Variational Equations of Schroedinger–Type in non–Cylindrical Domains *

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Abstract

Cauchy–Dirichlet problems are studied for linear Schroedinger–type P.D.E. in non–cylindrical domains, by assuming a monotonicity condition on their sections w.r.t. time. Existence and uniqueness results are obtained, as a consequence of new general results we prove here for a class of abstract Schroedinger–type differential equations in Hilbert spaces.

1 Introduction.

1*a* ***Schroedinger–type equations in non–cylindrical domains.*** For a fixed final time $T > 0$, let Q be an open set of $\mathbb{R}^N \times]0, T[$, whose sections

$$Q_t := \Big\{ x \in \mathbb{R}^N : (x,t) \in Q \Big\}, \quad t \in]0, T[, \tag{1.1}$$

satisfy the *non-decreasing property*

$$s < t < T \quad \Rightarrow \quad Q_s \subseteq Q_t. \tag{1.2}$$

Let Σ, Q_0, Q_T be its lateral, initial, and final boundaries, defined by

$$\Sigma := \partial Q \cap \Big(\mathbb{R}^N \times]0, T[\Big), \quad Q_0 := \operatorname{int} \bigcap_{t>0} Q_t, \quad Q_T := \bigcup_{0<t<T} Q_t;$$

for the sake of simplicity, we will assume that $Q_0 \neq \emptyset$.

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For given $u_0 \in L^2(Q_0)$ and $f \in L^2(Q)$ (we will deal with spaces of *complex*-valued functions) we want to study the following Cauchy-Dirichlet boundary value problem for the Schroedinger–type equation

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} + iAu(x,t) = f(x,t) & \text{in } Q, \\ u(x,t) = 0 & \text{on } \Sigma, \\ u(x,0) = u_0(x) & \text{on } Q_0. \end{cases} \tag{S}$$

Here A is a linear second order differential operator with variable coefficients of the type

$$Au := -\sum_{i,j} \frac{\partial}{\partial x_i}\Big(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\Big) + c(x,t)u,$$

where

$$a_{ij} \in L^\infty(Q),\ c \in L^\infty(Q;\mathbb{R}) \text{ with their time derivatives,} \quad a_{ij} = \overline{a_{ji}}, \tag{1.3}$$

satisfy the *uniform ellipticity condition*

$$\exists\, \alpha > 0: \quad \sum_{i,j} a_{ij}(x,t)\xi_i\overline{\xi_j} \geq \alpha|\xi|^2 \quad \forall\, \xi \in \mathbb{C}^N, \text{ for a.e. } (x,t) \in Q. \tag{1.4}$$

Problems of this type were studied in [12, 4, 6]. In particular, in [12] and [6], existence and uniqueness results for (S) were obtained without any monotonicity condition on the sections Q_t of (1.1), but only for very special and suitably smooth domains Q and with $A := -\Delta_x$. On the other hand, in [4] "very weak" solutions to (S) were considered in the general case: an existence theorem was proved, in particular, under condition (1.2); moreover, a uniqueness result was obtained by assuming that the sets Q_t satisfy a non–increasing property.

In our present paper, we always consider general domains Q satisfying only the *non–decreasing condition* (1.2). Then, on the one hand, we approach the problem of uniqueness by exhibiting a particular selection principle, which allows us to single out a unique solution among all the possible weak ones (we call it "*conservative*", since it is characterized by a conservation identity: cf. (S_5) below). On the other hand, we improve the existence result in [4] by obtaining (under the same basic assumptions) stronger solutions satisfying our selection requirement together with various additional properties.

Let us remark that, for Cauchy–Dirichlet problems in non–cylindrical domains, bibliography is much wider in the cases of parabolic and hyperbolic P.D.E.'s than in the one of Schroedinger–type equations. Various important references can be found e.g., in the recent papers [10, 14] (for parabolic P.D.E.'s) and [5] (for hyperbolic P.D.E.'s).

1*b* ***Variational solutions with finite energy.*** Let us make precise what we mean for a variational solution of (S). Here we are interested in $L^2(Q)$–solutions with finite Dirichlet integral

$$\mathscr{E}(u) := \int_Q |\nabla_x u(x,t)|^2\,dx\,dt < +\infty, \tag{S_1}$$

which satisfy the differential equation of (S) in the usual weak sense in Q, i.e.

$$\int_Q \Big(- u\frac{\partial \phi}{\partial t} + i\sum_{i,j} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_j} + i\, c\, u\phi\Big)\, dx\, dt = \int_Q f\phi\, dx\, dt \tag{S_2}$$
$$\text{for every test function } \phi \in H^1_0(Q),$$

and the initial condition in the integral form

$$\lim_{t\downarrow 0} \fint_0^t \int_{Q_0} |u(x,\tau) - u_0(x)|^2 dx\, d\tau = 0. \tag{S_3}$$

Finally, the lateral boundary conditions are imposed by asking that

$$u(\cdot, t) \in H^1_0(Q_t) \quad \text{for a.e. } t \in]0,T[\,. \tag{S_4}$$

Conditions of this type are quite natural in the context of *cylindrical* domains of the form $Q = Q_0\times]0,T[$, i.e. $Q_0 = Q_t = Q_T$ for every $t \in]0,T[$. Since in this case the finite energy condition (S_1) and the equation (S_2) yield

$$u \in L^2(0,T;H^1(Q_0)), \quad \frac{\partial u}{\partial t} \in L^2(0,T;H^{-1}(Q_0)), \tag{1.5}$$

applying standard trace results [11, Thm. 3.1,Lemma 12.1] it is easy to see that solutions of (S) with finite energy (S_1) are continuous w.r.t. the time variable with values in $L^2(Q_0) = L^2(Q_T)$ and satisfy the conservation law

$$\int_{Q_T} |u(x,T)|^2\, dx = \int_{Q_0} |u_0(x)|^2\, dx + 2\,\mathrm{Re}\int_Q f(x,\tau)\overline{u(x,\tau)}\, dx\, d\tau. \tag{S_5}$$

In the case of a non–cylindrical set Q satisfying (1.2), (S_5) still makes sense, since u satisfies a regularity property analogous to (1.5)

$$u \in L^2(T',T;H^1(Q_t)), \quad \frac{\partial u}{\partial t} \in L^2(T',T;H^{-1}(Q_t)) \quad \forall T' < T,$$

and therefore the trace of a finite energy solution at the final time T belongs to $L^2(Q_{T'})$ for every $T' < T$ and in particular to $L^2_{loc}(Q_T)$; but (S_5) is no more a direct consequence of the equation. The main result of our paper for (S) shows that (S_5) is a crucial criterion, in order to select a particular solution (we will denote it as a *conservative solution*) among the variational ones of (S_1),. . .,(S_4).

Theorem 1 *Suppose that* (1.2), (1.3), *and* (1.4) *hold true. If*

$$u_0 \in L^2(Q_0), \quad f \in L^2(Q),$$

then Problem (S) *admits* at most *one* conservative *solution* u *satisfying* $(S_1), \ldots, (S_5)$; *the subset*

$$D := \big\{(u_0,f) \in L^2(Q_0) \times L^2(Q) : \exists\, u \textit{ conservative solution of } (S_1),\ldots,(S_5)\big\}$$

is a linear space and the induced map $(u_0,f) \in D \mapsto u$ *is linear. If furthermore*

$$u_0 \in H^1_0(Q_0), \quad \frac{\partial f}{\partial t} \in L^2(Q),$$

then such a solution exists; moreover, still denoting by u its trivial extension by 0 outside Q, we have that

u belongs to $C^{1/2}([0,T];L^2(\mathbb{R}^N))$,

$u(\cdot,t) \in H^1_0(Q_t)$ for every *$t \in [0,T]$,*

$t \mapsto u(\cdot,t)$ is (right) continuous w.r.t. the (strong) weak topology of $H^1(\mathbb{R}^N)$.

1*c* ***The abstract approach.*** The *proof* of the above result is an immediate application of the abstract theory that we develop in the next sections. Here we recall what are the main features of this approach.

We introduce the complex Hilbert space $H := L^2(Q_T)$ endowed with the usual scalar product, and we consider u as a function of the time with values in H: this requires a preliminary (trivial) extension of it to the whole of Q_T. Correspondingly, we also introduce suitable regularity preserving extensions of f and of the coefficients a_{ij}, c (still denoted by the same symbols) to the cylinder $Q_T \times]0,T[$: e.g., when for a.e. $x \in Q_T$ they are absolutely continuous w.r.t. t in the fiber $\{t \in [0,T] : (x,t) \in Q\}$, we can define

$$f(x,t) := f(x,h(x,t)), \quad a_{ij}(x,t) := a_{ij}(x,h(x,t)), \quad c(x,t) := c(x,h(x,t)),$$

where

$$h(x,t) := \inf\big\{\tau \in [t,T] : (x,\tau) \in Q\big\}.$$

Motivated by (S_1) and (S_4), we also introduce a family $\mathscr{V} := \{V_t\}_{t \in [0,T]}$ of closed subspaces of $V := H^1(Q_T)$; more precisely, we set

$$V_t := \Big\{v \in H^1(Q_T) : v_{|Q_t} \in H^1_0(Q_t)\Big\},$$

so that (S_1) and (S_4) are equivalent to require that

$$u \in L^2(0,T;\mathscr{V}) := \Big\{v \in L^2(0,T;V) : v(t) \in V_t \quad \text{for a.e. } t \in]0,T[\Big\}, \tag{1.6}$$

and the variational formulation of (S) reads

$$\int_0^T \Big(-\big(u(t),\phi'(t)\big)_H + ia(t;u(t),\phi(t))\Big)\,dt = \int_0^T \big(f(t),\phi(t)\big)_H\,dt,$$

for every function $\phi \in L^2(0,T;\mathscr{V})$ with $\phi' \in L^2(0,T;V)$ and $\phi(0) = \phi(T) = 0$. Here $a(t;\cdot,\cdot)$ is the sesquilinear form associated to the operator A

$$a(t;u,v) := \int_{Q_T} \Big(\sum_{i,j} a_{ij}(x,t)\frac{\partial u}{\partial x_i}\overline{\frac{\partial v}{\partial x_j}} + c(x,t)u(x)\overline{v(x)}\Big)\,dx.$$

Finally, (S_3) and (S_5) can be easily expressed in the abstract framework, by using the norm of H.

We will collect in the next section the precise formalization of the abstract machinery; here we want to stress that the crucial assumption (1.2) is equivalent to a monotonicity property for the family $\mathscr{V}$, i.e.

$$s < t \quad \Rightarrow \quad V_s \subseteq V_t.$$

1*d* ***Cauchy–mixed problems in cylindrical domains.*** The abstract theory we will develop in the next sections also applies to Cauchy–mixed problems for linear Schroedinger type P.D.E.'s in a domain $Q := Q_0 \times]0, T[$: in this case Dirichlet–Neumann boundary conditions can be considered on *non-cylindrical* regions of the lateral boundary $\partial Q_0 \times]0, T[$. For the sake of brevity, we only refer to [4], where this kind of application is carefully detailed.

1*e* ***Plan of the paper.*** In the next section we will introduce the abstract framework, we will recall some basic results we need in the sequel, and we will make precise our problem. § 3 is devoted to study the *uniqueness* of the variational solutions under the conservativity condition; the last section contains the statement and the proof of the basic existence result.

2 Abstract formulation of the problem.

2*f* ***Hilbert triplet and vector function spaces.*** Let (as, e.g., in [11])

$$V \subseteq H \equiv H^* \subseteq V^*, \text{ with } V \text{ separable}, \tag{2.1}$$

be a standard triplet of *complex* Hilbert spaces. $(\cdot, \cdot)$ denotes both the scalar product in H (with norm $|\cdot|$) and the anti-duality pairing between V^* and V. $(\!(\cdot, \cdot)\!)$ and $\|\cdot\|$ are the scalar product and the related norm in V, $\|\cdot\|_*$ is the dual norm of V^*, defined by

$$\|v^*\|_* := \max\Big\{ \big|(v^*, v)\big| : v \in V, \ \|v\| \le 1 \Big\} \quad \forall\, v^* \in V^*.$$

The Riesz surjective isomorphism $J : V \to V^*$ is then defined by

$$(Jv, w) := (\!(v, w)\!) \quad \forall v, w \in V. \tag{2.2}$$

For a given $T \in (0, +\infty)$ and a complex Banach space X, we will denote by $C^{k,\theta}([0, T]; X)$, $L^p(0, T; X)$, $W^{k,p}(0, T; X)$ the usual Banach spaces of X-valued functions, with $1 \le p \le +\infty$, $0 \le \theta \le 1$ and $k \in \mathbb{N}$.

If $1 \le p < +\infty$ and p' denotes the conjugate exponent of p, we identify the anti-dual space of $L^p(0, T; X)$ with $L^{p'}(0, T; X^*)$.

The space

$$W^{1,2}(0, T; V, V^*) := \Big\{ v \in L^2(0, T; V) : \exists \tfrac{d}{dt} v \in L^2(0, T; V^*) \Big\} \tag{2.3}$$

will play an important role in the following; we recall that

$$v \in W^{1,2}(0, T; V, V^*) \quad \Rightarrow \quad v \in C^0([0, T]; H), \tag{2.4}$$

the map $t \mapsto |v(t)|^2$ is absolutely continuous and the differentiation rule holds

$$\tfrac{d}{dt} |v(t)|^2 = 2 \operatorname{Re} \big(\tfrac{d}{dt} v(t), v(t) \big) \quad \text{a.e. in }]0, T[. \tag{2.5}$$

2g ***Sesquilinear forms and time dependent operators.*** For $t \in [0,T]$ let us give

$$\text{a measurable family of sesquilinear forms } a(t;\cdot,\cdot) : V \times V \to \mathbb{C} \qquad (A_1)$$

which are *hermitian*

$$a(t;v,w) = \overline{a(t;w,v)} \quad \forall\, v,w \in V, \text{ a.e. in }]0,T[\,, \qquad (A_2)$$

and uniformly *bounded*

$$\exists\, M > 0 : \quad |a(t;v,w)| \le M\|v\|\,\|w\| \quad \forall\, v,w \in V, \quad \text{for a.e. } t \in,]0,T[\,. \quad (A_3)$$

We will adopt the general convention of denoting by $a(t;\cdot)$ the associated quadratic forms

$$a(t;v) := a(t;v,v) \quad \forall\, v \in V, \qquad (2.6)$$

and we will assume that they satisfy the *weak uniform coercivity condition*

$$\exists\, \alpha > 0, \lambda \in \mathbb{R} : \quad a(t;v) \ge \alpha\|v\|^2 - \lambda|v|^2 \quad \forall v \in V, \text{ a.e. in }]0,T[\,. \qquad (A_4)$$

Finally, we will say that $a(t;\cdot,\cdot)$ is *absolutely continuous* if there exists $N \in L^1(0,T)$ s.t.

$$\begin{array}{c} \textit{the function } t \mapsto a(t;v,w) \textit{ belongs to } W^{1,1}(0,T) \text{ and} \\ |a'(t;v,w)| \le N(t)\, ||v||\, ||w||, \quad \forall\, v,w \in V, \quad \textit{a.e. in }]0,T[\,. \end{array} \qquad (A_5)$$

We associate to a the family of linear operators

$$A(t) : v \in V \mapsto A(t)v \in V^* \quad (A(t)v,w) := a(t;v,w) \quad \forall\, v,w \in V, \qquad (2.7)$$

and we observe that (A_1, A_3) yield

$$u \in L^2(0,T;V) \quad \Rightarrow \quad Au \in L^2(0,T;V^*), \qquad (2.8)$$

where we denoted by Au the map $t \mapsto A(t)u(t)$.

2h ***Abstract Schroedinger equations in a constant domain.*** We summarize in the following Proposition the basic existence result for the solution to the Cauchy problem for an abstract Schroedinger type equation in a fixed domain (see, e.g., [11, 9], and in particular [13]).

Proposition 2.1 *Let*

$$u_0 \in V \quad \textit{and} \quad f \in W^{1,1}(0,T;V^*) \qquad (2.9)$$

be given and let us assume $(A_1), \ldots, (A_5)$ *hold. Then there exists a unique solution*

$$u \in C^0([0,T];V) \cap C^1([0,T];V^*) \qquad (2.10)$$

to the Cauchy problem

$$\begin{cases} \frac{d}{dt}u(t) + iA(t)u(t) = f(t) \quad \textit{a.e. in }]0,T[\,, \\ \qquad\qquad u(0) = u_0. \end{cases} \qquad (2.11)$$

Moreover, for every $t \in [0,T]$, u satisfies the conservation identities

$$\tfrac{1}{2}|u(t)|^2 = \tfrac{1}{2}|u_0|^2 + \mathrm{Re}\int_0^t (f(\tau), u(\tau))\, d\tau, \tag{2.12}$$

$$\begin{aligned} \tfrac{1}{2}a(t;u(t)) - \mathrm{Im}(f(t),u(t)) = \tfrac{1}{2}a(0;u_0) - \mathrm{Im}(f(0),u_0) + \\ + \int_0^t \Big(\tfrac{1}{2}a'(\tau;u(\tau)) - \mathrm{Im}(f'(\tau),u(\tau))\Big)\, d\tau. \end{aligned} \tag{2.13}$$

Remark 2.2 The first conservation property (2.12) is an immediate consequence of (2.4), (2.5), and (A_2); it is useful to prove the *uniqueness* of the solution to (2.11) under weaker assumptions. It is easy to see that if $u_0 \in H$ and $f \in L^2(0,T;V^*)$, then any two corresponding solutions $u_1, u_2 \in W^{1,2}(0,T;V,V^*)$ of (2.11) must coincide. $\square$

2*i* ***Time-dependent Hilbert spaces.*** We are given

$$\text{a family } \mathscr{V} := \{V(t)\}_{t\in[0,T]} \text{ of closed subspaces of } V, \tag{V_1}$$

and we will always assume that $\mathscr{V}$ is *non–decreasing*, i.e.

$$s \le t \quad \Rightarrow \quad V(s) \subseteq V(t) \quad \forall s,t \in [0,T]. \tag{V_2}$$

In particular, (V_2) implies that $\mathscr{V}$ is a *measurable* family (cf. [8, Thm. 3.9]) and the orthogonal projection operators

$$\pi(t) : V \to V(t), \quad (\!(\pi(t)v, w)\!) := (\!(v,w)\!) \quad \forall\, v \in V,\ w \in V(t) \tag{2.14}$$

are *weakly* measurable, i.e.

$$t \in [0,T] \mapsto (\!(\pi(t)v, w)\!) \quad \text{is measurable for every } v, w \in V. \tag{2.15}$$

We associate to $\{\pi(t)\}_{t\in[0,T]}$ the adjoint family $\{\pi^*(t)\}_{t\in[0,T]}$ of linear operators on V^* with respect to the anti-duality pairing $(\cdot,\cdot)$ between V^* and V; they are defined by

$$(\pi^*(t)v^*, v) := (v^*, \pi(t)v) \quad \forall v^* \in V^*,\ v \in V,\ t \in [0,T]. \tag{2.16}$$

It is readily seen that $\pi^*(t)$ is a projection operator on V^* and satisfies $\pi^*(t) = J\pi(t)J^{-1}$; we define

$$V^*(t) := \pi^*(t)V^*, \quad \mathscr{V}^* := \{V^*(t)\}_{t\in[0,T]},$$

and we observe that $V^*(t)$ is isomorphic to the anti-dual space of $V(t)$. For $p \in [1,+\infty]$, we denote by $L^p(0,T;\mathscr{V})$ the closed subspace of $L^p(0,T;V)$ defined by

$$L^p(0,T;\mathscr{V}) := \big\{v \in L^p(0,T;V) : v(t) \in V(t), \text{ a.e. in } (0,T)\big\}; \tag{2.17}$$

an analogous definition holds for $L^p(0,T;\mathscr{V}^*)$. (2.15) ensures that the linear operators

$$\begin{aligned} \Pi : v \in L^p(0,T;V) \mapsto v_{\mathscr{V}}, \quad v_{\mathscr{V}}(t) := \pi(t)v(t), \\ \Pi^* : v^* \in L^p(0,T;V^*) \mapsto v^*_{\mathscr{V}^*}, \quad v^*_{\mathscr{V}^*}(t) := \pi^*(t)v^*(t) \end{aligned} \tag{2.18}$$

are linear surjections on $L^p(0,T;\mathscr{V})$ and $L^p(0,T;\mathscr{V}^*)$ respectively. Similarly, we define

$$A_{\mathscr{V}^*}(t) := \pi^*(t)A(t), \quad (A_{\mathscr{V}^*}(t)v,w) = a(t;v,\pi(t)w) \quad \forall\, v,w \in V, \tag{2.19}$$

and we observe that if $u \in L^2(0,T;V)$ then the function $t \mapsto A_{\mathscr{V}^*}(t)u(t)$ belongs to $L^2(0,T;\mathscr{V}^*)$.

Remark 2.3 The choice of the scalar product of V induces an isomorphism between the anti-dual space of $L^p(0,T;\mathscr{V})$ and $L^{p'}(0,T;\mathscr{V}^*)$. In order to avoid the notion of "direct integrals of Hilbert spaces" (cf. [11, Ch.I, 2.3]), we will directly deal with $L^{p'}(0,T;\mathscr{V}^*)$, even if the above isomorphism is not intrinsic. $\square$

2j ***Weak derivative w.r.t. $\mathscr{V}^*$.*** In order to formulate the abstract differential equation in time-dependent domains we need firstly to define a suitable notion of weak derivative with respect to the family $\mathscr{V}^*$; for simplicity we limit us to consider the case $p = 2$.

Definition 2.4 *We say that $\xi^* \in L^2(0,T;V^*)$ is a* weak *derivative of $v^* \in L^2(0,T;V^*)$ w.r.t. $\mathscr{V}^*$ if*

$$\int_0^T (v^*(t), \tfrac{d}{dt}\phi(t))\,dt = -\int_0^T (\xi^*(t),\phi(t))\,dt \tag{2.20}$$

for every function $\phi \in W^{1,2}(0,T;V) \cap L^2(0,T;\mathscr{V})$ with $\phi(0) = \phi(T) = 0$.
If ξ^* is a weak *derivative of v^* w.r.t. $\mathscr{V}^*$ we set $\frac{d^*}{dt}v^* := \Pi^*\xi^*$. As in* (2.3) *we denote by $W^{1,2}(0,T;\mathscr{V},\mathscr{V}^*)$ the Hilbert space*

$$W^{1,2}(0,T;\mathscr{V},\mathscr{V}^*) := \Big\{v \in L^2(0,T;\mathscr{V}) : \exists \tfrac{d^*}{dt}v \in L^2(0,T;\mathscr{V}^*)\Big\} \tag{2.21}$$

endowed with the natural norm.

Remark 2.5 It is clear that $\frac{d^*}{dt}v^*$ is independent of the choice of ξ^* among all the weak derivatives of v^* w.r.t. $\mathscr{V}^*$; in §3 we will give an alternative carachterization of $\frac{d^*}{dt}v^*$ in terms of suitable limits of *right difference quotients.*
Moreover, if $v^* \in W^{1,2}(0,T;V^*)$ then it is easily seen that $\frac{d^*}{dt}v^*$ exists and coincides with the Π^*-projection of the usual time derivative

$$\tfrac{d^*}{dt}v^* = \Pi^*\tfrac{d}{dt}v^* \quad \forall\, v^* \in W^{1,2}(0,T;V^*);$$

the above formula justifies the notation we have introduced for $\frac{d^*}{dt}v^*$. Finally, if v_n^* is a sequence in $W^{1,2}(0,T;V^*)$, then

$$v_n^* \rightharpoonup v^*,\ \Pi^*\tfrac{d}{dt}v_n^* \rightharpoonup \xi^* \text{ in } L^2(0,T;V^*) \quad \Rightarrow \quad \exists\, \tfrac{d^*}{dt}v^* = \xi^* \in L^2(0,T;\mathscr{V}^*). \quad \square \tag{2.22}$$

Remark 2.6 As we will see in the next section, in general we cannot replace $W^{1,2}(0,T;V,V^*)$ with $W^{1,2}(0,T;\mathscr{V},\mathscr{V}^*)$ in (2.4) and (2.5); the non-decreasing condition (V_2) and $v \in W^{1,2}(0,T;\mathscr{V},\mathscr{V}^*)$ will only imply that

$$\tfrac{d}{dt}|v(t)|^2 \geq 2\,\mathrm{Re}\left(\tfrac{d^*}{dt}v(t), v(t)\right) \quad \text{in the sense of distributions on }]0,T[. \quad \square \tag{2.23}$$

2k ***The abstract Cauchy problem; conservative solutions.*** *Let us give*

$$u_0 \in \overline{V(0)}^H, \quad f \in L^2(0,T;V^*). \tag{CP_1}$$

We ask for

$$u \in W^{1,2}(0,T;\mathscr{V},\mathscr{V}^*) \tag{CP_2}$$

satisfying the initial Cauchy condition "$u(0) = u_0$" in the integral sense

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-1} \int_0^\varepsilon |u(t) - u_0|^2 \, dt = 0, \tag{CP_3}$$

and such that

$$\tfrac{d^*}{dt} u(t) + iA_{\mathscr{V}^*}(t)u(t) = f_{\mathscr{V}^*}(t) \quad \text{a.e. in }]0,T[\,. \tag{CP_4}$$

Furthermore, we say that u is a conservative solution (cf. (2.12)) *if it also satisfies*

$$|u(t)|^2 = |u_0|^2 + 2\,\mathrm{Re} \int_0^t \big(f(\tau), u(\tau)\big)\, d\tau \quad \text{for a.e. } t \in]0,T[\,. \tag{CP_5}$$

Remark 2.7 $u \in W^{1,2}(0,T;\mathscr{V},\mathscr{V}^*)$ and (V_2) do not imply, in general, that $u \in C^0([0,T];H)$ and even $u \in L^\infty(0,T;H)$ (cf. Remark 2.6); this fact motivates the weak form (CP_3) of the Cauchy condition and the conservative condition (CP_5). On the other hand, notice that conservative solutions belong to $L^\infty(0,T;H)$. $\square$

We can easily write an equivalent variational formulation of (CP_4) in an integral form: in fact $u \in L^2(0,T;\mathscr{V})$ satisfies (CP_4) iff

$$\int_0^T \Big(-\big(u(t), \phi'(t)\big) + ia\big(t; u(t), \phi(t)\big)\Big)\, dt = \int_0^T \big(f(t), \phi(t)\big)\, dt \tag{2.24}$$

for every test function $\phi \in W^{1,2}(0,T;V) \cap L^2(0,T;\mathscr{V})$ with $\phi(0) = \phi(T) = 0$.

3 Uniqueness of conservative solutions and partial regularity.

In this section we will prove that the set of conservative solutions of the Cauchy problems $(CP_2), \ldots, (CP_4)$ with respect to every choice of data u_0, f as in (CP_1) is in fact a *linear* space, which could be characterized by means of a suitable interpolation space of Besov type. In particular, since the only conservative solution of the homogeneous problem (corresponding to $u_0 = 0, f = 0$) is the trivial one, we will prove the following uniqueness result.

Theorem 2 *Let us assume (V_1, V_2) and $(A_1), \ldots, (A_3)$; if u_1, u_2 are* conservative solutions *of the Cauchy problem $(CP_1), \ldots, (CP_5)$, then $u_1(t) = u_2(t)$ a.e. in $(0,T)$.*

We will split our arguments in some steps.

3l ***Weak derivatives and difference quotients.*** An alternative characterization of the existence of a weak derivative w.r.t. $\mathscr{V}^*$ can be given by taking the limit of right difference quotients.

Lemma 3.1 *Let us assume that $v \in L^2(0,T;V^*)$ has a weak derivative $\xi^* \in L^2(0,T;V^*)$ w.r.t. $\mathscr{V}^*$, and let us define v_h, for $0 < h < T$, as*

$$v_h(t) := \begin{cases} h^{-1}(v(t+h) - v(t)) & \textit{a.e. in }]0, T-h[\\ 0 & \textit{in } [T-h, T[\,. \end{cases}$$

Then $\Pi^* v_h$ converges strongly to $\Pi^*\xi^* = \frac{d^*}{dt} v$ in $L^2(0,T;V^*)$ as $h \downarrow 0$.

Proof. Let us take $\varepsilon \in]0, T/2[$ and $\phi \in L^2(0,T;V)$ with $\phi(t) \equiv 0$ a.e. in $]0,\varepsilon[\cup]T-\varepsilon, T[$, and for $0 < h < \varepsilon$, let us set

$$\psi := \Pi\phi, \quad \Psi_h(t) := \begin{cases} h^{-1}\int_{t-h}^t \psi(s)\,ds & \text{if } h \le t \le T, \\ 0 & \text{if } 0 \le t < h. \end{cases}$$

Let us observe that $\Psi_h \in W_0^{1,2}(0,T;V) \cap L^2(0,T;\mathscr{V})$ and $\lim_{h\downarrow 0} \Psi_h = \psi$ in $L^2(0,T;V)$. We have

$$\begin{aligned} \int_0^T (\pi^*(t)v_h(t), \phi(t))\,dt &= \int_0^{T-h} (v_h(t), \psi(t))\,dt = \\ &= h^{-1}\Big(\int_h^T (v(t), \psi(t-h))\,dt - \int_0^{T-h} (v(t), \psi(t))\,dt \Big) = \\ &= h^{-1} \int_h^T (v(t), \psi(t-h) - \psi(t))\,dt = \\ &= -\int_h^T (v(t), \Psi_h'(t))\,dt = \\ &= -\int_0^T (v(t), \Psi_h'(t))\,dt = \int_0^T (\xi^*(t), \Psi_h(t))\,dt. \end{aligned}$$

Passing to the limit as $h \downarrow 0$ we get

$$\begin{aligned} \lim_{h\downarrow 0} \int_0^T (\pi^*(t)v_h(t), \phi(t))\,dt &= \int_0^T (\xi^*(t), \psi(t))\,dt = \int_0^T (\xi^*(t), \pi(t)\phi(t))\,dt = \\ &= \int_0^T (\pi^*(t)\xi^*(t), \phi(t))\,dt. \end{aligned}$$

As ε and ϕ are arbitrary, we conclude that $\pi^*(t)v_h(t)$ converges weakly to $\frac{d^*}{dt}v$ in $L^2(0,T;V^*)$. Since, by Jensen's inequality,

$$\|\Psi_h\|_{L^2(0,T;V)} \le \|\psi\|_{L^2(0,T;V)},$$

the previous calculations show that for every $h \in (0,T)$

$$\begin{aligned} \left| \int_0^T (\pi^*(t)v_h(t), \phi(t))\,dt \right| &\le \|\xi^*\|_{L^2(0,T;V^*)} \|\Psi_h\|_{L^2(0,T;V)} \le \\ &\le \|\xi^*\|_{L^2(0,T;V^*)} \|\psi\|_{L^2(0,T;V)} \le \\ &\le \|\xi^*\|_{L^2(0,T;V^*)} \|\phi\|_{L^2(0,T;V)}, \end{aligned}$$

so that

$$\|\Pi^* v_h\|_{L^2(0,T;V^*)} \le \|\tfrac{d^*}{dt} v\|_{L^2(0,T;V^*)}.$$

This relation and the lower semicontinuity of the $L^2(0,T;V^*)$-norm w.r.t. weak convergence yield

$$\lim_{h\downarrow 0} \|\Pi^* v_h\|_{L^2(0,T;V^*)} = \|\tfrac{d^*}{dt} v\|_{L^2(0,T;V^*)},$$

and therefore we deduce the strong convergence

$$\Pi^* v_h \to \tfrac{d^*}{dt} v \quad \text{in } L^2(0,T;V^*). \quad \blacksquare$$

Now we want to apply the previous Lemma to find an explicit formula for the time derivative of the function $t \mapsto |v(t)|^2$ when $v \in W^{1,2}(0,T;\mathscr{V},\mathscr{V}^*)$.

Lemma 3.2 *Let v be given in $W^{1,2}(0,T;\mathscr{V},\mathscr{V}^*)$; for every couple of Lebesgue points $s < t$ of v w.r.t. H we have*

$$\begin{aligned} |v(t)|^2 - |v(s)|^2 - 2\,\mathrm{Re} \int_s^t (\tfrac{d^*}{dt} v(\tau), v(\tau))\, d\tau &= \\ = \lim_{h\downarrow 0} \frac{1}{h} \int_s^{t-h} |v(\tau+h) - v(\tau)|^2\, d\tau. \end{aligned} \tag{3.1}$$

Proof. A straightforward computation shows that

$$2\,\mathrm{Re}(v(\tau) - v(\tau+h), v(\tau)) = |v(\tau+h) - v(\tau)|^2 - |v(\tau+h)|^2 + |v(\tau)|^2.$$

Now, for any h such that $0 < h < t - s$, it follows easily that

$$\begin{aligned} &2\,\mathrm{Re} \int_s^{t-h} (v(\tau) - v(\tau+h), v(\tau))\, d\tau = \\ &= \int_s^{t-h} |v(\tau+h) - v(\tau)|^2\, d\tau - \int_{s+h}^t |v(\tau)|^2\, d\tau + \int_s^{t-h} |v(\tau)|^2\, d\tau = \\ &= \int_s^{t-h} |v(\tau+h) - v(\tau)|^2\, d\tau - \int_{t-h}^t |v(\tau)|^2\, d\tau + \int_s^{s+h} |v(\tau)|^2\, d\tau, \end{aligned}$$

so that

$$\begin{aligned} &⨍_{t-h}^t |v(\tau)|^2\, d\tau - ⨍_s^{s+h} |v(\tau)|^2\, d\tau = \\ &= 2\,\mathrm{Re} \int_s^{t-h} \left(\frac{v(\tau+h) - v(\tau)}{h}, v(\tau) \right) d\tau + \frac{1}{h} \int_s^{t-h} |v(\tau+h) - v(\tau)|^2\, d\tau. \end{aligned} \tag{3.2}$$

Taking as s, t arbitrary Lebesgue points of v and using Lemma 3.1 we get (3.1). $\blacksquare$

The right-hand side of (3.1) is a sort of measure of the deviation of v from satisfaction of the identity $\frac{d}{dt}|v|^2 = 2\,\mathrm{Re}(\frac{d^*}{dt} v, v)$. Since it will play an important role in the following, we introduce the notation

$$\eta_v(t) := \limsup_{h\downarrow 0} h^{-1} \int_0^{t-h} |v(\tau+h) - v(\tau)|^2\, d\tau; \tag{3.3}$$

we observe that $t \mapsto \eta_v(t)$ is a non-decreasing function

$$s \le t \quad \Rightarrow \quad \eta_v(s) \le \eta_v(t), \tag{3.4}$$

so that its distributional derivative

$$\frac{d}{dt}\eta_v(t) \quad \textit{is a non-negative Radon measure}; \tag{3.5}$$

in particular, thanks to (3.1) we find (2.23).

Corollary 3.3 *If u is a solution to the Cauchy problem $(CP_1), \ldots, (CP_4)$ then for every Lebesgue point $t \in]0, T[$ of u w.r.t. H we have*

$$\begin{aligned} |u(t)|^2 - |u_0|^2 - 2\,\mathrm{Re} \int_0^t (f(\tau), u(\tau))\, d\tau = \\ = \lim_{h\downarrow 0} \frac{1}{h} \int_0^{t-h} |u(\tau+h) - u(\tau)|^2\, d\tau = \eta_u(t). \end{aligned} \tag{3.6}$$

Proof. From (CP_4) and (A_2) we get

$$\mathrm{Re}\left(\tfrac{d^*}{dt}u(t), u(t)\right) = \mathrm{Re}\left(f(t), u(t)\right) \quad \text{a.e. in } (0,T). \tag{3.7}$$

Substituting this relation in (3.1) for $s := 0$ we conclude. ■

3m H-boundedness and Besov regularity of conservative solutions.

Proposition 3.4 *If u is a solution to the Cauchy problem $(CP_1), \ldots, (CP_4)$ then the following three properties are equivalent:*

$$u \in L^\infty(0,T;H), \tag{3.8a}$$

$$\eta_u(T) = \limsup_{h\downarrow 0} \frac{1}{h} \int_0^{T-h} |u(t+h) - u(t)|^2\, dt < +\infty, \tag{3.8b}$$

$$\begin{gathered} \exists \textit{ an increasing sequence } n \mapsto t_n \textit{ of Lebesgue points of } u \textit{ such that} \\ \lim_{n\uparrow+\infty} t_n = T, \quad \sup_{n\in\mathbb{N}} |u(t_n)| < +\infty, \end{gathered} \tag{3.8c}$$

in particular, u belongs to $L^\infty(0,T';H)$ for every $T' < T$. Moreover, u is conservative if and only if

$$\eta_u(T) = \lim_{h\downarrow 0} \frac{1}{h} \int_0^{T-h} |u(t+h) - u(t)|^2\, dt = 0. \tag{3.9}$$

Proof. It is obvious that (3.8b) is equivalent to (3.8a) thanks to (3.6) and (3.4), and that they imply (3.8c).

To check the inverse implication, we observe that (3.8c) yields

$$\sup_{n\in\mathbb{N}} \eta_u(t_n) < +\infty,$$

and therefore $u \in L^\infty(0,t_n;H)$ with uniform bound of the norm. Since $t_n \uparrow T$, it follows that $u \in L^\infty(0,T;H)$.

In order to prove (3.9), let us observe that from (CP_5) we deduce

$$\lim_{h\downarrow 0} \fint_{T-h}^{T} |u(\tau)|^2\, d\tau = |u_0|^2 + 2\,\mathrm{Re} \int_0^T \big(f(\tau), u(\tau)\big)\, d\tau$$

since the map

$$t \mapsto |u_0|^2 + 2\,\mathrm{Re} \int_0^t \big(f(\tau), u(\tau)\big)\, d\tau$$

is continuous; inserting this formula in (3.2) with $t := T$ and $s := 0$, and recalling (3.7) again, we get (3.9). ■

Remark 3.5 The previous argument shows that every solution u of $(CP_1), \ldots, (CP_4)$ satisfies

$$\eta_u(t) = 0 \quad \forall\, t < T \quad \Rightarrow \quad \eta_u(T) = 0. \tag{3.10}$$

This property does not hold for general functions: take, e.g., $H := \mathbb{R}$, $u(t) := (T-t)^{1/2}$. □

Corollary 3.6 *If u, v are conservative solutions to Cauchy problem (CP_2),..., (CP_5) w.r.t. data u_0, v_0 and f, g as in (CP_1), then for every $\alpha, \beta \in \mathbb{C}$ the linear combination $\alpha u + \beta v$ is a conservative solution w.r.t. $\alpha u_0 + \beta v_0$ and $\alpha f + \beta g$.*

Proof. We simply apply (3.9) by noticing that

$$\eta_u(T) = \eta_v(T) = 0 \quad \Rightarrow \quad \eta_{\alpha u + \beta v}(T) = 0 \quad ■$$

The proof of Theorem 2 is now complete.

Proposition 3.4 suggests a different way to select conservative solutions among the other ones.

For $0 < h < T$ let us introduce the family of seminorms in $L^2(0, T; H)$

$$[v]_h := \|v(t+h) - v(t)\|_{L^2(0,T-h;H)} = \Big(\int_0^{T-h} |v(t+h) - v(t)|^2\, dt\Big)^{1/2};$$

we recall that the Besov space $B^\theta_{2,\infty}(0, T; H)$, $0 < \theta < 1$, can be defined by [3, 16]

$$B^\theta_{2,\infty}(0, T; H) := \Big\{ v \in L^2(0, T; H) : \sup_{0<h<T} h^{-\theta}[v]_h < +\infty \Big\} \tag{3.11}$$

equipped by the natural norm. $W^{1,2}(0, T; H)$ is continuously embedded into $B^\theta_{2,\infty}(0, T; H)$ for every $\theta \in (0, 1)$; its closure is called $B^{\theta,0}_{2,\infty}(0, T; H)$ and it can be characterized as follows

$$B^{\theta,0}_{2,\infty}(0, T; H) = \Big\{ v \in B^\theta_{2,\infty}(0, T; H) : \lim_{h\downarrow 0} h^{-\theta}[v]_h = 0 \Big\}. \tag{3.12}$$

Since in the case $\theta = 1/2$ we find

$$\lim_{h\downarrow 0} h^{-\theta}[v]_h = \Big(\eta_v(T)\Big)^{1/2}$$

it is not surprising that Proposition 3.4 can be restated in the following way.

Corollary 3.7 *A solution u of the Cauchy problem $(CP_1), \ldots, (CP_4)$ belongs to $L^\infty(0,T;H)$ if and only if it belongs to $B^{1/2}_{2,\infty}(0,T;H)$; moreover, it is conservative if and only if it belongs to $B^{1/2,0}_{2,\infty}(0,T;H)$.*

We conclude this section with another simple criterion of conservativity:

Lemma 3.8 *Let u be a solution to $(CP_1), \ldots, (CP_4)$; if there exists a sequence $n \in \mathbb{N} \mapsto t_n \in [0,T]$ of H-Lebesgue points of u converging to T as $n \uparrow +\infty$ and satisfying*

$$\limsup_{n\uparrow+\infty} |u(t_n)|^2 \le |u_0|^2 + 2\,\mathrm{Re} \int_0^T \big(f(\tau), u(\tau)\big)\, d\tau, \tag{3.13}$$

then u satisfies (CP_5), too.

Proof. By Remark 3.5, it is sufficient to prove that $\eta_u(t) = 0$ for every $t < T$. Since $t_n \uparrow T$, by (3.4) and (3.6), we have easily for $t < T$

$$\begin{aligned}
\eta_u(t) &\le \limsup_{n\uparrow+\infty} \eta_u(t_n) = \\
&= \limsup_{n\uparrow+\infty} \Big(|u(t_n)|^2 - |u_0|^2 - 2\,\mathrm{Re} \int_0^{t_n} \big(f(\tau), u(\tau)\big)\, d\tau \Big) = \\
&= \Big(\limsup_{n\uparrow+\infty} |u(t_n)|^2 \Big) - |u_0|^2 - 2\,\mathrm{Re} \int_0^T \big(f(\tau), u(\tau)\big)\, d\tau \le 0. \quad \blacksquare
\end{aligned}$$

4 Existence of conservative solutions and further regularity.

In this section we will show that Proposition 2.1 can be suitably extended to the case of non-decreasing domains.

Theorem 3 *Let us assume that (V_1, V_2) and $(A_1), \ldots, (A_5)$ hold. Then, for any*

$$u_0 \in V(0), \quad f \in W^{1,1}(0,T;V^*) \tag{4.1}$$

there exists a unique (conservative) solution u to the Cauchy problem $(CP_2), \ldots, (CP_5)$. Moreover,

u belongs to $C^{1/2}([0,T];H)$,
$u(t) \in V(t)$ for every $t \in [0,T]$,
u is continuous w.r.t. the weak topology of V,
u is right-continuous w.r.t. the strong one at every point $t \in [0,T[$.

We will carry out the proof of Theorem 3 in several steps, by using, as a main tool, a suitable procedure of penalization.

4*n* ***Regularized projections.*** Firstly, we extend the definition of $\mathscr{V}$ (and, correspondingly, of π, π^*, etc.) for $t > T$ by setting

$$V(t) := V(T), \quad \forall t > T, \tag{4.2}$$

and we observe that also this extended family is non-decreasing with t.

Let us consider the operator function $t \in [0, +\infty[\mapsto P(t) \in \mathcal{L}(V)$ defined by

$$P(t) : v \in V \mapsto v - \pi(t)v \quad \big(\text{hence} \quad v \in V(t) \quad \Leftrightarrow \quad P(t)v = 0\big), \tag{4.3}$$

where π is defined by (2.14), together with the associated hermitian sesquilinear forms

$$\wp(t; v, w) := (\!(P(t)v, w)\!) = (\!(P(t)v, P(t)w)\!) = (\!(v, P(t)w)\!) \quad \forall\, v, w \in V. \tag{4.4}$$

$P(t)$ is the orthogonal projection onto

$$V(t)^{\perp} := \Big\{ v \in V : (\!(v, w)\!) = 0 \quad \forall\, w \in V(t) \Big\}.$$

As $\mathscr{V}$ is non-decreasing, it is obvious that

$$\mathscr{V}^{\perp} := \{V(t)^{\perp}\}_{t>0} \quad \text{is a non-increasing family with } t. \tag{4.5}$$

In particular, the following relations hold

$$s \le t \quad \Rightarrow \quad P(s)P(t) = P(t)P(s) = P(t), \quad \pi(s)P(t) = P(t)\pi(s) = 0, \tag{4.6}$$

and (cf. (2.6))

$$s \le t \quad \Rightarrow \quad \wp(s; v) = \|P(s)v\|^2 \ge \wp(t; v) = \|P(t)v\|^2 \quad \forall\, v \in V. \tag{4.7}$$

We need to approximate P by more regular maps w.r.t. time. Observe that the monotonicity property (4.7) and the well known polarization formula for hermitian sesquilinear forms ensure that $t \mapsto \wp(t; v, w)$ is *measurable* for every $v, w \in V$. We define, for $k \in \mathbb{N}^+$, $t \ge 0$,

$$\wp_k(t; v, w) := \fint_0^{1/k} \wp(t + \tau; v, w)\, d\tau \quad \forall\, v, w \in V, \tag{4.8}$$

and we denote by $P_k(t) : V \to V$ the associated linear operators defined by

$$(\!(P_k(t)v, w)\!) := \wp_k(t; v, w) \quad \forall\, v, w \in V. \tag{4.9}$$

Lemma 4.1 *Let us assume (V_1, V_2) hold, and let $\wp_k$ be defined by (4.8). Then, for any integers $1 \le h \le k$, for any $t \in [0, +\infty[$, and for every $v, w \in V$ it results that*

$$\wp_k(t; v, w) = \overline{\wp_k(t; w, v)}; \tag{4.10a}$$

$$\wp_h(t; v) \le \wp_k(t; v) \le \wp(t; v) = \|P(t)v\|^2 \le \|v\|^2; \tag{4.10b}$$

$$\wp_k'(t; v) := \frac{d}{dt}\wp_k(t; v) = k\big(\wp(t + 1/k; v) - \wp(t; v)\big) \le 0; \tag{4.10c}$$

$$(\!(P_k(t)v, z)\!) = 0 \quad \forall\, z \in V(t). \tag{4.10d}$$

Moreover, for every sequence $v_k \in L^2(0, T; V)$ weakly converging to v in $L^2(0, T; V)$ we have:

$$\liminf_{k\uparrow+\infty} \int_0^T \wp_k(t; v_k(t))\, dt \ge \int_0^T \wp(t; v(t))\, dt. \tag{4.10e}$$

Proof. (4.10a) follows from (4.4), (4.8), and the corresponding property of the scalar product $(\!(\cdot,\cdot)\!)$.

(4.10b) follows immediately by (4.7), (4.10c) is a direct consequence of (4.7) and (4.8), while (4.10d) follows from (4.5), (4.8), and (4.9)

In order to prove (4.10e), let us observe that Lebesgue differentiation Theorem, (4.10b), and the separability of V yields

$$\lim_{h\uparrow+\infty} \wp_h(t;w) = \wp(t;w) \quad \forall\, w \in V, \quad \text{a.e. in }]0,T[\,.$$

Via Lebesgue dominated convergence Theorem, we obtain

$$\lim_{h\uparrow+\infty} \int_0^T \wp_h(t;v(t))\,dt = \int_0^T \wp(t;v(t))\,dt \quad \forall\, v \in L^2(0,T;V). \tag{4.11}$$

Therefore, if $v_k \rightharpoonup v$ in $L^2(0,T;V)$, by (4.10b) and the weak lower semicontinuity of convex functionals, we have for every $h \in \mathbb{N}^+$

$$\liminf_{k\uparrow+\infty} \int_0^T \wp_k(t;v_k(t))\,dt \geq \liminf_{k\uparrow+\infty} \int_0^T \wp_h(t;v_k(t))\,dt \geq \int_0^T \wp_h(t;v(t))\,dt.$$

Since h is arbitrary, by applying (4.11) we get (4.10e). ∎

4o The penalized problem. Let us choose an increasing sequence $\{\varrho_k\}_{k\in\mathbb{N}^+}$ of positive real numbers, and let us define the family of penalizing sesquilinear forms

$$a_k(t;v,w) := a(t;v,w) + \varrho_k \wp_k(t;v,w) \quad \forall\, v,w \in V, \tag{4.12}$$

corresponding to the linear operators

$$A_k(t) := A(t) + \varrho_k\, JP_k(t).$$

It is easy to see that a_k satisfies $(A_1),\ldots,(A_5)$ w.r.t. the constants

$$\alpha_k = \alpha, \quad \lambda_k = \lambda, \quad M_k = M + \varrho_k, \quad N_k(t) = N(t) + 2k\varrho_k.$$

We consider the corresponding family of Cauchy problem (2.11), where we replace A by A_k, i.e. we are looking for the solution $u_k \in C^0([0,T];V) \cap C^1([0,T];V^*)$ of

$$\begin{cases} \frac{d}{dt}u_k(t) + iA_k(t)u_k(t) = f(t) \quad \text{a.e. in }]0,T[\,, \\ \qquad\qquad u_k(0) = u_0 \in V(0). \end{cases} \tag{4.13}$$

By Proposition 2.1 u_k is well defined and satisfies the conservation identities

$$\tfrac{1}{2}|u_k(t)|^2 = \tfrac{1}{2}|u_0|^2 + \operatorname{Re} \int_0^t (f(\tau), u_k(\tau))\,d\tau, \tag{4.14}$$

$$\begin{aligned} &\tfrac{1}{2}\Big(a\big(t;u_k(t)\big) + \varrho_k\wp_k\big(t;u_k(t)\big)\Big) - \operatorname{Im}\big(f(t),u_k(t)\big) = \\ &\quad = \tfrac{1}{2}a(0;u_0) - \operatorname{Im}(f(0),u_0) + \\ &\quad + \int_0^t \Big[\tfrac{1}{2}\Big(a'\big(\tau;u_k(\tau)\big) + \varrho_k\wp_k'\big(t;u_k(\tau)\big)\Big) - \operatorname{Im}\big(f'(\tau),u_k(\tau)\big)\Big]\,d\tau. \end{aligned} \tag{4.15}$$

By taking into account (4.10c) and by using a generalized Gronwall lemma (see, e.g., [1, 13]), we infer from (4.14) and (4.15) the following uniform estimates

$$\exists K > 0: \ \|u_k(t)\| \leq K, \quad \varrho_k\wp_k\big(t;u_k(t)\big) \leq K^2, \quad \forall t \in [0,T],\ \forall k \in \mathbb{N}^+; \tag{4.16}$$

in particular, the sequence $\{u_k\}_{k\geq1}$ is uniformly bounded in $L^\infty(0,T;V)$.

4p ***Hölder equicontinuity.*** We now adapt a technique developed in [2, 15] for parabolic problems and we derive another estimate for u_k, which gives the Hölder equicontinuity of order $\frac{1}{2}$ on $[0, T]$ with respect to the H-norm.

Lemma 4.2 *Let u_k be the solution of* (4.13)*; then there exists a constant $C > 0$ independent of k such that*

$$|u_k(t) - u_k(s)| \le C|t - s|^{1/2} \quad \forall\, s, t \in [0, T]. \tag{4.17}$$

Proof. Let us fix $s \in [0, T[$ and $\tau \in]s, T]$; we "multiply" both sides of (4.13), written at $t := \tau$, by $u_k(\tau) - u_k(s)$ and we take the *real parts*. So, we get

$$\begin{aligned} \mathrm{Re}(u_k'(\tau), u_k(\tau) - u_k(s)) - \mathrm{Im}\Big(a(\tau; u_k(\tau), u_k(\tau) - u_k(s)) + \\ + \varrho_k \wp_k(\tau; u_k(\tau), u_k(\tau) - u_k(s))\Big) = \\ = \mathrm{Re}(f(\tau), u_k(\tau) - u_k(s)), \end{aligned} \tag{4.18}$$

i.e., thanks also to (A_2), (4.10a),

$$\begin{aligned} \frac{1}{2}\frac{d}{d\tau}|u_k(\tau) - u_k(s)|^2 &= -\,\mathrm{Im}\, a(\tau; u_k(\tau), u_k(s)) \\ &\quad -\varrho_k\, \mathrm{Im}\, \wp_k(\tau; u_k(\tau), u_k(s)) + \mathrm{Re}(f(\tau), u_k(\tau) - u_k(s)). \end{aligned} \tag{4.19}$$

By considering the right-hand side of (4.19), we can estimate at once the first term and the third one. In fact, by using (4.16), we obtain the following uniform bound for a suitable constant C_1 (independent of k)

$$|a(\tau; u_k(\tau), u_k(s))| + |(f(\tau), u_k(\tau) - u_k(s))| \le C_1 \quad \forall \tau, s \in [0, T]. \tag{4.20}$$

Consider now the second term at the right-hand side of (4.19); by Schwarz inequality, (4.10c), (4.16), and $s < \tau$, we get

$$\begin{aligned} \varrho_k \Big|\wp_k\big(\tau; u_k(\tau), u_k(s)\big)\Big| &\le \varrho_k \Big(\wp_k\big(\tau; u_k(\tau)\big)\, \wp_k\big(\tau, u_k(s)\big)\Big)^{1/2} \le \\ &\le \varrho_k \Big(\wp_k\big(\tau; u_k(\tau)\big)\, \wp_k\big(s; u_k(s)\big)\Big)^{1/2} \le K^2. \end{aligned}$$

If we integrate both sides of (4.19) w.r.t. τ from s to $t > s$ we obtain

$$\frac{1}{2}|u_k(t) - u_k(s)|^2 \le (C_1 + K^2)(t - s). \quad \blacksquare$$

4q ***Passage to the limit.*** In order to pass to the limit, we apply Ascoli-Arzelà Theorem to the sequence $u_k : [0, T] \to E$, where (E, d) is the compact metrizable space

$$E := \{v \in V : \|v\| \le K\} \quad \text{endowed with the } \textit{weak} \text{ topology.}$$

The distance d in E can be defined as in [7, Thm. III.25]: we choose a sequence $\{h_n\}_{n\in\mathbb{N}}$ strongly dense in the unit ball of H and we set

$$d(v, w) := \sum_{n=1}^{\infty} 2^{-n}\big|(h_n, v - w)\big| \quad \forall\, v, w \in E. \tag{4.21}$$

d induces the weak topology on every bounded set of V and it satisfies

$$d(v,w) \le |v-w|. \tag{4.22}$$

We have

Corollary 4.3 *Let $u_k \in C^0([0,T];H)$ be a uniformly bounded family satisfying* (4.17)*; then there exists a subsequence (still denoted by u_k) and a function $u \in C^{1/2}([0,T];H)$, continuous w.r.t. the weak topology of V, such that*

$$u_k(t) \rightharpoonup u(t) \quad \textit{in } V, \quad \textit{as } k\uparrow+\infty, \quad \forall\, t\in[0,T]. \tag{4.23}$$

PROOF. (4.17) and (4.22) show that $\{u_k\}_{k\ge1}$ is an equicontinuous family w.r.t. the distance d; since E is compact, Ascoli-Arzelà Theorem yields a subsequence converging at every point t w.r.t. d to a d-continuous function u. Since d induces the weak topology of V, we get (4.23).

On E the weak topology of V and of H coincide; by the weak lower semicontinuity of the H-norm, (4.17) is preserved in the limit, so that u belongs to $C^{1/2}([0,T];H)$, too. ■

Now we can conclude our proof: we will assume that $\lim_{k\uparrow+\infty} \varrho_k = +\infty$ and we will still denote by u_k, u the functions of (4.23). Since by the previous Corollary $u_k \rightharpoonup u$ in $L^2(0,T;V)$, (4.10e) and the *a priori* estimate (4.16) obviously imply

$$\int_0^T \wp(t;u(t)\,dt \le \liminf_{k\uparrow+\infty} \int_0^T \wp_k(t;u_k(t))\,dt = 0,$$

so that

$$\wp(t;u(t)) = 0 \quad \text{a.e. in }]0,T[\,, \text{ i.e. } u\in L^2(0,T;\mathscr{V}). \tag{4.24}$$

Applying the operator Π^* to the equation (4.13) we get

$$\pi^*(t)u_k'(t) + iA_{\mathscr{V}^*}(t)u_k(t) = \pi^*(t)f(t) = f_{\mathscr{V}^*}(t)$$

since by (4.10d)

$$(\pi^*(t)JP_k(t)v,w) = (JP_k(t)v,\pi(t)w) = (\!(P_k(t)v,\pi(t)w)\!) = 0 \quad \forall\, v,w\in V.$$

By (2.22) we deduce that $\Pi^* u_k'$ weakly converges in $L^2(0,T;V^*)$ to $\frac{d^*}{dt}u$, so that u satisfies $(CP_2),\dots,(CP_4)$.

In order to show that u is also conservative, we pass to the limit in (4.14) for $t := T$ and we obtain by the lower semicontinuity of the H-norm

$$\tfrac12|u(T)|^2 \le \tfrac12|u_0|^2 + \mathrm{Re}\int_0^T (f(\tau),u(\tau))\,d\tau;$$

finally we apply Lemma 3.8.

The last property of Theorem 3 follows as in [14]. ■

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