogies with a diffusion which is linear near to 0. In this case we have the immediate strict positivity of the solution also starting from compactly supported initial data.

On the contrary, the case $\mathfrak{d} < +\infty$ corresponds to a slow diffusion near to 0. In this case, starting from compactly supported initial data the solution could remain compactly supported for all time and it may happen that as $t \to +\infty$ the solution converges to a stationary solution which is not a global minimum of $\mathcal{F}$.

⟨rem:singularex⟩ **Remark 2.23** (Examples of singular solutions)**.** Let $\rho := u\mathscr{L}^1 + \rho^\perp \in \mathscr{M}_2^c(\mathbb{R}, \mathfrak{m})$ be a stationary solution of (1.DDE) with $\rho^\perp \neq 0$: e.g., one can consider the case when $\mathfrak{m}_c < +\infty$ and take a minimizer of $\mathcal{F}$ with $\mathfrak{m} > \mathfrak{m}_c$. If $\tilde{\rho}_0 = \tilde{u}\mathscr{L}^1 + \rho^\perp \in \mathscr{M}_2^c(\mathbb{R}, \mathfrak{m})$ with $\tilde{u} \geq u$ then the comparison principle shows that the Wasserstein solution $\tilde{\rho}_t$ of (1.DDE) with initial datum $\tilde{\rho}_0$ is singular and its singular part is $\rho^\perp$ for every $t \geq 0$.

⟨subsec:asymptotic⟩ 2.6. **Asymptotic behaviour.** Let us first consider the case of a convex potential V. Here we can apply the general results about the asymptotic behavior for displacement convex functionals (see [2]).

Moreover, as we observed in Remark 2.13, the specific form of the functional $\mathcal{F}$ yields that it has only one minimizer $\rho_{\min}$ in each class $\mathscr{M}_2^c(\mathbb{R}, \mathfrak{m})$ which is also the unique stationary solution by Theorem 2.15 and Corollary 2.18: the study of the asymptotic behaviour is therefore greatly simplified.

?⟨thm:main6⟩? **Theorem 2.24** (Asymptotic behavior I: the convex case)**.** *Let us assume that* (2.int) *holds with $\alpha = 2$, that the potential V is convex (i.e.* (1.V) *is satisfied with $\lambda = 0$) and let $\rho_{\min}$ be the unique minimizer of $\mathcal{F}$ in $\mathscr{M}_2(\mathbb{R}, \mathfrak{m})$. If ρ is a Wasserstein solution to* (1.DDE) *in $\mathscr{M}_2(\mathbb{R}, \mathfrak{m})$ then ρ_t weakly converges to $\rho_{\min}$ as $t \to +\infty$ in the duality with continuous and bounded functions. Moreover, for every $t \in (0, +\infty)$*

$$\mathcal{F}(\rho_t) - \mathcal{F}(\rho_{\min}) \leq \frac{W_2^2(\rho_0, \rho_{\min})}{2t}, \quad \mathcal{I}(\rho_t) \leq \frac{W_2^2(\rho_0, \rho_{\min})}{t^2}.$$

If the potential V also satisfies (1.V) *with $\lambda > 0$, then for every $t > 0$ we have the exponential estimates*

$$W_2(\rho_t, \rho_{\min}) \leq \mathrm{e}^{-\lambda t} W_2(\rho_0, \rho_{\min}),$$

$$\mathcal{F}(\rho_t) - \mathcal{F}(\rho_{\min}) \leq \mathrm{e}^{-2\lambda t}\big(\mathcal{F}(\rho_0) - \mathcal{F}(\rho_{\min})\big), \quad \mathcal{I}(\rho_t) \leq \mathrm{e}^{-\lambda t} \frac{W_2^2(\rho_0, \rho_{\min})}{t^2}.$$

The last result concerns more general potentials V: a simple characterization of the asymptotic behaviour of a Wasserstein solution is possible only when there exists a unique stationary solution for (1.DDE) (which therefore coincides with the minimizer of $\mathcal{F}$): this is the case when $\mathfrak{d} = +\infty$ and (2.int) holds with $\alpha = 2$ and V satisfies (2.coer), or when $\mathfrak{d} < +\infty$ and V satisfies the conditions of Corollary 2.18.

⟨thm:main5⟩ **Theorem 2.25** (Asymptotic behavior II)**.** *Let us suppose that* (2.int) *holds, V satisfies* (2.coer) *and let us assume that there exists a unique stationary solution*

$\bar\rho \in \mathscr{M}_+(\mathbb{R}, \mathfrak{m})$ with $\mathcal{I}(\bar\rho) = 0$ ($\bar\rho$ is also the unique minimizer of $\mathcal{F}$ in $\mathscr{M}_+(\mathbb{R}, \mathfrak{m})$). If ρ is a Wasserstein solution to (1.DDE) in $\mathscr{M}_2(\mathbb{R}, \mathfrak{m})$ then

$$\rho_t \rightharpoonup \bar\rho \quad \text{weakly as } t \to +\infty, \qquad \lim_{t\uparrow+\infty} \mathcal{I}(\rho_t) = 0.$$

In particular the continuous density u_t converges to $\bar u$ uniformly on the compact sets of $\mathsf{D}(\bar u)$; if moreover the support of $\rho_0^\perp$ is compact and $\mathfrak{m} < \mathfrak{m}_c$, then there exists a finite time $T > 0$ such that $\rho_t \ll \mathscr{L}^1$ for every $t \geq T$.

⟨sec:Wass⟩ 3. **Wasserstein distance and differential calculus.** In this Section we recall the definition and the main properties of the Wasserstein distance and differential calculus in Wasserstein spaces (we refer the interested reader to [18], [19], [2] for more details). Also, the subdifferential of the energy functional $\mathcal{F}$ will be characterized and discussed.

3.1. **Transport of measures, Wasserstein distance, and differential calculus.** If $\rho \in \mathscr{M}_+(\mathbb{R}^d, \mathfrak{m})$ and $\boldsymbol{r} : \mathbb{R}^d \to \mathbb{R}^h$ is a Borel map, the push-forward of ρ through $\boldsymbol{r}$ is the measure $\mu = \boldsymbol{r}_\#\rho \in \mathscr{M}_+(\mathbb{R}^h, \mathfrak{m})$ defined by

$$\mu(A) := \rho(\boldsymbol{r}^{-1}(A)) \quad \text{for every Borel subset } A \subset \mathbb{R}^h.$$

It can also be characterized by the change-of-variable formula

$$\int_{\mathbb{R}^h} \varphi(y)\,\mathrm{d}\mu(y) = \int_{\mathbb{R}^d} \varphi(\boldsymbol{r}(x))\,\mathrm{d}\rho(x),$$

for every bounded or nonnegative Borel function $\varphi : \mathbb{R}^h \to \mathbb{R}$.
According to this definition, the marginals $\rho^i \in \mathscr{M}_+(\mathbb{R}, \mathfrak{m})$, $i = 1, 2$, of $\boldsymbol{\rho} \in \mathscr{M}_+(\mathbb{R} \times \mathbb{R}, \mathfrak{m})$ can be defined by $\rho^i = (\pi^i)_\#\boldsymbol{\rho}$, where $\pi^i(x^1, x^2) = x^i$ is the projection on the i-th component in $\mathbb{R}\times\mathbb{R}$. In this case we say that $\boldsymbol{\rho}$ is a coupling between ρ^1, ρ^2 and we denote by $\Gamma(\rho^1, \rho^2)$ the (weakly) closed convex subset of $\mathscr{M}_+(\mathbb{R}\times\mathbb{R}, \mathfrak{m})$ consisting of such couplings. We recall that a sequence of measures $\rho_n \in \mathscr{M}_+(\mathbb{R}^d, \mathfrak{m})$ weakly converges to $\rho \in \mathscr{M}_+(\mathbb{R}^d, \mathfrak{m})$ if $\lim_{n\to+\infty} \int_{\mathbb{R}^d} \varphi(y)\,\mathrm{d}\rho_n(y) = \int_{\mathbb{R}^d} \varphi(y)\,\mathrm{d}\rho(y)$ for every continuous, bounded function $\varphi \in C^0_{\mathrm{b}}(\mathbb{R}^d)$.

For every couple of measures $\rho^1, \rho^2 \in \mathscr{M}_2(\mathbb{R}, \mathfrak{m})$ the L^2-Wasserstein distance is defined by

$$W_2^2(\rho^1, \rho^2) := \min\Big\{ \int_{\mathbb{R}\times\mathbb{R}} |x^1 - x^2|^2\,\mathrm{d}\boldsymbol{\rho}(x^1, x^2) : \boldsymbol{\rho} \in \Gamma(\rho^1, \rho^2)\Big\}. \qquad (3.1)$$ eq:35

The space $\mathscr{M}_2(\mathbb{R}, \mathfrak{m})$ endowed with the distance W_2 is a complete separable metric space and a sequence ρ_n converges to ρ in $\mathscr{M}_2(\mathbb{R}, \mathfrak{m})$ if and only if (2.5) holds (see e.g. [18]).

There exists a unique optimal coupling $\boldsymbol{\rho}_{\mathrm{opt}}$ realizing the minimum in (3.1): it admits a nice representation in terms of the cumulative distribution functions M_{ρ^i} of ρ^1, ρ^2 and of their pseudo-inverses Y_{ρ^i}.

Let us first recall their definitions in the case of $\sigma \in \mathscr{M}_+(\mathbb{R}, \mathfrak{m})$

$$M_\sigma(x) := \sigma\big((-\infty, x]\big) \quad x \in \mathbb{R};$$

$$Y_\sigma(w) := \inf\Big\{x \in \mathbb{R} : M_\sigma(x) \geq w\Big\}, \quad w \in (0, \mathfrak{m}).$$

Notice that M_σ is a right-continuous and nondecreasing map from $\mathbb{R}$ to $[0, \mathfrak{m}]$; if we denote by $\lambda_{\mathfrak{m}} = \mathscr{L}^1|_{(0,\mathfrak{m})}$ the restriction of the Lebesgue measure to the interval

$(0,\mathfrak{m})$, it is possible to show that

$$\big(Y_{\rho^i}\big)_{\#}\lambda_{\mathfrak{m}} = \rho^i, \quad \big(Y_{\rho^1}, Y_{\rho^2}\big)_{\#}\lambda_{\mathfrak{m}} = \boldsymbol{\rho}_{\rm opt}$$

so that

$$W_2^2(\rho^1,\rho^2) = \int_0^{\mathfrak{m}} \big|Y_{\rho^1}(w) - Y_{\rho^2}(w)\big|^2 \,\mathrm{d}w = \|Y_{\rho^1} - Y_{\rho^2}\|^2_{L^2(0,\mathfrak{m})}.$$

The map $\rho \mapsto Y_\rho$ provides an isometry between $\mathscr{M}_2(\mathbb{R},\mathfrak{m})$ and the cone of nondecreasing function in $L^2(0,\mathfrak{m})$.

Displacement interpolation and displacement convexity. Let $\rho^0,\rho^1 \in \mathscr{M}_2(\mathbb{R},\mathfrak{m})$. Their *displacement interpolation* is the path $\rho^\vartheta \in \mathscr{M}_2(\mathbb{R},\mathfrak{m})$ with $\vartheta \in [0,1]$, defined by

$$\rho^\vartheta := \big((1-\vartheta)Y_{\rho^0} + \vartheta Y_{\rho^1}\big)_{\#}\lambda_{\mathfrak{m}} = \big((1-\vartheta)\pi^1 + \vartheta\pi^2\big)_{\#}\boldsymbol{\rho}_{\rm opt}. \tag{3.2}$$ eq:38

The curve $\vartheta \mapsto \rho^\vartheta$ is the unique (minimal, constant speed) geodesic connecting ρ^0 to ρ^1 in $\mathscr{M}_2(\mathbb{R},\mathfrak{m})$ and it corresponds to the segment connecting Y_{ρ^0} to Y_{ρ^1} in $L^2(0,\mathfrak{m})$.

We say that a functional $\mathcal{G} : \mathscr{M}_2(\mathbb{R},\mathfrak{m}) \to (-\infty,+\infty]$ is displacement λ-convex if for every ρ^0,ρ^1 in its proper domain we have

$$\mathcal{G}(\rho^\vartheta) \le (1-\vartheta)\mathcal{G}(\rho^0) + \vartheta\mathcal{G}(\rho^1) - \frac{\lambda}{2}\vartheta(1-\vartheta)W_2^2(\rho^0,\rho^1).$$

In the one-dimensional case, the displacement convexity of the internal functional $\mathcal{E}$ is equivalent to the convexity of the energy density E and it coincides with convexity along generalized geodesics (see [2, Definition 9.2.4]).

⟨prop:lscF⟩ **Proposition 3.1** (Displacement λ-convexity and lower semicontinuity of $\mathcal{F}$)**.** *$\mathcal{F}$ is lower semicontinuous with respect to the Wasserstein distance in $\mathscr{M}_2(\mathbb{R},\mathfrak{m})$ and displacement λ-convex. Moreover $\mathcal{F}$ satisfies the following coercivity property*

$$\inf\Big\{\mathcal{F}(\rho) : \rho \in \mathscr{M}_2(\mathbb{R},\mathfrak{m}), \quad \int_{\mathbb{R}} |x|^2 \,\mathrm{d}\rho(x) \le C\Big\} > -\infty \quad \textit{for every } C > 0.$$

Proof. Since E is convex and sublinear, by [9] it follows that $\mathcal{E}$ is lower semicontinuous with respect to the weak convergence in $\mathscr{M}_+(\mathbb{R},\mathfrak{m})$. In the one-dimensional case the convexity of E is equivalent to the displacement convexity. The functional $\rho \mapsto \int_{\mathbb{R}} V(x)\,\mathrm{d}\rho(x)$ is displacement λ-convex if and only if V is λ-convex; it is also lower semicontinuous with respect to convergence in $\mathscr{M}_2(\mathbb{R},\mathfrak{m})$ since V is continuous and quadratically bounded from below.

By (2.7) and the strict convexity of E we have that there exist $r_0 \in (0,1)$ and $c_1 > 0$ such that

$$E(r) \ge \begin{cases} c_1\, r\log r & \text{if } r < r_0 \\ \dfrac{E(r_0)}{r_0}\, r & \text{if } r \ge r_0 \end{cases}. \tag{3.3}$$ integrabilityE

It follows that if $\rho \in \mathscr{M}_2(\mathbb{R}, \mathfrak{m})$ then

$$\begin{aligned}
\mathcal{E}(\rho) &\geq c_1 \int_{\{0<u<r_0\}} u \log u \,\mathrm{d}x + \frac{E(r_0)}{r_0} \int_{\{u \geq r_0\}} u \,\mathrm{d}x \\
&\geq -c_1 \Big(\int_{\mathbb{R}} (1+|x|)^2 u \,\mathrm{d}x \Big)^{\frac{1}{2}} \Big(\int_{\{0<u<r_0\}} \frac{u \log^2 u}{(1+|x|)^2} \,\mathrm{d}x \Big)^{\frac{1}{2}} \\
&\quad + \frac{E(r_0)}{r_0} \int_{\{u \geq r_0\}} u \,\mathrm{d}x > -\infty.
\end{aligned} \tag{3.4}$$

integrabilityEbis

The coercivity property follows from (3.4) and (1.V). □

?⟨def:subdifferential⟩? **Definition 3.2** (Subdifferential and slope)**.** Let $\mathcal{G} : \mathscr{M}_2(\mathbb{R}, \mathfrak{m}) \to (-\infty, +\infty]$ be a displacement λ-convex and lower semcontinuous functional, let $\rho^0 \in \mathscr{M}_2(\mathbb{R}, \mathfrak{m})$ with $\mathcal{G}(\rho^0) < +\infty$ and $\boldsymbol{\xi} \in L^2(\rho^0)$. We say that $\boldsymbol{\xi}$ belongs to the W_2-subdifferential of $\mathcal{G}$ at the point ρ^0, and we write $\boldsymbol{\xi} \in \partial\mathcal{G}(\rho^0)$, if for every $\rho^1 \in \mathscr{M}_2(\mathbb{R}, \mathfrak{m})$ the optimal coupling $\boldsymbol{\rho}_{\mathrm{opt}}$ between ρ^0 and ρ^1 satisfies

$$\mathcal{G}(\rho^1) - \mathcal{G}(\rho^0) \geq \int_{\mathbb{R}\times\mathbb{R}} \Big(\boldsymbol{\xi}(x)(y-x) + \frac{\lambda}{2}|y-x|^2 \Big) \,\mathrm{d}\boldsymbol{\rho}_{\mathrm{opt}}(x,y). \tag{3.5}$$

eq:40

$\partial\mathcal{G}(\rho^0)$ is a closed convex (and possibly empty) subset of $L^2(\rho^0)$. We say that $\boldsymbol{\xi}$ is a strong W_2-subdifferential of $\mathcal{G}$ at the point ρ^0, and we write $\boldsymbol{\xi} \in \partial_S\mathcal{G}(\rho^0)$ if (3.5) holds for every coupling $\boldsymbol{\rho} \in \Gamma(\rho^0, \rho^1)$. When $\partial\mathcal{G}(\rho^0)$ is not empty we denote by $\partial^\circ\mathcal{G}(\rho^0) \in L^2(\rho^0)$ its (unique) element of minimal $L^2(\rho^0)$-norm.

The (metric) slope of $\mathcal{G}$ is defined as

$$|\partial\mathcal{G}|(\rho^0) = \limsup_{W_2(\rho,\rho^0)\to 0} \frac{\big(\mathcal{G}(\rho^0) - \mathcal{G}(\rho)\big)^+}{W_2(\rho,\rho^0)} = \sup_{\rho\neq\rho^0} \Big(\frac{\big(\mathcal{G}(\rho^0) - \mathcal{G}(\rho)\big)^+}{W_2(\rho,\rho^0)} + \frac{\lambda}{2} W_2(\rho,\rho^0) \Big)^+. \tag{3.6}$$

eq:42

For general displacement λ-convex functionals, one has

$$|\partial\mathcal{G}|(\rho) \leq \|\partial^\circ\mathcal{G}(\rho)\|_{L^2(\rho)}. \tag{3.7}$$

eq:76

When the functional $\mathcal{G}$ satisfies the regularity condition

$$|\partial\mathcal{G}|(\rho^0) < +\infty \quad \Rightarrow \quad \rho^0 \ll \mathscr{L}^1,$$

then the metric slope (3.6) coincides with $\|\partial^\circ\mathcal{G}(\rho^0)\|_{L^2(\rho^0)}$.

⟨subsec:slopefisher⟩ 3.2. **Slope and Fisher dissipation in the super-linear case.** Let us consider the perturbed family of energy densities $E^\varepsilon(r) := E(r) + \varepsilon r \log r$ associated to the energy functionals

$$\mathcal{F}^\varepsilon(\rho) = \int_{\mathbb{R}} E^\varepsilon(u(x)) \,\mathrm{d}x + \int_{\mathbb{R}} V^\varepsilon(x) \,\mathrm{d}\rho(x) \quad \text{if } \rho = u\mathscr{L}^1; \quad \mathcal{F}^\varepsilon(\rho) = +\infty \quad \text{if } \rho \not\ll \mathscr{L}^1,$$

where V^ε is the family of potentials satisfying the properties (2.13a), (2.13b) and (2.13c). Notice that $(rE^\varepsilon)''(r) = \beta'(r) + \varepsilon = (\beta^\varepsilon)'(r)$, where β^ε is defined in (2.12). Since E^ε has a super-linear growth, the slope $|\partial\mathcal{F}^\varepsilon|$ can be explicitly characterized [2, Theorem 10.4.13] and it coincides with the square root of the associated Fisher-dissipation

$$\mathcal{I}^\varepsilon(\rho) := \int_{\mathbb{R}} \Big| \frac{\partial_x \beta^\varepsilon(u)}{u} + (V^\varepsilon)' \Big|^2 u \,\mathrm{d}x \qquad \text{if } \rho = u\mathscr{L}^1, \quad u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}).$$

As usual $\mathcal{I}^\varepsilon(\rho) = +\infty$ if $u \notin W^{1,1}_{\rm loc}(\mathbb{R})$ or even $\rho \not\ll \mathscr{L}^1$. Thus we have

$$|\partial\mathcal{F}^\varepsilon|^2(\rho) = \mathcal{I}^\varepsilon(\rho)$$

and the minimal subdifferential $\boldsymbol{\xi}^\varepsilon = \partial^\circ\mathcal{F}^\varepsilon(\rho) \in L^2(\rho)$ is characterized as ([2, Theorem 10.4.13])

$$\boldsymbol{\xi}^\varepsilon\rho = \partial_x\beta^\varepsilon(u)\mathscr{L}^1 + \rho\,(V^\varepsilon)' \quad \text{if } \rho = u\mathscr{L}^1 \in D(\mathcal{I}^\varepsilon). \tag{3.8}$$ eq:48

In this case a strong subdifferential coincides with $\partial^\circ\mathcal{F}^\varepsilon(\rho)$.

⟨prop:strongmin⟩ **Proposition 3.3.** *If $\boldsymbol{\xi} \in \partial_S\mathcal{F}^\varepsilon(\rho)$, then $\boldsymbol{\xi} = \partial^\circ\mathcal{F}^\varepsilon(\rho)$.*

Proof. Let $\zeta \in C^1_{\rm c}(\mathbb{R})$. Denoting by $\mathfrak{i}$ the identity map in $\mathbb{R}$ and considering the measures $\rho_t := (\mathfrak{i} + t\zeta)_\#\rho$, for $t > 0$, we obtain that

$$\lim_{t\downarrow 0}\frac{1}{t}\big(\mathcal{F}^\varepsilon(\rho_t)-\mathcal{F}^\varepsilon(\rho)\big) = -\int_{\mathbb{R}}\beta^\varepsilon(u)\zeta'\,\mathrm{d}x+\int_{\mathbb{R}}(V^\varepsilon)'u\zeta\,\mathrm{d}x = \int_{\mathbb{R}}\big(\partial_x\beta^\varepsilon(u)+(V^\varepsilon)'u\big)\zeta\,\mathrm{d}x. \tag{3.9}$$ b

On the other hand, using the couplings $\boldsymbol{\rho}_t := (\mathfrak{i}, \mathfrak{i} + t\zeta)_\#\rho$ in (3.5) we obtain

$$\mathcal{F}^\varepsilon(\rho_t) - \mathcal{F}^\varepsilon(\rho) \geq t\int_{\mathbb{R}}\boldsymbol{\xi}\zeta u\,\mathrm{d}x + \frac{\lambda}{2}t^2\int_{\mathbb{R}}|\zeta|^2 u\,\mathrm{d}x. \tag{3.10}$$ a

Dividing by t, passing to the limit as $t \downarrow 0$ in (3.10) and combining with (3.9) we obtain that

$$\int_{\mathbb{R}}\big(\partial_x\beta^\varepsilon(u) + (V^\varepsilon)'u - \boldsymbol{\xi}u\big)\zeta\,\mathrm{d}x \geq 0, \qquad \forall\zeta \in C^1_{\rm c}(\mathbb{R})$$

and we conclude. □

The following compactness and lower semicontinuity property will play a crucial role in the sequel.

⟨thm:lsc-dissipation⟩ **Theorem 3.4.** *If $\rho^\varepsilon = u^\varepsilon\mathscr{L}^1 \in D(\mathcal{I}^\varepsilon)$, $\varepsilon > 0$, with $u^\varepsilon(x) > 0$ for all $x \in \mathbb{R}$, is a family of measures satisfying*

$$\rho^\varepsilon \rightharpoonup \rho \quad \textit{weakly in } \mathscr{M}_+(\mathbb{R},\mathfrak{m}) \textit{ as } \varepsilon\downarrow 0, \quad \limsup_{\varepsilon\downarrow 0}\mathcal{I}^\varepsilon(\rho^\varepsilon) < +\infty, \tag{3.11}$$ wconv

then we have

$$\rho = u\,\mathscr{L}^1 + \rho^\perp \in D(\mathcal{I}) \subset \mathscr{M}^{\rm c}_2(\mathbb{R},\mathfrak{m}),$$
$$\mathcal{I}(\rho) \leq \liminf_{\varepsilon\downarrow 0}\mathcal{I}^\varepsilon(\rho^\varepsilon),$$
$$u^\varepsilon \textit{ converges to } u \textit{ uniformly on compact sets of } \mathsf{D}(u). \tag{3.12}$$ eq:219

Moreover, if $\boldsymbol{\xi}^\varepsilon = \partial^o\mathcal{F}^\varepsilon(\rho^\varepsilon)$ as in (3.8), we have

$$\boldsymbol{\xi}^\varepsilon\rho^\varepsilon \rightharpoonup \boldsymbol{\xi}\rho = \partial_x\beta(u)\mathscr{L}^1 + V'\rho, \quad \textit{in the duality with } C^0_{\rm c}(\mathbb{R}).$$

Finally, if $f : [0,+\infty) \to \mathbb{R}$ is a continuous function such that $\lim_{r\uparrow+\infty}\dfrac{f(r)}{r} = f_\infty \in \mathbb{R}$, then

$$f(u^\varepsilon)\mathscr{L}^1 \rightharpoonup f(u)\mathscr{L}^1 + f_\infty\rho^\perp \quad \textit{in the duality with } C^0_{\rm c}(\mathbb{R}). \tag{3.13}$$ eq:65

Proof. Since $\mathcal{I}^\varepsilon(\rho^\varepsilon) = \int_{\mathbb{R}}|\boldsymbol{\xi}^\varepsilon|^2\mathrm{d}\rho^\varepsilon$, by (3.11) (see [2, Theorem 5.4.4]) there exists $\boldsymbol{\xi} \in L^2(\rho)$ such that

$$\boldsymbol{\xi}^\varepsilon\rho^\varepsilon \rightharpoonup \boldsymbol{\xi}\rho, \quad \text{in the duality with } C^0_{\rm c}(\mathbb{R}), \tag{3.14}$$ eq:401

and

$$\int_{\mathbb{R}} |\boldsymbol{\xi}|^2 \mathrm{d}\rho \le \liminf_{\varepsilon \downarrow 0} \int_{\mathbb{R}} |\boldsymbol{\xi}^\varepsilon|^2 \mathrm{d}\rho^\varepsilon.$$

From (3.11) and (2.13b) it follows that

$$(V^\varepsilon)' u^\varepsilon \mathscr{L}^1 \rightharpoonup V'\rho \quad \text{ in the duality with } C^0_{\mathrm{c}}(\mathbb{R}). \tag{3.15} \quad \boxed{\texttt{eq:402}}$$

Since by (3.8)

$$\partial_x \beta^\varepsilon(u^\varepsilon) \mathscr{L}^1 = \boldsymbol{\xi}^\varepsilon u^\varepsilon \mathscr{L}^1 - (V^\varepsilon)' u^\varepsilon \mathscr{L}^1, \tag{3.16} \quad \boxed{\texttt{eq:80}}$$

(3.14) and (3.15) imply that

$$\partial_x \beta^\varepsilon(u^\varepsilon) \mathscr{L}^1 \rightharpoonup \boldsymbol{\xi}\rho - V'\rho \quad \text{ in the duality with } C^0_{\mathrm{c}}(\mathbb{R}).$$

Let us now prove that $\rho \in \mathscr{M}^c_2(\mathbb{R}, \mathfrak{m})$, $\beta(u) \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$ and $\partial_x \beta(u) = \boldsymbol{\xi}\rho - V'\rho$. We introduce the functions

$$G(r) = \int_0^r \frac{\beta'(s)}{\sqrt{s}} \,\mathrm{d}s, \qquad G^\varepsilon(r) = G(r) + 2\varepsilon\sqrt{r}.$$

Since $u^\varepsilon \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$, $u^\varepsilon(x) > 0$ and G is locally Lipschitz in $(0, +\infty)$, we have

$$\partial_x G^\varepsilon(u^\varepsilon) = \frac{\partial_x(\beta^\varepsilon(u^\varepsilon))}{\sqrt{u^\varepsilon}}. \tag{3.17} \quad \boxed{\texttt{sss}}$$

Let $I = (a, b)$ be an arbitrary bounded interval of $\mathbb{R}$. Since $\beta'(0^+) < +\infty$ we have that $G^\varepsilon(r) \le M\sqrt{r}$, for some $M > 0$. Therefore

$$\sup_\varepsilon \int_I |G^\varepsilon(u^\varepsilon)|^2 \,\mathrm{d}x < +\infty. \tag{3.18} \quad \boxed{\texttt{bound-11}}$$

By (3.16) and (3.11) we have

$$\int_I \left| \frac{\partial_x(\beta^\varepsilon(u^\varepsilon))}{\sqrt{u^\varepsilon}} \right|^2 \mathrm{d}x = \int_I |\boldsymbol{\xi}^\varepsilon - (V^\varepsilon)'|^2 u^\varepsilon \mathrm{d}x \le 2 \int_I |\boldsymbol{\xi}^\varepsilon|^2 u^\varepsilon \mathrm{d}x + 2 \int_I |(V^\varepsilon)'|^2 u^\varepsilon \mathrm{d}x$$

so that

$$\sup_{\varepsilon > 0} \int_I \left| \frac{\partial_x(\beta^\varepsilon(u^\varepsilon))}{\sqrt{u^\varepsilon}} \right|^2 \mathrm{d}x < +\infty. \tag{3.19} \quad \boxed{\texttt{eq:84bis}}$$

By (3.18), (3.19) and (3.17), we infer that the family $\{G^\varepsilon(u^\varepsilon)\}_{\varepsilon>0}$ is bounded in $H^1_{\mathrm{loc}}(\mathbb{R})$. Thus, for every sequence $\varepsilon_j \to 0$ we can extract a sub-sequence, still denoted by $\{\varepsilon_j\}$, such that $G^{\varepsilon_j}(u^{\varepsilon_j})$ converges weakly in $H^1_{\mathrm{loc}}(\mathbb{R})$, and uniformly on the compact sets of $\mathbb{R}$, to some continuous function $g \in H^1_{\mathrm{loc}}(\mathbb{R})$. Since

$$\begin{aligned} \sup_\varepsilon \int_I |\partial_x(\beta^\varepsilon(u^\varepsilon))| \,\mathrm{d}x &= \sup_\varepsilon \int_I |\boldsymbol{\xi}^\varepsilon - (V^\varepsilon)'| u^\varepsilon \,\mathrm{d}x \\ &\le \sup_\varepsilon \sqrt{\mathfrak{m}} \left(\int_I |\boldsymbol{\xi}^\varepsilon - (V^\varepsilon)'|^2 u^\varepsilon \,\mathrm{d}x \right)^{\frac{1}{2}} < +\infty, \end{aligned}$$

and $\{\beta^\varepsilon(u^\varepsilon)\}_{\varepsilon>0}$ is bounded in $L^1(\mathbb{R})$, the family $\{\beta^\varepsilon(u^\varepsilon)\}_{\varepsilon>0}$ is bounded in $L^\infty(I)$. Therefore the family $\{\varepsilon u^\varepsilon = \beta^\varepsilon(u^\varepsilon) - \beta(u^\varepsilon)\}_{\varepsilon>0}$ is bounded in $L^\infty(I)$. Since $0 \le G^\varepsilon(u^\varepsilon) - G(u^\varepsilon) = 2\sqrt{\varepsilon}\sqrt{\varepsilon u^\varepsilon}$, we conclude that $G(u^{\varepsilon_j})$ converges uniformly on the compact sets of $\mathbb{R}$ to g, as $j \to +\infty$. The inequality

$$0 \le G \le G_\infty = \int_0^{+\infty} \frac{\beta'(s)}{\sqrt{s}} \mathrm{d}s,$$

together with the previous observations, gives $0 \le g \le G_\infty$. Since G is strictly increasing and $G_\infty < +\infty$, we can define the function

$$u(x) := \begin{cases} G^{-1}(g(x)) & \text{if } g(x) < G_\infty, \\ +\infty & \text{if } g(x) = G_\infty \end{cases}$$

which turns out to be continuous on the open set $\mathsf{D}(u) := \{x \in \mathbb{R} : g(x) < G_\infty\}$. Since $G(u^{\varepsilon_j}) \to g$ uniformly on the compact sets of $\mathbb{R}$, we have that $u^{\varepsilon_j} = G^{-1}(G(u^{\varepsilon_j})) \to u$ on the compact sets of $\mathsf{D}(u)$ and $u^{\varepsilon_j}(x) \to +\infty$ for every $x \in \mathbb{R} \setminus \mathsf{D}(u)$. By Fatou's Lemma we obtain that $u \in L^1(\mathbb{R})$ and $\mathscr{L}^1(\mathbb{R} \setminus \mathsf{D}(u)) = 0$. For every $\psi \in C^0_{\rm c}(\mathsf{D}(u))$, using (3.11) we have

$$\int_{\mathbb{R}} \psi(x) u(x) \, \mathrm{d}x = \lim_{j \to +\infty} \int_{\mathbb{R}} \psi(x) \, \mathrm{d}\rho^{\varepsilon_j} = \int_{\mathbb{R}} \psi(x) \, \mathrm{d}\rho.$$

Thus

$$\rho_{|\mathsf{D}(u)} = u\mathscr{L}^1 \quad \text{and} \quad \rho_{|\mathbb{R} \setminus \mathsf{D}(u)} = \rho^\perp. \tag{3.20}$$ eq:decomp

This shows that $\rho \in \mathscr{M}^c_2(\mathbb{R}, \mathfrak{m})$. Moreover, we deduce that the whole family u^ε converges to u uniformly on compact sets of $\mathsf{D}(u)$, as $\varepsilon \downarrow 0$.

For any bounded interval $I = (a, b)$, we have proved that $\{\beta^\varepsilon(u^\varepsilon)\}_{\varepsilon > 0}$ is bounded in $W^{1,1}(I)$. Then, by BV compactness (see e.g. [1]) there exists $h \in \mathrm{BV}_{\rm loc}(\mathbb{R})$ such that, up to subsequences as before, $\beta^\varepsilon(u^\varepsilon) \to h$ in $L^1_{\rm loc}(\mathbb{R})$ and $\mathscr{L}^1$-a.e. and $\partial_x \beta^\varepsilon(u^\varepsilon)\mathscr{L}^1 \rightharpoonup \partial_x h$ in duality with $C^0_{\rm c}(\mathbb{R})$. Since $0 \le \beta^\varepsilon(u^\varepsilon) - \beta(u^\varepsilon) = \varepsilon u^\varepsilon$ and $\varepsilon u^\varepsilon(x) \to 0$ pointwise in $\mathsf{D}(u)$, we have that $\beta(u^\varepsilon) \to h$, $\mathscr{L}^1$-a.e. On the other hand, by the continuity of β, $\beta(u^\varepsilon) \to \beta(u)$ $\mathscr{L}^1$-a.e. Hence $h = \beta(u)$. Moreover, by using (3.16), it is easy to see that $\partial_x \beta(u) \mathscr{L}^1 = \boldsymbol{\xi}\rho - V'\rho$. The last identity and (3.20) yield $\beta(u) \in \mathrm{BV}_{\rm loc}(\mathbb{R}) \cap W^{1,1}_{\rm loc}(\mathsf{D}(u))$.

Finally, we prove that $\beta(u) \in W^{1,1}_{\rm loc}(\mathbb{R})$ and $\partial_x(\beta(u)) = \partial_x(\beta(u))_{|\mathsf{D}(u)}$. Since $\mathsf{D}(u)$ is open, we can write

$$\mathsf{D}(u) = \bigcup_{n \in \mathbb{N}} (a_n, b_n)$$

where the intervals are pairwise disjoint; recalling that $\beta(u(a_n)) = \beta(u(b_n)) = \beta_\infty$, we have for every $\zeta \in C^\infty_{\rm c}(\mathbb{R})$

$$\begin{aligned}
\int_{\mathbb{R}} \zeta' \beta(u) \, \mathrm{d}x &= \int_{\mathsf{D}(u)} \zeta' \, \beta(u) \, \mathrm{d}x + \int_{\mathbb{R} \setminus \mathsf{D}(u)} \zeta' \, \beta(u) \, \mathrm{d}x \\
&= \sum_n \int_{a_n}^{b_n} \zeta' \beta(u) \, \mathrm{d}x + \beta_\infty \int_{\mathbb{R} \setminus \mathsf{D}(u)} \zeta' \, \mathrm{d}x \\
&= \sum_n \Big(- \int_{a_n}^{b_n} \zeta \, \partial_x(\beta(u)) \, \mathrm{d}x + \big(\zeta(b_n) - \zeta(a_n)\big)\beta_\infty \Big) + \beta_\infty \int_{\mathbb{R} \setminus \mathsf{D}(u)} \zeta' \, \mathrm{d}x \\
&= - \int_{\mathsf{D}(u)} \zeta \, \partial_x(\beta(u)) \, \mathrm{d}x + \sum_n \beta_\infty \int_{a_n}^{b_n} \zeta' \, \mathrm{d}x + \beta_\infty \int_{\mathbb{R} \setminus \mathsf{D}(u)} \zeta' \, \mathrm{d}x \\
&= - \int_{\mathsf{D}(u)} \zeta \, \partial_x(\beta(u)) \, \mathrm{d}x + \beta_\infty \int_{\mathbb{R}} \zeta' \, \mathrm{d}x = - \int_{\mathsf{D}(u)} \zeta \, \partial_x(\beta(u)) \, \mathrm{d}x.
\end{aligned}$$

We eventually prove (3.13). By possibly substituting $f(r)$ with $f(r)-f_\infty r$ it is not restrictive to assume $f_\infty=0$, i.e.

$$\lim_{r\to+\infty}\frac{f(r)}{r}=0\quad\text{or, equivalently,}\quad\forall\,\eta>0\;\exists\,M_\eta:\quad |f(r)|\le M_\eta+\eta r\quad\forall r\ge 0. \tag{3.21}$$ eq:74

Property (3.21) easily shows that the family $\{f(u^\varepsilon)\}_{\varepsilon>0}$ is equi-integrable in $\mathbb{R}$: for every $\delta>0$ and choosing $\eta:=\delta/2\mathfrak{m}$, every Borel set A with measure $\mathscr{L}^1(A)\le\delta/2M_\eta$ satisfies

$$\int_A|f(u^\varepsilon(x))|\,\mathrm{d}x\le\int_A\Big(M_\eta+\eta u^\varepsilon(x)\Big)\,\mathrm{d}x\le M_\eta\,\mathscr{L}^1(A)+\eta\,\mathfrak{m}\le\delta\quad\text{for every }\varepsilon>0.$$

The previous equi-integrability estimate and the tightness of ρ^ε given by (3.11) show that the family $f(u^\varepsilon)$ is weakly compact in $L^1(\mathbb{R})$. On the other hand, $f(u^\varepsilon)\to f(u)$ locally uniformly in $\mathsf{D}(u)$. Since $\mathscr{L}^1(\mathbb{R}\setminus\mathsf{D}(u))=0$ it follows that $f(u)$ is also the weak limit of $f(u^\varepsilon)$ in $L^1(\mathbb{R})$.

□

By a similar and even simpler argument it is possible to prove the following lower semicontinuity result for the Fisher dissipation $\mathcal{I}$ with respect to weak convergence. Lower semicontinuity with respect to Wasserstein convergence will follow by (3.6) and the representation (3.6) of the metric slope for a displacement λ-convex functional [2, Corollary 2.4.10].

⟨thm:lsc-dissipation2⟩ **Theorem 3.5** (Lower semicontinuity of $\mathcal{I}$)**.** *If $\rho_n=u_n\,\mathscr{L}^1+\rho_n^\perp\in D(\mathcal{I})$ is a sequence of measures weakly convergent to a measure ρ and satisfying*

$$\limsup_{n\to+\infty}\mathcal{I}(\rho_n)<+\infty,$$

then we have

$$\rho=u\,\mathscr{L}^1+\rho^\perp\in D(\mathcal{I})\subset\mathscr{M}_2^{\mathrm{c}}(\mathbb{R},\mathfrak{m}),\qquad \mathcal{I}(\rho)\le\liminf_{n\to+\infty}\mathcal{I}(\rho_n).$$

Moreover

u_n converges to u uniformly on compact sets of $\mathsf{D}(u)$.

3.3. Characterization of the Wasserstein subdifferential of $\mathcal{F}$.

⟨th:charsubdiff⟩ **Theorem 3.6** (Characterization of $\partial\mathcal{F}$)**.** *Let $\rho=u\mathscr{L}^1+\rho^\perp\in\mathscr{M}_2(\mathbb{R},\mathfrak{m})$ with $\mathcal{F}(\rho)<+\infty$ and $\boldsymbol{\xi}\in L^2(\rho)$.*
$\boldsymbol{\xi}=\partial^o\mathcal{F}(\rho)$ (and, in particular, $\partial\mathcal{F}(\rho)$ is not empty) if and only if

$$\rho\in\mathscr{M}_2^{\mathrm{c}}(\mathbb{R},\mathfrak{m}),\quad \mathcal{I}(\rho)<+\infty,\quad \boldsymbol{\xi}\rho=\partial_x\beta(u)\mathscr{L}^1+V'\rho. \tag{3.22}$$ eq:50

In this case

$$|\partial\mathcal{F}|^2(\rho)=\int_{\mathbb{R}}|\boldsymbol{\xi}|^2\,\mathrm{d}\rho=\mathcal{I}(\rho).$$

Proof. Let us first suppose that $\boldsymbol{\xi}=\partial^\circ\mathcal{F}(\rho)$ and we prove that (3.22) holds. Since in particular $\partial^\circ\mathcal{F}(\rho)$ is not empty, we apply Proposition 3.8. By (3.33), (3.34) combined with Proposition 3.3 and (3.36) we can apply Theorem 3.4 and we conclude that (3.22) holds and

$$\mathcal{I}(\rho)=\int_{\mathbb{R}}|\boldsymbol{\xi}|^2\,\mathrm{d}\rho\le|\partial\mathcal{F}|^2(\rho). \tag{3.23}$$?eq:78?

Let us now suppose that $\rho, \boldsymbol{\xi}$ satisfy (3.22) and let us prove that $\boldsymbol{\xi} \in \partial\mathcal{F}(\rho)$, i.e. (3.5) holds with $\rho^0 := \rho$; in particular, recalling (3.7), this also shows that

$$|\partial\mathcal{F}|^2(\rho) \le \int_{\mathbb{R}} |\boldsymbol{\xi}|^2 \,\mathrm{d}\rho = \mathcal{I}(\rho) < +\infty. \tag{3.24}$$?eq:77?

It is not restrictive to assume $\lambda = 0$. By a standard regularization and stability of the optimal couplings with respect to weak convergence, we can also suppose that $\rho^1 = u^1\,\mathscr{L}^1$ with $u^1 \in C^1(\mathbb{R})$ supported in the bounded interval $[a,b]$ with $u^1(x) > 0$ for every $x \in (a,b)$. In this case $M_{\rho^1} \in C^2(\mathbb{R})$, the monotone rearrangement map $Y_{\rho^1} \in C^0([0,\mathfrak{m}])$ satisfies $Y_{\rho^1}(0) = a,\ Y_{\rho^1}(\mathfrak{m}) = b$ and its restriction to $(0,\mathfrak{m})$ is of class C^2. We set

$$\begin{cases} \boldsymbol{r}(x) := Y_{\rho^1}(M_\rho(x)),\ \boldsymbol{r}^\vartheta(x) := (1-\vartheta)x + \vartheta\boldsymbol{r}(x) \\ \boldsymbol{s}(y) := Y_\rho(M_{\rho^1}(y)),\ \boldsymbol{s}^\vartheta(y) := \vartheta y + (1-\vartheta)\boldsymbol{s}(y) \end{cases} \quad \text{for every } x,y \in \mathbb{R},\ \vartheta \in [0,1],$$

and we observe that $\boldsymbol{r}^\vartheta|_{\mathsf{D}(u)}$ is C^1. We introduce the sets

$$\begin{aligned} &\mathsf{D} := \mathsf{D}(u), \quad \mathsf{D}_+ := \{x \in \mathsf{D}(u) : u(x) > 0\}, \quad \tilde{\mathsf{D}} := \mathbb{R} \setminus \mathsf{D}, \\ &\mathsf{G} := \boldsymbol{r}(\mathsf{D}) = \boldsymbol{r}(\mathsf{D}_+), \quad \tilde{\mathsf{G}} := (a,b) \setminus \mathsf{G}, \end{aligned}$$

and we have

$$\begin{gathered} (\boldsymbol{\rho}_{\rm opt})|_{\mathsf{D}\times\mathbb{R}} = (\boldsymbol{i} \times \boldsymbol{r}^1)_\#(u\mathscr{L}^1) = (\boldsymbol{s}^0 \times \boldsymbol{i})_\#(u^1\mathscr{L}^1|_{\mathsf{G}}), \\ (\boldsymbol{\rho}_{\rm opt})|_{\tilde{\mathsf{D}}\times\mathbb{R}} = (\boldsymbol{s}^0 \times \boldsymbol{i})_\#(u^1\mathscr{L}^1|_{\tilde{\mathsf{G}}}) \\ \rho^\vartheta|_{\boldsymbol{r}^\vartheta(\mathsf{D})} = \boldsymbol{r}^\vartheta_\#(u\mathscr{L}^1), \quad \rho^\vartheta|_{\mathbb{R}\setminus\boldsymbol{r}^\vartheta(\mathsf{D})} = \boldsymbol{s}^\vartheta_\#(u^1\mathscr{L}^1|_{\tilde{\mathsf{G}}}), \\ u^\vartheta(\boldsymbol{r}^\vartheta(x))(\boldsymbol{r}^\vartheta)'(x) = u(x), \quad u^\vartheta(\boldsymbol{s}^\vartheta(y))(\boldsymbol{s}^\vartheta)'(y) = u^1(y) \text{ for every } x \in \mathsf{D},\ y \in (a,b). \end{gathered}$$

Since $(\boldsymbol{s}^0)'(y) = 0$ for every $y \in \tilde{G}$

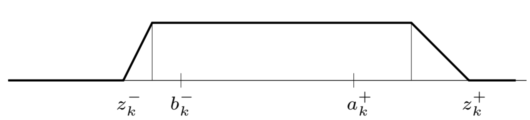
$$\mathcal{E}(\rho^\vartheta) = \int_{\mathsf{D}_+} E\Big(\frac{u(x)}{(1-\vartheta) + \vartheta\boldsymbol{r}'(x)}\Big)(1 - \vartheta + \vartheta\boldsymbol{r}'(x))\,\mathrm{d}x + \int_{\tilde{\mathsf{G}}} E\Big(\frac{u^1(y)}{\vartheta}\Big)\vartheta\,\mathrm{d}y.$$

Therefore, owing to the convexity of the maps $\vartheta \mapsto \mathcal{E}(\rho^\vartheta)$ and $s \mapsto sE(\alpha/s)$ for every $\alpha \ge 0$,

$$+\infty > \mathcal{E}(\rho^1) - \mathcal{E}(\rho) \ge \lim_{\vartheta\downarrow 0} \vartheta^{-1}\Big(\mathcal{E}(\rho^\vartheta) - \mathcal{E}(\rho)\Big) = -\int_{\mathsf{D}} \beta(u)(\boldsymbol{r}' - 1)\,\mathrm{d}x - \beta_\infty\mathscr{L}^1(\tilde{\mathsf{G}}).$$

Let us now choose two sequences $z_k^- \to -\infty$, $z_k^+ \to +\infty$ in D. Let (a_k^-, b_k^-) and (a_k^+, b_k^+) be the connected components of D containing z_k^- and z_k^+ respectively, and let $I_k^n := (a_k^n, b_k^n)$, $n \in \Lambda_k \subset \mathbb{N}$ be the (at most countable) connected components of $\mathsf{D} \cap (b_k^-, a_k^+)$. We consider a continuous function $\psi_k : \mathbb{R} \to [0,1]$ satisfying

$$\begin{aligned} &\psi_k(x) = 0 \text{ in } \mathbb{R} \setminus [z_k^-, z_k^+], \quad \psi_k(x) \equiv 1 \text{ if } x \in [\tfrac12(z_k^- + b_k^-), \tfrac12(z_k^+ + a_k^+)], \\ &\psi_k|_{[z_k^-, z_k^+]} \text{ is concave.} \end{aligned}$$



For sufficiently big k we have $\psi_k \equiv 1$ on (a,b). Then

$$\begin{aligned}\beta(u(x))(\boldsymbol{r}(x)-x)\psi_k'(x) &\geq \beta(u(x))\big(\psi_k(\boldsymbol{r}(x))-\psi_k(x)\big)\\ &\geq \beta(u(x))(1-\psi_k(x)) \geq 0 \quad \text{for every } x\in[z_k^-,z_k^+];\end{aligned}$$

$$-\beta_\infty \mathscr{L}^1(\tilde{\mathsf{G}}) = \lim_{k\to\infty} \mathscr{L}^1(\tilde{\mathsf{G}}\cap(\boldsymbol{r}(b_k^-),\boldsymbol{r}(a_k^+))).$$

Moreover

$$-\int_{\mathsf{D}} \beta(u)(\boldsymbol{r}'-1)\,\mathrm{d}x = \lim_{k\uparrow+\infty} -\int_{\mathsf{D}} \beta(u)(\boldsymbol{r}'-1)\,\psi_k(x)\,\mathrm{d}x$$

and

$$\begin{aligned}-\int_{\mathsf{D}}\beta(u)(\boldsymbol{r}'-1)\,\psi_k(x)\,\mathrm{d}x &\geq \int_{a_k^+}^{z_k^+} \partial_x\beta(u)\,(\boldsymbol{r}(x)-x)\psi_k(x)\,\mathrm{d}x\\ &+\int_{z_k^-}^{b_k^-}\partial_x\beta(u)\,(\boldsymbol{r}(x)-x)\psi_k(x)\,\mathrm{d}x + \sum_{n\in\Lambda_k}\int_{a_k^n}^{b_k^n}\partial_x\beta(u)\,(\boldsymbol{r}(x)-x)\,\mathrm{d}x\\ &+\beta_\infty\Big[(\boldsymbol{r}(a_k^+)-a_k^+)-(\boldsymbol{r}(b_k^-)-b_k^-)-\sum_{n\in\Lambda_k}\big(\boldsymbol{r}(b_k^n)-\boldsymbol{r}(a_k^n)-(b_k^n-a_k^n)\big)\Big]\\ &=\int_{\mathbb{R}}\partial_x\beta(u)(\boldsymbol{r}(x)-x)\psi_k(x)\,\mathrm{d}x+\beta_\infty\mathscr{L}^1(\tilde{\mathsf{G}}\cap(\boldsymbol{r}(b_k^-),\boldsymbol{r}(a_k^+))),\end{aligned}$$

where we used the fact that $\mathscr{L}^1\big((b_k^-,a_k^+)\setminus\mathsf{D}\big)=0$.

Combining all these estimates we get

$$+\infty > \mathcal{E}(\rho^1)-\mathcal{E}(\rho) \geq \limsup_{k\uparrow+\infty}\int_{\mathbb{R}}\partial_x\beta(u)(\boldsymbol{r}(x)-x)\psi_k(x)\,\mathrm{d}x.$$

On the other hand

$$\begin{aligned}+\infty > \int_{\mathbb{R}} V(y)\,\mathrm{d}\rho^1(y)-\int_{\mathbb{R}}V(x)\,\mathrm{d}\rho(x) &= \int_{\mathbb{R}\times\mathbb{R}}\Big(V(y)-V(x)\Big)\,\mathrm{d}\boldsymbol{\rho}_{\mathrm{opt}}(x,y)\\ \geq \int_{\mathbb{R}\times\mathbb{R}} V'(x)(y-x)\,\mathrm{d}\boldsymbol{\rho}_{\mathrm{opt}}(x,y) &\geq \limsup_{k\uparrow+\infty}\int_{\mathbb{R}\times\mathbb{R}}V'(x)(y-x)\psi_k(x)\,\mathrm{d}\boldsymbol{\rho}_{\mathrm{opt}}(x,y)\end{aligned}$$

Summing up the two contributions we have

$$\begin{aligned}\mathcal{F}(\rho^1)-\mathcal{F}(\rho) &\geq \limsup_{k\uparrow+\infty}\int_{\mathbb{R}\times\mathbb{R}}\boldsymbol{\xi}(x)(y-x)\psi_k(x)\,\mathrm{d}\boldsymbol{\rho}_{\mathrm{opt}}(x,y)\\ &=\int_{\mathbb{R}\times\mathbb{R}}\boldsymbol{\xi}(x)(y-x)\,\mathrm{d}\boldsymbol{\rho}_{\mathrm{opt}}(x,y).\end{aligned}$$

Since $\boldsymbol{\xi}\in\partial\mathcal{F}(\rho)$ is not empty, let $\boldsymbol{\eta}=\partial^\circ\mathcal{F}(\rho)$. For the first part of the proof we have that $\boldsymbol{\eta}$ satisfies (3.22) and then $\boldsymbol{\xi}=\boldsymbol{\eta}$ in $L^2(\rho)$.

□

3.4. **Γ-convergence of $\mathcal{F}^\varepsilon$ to $\mathcal{F}$.** The following lemma shows that the family of functionals $\mathcal{F}^\varepsilon$ defined in Subsection 3.2 converges to $\mathcal{F}$ in a kind of Γ-convergence way (with different convergence in the lim inf and the lim sup inequalities). We refer to [5] and [8] for the basic theory on Γ-convergence.

⟨le:Gamma-convergence⟩ **Lemma 3.7.** *As $\varepsilon\downarrow 0$ the family of functionals $\mathcal{F}^\varepsilon$ converge to $\mathcal{F}$ according to the following two properties:*

(i) *For every family* $\{\rho^\varepsilon\}\subset \mathscr{M}_2(\mathbb{R},\mathfrak{m})$ *such that* $\rho^\varepsilon \rightharpoonup \rho$*, as* $\varepsilon\downarrow 0$*, in duality with* $C^0_{\rm b}(\mathbb{R})$*, and*

$$M_2 := \limsup_{\varepsilon\downarrow 0} \mathfrak{m}_2(\rho^\varepsilon) < +\infty, \tag{3.25}$$ boundsecmom

one has

$$\liminf_{\varepsilon\downarrow 0} \mathcal{F}^\varepsilon(\rho^\varepsilon) \ge \mathcal{F}(\rho).$$

(ii) *For every* $\rho\in\mathscr{M}_2(\mathbb{R},\mathfrak{m})$ *there exists a family of measures* $\{\rho^\varepsilon\}\subset\mathscr{M}_2(\mathbb{R},\mathfrak{m})$ *such that* $W_2(\rho^\varepsilon,\rho)\to 0$ *as* $\varepsilon\downarrow 0$ *and*

$$\limsup_{\varepsilon\downarrow 0} \mathcal{F}^\varepsilon(\rho^\varepsilon) \le \mathcal{F}(\rho).$$

Proof. (i). In order to prove the "liminf" inequality for the potential energy $\mathcal{V}^\varepsilon(\rho) := \int_{\mathbb{R}} V^\varepsilon\,\mathrm{d}\rho$ under weak convergence and (3.25), we observe that from (2.13c) and (2.13b), for every $\delta>0$ there exist $R>\delta^{-1}$ and $\varepsilon_0>0$ such that

$$V^\varepsilon(x) \ge -\delta|x|^2 \quad \text{for every } x\in\mathbb{R}\setminus[-R,R],$$
$$V^\varepsilon(x) \ge V(x)-\delta \quad \text{for every } x\in[-2R,2R],\ 0<\varepsilon<\varepsilon_0.$$

Then for every $0<\varepsilon<\varepsilon_0$ and every smooth function

$$\chi:\mathbb{R}\to[0,1] \text{ with } \chi(x)=1 \text{ if } |x|\le 1 \text{ and } \chi(x)=0 \text{ if } |x|\ge 2 \tag{3.26}$$ eq:88

we have

$$\begin{aligned}\mathcal{V}^\varepsilon(\rho^\varepsilon) &= \int_{\mathbb{R}} V^\varepsilon(x)\chi(x/R)\,\mathrm{d}\rho^\varepsilon + \int_{\mathbb{R}} V^\varepsilon(x)(1-\chi(x/R))\,\mathrm{d}\rho^\varepsilon\\ &\ge \int_{\mathbb{R}} \Big(V(x)-\delta\Big)\chi(x/R)\,\mathrm{d}\rho^\varepsilon - \delta\int_{\mathbb{R}}|x|^2\,\mathrm{d}\rho^\varepsilon\end{aligned}$$

so that

$$\liminf_{\varepsilon\to 0}\mathcal{V}^\varepsilon(\rho^\varepsilon) \ge \int_{\mathbb{R}} \chi(x/R)V(x)\,\mathrm{d}\rho(x) - \delta(\mathfrak{m}+M_2).$$

Since $R\ge\delta^{-1}$ and the previous inequality is valid for arbitrary $\delta>0$, passing to the limit as $\delta\to 0$ we obtain

$$\liminf_{\varepsilon\to 0}\mathcal{V}^\varepsilon(\rho^\varepsilon)\ge\mathcal{V}(\rho).$$

Let us now prove the "liminf" inequality for $\mathcal{E}^\varepsilon$: recalling the usual decomposition $\rho^\varepsilon = u^\varepsilon\mathscr{L}^1 + (\rho^\varepsilon)^\perp$, thanks to the definition of $\mathcal{E}^\varepsilon$ we get

$$\mathcal{E}^\varepsilon(\rho^\varepsilon) = \mathcal{E}(\rho^\varepsilon) + \varepsilon\int_{\mathbb{R}} u^\varepsilon\log u^\varepsilon\mathrm{d}x \ge \mathcal{E}(\rho^\varepsilon) + \varepsilon\int_{\{0<u^\varepsilon<1\}} u^\varepsilon\log u^\varepsilon\mathrm{d}x.$$

By Cauchy-Schwarz inequality and (3.25) we obtain

$$\begin{aligned}&\limsup_{\varepsilon\downarrow 0}\left|\int_{\{0<u^\varepsilon<1\}} u^\varepsilon\log u^\varepsilon\,\mathrm{d}x\right|\\ &\le \limsup_{\varepsilon\downarrow 0}\left(\int_{\mathbb{R}}(1+|x|)^2u^\varepsilon\,\mathrm{d}x\right)^{\frac12}\left(\int_{\{0<u^\varepsilon<1\}}\frac{u^\varepsilon\log^2 u^\varepsilon}{(1+|x|)^2}\,\mathrm{d}x\right)^{\frac12} < +\infty.\end{aligned}$$

Hence

$$\liminf_{\varepsilon\downarrow 0}\mathcal{E}^\varepsilon(\rho^\varepsilon) \ge \liminf_{\varepsilon\downarrow 0}\mathcal{E}(\rho^\varepsilon),$$

and (i) follows by the lower semicontinuity of $\mathcal{E}$ with respect to the weak convergence.

(ii). Let $\rho = u\mathscr{L}^1 + \rho^\perp \in \mathscr{M}_2(\mathbb{R}, \mathfrak{m})$ with $\mathcal{F}(\rho) < +\infty$ (the case $\mathcal{F}(\rho) = +\infty$ is trivial). Defining $c^\varepsilon := \mathfrak{m}/\rho([-1/\varepsilon, 1/\varepsilon])$, and $h^\varepsilon := c^\varepsilon \chi_{[-1/\varepsilon,1/\varepsilon]}$, we set

$$\rho^\varepsilon := h^\varepsilon \rho = h^\varepsilon u \mathscr{L}^1 + h^\varepsilon \rho^\perp.$$

Since $\lim_{\varepsilon\downarrow 0} h^\varepsilon = 1$ pointwise, for every $W : \mathbb{R} \to \mathbb{R}$ such that $\int_{\mathbb{R}} |W(x)| \,\mathrm{d}\rho(x) < +\infty$, the dominated convergence theorem shows that

$$\lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}} W(x) \mathrm{d}\rho^\varepsilon(x) = \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}} W(x) h^\varepsilon(x) \mathrm{d}\rho(x) = \int_{\mathbb{R}} W(x)\, \mathrm{d}\rho(x). \tag{3.27}$$

contW

In particular, choosing $W = \varphi$ as in (2.5) we obtain that $W_2(\rho^\varepsilon, \rho) \to 0$. If $\int_{\mathbb{R}} |V(x)| \,\mathrm{d}\rho(x) < +\infty$, then we may use (3.27) with $W = V$ so that

$$\lim_{\varepsilon\downarrow 0} \mathcal{V}(\rho^\varepsilon) = \mathcal{V}(\rho).$$

If $\int_{\mathbb{R}} |V(x)| \,\mathrm{d}\rho(x) = +\infty$, which is equivalent to $\int_{\mathbb{R}} V(x) \,\mathrm{d}\rho(x) = +\infty$ by condition (1.V), recalling also (3.4), we obtain $\mathcal{F}(\rho) = +\infty$. This was however the trivial case. On the other hand, since $E \le 0$ is continuous, by Fatou's Lemma we have

$$\limsup_{\varepsilon\downarrow 0} \int_{\mathbb{R}} E(u^\varepsilon(x)) dx \le \int_{\mathbb{R}} E(u(x)) dx.$$

Denoting by $\mathscr{M}_+^{\mathrm{comp}}(\mathbb{R}, \mathfrak{m})$ the set of nonnegative measures with compact support and total mass $\mathfrak{m}$, we have just proved that

$$\forall\, \rho \in \mathscr{M}_2(\mathbb{R}, \mathfrak{m}) \cap D(\mathcal{F}) \;\; \exists \{\rho^\varepsilon\} \subset \mathscr{M}_+^{\mathrm{comp}}(\mathbb{R}, \mathfrak{m}) : W_2(\rho^\varepsilon, \rho) \to 0,\; \lim_{\varepsilon\downarrow 0} \mathcal{F}(\rho^\varepsilon) = \mathcal{F}(\rho).$$

A standard density argument for Γ-convergence (see e.g. Remark 1.29 in [5]) shows that (ii) can be reduced to prove

$$\forall\, \rho \in \mathscr{M}_+^{\mathrm{comp}}(\mathbb{R}, \mathfrak{m}),\; \exists \{\rho^\varepsilon\} \subset \mathscr{M}_+^{\mathrm{comp}}(\mathbb{R}, \mathfrak{m}) : W_2(\rho^\varepsilon, \rho) \to 0,\; \limsup_{\varepsilon\downarrow 0} \mathcal{F}^\varepsilon(\rho^\varepsilon) \le \mathcal{F}(\rho). \tag{3.28}$$

GammaLimsupComp

Let $\rho = u\mathscr{L}^1 + \rho^\perp \in \mathscr{M}_+^{\mathrm{comp}}(\mathbb{R}, \mathfrak{m})$; denoting by $k^\varepsilon = \varepsilon^{-1} k(\cdot/\varepsilon)$ a standard family of symmetric and nonnegative mollifiers with support $[-\varepsilon, \varepsilon]$, we set $u^\varepsilon(x) = (k^\varepsilon * \rho)(x) = \int_{\mathbb{R}} k^\varepsilon(x - y) \,\mathrm{d}\rho(y)$ and $\rho^\varepsilon = u^\varepsilon \mathscr{L}^1$. Then $\rho^\varepsilon \in \mathscr{M}_+^{\mathrm{comp}}(\mathbb{R}, \mathfrak{m})$ and the supports of ρ^ε are all contained in a compact set of $\mathbb{R}$. By definition of convolution and Fubini's theorem we have

$$\begin{aligned} \int_{\mathbb{R}} \varphi(x) \,\mathrm{d}\rho^\varepsilon(x) &= \int_{\mathbb{R}} \varphi(x) \int_{\mathbb{R}} k^\varepsilon(x - y) \,\mathrm{d}\rho(y) \,\mathrm{d}x \\ &= \int_{\mathrm{supp}(\rho)} \int_{[-1,1]} \varphi(y + \varepsilon z) k(z) \,\mathrm{d}z \,\mathrm{d}\rho(y). \end{aligned}$$

If φ is as in (2.5), then by the dominated convergence theorem it follows that $W_2(\rho^\varepsilon, \rho) \to 0$. Choosing $\varphi = V$, by the continuity of V

$$\lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}} V(x) \mathrm{d}\rho^\varepsilon(x) = \int_{\mathbb{R}} V(x) \mathrm{d}\rho(x). \tag{3.29}$$

contV2

Moreover, writing

$$\int_{\mathbb{R}} V^\varepsilon(x) \mathrm{d}\rho^\varepsilon(x) - \int_{\mathbb{R}} V(x) \mathrm{d}\rho(x) = \int_{\mathbb{R}} \big(V^\varepsilon(x) - V(x)\big) \mathrm{d}\rho^\varepsilon(x) + \int_{\mathbb{R}} V(x) \mathrm{d}(\rho^\varepsilon - \rho)(x), \tag{3.30}$$

contVeps

we easily have that $\lim_{\varepsilon\downarrow 0}\mathcal{V}^\varepsilon(\rho^\varepsilon)=\mathcal{V}(\rho)$, as the two integrals converge both to 0, by (2.13b) and (3.29), respectively.
Recalling that E is strictly decreasing and applying Jensen's inequality to the probability measure $k^\varepsilon(x-y)\mathscr{L}^1(y)$ and the convex function E we get

$$E(u^\varepsilon(x))=E\Big(\int_{\mathbb{R}}k^\varepsilon(x-y)\,\mathrm{d}\rho(y)\Big)\le E\Big(\int_{\mathbb{R}}u(y)k^\varepsilon(x-y)\,\mathrm{d}y\Big)$$
$$\le\int_{\mathbb{R}}E(u(y))k^\varepsilon(x-y)\,\mathrm{d}y.$$

Integrating with respect to x and using Fubini's theorem we obtain

$$\int_{\mathbb{R}}E(u^\varepsilon(x))\,\mathrm{d}x\le\int_{\mathbb{R}}E(u(x))\,\mathrm{d}x. \tag{3.31}$$ supE

Finally, since $k^\varepsilon\le 1/\varepsilon$ and $u^\varepsilon(x)\le\varepsilon^{-1}\mathfrak{m}$, we have

$$\varepsilon\int_{\mathbb{R}}u^\varepsilon\log u^\varepsilon\,\mathrm{d}x\le\mathfrak{m}\,\varepsilon\log\frac{\mathfrak{m}}{\varepsilon}. \tag{3.32}$$ estEntr

Therefore (3.30), (3.31) and (3.32) yield (3.28). □

The Γ-convergence Lemma 3.7 implies the following Proposition.

⟨thm:GammaConv⟩ **Proposition 3.8.** *For every ρ satisfying $\partial\mathcal{F}(\rho)\neq\emptyset$ there exist sequences $\varepsilon_n\downarrow 0$, ρ^{ε_n} and $\boldsymbol{\xi}^n\in L^2(\rho^{\varepsilon_n})$ such that*

$$\rho^{\varepsilon_n}\to\rho\ \text{in}\ \mathscr{M}_2(\mathbb{R},\mathfrak{m}), \tag{3.33}$$ G1
$$|\partial\mathcal{F}^{\varepsilon_n}|(\rho^{\varepsilon_n})<+\infty,\qquad \boldsymbol{\xi}^n\in\partial_S\mathcal{F}^{\varepsilon_n}(\rho^{\varepsilon_n}), \tag{3.34}$$ G2
$$\boldsymbol{\xi}^n\rho^{\varepsilon_n}\rightharpoonup\partial^o\mathcal{F}(\rho)\rho\quad\text{in duality with}\ C^0_{\mathrm{b}}(\mathbb{R}), \tag{3.35}$$?G3?
$$\int_{\mathbb{R}}|\boldsymbol{\xi}^n|^2\mathrm{d}\rho^{\varepsilon_n}\to\int_{\mathbb{R}}|\partial^o\mathcal{F}(\rho)|^2\mathrm{d}\rho. \tag{3.36}$$ G4

The proof of Theorem 3.8 is a variation of the proof of Lemma 10.3.16 and Lemma 10.3.17 of [2] and Theorem 4.10 of [3].

4. Proofs of the main Theorems.

⟨sec:gf⟩ ### 4.1. Subdifferential characterization of the gradient flow of $\mathcal{F}$ and existence result.

Proof of Theorem 2.6. The proof of Theorem 2.6 is based on the general results about the generation of gradient flows for displacement λ-convex functionals in $\mathscr{M}_2(\mathbb{R},\mathfrak{m})$ established in [2] (notice that all the theory in [2] can be applied to the space $\mathscr{M}_2(\mathbb{R},\mathfrak{m})$ and not only to the space $\mathscr{M}_2(\mathbb{R},1)$ considered in [2]).

By Proposition 3.1 the functional $\mathcal{F}$ is displacement λ-convex (in dimension 1 generalized geodesics [2, Definition 9.2.2] coincide with the displacement interpolations (3.2)) and we can apply the general theory summarized in Theorem 11.2.1 [2].

Since $D(\mathcal{F})=\{\rho\in\mathscr{M}_2(\mathbb{R},\mathfrak{m}):\mathcal{F}(\rho)<+\infty\}$ is dense in $\mathscr{M}_2(\mathbb{R},\mathfrak{m})$, the evolution is well defined starting from an arbitrary element of $\mathscr{M}_2(\mathbb{R},\mathfrak{m})$. Therefore, by [2, Theorem 11.2.1], for every $\rho_0 \in \mathscr{M}_2(\mathbb{R}, \mathfrak{m})$ there exists a unique curve ρ belonging to $C^0([0,+\infty); \mathscr{M}_2(\mathbb{R}, \mathfrak{m}))$ such that $\rho_t \in D(\mathcal{I}) \subset D(\mathcal{F})$ for every $t > 0$ and

$$\partial_t \rho_t + \partial_x(\rho_t\, \boldsymbol{v}_t) = 0, \qquad \text{in } \mathscr{D}'(\mathbb{R} \times (0,+\infty)), \tag{4.1} \quad \boxed{\texttt{ce}}$$

$$\boldsymbol{v}_t = -\partial^\circ \mathcal{F}(\rho_t), \qquad \text{for } \mathscr{L}^1\text{-a.e. } t \in (0,+\infty), \tag{4.2} \quad \boxed{\texttt{nlrel}}$$

$$\mathcal{F}(\rho_{t_0}) - \mathcal{F}(\rho_{t_1}) = \int_{t_0}^{t_1} \int_{\mathbb{R}} |\boldsymbol{v}_t|^2\, \rho_t(x)\, \mathrm{d}t \qquad 0 \le t_0 < t_1,$$

Moreover the map $\rho_0 \mapsto S_t(\rho_0) := \rho_t$ defines a continuous semigroup satisfying the λ-contraction property (2.11). From [2, Theorem 2.4.15] the map $t \mapsto e^{\lambda t} |\partial \mathcal{F}|^2(\rho_t)$ is non-increasing, and then (2.10) holds. The regularization estimate (2.9) (which implies (2.4c)) still follows by Theorem 11.2.1 and by [16] in the case $\lambda \neq 0$. From (4.2) and Theorem 3.6 we have (2.4b). Finally (4.1), (4.2), and (3.22) yield (2.4d). The comparison result follows from Theorem 2.8 and the corresponding property for solution of the viscous regularization. □

Proof of Theorem 2.8. The part concerning existence of solutions to problem (2.14) for a measure initial datum, is similar to the part concerning existence for problem (1.DDE), taking into account the characterization of the subdifferential of $\mathcal{F}^\varepsilon$ (3.8).

The stability with respect to the convergence in $\mathscr{M}_2(\mathbb{R}, \mathfrak{m})$ follows from Lemma 3.7 and Theorem 11.2.1 of [2]. The uniform convergence follows from Theorem 3.4 (3.12). □

⟨sec:estimates⟩ 4.2. **Localized entropy estimates and propagation of singularities.** Let us consider

$$\text{a smooth convex function } \psi : [0,+\infty) \to \mathbb{R} \text{ with } \psi(0) = 0, \tag{4.3a} \quad \boxed{\texttt{eq:93}}$$

and let us set (recall that $\beta^\varepsilon(r) = \beta(r) + \varepsilon r$)

$$\eta(r) := r\psi'(r) - \psi(r), \quad \gamma(r) := \int_0^r \beta'(s)\psi'(s)\, \mathrm{d}s, \tag{4.3b} \quad \boxed{\texttt{eq:63}}$$

$$\gamma^\varepsilon(r) := \gamma(r) + \varepsilon\psi(r) = \int_0^r (\beta^\varepsilon)'(s)\psi'(s)\, \mathrm{d}s. \tag{4.3c} \quad \boxed{\texttt{eq:63bis}}$$

⟨thm:propagsing⟩ **Theorem 4.1.** *If u^ε is a smooth bounded solution to (2.14) and $\psi, \eta, \gamma^\varepsilon$ satisfy (4.3a), (4.3b) and (4.3c), then $\psi(u^\varepsilon)$ is a classical solution to*

$$\partial_t \psi(u^\varepsilon) - \partial_x\big(\partial_x \gamma^\varepsilon(u^\varepsilon) + \psi(u^\varepsilon)(V^\varepsilon)'\big) \le \eta(u^\varepsilon)(V^\varepsilon)''. \tag{4.4} \quad \boxed{\texttt{eq:59bis}}$$

In particular, for every nonnegative $\phi \in C^2_{\mathrm{c}}(\mathbb{R} \times [0,T])$ it holds

$$\begin{aligned} &\int_{\mathbb{R}} \psi(u^\varepsilon(x,T))\phi(x,T)\, \mathrm{d}x + \int_0^T \int_{\mathbb{R}} \psi(u^\varepsilon)\big(- \partial_t\phi + \partial_x\phi(V^\varepsilon)'\big)\, \mathrm{d}x\, \mathrm{d}t \\ &\quad - \int_0^T \int_{\mathbb{R}} \big(\gamma^\varepsilon(u^\varepsilon)\partial_x^2\phi + \eta(u^\varepsilon)\phi(V^\varepsilon)''\big)\, \mathrm{d}x\, \mathrm{d}t \le \int_{\mathbb{R}} \psi(u^\varepsilon(x,0))\phi(x,0)\, \mathrm{d}x. \end{aligned} \tag{4.5} \quad \boxed{\texttt{eq:59}}$$

Proof. By straightforward computations we obtain that

$$\partial_t\psi(u^\varepsilon) - \partial_x\big(\partial_x\gamma^\varepsilon(u^\varepsilon) + \psi(u^\varepsilon)(V^\varepsilon)'\big) = \eta(u^\varepsilon)(V^\varepsilon)'' - (\beta^\varepsilon)'(u^\varepsilon)\psi''(u^\varepsilon)(\partial_x u^\varepsilon)^2.$$

Since ψ is convex and β^ε is strictly increasing, $(\beta^\varepsilon)'(u^\varepsilon)\psi''(u^\varepsilon)(\partial_x u^\varepsilon)^2 \ge 0$. This implies (4.4). □

We will now prove the *a priori* estimate (2.2).

⟨cor:linfty_bound⟩ **Corollary 4.2.** *Let us assume that* (2.1) *holds and that* $\rho_0 = u_0\mathscr{L}^1$ *has a bounded density. Then* (2.2) *holds.*

Proof. By Theorem 2.8 it is sufficient to show (2.2) for the (bounded and integrable) solutions $\rho^\varepsilon = u^\varepsilon\mathscr{L}^1$ of (2.14) with initial datum ρ_0. Let us apply (4.5) with $\psi(r) = r^p$, $p \geq 2$, and $\phi(x) = \chi(x/n)$, where χ satisfies (3.26). Since $(V^\varepsilon)'$ is bounded and $(V^\varepsilon)'' \leq \mathsf{c}$, it is not difficult to pass to the limit as $n \to +\infty$, getting

$$\int_{\mathbb{R}} u^\varepsilon(x,T)^p \,\mathrm{d}x \leq \int_{\mathbb{R}} u_0^p(x)\,\mathrm{d}x + \mathsf{c}(p-1)\int_0^T\int_{\mathbb{R}} u^\varepsilon(x,t)^p\,\mathrm{d}x\,\mathrm{d}t.$$

From Gronwall's Lemma it follows that

$$\int_{\mathbb{R}} u^\varepsilon(x,T)^p\,\mathrm{d}x \leq \mathrm{e}^{\mathsf{c}(p-1)T}\int_{\mathbb{R}} u_0^p(x)\,\mathrm{d}x, \qquad \text{for all } T>0.$$

Raising to the power $1/p$ and letting $p \uparrow +\infty$ we get estimate (2.2) for ρ^ε. □

The following corollary of Theorem 4.1 is a preliminary step for the proof of Theorem 2.10 on the propagation of the singularities.

⟨cor3⟩ **Corollary 4.3.** *Let* ψ,η,γ *be as in* (4.3a) *and* (4.3b), *with* $\lim_{r\uparrow+\infty}\psi'(r) = \psi'_\infty \in (0,+\infty)$. *If* $\rho = u\mathscr{L}^1 + \rho^\perp$ *is the measure-valued solution to* (1.DDE) *and* $\psi(\rho) := \psi(u)\mathscr{L}^1 + \psi'_\infty\,\rho^\perp$, *we have*

$$\partial_t\psi(\rho) - \partial_x\big(\psi(\rho)V'\big) \leq \partial_x^2(\gamma(u)) + \eta(u)V'' \quad \textit{in the sense of distributions.}$$

Proof. It is sufficient to pass to the limit in (4.5), recalling (3.13) and applying the dominated convergence theorem with the estimate $|\psi(r)| \leq \|\psi'\|_{L^\infty((0,+\infty))}r$.

Notice that

$$\lim_{r\to+\infty}\frac{\eta(r)}{r} = \lim_{r\to+\infty}\Big(\psi'(r) - \frac{\psi(r)}{r}\Big) = 0$$

and

$$\lim_{r\to+\infty}\frac{\gamma(r)}{r} = \lim_{r\to+\infty}\frac{1}{r}\Big(\beta(r)\psi'(r) - \beta(0)\psi'(0) - \int_0^r \beta(s)\psi''(s)\,\mathrm{d}s\Big) = 0,$$

since $\lim_{r\uparrow+\infty}\beta(r) = \beta_\infty < +\infty$ and we estimate the integral as follows

$$0 \leq \frac{1}{r}\int_0^r \beta(s)\psi''(s)\,\mathrm{d}s \leq \frac{\beta_\infty}{r}\Big(\psi'(r) - \psi'(0)\Big).$$

□

Proof of Theorem 2.10. Let us fix a nonnegative function $\zeta \in C^\infty_{\mathrm{c}}(\mathbb{R})$ with compact support in $[0,1]$ and integral equal to 1. We set $\zeta_k(r) := \zeta(r-k)$, $Z_k(r) := \int_0^r \zeta_k(s)\,\mathrm{d}s$, $\psi_k(r) = \int_0^r Z_k(s)\,\mathrm{d}s$. It is immediate to check that ψ_k satisfies the assumptions of Corollary 4.3. It follows that for every nonnegative $\phi \in C^2_{\mathrm{c}}(\mathbb{R}\times[0,T])$

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}}\psi_k(u)\big(-\partial_t\phi + V'\partial_x\phi\big)\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\mathbb{R}}\big(-\partial_t\phi + V'\partial_x\phi\big)\mathrm{d}\rho_t^\perp(x)\mathrm{d}t\\
&\quad+\int_{\mathbb{R}}\psi_k(u(x,T))\phi(x,T)\,\mathrm{d}x - \int_{\mathbb{R}}\psi_k(u(x,0))\phi(x,0)\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}}\phi(x,T)\,\mathrm{d}\rho_T^\perp(x) - \int_{\mathbb{R}}\phi(x,0)\,\mathrm{d}\rho_0^\perp(x)\\
&\leq \int_0^T\int_{\mathbb{R}}\gamma_k(u)\partial_x^2\phi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\mathbb{R}}\eta_k(u)V''\phi\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{4.6}$$

eq:66bis

By construction, the functions $\psi_k(r)$ and the corresponding functions $\gamma_k(r)$ and $\eta_k(r)$ are uniformly bounded by Cr and converge to 0 pointwise as $k \to +\infty$. Passing to the limit in (4.6) as $k \uparrow +\infty$ and using the dominated convergence theorem, we obtain (2.16). Now, set $\mu_t = (\mathsf{X}_t)_\# \rho_0^\perp$. It is well known that μ_t solves $\partial_t \mu_t - \partial_x(\mu_t V') = 0$. Then the family of measures $\sigma_t = \rho_t^\perp - \mu_t$ satisfies $\partial_t \sigma_t - \partial_x(\sigma_t V') \le 0$ with $\sigma_0 \le 0$. By a simple variant of Proposition 8.1.7 of [2] we deduce that $\sigma_t \le 0$ for every $t \ge 0$. Therefore for every Borel set $A \subset \mathbb{R}$, $\rho_t^\perp(A) \le \rho_0^\perp\big(\mathsf{X}_t^{-1}(A)\big)$. Choosing $A = D_t$, the inclusion $\mathsf{J}(u_t) \subset \mathsf{J}_t$ follows. □

?⟨sec:min⟩? 4.3. Minimizers, stationary solutions, and asymptotic properties.

Proof of Theorem 2.12. Let us first show that every measure $\rho_{\min} = u_{\min}\mathscr{L}^1 + \rho_{\min}^\perp$ satisfying (2.18) is a minimizer for $\mathcal{F}$.

Notice that by construction $\rho_{\min} \in \mathscr{M}_+^{\mathrm{c}}(\mathbb{R}, \mathfrak{m})$. We prove that $\mathcal{F}(\rho_{\min}) > -\infty$. From (2.coer) it follows that $\mathcal{V}(\rho_{\min}) \ge \mathfrak{m}\, V_{\min}$. From (3.3) it follows that

$$\mathcal{E}(\rho_{\min}) \ge c_1 \int_{\{0<u_{\min}<r_0\}} u_{\min} \log u_{\min} \,\mathrm{d}x + \frac{E(r_0)}{r_0} \int_{\{u_{\min} \ge r_0\}} u_{\min} \,\mathrm{d}x. \qquad (4.7)\ \{?\}$$

The second integral is finite. In order to estimate the first integral we observe that the set $\{0 < u_{\min} < r_0\}$ is contained in $\mathbb{R} \setminus \tilde{Q}$, where $\tilde{Q}$ is a (possibly different) open neighborhood of Q. Therefore

$$\int_{\{0<u_{\min}<r_0\}} (1+|x|)^\alpha u_{\min} \,\mathrm{d}x < +\infty \qquad (4.8)\ \{?\}$$

by (2.int). Let us choose $\gamma \in \big(\frac{\alpha}{\alpha+1}, \alpha\big)$. Then using Hölder's inequality with exponent $p = \frac{\alpha}{\gamma}$, we have

$$\begin{aligned}
&\left| \int_{\{0<u_{\min}<r_0\}} u_{\min} \log u_{\min} \,\mathrm{d}x \right| \\
=&\left| \int_{\{0<u_{\min}<r_0\}} (1+|x|)^\gamma u_{\min}^{\frac{\gamma}{\alpha}}\ (1+|x|)^{-\gamma} u_{\min}^{1-\frac{\gamma}{\alpha}} \log u_{\min} \,\mathrm{d}x \right| \\
\le&\left(\int_{\{0<u_{\min}<r_0\}} (1+|x|)^\alpha u_{\min} \,\mathrm{d}x \right)^{\frac{\gamma}{\alpha}} \left(\int_{\{0<u_{\min}<r_0\}} \frac{|u_{\min}^{1-\frac{\gamma}{\alpha}} \log u_{\min}|^{\frac{\alpha}{\alpha-\gamma}}}{(1+|x|)^{\frac{\alpha\gamma}{\alpha-\gamma}}} \,\mathrm{d}x \right)^{1-\frac{\gamma}{\alpha}}.
\end{aligned}$$

Since $r^{1-\frac{\gamma}{\alpha}} \log r$ is bounded in $[0, r_0]$ and $\frac{\alpha\gamma}{\alpha-\gamma} > 1$, we obtain that $\mathcal{E}(\rho_{\min}) > -\infty$.

Let $\rho = u\mathscr{L}^1 + \rho^\perp$ be an arbitrary measure in $\mathscr{M}_+(\mathbb{R}, \mathfrak{m})$. If $A = \{x \in \mathbb{R} : V(x) - \mathfrak{v} < \mathfrak{d}\}$ and $B = \mathbb{R} \setminus A$ denotes its complement,

$$u_{\min}(x) = \begin{cases} H(V(x) - \mathfrak{v}) & \text{if } x \in A, \\ 0 & \text{if } x \in B. \end{cases}$$

Since

$$E'(H(v)) = \begin{cases} -v & \text{if } v \in (0, \mathfrak{d}), \\ -\mathfrak{d} & \text{if } v \in [\mathfrak{d}, +\infty), \end{cases}$$

and E is convex, we get

$$\begin{aligned}
\mathcal{E}(\rho) - \mathcal{E}(\rho_{\min}) &= \int_{\mathbb{R}} \big(E(u(x)) - E(u_{\min}(x))\big)\,\mathrm{d}x \\
&\geq \int_{\mathbb{R}} E'(u_{\min}(x))\big(u(x) - u_{\min}(x)\big)\,\mathrm{d}x \\
&= \int_A (\mathfrak{v} - V(x))\big(u(x) - u_{\min}(x)\big)\,\mathrm{d}x - \mathfrak{d}\int_B u(x)\,\mathrm{d}x.
\end{aligned}$$

Moreover, since $V(x) - \mathfrak{v} \geq \mathfrak{d}$ for every $x \in B$,

$$\begin{aligned}
\mathcal{F}(\rho) - \mathcal{F}(\rho_{\min}) &= \mathcal{E}(\rho) - \mathcal{E}(\rho_{\min}) + \int_{\mathbb{R}} V\,\mathrm{d}\rho - \int_{\mathbb{R}} V\,\mathrm{d}\rho_{\min} \\
&\geq \int_A (\mathfrak{v} - V(x))\big(u(x) - u_{\min}(x)\big)\,\mathrm{d}x + \int_B (V(x) - \mathfrak{d})u(x)\,\mathrm{d}x \\
&\quad + \int_A V(x)\big(u(x) - u_{\min}(x)\big)\,\mathrm{d}x + \int_{\mathbb{R}} V\,\mathrm{d}\rho^\perp - \int_{\mathbb{R}} V\,\mathrm{d}\rho_{\min}^\perp \\
&\geq \int_{\mathbb{R}} \mathfrak{v}\big(u(x) - u_{\min}(x)\big)\,\mathrm{d}x + \int_{\mathbb{R}} V\,\mathrm{d}\rho^\perp - \int_{\mathbb{R}} V\,\mathrm{d}\rho_{\min}^\perp.
\end{aligned}$$

Hence, owing to the identity

$$\rho(\mathbb{R}) = \rho_{\min}(\mathbb{R}), \quad \text{so that} \quad \int_{\mathbb{R}} u\,\mathrm{d}x - \int_{\mathbb{R}} u_{\min}\,\mathrm{d}x = \int_{\mathbb{R}} \mathrm{d}\rho_{\min}^\perp - \int_{\mathbb{R}} \mathrm{d}\rho^\perp,$$

and recalling that $\rho_{\min}^\perp$ is concentrated in Q, we obtain

$$\begin{aligned}
\mathcal{F}(\rho) - \mathcal{F}(\rho_{\min}) &\geq \int_{\mathbb{R}} \mathfrak{v}\big(u(x) - u_{\min}(x)\big)\,\mathrm{d}x + \int_{\mathbb{R}} V\,\mathrm{d}\rho^\perp - \int_{\mathbb{R}} V\,\mathrm{d}\rho_{\min}^\perp \\
&= \int_{\mathbb{R}} (V - \mathfrak{v})\,\mathrm{d}\rho^\perp - \int_{\mathbb{R}} (V - \mathfrak{v})\,\mathrm{d}\rho_{\min}^\perp \geq -\int_{\mathbb{R}} (V - \mathfrak{v})\,\mathrm{d}\rho_{\min}^\perp = 0.
\end{aligned}$$

This shows that $\mathcal{F}(\rho) \geq \mathcal{F}(\rho_{\min})$ for every $\rho \in \mathscr{M}_+(\mathbb{R}, \mathfrak{m})$.

We prove now that every minimizer $\rho = u\mathscr{L}^1 + \rho^\perp \in \mathscr{M}_+(\mathbb{R}, \mathfrak{m})$ of $\mathcal{F}$ in $\mathscr{M}_+(\mathbb{R}, \mathfrak{m})$ satisfies (2.18). We consider another minimizer $\rho_{\min}$ given by (2.18) so that equalities hold in all the previous inequalities and in particular we have

$$0 = \mathcal{F}(\rho) - \mathcal{F}(\rho_{\min}) = \int_{\mathbb{R}} (V - \mathfrak{v})\,\mathrm{d}\rho^\perp.$$

It follows that $\rho^\perp$ is concentrated on Q and $\rho^\perp = 0$ when $\mathfrak{m} < \mathfrak{m}_c$ (recall that $V(x) - \mathfrak{v} \geq 0$ and equality holds if and only if $\mathfrak{v} = V_{\min}$ and $x \in Q$). If $u \neq u_{\min}$, then, by the strict convexity of E, $\mathcal{F}((1-\theta)\rho + \theta\rho_{\min}) < \mathcal{F}(\rho_{\min})$ for every $\theta \in (0,1)$. Taking the continuity of u into account, it follows that $u(x) = u_{\min}(x)$ for every $x \in \mathbb{R}$. Consequently $\rho^\perp(\mathbb{R}) = \rho_{\min}^\perp(\mathbb{R})$ and we conclude. □

Proof of Theorem 2.14. It follows easily by [2, Theorem 11.1.3], which shows in particular that ρ is a stationary solution of the Wasserstein gradient flow of a displacement λ-convex functional $\mathcal{F}$ if and only if $|\partial\mathcal{F}|(\rho) = 0$. We can then invoke Theorem 3.6. □

The proof of Theorems 2.15 and 2.17 is based on the following lemma:

⟨le:char⟩ **Lemma 4.4.** *Let $\rho = u\mathscr{L}^1 + \rho^\perp \in \mathscr{M}_+^{\mathrm{c}}(\mathbb{R})$ be a measure satisfying $\mathcal{I}(\rho) = 0$, and let us consider the open set $\Omega_+(u) := \big\{x \in \mathbb{R} : u(x) > 0\big\}$. If I is a connected component of $\Omega_+(u)$ then there exists a constant c_I such that*

$$E'(u(x)) + V(x) = c_I \quad \textit{for every } x \in I. \tag{4.9}$$ eq:64

Proof. Let us first show that the function $E' \circ u$ belongs to $W^{1,1}_{\mathrm{loc}}(\Omega_+(u))$ with

$$\partial_x\big(E' \circ u\big) = \frac{\partial_x\big(\beta \circ u\big)}{u} \quad \text{in } \Omega_+(u). \tag{4.10}$$ eq:100

We can simply write $E' \circ u = L \circ (\beta \circ u)$ where $L := E' \circ \beta^{-1}$ and $\beta \circ u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$. The function L belongs to $C^1(0, \beta_\infty)$ and can be extended to β_∞ by continuity setting $L(\beta_\infty) = 0$; it is easy to check that this extension belongs to $C^1(0, \beta_\infty]$, since

$$L'(r) = \frac{E'' \circ \beta^{-1}}{\beta' \circ \beta^{-1}} = \frac{1}{\beta^{-1}}, \qquad \lim_{r \uparrow \beta_\infty} L'(r) = 0.$$

(4.10) then follows by the chain rule for the composition of a C^1 with a Sobolev function.

If I is a connected component of $\Omega_+(u)$, we have

$$0 = \frac{\partial_x \beta(u(x))}{u(x)} + V'(x) = \partial_x(E'(u(x)) + V(x)) \quad \text{in } I,$$

so that there exists a constant c_I such that (4.9) holds. □

Proof of Theorem 2.15. We have to prove only the "right" implication ⇒.

A simple argument by contradictions shows that $\Omega_+(u) = \mathbb{R}$: otherwise, if the interval $I = (a, b)$ is a connected component of $\Omega_+(u)$ and one of its extremes, say a, is finite, we should have

$$\lim_{x \downarrow a} u(x) = 0, \qquad -\mathfrak{d} = \lim_{x \downarrow a} E'(u(x)) = c_I - V(a) > -\infty.$$

Since $\Omega_+(u) = \mathbb{R}$ Lemma 4.4 yields $V(x) \geq c_I$ for every $x \in \mathbb{R}$ and $u(x) = H(V(x) - c_I)$. Since $\rho \in \mathscr{M}_+^{\mathrm{c}}(\mathbb{R}, \mathfrak{m})$ we conclude that (2.18) holds and ρ is a minimizer of $\mathcal{F}$ by Theorem 2.12. □

Proof of Theorem 2.17. Let $\rho = u\mathscr{L}^1 + \rho^\perp \in \mathscr{M}_+^{\mathrm{c}}(\mathbb{R}, \mathfrak{m})$ with $\mathcal{I}(\rho) = 0$ and let $I = (a, b)$ be a connected component of the open set $\Omega_+(u)$. Since the range of the function $r \mapsto -E'(r)$ for $r \in (0, +\infty]$ is the bounded interval $(0, \mathfrak{d}]$ and $\lim_{|x| \to \infty} V(x) = +\infty$ we deduce from Lemma 4.4 that I is bounded.

It follows that $u(a) = u(b) = 0$ and therefore $\lim_{x \downarrow a} E'(u(x)) = \lim_{x \uparrow b} E'(u(x)) = -\mathfrak{d}$, $c_I = V(a) - \mathfrak{d} = V(b) - \mathfrak{d}$. We thus obtain (2.20) and the representation (2.22), which also yields (2.21) since u is integrable in $\mathbb{R}$. Since for every $x \in I$ $u(x) = +\infty$ if and only if $V(x) = V(a) - \mathfrak{d}$, i.e. $x \in Q_I$, we obtain (2.23).

Conversely, if $\rho = u\mathscr{L}^1 + \rho^\perp \in \mathscr{M}_+^{\mathrm{c}}(\mathbb{R}, \mathfrak{m})$ satisfies the three conditions of Theorem 2.17, we immediately have that $\mathcal{I}(\rho) = 0$. In fact, the first integral of the definition of $\mathcal{I}$ in (2.3) vanishes by (4.9) and (4.10); the second integral, corresponding to the singular part of ρ vanishes since $\rho^\perp$ is concentrated on $Q(u)$ and V' vanishes in each point of Q_I, which is a local minimizer of V. □

Proof of Corollary 2.18. Remark 2.13 shows that the minimizer of $\mathcal{F}$ is unique. We have just to check the case when $\mathfrak{d} < +\infty$. By the assumption on the first derivative of V is immediate to check that the set $\Omega_+(u)$ contains just one connected component $I = (a,b)$ with $a < q_- < q_+ < b$. Theorem 2.12 shows that ρ is a minimizer of $\mathcal{F}$. □

Proof of Theorem 2.25. We use the dissipation identity (2.8) to obtain the inequality

$$\int_{t_0}^{t_1} \mathcal{I}(\rho_t)\,\mathrm{d}t = \mathcal{F}(\rho_{t_0}) - \mathcal{F}(\rho_{t_1}) \le \mathcal{F}(\rho_{t_0}) - \mathcal{F}(\bar\rho) < +\infty \text{ for every } 0 < t_0 < t_1 < +\infty.$$

Passing to the limit as $t_1 \uparrow +\infty$ we get $\mathcal{I}(\rho_t) \in L^1(t_0, +\infty)$. If $\lambda \ge 0$, then, by (2.10), $\mathcal{I}(\rho_t)$ is decreasing with respect to t so that $\lim_{t\uparrow+\infty} \mathcal{I}(\rho_t) = 0$.
If $\lambda < 0$, then (2.10) yields $\mathcal{I}(\rho_t) \ge \mathrm{e}^{2\lambda}\mathcal{I}(\rho_n)$, for any $t \in (n-1, n)$. We obtain that

$$\sum_{n=2}^{+\infty} \mathcal{I}(\rho_n) \le e^{-2\lambda} \sum_{n=2}^{+\infty} \int_{n-1}^{n} \mathcal{I}(\rho_t)\,\mathrm{d}t = e^{-2\lambda} \int_1^{+\infty} \mathcal{I}(\rho_t)\,\mathrm{d}t < +\infty.$$

Therefore $\lim_{n\uparrow+\infty} \mathcal{I}(\rho_n) = 0$. A further application of (2.10) yields $\mathcal{I}(\rho_n) \ge \mathrm{e}^{2\lambda}\mathcal{I}(\rho_t)$ for any $t > n$ and then $\lim_{t\uparrow+\infty} \mathcal{I}(\rho_t) = 0$.
Since $\mathcal{F}(\rho_t) \le \mathcal{F}(\rho_{t_0})$ for every $t \ge t_0$, by (2.coer) we infer that $\{\rho_t\}_{t\ge t_0}$ is tight; by Theorem 3.5 any weak limit point ρ_∞ of ρ_t as $t \uparrow +\infty$ satisfies $\mathcal{I}(\rho_\infty) = 0$ and therefore $\rho_\infty = \bar\rho$. It follows that $\rho_t \rightharpoonup \bar\rho$ weakly as $t \uparrow +\infty$.

Theorem 3.5 yields the uniform convergence of u_t to $\bar u$ on compact sets of $\mathsf{D}(\bar u)$ as $t \to +\infty$. When $\mathfrak{m} < \mathfrak{m}_c$, $\bar\rho$ has a bounded density and therefore for every compact subset $K \subset \mathbb{R}$ there exists a time $T > 0$ such that ρ_t is bounded on K for every $t \ge T$. Choosing as $K := \big\{x \in \mathbb{R} : V(x) \le c\big\}$ for a constant c sufficiently big so that K contains the support of $\rho_0^\perp$, Theorem 2.10 shows that the support of $\rho_t^\perp$ is contained in K for every $t > 0$ and therefore $\rho_t^\perp = 0$ for $t \ge T$. □

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