THE ASYMPTOTIC PROFILE OF SOLUTIONS OF A CLASS OF DOUBLY NONLINEAR EQUATIONS

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Abstract. Let Ω be a bounded domain of $\mathbf{R}^N$ and let u_0 belong to $L^{\frac{m+2p-3}{p-1}}(\Omega)$, where $p > 1$, $2 < m + p < 3$ and $p\,\frac{m+2p-3}{m-3+(4-m)p-p^2} > N$. We study the asymptotic behaviour of the solutions of the initial-boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}(|u|^{m-1}|Du|^{p-2}Du) & \text{in } \Omega\times]0,\infty[, \\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases}$$

under Dirichlet, variational and mixed boundary conditions.

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1. Introduction and main results.

In the last few years several authors studied the regularity and the asymptotic behaviour of nonnegative weak solutions of the boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u^m = 0 & \text{in } \Omega\times]0,T[, \\ u \in C(0,T;L^2(\Omega)), \quad u^m \in L^2(0,T;W_0^{1,2}(\Omega)) & \\ u(x,0) = u_0(x) \in L^{1+m}(\Omega) & \end{cases} \tag{1,1}$$

with $0 < m < 1$.

Among them, we quote [2] where the asymptotic behaviour for separable classical solutions is studied (we refer the reader also to [1] where the physical motivation of $(1,1)$ is pointed out).

In [12] the result of [2] is extended to the case of weak solutions. In [8] it is proved that if $\frac{(N-2)_+}{N+2} < m < 1$ then the solution is analytic in the interior of Ω with respect to the space variables and Lipschitz continuous with respect to t up to the extinction time.

The aim of this note is to extend the previous results to a larger class of equations under more general boundary conditions dropping the hypothesis of nonnegativity of the initial datum and weakening the regularity assumptions on the boundary of Ω (in the above mentioned papers $\partial\Omega$ is assumed to be of class C^2).

We consider the case of a bounded domain $\Omega \subset \mathbf{R}^N$ with boundary of class $C^{1,\alpha}$ for some $\alpha > 0$. The initial datum u_0 belongs to $L^{\frac{m+2p-3}{p-1}}(\Omega)$, where:

$$p > 1, \quad 2 < m + p < 3, \quad p\frac{m+2p-3}{m-3+(4-m)p-p^2} > N \tag{1,2}$$

We study consider the weak solutions of:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}(|u|^{m-1}|Du|^{p-2}Du) & \text{in } \Omega\times]0,\infty[, \\ u(x,0) = u_0(x) & \text{in } \Omega \\ u \in C(0,\infty;L^2(\Omega)), \quad u^{\frac{m-1}{p-1}}Du \in L^p(\Omega\times]0,\infty[) & \end{cases} \tag{1,3}$$

satisfying Dirichlet, variational or mixed conditions.

Here the notion of weak solution is standard and we refer the reader to [7] for details.

Equations of the type $(1,3)$ are classified as "doubly nonlinear" ([14]) or "with implicit nonlinearity" ([11]). Many authors studied this kind of equations on account of their physical and mathematical interest. We refer the reader to the review paper [11] and to the introduction of [16].

Before stating the main results, let us make some remarks about the range of m and p. If $m + p \leq 2$, it is not known whether L^∞−estimates and regularity results hold. See the detailed discussion in [16].

If $m+p<3$, then the solution extinguishes after a finite time (see [5]), while this does not happen if $m+p\geq 3$. Different techniques are required in the latter case, which will be studied in a forthcoming paper.

The condition $p\frac{m+2p-3}{m-3+(4-m)p-p^2}>N$ implies that a Sobolev imbedding inequality holds.

Main results.

1.1 THEOREM. *Let $u_0\not\equiv 0$, and let u be a weak solution of $(1,3)$ satisfying Dirichlet or mixed boundary conditions, and let $T>0$ be the extinction time. Let $\bar{u}(x,t)=u(x,t)(T-t)^{\frac{-1}{3-m-p}}$. Then there exists a sequence $t_n\nearrow T$ such that $\bar{u}(x,t)\to w(x)$, where w is a nontrivial solution of the equation:*

$$\mathrm{div}(|w|^{m-1}|Dw|^{p-2}Dv)=\frac{1}{3-m-p}w,\quad \text{in } \Omega \tag{1,4}$$

satisfying Dirichlet or mixed boundary conditions, respectively. ■

1.2 THEOREM. *Let u be a weak solution of $(1,3)$ satisfying variational boundary conditions.*
If $\int_\Omega u_0(x)\,dx\neq 0$, then:

$$\lim_{t\to\infty}u(x,t)=|\Omega|^{-1}\int_\Omega u_0(x)\,dx,\qquad \text{in } W^{1,p}(\Omega) \tag{1,5}$$

If $\int_\Omega u_0(x)\,dx=0$ and $u_0\not\equiv 0$ then the conclusion of Theorem 1.1 holds. ■

1.3 REMARK. Result $(1,5)$ holds for $2<m+p<3$ and $p>1$.

1.4 REMARK. Theorem 1.1 above implies the existence of a nontrivial weak solution of the problem $(1,4)$ satisfying Dirichlet or mixed boundary conditions. A straightforward consequence of theorem 1.2 is that the average of any solution of $(1,4)$ satisfying variational boundary conditions is zero. Indeed if $u(x,t)$ solves $(1,3)$ under variational boundary conditions then $\int_\Omega u(x,t)\,dx$ is constant. Now, assume the existence of a solution $w(x)$ of (1.4) satisfying variational boundary conditions with $\int_\Omega w(x)\,dx\neq 0$. Set $u(x,t)=w(x)(1-t)^{\frac{1}{3-m-p}}$. The function $u(x,t)$ solves $(1,3)$, it satisfies variational boundary conditions and its average is not constant in time. Contradiction.

If u is a nonnegative local solution of $(1,3)$, i.e.:

$$\begin{cases} u\in C_{loc}(0,T;L^2_{loc}(\Omega)), \quad u^{\frac{m-1}{p-1}}Du\in L^p_{loc}(\Omega\times[0,T[) & \\ \dfrac{\partial u}{\partial t}=\mathrm{div}(|u|^{m-1}|Du|^{p-2}Du) & \text{in } \Omega\times]0,T[\end{cases}$$

where $p > 1$, $m + p > 2$, $m + p + \frac{p}{N} > 3$, we can state more precise results.

Fix any $(x_0, t_0) \in \Omega_T = \Omega \times]0, T[$, assume that $u(x_0, t_0) > 0$ and let $1 > \rho > 0$ be so small such that the cylindrical domain:

$$\{|x - x_0| < \rho\} \times (t_0 - u(x_0, t_0)^{3-m-p}\rho^p, t_0 + u(x_0, t_0)^{3-m-p}\rho^p)$$

is contained in Ω_T.

1.5 THEOREM. *There exists a constant C depending only upon N, m and p and independent of u such that:*

$$|Du(x_0, t_0)| \leq C\rho^{-1}u(x_0, t_0) \quad \blacksquare \tag{1,6}$$

1.6 REMARK. Note that estimate $(1,6)$ is false when $m + p = 3$. For instance, the function:

$$u(x, t) = ct^{-\frac{N}{p}} \exp\left[(p-1)^2 \left(\frac{|x|t^{\frac{-1}{p}}}{p}\right)^{\frac{p-1}{p}}\right]$$

solves the equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}(|u|^{2-p}|Du|^{p-2}Du) & \text{in } \mathbf{R}^N \times]0, \infty[\\ u(0) = c_1\delta(0) \end{cases}$$

The counterexample can be easily built choosing $t = 1$ and a sequence of points x_n such that $|x_n| \to \infty$.

Now we focus our attention on the case $u_0 \geq 0$ and Dirichlet boundary condition.

1.7 THEOREM. *For any $\epsilon \in (0, T)$, there exist constants γ_i, $i = 1, 2, 3$, depending only upon $N, m, p, \|u_0\|_{L^{\frac{m+2p-3}{p-1}}(\Omega)}, \Omega$ and ϵ such that for all $(x, t) \in \Omega_T$, $t > \epsilon$:*

$$\gamma_1 \operatorname{dist}(x, \partial\Omega)^{\frac{1-m}{m+p-2}}(T-t)^{\frac{1}{3-m-p}} \leq u(x, t) \leq \gamma_2 \operatorname{dist}(x, \partial\Omega)^{\frac{1-m}{m+p-2}}(T-t)^{\frac{1}{3-m-p}} \tag{1,7}$$

$$|Du(x, t)| \leq \gamma_3 \operatorname{dist}(x, \partial\Omega)^{\frac{1-m}{m+p-2}}(T-t)^{\frac{1}{3-m-p}} \quad \blacksquare \tag{1,8}$$

1.8 REMARK. As $p\frac{m+2p-3}{m=3+(4-m)p-p^2} \searrow N$ the constants γ_i blow up.

1.9 Remark. Note that $(1,7)$ and $(1,8)$ imply that any nontrivial nonnegative solution of:

$$\begin{cases} \operatorname{div}(w^{m-1}|Dw|^{p-2}Dw) = \frac{1}{3-m-p}w & \text{in } \Omega, \\ w|_{\partial\Omega} = 0, \end{cases}$$

satisfies:

$$\gamma_1 \operatorname{dist}(x,\partial\Omega)^{\frac{1-m}{m+p-2}} \le w(x,t) \le \gamma_2 \operatorname{dist}(x,\partial\Omega)^{\frac{1-m}{m+p-2}}; \tag{1,9}$$

$$|Du(x,t)| \le \gamma_3 \operatorname{dist}(x,\partial\Omega)^{\frac{1-m}{m+p-2}}. \tag{1,10}$$

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2. Asymptotic results in the general case.

As in [2], we suppose that u is a regular solution. The general case can be studied by approximating the equations with nonsingular ones (see [12]).

Let

$$a = \frac{p-1}{m+p-2}, \qquad a > p-1 > 0; \qquad u = v^a.$$

2.1 LEMMA. *The quotient:*

$$\mathcal{E}[v](t) = \left(\frac{\|Dv(t)\|_{L^p(\Omega)}}{\|v(t)\|_{L^{a+1}(\Omega)}} \right)^p \tag{2,1}$$

is non increasing in time.

PROOF. Note that v satisfies the equation:

$$\frac{\partial}{\partial t}(v^a) = a^{p-1} \operatorname{div}(|Dv|^{p-2} Dv) \tag{2,2}$$

in $\Omega \times]0, \infty[$; multiplying by v and integrating over Ω we obtain:

$$\frac{1}{a+1} \frac{d}{dt} \int_\Omega v^{a+1}\, dx + a^{p-2} \int_\Omega |Dv|^p\, dx = 0. \tag{2,3}$$

Setting $\Delta_p v = \operatorname{div}(|Dv|^{p-2} Dv)$, we get:

$$\int_\Omega |Dv|^p\, dx = -\int_\Omega v \Delta_p v\, dx \leq \left(\int_\Omega v^{a+1}\, dx \right)^{1/2} \left(\int_\Omega \frac{|\Delta_p v|^2}{v^{a-1}}\, dx \right)^{1/2}. \tag{2,4}$$

On the other hand,

$$\frac{d}{dt} \int_\Omega |Dv|^p\, dx = p \int_\Omega |Dv|^{p-2}\, Dv \cdot Dv_t\, dx = -p \int_\Omega v_t \Delta_p v\, dx, \tag{2,5}$$

and from equation (2.2) we have:

$$v_t = a^{p-2} \frac{\Delta_p v}{v^{a-1}}.$$

Hence:

$$\frac{d}{dt} \int_\Omega |Dv|^p\, dx = -p a^{p-2} \int_\Omega \frac{|\Delta_p v|^2}{v^{a-1}}\, dx, \tag{2,6}$$

which gives, together with $(2,4)$:

$$\frac{d}{dt} \int_\Omega |Dv|^p\, dx \leq -p a^{p-2} \frac{\left(\int_\Omega |Dv|^p\, dx \right)^2}{\int_\Omega v^{a+1}\, dx}. \tag{2,7}$$

Finally, by $(2,2)$

$$\frac{1}{p}\frac{\frac{d}{dt}\int_\Omega |Dv|^p\,dx}{\int_\Omega |Dv|^p\,dx} \le \frac{1}{a+1}\frac{\frac{d}{dt}\int_\Omega v^{a+1}\,dx}{\int_\Omega v^{a+1}\,dx}$$

and:

$$\frac{d}{dt}\log\left(\int_\Omega |Dv|^p\,dx\right)^{1/p} \le \frac{d}{dt}\log\left(\int_\Omega v^{a+1}\,dx\right)^{\frac{1}{a+1}}$$

which means that the function $\mathcal{E}[v](t)$ does not increase in time. ∎

2.2 Remark. By $(2,3)$ we get

$$\frac{d}{dt}\int_\Omega v^{a+1}\,dx = -(a+1)a^{p-2}\int_\Omega |Dv|^p\,dx = -(a+1)a^{p-2}\mathcal{E}[v]\left(\int_\Omega v^{a+1}\right)^{\frac{p}{a+1}}$$

that is

$$\frac{d}{dt}\|v\|^{a+1-p}_{L^{a+1}(\Omega)} = -(a+1-p)a^{p-2}\mathcal{E}[v], \qquad a+1-p>0.$$

Integrating the O.D.E. we have:

$$\|v(t)\|^{a+1-p}_{L^{a+1}(\Omega)} = -(a+1-p)a^{p-2}\int_0^t \mathcal{E}[v](s)\,ds + \|v_0\|^{a+1-p}_{L^{a+1}(\Omega)}. \qquad (2,8)$$

2.3 Remark. Assume that the solution v satisfies a Sobolev type inequality, i. e. :

$$\exists B = B_{m,p}(\Omega) > 0 : \left(\int_\Omega v^{a+1}\,dx\right)^{\frac{p}{a+1}} \le B\int_\Omega |Dv|^p\,dx \qquad (2,9)$$

Then: $\mathcal{E}[v](0) \ge \mathcal{E}[v](t) \ge B$ and by $(2,8)$ we get that u becomes extinct after a finite time T, with rate of extinction $(T-t)^{\frac{1}{a+1-p}}$.

We make the substitution $t = T - Te^{-\tau}$ and consider the function $\tilde{v}$:

$$\tilde{v}(\cdot,\tau) = \frac{v(\cdot, T-Te^{-\tau})}{(Te^{-\tau})^{\frac{1}{a+1-p}}}$$

that satisfies the equation:

$$\frac{\partial}{\partial t}(\tilde{v}^a) - a^{p-1}\Delta_p\tilde{v} = \frac{a}{a+1-p}\tilde{v}^a \qquad (2,10)$$

2.4 Lemma. *Under the hypotheses of lemma 2.1, the function:*

$$F[\tilde{v}](\tau) = a^{p-2}\int_\Omega |D\tilde{v}|^p\,dx - \frac{p}{(a+1)(a+1-p)}\int_\Omega \tilde{v}^{a+1}\,dx \qquad (2,11)$$

is not increasing.

From $(2,5)$ and $(2,10)$, we get:

$$a^{p-2}\frac{d}{dt}\int_\Omega |D\tilde{v}|^p\,dx = -\,p\int_\Omega (\tilde{v}_t)^2\tilde{v}^{a-1}\,dx + \frac{p}{a+1-p}\int_\Omega \tilde{v}_t\tilde{v}^a\,dx =$$
$$= -\,p\int_\Omega (\tilde{v}_t)^2\tilde{v}^{a-1}\,dx + \frac{p}{(a+1)(a+1-p)}\frac{d}{dt}\int_\Omega \tilde{v}^{a+1}\,dx \quad \blacksquare$$

2.5 COROLLARY. *There exists a sequence $\tau_n \to \infty$ such that:*

$$\lim_{n\to\infty}\frac{d}{d\tau}F[\tilde{v}](\tau_n) = 0$$

PROOF. It is sufficient to remark that $F(\tau)$ is bounded from below. $\blacksquare$

Proof of Theorem 1.1.

In the cases of mixed or Dirichlet conditions a Sobolev inequality holds and the results following remark 2.3 are verified. Therefore, arguing as in [2], we get that there is a sequence $t_n \nearrow T$ such that:

$$\frac{u(\cdot,t_n)}{(T-t_n)^{\frac{1}{3-m-p}}} \to w$$

where w is a solution of the equation:

$$\mathrm{div}(|w|^{m-1}|Dw|^{p-2}Dw) = \frac{1}{3-m-p}w$$

satisfying the corresponding boundary conditions. On the other hand, from $(2,8)$ we get:

$$c_1 \le \frac{\|v(t_n)\|_{L^{a+1}(\Omega)}}{(T-t_n)^{\frac{1}{a+1-p}}} \le c_2$$

Hence by lemma 2.1 and remark 2.3 we have

$$Bc_1 \le \frac{\|Dv(t_n)\|_{L^p(\Omega)}}{(T-t_n)^{\frac{1}{a+1-p}}} \le \mathcal{E}[v](0)c_2$$

Therefore a suitable subsequence of $\left\{\frac{v(t_n)}{(T-t_n)^{\frac{1}{a+1-p}}}\right\}_{n\in\mathbf{N}}$ strongly converges in $L^{a+1}(\Omega)$ to a function $w^{\frac{1}{a}} \not\equiv 0$. $\blacksquare$

Proof of Theorem 1.2.

If $\int_\Omega u_0\,dx = 0$, then a Sobolev inequality holds for each $t \geq 0$ and we may repeat the previous arguments.

In the case $\int_\Omega u_0\,dx \neq 0$, there is not extinction time. Recalling that $\|v\|_{L^{a+1}(\Omega)}$ is non increasing by Remark 2.2, from $(2,7)$ we deduce:

$$\frac{d}{dt}\int_\Omega |Dv|^p\,dx \leq -pa^{p-2}\frac{\left(\int_\Omega |Dv|^p\,dx\right)^2}{\int_\Omega v_0^{a+1}\,dx} \tag{2,12}$$

which implies

$$\int_\Omega |Dv(x,t)|^p\,dx \leq \frac{c}{t} \tag{2,13}$$

On the other hand we have:

$$\int_\Omega u(x,t)\,dx = \int_\Omega u_0\,dx \tag{2,14}$$

Therefore

$$u(\cdot,t) \to |\Omega|^{-1}\int_\Omega u_0\,dx \quad \text{in } W^{1,p}(\Omega) \quad \blacksquare$$

3. Regularity results for nonnegative local solutions.

In order to prove theorem 1.5, let us recall a result of [16].

3.1 PROPOSITION. *Let u be a nonnegative local solution of* $(1,3)$. *Fix* $(x_0, t_0) \in \Omega_T$ *and set* $u_0 = u(x_0, t_0)$*; choose* $\rho > 0$ *so small that the cylindrical domain:*

$$C_{4\rho} = \{|x - x_0| < 4\rho\} \times (t_0 - u_0^{3-m-p}(4\rho)^p, t_0 + u_0^{3-m-p}(4\rho)^p)$$

is contained in Ω_T. *There exist constants* γ, γ_1 *depending only upon* N, m, p *such that:*

$$\inf_{|x-x_0|<\rho} u(x, t_0) \geq \gamma \sup_{|x-x_0|<\rho} u(x, t_0) \tag{3,1}$$

and:

$$\inf_{C_\rho} u(x,t) \geq \gamma_1 \sup_{C_\rho} u(x,t) \tag{3,2}$$

where $m + p < 3$ *and* $m + p + \frac{p}{N} > 3$. ∎

3.2 REMARK. As pointed out in [17], inequalities $(3,1)$ and $(3,2)$ do not hold in the case $m + p = 3$.

Another tool is the following regularity result:

3.3 PROPOSITION. *Let* $Q_s = \{|x| < s\} \times \{|t| < s\}$ *and let u be a nonnegative local solution of the equation:*

$$\frac{\partial u}{\partial t} = \operatorname{div}(|u|^{m-1}|Du|^{p-2}Du) \quad \text{in } Q_1 \tag{3,3}$$

Moreover assume $\frac{1}{2} \leq u \leq 2$ *in* Q_1*; then there exists* $\alpha > 0$ *such that* $u \in C^{1,\alpha}(\overline{Q_{1/2}})$ *and:*

$$\|Du\|_{L^\infty(Q_{1/2})} \leq K \tag{3,4}$$

where K depends only upon N, m and p.

PROOF. Let $(x_0, t_0) \in Q_{1/2}$, $u_0 = u(x_0, t_0)$ and for $r < 1/2$ denote by Q'_r the cylinder:

$$\{|x - x_0| < r\} \times]t_0 - r, t_0[\subset Q_1.$$

Let v the solution of:

$$\begin{cases} \dfrac{\partial v}{\partial t} = \operatorname{div}(u_0^{m-1}|Dv|^{p-2}Dv) & \text{in } Q'_r \\ v = u & \text{on the parabolic boundary of } Q'_r \end{cases}$$

In [13] (see also [9]) it is proved that there is $\alpha > 0$ such that for each $0 < \rho < r$:

$$\iint_{Q'_\rho} |Dv - (Dv)_\rho|^p \, dx\, dt \le c_\alpha \left(\frac{\rho}{r}\right)^{N+\alpha} \iint_{Q'_r} |Dv - (Dv)_r|^p \, dx\, dt$$

where $(Dv)_s = \frac{1}{|Q'_s|} \iint_{Q'_s} Dv \, dx\, dt$.

Hence:

$$\frac{\partial}{\partial t}(v-u) - \operatorname{div}[u_0^{m-1}(|Dv|^{p-2}Dv - |Du|^{p-2}Du)] = \\ = \operatorname{div}[(u^{m-1} - u_0^{m-1})|Du|^{p-2}Du]$$

Therefore, since u is Hölder continuous ([15], [16]), arguing as in [5] one gets:

$$\iint_{Q'_r} |Dv - Du|^p \, dx\, dt \le \gamma r^\delta \iint_{Q'_r} |Du|^p \, dx\, dt$$

and consequently:

$$\begin{aligned}
&\iint_{Q'_\rho} |Du - (Du)_\rho|^p \, dx\, dt \le \\
&\quad \le c \iint_{Q'_\rho} \{|Dv - (Dv)_\rho|^p + |Du - Dv|^p + |(Du)_\rho - (Dv)_\rho|^p\} \, dx\, dt \le \\
&\quad \le c\gamma r^\delta \iint_{Q'_r} |Du|^p \, dx\, dt + c c_\alpha \left(\frac{\rho}{r}\right)^{N+\alpha} \iint_{Q'_r} |Dv - (Dv)_r|^p \, dx\, dt \le \\
&\quad \le K r^\delta \iint_{Q'_r} |Du|^p \, dx\, dt + K \left(\frac{\rho}{r}\right)^{N+\alpha} \iint_{Q'_r} |Du - (Du)_r|^p \, dx\, dt
\end{aligned}$$

The last inequality implies that the gradient of u is Hölder continuous (see [4], [10]). ■

Proof of theorem 1.5.

Fix $(x_0, t_0) \in \Omega_T$, $u_0 = u(x_0, t_0)$ and for $\theta > 0$ consider the cylinder

$$C = \{|x - x_0| < 8r\} \times (t_0 - u_0^{3-m-p} 8\theta r^p, t_0 + u_0^{3-m-p} 8\theta r^p) \subset \Omega_T.$$

The change of variables

$$x \to \frac{x - x_0}{r}, \qquad t \to \frac{t - t_0}{u_0^{3-m-p} r^p}, \qquad v = \frac{u}{u_0}$$

maps C in $B_8 \times (-8\theta, 8\theta)$ and v satisfies:

$$\begin{cases} \frac{\partial}{\partial t} v - \operatorname{div}(v^{m-1}|Dv|^{p-2}Dv) = 0 & \text{in } B_8 \times (-8\theta, 8\theta) \\ v(0,0) = 1 \end{cases}$$

By $(3,2)$:

$$\sup_{B_4 \times (-4\theta, 4\theta)} v \le \gamma_1^{-1}, \qquad \inf_{B_4 \times (-4\theta, 4\theta)} v \ge \gamma_1$$

Hence by proposition 3.3 $v \in C^{1,\alpha}(B_4 \times (-4\theta, 4\theta))$ and $|Dv(0,0)| \le K$.

Estimate $(1,6)$ follows by recalling the definition of v ■

4. Asymptotic results for nonnegative Dirichlet solutions.

Here we follow the approach introduced in [8]. Our main goal is to weaken the regularity assumptions on the boundary. The proof of Theorem 1.7 is based on the following two proposition.

4.1 PROPOSITION. *Let u a nonnegative bounded solution of:*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}(|u|^{m-1}|Du|^{p-2}Du) & \text{in } \Omega\times]s,T[, \\ u \in C(s,T;L^2(\Omega)), \quad u^{\frac{m-1}{p-1}}Du \in L^p(\Omega\times]s,T[) \\ u \le M \end{cases} \tag{4,1}$$

for some $s \in]0,T[$ and some $M > 0$. For every $\nu > 0$, there exists a constant γ depending only upon N, m, p, ν and $\|\partial\Omega\|_{1,\alpha}$ such that, for all $t - s \ge \nu M^{3-m-p}$:

$$u(x,t) \le \gamma M[\operatorname{dist}(x,\partial\Omega)]^a, \quad a = \frac{p-1}{m+p-2} \tag{4,2}$$

4.2 PROPOSITION. *Let u be a non-negative bounded solution of* $(4,1)$ *for some $s \in (0,T)$ and some $M > 0$. For $r > 0$ let:*

$$\Omega^r = \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) \ge r\}, \quad \Omega^{r,t} = \Omega^r \times (s,t), \quad \mu(r) = \inf_{\Omega^{r,t}} u \tag{4,3}$$

For every $\nu > 0$ there exist constants r_0 and γ depending only upon N, m, p, ν and Ω such that for all $t - s > \nu M^{3-m-p}$:

$$u(x,t) \ge \gamma\mu(r_0)[d(x)]^a, \quad a = \frac{p-1}{m+p-2} \ge 1 \tag{4,4}$$

Denote by $d(x)$ the distance $\operatorname{dist}(x,\partial\Omega)$ and by Ω_s, $s > 0$, the subset of Ω where $d(x) < s$. For a point $x \in \Omega$ denote by Γ_x the set of the elements of $\partial\Omega$ closest to x: $\Gamma_x = \{y \in \partial\Omega : |x-y| = d(x)\} \ne \emptyset$. For every point $y \in \Gamma_x$ set:

$$y_\lambda = y + \lambda\frac{y-x}{|y-x|}, \quad d_\lambda = d(y_\lambda), \quad \lambda \ge 0$$

For $C^{1,\alpha}$ open sets, we have the following geometrical lemma, whose easy proof is left to the reader.

4.3 Lemma. *Let us fix two constants $\beta \in]0,\alpha[$ and $\epsilon > 0$. Then there exist constants $\bar{s}, \bar{\lambda} > 0$ depending only upon Ω such that for $x \in \Omega_{\bar{s}}$, $y \in \Gamma_x$, $\lambda \in]0,\bar{\lambda}[$ we have:*

$$\lambda - d_\lambda \leq \epsilon\lambda^{1+\beta}, \qquad d_\lambda \geq \frac{\lambda}{2} \quad \blacksquare \tag{4,5}$$

Consider now the function $v = u^{\frac{1}{a}}$ solution of $(2,2)$, set $\tilde{a} = (a^{-1}-1)^+$ and fix a number $\nu > 0$. We call I_ν the interval $]s + \nu\, M^{3-m-p}, T[$ and we set

$$M_r = \sup_{\Omega_r \times I_\nu} v(x,t) \leq M^{\frac{1}{a}}.$$

We prove the following:

4.4 Lemma. *Let us fix $\beta \in]0,\alpha[$; there exist $c, \bar{r} > 0$ such that for $0 < r < \bar{r}$ we have*

$$M_r \leq r^{\frac{\beta}{1+\beta}}\, M_{cr^{\frac{1}{1+\beta}}} \tag{4,6}$$

Proof. Let $(x_0,t_0) \in \Omega_r \times I_\nu$, $y \in \Gamma_{x_0}$ $I_\nu(t_0) =]t_0 - \nu\, M^{3-m-p}, t_0[\subset]0,T[$.

Following [8], we construct a barrier $\psi_{k,\lambda}$ of the type:

$$\psi_{k,\lambda}(x,t) = CM_{1/k}(1 - \eta_{k,\lambda}(x,t))$$

$$\eta_{k,\lambda}(x,t) = \exp[-k(|x-y_\lambda| - d_\lambda]\exp[M^{p+m-3}(t-t_0)^{1+\tilde{a}}] \tag{4,7}$$

We impose conditions on k, λ in order to have $v \leq \psi_{k,\lambda}$ in the set

$$\mathcal{N}_{k,\lambda} = \{x \in \Omega : \ |x - y_\lambda| - d_\lambda \leq \frac{1}{k}\} \times I_\nu(t_0) \subset \Omega_{1/k} \times I_\nu(t_0). \tag{4,8}$$

In view of the maximum principle, it is sufficient that

$$v(x,t) \leq \psi_{k,\lambda}(x,t), \qquad \text{on the parabolic boundary of } \mathcal{N}_{k,\lambda}, \tag{4,9}$$

$$\frac{\partial}{\partial t}[\psi^a_{k,\lambda}] - \Delta_p \psi \geq 0, \qquad \text{in } \mathcal{N}_{k,\lambda}. \tag{4,10}$$

Inequality $(4,9)$ is verified if

$$C = \max\{(1-e^{-1})^{-1}, (1-e^{-\nu})^{-1}\}.$$

Indeed, $\psi_{k,\lambda} \geq 0$ in $\mathcal{N}_{k,\lambda}$. Moreover in the set

$$\{x \in \Omega : \ |x - y_\lambda| - d_\lambda = 1/k\} \times I_\nu(t_0)$$

it holds $\psi_{k,\lambda} \geq CM_{1/k}(1-e^{-1}) \geq C(1-e^{-1})v$. Finally, in

$$\Omega_{1/k} \times \{t_0 - \nu M^{3-m-p}\},$$

we have $\psi_{k,\lambda} \geq CM_{1/k}(1-e^{-\nu})$.

By direct calculations, $(4,10)$ is equivalent to:

$$(p-1)k^p - \frac{N-1}{|x-y_\lambda|}k^{p-1} - (1+\tilde{a})a^{2-p}(CM_{1/k})^{a+1-p}M^{m+p-3}(1-\eta_{k,\lambda})^{a-1}(t-t_0)^{\tilde{a}}\eta^{2-p} \geq 0 \tag{4,11}$$

By recalling that

$$\inf_{x\in\Omega}|x-y_\lambda| = d_\lambda, \quad M_{1/k}^{a+1-p} \leq M^{\frac{a+1-p}{a}} = M^{3-m-p}, \quad 0 \leq \eta_{k,\lambda} \leq 1,$$

inequality $(4,11)$ holds if:

$$k \geq C_{m,p,\nu} + 2\frac{N-1}{(p-1)d_\lambda} \tag{4,12}$$

Let $1/k = 2Cr^{\frac{1}{1+\beta}}$, $\lambda = 8C\left(\frac{N-1}{p-1}\right)r^{\frac{1}{1+\beta}}$ and $(x_0,t_0) \in \Omega_r \times I_\nu$. Choose $\bar{r}$ small enough (depending only on m,p,N,ν,r). By the maximum principle

$$v(x_0,t_0) \leq \psi_{k,\lambda}(x_0,t_0).$$

Hence:

$$v(x_0,t_0) \leq CM_{\frac{1}{k}}k(r+\lambda-d_\lambda) \leq M_{2Cr^{\frac{1}{1+\beta}}}r^{\frac{\beta}{1+\beta}} \tag{4,13}$$

Estimate $(4,13)$ implies $(4,6)$ because the constants are independent of the choice of (x_0,t_0). ∎

Proof of proposition 4.1.

It is sufficient to prove $(4,1)$ for $x \in \Omega_{\bar{r}}$ with $\bar{r}$ given by the previous lemma. Moreover, $(4,1)$ is equivalent to:

$$\forall r < \bar{r}: \qquad M_r \leq \gamma M r \tag{4,14}$$

Let $\delta = \frac{1}{1+\beta}$, $C' = (2C)^{\frac{\beta}{\beta+1}}$ and let $n \in \mathbf{N}$ be such that:

$$C'r^{(\delta^n)} \leq \bar{r} \leq C'r^{(\delta^{n+1})}, \qquad \text{for } n \in \mathbf{N} \tag{4,15}$$

By $(4,13)$, we get:

$$M_r \leq (M_{C'r^{\delta^n}})\, r^{\beta\delta\frac{1-\delta^n}{1-\delta}} \leq C'M_{\bar{r}}\, r\,\frac{1}{\bar{r}^{1+\beta}} \leq \frac{C'}{\bar{r}^{\frac{1}{1+\beta}}}\,M\,r \quad ∎$$

Before proving theorem 1.7 let us introduce some notation. Let $x \in \Omega$, $y \in \Gamma_x$ and denote by y^λ the point

$$y^\lambda = x - \lambda\frac{y-x}{|y-x|}, \qquad d^\lambda = d(y^\lambda)$$

As before, the following statement can be easily proved:

4.5 LEMMA. *Let us fix two constants $\beta \in]0, \alpha[$ and $\epsilon > 0$. Then there exist constants $\bar{s}, \bar{\lambda} > 0$ depending only upon Ω such that for $x \in \Omega^{\bar{s}}$, $y \in \Gamma_x$, $\lambda \in]0, \bar{\lambda}[$ we have*

$$d(x) + \lambda - d^\lambda \leq \epsilon \lambda^{1+\beta}, \qquad d^\lambda \geq d(x) + \frac{\lambda}{2} \quad \blacksquare \tag{4,16}$$

PROOF OF PROPOSITION 4.2.

Let (x_0, t_0) be a point in $\Omega^r \times I_\nu$, let y belong to Γ_{x_0}, and let $I_\nu(t_0)$ be equal to $]s + \nu M^{3-m-p}, t_0[$.

Consider the barrier:

$$\tilde{\psi}_{k,\lambda} = e^{-1}\mu_{1/k}(\tilde{\eta}_{k,\lambda} - 1), \quad \tilde{\eta}_{k,\lambda}(x,t) = \exp[-k(|x - y_\lambda| - d_\lambda)] \exp\left(\frac{t - t_0}{t_0 - s}\right)^{1+\tilde{a}}$$

in:

$$\tilde{\mathcal{N}}_{k,\lambda} = \{x \in \Omega : \ d^\lambda - 1/k \leq |x - y^\lambda| \leq d^\lambda\} \times I_\nu(t_0).$$

$v \geq u$ in $\tilde{\mathcal{N}}_{k,\lambda}$ by means of the comparison principle as in lemma 4.3.

Actually, in the set

$$\{|x - y^\lambda| = d^\lambda\} \times I_\nu(t_0)$$

we have $\tilde{\psi}_{k,\lambda} \leq 0$, in the set

$$\{|x - y^\lambda| = d^\lambda - 1/k\} \times I_\nu(t_0) \subset \Omega^{1/k} \times I_\nu(t_0)$$

we have $\tilde{\psi}_{k,\lambda} \leq \frac{e-1}{e}\mu_{1/k} \leq v$ and in the bottom of $\tilde{\mathcal{N}}_{k,\lambda}$ we have $\tilde{\eta}_{k,\lambda} \leq 1$ and $\tilde{\psi}_{k,\lambda} \leq 0$.

Moreover, by direct calculations it is possible to check that:

$$\frac{\partial}{\partial t}[\psi^a_{k,\lambda}] - \Delta_p \psi \leq 0, \qquad \text{in } \mathcal{N}_{k,\lambda}$$

if:

$$k \geq C_{m,p,\nu} + 2\frac{N-1}{(p-1)d^\lambda}$$

where C depends only on m, p, ν. Choosing $1/k = \frac{1}{2e} r^{\frac{1}{1+\beta}}$, $\lambda = 2\frac{N-1}{p-1} r^{\frac{1}{1+\beta}}$ and $\bar{r}$ sufficiently small, we get

$$v(x_0, t_0) \geq e^{-1}\mu_{1/k} k[d(x_0) - (d(x_0) + \lambda - d^\lambda)] \geq 2\mu_{1/k} r^{\frac{-1}{1+\beta}}[r - \epsilon\lambda^{1+\beta}] \geq \mu_{2er^{\frac{1}{1+\beta}}} r^{\frac{\beta}{1+\beta}}$$

The last inequality implies $(4,15)$ through arguments similar to the ones of Proposition 4.1. $\blacksquare$

Proof of Theorem 1.7.

As the proof is similar to the one followed in [8], we focus our attention only on what is really new and we refer the reader to section 6 of [8].

By the $L^\infty-$estimates proved in section 3 of [17], and by the Raleigh quotient one may deduce that $\forall \epsilon < t < T$:

$$c_0(T-t)^{\frac{1}{3-m-p}} \leq \|u(\cdot,t)\|_{L^\infty(\Omega)} \leq C_0(\epsilon)(T-t)^{\frac{1}{3-m-p}} \tag{4,17}$$

where C_0 depends on the initial datum and on ϵ.

Consider the change of coordinates: $T-t=(T-t_0)e^{-\tau}$. Let

$$w(\cdot,\tau)=\frac{u(\cdot,T-(T-t_0)e^{-\tau})}{(T-t_0)^{\frac{1}{3-m-p}}e^{\frac{\tau}{3-m-p}}}$$

The function w is a non-negative bounded weak solution of:

$$\begin{cases} \frac{\partial}{\partial t}w-\operatorname{div}(w^{m-1}|Dw|^{p-2}Dw)=\frac{1}{3-m-p}w, & \text{in } \Omega\times(0,\infty) \\ c_0 \leq \|w(\cdot,t)\|_{L^\infty(\Omega)} \leq C_0 \end{cases} \tag{4,18}$$

The next step is to show that for every $\sigma>0$ there exist two positive constants $\lambda<\Lambda$ depending only upon $N,m,p,\epsilon,\|\partial\Omega\|_{1,\alpha}$ and $\sigma>0$ such that $\forall t>\sigma$, $\forall x\in\Omega$:

$$\lambda d(x)^a \leq w(x,t) \leq \Lambda d(x)^a \tag{4,19}$$

Now by $(4,2)$

$$w(x,t)\leq \Lambda d(x)^q \tag{4,20}$$

Note that $(4,18)$ and $(4,20)$ imply that the maximum is achieved at some point (x_0,t_0) satisfying:

$$d(x_0)\geq\left(\frac{c_0}{\Lambda}\right)^{\frac{1}{a}}=r$$

Estimate $(4,19)$ follows by arguing as in lemma 6.1 of [8] and by choosing as a subsolution:

$$\psi=\frac{cv^{\frac{1}{m+p-2}}}{\mathcal{F}^{\frac{\theta}{3-m-p}}}$$

where $\mathcal{F}=1+c^{\frac{3-m-p}{p-1}}b\left(\frac{|x|^p}{t}\right)^{\frac{1}{p-1}}$, $\quad v=(R^\beta-|x|^\beta)^p$, c is a positive constant and θ,β,b are positive numbers that can be determined a priori only in terms of N,m,p and c.

Now estimates $(1,7)$ and $(1,8)$ follow by $(4,19)$ by means of the regularity result of section 3.

4.6 Remark. All the results of this section are based on the compact imbedding of $W^{1,p}(\Omega)$ in $L^{\frac{m+2p-3}{m+p-2}}(\Omega)$; by the Rellich-Kondrachov theorem, this holds true if:

$$\frac{m+p-2}{m+2p-3} > \frac{1}{p} - \frac{1}{N} \Leftrightarrow \frac{p(m+2p-3)}{m-3+(4-m)p-p^2} > N$$

Theorem 1.7 does not hold if the above condition on m, p is not satisfied. Actually, let $\Omega = \{x \in \mathbf{R}^N : |x| \leq 1\}$, $p = 2$, $N \geq 3$ and $m = \frac{N-2}{N+2}$. Assume that Theorem 1.7 holds. Then there exists a nontrivial nonnegative solution of the equation:

$$\begin{cases} \Delta u = u^{\frac{N+2}{N-2}} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4,21}$$

This is a contradiction because problem (4.21) admits only the trivial solution (see for instance [3]).

References.

[1] J.G. Berryman, *Evolution of a stable profile for a class of nonlinear diffusion equations with fixed boundaries*, J. Math. Physics **18** (1977), 2108-2115.

[2] J.G. Berryman, C.J. Holland, *Stability of the separable solutions for fast diffusion equation*, Arch. Rat. Mech. Anal. **76** (1980), 379-388.

[3] H. Brezis, *Elliptic equations with limiting Sobolev exponent. The impact of topology*, Comm. Pure Appl. Math. **39** (1986), 517-539.

[4] S. Campanato, *Equazioni paraboliche del secondo ordine e spazi $\mathcal{L}^{(2,\theta)}(\Omega,\delta)$*, Ann. Mat. Pura Appl. **73** (1966), 55-102.

[5] Y.Z. Chen, E. Di Benedetto, *On the local behaviour of solutions of singular parabolic equations*, Arch. Rat. Mech. Anal. **104** (1988), 319-345.

[6] Y.Z. Chen, E. Di Benedetto, *Boundary regularity for nonlinear parabolic systems*, J. Reine Angew. Math. **385** (1989), 102-131.

[7] E. Di Benedetto, *Topics in quasilinear degenerate and singular parabolic equations*, Lipshitz Vorlesung, Institut für Angewandte Mathematik **20** (1992).

[8] E. Di Benedetto, Y.C. Kwong, V. Vespri, *Local space analiticity and asymptotic behaviour of certain singular parabolic eqations*, Indiana Univ. Math. J. **40** (1991), 741-765.

[9] E. Di Benedetto, J.J. Manfredi, *On the higher integrability of the gradient of weak solutions of certain degenerate elliptic systems*, to appear in American Math. J.

[10] M. Giaquinta, *Multiple integrals in the calculus of variations* (1983), Princeton University Press, Princeton.

[11] A.S. Kalashnikov, *Some problems of the qualitative theory of nonlinear degenerate second order parabolic equations*, Russian Math. Surveys **42** (1987), 169-222.

[12] Y.C. Kwong, *Asymptotic behaviour of a plasma type equation with finite extinction*, Arch. Rat. Mech. Anal. **104** (1988), 277-294.

[13] G.M. Liebermann, *Mean oscillation estimate and Hölder regularity for the gradient of solutions of degenerate parabolic problems,* to appear.

[14] J.L. Lions, *Quelque methodes de resolution des problemes aux limites nonlineaire* (1972), Springer Verlag, Berlin.

[15] M.M. Porzio, V. Vespri, *Hölder estimate for local solutions of some doubly nonlinear degenerate parabolic equations*, to appear in J. Diff. Eq.

[16] V. Vespri, *On the local behaviour of a certain class of doubly nonlinear parabolic equations*, Manuscripta Math. **75** (1992), 65-80.

[17] V. Vespri, *Harnack type inequalities for solutions of certain doubly nonlinear parabolic equations*, to appear in J. Math. Anal. Appl.