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Domain perturbations and estimates for the solutions of second order elliptic equations [☆]

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Abstract

We study the dependence of the variational solution of the inhomogeneous Dirichlet problem for a second order elliptic equation with respect to perturbations of the domain. We prove optimal L^2 and energy estimates for the difference of two solutions in two open sets in terms of the "distance" between them and suitable geometrical parameters which are related to the regularity of their boundaries. We derive such estimates when at least one of the involved sets is uniformly Lipschitz: due to the connection of this problem with the regularity properties of the solutions in the L^2 family of Sobolev–Besov spaces, the Lipschitz class is the reasonably weakest one compatible with the optimal estimates.

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1. Introduction

Let us fix a function $f \in L^2(\mathbb{R}^N)$ and, for given open sets $\Omega_i \subset D \subset \mathbb{R}^N$, $i = 1, 2$, let us consider the variational solutions $u_i := u_{\Omega_i} \in H_0^1(\Omega_i)$ of the elliptic problem with homogeneous Dirichlet boundary conditions:

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$$\begin{cases} \mathcal{A}u_i = f & \text{in } \Omega_i, \\ u_i = 0 & \text{on } \partial\Omega_i, \end{cases} \tag{1.1}$$

where $\mathcal{A}$ is a usual uniformly elliptic second order operator (see Section 2.1) satisfying suitable coercivity conditions on $H_0^1(D)$

$$\mathcal{A}u := -\sum_{i,j} \frac{\partial}{\partial x_j}\left(a^{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_i b^i(x)\frac{\partial u}{\partial x_i} + c(x)u \tag{1.2}$$

and whose coefficients are defined in D.

The aim of this paper is to study the dependence of u_Ω with respect to Ω: more precisely, if we trivially extend the u_i to 0 outside their domains, we want to give a precise estimate of the L^2 and the energy norm $\|\nabla u_1 - \nabla u_2\|_{L^2(\mathbb{R}^N;\mathbb{R}^N)}$ in terms of some "distance" between Ω_1 and Ω_2 and suitable geometrical parameters which are related to the regularity of their boundaries.

Problems of this kind have been widely studied under many points of view: without claiming any completeness, among the various contributions here we quote:

(a) the theory of stability for the solution of the Dirichlet problem (see [18,23], the expository papers [19,20], and the references therein),
(b) the variational approach by Γ- and Mosco-convergence (see [5,13–16,25,26]),
(c) the problems of shape optimization [6–8,31],
(d) the numerical analysis of the Dirichlet problem in curved domains by finite element methods [11,24].

In all these approaches, instead of fixing two open sets, it is often considered the behavior of a *family* (which could also depend on a continuous parameter) of domains Ω_n, $n \in \mathbb{N}$, together with the respective solutions $u_n := u_{\Omega_n}$: when the sequence Ω_n admits a "limit" set Ω as $n \uparrow +\infty$ with respect to a suitable notion of convergence, (a) (respectively (b)) provides general conditions on Ω (respectively on Ω_n) for the convergence of u_n to the solution $u := u_\Omega$ in the limit open set Ω. When the limit behavior of Ω_n is not known *a priori*, the questions of compactness (c) and of the general characterization of the limit ((b) and (c)) are then investigated. Error estimates between u and u_n when Ω_n are suitable finite element triangulations approaching Ω are presented in (d), whereas shape analysis (c) often deals with families originating from deformations of a fixed set Ω.

In our paper we are interested to obtain some *explicit* and *quantitative* measure of the error between u and u_n in terms of a suitably chosen *distance* between Ω and Ω_n, under the reasonably weakest regularity assumptions and without requiring any structural relations between the open sets.

In order to understand what kind of distance should be considered and which regularity should be assumed, let us briefly recall the simple arguments which are preliminary to the stability analysis and which provide the continuous dependence of u_Ω on Ω (see e.g. [27, Chapter 3, Sections 6.4–6.6]).

If Ω_n, Ω are given open sets confined in a fixed ball $D \subset \mathbb{R}^N$, and u_n, u are the corresponding solutions of (1.1) e.g. with $\mathcal{A} := -\Delta$, the convergence of u_n to u in $H_0^1(D)$ for every choice of $f \in H^{-1}(D)$ is in fact equivalent to the convergence in the sense of Mosco [25] of the closed subspaces $H_0^1(\Omega_n)$ to $H_0^1(\Omega)$ in $H_0^1(D)$ (here we adopt the natural convention to extend each function to 0 outside its domain of definition). By definition, this convergence means that the following two properties are satisfied:

$$\forall v \in H_0^1(\Omega) \quad \exists v_n \in H_0^1(\Omega_n): \ v_n \to v \quad \text{in } H_0^1(D), \tag{1.3}$$

$$v_{n_k} \in H_0^1(\Omega_{n_k}), \quad v_{n_k} \rightharpoonup v \text{ in } H_0^1(D) \quad \Rightarrow \quad v \in H_0^1(\Omega). \tag{1.4}$$

It is easy to see that (1.3) surely holds if every compact subset $K \Subset \Omega$ is absorbed by the sequence Ω_n, i.e.

$$\forall K \Subset \Omega \quad \exists n_0 \in \mathbb{N}: K \subset \Omega_n \quad \forall n \geqslant n_0; \tag{1.5}$$

if we set:

$$\check{e}(\Omega, \Omega_n) := \sup_{x \in \Omega \setminus \Omega_n} d\big(x, \mathbb{R}^N \setminus \Omega\big) = \sup_{x \in \Omega \setminus \Omega_n} d(x, \partial\Omega), \tag{1.6}$$

then (1.5) can rephrased as

$$\lim_{n \uparrow +\infty} \check{e}(\Omega, \Omega_n) = 0. \tag{1.7}$$

Analogously, if every compact set $K' \Subset \overline{D} \setminus \Omega$ definitively has empty intersection with Ω_n, i.e.

$$\forall K' \Subset \overline{D} \setminus \Omega \quad \exists n_0 \in \mathbb{N}: K' \cap \Omega_n = \emptyset \quad \forall n \geqslant n_0, \tag{1.8}$$

then (1.4) holds, *provided* Ω is sufficiently regular (in this case we say that Ω is *stable*), e.g. if Ω satisfies an exterior cone condition (see [20, Section 2.4], and the next Section 2.5) or even if it has a continuous boundary ([17, Theorem 1.4.2.2]: the minimal assumptions, due to Keldyš [23], can be expressed in terms of capacity [18]); again, (1.8) is equivalent to

$$\lim_{n \uparrow +\infty} e(\Omega_n, \Omega) = 0, \tag{1.9}$$

where

$$e(\Omega_n, \Omega) := \sup_{x \in \Omega_n} d(x, \Omega) = \sup_{x \in \Omega_n \setminus \Omega} d(x, \partial\Omega). \tag{1.10}$$

Therefore, we can say that if Ω is stable and if we can control both the quantities $\check{e}(\Omega_n, \Omega)$, $e(\Omega, \Omega_n)$, i.e.

$$\lim_{n\uparrow+\infty} e(\Omega \Delta \Omega_n, \partial\Omega) = \lim_{n\uparrow+\infty}\Bigl(\sup_{x\in\Omega\Delta\Omega_n} d(x, \partial\Omega)\Bigr) = 0, \tag{1.11}$$

then (1.3), (1.4) hold and $\lim_{n\uparrow+\infty} u_n = u$ strongly in $H^1_0(D)$.

These considerations point out that a reasonable measure of the difference between Ω_n and Ω should be $e(\Omega\Delta\Omega_n, \partial\Omega)$; this is not a symmetric quantity and does not provide a true distance between Ω_n and Ω: this reflects the fact that stability is imposed only on Ω. On the other hand, as it is observed by [20], interesting counterexamples coming from homogenization theory [12] show that, even if Ω is stable, asymptotic conditions expressed in terms of the usual Hausdorff distance or of its complementary version (see the next Section 2.3) like

$$\lim_{n\uparrow+\infty} d_H(\Omega, \Omega_n) = 0 \quad \text{or} \quad \lim_{n\uparrow+\infty} d_H\bigl(\mathbb{R}^N\setminus\Omega, \mathbb{R}^N\setminus\Omega_n\bigr) = 0, \tag{1.12}$$

are not sufficient to yield the convergence of u_n to u.

Therefore, it is natural to look for estimates of the type:

$$\|\nabla u - \nabla u_n\|_{L^2(\mathbb{R}^N;\mathbb{R}^N)} \leqslant C e(\Omega\Delta\Omega_n, \partial\Omega)^{\varsigma}, \quad \text{for some } \varsigma > 0, \tag{1.13}$$

where C is a constant which could depend only on (the norm of) f and on the regularity of Ω. It is not difficult to see (cf. Section 2.5) that whenever an estimate like (1.13) holds true, the solution u of the Dirichlet problem gains more regularity: more precisely, if (1.13) holds, e.g., for $f := 1$, then (the trivial extension of) u belongs to the Besov space $B^{1+\varsigma}_{2,\infty}(\mathbb{R}^N)$.

Simple one-dimensional examples show that this is possible only for $\varsigma \leqslant 1/2$: therefore it is quite natural to deal with domains where the solution of the homogeneous Dirichlet problem has such threshold regularity: up to now, the widest class of domains yielding such regularity is provided by the (*uniformly*) *Lipschitz open sets* [21,22,30]. It is well known that such sets can also be characterized by two nonnegative real parameters ρ, θ through the *uniform cone property* (see Section 2.6): in a ball of radius ρ around each point of their boundary it is possible to find an outward cone of directions whose opening angle is θ.

We will show (Example 1, Section 3) that if Ω is a (ρ, θ)-Lipschitz open set, $\rho \leqslant 1$, and u_n, u are the solutions of the Laplace equation in Ω_n, Ω, then there exists a constant $\ell > 0$ *depending only on the dimension* N and on the diameter of D such that

$$\|\nabla u - \nabla u_n\|_{L^2(\mathbb{R}^N;\mathbb{R}^N)} \leqslant \ell \|f\|^{1/2}_{L^2(D)} \|f\|^{1/2}_{H^{-1}(D)} \left(\frac{e(\Omega \vartriangle \Omega_n, \partial\Omega)}{\rho\sin\theta}\right)^{1/2}; \tag{1.14}$$

in particular, the best exponent $\varsigma = 1/2$ is effectively allowed.

We will also exhibit the analogous $L^2(\mathbb{R}^N)$ estimate for $u - u_n$, which will turn out to be of order 1. Finally, if also Ω_n is a (family of) Lipschitz open sets with uniformly bounded constants, then we will see that the quantity $e(\Omega \vartriangle \Omega_n, \partial\Omega)$ in (1.14) could be replaced by $d_H(\Omega, \Omega_n)$ or $d_H(\mathbb{R}^N\setminus\Omega, \mathbb{R}^N\setminus\Omega_n)$ with a careful analysis of the dependence of the related constants on the geometry of the open sets. A simple interpolation argument allows us to weaken the regularity assumptions on f and to extend the estimates to the Dirichlet problem with non-homogeneous boundary conditions.

L^2 and energy estimates could be the starting point of a quantitative analysis of the dependence of the eigenfunctions and the eigenvalues of the homogeneous Dirichlet problem in terms of Ω, following the general variational theory developed e.g. in [2]. Estimates like (1.14) (for different type of boundary conditions, see [29]) are also crucial for studying maximal regularity properties for the solutions of parabolic equations in non cylindrical domains, applying the abstract results of [28]: this topic will be addressed in a forthcoming paper.

After a preliminary section devoted to fix some notation, to recall some basic notions we need in the sequel, and to discuss the link between estimates like (1.13) and regularity, we will formalize our main results in Section 3. Section 4 presents some refinement of the classical variational framework which will be useful to derive our estimates and Section 5 provides the standard technical tools needed to “localize” them. The core of our arguments is developed in Section 6 whereas Section 7 deals with simple geometrical properties of Lipschitz open sets, which supply the required link between set distances and the regularity estimates of Section 6.

Finally, the last section collects all these results and completes the proofs of the main theorems, providing a finer descriptions of the estimates and their dependence on the various constants which characterize our problem.

2. Notation and problems

2.1. Second order variational elliptic equations

Let D be an open subset of $\mathbb{R}^N$ and for $1 \leqslant i, j \leqslant N$ let us be given coefficients $a^{ij} \in W^{1,\infty}(D)$, $b^i, c \in L^\infty(D)$ and nonnegative constants $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{L}$ (we are normalizing to 1 the ellipticity constant) satisfying:

$$a^{ij}(x) = a^{ji}(x) \quad \forall x \in D, \tag{A1}$$

$$|\xi|^2 \leqslant \mathfrak{a}(x,\xi) := \sum_{ij} a^{ij}(x)\xi_i\xi_j \leqslant \mathsf{a}|\xi|^2 \quad \forall x \in D,\ \xi \in \mathbb{R}^N, \tag{A2}$$

$$\sum_i \left|b^i(x)\right|^2 \leqslant \mathsf{b}^2, \quad \mathsf{c} \leqslant c(x) \leqslant \mathsf{c} + \mathsf{d} \quad \text{for a.e. } x \in D, \tag{A3}$$

$$\left|\mathfrak{a}(x,\xi) - \mathfrak{a}(y,\xi)\right| \leqslant \mathsf{L}\,|x-y||\xi|^2 \quad \forall x, y \in D,\ \xi \in \mathbb{R}^N. \tag{A4}$$

We introduce the bilinear form on $V \times V$:

$$a(u,v) := \int_D \left\{ \sum_{i,j} a^{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_i b^i \frac{\partial u}{\partial x_i} v + c\,uv \right\} \mathrm{d}x, \tag{A5}$$

which we will suppose *coercive* on the Hilbert space $V := H_0^1(D)$; more precisely, denoting by $\mathsf{p}_D \geqslant 0$ the squared inverse of the best constant for the Poincaré-type inequality in D

$$\mathsf{p}_D := \inf\left\{\int_D \left|\nabla\zeta(x)\right|^2 \mathrm{d}x\colon\ \zeta \in \mathcal{D}(D),\ \|\zeta\|^2_{L^2(D)} = 1\right\},$$

we suppose that

$$\mathsf{p} := \mathsf{p}_D + \mathsf{c} > 0 \tag{A6}$$

so that we can equip the Hilbert spaces $V := H_0^1(D)$, $H := L^2(D)$ with the equivalent norms:

$$\begin{aligned}\|u\|_V^2 &:= \int_D \mathfrak{a}\big(x, \nabla u(x)\big)\,\mathrm{d}x + \mathsf{c}\int_D |u|^2\,\mathrm{d}x \\ &\geqslant \int_D \left|\nabla u(x)\right|^2 \mathrm{d}x + \mathsf{c}\int_D |u|^2\,\mathrm{d}x \geqslant \mathsf{p}\int_D |u|^2\,\mathrm{d}x =: \|u\|_H^2.\end{aligned} \tag{A7}$$

We will measure the coercivity of a with respect to this norm, i.e.

$$\exists\alpha \in (0,1]\colon\quad a(u,u) \geqslant \alpha\|u\|_V^2 \quad \forall u \in V. \tag{A8}$$

When we consider the transposed bilinear form of a in order to derive L^2 estimates, it will be useful to assume that (a slight variant of) the so called Picard's condition [27, Chapter 1, Example 3.15] holds:

$$b^i \in W^{1,1}(D), \quad 0 \leqslant -\frac{1}{2}\sum_i \frac{\partial b^i}{\partial x_i} \leqslant \bar{\mathsf{b}} \quad \text{a.e. in } D. \tag{A9}$$

We adopt the convention to trivially extend to 0 each function defined in an open subset $\Omega \subset \mathbb{R}^N$ so that, if $\Omega \subset D$, we can identify $H_0^1(\Omega)$ with the closed subspace V_Ω of V

$$V_\Omega := \overline{\{\zeta \in \mathcal{D}(D)\colon \operatorname{supp}(\zeta) \subset \Omega\}}^{\,V}; \tag{2.1}$$

of course

$$\Omega_1 \subset \Omega_2 \quad \Rightarrow \quad V_{\Omega_1} \subset V_{\Omega_2}; \quad V_D = V.$$

The Lax–Milgram Lemma ensures that for every $f \in V'$ (which we can identify with $H^{-1}(D)$) there exists a unique solution $u := \mathcal{G}(f; V_\Omega) \in V_\Omega$ of the problem:

$$u \in V_\Omega, \quad a(u,v) = \langle f, v\rangle \quad \text{for all } v \in V_\Omega, \tag{2.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between V' and V. If we introduce the dual norm

$$\|f\|_{V'} := \sup\bigl\{\langle f, v\rangle \colon\ \|v\| \leqslant 1\bigr\} \quad \forall f \in V', \tag{2.3}$$

we have the obvious bound:

$$\bigl\|\mathcal{G}(f; V_\Omega)\bigr\|_V \leqslant \alpha^{-1}\,\|f\|_{V'} \quad \forall f \in V'. \tag{2.4}$$

2.2. Coercivity and natural splitting of bilinear forms

We are making precise some easy facts about the bilinear form a which we will need in the sequel. First of all we separate the “first order” part

$$b(u, v) := \int_D \sum_i b^i \frac{\partial u}{\partial x_i} v \,\mathrm{d}x \tag{2.5}$$

which we decompose into the sum of its symmetric and antisymmetric parts b_s, b_a:

$$b_s(u, v) := \frac{1}{2}\bigl(b(u, v) + b(v, u)\bigr) = \frac{1}{2}\int_D \sum_i b^i \frac{\partial(uv)}{\partial x_i}\,\mathrm{d}x, \tag{2.6}$$

$$b_a(u, v) := \frac{1}{2}\bigl(b(u, v) - b(v, u)\bigr) = \frac{1}{2}\int_D \sum_i b^i \left(\frac{\partial u}{\partial x_i} v - \frac{\partial v}{\partial x_i} u\right)\mathrm{d}x. \tag{2.7}$$

We observe that

$$\begin{aligned} \max\bigl[b_a(u, v), b(u, v)\bigr] &\leqslant \beta\|u\|_V\|v\|_V \\ b_s(u, v) &\leqslant \beta_s\|u\|_V\|v\|_V \end{aligned}, \quad \text{where } \beta_s, \beta := \frac{\mathsf{b}}{\sqrt{\mathsf{p}}}. \tag{2.8}$$

We can therefore split a as the sum of

$$\begin{gathered} a := a_0 + b = a_0 + b_s + b_a, \\ a_0(u, v) := \int_D \left\{\sum_{i,j} a^{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + c\,uv\right\}\mathrm{d}x \end{gathered} \tag{2.9}$$

and we observe that

$$a_0(u, v) \leqslant (1 + \gamma)\|u\|_V\,\|v\|_V, \quad \text{where } \gamma := \frac{\mathsf{d}}{\mathsf{p}}. \tag{2.10}$$

Combining all these easy bounds, we get:

$$\begin{aligned} a_s(u,v) &= a_0(u,v) + b_s(u,v) \\ &\leqslant (1+\beta_s+\gamma)\|u\|_V\,\|v\|_V \quad \forall u,v \in V, \end{aligned} \tag{2.11}$$

$$a_a(u,v) = b_a(u,v) \leqslant \beta\|u\|_V\,\|v\|_V \quad \forall u,v \in V. \tag{2.12}$$

The coercivity of a then follows if we assume that b is sufficiently small, e.g. if

$$\beta < 1; \quad \text{in this case we can choose } \alpha := 1-\beta. \tag{2.13}$$

When Picard's condition (A9) holds, then b_s behaves better, since

$$b_s(u,u) = b(u,u) \geqslant 0 \quad \forall u \in V \tag{2.14}$$

and therefore we can choose $\alpha = 1$ in (A8); moreover

$$b_s(u,v) \leqslant \beta_s\|u\|_H\|v\|_H \quad \forall u,v \in V, \quad \text{where now } \beta_s := \frac{\bar{\mathsf{b}}}{\mathsf{p}}; \tag{2.15}$$

in particular, we can use this last value of β_s in (2.11). We conclude these remarks by noticing that the adjoint bilinear form

$$\hat{a}(u,v) := a(v,u) = a_s(u,v) + b_s(u,v) - b_a(u,v) \quad \forall u,v \in V \tag{2.16}$$

satisfies the same bounds (2.11), (2.12), (2.15); moreover, when (A9) holds, then $\hat{a}$ admits the representation:

$$\hat{a}(u,v) = \int_D \left\{ \sum_{i,j} \hat{a}^{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_i \hat{b}^i \frac{\partial u}{\partial x_i} v + \hat{c}\, uv \right\} \mathrm{d}x \quad \forall u,v \in V, \tag{2.17}$$

where

$$\hat{a}^{ij} = a^{ij}, \qquad \hat{b}^i = -b^i, \qquad \hat{c} = c - \sum_i \frac{\partial b^i}{\partial x_i}. \tag{2.18}$$

Therefore, $\hat{a}$ satisfies (A1)–(A9) with respect to the coefficients $\hat{\alpha}$, $\hat{\mathsf{a}}$, $\hat{\mathsf{b}}$, $\hat{\mathsf{c}}$, $\hat{\mathsf{d}}$, $\hat{\mathsf{L}}$, $\hat{\mathsf{p}}$ with

$$\begin{aligned} &\hat{\alpha} = \alpha = 1, \qquad \hat{\mathsf{a}} = \mathsf{a}, \qquad \hat{\mathsf{b}} = \mathsf{b}, \qquad \hat{\mathsf{c}} = \mathsf{c}, \\ &\hat{\mathsf{L}} = \mathsf{L}, \qquad \hat{\mathsf{p}} = \mathsf{p}, \qquad \hat{\mathsf{d}} = \mathsf{d} + 2\bar{\mathsf{b}}. \end{aligned} \tag{2.19}$$

2.3. *Excess and Hausdorff distance between subsets of* $\mathbb{R}^N$

We shall denote by $B_\varepsilon(x)$ the *open* ball of center x and radius ε; as usual, for every $x \in \mathbb{R}^N$, $Y \subset \mathbb{R}^N$, we set:

$$d(x, Y) := \inf_{y \in Y} |x - y| = \min_{y \in \overline{Y}} |x - y|. \tag{2.20}$$

For every $\varepsilon > 0$ and $X \subset \mathbb{R}^N$, the ε-neighborhood of X is

$$X^\varepsilon := \bigl\{x \in \mathbb{R}^N\colon\ d(x, X) < \varepsilon\bigr\} = \bigcup_{x \in X} B_\varepsilon(x), \tag{2.21}$$

whereas

$$X^{-\varepsilon} := \bigl\{x \in X\colon\ B_\varepsilon(x) \subset X\bigr\} = \mathbb{R}^N \Big\backslash \bigl(\mathbb{R}^N \backslash X\bigr)^\varepsilon; \tag{2.22}$$

if X is a subset of Y, the *internal gap* $\delta(X, Y)$ is defined as:

$$\delta(X, Y) := \sup\bigl\{\varepsilon \geqslant 0\colon\ X^\varepsilon \subset Y\bigr\} = \sup\bigl\{\varepsilon \geqslant 0\colon\ X \subset Y^{-\varepsilon}\bigr\}. \tag{2.23}$$

Given two subsets $X, Y \subset \mathbb{R}^N$, we define the *excess* and the *complementary excess* of X from Y as

$$e(X, Y) := \inf\bigl\{\varepsilon > 0\colon\ X \subset Y^\varepsilon\bigr\} = \sup_{x \in X} d(x, Y), \tag{2.24}$$

$$\check{e}(X, Y) := \inf\bigl\{\varepsilon > 0\colon\ X^{-\varepsilon} \subset Y\bigr\} = e\bigl(\mathbb{R}^N \backslash Y, \mathbb{R}^N \backslash X\bigr); \tag{2.25}$$

it is easy to check that

$$e(X, Y) = e(X \backslash Y, Y) = e(X \backslash Y, \partial Y), \tag{2.26}$$

$$\check{e}(X, Y) = e\bigl(X \backslash Y, \mathbb{R}^N \backslash X\bigr) = e(X \backslash Y, \partial X). \tag{2.27}$$

The *Hausdorff distance* between X and Y is defined by:

$$d_H(X, Y) := \max\bigl\{e(X, Y), e(Y, X)\bigr\}, \tag{2.28}$$

and its *complementary* version is

$$\check{d}_H(X, Y) := \max\bigl\{\check{e}(X, Y), \check{e}(Y, X)\bigr\} = d_H\bigl(\mathbb{R}^N \backslash X, \mathbb{R}^N \backslash Y\bigr); \tag{2.29}$$

we also set:

$$\mathfrak{d}(X, Y) := \max\bigl\{d_H(X, Y), \check{d}_H(Y, X)\bigr\}. \tag{2.30}$$

It follows from (2.26) and (2.27) that

$$\max\{e(X,Y),\check{e}(Y,X)\} = e(X \triangle Y, \partial Y), \tag{2.31a}$$

$$\begin{aligned}\mathfrak{d}(X,Y) &= \max\{e(X,Y), e(Y,X), \check{e}(X,Y), \check{e}(Y,X)\} \\ &= \max\{e(X \triangle Y, \partial Y), e(X \triangle Y, \partial X)\}\end{aligned} \tag{2.31b}$$

so that $\mathfrak{d}(X,Y)$ provides the strongest way to measure the distance between X, Y.

For the convenience of the reader we provide a brief proof of the previous formula. (2.21) is immediate; for (2.22) observe that

$$\begin{aligned}x \in X^{-\varepsilon} \quad &\Leftrightarrow \quad B_\varepsilon(x) \subset X \quad \Leftrightarrow \quad B_\varepsilon(x) \subset \mathbb{R}^N \setminus (\mathbb{R}^N \setminus X) \\ &\Leftrightarrow \quad d(x, \mathbb{R}^N \setminus X) \geqslant \varepsilon \quad \Leftrightarrow \quad x \notin (\mathbb{R}^N \setminus X)^\varepsilon.\end{aligned}$$

(2.24) is a simple consequence of the following fact:

$$\text{for every } \varepsilon' > \delta := \sup_{x \in X} d(x,Y) > \varepsilon \text{ we have } X \not\subset Y^\varepsilon, \quad X \subset Y^{\varepsilon'}.$$

(2.25) follows from (2.22) by taking the complement of the sets in the definition (2.24):

$$\begin{aligned}\check{e}(X,Y) &:= \inf\{\varepsilon > 0\colon X^{-\varepsilon} \subset Y\} = \inf\{\varepsilon > 0\colon \mathbb{R}^N \setminus (\mathbb{R}^N \setminus X)^\varepsilon \subset Y\} \\ &= \inf\{\varepsilon > 0 : \mathbb{R}^N \setminus Y \subset (\mathbb{R}^N \setminus X)^\varepsilon\} = e(\mathbb{R}^N \setminus Y, \mathbb{R}^N \setminus X);\end{aligned}$$

In order to show (2.26), first of all we observe that

$$e(X,Y) = \sup_{x \in X} d(x,Y) = \sup_{x \in X \setminus Y} d(x,Y) = e(X \setminus Y, Y)$$

since $d(x,Y) = 0$ if $x \in Y$. The last relation of (2.26) follows by observing that, if $Z \cap Y = \emptyset$,

$$e(Z,Y) = e(Z, \overline{Y}) = \sup_{z \in Z} \min_{y \in \overline{Y}} |z - y| = \sup_{z \in Z} \min_{y \in \partial Y} |z - y| = e(Z, \partial Y);$$

(2.27) is a consequence of (2.26) and (2.25). Finally, (2.26) and (2.27) yield (2.31a) and (2.31b).

2.4. Interpolation and Intermediate Sobolev–Besov spaces

Let us briefly recall the definition and the basic properties of the intermediate Sobolev–Besov spaces we need (for a complete treatment of the relative theory, we refer to [3,9,33]). It will be useful to adopt the point of view of interpolation theory: if $E_1 \subset E_0$ (with continuous imbedding) are Banach spaces, $s \in (0,1)$, $q \in [1,+\infty]$, we will denote by $(E_0, E_1)_{s,q} = (E_1, E_0)_{1-s,q}$ the Banach space constructed by the real interpolation method and normed by means of the K-Peetre's functional [33, Section 1.3.2]. This family of Banach spaces satisfies the monotonicity properties

$$s_1 > s_2 \quad \Rightarrow \quad (E_0, E_1)_{s_1,q_1} \subset (E_0, E_1)_{s_2,q_2} \quad \forall q_1, q_2 \in [0, +\infty];$$
$$q_1 < q_2 \quad \Rightarrow \quad (E_0, E_1)_{s,q_1} \subset (E_0, E_1)_{s,q_2} \quad \forall s \in (0, 1). \tag{2.32}$$

Let D be an open subset of $\mathbb{R}^N$; for $s \in (0,1)$, $q \in [1, \infty]$ we define:

$$B^s_{2,q}(D) := \big(L^2(D), H^1(D)\big)_{s,q}, \qquad B^{1+s}_{2,q}(D) := \big(H^1(D), H^2(D)\big)_{s,q},$$
$$B^{-s}_{2,q}(D) := \big(L^2(D), H^{-1}(D)\big)_{s,q} \tag{2.33}$$

with the well known particular cases $H^s(D) = B^s_{2,2}(D)$ for $s \in (-1, 2)$, and the obvious continuous inclusions yielded by (2.32).

We will use the following characterizations of $B^{1+s}_{2,\infty}(\mathbb{R}^N)$, $0 < s < 1$, by difference quotients: for a function v defined in $\mathbb{R}^N$ and a vector $h \in \mathbb{R}^N$, let us first denote by v_h the shifted function $v_h(x) := v(x + h)$. A function $v \in H^1(\mathbb{R}^N)$ belongs to the Besov space $B^{1+\varsigma}_{2,\infty}(\mathbb{R}^N)$ for $\varsigma \in (0, 1)$ if and only if there exist constants $c, h_0 > 0$, such that

$$\|\nabla v - \nabla v_h\|_{L^2(\mathbb{R}^N)} \leqslant c|h|^s \quad \forall h \in \mathbb{R}^N,\ |h| \leqslant h_0. \tag{2.34}$$

We conclude by recalling a useful property which follows by the same arguments of [3, 3.5(b)].

Proposition 2.1. *Suppose that $E_1 \subset E_0$ is a couple of Banach spaces, the inclusion being continuous, and suppose that $\mathcal{T}$ is a linear bounded operator mapping E_1 into a Banach space F and there exist $L > 0$ and $s \in]0, 1[$ such that*

$$\|\mathcal{T} e\|_F \leqslant L\|e\|^{1-s}_{E_0}\|e\|^s_{E_1}, \quad \forall e \in E_1. \tag{2.35}$$

Then $\mathcal{T}$ can be continuously extended to a bounded linear operator between $(E_0, E_1)_{s,1}$ and F and there exists a constant $c = c_s$ such that

$$\|\mathcal{T}\|_{(E_0,E_1)_{s,1}\to F} \leqslant cL. \tag{2.36}$$

2.5. A general perturbation problem

We can now try a first formalization of the type of problems we are mainly interested in.

Problem (P). *Let Ω_1, Ω_2 be two open subsets of D and let W be a Banach space with $L^2(D) \subset W \subset V'$. For every $f \in W$ we consider the solutions u_1, u_2 of the Dirichlet problems* (2.2) *in Ω_1, Ω_2, respectively*

$$u_i = \mathcal{G}(f; V_{\Omega_i}) \in V_{\Omega_i}, \tag{2.37}$$

and we look for estimates of the type

$$\|u_1 - u_2\|_V \leqslant C\|f\|_W \mathfrak{d}(\Omega_1, \Omega_2)^\varsigma \quad \textit{for some } \varsigma > 0, \tag{2.38}$$

where C is a constant which should depend only on the bilinear form a and on suitable parameters which measure the regularity properties of the open sets Ω_1, Ω_2.

Remark 2.2. Since we have

$$\|u_1 - u_2\|_V \leqslant 2\alpha^{-1}\|f\|_{V'} \tag{2.39}$$

independently of Ω_1, Ω_2, it is interesting (and sufficient) to check (2.38) only for small distances $\mathfrak{d}(\Omega_1, \Omega_2)$; therefore it will be not restrictive to assume $\mathfrak{d}(\Omega_1, \Omega_2)$ is less than some parameter which will be fixed each time.

We have chosen *a priori* $\mathfrak{d}(\Omega_1, \Omega_2)$ in (2.38) since we already observed that it dominates the other set-distances between Ω_1, Ω_2, but we are also interested to find analogous estimates with respect to any of the other quantities introduced in the paragraph above.

It is not surprising that the constant C in (2.38) should also depend on the regularity of the open sets Ω_1, Ω_2, since (2.38) implies some extra regularity on the solution of the Dirichlet problem.

Proposition 2.3. *Let us suppose that $D := \mathbb{R}^N$ and $\mathcal{A}u := -\Delta u + u$, so that $V = H^1(\mathbb{R}^N)$. If $\Omega := \Omega_1$ is an admissible open subset for* (2.38), *then* (2.38) *yields the following regularity result*:

$$f \in L^2(\mathbb{R}^N) \quad \Rightarrow \quad u := \mathcal{G}(f; V_\Omega) \in B^{1+\varsigma}_{2,\infty}(\mathbb{R}^N). \tag{2.40}$$

Proof. Since the constant C in (2.38) should depend only on the geometric properties of Ω, then for every vector $h \in \mathbb{R}^N$ the translated set $\Omega_2 := \Omega - h$ is admissible, too, and the constant C does not depend on h. Therefore, if $u := \mathcal{G}(f; V_\Omega)$,

$$f_h(t) := f(t+h), \qquad u_h(t) := u(t+h) \qquad \forall t \in \mathbb{R}^N,$$

and $v^h := \mathcal{G}(f_h; V_\Omega)$, then $u_h = \mathcal{G}(f_h; V_{\Omega - h})$ and (2.38) yields that

$$\left\|v^h - u_h\right\|_{H^1(\mathbb{R}^N)} \leqslant C\|f_h\|_{L^2(\mathbb{R}^N)}|h|^\varsigma = C\|f\|_{L^2(\mathbb{R}^N)}|h|^\varsigma, \tag{2.41}$$

since $\mathfrak{d}(\Omega, \Omega - h) \leqslant |h|$. On the other hand,

$$\left\|u - v^h\right\|_{H^1(\mathbb{R}^N)} \leqslant \|f - f_h\|_{H^{-1}(\mathbb{R}^N)} \leqslant |h|\|f\|_{L^2(\mathbb{R}^N)}. \tag{2.42}$$

Combining (2.41) and (2.42), since $\varsigma \leqslant 1$ we get:

$$\|u - u_h\|_{H^1(\mathbb{R}^N)} \leqslant (C+1)\|f\|_{L^2(\mathbb{R}^N)}|h|^\varsigma \quad \forall h \in \mathbb{R}^N,\ |h| \leqslant 1. \tag{2.43}$$

Thanks to (2.34) we conclude that $u \in B^{1+\varsigma}_{2,\infty}(\mathbb{R}^N)$. $\square$

Remark 2.4. (2.38) is possible only if

$$\varsigma \leqslant 1/2. \tag{2.44}$$

This fact can be easily seen by the following one-dimensional example: we choose:

$$D := \mathbb{R}, \qquad \mathcal{A}u := -u'' + u, \qquad f(x) := \chi_{(-2,2)}(x), \tag{2.45}$$

$$\Omega_1 := (-1,1), \qquad \Omega_2 := (-1-h, 1-h), \quad 0 < h < 1. \tag{2.46}$$

We have:

$$u_1(x) = \left(1 - \frac{\cosh x}{\cosh 1}\right)^+, \qquad u_2(x) = \left(1 - \frac{\cosh(x+h)}{\cosh 1}\right)^+ = u_1(x+h), \tag{2.47}$$

and a simple direct computation shows (2.40) for $\varsigma = 1/2$. If (2.38) holds for $\varsigma > 1/2$ by (2.34) we get $u_1 \in B^{1+\varsigma}_{2,\infty}(\mathbb{R}) \subset H^{3/2}(\mathbb{R})$, but this is impossible since u_1' has jump discontinuities at $x = \pm 1$.

Remark 2.5. By analogous (but easier) arguments it is possible to check that the best possible estimate for the L^2 norm of the difference between u_1 and u_2 is of order 1, i.e. the corresponding version of (2.38)

$$\|u_1 - u_2\|_{L^2(\mathbb{R}^N)} \leqslant C\|f\|_W \mathfrak{d}(\Omega_1, \Omega_2)^\varsigma \quad \text{is in general false if } \varsigma > 1. \tag{2.48}$$

Since we need a quantitative way to measure the regularity of the open sets Ω_1, Ω_2, and Proposition 2.3 enlightens the link between Problem (P) and the regularity properties of the solutions of (2.2), it is natural to focus our attention to the class of Lipschitz domains.

2.6. *Lipschitz open sets and cone conditions*

In this paper we will mainly deal with *uniform Lipschitz domains* of $\mathbb{R}^N$, i.e. open sets satisfying the minimal smooth condition [32, Section VI, 3.2–3.3]. Here we recall an equivalent geometric characterization of such sets, which is well adapted to our regularity analysis (cf. [30, Section 3]).

First of all for every angle $\theta \in\,]0,\pi]$, radius $\rho > 0$, and unitary vector $\boldsymbol{n} \in \mathbb{S}^{N-1}$, we will consider the open cone with vertex at 0, height ρ, opening θ, and the axis pointing toward $\boldsymbol{n}$:

$$\mathcal{C}_{\rho,\theta}(\boldsymbol{n}) := \left\{h \in \mathbb{R}^N \colon\ 0 < |h| < \rho,\ h \cdot \boldsymbol{n} > |h| \cos\theta\right\}. \tag{2.49}$$

Definition 2.6. Let $\Omega \subset \mathbb{R}^N$, $\rho > 0$, and $\theta \in\,]0,\pi]$ be given; for every $x_0 \in \mathbb{R}^N$ we call $\mathcal{N}_{\rho,\theta}(x_0, \Omega)$ the (possibly empty) set of the vectors $\boldsymbol{n} \in \mathbb{S}^{N-1}$ satisfying:

$$\left.\begin{array}{l}\big(B_{3\rho}(x_0) \cap \Omega\big) - h \subset \Omega, \\ \big(B_{3\rho}(x_0) \setminus \Omega\big) + h \subset \mathbb{R}^N \setminus \Omega\end{array}\right\} \quad \forall h \in \mathcal{C}_{\rho,\theta}(\boldsymbol{n}). \tag{2.50}$$

We say that Ω satisfies a (ρ,θ) *cone condition* on $\Gamma\subset\mathbb{R}^N$ if

$$\mathcal{N}_{\rho,\theta}(x_0,\Omega)\neq\emptyset\quad\forall x_0\in\Gamma;\tag{2.51}$$

finally, we say that Ω satisfies a *uniform* (ρ,θ) *cone condition* or it is a *uniform* (ρ,θ)*-set*, if (2.51) holds for $\Gamma\equiv\mathbb{R}^N$.

Remark 2.7. When Ω is the epigraph of a Lipschitz function $g:\mathbb{R}^{N-1}\to\mathbb{R}$, i.e.,

$$\Omega:=\bigl\{x=(x_1,\dots,x_{N-1},x_N)\in\mathbb{R}^N:\ x_N>g(x_1,\dots,x_{N-1})\bigr\},\tag{2.52}$$

then for every $x_0\in\mathbb{R}^N$ the vector $\boldsymbol{n}:=(0,\dots,0,-1)$ belongs to $\mathcal{N}_{\rho,\theta}(x_0,\Omega)$ with $\rho:=+\infty$ and $\theta:=\arctan(1/\operatorname{Lip}g)$ (here $\operatorname{Lip}g$ denotes the Lipschitz constant of g).

If Ω is a *minimally smooth open set* as defined in [32] (in particular, if it is a bounded *Lipschitz* open set), it is well known that Ω satisfies the previous *cone condition* on its boundary $\partial\Omega$ for some admissible couple (ρ,θ) (see [17, Definition 1.2.2.1 and Theorem I.2.2.2]); the following remark shows that in this case it is not restrictive to suppose that Ω satisfies an analogous condition on the whole $\mathbb{R}^N$, as we will always assume in the following.

Remark 2.8. It is easy to see that if Ω satisfies a (ρ,θ)-cone condition on $\partial\Omega$, then Ω is a uniform $(\rho/3,\theta)$-set. Indeed, if $B_{2\rho}(y)\cap\partial\Omega\neq\emptyset$ then there exists $x_0\in\partial\Omega$ such that $B_\rho(y)\subset B_{3\rho}(x_0)$; by (2.50) we have:

$$\mathcal{N}_{\rho/3,\theta}(y,\Omega)\supset\mathcal{N}_{\rho,\theta}(x_0,\Omega).$$

If $B_{2\rho}(y)\cap\partial\Omega=\emptyset$ then $B_{2\rho}(y)\subset\Omega$ or $B_{2\rho}(y)\subset\mathbb{R}^N\setminus\Omega$. In both cases $\mathcal{N}_{\rho/3,\theta}(y,\Omega)=\mathbb{S}^{N-1}$.

2.7. Scaling invariance

It is natural to study how the estimates of Problem (P) depend on dilations of $\mathbb{R}^N$.

If, for $\kappa>0$, we denote by:

$$D^\kappa:=\kappa D,\quad \Omega_i^\kappa:=\kappa\Omega_i,\quad \begin{array}{l} u_i^\kappa(x):=\kappa^{-N/2+1}u_i(x/\kappa),\\ f^\kappa(x):=\kappa^{-N/2-1}f(x/\kappa),\end{array}\tag{2.53}$$

then it is easy to see that u_i^κ solve the Dirichlet problems:

$$\mathcal{A}^\kappa u_i^\kappa=f^\kappa\quad\text{in }\Omega_i^\kappa,\qquad u_i^\kappa=0\quad\text{on }\partial\Omega_i^\kappa,\tag{2.54}$$

where the coefficients $a^{ij,\kappa}$, $b^{i,\kappa}$, c^κ of $\mathcal{A}^\kappa$ are given by:

$$a^{ij,\kappa}(x) := a^{ij}(x/\kappa), \quad b^{i,\kappa}(x) := \kappa^{-1} b^i(x/\kappa),$$
$$c^\kappa(x) := \kappa^{-2} c(x/\kappa), \tag{2.55}$$

so that the respective bounds satisfy:

$$\mathsf{a}^\kappa = \mathsf{a}; \qquad \mathsf{b}^\kappa = \kappa^{-1}\mathsf{b}, \qquad \mathsf{L}^\kappa = \kappa^{-1}\mathsf{L};$$
$$\mathsf{c}^\kappa = \kappa^{-2}\mathsf{c}, \qquad \mathsf{d}^\kappa = \kappa^{-2}\mathsf{d}, \qquad \bar{\mathsf{b}}^\kappa = \kappa^{-2}\bar{\mathsf{b}}.$$

Simple calculations show that the Poincaré like constant p^κ rescales as

$$\mathsf{p}_D^\kappa := \mathsf{p}_{D^\kappa} = \kappa^{-2}\mathsf{p}_D, \qquad \mathsf{p}^\kappa = \kappa^{-2}\mathsf{p}. \tag{2.56}$$

It follows that the bilinear form a and the Hilbert space norms we introduced in (A7) are invariant with respect to (2.53), since, with obvious meaning of a^κ, V^κ, H^κ,

$$a^\kappa\left(u^\kappa, v^\kappa\right) = a(u, v), \qquad {}_{(V^\kappa)'}\left\langle f^\kappa, u^\kappa\right\rangle_{V^\kappa} = {}_{V'}\langle f, v\rangle_V,$$
$$\left\|u_i^\kappa\right\|_{V^\kappa} = \|u_i\|_V, \qquad \left\|u_i^\kappa\right\|_{H^\kappa} = \|u_i\|_H,$$
$$\left\|f^\kappa\right\|_{(H^\kappa)'} = \|f\|_{H'}, \qquad \left\|f^\kappa\right\|_{(V^\kappa)'} = \|f\|_{V'}.$$

Since geometric quantities behave like

$$\mathfrak{d}\left(\Omega_1^\kappa, \Omega_2^\kappa\right) = \kappa\,\mathfrak{d}(\Omega_1, \Omega_2), \qquad \rho^\kappa = \kappa\rho, \qquad \theta^\kappa = \theta,$$

we can guess that the right invariant form for estimates like (2.38) should be

$$\|u_1 - u_2\|_V \leqslant C(\mathsf{a}, \alpha, \beta, \beta_s, \gamma, \lambda, \theta, \pi_\rho, N)\|f\|_W\left(\frac{\mathfrak{d}(\Omega_1, \Omega_2)}{\rho}\right)^\varsigma, \tag{2.57}$$

where $\|\cdot\|_W$ is an invariant norm (as for H', V') and (see (2.8), (2.10)),

$$\beta := \frac{\mathsf{b}}{\mathsf{p}^{1/2}}, \qquad \beta_s := \frac{\bar{\mathsf{b}}}{\mathsf{p}}, \qquad \gamma := \frac{\mathsf{d}}{\mathsf{p}}, \qquad \lambda := \frac{\mathsf{L}}{\mathsf{p}^{1/2}}, \qquad \pi_\rho := \mathsf{p}^{1/2}\rho \tag{2.58}$$

are invariant parameters associated to a, Ω, D.

3. Main results

In our first result, we are assuming that only Ω_1 satisfies a (ρ, θ) cone condition on D: we will show that the regularity of Ω_1 affects the estimates through the constant $\rho\sin\theta$ and the relevant geometric "distance" between Ω_1, Ω_2 is $e(\Omega_1 \triangle \Omega_2, \partial\Omega_1)$. Let us recall that

$$e(\Omega_1 \triangle \Omega_2, \partial\Omega_1) \leqslant \varepsilon \quad \Leftrightarrow \quad \Omega_1^{-\varepsilon} \subset \Omega_2 \subset \Omega_1^\varepsilon \quad \forall \varepsilon > 0.$$

Notation 3.1. We say that a constant C is *admissible* if it depends only on the invariant constants $\mathsf{a}, \alpha^{-1}, \beta, \gamma, \lambda, \pi_\rho, N$ (see (A2), (A8), and (2.58)) and it is not decreasing with respect to each parameter. When Picard's condition (A9) holds, we admit the dependence on β_s, too.

Theorem 1. *Let us suppose that* (A1)–(A8) *hold, let Ω_1, Ω_2 be two open subsets of D, and let Ω_1 satisfy a uniform (ρ_1, θ_1) cone condition on D. Let $f \in L^2(D)$ and let*

$$u_i = \mathcal{G}(f; V_{\Omega_i}) \in V_{\Omega_i} \quad \text{be the solutions to (2.2).} \tag{3.1}$$

There exists an admissible *constant $C_1 > 0$ such that*

$$\|u_1 - u_2\|_V^2 \leqslant C_1 \|f\|_{H'} \|f\|_{V'} \frac{e(\Omega_1 \triangle \Omega_2, \partial\Omega_1)}{\rho_1 \sin\theta_1}, \tag{3.2}$$

provided

$$e(\Omega_2, \Omega_1) \leqslant \delta(\Omega_1, D). \tag{3.3}$$

If also Ω_2 is a (ρ_2, θ_2) uniform set, then for an admissible constant C_1',

$$\|u_1 - u_2\|_V^2 \leqslant C_1' \|f\|_{H'} \|f\|_{V'} \left\{ \frac{\check{e}(\Omega_1, \Omega_2)}{\rho_1 \sin\theta_1} + \frac{\check{e}(\Omega_2, \Omega_1)}{\rho_2 \sin\theta_2} \right\}. \tag{3.4}$$

Finally, if

$$e(\Omega_2, \Omega_1) \leqslant \delta(\Omega_1, D), \qquad e(\Omega_1, \Omega_2) \leqslant \delta(\Omega_2, D), \tag{3.5}$$

then

$$\|u_1 - u_2\|_V^2 \leqslant C_1'' \|f\|_{H'} \|f\|_{V'} \left\{ \frac{e(\Omega_2, \Omega_1)}{\rho_1 \sin\theta_1} + \frac{e(\Omega_1, \Omega_2)}{\rho_2 \sin\theta_2} \right\}. \tag{3.6}$$

Remark 3.2. (3.3) and (3.5) simply mean that D is sufficiently far from the boundaries of Ω_1 and Ω_2, whereas (3.4) holds without this restriction: of course, when $D = \mathbb{R}^N$ this condition is always satisfied; in the general case it would not be difficult to extend the coefficients $a^{ij},\ b^i,\ c$ of the bilinear form a to some neighborhood D' of D and to replace D by D', so that conditions (3.3) and (3.5) become irrelevant. On the other hand, this extension could affect the values of the various constants $\mathsf{a}, \mathsf{b}, \ldots$ and we preferred to keep them as accurate as possible in the estimates: a precise formula for the constants C_1 (and the next C_2) will be presented in the last section of this paper.

Theorem 2. *Let* (A1)–(A9) *hold, let Ω_1, Ω_2 be two open subsets of $\mathbb{R}^N$ satisfying* (3.3), *let $f \in L^2(D)$, and let u_i be defined as in* (3.1). *If Ω_1 satisfies a uniform (ρ_1, θ_1) cone condition on D, then there exists an* admissible *constant $C_2 > 0$ such that*

$$\|u_1 - u_2\|_H^2 \leqslant C_2 \|f\|_{H'} \|f\|_{V'} \left(\frac{e(\Omega_1 \triangle \Omega_2, \partial\Omega_1)}{\rho_1 \sin\theta_1} \right)^2. \tag{3.7}$$

If also Ω_2 is a (ρ_2, θ_2) uniform set, then for an admissible constant C_2', we have:

$$\|u_1 - u_2\|_H^2 \leqslant C_2' \|f\|_{H'} \|f\|_{V'} \left\{ \frac{\check{e}(\Omega_1, \Omega_2)}{\rho_1 \sin\theta_1} + \frac{\check{e}(\Omega_2, \Omega_1)}{\rho_2 \sin\theta_2} \right\}^2. \tag{3.8}$$

Finally, if (3.5) *is satisfied, then*

$$\|u_1 - u_2\|_H^2 \leqslant C_2'' \|f\|_{H'} \|f\|_{V'} \left\{ \frac{e(\Omega_2, \Omega_1)}{\rho_1 \sin\theta_1} + \frac{e(\Omega_1, \Omega_2)}{\rho_2 \sin\theta_2} \right\}^2. \tag{3.9}$$

The particular structure of the estimates of Theorems 1 and 2 allows us to prove analogous results under weaker conditions on f.

Corollary 3. *Theorems* 1 *and* 2 *hold even for* $f \in B^{-1/2}_{2,1}(D) \supset H^{-s}(D)$, $0 \leqslant s < 1/2$, *simply by substituting the terms*

$$\|f\|_{H'} \|f\|_{V'} \quad \textit{with} \quad \|f\|^2_{(H',V')_{1/2,1}} \approx \|f\|^2_{B^{-1/2}_{2,1}(D)} \tag{3.10}$$

in formulas (3.2), (3.4), (3.6)–(3.9).

Proof. We simply apply Proposition 2.1 to the linear operator

$$\mathcal{T}_{\Omega_1,\Omega_2} : f \in V' \mapsto u_1 - u_2 \in V. \qquad \square \tag{3.11}$$

A simple consequence of this corollary is the following application to the Dirichlet problem with non homogeneous boundary conditions: so, let us choose $f \in B^{-1/2}_{2,1}(D)$ and $g \in B^{3/2}_{2,1}(D)$ and let us consider the solutions u_i of

$$\begin{cases} \mathcal{A}u_i = f & \text{in } \Omega_i, \\ u_i = g & \text{on } \partial\Omega_i. \end{cases} \tag{3.12}$$

Corollary 4. *Theorems* 1 *and* 2 *hold even for the solutions* u_1, u_2 *of* (3.12) *with* $f \in B^{-1/2}_{2,1}(D)$, $g \in B^{3/2}_{2,1}(D)$, *simply by substituting the terms*

$$\|f\|_{H'} \|f\|_{V'} \quad \textit{with} \quad \|f\|^2_{(H',V')_{1/2,1}} + \|\mathcal{A}g\|^2_{(H',V')_{1/2,1}} \tag{3.13}$$

in formulas (3.2), (3.4), (3.6)–(3.9).

Proof. We simply apply the previous corollary to the couple of functions

$$\tilde{u}_i := u_i - g \quad \text{which solve (1.1) w.r.t. } \tilde{f} := f - \mathcal{A}g \in B^{-1/2}_{2,1}(D). \qquad \square$$

Remark 3.3. One could be disappointed by the use of the seemingly non optimal spaces $B^{-1/2}_{2,1}(D)$ for f and the related $B^{3/2}_{2,1}(D)$ for g in the previous corollary, instead of the more "elegant" $H^{-1/2}(D)$ and $H^{3/2}(D)$. However, these choices are not due to technical difficulties nor to lack of regularity of Ω_1, but are necessary to recover the optimal estimates of Theorems 1: as we already observed in Section 2.5, (3.2) implies that (the trivial extension of) u_1 belongs to $B^{3/2}_{2,\infty}(\mathbb{R}^N)$ and in general, even in the case of a smooth open set, this property does not hold if $u_1 \in H^1_0(\Omega_1) \cap B^{3/2}_{2,q}(\Omega_1)$ for some $q > 1$ but $u_1 \notin B^{3/2}_{2,1}(\Omega_1)$. Therefore, the functional setting of Corollary 4 is optimal for (3.2). One should also take into account that in Lipschitz open sets it may happen that the solution of the Dirichlet problem with $f \in H^{-1/2}(\Omega_1)$ does not belong to $H^{3/2}(\Omega_1)$ and the trace of a $H^{3/2}(\mathbb{R}^N)$ function on $\partial\Omega_1$ does not belong to $H^1(\partial\Omega_1)$: we refer to [22] for these and other deep counterexamples.

When f belongs only to $H^{-1}(D)$ we can prove weaker estimates in L^2-norm: indeed, we limit ourselves to consider the case corresponding to (3.7) since the other geometric situations lead to results analogous to (3.8), (3.9).

Theorem 5. *Let* (A1)–(A9) *hold, let* Ω_1, Ω_2 *be two open subsets of* $\mathbb{R}^N$ *satisfying* (3.3), *let* $f \in H^{-1}(D)$, *and let* u_i *be defined as in* (3.1). *If* Ω_1 *satisfies a uniform* (ρ_1, θ_1) *cone condition on* D, *then there exists an* admissible *constant* $C_3 > 0$ *such that*

$$\|u_1 - u_2\|_H \leqslant C_3 \|f\|_{V'} \left\{ \frac{e(\Omega_1 \triangle \Omega_2, \partial\Omega_1)}{\rho_1 \sin\theta_1} \right\}^{1/2}, \tag{3.14}$$

provided (3.3) *holds.*

Remark 3.4. One could think that (3.14) is not optimal, since even for $f \in H^{-1}(D)$ u_1 belongs to $H^1_0(\Omega_1)$ and therefore one could expect a order one estimate. Actually, (3.14) is optimal and in general the exponent $1/2$ cannot be improved. Here is a significant one-dimensional example: consider the family of open intervals $\Omega_h :=]-1, h[$ for $0 \leqslant h < 1$, and

$$u_h : \Omega_h \to \mathbb{R}, \qquad u_h(x) := c_h(1 + x) + \frac{x^{1/2}}{\log x} \chi_{]0,h]}(x),$$

where $\chi_{]0,h]}(x)$ denotes the characteristic function of $]0, h[$ and the constant c_h is chosen in such a way that the function u_h satisfies the conditions $u_h(-1) = u_h(h) = 0$. One can directly check that u_h solves an elliptic problem of the form

$$u_h \in H^1_0(\Omega_h), \qquad -u_h'' = f \quad \text{in } H^{-1}(\Omega_h),$$

where the function $f \in H^{-1}(D)$, independent of h, might be explicitly computed. Upon noticing that $u_0 = 0$, a simple calculation yields:

$$\|u_h - u_0\|^2_{L^2(D)} \geqslant \|u_h\|^2_{L^2(\Omega_0)} = \frac{c_h^2}{3} = \frac{h}{3(1+h)^2 \log^2 h}.$$

Since $\|f\|_{H^{-1}(D)}$ is constant and $\mathfrak{d}(\Omega_h, \Omega_0) = h$, we see that the exponent $1/2$ is optimal.

Examples. It could be interesting to show in some particular cases how look the various constants in the previous estimates (3.2), (3.7). We are assuming that the hypotheses of Theorems 1 and 2 are satisfied, we are keeping the same notation of Problem (P), and we will denote by ℓ all the constants which depend only on the space dimension N. The following bounds are a consequence of the explicit formulas of Theorems 8.3 and 8.4.

Example 1. Let us first consider the case of the equation

$$-\sum_{i,j}\frac{\partial}{\partial x_j}\left(a^{ij}(x)\frac{\partial u}{\partial x_i}\right)=f,\quad \Omega_1,\Omega_2\subset B_R(0),$$

with $\rho_1\leqslant 1$; in this case $\alpha=1$, $\beta=\beta_s=\gamma=0$, so that, denoting by G_R the solution of

$$-\Delta G_R=f\quad\text{in } B_R(0),\qquad G_R=0\quad\text{on } \partial B_R(0),$$

we have $\|f\|_{V'}\leqslant\|\nabla G_R\|_{L^2(B_R(0))}$ and

$$\begin{aligned}&\|\nabla u_1-\nabla u_2\|^2_{L^2(\mathbb{R}^N)}\\&\quad\leqslant \ell\big((\mathsf{a}+\mathsf{L})\|\nabla G_R\|^2_{L^2(B_R(0))}+\|f\|_{L^2(B_R(0))}\|\nabla G_R\|_{L^2(B_R(0))}\big)\frac{e(\Omega_1\vartriangle\Omega_2,\partial\Omega_1)}{\rho_1\sin\theta_1}.\end{aligned}$$

Example 2. Let us now consider the equation:

$$-\Delta u+\mathsf{c}\,u=f,\quad \mathsf{c}>0,\quad \Omega_1\text{ is a Lipschitz epigraph as in (2.52).}$$

In this case $\alpha=1$, $\beta=\beta_s=\gamma=\lambda=0$, $\mathsf{p}=\mathsf{c}$ and we can take ρ_1 arbitrarily large: passing to the limit as $\rho_1\uparrow+\infty$, we get:

$$\begin{aligned}&\|\nabla u_1-\nabla u_2\|^2_{L^2(\mathbb{R}^N)}+\mathsf{c}\|u_1-u_2\|^2_{L^2(\mathbb{R}^N)}\\&\quad\leqslant \ell\frac{1+\operatorname{Lip} g}{\sqrt{\mathsf{c}}}\|f\|^2_{L^2(\mathbb{R}^N)}e(\Omega_1\vartriangle\Omega_2,\partial\Omega_1)\end{aligned}$$

and

$$\|u_1-u_2\|_{L^2(\mathbb{R}^N)}\leqslant \ell\frac{1+\operatorname{Lip} g}{\sqrt{\mathsf{c}}}\|f\|_{L^2(\mathbb{R}^N)}e(\Omega_1\vartriangle\Omega_2,\partial\Omega_1).$$

Example 3. Finally, we are considering the nonsymmetric case in $D:=\mathbb{R}^N$,

$$-\Delta u+\sum_{i=1}^n b^i(x)\frac{\partial u}{\partial x_i}+u(x)=f,$$

under Picard's assumptions

$$-\sum_i \frac{\partial b^i(x)}{\partial x_i} = 0. \tag{3.15}$$

In this case $V = H^1(\mathbb{R}^N)$, $\alpha = 1$, $\beta_s = \gamma = \lambda = 0$, $\mathsf{p} = 1$, $\pi_\rho\beta = \mathsf{b}$; assuming $\rho_1 \leqslant 1$ we get:

$$\begin{aligned}&\|u_1 - u_2\|^2_{H^1(\mathbb{R}^N)}\\ &\quad\leqslant \ell(1+\mathsf{b})^2\big((1+\mathsf{b})\|f\|^2_{H^{-1}(\mathbb{R}^N)} + \|f\|_{L^2(\mathbb{R}^N)}\|f\|_{H^{-1}(\mathbb{R}^N)}\big)\frac{e(\Omega_1 \,\triangle\, \Omega_2, \partial\Omega_1)}{\rho_1 \sin\theta_1}.\end{aligned}$$

Sketch of the proofs. We split the proofs of the previous theorems in 4 parts (the next 4 sections of this paper), which are of independent interest.

In the next section we present some simple abstract results, which we will apply to derive our estimates; these results show the importance to know a precise evaluation of the H^1-distance of a function v in $V \approx H^1_0(D)$ from the subspace $V_{\Omega_1} \approx H^1_0(\Omega_1)$, as Ω_1 varies among the open subsets of D. Actually, in Section 5 we will develop a general localization formula to calculate this distance.

The key point is that v is not only a generic element of V but we can suppose that it solves a Dirichlet problem (2.2) in another Lipschitz open set Ω_2: we should be able to exploit this fact to obtain extra *regularity* and *geometric* informations on v.

Therefore, in Section 6, following the ideas of [30], we will derive the basic regularity estimate for such a solution. The geometric tools linking localization arguments and regularity in our Lipschitz setting will be developed in Section 7, where some elementary but useful geometric properties of Lipschitz domains will be detailed. Then, the proofs of the main theorems will be completed in the last Section 8.

4. Abstract framework

In the (general) Hilbert space V let

$$V_1, V_2 \quad \text{be two } \textit{closed} \text{ subspaces of } V, \tag{4.1}$$

$$a : V \times V \to \mathbb{R} \quad \text{be a continuous bilinear form,} \tag{4.2}$$

decomposed into the sum of its *symmetric* and *antisymmetric* parts

$$a(u, v) := a_s(u, v) + a_a(u, v) \quad \forall u, v \in V, \tag{4.3}$$

satisfying, for given $\mu_s, \mu_a, \alpha > 0$,

$$\alpha\|u\|^2_V \leqslant a_s(u, u) = a(u, u) \leqslant \mu_s\|u\|^2_V, \quad \forall u \in V, \tag{4.4}$$

$$a_a(u, u) = 0, \qquad a_a(u, v) \leqslant \mu_a\|u\|_V\|v\|_V \quad \forall u, v \in V. \tag{4.5}$$

We denote by $\mu := \mu_s + \mu_a$ the continuity constant of a.

Remark 4.1. Observe that if a is defined as in (A5) of Section 2.1 in the Hilbert space $V = H_0^1(D)$ endowed with the norm (A7), then Section 2.2 yields

$$\mu_s \leqslant 1 + \beta_s + \gamma, \quad \mu_a \leqslant \beta. \tag{4.6}$$

When Picard's condition (A9) holds, then $\alpha = 1$ and β_s takes the value of (2.15).

Thanks to the Lax–Milgram Lemma, to every $f \in V'$ we associate the solutions $u_i := \mathcal{G}(f; V_i)$ of

$$u_i \in V_i, \quad a(u_i, v) = \langle f, v\rangle \quad \forall v \in V_i, \quad i = 1, 2. \tag{4.7}$$

The aim of this section is to provide an estimate of the difference between u_1 and u_2; let us remark that in the case $V_1 \subset V_2$, u_1 is the "elliptic projection" of u_2 on V_1 and (a simple variant of) the classical "Cea's lemma trick" [10, Theorem 2.4.1] provides the following best approximation and stability result:

Lemma 4.2. *Let u_i be as in* (4.7) *and let us suppose that $V_1 \subset V_2$. Then*

$$\|u_1 - u_2\|_V \leqslant \sigma d(u_2, V_1), \tag{4.8}$$

$$\|u_1 - v_1\|_V \leqslant \sigma \|u_2 - v_1\| \quad \forall v_1 \in V_1, \tag{4.9}$$

where we have set

$$\sigma := \sqrt{\tfrac{\mu_s}{\alpha}} + \tfrac{\mu_a}{\alpha}, \qquad d(v, W) := \inf_{w \in W} \|v - w\|_V \quad \forall v \in V. \tag{4.10}$$

Proof. Let us denote by $a_s(\cdot) = a(\cdot)$ the quadratic form associated to a_s, a. Since

$$a(u_1, v) = \langle f, v\rangle = a(u_2, v), \qquad a(u_2 - u_1, v) = 0 \qquad \forall v \in V_1, \tag{4.11}$$

for every choice of $v_1 \in V_1$, we have:

$$\begin{aligned} a(u_2 - u_1) &= a(u_2 - u_1, u_2 - u_1) = a(u_2 - u_1, u_2 - v_1) \\ &= a_s(u_2 - u_1, u_2 - v_1) + a_a(u_2 - u_1, u_2 - v_1) \\ &\leqslant a_s^{1/2}(u_2 - u_1) a_s^{1/2}(u_2 - v_1) + \mu_a \|u_2 - u_1\|_V \, \|u_2 - v_1\|_V. \end{aligned}$$

Dividing by $(\alpha a(u_2 - u_1))^{1/2}$ and recalling that

$$\|u_2 - u_1\|_V \leqslant \frac{1}{\sqrt{\alpha}} a^{1/2}(u_2 - u_1), \tag{4.12}$$

we obtain:

$$\|u_2 - u_1\|_V \leqslant \frac{a_s^{1/2}(u_2 - v_1)}{\sqrt{\alpha}} + \frac{\mu_a \|u_2 - v_1\|_V}{\alpha} \leqslant \sigma \|u_2 - v_1\|_V. \tag{4.13}$$

Taking the infimum as v_1 varies in V_1, we get (4.8).

(4.9) follows by the same argument, starting from

$$\begin{aligned} a(u_1 - v_1) &= a(u_1 - v_1, u_1 - v_1) = a(u_2 - v_1, u_1 - v_1) \\ &= a_s(u_2 - v_1, u_1 - v_1) + a_a(u_2 - v_1, u_1 - v_1) \\ &\leqslant a_s^{1/2}(u_2 - v_1) a_s^{1/2}(u_1 - v_1) + \mu_a \|u_2 - v_1\|_V \|u_1 - v_1\|_V. \end{aligned}$$

and dividing by $(\alpha a(u_1 - v_1))^{1/2}$. $\square$

When V_1, V_2 are arbitrary closed subspaces of V we have a slightly different formula, involving $V_1 \cap V_2$ or a new subspace $V^{1,2}$ containing $V_1 \cup V_2$.

Corollary 4.3. *The solutions u_1, u_2 of* (4.7) *satisfy the estimate*:

$$\|u_1 - u_2\|_V \leqslant \sigma\big(d(u_1, V_1 \cap V_2) + d(u_2, V_1 \cap V_2)\big). \tag{4.14}$$

Moreover, if $V^{1,2}$ is a closed subspace of V satisfying

$$V_1 \cup V_2 \subset V^{1,2} \subset V \tag{4.15}$$

and $u^{1,2} = \mathcal{G}(f; V^{1,2})$ is the solution of

$$u^{1,2} \in V^{1,2} \quad a\big(u^{1,2}, v\big) = \langle f, v\rangle \quad \forall v \in V^{1,2}, \tag{4.16}$$

then

$$\|u_1 - u_2\|_V \leqslant \sigma\big(d\big(u^{1,2}, V_1\big) + d\big(u^{1,2}, V_2\big)\big). \tag{4.17}$$

Proof. (4.14) and (4.17) are easy consequences of Cea's Lemma: setting $u_{1,2} := \mathcal{G}(f; V_1 \cap V_2)$ we have:

$$\begin{aligned} \|u_1 - u_2\|_V &\leqslant \|u_1 - u_{1,2}\|_V + \|u_2 - u_{1,2}\|_V \\ &\leqslant \sigma d(u_1, V_1 \cap V_2) + \sigma\, d(u_2, V_1 \cap V_2). \end{aligned} \tag{4.18}$$

Analogously,

$$\begin{aligned} \|u_1 - u_2\|_V &\leqslant \big\|u_1 - u^{1,2}\big\|_V + \big\|u_2 - u^{1,2}\big\|_V \\ &\leqslant \sigma d\big(u^{1,2}, V_1\big) + \sigma d\big(u^{1,2}, V_2\big). \qquad \square \end{aligned} \tag{4.19}$$

Remark 4.4. If $\sigma = 1$ (i.e. if the bilinear form a is (a multiple of) the scalar product of V), then the best choice for $V^{1,2}$ is the closed subspace generated by $V_1 \cup V_2$ since (4.9) shows that the function

$$V^{1,2} \mapsto d\big(u^{1,2}, V_1\big) \quad \text{is increasing w.r.t. the inclusion of subspaces.} \tag{4.20}$$

In the general case (4.20) does not hold, so that the possibility to choose freely $V^{1,2}$ could be useful to obtain better estimates.

Corollary 4.5. *Let $V^{1,2}$, $V^{2,1}$ be two closed subspaces of V such that*

$$V_1 \cup V_2 \subset V^{1,2}, \quad V^{2,1} \subset V, \tag{4.21}$$

and let $u^{1,2} \in V^{1,2}$, $u^{2,1} \in V^{2,1}$ be as in (4.16)*. Then the solutions u_1, u_2 of* (4.7) *satisfy the relation*

$$\|u_1 - u_2\|_V \leqslant \sigma^2\big(d\big(u^{1,2}, V_1\big) + d\big(u^{2,1}, V_2\big)\big). \tag{4.22}$$

Proof. Applying (4.17) to the closed subspace $W := V^{1,2} \cap V^{2,1}$ we get:

$$\|u_1 - u_2\|_V \leqslant \sigma\big(d(w, V_1) + d(w, V_2)\big), \quad w := \mathcal{G}(f; W). \tag{4.23}$$

(4.9) yields

$$d(w, V_1) = \inf_{v_1 \in V_1} \|w - v_1\| \leqslant \inf_{v_1 \in V_1} \sigma \big\|u^{1,2} - v_1\big\| = \sigma d\big(u^{1,2}, V_1\big). \tag{4.24}$$

Inserting the above inequality (and the corresponding one for $u^{2,1}, V_2$) in (4.23) we get (4.22). □

Remark 4.6. The constant of (4.22) is worse than the corresponding one of (4.17) (at least in the nonsymmetric case), but (4.22) allows more flexibility in the choice of $V^{2,1}$, which could be different from $V^{1,2}$.

The next lemma will be useful to obtain estimates for $u_1 - u_2$ w.r.t. weaker norms; since we will use a standard duality technique, we introduce the *adjoint* bilinear form $\hat{a}$:

$$\hat{a}(u, v) := a(v, u) \quad \forall u, v \in V \tag{4.25}$$

and we denote by $\widehat{\mathcal{G}}(\cdot\,; V_i)$ the corresponding Green operators

$$\hat{v}_i = \widehat{\mathcal{G}}(g; V_i) \quad \Rightarrow \quad \hat{v}_i \in V_i,$$
$$\hat{a}(\hat{v}_i, w) = a(w, \hat{v}_i) = \langle g, w\rangle \quad \forall w \in V_i.$$

Of course, when a is *symmetric*, we have $\hat{a} = a$, $\widehat{\mathcal{G}} = \mathcal{G}$.

We also introduce the "residual" functionals associated to the closed subspaces W of V:

Definition 4.7. If W is a closed subspace of V, we will denote by $\mathcal{R}_W : V' \times V \to \mathbb{R}$ the bilinear form:

$$\mathcal{R}_W(f, v) := \langle f, v\rangle - a\big(\mathcal{G}(f; W), v\big) \quad \forall f \in V',\ v \in V. \tag{4.26}$$

As before, we denote by $\widehat{\mathcal{R}}_W(\cdot\,,\cdot)$ the analogous residual associated to the adjoint bilinear form $\hat{a}$:

$$\widehat{\mathcal{R}}_W(g, v) := \langle g, v\rangle - \hat{a}\big(\widehat{\mathcal{G}}(g; W), v\big) \quad \forall g \in V',\ v \in V. \tag{4.27}$$

Lemma 4.8. *For a closed subspace W in V and a given $g \in V'$, let $\widehat{w} := \widehat{\mathcal{G}}(g; W)$. Then, for every $f \in V'$, the solutions $u_i = \mathcal{G}(f; V_i)$ satisfy*

$$\langle g, u_1 - u_2\rangle = \widehat{\mathcal{R}}_W(g, u_1 - u_2) + \mathcal{R}_{V_2}(f, \widehat{w}) - \mathcal{R}_{V_1}(f, \widehat{w}) \tag{4.28}$$

Proof. We have:

$$\begin{aligned}
\langle g, u_1 - u_2\rangle &= \widehat{\mathcal{R}}_W(g, u_1 - u_2) + \hat{a}(\widehat{w}, u_1 - u_2)\\
&= \widehat{\mathcal{R}}_W(g, u_1 - u_2) + a(u_1 - u_2, \widehat{w})\\
&= \widehat{\mathcal{R}}_W(g, u_1 - u_2) + a(u_1, \widehat{w}) - \langle f, \widehat{w}\rangle + \langle f, \widehat{w}\rangle - a(u_2, \widehat{w})\\
&= \widehat{\mathcal{R}}_W(g, u_1 - u_2) - \mathcal{R}_{V_1}(f, \widehat{w}) + \mathcal{R}_{V_2}(f, \widehat{w}).
\end{aligned}$$

Lemma 4.9. *Let $V_1 \subset V^{1,2}$ be closed subspaces of V and, for $f \in V'$, let $u^{1,2} = \mathcal{G}(f; V^{1,2}) \in V^{1,2}$ as in* (4.16). *Then for every $v \in V^{1,2}$*

$$\mathcal{R}_{V_1}(f, v) \leqslant \sigma\mu d\big(u^{1,2}, V_1\big) d(v, V_1). \tag{4.29}$$

Proof. By (4.15) and (4.16), for every choice of $v_1 \in V_1$ we have:

$$\begin{aligned}
\mathcal{R}_{V_1}(f, v) &= \langle f, v\rangle - a(u_1, v) = a\big(u^{1,2}, v\big) - a(u_1, v)\\
&= a\big(u^{1,2} - u_1, v\big) = a\big(u^{1,2} - u_1, v - v_1\big)\\
&\leqslant \mu \big\|u^{1,2} - u_1\big\|_V \|v - v_1\|_V .
\end{aligned} \tag{4.30}$$

Since $V_1 \subset V^{1,2}$, Lemma 4.2 shows that

$$\big\|u^{1,2} - u_1\big\|_V \leqslant \sigma d\big(u^{1,2}, V_1\big). \tag{4.31}$$

Plugging (4.31) into (4.30) we get:

$$\mathcal{R}_{V_1}(f, v) \leqslant \sigma\mu d\big(u^{1,2}, V_1\big)\|v - v_1\|_V \quad \forall v_1 \in V_1. \tag{4.32}$$

Taking the infimum as v_1 varies in V_1, we get (4.29). □

Corollary 4.10. *Let g be given in V', let $V_{1,2} := V_1 \cap V_2$, let $V^{1,2}$, $\widehat{V}^{1,2}$, $V^{2,1}$, $\widehat{V}^{2,1}$ be closed subspaces of V satisfying*

$$V_1, V_2 \subset V^{1,2} \cap V^{2,1}, \qquad V_1, V_2 \subset \widehat{V}^{1,2} \cap \widehat{V}^{2,1}, \tag{4.33}$$

and let us set $\hat{v}_i := \widehat{\mathcal{G}}(g; V_i)$, $\hat{v}^{1,2} := \widehat{\mathcal{G}}(g; \widehat{V}^{1,2})$, $\hat{v}^{2,1} := \widehat{\mathcal{G}}(g; \widehat{V}^{2,1})$. Then for every $f \in V'$ we have:

$$\langle g, u_1 - u_2\rangle \leqslant \begin{cases} \sigma^2\mu\big(d\big(u^{1,2}, V_1\big)d\big(\hat{v}^{1,2}, V_1\big) + d\big(u^{2,1}, V_2\big)d\big(\hat{v}^{2,1}, V_2\big)\big) \\ \sigma\mu\big(d\big(u^{1,2}, V_2\big)\|\hat{v}_1 - \hat{v}_2\|_V + d\big(\hat{v}^{1,2}, V_1\big)\|u_2 - u_1\|_V\big) \\ \sigma\mu\|u_2 - u_1\|_V d\big(\hat{v}^{1,2}, V_{1,2}\big) \\ \sigma\mu\big(d(\hat{v}_1, V_{1,2})d(u_1, V_{1,2}) + d(\hat{v}_2, V_{1,2})d(u_2, V_{1,2})\big). \end{cases} \tag{4.34}$$

Proof. The four inequalities above follow directly from (4.28) and (4.29) upon choosing suitably W: more precisely, one has to take $W := \widehat{V}^{1,2} \cap \widehat{V}^{2,1}$, $W := V_1$, $W := V_{1,2}$, and $W := V_{1,2}$ again, respectively for the four cases, and to possibly work on the solutions of the dual problems.

In the first case, choosing $W := \widehat{V}^{1,2} \cap \widehat{V}^{2,1}$ which contains both u_1 and u_2 we get:

$$\begin{aligned} &\widehat{\mathcal{R}}_W(g, u_1 - u_2) = 0, \quad \text{so that} \\ \langle g, u_1 - u_2\rangle &= \mathcal{R}_{V_2}(f, \widehat{w}) - \mathcal{R}_{V_1}(f, \widehat{w}) \\ &\leqslant \sigma\mu\big(d\big(u^{2,1}, V_2\big)d(\widehat{w}, V_2) + d\big(u^{1,2}, V_1\big)d(\widehat{w}, V_1)\big) \\ &\leqslant \sigma\mu\big(d\big(u^{2,1}, V_2\big)\sigma d\big(\hat{v}^{2,1}, V_2\big) + d\big(u^{1,2}, V_1\big)\sigma d\big(\hat{v}^{1,2}, V_1\big)\big) \end{aligned}$$

where, in the last inequality, we applied (4.9) as in (4.24).

The second inequality follows by noticing that $W := V_1$ yields:

$$\begin{aligned} \widehat{\mathcal{R}}_W(g, u_1 - u_2) &= \widehat{\mathcal{R}}_{V_1}(g, u_1 - u_2) \leqslant \sigma\mu d\big(\hat{v}^{1,2}, V_1\big)d(u_1 - u_2, V_1) \\ &\leqslant \sigma\mu d\big(\hat{v}^{1,2}, V_1\big)\|u_1 - u_2\|_V \end{aligned}$$

and, since $\widehat{w} = \hat{v}_1 \in V_1$,

$$\begin{gathered} \mathcal{R}_{V_1}(f, \widehat{w}) = 0, \\ \mathcal{R}_{V_2}(f, \widehat{w}) \leqslant \sigma\mu d\big(u^{1,2}, V_2\big)d(\widehat{w}, V_2) \leqslant \sigma\mu d\big(u^{1,2}, V_2\big)\|\hat{v}_1 - \hat{v}_2\|_V. \end{gathered}$$

The proofs of the third and fourth inequalities correspond to $W := V_{1,2} = V_1 \cap V_2$: one has $\widehat{w} = \hat{v}_{1,2}$, so that

$$\begin{aligned} \langle g, u_1 - u_2\rangle &= \widehat{\mathcal{R}}_{V_{1,2}}(g, u_1 - u_2) \\ &\leqslant \sigma\mu\big(d\big(\hat{v}^{1,2}, V_{1,2}\big)d(u_1 - u_2, V_{1,2})\big) \\ &\leqslant \sigma\mu\big(d\big(\hat{v}^{1,2}, V_{1,2}\big)\|u_1 - u_2\|_V\big), \end{aligned}$$

that gives the third inequality. The proof of the fourth one is similar: one just has to observe that, by linearity,

$$\widehat{\mathcal{R}}_{V_{1,2}}(g, u_1 - u_2) = \widehat{\mathcal{R}}_{V_{1,2}}(g, u_1) - \widehat{\mathcal{R}}_{V_{1,2}}(g, u_2)$$

and evaluate the above right hand side by using (4.29). □

5. Localization estimates

Theorem 4.3 and Corollary 4.10 show the importance to estimate the H^1-distance of an element $v \in V_Y$ (recall (2.1)) from V_X, X, Y being open subsets of $D \subset \mathbb{R}^N$. In this section we perform a standard localization technique to deduce a global estimate from local ones, trying to obtain a precise control of the size of the constants involved. We recall that $\mathfrak{a}(x, \cdot)$ is the quadratic form associated to the principal part of a as defined in (A2).

Proposition 5.1. *Let $X, Y \subset D$ be two open subsets of $\mathbb{R}^N$ and let $v \in V_Y$. Setting $\Lambda := Y \backslash X$, we assume that for every $y \in \Lambda$ there exists a vector $\boldsymbol{\nu}(y)$ such that*

$$x \in B_\rho(y) \backslash X \quad \Rightarrow \quad x + \boldsymbol{\nu}(y) \notin Y, \tag{5.1}$$

and that there exists two nonnegative measurable density functions G, H such that for every $y \in \Lambda^{3\rho}$

$$\int_{B_\rho(y)} \left| v\big(x + \boldsymbol{\nu}(y)\big) - v(x) \right|^2 \mathrm{d}x \leqslant \int_{B_{3\rho}(y)} G(x)\, \mathrm{d}x, \tag{5.2}$$

$$\int_{B_\rho(y)} \mathfrak{a}\big(x, \nabla v\big(x + \boldsymbol{\nu}(y)\big) - \nabla v(x)\big)\, \mathrm{d}x \leqslant \int_{B_{3\rho}(y)} H(x)\, \mathrm{d}x. \tag{5.3}$$

Then there exists a constant ℓ_1 depending only on N and a function $w \in V_X$ (independent of the choice of G and H) such that

$$\|v - w\|^2_{L^2(\mathbb{R}^N)} \leqslant \ell_1 \int_{\Lambda^{3\rho}} G(x)\, \mathrm{d}x, \tag{5.4}$$

$$\int_D \mathfrak{a}\big(x, \nabla v(x) - \nabla w(x)\big)\, \mathrm{d}x \leqslant \ell_1 \int_{\Lambda^{3\rho}} \big(\mathsf{a}\, \rho^{-2} G(x) + H(x)\big)\, \mathrm{d}x. \tag{5.5}$$

Proof. An easy iterative construction (cf. e.g. [1, p. 49]) shows that there exists a (at most) countable set $\{x_j\}_{j \in J \subset \mathbb{N}} \subset Y \backslash X$ such that

$$i \neq j \quad \Rightarrow \quad B_{\rho/4}(x_i) \cap B_{\rho/4}(x_j) = \emptyset, \quad \forall i, j \in J; \tag{5.6}$$

$$\Lambda \subset \bigcup_{j \in J} B_{\rho/2}(x_j), \quad \Lambda^{\rho/4} = \bigcup_{x \in Y \setminus X} B_{\rho/4}(x) \subset \bigcup_{j \in J} B_{3\rho/4}(x_j). \tag{5.7}$$

For $k > 0$ and $x \in \mathbb{R}^N$ it is easy to see that

$$\#\bigl\{j \in J\colon\ x \in B_{k\rho}(x_j)\bigr\} = \sum_{j \in \mathbb{N}} \chi_{B_{k\rho}(x_j)}(x) \leqslant (4k+1)^N; \tag{5.8}$$

in fact,

$$\chi_{B_{k\rho}(x_j)}(x) = 1 \quad \Rightarrow \quad B_{\rho/4}(x_j) \subset B_{(k+1/4)\rho}(x) \tag{5.9}$$

so that (5.6) implies

$$\bigl|B_{(k+1/4)\rho}(x)\bigr| = \omega_N (k+1/4)^N \rho^N \geqslant \#\bigl\{j \in J\colon\ x \in B_{k\rho}(x_j)\bigr\} \rho^N \omega_N / 4^N,$$

where ω_N is the Lebesgue measure of the N-dimensional unit ball. Let us now define $\phi_j(x)$, $j = 1, 2, \ldots,$ as

$$\phi_j(x) := \min\bigl\{1, (4 - 4|x - x_j|/\rho)^+\bigr\} \quad \forall x \in \mathbb{R}^N, \tag{5.10}$$

which satisfy for every $x \in \mathbb{R}^N$

$$0 \leqslant \phi_j(x) \leqslant 1, \quad \operatorname{supp}(\phi_j) \subset \overline{B_\rho(x_j)}, \quad \phi_j(x) \equiv 1 \text{ in } B_{3\rho/4}(x_j), \tag{5.11}$$

$$\bigl|\nabla \phi_j(x)\bigr| \leqslant 4\rho^{-1} \chi_{B_\rho(x_j)}(x), \tag{5.12}$$

and

$$\phi_0(x) := \min\bigl\{1, 4d(x, \Lambda)/\rho\bigr\} \quad \forall x \in \mathbb{R}^N,$$

so that, also on account of (5.7),

$$1 \leqslant \sum_{j=0}^{+\infty} \phi_j(x) \leqslant 1 + \sum_{j=1}^{+\infty} \chi_{B_\rho(x_j)}(x) \leqslant 5^N + 1 =: c_1 \quad \forall x \in \mathbb{R}^N.$$

Correspondingly we set

$$\psi_j(x) := \frac{\phi_j(x)}{\sum_{k=0}^{+\infty} \phi_k(x)} \quad \forall j \in \mathbb{N},\ x \in \mathbb{R}^N,$$

and it is easy to see that for every $j \in \mathbb{N}$

$$0 \leqslant \psi_j(x) \leqslant 1, \quad \sum_{j \in \mathbb{N}} \psi_j \equiv 1, \quad |\nabla \psi_j| \leqslant c_2/\rho, \quad c_2 := 4(c_1 + 1),$$

with $\operatorname{supp}(\psi_j) \subset \overline{B_\rho(x_j)}$ for $j \geqslant 1$. Now we set

$$\boldsymbol{v}_0 := 0, \qquad \boldsymbol{v}_j := \boldsymbol{v}(x_j),\ \ j \geqslant 1, \tag{5.13}$$

$$\widetilde{v}_j(x) := v(x + \boldsymbol{v}_j), \qquad w(x) := \sum_{j=0}^{+\infty} \psi_j(x)\widetilde{v}_j(x), \tag{5.14}$$

so that $w \in V_X$ by (5.1). A standard convex inequality yields:

$$\begin{aligned}
\int_{\mathbb{R}^N} \big|v(x) - w(x)\big|^2 \,\mathrm{d}x &= \int_{\mathbb{R}^N} \left| \sum_{j=0}^{+\infty} \psi_j(x)\big(v(x) - \widetilde{v}_j(x)\big) \right|^2 \mathrm{d}x \\
&\leqslant \int_{\mathbb{R}^N} \sum_{j=0}^{+\infty} \psi_j(x)\big|v(x) - \widetilde{v}_j(x)\big|^2 \,\mathrm{d}x \\
&\leqslant \sum_{j=1}^{+\infty} \int_{B_\rho(x_j)} \big|v(x) - v(x + \boldsymbol{v}_j)\big|^2 \,\mathrm{d}x \\
&\leqslant \sum_{j=1}^{+\infty} \int_{B_{3\rho}(x_j)} G(x)\,\mathrm{d}x \leqslant 13^N \int_{\Lambda^{3\rho}} G(x)\,\mathrm{d}x;
\end{aligned}$$

from this inequality we get (5.4). Analogously, we have:

$$\begin{aligned}
&\int_D \mathfrak{a}(x, \nabla v - \nabla w)\,\mathrm{d}x \\
&\quad= \int_D \mathfrak{a}\left(x, \nabla \sum_{j=0}^{+\infty} \psi_j(v - \widetilde{v}_j)\right) \mathrm{d}x \\
&\quad= \int_D \mathfrak{a}\left(x, \sum_{j=0}^{+\infty} \nabla\psi_j(v - \widetilde{v}_j) + \psi_j(\nabla v - \nabla\widetilde{v}_j)\right) \mathrm{d}x \\
&\quad\leqslant 2c_1 \sum_{j=0}^{+\infty} \int_D \mathfrak{a}(x, \nabla\psi_j)|v - \widetilde{v}_j|^2 \,\mathrm{d}x + 2\sum_{j=0}^{+\infty} \int_D \psi_j\, \mathfrak{a}(x, \nabla v - \nabla\widetilde{v}_j)\,\mathrm{d}x \\
&\quad\leqslant 2c_1 c_2^2 \mathsf{a}/\rho^2 \sum_{j=1}^{+\infty} \int_{B_\rho(x_j)} \big|v(x) - v(x + \boldsymbol{v}_j)\big|^2 \,\mathrm{d}x
\end{aligned}$$

$$+\,2\sum_{j=1}^{+\infty}\int_{B_\rho(x_j)} \mathfrak{a}\big(x, \nabla v(x)-\nabla v(x+\boldsymbol{\nu}_j)\big)\,\mathrm{d}x$$

$$\leqslant 2\sum_{j=1}^{+\infty}\int_{B_{3\rho}(x_j)} \big(c_1c_2^2\mathsf{a}/\rho^2 G(x)+H(x)\big)\,\mathrm{d}x \leqslant \ell_1\int_{\Lambda^{3\rho}}\big(\mathsf{a}\rho^{-2}G(x)+H(x)\big)\,\mathrm{d}x$$

and a common choice for ℓ_1 in (5.4), (5.5) could be $\ell_1 := 13^N(2+2c_1c_2^2)$. □

6. Regularity in Lipschitz domains

In order to apply Proposition 5.1 we need two kind of information:

(1) a precise bound for G, H in (5.2) and (5.3), once the size of $\boldsymbol{\nu}(y)$ is known;
(2) a geometric link between the open sets X, Y in order to verify (5.1).

We postpone the analysis of the second question to the next section and now we try to derive the estimates for G and H for suitable classes of vectors $\boldsymbol{\nu}(y)$; the first one can be easily deduced from the $H^1(\mathbb{R}^N)$-regularity of v, as the next Lemma shows (see the proof of [4, Proposition IX.3]):

Lemma 6.1. *If $v\in H^1(\mathbb{R}^N)$ and $x_0\in\mathbb{R}^N$, then for every $h\in\mathbb{R}^N$, $|h|<\rho$,*

$$\int_{B_{2\rho}(x_0)} \big|v(x+h)-v(x)\big|^2\,\mathrm{d}x \leqslant |h|^2\int_{B_{3\rho}(x_0)}\big|\nabla v(x)\big|^2\,\mathrm{d}x; \tag{6.1}$$

(5.3) requires finer regularity properties of v: let us recall that if Ω is a *uniform* (ρ,θ) open set and $f\in L^2(D)$, then [30, Theorem 2] shows that

$$u=\mathcal{G}(f;V_\Omega)\in B^{3/2}_{2,\infty}\big(\mathbb{R}^N\big);$$

here we will reproduce the key estimation related to this regularity result, trying to take care of the various constants involved in the calculations. It will be useful to introduce the scalar product of V:

$$((u,v)) := \int_D \sum_{ij} a^{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}+\mathsf{c}u(x)v(x)\,\mathrm{d}x \tag{6.2}$$

and a localized version of its norm in $\Omega\subset D$

$$[u]^2_\Omega := \int_\Omega \mathfrak{a}\big(x,\nabla u(x)\big)\,\mathrm{d}x+\mathsf{c}\int_\Omega |u|^2\,\mathrm{d}x. \tag{6.3}$$

Theorem 6.2. *Let us suppose that $\Omega \subset D$ is a uniform (ρ, θ) open set and $f \in L^2(D)$. If*

$$u := \mathcal{G}(f; V_\Omega) \in V_\Omega \quad \text{is the solution to } (2.2), \tag{6.4}$$

$$x_0 \in \mathbb{R}^N, \quad \boldsymbol{n} \in \mathcal{N}_{\rho,\theta}(x_0, \Omega), \quad h \in \mathcal{C}_{\rho,\theta}(\boldsymbol{n}), \tag{6.5}$$

then the shifted function $u_h(x) := u(x+h)$ satisfies:

$$[u - u_h]^2_{B_\rho(x_0)} \leqslant |h| \left\{ \left(\frac{5\mathsf{a}}{\rho} + 2\mathsf{b} + \mathsf{L} \right) [u]^2_{B_{3\rho}(x_0)} \right. \\ \left. + 2 \|\tilde{f}\|_{L^2(B_{2\rho}(x_0) \cap \Omega)} \|\nabla u\|_{L^2(B_{3\rho}(x_0); \mathbb{R}^N)} \right\}, \tag{6.6}$$

where

$$\tilde{f}(x) := f(x) - \big(c(x) - \mathsf{c}\big) u(x). \tag{6.7}$$

Proof. We choose a Lipschitz "cut-off" function ϕ centered at x_0 with support contained in $\overline{B_{2\rho}(x_0)}$, e.g.,

$$\phi(x) := \min\big\{1, \big(2 - \rho^{-1}|x - x_0|\big)^+\big\}, \tag{6.8}$$

which satisfies

$$0 \leqslant \phi(x) \leqslant 1, \quad \big|\nabla\phi(x)\big| \leqslant \rho^{-1}, \quad \phi(x) \equiv 1 \text{ in } B_\rho(x_0), \tag{6.9}$$

and we define for every function $v \in V \subset H^1(\mathbb{R}^N)$:

$$v_h(x) := v(x+h), \qquad T_h v(x) := \phi(x) v(x+h) + \big(1 - \phi(x)\big) v(x); \tag{6.10}$$

notice that

$$T_h v(x) - v(x) = \phi(x)\big(v(x+h) - v(x)\big) \quad \forall x \in \mathbb{R}^N. \tag{6.11}$$

By (6.5), the property (2.50) shows that

$$v \in V_\Omega \quad \Rightarrow \quad T_h v \in V_\Omega \subset V, \quad \operatorname{supp}(v),\ \operatorname{supp}(T_h v) \subset \overline{\Omega}, \tag{6.12}$$

so that, by (6.11) and (6.9), we infer:

$$\int_{B_\rho(x_0)} \mathfrak{a}\big(x, \nabla\big(u(x+h) - u(x)\big)\big) + \mathsf{c}\big|u(x+h) - u(x)\big|^2 \,\mathrm{d}x \\ = \int_{B_\rho(x_0)} \mathfrak{a}\big(x, \nabla(T_h u - u)\big) + \mathsf{c}|T_h u - u|^2 \,\mathrm{d}x$$

$$
\begin{aligned}
&\leqslant ((T_h u - u, T_h u - u)) = ((T_h u, T_h u)) - ((u,u)) + 2((u, u - T_h u)) \\
&= ((T_h u, T_h u)) - ((u,u)) + 2a(u, u - T_h u) \\
&\quad - 2\int_\Omega \left((c(x) - \mathsf{c})u + \sum_i b^i \frac{\partial u}{\partial x_i} \right)(u - T_h u)\, \mathrm{d}x \\
&= ((T_h u, T_h u)) - ((u,u)) + 2\int_\Omega \left(\tilde f - \sum_i b^i \frac{\partial u}{\partial x_i} \right)(u - T_h u)\, \mathrm{d}x, \qquad (6.13)
\end{aligned}
$$

where $\tilde f$ is given by (6.7). By (6.1) the last term in (6.13) can be bounded by (from now on, we denote the norm in $L^2(X;\mathbb{R}^N)$ simply by $\|\cdot\|_{L^2(X)}$)

$$
\begin{aligned}
&2\int_\Omega \left(\tilde f - \sum_i b^i \frac{\partial u}{\partial x_i} \right)(u - T_h u)\, \mathrm{d}x \\
&\quad \leqslant 2|h| \|\nabla u\|_{L^2(B_{3\rho}(x_0))} \left\{ \|\tilde f\|_{L^2(\Omega \cap B_{2\rho}(x_0))} + \mathsf{b}\|\nabla u\|_{L^2(B_{2\rho}(x_0))} \right\}. \qquad (6.14)
\end{aligned}
$$

Theorem 1 will follow from (6.13), (6.14), and the next lemma.

Lemma 6.3. *Under the same notation and assumptions of Theorem* 6.2*, for every* $v \in V_\Omega$ *we have*:

$$
((T_h v, T_h v)) - ((v,v)) \leqslant |h| \int_{B_{3\rho}(x_0)} \left(\frac{5\mathsf{a}}{\rho} + \mathsf{L} \right) |\nabla v|^2 + \frac{\mathsf{c}}{\rho} |v|^2 \, \mathrm{d}x. \qquad (6.15)
$$

Proof. Since the gradient of $T_h v$ is

$$
\nabla[T_h v] = \phi \nabla v_h + (1-\phi)\nabla v + \nabla\phi(v_h - v) = T_h \nabla v + \nabla \phi (v_h - v), \qquad (6.16)
$$

(6.12) yields

$$
\begin{aligned}
&((T_h v, T_h v)) - ((v,v)) \\
&\quad \leqslant \int_\Omega \left[\mathfrak{a}\big(x, T_h \nabla v + \nabla\phi(v_h - v)\big) - \mathfrak{a}(x, T_h \nabla v) \right] \mathrm{d}x \qquad (6.17) \\
&\qquad + \int_\Omega \left[\mathfrak{a}(x, T_h \nabla v) - \mathfrak{a}(x, \nabla v) \right] \mathrm{d}x \qquad (6.18) \\
&\qquad + \mathsf{c} \int_\Omega \left[|T_h v|^2 - |v|^2 \right] \mathrm{d}x. \qquad (6.19)
\end{aligned}
$$

We estimate separately these three last integrals.

- The first one (6.17) can be estimated from above, by a simple application of the Cauchy–Schwarz inequality

$$\begin{aligned}\mathfrak{a}(x,\xi+\eta)-\mathfrak{a}(x,\xi) &\leqslant \big(\mathfrak{a}(x,\eta)\mathfrak{a}(x,2\xi+\eta)\big)^{1/2}\\ &\leqslant \mathsf{a}|\eta|\big(2|\xi|+|\eta|\big) \quad \forall\xi,\eta\in\mathbb{R}^N;\end{aligned}$$

recalling that $\operatorname{supp}(\nabla\phi)\subset\overline{B_{2\rho}(x_0)}$, $\operatorname{supp}(T_h v)\subset\overline{\Omega}$ we get:

$$\begin{aligned}&\int_\Omega \big[\mathfrak{a}\big(x,T_h\nabla v+\nabla\phi(v_h-v)\big)-\mathfrak{a}(x,T_h\nabla v)\big]\,\mathrm{d}x\\ &\quad=\int_{\Omega\cap B_{2\rho}(x_0)} \big[\mathfrak{a}\big(x,T_h\nabla v+\nabla\phi(v_h-v)\big)-\mathfrak{a}(x,T_h\nabla v)\big]\,\mathrm{d}x\\ &\quad\leqslant \frac{\mathsf{a}}{\rho}\|v-v_h\|_{L^2(B_{2\rho}(x_0))}\big(\rho^{-1}\|v-v_h\|_{L^2(B_{2\rho}(x_0))}+2\|T_h\nabla v\|_{L^2(B_{2\rho}(x_0))}\big).\end{aligned}$$

Since

$$2\|T_h\nabla v\|_{L^2(B_{2\rho}(x_0))}\leqslant 3\|\nabla v\|_{L^2(B_{3\rho}(x_0))} \tag{6.20}$$

and $|h|<\rho$, we deduce by (6.1),

$$\begin{aligned}&\int_\Omega \big[\mathfrak{a}\big(x,T_h\nabla v+\nabla\phi(v_h-v)\big)-\mathfrak{a}(x,T_h\nabla v)\big]\,\mathrm{d}x\\ &\quad\leqslant \frac{\mathsf{a}}{\rho}\left(3+\frac{|h|}{\rho}\right)|h|\|\nabla v\|^2_{L^2(B_{3\rho}(x_0))}\leqslant \frac{4\mathsf{a}}{\rho}|h|\|\nabla v\|^2_{L^2(B_{3\rho}(x_0))}.\end{aligned}$$

- The second integral (6.18) can be estimated thanks to the convexity of $\mathfrak{a}$, which yields

$$\begin{aligned}&\mathfrak{a}\big(x,T_h\nabla v(x)\big)-\mathfrak{a}\big(x,\nabla v(x)\big)\\ &\quad\leqslant \big(1-\phi(x)\big)\mathfrak{a}\big(x,\nabla v(x)\big)+\phi(x)\mathfrak{a}\big(x,\nabla v_h(x)\big)-\mathfrak{a}\big(x,\nabla v(x)\big)\\ &\quad=\phi(x)\big[\mathfrak{a}\big(x,\nabla v_h(x)\big)-\mathfrak{a}\big(x,\nabla v(x)\big)\big].\end{aligned}$$

Since in $B_{3\rho}(x_0)\backslash\Omega$ we have $\nabla v_h(x)\equiv\nabla v(x)\equiv 0$, recalling the support property of ϕ and integrating in Ω we get:

$$\begin{aligned}&\int_\Omega \big[\mathfrak{a}\big(x,T_h\nabla v(x)\big)-\mathfrak{a}\big(x,\nabla v(x)\big)\big]\,\mathrm{d}x\\ &\quad\leqslant \int_{\Omega\cap B_{2\rho}(x_0)} \phi(x)\big[\mathfrak{a}\big(x,\nabla v_h(x)\big)-\mathfrak{a}\big(x,\nabla v(x)\big)\big]\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= \int\limits_{\Omega\cap B_{2\rho}(x_0)+h} \phi(x-h)\mathfrak{a}\big(x-h,\nabla v(x)\big)\,\mathrm{d}x \\
&\quad - \int\limits_{\Omega\cap B_{2\rho}(x_0)} \phi(x)\mathfrak{a}\big(x,\nabla v(x)\big)\,\mathrm{d}x \\
&\leqslant \int\limits_{\Omega\cap B_{3\rho}(x_0)} \big[\phi(x-h)\mathfrak{a}\big(x-h,\nabla v(x)\big)-\phi(x)\mathfrak{a}\big(x-h,\nabla v(x)\big)\big]\,\mathrm{d}x \\
&\quad + \int\limits_{\Omega\cap B_{2\rho}(x_0)} \phi(x)\big[\mathfrak{a}\big(x-h,\nabla v(x)\big)-\mathfrak{a}\big(x,\nabla v(x)\big)\big]\,\mathrm{d}x \\
&\leqslant \big(\rho^{-1}\mathsf{a}+\mathsf{L}\big)|h|\|\nabla v\|^2_{L^2(\Omega\cap B_{3\rho}(x_0))}. \qquad (6.21)
\end{aligned}
$$

- The last integral (6.19) can be estimated in the same way:

$$
\int\limits_{\Omega} \big[\big|T_h v(x)\big|^2-\big|v(x)\big|^2\big]\,\mathrm{d}x \qquad (6.22)
$$

$$
\begin{aligned}
&\leqslant \int\limits_{\Omega\cap B_{2\rho}(x_0)} \phi(x)\big[\big|v_h(x)\big|^2-\big|v(x)\big|^2\big]\,\mathrm{d}x \\
&= \int\limits_{\Omega\cap B_{2\rho}(x_0)+h} \phi(x-h)\big|v(x)\big|^2\,\mathrm{d}x - \int\limits_{\Omega\cap B_{2\rho}(x_0)} \phi(x)\big|v(x)\big|^2\,\mathrm{d}x \\
&\leqslant \rho^{-1}|h| \int\limits_{\Omega\cap B_{3\rho}(x_0)} \big|v(x)\big|^2\,\mathrm{d}x. \qquad \square \qquad (6.23)
\end{aligned}
$$

7. Set distance and neighborhoods of Lipschitz sets

As we already mentioned at the beginning of Section 6, in this section we will investigate the relationships between the notion of the excess and Hausdorff distance (which we introduced in Section 2.3) and the uniform cone condition (2.51), in order to find admissible vectors $\boldsymbol{\nu}$ for (5.1). Let us collect some preliminary geometric properties of (ρ,θ) open sets which will turn out to be useful in the following.

Lemma 7.1. *Suppose that $\Omega\subset\mathbb{R}^N$ is a uniform (ρ,θ) open set; then for every $\varepsilon\leqslant\rho$ its neighborhood Ω^ε is a uniform $(\rho/2,\theta)$ open set and*

$$
\mathcal{N}_{\rho,\theta}(x,\Omega)\subset\mathcal{N}_{\rho/2,\theta}\big(x,\Omega^\varepsilon\big)\quad \forall x\in\mathbb{R}^N. \qquad (7.1)
$$

Proof. Let us fix $x \in \mathbb{R}^N$, $\boldsymbol{n} \in \mathcal{N}_{\rho,\theta}(x, \Omega)$, $h \in \mathcal{C}_{\rho/2,\theta}(\boldsymbol{n})$, and $y \in B_{3\rho/2}(x)$. If $y \in \Omega^\varepsilon$ then there exists $z \in \Omega$ such that

$$|z - y| < \varepsilon \leqslant \rho.$$

Since $|z - x| < \rho + \frac{3}{2}\rho < 3\rho$, (2.50) yields

$$z - h \in \Omega, \quad \text{and} \quad d(y - h, \Omega) \leqslant \big|y - h - (z - h)\big| = |y - z| < \varepsilon,$$

i.e. $y - h \in \Omega^\varepsilon$.

Conversely, if $y \notin \Omega^\varepsilon$ we must prove that $y + h \notin \Omega^\varepsilon$. We argue by contradiction, assuming that $y + h \in \Omega^\varepsilon$: then, as before, we can find

$$z \in \Omega: \quad \big|z - (y + h)\big| < \varepsilon \leqslant \rho.$$

Again, since $|z - x| < \rho + |h| + \frac{3}{2}\rho < 3\rho$, then (2.50) yields

$$z - h \in \Omega \quad \text{and} \quad d(y, \Omega) \leqslant \big|y - (z - h)\big| = \big|(y + h) - z\big| < \varepsilon. \qquad \square$$

Remark 7.2. By making a simple geometric construction, it is easy to check that

$$0 \leqslant \varepsilon \leqslant \frac{1}{2}\rho \sin\theta \quad \Rightarrow \quad B_\varepsilon\left(\frac{\varepsilon}{\sin\theta}\boldsymbol{n}\right) \subset \mathcal{C}_{\rho,\theta}(\boldsymbol{n}) \quad \forall \boldsymbol{n} \in \mathbb{S}^{N-1}. \tag{7.2}$$

Lemma 7.3. *If Ω is a uniform (ρ, θ) open set and*

$$-\frac{1}{2}\rho \sin\theta \leqslant \eta \leqslant 0 \leqslant \varepsilon \leqslant \frac{1}{2}\rho \sin\theta, \tag{7.3}$$

then for every $y \in \mathbb{R}^N$, $\boldsymbol{n} \in \mathcal{N}_{\rho,\theta}(y, \Omega)$

$$x \in B_{2\rho}(y) \setminus \Omega^\eta \quad \Rightarrow \quad x_{\varepsilon-\eta} = x + \frac{\varepsilon - \eta}{\sin\theta}\boldsymbol{n} \notin \Omega^\varepsilon. \tag{7.4}$$

Proof. Let us first prove (7.4) for $\eta = 0$, i.e.,

$$x \in B_{3\rho}(y) \setminus \Omega \quad \Rightarrow \quad x_\varepsilon := x + \frac{\varepsilon}{\sin\theta}\boldsymbol{n} \notin \Omega^\varepsilon. \tag{7.5}$$

We observe that by (2.50) the cone $x + \mathcal{C}_{\rho,\theta}(\boldsymbol{n})$ is contained in the complement of Ω; therefore, by Remark 7.2,

$$B_\varepsilon(x_\varepsilon) = B_\varepsilon\left(x + \frac{\varepsilon}{\sin\theta}\boldsymbol{n}\right) \subset x + \mathcal{C}_{\rho,\theta}(\boldsymbol{n}) \subset \mathbb{R}^N \setminus \Omega,$$

i.e., $d(x_\varepsilon, \Omega) \geqslant \varepsilon$.

In the case $\varepsilon = 0$ (7.4) follows by contradiction: if $x_{-\eta} \in \Omega$, since $x_{-\eta} \in B_{3\rho}(y)$, then, using again Remark 7.2,

$$B_{|\eta|}(x) = B_{|\eta|}\left(x_{-\eta} - \frac{|\eta|}{\sin\theta}\boldsymbol{n}\right) \subset x_{-\eta} - \mathcal{C}_{\rho,\theta}(\boldsymbol{n}) \subset \Omega,$$

i.e. $x \in \Omega^\eta$.

Formula (7.4) for arbitrary ε follows now by a further application of (7.5). □

Lemma 7.4. *Let Ω_1, Ω_2 be open sets and let Ω_1 satisfy a uniform (ρ,θ) cone condition. If $e(\Omega_2,\Omega_1) \leqslant \frac{1}{2}\rho\sin\theta$, then*

$$\check{e}(\Omega_2,\Omega_1) \leqslant \frac{e(\Omega_2,\Omega_1)}{\sin\theta}. \tag{7.6}$$

In particular, if also Ω_2 is a uniform (ρ,θ)-set and $d_H(\Omega_1,\Omega_2) \leqslant \frac{1}{2}\rho\sin\theta$, then

$$\mathfrak{d}(\Omega_1,\Omega_2) \leqslant \frac{d_H(\Omega_1,\Omega_2)}{\sin\theta}. \tag{7.7}$$

Proof. Let $\lambda = e(\Omega_2,\Omega_1)$, so that $\Omega_2 \subset \Omega_1^\lambda$, and let $y \in \Omega_2 \setminus \Omega_1$. Since $\lambda \leqslant \frac{1}{2}\rho\sin\theta$ then, by the previous lemma, for every $\boldsymbol{n} \in \mathcal{N}_{\rho,\theta}(y,\Omega_1)$ it is

$$y + \frac{\lambda}{\sin\theta}\boldsymbol{n} \in \mathbb{R}^N \setminus \Omega_1^\lambda \subset \mathbb{R}^N \setminus \Omega_2.$$

Hence,

$$d\big(y, \mathbb{R}^N \setminus \Omega_2\big) \leqslant \frac{\lambda}{\sin\theta}.$$

Since y is arbitrary, we conclude that

$$\check{e}(\Omega_2,\Omega_1) = \sup_{y \in \Omega_2 \setminus \Omega_1} d\big(y, \mathbb{R}^N \setminus \Omega_2\big) \leqslant \frac{\lambda}{\sin\theta}. \qquad \square$$

8. Proofs of Theorems 1–3

The next result contains the fundamental consequence of the theory developed in the previous sections.

Lemma 8.1. *Let Ω be a uniform (ρ,θ) open subset of $\mathbb{R}^N$ and let ε, η satisfy*:

$$-\frac{1}{2}\rho\sin\theta \leqslant \eta \leqslant 0 \leqslant \varepsilon \leqslant \min\left[\delta(\Omega, D), \frac{1}{2}\rho\sin\theta\right].$$

For every $f \in L^2(D)$ *and*

$$u_\varepsilon := \mathcal{G}(f; V_{\Omega^\varepsilon}) \in V_{\Omega^\varepsilon}, \quad \textit{solution to } (2.2), \quad \Lambda := \Omega^\varepsilon \setminus \Omega^\eta,$$

there exists $w_\eta \in V_{\Omega^\eta}$ *such that*

$$\|u_\varepsilon - w_\eta\|^2_{L^2(\mathbb{R}^N)} \leqslant \ell_1 \frac{(\varepsilon-\eta)^2}{\sin^2\theta} \|\nabla u_\varepsilon\|^2_{L^2(\Lambda^{3\rho})} \leqslant \ell_1 \rho \frac{\varepsilon-\eta}{\sin\theta} \|\nabla u_\varepsilon\|^2_{L^2(\Lambda^{3\rho})}, \tag{8.1}$$

$$\|u_\varepsilon - w_\eta\|^2_V \leqslant \ell_2 \frac{\varepsilon-\eta}{\sin\theta} \left\{ \left(\frac{\mathsf{a}}{\rho} + \mathsf{b} + \mathsf{L} \right) [u_\varepsilon]^2_{\Lambda^{3\rho}} + \|\tilde{f}_\varepsilon\|_{L^2(\Lambda^{3\rho})} \|\nabla u_\varepsilon\|_{L^2(\Lambda^{3\rho})} \right\}, \tag{8.2}$$

where $\tilde{f}_\varepsilon$ *is given by* (6.7) *and, as usual, the constant* ℓ_2 *only depends on the dimension* N.

Proof. We apply Proposition 5.1 with the choices

$$\begin{gathered} Y := \Omega^\varepsilon, \qquad X := \Omega^\eta, \qquad v := u_\varepsilon, \\ \boldsymbol{\nu}(y) := \frac{\varepsilon-\eta}{\sin\theta} \boldsymbol{n}, \quad \boldsymbol{n} \in \mathcal{N}_{\rho,\theta}(y, \Omega). \end{gathered} \tag{8.3}$$

Recalling that $\varepsilon - \eta \leqslant \rho \sin\theta$ and taking (7.4) into account, we observe that by (6.1) we can choose

$$G(x) = \frac{(\varepsilon-\eta)^2}{\sin^2\theta} \big|\nabla u_\varepsilon(x)\big|^2 \leqslant \rho \frac{\varepsilon-\eta}{\sin\theta} \big|\nabla u_\varepsilon(x)\big|^2, \tag{8.4}$$

which provides (8.1). On the other hand, Theorem 6.2 suggests the choice, depending on the parameter $\kappa > 0$,

$$H := \frac{\varepsilon-\eta}{\sin\theta} \big\{ (5\mathsf{a}/\rho + 2\mathsf{b} + \mathsf{L}) \big(\mathfrak{a}(x, \nabla u_\varepsilon) + \mathsf{c}|u_\varepsilon|^2 \big) + \kappa |\nabla u_\varepsilon|^2 + 1/\kappa \big|\tilde{f}_\varepsilon\big|^2 \big\}. \tag{8.5}$$

Integrating in $\Lambda^{3\rho}$, by (5.5) we get:

$$\begin{aligned} &\int_D \mathfrak{a}\big(x, \nabla u_\varepsilon - \nabla w_\eta\big)\,\mathrm{d}x \\ &\quad \leqslant \ell_1 \frac{\varepsilon-\eta}{\sin\theta} \big\{ (6\mathsf{a}/\rho + 2\mathsf{b} + \mathsf{L}) [u]^2_{\Lambda^{3\rho}} + \kappa \|\nabla u_\varepsilon\|^2_{L^2(\Lambda^{3\rho})} + (1/\kappa) \|\tilde{f}_\varepsilon\|^2_{L^2(\Lambda^{3\rho})} \big\}. \end{aligned} \tag{8.6}$$

Choosing now, according to (6.6),

$$\mathsf{c}G := \frac{\varepsilon-\eta}{\sin\theta} \big\{ (5\mathsf{a}/\rho + 2\mathsf{b} + \mathsf{L}) \big(\mathfrak{a}(x, \nabla u_\varepsilon) + \mathsf{c}|u_\varepsilon|^2 \big) + \kappa |\nabla u_\varepsilon|^2 + 1/\kappa \big|\tilde{f}_\varepsilon\big|^2 \big\}, \tag{8.7}$$

we get from (5.4)

$$\begin{aligned}&\mathsf{c}\|u_\varepsilon - w_\eta\|^2_{L^2(\mathbb{R}^N)}\\ &\quad\leqslant \ell_1\frac{\varepsilon-\eta}{\sin\theta}\bigl\{(5\mathsf{a}/\rho+2\mathsf{b}+\mathsf{L})[u_\varepsilon]^2_{\Lambda^{3\rho}}+\kappa\|\nabla u_\varepsilon\|^2_{L^2(\Lambda^{3\rho})}+(1/\kappa)\bigl\|\tilde f_\varepsilon\bigr\|^2_{L^2(\Lambda^{3\rho})}\bigr\}. \end{aligned}\tag{8.8}$$

Since $\kappa > 0$ is arbitrary, summing up (8.6) and (8.8) it is easy to infer:

$$\|u_\varepsilon - w_\eta\|^2_V \leqslant \ell_2\frac{\varepsilon-\eta}{\sin\theta}\bigl\{(\mathsf{a}/\rho+\mathsf{b}+\mathsf{L})[u_\varepsilon]^2_{\Lambda^{3\rho}}+\bigl\|\tilde f_\varepsilon\bigr\|_{L^2(\Lambda^{3\rho})}\|\nabla u_\varepsilon\|_{L^2(\Lambda^{3\rho})}\bigr\},\tag{8.9}$$

for $\ell_2 := 11\ell_1$. □

In order to write in a compact way our estimates, we recall the "re-normalized" norms $\|\cdot\|_H$, $\|\cdot\|_{H'}$ of $L^2(D)$ we introduced in (A7) and we denoted by

$$\|u\|^2_H := \mathsf{p}\int_D \bigl|u(x)\bigr|^2\,\mathrm{d}x,\qquad \|f\|^2_{H'} := \mathsf{p}^{-1}\int_D\bigl|f(x)\bigr|^2\,\mathrm{d}x;\tag{8.10}$$

observe that, if $u = \mathcal{G}(f; V_\Omega)$, $\Omega\subset D$,

$$\|u\|_H \leqslant \|u\|_V \leqslant \frac{1}{\alpha}\|f\|_{V'} \leqslant \frac{1}{\alpha}\|f\|_{H'}.\tag{8.11}$$

Moreover,

$$\begin{aligned}\bigl\|\tilde f_\varepsilon\bigr\|_{L^2(\Lambda^{3\rho})} &\leqslant \|f\|_{L^2(D)}+\mathsf{d}\|u_\varepsilon\|_{L^2(D)}\\ &\leqslant \mathsf{p}^{1/2}\|f\|_{H'}+\mathsf{p}^{1/2}\gamma\|u_\varepsilon\|_H \leqslant \mathsf{p}^{1/2}\biggl(\|f\|_{H'}+\frac{\gamma}{\alpha}\|f\|_{V'}\biggr).\end{aligned}\tag{8.12}$$

Corollary 8.2. *For $f\in L^2(D)$ let us introduce the quantity*:

$$\begin{aligned}\Gamma^2_\rho[f] &:= \frac{\mathsf{a}+\pi_\rho(\beta+\gamma+\lambda)}{\alpha^2}\|f\|^2_{V'}+\frac{\pi_\rho}{\alpha}\|f\|_{V'}\|f\|_{H'}\\ &\leqslant \frac{\mathsf{a}+\pi_\rho(\alpha+\beta+\gamma+\lambda)}{\alpha^2}\|f\|_{V'}\|f\|_{H'} =: \Gamma^2_\rho\,\|f\|_{V'}\,\|f\|_{H'}.\end{aligned}\tag{8.13}$$

Then

$$d(u_\varepsilon, V_{\Omega^\eta})^2 \leqslant \|u_\varepsilon - w_\eta\|^2_V \leqslant \ell_2\frac{\varepsilon-\eta}{\rho\sin\theta}\Gamma^2_\rho[f].\tag{8.14}$$

Before stating our first result, it could be useful to recall the meaning of the constants we introduced before (cf. (2.11), (4.6), (4.8)):

$$\sigma = \sqrt{\frac{1+\beta_s+\gamma}{\alpha}}+\frac{\beta}{\alpha},\qquad \mu = 1+\beta+\gamma.\tag{8.15}$$

We set

$$\delta_i := \delta(\Omega_i, D),\ i = 1, 2, \quad \text{so that} \quad \varepsilon \leqslant \delta_i \quad \Rightarrow \quad \Omega_i^\varepsilon \subset D. \tag{8.16}$$

Theorem 8.3. *Let* (A1)–(A8) *hold, let* Ω_1, Ω_2 *be two open subsets of* D*, let* $f \in L^2(D)$*,* $\ell_3 := 8\ell_2$ *be a constant dependent only on* N*, and let*

$$u_i = \mathcal{G}(f; V_{\Omega_i}) \in V_{\Omega_i} \quad \textit{be the solutions to } (2.2).$$

- *Suppose that* Ω_1 *satisfies a uniform* (ρ_1, θ_1) *cone condition on* D *and*

$$e(\Omega_1 \triangle \Omega_2, \partial\Omega_1) \leqslant \frac{1}{2}\rho_1 \sin\theta_1, \quad e(\Omega_2, \Omega_1) \leqslant \delta_1. \tag{8.17}$$

  Then,

$$\|u_1 - u_2\|_V^2 \leqslant \ell_3 \sigma^2 \Gamma_{\rho_1}^2[f] \frac{e(\Omega_2 \triangle \Omega_1, \partial\Omega_1)}{\rho_1 \sin\theta_1}, \tag{8.18}$$

  where $\Gamma_{\rho_1}^2[f], \sigma$ *are defined by* (8.13) *and* (8.15)*.*
- *If also* Ω_2 *is a* (ρ_2, θ_2) *uniform set and*

$$\check{e}(\Omega_1, \Omega_2) \leqslant \frac{1}{2}\rho_1 \sin\theta_1, \qquad \check{e}(\Omega_2, \Omega_1) \leqslant \frac{1}{2}\rho_2 \sin\theta_2, \tag{8.19}$$

  then

$$\|u_1 - u_2\|_V^2 \leqslant \ell_3 \sigma^2 \left\{ \Gamma_{\rho_1}^2[f] \frac{\check{e}(\Omega_1, \Omega_2)}{\rho_1 \sin\theta_1} + \Gamma_{\rho_2}^2[f] \frac{\check{e}(\Omega_2, \Omega_1)}{\rho_2 \sin\theta_2} \right\}. \tag{8.20}$$

- *Finally, if*

$$\begin{cases} e(\Omega_1, \Omega_2) \leqslant \min\bigl(\delta_2, \frac{1}{2}\rho_2 \sin\theta_2\bigr), \\ e(\Omega_2, \Omega_1) \leqslant \min\bigl(\delta_1, \frac{1}{2}\rho_1 \sin\theta_1\bigr), \end{cases} \tag{8.21}$$

  then

$$\|u_1 - u_2\|_V^2 \leqslant \ell_3 \sigma^4 \left\{ \Gamma_{\rho_1}^2[f] \frac{e(\Omega_2, \Omega_1)}{\rho_1 \sin\theta_1} + \Gamma_{\rho_2}^2[f] \frac{e(\Omega_1, \Omega_2)}{\rho_2 \sin\theta_2} \right\}. \tag{8.22}$$

Proof. Let us set

$$\begin{aligned} \varepsilon &:= e(\Omega_2, \Omega_1), \qquad \check{\varepsilon} := \check{e}(\Omega_2, \Omega_1), \\ \eta &:= e(\Omega_1, \Omega_2), \qquad \check{\eta} := \check{e}(\Omega_1, \Omega_2). \end{aligned} \tag{8.23}$$

To show (8.18), we aim to apply (4.17) of Corollary 4.3 with

$$V_1 := V_{\Omega_1}, \quad V_2 := V_{\Omega_2}, \quad V^{1,2} := V_{\Omega_1^\varepsilon}. \tag{8.24}$$

Since

$$V_{\Omega_1^{-\check\eta}} \subset V_1 \cap V_2, \qquad d\big(u^{1,2}, V_2\big) \leqslant d\big(u^{1,2}, V_{\Omega_1^{-\check\eta}}\big), \tag{8.25}$$

the above corollary (applied with $\Omega = \Omega_1$) yields (8.18).

In order to prove (8.20), we apply (4.14) of Corollary 4.3: it is easy to see by (8.14) that

$$V_{\Omega_1^{-\check\eta}} \subset V_1 \cap V_2, \quad d(u_1, V_1 \cap V_2)^2 \leqslant d(u_1, V_{\Omega_1^{-\check\eta}})^2 \leqslant \ell_2 \frac{\check\eta}{\rho_1 \sin\theta_1} \Gamma_{\rho_1}^2[f],$$
$$V_{\Omega_2^{-\check\varepsilon}} \subset V_1 \cap V_2, \quad d(u_2, V_1 \cap V_2)^2 \leqslant d(u_2, V_{\Omega_2^{-\check\varepsilon}})^2 \leqslant \ell_2 \frac{\check\varepsilon}{\rho_2 \sin\theta_2} \Gamma_{\rho_2}^2[f].$$

Finally, (8.22) follows from Corollary 4.5 and the estimates of Corollary 8.2, by choosing

$$V^{1,2} := V_{\Omega_1^\varepsilon}, \qquad V^{2,1} := V_{\Omega_2^\eta}. \qquad \square \tag{8.26}$$

We are now interested to derive a L^2-estimate for $u_1 - u_2$, by using the duality argument of Corollary 4.10. Recalling (2.17), (2.18) and (2.19) of Section 2.2, we can correspondingly define $\widehat\Gamma_\rho^2[f]$, $\widehat\Gamma_\rho^2$ as in (8.13), the only difference being in the constant

$$\hat\gamma := \frac{\mathsf{d} + 2\bar{\mathsf{b}}}{\mathsf{p}} = \gamma + 2\beta_s. \tag{8.27}$$

Since the coercivity constant of $a, \hat a$ is $\alpha = 1$, and (8.11) holds, $\widehat\Gamma_\rho^2[f]$ can be bounded by

$$\widehat\Gamma_\rho^2[f] \leqslant \widehat\Gamma_\rho^2 \|f\|_{H'}^2, \quad \widehat\Gamma_\rho^2 := \mathsf{a} + \pi_\rho(1 + \beta + \hat\gamma + \lambda). \tag{8.28}$$

We set

$$\overline\Gamma_\rho[f] := \widehat\Gamma_\rho \Gamma_\rho^2[f]. \tag{8.29}$$

Theorem 8.4. *Let* (A1)–(A9) *hold, let Ω_1, Ω_2 be two open subsets of D, let $f \in L^2(D)$, let ℓ_3 be the same constant* (*dependent only on N*) *of the previous theorem, and let*

$$u_i = \mathcal{G}(f; V_{\Omega_i}) \in V_{\Omega_i} \quad \textit{be the solutions to } (2.2).$$

- *If Ω_1 satisfies a uniform (ρ_1, θ_1) cone condition on D and* (8.17):

$$\|u_1 - u_2\|_H \leqslant \ell_3 \sigma^2 \mu \overline\Gamma_{\rho_1}[f] \frac{e(\Omega_2 \triangle \Omega_1, \partial\Omega_1)}{\rho_1 \sin\theta_1}, \tag{8.30}$$

  where $\overline\Gamma_{\rho_1}[f], \sigma, \mu$ are defined by (8.13), (8.15), *and* (8.29).

- *If also* Ω_2 *is a* (ρ_2, θ_2) *uniform set and* (8.19) *holds, then*

$$\|u_1 - u_2\|_H \leqslant \ell_2 \sigma \mu \left\{ \overline{\Gamma}_{\rho_1}[f] \frac{\check{e}(\Omega_1, \Omega_2)}{\rho_1 \sin\theta_1} + \overline{\Gamma}_{\rho_2}[f] \frac{\check{e}(\Omega_2, \Omega_1)}{\rho_2 \sin\theta_2} \right\}. \tag{8.31}$$

- *Finally, if* (8.21) *holds, then*

$$\|u_1 - u_2\|_H \leqslant \ell_2 \sigma^2 \mu \left\{ \overline{\Gamma}_{\rho_1}[f] \frac{e(\Omega_2, \Omega_1)}{\rho_1 \sin\theta_1} + \overline{\Gamma}_{\rho_2}[f] \frac{e(\Omega_1, \Omega_2)}{\rho_2 \sin\theta_2} \right\}. \tag{8.32}$$

Proof. Taking again the choices (8.23), (8.24) and setting, for $u_1 \neq u_2$,

$$g := \sqrt{\mathsf{p}} \frac{u_1 - u_2}{\|u_1 - u_2\|_{L^2(D)}}, \quad \|g\|_{H'} = 1, \quad \langle g, u_1 - u_2 \rangle = \|u_1 - u_2\|_H, \tag{8.33}$$

we now apply Corollary 4.10 and have to use Lemma 8.1 and Theorem 8.3 in order to control the terms on the right hand side of (4.34).

For the sake of clarity, let us detail the proof of (8.30). Referring to the third formula in (4.34), by (8.24) and (8.25) we see that

$$\|u_1 - u_2\|_H \leqslant \sigma \mu \|u_1 - u_2\|_V \, d\big(\hat{v}^\varepsilon, V_{\Omega_1^{-\check{\eta}}}\big), \tag{8.34}$$

where $\hat{v}^\varepsilon = \widehat{\mathcal{G}}(g; V_{\Omega_1^\varepsilon})$ is the solution of the adjoint problem in Ω_1^ε w.r.t. g. (8.14) and (8.28) provide the estimate

$$d\big(\hat{v}^\varepsilon, V_{\Omega_1^{-\check{\eta}}}\big)^2 \leqslant 2\ell_2 \frac{e(\Omega_2 \triangle \Omega_1, \partial\Omega_1)}{\rho_1 \sin\theta_1} \widehat{\Gamma}_\rho^2, \tag{8.35}$$

which, combined with (8.18), yields (8.30).

The proof of (8.31) follows by an analogous argument, applying the fourth formula of (4.34) and (8.14). Finally, (8.32) is a consequence of the first estimate of (4.34), by choosing

$$V^{1,2} := V_{\Omega_1^\varepsilon}, \qquad V^{2,1} := V_{\Omega_2^\eta},$$

and applying (8.14) again. □

We conclude with a weaker estimate for $\|u_1 - u_2\|_{L^2(D)}$ in the case f belongs only to $H^{-1}(D)$, i.e., in the framework of Theorem 5. We limit ourselves to impose regularity only to Ω_1 since the estimates in the same spirit of (8.31), (8.32) could be easily deduced in the same way.

Theorem 8.5. *Let* Ω_1, Ω_2 *be two open subsets of* D, *let* $f \in H^{-1}(D)$, *and let*

$$u_i = \mathcal{G}(f; V_{\Omega_i}) \in V_{\Omega_i} \quad \textit{be the solutions to } (2.2).$$

Suppose that Ω_1 satisfies a uniform (ρ_1, θ_1) cone condition on D and (8.17). *Then*

$$\|u_1 - u_2\|_H \leqslant \ell_3^{1/2} \sigma \mu \widehat{\Gamma}_{\rho_1} \|f\|_{V'} \left(\frac{e(\Omega_2 \triangle \Omega_1, \partial\Omega_1)}{\rho_1 \sin\theta_1} \right)^{1/2}. \tag{8.36}$$

Proof. We repeat the choices (8.23), (8.24) taking (8.33) for g and exploiting the third inequality in (4.34). Thus we end up with (8.34); thanks to $\alpha = 1$, (2.39), and (8.35) we readily see that (8.36) holds. □

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