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SOME RESULTS ON MINIMIZING MOVEMENTS

Abstract. In the framework of minimizing movements, recently suggested by E. De Giorgi, we consider a "gradient flow type" problem associated to an abstract functional and prove some existence, uniqueness and convergence results for the related variational evolution inequalities. These results properly extend and confirm some conjectures presented in [DG].

ALCUNI RISULTATI SUI MOVIMENTI MINIMIZZANTI

Sommario. Nell'ambito dei "Movimenti Minimizzanti", recentemente proposti da E. De Giorgi, consideriamo un problema di evoluzione associato ad un funzionale astratto e dimostriamo alcuni risultati di esistenza, unicitá e regolaritá per le relative disequazioni variazionali. Questi teoremi estendono e confermano alcune congetture presentate in [DG].

0. - Introduction

In a recent paper (see [DG]) E. De Giorgi has proposed the idea of *minimizing movements* as a useful tool to unify many different problems in Calculus of Variations, Differential Equations and Geometric Measure Theory. By combining minimization and iteration a lot of interesting examples receive a variational formulation which allows to get faster to the solution.

In particular he suggested some conjectures, which very naturally lead to approximation problems for variational inequalities of evolution; we study them in an abstract formulation of the type

$$(0.1)\quad a\big(u(t),u'(t)-v\big)+b\big(u(t),u'(t)-v\big)+\varphi(u'(t))-\varphi(v)\le 0,\ \forall\, v\in V \qquad u(0)=u_0\in V$$

where V is a Hilbert space, $\varphi : V \mapsto]-\infty,+\infty]$ is a proper, convex, lower semicontinuous function, $a,\ b : V\times V\mapsto \mathbf{R}$ are continuous bilinear forms, and a is symmetric and positive but in general not coercive.

De Giorgi's approach to this equation is to choose a time step $\tau = 1/\lambda > 0$ and to look for a step function $u_\tau(t)$, constant on each interval $[k\tau,(k+1)\tau[$, $k\in\mathbf{N}$, such that $u_\tau(0) = u_0$ and $u_\tau((k+1)\tau)$ solves the minimum problem:

$$(0.2)\qquad \min_{v\in V}\left[\frac{1}{2}a(v,v)+b(u_\tau(k\,\tau),v)+\tau\,\varphi\Big(\frac{v-u_\tau(k\,\tau)}{\tau}\Big)\right].$$

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If

$$\exists \lim_{\tau\to 0} u_\tau(t) = u(t), \quad \text{in } V, \quad \forall\, t \in [0, +\infty[\tag{0.3}$$

we call u a minimizing movement associated to the functional (0.2) and the equation (0.1).

Under suitable hypotheses on V, a, φ we prove some existence, uniqueness and convergence results from which De Giorgi's conjectures can be obtained as a special case.

Equations of this kind have been largely studied in these last years; just to quote a few examples, in [DL] and [Br2] a non zero datum $f(t)$ is considered while in [CV] and [Co] a double nonlinearity is studied and a fairly complete list of works on the argument is given.

It doesn't seem, however, that anything like we did already exists in the literature, due to the lack of coerciveness of the bilinear form a, to the unboundedness of φ and above all to the strong pointwise estimate required by (0.2).

We mainly use maximal monotone operators theory and convex analysis, whose techniques are developped for example in [Bar], [Br1] and [ET] and to which we refer for additional related material. On the other hand the stability and convergence estimates are typical of the abstract numerical analysis of parabolic variational inequalities and they are developped for a different problem in [Ba].

All these results have been previously announced in a joint note with M. Gobbino [GGS], where further bibliographical references on the subject can be found.

In the first paragraph we discuss the problem and present our results in theorems 1–3; the proofs are given in the following sections.

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1. - General definitions and statement of the results

In the following $\bar{\mathbf{R}}$ will be the extended real line ($\bar{\mathbf{R}} = \mathbf{R} \bigcup \{-\infty, +\infty\}$). Let S be a topological space. In [DG] De Giorgi proposed the following

1.1 Definition. *Let* $F :]1, +\infty[\times \mathbf{Z} \times S \times S \to \bar{\mathbf{R}}$ *and* $u : \mathbf{R} \to S$*; we say that* u *is a* minimizing movement associated to F and S *and we write* $u \in MM(F, S)$ *if there exists* $w :]1, +\infty[\times \mathbf{Z} \to S$ *such that for any* $t \in \mathbf{R}$

$$\lim_{\lambda \to +\infty} w(\lambda, [\lambda t]) = u(t) \tag{1.1}$$

and for any $\lambda \in]1, +\infty[, k \in \mathbf{Z}$

$$F(\lambda, k, w(\lambda, k+1), w(\lambda, k)) = \min_{s \in S} F(\lambda, k, s, w(\lambda, k)) \quad \square \tag{1.2}$$

Let us now suppose that $S = H^1(\mathbf{R}^n)$, u_0 is a given function in this space and F is an integral functional as follows; in this case we deal with *gradient flow* type problems and De Giorgi [DG] suggested the following

Conjecture 1. *Let* $\beta \in]1, 2]$ *be given; if we set*

$$F(\lambda, k, f, g) = \begin{cases} \displaystyle\int_{\mathbf{R}^n} |f - u_0|^2 \, dx & \text{if } k \le 0 \\ \displaystyle\int_{\mathbf{R}^n} \lambda |\nabla f|^2 + (\lambda |f - g|)^\beta \, dx & \text{if } k > 0 \end{cases} \tag{1.3}$$

then $u \in MM(F, S)$ *if and only if* $u : \mathbf{R} \to S$ *is continuous,* $u(t) = u_0$ *for any* $t \le 0$ *and, setting* $v(x, t) = u(t)(x)$, v *solves*

$$\frac{\beta}{2} \frac{\partial v}{\partial t} = \left| \frac{\partial v}{\partial t} \right|^{2-\beta} \Delta_x v \qquad \text{in } \mathbf{R}^n \times]0, +\infty[. \tag{1.4}$$

Conjecture 2. *Let* $a_1, \ldots, a_n \in L^\infty(\mathbf{R}^n)$ *be given; if we define*

$$F(\lambda, k, f, g) = \begin{cases} \displaystyle\int_{\mathbf{R}^n} |f - u_0|^2 \, dx & \text{if } k \le 0 \\ \displaystyle\int_{\mathbf{R}^n} |\nabla f|^2 - 2 \sum_{i=1}^n a_i \frac{\partial g}{\partial x_i} f + \lambda |f - g|^2 \, dx & \text{if } k > 0 \end{cases} \tag{1.5}$$

then $u \in MM(F, S)$ *if and only if* $u : \mathbf{R} \to S$ *is continuous,* $u(t) = u_0$ *for any* $t \le 0$ *and, setting* $v(x, t) = u(t)(x)$, v *solves*

$$\frac{\partial v}{\partial t} = \Delta_x v + \sum_{i=1}^n a_i \frac{\partial v}{\partial x_i} \qquad \text{in } \mathbf{R}^n \times]0, +\infty[. \tag{1.6}$$

Conjecture 3. *If we put*

$$(1.7)\qquad F(\lambda,k,f,g)=\begin{cases}\displaystyle\int_{\mathbf{R}^n}|f-u_0|^2\,dx & \text{if } k\le 0\\ \displaystyle\int_{\mathbf{R}^n}|\nabla f|^2+\lambda|f-g|^2\,dx & \text{if } k>0,\ f\ge g\\ +\infty & \text{otherwise}\end{cases}$$

then $u\in MM(F,S)$ *if and only if* $u:\mathbf{R}\to S$ *is continuous,* $u(t)=u_0$ *for any* $t\le 0$ *and, setting* $v(x,t)=u(t)(x)$*,* v *solves*

$$(1.8)\qquad \frac{\partial v}{\partial t}=(\Delta_x v)^+\qquad \text{in } \mathbf{R}^n\times]0,+\infty[.$$

1.2 Remark. In the above stated conjectures it must be explained what is the weak sense in which equations (1.4) and (1.8) have to be understood. □

1.3 Remark. The above written (1.3), (1.5), (1.7) can all be seen as generalization of the basic functional

$$(1.9)\qquad F(\lambda,k,f,g)=\begin{cases}\displaystyle\int_{\mathbf{R}^n}|f-u_0|^2\,dx & \text{if } k\le 0\\ \displaystyle\int_{\mathbf{R}^n}|\nabla f|^2+\lambda|f-g|^2\,dx & \text{if } k>0\end{cases}$$

Let us observe that in these cases it is sufficient to study the limit (1.1) for $t\ge 0$. □

Actually all these conjectures naturally fit into a more general and abstract framework. Precisely we consider

$$(1.10)\qquad \begin{cases}V \text{ Hilbert space with norm } \|\cdot\|\\ V' \text{ its dual, } <\cdot,\cdot> \text{ the duality pairing}\end{cases}$$

$$(1.11)\qquad \begin{cases}\varphi:V\to\mathbf{R}\cup\{+\infty\} \text{ proper, lower semicontinuous, convex}\\ D(\varphi) \text{ its domain, } \partial\varphi \text{ its subdifferential, } \varphi^* \text{ its conjugate } (^1)\end{cases}$$

(^{1}) Let us recall that $D(\varphi)=\{v\in V:\varphi(v)<+\infty\}\ne\emptyset$ and that $\forall v\in V,\ w\in V'$

$$w\in\partial\varphi(v)\quad\Leftrightarrow\quad <w,z-v>\le\varphi(z)-\varphi(v)\quad\forall z\in V$$

$\varphi^*:V'\mapsto\bar{\mathbf{R}}$ is a proper l.s.c. convex function defined by

$$\varphi^*(w)=\sup_{v\in V}<w,v>-\varphi(v),\qquad \forall w\in V'$$

$$(1.12) \qquad \begin{cases} a(\cdot,\cdot),\ b(\cdot,\cdot) : V \times V \to \mathbf{R} \text{ continuous bilinear forms} \\ \text{to which the linear continuous operators } A, B : V \to V' \text{ are associated } (^2) \end{cases}$$

Moreover, $a(\cdot,\cdot)$ is symmetric and the associated quadratic form, which we e by $a(\cdot)$, is positive:

$$a(u,v) = a(v,u); \qquad a(u) = a(u,u) \geq 0 \quad \forall u, v \in V.$$

We can then think to the following problem

Problem P. *Let $u_0 \in V$ be given and consider*

$$(1.13) \qquad F(\lambda,k,v,w) = \begin{cases} \|v-u_0\| & \text{if } k \leq 0 \\ \dfrac{1}{2}a(v,v) + b(v,w) + \dfrac{1}{\lambda}\varphi[\lambda(v-w)] & \text{otherwise.} \end{cases}$$

We want to find conditions on a, b, φ, u_0 such that there exists a unique $u : \mathbf{R} \to V$ with $u \in$ $\mathrm{MM}(F,V)$ and

$$(1.14) \qquad \begin{cases} u \in C^0(\mathbf{R}, V) \cap AC_{loc}(]0,+\infty[; V) \\ u(t) \equiv u_0 \quad \forall t \leq 0 \\ 0 \in Au + Bu + \partial\varphi(u') \quad \text{a.e. in }]0,+\infty[. \end{cases}$$

Let us start and consider the case with

$$(1.15) \qquad b \equiv 0$$

If the form $a(\cdot,\cdot)$ is coercive on V, that is, if it satisfies

$$(1.16) \qquad \exists \alpha > 0 : \quad \forall u \in V \quad a(u,u) \geq \alpha\|u\|^2,$$

then Problem P has a very general answer. As a matter of fact we have the following

Theorem 1. *Under hypotheses* (1.15) *and* (1.16), *if* (^{3})

$$(1.17) \qquad -Au_0 \in \overline{D(\varphi^*)}^{V'}$$

(^{2}) As it is usual, $\forall u, v \in V$

$$< Au, v >= a(u,v) \leq M_a \|u\| \, \|v\|; \qquad < Bu, v >= b(u,v).$$

(^{3}) $\exists! u \in MM(F,V)$ even if this condition is not satisfied; the solution u still verifies the differential inclusion but it is not continuous at $t=0$; in fact we have $\lim_{t\to 0^+} u(t) = u_0^+$ and $-u_0^+$ is the projection of u_0 on the convex $A^{-1}\big(D(\varphi^*)\big)$ with respect to the scalar product defined by $a(\cdot,\cdot)$.

Problem P admits a solution u that satisfies (1.14) *and*

$$u \in W^{1,+\infty}_{loc}(]0,+\infty[;V). \tag{1.18}$$

Moreover, if we define $\check{\varphi}$ *as the function* $v \mapsto \varphi(-v)$, *u is the solution to*

$$\begin{cases} u(0) = u_0 \\ u' + \partial\check{\varphi}^*(Au) \ni 0 \qquad \text{a.e. in }]0,+\infty[\end{cases} \tag{1.19}$$

and belongs to $\mathrm{MM}(F^*,V)$, *where*

$$F^*(\lambda,k,v,w) = \begin{cases} \|v-u_0\| & \text{if } k \le 0 \\ \dfrac{\lambda}{2}a(v-w,v-w) + \check{\varphi}^*(Av) & \text{otherwise.} \end{cases} \tag{1.20}$$

1.4 Remark. The minimization of functionals F, F^* given by (1.13) and (1.20) leads to backward Euler scheme for the approximation of (1.14) and (1.19) respectively; this last case, in particular, is deeply developped in [Br 2] in a slightly different framework and our proof is an easy consequence of that theory. On the other hand, using the same procedures as in section 3 and 4 we could give a direct proof of this result, which would be more related to the minimizing movement idea; we restricted ourselves to the results which seem to be new. □

1.5 Remark. Let us substitute $\mathbf{R}^n$ with a bounded domain Ω and choose $S = V = H^1_0(\Omega)$ in conjecture 1. We define

$$a(u,v) = \int_\Omega (\nabla u, \nabla v)\,dx, \qquad \varphi(u) = \frac{1}{2}\int_\Omega |u|^\beta\,dx, \quad \beta \in]1,+\infty[. \tag{1.21}$$

Then

$$Au = -\Delta u, \quad \partial\varphi(u) = \frac{\beta}{2}|u|^{\beta-2}u$$

and

$$D(\varphi^*) = L^{\beta'}(\Omega) \hookrightarrow H^{-1}(\Omega), \qquad \frac{1}{\beta} + \frac{1}{\beta'} = 1$$

where the inclusion is dense. It is easy to see that condition (1.17) is satisfied for any choice of u_0. □

1.6 Remark. An analogous work can be done with conjecture 3 once we put

$$\varphi(u) = \frac{1}{2}\int_\Omega |u|^2\,dx + I_{\{u\ge 0\}}, \quad I_{\{u\ge 0\}} = \begin{cases} 0 & \text{if } u \ge 0 \text{ a.e.} \\ +\infty & \text{otherwise.} \end{cases} \tag{1.22}$$

Under these hypotheses u satisfies the variational inequality

$$\int_\Omega \frac{\partial u}{\partial t}\left(\frac{\partial u}{\partial t} - v\right) + \int_\Omega \left(\nabla u, \nabla\Big(\frac{\partial u}{\partial t} - v\Big)\right) \le 0, \ \ \forall v \in H^1_0(\Omega),\ v \ge 0 \text{ in } \Omega \quad \text{a.e. in }]0,+\infty[. \ \square$$

If $a(\cdot,\cdot)$ is not coercive (which is the case of our conjectures), the situation is more complicated. We will suppose that a *uniformly convex* Banach space W is given, which is compatible with V in the sense that
(1.23)
$$\begin{cases} V \text{ and } W \text{ are both continuously embedded in a Hausdorff topological vector space } \mathcal{T} \\ V \cap W \text{ is dense in } V \text{ and in } W \ (^4) \end{cases}$$

Moreover we assume that ([5])

$$\exists l,\, \alpha_l > 0 : \quad \sqrt{a(u,u)} + l\|u\|_W \geq \alpha_l \|u\|_V, \ \forall u \in V \cap W \tag{1.24}$$

φ is linked to W in the following sense:

$$\begin{cases} \exists \beta \in]1,+\infty[, \ \exists \psi : V \cap W \mapsto \mathbf{R} \cup \{+\infty\} \ \text{ proper, convex, l.s.c.} \\ \varphi(u) = \|u\|_W^\beta + \psi(u), \ \forall u \in V \cap W. \end{cases} \tag{1.25}$$

We have now

Theorem 2. *Under the preceding hypotheses* (1.15)*,* (1.23)*–*(1.25)*,* $\forall u_0 \in V$ *Problem* P *has a unique solution* u*. Moreover,* $\forall\, T > 0$*,*

$$\begin{cases} u \in W^{1,\beta}_{loc}(]0,+\infty[;V), \quad u' \in L^\beta(0,T;W) \\ \varphi^*(-Au), \quad \psi(u') \ \in L^1(0,T) \end{cases} \tag{1.26}$$

and u *satisfies also* (1.19)*.*

1.7 Remark. If ψ satisfies:

$$0 \in D(\psi), \qquad \psi(v) \geq 0, \ \ \forall\, v \in D(\psi) \tag{1.27}$$

the previous result holds for $T = +\infty$ and the proof is sligthly easier. In this case $\check{\varphi}^*$ satisfies (1.27), too. □

1.8 Remark. Let us choose $W = L^\beta(\mathbf{R}^n)$, with $\beta \in]1,2]$, $\psi = 0$ and consequently

$$a(u,v) = \int_{\mathbf{R}^n} (\nabla u, \nabla v)\, dx, \qquad \varphi(u) = \frac{1}{2}\int_{\mathbf{R}^n} |u|^\beta\, dx.$$

We are dealing with conjecture 1 and our hypotheses are trivially satisfied. In this case

$$-\Delta u \in L^{\beta'}(\mathbf{R}^n \times]0,+\infty[), \text{ and } \frac{\partial u}{\partial t} \in L^\beta(\mathbf{R}^n \times]0,+\infty[) \quad \square$$

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(4) This in particular implies that V' and W' can be continuously and densely imbedded into $(V \cap W)'$, which will coincide with their sum.
(5) As $a(\cdot,\cdot)$ is positive, this is the same as

$$\forall \epsilon > 0 \ \ \exists \alpha_\epsilon > 0 : \quad \sqrt{a(u,u)} + \epsilon\|u\|_W \geq \alpha_\epsilon \|u\|_V, \ \ \forall u \in V \cap W.$$

1.9 Remark. As in the previous case, let us choose

$$W = L^\beta(\mathbf{R}^n),\ \psi = I_K,\ K = \{u \geq 0\},\ \varphi(u) = \frac{1}{\beta}\int_{\mathbf{R}^n} |u|^\beta\, dx + \psi(u)$$

then conjecture 3 follows from the abstract theory for $\beta = 2$; moreover, denoting by $\mathcal{M}(\mathbf{R}^n)$ the topological vector space of Radon measures on $\mathbf{R}^n$, we have

$$D(\check{\varphi}^*) = \{v \in H^{-1}(\mathbf{R}^n) \cap \mathcal{M}(\mathbf{R}^n) : v^- \in L^{\beta'}(\mathbf{R}^n)\}$$

$$\check{\varphi}^*(u) = \frac{1}{\beta'}\int_{\mathbf{R}^n} |u^-|^{\beta'}\, dx, \qquad \partial\check{\varphi}^*(u) = -(u^-)^{\beta'-1}.$$

For a.e. t, Δu is a measure whose positive part is in $L^{\beta'}(\mathbf{R}^n)$ and u satisfies

$$\frac{\partial u}{\partial t} = [(\Delta u)^+]^{\beta'-1} \quad \square$$

1.10 Remark. (Neumann's problem) As in Remark 1.5 we choose $V = H^1(\Omega)$ and u is the solution to

$$\begin{cases} \frac{\beta}{2}|u'|^{\beta-2}u' = \Delta u & \text{in } \Omega\times]0,+\infty[\\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega\times]0+\infty[\\ u = u_0 & \text{on } \Omega\times\{0\}. \end{cases}$$

As Δu belongs to $L^{\beta'}(\Omega)$ the Neumann condition makes sense. $\square$

Finally, let us consider $b \not\equiv 0$; we will suppose that

$$(1.28)\quad \begin{cases} W = H \text{ is a Hilbert space and } V \text{ is continuously densely imbedded in } H, \\ b \text{ can be extended to a continuous bilinear form on } V \times H \mapsto \mathbf{R}, \text{ that is} \\ \exists M_b > 0 : \quad b(u,v) \leq M_b\, \|u\|_V \|v\|_H, \quad \forall\, u,v \in V \qquad \varphi(u) = \frac{1}{2}\|u\|_H^2 + \psi(u) \end{cases}$$

Theorem 3. *Problem P admits a unique solution u, which belongs also to $H^1(0,T;H)$ $\forall T > 0$.*

1.11 Remark. We end up with conjecture 2 if we suppose that $\psi \equiv 0$, $H = L^2(\mathbf{R}^n)$ and

$$b(u,v) = \sum_{i=1}^n \int_{\mathbf{R}^n} a_i \frac{\partial u}{\partial x_i} v\, dx \quad \square$$

2. - Proof of theorem 1

2.1 Lemma. *For any $v \in V$ there exists a unique $w = \sigma(\lambda, v) \in V$ such that*

$$\frac{1}{2}a(w,w) + \frac{1}{\lambda}\varphi(\lambda(w-v)) = \min_{z\in V}[\frac{1}{2}a(z,z) + \frac{1}{\lambda}\varphi(\lambda(z-v))] \tag{2.1}$$

and w solves the following variational inequality

$$a(w, w-z) + \frac{1}{\lambda}\varphi(\lambda(w-v)) - \frac{1}{\lambda}\varphi(\lambda(z-v)) \leq 0 \qquad \forall z \in V. \tag{2.2}$$

Moreover w also solves

$$\lambda a(w-v, w-z) + \check{\varphi}^*(Aw) - \check{\varphi}^*(Az) \leq 0 \qquad \forall z \in V \tag{2.3}$$

and the dual problem

$$\frac{\lambda}{2}a(w-v, w-v) + \check{\varphi}^*(Aw) = \min_{z\in V}[\frac{\lambda}{2}a(z-v, z-v) + \check{\varphi}^*(Az)]. \tag{2.4}$$

Proof The first two relations are obvious since we are looking for the minimum of the sum of convex functions, one of which is coercive and Frechèt-differentiable (see [V], [L]). If we define $\tilde{z} = \lambda(z-v)$ we can rewrite (2.2) in this way

$$a(w, \lambda(w-v) - \tilde{z}) + \varphi(\lambda(w-v)) - \varphi(\tilde{z}) \leq 0 \qquad \forall \tilde{z} \in V; \tag{2.5}$$

that is:

$$-Aw \in \partial\varphi(\lambda(w-v))$$

Recalling that $\partial\varphi^*$ is the inverse operator of $\partial\varphi$ (see [Br2], [Bar]) we get:

$$\lambda(w-v) \in \partial\varphi^*(-Aw) = -\partial\check{\varphi}^*(Aw) \tag{2.6}$$

that is (2.3) and (2.4). ∎

2.2 Remark. By (2.6) the function $w = \sigma(\lambda, v)$ satisfies:

$$w + \frac{1}{\lambda}\partial\check{\varphi}^*(Aw) \ni v; \tag{2.7}$$

the multivalued operator $V \ni v \mapsto \partial\check{\varphi}^* \circ A \subset V$ is then maximal and monotone, when V is endowed with the scalar product $a(\cdot,\cdot)$. It is also easy to verify that we are dealing with the subdifferential of the conjugate function of φ once we identify V and V' through a. Its resolvent is given by

$$J_\tau v = \sigma(\frac{1}{\tau}, v), \quad V \mapsto V \quad \square \tag{2.8}$$

We can then finally come to the *proof* of theorem 1; if we apply the theory developped in [Br2] we can see immediately that

$$(2.9) \qquad \begin{cases} u' + \partial\check{\varphi}^*(Au) \ni 0 & \text{a.e. in }]0,+\infty[\\ u(0) = u_0 \end{cases}$$

has a unique solution u in $C^0([0,+\infty[;V) \cap W^{1,+\infty}_{loc}(]0,+\infty[;V)$ if $-Au_0 \in \overline{D(\varphi^*)}^{V'}$. Moreover, thanks to the exponential formula we have

$$(2.10) \qquad u(t) = \lim_{n\to+\infty} (\cdot + \frac{t}{n}\partial\check{\varphi}^*(A\,\cdot))^{-n} u_0 = \lim_{n\to+\infty} J^n_{t/n} u_0 \qquad \forall t > 0$$

uniformly on the compact subsets of $[0,+\infty[$. If we now define $w(\lambda,k)$ as in (1.2) we have

$$(2.11) \qquad w(\lambda,k) = u_0 \quad \text{if } k \le 0$$

$$(2.12) \qquad w(\lambda,k+1) = \sigma(\lambda, w(\lambda,k)) = J_{\frac{1}{\lambda}} w(\lambda,k) = J^{k+1}_{\frac{1}{\lambda}} u_0.$$

Choosing $1/\lambda = t/n$, it follows that, $\forall t > 0$,

$$(2.13) \qquad \lim_{n\to+\infty} w(\frac{n}{t}, n) = u(t) \qquad \forall t > 0$$

uniformly on the compact subsets of $]0,+\infty[$. Finally, (1.1) is a direct consequence of the following

2.3 Lemma. *Let (S,d) be a metric space and $w : [0,+\infty[\times\mathbf{N} \to S$ a function such that*

$$\exists \lim_{n\to+\infty} w(\frac{n}{t}, n) = u(t)$$

uniformly on the compact subsets of $]0,+\infty[$. *If $u(t)$ is continuous, then*

$$\lim_{\lambda\to+\infty} w(\lambda, [\lambda t]) = u(t) \quad \forall t > 0.$$

Proof It is enough to observe that

$$(2.14) \qquad |\frac{[\lambda t]}{\lambda} - t| \le \frac{1}{\lambda}$$

and

$$(2.15) \qquad d(w(\lambda,[\lambda t]), u(t)) \le d(w(\lambda,[\lambda t]), u(\frac{[\lambda t]}{\lambda})) + d(u(\frac{[\lambda t]}{\lambda}), u(t)) \quad \blacksquare$$

3. - Proof of theorem 2

We make a slight change in the notations, using the same as in [Ba]. We put

$$\tau = \frac{1}{\lambda} \in]0,1[, \qquad u_\tau^k = w(1/\tau, k) \in V. \tag{3.1}$$

We have already seen that $\{u_\tau^k\}$ is the sequence such that

$$u_\tau^k \equiv u_0 \quad \text{for} \quad k \leq 0 \tag{3.2}$$

$$\begin{cases} \delta_\tau^k = \dfrac{u_\tau^k - u_\tau^{k-1}}{\tau} \in D(\varphi) \\ a(u_\tau^k, \delta_\tau^k - v) + \varphi(\delta_\tau^k) - \varphi(v) \leq 0 \qquad \forall v \in V \quad k \geq 1 \end{cases} \tag{3.3}$$

$$\begin{cases} Au_\tau^k \in D(\check{\varphi}^*) \\ < \delta_\tau^k, Au_\tau^k - v > + \check{\varphi}^*(Au_\tau^k) - \check{\varphi}^*(v) \leq 0 \qquad \forall v \in V' \quad k \geq 1 \end{cases} \tag{3.4}$$

where this last relation is the dual of the previous one. Let us now divide $\mathbf{R}$ into the intervals $I_\tau^k = [\tau(k-1), \tau k[$ and consider the function u_τ piecewise constant whose value in I_τ^k is u_τ^{k-1}. Our aim is to show that

$$\lim_{\tau \to 0^+} u_\tau(t) = u(t) \quad \text{in } V \quad \forall t \geq 0 \tag{3.5}$$

and that $u(t)$ satisfies (1.14). It is convenient to divide the proof in different steps:

1) *Stability estimates* in suitable functional spaces on the bounded intervals $]0,T[$, $T > 0$.
2) *Stationary estimates* for the singular perturbation problem (3.3) when τ goes to zero.
3) *Cauchy estimate* with respect to the seminorm induced by a.
4) *Passage to the limit* for a subsequence through weak compactness and further reinforcement of the convergence thanks to the uniform reflexivity of W.
5) *Uniqueness of the solution* to (1.14) and therefore existence of the limit (3.5).

Step 1: stability estimates.

Let us fix $T > 0$ and choose $N \in \mathbf{N}$ such that $T \in I_\tau^N$. It is also useful to consider the piecewise linear function $\hat{u}_\tau$ such that:

$$\hat{u}_\tau(k\tau) = u_\tau^k; \qquad \hat{u}_\tau(t) = \big(t/\tau - (k-1)\big)\, u_\tau^k + (k - t/\tau)\, u_\tau^{k-1}, \;\; \text{on } I_\tau^k \tag{3.6}$$

We then have

3.1 Theorem. *There exist constants $C_0, C_1 > 0$, independent from τ, such that*

$$\|u_\tau\|_{L^\infty(0,T;V)} \le \|\hat{u}_\tau\|_{L^\infty(0,T;V)} \le C_0 \tag{3.7}$$

$$\|\hat{u}'_\tau\|_{L^\beta(0,T;W)} \le C_1 \tag{3.8}$$

$$\left.\begin{array}{r} \displaystyle\int_0^T \psi(\hat{u}'_\tau(t))\,dt \\ \displaystyle\int_0^T \check{\varphi}^*(Au_\tau(t+\tau))\,dt \end{array}\right\} \le \frac{1}{2}a(u_0). \tag{3.9}$$

Moreover, if (1.27) *holds true, then we can choose* $C_1 = \frac{1}{2}a(u_0)$.

Proof Since in general

$$w \in \partial\varphi(v) \quad \Rightarrow \quad (w, v) = \varphi(v) + \varphi^*(w)$$

we observe that

$$< Au_\tau^k, \delta_\tau^k > + \|\delta_\tau^k\|_W^\beta + \psi(\delta_\tau^k) + \check{\varphi}^*(Au_\tau^k) = 0. \tag{3.10}$$

Adding up for $k = 1 \ldots\ m \le N$(6)

$$\frac{1}{2}a(u_\tau^m) + \frac{\tau^2}{2}\sum_{k=1}^m a(\delta_\tau^k) + \tau\sum_{k=1}^m \Big[\|\delta_\tau^k\|_W^\beta + \psi(\delta_\tau^k) + \check{\varphi}^*(Au_\tau^k)\Big] = \frac{1}{2}a(u_0). \tag{3.11}$$

Recalling that $\hat{u}'_\tau(t) \equiv \delta_\tau^k$ on I_τ^k, (3.9) follows immediately.

We already observed that (1.27) implies

$$\check{\varphi}^*(v) \ge 0, \qquad \forall\, v \in V' \tag{3.12}$$

and consequently (3.8); since

$$\tau\sum_{k=1}^m \|\delta_\tau^k\|_W^\beta \ge \frac{1}{(T+\tau)^{\frac{\beta}{\beta'}}}\|u_\tau^m - u_0\|_W^\beta \tag{3.13}$$

via (1.24) we get also (3.7).

(6) We often use the elementary equality:

$$a(v, v-w) = \frac{1}{2}\big[a(v) + a(v-w) - a(w)\big]$$

Otherwise we recall that convex lower semicontinuous functions are bounded from below by affine functions

$$\left\{\begin{array}{l}\exists \xi_\psi \in V' + W',\ \xi_{\check\varphi^*} \in V,\ \eta \in \mathbf{R}: \\ \psi(v) \ge -<\xi_\psi, v> -\eta \\ \check\varphi^*(v) \ge -<\xi_{\check\varphi^*}, v> -\eta \end{array}\right. . \tag{3.14}$$

Therefore

$$\begin{aligned} &\frac{1}{2}a(u_\tau^m) + \frac{\tau^2}{2}\sum_{k=1}^m a(\delta_\tau^k) + \tau\sum_{k=1}^m \|\delta_\tau^k\|_W^\beta \le \\ &\le \frac{1}{2}a(u_0) + <\xi_\psi, u_\tau^m - u_0> + \tau\sum_{k=1}^m <\xi_{\check\varphi^*}, Au_\tau^k> + 2(\tau m)\,\eta \le \\ &\le \frac{1}{2}a(u_0) + \|\xi_\psi\|_{V'+W'}\cdot\|u_\tau^m - u_0\|_{V\cap W} + (T+\tau)[a(\xi_{\check\varphi^*}) + 2\eta] + \frac{\tau}{4}\sum_{k=1}^m a(u_\tau^k) \end{aligned} \tag{3.15}$$

By (3.13) there exists a constant $C > 0$ such that

$$\frac{1}{2}a(u_\tau^m) + \tau\sum_{k=1}^m \|\delta_\tau^k\|_W^\beta \le C[1 + \tau\sum_{k=1}^m a(u_\tau^k)].$$

From here we obtain the uniform boundedness of $a(u_\tau^m)$ (it is a discrete version of Gronwall lemma) and (3.8), (3.7). ∎

To go to the limit in the weak sense we need to control $a(\hat u_\tau')$. This is done precisely by

3.2 Theorem. *$\exists C_2 > 0$ independent of τ such that*

$$\int_0^T t a(\hat u_\tau'(t))\,dt \le C_2. \tag{3.16}$$

Proof It is enough to choose $v = Au_\tau^{k-1},\ k \ge 2$ in (3.4); if we multiply for $k\tau$ and sum from 2 up to N we have:

$$\tau\sum_{k=2}^N k\tau\, a(\delta_\tau^k) + \tau\sum_{k=2}^N \left[k\check\varphi^*(Au_\tau^k) - (k-1)\check\varphi^*(Au_\tau^{k-1})\right] - \tau\sum_{k=2}^N \check\varphi^*(Au_\tau^{k-1}) \le 0 \tag{3.17}$$

from which

$$\tau\sum_{k=2}^N k\tau\, a(\delta_\tau^k) + N\tau\,\check\varphi^*(Au_\tau^N) - \tau\check\varphi^*(Au_\tau^1) \le \tau\sum_{k=1}^{N-1}\check\varphi^*(Au_\tau^k)$$

Summing (3.11) written with $m = 1$ we have

$$\tau\sum_{k=1}^N k\tau\, a(\delta_\tau^k) \le \tau\sum_{k=2}^{N-1}\check\varphi^*(Au_\tau^k) + a(u_0) - \tau\psi(\delta_\tau^1) - N\tau\check\varphi^*(Au_\tau^N) \le C_2$$

from which we obtain (3.16). ∎

3.3 Lemma. *The sequences*

$$k \mapsto a(\delta_\tau^k); \quad k \mapsto \check{\varphi}^*(Au_\tau^k) \qquad k \geq 1 \tag{3.18}$$

are not increasing.

Proof In (3.3) we choose $v = \delta_\tau^{k-1}$ and in the one relative to the preceding step we choose $v = \delta_\tau^k$; summing up we get

$$a(\delta_\tau^k, \delta_\tau^k - \delta_\tau^{k-1}) \leq 0 \tag{3.19}$$

and this is precisely the first relation; for the second one it is sufficient to choose $v = Au_\tau^{k-1}$ in (3.4). ∎

3.4 Corollary. *As a simple consequence*

$$a(\hat{u}'_\tau(t)) \leq \frac{2C_2}{t^2}, \quad \forall t \leq T \quad ∎ \tag{3.20}$$

Step 2: stationary estimates.

3.5 Lemma. *We have*

$$\lim_{\tau\to 0^+} \|u_\tau^1 - u_0\|_W + a(u_\tau^1 - u_0) = 0; \qquad \lim_{\tau\to 0} \tau\, \check{\varphi}^*(Au_\tau^1) = 0. \tag{3.21}$$

Proof From (3.8) we immediately obtain

$$\|u_\tau^1 - u_0\|_W^\beta \leq C_1\, \tau^{\beta-1} \to 0 \quad \text{when } \tau \to 0^+; \tag{3.22}$$

moreover, since u_τ^1 is bounded in V, compatibility requires that $u_\tau^1 \rightharpoonup u_0$ in V. If we write (3.11) with $m = 1$ we can see that

$$\limsup_{\tau\to 0^+} a(u_\tau^1) \leq a(u_0) \tag{3.23}$$

and taking into account the positivity of a we have

$$\lim_{\tau\to 0^+} a(u_\tau^1 - u_0) = 0.$$

Finally, from (3.11) we have:

$$\tau\check{\varphi}^*(Au_\tau^1) \leq \frac{1}{2}[a(u_0) - a(u_\tau^1)] + < \xi_\psi, u_\tau^1 - u_0 > + \tau\,\eta$$

obtaining the second of (3.21). ∎

3.6 Corollary. *We have*

$$\lim_{\tau\to 0^+} a(u_\tau^{k+1}-u_\tau^k)=0 \tag{3.24}$$

uniformly with respect to k.

Proof It is immediate if we combine the previous result with lemma 3.3. ∎

3.7 Corollary. *We find*

$$\left.\begin{array}{l}\lim\limits_{\tau\to 0^+}\sup\limits_{t>0} a(u_\tau(t+\tau)-u_\tau(t))\\ \lim\limits_{\tau\to 0^+}\sup\limits_{t>0} a(u_\tau(t)-\hat{u}_\tau(t))\\ \lim\limits_{\tau\to 0^+}\sup\limits_{t>0} a(\hat{u}_\tau(t+\tau)-\hat{u}_\tau(t))\end{array}\right\}=0 \quad ∎ \tag{3.25}$$

Step 3: Cauchy estimate.

3.8 Theorem. *We have*

$$\lim_{\tau,\rho\to 0^+}\sup_{t\in[0,T]} a(\hat{u}_\tau(t)-\hat{u}_\rho(t))=0. \tag{3.26}$$

with $T=+\infty$ if (1.27) holds.

Proof It follows from the following lemma:

3.9 Lemma. *Let $\ell_\tau(t)$ be the $\tau-$periodic function that is equal to t/τ in $[0,\tau[$ and U_τ, $\hat{U}_\tau$ the $\tau-$translated functions:*

$$U_\tau(t)=u_\tau(t+\tau), \qquad \hat{U}_\tau(t)=\hat{u}_\tau(t+\tau) \tag{3.27}$$

then $\hat{U}_\tau$ satisfies the inequality

$$a(\hat{U}_\tau',\hat{U}_\tau-v)+\check{\varphi}^*(A\hat{U}_\tau)-\check{\varphi}^*(Av)\le(1-\ell_\tau)[\check{\varphi}^*(AU_\tau)-\check{\varphi}^*(AU_\tau(t+\tau))]. \tag{3.28}$$

We choose in (3.28) $v=\hat{U}_\rho$; changing the role of τ and ρ, summing up and integrating between 0 and t, we obtain in the left hand member the difference:

$$\frac{1}{2}a(\hat{U}_\tau(t)-\hat{U}_\rho(t))-\frac{1}{2}a(\hat{U}_\tau(0)-\hat{U}_\rho(0))$$

while in the right hand member, due to the $\tau-$periodicity of $\ell_\tau(t)$:

$$\int_0^\tau(1-\ell_\tau)\Big[\check{\varphi}^*(AU_\tau(t))-\check{\varphi}^*(AU_\tau(T+t))\Big]\,dt+\int_0^\rho(1-\ell_\rho)\Big[\check{\varphi}^*(AU_\rho(t))-\check{\varphi}^*(AU_\rho(T+t))\Big]\,dt$$

so that:

$$a(\hat{U}_\tau(t)-\hat{U}_\rho(t))\le a(u_\tau^1-u_\rho^1)+\tau\check{\varphi}^*(Au_\tau^1)+\rho\check{\varphi}^*(Au_\rho^1)+2C_0\,(\tau+\rho)(\|A\xi_{\check{\varphi}^*}\|_*+\eta)$$

By lemma 3.5 and its corollary we deduce (3.26). ∎

Proof of Lemma 3.9 Let us observe that

$$\hat{u}_\tau(t) = \ell_\tau(t) u_\tau(t+\tau) + (1-\ell_\tau(t)) u_\tau(t) \tag{3.29}$$

$$u_\tau(t+\tau) - \hat{u}_\tau(t) = \tau(1-\ell_\tau(t))\hat{u}'_\tau(t) \tag{3.30}$$

with $0 \le \ell_\tau(t) \le 1$. Now, thanks to (3.4) we have

$$< \hat{u}'_\tau(t), Au_\tau(t+\tau) - v > + \check{\varphi}^*(Au_\tau(t+\tau)) - \check{\varphi}^*(v) \le 0 \quad \forall v \in V' \tag{3.31}$$

and consequently

$$\begin{aligned} & a(\hat{u}'_\tau(t), \hat{u}_\tau(t) - v) + \check{\varphi}^*(A\hat{u}_\tau(t)) - \check{\varphi}^*(Av) \le \\ & \qquad \le a(\hat{u}'_\tau(t), \hat{u}_\tau(t) - u_\tau(t+\tau)) + \check{\varphi}^*(A\hat{u}_\tau(t)) - \check{\varphi}^*(Au_\tau(t+\tau)). \end{aligned} \tag{3.32}$$

Thanks to (3.30)

$$a(\hat{u}'_\tau(t), \hat{u}_\tau(t) - u_\tau(t+\tau)) = -\tau(1-\ell_\tau) a(\hat{u}'_\tau) \le 0$$

while, due to convexity,

$$\check{\varphi}^*(A\hat{u}_\tau(t)) \le (1-\ell_\tau)\check{\varphi}^*(Au_\tau(t)) + \ell_\tau \check{\varphi}^*(Au_\tau(t+\tau)). \tag{3.33}$$

Taking account of these two inequalities in (3.32), we obtain (3.28). ■

Step 4: passage to the limit.

The previous estimates allow us to prove

3.10 Theorem. *There exists a subsequence $\{\tau_n\}$ with $\tau_n \searrow 0$ and a function*

$$u \in C^0([0,T];V) \cap W^{1,\beta}_{loc}(]0,T];V)$$

such that

$$\hat{u}_{\tau_n}(t) \rightharpoonup u(t) \text{ in } V, \quad \forall t \in [0,T] \qquad \lim_{n\to+\infty} \sup_{t\in[0,T]} a(\hat{u}_{\tau_n}(t) - u(t)) = 0. \tag{3.34}$$

Moreover $u'(t) \in L^\beta(0,T;W)$ while $\psi(u'(t))$ and $\check{\varphi}^*(Au(t))$ are in $L^1(0,T)$ and u satisfies

$$u'(t) + \partial\check{\varphi}^*(Au(t)) \ni 0, \quad u(0) = u_0. \tag{3.35}$$

Proof The proof is standard: once we extract a subsequence $\{\tau_n\}$ such that $\hat{u}_{\tau_n}$ weak*-converges to u in $L^\infty(0,T;V)$ and $\hat{u}'_{\tau_n} \rightharpoonup u'$ in $L^\beta(0,T;W)$ and in $L^\beta_{loc}(]0,T];V)$, it is immediate to verify that

$$\hat{u}_{\tau_n}(t) \rightharpoonup u(t) \text{ in } V \quad \forall t \in [0,T] \tag{3.36}$$

and that

$$\lim_{n\to+\infty} \sup_{t\in[0,T]} a(\hat{u}_{\tau_n}(t)-u(t))=0. \tag{3.37}$$

thanks to the lower semicontinuity of the quadratic form $a(\cdot)$.

It follows that also U_{τ_n} uniformly converges to u in the seminorm induced by a and therefore

$$Au_{\tau_n} \to Au \text{ in } L^\infty(0,T;V') \tag{3.38}$$ [7]

Since by (3.31)

$$\hat{u}'_{\tau_n} + \partial\check{\varphi}^*(Au_{\tau_n}) \ni 0 \quad \text{ a.e. in }]0,T[\tag{3.39}$$

the inclusion is still true for u:

$$u'(t) + \partial\check{\varphi}^*(Au) \ni 0 \quad \text{ a.e. in }]0,T[. \tag{3.40}$$

We still have to check the continuity in 0; on one hand it is easy to see that

$$a(u(\epsilon)-u_0) = a(u(\epsilon)-u_\tau(\epsilon)) \quad \forall\tau>\epsilon.$$

In fact, if we choose $n_\epsilon = \max\{k : \tau_k > \epsilon\}$, n_ϵ grows to infinity when ϵ goes to 0 and therefore

$$\lim_{\epsilon\to0^+} a(u(\epsilon)-u_0) = \lim_{\epsilon\to0^+} a(u(\epsilon)-u_{\tau_{n_\epsilon}}(\epsilon)) \le \lim_{\epsilon\to0^+} \sup_{t\in[0,1]} a(u(t)-u_{\tau_{n_\epsilon}}(t))=0.$$

On the other hand

$$\|u(\epsilon)-u_0\|_W \le \liminf_{n\to+\infty} \|\hat{u}_{\tau_n}(\epsilon)-u_0\|_W \le C\epsilon^{\frac{1}{\beta'}}$$

and we are done. ■

As a matter of fact the convergence is actually stronger. At this regard we have

---

(7) Observe that the Schwartz inequality applied to a gives:

$$\|Av\|_* = \sup_{\|w\|=1} < Av, w > \le \sup_{\|w\|=1} \sqrt{a(v)}\,\sqrt{a(w)} \le M_a \sqrt{a(v)}$$

3.11 THEOREM. *Let τ_n and u be as in the previous proposition. Then*

$$\hat{u}'_{\tau_n} \to u' \quad \text{in } L^\beta(0,T;W). \tag{3.41}$$

In particular

$$\lim_{n\to+\infty} \hat{u}_{\tau_n}(t) = u(t) \qquad \text{in } V$$

uniformly with respect to t on any bounded interval.

Proof From (3.40) we get

$$a(u'(t),u(t)) + \check{\varphi}^*(Au(t)) + \varphi(u'(t)) = 0 \qquad \text{a.e. in }]0,+\infty[\tag{3.42}$$

while from (3.39) we have

$$a(\hat{u}'_\tau(t),\hat{u}_\tau(t)) + \check{\varphi}^*(AU_\tau(t)) + \varphi(\hat{u}'_\tau(t)) \leq 0. \tag{3.43}$$

Let us now integrate from 0 to T (8); we find

$$\frac{1}{2}a(u(T)) + \int_0^T \check{\varphi}^*(Au(t))\,dt + \int_0^T [\|u'(t)\|_W^\beta + \psi(u'(t))]\,dt = \frac{1}{2}a(u_0) \tag{3.44}$$

and

$$\frac{1}{2}a(\hat{u}_{\tau_n}(T)) + \int_0^T \check{\varphi}^*(AU_{\tau_n}(t))\,dt + \int_0^T [\|\hat{u}'_{\tau_n}(t)\|_W^\beta + \psi(\hat{u}'_{\tau_n}(t))]\,dt \leq \frac{1}{2}a(u_0). \tag{3.45}$$

Passing to the limit for $\tau_n \to 0$ we have

$$\limsup_{n\to+\infty} \int_0^T [\check{\varphi}(A\,U_{\tau_n}(t)) + \|\hat{u}'_{\tau_n}(t)\|_W^\beta + \psi(\hat{u}'_{\tau_n}(t))]\,dt \leq$$

$$\leq \int_0^T [\check{\varphi}^*(Au(t)) + \|\hat{u}'(t)\|_W^\beta + \psi(u'(t))]\,dt. \tag{3.46}$$

On the other hand the three functions $\check{\varphi}^*$, $\|\cdot\|_W^\beta$ and ψ are lower semicontinuous. Therefore

$$\lim_{n\to+\infty} \int_0^T \|\hat{u}'_{\tau_n}(t)\|_W^\beta\,dt = \int_0^T \|u'(t)\|_W^\beta\,dt. \tag{3.47}$$

As $L^\beta(0,T;W)$ is uniformly convex (9),

$$\hat{u}'_{\tau_n} \to u' \qquad \text{in } L^\beta(0,T;W) \quad \blacksquare$$

Step 5: uniqueness of the solution.

(8) Thanks to (3.40) we integrate from $\epsilon > 0$ to T and then we let ϵ go to zero, using the continuity of u in W.

(9) For the uniform convexity of $L^\beta(W)$ with W uniformly convex, see [D].

Since $u_\tau(t) = \hat{u}_\tau(\tau[\frac{t}{\tau}])$ we also have $u_{\tau_n}(t) \to u(t)$. To be sure of the convergence of the whole family $\{u_\tau\}$ we must prove the uniqueness of the limit.

3.12 Proposition. *Let* $u_1,\ u_2 \in C^0([0,+\infty[;V) \cap AC_{loc}(]0,+\infty[;V)$ *be two solutions of the Cauchy problem*

$$\begin{cases} Au + \partial\varphi(u') \ni 0 \\ u(0) = u_0 \end{cases}.$$

If φ *is strictly convex, then* $u_1 \equiv u_2$.

Proof Usually, thanks to the monotonicity, we obtain that $t \mapsto a(u_1(t) - u_2(t))$ is not increasing, so that

$$a(u_1(t) - u_2(t)) \equiv 0.$$

Moreover, if $u_1' \neq u_2'$, then for the strict monotonicity of the subdifferential we should have

$$a(u_1(T) - u_2(T)) < a(u_1(\epsilon) - u_2(\epsilon)) = 0$$

for some $0 < \epsilon < T$ which is absurd. It follows that $u_1' = u_2'$ and, for the Cauchy condition, $u_1 \equiv u_2$. ∎

4. - Proof of Theorem 3

We repeat faster the previuos sequence, enhancing all the changes that are to be made and underlining the specific difficulties of this case.
With regard to the notation, we will identify H with its dual so that we have the following continuous and dense inclusions:

$$V \hookrightarrow H \equiv H' \hookrightarrow V'; \tag{4.1}$$

in this way the scalar product $(\cdot,\cdot)$ of H may be extended to the duality between V and V', which we will denote with the same symbol. Finally, $|\cdot|$ will be the norm in H.

The sequence u_τ^k satisfies now the inequalities $(k \geq 1)$

$$\begin{cases} \delta_\tau^k \in D(\psi) \\ (\delta_\tau^k + Au_\tau^k + Bu_\tau^{k-1}, \delta_\tau^k - v) + \psi(\delta_\tau^k) - \psi(v) \leq 0 \end{cases} \tag{4.2}$$

$$\begin{cases} \delta_\tau^k + Au_\tau^k + Bu_\tau^{k-1} \in D(\check{\psi}^*) \\ (\delta_\tau^k, \delta_\tau^k + Au_\tau^k + Bu_\tau^{k-1} - v) + \check{\psi}^*(\delta_\tau^k + Au_\tau^k + Bu_\tau^{k-1}) - \check{\psi}^*(v) \leq 0 \end{cases} \tag{4.3}$$

and also

$$(\delta_\tau^k, \delta_\tau^k + Au_\tau^k + Bu_\tau^{k-1}) + \check{\psi}^*(\delta_\tau^k + Au_\tau^k + Bu_\tau^{k-1}) + \psi(\delta_\tau^k) = 0. \tag{4.4}$$

As for the stability estimates we have

4.1 Theorem. *$\exists C > 0$ independent from τ such that*

$$\|u_\tau\|_{L^\infty(0,T;V)} \leq \|\hat{u}_\tau\|_{L^\infty(0,T;V)} \leq C \tag{4.5}$$

$$\|\hat{u}_\tau'\|_{L^2(0,T;H)} \leq C \tag{4.6}$$

$$\int_0^T \psi(\hat{u}_\tau'(t))\, dt \leq C \tag{4.7}$$

$$\int_0^T \check{\psi}^*(\hat{u}'(t) + Au_\tau(t+\tau) + Bu_\tau(t))\, dt \leq C \tag{4.8}$$

$$\int_0^T ta(\hat{u}_\tau'(t))\, dt \leq C \tag{4.9}$$

Proof We work as in the previous paragraph and we obtain from (4.4)

$$
\begin{aligned}
&\tau\sum_{k=1}^{m}\Big(|\delta_\tau^k|_H^2+\frac{\tau}{2}a(\delta_\tau^k)\Big)+\frac{1}{2}a(u_\tau^m)\le \\
&\le\frac{1}{2}a(u_0)+2m\tau\eta+(\xi_\psi+\xi_{\check{\psi}^*},u_\tau^m-u_0)+\tau\sum_{k=1}^{m}b(u_\tau^{k-1},\delta_\tau^k+\xi_{\check{\psi}^*})+a(u_\tau^k,\xi_{\check{\psi}^*}).
\end{aligned}
\tag{4.10}
$$

We get to (4.5) thanks to Gronwall lemma and to the boundedness of the form b in $V\times H$ that leads to the following estimate

$$
\tau\sum_{k=1}^{m}b(u_\tau^{k-1},\delta_\tau^k+\xi_{\check{\psi}^*})\le\frac{1}{2}\tau\sum_{k=1}^{m}|\delta_\tau^k|^2+\frac{m\tau}{2}|\xi_{\check{\psi}^*}|^2+\frac{M_b^2}{2}\tau\sum_{k=0}^{m-1}\|u_\tau^k\|^2.
$$

(4.6), (4.7) and (4.8) are immediate and (4.9) comes in a similar way from theorem 3.2. ■

The stationary estimates are evaluated as in the case of theorem 1.6. In fact

4.2 Lemma. *It results*

$$
\lim_{\tau\to0^+}\|u_\tau^1-u_0\|=0;
\tag{4.11}
$$

$$
\lim_{\tau\to0^+}\tau\check{\psi}^*(\delta_\tau^1+Au_\tau^1+Bu_0)=0\quad\blacksquare
\tag{4.12}
$$

It is more delicate the following

4.3 Proposition. *It exists a constant $C>0$, indipendent from τ such that*

$$
\sup_{t\in[0,T]}\|u_\tau(t)-u_\tau(t+\tau)\|_V\le C\,\|u_\tau^1-u_0\|
\tag{4.13}
$$

$$
\int_0^T|\hat{u}'_\tau(t)-\hat{u}'_\tau(t+\tau)|_H^2\le C\|u_\tau^1-u_0\|^2\quad\blacksquare
\tag{4.14}
$$

Proof If we work as in Lemma 3.3 we have

$$
\big(\delta_\tau^k-\delta_\tau^{k-1},\delta_\tau^k-\delta_\tau^{k-1}+\tau A\delta_\tau^k+\tau B\delta_\tau^{k-1}\big)\le0;
$$

summing to each member the quantity $\tau(\delta_\tau^k,\delta_\tau^k-\delta_\tau^{k-1})$ and dividing by τ we get:

$$
\begin{aligned}
&\frac{1}{\tau}|\delta_\tau^k-\delta_\tau^{k-1}|^2+\frac{1}{2}\Big[a(\delta_\tau^k)+|\delta_\tau^k|^2\Big]\le \\
&\quad\le\frac{1}{2}\Big[a(\delta_\tau^{k-1})+|\delta_\tau^{k-1}|^2\Big]-b(\delta_\tau^{k-1},\delta_\tau^k-\delta_\tau^{k-1})+(\delta_\tau^k,\delta_\tau^k-\delta_\tau^{k-1})\le \\
&\quad\le\frac{1}{2}\Big[a(\delta_\tau^{k-1})+|\delta_\tau^{k-1}|^2\Big]+\frac{1}{2\tau}|\delta_\tau^k-\delta_\tau^{k-1}|^2+\tau\Big[M_b^2\|\delta_\tau^{k-1}\|^2+|\delta_\tau^k|^2\Big].
\end{aligned}
\tag{4.15}
$$

If we define

$$x_\tau^k = a(u_\tau^k - u_\tau^{k-1}) + |u_\tau^k - u_\tau^{k-1}|^2 \geq \alpha\|u_\tau^k - u_\tau^{k-1}\|^2$$

summing up (4.15) from $k = 2$ to m we obtain

$$x_\tau^m \leq x_\tau^1 + C\tau \sum_{k=1}^{m} x_\tau^k$$

from which

$$x_\tau^m \leq x_\tau^1 \cdot e^{C(T+\tau)}$$

Weak coercivity of a assures (4.13) and (4.14) consequently follows from (4.15). ∎

Before coming to the wanted convergence, let us define a new function $\tilde{u}_\tau(t)$ that interpolates the sequence $\{u_\tau^k\}$ quadratically:

$$\tilde{u}_\tau(t) = \frac{1}{2}u_\tau(t+\tau) + \frac{1}{2}[\ell_\tau(t)\hat{u}_\tau(t+\tau) + (1-\ell_\tau(t))\hat{u}(t)] \tag{4.16}$$

with

$$\tilde{u}'_\tau(t) = \frac{\hat{u}'_\tau(t+\tau) - \hat{u}'_\tau(t)}{\tau} = \ell_\tau \hat{u}'_\tau(t+\tau) + (1-\ell_\tau)\hat{u}'_\tau(t) \tag{4.17}$$

Thanks to proposition 4.3 we can see easily that the families u_τ, $\hat{u}_\tau$ and $\tilde{u}_\tau$ have the same limit points in $L^\infty(0,T;V) \cap H^1(0,T;H)$ just like their translated versions U_τ, $\hat{U}_\tau$ and $\tilde{U}_\tau$ (see (3.27)).

Let us set for semplicity

$$V_\tau = \hat{U}'_\tau + AU_\tau + Bu_\tau; \qquad \hat{V}_\tau = \tilde{U}'_\tau + A\hat{U}_\tau + B\hat{u}_\tau \tag{4.18}$$

Argueing as in lemma 3.9 we have:

4.4 Lemma. *For every $v \in D(\check{\psi}^*)$ it holds:*

$$\begin{aligned} &\big(\hat{U}'_\tau, \tilde{U}'_\tau + A\hat{U}_\tau + B\hat{u}_\tau - v\big) + \check{\psi}^*(\hat{V}_\tau) - \check{\psi}^*(v) \leq \\ &\quad \leq (1-\ell_\tau)\big[\check{\psi}^*(V_\tau) - \check{\psi}^*(V_\tau(t+\tau))\big] + b(\hat{u}_\tau - u_\tau, \hat{U}'_\tau) + (\hat{U}'_\tau, \tilde{U}'_\tau - \hat{U}'_\tau) \end{aligned} \tag{4.19}$$ ∎

4.5 Lemma. *$\forall \tau,\ \rho > 0,\ s \in [0,T]$ we have*

$$\begin{aligned} &\int_0^s |\tilde{U}'_\tau(t) - \tilde{U}'_\rho(t)|^2\, dt + a(\hat{U}_\tau(s) - \hat{U}_\rho(s)) \leq \\ &\quad \leq a(u_\tau^1 - u_\rho^1) + E_\tau^2(0,s) + E_\rho^2(0,s) + \int_0^s b(\hat{u}_\tau(t) - \hat{u}_\rho(t), \hat{U}'_\tau(t) - \hat{U}'_\rho(t))\, dt \end{aligned} \tag{4.20}$$

where:

$$\lim_{\tau\to 0^+} E_\tau(0,s) = 0 \qquad \textit{uniformly for } s \in [0,T]$$

Proof Let us choose $v = \hat{V}_\rho$ in (4.19); by usual calculations we obtain:

$$\int_0^s \Big[(\tilde{U}'_\tau, \tilde{U}'_\tau - \tilde{U}'_\rho) + a(\hat{U}'_\tau, \hat{U}_\tau - \hat{U}_\rho)\Big]\, dt + \int_0^s \check{\psi}^*(\hat{V}_\tau) - \check{\psi}^*(\hat{V}_\rho)\, dt \leq$$
$$\leq \int_0^s \Big\{(1-\ell_\tau)\Big[\check{\psi}^*(V_\tau) - \check{\psi}^*(V_\tau(t+\tau))\Big] + \frac{1}{2}\Big[|\tilde{U}'_\tau|^2 - |\hat{U}'_\tau|^2\Big]\Big\}\, dt +$$
$$+ \int_0^s \Big\{M_b \, \|\hat{u}_\tau - u_\tau\| \, |\hat{U}'_\tau| + \frac{1}{2}|\tilde{U}'_\tau - \hat{U}'_\tau|^2 + \frac{1}{4}|\tilde{U}'_\tau - \tilde{U}'_\rho|^2 - b(\hat{u}_\tau - \hat{u}_\rho, \hat{U}'_\tau)\Big\}\, dt$$

Changing the role of τ and ρ and summing up, we obtain (4.20) with:

$$E_\tau^2(0,s) =$$
$$\int_s^{s+\tau} \Big\{|\hat{U}'_\tau|^2 - 2\check{\psi}^*(V_\tau)\Big\}\, dt + \int_0^s \Big\{|\tilde{U}'_\tau - \hat{U}'_\tau|^2 + 2M_b\, \|\hat{u}_\tau - u_\tau\|\, |\hat{U}'_\tau|\Big\}\, dt + \tau\check{\psi}^*(V_\tau(0)) \quad \blacksquare$$

By a further application of Gronwall lemma we conclude:

4.6 Corollary. *$\{\hat{u}_\tau\}$ and $\{\tilde{u}_\tau\}$ are Cauchy's families in $C^0([0,T];V) \cap H^1(0,T;H)$, for every $T > 0$.* $\blacksquare$

5. - References.

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