SINGULAR PERTURBATION AND INTERPOLATION

C. BAIOCCHI

Math. Dep. "Guido Castelnuovo"

Università "La Sapienza", Piazzale A.Moro, Roma (Italy)

G. SAVARÉ

Istituto di Analisi Numerica del C.N.R.

Corso Carlo Alberto 5, Pavia (Italy)

It is well known that the rate of convergence of the solution u_ϵ of a singular perturbed problem to the solution u of the unperturbed equation can be measured in terms of the "smoothness" of u; smoothness which, in turn, can be expressed in terms of Linear Interpolation Theory. We want to prove a more close relationship between Interpolation and Singular perturbations, showing that interpolate spaces can be *characterized* by such a rate of convergence. Furthermore, with respect to a suitable (quite natural) definition of interpolation between *convex sets*, such a characterization holds true also in the framework of Variational Inequalities.

1. The problem

We are given two Hilbert spaces, V, H with $V \underset{dense}{\subset} H$; and a bilinear continuous form $a : V \times V \to \mathbf{R}$, coercive on $V \times V$. For any $u \in H, \epsilon > 0$, let u_ϵ be the solution of:

$$u_\epsilon \in V; \quad (u_\epsilon, v)_H + \epsilon^2 a(u_\epsilon, v) = (u, v)_H \ \forall v \in V \tag{1.1}$$

We want to study the behavior of u_ϵ as $\epsilon \longrightarrow 0$. One has:

Theorem 1.1 *For any $\theta \in]0,1[$, the following properties are equivalent* [a]*:*

$$\| u_\epsilon - u \|_H = O(\epsilon^\theta) \tag{1.2}$$

$$\| u_\epsilon \|_V = O(\epsilon^{\theta-1}) \tag{1.3}$$

$$u \in (V, H)_{1-\theta,\infty} \tag{1.4}$$

[a] We will use the notation $(V, H)_{\sigma,p}$ for the interpolate space constructed through a *real* method. A definition and some properties of such spaces are given in Section 2 later on; for a general treatment we refer to [LP], [BL]

More generally, for p given with $p \in [1, +\infty]$, let us denote by $L^p_*(0, +\infty; X)$ the space of functions defined (a.e.) in $(0, +\infty)$, valued in X which are in L^p with respect to the Haar measure $\frac{d\epsilon}{\epsilon}$. Then the following properties:

$$\epsilon^{-\theta}(u_\epsilon - u) \in L^p_*(0, +\infty; H) \tag{1.5}$$

$$\epsilon^{1-\theta} u_\epsilon \in L^p_*(0, +\infty; V) \tag{1.6}$$

$$u \in (V, H)_{1-\theta,p} \tag{1.7}$$

are equivalent. ∎

We will also prove that similar properties hold true for the solutions of VI (Variational Inequalities): let K be a closed convex subset of V, with $0 \in K$; and let u_ϵ be the solution of the VI:

$$u_\epsilon \in K;\ (u_\epsilon, u_\epsilon - v)_H + \epsilon^2 a(u_\epsilon, u_\epsilon - v) \leq (u, u_\epsilon - v)_H\ \forall v \in K \tag{1.8}$$

where u is now given with:

$$u \in \mathcal{K} := \text{ the closure of } K \text{ in } H. \tag{1.9}$$

With respect to a suitable definition for the set $(K, \mathcal{K})_{\sigma,p}$ [b], we will prove:
Theorem 1.2 *Let u be given in H and let u_ϵ be the solution of (1.8). The relation:*

$$u \in (K, \mathcal{K})_{\sigma,p} \tag{1.10}$$

holds if and only if (1.6) holds true; and if and only if both (1.5), (1.9) hold true. ∎

Furthermore, under a suitable "compatibility" assumptions, our definition for the set $(K, \mathcal{K})_{\theta,p}$ will provide both an "interpolation of intersections" result, say:

$$(K, \mathcal{K})_{\theta,p} = \mathcal{K} \bigcap (V, H)_{\theta,p}, \tag{1.11}$$

and a density result:

$$(K, \mathcal{K})_{\theta,p} = \overline{K}^{(V,H)_{\theta,p}}, \qquad 1 \leq p < +\infty. \tag{1.12}$$

2. Notations

Let us recall some basic properties and definitions in Interpolation theory. Confining ourselves to the case we will deal with, and following the notations of [LP], let us assume we are given two Banach spaces, A_0, A_1 with $A_0 \underset{dense}{\subset} A_1$. For $\theta_0, \theta_1, p_0, p_1$ given with $\theta_j \in \mathbf{R}, 1 \leq p_j \leq +\infty$, one denotes by $W(p_0, \theta_0, A_0; p_1, \theta_1, A_1)$ the space

[b] of course such a definition, when applied in the case of K subspace, will give rise to the usual interpolate space

of functions $u : t \mapsto u(t)$ defined (a.e.) in $\mathbf{R}$, valued into A_0, such that for $j = 0, 1$ one has $e^{t\theta_j} \cdot u(t) \in L^{p_j}(-\infty, +\infty; A_j)$. Provided with the obvious norm, it is a Banach space. Under the hypothesis (always assumed in the sequel) that $\theta_0 \cdot \theta_1 < 0$, one can show that

$$W(p_0, \theta_0, A_0; p_1, \theta_1, A_1) \subset L^1(-\infty, +\infty; A_1)$$

so that the following notation defines a subset of A_1:

$$S(p_0, \theta_0, A_0; p_1, \theta_1, A_1) := \left\{ \int_{\mathbf{R}} u(t)\, dt \;\middle|\; u \in W(p_0, \theta_0, A_0; p_1, \theta_1, A_1) \right\}.$$

Provided by the quotient norm such a space is a Banach space; in the sequel we will assume always $p_0 = p_1$, and we will shorten the notations by putting

$$(A_0, A_1)_{\theta,p} := S(p, \theta, A_0; p, \theta - 1, A_1) \quad (1 \le p \le +\infty, 0 < \theta < 1).$$

The essential feature of such a construction is the so called "linear interpolation property": given a second couple B_0, B_1, each linear continuous map $\mathcal{T}$ from A_1 to B_1 which [c] is linear continuous from A_0 to B_0, maps continuously $(A_0, A_1)_{\theta,p}$ into $(B_0, B_1)_{\theta,p}$; furthermore one has the following norm–estimate:

$$\| \mathcal{T} \|_{\mathcal{L}((A_0,A_1)_{\theta,p},(B_0,B_1)_{\theta,p})} \le C \cdot \| \mathcal{T} \|^{1-\theta}_{\mathcal{L}(A_0,B_0)} \cdot \| \mathcal{T} \|^{\theta}_{\mathcal{L}(A_1,B_1)} \tag{2.1}$$

Remark 2.1 For the solution u_ϵ of (1.1) one easily checks the estimates:

$$\| u_\epsilon \|_V \le C \cdot \| u \|_V ; \quad \| u_\epsilon \|_V \le C \cdot \epsilon^{-1} \cdot \| u \|_H$$

so that from (2.1) [d] we get $\| u_\epsilon \|_{(V,H)_{\theta,+\infty}} \le C \cdot \epsilon^{\theta-1} \| u \|_H$; this proves the implication $\{(1.4) \Longrightarrow (1.3)\}$. In a similar way, starting from the easy inequalities:

$$\| u_\epsilon - u \|_V \le C \cdot \| u \|_V ; \quad \| u_\epsilon - u \|_H \le C \cdot \epsilon \cdot \| u \|_V \quad \forall u \in V$$

we get the implication $\{(1.4) \Longrightarrow (1.2)\}$. ∎

Let us end up this section by recalling some further properties of our spaces, for which we refer to [BL], [LP], [Ta]:

- *Homogeneity*: for any $\lambda \ne 0$ one has [e]

$$(A_0, A_1)_{\theta,p} = S(p_0, \lambda\theta, A_0; p_1, \lambda(\theta - 1), A_1)$$

- *Haar measure*: in the definition of spaces $W(...), S(...)$ the change of variable $\epsilon := \exp(t)$ transforms the spaces $L^p(-\infty, +\infty)$ with respect to the Lebesgue measure dt and weights $e^{t\theta_j}$ into spaces $L^p(0, +\infty)$ with respect to the Haar measure $\frac{d\epsilon}{\epsilon}$ and weights ϵ^{θ_j}

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[c] of course: is the restriction $\mathcal{T}\big|_{A_0}$ of $\mathcal{T}$ to A_0 which is continuous from A_0 to B_0; in the sequel we will not explicitly mention such a restriction

[d] where, of course, $\mathcal{T}$ is the map $u \mapsto u_\epsilon$; and we choose $A_0 = H, A_1 = B_0 = B_1 = V$. The choice $p = +\infty$ corresponds, for each θ, to the greatest interpolate space

[e] here and in the following, "=" means algebraically identical, with equivalent norms

- *"Dual" definition*: one has

$$S(p_0,\theta_0,A_0;p_1,\theta_1,A_1) \ =$$
$$\left\{a \in A_1 \,\middle|\, a = a_0(t)+a_1(t) \text{ for suitable } a_j(t) \text{ with } e^{t\cdot\theta_j}a_j(t) \in L^{p_j}(-\infty,+\infty;A_j)\right\}$$

- *Traces*: let $x:\epsilon \mapsto x(\epsilon)$ be a differentiable function from $(0,+\infty)$ into A_0 such that:
$$\epsilon^{2-\theta}\frac{dx}{d\epsilon} \in L^p_*(0,+\infty;A_0);\epsilon^{1-\theta}\frac{dx}{d\epsilon} \in L^p_*(0,+\infty;A_1) \tag{2.2}$$
then the decomposition:
$$\begin{cases} x(0) = x_0(\epsilon)+x_1(\epsilon); \\ x_0(\epsilon) := \int_\epsilon^{+\infty}\left\{-\tau\frac{dx(\tau)}{d\tau}\right\}\frac{d\tau}{\tau}; \\ x_1(\epsilon) := \int_0^\epsilon\left\{-\tau\frac{dx(\tau)}{d\tau}\right\}\frac{d\tau}{\tau} \end{cases} \tag{2.3}$$
gives $x(0) \in (A_0,A_1)_{1-\theta,p}$ [f].
- *Non linear interpolation*: if $\mathcal{T}$ is a Lipschitz map from A_0 into B_0 which is bounded from A_1 into B_1, then $\mathcal{T}$ maps $(A_0,A_1)_{\theta,p}$ into $(B_0,B_1)_{\theta,p}$

3. The linear case

We will denote by $\|\cdot\|, |\cdot|, \|\cdot\|_*$ the norms in V, H, V^* respectively (V^* being the dual space of V); the scalar product in H, as well as the pairing V^*-V, will be denoted $(\cdot,\cdot)$. We associate to the form $a(\cdot,\cdot)$ the operator $A:V\to V^*$ defined by $(Au,v)=a(u,v)\forall u,v\in V$; so that (1.1) can be written in the form:

$$u_\epsilon \in V; \quad u_\epsilon + \epsilon^2 Au_\epsilon = u \tag{3.1}$$

We will also denote by $D=D(A,H)$ the set of $v\in V$ such that $Av\in H$; equipped with the norm $\|v\|_D := |Av|$ is a Hilbert space. Let us state some inclusions we will need [g]:
$$(V,V^*)_{\theta,p} \subset (V,H)_{2\theta,p} \quad \text{if } \theta < \frac{1}{2} \tag{3.3}$$
$$(V,V^*)_{\theta,p} \subset (H,V^*)_{2\theta-1,p} \quad \text{if } \theta > \frac{1}{2} \tag{3.4}$$
$$(D,H)_{\theta,p} \subset (V,H)_{2\theta-1,p} \quad \text{if } \theta > \frac{1}{2} \tag{3.5}$$

[f] because of the Hardy–inequality: we are in the framework of the dual definition with the Haar measure. For more general results concerning trace–interpolation see again [LP]

[g] they are particular cases of the "Reiteration Theorem"; see [LP] again

In fact, for $u \in (V,V^*)_{\theta,p}$, let $u(t) \in W(p,\theta,V;p,\theta-1,V^*)$ be such that $u = \int_{\mathbf{R}} u(t)\,dt$. The assumption $u(t) \in W$ reads:

$$e^{t\cdot\theta}u(t) \in L^p(-\infty,+\infty;V);\ e^{t\cdot(\theta-1)}u(t) \in L^p(-\infty,+\infty;V^*). \tag{3.5}$$

From the estimate

$$|\,u\,|^2 \leq \|\,u\,\|_* \cdot \|\,u\,\| \tag{3.6}$$

we then get

$$e^{t\cdot(\theta-\frac{1}{2})}u(t) \in L^p(-\infty,+\infty;H); \tag{3.7}$$

if $\theta < \frac{1}{2}$ we can couple the (3.7) with the first one of (3.5), thus getting (3.2) [h]; if $\theta > \frac{1}{2}$ we will couple (3.7) with the second one of (3.5) and we get (3.3). Relation (3.4) follows in a similar manner, the only change being the replacement of (3.6) by:

$$\|\,u\,\|^2 \leq C\cdot\|\,v\,\|_D \cdot |\,v\,|$$

which follows from the coerciveness:

$$\alpha\,\|\,u\,\|^2 \leq a(v,v) = (Av,v) \leq |\,Av\,|\cdot|\,v\,|$$

Proof of Theorem 1.1 First of all, remark that (1.2), (1.3) are nothing but a reformulation of (1.5), (1.6) when $p=+\infty$.
We put $t := \ln(\epsilon)$ and, for u_ϵ solution of (3.1), $u(t) := u_\epsilon\big|_{\epsilon=e^t}$, so that:

$$e^{2t}Au(t) = u - u(t) \tag{3.7}$$

and we can split u in the form:

$$u = u_0(t) + u_1(t);\quad u_0(t) := u(t),\ u_1(t) := u - u(t)$$

so that, because of (3.7), we have:

$$e^{\sigma t}u_0(t) \in L^p(-\infty,+\infty;V) \Longleftrightarrow e^{(\sigma-2)t}u_1(t) \in L^p(-\infty,+\infty;V^*)$$

$$e^{\sigma t}u_0(t) \in L^p(-\infty,+\infty;D) \Longleftrightarrow e^{(\sigma-2)t}u_1(t) \in L^p(-\infty,+\infty;H).$$

Using the "dual" representation of interpolate spaces and the homogeneity property, we have:

$$e^{\sigma t}u_0(t) \in L^p(-\infty,+\infty;V) \Longrightarrow u \in (V,V^*)_{\frac{\sigma}{2},p}\quad 0<\sigma<2$$

$$e^{(\sigma-2)t}u_1(t) \in L^p(-\infty,+\infty;H) \Longrightarrow u \in (D,H)_{\frac{\sigma}{2},p}\quad 0<\sigma<2.$$

Inserting the variable $t=\ln(\epsilon)$ into (1.5), (1.6), and using the inclusions (3.3), (3.5), we then get that either (1.5), (1.6) implies (1.7).

[h] because of the homogeneity property

Conversely, let us work directly in the framework of inequalities; let u_ϵ be the solution of (1.8), and assume we can decompose the datum u in the form [i]:

$$u = u_{0,\epsilon} + u_{1,\epsilon}; \begin{cases} \epsilon^{1-\theta} u_{0,\epsilon} \in L^p_*(0,+\infty;V), \ u_{0,\epsilon} \in K; \\ \epsilon^{-\theta} u_{1,\epsilon} \in L^p_*(0,+\infty;H). \end{cases} \tag{3.6}$$

We choose in (1.8) $v := u_{0,\epsilon}$ thus getting:

$$(u_\epsilon - u, u_\epsilon - u_{0,\epsilon}) + \epsilon^2 a(u_\epsilon, u_\epsilon - u_{0,\epsilon}) \le 0;$$

inserting the decomposition $u = u = u_{0,\epsilon} + u_{1,\epsilon}$ we then get:

$$(u_\epsilon - u, u_\epsilon - u) + \epsilon^2 a(u_\epsilon - u_{0,\epsilon}, u_\epsilon - u_{0,\epsilon}) \le (u_\epsilon - u, -u_{1,\epsilon}) + \epsilon^2 a(-u_{0,\epsilon}, u_\epsilon - u_{0,\epsilon})$$

from which we derive:

$$|\, u_\epsilon - u \,| + \epsilon \,\|\, u_\epsilon - u_{0,\epsilon} \,\| \le C_{\alpha,M} \,\{|\, u_{1,\epsilon} \,| + \epsilon \,\|\, u_{0,\epsilon} \,\|\}$$

where $C_{\alpha,M}$ depends only from the form a, say from its constants α (of coerciveness) and M (of continuity). From such an inequality, starting from the assumptions on $u_{j,\epsilon}$, we get both (1.5) and (1.6) [j]. ∎

4. Interpolation of convex sets

Let us summarize some (mostly well known) properties of the solution u_ϵ of (1.8); in the following Lemma we use notation (1.9) and we denote by $P_{\mathcal{K}}$ the projection (with respect to the H–norm) of H onto $\mathcal{K}$:

Lemma 4.1 *There exists $C = C_{\alpha,M}$ such that, for each $u \in H$, the solution u_ϵ of (1.8) satisfies:*

$$|\, u_\epsilon \,| + \epsilon \cdot \|\, u_\epsilon \,\| \le C \cdot |\, u \,| ; \tag{4.1}$$

$$\textit{as } \epsilon \mapsto 0, \ \textit{it is} \begin{cases} \epsilon u_\epsilon \mapsto 0 & \textit{weakly in } V; \\ u_\epsilon \mapsto P_{\mathcal{K}}(u) & \textit{weakly in } H; \end{cases} \tag{4.2}$$

Furthermore the map $\epsilon \mapsto u_\epsilon$ is differentiable; and denoting by u_ϵ' its derivative, one has:

$$|\, u_\epsilon' |^2 + \epsilon^2 \,\|\, u_\epsilon' \|^2 \le -2\epsilon a(u_\epsilon, u_\epsilon'); \tag{4.3}$$

$$(u_\epsilon - u, u_\epsilon') + \epsilon^2 a(u_\epsilon, u_\epsilon') = 0. \tag{4.4}$$

Proof We first choose in (1.8) $v = 0$; from the coerciveness and the continuity of a we get (4.1). In particular, for $\epsilon \mapsto 0$, the family $\{\epsilon u_\epsilon\}$ will converge (strongly in

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[i] this assumption, in the case $K = V$, coincides with (1.7): use the dual representation and the Haar measure. For a generic convex K, (3.6) will be the *definition* of $(K, \mathcal{K})_{1-\theta,p}$; see Def. 4.1 later on

[j] of course we don't care about the measurability of $\epsilon \mapsto u_\epsilon \ldots$

H) to 0; and [k] $\{\epsilon u_\epsilon\}$ converges to 0 weakly in V, and u_ϵ converges weakly in H to some $u_0 \in H$, or better $u_0 \in \mathcal{K}$ because of $u_\epsilon \in K$. In order to complete the proof of (4.2) we only need the inequality:

$$(u_0 - u, u_0 - v) \leq 0 \quad \forall v \in K$$

(the definition of $P_\mathcal{K}$ would require the inequality for $v \in \mathcal{K}$, but we can confine ourselves to prove the inequality only for $v \in K$ because of the density). Such inequality follows from (1.8), by passing to the limit as $\epsilon \to 0$.

In order to prove the "smoothness" of u_ϵ , we can use the method of the differential quotients; more precisely, let us first choose in (1.8) $v = u_{\tilde\epsilon}$ with $\tilde\epsilon = \epsilon + \Delta\epsilon$; then we write (1.8) in the point $\tilde\epsilon$ and we choose in it $v = u_\epsilon$. Summing up those inequalities we get:

$$\left|\frac{\Delta u_\epsilon}{\Delta\epsilon}\right|^2 + \epsilon^2 a(\frac{\Delta u_\epsilon}{\Delta\epsilon}, \frac{\Delta u_\epsilon}{\Delta\epsilon}) \leq (\epsilon + \tilde\epsilon) a(-u_{\tilde\epsilon}, \frac{\Delta u_\epsilon}{\Delta\epsilon})$$

in which we can pass to the limit, thus obtaining the existence of the derivative of u_ϵ as well as inequality (4.3).

Once the existence of $u_\epsilon{}'$ is known, writing (1.8) with $v = u_{\tilde\epsilon}$ and dividing by $\epsilon - \tilde\epsilon$ (which can have either sign) one can pass to the limit, thus obtaining (4.4). ∎

Remark 4.1 From (4.3), we get

$$|\, u_\epsilon{}' | + \epsilon \,\|\, u_\epsilon{}' \| \leq C_{\alpha,M} \,\|\, u_\epsilon \,\| \tag{4.5}$$

while, inserting (4.4) into (4.3), we get:

$$|\, u_\epsilon{}' | + \epsilon \,\|\, u_\epsilon{}' \| \leq C_{\alpha,M} \frac{1}{\epsilon} \,|\, u_\epsilon - u \,| \,. \tag{4.6}$$

Also remark that convergencies in (4.2) are in fact *strong*; however we will not use such a feature. ∎

In order to prove Thm. 1.2 we need of course a definition for the set $(K, \mathcal{K})_{\sigma,p}$; following the "dual" definition of interpolate spaces we will put:

Definition 4.1 *Let K be a convex closed subset of V, containing the origin* [l]*; and let $\mathcal{K}$ be the closure of K in H. For $\sigma \in (0,1)$ and $p \in [1, +\infty]$ we denote by $(K, \mathcal{K})_{\sigma,p}$ the set of the u's which admit a decomposition $u = u_0(t) + u_1(t)$ for suitable $u_j(t)$ with:*

$$\begin{cases} e^{t\sigma} u_0(t) \in L^p(-\infty, +\infty; V),\ u_0(t) \in K; \\ e^{t(\sigma-1)} u_1(t) \in L^p(-\infty, +\infty; H). \end{cases} \tag{4.7}$$

Of course the use of the Haar's measure allows to write such a definition in the form:

$$u = u_{0,\epsilon} + u_{1,\epsilon}; \begin{cases} \epsilon^\sigma u_{0,\epsilon} \in L^p_*(0, +\infty; V),\ u_{0,\epsilon} \in K; \\ \epsilon^{\sigma-1} u_{1,\epsilon} \in L^p_*(0, +\infty; H). \end{cases} \tag{4.8}$$

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[k] possibly by extraction of a subsequence; but the entire family will converge, because of the uniqueness of the limit

[l] of course, via a translation, we can adapt our definition to any non–empty closed convex set

Proof of Theorem 1.2 Let us assume that, for u_ϵ solution of (1.8), we have either both (1.5) and (1.10); or just (1.6) (which in turn implies (1.10)). In either case (because of (4.5) or (4.6)) we will have:

$$\epsilon^{1-\theta}\{|\, u_\epsilon'| + \epsilon \,\|\, u_\epsilon' \|\} \in L^p_*(0, +\infty). \tag{4.9}$$

Then, using also (4.2) and $P_{\mathcal{K}} u = u$, we decompose u in the form

$$u = [u_\epsilon] + [u - u_\epsilon] = [\int_\epsilon^{+\infty} -u_\tau' d\tau] + [\int_0^\epsilon -u_\tau' d\tau]$$

so that, because of (4.9), we are in the framework of (2.2), (2.3) and formula (4.8) holds true.

The converse implications being already proved in Section 3, Theorem 1.2 is completely proved. ∎

Remark 4.2 Of course, this gives a second proof of Thm. 1.1. ∎

5. Further properties of the interpolated convex sets

Apart from the equivalencies stated in Thm. 1.2, Def. 4.1 seems quite natural for a number of reasons; let us state, without proofs, the easiest ones:

- $(K, \mathcal{K})_{\sigma,p}$ is a convex set.
- When $K = V$ we get the usual interpolate *space*.
- Let V_1, H_1 be a second couple of Hilbert spaces, let K_1 be a convex closed subset of V_1 containing the origin, and let $\mathcal{T}$ be given with:

$$\begin{cases} \mathcal{T} : H \to H_1 & \text{is Lipschitz continuous;} \\ \mathcal{T}(K) \subseteq K_1 & \text{and } \|\, \mathcal{T}(k)\,\|_{V_1} \le C \cdot \|\, k\,\| \,\forall k \in K: \end{cases} \tag{5.1}$$

then (denoting by $\mathcal{K}_1$ the closure in H_1 of K_1) $\mathcal{T}$ maps $(K, \mathcal{K})_{\theta,p}$ into $(K_1, \mathcal{K}_1)_{\theta,p}$.

If we study the relations between $(K, \mathcal{K})_{\sigma,p}$ and the corresponding interpolation space $(V, H)_{\sigma,p}$, we have in general the inclusions:

Lemma 5.1 *For* $\sigma \in]0, 1[$ *and* $p \in [1, +\infty[$

$$(K, \mathcal{K})_{\sigma,p} \subseteq \overline{K}^{(V,H)_{\sigma,p}} \subseteq \mathcal{K} \bigcap (V, H)_{\sigma,p} \tag{5.2}$$

Proof The second inclusion being obvious, we only consider the first one.

Recall that, if $\tau \mapsto x(\tau)$ is given with

$$\tau^\sigma x(\tau) \in L^p_*(0, +\infty; V), \quad \tau^{\sigma-1} x(\tau) \in L^p_*(0, +\infty; H) \tag{5.3}$$

then for $x = \int_0^\infty x(\tau)\, \frac{d\tau}{\tau}$ one has

$$\|x\|_{(V,H)_{\sigma,p}} \le C \cdot \|\tau^\sigma x(\tau)\|^{1-\sigma}_{L^p_*(0,+\infty;V)} \cdot \|\tau^{\sigma-1} x(\tau)\|^{\sigma}_{L^p_*(0,+\infty;H)} \tag{5.4}$$

We choose $x(\tau) = \tau\, u'_\tau \cdot \chi_{]0,\epsilon[}(\tau)$, so that $x = u_\epsilon - u$ and then:

$$\|u - u_\epsilon\|^p_{(V,H)_{\sigma,p}} \le C \cdot \left\{ \int_0^\epsilon \left(\tau^{1+\sigma} \|u'_\tau\|\right)^p \frac{d\tau}{\tau} \right\}^{1-\sigma} \cdot \left\{ \int_0^\epsilon \left(\tau^\sigma |u'_\tau|\right)^p \frac{d\tau}{\tau} \right\}^\sigma \tag{5.5}$$

By (4.9), with $\sigma = 1 - \theta$, this quantity tends to 0 when $\epsilon \to 0$. [m] ∎

We can reverse the inclusions (5.2) (and consequently prove (1.11) and (1.12)) under the assumption:

$$P_{\mathcal{K}}(V) \subseteq K; \;\; \| P_{\mathcal{K}}(v) \| \le C \| v \| \quad \forall v \in V. \tag{5.6}$$

Theorem 5.1 *Under the assumption (5.6) one has (1.11) and (1.12).*
Proof Starting from any $u \in \mathcal{K} \bigcap (V,H)_{\sigma,p}$ and from a corresponding decomposition $u = u_0(t) + u_1(t)$ as in the dual representation of interpolate spaces, we put $u = P_{\mathcal{K}}(u_0(t)) + [u - P_{\mathcal{K}}(u_0(t)]$.
Because of (5.6) the first term is in K and has a norm in V bounded by $C \| u_0(t) \|$; the second term, taking into account $u \in \mathcal{K}$, can be written and estimated through:

$$| P_{\mathcal{K}}(u) - P_{\mathcal{K}}(u_0(t)) | \le | u - u_0(t) | = | u_1(t) |$$

so that we have a decomposition like (4.7) and the theorem is proved. ∎

A sufficient condition for this relation is related to an "abstract maximum principle" introduced in [BS] as a "compatibility" relation between K and A: the couple $\{A, K\}$ is said compatible if:

$$\forall \eta > 0 \;\text{ one has }\; (I + \eta A)^{-1}(K) \subset K. \tag{5.7}$$

Let us show a consequence of (5.7):

Lemma 5.2 *Under the assumption (5.7), $\forall u \in H$ the solution u_ϵ of (1.8) satisfies:*

$$Au_\epsilon \in H, \quad (u_\epsilon - u, Au_\epsilon) + \epsilon^2 \, | Au_\epsilon |^2 \le 0. \tag{5.8}$$

Proof For fixed ϵ , and with $\eta > 0$ which will tend to 0, we put $v_\eta := (I + \eta A)^{-1} u_\epsilon$ so that

$$u_\epsilon - v_\eta = \eta A v_\eta \tag{5.9}$$

and, because of (5.7), we can choose $v = v_\eta$ in (1.8) thus getting:

$$(u_\epsilon - u, Av_\eta) + \epsilon^2 \, | Av_\eta | \le 0. \tag{5.10}$$

In order to prove (5.8), we only need to prove that

$$| Av_\eta | \text{ remains bounded as } \eta \mapsto 0 \tag{5.11}$$

[m] Of course, we also have $(K, \mathcal{K})_{\sigma,\infty} \subseteq \mathcal{K} \cap (V,H)_{\sigma,\infty}$; in general, the density of K in $(K, \mathcal{K})_{\sigma,\infty}$ and in $\mathcal{K} \cap (V,H)_{\sigma,p}$ is false even if K is a subspace.

because from (5.9) we then get $v_\eta \mapsto u_\epsilon$ so that Av_η must converge (in H) to Au_ϵ; (5.8) will then follow by passing to the limit in η into (5.10).
In order to get (5.11) we start from the Minty's formulation of (1.8):

$$(v-u, u_\epsilon - v) + \epsilon^2 a(v, u_\epsilon - v) \leq 0 \quad \forall v \in K$$

and we choose $v = v_\eta$, so that:

$$(v_\eta - u, Av_\eta) + \epsilon^2 \mid Av_\eta \mid^2 \leq 0$$

which obviously implies (5.11). ∎

Corollary 5.1 *A sufficient condition in order to have (5.6) is that:*
there exists a coercive $A \in \mathcal{L}(V, V^*)$ such that $\{A, K\}$ is compatible.

Proof In fact, for $u \in V$, we can represent $P_{\mathcal{K}}(u)$ as the limit in H of the family $\{u_\epsilon\}\big|_{\epsilon \mapsto 0}$; (5.8) now gives

$$a(u_\epsilon, u_\epsilon) + \epsilon^2 \mid Au_\epsilon \mid^2 \leq a(u, u_\epsilon)$$

so that $\alpha \| u_\epsilon \| \leq M \| u \|$; passing to the limit in ϵ and using (4.2), we get (5.6). ∎

6. Domains of convex function

We can ask what kind of structure (besides the convexity, of course) are preserved by the interpolation of convex sets. The simplest one is perhaps the following: our convex K can be regarded as the domain of the convex function $\phi : u \mapsto \|u\| + I_K(u)$ [n] which is lower semi continuous with respect to the $H-$topology; we want to show that $(K, \mathcal{K})_{\sigma,p}$ is such a domain, too.

Let us start from $u \in (K, \mathcal{K})_{\sigma,p}$ and a decomposition $u = u_{0,\epsilon} + u_{1,\epsilon}$ as in (4.8); we can evaluate its growth taking the sum of the two norms in the corresponding L^p_* spaces, or, what is equivalent, considering the integral: [o]

$$\left\{ \int_0^\infty \left[\epsilon^{(\sigma-1)} \left(\epsilon^2 \|u_{0,\epsilon}\|^2 + |u_{1,\epsilon}|^2 \right)^{1/2} \right]^p \frac{d\epsilon}{\epsilon} \right\}^{\frac{1}{p}} \tag{6.1}$$

As is usual in Interpolation Theory, we try to minimize (6.1) over all the admissible decompositions of u; this lead to minimize the integrand and to introduce the quantity: [p]

$$Y(\epsilon, u) := \inf \left\{ \left(\epsilon^2 \|u_0\|^2 + |u_1|^2 \right)^{1/2}; \quad u_0 \in K, \ u_1 \in H, \ u_0 + u_1 = u \right\} \tag{6.2}$$

---

[n] I_k is the indicatrix function of K: $I_K(u) = \begin{cases} 0 & \text{if } u \in K; \\ +\infty & \text{if } u \notin K. \end{cases}$

[o] with obvious changes, when $p = +\infty$

[p] which is strictly related to the Peetre's $K-$functional and to the Yosida approximation of ϕ.

or:

$$Y(\epsilon,u)=\inf_{w\in K}\left\{|u-w|^2+\epsilon^2\|w\|^2\right\}^{1/2}=\inf_{w\in H}\left\{|u-w|^2+\epsilon^2\phi^2(w)\right\}^{1/2} \tag{6.3}$$

Let us choose

$$a(\cdot,\cdot):=\text{ scalar product of } V;$$

then, for the solution $\tilde{u}_\epsilon$ of (1.8) we get:

$$Y(\epsilon,u)=\left\{|u-\tilde{u}_\epsilon|^2+\epsilon^2\|\tilde{u}_\epsilon\|^2\right\}^{1/2} \tag{6.4}$$

so that the infimum (6.3) is really a minimum.

We recall three simple properties of $Y(\epsilon,u)$:

Lemma 6.1 *For every fixed $\epsilon>0$, the functional $u\mapsto Y(\epsilon,u)$ is a positive convex Lipschitz function defined on H, with Lipschitz constant ≤ 1; once u is fixed, $Y(\epsilon,u)$ is increasing in ϵ.*

Proof Observe that, for every $v\in H$ and $\epsilon>0$, the map:

$$u\in H\mapsto\left(|u-v|^2+\epsilon^2\|v\|^2\right)^{1/2}$$

is convex; so $Y(\epsilon,u)$ is convex with respect to u, being the infimum of a family of convex functions. If u,v are in H, we have:

$$Y(\epsilon,v)-Y(\epsilon,u)\le\left\{|v-\tilde{u}_\epsilon|^2+\epsilon^2\|\tilde{u}_\epsilon\|^2\right\}^{1/2}-\left\{|u-\tilde{u}_\epsilon|^2+\epsilon^2\|\tilde{u}_\epsilon\|^2\right\}^{1/2}\le|v-u|$$

Changing the role of u and v, we prove the contraction property. The last statement is obvious. ∎

A straightforward use of Fatou's Lemma gives:

Corollary 6.1 *The integral:*

$$\phi_{\sigma,p}(u)=\left\{\int_0^\infty\left[\epsilon^{\sigma-1}Y(\epsilon,u)\right]^p\frac{d\epsilon}{\epsilon}\right\}^{1/p}=\|\epsilon^{\sigma-1}Y(\epsilon,u)\|_{L^p_*(0,+\infty)} \tag{6.5}$$

is well defined; $\phi_{\sigma,p}$ is a proper convex l.s.c. function whose domain is $(K,\mathcal{K})_{\sigma,p}$. ∎

Remark 6.1 Let $J:V\mapsto V^*$ be the duality mapping induced by the scalar product of V; if K and J are compatible (in the sense of (5.3)), then $\phi_{\sigma,p}$ has the same structure of ϕ, that is:

$$\phi_{\sigma,p}(u)=\|u\|_{(V,H)_{\sigma,p}}+I_{(K,\mathcal{K})_{\sigma,p}}(u) \tag{6.6}$$

since the minima of (6.3) in K and in V are the same, if $u\in\mathcal{K}$. Note that this hypothesis is equivalent to (5.6) with $C=1$; when $C>1$ we have the bound:

$$\|u\|_{(V,H)_{\sigma,p}}\le\phi_{\sigma,p}(u)\le C\,\|u\|_{(V,H)_{\sigma,p}}+I_{(K,\mathcal{K})_{\sigma,p}}(u). \quad ∎$$

In this context, we can give an obvious refinement of the nonlinear interpolation theorem: if $\mathcal{T}$ is as in (5.1) with Lipschitz constant L and $\psi = \|\cdot\|_{V_1} + I_{K_1}$, then we have:

$$\forall u \in (K, \mathcal{K})_{\sigma,p}, \qquad \psi_{\sigma,p}\big(\mathcal{T}u\big) \le L^\sigma\, C^{1-\sigma}\, \phi_{\sigma,p}(u) \tag{6.7}$$

As a simple application, we can prove a "density" result which holds also for $p = +\infty$:

Corollary 6.2 *If $u \in (K, \mathcal{K})_{\sigma,p}$ and $\tilde u_\epsilon \in K$ is given by (6.4), then:*

$$\lim_{\epsilon\to 0} \phi_{\sigma,p}(\tilde u_\epsilon) = \phi_{\sigma,p}(u) \tag{6.8}$$

Proof We know from Lemma 4.1 and the lower semi continuity of $\phi_{\sigma,p}$ that:

$$\liminf_{\epsilon\to 0} \phi_{\sigma,p}(\tilde u_\epsilon) \ge \phi_{\sigma,p}(u) \tag{6.9}$$

(6.8) will be proved if we show that:

$$\phi_{\sigma,p}(\tilde u_\epsilon) \le \phi_{\sigma,p}(u), \quad \forall \epsilon > 0 \tag{6.10}$$

We already observed that the map $u \mapsto \tilde u_\epsilon$ is a contraction of H; on the other side, if $u \in K$ we get $\|\tilde u_\epsilon\| \le \|u\|$ by the minimum property. Applying (6.7) we conclude our proof. ∎

At this point, it seems to be natural the following:

Definition 6.1 *Let us given a convex l.s.c. function $\phi : H \mapsto [0, +\infty]$ with $\phi(0) = 0$ and domain $D_\phi = \{u \in H : \phi(u) < +\infty\}$. We denote by $(D_\phi, H)_{\sigma,p}$ the subset of H whose elements u can be approximated from D_ϕ in the usual way:*

$$\exists\, \epsilon \mapsto x_\epsilon \in D_\phi : \quad \epsilon^{\sigma-1}(x_\epsilon - u) \in L^p_*(0, +\infty; H), \quad \epsilon^\sigma \phi(x_\epsilon) \in L^p_*(0, +\infty) \tag{6.11}$$

Setting $Y_\phi(\epsilon, u)$ as in (6.3), $(D_\phi, H)_{\sigma,p}$ is the domain of the convex l.s.c. function $\phi_{\sigma,p}$ given by (6.5). ∎

The problem of minimizing (6.3) is now associated to the variational inequality:

$$u_\epsilon \in D_\phi; \quad 2(u_\epsilon - u, u_\epsilon - v) + \epsilon^2\big[\phi^2(u_\epsilon) - \phi^2(v)\big] \le 0, \quad \forall\, v \in D_\phi \tag{6.12}$$

It is interesting that a result like Thm. 1.2 holds true in this case, too.

Theorem 6.2 *Let us given ϕ as in definition 6.1, $\theta \in]0, 1[$ and $p \in [1, +\infty]$; for every $u \in \overline{D_\phi}^H$ the following properties are equivalent:*

$$\epsilon^{-\theta}(u_\epsilon - u) \in L^p_*(0, +\infty; H) \tag{6.13}$$

$$\epsilon^{1-\theta}\phi(u_\epsilon) \in L^p_*(0, +\infty) \tag{6.14}$$

$$u \in (D_\phi, H)_{1-\theta,p} \tag{6.15}$$

where u_ϵ is the solution of (6.12).

Proof Setting $x_\epsilon = \phi^2(u_\epsilon)$ and arguing as in Lemma 4.1, we get:

$$\left|\frac{\Delta u_\epsilon}{\Delta \epsilon}\right|^2 + \epsilon \frac{\Delta x_\epsilon}{\Delta \epsilon} + \frac{\Delta x_\epsilon}{2} \le 0, \qquad 2(u_\epsilon - u, \Delta u_\epsilon) + \epsilon^2 \Delta x_\epsilon \le 0$$

From the first inequality we deduce that x_ϵ is decreasing and u_ϵ is absolutely continuous on the compact subsets of $]0,+\infty[$ with values in H; from the second one we find that $|u_\epsilon - u|$ is increasing and x_ϵ is absolutely continuous, too. Passing to the limit for $\Delta\epsilon \to 0$ we have:

$$|u_\epsilon'|^2 + \epsilon\, x_\epsilon' \le 0, \qquad \frac{d}{d\epsilon}|u_\epsilon - u|^2 = -\epsilon^2\, x_\epsilon' \tag{6.16}$$

collecting these relations we have:

$$\epsilon^2\, x_\epsilon' \le \frac{2}{\epsilon}|u - u_\epsilon|^2 \tag{6.17}$$

and

$$|u_\epsilon - u|^2 = -\int_0^\epsilon \tau^2\, x_\tau'\, d\tau \le 2\int_0^\epsilon \tau\, x_\tau\, d\tau \tag{6.18}$$

From (6.17) we find:

$$\epsilon^{2(1-\theta)} x_\epsilon \le 2\int_\epsilon^\infty \left(\frac{\epsilon}{\tau}\right)^{2(1-\theta)} \tau^{-2\theta}|u - u_\tau|^2\, \frac{d\tau}{\tau}$$

so that: [q]

$$\tau^{-2\theta}|u - u_\tau|^2 \in L_*^{p/2}(0,+\infty) \Rightarrow \epsilon^{2(1-\theta)} x_\epsilon \in L_*^{p/2}(0,+\infty)$$

and from (6.13) we can obtain (6.14).

Analogously, (6.18) gives:

$$\epsilon^{-2\theta}|u - u_\epsilon|^2 \le 2\int_0^\epsilon \left(\frac{\epsilon}{\tau}\right)^{-2\theta} \tau^{2(1-\theta)} x_\tau\, \frac{d\tau}{\tau}$$

showing the converse. ■

Remark 6.2 In the framework of interpolation between domains of convex l.s.c. functions, one can try to reproduce some fundamental results which hold for Banach spaces. Let us state without proof a simple reiteration formula for convex sets with obvious notations:

$$\big((K,\mathcal{K})_{\sigma,p},\mathcal{K}\big)_{\eta,q} = (K,\mathcal{K})_{\bar\eta,q}, \qquad \bar\eta = (1-\eta)\sigma + \eta$$

[q] it is a Hardy–type inequality, easy to proof also for $1 \le p < 2$ because of the monotonicity property of $\epsilon \mapsto x_\epsilon$ and $\epsilon \mapsto |u_\epsilon - u|$

Such type of results as well as further related properties will be detailed In a forthcoming paper. ∎

7 Final Remarks

We exploited a strict relation between interpolate spaces and singular perturbations; we confined ourselves to perturb the Identity operator, but at a first look it seems not difficult, *at least for linear problems*, to adapt our theory to more general situations, like the ones studied in [Li]. Non linear interpolation theory was proposed by [Ta], where in particular interpolation between *open* sets was investigated; we propose here a definition concerning *closed convex* sets. Completely satisfactory results are obtained under a compatibility condition; in the general case, the theory still presents open problems.

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REFERENCES

[BL] J.Bergh and J.Löfstrom, Interpolation Spaces, an Introduction. Springer, Berlin, 1976.

[BS] H.Brézis and G.Stampacchia, *Sur la régularité des solutions d'inéquations elliptiques*, Bull. Soc. Math. France **96** (1973) 129–132

[Li] J.L.Lions, Perturbations Singulières dans les Problèmes aux Limites et en Contrôle Optimal. Springer, Berlin, 1973

[LP] J.L.Lions and J.Peetre, *Sur une classe d'espaces d'interpolation*, Pubbl. I.H.E.S. **19** (1964) 5–68

[Ta] L.Tartar, *Interpolation non linéaire et régularité*, J. Funct. Anal. **9** (1972) 469–489