A(Θ)-STABLE APPROXIMATIONS OF ABSTRACT CAUCHY PROBLEMS

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Summary. We study the approximation of linear parabolic Cauchy problems by means of Galerkin methods in space and $A(\Theta)-$stable multistep schemes of arbitrary order in time. The error is evaluated in the norm of $L^2_t(H^1_x) \cap L^\infty_t(L^2_x)$.

0 Introduction

The aim of this paper is to analyse the approximation of a linear parabolic Cauchy problem of the type:

$$\begin{cases} \dfrac{\partial u}{\partial t} + Au = f & \text{in } \Omega \times]0, \infty[\\ u(x,0) = u_0(x) & \text{in } \Omega \\ u(x,t) = 0 & \text{in } \partial\Omega \times]0, \infty[, \end{cases} \tag{0.1}$$

by using a Galerkin method in space and an $A(\Theta)-$stable linear multistep method of order $q \geq 1$ in time. The use of a generic $A(\Theta)-$stable method (introduced by Widlund in [13]) allows us to discuss separetely the space and the time discretization, and to overcome the second order Dahlquist barrier of the $A-$stable methods (see [5]).

We write (0.1) as an abstract Cauchy problem in an usual Hilbert triple $V \subset H \subset V^*$:

$$u(0) = u_0; \qquad u'(t) + A(t)u(t) = f(t), \quad \text{for } t > 0, \tag{0.2}$$

and we study the error in the norm of $L^2(0, \infty; V) \cap L^\infty(0, \infty; H)$.

The time discretization by means of an implicit Euler scheme was studied in [12]. The error analysis in the case $u_0 = 0$ for Euler and Crank–Nicolson methods was carried out in [4], whose outline we follow. For a different approach see e. g. [3], [7].

We choose a Galerkin approximation family $\{V_h\}$ of V and a couple (ρ, σ) of polynomials which define the multistep method:

$$\rho(z) = \sum_{j=0}^{g} \alpha_j z^j, \qquad \sigma(z) = \sum_{j=0}^{g} \beta_j z^j \quad \in \mathbf{C}[z].$$

For a discretization step $k > 0$ and a suitable choice of g initial values ${}^h u_0^k, \dots, {}^h u_{g-1}^k$ in V_h, the fully discretized problem consists in the sequence of linear equations in the unknown ${}^h u_{n+g}^k \in V_h$:

$$\frac{1}{k} \sum_{j=0}^{g} \alpha_j \left({}^h u_{n+j}^k, v \right) + \sum_{j=0}^{g} \beta_j \, a_{n+j}^k \left({}^h u_{n+j}^k, v \right) = \sum_{j=0}^{g} \beta_j \left(f_{n+j}^k, v \right), \quad \forall v \in V_h, \ \forall n \geq 0,$$

where $f_n^k = f(kn)$ and $a_n^k(u,v) = {}_{V^*}\langle A(kn)u, v \rangle_V$.

In particular we get the stabilty estimate:

$$k \sum_{n \in \mathbf{N}} \| {}^h u_n^k \|_V^2 + \sup_{n \in \mathbf{N}} \| {}^h u_n^k \|_H^2 \leq C \left\{ k \sum_{n \in \mathbf{N}} \| f_n^k \|_{V^*}^2 + \sum_{j=0}^{g-1} \left(| {}^h u_j^k |_H^2 + k \| {}^h u_j^k \|_V^2 \right) \right\}.$$

If the multistep method is of order q and the data $\{f, u_0\}$ are sufficiently smooth and compatible, so that u belongs to $H^q(0, \infty; V) \cap H^{q+1}(0, \infty; V^*)$ and the initial values may be chosen opportunely, we have the error estimate:

$$\begin{aligned} &\left\{ k \sum_{n \in \mathbf{N}} \| u(kn) - {}^h u_n^k \|^2 \right\}^{1/2} + \sup_{n \in \mathbf{N}} | u(kn) - {}^h u_n^k | \leq \\ &\qquad\qquad \leq C \left\{ e_h[u] + k^q \| u \|_{H^q(0,\infty;V) \cap H^{q+1}(0,\infty;V^*)} \right\}, \end{aligned}$$

where $e_h[u]$ is the best approximation error:

$$e_h[u] = \inf \left\{ \|u - {}^h v\|_{L^2(0,\infty;V)\cap L^\infty(0,\infty;H)}; \quad {}^h v \in L^2(0,\infty;V_h) \cap L^\infty(0,\infty;H) \right\}. \quad (0.3)$$

The paper can be outlined as follows: in section 1 we make precise our hypotheses and state the theorems about stability and convergence in the "energy norm"; proofs are given in section 2 and 3.

Error estimates in norms of type $L^2(0,\infty;V) \cap H^{1/2}(0,\infty;H)$ as showed in [11], are contained in a forthcoming paper.

1 The continuous problem and its discretization.

Notations.

Let:

$$V \hookrightarrow^{ds} H \equiv H^* \hookrightarrow^{ds} V^*$$

be a triple of separable Hilbert spaces, $\|\cdot\|$ the norm of V and $|\cdot|$ the norm of H, induced by the scalar products $((\cdot,\cdot))$ and $(\cdot,\cdot)$ respectively; we identify H and H^* and denote by $(\cdot,\cdot)$ again the antiduality between V^* and V. A density argument allows us to consider V^* as the completion of H with respect to the dual norm:

$$\|\cdot\|_* = \sup_{v\in V, \|v\|=1} (\cdot, v).$$

We shall also assume, without loss of generality, that $|v| \leq \|v\|, \quad \forall v \in V$.

Let $\mathcal{B}$ be a Banach space and let $n \in \mathbf{N}$. $H^n_+(\mathcal{B})$ and $W^{n,\infty}_+(\mathcal{B})$ are the usual Sobolev space of $\mathcal{B}-$valued distributions on the real half line $]0, +\infty[$.

We set also, for $n \in \mathbf{N}$:

$$H^{n+1}_+(V, V^*) = H^n_+(V) \cap H^{n+1}_+(V^*),$$

and we recall the continuous imbedding $H^{n+1}_+(V, V^*) \hookrightarrow W^n_+(H)$.

The continuous problem.

Assume that we are given, for $t > 0$, a measurable family of linear continuous operators $A(t)$ from V to V^* and five constants $M, L, \alpha, \Theta, \delta > 0, \quad \delta < \Theta \leq \pi/2,$ such that, for every $v \in V$, $t \in \mathbf{R}^+$:

$$(A1) \qquad \|A(t)v\|_* \leq M\|v\|, \qquad \mathrm{Re}\,\big(A(t)v, v\big) \geq \alpha\|v\|^2;$$

$$(A2) \qquad |\arg(A(t)v, v)| \leq \Theta - \delta;$$

$$(A3) \qquad \sum_{j\in\mathbf{N}} \|A(t_{j+1}) - A(t_j)\|_{\mathcal{L}(V,V^*)} \leq L, \quad \forall t_0 < t_1 < \ldots < t_n < \ldots \in \mathbf{R}^+.$$

Remark 1.1. The values of Θ and δ influence the choice of the multistep method we consider; hypothesis $(A1)$, which ensures the well-posedness of the successive Cauchy problem, implies that $(A2)$ holds at least for $\Theta = \arccos(\alpha/M) + \delta$. $(A3)$ is a supplementary hypothesis required by the stability of the discretizations; it simply means that A is of bounded variation.

For every $f \in L^2_+(V^*)$, $u_0 \in H$, we shall construct and study a family of approximations of the solution u of the abstract Cauchy problem:

$$u(0) = u_0; \qquad u'(t) + A(t)u(t) = f(t), \quad \text{for } t > 0. \tag{1.1}$$

This function belongs to $H^1_+(V, V^*)$ and satisfies the "energy inequality" (see [2], for example):

$$\|u\|_{L^2_+(V) \cap L^\infty_+(H)} \leq C\{\|f\|_{L^2_+(V^*)} + |u_0|\}. \tag{1.2}$$

Moreover, when f belongs to $H^q_+(V^*)$, A belongs to $W^{q,\infty}_+(\mathcal{L}(V, V^*))$ and $\{f, A, u_0\}$ are related by suitable compatibility conditions, then u belongs to $H^{q+1}_+(V, V^*)$. These relations may be easily deduced by $q-$times differentiation of equation (1.1) and are expressed in terms of a vector $\mathbf{c}_q(f, u_0) = (c_0, \dots, c_q)$ whose components are so defined:

$$c_0 = u_0, \quad c_{m+1} = f^{(m)}(0) - \sum_{j=0}^{m} \binom{m}{j} A^{(j)}(0)\, c_{m-j}; \qquad 0 \leq m < q. \tag{1.3}$$

If we ask that $\mathbf{c}_q \in V^q \times H$ we obtain:

$$\begin{cases} u \in H^{q+1}_+(V, V^*), \quad u^{(j)}(0) = c_j(f, u_0), \quad 0 \leq j \leq q \\ \|u\|_{H^{q+1}_+(V,V^*)} \leq C\{\|f\|_{H^q_+(V^*)} + \|\mathbf{c}_q(f, u_0)\|_{V^q \times H}\}, \end{cases} \tag{1.4}$$

so that we may summarize our regularity hypotheses:

$(A4)$ $\qquad f \in H^q_+(V^*), \quad A \in W^{q,\infty}_+(\mathcal{L}(V, V^*)), \quad \mathbf{c}_q(f, u_0) \in V^q \times H; \quad q \geq 1.$

The method.

We discretize problem (1.1) by a $g-$step linear method. More precisely, we assign $2g + 2$ coefficients $\{\alpha_j, \beta_j\}_{j=0,\dots,g}$ and we set, for every time step $k > 0$,

$$f^k_n = f(nk), \quad A^k_n = A(kn); \qquad n \in \mathbf{N} \ (^1) \tag{1.5}$$

Choosing g initial values $u^k_0, \dots, u^k_{g-1} \in V$, we intend to construct an approximation u^k_n of the solution $u(nk)$ by the following algorithm:

$$\begin{cases} \forall n \geq 0, \quad \text{find} \quad u^k_{n+g} \in V \quad \text{such that:} \\ \dfrac{1}{k} \displaystyle\sum_{j=0}^{g} \alpha_j u^k_{n+j} + \sum_{j=0}^{g} \beta_j\, A^k_{n+j} u^k_{n+j} = \sum_{j=0}^{g} \beta_j f^k_{n+j}. \end{cases} \tag{1.6}$$

If $\mathrm{Re}[\alpha_g \bar{\beta}_g] > 0$ (^{2}), by $(A1)$ and the Lax–Milgram lemma we can invert the operator:

$$\frac{1}{k}\alpha_g + \beta_g A^k_{n+g}, \tag{1.7}$$

(^{1}) By $(A4)$ f and A are continuous, so this setting makes sense.

(^{2}) By $(A2)$, $\alpha_g \bar{\beta}_g \neq 0$, $\arg[\alpha_g \bar{\beta}_g] \leq \pi - \Theta$ would suffice. In fact these conditions are equivalent if the coefficients are real.

for every $n \in \mathbf{N}$ and we can solve (1.6) with respect to u^k_{n+g}, once

$$u^k_n, \dots, u^k_{n+g-1}, \quad f^k_n, \dots, f^k_{n+g}$$

are given. By induction we obtain existence and uniqueness for the sequence $\{u^k_n\}_{n\in\mathbf{N}}$.

To solve (1.6) from the numerical point of view we introduce a Galerkin family $\{V_h\}$ of closed subspaces of V ([3]), and consider the fully discretized problem:

$$\begin{cases} \text{Given } \ {}^h u^k_0, {}^h u^k_1, \dots, {}^h u^k_{g-1} \in V_h, \quad \text{find } \ \{{}^h u^k_{n+g}\}_{n\in\mathbf{N}} \subset V_h \ \text{ such that:} \\ \Big(\dfrac{1}{k}\displaystyle\sum_{j=0}^{g} \alpha_j \, {}^h u^k_{n+j} + \sum_{j=0}^{g} \beta_j A^k_{n+j} \, {}^h u^k_{n+j} - \sum_{j=0}^{g} \beta_j f^k_{n+j}, {}^h w\Big) = 0 \quad \forall \ {}^h w \in V_h. \end{cases} \tag{1.8}$$

The stability and convergence properties of these methods (in the finite dimensional case) may be briefly expressed in terms of the two polynomials:

$$\rho(z) = \sum_{j=0}^{g} \alpha_j z^j, \qquad \sigma(z) = \sum_{j=0}^{g} \beta_j z^j \quad \in \mathbf{C}[z]; \qquad |\alpha_g|^2 + |\beta_g|^2 > 0, \tag{1.9}$$

which we may suppose prime. On (ρ, σ) we shall impose the following conditions (see for instance [10]):

$(P1)$ *strong* $A(\Theta)-$*stability*: for $|z| \geq 1$ $\sigma(z)$ is different from 0 and the quotient $\rho(z)/\sigma(z)$ is contained in the open sector:

$$\mathsf{S}_{\pi-\Theta} = \{\xi \in \mathbf{C} \setminus \{0\} : |\arg \xi| < \pi - \Theta\}, \qquad 0 < \Theta \leq \pi/2. \tag{1.10}$$

$(P2)$ *order* q: when $z \to 0$ we have

$$\rho(e^z) - z\sigma(e^z) = O(z^{q+1}) \tag{1.11}$$

for an integer $q \geq 1$; in particular this implies the consistency, i. e.:

$$\rho(1) = 0, \qquad \rho'(1) = \sigma(1) \neq 0 \tag{1.12}$$

Remark 1.2. $(P1)$ implies that $\alpha_g \bar{\beta}_g$ is different from 0 and is contained in $\mathsf{S}_{\pi-\Theta}$; in other words, the method must be implicit ((ρ, σ) have degree g) and (1.7) can be inverted. Moreover, the possible unitary roots of ρ are simple.

([3]) In practice, V_h are finite–dimensional.

Remark 1.3. When $\Theta = \pi/2$ we are dealing with an $A-$stable method, whose stability properties are well known (see [3], [5]). On the other hand, for these methods the "Dahlquist Barrier" forces $q \leq 2$, so that the use of more general $A(\Theta)-$stable methods with $\Theta < \pi/2$ becomes necessary if we want to reach higher orders. We recall, for example, the Backward Differentiation Schemes of orders ≤ 5.

From now on we assume that (P1) and (P2) are satisfied for fixed Θ and q.

Stability estimates and approximation results.

Theorem 1.4. *Let us assume that properties* $(A1-3)$ *and* $(P1)$ *hold; then the solution* ${}^h u_n^k$ *of* (1.8) *satisfies:*

$$k \sum_{n\in\mathbf{N}} \|{}^h u_n^k\|^2 + \sup_{n\in\mathbf{N}} |{}^h u_n^k|^2 \leq C\Big\{k \sum_{n\in\mathbf{N}} \|f_n^k\|_*^2 + \sum_{j=0}^{g-1} \big(k\|{}^h u_j^k\|^2 + |{}^h u_j^k|^2\big)\Big\}, \tag{1.13}$$

where C *depends only on the constants* $M, L, \alpha, \Theta, \delta$ *and on* (ρ, σ).(4)

Remark 1.5. We have the estimate:

$$k \sum_{n\in\mathbf{N}} \|f_n^k\|_*^2 \leq 2\|f\|^2_{H^1_+(V^*)}; \tag{1.14}$$

so, by $(A4)$ the righthand member of (1.13) is finite.

We denote with H_h the closure of V_h in the $H-$norm and with V_h^* the antidual of V_h, so that V_h, H_h, V_h^* is a new Hilbert triple; P_h is the surjective "restriction" of V^* on V_h^*:

$${}_{V_h^*}\big\langle P_h v, {}^h w\big\rangle_{V_h} = (v, {}^h w), \qquad \|P_h v\|_{V_h^*} \leq \|v\|_*, \qquad \forall v \in V^*, \quad \forall {}^h w \in V_h. \tag{1.15}$$

Moreover, we have the best approximation result:

$$\forall v \in H, \quad P_h v \in H_h, \quad |v - P_h v| = \min_{{}^h w\in H_h} |v - {}^h w|.$$

We assume that:

$$(G1) \qquad P_h(V) \subset V_h; \qquad \exists C > 0: \quad \|P_h v\| \leq C\|v\|, \quad \forall v \in V$$

for a constant C independent of h. In particular, this implies that:

$$\forall {}^h w \in V_h, \quad \|v - P_h v\| \leq \|v - {}^h w\| + \|{}^h w - P_h v\| = \|v - {}^h w\| + \|P_h({}^h w - v)\| \leq (1+C)\|v - {}^h w\|,$$

(4) From now on, we always denote with C such constants.

so that P_h realizes:

$$\|v - P_h v\| \leq C' \min_{{}^h w \in V_h} \|v - {}^h w\|, \tag{1.16}$$

and, for a function u in $L^2_+(V) \cap L^\infty_+(H)$:

$$\|u - P_h u\|_{L^2_+(V) \cap L^\infty_+(H)} \leq C e_h[u], \tag{1.17}$$

$e_h[u]$ given by (0.3). We denote the error on the initial values by:

$$\epsilon^2[u; {}^h u^k_0, \ldots, {}^h u^k_{g-1}] = \max_{0 \leq j < g} |P_h u(kj) - {}^h u^k_j|^2 + k \sum_{j=0}^{g-1} \|P_h u(kj) - {}^h u^k_j\|^2 \tag{1.18}$$

and we may suppose that the choice of the initial values satisfies the following requirement:

$$(I1) \qquad \epsilon[u; {}^h u^k_0, \ldots, {}^h u^k_{g-1}] \leq C k^q \big[\|f\|_{H^q(0,kg;V^*)} + \|\mathbf{c}_q\|_{V^q \times H}\big].$$

Remark 1.6. By $(A4)$ we know from the equation the Taylor expansion of u around 0 up to the order q; so, a possible choice of the initial values is:

$$u^k_j = \sum_{\ell=0}^{q-1} \frac{c_\ell}{\ell!} (jk)^\ell, \qquad {}^h u^k_j = P_h u^k_j; \qquad 0 \leq j < g. \tag{1.19}$$

We have:

Theorem 1.7. *Assume that $(A1-4)$, $(P1-2)$, $(G1)$ and $(I1)$ hold; then the solution ${}^h u^k_n$ of (1.8) satisfies:*

$$\begin{aligned}
\big\{k \sum_{n \in \mathbf{N}} \|u(kn) - {}^h u^k_n\|^2\big\}^{1/2} + \sup_{n \in \mathbf{N}} |u(kn) - {}^h u^k_n| \leq \\
C \left\{k^q \|u\|_{H^{q+1}_+(V,V^*)} + \|u - P_h u\|_{L^2_+(V) \cap L^\infty_+(H)} + \epsilon[u; {}^h u^k_0, \ldots, {}^h u^k_{g-1}]\right\} \leq \\
C \left\{k^q \left[\|f\|_{H^q_+(V^*)} + \|\mathbf{c}_q(f, u_0)\|_{V^q \times H}\right] + e_h[u]\right\},
\end{aligned}$$

with C depending only on the various constants introduced but not on h, k.

2 Proof of the theorems: stability.

Preliminary outline; sequences spaces.

We try to find the estimates of the preceding theorems by rewriting equations (1.6) and (1.8) in a different form. Setting ${}^hA = P_h A$, equation (1.8) becomes formally equivalent to (1.6) in the new Hilbert triple V_h, H_h, V_h^*:

$$\frac{1}{k}\sum_{j=0}^{g} \alpha_j \, {}^hu^k_{n+j} + \sum_{j=0}^{g} \beta_j \, {}^hA^k_{n+j} \, {}^hu^k_{n+j} = \sum_{j=0}^{g} \beta_j \, P_h f^k_{n+j}, \qquad n \geq 0; \tag{2.1}$$

moreover, the operator ${}^hA$ satisfies in this framework the same conditions $(A1-3)$ and by (1.15) P_h is a contraction from V^* to V_h^*; so, concerning the study of stability, we may limit ourselves to consider equation (1.6), suppressing the index h.

We denote vector valued sequences with bold characters and suppress the index k too when this fact does not generate mistakes. If $\mathcal{H}$ is an Hilbert space, we introduce the operator E on $\mathcal{H}^{\mathbf{N}}$:

$$(\mathrm{E}\mathbf{v})_n = v_{n+1}, \tag{2.2}$$

with its powers:

$$(\mathrm{E}^j\mathbf{v})_n = v_{n+j}, \qquad (\mathrm{E}^{-j}\mathbf{v})_n = \begin{cases} v_{n-j}, & \text{if } n \geq j \\ 0, & \text{if } n < j \end{cases} \qquad \forall\, j \in \mathbf{N}. \tag{2.3}$$

E^{-j} is the right inverse of E^j: $\mathrm{E}^j\mathrm{E}^{-j}\mathbf{v} = \mathbf{v}$, for every sequence $\mathbf{v}$. For every polynomial $\tau(z) = \sum_{j=0}^{g} \gamma_j z^j$ we have consequently:

$$(\tau(\mathrm{E})\mathbf{v})_n = \sum_{j=0}^{g} \gamma_j v_{n+j}. \tag{2.4}$$

Setting $(\mathbf{A}\mathbf{v})_n = A_n v_n$, for $\mathbf{v} \in V^{\mathbf{N}}$, we write:

$$\frac{1}{k}\sum_{j=0}^{g} \alpha_j v_{n+j} + \sum_{j=0}^{g} \beta_j A_{n+j} v_{n+j} = \left(\frac{\rho(\mathrm{E})}{k}\mathbf{v} + \sigma(\mathrm{E})\mathbf{A}\mathbf{v} \right)_n, \quad \forall n \in \mathbf{N}.$$

We set also:

$$\forall\, \mathbf{v} \in \mathcal{H}^{\mathbf{N}}, \qquad \mathbf{v}|_j = \begin{cases} v_n, & \text{if } n \leq j \\ 0, & \text{if } n > j \end{cases} \tag{2.5}$$

so that, if $\underline{\mathbf{u}} = (u_0, \ldots, u_{g-1}) \in V^g \subset V^{\mathbf{N}}$ is the vector of the initial values, (1.6) becomes:

$$\begin{cases} \mathbf{u}|_{g-1} = \underline{\mathbf{u}}, \\ \dfrac{\rho(\mathrm{E})}{k}\mathbf{u} + \sigma(\mathrm{E})\mathbf{A}\mathbf{u} = \sigma(\mathrm{E})\mathbf{f} \end{cases} \tag{2.6}$$

Finally, we call $T_k = k^{-1}\rho(\mathrm{E}) + \sigma(\mathrm{E})\mathbf{A}$, and write (2.6) in the compact form:

$$T_k \mathbf{u} = \sigma(\mathrm{E})\mathbf{f}, \qquad \mathbf{u}_{|_{g-1}} = \underline{\mathbf{u}}. \tag{2.7}$$

By linearity we may enclose the initial conditions in the equation and write it in terms of $\mathbf{u}^+ = \mathbf{u} - \underline{\mathbf{u}}$:

$$T_k \mathbf{u}^+ = \sigma(\mathrm{E})\mathbf{f} + T_k\underline{\mathbf{u}}, \quad \mathbf{u}^+{}_{|_{g-1}} = 0. \tag{2.8}$$

To complete our formulation, we specify the spaces where we set (2.8), taking into account the quantities arising in (1.13) which we shall deal with.

We call $l^p_k(\mathcal{H})$ the Banach space of the $\mathcal{H}-$valued sequences $\mathbf{v}$ such that:

$$\|\mathbf{v}\|^p_{l^p_k(\mathcal{H})} = k\sum_{n\in\mathbf{N}} \|v_n\|^p_{\mathcal{H}} < \infty, \qquad 1 \le p < \infty, \tag{2.9}$$

and $l^\infty_k(H) = l^\infty(\mathcal{H})$ the Banach space of the bounded sequences with the *sup*$-$norm; we observe that there is a natural antiduality between $l^p_k(\mathcal{H})$ and $l^{p'}_k(\mathcal{H}^*)$:

$$_{l^p_k(\mathcal{H})}\langle \mathbf{v}, \mathbf{w}\rangle_{l^{p'}_k(\mathcal{H}^*)} = k\sum_{n\in\mathbf{N}} {}_{\mathcal{H}}\big(v_n, w_n\big)_{\mathcal{H}^*}; \quad \frac{1}{p} + \frac{1}{p'} = 1; \tag{2.10}$$

finally, we indicate with $\dot{l}^p_k(\mathcal{H})$ the closed subspace of $l^p_k(\mathcal{H})$ given by the sequences $\mathbf{v}$ with $\mathbf{v}_{|_{g-1}} = 0$. The operator E is well defined on these spaces and its norm is 1.

Theorem 1.4 may be so restated:

Theorem 2.1. *Assume that* $\mathbf{u}^+$ *is a solution of* (2.8) *with* $\mathbf{f} \in l^2_k(V^*)$. *Then* $\mathbf{u}^+$ *satisfies the stability estimate:*

$$\|\mathbf{u}^+\|_{l^2_k(V)\cap l^\infty(H)} \le C\{\|\mathbf{f}\|_{l^2_k(V^*)} + \|\underline{\mathbf{u}}\|_{l^\infty(H)\cap l^2_k(V)}\}. \tag{2.11}$$

Remark 2.2. As we have already noticed, this result give an analogous bound for the solution of (2.1): we call ${}^hT_k$ the operator P_hT_k and consider ${}^h\mathbf{u}^+$, solution of:

$${}^hT_k\,{}^h\mathbf{u}^+ = \sigma(\mathrm{E})P_h\mathbf{f} + {}^hT_k{}^h\underline{\mathbf{u}},$$

we have:

$$\|{}^h\mathbf{u}^+\|_{l^2_k(V)\cap l^\infty(H)} \le C\{\|\mathbf{f}\|_{l^2_k(V^*)} + \|{}^h\underline{\mathbf{u}}\|_{l^\infty(H)\cap l^2_k(V)}\}. \tag{2.12}$$

Up to now we have only changed our notations; we shall show how these are really more convenient. The basic tool of our proof is explained in the following section; we state first a lemma on inversion of operators like (2.4):

Lemma 2.3. *Assume that the roots of the polynomial* $\tau(z) = \sum_{j=0}^{g} \gamma_j z^j$ *have modulus* < 1*; then there exists a sequence of complex numbers* $\{\gamma'_j\}_{j\in\mathbf{N}+g}$ *such that:*

$$\sum_{j\geq g} |\gamma'_j| = |\tau^{-1}| < \infty,$$

and $\forall\, \mathbf{w} \in \mathcal{H}^{\mathbf{N}}$*:*

$$\mathbf{v}\big|_{g-1} = 0, \quad \tau(\mathrm{E})\mathbf{v} = \mathbf{w} \;\Leftrightarrow\; v_n = \sum_{j=g}^{n} \gamma'_j w_{n-j}, \quad \forall n \geq g. \tag{2.13}$$

Moreover:

$$\mathbf{w} \in l^p_k(\mathcal{H}) \;\Rightarrow\; \mathbf{v} \in l^p_k(\mathcal{H}), \quad \|\mathbf{v}\|_{l^p_k(\mathcal{H})} \leq |\tau^{-1}|\, \|\mathbf{w}\|_{l^p_k(\mathcal{H})}. \tag{2.14}$$

Proof. Thanks to the hypothesis on τ, $\tau(z)^{-1}$ is a holomorphic function in $|z| > 1-\epsilon$ for an $\epsilon > 0$ and we can write its power series development around ∞:

$$\tau(z)^{-1} = \sum_{j\geq g} \gamma'_j z^{-j}, \qquad \sum_{j\geq g} |\gamma'_j| = |\tau^{-1}| < \infty. \tag{2.15}$$

We denote with $\tau^{-1}(\mathrm{E})$ the linear operator:

$$\mathbf{w} \to \tau^{-1}(\mathrm{E})\mathbf{w} = \mathbf{v}, \qquad v_n = \sum_{j=g}^{n} \gamma'_j w_{n-j}$$

which is uniformly bounded in every $l^p_k(\mathcal{H})$ by $|\tau^{-1}|$.

It remains to prove (2.13); by definition, the coefficients γ'_j satisfy the algebraic relations:

$$\sum_{j=0}^{g} \gamma_j \gamma'_{n+j} = \delta_{0,n} = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{if } n > 0 \end{cases} \qquad \forall\, n \in \mathbf{N},$$

which imply that:

$$\begin{aligned} \big(\tau(\mathrm{E})\,\tau^{-1}(\mathrm{E})\mathbf{w}\big)_n &= \sum_{j=0}^{g} \gamma_j (\tau(\mathrm{E})^{-1}\mathbf{w})_{n+j} = \sum_{j=0}^{g} \gamma_j \sum_{i=j}^{n+j} \gamma'_i\, w_{n+j-i} = \\ (i = j+l) \qquad &= \sum_{l=0}^{n} \left(\sum_{j=0}^{g} \gamma_j \gamma'_{j+l} \right) w_{n-l} = w_n \quad \blacksquare \end{aligned}$$

Remark 2.4. It's obvious that $\tau(\mathrm{E})$ is bounded on every $l^p_k(\mathcal{H})$, with norm $\leq |\tau| = \sum_{j=0}^{g} |\gamma_j|$.

Corollary 2.5. *Suppose that* $\mathbf{v}$ *satisfies:*

$$\mathbf{v}|_{g-1} = \underline{\mathbf{v}}, \qquad \tau(\mathrm{E})\mathbf{v} = \mathbf{w} \ \in l^p_k(\mathcal{H}).$$

Then we have:

$$\|\mathbf{v}\|_{l^p_k(\mathcal{H})} \le |\tau^{-1}|\,\|\mathbf{w}\|_{l^p_k(\mathcal{H})} + |\tau^{-1}|\,|\tau|\,\|\underline{\mathbf{v}}\|_{l^p_k(\mathcal{H})} \tag{2.16}$$

Proof. Writing $\mathbf{v}^+ = \mathbf{v} - \underline{\mathbf{v}}$ we observe that $\mathbf{v}^+$ satisfies:

$$(\mathbf{v}^+)|_{g-1} = 0; \qquad \tau(\mathrm{E})\mathbf{v}^+ = \mathbf{w} + \tau(\mathrm{E})\underline{\mathbf{v}},$$

and conclude by the previous lemma. ∎

A basic isomorphism.

Let $\mathbf{U}$ be the subset of the extended complex plane: $\{z \in \mathbf{C} : \ |z| > 1\} \cup \{\infty\}$ and consider the Hardy space $H^2(\mathbf{U};\mathcal{H})$ of the $\mathcal{H}-$valued holomorphic functions g on $\mathbf{U}$ such that:

$$\exists \lim_{r\to 1^+} \frac{1}{2\pi}\int_{-\pi}^{\pi} \|g(re^{i\theta})\|^2_{\mathcal{H}}\, d\theta = \|g\|^2_{H^2(\mathbf{U};\mathcal{H})}. \tag{2.17}$$

Every g in $H^2(\mathbf{U};\mathcal{H})$ admits a trace (still denoted with g) on $\partial\mathbf{U} = \{z \in \mathbf{C} : |z| = 1\}$ which belongs to $L^2(\partial\mathbf{U};\mathcal{H})$. The coefficients of the Laurent expansion around ∞ are given by the Fourier coefficients of g in $L^2(\partial\mathbf{U};\mathcal{H})$:

$$g_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(e^{i\theta})e^{in\theta}\, d\theta, \qquad g(z) = \sum_{n\in\mathbf{N}} g_n z^{-n}. \tag{2.18}$$

We have the fundamental relation:

$$\|g\|^2_{H^2(\mathbf{U};\mathcal{H})} = \|g\|^2_{L^2(\partial\mathbf{U};\mathcal{H})} = \sum_{n\in\mathbf{N}} \|g_n\|^2_{\mathcal{H}}. \tag{2.19}$$

So, $H^2(\mathbf{U};\mathcal{H})$ is a Hilbert space isomorphic to $l^2_k(\mathcal{H})$ by the transformation:

$$\mathbf{g} \in l^2_k(\mathcal{H}) \ \to \ \hat{g}(z) = \sum_{n\in\mathbf{N}} g_n z^{-n}; \qquad \|g\|^2_{l^2_k(\mathcal{H})} = k\|\hat{g}\|^2_{H^2(\mathbf{U};\mathcal{H})}. \tag{2.20}$$

The most interesting fact for us is given by the following rules:

$$\text{if } g_0 = 0 \ \text{ then } \ \widehat{\mathrm{E}\mathbf{g}}(z) = z\hat{g}(z); \tag{2.21}$$

$$\mathbf{A} \equiv A \text{ constant } \ \Rightarrow \ \widehat{A\mathbf{g}}(z) = A\hat{g}(z). \tag{2.22}$$

For a sequence $\mathbf{v} \in l_k^2(V)$ we have:

$$\widehat{\rho(\mathrm{E})\mathbf{v}}(z) = \rho(z)\hat{v}(z), \qquad \widehat{\sigma(\mathrm{E})\mathbf{v}}(z) = \sigma(z)\hat{v}(z),$$

and:

$$\widehat{T_k\mathbf{v}}(z) = \frac{\rho(z)}{k}\hat{v}(z) + A\sigma(z)\hat{v}(z) = \hat{T}_k\hat{v}(z) \tag{2.23}$$

when $\mathbf{A}$ is constant.

Proof of theorem 2.1: the case $\mathbf{A} \equiv A$ constant.

We call $\mathbf{g}_1 = \sigma(\mathrm{E})[\mathbf{f} + A\underline{\mathbf{u}}]$, $\mathbf{g}_2 = k^{-1}\rho(\mathrm{E})\underline{\mathbf{u}}$ with the obvious bounds:

$$\|\mathbf{g}_1\|_{l_k^2(V^*)} \le |\sigma| \left\{ \|\mathbf{f}\|_{l_k^2(V^*)} + M\|\underline{\mathbf{u}}\|_{l_k^2(V)} \right\}, \qquad \|\mathbf{g}_2\|_{l_k^1(H)} \le g|\rho| \, \|\underline{\mathbf{u}}\|_{l^\infty(H)}$$

We split correspondingly $\mathbf{u}^+$ into the sum $\mathbf{u}_1 + \mathbf{u}_2$, with:

$$(\mathbf{u}_j)_{|g-1} = 0, \qquad T_k\mathbf{u}_j = \mathbf{g}_j, \qquad j = 1, 2$$

and study separately these sequences.

Claim 2.6.

$$\|\mathbf{u}_1\|_{l_k^2(V)} \le C\|\mathbf{g}_1\|_{l_k^2(V^*)}. \tag{2.24}$$

By (2.23), $\mathbf{u}_1$ belongs to $l_k^2(V)$ if and only if there exists a solution $\hat{u}_1$ in $H^2(\mathbf{U}; V)$ of the equation:

$$\hat{T}_k\hat{u}_1(z) = \frac{\rho(z)}{k}\hat{u}(z) + A\sigma(z)\hat{u}(z) = \hat{g}_1(z). \tag{2.25}$$

We know that, for $|z| \ge 1$, is $\sigma(z) \ne 0$; denoting by $\gamma(z)$ the rational function $\frac{\rho(z)}{\sigma(z)}$, $\gamma(z)$ is holomorphic in $\mathbf{U}$ and continuous on $\partial\mathbf{U}$. We may rewrite (2.25) as follows:

$$\frac{\gamma(z)}{k}\hat{u}_1(z) + A\hat{u}(z) = \sigma(z)^{-1}\hat{g}_1(z). \tag{2.26}$$

If $\hat{g}_1(z)$ is in $H^2(\mathbf{U}; V^*)$ also $\frac{\hat{g}_1(z)}{\sigma(z)}$ belongs to $H^2(\mathbf{U}; V^*)$ and its norm is bounded by $C_\sigma\|\mathbf{g}_1\|_{H^2(\mathbf{U};V^*)}$, with: $C_\sigma = \max_{|z|=1} |\sigma(z)^{-1}|$.

It remains to study the invertibility of $\gamma(z) + A$. But $(A1-2)$ imply that the operator $\zeta + A$ is invertible from V^* to V if $\zeta \in \overline{\mathsf{S}}_{\pi-\Theta}$ with the bound:

$$\zeta v + Av = f \;\Rightarrow\; \|v\| \le \frac{1}{\alpha \sin\delta}\|f\|_* \tag{2.27}$$

By $(P1)$ $\gamma(z)$ belongs to $\overline{\mathsf{S}}_{\pi-\Theta}$ when $|z| \geq 1$, so the mapping:

$$z \to \left[\frac{\gamma(z)}{k} + A\right]^{-1}$$

is well defined, bounded and continuous from $\overline{\mathbf{U}}$ to $\mathcal{L}(V^*, V)$ and holomorphic in $\mathbf{U}$. It follows that $[k^{-1}\gamma(z) + A]^{-1}\sigma(z)^{-1}\hat{g}_1(z)$ is holomorphic in $\mathbf{U}$, has a 0 of order g in ∞ and satisfies the estimate:

$$\|\hat{u}(z)\| \leq \frac{1}{\alpha \sin\delta|\sigma(z)|}\|\hat{g}_1(z)\|_*. \tag{2.28}$$

Because of (2.19) we get:

$$\|\mathbf{u}\|_{l^2_k(V)} \leq \frac{C_\sigma}{\alpha \sin\delta}\|\mathbf{g}_1\|_{l^2_k(V^*)},$$

that is (2.24). ∎

Claim 2.7. *There exists a polynomial $\lambda(z)$ of degree g such that:*

$$\sup_{n\in\mathbf{N}} \left\{ \mathrm{Re}\ {}_{l^2_k(H)}\Big\langle \frac{\rho(\mathrm{E})\mathbf{v}}{k}, [\lambda(\mathrm{E})\mathbf{v}]_{|n} \Big\rangle_{l^2_k(H)} \right\} \geq \|\mathbf{v}\|^2_{l^\infty(H)}, \qquad \forall\, \mathbf{v} \in \dot{l}^2_k(H); \tag{2.29}$$

in particular, this implies:

$$\|\mathbf{u}_1\|_{l^\infty(H)} \leq C\|\mathbf{g}_1\|_{l^2_k(V^*)}. \tag{2.30}$$

We denote with Z_ρ the set of the unitary roots of ρ, and set:

$$\rho_u(z) = \prod_{\xi\in Z_\rho}(z-\xi), \quad \rho_0 = \rho/\rho_u, \quad \rho_\xi(z) = \frac{\rho_u(z)}{z-\xi}.$$

We call $\mathbf{w} = \rho_0(\mathrm{E})\mathbf{v}$; by lemma 2.3 there exists a constant $\beta = |\rho_0^{-1}| > 0$ only depending on ρ_0 such that:

$$\|\mathbf{v}\|_{l^\infty(H)} \leq \beta\|\mathbf{w}\|_{l^\infty(H)}. \tag{2.31}$$

We note that, by remark 1.2, there exist constants $\{c_\xi\}_{\xi\in Z_\rho}$ such that:

$$1 = \sum_{\xi\in Z_\rho} c_\xi \rho_\xi(z) \ \Rightarrow\ \mathbf{w} = \sum_{\xi\in Z_\rho} c_\xi\rho_\xi(\mathrm{E})\mathbf{w};$$

setting $c = \sum_{\xi\in Z_\rho}|c_\xi|^2$ and $\mathbf{v}^\xi = \rho_\xi(\mathrm{E})\mathbf{w} = [\rho_\xi\rho_0](\mathrm{E})\mathbf{v}$, we have:

$$|w_n|^2 \leq c\sum_{\xi\in Z_\rho}|v^\xi_n|^2; \qquad \|\mathbf{w}\|^2_{l^\infty(H)} \leq c\sup_{n\in\mathbf{N}}\sum_{\xi\in Z_\rho}|v^\xi_n|^2. \tag{2.32}$$

We say that:

$$\lambda(z) = 2\beta c\, z\rho_0(z) \sum_{\xi\in Z_\rho} \rho_\xi(z), \qquad \lambda(\mathrm{E})\mathbf{v} = 2\beta c \sum_{\xi\in Z_\rho} \rho_\xi(\mathrm{E})\mathrm{E}\mathbf{w} = 2\beta c \sum_{\xi\in Z_\rho} \mathrm{E}\mathbf{v}^\xi \tag{2.33}$$

is a good choice for (2.29). Recalling that $\rho(\mathrm{E})\mathbf{v} = \mathrm{E}\mathbf{v}^\xi - \xi\mathbf{v}^\xi$ and observing that $\rho_\xi\rho_0$ has degree $g-1$ and consequently $v_0^\xi = 0$, we have:

$$\begin{aligned}
\mathrm{Re}\ {}_{l_k^2(H)}\left\langle \frac{\rho(\mathrm{E})\mathbf{v}}{k}, [\lambda(\mathrm{E})\mathbf{v}]|_n \right\rangle_{l_k^2(H)} &= \frac{2\beta c}{k} \mathrm{Re} \sum_{\xi\in Z_\rho} {}_{l_k^2(H)}\left\langle \mathrm{E}\mathbf{v}^\xi - \xi\mathbf{v}^\xi, (\mathrm{E}\mathbf{v}^\xi)|_n \right\rangle_{l_k^2(H)} = \\
&= 2\beta c\ \mathrm{Re} \sum_{\xi\in Z_\rho} \sum_{j=0}^{n} \left(v_{j+1}^\xi - \xi v_j^\xi, v_{j+1}^\xi\right) \geq \\
&\geq \beta c \sum_{\xi\in Z_\rho} \sum_{j=0}^{n} |v_{j+1}^\xi|^2 - |v_j^\xi|^2 = \beta c \sum_{\xi\in Z_\rho} |v_{n+1}^\xi|^2.
\end{aligned}$$

By (2.32) and (2.31) we get (2.29); (2.30) follows by taking the duality of equation $T_k\mathbf{u}_1 = \mathbf{g}_1$ with $\lambda(\mathrm{E})\mathbf{u}_1|_n$ and recalling (2.24). ∎

Claim 2.8.

$$\|\mathbf{u}_2\|_{l_k^2(V)} \leq C\|\mathbf{g}_2\|_{l_k^1(H)}. \tag{2.34}$$

We use a duality argument; first we establish a transposition formula. Suppose that $\mathbf{u}|_{g-1} = \mathbf{v}|_{g-1} = 0$ and consider the symmetry:

$$\mathrm{S}_N:\ \mathbf{w} \to \mathrm{S}_N\mathbf{w} = \mathbf{w}', \qquad (\mathbf{w}')_n = \begin{cases} w_{N-n} & \text{if } 0 \leq n \leq N \\ 0 & \text{if } n > N \end{cases}.$$

For a polynomial $\tau(z) = \sum_{j=0}^{g} \gamma_j z^j$ we have:

$$ {}_{l_k^2(\mathcal{H})}\langle \tau(\mathrm{E})\mathbf{u}, \mathbf{v}' \rangle_{l_k^2(\mathcal{H})} \ = \ {}_{l_k^2(\mathcal{H})}\langle \mathbf{u}', \bar\tau(\mathrm{E})\mathbf{v} \rangle_{l_k^2(\mathcal{H})}, \tag{2.35}$$

where we called $\bar\tau(z) = \overline{\tau(\bar z)} = \sum_{j=0}^{g} \bar\gamma_j z^j$. In fact we have:

$$\begin{aligned}
{}_{l_k^2(\mathcal{H})}\langle \tau(\mathrm{E})\mathbf{u}, \mathbf{v}' \rangle_{l_k^2(\mathcal{H})} &= k\sum_{n=0}^{N} \left(\sum_{j=0}^{g} \gamma_j u_{n+j}, w_{N-n} \right) = k \sum_{n=0}^{N} \sum_{j=0}^{g} (u_{n+j}, \bar\gamma_j w_{N-n}) = \\
(n = N-m-j) \quad &= k\sum_{j=0}^{g} \sum_{n=0}^{N-g} (u_{n+j}, \bar\gamma_j w_{N-n}) = k \sum_{j=0}^{g} \sum_{m=0}^{N-g} (u_{N-m}, \bar\gamma_j w_{m+j}) = \\
&= k\sum_{j=0}^{g} \sum_{m=0}^{N} (u'_m, \bar\gamma_j w_{m+j}) = {}_{l_k^2(\mathcal{H})}\langle \mathbf{u}', \bar\tau(\mathrm{E})\mathbf{w} \rangle_{l_k^2(\mathcal{H})}.
\end{aligned}$$

Consider now A^*, the adjoint of A, and set:

$$\bar{T}_k = \frac{\bar{\rho}(\mathrm{E})}{k} + \bar{\sigma}(\mathrm{E})A^*;$$

$\bar{T}_k$ has the same property of T_k, since $(\bar{\rho}, \bar{\sigma})$ satisfies $(P1)$ and A^* satisfies $(A1-2)$. In particular:

$$\|\mathbf{w}\|_{l^\infty(H)} \le C\|\bar{T}_k \mathbf{w}\|_{l^2_k(V^*)}, \qquad \forall\, \mathbf{w} \in \mathring{l}^2_k(V).$$

and, by (2.35):

$$_{l^2_k(V^*)}\langle T_k \mathbf{u}, \mathrm{S}_N \mathbf{v}\rangle_{\,l^2_k(V)} = {}_{l^2_k(V)}\langle \mathrm{S}_N \mathbf{u}, \bar{T}_k \mathbf{v}\rangle_{\,l^2_k(V^*)} \tag{2.36}$$

On the other hand we have:

$$\|\mathbf{u}_2\|_{l^2_k(V)} = \sup_{N\in\mathbf{N}} \|\mathrm{S}_N \mathbf{u}_2\|_{l^2_k(V)},$$

and:

$$\begin{aligned}
\|\mathrm{S}_N \mathbf{u}_2\|_{l^2_k(V)} &= \sup_{\mathbf{w}\in \mathring{l}^2_k(V)\setminus\{0\}} \frac{_{l^2_k(V)}\langle \mathrm{S}_N \mathbf{u}_2, \bar{T}_k \mathbf{w}\rangle_{\,l^2_k(V^*)}}{\|\bar{T}_k \mathbf{w}\|_{l^2_k(V^*)}} = \\
&= \sup_{\mathbf{w}\in \mathring{l}^2_k(V)\setminus\{0\}} \frac{_{l^2_k(V^*)}\langle T_k \mathbf{u}_2, \mathrm{S}_N \mathbf{w}\rangle_{\,l^2_k(V)}}{\|\bar{T}_k \mathbf{w}\|_{l^2_k(V^*)}} = \\
&= \sup_{\mathbf{w}\in \mathring{l}^2_k(V)\setminus\{0\}} \frac{_{l^1_k(H)}\langle \mathbf{g}_2, \mathrm{S}_N \mathbf{w}\rangle_{\,l^\infty(H)}}{\|\bar{T}_k \mathbf{w}\|_{l^2_k(V^*)}} \le \\
&\le \|\mathbf{g}_2\|_{l^1_k(H)} \sup_{\mathbf{w}\in \mathring{l}^2_k(V)\setminus\{0\}} \frac{\|\mathbf{w}\|_{l^\infty(H)}}{\|\bar{T}_k \mathbf{w}\|_{l^2_k(V^*)}} \le C\|\mathbf{g}_2\|_{l^1_k(H)} \quad \blacksquare
\end{aligned}$$

Claim 2.9.

$$\|\mathbf{u}_2\|_{l^\infty(H)} \le \|\mathbf{g}_2\|_{l^1_k(H)}.$$

We repeat the same technique of 2.7. ■

Remark 2.10. It may seem that notations like $\|\cdot\|_{l^2_k(V)\cap l^\infty(H)}$ are superfluous, being $l^2_k(V) \hookrightarrow l^\infty(H)$; actually the norm of this immersion tends to ∞ when k goes to 0, whereas our constants C are independent of k.

Proof of theorem 2.1*: A depending on time.*

The discussion of this more general case is based on the simple remark that the values of the truncated sequence $\mathbf{u}_{|N}$ of (2.7) depend only on $\underline{\mathbf{u}}$ and $\mathbf{f}_{|N}$. Observing that $\mathbf{u}$ satisfies:

$$\frac{\rho(\mathrm{E})}{k}\mathbf{u} + \sigma(\mathrm{E})A_N \mathbf{u} = \sigma(\mathrm{E})\mathbf{f} + T_k \underline{\mathbf{u}} + \sigma(\mathrm{E})\left[(A_N - \mathbf{A})\mathbf{u}\right], \qquad \forall N \in \mathbf{N}, \tag{2.37}$$

we get consequently the estimate:

$$\|\mathbf{u}_{|N}\|^2_{l^2_k(V)\cap l^\infty(H)} \le C\left[\|\mathbf{f}_{|N}\|^2_{l^2_k(V^*)} + \|(A_N - \mathbf{A})\mathbf{u}_{|N}\|^2_{l^2_k(V^*)} + \|\underline{\mathbf{u}}\|^2_{l^2_k(V)\cap l^\infty(H)}\right]; \quad (2.38)$$

the last term may be controlled in the following way (we set $\mathbf{u}_{|-1} = 0$):

$$\begin{aligned}
\|(A_N - \mathbf{A})\mathbf{u}_{|N}\|^2_{l^2_k(V^*)} &\le k\sum_{j=0}^{N}\|A_N - A_j\|^2 \cdot \|u_j\|^2 \le \\
&\le \sum_{j=0}^{N}\|A_N - A_j\|^2 \cdot (\|\mathbf{u}_{|j}\|^2_{l^2_k(V)} - \|\mathbf{u}_{|j-1}\|^2_{l^2_k(V)}) \le \\
&\le \sum_{j=0}^{N-1}\|A_N - A_j\|^2 \cdot \|\mathbf{u}_{|j}\|^2_{l^2_k(V)} - \sum_{j=0}^{N-1}\|A_N - A_{j+1}\|^2 \cdot \|\mathbf{u}_{|j}\|^2_{l^2_k(V)} \le \\
&\le 4M\sum_{j=0}^{N-1}\Big|\|A_N - A_j\| - \|A_N - A_{j+1}\|\Big| \cdot \|\mathbf{u}_{|j}\|^2_{l^2_k(V)} \\
&\le 4M\sum_{j=0}^{N-1}\|A_j - A_{j+1}\| \cdot \|\mathbf{u}_{|j}\|^2_{l^2_k(V)}
\end{aligned}$$

From (2.38), denoting with X_N the square of the norm of $\mathbf{u}_{|N}$ in $l^2_k(V)\cap l^\infty(H)$, we get the recurrent relation:

$$X_N \le C\{\|\mathbf{f}\|^2_{l^2_k(V^*)} + \|\underline{\mathbf{u}}\|^2_{l^2_k(V)\cap l^\infty(H)}\} + \sum_{j=0}^{N-1} a_j X_j, \qquad a_j = 4M\|A_{j+1} - A_j\|_{\mathcal{L}(V,V^*)}. \quad (2.39)$$

Since $\sum_{j\in\mathbf{N}} a_j \le 4ML < \infty$, by an easily application of a Gronwall–like lemma, we have:

$$\|\mathbf{u}\|_{l^2_k(V)\cap l^\infty(H)} \le C\left\{\|\mathbf{f}\|_{l^2_k(V^*)} + \|\underline{\mathbf{u}}\|_{l^2_k(V)\cap l^\infty(H)}\right\} \quad \blacksquare$$

3 Proof of the theorems: convergence.

Approximation lemmata.

We shall compare the approximate solution ${}^h\mathbf{u}$ of (1.8) with the discretized continuous solution u; we set:

$$\big(\Pi u\big)_n = u(kn), \qquad \big({}^h\Pi u\big)_n = P_h u(kn) = \big(\Pi P_h u\big)_n. \tag{3.1}$$

On Π we have the following results (see [1], [9]):

Lemma 3.1. *There exists a constant $C>0$ such that:*

$$\forall\, v \in H^q_+(\mathcal{H}), \quad \|\Pi v\|_{l^2_k(\mathcal{H})} \le C\left\{\|v\|_{L^2_+(\mathcal{H})} + k^q\|D^q v\|_{L^2_+(\mathcal{H})}\right\} \quad \blacksquare \tag{3.2}$$

Corollary 3.2. *If v belongs to $H^q_+(V)$ and $(G1)$ holds true, we have:*

$$\|\Pi v - {}^h\Pi v\|_{l^2_k(V)\cap l^\infty(H)} \le C\left\{k^q\|v\|_{H^q_+(V)} + \|v-P_h v\|_{L^2_+(V)\cap L^\infty(H)}\right\} \quad \blacksquare \tag{3.3}$$

Lemma 3.3. *Assume that $v \in H^{q+1}_+(\mathcal{H})$ and consider the local truncation error:*

$$G_k[v](t) = \frac{1}{k}\sum_{j=0}^{g}\alpha_j v(t+jk) - \sum_{j=0}^{g}\beta_j v'(t+jk), \qquad t \ge 0. \tag{3.4}$$

There exists a constant $C>0$ such that:

$$\|G_k[v]\,\|_{L^2_+(\mathcal{H})} + k^q\|D^q G_k[v]\,\|_{L^2_+(\mathcal{H})} \le C\,k^q\|v\|_{H^{q+1}_+(\mathcal{H})}, \tag{3.5}$$

and:

$$\|\Pi G_k[v]\|_{l^2_k(\mathcal{H})} \le C\,k^q\,\|u\|_{H^{q+1}_+(\mathcal{H})}. \tag{3.6}$$

Proof. (3.6) is an immediate consequence of (3.5) and (3.2); so, we may limit ourselves to prove (3.5), or equivalently:

$$\|D^j G_k[v]\,\|_{L^2_+(\mathcal{H})} \le C\,k^{q-j}\|v\|_{H^{q+1}_+(\mathcal{H})}, \qquad 0 \le j \le q. \tag{3.7}$$

Let $r_{[0,\infty[}$ be the restriction operator from $L^2(\mathcal{H})$ to $L^2_+(\mathcal{H})$ and let p be a linear extension operator with the properties:

$$p \in \mathcal{L}(L^2_+(\mathcal{H}), L^2(\mathcal{H})) \cap \mathcal{L}(H^{q+1}_+(\mathcal{H}), H^{q+1}(\mathcal{H})); \quad \forall f \in L^2_+(H),\ r_{[0,\infty[}(pf) = f. \tag{3.8}$$

Still denoting by G_k the operator (3.4) on the whole real line, we have:

$$r_{[0,\infty[} G_k[p(v)] = G_k[v],$$

so that:

$$\|G_k[v]\|_{L^2_+(\mathcal{H})} = \left\| r_{[0,\infty[}\big[G_k[p(v)]\big] \right\|_{L^2_+(\mathcal{H})} \leq \|G_k[p(v)]\|_{L^2(\mathcal{H})};$$

therefore we have only to prove (3.7) for $\mathbf{R}-$defined functions.

By applying the Fourier transform ([5]) to $G_k[v]$ we obtain:

$$\mathcal{F}\big[G_k[v]\big](\xi) = k^{-1}\left\{\rho(e^{2\pi i\, k\xi}) - 2\pi i\, k\xi\sigma(e^{2\pi i\, k\xi})\right\}\mathcal{F}[v](\xi).$$

By $(P2)$ we get:

$$|\rho(e^{ix}) - ix\sigma(e^{ix})| \leq C\,|x|^{q+1}, \qquad x \in \mathbf{R},$$

so that:

$$\left\|\mathcal{F}\big[G_k[v]\big](\xi)\right\|_{L^2(\mathcal{H})} \leq C\,k^q \left\| |\xi|^{q+1}\mathcal{F}[v](\xi)\right\|_{L^2(\mathcal{H})} \leq C\,k^q \|v\|_{H^{q+1}(\mathcal{H})}.$$

(3.7) for $j > 0$ follows immediately by the identity $D^j G_k[v] = G_k[D^j v]$. ∎

Remark 3.4. We observe that:

$$\Pi G_k[v] = \frac{\rho(\mathrm{E})}{k}\Pi v - \sigma(\mathrm{E})\Pi v'.$$

Convergence theorem.

With new notations, theorem 1.7 becomes:

([5]) We denote with $\mathcal{F}$ the Fourier transform in $L^2(\mathcal{H})$:

$$\mathcal{F}[v](\xi) = \int_{\mathbf{R}} e^{-2\pi i\xi t} v(t)\,dt; \qquad \|\mathcal{F}[v]\|_{L^2(\mathcal{H})} = \|v\|_{L^2(\mathcal{H})}.$$

Theorem 3.5. *Assume that* $(A1-4)$, $(P1-2)$, $(G1)$ *and* $(I1)$ *hold true; the solution* ${}^h\mathbf{u}$ *of:*

$$ {}^hT_k\,{}^h\mathbf{u} = \sigma(\mathrm{E})^h\Pi f, \quad {}^h\mathbf{u}_{|g-1} = {}^h\underline{\mathbf{u}} \tag{3.9} $$

satisfies:

$$ \begin{aligned} \|{}^h\mathbf{u} - \Pi u\|_{l^2_k(V)\cap l^\infty(H)} &\le C\Big\{k^q\|u\|_{H^{q+1}_+(V,V^*)} + \|u - P_h u\|_{L^2_+(V)\cap L^\infty_+(H)} + \epsilon[u;{}^h\underline{\mathbf{u}}]\Big\} \le \\ &\le C\left\{k^q\left[\|f\|_{H^q_+(V^*)} + \|\mathbf{c}_q(f,u_0)\|_{V^q\times H}\right] + e_h[u]\right\} \end{aligned} \tag{3.10} $$

Proof. We have the following decomposition:

$$ \Pi u - {}^h\mathbf{u} = \left(\Pi u - {}^h\Pi u\right) + \left({}^h\Pi u - {}^h\mathbf{u}\right) $$

so that, by applying corollary 3.2, it remains to study the difference ${}^h\mathbf{d} = {}^h\Pi u - {}^h\mathbf{u}$ which is contained in $l^2_k(V_h)\cap l^\infty(H_h)$.

Our purpose is to write a difference equation satisfied by ${}^h\mathbf{d}$ and to apply the preceding stability estimates. We observe that:

$$ \|{}^h\mathbf{d}_{|g-1}\|_{l^2_k(V)\cap l^\infty(H)} = \|({}^h\Pi u)_{|g-1} - {}^h\underline{\mathbf{u}}\|_{l^2_k(V)\cap l^\infty(H)} = \epsilon[u;{}^h\underline{\mathbf{u}}] $$

so that, by $(I1)$:

$$ \|{}^h\mathbf{d}_{|g-1}\|_{l^2_k(V)\cap l^\infty(H)} \le k^q\left[\|f\|_{H^q(0,kg;V^*)} + \|\mathbf{c}_q\|_{V^q\times H}\right]. \tag{3.11} $$

If we apply operator ${}^h\Pi$ to (1.1), we obtain ${}^h\Pi u' + {}^h\mathbf{A}\Pi u = {}^h\mathbf{f}$, with ${}^h\mathbf{A} = P_h\mathbf{A}$, ${}^hf = P_hf$, and:

$$ {}^hT_k\left[{}^h\Pi u\right] = \sigma(\mathrm{E})^h\Pi f + P_h\left\{\frac{\rho(\mathrm{E})}{k}\Pi u - \sigma(\mathrm{E})\Pi u'\right\} + \sigma(\mathrm{E})^h\mathbf{A}\Pi(P_hu - u). $$

Taking the difference with (3.9), we get:

$$ {}^hT_k\,{}^h\mathbf{d} = {}^h\Pi G_k[u] + \sigma(\mathrm{E})^h\mathbf{A}\Pi(P_hu - u). $$

By lemma 3.3

$$ \|{}^h\Pi G_k[u]\,\|_{l^2_k(V_h^*)} \le C\,k^q\|u\|_{H^{q+1}_+(V^*)}, $$

and by corollary 3.2 we have:

$$ \|{}^h\mathbf{A}\Pi(P_hu - u)\|_{l^2_k(V_h^*)} \le M\|{}^h\Pi u - \Pi u\|_{l^2_k(V)} \le C\left\{\|P_hu - u\|_{L^2_+(V)} + k^q\|u\|_{H^q_+(V)}\right\}; $$

taking into account (3.11) and applying Theorem 1.4, we conclude our proof. ∎

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References.

1. J. P. Aubin (1972): Approximation of Elliptic Boundary Value Problems. Wiley, New York.

2. C. Baiocchi (1971): Teoremi di regolarità per le soluzioni di un'equazione differenziale astratta. Ist. Naz. Alta Mat. Symp. Mat. **7**, 269-323.

3. C. Baiocchi: Discretizzazione di problemi parabolici. To appear on Ricerche di Matematica.

4. C. Baiocchi, F. Brezzi (1983): Optimal error estimates for linear parabolic problems under minimal regularity assumptions. Calcolo **20**, 143-176.

5. G. Dahlquist (1963): A special problem for linear multistep methods. BIT **3**, 27-43.

6. J. L. Lions, E. Magenes (1972): Non homogeneus boundary value problems and applications I. Springer Verlag, Berlin.

7. O. Nevanlinna, F. Odeh (1981): Multiplier techniques for linear multistep methods. Numer. funct. anal. and optimiz. **3**, 377-423.

8. W. Rudin (1974): Real and Complex Analysis. McGraw-Hill, New York.

9. G. Savaré (1991): Discretizzazioni $A(\Theta)-$stabili di equazioni differenziali astratte. Calcolo **28**, 205-247.

10. H. J. Stetter (1973): Analysis of discretization methods for ordinary differential equations. Springer Verlag, Berlin.

11. F. Tomarelli (1983): Weak solutions for an abstract Cauchy problem of parabolic type. Annali di Matematica Pura e Applicata IV **130**, 93-123.

12. F. Tomarelli (1984): Regularity theorems and optimal error estimates for linear parabolic Cauchy problems. Numerische Mathematik **45**, 23-50.

13. O. B. Widlund (1967): A note on unconditionally stable linear multistep methods. BIT **7**, 65-70.