Diffusion, optimal transport and Ricci curvature

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Abstract. Starting from the pioneering paper of Otto-Villani [73], the link between Optimal Transport and Ricci curvature in smooth Riemannian geometry has been deeply studied [35, 86]. Among the various functional and analytic applications, the point of view of Optimal Transport has played a crucial role in the Lott-Sturm-Villani [84, 85, 69, 87] formulation of a "synthetic" notion of lower Ricci curvature bound, which has been extended from the realm of smooth Riemannian manifold to the general framework of metric measure spaces $(X, \mathsf{d}, \mathfrak{m})$, i.e. (separable, complete) metric spaces endowed with a finite or locally finite Borel measure $\mathfrak{m}$.

Lower Ricci curvature bounds can also be captured by the celebrated Bakry-Émery [21] approach based on Markov semigroups, diffusion operators and Γ-calculus for strongly local Dirichlet forms [22].

We will discuss a series of recent contributions [7, 8, 9, 5, 38, 12] showing the link of both the approaches with the metric-variational theory of gradient flows [6] and diffusion equations. As a byproduct, when the Cheeger energy on $(X, \mathsf{d}, \mathfrak{m})$ is quadratic (or, equivalently, the Sobolev space $W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is Hilbertian), we will show that the two approaches lead to essentially equivalent definitions and to a nice geometric framework suitable for deep analytic results.

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Introduction. This paper provides a brief and informal introduction to the recent series of contributions [7, 8, 9, 5, 38, 12], showing the equivalence of the Bakry-Émery [21] and of the Lott-Sturm-Villani [84, 85, 69, 87] approaches to lower Ricci curvature bounds in metric measure/energy spaces. Starting by the simpler, still relevant, smooth Riemannian case, we will discuss the basic features of two formalisms and we will show how the variational/metric theory of gradient flows provide a unifying point of view, which essentially lies behind the equivalence proof.

1. The Bakry-Émery approach to lower Ricci bounds

In this section we will illustrate the Bakry-Émery approach to lower Ricci curvature bound, leading to the so-called $\mathrm{BE}(K, N)$ curvature-dimension condition. We first start from a simple example in the Euclidean space.

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1.1. Pointwise gradient estimates along the Euclidean Heat flow. Let H_t be the Heat flow in $\mathbb{R}^d$, expressing the solution $f_t = \mathsf{H}_t f$ of the heat equation

$$\partial_t f_t = \Delta f_t \quad \text{in } (0,\infty)\times\mathbb{R}^d, \quad \lim_{t\downarrow 0} f_t(x) = f(x),$$

which can also be obtained through the convolution formula

$$f_t(x) = \int_{\mathbb{R}^d} f(y) g_t(x-y)\,\mathrm{d}y, \quad g_t(x) := \frac{1}{(4\pi t)^{d/2}}\exp(-|x|^2/4t). \tag{1}$$

Whenever $f \in \mathrm{C}^1_b(\mathbb{R}^d)$, (1) yields

$$|\nabla f_t(x)| \le \int_{\mathbb{R}^d} |\nabla f(y)| g_t(x-y)\,\mathrm{d}y = \mathsf{H}_t(|\nabla f|)(x),$$

and Jensen's inequality immediately gives for every $p \ge 1$

$$\big|\nabla \mathsf{H}_t f\big|^p \le \mathsf{H}_t\big(|\nabla f|^p\big) \quad \text{in } \mathbb{R}^d. \tag{2}$$

Following Bakry and Ledoux, we can also derive the elementary inequality (2) for $p = 2$ by a differential argument and the basic identity

$$\frac{1}{2}\Delta\big(|\nabla u|^2\big) - \langle \nabla u, \nabla\Delta u\rangle = \big|\nabla^2 u\big|^2, \quad u \in \mathrm{C}^\infty(\mathbb{R}^d). \tag{3}$$

For a fixed time $t > 0$ and a given $f \in \mathrm{C}^1_b(\mathbb{R}^d)$ one considers the function

$$\mathfrak{b}(s) := \mathsf{H}_s\big(|\nabla \mathsf{H}_{t-s} f|^2\big) \quad s \in [0,t], \tag{4}$$

observing that

$$\mathfrak{b}(0) = |\nabla \mathsf{H}_t f|^2, \quad \mathfrak{b}(t) = \mathsf{H}_t\big(|\nabla f|^2\big),$$

so that (2) will follows if we can show that $s \mapsto \mathfrak{b}(s)$ is nondecreasing. Evaluating the derivative of $\mathfrak{b}$ for $s \in (0,t)$ and using the fact that $\frac{\mathrm{d}}{\mathrm{d}s}\mathsf{H}_s f = \Delta \mathsf{H}_s f$ we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\mathfrak{b}(s) &= \Delta\mathsf{H}_s\big(|\nabla \mathsf{H}_{t-s} f|^2\big) - 2\mathsf{H}_s\Big(\langle \nabla \mathsf{H}_{t-s} f, \nabla\Delta \mathsf{H}_{t-s} f\rangle\Big)\\ &= 2\mathsf{H}_s\Big(\frac{1}{2}\Delta\big(|\nabla \mathsf{H}_{t-s} f|^2\big) - \langle \nabla \mathsf{H}_{t-s} f, \nabla\Delta \mathsf{H}_{t-s} f\rangle\Big) \overset{(3)}{=} 2\mathsf{H}_s\Big(\big|\nabla^2 \mathsf{H}_{t-s} f\big|^2\Big) \ge 0.\end{aligned}$$

1.2. Drift-diffusion equations and the Riemannian framework. One of the main advantages of the Bakry-Ledoux method (with striking applications to deep geometric and functional analytic results [24, 20, 25, 67, 88]) is that it can be easily extended to more general equations, where the Laplacian is substituted by a diffusion operator in divergence form [22, Sect. 1.11.3] with a drift term.

The natural framework for such equations is indeed a complete Riemannian manifold $(\mathbb{M}, \mathsf{g})$ of dimension d with a smooth potential $V : \mathbb{M} \to \mathbb{R}$, inducing the diffusion operator

$$\mathsf{A} f := \Delta_{\mathsf{g}} f - \langle \nabla V, \nabla f\rangle_{\mathsf{g}},$$

where Δ_{g} denotes the Laplace-Beltrami operator. In this setting, (3) corresponds to the Bochner-Lichnerowicz formula

$$\frac{1}{2}\Delta_{\mathsf{g}}\big(|\nabla u|_{\mathsf{g}}^2\big) - \langle \nabla u, \nabla \Delta_{\mathsf{g}} u\rangle_{\mathsf{g}} = \big|\nabla^2 u\big|_{\mathsf{g}}^2 + \mathrm{Ric}_{\mathsf{g}}(\nabla u, \nabla u), \tag{5}$$

where $\mathrm{Ric}_{\mathsf{g}}$ denotes the Ricci curvature tensor of $(\mathbb{M}, \mathsf{g})$; by taking into account the drift contribution we end up with

$$\frac{1}{2}\mathsf{A}\big(|\nabla u|_{\mathsf{g}}^2\big) - \langle \nabla u, \nabla \mathsf{A} u\rangle_{\mathsf{g}} = \big|\nabla^2 u\big|_{\mathsf{g}}^2 + \mathrm{Ric}_{\mathsf{g}}(\nabla u, \nabla u) + \nabla_{\mathsf{g}}^2 V(\nabla u, \nabla u). \tag{6}$$

If for some $N \in (d, +\infty]$ the Ricci tensor and the Hessian of V satisfy the condition

$$\mathrm{Ric}_{\mathsf{g}}(\xi,\xi) + \nabla_{\mathsf{g}}^2 V(\xi,\xi) \geq K|\xi|_{\mathsf{g}}^2 + \frac{1}{N-d}\langle \nabla V, \xi\rangle_{\mathsf{g}}^2 \quad \text{for every tangent vector } \xi, \tag{7}$$

then we can adapt the previous strategy to the solution $f_t = \mathsf{P}_t f$ of the equation

$$\partial_t f_t = \mathsf{A} f_t \quad \text{in } (0,\infty)\times\mathbb{M}, \quad \lim_{t\downarrow 0} f_t = f. \tag{8}$$

If we take into account the dimension d of $\mathbb{M}$ and the positive contribution of

$$|\nabla^2 u|_{\mathsf{g}}^2 \geq \frac{1}{d}(\Delta_{\mathsf{g}} u)^2,$$

we obtain the following refined Bakry-Ledoux-Wang [24, 88] estimate, where we set

$$\mathrm{I}_\lambda(t) := \int_0^t \mathrm{e}^{\lambda s}\,\mathrm{d}s, \quad \mathrm{I}_\lambda(t) = \frac{\mathrm{e}^{\lambda t}-1}{\lambda} \text{ if } \lambda \neq 0, \quad \mathrm{I}_0(t) = t. \tag{9}$$

Theorem 1.1. *If $f_t = \mathsf{P}_t f$ is a smooth solution of* (8) *then*

$$|\nabla \mathsf{P}_t f|_{\mathsf{g}}^2 + \frac{2}{N}\mathrm{I}_{-2K}(t)\big(\mathsf{A}\mathsf{P}_t f\big)^2 \leq \mathrm{e}^{-2Kt}\,\mathsf{P}_t\big(|\nabla f|_{\mathsf{g}}^2\big) \quad \textit{in } \mathbb{M}, \textit{ for every } t \geq 0. \tag{10}$$

The *proof* shows the joint role of the lower bound (7) and of the Bochner-Lichnerowicz formula (5), whose intimate link with the diffusion operator A can be nicely expressed through the Bakry-Émery Γ-formalism. One first defines for smooth functions $u, v : \mathbb{M} \to \mathbb{R}$ the bilinear form

$$\Gamma(u,v) := \frac{1}{2}\Big(\mathsf{A}(uv) - u\mathsf{A}v - v\mathsf{A}u\Big), \tag{11}$$

observing that Γ encodes the geometric property of the manifold since

$$\Gamma(u,v) = \langle \nabla u, \nabla v\rangle_{\mathsf{g}}.$$

In a similar way one can introduce the Γ^2-tensor

$$\Gamma^2(u,v) := \frac{1}{2}\Big(\mathsf{A}\Gamma(u,v) - \Gamma(u,\mathsf{A}v) - \Gamma(v,\mathsf{A}u)\Big); \tag{12}$$

combining (6) and (7), Γ^2 can be bounded from below by

$$\begin{aligned}\Gamma^2(u,u) &\overset{(6)}{=} \big|\nabla^2 u\big|^2_{\mathsf{g}} + \mathrm{Ric}_{\mathsf{g}}(\nabla u,\nabla u) + \nabla^2_{\mathsf{g}} V(\nabla u,\nabla u)\\ &\overset{(7)}{\geq} K\Gamma(u,u) + \frac{1}{N}(\mathsf{A}u)^2.\end{aligned} \tag{13}$$

On the other hand, setting as in (4)

$$\mathfrak{b}(s) := \mathsf{P}_s\big(\Gamma(\mathsf{P}_{t-s}f,\mathsf{P}_{t-s}f)\big),\quad \mathfrak{c}(s) := \mathsf{P}_s\big(\mathsf{A}\,\mathsf{P}_{t-s}f\big)^2,$$

we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\mathfrak{b}(s) &= 2\mathsf{P}_s\Gamma_2(\mathsf{P}_{t-s}f,\mathsf{P}_{t-s}f) \geq 2\mathsf{P}_s\Big(K\Gamma(\mathsf{P}_{t-s}f,\mathsf{P}_{t-s}f) + \frac{2}{N}(\mathsf{A}\mathsf{P}_{t-s}f)^2\Big)\\ &= 2K\,\mathfrak{b}(s) + \frac{2}{N}\mathfrak{c}(s),\end{aligned} \tag{14}$$

and therefore, since $s\mapsto\mathfrak{c}(s)$ is nondecreasing,

$$\begin{aligned}\mathrm{e}^{-2Kt}\mathsf{P}_t\big(|\nabla f|^2_{\mathsf{g}}\big) = \mathrm{e}^{-2Kt}\mathfrak{b}(t) &\geq \mathfrak{b}(0) + \frac{2}{N}\int_0^t \mathrm{e}^{-2Ks}\mathfrak{c}(s)\,\mathrm{d}s\\ &\geq |\nabla\mathsf{P}_t f|^2_{\mathsf{g}} + \frac{2}{N}\mathrm{I}_{-2K}(t)\big(\mathsf{A}\mathsf{P}_t f\big)^2\end{aligned}$$

which is precisely the pointwise estimate (10). It is worth noticing that (10) is in fact *equivalent* to the estimate (13) (see [88]).

1.3. The Bakry-Émery condition BE(K,N) for Markov diffusions. The argument discussed in the previous section can be nicely generalized to Markov diffusion semigroups. The natural abstract setting is provided by a strongly local, symmetric Dirichlet form $\mathcal{E}$ on a Polish topological space (X,τ) endowed with a σ-finite Borel measure $\mathfrak{m}$ with full support (see e.g. [27, 22]).

Recall that $\mathcal{E}$ is a closed, symmetric and nonnegative bilinear form defined in a dense subspace $D(\mathcal{E})$ of $L^2(X,\mathfrak{m})$ satisfying for every $u,v\in D(\mathcal{E})$

$$\mathcal{E}(u_+,u_+)\leq\mathcal{E}(u,u);\quad \mathcal{E}(u,v)=0\quad\text{whenever } (u+a)v=0\ \mathfrak{m}\text{-a.e., } a\in\mathbb{R}.$$

$\mathcal{E}$ induces a Markov semigroup $(\mathsf{P}_t)_{t\geq0}$ on $L^2(X,\mathfrak{m})$ which satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{P}_t f = \mathsf{A}\mathsf{P}_t f\quad\text{for every } t>0,$$

where $\mathsf{A}: D(\mathsf{A})\to L^2(X,\mathfrak{m})$ is the linear self-adjoint operator generated by $\mathcal{E}$

$$w=-\mathsf{A}u\quad\Leftrightarrow\quad\int_X vw\,\mathrm{d}\mathfrak{m} = \mathcal{E}(u,v)\quad\text{for every } v\in D(\mathcal{E}).$$

We will also assume that P is mass preserving, i.e. $\displaystyle\int_X \mathsf{P}_t f\,\mathrm{d}\mathfrak{m} = \int_X f\,\mathrm{d}\mathfrak{m}$, a property which is equivalent to $1\in D(\mathcal{E})$ when $\mathfrak{m}(X)<\infty$.

Starting from A, one can try to recover a Γ-calculus as in (11) and (12). A first approach (see e.g. [22]) is to assume the existence of a "nice" algebra of functions in $D(\mathsf{A})$, invariant with respect to the application of the operator A and the composition with C^∞ real functions.

A weaker approach, which will be well adapted to lower curvature bounds in a nonsmooth setting, consists in writing suitable integral formulations of the pointwise definitions of Γ and Γ^2. We will assume that that $\mathcal{E}$ can be represented by a *Carré du champ* $\Gamma : D(\mathcal{E}) \times D(\mathcal{E}) \to L^1(X, \mathfrak{m})$: a symmetric bilinear and continuous operator satisfying the "weak formulation" of (11)

$$\int_X \Gamma(u,v)\varphi \,\mathrm{d}\mathfrak{m} = \frac{1}{2}\Big(\mathcal{E}(u\varphi, v) + \mathcal{E}(u, v\varphi) - \mathcal{E}(uv, \varphi)\Big)$$

for every $u, v, \varphi \in D(\mathcal{E}) \cap L^\infty(X, \mathfrak{m})$. In particular,

$$\mathcal{E}(u,v) = \int_X \Gamma(u,v)\,\mathrm{d}\mathfrak{m}$$

so that $\Gamma(u,v)$ plays the role of $\langle \nabla u, \nabla v\rangle_{\mathsf{g}}$ in the smooth Riemannian case.

Defining the Γ^2 tensor is more involved, since it is not clear, in general, if the set of functions u with $\Gamma(u,u) \in D(\mathsf{A})$ is sufficiently rich. However, one can define [48] a trilinear form $\mathbf{\Gamma}^2[u,v;\varphi]$, formally acting as $\int_X \Gamma^2(u,v)\varphi\,\mathrm{d}\mathfrak{m}$, by

$$\mathbf{\Gamma}^2[u,v;\varphi] := \frac{1}{2}\int_X \Big(\Gamma(u,v)\mathsf{A}\varphi - \big(\Gamma(u,\mathsf{A}v) - \Gamma(\mathsf{A}u, v)\big)\varphi\Big)\,\mathrm{d}\mathfrak{m}$$

for every function $u, v, \varphi \in D(\mathsf{A})$ with $\mathsf{A}u, \mathsf{A}v \in D(\mathcal{E})$ and $\mathsf{A}\varphi \in L^\infty(X, \mathfrak{m})$.

It turns out that the Γ^2-formulation of the Bakry-Émery condition $\mathrm{BE}(K,N)$

$$\Gamma^2(f,f) \geq K\Gamma(f,f) + \frac{1}{N}(\mathsf{A}f)^2 \tag{15}$$

can be defined in the following equivalent way, where we suppose that the Dirichlet space $(X, \tau, \mathcal{E})$ complies with the assumptions we have just detailed.

Definition 1.2 (Weak formulations of $\mathrm{BE}(K,N)$)**.** Let $K \in \mathbb{R}$, $N \in [1,\infty]$. We say that the Dirichlet space $(X,\tau,\mathcal{E})$ satisfies the $\mathrm{BE}(K,N)$ condition if one of the following equivalent conditions are satisfied [9]:

Weak Γ^2-formulation:

$$\mathbf{\Gamma}^2[f,f;\varphi] \geq K\int_X \Gamma(f,f)\varphi\,\mathrm{d}\mathfrak{m} + \frac{1}{N}\int_X \big(\mathsf{A}f\big)^2\varphi\,\mathrm{d}\mathfrak{m} \tag{16}$$

for every $f, \varphi \in D(\mathsf{A})$ with $\mathsf{A}f \in D(\mathcal{E})$, $\mathsf{A}\varphi \in L^\infty(X,\mathfrak{m})$, and $\varphi \geq 0$.

Pointwise gradient bounds:

$$\Gamma(\mathsf{P}_t f) + \frac{2}{N}\mathrm{I}_{-2K}(t)\big(\mathsf{A}\mathsf{P}_t f\big)^2 \leq \mathrm{e}^{-2Kt}\mathsf{P}_t\big(\Gamma(f)\big) \quad \text{for every } f \in D(\mathcal{E}). \tag{17}$$

Integrated Bakry-Ledoux: for every $f \in L^2(X,\mathfrak{m}), \varphi \in L^2 \cap L^\infty(X,\mathfrak{m})$ with $\varphi \geq 0$ and for every $t > 0$ the maps

$$\mathfrak{a}(s) := \int_X \big(\mathsf{P}_{t-s} f\big)^2 \mathsf{P}_s \varphi \,\mathrm{d}\mathfrak{m}, \quad \mathfrak{c}(s) := \int_X \big(\mathsf{A}\mathsf{P}_{t-s} f\big)^2 \mathsf{P}_s \varphi \,\mathrm{d}\mathfrak{m},$$

satisfy $\quad \mathfrak{a}'' \geq 2K\mathfrak{a}' + \mathfrak{a}41N\mathfrak{c} \quad$ in $\mathscr{D}'(0,t)$.

2. The Lott-Sturm-Villani approach to lower Ricci curvature bounds

Ricci curvature plays a crucial role in a second important class of functional inequalities in Riemannian manifold, which can be considered as a natural extension of the Euclidean Brunn-Minkowski inequality.

2.1. Brunn-Minkowski and weighted Prékopa-Leindler inequalities. The Brunn-Minkowski inequality (in its dimension free formulation) simply states that for every Borel sets $A, B \subset \mathbb{R}^d$ and for every $\vartheta \in (0,1)$ we have

$$\mathrm{Vol}\,\big((1-\vartheta)A + \vartheta B\big) \geq \big(\,\mathrm{Vol}\,A\big)^{1-\vartheta}\big(\,\mathrm{Vol}\,B\big)^{\vartheta} \tag{18}$$

where $(1-\vartheta)A + \vartheta B := \Big\{(1-\vartheta)a + \vartheta b : a \in A,\ b \in B\Big\}$.

In order to generalise it to a Riemannian manifold $(\mathbb{M}, \mathsf{g})$, it is natural to consider the Riemannian distance d_g, the (minimal, constant speed) geodesics, i.e. curves $\mathsf{x} : [0,1] \to \mathbb{M}$ satisfying

$$\mathsf{d}_\mathsf{g}(\mathsf{x}(s), \mathsf{x}(t)) = |t-s|\mathsf{d}_\mathsf{g}(\mathsf{x}(0), \mathsf{x}(1)) \quad \text{for every } s,t \in [0,1], \tag{19}$$

and the set $Z_\vartheta(x_0, x_1)$ of ϑ-barycenters in $\mathbb{M}$ between $x_0, x_1 \in \mathbb{M}$, $\vartheta \in [0,1]$:

$$Z_\vartheta(x_0,x_1) := \big\{x \in \mathbb{M} : \mathsf{d}_\mathsf{g}(x_0,x) = \vartheta \mathsf{d}_\mathsf{g}(x_0,x_1),\ \mathsf{d}_\mathsf{g}(x,x_1) = (1-\vartheta)\mathsf{d}_\mathsf{g}(x_0,x_1)\big\}.$$

We also denote by Vol_g the Riemannian volume measure.

Theorem 2.1 (Weighted Prékopa-Leindler inequality [35, 36])**.** *Let us consider the Borel measure $\mathfrak{m} = \mathrm{e}^{-V}\,\mathrm{Vol}_\mathsf{g}$, where V is a smooth potential satisfying*

$$\mathrm{Ric}_\mathsf{g} + \nabla^2_\mathsf{g} V \geq K\mathsf{g}, \tag{20}$$

and let $f_0, f_1, f : \mathbb{M} \to [0,\infty)$ be Borel functions satisfying

$$f(x) \geq \exp\Big(-\frac{K}{2}\mathsf{d}(x_0,x)\mathsf{d}(x,x_1)\Big) f_0(x_0)^{1-\vartheta} f_1(x_1)^{\vartheta} \tag{21}$$

for some $\vartheta \in [0,1]$ and every $x_0, x_1 \in \mathbb{M},\ x \in Z_\vartheta(x_0,x_1)$. Then

$$\int_\mathbb{M} f \,\mathrm{d}\mathfrak{m} \geq \Big(\int_\mathbb{M} f_0 \,\mathrm{d}\mathfrak{m}\Big)^{1-\vartheta}\Big(\int_\mathbb{M} f_1 \,\mathrm{d}\mathfrak{m}\Big)^{\vartheta}. \tag{22}$$

Notice that in the case $K \geq 0$, by choosing the characteristic functions $f_0 := \chi_A$, $f_1 := \chi_B$ and $f := \chi_{Z_\vartheta(A,B)}$ with $Z_\vartheta(A,B) := \cup_{a\in A, b\in B} Z_\vartheta(a,b)$, we obtain

$$\mathfrak{m}\Big(Z_v\vartheta(A,B))\Big) \geq \big(\mathfrak{m}(A)\big)^{1-\vartheta}\big(\mathfrak{m}(B)\big)^{\vartheta} \tag{23}$$

(22) has a beautiful geometric interpretation in terms of entropy and optimal transport of probability measures, first discovered by [70, 73]. It requires a few notions, that we will briefly recall (we refer to [6, 87] for more details).

2.2. Probability measures, couplings, dynamic plans and Entropy. Let (X, d) be a complete and separable metric space. $\mathscr{P}(X)$ will denote the set of Borel probability measures in X and (here x_o is an arbitrary point of X)

$$\mathscr{P}_2(X) := \Big\{\mu \in \mathscr{P}(X) : \int_X \mathsf{d}^2(x, x_o)\,\mathrm{d}\mu(x) < \infty\Big\}.$$

Geodesics curves $\mathsf{x} : [0,1] \to X$ can be defined as for Riemannian manifold (see (19)): they form a closed subset of the complete metric space $\mathrm{C}([0,1];X)$ (w.r.t. the sup distance) that we will denote by $\mathrm{Geo}(X)$. We will say that X is a *geodesic space* if every couple of points $x_0, x_1 \in X$ can be connected by a geodesic $\mathsf{x} \in \mathrm{Geo}(X)$. A *coupling* between $\mu_0, \mu_1 \in \mathscr{P}(X)$ is a measure $\boldsymbol{\mu} \in \mathscr{P}(X \times X)$ such that

$$(\pi^i)_\sharp \boldsymbol{\mu} = \mu_i \quad \text{where } \pi^i : X \times X \to X \text{ are the projections } \pi^i(x_0, x_1) := x_i; \tag{24}$$

recall that the push-forward $\mathsf{p}_\sharp : \mathscr{P}(X) \to \mathscr{P}(Y)$ associated to a Borel map $\mathsf{p} : X \to Y$ between separable metric spaces, is defined as

$$\mathsf{p}_\sharp(\mu)(B) := \mu(\mathsf{p}^{-1}(B)) \quad \text{for every Borel subset } B \subset Y.$$

In a similar way, a *geodesic coupling* between $\mu_0, \mu_1 \in \mathscr{P}(X)$ is a Borel measure $\boldsymbol{\pi} \in \mathscr{P}(\mathrm{Geo}(X))$ such that

$$(\mathsf{e}^i)_\sharp \boldsymbol{\pi} = \mu_i, \quad \text{where } \mathsf{e}^\vartheta : \mathrm{C}([0,1];X) \to X \text{ are the evaluation maps } \mathsf{e}^\vartheta(\mathsf{x}) := \mathsf{x}(\vartheta).$$

Many geometric notions can be lifted from X to $\mathscr{P}_2(X)$ by introducing the (squared) Kantorovich-Rubinstein-Wasserstein distance between measures $\mu_0, \mu_1 \in \mathscr{P}_2(X)$:

$$W_2^2(\mu_0, \mu_1) := \min\Big\{\int \mathsf{d}^2(x_0, x_1)\,\mathrm{d}\boldsymbol{\mu} : \boldsymbol{\mu} \text{ is a coupling between } \mu_0, \mu_1\Big\}.$$

Every geodesic $(\mu_\vartheta)_{\vartheta\in[0,1]}$ in $\mathscr{P}_2(X)$, thus satisfying

$$W_2(\mu_s, \mu_t) = |t-s| W_2(\mu_0, \mu_1) \quad \text{for every } s, t \in [0,1], \tag{25}$$

can be represented by *an optimal geodesic coupling* $\boldsymbol{\pi} \in \mathscr{P}(\mathrm{Geo}(X))$ such that

$$W_2^2(\mu_0, \mu_1) = \int \mathsf{d}^2(\mathsf{x}(0), \mathsf{x}(1))\,\mathrm{d}\boldsymbol{\pi}(\mathsf{x}),$$

so that $\mu_\vartheta := \mathsf{e}^\vartheta_\sharp \boldsymbol{\pi}$ for every $\vartheta \in [0,1]$. Notice that if μ_i are concentrated on the Borel sets $A_i \subset X$, then μ_ϑ is concentrated on $Z_\vartheta(A_0, A_1)$.

Let us eventually introduce the relative Entropy functional of a measure $\mu \in \mathscr{P}_2(X)$ with respect to a nonnegative Borel measure $\mathfrak{m}$:

$$\mathrm{Ent}_\mathfrak{m}(\mu) := \int_X \varrho \log(\varrho)\,\mathrm{d}\mathfrak{m} \quad \text{if } \mu = \varrho\mathfrak{m} \ll \mathfrak{m}, \quad \mathrm{Ent}_\mathfrak{m}(\mu) := +\infty \text{ if } \mu \not\ll \mathfrak{m}. \tag{26}$$

We set $D(\mathrm{Ent}_\mathfrak{m}) := \{\mu \in \mathscr{P}_2(X) : \mathrm{Ent}_\mathfrak{m} < +\infty\}$; if $\mathfrak{m}$ satisfies the growth condition

$$\mathfrak{m}(B_r(x_o)) \le \mathsf{c}_1 \exp(\mathsf{c}_2 r^2) \quad \text{for every } r > 0, \tag{27}$$

then (26) is well defined (i.e. $\varrho(\log \varrho)_- \in L^1(X, \mathfrak{m})$) for every $\mu \in \mathscr{P}_2(X)$; if moreover $\mathfrak{m}(X) = Z < \infty$ then $\mathrm{Ent}_\mathfrak{m}(\mu) \ge -\log(Z)$.

We can thus state the crucial K-convexity interpolation inequality for the relative entropy functional [73, 86, 36], a deep result that combines many ideas and subtle properties of optimal transportation in Riemannian geometry.

Theorem 2.2. *Let* $(\mathbb{M}, \mathrm{g})$ *be a complete Riemannian manifold and us consider the Borel measure* $\mathfrak{m} = \mathrm{e}^{-V}\,\mathrm{Vol}_\mathsf{g}$, *where* V *is a smooth potential satisfying* (20). *Then the entropy functional* $\mu \mapsto \mathrm{Ent}_\mathfrak{m}(\mu)$ *is (strongly)* K*-displacement convex: for every geodesic* $(\mu_\vartheta)_{\vartheta\in[0,1]}$ *in* $\mathscr{P}_2(X)$ *as in* (25) *with* $\mathrm{Ent}_\mathfrak{m}(\mu_i) < \infty$, $i = 0, 1$,

$$\textit{the map } \vartheta \mapsto \mathrm{Ent}_\mathfrak{m}(\mu_\vartheta) \quad \textit{is } K\textit{-convex, i.e. } \frac{\mathrm{d}^2}{\mathrm{d}\vartheta^2}\mathrm{Ent}_\mathfrak{m}(\mu_\vartheta) \ge K W_2^2(\mu_0, \mu_1).$$

Equivalently we have

$$\mathrm{Ent}_\mathfrak{m}(\mu_\vartheta) \le (1-\vartheta)\,\mathrm{Ent}_\mathfrak{m}(\mu_0) + \vartheta\,\mathrm{Ent}_\mathfrak{m}(\mu_1) - \frac{K}{2}\vartheta(1-\vartheta) W_2^2(\mu_0, \mu_1). \tag{28}$$

It is interesting to note that Theorem 2.2 implies the weighted Prékopa-Leindler inequality (22). In fact, by an homogeneity argument, it is sufficient to check (22) when $\int f_0\,\mathrm{d}\mathfrak{m} = \int f_1\,\mathrm{d}\mathfrak{m} = 1$; (22) is then equivalent to prove

$$\log F \ge 0 \quad \text{where } F := \int_X f\,\mathrm{d}\mathfrak{m}. \tag{29}$$

We can introduce $\mu_i = f_i\mathfrak{m}$ (by a truncation argument it is not restrictive to assume $\mu_i \in \mathscr{P}_2(\mathbb{M})$) and we then consider an optimal geodesic coupling $\boldsymbol{\pi}$ between μ_0 and μ_1, with the corresponding interpolation measures $\mu_\vartheta = \mathsf{e}^\vartheta_\sharp\boldsymbol{\pi}$ and the measure $\nu := F^{-1} f\mathfrak{m} \in \mathscr{P}(X)$ (if $F = \infty$ nothing has to be proved). (21) shows that $f > 0$ μ_ϑ-a.e. so that a simple calculation and Jensen inequality yields

$$\log F = \int f\,\mathrm{d}\mu_\vartheta - \mathrm{Ent}_\mathfrak{m}(\mu_\vartheta) + \mathrm{Ent}_\nu(\mu_\vartheta) \ge \int \log(f)\,\mathrm{d}\mu_\vartheta - \mathrm{Ent}_\mathfrak{m}(\mu_\vartheta).$$

On the other hand, (21) yields

$$\begin{aligned}\int \log f \,\mathrm{d}\mu_\vartheta &= \int \log(f(\mathsf{x}(\vartheta)))\,\mathrm{d}\boldsymbol{\pi}(\mathsf{x})\\ &\geq -\frac{K}{2}\vartheta(1-\vartheta)\int \mathsf{d}^2(\mathsf{x}(0),\mathsf{x}(1))\,\mathrm{d}\boldsymbol{\pi}(\mathsf{x})\\ &\quad + (1-\vartheta)\int \log(f_0(\mathsf{x}(0)))\,\mathrm{d}\boldsymbol{\pi}(\mathsf{x}) + \vartheta\int \log(f_1(\mathsf{x}(1))\,\mathrm{d}\boldsymbol{\pi}(\mathsf{x})\\ &= -\frac{K}{2}\vartheta(1-\vartheta)W_2^2(\mu_0,\mu_1) + (1-\vartheta)\int f_0\log f_0\,\mathrm{d}\mathfrak{m} + \vartheta\int f_1\log f_1\,\mathrm{d}\mathfrak{m}\\ &= -\frac{K}{2}\vartheta(1-\vartheta)W_2^2(\mu_0,\mu_1) + (1-\vartheta)\,\mathrm{Ent}_{\mathfrak{m}}(\mu_0) + \vartheta\,\mathrm{Ent}_{\mathfrak{m}}(\mu_1).\end{aligned}$$

Applying (28) we eventually get (29).

As for the BE conditions, in the Riemannian setting it is possible to prove that the properties stated in Theorem 2.2 are in fact *equivalent* to (20) [86].

2.3. The Lott-Sturm-Villani condition CD(K,N) in metric measure spaces. Taking inspiration from Theorem 2.2, a new metric approach to find synthetic notions of lower Ricci bounds has been proposed by Sturm [84, 85] and LottVillani [69]. Many previous contributions (see in particular [39, 33, 57]) had clarified that the natural setting should be provided by metric measure spaces $(X,\mathsf{d},\mathfrak{m})$, i.e. complete and separable metric spaces (X,d) equipped with a nonnegative Borel reference measure $\mathfrak{m}$ with full support and satisfying the growth condition (27).

The goal was then to find a notion consistent with the smooth Riemannian case and stable under measured Gromov-Hausdorff limits. Having at our disposal the notions of relative Entropy, Wasserstein distance, and geodesic interpolation (see Section 2.2), the property of Theorem 2.2 can be adopted as a metric definition.

Definition 2.3 (The Lott-Sturm-Villani CD(K,∞) condition)**.** A metric measure space $(X,\mathsf{d},\mathfrak{m})$ satisfies the CD(K,∞) condition if the Entropy functional $\mathrm{Ent}_{\mathfrak{m}}$ is geodesically K-convex in $\mathscr{P}_2(X)$: every couple $\mu_0,\mu_1\in D(\mathrm{Ent}_{\mathfrak{m}})$ can be connected by a geodesic $(\mu_\vartheta)_{\vartheta\in[0,1]}$ as in (17) such that for every $\vartheta\in[0,1]$

$$\mathrm{Ent}_{\mathfrak{m}}(\mu_\vartheta)\leq(1-\vartheta)\,\mathrm{Ent}_{\mathfrak{m}}(\mu_0)+\vartheta\,\mathrm{Ent}_{\mathfrak{m}}(\mu_1)-\frac{K}{2}\vartheta(1-\vartheta)W_2^2(\mu_0,\mu_1). \tag{30}$$

If (30) is satisfied along *any* geodesic connecting μ_0 to μ_1 then we say that $(X,\mathsf{d},\mathfrak{m})$ is a *strong* CD(K,∞) space.

Besides many useful geometric and functional applications of this notion [84, 85, 87, 68, 75, 41, 45], one of its strongest features is its stability under measured Gromov-Hausdorff convergence [39, 57], also in the weaker transport-formulation proposed by Sturm [84] or in the localized pointed version studied by [51].

The case when the dimension N is finite requires a more subtle definition. Perhaps the simplest condition is the entropic formulation recently proposed by

[38], that we consider here in the strong form. To this aim, we introduce the N-dimensional Entropy power

$$\mathrm{H}_N(\mu) := \exp\Big(-\frac{1}{N}\,\mathrm{Ent}_{\mathfrak m}(\mu)\Big) \tag{31}$$

and the distortion interpolating coefficients

$$\sigma^{(t)}_\kappa(\delta) := \begin{cases} \dfrac{\sin(\sqrt{\kappa}\delta t)}{\sin(\sqrt{\kappa}\delta)} & \text{if } \kappa\,\delta^2 \in (0,\pi^2),\\ t & \text{if } \kappa = 0,\\ \dfrac{\sinh(\sqrt{-\kappa}\delta t)}{\sinh(\sqrt{-\kappa}\delta)} & \text{if } \kappa\,\delta^2 < 0. \end{cases} \qquad \sigma^{(t)}_\kappa(\delta) = +\infty \text{ if } \kappa\,\delta^2 \ge \pi^2. \tag{32}$$

Definition 2.4 ([38])**.** $(X,\mathsf{d},\mathfrak m)$ satisfies the (strong) $\mathrm{CD}^e(K,N)$ condition, $N \in [1,\infty)$, if every geodesic $(\mu_\vartheta)_{\vartheta\in[0,1]}$ connecting a couple $\mu_0,\mu_1 \in D(\mathrm{Ent}_{\mathfrak m})$ satisfies

$$\mathrm{H}_N(\mu_\vartheta) \ge \sigma^{1-\vartheta}_{K/N}\big(W_2(\mu_0,\mu_1)\big)\mathrm{H}_N(\mu_0) + \sigma^{\vartheta}_{K/N}\big(W_2(\mu_0,\mu_1)\big)\mathrm{H}_N(\mu_1). \tag{33}$$

Other characterizations of lower Ricci curvature bounds involve more general classes of entropies of the form

$$\mathcal U(\mu) := \begin{cases} \displaystyle\int_X U(\varrho)\,\mathrm d\mathfrak m & \text{if } \mu = \varrho\mathfrak m \ll \mathfrak m,\\ +\infty & \text{otherwise} \end{cases} \tag{34}$$

where $U \in \mathrm C^0([0,\infty)) \cap \mathrm C^2((0,\infty))$ with $U(0) = 1$ is a convex and regular entropy density, satisfying the following McCann condition.

Definition 2.5 ([70],[87, Def. 17.1],[12])**.** The entropy functional $\mathcal U$ defined as in (34) belongs to the McCann's class $\mathrm{DC}_{reg}(N)$, $N \in [1,\infty]$, if U is convex and the corresponding pressure function $P(r) := rU'(r) - U(r)$, $r \in (0,\infty)$ satisfies for every $r \in (0,\infty)$

$$\lim_{r\downarrow 0} P(r) = 0, \quad 0 < a \le P'(r) \le a^{-1} < \infty, \quad rP'(r) \ge \big(1 - \frac{1}{N}\big)P(r).$$

If $\mathcal U \in \mathrm{DC}_{reg}(N)$ and $\mu_\vartheta = \varrho_\vartheta\mathfrak m = \mathsf e^\vartheta_\sharp\boldsymbol\pi$ is a geodesic associated with an optimal geodesic plan $\boldsymbol\pi \in \mathscr P(\mathrm{Geo}(X))$, we define the weighted actions

$$\mathcal A^\vartheta_U(\boldsymbol\pi) := \int_0^1 G_\vartheta(s)\int Q(\varrho_s(\mathsf x(s)))\mathsf d^2(\mathsf x(0),\mathsf x(1))\,\mathrm d\boldsymbol\pi(\mathsf x)\,\mathrm ds, \tag{35}$$

$$\mathcal B_U(\boldsymbol\pi) := \int_0^1 (1-s)\int Q(\varrho_s(\mathsf x(s)))\mathsf d^2(\mathsf x(0),\mathsf x(1))\,\mathrm d\boldsymbol\pi(\mathsf x)\,\mathrm ds, \tag{36}$$

where

$$G_\vartheta(s) := \begin{cases} (1-\vartheta)s & \text{if } 0 \le s \le \vartheta,\\ \vartheta(1-s) & \text{if } \vartheta \le s \le 1, \end{cases} \quad \text{and} \quad Q(r) := P(r)/r. \tag{37}$$

When there exists only one optimal geodesic plan $\boldsymbol{\pi}_{\mu_0\to\mu_1}$ connecting μ_0 to μ_1 we will simply write $\mathcal{A}^{\vartheta}_U(\mu_0,\mu_1)$ for $\mathcal{A}^{\vartheta}_U(\boldsymbol{\pi}_{\mu_0\to\mu_1})$. Notice that the relative Entropy function $\mathrm{Ent}_{\mathfrak{m}}$ corresponds to $P(r)=r$, $N=+\infty$; in this case $\mathcal{A}^{\vartheta}_U$ and $\mathcal{B}_U$ take the form

$$\mathcal{A}^{\vartheta}_U(\mu_0,\mu_1)=\frac12\vartheta(1-\vartheta)W_2^2(\mu_0,\mu_1),\quad \mathcal{B}_U(\mu_0,\mu_1)=\frac12 W_2^2(\mu_0,\mu_1).$$

Theorem 2.6 (Equivalent formulation of the strong $\mathrm{CD}(K,N)$ condition [38, 12])**.** *The following conditions are equivalent:*

(1) $(X,\mathsf{d},\mathfrak{m})$ *satisfies the strong* $\mathrm{CD}^e(K,N)$ *condition.*

(2) $(X,\mathsf{d},\mathfrak{m})$ *satisfies the strong* $\mathrm{CD}^*(K,N)$ *of* [19]*: for every $M\ge N$ we have*

$$\int \varrho_\vartheta^{1-1/M}\,\mathrm{d}\mathfrak{m}\ge\int\Big(\sigma^{1-\vartheta}_{K/M}(\mathsf{d}(x_0,x_1))\varrho_0^{-1/M}(x_0)+\sigma^{\vartheta}_{K/M}(\mathsf{d}(x_0,x_1))\varrho_1^{-1/M}(x_1)\Big)\,\mathrm{d}\boldsymbol{\mu}$$

holds along any geodesic $\mu_\vartheta=\varrho_\vartheta\mathfrak{m}$ induced by the optimal coupling $\boldsymbol{\mu}$ connecting two measures $\mu_i=\varrho_i\mathfrak{m}$ with bounded support.

(3) *(See* [12]*) For every entropy $\mathcal{U}\in\mathrm{DC}_{reg}(N)$ and every optimal geodesic coupling $\boldsymbol{\pi}\in\mathscr{P}(\mathrm{Geo}(X))$ connecting $\mu_0,\mu_1\in D(\mathcal{U})$*

$$\mathcal{U}(\mu_\vartheta)\le(1-\vartheta)\mathcal{U}(\mu_0)+\vartheta\mathcal{U}(\mu_1)-K\mathcal{A}^{\vartheta}_U(\boldsymbol{\pi})\quad \vartheta\in[0,1],\ \mu_\vartheta=\mathsf{e}^{\vartheta}_\sharp\boldsymbol{\pi}.$$

3. The RCD(K,∞) condition and the equivalence between BE and CD

In Riemannian manifolds [86] showed that the Bakry-Émery condition $\mathrm{BE}(K,\infty)$ is equivalent to the Lott-Sturm-Villani condition $\mathrm{CD}(K,\infty)$, since both are in fact equivalent to the pointwise differential condition (20). It has been a relevant question to prove their equivalence in nonsmooth contexts: this would allow to use both the tools of the two frameworks, extending many deep results available for Riemannian manifolds to the metric-measure case (see the impressive list of contributions at the end of the paper).

3.1. The Cheeger energy and RCD(K,∞). By adopting the LSV point of view of metric measure spaces, one immediately has to deal with the problem to generate a canonical Markov semigroup, or, equivalently, a Dirichlet form. A natural starting point is provided by the Cheeger Energy [32], which can be defined by the following relaxation procedure.

Recall that the metric slopes of a function $f:X\to\mathbb{R}\cup\{\pm\infty\}$ at an accumulation point $x\in D(f)\subset X$ are defined by

$$|\mathrm{D}^{\pm}f|(x):=\limsup_{y\to x}\frac{(f(y)-f(x))_\pm}{\mathsf{d}(x,y)},\quad |\mathrm{D}f|(x)=\max\big(|\mathrm{D}^+f|(x),|\mathrm{D}^-f|(x)\big)\tag{38}$$

and by 0 if x is isolated. $\mathrm{Lip}_b(X)$ is the space of bounded Lipschitz real functions.

Definition 3.1 (The Cheeger energy)**.** For every $f \in L^2(X, \mathfrak{m})$ we define

$$\mathrm{Ch}(f) := \inf\Big\{ \liminf_{n\to\infty} \frac{1}{2}\int_X |\mathrm{D} f_n|^2 \,\mathrm{d}\mathfrak{m} : f_n \in \mathrm{Lip}_b(X),\ f_n \stackrel{L^2}{\to} f \Big\}. \tag{39}$$

The proper domain of Ch is $D(\mathrm{Ch}) := \{f \in L^2(X, \mathfrak{m}) : \mathrm{Ch}(f) < \infty\}$.

It is possibile to prove (see e.g. [7]) that $\mathrm{Ch} : L^2(X, \mathfrak{m}) \to [0, \infty]$ is a convex, 2-homogeneous, l.s.c. functional. If $f \in D(\mathrm{Ch})$ then the infimum in (39) is attained by some optimal sequence $(f_n)_n$ in $\mathrm{Lip}_b(X)$ and in this case $|\mathrm{D}f_n|$ strongly converges in $L^2(X, \mathfrak{m})$ to a unique limit which is called $|\mathrm{D}f|_w$. The map $f \mapsto |\mathrm{D}f|_w$ is 1-homogeneous and subadditive from $D(\mathrm{Ch})$ to $L^2(X, \mathfrak{m})$, and represents Ch through the formula

$$\mathrm{Ch}(f) = \frac{1}{2}\int_X |\mathrm{D}f|_w^2 \,\mathrm{d}\mathfrak{m}. \tag{40}$$

Applying the general results concerning gradient flows of l.s.c. and convex functionals in Hilbert spaces it is possible to define the L^2-gradient flow of Ch, a contraction semigroup that we will still denote by $(\mathsf{P}_t)_{t\ge 0}$. Starting from every initial function $f \in L^2(X, \mathfrak{m})$ the semigroup P_t provides a curve $t \mapsto f_t = \mathsf{P}_t f$ which is locally Lipschitz in $(0, \infty)$ with values in $L^2(X, \mathfrak{m})$ and solves

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|f_t - h\|_{L^2}^2 \le \mathrm{Ch}(h) - \mathrm{Ch}(f_t) \quad \text{a.e. in } (0, \infty) \text{ for every } h \in D(\mathrm{Ch}). \tag{41}$$

If $\mathrm{CD}(K, \infty)$ holds, then P_t can be equivalently characterized as the unique Wasserstein gradient flow of the Entropy functional in the metric sense [6, 41, 48], a result also relying on the crucial estimates of [65].

Theorem 3.2 ([7])**.** *Let us suppose that $(X, \mathsf{d}, \mathfrak{m})$ satisfies the $\mathrm{CD}(K, \infty)$ condition. For every $\mu = f\mathfrak{m} \in \mathscr{P}_2(X)$ with $\mathrm{Ent}_\mathfrak{m}(f) < +\infty$ the curve $\mu_t := (\mathsf{P}_t f)\mathfrak{m}$ is locally Lipschitz in $\mathscr{P}_2(X)$, the map $t \mapsto \mathrm{Ent}_\mathfrak{m}(\mu_t)$ is locally Lipschitz and for every $t > 0$*

$$-\frac{\mathrm{d}}{\mathrm{d}t_+}\mathrm{Ent}_\mathfrak{m}(\mu_t) = |\mathrm{D}^-\mathrm{Ent}_\mathfrak{m}|^2(\mu_t) = \lim_{h\downarrow 0}\frac{W_2^2(\mu_{t+h}, \mu_t)}{h^2}.$$

Moreover, for every $\mu = f\mathfrak{m} \in D(\mathrm{Ent}_\mathfrak{m})$ the descending slope of $\mathrm{Ent}_\mathfrak{m}$ coincides with the Fisher information and we have

$$|\mathrm{D}^-\mathrm{Ent}_\mathfrak{m}|^2(\mu) = \mathrm{F}(f) = \int_{f>0}\frac{|\mathrm{D}f|_w^2}{f}\,\mathrm{d}\mathfrak{m} = 4\int_X |\mathrm{D}\sqrt{f}|_w^2\,\mathrm{d}\mathfrak{m}.$$

It is remarkable that whenever Ch is a quadratic form, than Ch is a strongly local Dirichlet form admitting a Carré du Champ Γ coinciding with $|\mathrm{D}f|_w^2$.

Theorem 3.3 (Cheeger energy and Dirichlet forms [8])**.** *If the Cheeger energy form is quadratic, i.e.*

$$\mathrm{Ch}(f+g) + \mathrm{Ch}(f-g) = 2\Big(\mathrm{Ch}(f) + \mathrm{Ch}(g)\Big) \quad \textit{for every } f, g \in D(\mathrm{Ch}), \tag{42}$$

then $\mathcal{E}(f,g) := \mathrm{Ch}(f+g) - \mathrm{Ch}(f) - \mathrm{Ch}(g)$ *is a strongly local Dirichlet form in* $L^2(X,\mathfrak{m})$ *admitting a Carré du Champ* Γ *which coincides with* $|\cdot|_w^2$*, i.e.*

$$\Gamma(f,f) = |\mathrm{D}f|_w^2 \quad \mathfrak{m}\text{-}a.e.\ for\ every\ f \in D(\mathrm{Ch}).$$

Since the Bakry-Émery condition assumes a priori that the energy form is quadratic so that the corresponding gradient flow is linear, it is natural to restrict the investigation to metric measure spaces where the Cheeger energy is quadratic as well. Metric measure spaces with *Riemannian* Ricci curvature bounded from below can thus be defined combining the LSV $\mathrm{CD}(K,\infty)$ condition with the quadratic property of the Cheeger energy.

Definition 3.4 ($\mathrm{RCD}(K,\infty)$ spaces [8])**.** We say that $(X,\mathsf{d},\mathfrak{m})$ is a $\mathrm{RCD}(K,\infty)$ if it satisfies the $\mathrm{CD}(K,\infty)$ condition and the induced Cheeger energy Ch is quadratic, according to (42).

3.2. From RCD to BE: the dual characterization of the Markov semigroup as a K-flow of the entropy functional. The BE condition involves pointwise gradient estimates for the Markov semigroup generated by the Cheeger energy; the converse implication amounts to prove a convex property of the Entropy functional along geodesics in the Wasserstein space. In both cases one needs a better understanding of the relation between the diffusion semigroup generated by the Cheeger energy and the Entropy functional.

The crucial notion that connects the two points of view is provided by the so called EVI formulation of gradient flows for K-convex functionals. In order to state it, it is useful to recall that the classical solution $u : [0,\infty) \to \mathbb{R}^d$ of an Euclidean gradient flow

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = -\nabla\Phi(u(t))$$

generated by a smooth function $\Phi : \mathbb{R}^d \to \mathbb{R}$ with $\mathrm{D}^2\Phi \geq KI$, can be characterized by the system of inequalities (compare with (41) in the case $K = 0$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}|u(t)-v|^2 + \frac{K}{2}|u(t)-v|^2 \leq \Phi(v) - \Phi(u(t)) \quad \text{for every } t \geq 0,\ v \in \mathbb{R}^d. \quad (43)$$

Motivated by (43) we introduce the following definition (equivalent weaker formulations are still possible), see [78, 37].

Definition 3.5 (EVI K-flow of the Entropy in $\mathscr{P}_2(X)$)**.** We say that the relative Entropy functional $\mathrm{Ent}_{\mathfrak{m}}$ admits a K-flow in $\mathscr{P}_2(X)$ if there exists a semigroup $(\mathsf{S}_t)_{t\geq 0}$ in $\mathscr{P}_2(X)$ satisfying the following properties:

(1) for every $\mu \in \mathscr{P}_2(X)$ the curve $t \mapsto \mathsf{S}_t\mu = \mu_t$ is locally Lipschitz in $(0,\infty)$ with $\lim_{t\downarrow 0}\mu_t = \mu$ in $\mathscr{P}_2(X)$;

(2) $t \mapsto \mathrm{Ent}_{\mathfrak{m}}(\mu_t)$ is finite and nonincreasing in $(0,\infty)$;

(3) for every $\mu \in \mathscr{P}_2(X)$ and every $\nu \in D(\mathrm{Ent}_{\mathfrak{m}})$ $\mu_t = \mathsf{S}_t\mu$ satisfies the *Evolution Variational Inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t,\nu) + \frac{K}{2}W_2^2(\mu_t,\nu) + \mathrm{Ent}_{\mathfrak{m}}(\mu_t) \le \mathrm{Ent}_{\mathfrak{m}}(\nu) \quad \text{for a.e. every } t > 0. \tag{EVI$_K$}$$

The existence of an EVI K-flow for the entropy simultaneously implies the $\mathrm{RCD}(K,\infty)$ and the $\mathrm{BE}(K,\infty)$ conditions, thus it provides a unifying characterization of both the conditions.

Theorem 3.6 (EVI K-flows for the Entrpy [8])**.** *Let us suppose that the relative Entropy functional* $\mathrm{Ent}_{\mathfrak{m}}$ *admits an* EVI *K-flow* $(\mathsf{S}_t)_{t\ge 0}$ *in* $\mathscr{P}_2(X)$ *according to the previous definition 3.5. Then*

(1) *The relative entropy functional is strongly K-convex in $\mathscr{P}_2(X)$, i.e.* $(X,\mathsf{d},\mathfrak{m})$ *is a strong* $\mathrm{CD}(K,\infty)$ *metric measure space.*

(2) S_t *satisfies the K-contraction property*

$$W_2(\mathsf{S}_t\mu,\mathsf{S}_t\nu) \le \mathrm{e}^{-Kt}W_2(\mu,\nu) \quad \text{for every } \mu,\nu \in \mathscr{P}_2(X).$$

(3) S_t *satisfies the linearity property*

$$\mathsf{S}_t(\alpha\mu+\beta\nu) = \alpha\mathsf{S}_t\mu + \beta\mathsf{S}_t\nu,$$

for every $\mu,\nu \in \mathscr{P}_2(X),\ \alpha,\beta \in [0,1],\ \alpha+\beta = 1.$

(4) *The Cheeger energy is quadratic so that* $(X,\mathsf{d},\mathfrak{m})$ *is a* $\mathrm{RCD}(K,\infty)$ *metric measure space.*

(5) *For every* $\mu = f\mathfrak{m}$ *with* $f \in L^2(X,\mathfrak{m})$ *we have*

$$\mathsf{S}_t(\mu) = (\mathsf{P}_t f)\mathfrak{m} \quad t \ge 0,$$

where $(\mathsf{P}_t)_{t\ge 0}$ *is the Markov semigroup generated by the Dirichlet form* $\mathcal{E}$ *induced by the Cheeger energy.*

(6) $(\mathsf{P}_t)_{t\ge 0}$ *satisfies the pointwise gradient bound*

$$|\mathrm{D}\mathsf{P}_t f|_w^2 \le \mathrm{e}^{-2Kt}\mathsf{P}_t\big(|\mathrm{D}f|_w^2\big) \quad \mathfrak{m}\text{-}a.e.$$

so that the Dirichlet form $\mathcal{E}$ *satisfies the Bakry-Émery condition* $\mathrm{BE}(K,\infty)$.

(7) *Every function* $f \in D(\mathrm{Ch}) \cap L^\infty(X,\mathfrak{m})$ *with* $|\mathrm{D}f|_w \le 1$ $\mathfrak{m}$*-a.e. admits a continuous representative* $\tilde f$ *which is* 1*-Lipschitz and it is possible to reconstruct the distance through the Biroli-Mosco formula*

$$\mathsf{d}(x,y) = \sup\Big\{\tilde f(x) - \tilde f(y) : f \in D(\mathcal{E}) \cap L^\infty(X,\mathfrak{m}),\ |\mathrm{D}f|_w \le 1\Big\}. \tag{44}$$

Moreover, the notion of EVI K*-flow (and in particular its linearity) is stable for measured Gromov-Hausdorff convergence.*

It is clear from this result that the equivalence between RCD and BE spaces can be obtained by exhibiting an EVI K-flow of the Entropy functional. The first implication is stated in the next Theorem.

Theorem 3.7 (RCD $\Rightarrow$ BE [8, 5])**.** *If* $(X, \mathsf{d}, \mathfrak{m})$ *is a* $\mathrm{RCD}(K,\infty)$ *metric measure space then the* L^2*-gradient flow of the Cheeger energy induces an EVI* K*-flow of the Entropy functional; in particular the Cheeger energy satisfies the* $\mathrm{BE}(K,\infty)$ *condition according to Definition 1.2.*

3.3. From BE to RCD. Let us now consider the converse point of view: starting from an energy measure space $(X, \tau, \mathcal{E})$ as in Section 1.3 satisfying the $\mathrm{BE}(K,\infty)$ condition, recover the $\mathrm{RCD}(K,\infty)$ property. Clearly, one has to settle the question of defining a natural distance $\mathsf{d}_\mathcal{E}$ which should be compatible with the energy structure. We can use a formula analogous to (44) [26], provided $\mathcal{E}$ satisfies two further compatibility condition with the topology τ of the space:

a) every function $f \in D(\mathcal{E}) \cap L^\infty(X,\mathfrak{m})$ with $\Gamma(f,f) \le 1$ $\mathfrak{m}$-a.e. admits a τ-continuous representative $\tilde f$;

b) there exists a function $\theta \in \mathrm{C}_b(X)$ such that $\theta_k := \mathrm{S_k}(\theta)$ satisfies $\theta_k \in D(\mathcal{E})$ with $\Gamma(\theta_k,\theta_k) \le 1$ for every $k \in \mathbb{N}$.

Here $\mathrm{S_k}(a) = k\,\mathrm{S}(a/k)$ for an arbitrary truncation function $\mathrm{S} \in \mathrm{C}^1(\mathbb{R})$ with

$$|\,\mathrm{S}'(a)| \le 1, \quad \mathrm{S}(a) = \begin{cases} 1 & \text{if } |a| \le 1,\\ 0 & \text{if } |a| \ge 3 \end{cases} \qquad \text{for every } a \in \mathbb{R}.$$

We can thus define

$$\mathsf{d}_\mathcal{E}(x,y) = \sup\Big\{\tilde f(x) - \tilde f(y) : f \in D(\mathcal{E}) \cap L^\infty(X,\mathfrak{m}),\ \Gamma(f,f) \le 1\Big\}, \tag{45}$$

and state our second main result.

Theorem 3.8 (BE $\Rightarrow$ RCD [9])**.** *Let* (X,τ) *be a Polish space and let* $\mathfrak{m}$ *be a* σ*-finite Borel measure in* X*. Let* $\mathcal{E} : L^2(X,\mathfrak{m}) \to [0,\infty]$ *be a strongly local, symmetric Dirichlet form generating a mass preserving Markov semigroup* $(\mathsf{P}_t)_{t\ge 0}$ *in* $L^2(X,\mathfrak{m})$*, and satisfying conditions a), b) above. If* $\mathsf{d}_\mathcal{E}$ *is a complete distance on* X *which induces the topology* τ *and satisfies volume growth condition* (27)*, and the Bakry-Émery* $\mathrm{BE}(K,\infty)$ *condition holds, then* P *induces a* EVI K*-flow for the Entropy functional with respect to the* $\mathsf{d}_\mathcal{E}$*-Wasserstein distance and therefore* $(X, \mathsf{d}_\mathcal{E}, \mathfrak{m})$ *is a* $\mathrm{RCD}(K,\infty)$ *space. Moreover, the Cheeger energy* Ch *induced by* $\mathsf{d}_\mathcal{E}$ *coincides with* $\frac{1}{2}\mathcal{E}$*.*

3.4. The equivalence between RCD(K, N) and BE(K, N). By the identification result of the previous section, it is natural to ask if for a RCD(K, ∞) metric measure space $(X, \mathsf{d}, \mathfrak{m})$ the conditions $\mathrm{CD}^*(K, N)$ and $\mathrm{BE}(K, N)$ are equivalent.

Theorem 3.9 ($\mathrm{RCD}(K, N) \Leftrightarrow \mathrm{BE}(K, N)$)**.** *Let $K \in \mathbb{R}$, $N \in [1, \infty)$; if $(X, \mathsf{d}, \mathfrak{m})$ has a quadratic Cheeger energy, then it satisfies the $\mathrm{CD}^*(K, N)$ (or, equivalently, $\mathrm{CD}^e(K, N)$) curvature-dimension condition if and only if Ch satisfies the $\mathrm{BE}(K, N)$ condition and every function $f \in D(\mathrm{Ch})$ with $|\mathrm{D}f|_w \le 1$ has a 1-Lipschitz representative.*
In both cases, $(X, \mathsf{d}, \mathfrak{m})$ is $\mathrm{RCD}(K, \infty)$ space.

This fundamental result has been proved by [38], still by using a nice characterization of both conditions in term of the existence of an EVI (K, N)-flow for the Entropy power functional H_N (31): this means that for every initial datum $\mu \in D(\mathrm{H}_N)$ there exists a locally Lipschitz curve $t \mapsto \mu_t$ solving

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{s}_{K/N}\Big(\frac{1}{2}W_2^2(\mu_t, \nu)\Big) + K\mathsf{s}_{K/N}\Big(\frac{1}{2}W_2^2(\mu_t, \nu)\Big) \le \frac{N}{2}\Big(1 - \frac{\mathrm{H}_N(\nu)}{\mathrm{H}_N(\mu_t)}\Big)$$

for a.e. $t > 0$ and every $\nu \in D(\mathrm{H}_N)$, where

$$\mathsf{s}_\kappa(\delta) := \begin{cases} \frac{1}{\sqrt{\kappa}}\sin(\sqrt{\kappa}\delta) & \text{if } \kappa > 0, \\ \delta & \text{if } \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}}\sinh(\sqrt{-\kappa}\delta) & \text{if } \kappa < 0. \end{cases}$$

The deep arguments of [38] then show that an EVI (K, N)-flow for H_N is provided by the L^2-gradient flow of the (quadratic) Cheeger energy in the RCD case and by the Markov semigroup generated by $\mathcal{E}$ in the BE case.

A different approach has been followed by [12], by considering solutions $t \mapsto \mu_t$ of the modified EVI flow (recall (36))

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t, \nu) + K\mathcal{B}_U(\mu_t, \nu) \le \mathcal{U}(\nu) - \mathcal{U}(\mu_t) \quad \text{for a.e. } t > 0,\ \forall\, \nu \in D(\mathcal{U}), \tag{46}$$

for an arbitrary regular entropy functional $\mathcal{U} \in \mathrm{DC}_{reg}(N)$. The existence of such a flow still implies the $\mathrm{CD}^*(K, N)$ condition and the $\mathrm{BE}(K, N)$ condition, in the nonlinear equivalent form

$$\mathbf{\Gamma}^2(f, f; P(\varphi)) + \int_X \big(\varphi P'(\varphi) - P(\varphi)\big)\big(\mathsf{A}f\big)^2\,\mathrm{d}\mathfrak{m} \ge K\int_X \Gamma(f, f)P(\varphi)\,\mathrm{d}\mathfrak{m}$$

for every $f \in D(\mathsf{A}) \cap L^\infty(X, \mathfrak{m})$ with $\mathsf{A}f \in L^\infty(X, \mathfrak{m})$ and every nonnegative and bounded $\varphi \in D(\mathsf{A})$ with $\mathsf{A}P(\varphi) \in L^\infty(X, \mathfrak{m})$; recall that $P(r) = rU'(r) - U(r)$.

Existence of solutions to (46) have been provided by solving the nonlinear diffusion equations

$$\partial_t f_t = \mathsf{A}(P(f_t)), \quad \lim_{t \downarrow 0} f_t = f,$$

with initial datum $f \in L^2(X, \mathfrak{m})$ and constructing a nonlinear semigroup $(\mathsf{N}_t)_{t \ge 0}$ in $\mathscr{P}_2(X)$ which exhibits a solution to (46).

3.5. Further developments. It is very difficult to give even a short description of the ongoing striking developments of the metric theory of RCD spaces. We just quote an (incomplete) list of main results with the corresponding references.
- *Tensorization:* [9, 15]
- *Global-to-local and local-to-global:* [8, 13]
- *Quasi-regularity and construction of a diffusion process:* [8]
- *Stability w.r.t. measured Gromov convergence:* [51, 16, 10]
- *Improved Bakry-Émery condition:* [79]
- *The splitting theorem:* [42, 43]
- *Nonsmooth Differential and Riemannian geometry:* [52, 71, 45, 44]
- *Sharp isoperimetric and functional inequalities:* [59, 30, 31, 56, 53, 74, 61, 11]
- *Structure of volume, tangent and metric cones:* [50, 62, 46]
- *Regularity of spaces with bounded Ricci curvature:* [72, 34]
- *Ricci flow:* [49, 83, 63]
- *Ricci tensor and variable Ricci bounds:* [81, 82]
- *Displacement interpolation and nonbanching:* [75, 77]
- *Local Poincaré inequalities:* [76]
- *Harnack, Li-Yau, and refined gradient estimates:* [40, 58, 23, 60]
- *Wasserstein duality for pointwise gradient estimates:* [66]
- *Properties of metric Sobolev spaces:* [1, 14, 2, 3]
- *Lagrangian flows, continuity equation and ODE's:* [17, 80, 47, 18]
- *Dirichlet forms, infinite dimensional analysis:* [64, 4]
- *Monge and optimal transportation problems:* [54, 29, 28]
- *Hamilton-Jacobi equation in metric spaces:* [55]

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