Gradient flows and Evolution Variational Inequalities in metric spaces. I: Structural properties

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A R T I C L E   I N F O

A B S T R A C T

This is the first of a series of papers devoted to a thorough analysis of the class of gradient flows in a metric space (X, d) that can be characterized by Evolution Variational Inequalities (EVI). We present new results concerning the structural properties of solutions to the EVI formulation, such as contraction, regularity, asymptotic expansion, precise energy identity, stability, asymptotic behavior and their link with the geodesic convexity of the driving functional.

Under the crucial assumption of the existence of an EVI gradient flow, we will also prove two main results:

– the equivalence with the De Giorgi variational characterization of curves of maximal slope;
– the convergence of the Minimizing Movement-JKO scheme to the EVI gradient flow, with an explicit and uniform error estimate of order 1/2 with respect to the step size, independent of any geometric hypothesis (as upper or lower curvature bounds) on d.

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In order to avoid any compactness assumption, we will also introduce a suitable relaxation of the Minimizing Movement algorithm obtained by the Ekeland variational principle, and we will prove its uniform convergence as well.

Contents

1. Introduction

This is the first of a series of papers devoted to a thorough analysis of the class of gradient flows in metric spaces that can be characterized by Evolution Variational Inequalities (EVI, in short).

Gradient flows govern a wide range of important evolution problems. Perhaps the most popular and well-known theory, with relevant applications to various classes of Partial Differential Equations, concerns the evolution in a Hilbert space X driven by a lower semicontinuous and convex (or λ-convex) functional $\phi : X \to (-\infty, +\infty]$ with non empty proper domain $\mathrm{Dom}(\phi) := \{w \in X : \phi(w) < \infty\}$. The evolution can be described by a locally Lipschitz curve $u : (0, +\infty) \to \mathrm{Dom}(\phi)$ solving the differential inclusion

$$u'(t) \in -\partial\phi(u(t)) \quad \text{for } \mathscr{L}^1\text{-a.e. } t > 0, \qquad \lim_{t\downarrow 0} u(t) = u_0 \in \overline{\mathrm{Dom}(\phi)}, \qquad (1.1)$$

List of main notations

(X, d)	the reference metric space
ϕ	a l.s.c. functional on X with values in $(-\infty, +\infty]$
$\mathrm{AC}^p(I; X)$	p-absolutely continuous curves $\mathsf{x} : I \to X$, Definition 2.1
$\|\dot{\mathsf{x}}\|$, $\mathrm{Length}[\mathsf{x}]$	metric velocity and length of an AC curve x, (2.2) and (2.3)
$\mathsf{d}_\ell, \mathsf{d}_{D,\ell}$	length distance induced by d (in a subset D), (2.5)
$\mathrm{Geo}(D), \mathrm{Geo}_D[x_0 \to x_1]$	geodesics in a subset D, Definition 2.3
$\mathrm{Dom}(\phi)$	domain of a proper functional ϕ, (2.12)
$\|\partial\phi\|(x)$, $\mathfrak{L}_\lambda[\phi](x)$	metric and global slopes of ϕ, Definition 2.7
$\mathsf{J}_{\tau,\eta}[x]$	Moreau-Yosida-Ekeland resolvent, Definition 2.9
$\phi'(\mathsf{x}_0, \mathsf{x})$	directional derivative of ϕ along a geodesic x, Definition 2.15
$\mathbb{X} = (X, \mathsf{d}, \phi)$	a metric-functional system, (3.1)
EVI_λ	Evolution Variational Inequalities characterizing the flow, Definition 3.1
$\mathsf{E}_\lambda(t)$	the primitive of the function $\mathrm{e}^{\lambda t}$ s.t. $\mathsf{E}_\lambda(0) = 0$, (3.3)
$(U_{\tau,\eta}^n)_{n \in \mathbb{N}}$	a sequence in X generated by the Minimizing Movement algorithm, Definition 5.1
τ, η	step size and Ekeland relaxation parameter of a Minimizing Movement, Definition 5.1
$\overline{U}_{\tau,\eta}(t)$	piecewise-constant interpolant of a discrete Minimizing Movement, (5.3)
$\mathrm{MM}(X, \mathsf{d}, \phi; u_0)$	Minimizing Movements starting from u_0, Definition 5.2
$\mathrm{GMM}(X, \mathsf{d}, \phi; u_0)$	Generalized Minimizing Movements starting from u_0, Definition 5.2

where $\partial\phi$ denotes the Fréchet subdifferential of convex analysis. Existence, uniqueness, well-posedness, approximation and regularity properties of solutions to (1.1) have been deeply studied by the pioneering papers of KOMURA, CRANDALL-PAZY, DORROH, KATO, BRÉZIS, see e.g. the book [21, Chapters III, IV] and the references therein.

In the Hilbert framework solutions to (1.1) enjoy many important properties: they give rise to a continuous semigroup $(\mathsf{S}_t)_{t \geq 0}$ of λ-contracting maps $\mathsf{S}_t : \overline{\mathrm{Dom}(\phi)} \to \mathrm{Dom}(\phi)$, in the sense that $u(t) := \mathsf{S}_t(u_0)$ is the unique solution to (1.1) and

$$\|\mathsf{S}_t(u_0) - \mathsf{S}_t(v_0)\|_X \leq \mathrm{e}^{-\lambda t}\|u_0 - v_0\|_X \quad \text{for every } u_0, v_0 \in \mathrm{Dom}(\phi). \tag{1.2}$$

Among the many important properties of $(\mathsf{S}_t)_{t \geq 0}$, we recall that the energy map $t \mapsto \phi(u(t))$ is absolutely continuous and satisfies the energy-dissipation identity (here in a differential form, where $\|\partial\phi(\cdot)\|_X$ denotes the minimal norm of the elements in $\partial\phi(\cdot)$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(u(t)) = -\|u'(t)\|_X^2 = -\|\partial\phi(u(t))\|_X^2 \quad \text{for } \mathscr{L}^1\text{-a.e. } t > 0; \tag{1.3}$$

moreover, every solution u arises from the locally-uniform limit of the discrete approximations obtained by the Implicit Euler Method

$$U_\tau^n = \mathrm{J}_\tau(U_\tau^{n-1}), \quad n \in \mathbb{N}, \quad U_\tau^0 := u_0, \tag{1.4a}$$

where the resolvent map $\mathrm{J}_\tau : X \to X$ is defined by

$$U = \mathrm{J}_\tau(V) \quad \Leftrightarrow \quad \frac{U - V}{\tau} \in -\partial\phi(U). \tag{1.4b}$$

In fact (1.4a) recursively defines a family of sequences $(U_\tau^n)_{n\in\mathbb{N}}$ depending on a sufficiently small time step $\tau > 0$ and inducing a piecewise constant interpolant

$$\overline{U}_\tau(t) := U_\tau^n = \mathrm{J}_\tau^n(u_0) \quad \text{whenever } t \in ((n-1)\tau, n\tau], \tag{1.5}$$

converging to the continuous solution u of (1.1) as $\tau \downarrow 0$.

Under the initial impulse of DE GIORGI, DEGIOVANNI, MARINO, TOSQUES [37,38, 61], the abstract theory has been extended towards two main directions: a relaxation of the convexity assumptions on ϕ (see e.g. [67,80]) and a broadening of the structure of the ambient space, from Hilbert to Banach spaces (for the theory of doubly nonlinear evolution equations, see e.g. [15,28]) or to more general metric and topological spaces [36]. It is remarkable that the original approach by De Giorgi and his collaborators encompasses both these directions.

Here we focus on the second direction and we consider the metric side of the theory, referring also to [35,81] for recent overviews. There are (at least) three different formulations of gradient flows in a metric space (X, d): the first one, introduced by [37], got inspiration from (1.3), which is in fact an equivalent characterization of (1.1) in Hilbert spaces. One can give a metric meaning to the (scalar) velocity of a curve $u : [0, \infty) \to X$ by the so called *metric derivative*

$$|\dot{u}|(t) := \lim_{h \to 0} \frac{\mathsf{d}(u(t+h), u(t))}{|h|},$$

and to the norm of the (minimal selection in the) subdifferential by the *metric slope*

$$|\partial\phi|(u) := \limsup_{v \to u} \frac{(\phi(u) - \phi(v))_+}{\mathsf{d}(u, v)}.$$

A *curve of maximal slope* is an absolutely continuous curve $u : [0, +\infty) \to X$ satisfying the energy-dissipation identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(u(t)) = -|\dot{u}|^2(t) = -|\partial\phi|^2(u(t)) \quad \text{for } \mathscr{L}^1\text{-a.e. } t > 0, \tag{1.6}$$

which is the metric formulation of (1.3).

A second approach is related to a variational formulation of (1.4b) (since the latter does not make sense in a pure metric setting), which can be considered as the first-order optimality condition for the variational problem

$$U \in \mathsf{J}_\tau(V) \quad \Leftrightarrow \quad U \in \operatorname*{argmin}_{W \in X} \frac{1}{2\tau} \mathsf{d}^2(W, V) + \phi(W), \qquad (1.7)$$

whose minimizers define a multivalued map still denoted by J_τ. A recursive selection of U_τ^n among the minimizers $\mathsf{J}_\tau(U^{n-1})$ of (1.7) yields a powerful algorithm to approximate gradient flows. Pointwise limits up to subsequences of the interpolants $\overline{U}_\tau$ defined by (1.5) as $\tau \downarrow 0$ are denoted by $\mathsf{GMM}(X, \mathsf{d}, \phi; u_0)$ and called *Generalized Minimizing Movements*, a particular important case of a general framework introduced in [36]. Let us remark that the extensive application of (1.7) in the area of Optimal Transport has been independently initiated by the celebrated paper of JORDAN-KINDERLEHRER-OTTO [51]; their approach has been so influential, that the realization of (1.7) in Kantorovich-Rubinstein-Wasserstein spaces (see the examples below and Section 3.4.3) is commonly called *JKO scheme*.

It is not difficult to prove existence of generalized minimizing movements, under suitable compactness conditions (see [2,5]); if moreover the slope of ϕ is a lower semi-continuous upper gradient (see [5, Definitions 1.2.1, 1.2.2]) then generalized minimizing movements are also curves of maximal slope, according to (1.6).

Even if ϕ is geodesically λ-convex (the natural extension of convexity in metric spaces), however, the evolution does not enjoy all the nice semigroup properties of the Hilbertian case: it can be easily checked even in finite-dimensional spaces, endowed with a non-quadratic norm [74]. For this to hold, we need to ask something more. Indeed, the most powerful (and demanding) notion of gradient flow can be obtained by a metric formulation of (1.1), which may be written as a system of *evolution variational inequalities*, observing that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t) - w\|_X^2 = \langle u'(t), u(t) - w \rangle_X$$

$$\leq \phi(w) - \phi(u(t)) - \frac{\lambda}{2}\|u(t) - w\|_X^2 \quad \text{for every } w \in \mathrm{Dom}(\phi),$$

thanks to the λ-convexity of ϕ and the properties of the subdifferential.

In metric spaces one may similarly look for curves $u : [0, \infty) \to \mathrm{Dom}(\phi)$ satisfying

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{d}^2(u(t), w) + \frac{\lambda}{2}\mathsf{d}^2(u(t), w) \leq \phi(w) - \phi(u(t)) \quad \text{for every } w \in \mathrm{Dom}(\phi); \qquad (1.8)$$

if a solution exists for every initial datum $u_0 \in \mathrm{Dom}(\phi)$, it gives rise to a λ-contracting semigroup $(\mathsf{S}_t)_{t \geq 0}$ in $\mathrm{Dom}(\phi)$ satisfying the analogue of the stability property (1.2). Let us recall that (1.8), which resembles previous formulations of [14,16,20] in Hilbert and

Banach spaces, has been introduced in the metric framework by [5] and it is called the EVI_λ-formulation of the gradient flow driven by ϕ.

In fact (1.8) is the most restrictive definition of gradient flow for λ-convex functionals. Differently from the Hilbertian case, in arbitrary metric spaces X some "Riemannian-like" structure for X should also be required [74,90]. Among the most interesting examples of spaces and λ-convex functionals where an EVI_λ-gradient flow exists we can quote:

▷ Hilbert spaces,
▷ complete and smooth Riemannian manifolds,
▷ Hadamard non-positively curved (NPC) spaces, or more generally, metric spaces with an upper curvature bound [5,13,52,63],
▷ PC spaces, or more generally Alexandrov spaces with a uniform lower curvature bound [22,23,70,72,77–79,82],
▷ the Wasserstein space $(\mathscr{P}_2(X), \mathsf{d}_W)$, where X is a complete and smooth Riemannian manifold, a compact Alexandrov space or a Hilbert space [5,6,41,47,72,82,89],
▷ RCD(K,∞) metric measure spaces [8,12,45],
▷ $(\mathscr{P}_2(X), \mathsf{d}_W)$, where X is an RCD(K,∞) metric-measure space and ϕ is the relative entropy functional [69,88],

and there is an intensive research to construct EVI gradient flows in ad-hoc geometric settings for reaction-diffusion equations [25,54,55,57] and systems [48,56,65], nonlinear viscoelasticity [68], Markov chains [44,60,66], jump processes [42], configuration and Wiener spaces [10,43]. We refer to Section 3.4 for a more detailed discussion of the main classes of metric structures.

We think that all these examples and the developments of the metric theory justify a systematic study of the EVI_λ-formulation of gradient flows. This is in fact the main contribution of our investigation. In the present paper we will assume that an EVI_λ-flow exists and we will deeply examine its main structural properties, independently of the construction method. In the following companion papers, we will study the natural stability properties related to perturbation of the functional ϕ and of the distance d under suitable variational notions of convergence (as Γ, Mosco, or Gromov-Hausdorff convergence) and the generation results under the weakest assumptions on the metric and on the functional. In all our analyses we will make a considerable effort to avoid ad hoc hypotheses, in particular concerning compactness, in order to cover also important infinite-dimensional examples.

Plan of the paper and main results

Let us now quickly describe the structure and the main results of the present paper.

In the preliminary Section 2 we will briefly recall the main metric notions we will extensively deal with. Since we want to avoid compactness assumptions, *a particular attention is devoted to approximate conditions*: length properties (Definition 2.4 and

Lemma 2.6), a new relaxed formulation of the Moreau-Yosida regularization inspired by the Ekeland variational principle [39] (Section 2.2), and an approximate notion of λ-convexity (Definition 2.12). Theorem 2.10 provides a very useful *approximation by points with finite slope* and a crucial estimate, which lies at the core of the relaxed version of the Minimizing Movement scheme of Section 5. Theorem 2.17 shows that *approximately convex (resp. λ-convex) functions are linearly (resp. quadratically) bounded from below*, a property which is well known in Banach spaces. Both these results only depend on the completeness of the sublevels of ϕ.

Section 3 contains the main structural properties of solutions to the Evolution Variational Inequalities EVI_λ; first of all, in Section 3.1 we will consider various formulations of (1.8), showing their equivalence. It is clear that classical formulations in terms of pointwise derivatives are quite useful to obtain contraction and regularity estimates, whereas integral or distributional characterizations are important when stability issues are involved.

Our first main result is *Theorem 3.5*, which collects *all the fundamental properties of solutions* of the EVI_λ-formulation, mainly inspired by the Hilbert framework. Some results (as λ-contraction, regularization or asymptotic behavior) had previously been obtained in [5], occasionally using stronger structural properties on ϕ. Here we will show that all the relevant properties are consequence of the Evolution Variational Inequalities; in particular we will obtain precise pointwise versions of the energy identity, refined regularization estimates, and sharp asymptotic expansions, which will play a crucial role in the analysis of the curves of maximal slopes and of the minimizing movements in Sections 4 and 5. Section 3.3 collects two basic properties involving EVI_λ-flows and λ-convexity: first of all, we extend the result by [34] showing that the existence of an EVI_λ-flow entails approximate λ-convexity if $\mathrm{Dom}(\phi)$ is a length (also called intrinsic) subset. Differently from the geodesic property, the length assumption should not be considered restrictive, since a result of [10] shows that an EVI_λ-flow in (X, d) is also an EVI_λ-flow if we replace d by the length distance d_ℓ; Theorem 3.10 then implies that ϕ is always λ-convex in the modified intrinsic geometry. We will also show that the constants λ of the EVI_λ-formulation and of the λ-convexity coincide.

In the last two sections of the paper we will compare the notions of EVI_λ-flow, of curves of maximal slope and of generalized Minimizing Movements. Our main assumption is that an EVI_λ-flow exists; in this case, in Section 4 we prove that any curve of maximal slope, even in a weaker integral sense than (1.6), is a solution to the EVI_λ formulation; in particular, we get also uniqueness of curves of maximal slope.

In Section 5 we will study the Minimizing Movement scheme, in a new and suitably relaxed form where we use approximate minimizers obtained at each step via the Ekeland variational principle. In this way, we can establish the existence of a discrete approximating sequence without invoking compactness arguments, and we will prove that every sequence of discrete Minimizing Movements converges to the solution of the EVI_λ-flow. As a byproduct of our analysis, we will also prove explicit error estimates between discrete Minimizing Movements and continuous solutions, assuming different regularity on

the initial data and allowing for a further relaxation of the minimality condition: when u_0 has finite slope, in the case of a convex functional with no need for the Ekeland regularization, we will obtain the uniform error estimate of order $1/2$

$$\mathsf{d}(\mathsf{S}_t(u_0), \mathsf{J}_\tau^n(u_0)) \leq \frac{t}{\sqrt{n}} |\partial\phi|(u_0), \quad \tau = t/n, \tag{1.9}$$

which reproduces in our metric setting (with a better constant) the celebrated Crandall-Liggett estimates for contraction semigroups in Banach spaces [32], previously obtained in specific metric settings under much stronger assumptions on d [5,13,26,31,63], the latter implying suitable contraction properties of J_τ.

These estimates are completely new; their strength relies on the fact that they do not depend on any upper or lower curvature bound on the distance d, but only on the existence of an EVI_λ-flow. In particular, they can be applied to the setting of $\mathrm{RCD}(K, \infty)$ metric spaces or to Wasserstein spaces, showing that the JKO-Minimizing Movement Scheme is always at least of order $1/2$ whenever it is applied to an EVI_λ-gradient flow. Apart from their intrinsic interest, these estimates will also be crucial in the study of the convergence of EVI_λ-flows, which will be investigated in the next paper [69]. For these reasons, we tried to reach the greatest level of generality.

2. Preliminaries

Let us first briefly recall some basic definitions and tools we will extensively use in the forthcoming sections, referring to [5] for a more detailed introduction to the whole subject. Throughout the present paper we will refer to a metric space (X, d). We will often use the symbol D to denote a distinguished subset of X, which inherits the distance d from X.

2.1. Absolutely continuous curves, length subsets and geodesics

Definition 2.1 *(Absolutely continuous curves and metric derivative).* Let $I \subset \mathbb{R}$ be an interval and $p \in [1, +\infty]$. A curve $\mathsf{x} : I \to X$ belongs to $\mathrm{AC}^p(I; X)$ if there exists $m \in L^p(I)$ such that

$$\mathsf{d}(\mathsf{x}_s, \mathsf{x}_t) \leq \int_s^t m(r)\, \mathrm{d}r \quad \text{for every } s, t \in I \text{ with } s \leq t. \tag{2.1}$$

The *metric derivative* of x is defined, where it exists, as

$$|\dot{\mathsf{x}}|(t) := \lim_{h \to 0} \frac{\mathsf{d}(\mathsf{x}_{t+h}, \mathsf{x}_t)}{|h|}. \tag{2.2}$$

The proof of the following result can be found, e.g., in [5, Theorem 1.1.2].

Theorem 2.2 *(Absolutely continuous curves and metric derivative). If* $\mathsf{x} \in \mathrm{AC}^p(I; X)$, *then* x *is uniformly continuous, its metric derivative exists at* $\mathscr{L}^1$-*a.e.* $t \in I$, *belongs to* $L^p(I)$ *and provides the minimal function* m *satisfying* (2.1), *i.e.* $|\dot{\mathsf{x}}|$ *complies with* (2.1) *and every function* m *as in* (2.1) *satisfies* $m(t) \geq |\dot{\mathsf{x}}|(t)$ *for* $\mathscr{L}^1$-*a.e.* $t \in I$.

Still by means of similar techniques to those used in the proof of [5, Theorem 1.1.2], it is not difficult to show that a curve belongs to $\mathrm{AC}^1(I; X)$ (resp. $\mathrm{AC}^\infty(I; X)$) if and only if it is *absolutely continuous* (resp. *Lipschitz continuous*). Hence from now on $\mathrm{AC}^1 \equiv \mathrm{AC}$. The length of a curve $\mathsf{x} \in \mathrm{AC}(I; X)$ is

$$\mathrm{Length}[\mathsf{x}] := \int_I |\dot{\mathsf{x}}|(t)\,\mathrm{d}t. \tag{2.3}$$

A well-known reparametrization result (see e.g. [5, Lemma 1.1.4]) shows that for every $\mathsf{x} \in \mathrm{AC}([a, b]; X)$ with length L, there exist unique maps $\mathsf{y} \in \mathrm{AC}^\infty([0, 1]; X)$ and $\sigma : [a, b] \to [0, 1]$ continuous and (weakly) increasing such that

$$\mathsf{x} = \mathsf{y} \circ \sigma, \quad |\dot{\mathsf{y}}| = L \quad \mathscr{L}^1\text{-a.e. in } [0, 1]. \tag{2.4}$$

Given a subset $D \subset X$, we can then define the *length distance* induced by d in D:

$$\mathsf{d}_{D,\ell}(x_0, x_1) := \inf \Big\{ \mathrm{Length}[\mathsf{x}] : \mathsf{x} \in \mathrm{AC}([0, 1]; D),\ \mathsf{x}(i) = x_i,\ i = 0, 1 \Big\}, \tag{2.5}$$

and we will simply use the symbol d_ℓ when $D = X$.

If D is *Lipschitz connected*, i.e. each couple of points $x_0, x_1 \in D$ can be connected by a curve $\mathsf{x} \in \mathrm{AC}([0, 1]; D)$, then $(D, \mathsf{d}_{D,\ell})$ is a metric space. In general, we adopt the usual convention to set $\mathsf{d}_{D,\ell}(x_0, x_1) = +\infty$ if there are no absolutely continuous curves connecting x_0 to x_1 in D. In this case $\mathsf{d}_{D,\ell} : D \times D \to [0, +\infty]$ is an *extended* distance on D, i.e. it satisfies all the axioms of a distance function, possibly assuming the value $+\infty$. By partitioning D in the equivalence classes with respect to the relation $\sim_{D,\ell}$ defined by

$$x \sim_{D,\ell} y \quad \Leftrightarrow \quad \mathsf{d}_{D,\ell}(x, y) < \infty, \tag{2.6}$$

each class

$$[\bar{x}]_{D,\ell} := \Big\{ x \in D : \mathsf{d}_{D,\ell}(x, \bar{x}) < \infty \Big\}, \quad \bar{x} \in D,$$

endowed with the distance $\mathsf{d}_{D,\ell}$ becomes a metric space in the usual sense and it is closed and open in $(D, \mathsf{d}_{D,\ell})$. Since $\mathsf{d} \leq \mathsf{d}_{D,\ell}$, the topology induced by $\mathsf{d}_{D,\ell}$ is stronger than the original topology induced by d; moreover, if (D, d) is complete, then $(D, \mathsf{d}_{D,\ell})$ is complete as well.

Definition 2.3 *(Geodesics and geodesic subsets).* A *constant speed, minimal geodesic* (in short, *geodesic*) in $D \subset X$ is a (Lipschitz) curve $\mathsf{x} : [0, 1] \to D$ such that

$$\frac{\mathsf{d}(\mathsf{x}_s, \mathsf{x}_t)}{|s - t|} = \mathsf{d}(\mathsf{x}_0, \mathsf{x}_1) =: |\dot{\mathsf{x}}| \quad \text{for every } 0 \le s < t \le 1.$$

We denote by $\mathrm{Geo}(D) \subset \mathrm{AC}^\infty([0, 1]; X)$ the collection of all of the geodesics in D and by $\mathrm{Geo}_D[x_0 \to x_1]$ the (possibly empty) collection of the geodesics in D satisfying $\mathsf{x}_i = x_i$, $i = 0, 1$. If $\mathrm{Geo}_D[x_0 \to x_1] \ne \emptyset$ for every $x_0, x_1 \in D$, we say that D is a *geodesic subset*.

So geodesics are absolutely continuous curves along which the space swept between any two points (x_0 and x_1 are enough actually) is equal to their distance. By the arc-length reparametrization (2.4), it is not restrictive to assume that they have constant speed. In particular, if X is a geodesic space then $\mathsf{d} = \mathsf{d}_\ell$ and the inf in (2.5) is attained.

Since in general metric spaces the existence of geodesics is not for granted, one can consider weaker notions; the first one we provide is the length (or intrinsic) property.

Definition 2.4 *(Length (or intrinsic) property).* D is a length (or intrinsic) subset of X if for every $x_0, x_1 \in D$ and every $d > \mathsf{d}(x_0, x_1)$ there exists a curve $\mathsf{x} \in \mathrm{AC}([0, 1]; D)$ connecting x_0 to x_1 such that $\mathrm{length}[\mathsf{x}] \le d$.

Note that D is intrinsic if and only if $\mathsf{d}_{D,\ell} = \mathsf{d}_\ell = \mathsf{d}$ on $D \times D$.

A second, even weaker property, is related to the existence of ε-approximate intermediate points.

Definition 2.5 *(Approximate length subsets).* Let $x_0, x_1 \in X$, $\vartheta \in (0, 1)$, $\varepsilon \in (0, 1)$. We say that x is a (ϑ, ε)-intermediate point (ε-midpoint if $\vartheta = 1/2$) between x_0 and x_1 if

$$\frac{\mathsf{d}^2(x_0, x)}{\vartheta} + \frac{\mathsf{d}^2(x, x_1)}{1 - \vartheta} \le \mathsf{d}^2(x_0, x_1)(1 + \varepsilon^2 \vartheta(1 - \vartheta)). \tag{2.7}$$

We say that $D \subset X$ is an *approximate length subset* if for every $x_0, x_1 \in D$ and every $\varepsilon \in (0, 1)$ there exists an ε-midpoint $x \in D$ between x_0 and x_1.

Note that in the case of ε-midpoints (2.7) reads

$$\mathsf{d}^2(x_0, x) + \mathsf{d}^2(x, x_1) \le \frac{1}{2}\mathsf{d}^2(x_0, x_1)(1 + \varepsilon^2/4).$$

The above definition comes from the fact that geodesic points x_ϑ can be characterized as minimizers of the functional in the l.h.s. of (2.7) for which the minimum coincides with $\mathsf{d}^2(x_0, x_1)$, see [22, Lemma 2.4.8]; by introducing ε we admit an arbitrarily small error with respect to exact minima. It is immediate to check that if

$$\mathsf{d}(x_0, x) \le \vartheta \mathsf{d}(x_0, x_1)(1 + \delta) \quad \text{and} \quad \mathsf{d}(x, x_1) \le (1 - \vartheta)\mathsf{d}(x_0, x_1)(1 + \delta) \tag{2.8}$$

with (for instance) $\delta \leq \frac{1}{3}\varepsilon^2 \vartheta(1-\vartheta)$, then (2.7) holds. On the other hand, recalling the elementary identity

$$(\vartheta a + (1-\vartheta)b)^2 = \vartheta a^2 + (1-\vartheta)b^2 - \vartheta(1-\vartheta)(a-b)^2,$$

$\ell := \mathsf{d}(x_0, x) + \mathsf{d}(x, x_1)$ satisfies

$$\mathsf{d}^2(x_0, x_1) \leq \ell^2 = \frac{\mathsf{d}^2(x_0, x)}{\vartheta} + \frac{\mathsf{d}^2(x, x_1)}{1-\vartheta} - \vartheta(1-\vartheta)\left(\frac{\mathsf{d}(x_0, x)}{\vartheta} - \frac{\mathsf{d}(x, x_1)}{1-\vartheta}\right)^2,$$

so that if (2.7) holds we get

$$\left|\frac{\mathsf{d}(x_0, x)}{\vartheta} - \frac{\mathsf{d}(x, x_1)}{1-\vartheta}\right| \leq \mathsf{d}(x_0, x_1)\varepsilon \qquad \text{and} \qquad \ell \leq \mathsf{d}(x_0, x_1)\sqrt{1 + \varepsilon^2\vartheta(1-\vartheta)}; \quad (2.9)$$

from (2.9) it is not difficult to deduce that (2.8) is satisfied e.g. with $\delta = \varepsilon$.

We collect in the following lemma a list of useful properties; to our purposes, let us denote by $\mathbb{D} := \{k2^{-n} : k, n \in \mathbb{N}, \ 0 \leq k \leq 2^n\}$ the set of all dyadic points in $[0, 1]$.

Lemma 2.6.

L1: *If $D \subset X$ is a length subset then it is also an approximate length subset.*

L2: *$D \subset X$ is an approximate length subset if and only if $\overline{D}$ is.*

L3: *If D is an approximate length subset of X then for every $x_0, x_1 \in D$ and every $L > \mathsf{d}(x_0, x_1)$ there exists $\{\mathsf{x}_\vartheta\}_{\vartheta \in \mathbb{D}} \subset D$ such that $\mathsf{x}_0 = x_0$, $\mathsf{x}_1 = x_1$ and*

$$\mathsf{d}(\mathsf{x}_{\vartheta'}, \mathsf{x}_{\vartheta''}) \leq L|\vartheta' - \vartheta''| \quad \text{for every } \vartheta', \vartheta'' \in \mathbb{D}.$$

In particular, for every $\vartheta, \varepsilon \in (0, 1)$ there is a (ϑ, ε)-intermediate point $x \in D$ between x_0 and x_1.

L4: *If D is a complete and approximate length subset of X then it is a length subset of X.*

Proof. L1 is a consequence of the discussion following Definition 2.5. The density of D in $\overline{D}$ and the triangle inequality yield L2.

L3 can be proved by adapting the arguments of [22, Section 2.4.4]: we briefly sketch the main steps.

Let $\delta > 0$ be such that $L = \mathsf{d}(x_0, x_1)(1 + \delta)$ and let us fix a sequence $(\varepsilon_n)_{n \in \mathbb{N}} \subset (0, 1)$ satisfying $\sum_{n=0}^{\infty} \varepsilon_n \leq \log(1 + \delta)$. We can parametrize all the points in $\mathbb{D}$ by two integers $n \in \mathbb{N}$ and $h \in [0, 2^n]$ by the surjective (but not injective) map $(h, n) \mapsto \theta_{h,n} = h2^{-n}$. We call $\mathbb{D}_n := \{\theta_{h,n} : h \in [0, 2^n]\}$. It is clear that, for any odd integer $h = 2k + 1$, $\theta_{h,n+1}$ is the midpoint between $\theta_{k,n}$ and $\theta_{k+1,n}$, belonging to $\mathbb{D}_n$, whereas for an even integer $h = 2k$, the point $\theta_{h,n+1}$ belongs to $\mathbb{D}_n$.

We can construct a map $\vartheta \mapsto \mathsf{x}(\vartheta) = \mathsf{x}_\vartheta$ in $\mathbb{D}$ satisfying

$$\mathsf{d}(\mathsf{x}(\vartheta), \mathsf{x}(\vartheta')) \le \mathsf{d}(x_0, x_1)|\vartheta - \vartheta'| \exp\Big(\sum_{m=0}^n \varepsilon_m \Big) \quad \text{for every } \vartheta, \vartheta' \in \mathbb{D}_n, \tag{2.10}$$

by induction with respect to n: note that (2.10) surely holds, by the triangle inequality, if

$$\mathsf{d}(\mathsf{x}((k+1)2^{-n}), \mathsf{x}(k2^{-n})) \le \frac{1}{2^n}\mathsf{d}(x_0, x_1) \exp\Big(\sum_{m=0}^n \varepsilon_m \Big) \quad \text{for every integer } k \in [0, 2^n].$$

The case $n = 0$ is trivial; assuming that x has already been defined on $\mathbb{D}_n$ we can extend it to every point $\theta_{2k+1,n+1} \in \mathbb{D}_{n+1} \setminus \mathbb{D}_n$ by choosing an ε_{n+1}-midpoint between $\mathsf{x}(\theta_{k,n})$ and $\mathsf{x}(\theta_{k+1,n})$; it is clear that (recall (2.8))

$$\begin{aligned}
\mathsf{d}(\mathsf{x}(\theta_{2k+1,n+1}), \mathsf{x}(\theta_{2k,n+1})) &\le \frac{1}{2}\mathsf{d}(\mathsf{x}(\theta_{k,n}), \mathsf{x}(\theta_{k+1,n}))(1 + \varepsilon_{n+1}) \\
&\le \frac{1}{2^{n+1}}\mathsf{d}(x_0, x_1) \exp\Big(\sum_{m=0}^n \varepsilon_m \Big) \exp(\varepsilon_{n+1}) \\
&= \frac{1}{2^{n+1}}\mathsf{d}(x_0, x_1) \exp\Big(\sum_{m=0}^{n+1} \varepsilon_m \Big),
\end{aligned}$$

and an analogous estimate holds for $\mathsf{d}(\mathsf{x}(\theta_{2k-1,n+1}), \mathsf{x}(\theta_{2k,n+1}))$.

L4 follows immediately from L3 and the completeness of D. $\quad\square$

In analogy with (2.5), it is not difficult to show (for instance by using L3) that D is an approximate length subset if and only if for all $x, y \in D$ and $\varepsilon > 0$ there holds

$$\mathsf{d}(x, y)$$
$$= \inf\Big\{ \sum_{k=1}^N \mathsf{d}(x_k, x_{k-1}) : N \in \mathbb{N}, \{x_k\} \subset D, \ x_0 = x, \ x_N = y, \ \max_k \mathsf{d}(x_k, x_{k-1}) \le \varepsilon \Big\}. \tag{2.11}$$

2.2. Moreau-Yosida regularizations, slopes and Ekeland's variational principle

On X we will be considering *proper* functionals $\phi : X \to (-\infty, +\infty]$, where

$$\mathrm{Dom}(\phi) := \big\{ x \in X : \phi(x) < +\infty \big\} \tag{2.12}$$

denotes the (non-empty) domain of ϕ. We say that ϕ is *quadratically bounded from below* if there exist $o \in X$, $\phi_o, \kappa_o \in \mathbb{R}$ such that

$$\phi(x) + \frac{\kappa_o}{2}\mathsf{d}^2(x,o) \geq \phi_o \quad \text{for every } x \in X. \tag{2.13}$$

Similarly, we say that ϕ is *linearly bounded from below* if there exist $o \in X$, $\phi_o, \ell_o \in \mathbb{R}$ such that

$$\phi(x) + \ell_o\mathsf{d}(x,o) \geq \phi_o \quad \text{for every } x \in X, \tag{2.14}$$

which in particular implies that it is quadratically bounded from below for all $\kappa_o > 0$.

The (quadratic) *Moreau-Yosida* regularizations of ϕ (we refer e.g. to [5, Section 3.1] or [33, Chapter 9]) are the functionals $\phi_\tau : X \to \mathbb{R}$ defined by

$$\phi_\tau(x) := \inf_{y \in X} \phi(y) + \frac{1}{2\tau}\mathsf{d}^2(y,x) \quad \text{for every } x \in X, \tag{2.15}$$

and we set

$$\tau_o := \sup\Big\{\tau > 0 : \operatorname{Dom}(\phi_\tau) \neq \emptyset\Big\}. \tag{2.16}$$

Note that $\tau_o > 0$ if and only if ϕ is quadratically bounded from below and $\operatorname{Dom}(\phi_\tau) = X$ for every $\tau \in (0, \tau_o)$. If ϕ satisfies (2.13) then

$$\tau_o \geq \kappa_o^{-1} \quad \text{if } \kappa_o > 0; \qquad \tau_o = +\infty \quad \text{if } \kappa_o \leq 0.$$

Because we will mainly deal with functionals that are quadratically bounded from below (see for instance Theorem 2.17 and Theorem 3.5 below), the regularizations given by (2.15) come naturally into play; they are also strictly related to the Minimizing Movement approach to gradient flows, see the Introduction or Section 5. However, it is also possible to deal with more general regularizations: this will be addressed in detail in [69].

Definition 2.7 *(Metric slopes).* The *metric slope* of ϕ at $x \in X$ is

$$|\partial\phi|(x) := \begin{cases} +\infty & \text{if } x \notin \operatorname{Dom}(\phi), \\ 0 & \text{if } x \in \operatorname{Dom}(\phi) \text{ is isolated}, \\ \limsup_{y \to x} \dfrac{(\phi(x) - \phi(y))^+}{\mathsf{d}(x,y)} & \text{otherwise.} \end{cases}$$

As usual, we set $\operatorname{Dom}(|\partial\phi|) := \{x \in X : |\partial\phi|(x) < +\infty\}$. For $\lambda \in \mathbb{R}$ and $x \in X$ we then introduce the global λ-slope

$$\mathfrak{L}_\lambda[\phi](x) := \sup_{y \neq x} \frac{\big(\phi(x) - \phi(y) + \frac{\lambda}{2}\mathsf{d}^2(x,y)\big)^+}{\mathsf{d}(x,y)},$$

understood to be $+\infty$ if $x \notin \operatorname{Dom}(\phi)$. Similarly, we set $\operatorname{Dom}(\mathfrak{L}_\lambda[\phi]) := \{x \in X : \mathfrak{L}_\lambda[\phi](x) < +\infty\}$.

Finally, we will denote by $|\partial\phi|_\ell$ the metric slope of ϕ evaluated w.r.t. the length distance d_ℓ (2.5), i.e. in the extended metric space (X, d_ℓ).

Note that for every $S \in [0, +\infty)$ and $x \in X$ there holds

$$|\partial\phi|(x) \le S \quad \Leftrightarrow \quad x \in \mathrm{Dom}(\phi), \quad \phi(y) \ge \phi(x) - S\mathsf{d}(y, x) + o(\mathsf{d}(y, x)) \quad \text{as } \mathsf{d}(y, x) \to 0, \tag{2.17}$$

whereas

$$\mathfrak{L}_\lambda[\phi](x) \le S \quad \Leftrightarrow \quad \phi(y) \ge \phi(x) - S\mathsf{d}(y, x) + \frac{\lambda}{2}\mathsf{d}^2(y, x) \quad \text{for every } y \in X. \tag{2.18}$$

In particular,

$$|\partial\phi|(x) \le \mathfrak{L}_\lambda[\phi](x) \quad \text{for every } x \in X, \ \lambda \in \mathbb{R}. \tag{2.19}$$

We will mostly deal with proper functionals ϕ that are *lower semicontinuous* (l.s.c. for short): under such an assumption, it is easy to check that the global λ-slope is also lower semicontinuous. Contrarily, the metric slope is in general not lower semicontinuous, though it can be seen as a pointwise limit of lower semicontinuous functionals:

$$|\partial\phi|(x) = \lim_{n\to\infty} \sup_{\substack{y \in \mathrm{Dom}(\phi)\setminus\{x\} \\ \mathsf{d}(x,y) < \frac{1}{n}}} \frac{(\phi(x) - \phi(y))^+}{\mathsf{d}(x, y)}, \tag{2.20}$$

where the supremum is understood to be eventually 0 if x is isolated in $\mathrm{Dom}(\phi)$ and $+\infty$ if $x \notin \mathrm{Dom}(\phi)$.

It turns out that the metric slope is always a *weak upper gradient* for ϕ, whereas the global λ-slope is a *strong upper gradient* for ϕ provided the latter is lower semicontinuous: see [5, Definitions 1.2.1, 1.2.2 and Theorem 1.2.5] for more details.

When ϕ satisfies suitable coercivity assumptions (e.g. its sublevels are locally compact), it is not difficult to check that $\mathrm{Dom}(|\partial\phi|)$ is dense in $\mathrm{Dom}(\phi)$, and in particular is not empty. This can be proved [5, Lemma 3.1.3] by studying the properties of the minimizers $y \in X$ of the variational problem (2.15). More generally we will see that, thanks to *Ekeland's variational principle*, the density of $\mathrm{Dom}(|\partial\phi|)$ in $\mathrm{Dom}(\phi)$ holds if the sublevels of ϕ are merely *complete* (which in particular implies that ϕ is l.s.c.). As a byproduct, if in addition ϕ is quadratically bounded from below, $\mathfrak{L}_\lambda[\phi]$ is also a proper functional for all $\lambda \le -\kappa_o$ (it is enough that $x \mapsto \phi(x) - \frac{\lambda}{2}\mathsf{d}^2(x, o)$ is linearly bounded from below actually).

Let us first recall the celebrated variational principle of Ekeland [39,40].

Theorem 2.8 *(Ekeland's variational principle). Let $\varphi : X \to (-\infty, +\infty]$ be a functional bounded from below, with complete sublevels $\{x \in X : \varphi(x) \le c\}$ for all $c \in \mathbb{R}$. For every $x \in \mathrm{Dom}(\varphi)$ and for every $\eta > 0$ there exists a point $x_\eta \in \mathrm{Dom}(\varphi)$ such that*

$$\varphi(x_\eta) < \varphi(y) + \eta \, \mathsf{d}(y, x_\eta) \quad \text{for every } y \in X \setminus \{x_\eta\}, \tag{2.21a}$$

$$\varphi(x_\eta) + \eta \, \mathsf{d}(x, x_\eta) \le \varphi(x). \tag{2.21b}$$

In the next definition we introduce a relaxed version of the variational problem that underlies (2.15), suitable to deal with possible lack of coercivity.

Definition 2.9 *(Moreau-Yosida-Ekeland resolvent).* Let $\phi : X \to (-\infty, +\infty]$ be a proper functional. For every $x \in X$, $\tau > 0$ and $\eta \ge 0$ we denote by $\mathsf{J}_{\tau,\eta}[x]$ the (possibly empty) set of points $x_{\tau,\eta} \in \mathrm{Dom}(\phi)$ characterized by the following two conditions:

$$\frac{1}{2\tau} \mathsf{d}^2(x, x_{\tau,\eta}) + \phi(x_{\tau,\eta}) \le \frac{1}{2\tau} \mathsf{d}^2(x, y) + \phi(y) + \frac{\eta}{2} \, \mathsf{d}(x, x_{\tau,\eta}) \, \mathsf{d}(y, x_{\tau,\eta}) \quad \text{for every } y \in X, \tag{2.22a}$$

$$\frac{1}{2\tau} \mathsf{d}^2(x, x_{\tau,\eta}) + \phi(x_{\tau,\eta}) \le \phi(x). \tag{2.22b}$$

Note that either ϕ is *not* quadratically bounded from below and therefore the infimum in (2.15) is $-\infty$ for all $\tau > 0$, or ϕ is quadratically bounded from below and for every family $x_{\tau,\eta} \in \mathsf{J}_{\tau,\eta}[x]$, $\eta > 0$, $\tau \in (0, \tau_o)$ and $\bar{x} \in X$ the following properties hold:

$$\mathsf{d}^2(x_{\tau,\eta}, x) \le \frac{2\tau\tau_o}{\tau_o - \tau} \left(\phi(x) - \phi_{\tau_o}(x) \right), \qquad \lim_{\tau \downarrow 0} \mathsf{d}(x_{\tau,\eta}, x) = 0 \quad \text{provided } x \in \mathrm{Dom}(\phi), \tag{2.23}$$

$$\lim_{\eta \downarrow 0} \frac{1}{2\tau} \mathsf{d}^2(x, x_{\tau,\eta}) + \phi(x_{\tau,\eta}) = \phi_\tau(x); \qquad \phi_\tau(x) = \frac{1}{2\tau} \mathsf{d}^2(x, \bar{x}) + \phi(\bar{x}) \quad \Leftrightarrow \quad \bar{x} \in \mathsf{J}_{\tau,0}[x].$$

In particular, $\mathsf{J}_{\tau,\eta}[x]$ is a useful substitute (for $\eta > 0$) of $\mathsf{J}_{\tau,0}[x]$ when this set is empty, i.e. when the infimum in (2.15) is not attained. More importantly, the slopes of its points can be estimated in a quantitative way.

Theorem 2.10 *(Ekeland resolvent and slopes).* *Let* $\phi : X \to (-\infty, +\infty]$ *be a proper functional and* $x \in X$, $\tau > 0$, $\eta \ge 0$. *Then every* $x_{\tau,\eta} \in \mathsf{J}_{\tau,\eta}[x]$ *belongs to* $\mathrm{Dom}(|\partial\phi|)$ *and satisfies the bound*

$$|\partial\phi|(x_{\tau,\eta}) \le \mathfrak{L}_{-1/\tau}[\phi](x_{\tau,\eta}) \le (1 + \tfrac{1}{2}\eta\tau) \frac{\mathsf{d}(x, x_{\tau,\eta})}{\tau}. \tag{2.24}$$

If moreover ϕ has complete sublevels (thus in particular it is l.s.c.) and is quadratically bounded from below according to (2.13), then for every choice of $x \in \mathrm{Dom}(\phi)$, $\tau \in (0, \tau_o)$ and $\eta > 0$, the set $\mathsf{J}_{\tau,\eta}[x]$ is not empty. In particular, $\mathrm{Dom}(|\partial\phi|)$ is dense in $\mathrm{Dom}(\phi)$.

Proof. Let us first check (2.24): if $x_{\tau,\eta} \in \mathsf{J}_{\tau,\eta}[x]$ then (2.22a) yields, for every $y \in X$,

$$\phi(y) \ge \phi(x_{\tau,\eta}) - \frac{\eta}{2} \, \mathsf{d}(x, x_{\tau,\eta}) \mathsf{d}(y, x_{\tau,\eta}) + \frac{1}{2\tau} \left(\mathsf{d}^2(x, x_{\tau,\eta}) - \mathsf{d}^2(x, y) \right)$$

$$\geq \phi(x_{\tau,\eta}) - \frac{\eta}{2}\mathsf{d}(x,x_{\tau,\eta})\mathsf{d}(y,x_{\tau,\eta}) - \mathsf{d}(y,x_{\tau,\eta})\frac{1}{2\tau}\Big(\mathsf{d}(x,x_{\tau,\eta}) + \mathsf{d}(x,y)\Big)$$

$$= \phi(x_{\tau,\eta}) - \Big(\frac{\eta}{2} + \tau^{-1}\Big)\mathsf{d}(x,x_{\tau,\eta})\mathsf{d}(y,x_{\tau,\eta}) - \mathsf{d}(y,x_{\tau,\eta})\frac{1}{2\tau}\Big(\mathsf{d}(x,y) - \mathsf{d}(x,x_{\tau,\eta})\Big)$$

$$\geq \phi(x_{\tau,\eta}) - \Big(\frac{\eta}{2} + \tau^{-1}\Big)\mathsf{d}(x,x_{\tau,\eta})\mathsf{d}(y,x_{\tau,\eta}) - \frac{1}{2\tau}\mathsf{d}^2(y,x_{\tau,\eta}).$$

Recalling (2.18) and (2.19) we obtain (2.24).

Let us now prove that if ϕ satisfies (2.13), its sublevels are complete, $x \in \mathrm{Dom}(\phi)$, $\tau < \tau_o$ and $\eta > 0$, then $\mathrm{J}_{\tau,\eta}[x]$ is not empty. We want to apply Ekeland's variational principle (Theorem 2.8) to the function

$$\varphi(y) := \frac{1}{2\tau}\mathsf{d}^2(x,y) + \phi(y) \quad y \in X,$$

choosing $x_0 = x$. Note that φ is bounded from below by (2.13). In case x itself is a minimizer of φ, we can simply pick $x_{\tau,\eta} = x$. Otherwise, $\phi(x) = \varphi(x) > \inf_{y \in X} \varphi(y)$ and we can set $\varepsilon = 2(\varphi(x) - \inf_{y \in X} \varphi(y)) > 0$. Upon choosing a vanishing sequence $\eta_n \downarrow 0$, Ekeland's variational principle provides a sequence $\{x_n\} \subset \mathrm{Dom}(\phi)$ such that

$$\varphi(x_n) = \frac{1}{2\tau}\mathsf{d}^2(x,x_n) + \phi(x_n) \overset{(2.21b)}{\leq} \varphi(x) = \phi(x) \tag{2.25}$$

and

$$\varphi(x_n) \overset{(2.21a)}{\leq} \varphi(y) + \eta_n\,\mathsf{d}(y,x_n) \quad \text{for every } y \in X. \tag{2.26}$$

Since x is not a minimizer of φ and ϕ is l.s.c., there exists $n_0 \in \mathbb{N}$ such that the quantity $\delta := \inf_{n \geq n_0}\mathsf{d}(x,x_n)$ is strictly positive and therefore there exists in turn an integer $\bar{n} \geq n_0$ such that $2\eta_{\bar{n}} \leq \delta\eta$. By choosing $x_{\tau,\eta} = x_{\bar{n}}$, inequalities (2.22b) and (2.22a) follow from (2.25) and (2.26), respectively. The last assertion is just a consequence of (2.23). $\quad\square$

2.3. λ-convex and approximately λ-convex functionals

We will study a class of gradient flows which is strictly related to suitable convexity properties of the driving functional. We therefore recall the meaning of convexity on geodesic subsets and introduce a slightly weaker property on (approximate) length subsets.

Definition 2.11 (λ-convexity). A functional $\phi : X \to (-\infty, +\infty]$ is λ-convex in $D \subset X$ for some $\lambda \in \mathbb{R}$ if for every couple of points $\mathsf{x}_0, \mathsf{x}_1 \in D \cap \mathrm{Dom}(\phi)$ there exists a geodesic $\mathsf{x} \in \mathrm{Geo}_D[\mathsf{x}_0 \to \mathsf{x}_1]$ such that

$$\phi(\mathsf{x}_t) \leq (1-t)\phi(\mathsf{x}_0) + t\phi(\mathsf{x}_1) - \frac{\lambda}{2}t(1-t)\mathsf{d}^2(\mathsf{x}_0,\mathsf{x}_1) \quad \text{for every } t \in [0,1]. \tag{2.27}$$

We denote by $\operatorname{Geo}_D^{\phi,\lambda}[x_0 \to x_1] \subset \operatorname{Geo}_D[x_0 \to x_1]$ the subset of geodesics enjoying (2.27). The functional ϕ is *strongly λ-convex* in D if it is λ-convex and (2.27) holds for *every* geodesic in $\operatorname{Geo}_D[x_0 \to x_1]$, i.e. $\emptyset \neq \operatorname{Geo}_D^{\phi,\lambda}[x_0 \to x_1] = \operatorname{Geo}_D[x_0 \to x_1]$.

Definition 2.12 *(Approximate λ-convexity).* A functional $\phi : X \to (-\infty, +\infty]$ is *approximately λ-convex* in $D \subset X$ for some $\lambda \in \mathbb{R}$ if for every couple of points $x_0, x_1 \in D \cap \operatorname{Dom}(\phi)$ and for every $\vartheta, \varepsilon \in (0, 1)$ there exists a (ϑ, ε)-intermediate point $x_{\vartheta,\varepsilon} \in D$ between x_0 and x_1 such that

$$\phi(x_{\vartheta,\varepsilon}) \leq (1 - \vartheta)\phi(x_0) + \vartheta\phi(x_1) - \frac{\lambda - \varepsilon}{2}\vartheta(1 - \vartheta)d^2(x_0, x_1). \tag{2.28}$$

When $D = X$ (or, equivalently, $D = \operatorname{Dom}(\phi)$) we just say that ϕ is (approximately) λ-convex, and if $\lambda = 0$ we say (approximately) "convex" instead of (approximately) "0-convex".

Note that an approximately convex function may be far from being convex. For example, consider the Dirichlet function in $[0, 1]$ which takes the values 0 in $\mathbb{Q}$ and 1 in $[0, 1] \setminus \mathbb{Q}$: the latter is clearly non-convex but turns out to be approximately convex. Nevertheless, if D is complete and ϕ is l.s.c. such kind of situations cannot occur (see Remark 2.14).

Lemma 2.13. *Let $D \subset X$ be a set whose intersection with the sublevels of ϕ, $\{\phi \leq c\}$ with $c \in \mathbb{R}$, are complete. If ϕ is approximately λ-convex in D then D is a length subset of X.*

Proof. We argue as in the proof of Lemma 2.6; we only have to check that the points constructed by the induction argument are contained in a common sublevel of ϕ. In fact, we will show that one can pick every point $x(k2^{-n})$ so that

$$\phi(x(k2^{-n})) \leq \varphi_n := \varphi_0 + a \sum_{h=0}^{n-1} 4^{-h} \leq \varphi_0 + 4a/3, \tag{2.29}$$

where $\varphi_0 := \max(\phi(x_0), \phi(x_1))$, $a := \frac{1}{2}(\lambda - 1)^- d^2(x_0, x_1)$. We argue by induction: the case $n = 0$ is trivial; assuming (2.29), it is sufficient to check the induction step for every $\vartheta \in \mathbb{D}_{n+1}$ of the form $\vartheta = x((2k + 1)2^{-(n+1)})$ for some integer $k \in [0, 2^n - 1]$. If we pick $x((2k + 1)2^{-(n+1)})$ as an ε_{n+1}-midpoint between $x(k2^{-n})$ and $x((k + 1)2^{-n})$ satisfying (2.28), we deduce that

$$\phi(x((2k + 1)2^{-(n+1)}))$$

$$\leq \varphi_n + \frac{1}{8}(\lambda - 1)^- d^2(x(k2^{-n}), x((k + 1)2^{-n}))$$

$$\leq \varphi_n + \frac{(\lambda - 1)^-}{8 \cdot 4^n} d^2(x_0, x_1)(1 + \delta)^2 \leq \varphi_n + \frac{(\lambda - 1)^-}{2 \cdot 4^n} d^2(x_0, x_1) \leq \varphi_{n+1},$$

where we used the fact that $1 + \delta \leq 2$. $\quad\square$

Remark 2.14 *(Approximate convexity along approximate geodesics).* Under the same hypotheses as in Lemma 2.13, one can show that for any given $\varepsilon \in (0,1)$ there exists a Lipschitz curve $\vartheta \in [0,1] \mapsto \mathsf{x}_{\vartheta,\varepsilon} \in D \cap \mathrm{Dom}(\phi)$ such that (2.28) holds and

$$\mathsf{d}(\mathsf{x}_{\vartheta',\varepsilon}, \mathsf{x}_{\vartheta'',\varepsilon}) \leq \mathsf{d}(x_0,x_1)\,(1+\varepsilon)\,|\vartheta'-\vartheta''| \quad \text{for every } \vartheta',\vartheta'' \in [0,1].$$

In order to prove it, one has to adapt the proof of Lemma 2.13 so as to make sure that (2.28) is satisfied at every induction step. We omit details, but let us mention that for this procedure to work it is necessary to require that each ε_n, at fixed n, is chosen depending also on k (just as a consequence of the fact that the ε-error allowed in (2.28) is weighted by $\vartheta(1-\vartheta)$).

For λ-convex functionals there is a way of characterizing $|\partial\phi|(x)$ as the modulus of the slope along the *direction of maximal slope*, to some extent.

Definition 2.15 *(Directional derivatives).* If x is a geodesic starting at $\mathsf{x}_0 \in \mathrm{Dom}(\phi)$ we set

$$\phi'(\mathsf{x}_0;\mathsf{x}) := \liminf_{t\downarrow 0} \frac{\phi(\mathsf{x}_t) - \phi(\mathsf{x}_0)}{t}.$$

It is not difficult to check that, if in addition $\mathsf{x} \in \mathrm{Geo}_X^{\phi,\lambda}[\mathsf{x}_0 \to \mathsf{x}_1]$, then

$$\phi(\mathsf{x}_1) \geq \phi(\mathsf{x}_0) + \phi'(\mathsf{x}_0;\mathsf{x}) + \frac{\lambda}{2}\mathsf{d}^2(\mathsf{x}_0,\mathsf{x}_1). \tag{2.30}$$

Indeed, (2.27) can be rewritten as

$$\phi(\mathsf{x}_1) \geq \phi(\mathsf{x}_0) + \frac{\phi(\mathsf{x}_t) - \phi(\mathsf{x}_0)}{t} + \frac{\lambda}{2}(1-t)\mathsf{d}^2(\mathsf{x}_0,\mathsf{x}_1);$$

passing to the limit as $t \downarrow 0$ we obtain (2.30). If ϕ is λ-convex it follows that [5, Theorem 2.4.9]

$$|\partial\phi|(x) = \sup_{y \in \mathrm{Dom}(\phi),\, \mathsf{x} \in \mathrm{Geo}_X^{\phi,\lambda}[x \to y]} \frac{(\phi'(x;\mathsf{x}))^-}{|\dot{\mathsf{x}}|} = \mathfrak{L}_\lambda[\phi](x),$$

where the supremum is understood to be zero if $x \in \mathrm{Dom}(\phi)$ is isolated in $\mathrm{Dom}(\phi)$, whereas it is set to $+\infty$ if $x \notin \mathrm{Dom}(\phi)$. Hence the slope of ϕ is lower semicontinuous if ϕ is, and it can be characterized by the global lower bound (2.18). In fact the identity between metric slope and global λ-slope also holds if ϕ is approximately λ-convex.

Proposition 2.16 *(Approximate λ-convexity and slopes). If $\phi : X \to (-\infty, +\infty]$ is an approximately λ-convex functional then*

$$|\partial\phi|(x) = \mathfrak{L}_\lambda[\phi](x) \quad \text{for every } x \in X. \tag{2.31}$$

If the sublevels of ϕ are complete, we also have

$$|\partial\phi|(x) = |\partial\phi|_\ell(x) \quad \text{for every } x \in X. \tag{2.32}$$

Proof. We can suppose with no loss of generality that $x = \mathsf{x}_0 \in \mathrm{Dom}(|\partial\phi|)$ and $y = \mathsf{x}_1 \in \mathrm{Dom}(\phi)$. By Definitions 2.5 (along with subsequent discussions) and 2.12, for every $\vartheta, \varepsilon \in (0,1)$ we can find a (ϑ, ε)-intermediate point $\mathsf{x}_{\vartheta,\varepsilon}$ between x_0 and x_1 such that

$$\phi(\mathsf{x}_1) \geq \phi(\mathsf{x}_0) + \frac{\phi(\mathsf{x}_{\vartheta,\varepsilon}) - \phi(\mathsf{x}_0)}{\vartheta} + \frac{\lambda - \varepsilon}{2}(1 - \vartheta)\mathsf{d}^2(\mathsf{x}_0, \mathsf{x}_1),$$
$$\vartheta(1-\varepsilon)\mathsf{d}(\mathsf{x}_0, \mathsf{x}_1) \leq \mathsf{d}(\mathsf{x}_0, \mathsf{x}_{\vartheta,\varepsilon}) \leq \vartheta(1+\varepsilon)\mathsf{d}(\mathsf{x}_0, \mathsf{x}_1), \tag{2.33}$$

whence

$$\phi(\mathsf{x}_1) \geq \phi(\mathsf{x}_0) - \frac{(\phi(\mathsf{x}_0) - \phi(\mathsf{x}_{\vartheta,\varepsilon}))^+}{\mathsf{d}(\mathsf{x}_0, \mathsf{x}_{\vartheta,\varepsilon})}\mathsf{d}(\mathsf{x}_0, \mathsf{x}_1)(1+\varepsilon) + \frac{\lambda - \varepsilon}{2}(1 - \vartheta)\mathsf{d}^2(\mathsf{x}_0, \mathsf{x}_1);$$

by passing to the limit as $\vartheta \downarrow 0$, we end up with

$$\phi(\mathsf{x}_1) \geq \phi(\mathsf{x}_0) - |\partial\phi|(\mathsf{x}_0)\mathsf{d}(\mathsf{x}_0, \mathsf{x}_1)(1+\varepsilon) + \frac{\lambda - \varepsilon}{2}\mathsf{d}^2(\mathsf{x}_0, \mathsf{x}_1).$$

Identity (2.31) follows by finally letting $\varepsilon \downarrow 0$, recalling (2.18)–(2.19).

In order to prove (2.32) it is sufficient to observe that $\mathrm{Dom}(\phi)$ is a length subset of X, thanks to Lemma 2.13, so that $\mathsf{d} = \mathsf{d}_\ell = \mathsf{d}_{\mathrm{Dom}(\phi),\ell}$ on $\mathrm{Dom}(\phi)$ and therefore $|\partial\phi| = |\partial\phi|_\ell$ by the characterization (2.20) of the metric slope. $\square$

Another important property of approximately λ-convex functionals ϕ is their quadratic boundedness from below, provided sublevels are complete. This is obvious if $\lambda > 0$ as long as standard convexity is concerned, since in such case ϕ is even bounded from below, see [5, Lemma 2.4.8]. We will also show that $x \mapsto \phi(x) - \frac{\lambda}{2}\mathsf{d}^2(x, o)$ is linearly bounded from below; in the case $\lambda = 0$ we thus find a metric analog of the well-known property of convex l.s.c. functionals in Banach spaces.

Theorem 2.17 *(Approximate λ-convexity and quadratic boundedness). If $\phi : X \to (-\infty, +\infty]$ is approximately λ-convex and has complete sublevels, then it is quadratically bounded from below. More precisely, for every $o \in \mathrm{Dom}(\phi)$ and $\kappa_o > -\lambda$ it satisfies the lower bound (2.13) with*

$$\phi_o := \phi(o) - \frac{\left(\phi(o) - m_o + \frac{\lambda^+}{2}\right)^2}{2(\lambda + \kappa_o)} - \frac{\lambda + \kappa_o}{2},$$
$$m_o := \inf\{\phi(x) : x \in X, \ \mathsf{d}(x, o) \leq 1\} > -\infty. \tag{2.34}$$

Furthermore, the functional $x \mapsto \phi(x) - \frac{\lambda}{2}\mathsf{d}^2(x,o)$ *is linearly bounded from below and satisfies the lower bound* (2.14) *with*

$$\ell_o := \phi(o) - m_o + \frac{\lambda}{2}, \quad \phi_o := m_o - \frac{\lambda^-}{2}. \tag{2.35}$$

Proof. Let us first prove that m_o given by (2.34) is finite: we argue by contradiction assuming that $m_o = -\infty$. Let Z be the following complete metric space:

$$Z := \{x \in X : \phi(x) \le \phi(o), \ \mathsf{d}(x,o) \le 1\}.$$

For all $\varepsilon \in (0,1)$, we can pick a general sequence $\{y_n\} \subset Z$ such that $\phi(y_n) \le -4^n$ and define inductively another sequence $\{x_n\}$ by setting $x_{n+1} := \mathsf{x}_{\vartheta,\varepsilon}$, where $\vartheta = 2^{-n}$ and $\mathsf{x}_{\vartheta,\varepsilon}$ is a (ϑ,ε)-intermediate point between x_n and y_n fulfilling (2.28) with $\mathsf{x}_0 = x_n$ and $\mathsf{x}_1 = y_n$. The sequence is supposed to start from $x_3 = o$, so that $\mathsf{d}(x_n,o) \le 1$ for all $n \ge 3$ and (note that $\mathsf{d}(x_n,y_n) \le 2$)

$$\begin{aligned}
\mathsf{d}(x_n,x_{n+1}) &\overset{(2.8)}{\le} 2^{-n+1}(1+\varepsilon), \\
\phi(x_{n+1}) - \phi(x_n) &\overset{(2.28)}{\le} 2^{-n}\big[\phi(y_n) - \phi(x_n) + 2(\lambda^- + \varepsilon)\big].
\end{aligned} \tag{2.36}$$

From the right-hand inequality of (2.36) it is not difficult to check that $\phi(x_n) \to -\infty$ and therefore $x_n \in Z$ eventually: because $\{x_n\}$ is a Cauchy sequence (hence converging to some $x_\infty \in Z$) and ϕ is l.s.c., it follows that $\phi(x_\infty) \le -\infty$, which is impossible since $-\infty$ is not in the range of ϕ.

In order to show the bound (2.13), it is not restrictive to assume $x \in \mathrm{Dom}(\phi)$. The latter is then a simple consequence of (2.33), understood with $\mathsf{x}_0 = o$ and $\mathsf{x}_1 = x$, which yields for every $\vartheta, \varepsilon \in (0,1)$

$$\phi(x) \ge \phi(o) - \vartheta^{-1}\big(\phi(o) - \phi(\mathsf{x}_{\vartheta,\varepsilon})\big) + \frac{\lambda - \varepsilon}{2}(1-\vartheta)\mathsf{d}^2(o,x). \tag{2.37}$$

When $\mathsf{d}(o,x) > 1$ we can pick $\vartheta^{-1} := \mathsf{d}(o,x)(1+\varepsilon)$, so that $\phi(\mathsf{x}_{\vartheta,\varepsilon}) \ge m_o$; in any case, we obtain for every $x \in X$

$$\phi(x) \ge \phi(o) - (\mathsf{d}(o,x) \vee 1)(1+\varepsilon)\big(\phi(o) - m_o + \lambda^+/2\big) + \frac{\lambda - \varepsilon}{2}\mathsf{d}^2(o,x),$$

and by letting $\varepsilon \downarrow 0$

$$\begin{aligned}
\phi(x) \ge &\phi(o) - (\mathsf{d}(o,x) \vee 1)\big(\phi(o) - m_o + \lambda^+/2\big) + \frac{\lambda}{2}\mathsf{d}^2(o,x) \\
\ge &\phi(o) - \frac{1}{2\delta}\big(\phi(o) - m_o + \lambda^+/2\big)^2 - \frac{\delta}{2} + \frac{\lambda - \delta}{2}\mathsf{d}^2(o,x)
\end{aligned}$$

for every $\delta > 0$. By choosing $\delta := \lambda + \kappa_o > 0$ we get (2.34). Linear boundedness from below with the choices (2.35) follows straightforwardly from (2.37) arguing similarly to above. $\square$

3. The metric approach to gradient flows

The goal of this section is to study the properties of λ-gradient flows, which will be associated with a metric-functional system

$$
\begin{aligned}
&\mathbb{X} = (X, \mathsf{d}, \phi), \text{ where } (X, \mathsf{d}) \text{ is a metric space and} \\
&\phi : X \to (-\infty, +\infty] \text{ is a proper l.s.c. functional.}
\end{aligned}
\tag{3.1}
$$

In the following, $\mathbb{X}$ and (X, d, ϕ) will always refer to (3.1).

3.1. Evolution Variational Inequalities (EVI)

The next (quite demanding) definition is modeled after the case of λ-convex functionals in Euclidean-like spaces, and has first been introduced in [5, Chapter 4]. In the sequel, by $\frac{\mathrm{d}^+}{\mathrm{d}t}$ and $\frac{\mathrm{d}}{\mathrm{d}t_+}$ we will denote right Dini derivatives (see Appendix A).

Definition 3.1 (EVI *and gradient flow*). Let $\lambda \in \mathbb{R}$ and $\mathbb{X} = (X, \mathsf{d}, \phi)$ be a metric-functional system as in (3.1). A solution of the *Evolution Variational Inequality* $\mathrm{EVI}_\lambda(X, \mathsf{d}, \phi)$ is a continuous curve $u : t \in (0, +\infty) \mapsto u_t \in \mathrm{Dom}(\phi)$ such that

$$
\frac{1}{2} \frac{\mathrm{d}^+}{\mathrm{d}t} \mathsf{d}^2(u_t, v) + \frac{\lambda}{2} \mathsf{d}^2(u_t, v) \le \phi(v) - \phi(u_t) \quad \text{for every } t \in (0, +\infty), \ v \in \mathrm{Dom}(\phi). \ (\mathrm{EVI}_\lambda)
$$

An EVI_λ-*gradient flow* of ϕ in $D \subset \overline{\mathrm{Dom}(\phi)}$ is a family of continuous maps $\mathsf{S}_t : D \to D$, $t \ge 0$, such that for every $u_0 \in D$

$$
\mathsf{S}_{t+h}(u_0) = \mathsf{S}_h(\mathsf{S}_t(u_0)) \quad \text{for every } t, h \ge 0, \qquad \lim_{t \downarrow 0} \mathsf{S}_t(u_0) = \mathsf{S}_0(u_0) = u_0, \tag{3.2a}
$$

$$
\text{the curve } t \mapsto \mathsf{S}_t(u_0) \text{ is a solution of } \mathrm{EVI}_\lambda(X, \mathsf{d}, \phi). \tag{3.2b}
$$

Remark 3.2 (*Upper, lower and distributional derivatives*). Thanks to Lemma A.1 from Appendix A, for *continuous* curves u the upper right derivative in (EVI_λ) can be replaced by the lower right derivative or the distributional one; in other words, (EVI_λ) is equivalent to

$$
\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t_+} \mathsf{d}^2(u_t, v) + \frac{\lambda}{2} \mathsf{d}^2(u_t, v) \le \phi(v) - \phi(u_t) \quad \text{for every } t \in (0, +\infty), \ v \in \mathrm{Dom}(\phi), \ (\mathrm{EVI}_\lambda^\ell)
$$

or to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{d}^2(u_t,v) + \frac{\lambda}{2}\mathsf{d}^2(u_t,v) \leq \phi(v) - \phi(u_t) \quad \text{in } \mathscr{D}'((0,+\infty)), \quad \text{for every } v \in \mathrm{Dom}(\phi)$$

$$(\mathrm{EVI}_\lambda^{\mathscr{D}})$$

along with the further requirement $\phi \circ u \in L^1_{\mathrm{loc}}((0,+\infty))$. If u is also *absolutely* continuous, a pointwise derivative almost everywhere (compare with [5, Theorem 4.0.4]) is sufficient: indeed, in such case, u satisfies $\mathrm{EVI}_\lambda(X,\mathsf{d},\phi)$ in the integral form (EVI'_λ) below (the fact that $\phi \circ u \in L^1_{\mathrm{loc}}((0,+\infty))$ easily follows from the lower semicontinuity of ϕ and (EVI_λ) itself).

The next result shows two different characterizations of solutions to (EVI_λ) (see also [27,34]). At the core of its proof is the fact that $t \mapsto \phi(u_t)$ is a nonincreasing function along any solution to (EVI_λ). To our purposes, from here on it is convenient to set

$$\mathsf{E}_\lambda(t) := \int_0^t e^{\lambda r}\,\mathrm{d}r = \begin{cases} \frac{e^{\lambda t}-1}{\lambda} & \text{if } \lambda \neq 0, \\ t & \text{if } \lambda = 0. \end{cases} \tag{3.3}$$

Theorem 3.3 *(Integral characterizations of* EVI*). A curve $u : (0,+\infty) \to \overline{\mathrm{Dom}(\phi)}$ is a solution of $\mathrm{EVI}_\lambda(X,\mathsf{d},\phi)$ according to Definition 3.1 if and only if it satisfies one of the following equivalent formulations.*

(IC1) *For every $v \in \mathrm{Dom}(\phi)$ the maps $t \mapsto \phi(u_t)$, $t \mapsto \mathsf{d}^2(u_t,v)$ belong to $L^1_{\mathrm{loc}}((0,+\infty))$ and for all $s,t \in (0,+\infty)$ with $s < t$ there holds*

$$\frac{1}{2}\mathsf{d}^2(u_t,v) - \frac{1}{2}\mathsf{d}^2(u_s,v) + \int_s^t \left(\phi(u_r) + \frac{\lambda}{2}\mathsf{d}^2(u_r,v)\right)\mathrm{d}r \leq (t-s)\phi(v). \quad (\mathrm{EVI}'_\lambda)$$

(IC2) *For every $s,t \in (0,+\infty)$ with $s < t$ and $v \in \mathrm{Dom}(\phi)$ there holds*

$$\frac{e^{\lambda(t-s)}}{2}\mathsf{d}^2(u_t,v) - \frac{1}{2}\mathsf{d}^2(u_s,v) \leq \mathsf{E}_\lambda(t-s)\Big(\phi(v) - \phi(u_t)\Big). \quad (\mathrm{EVI}''_\lambda)$$

Furthermore, the map $t \mapsto \phi(u_t)$ is nonincreasing (in particular it is right continuous).

Proof. We split the proof in various steps.

$(\mathrm{EVI}_\lambda) \Rightarrow (\mathrm{EVI}'_\lambda)$: *solutions of (EVI_λ) according to Definition 3.1 satisfy* (IC1). *It is enough to apply Lemma A.1 to the functions*

$$\zeta(t) = \frac{1}{2}\mathsf{d}^2(u_t,v), \quad \eta(t) = \frac{\lambda}{2}\mathsf{d}^2(u_t,v) + \phi(u_t) - \phi(v).$$

$(\mathrm{EVI}''_\lambda) \Rightarrow (\mathrm{EVI}_\lambda)$: *solutions of* (IC2) *satisfy (EVI_λ) according to Definition 3.1 (and $t \mapsto \phi(u_t)$ is nonincreasing).* First of all note that (EVI''_λ) yields $u_t \in \mathrm{Dom}(\phi)$ for all

$t > 0$. Hence by choosing $v = u_s$ in (EVI$''_\lambda$) we immediately get that $t \mapsto \phi(u_t)$ is nonincreasing, in particular it is locally bounded in $(0, +\infty)$. It follows that

$$\limsup_{t \downarrow t_0} \mathsf{d}(u_t, v) \le \mathsf{d}(u_{t_0}, v) \le \liminf_{t \uparrow t_0} \mathsf{d}(u_t, v) \quad \text{for every } t_0 > 0, \ v \in \mathrm{Dom}(\phi),$$

so that u is right continuous and the maps $t \mapsto \mathsf{d}(u_t, v)$, $t \mapsto \phi(u_t)$ are lower semicontinuous and satisfy (EVI$_\lambda$); arguing as in step (EVI$_\lambda$) $\Rightarrow$ (EVI$'_\lambda$) (only lower semicontinuity is needed) we can show that they satisfy (IC1) and by the next step u is also continuous.

(EVI$'_\lambda$) $\Rightarrow$ (EVI$_\lambda$): *solutions of* (IC1) *satisfy* (EVI$_\lambda$) *according to Definition* 3.1 *(and $t \mapsto \phi(u_t)$ is nonincreasing)*. To begin with, from (EVI$'_\lambda$) we immediately deduce the right continuity of u, since

$$\limsup_{t \downarrow s} \mathsf{d}^2(u_t, v) \le \mathsf{d}^2(u_s, v) \quad \text{for every } \ v \in \mathrm{Dom}(\phi)$$

and $u_s \in \overline{\mathrm{Dom}(\phi)}$. Because $t \mapsto \phi(u_t) \in L^1_{\mathrm{loc}}((0, +\infty))$ and u is right continuous, there exists a sequence $v_n \subset \mathrm{Dom}(\phi)$ which is dense in the image of u. In particular there holds

$$\mathsf{d}^2(u_t, u_s) = \sup_{n \in \mathbb{N}} |\mathsf{d}(u_t, v_n) - \mathsf{d}(u_s, v_n)|^2 \quad \forall s, t \in (0, +\infty),$$

so that the real function $(t, s) \mapsto \mathsf{d}^2(u_t, u_s)$ is Lebesgue measurable in $(0, +\infty) \times (0, +\infty)$. Given almost every $s > 0$ and any $h > 0$, we can choose $v = u_s$ and $t = s + h$ in (EVI$'_\lambda$) to get

$$\frac{1}{2} \mathsf{d}^2(u_{s+h}, u_s) + \frac{\lambda}{2} \int_0^h \mathsf{d}^2(u_{s+r}, u_s) \, \mathrm{d}r \le \int_0^h \Big(\phi(u_s) - \phi(u_{s+r}) \Big) \, \mathrm{d}r; \qquad (3.4)$$

a further integration w.r.t. s from $t_0 > 0$ to $t_1 > t_0$ yields

$$\int_{t_0}^{t_1} \mathsf{d}^2(u_{s+h}, u_s) \, \mathrm{d}s + \lambda \int_0^h \int_{t_0}^{t_1} \mathsf{d}^2(u_{s+r}, u_s) \, \mathrm{d}s \, \mathrm{d}r \le 2 \int_0^h \int_{t_0}^{t_1} \Big(\phi(u_s) - \phi(u_{s+r}) \Big) \, \mathrm{d}s \, \mathrm{d}r$$

$$= 2 \int_0^h \int_0^r \Big(\phi(u_{t_0+\xi}) - \phi(u_{t_1+\xi}) \Big) \, \mathrm{d}\xi \, \mathrm{d}r = 2 \int_0^h \int_0^h \Big(\phi(u_{t_0+\xi}) - \phi(u_{t_1+\xi}) \Big) \chi_{[0,r]}(\xi) \, \mathrm{d}\xi \, \mathrm{d}r$$

$$= 2 \int_0^h \Big(\phi(u_{t_0+\xi}) - \phi(u_{t_1+\xi}) \Big) (h - \xi) \, \mathrm{d}\xi = 2h^2 \int_0^1 \Big(\phi(u_{t_0+h\xi}) - \phi(u_{t_1+h\xi}) \Big) (1 - \xi) \, \mathrm{d}\xi.$$

Upon setting $x(h) := \int_{t_0}^{t_1} \mathsf{d}^2(u_{s+h}, u_s) \, \mathrm{d}s$, we therefore obtain

$$x(h) + \lambda \int_0^h x(r) \, \mathrm{d}r \le h^2 \, y(h), \qquad y(h) := 2 \int_0^1 \Big(\phi(u_{t_0+h\xi}) - \phi(u_{t_1+h\xi}) \Big)(1-\xi) \, \mathrm{d}\xi,$$

so that Gronwall's Lemma yields

$$h^{-2} \, x(h) \le \mathrm{e}^{\lambda^- h} \sup_{0 \le \delta \le h} y(\delta). \tag{3.5}$$

If t_0, t_1 are Lebesgue points of the map $s \mapsto \phi(u_s)$, then

$$\lim_{h \downarrow 0} y(h) = \lim_{h \downarrow 0} \sup_{0 \le \delta \le h} y(\delta) = \phi(u_{t_0}) - \phi(u_{t_1}),$$

whence from (3.5)

$$\limsup_{h \downarrow 0} \int_{t_0}^{t_1} \frac{\mathrm{d}^2(u_{s+h}, u_s)}{h^2} \, \mathrm{d}s \le \phi(u_{t_0}) - \phi(u_{t_1}). \tag{3.6}$$

It follows that the restriction of the map $t \mapsto \phi(u_t)$ to its Lebesgue set is not increasing; ϕ being l.s.c. and u right continuous, we get $\phi(u_t) \le \phi(u_{t_0})$ for every Lebesgue point $t_0 < t$, and in particular $u_t \in \mathrm{Dom}(\phi)$ for every $t > 0$. On the other hand, by choosing $t = s + h$ and $v = u_s$ in (EVI$'_\lambda$) (now we are allowed to do it for every $s > 0$) and recalling the right continuity of u, we end up with

$$\phi(u_t) \le \limsup_{h \downarrow 0} \frac{1}{h} \int_s^{s+h} \phi(u_r) \, \mathrm{d}r \le \phi(u_s) \quad \text{for every } t > s > 0,$$

i.e. $t \mapsto \phi(u_t)$ is nonincreasing in $(0, +\infty)$. As ϕ is lower semicontinuous, we also deduce that

$$\phi(u_t) \le \phi(u_s) = \lim_{h \downarrow 0} \phi(u_{s+h}) = \lim_{h \downarrow 0} \frac{1}{h} \int_s^{s+h} \phi(u_r) \, \mathrm{d}r \quad \text{for every } t > s > 0. \tag{3.7}$$

Moreover, (3.6) and the triangle inequality show that for every interval $(t_0, t_1) \subset (0, +\infty)$ there exists a constant $C_{t_0,t_1} > 0$ such that

$$\int_{t_0}^{t_1-h} \frac{\mathrm{d}^2(u_{s+h}, u_s)}{h^2} \, \mathrm{d}s \le C_{t_0,t_1} \quad \text{for every } h \in (0, t_1 - t_0). \tag{3.8}$$

Since $|\mathsf{d}(u_{s+h}, w) - \mathsf{d}(u_s, w)| \le \mathsf{d}(u_{s+h}, u_s)$, by (3.8) we can infer that the curve $t \mapsto \mathsf{d}(u_t, w)$ belongs to the Sobolev space $W^{1,2}_{\mathrm{loc}}((0, +\infty))$ for every $w \in X$. Because it is also right continuous, it coincides with its continuous representative and therefore $u \in C^0((0, +\infty); X)$. By differentiating (EVI$'_\lambda$) and using (3.7), we finally obtain (EVI$_\lambda$).

$(\text{EVI}_\lambda) \Rightarrow (\text{EVI}''_\lambda)$: *solutions of* (EVI_λ) *according to Definition* 3.1 *satisfy* (IC2). We multiply inequality (EVI_λ) by $\mathrm{e}^{\lambda t}$ obtaining

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\left(\frac{\mathrm{e}^{\lambda t}}{2}\mathsf{d}^2(u_t, v)\right) \leq \mathrm{e}^{\lambda t}\Big(\phi(v) - \phi(u_t)\Big). \tag{3.9}$$

By combining steps $(\text{EVI}_\lambda) \Rightarrow (\text{EVI}'_\lambda)$ and $(\text{EVI}'_\lambda) \Rightarrow (\text{EVI}_\lambda)$ we have established that $t \mapsto \phi(u_t)$ is nonincreasing, so that an integration of (3.9) through Lemma A.1 yields (EVI''_λ). □

Remark 3.4 *(EVI in* $[0, +\infty)$*)*. We can actually extend the above results down to $t = 0$. More precisely, a curve $u : [0, +\infty) \to \overline{\mathrm{Dom}(\phi)}$ is a solution of $\text{EVI}_\lambda(X, \mathsf{d}, \phi)$ according to Definition 3.1 (with the additional requirement that $u \in C^0([0, +\infty); X)$) if and only if it is a solution of (IC1) down to $s = 0$ (with the additional requirements that $t \mapsto \mathsf{d}^2(u_t, v) \in L^1_{\mathrm{loc}}([0, +\infty))$, $t \mapsto \phi^-(u_t) \in L^1_{\mathrm{loc}}([0, +\infty))$) and if and only if it is a solution of (IC2) down to $s = 0$. Accordingly, the map $t \mapsto \phi(u_t)$ is nonincreasing in $[0, +\infty)$. Furthermore, if $u_0 \in \mathrm{Dom}(\phi)$ then (EVI_λ) also holds at $t = 0$, whereas in the case $u_0 \notin \mathrm{Dom}(\phi)$ both the l.h.s. and the r.h.s. are $-\infty$.

In order to prove such properties, one can reproduce the proof of Theorem 3.3 and exploit the already established local (in $[0, +\infty)$) boundedness from below of $t \mapsto \phi(u_t)$.

3.2. The fundamental properties of EVI

The next theorem collects many useful results which illustrate some important consequences of $\text{EVI}_\lambda(X, \mathsf{d}, \phi)$, which can be considered as the metric version of the analogous properties for gradient flows of λ-convex functionals in Hilbert spaces, see e.g. [21]; partial results have also been obtained in the Wasserstein framework or under some joint convexity properties of (X, d, ϕ), see [4,5]. In particular, the energy identity (3.17) plays an important role in [3,8].

Theorem 3.5 *(Properties of* EVI_λ*)*. *Let* $u, u^1, u^2 \in C^0([0, +\infty); X)$ *be solutions of* $\text{EVI}_\lambda(X, \mathsf{d}, \phi)$. *Then the following claims hold.*

λ-contraction and uniqueness

$$\mathsf{d}(u^1_t, u^2_t) \leq \mathrm{e}^{-\lambda(t-s)}\mathsf{d}(u^1_s, u^2_s) \quad \textit{for every } 0 \leq s < t < +\infty. \tag{3.10}$$

In particular, for every $u_0 \in \overline{\mathrm{Dom}(\phi)}$ *there is at most one solution* u *of* $\text{EVI}_\lambda(X, \mathsf{d}, \phi)$ *satisfying the initial condition* $\lim_{t\downarrow 0} u_t = u_0$.

Regularizing effects

u is locally Lipschitz in $(0, +\infty)$ *and* $u_t \in \mathrm{Dom}(|\partial\phi|) \subset \mathrm{Dom}(\phi)$ *for every* $t > 0$;

the map $t \in [0, +\infty) \mapsto \phi(u_t)$ *is nonincreasing and*

$$(\text{locally semi- in } (0, +\infty), \text{ if } \lambda < 0) \text{ convex;} \tag{3.11}$$

$$\text{the map } t \in [0, +\infty) \mapsto \mathrm{e}^{\lambda t} |\partial\phi|(u_t) \text{ is nonincreasing and right continuous.} \tag{3.12}$$

A priori estimates *for every* $v \in \mathrm{Dom}(\phi)$, $t > 0$

$$\frac{\mathrm{e}^{\lambda t}}{2}\mathsf{d}^2(u_t, v) + \mathsf{E}_\lambda(t)\Big(\phi(u_t) - \phi(v)\Big) + \frac{\big(\mathsf{E}_\lambda(t)\big)^2}{2}|\partial\phi|^2(u_t) \le \frac{1}{2}\mathsf{d}^2(u_0, v), \tag{3.13}$$

and for every $v \in \mathrm{Dom}(\mathfrak{L}_\lambda[\phi])$

$$|\partial\phi|^2(u_t) \le \frac{1}{2\mathrm{e}^{\lambda t} - 1}\mathfrak{L}_\lambda^2[\phi](v) + \frac{1}{(\mathsf{E}_\lambda(t))^2}\mathsf{d}^2(u_0, v) \quad provided \; -\lambda t < \log 2. \tag{3.14}$$

Asymptotic expansions as $t \downarrow 0$ *if* $u_0 \in \mathrm{Dom}(|\partial\phi|)$ *and* $\lambda \le 0$ *then for every* $v \in \mathrm{Dom}(\phi)$, $t \ge 0$

$$\frac{\mathrm{e}^{2\lambda t}}{2}\mathsf{d}^2(u_t, v) - \frac{1}{2}\mathsf{d}^2(u_0, v) \le \mathsf{E}_{2\lambda}(t)\Big(\phi(v) - \phi(u_0)\Big) + \frac{t^2}{2}|\partial\phi|^2(u_0). \tag{3.15}$$

The functional $x \mapsto \phi(x) - \frac{\lambda}{2}\mathsf{d}^2(x, o)$ *is linearly bounded from below for all* $o \in \mathrm{Dom}(\phi)$, *and for every* $u_0, v \in \mathrm{Dom}(\phi)$, $t \ge 0$ *there holds*

$$\frac{\mathrm{e}^{\lambda t}}{2}\mathsf{d}^2(u_t, v) - \frac{1}{2}\mathsf{d}^2(u_0, v)$$

$$\le \mathsf{E}_\lambda(t)\Big(\phi(v) - \phi(u_0)\Big) + 2\mathrm{e}^{-\lambda t}\int\limits_0^t \mathrm{e}^{2\lambda s}\Big(\phi(u_0) - \phi_{\mathsf{E}_\lambda(s)}(u_0)\Big)\mathrm{d}s. \tag{3.16}$$

Right, left limits and energy identity *for every* $t > 0$ *the* right limits

$$|\dot{u}_{t+}| := \lim_{h\downarrow 0}\frac{\mathsf{d}(u_{t+h}, u_t)}{h}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\phi(u_{t+}) := \lim_{h\downarrow 0}\frac{\phi(u_{t+h}) - \phi(u_t)}{h}$$

exist finite, satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(u_{t+}) = -|\dot{u}_{t+}|^2 = -|\partial\phi|^2(u_t) = -|\partial\phi|_\ell^2(u_t) = -\mathfrak{L}_\lambda^2[\phi](u_t) \quad for \; every \; t > 0 \tag{3.17}$$

and define a right-continuous map. If in addition $u_0 \in \mathrm{Dom}(|\partial\phi|)$ *the same holds at* $t = 0$, *otherwise all the terms in* (3.17) *are* $-\infty$ *at* $t = 0$.

Moreover, there exists an at-most-countable set $\mathcal{C} \subset (0, +\infty)$ *such that the analogous identities for the* left limits *hold in* $(0, +\infty) \setminus \mathcal{C}$, *while in* $\mathcal{C}$ *the latter hold upon replacing* $|\partial\phi|(u_t)$, $|\partial\phi|_\ell(u_t)$ *and* $\mathfrak{L}_\lambda[\phi](u_t)$ *with* $\lim_{h\downarrow 0}|\partial\phi|(u_{t-h})$, $\lim_{h\downarrow 0}|\partial\phi|_\ell(u_{t-h})$ *and* $\lim_{h\downarrow 0}\mathfrak{L}_\lambda[\phi](u_{t-h})$, *respectively.*

Asymptotic behaviour as $t \to +\infty$ *if $\lambda > 0$ and ϕ has complete sublevels, then it admits a unique minimum point $\bar{u}$ and for every $t > t_0 \geq 0$ we have:*

$$\frac{\lambda}{2}\mathsf{d}^2(u_t, \bar{u}) \leq \phi(u_t) - \phi(\bar{u}) \leq \frac{1}{2\lambda}|\partial\phi|^2(u_t), \tag{3.18a}$$

$$\mathsf{d}(u_t, \bar{u}) \leq \mathsf{d}(u_{t_0}, \bar{u})\,\mathrm{e}^{-\lambda(t-t_0)}, \tag{3.18b}$$

$$\phi(u_t) - \phi(\bar{u}) \leq \Big(\phi(u_{t_0}) - \phi(\bar{u})\Big)\mathrm{e}^{-2\lambda(t-t_0)}, \quad \phi(u_t) - \phi(\bar{u}) \leq \frac{1}{2\mathsf{E}_\lambda(t-t_0)}\mathsf{d}^2(u_{t_0}, \bar{u}), \tag{3.18c}$$

$$|\partial\phi|(u_t) \leq |\partial\phi|(u_{t_0})\,\mathrm{e}^{-\lambda(t-t_0)}, \quad |\partial\phi|(u_t) \leq \frac{1}{\mathsf{E}_\lambda(t-t_0)}\mathsf{d}(u_{t_0}, \bar{u}). \tag{3.18d}$$

If $\lambda = 0$ and $\bar{u}$ is a minimum point of ϕ (if any), then

$$|\partial\phi|(u_t) \leq \frac{\mathsf{d}(u_0, \bar{u})}{t}, \quad \phi(u_t) - \phi(\bar{u}) \leq \frac{\mathsf{d}^2(u_0, \bar{u})}{2t} \qquad \text{for every } t > 0, \tag{3.19}$$

the map $t \mapsto \mathsf{d}(u_t, \bar{u})$ is nonincreasing.

If in addition ϕ has compact *sublevels, then u_t converges to a minimum point of ϕ as $t \to +\infty$.*

Stability *if $\{u^n\} \subset C^0([0, +\infty); X)$ is a sequence of solutions of $\mathrm{EVI}_\lambda(X, \mathsf{d}, \phi)$ such that $\lim_{n\to\infty} u_0^n = u_0$, then*

$$\lim_{n\to\infty} u_t^n = u_t \quad \text{for every } t \geq 0, \tag{3.20}$$

$$\lim_{n\to\infty} \phi(u_t^n) = \phi(u_t) \quad \text{for every } t > 0, \tag{3.21}$$

$$\lim_{n\to\infty} |\partial\phi|(u_t^n) = |\partial\phi|(u_t) \quad \text{for every } t \in (0, +\infty) \setminus \mathcal{C}. \tag{3.22}$$

Moreover, (3.20), (3.21), (3.22) *occur* locally uniformly *in $[0, +\infty)$, $(0, +\infty)$, $(0, +\infty) \setminus \mathcal{C}$, respectively, and*

$$|\partial\phi|(u_t) \leq \liminf_{n\to\infty} |\partial\phi|(u_t^n) \leq \limsup_{n\to\infty} |\partial\phi|(u_t^n) \leq \lim_{s\uparrow t} |\partial\phi|(u_s) \quad \text{for every } t \in \mathcal{C}. \tag{3.23}$$

Proof. Let us consider each statement in turn.

λ-contraction Since u^1, u^2 are solutions of (EVI_λ), by applying (EVI_λ'') with $u_t = u_t^1$ and $v = u_t^2$, after a further multiplication by $\mathrm{e}^{\lambda(t-s)}$ we get

$$\frac{\mathrm{e}^{2\lambda(t-s)}}{2}\mathsf{d}^2(u_t^1, u_t^2) - \frac{\mathrm{e}^{\lambda(t-s)}}{2}\mathsf{d}^2(u_s^1, u_t^2) \leq \mathrm{e}^{\lambda(t-s)}\mathsf{E}_\lambda(t-s)\Big(\phi(u_t^2) - \phi(u_t^1)\Big); \tag{3.24}$$

analogously, by applying (EVI_λ'') with $u_t = u_t^2$ and $v = u_s^1$, we obtain

$$\frac{e^{\lambda(t-s)}}{2}d^2(u_t^2, u_s^1) - \frac{1}{2}d^2(u_s^2, u_s^1) \le E_\lambda(t-s)\Big(\phi(u_s^1) - \phi(u_t^2)\Big). \qquad (3.25)$$

The sum of (3.24) and (3.25) yields

$$\frac{e^{2\lambda(t-s)}}{2}d^2(u_t^1, u_t^2) - \frac{1}{2}d^2(u_s^1, u_s^2)$$

$$\le \Big(e^{\lambda(t-s)} - 1\Big)E_\lambda(t-s)\Big(\phi(u_t^2) - \phi(u_t^1)\Big) + E_\lambda(t-s)\Big(\phi(u_s^1) - \phi(u_t^1)\Big)$$

and therefore, upon inverting the roles of u^1, u^2, multiplying by $e^{2\lambda s}$ and again summing up,

$$e^{2\lambda t}d^2(u_t^1, u_t^2) - e^{2\lambda s}d^2(u_s^1, u_s^2) \le e^{2\lambda s}E_\lambda(t-s)\Big(\phi(u_s^1) - \phi(u_t^1) + \phi(u_s^2) - \phi(u_t^2)\Big). \quad (3.26)$$

Dividing (3.26) by $t - s$ and passing to the limit as $t \downarrow s$ (using the lower semicontinuity of $t \mapsto \phi(u_t)$) we end up with

$$\frac{d^+}{dt}\Big(e^{2\lambda t}d^2(u_t^1, u_t^2)\Big) \le 0 \quad \text{for every } t > 0,$$

which yields (3.10) by Lemma A.1 (recall the continuity of u_t^1, u_t^2 down to $t = 0$).

Regularizing effects I: solutions are locally Lipschitz By choosing $u_t^1 = u_t$ and $u_t^2 = u_{t+h}$ in (3.10) (note that for every $h > 0$ the curve $t \mapsto u_{t+h}$ is still a solution of $\text{EVI}_\lambda(X, d, \phi)$), we find that

$$\text{the map } t \mapsto e^{2\lambda t}\frac{d^2(u_{t+h}, u_t)}{h^2} \text{ is nonincreasing for every } h > 0,$$

which together with (3.8) yields for every $t > 3t_0$ and $0 < h < t_0$ (it suffices to consider the case $\lambda \le 0$)

$$e^{2\lambda t}\frac{d^2(u_{t+h}, u_t)}{h^2} \le \frac{1}{t_0}\int\limits_{t_0}^{3t_0-h}\frac{d^2(u_{s+h}, u_s)}{h^2}\,ds \le \frac{C_{t_0, 3t_0}}{t_0}. \qquad (3.27)$$

Hence (3.27) ensures that u is locally Lipschitz in $(0, +\infty)$.

Right limits, energy identity and regularizing effects II at $t > 0$ By reasoning as above, estimate (3.10) yields

$$d(u_{t+h}, u_t) \le e^{-\lambda(t-t_0)}d(u_{t_0+h}, u_{t_0}) \quad \text{for every } 0 \le t_0 < t < +\infty. \qquad (3.28)$$

If we set

$$\delta_+(t) := \limsup_{h \downarrow 0}\frac{d(u_{t+h}, u_t)}{h}, \quad \delta_-(t) := \liminf_{h \downarrow 0}\frac{d(u_{t+h}, u_t)}{h} \qquad \text{for every } t \ge 0,$$

then from (3.28) we deduce that

$$\text{the map } t \in [0, +\infty) \mapsto \mathrm{e}^{\lambda t} \delta_+(t) \text{ is nonincreasing.} \tag{3.29}$$

We denote by $\mathcal{I}$ the subset of $(0, +\infty)$ where the metric derivative (2.2) of u_t exists finite. As u is locally Lipschitz, by Theorem 2.2 we know that $\mathscr{L}^1\big((0, +\infty) \setminus \mathcal{I}\big) = 0$ and

$$\delta_-(t) = \delta_+(t) = |\dot{u}|(t) < +\infty \quad \text{for every } t \in \mathcal{I}; \tag{3.30}$$

in particular, (3.29) and (3.30) guarantee that $\delta_+(t) \le M_{t_0} < +\infty$ for every $t > t_0 > 0$. We aim at showing that in fact

$$\delta_-(t) = \delta_+(t) = |\partial\phi|(u_t) = \mathfrak{L}_\lambda[\phi](u_t) \quad \text{for every } t > 0. \tag{3.31}$$

Dividing (EVI'_λ) by $t - s$, for all $v \in \mathrm{Dom}(\phi)$ and $0 < s < t$ we get

$$-\frac{\mathsf{d}(u_t, u_s)}{2(t-s)}\Big(\mathsf{d}(u_t, v) + \mathsf{d}(u_s, v)\Big) + \frac{1}{t-s}\int\limits_s^t \Big(\phi(u_r) + \frac{\lambda}{2}\mathsf{d}^2(u_r, v)\Big)\mathrm{d}r \le \phi(v).$$

Passing to the limit as $t \downarrow s$ and recalling (3.7), we obtain

$$\phi(v) \ge \phi(u_s) - \delta_-(s)\mathsf{d}(u_s, v) + \frac{\lambda}{2}\mathsf{d}^2(u_s, v) \quad \text{for every } v \in \mathrm{Dom}(\phi), \ s > 0,$$

which upon recalling (2.18) yields

$$|\partial\phi|(u_s) \le \mathfrak{L}_\lambda[\phi](u_s) \le \delta_-(s) \le \delta_+(s) \quad \text{for every } s > 0. \tag{3.32}$$

In particular, $u_t \in \mathrm{Dom}(|\partial\phi|)$ for all $t > 0$. Now let us fix $s > 0$. We know that $\delta_+(s) < +\infty$ and from (3.32) we can assume with no loss of generality that $\delta_+(s) > 0$; so, dividing (3.4) by h^2 and rescaling the integrand, we infer that for all $\varepsilon > 0$ and h small enough there holds

$$\frac{1}{2h^2}\mathsf{d}^2(u_{s+h}, u_s) \le \frac{1}{h}\int\limits_0^1 \Big(\phi(u_s) - \phi(u_{s+h\rho}) - \frac{\lambda}{2}\mathsf{d}^2(u_{s+h\rho}, u_s)\Big)\mathrm{d}\rho$$

$$\le \int\limits_0^1 \frac{(\phi(u_s) - \phi(u_{s+h\rho}))^+}{\mathsf{d}(u_{s+h\rho}, u_s)}\frac{\mathsf{d}(u_{s+h\rho}, u_s)}{h\rho}\rho\,\mathrm{d}\rho - \frac{\lambda}{2}\int\limits_0^1 \frac{\mathsf{d}^2(u_{s+h\rho}, u_s)}{h}\,\mathrm{d}\rho$$

$$\le \int\limits_0^1 \big(|\partial\phi|(u_s) + \varepsilon\big)\big(\delta_+(s) + \varepsilon\big)\rho\,\mathrm{d}\rho + \frac{\lambda^-}{2}\varepsilon\int\limits_0^1 \big(\delta_+(s) + \varepsilon\big)\rho\,\mathrm{d}\rho. \tag{3.33}$$

Letting first $h \downarrow 0$ and then $\varepsilon \downarrow 0$, we therefore obtain

$$\frac{1}{2}\delta_+^2(s) \le \frac{1}{2}|\partial\phi|(u_s)\,\delta_+(s) \quad \text{for every } s > 0, \tag{3.34}$$

which yields (3.31) in view of (3.32) and (3.12) (in the open interval $(0, +\infty)$) in view of (3.29): the right continuity of $t \mapsto \mathrm{e}^{\lambda t}|\partial\phi|(u_t)$ just follows from the continuity of u_t and the lower semicontinuity of the global λ-slope of ϕ.

Since the map $t \mapsto |\partial\phi|(u_t)$ is locally bounded in $(0, +\infty)$, inequality (2.18) together with the fact that u is also locally Lipschitz show that the map $t \mapsto \phi(u_t)$ is in turn locally Lipschitz continuous. Hence by combining (3.6), (2.18), (3.30) and (3.31) we get:

$$\int\limits_{t_0}^{t_1} |\dot{u}_{t+}|^2 \, \mathrm{d}t \le \phi(u_{t_0}) - \phi(u_{t_1}) \le |\partial\phi|(u_{t_0})\mathsf{d}(u_{t_0}, u_{t_1}) - \frac{\lambda}{2}\mathsf{d}^2(u_{t_0}, u_{t_1})$$

for every $t_1 > t_0 > 0$;

dividing by $t_1 - t_0$ and passing to the limit as $t_1 \downarrow t_0$, since $t \mapsto |\dot{u}_{t+}| = |\partial\phi|(u_t)$ is right continuous we obtain

$$|\dot{u}_{t_0+}|^2 = \lim_{t_1\downarrow t_0}\frac{1}{t_1 - t_0}\int\limits_{t_0}^{t_1}|\dot{u}_{t+}|^2\,\mathrm{d}t \le -\frac{\mathrm{d}}{\mathrm{d}t}\phi(u_t)\big|_{t=t_0+} \le |\partial\phi|(u_{t_0})|\dot{u}_{t_0+}|, \tag{3.35}$$

which yields the first two equalities of (3.17). Hence if $\lambda \ge 0$ the convexity of $t \mapsto \phi(u_t)$ (at least in $(0, +\infty)$) is just a consequence of the latter and the fact that $t \mapsto \mathrm{e}^{\lambda t}|\partial\phi|(u_t)$ is nonincreasing. More in general, the function

$$t \mapsto \mathrm{e}^{2\lambda t}\phi(u_t) - 2\lambda\int\limits_0^t \mathrm{e}^{2\lambda s}\phi(u_s)\,\mathrm{d}s$$

is convex, which yields the claim about the local semi-convexity of $t \mapsto \phi(u_t)$ in the case $\lambda < 0$.

Finally, in order to prove that $|\partial\phi|(u_t) = |\partial\phi|_\ell(u_t)$, it is sufficient to observe that

$$\mathsf{d}_\ell(u_t, u_{t+h}) \le \int\limits_t^{t+h}|\dot{u}_s|\,\mathrm{d}s = \int\limits_t^{t+h}|\partial\phi|(u_s)\,\mathrm{d}s,$$

where d_ℓ is either the length distance induced by $\mathrm{Dom}(\phi)$ or by X, so that if $|\partial\phi|(u_t) > 0$ (otherwise there is nothing to prove) then

$$|\partial\phi|_\ell(u_t) \ge \limsup_{h\downarrow 0}\frac{\phi(u_t) - \phi(u_{t+h})}{\mathsf{d}_\ell(u_{t+h}, u_t)} \ge \limsup_{h\downarrow 0}\left(\frac{\phi(u_t) - \phi(u_{t+h})}{\int_t^{t+h}|\partial\phi|(u_s)\,\mathrm{d}s}\right)$$

$$= \limsup_{h\downarrow 0} \left(\frac{\int_t^{t+h} |\partial\phi|^2(u_s)\,\mathrm{d}s}{\int_t^{t+h} |\partial\phi|(u_s)\,\mathrm{d}s} \right) \overset{\text{Hölder}}{\geq} \limsup_{h\downarrow 0} \sqrt{\frac{\int_t^{t+h} |\partial\phi|^2(u_s)\,\mathrm{d}s}{h}} = |\partial\phi|(u_t).$$
(3.36)

Since the converse inequality $|\partial\phi|_\ell \leq |\partial\phi|$ is always true, we conclude.

Right limits, energy identity and regularizing effects II at $t = 0$ Through Remark 3.4 we have already seen that $t \mapsto \phi(u_t)$ is in fact nonincreasing down to $t = 0$, which in particular ensures that it is continuous at $t = 0$ as well and therefore convex in the whole $[0, +\infty)$ if $\lambda \geq 0$.

As for the energy identity (3.17), let $u_0 \notin \mathrm{Dom}(|\partial\phi|)$. By combining (3.29), (3.31) and the lower semicontinuity of the global λ-slope, it follows that $\delta_-(0) = \delta_+(0) = +\infty$, whence (3.31) also holds at $t = 0$. If $u_0 \notin \mathrm{Dom}(\phi)$, it is apparent that $\frac{\mathrm{d}}{\mathrm{d}t}\phi(u_{0+}) = -\infty$, and all the slopes at u_0 are by definition $+\infty$. On the other hand, if $u_0 \in \mathrm{Dom}(\phi)$ the left-hand inequality in (3.35) does hold at $t_0 = 0$, and the l.h.s. is $+\infty$ since $\lim_{t\downarrow 0} |\partial\phi|(u_t) = +\infty$. Similarly, we have that $|\partial\phi|_\ell(u_0) = +\infty$ because in this case (3.36) still holds at $t = 0$. Suppose now that $u_0 \in \mathrm{Dom}(|\partial\phi|)$. In order to prove the validity of (3.17) at $t = 0$ it suffices to show that $\delta_+(0)$ is finite: then by arguing as above the key inequalities (3.32), (3.34), (3.35) and (3.36) still hold at zero. To this end, let us consider (3.33) at $s = 0$, which in particular yields

$$\mathsf{d}^2(u_h, u_0) \leq 2\big(|\partial\phi|(u_0) + \varepsilon\big) \int_0^h \mathsf{d}(u_\tau, u_0)\,\mathrm{d}\tau - \lambda \int_0^h \mathsf{d}^2(u_\tau, u_0)\,\mathrm{d}\tau \leq 4C \int_0^h \mathsf{d}(u_\tau, u_0)\,\mathrm{d}\tau$$

for a suitable $C > 0$ independent of h small enough. Upon letting $x(h) := \int_0^h \mathsf{d}(u_\tau, u_0)\,\mathrm{d}\tau$, an elementary ODE argument shows that $x(h) \leq Ch^2$, whence $\mathsf{d}(u_h, u_0) \leq 2Ch$ and the finiteness of $\delta_+(0)$ is proved. We can then conclude that (3.12) is true at $t = 0$ as well.

Left limits   We already know that $t \mapsto \mathrm{e}^{\lambda t} |\partial\phi|(u_t)$ is finite, nonincreasing and right continuous in $(0, +\infty)$; let us therefore denote by $\mathcal{C}$ its (at most countable) jump set, i.e.

$$(0, +\infty) \setminus \mathcal{C} = \Big\{ t \in (0, +\infty) : \lim_{h\downarrow 0} |\partial\phi|(u_{t-h}) = |\partial\phi|(u_t) \Big\}.$$

If $t_0 \in (0, +\infty) \setminus \mathcal{C}$ it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(u_t)\Big|_{t=t_0-} = \lim_{h\downarrow 0} \frac{\phi(u_{t_0}) - \phi(u_{t_0-h})}{h} = -|\partial\phi|^2(u_{t_0})$$

$$\overset{(2.18)}{\geq} -|\partial\phi|(u_{t_0}) \liminf_{h\downarrow 0} \frac{\mathsf{d}(u_{t_0}, u_{t_0-h})}{h},$$
(3.37)

and therefore

$$\liminf_{h\downarrow 0} \frac{\mathsf{d}(u_{t_0}, u_{t_0-h})}{h} \geq |\partial\phi|(u_{t_0});$$
(3.38)

on the other hand, by the $\mathscr{L}^1$-a.e. equality between $|\dot{u}|(t)$ and $|\partial\phi|(u_t)$ we get

$$
\frac{\mathsf{d}(u_{t_0-h}, u_{t_0})}{h} \leq \frac{1}{h} \int\limits_{t_0-h}^{t_0} |\dot{u}_s| \, \mathrm{d}s = \frac{1}{h} \int\limits_{t_0-h}^{t_0} |\partial\phi|(u_s) \, \mathrm{d}s \quad \Rightarrow
$$

$$
\limsup_{h\downarrow 0} \frac{\mathsf{d}(u_{t_0-h}, u_{t_0})}{h} \leq |\partial\phi|(u_{t_0}).
$$

(3.39)

If $t_0 \in \mathcal{C}$, just note that inequalities (3.37)–(3.39) still hold provided one replaces $|\partial\phi|(u_{t_0})$ with $\lim_{h\downarrow 0} |\partial\phi|(u_{t_0-h})$.

A priori estimates   In order to show (3.13) and (3.14), we can apply (3.12), the fact that $-\frac{\mathrm{d}}{\mathrm{d}t}\phi(u_t) = |\partial\phi|^2(u_t)$ outside $\mathcal{C}$, and finally (EVI$_\lambda$) to obtain

$$
\frac{1}{2}\big(\mathsf{E}_\lambda(t)\big)^2|\partial\phi|^2(u_t) = \frac{1}{2}\big(\mathsf{E}_{-\lambda}(t)\big)^2 \mathrm{e}^{2\lambda t}|\partial\phi|^2(u_t) \leq \int\limits_0^t \mathsf{E}_{-\lambda}(s)\mathrm{e}^{-\lambda s}\mathrm{e}^{2\lambda s}|\partial\phi|^2(u_s) \, \mathrm{d}s
$$

$$
= -\int\limits_0^t \mathsf{E}_{-\lambda}(s)\mathrm{e}^{\lambda s}\Big(\phi(u_s) - \phi(u_t)\Big)' \, \mathrm{d}s = \int\limits_0^t \mathrm{e}^{\lambda s}\Big(\phi(u_s) - \phi(u_t)\Big) \mathrm{d}s
$$

$$
\overset{\text{(EVI}_\lambda)}{\leq} \int\limits_0^t -\frac{1}{2}\Big(\mathrm{e}^{\lambda s}\mathsf{d}^2(u_s, v)\Big)' + \mathrm{e}^{\lambda s}\Big(\phi(v) - \phi(u_t)\Big) \mathrm{d}s
$$

$$
= \frac{1}{2}\mathsf{d}^2(u_0, v) + \mathsf{E}_\lambda(t)\Big(\phi(v) - \phi(u_t)\Big) - \frac{\mathrm{e}^{\lambda t}}{2}\mathsf{d}^2(u_t, v),
$$

where $\frac{\mathrm{d}}{\mathrm{d}s}$ has been replaced by $'$ for notational convenience. This proves (3.13). If $v \in \mathrm{Dom}(\mathfrak{L}_\lambda[\phi])$, thanks to (2.18) and Young's inequality, we can bound the difference $\phi(v) - \phi(u_t)$ by

$$
\mathsf{E}_\lambda(t)\big(\phi(v) - \phi(u_t)\big) \leq \mathsf{E}_\lambda(t)\Big(\mathfrak{L}_\lambda[\phi](v)\,\mathsf{d}(u_t, v) - \frac{\lambda}{2}\mathsf{d}^2(u_t, v)\Big)
$$

$$
\leq \frac{(\mathsf{E}_\lambda(t))^2}{2(2\mathrm{e}^{\lambda t} - 1)}\mathfrak{L}_\lambda^2[\phi](v) + \frac{2\mathrm{e}^{\lambda t} - 1}{2}\mathsf{d}^2(u_t, v) - \frac{\mathrm{e}^{\lambda t} - 1}{2}\mathsf{d}^2(u_t, v)
$$

$$
= \frac{(\mathsf{E}_\lambda(t))^2}{2(2\mathrm{e}^{\lambda t} - 1)}\mathfrak{L}_\lambda^2[\phi](v) + \frac{\mathrm{e}^{\lambda t}}{2}\mathsf{d}^2(u_t, v),
$$

at least when $2\mathrm{e}^{\lambda t} > 1$. Substituting this bound in (3.13) we obtain (3.14).

Asymptotic expansions   In order to show the validity of (3.15) we multiply (3.9) by $\mathrm{e}^{\lambda t}$, integrate and use the estimate

$$- \int_0^t e^{2\lambda r} \phi(u_r) \, \mathrm{d}r = \int_0^t \big(\mathsf{E}_{2\lambda}(t) - \mathsf{E}_{2\lambda}(r) \big) |\partial \phi|^2 (u_r) \, \mathrm{d}r - \mathsf{E}_{2\lambda}(t) \phi(u_0)$$

$$\leq |\partial \phi|^2 (u_0) \int_0^t \big(\mathsf{E}_{2\lambda}(t) - \mathsf{E}_{2\lambda}(r) \big) e^{-2\lambda r} \, \mathrm{d}r - \mathsf{E}_{2\lambda}(t) \phi(u_0)$$

along with the elementary inequality (for $\lambda \leq 0$)

$$\big(\mathsf{E}_{2\lambda}(t) - \mathsf{E}_{2\lambda}(r) \big) e^{-2\lambda r} \leq (t - r) \quad \text{for every } r \in [0, t].$$

The fact that $x \mapsto \phi(x) - \frac{\lambda}{2} \mathsf{d}^2(x, o)$ is linearly bounded from below is a simple consequence of (2.18) and the energy identity (3.17), which in particular ensures that $\mathrm{Dom}(\mathfrak{L}_\lambda[\phi])$ is not empty.

In order to prove (3.16), it is convenient to use an approach by *curves of maximal slope* (see the next Section 4). Namely, first of all one observes that the energy identity (3.17) implies

$$\frac{e^{\lambda s}}{2} |\dot{u}_s|^2 + \frac{e^{\lambda s}}{2} |\partial \phi|^2 (u_s) - \lambda e^{\lambda s} \phi(u_s) \leq -\frac{\mathrm{d}}{\mathrm{d}s} \big(e^{\lambda s} \phi(u_s) \big) \quad \text{for every } s \in (0, +\infty) \setminus \mathcal{C}. \tag{3.40}$$

An integration of (3.40) yields

$$\frac{1}{2} \int_0^t |\dot{u}_s|^2 \, e^{\lambda s} \mathrm{d}s + \frac{1}{2} \int_0^t |\partial \phi|^2 (u_s) \, e^{\lambda s} \mathrm{d}s - \lambda \int_0^t e^{\lambda s} \phi(u_s) \, \mathrm{d}s \leq \phi(u_0) - e^{\lambda t} \phi(u_t)$$

for every $t > 0$. \tag{3.41}

Since Hölder's inequality and the definition of metric derivative entail

$$\mathsf{d}^2(u_t, u_0) \leq \left(\int_0^t e^{-\lambda s} \, \mathrm{d}s \right) \left(\int_0^t |\dot{u}_s|^2 \, e^{\lambda s} \mathrm{d}s \right) = \mathsf{E}_{-\lambda}(t) \int_0^t |\dot{u}_s|^2 \, e^{\lambda s} \mathrm{d}s,$$

from (3.41) there follows, for every $t > 0$,

$$\frac{1}{2\mathsf{E}_{-\lambda}(t)} \mathsf{d}^2(u_t, u_0) + e^{\lambda t} \phi(u_t) + \frac{1}{2} \int_0^t |\partial \phi|^2 (u_s) \, e^{\lambda s} \mathrm{d}s - \lambda \int_0^t e^{\lambda s} \phi(u_s) \, \mathrm{d}s \leq \phi(u_0). \tag{3.42}$$

Recalling the definition of Moreau-Yosida approximation (2.15), by virtue of (3.42) we deduce

$$\int\limits_0^t |\partial\phi|^2(u_s)\, \mathrm{e}^{\lambda s}\mathrm{d}s - 2\lambda\int\limits_0^t \mathrm{e}^{\lambda s}\phi(u_s)\,\mathrm{d}s \le 2\Big(\phi(u_0) - \mathrm{e}^{\lambda t}\,\phi_{\mathsf{E}_\lambda(t)}(u_0)\Big) \quad \text{for every } t > 0.$$

$$(3.43)$$

Note that, ϕ being quadratically bounded from below for all $\kappa_o > -\lambda$ and $\lambda\mathsf{E}_\lambda(t) > -1$ for all $t > 0$, the r.h.s. of (3.43) is always finite. By exploiting again the energy identity, we can integrate by parts the first term in l.h.s. of (3.43) to get

$$-\mathrm{e}^{\lambda t}\phi(u_t) - \lambda\int\limits_0^t \mathrm{e}^{\lambda s}\phi(u_s)\,\mathrm{d}s \le \phi(u_0) - 2\mathrm{e}^{\lambda t}\,\phi_{\mathsf{E}_\lambda(t)}(u_0) \quad \text{for every } t > 0. \qquad (3.44)$$

Hence by integrating the differential inequality (3.44) w.r.t. the unknown $t \mapsto -\int_0^t \mathrm{e}^{\lambda s}\phi(u_s)\,\mathrm{d}s$, we end up with

$$-\int\limits_0^t \mathrm{e}^{\lambda s}\phi(u_s)\,\mathrm{d}s \le \mathrm{e}^{-\lambda t}\int\limits_0^t \mathrm{e}^{\lambda s}\Big(\phi(u_0) - 2\mathrm{e}^{\lambda s}\,\phi_{\mathsf{E}_\lambda(s)}(u_0)\Big)\mathrm{d}s$$

$$= -\mathsf{E}_\lambda(t)\phi(u_0) + 2\mathrm{e}^{-\lambda t}\int\limits_0^t \mathrm{e}^{2\lambda s}\Big(\phi(u_0) - \phi_{\mathsf{E}_\lambda(s)}(u_0)\Big)\mathrm{d}s$$

for every $t > 0$. (3.45)

Estimate (3.16) is therefore a consequence of (3.45) combined with the integral version of (3.9).

Stability　The convergence of $\{u_t^n\}$ to u_t for all $t > 0$ along with the fact that the latter is locally uniform in $[0, +\infty)$ are immediate consequences of (3.10). Moreover, (3.12) and (3.14) ensure that the sequence $|\partial\phi|(u_t^n)$ is uniformly (in n) bounded by a constant $M_t < +\infty$ for every $t > 0$. Hence thanks to (3.17) and (2.18) we get

$$\phi(u_t) \ge \phi(u_t^n) - M_t\,\mathsf{d}(u_t, u_t^n) - \frac{\lambda}{2}\mathsf{d}^2(u_t, u_t^n),$$

so that $\limsup_{n\to\infty} \phi(u_t^n) \le \phi(u_t)$. On the other hand, ϕ being lower semicontinuous, we also have that $\liminf_{n\to\infty} \phi(u_t^n) \ge \phi(u_t)$, which finally implies (3.21). Such convergence is locally uniform in $(0, +\infty)$ since $\{\phi(u_t^n)\}$ is a sequence of nonincreasing functions converging pointwise to the continuous function $\phi(u_t)$.

Let us turn to slopes. By virtue of (3.12) and (3.17), it is straightforward to deduce the following inequalities:

$$\frac{\phi(u_t^n) - \phi(u_{t+\tau}^n)}{\mathsf{E}_{-2\lambda}(\tau)} \le |\partial\phi|^2(u_t^n) \le \frac{\phi(u_{t-\tau}^n) - \phi(u_t^n)}{\mathsf{E}_{2\lambda}(\tau)} \quad \text{for every } t > 0,\ \tau \in (0, t);\quad (3.46)$$

(3.22) and (3.23) can therefore be proved by letting first $n \to \infty$ in (3.46) (using (3.21)) and then $\tau \downarrow 0$, exploiting the energy identities both for right and left limits. The reason why (3.22) occurs locally uniformly in $(0, +\infty) \setminus \mathcal{C}$ is again a consequence of the fact that $\{e^{\lambda t} |\partial \phi|(u_t^n)\}$ is a sequence of nonincreasing functions converging pointwise in $(0, +\infty) \setminus \mathcal{C}$ to $e^{\lambda t} |\partial \phi|(u_t)$, which is continuous in $(0, +\infty) \setminus \mathcal{C}$.

Asymptotic behavior   When $\lambda > 0$, (3.11) and (3.28) ensure that the sequence $k \mapsto u_k$ belongs to a fixed sublevel of ϕ and satisfies the Cauchy condition in X, since

$$\mathsf{d}(u_{k+1}, u_k) \le e^{-\lambda} \mathsf{d}(u_k, u_{k-1});$$

thus, it is convergent to some limit $\bar{u} \in \mathrm{Dom}(\phi)$. If we multiply (3.13) by $e^{-\lambda t}$, let $t = k$ and $k \to \infty$, thanks to the lower semicontinuity of ϕ we deduce that the constant curve $t \mapsto \bar{u}$ solves (EVI_λ) (if we let $v = u_t$ we get the left-hand inequality in (3.18a)), which ensures that $\bar{u}$ is the unique minimum point for ϕ, along with the validity of (3.18b). Inequality (3.18d) is just (3.12) and (3.14) with $v = \bar{u}$ (note that $|\partial \phi|(\bar{u}) = \mathfrak{L}_\lambda[\phi](\bar{u}) = 0$). In order to prove the right-hand inequality in (3.18a), observe that (2.18) and Young's inequality yield

$$\phi(u_t) - \phi(\bar{u}) \le |\partial \phi|(u_t) \, \mathsf{d}(u_t, \bar{u}) - \frac{\lambda}{2} \mathsf{d}^2(u_t, \bar{u}) \le \frac{1}{2\lambda} |\partial \phi|^2(u_t). \qquad (3.47)$$

The first estimate of (3.18c) now follows upon noticing that (3.47) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\phi(u_t) - \phi(\bar{u}) \Big) = -|\partial \phi|^2(u_t) \le -2\lambda \Big(\phi(u_t) - \phi(\bar{u}) \Big);$$

on the other hand, the second estimate of (3.18c) easily follows from (3.13) with $v = \bar{u}$.

If $\lambda = 0$ and $\bar{u}$ is a minimum point of ϕ, the map $t \mapsto \bar{u}$ trivially solves (EVI_λ), so that $t \mapsto \mathsf{d}(u_t, \bar{u})$ is nonincreasing by virtue of (3.10). The two estimates in (3.19) follow similarly from (3.13) and (3.14), upon observing again that $\mathfrak{L}_0[\phi](\bar{u}) = 0$. If in addition ϕ has compact sublevels, then by (3.13) and the lower semicontinuity of ϕ we easily deduce that there exists a sequence $\{u_{t_n}\}$ converging to some $\bar{u}$ such that

$$\phi(\bar{u}) \le \liminf_{n \to \infty} \phi(u_{t_n}) \le \phi(v) \quad \text{for every } v \in X,$$

whence $\bar{u}$ is a minimum point of ϕ. On the other hand, in this case we know that $t \mapsto \mathsf{d}(u_t, \bar{u})$ is nonincreasing, so that the whole curve u_t is forced to converge to $\bar{u}$ as $t \to +\infty$.   □

As a consequence of Theorem 3.5, we have a finer equivalence between the notions of slope on points that belong to the domain of the gradient flow, which is basically a consequence of the fact that solutions of the EVI are curves of maximal slope (see Section 4 below).

Proposition 3.6 *(Equivalence of slopes). Let $\phi : X \to (-\infty, +\infty]$ be a proper l.s.c. functional which admits an EVI_λ-gradient flow in $D \neq \emptyset$. Then*

$$\limsup_{\substack{y \to x \\ y \in D \cap \mathrm{Dom}(|\partial\phi|)}} \frac{(\phi(x) - \phi(y))^+}{\mathsf{d}(x, y)} = |\partial\phi|(x) = |\partial\phi|_\ell(x) = \mathfrak{L}_\lambda[\phi](x) \quad \text{for every } x \in D. \tag{3.48}$$

Proof. The identity $|\partial\phi|(x) = |\partial\phi|_\ell(x) = \mathfrak{L}_\lambda[\phi](x)$ is an immediate consequence of Theorem 3.5 (precisely, the statement immediately below (3.17)).

In order to prove that these quantities also coincide with the left-hand side of (3.48), it is enough to notice that, in the proof of Theorem 3.5, when showing the energy identity (3.17) (see in particular inequalities (3.33)–(3.34)) the metric slope at $x = u_s$ can in fact be replaced by the $\limsup$ of $(\phi(u_s) - \phi(u_t))/\mathsf{d}(u_s, u_t)$ as $t \downarrow s$, which is clearly bounded from above by the l.h.s. of (3.48).

Alternatively, one could just observe that u_s is also a solution of the EVI associated with ϕ_D, the latter functional being the same as ϕ on D and $+\infty$ elsewhere, so that (3.48) is a direct consequence of both the energy identities for $\mathrm{EVI}_\lambda(X, \mathsf{d}, \phi)$ and $\mathrm{EVI}_\lambda(X, \mathsf{d}, \phi_D)$. Here the fact that ϕ_D may not be lower semicontinuous in $\mathrm{Dom}(\phi) \setminus D$ is inessential. $\quad\square$

Remark 3.7 *(EVI, slopes and Moreau-Yosida regularizations).* Because the function $\tau \mapsto \phi_\tau(u_0)$ is nonincreasing, one can simply bound the integral remainder in the r.h.s. of (3.16) by the *pointwise* remainder

$$2\mathrm{e}^{-\lambda t}\mathsf{E}_{2\lambda}(t)\Big(\phi(u_0) - \phi_{\mathsf{E}_\lambda(t)}(u_0)\Big). \tag{3.49}$$

On the other hand, from [5, Lemma 3.1.5] we have the identity

$$\limsup_{\tau \downarrow 0} \frac{\phi(u_0) - \phi_\tau(u_0)}{\tau} = \frac{1}{2}|\partial\phi|^2(u_0), \tag{3.50}$$

so that if $u_0 \in \mathrm{Dom}(|\partial\phi|)$ then (3.49) reproduces (as $t \downarrow 0$) the remainder in (3.15) up to a factor 2. However, in order to get asymptotically the same estimate (i.e. with the same factor), it is essential to keep such remainder in the integral form (3.16). Indeed, from (3.16) itself we easily deduce (use the fact that v is arbitrary)

$$\mathsf{E}_\lambda(t)\Big(\phi(u_0) - \phi_{\mathsf{E}_\lambda(t)}(u_0)\Big) \leq 2\mathrm{e}^{-\lambda t}\int\limits_0^t \mathrm{e}^{2\lambda s}\Big(\phi(u_0) - \phi_{\mathsf{E}_\lambda(s)}(u_0)\Big)\mathrm{d}s, \tag{3.51}$$

which implies in turn that

$$\text{the map} \quad t \in (0, +\infty) \mapsto \frac{\int_0^t \mathrm{e}^{2\lambda s}\big(\phi(u_0) - \phi_{\mathsf{E}_\lambda(s)}(u_0)\big)\,\mathrm{d}s}{(\mathsf{E}_\lambda(t))^2} \quad \text{is nonincreasing.} \tag{3.52}$$

Hence by gathering (3.52) and (3.50), we end up with

$$\frac{\int_0^t e^{2\lambda s}\left(\phi(u_0) - \phi_{\mathsf{E}_\lambda(s)}(u_0)\right)\mathrm{d}s}{(\mathsf{E}_\lambda(t))^2} \le \limsup_{h\downarrow 0} \frac{\int_0^h e^{2\lambda s}\left(\phi(u_0) - \phi_{\mathsf{E}_\lambda(s)}(u_0)\right)\mathrm{d}s}{(\mathsf{E}_\lambda(h))^2} \le \frac{1}{4}|\partial\phi|^2(u_0); \tag{3.53}$$

in particular (compare with [5, Theorem 3.1.6] in the λ-convex case), (3.50), (3.51), (3.53) yield

$$\sup_{\tau>0:\,\lambda\tau>-1}(1+\lambda\tau)\frac{\phi(u_0) - \phi_\tau(u_0)}{\tau} = \frac{1}{2}|\partial\phi|^2(u_0).$$

Nevertheless, we point out that in order to obtain the correct asymptotic expansion for $\mathsf{d}(u_t, u_0)$ (let $\lambda = 0$ for simplicity) it is enough to use the energy identity as follows:

$$\mathsf{d}^2(u_t, u_0) \le t \int_0^t |\dot{u}_s|^2\,\mathrm{d}s = t\left(\phi(u_0) - \phi(u_t)\right),$$

from which it is easy to deduce that

$$\frac{1}{2}\mathsf{d}^2(u_t, u_0) \le t\left(\phi(u_0) - \phi_t(u_0)\right).$$

In any case, the integral remainder is necessary in (3.16), which holds for all $v \in \mathrm{Dom}(\phi)$.

Corollary 3.8 *(Construction of* EVI_λ*-gradient flows). Suppose that for every initial value* $u_0 \in D_0 \subset \overline{\mathrm{Dom}(\phi)}$ *there exists a solution* u_t *of* $\mathrm{EVI}_\lambda(X, \mathsf{d}, \phi)$ *such that* $\lim_{t\downarrow 0} u_t = u_0$. *Then, if we set*

$$D := \bigcup_{t\ge 0,\, u_0\in D_0}\left\{u_t :\ u \text{ is the solution of } (\mathrm{EVI}_\lambda) \text{ starting from } u_0\right\},$$

there exists a unique EVI_λ*-gradient flow* $\mathsf{S}_t : D \to D$ *of* ϕ *according to Definition 3.1, which satisfies the* λ*-contraction property*

$$\mathsf{d}(\mathsf{S}_t(u_0), \mathsf{S}_t(v_0)) \le e^{-\lambda t}\mathsf{d}(u_0, v_0)\qquad \text{for every } u_0, v_0 \in D,\ t \ge 0. \tag{3.54}$$

If in addition ϕ *has complete sublevels (in particular, if* X *is complete), then* S_t *can always be extended by density to an* EVI_λ*-gradient flow of* ϕ *in* $\overline{D}$.

Proof. We just check the last statement, since the other ones are straightforward consequences of the λ-contraction and uniqueness properties entailed by Theorem 3.5. Thanks to the latter, it is easy to extend S_t from D to its closure: for every $u_0 \in \overline{D}$ one can simply take an arbitrary sequence $\{u_0^n\} \subset D$ converging to u_0 and set $\mathsf{S}_t(u_0) := \lim_{n\to\infty} \mathsf{S}_t(u_0^n)$

at every $t > 0$. The limit does exist since $\{S_t(u_0^n)\}$ is a Cauchy sequence by (3.10), and in view of (3.13)

$$\phi(S_t(u_0^n)) \leq \frac{1}{2E_\lambda(t)} d^2(u_0^n, v) + \phi(v) \quad \text{for every } v \in \text{Dom}(\phi),$$

so that it also belongs to a fixed sublevel of ϕ, which is complete by assumption. Estimate (3.10) itself shows that the limit is independent of the chosen sequence $\{u_0^n\}$. Thanks to the lower semicontinuity of ϕ, it is immediate to check that each trajectory u_t of the extended flow still satisfies e.g. the integral formulation (EVI_λ'') down to $s = 0$. Finally, the semigroup property (3.2a) is inherited as well by the extended flow since, as mentioned above, the limit trajectory is independent of the particular sequence $\{u_0^n\}$. $\quad\square$

3.3. EVI_λ-gradient flows, the length distance, and λ-convexity

Let us first recall a result of [10, Theorem 3.5].

Theorem 3.9 *(Self-improvement of* EVI_λ*-gradient flows). Let* $(S_t)_{t\geq 0}$ *be an* EVI_λ*-gradient flow of* ϕ *in* $X = \overline{\text{Dom}(\phi)}^{d_\ell}$*. Then* $(S_t)_{t\geq 0}$ *is also an* EVI_λ*-gradient flow in* X *w.r.t. the length distance* d_ℓ *on each equivalence class defined by* $\sim_\ell$ *in (2.6).*

The restriction to equivalence classes in the previous Theorem is due to the fact that our definition of EVI_λ-gradient flow just refers to distances. If X is Lipschitz connected, then X contains only one equivalence class of $\sim_\ell$ and therefore $(S_t)_{t\geq 0}$ is an EVI_λ-gradient flow of ϕ in X w.r.t. d_ℓ. Note that Theorem 3.9, as a byproduct, provides an alternative way to prove the identity $|\partial\phi|(u_t) = |\partial\phi|_\ell(u_t)$ (at least under the corresponding assumptions).

A result of [34] (see Theorem 3.2 there) shows, in particular, that if $D \subset \overline{\text{Dom}(\phi)}$ is a geodesic subset and ϕ admits an EVI_λ-gradient flow in D, then ϕ is strongly λ-convex in D. An analogous property is enjoyed by approximate length subsets.

Theorem 3.10 *(*EVI_λ*-gradient flows entail* λ*-convexity). Suppose that* ϕ *admits an* EVI_λ*-gradient flow in the subset* $D \subset \overline{\text{Dom}(\phi)}$*. Then the following hold:*

(1) if $x_0, x_1 \in D \cap \text{Dom}(\phi)$*,* $\vartheta, \varepsilon \in (0, 1)$ *and* $x_{\vartheta,\varepsilon} \in D$ *is a* (ϑ, ε)*-intermediate point between them,*

$$\phi(S_t(x_{\vartheta,\varepsilon})) \leq (1 - \vartheta)\phi(x_0) + \vartheta\phi(x_1) - \frac{1}{2}\left(\lambda - \frac{\varepsilon^2}{E_\lambda(t)}\right)\vartheta(1 - \vartheta)d^2(x_0, x_1)$$

for every $t > 0$; $\hspace{12em}$ (3.55)

(2) if D *is an approximate length subset then* ϕ *is approximately* λ*-convex in* D*;*

(3) if $D = X = \overline{\mathrm{Dom}(\phi)}^{\mathsf{d}_\ell}$, then ϕ is approximately λ-convex w.r.t. the distance d_ℓ on each equivalence class of $\sim_\ell$;

(4) if $\mathsf{x}_0, \mathsf{x}_1 \in D \cap \mathrm{Dom}(\phi)$ and $\mathsf{x} \in \mathrm{Geo}_D[\mathsf{x}_0 \to \mathsf{x}_1]$ then $\mathsf{x} \in \mathrm{Geo}_D^{\phi,\lambda}[\mathsf{x}_0 \to \mathsf{x}_1]$; in particular, if for every $\mathsf{x}_0, \mathsf{x}_1 \in D \cap \mathrm{Dom}(\phi)$ the set $\mathrm{Geo}_D[\mathsf{x}_0 \to \mathsf{x}_1]$ is not empty, then ϕ is strongly λ-convex in D.

Proof. Property (4) follows from (2), and in fact had already been established in [34, Theorem 3.2], from which one can borrow tools to prove (1). The main idea consists in evaluating (3.13) with $u_t = \mathsf{S}_t(\mathsf{x}_{\vartheta,\varepsilon})$ and $v = \mathsf{x}_0$, multiply it by $(1 - \vartheta)$ and sum it to the analogous estimate one obtains by plugging $v = \mathsf{x}_1$ instead, multiplied by ϑ. By exploiting (2.7), it is then not difficult to deduce the validity of (3.55).

As for (2), just note that $\mathsf{d}(\mathsf{S}_t(\mathsf{x}_{\vartheta,\delta}), \mathsf{x}_i) \le e^{-\lambda t}\mathsf{d}(\mathsf{x}_{\vartheta,\delta}, \mathsf{x}_i) + \mathsf{d}(\mathsf{S}_t(\mathsf{x}_i), \mathsf{x}_i)$ $(i = 0, 1)$. Hence for any fixed $\varepsilon \in (0, 1)$, one sees that $\mathsf{S}_t(\mathsf{x}_{\vartheta,\delta})$ is a (ϑ, ε)-intermediate point up to picking $\delta \le \varepsilon/2$ and $t > 0$ sufficiently small independently of δ. Choosing now $\delta \le (\varepsilon \mathsf{E}_\lambda(t))^{1/2}$, thanks to (3.55) we see that $\mathsf{S}_t(\mathsf{x}_{\vartheta,\delta})$ also satisfies (2.28).

Finally, (3) is an immediate consequence of (2) and Theorem 3.9. $\quad\square$

We conclude this subsection by an alternative *local* formulation of gradient flows, which becomes a characterization in case ϕ is geodesically λ-convex for some $\lambda \in \mathbb{R}$. This leads to a definition which is actually independent of λ. To this aim let us first introduce, for a given geodesic v starting from $\mathsf{v}_0 = u_{t_0}$, the following quantity:

$$
[\dot{u}, \mathsf{v}]_{t_0} := \lim_{s \downarrow 0} \frac{1}{2s} \frac{\mathrm{d}^+}{\mathrm{d}t} \mathsf{d}^2(u_t, \mathsf{v}_s)\big|_{t=t_0} = \sup_{0 < s \le 1} \frac{1}{2s} \frac{\mathrm{d}^+}{\mathrm{d}t} \mathsf{d}^2(u_t, \mathsf{v}_s)\big|_{t=t_0}
$$
$$
= \sup_{0 < s \le 1} \limsup_{t \downarrow t_0} \frac{\mathsf{d}^2(u_t, \mathsf{v}_s) - \mathsf{d}^2(u_{t_0}, \mathsf{v}_s)}{2s(t - t_0)}.
\tag{3.56}
$$

The identity between sup and lim in (3.56) will be justified through Lemma 3.13 below.

Proposition 3.11 (*Local characterization of* EVI). *Let* $u : (0, +\infty) \to \mathrm{Dom}(\phi)$ *be a continuous curve. If* u *is a solution of* $\mathrm{EVI}_\lambda(X, \mathsf{d}, \phi)$ *according to Definition 3.1, then for every* $t > 0$ *and every geodesic* v *emanating from* $\mathsf{v}_0 = u_t \in \mathrm{Dom}(\phi)$ *there holds*

$$
[\dot{u}, \mathsf{v}]_t \le \phi'(u_t; \mathsf{v}).
\tag{3.57}
$$

Conversely, if u *satisfies* (3.57) *and* ϕ *is* λ-convex, then u *is a solution of* $\mathrm{EVI}_\lambda(X, \mathsf{d}, \phi)$.

Proof. Let the continuous curve $u : (0, +\infty) \to \mathrm{Dom}(\phi)$ satisfy $\mathrm{EVI}_\lambda(X, \mathsf{d}, \phi)$ and v be a geodesic emanating from u_{t_0}, for any $t_0 > 0$. By plugging $v = \mathsf{v}_s$ in (EVI_λ) (one can assume with no loss of generality that $\mathsf{v}_s \in \mathrm{Dom}(\phi)$), for every $s \in (0, 1]$ we get

$$
\frac{1}{2} \frac{\mathrm{d}^+}{\mathrm{d}t} \mathsf{d}^2(u_t, \mathsf{v}_s)\big|_{t=t_0} \le \phi(\mathsf{v}_s) - \phi(u_{t_0}) - \frac{\lambda s^2}{2} \mathsf{d}^2(u_{t_0}, \mathsf{v}_1).
$$

Dividing by s and passing to the limit as $s \downarrow 0$, we end up with

$$[\dot{u}, \mathsf{v}]_{t_0} = \lim_{s \downarrow 0} \frac{1}{2s} \frac{\mathrm{d}^+}{\mathrm{d}t} \mathsf{d}^2(u_t, \mathsf{v}_s)|_{t=t_0} \le \liminf_{s \downarrow 0} \left(\frac{\phi(\mathsf{v}_s) - \phi(u_{t_0})}{s} - \frac{\lambda s}{2} \mathsf{d}^2(u_{t_0}, \mathsf{v}_1) \right)$$
$$= \phi'(u_{t_0}; \mathsf{v}).$$

Suppose now that u satisfies (3.57) and ϕ is λ-convex; for every $t_0 > 0$ and $v \in \mathrm{Dom}(\phi)$, let v be an admissible geodesic in $\mathrm{Geo}_X^{\phi, \lambda}[u_{t_0} \to v]$. By the definition of $[\dot{u}, \mathsf{v}]_t$ and (2.30), we obtain:

$$\frac{1}{2} \frac{\mathrm{d}^+}{\mathrm{d}t} \mathsf{d}^2(u_t, v)|_{t=t_0} \overset{(3.56)}{\le} [\dot{u}, \mathsf{v}]_{t_0} \le \phi'(u_{t_0}; \mathsf{v}) \le \phi(v) - \phi(u_{t_0}) - \frac{\lambda}{2} \mathsf{d}^2(u_{t_0}, v). \quad \square$$

Corollary 3.12 *(EVI$_\lambda$ with different λ). Let $\lambda_1 < \lambda_2$ and let u be a solution of EVI$_{\lambda_1}(X, \mathsf{d}, \phi)$ according to Definition 3.1. If ϕ is λ_2-convex, then u is also a solution of EVI$_{\lambda_2}(X, \mathsf{d}, \phi)$.*

Lemma 3.13. *Let $u : (0, +\infty) \to X$ be a continuous curve and v be a given geodesic emanating from $\mathsf{v}_0 := u_t$ for some $t > 0$. Then the map*

$$s \mapsto s^{-1} \frac{\mathrm{d}^+}{\mathrm{d}t} \mathsf{d}^2(u_t, \mathsf{v}_s) \text{ is nonincreasing in } (0, 1].$$

Proof. Thanks to the continuity of u, we have:

$$s^{-1} \frac{\mathrm{d}^+}{\mathrm{d}t} \mathsf{d}^2(u_t, \mathsf{v}_s) = 2s^{-1} \mathsf{d}(u_t, \mathsf{v}_s) \limsup_{h \downarrow 0} \frac{\mathsf{d}(u_{t+h}, \mathsf{v}_s) - \mathsf{d}(u_t, \mathsf{v}_s)}{h} = 2|\dot{\mathsf{v}}| \frac{\mathrm{d}^+}{\mathrm{d}t} \mathsf{d}(u_t, \mathsf{v}_s).$$

We are therefore left with proving that $s \mapsto \frac{\mathrm{d}^+}{\mathrm{d}t} \mathsf{d}(u_t, \mathsf{v}_s)$ is not increasing with respect to s. To this aim, note that if $s_1 < s_2$ the triangular inequality and the geodesic property of v yield

$$\mathsf{d}(u_{t+h}, \mathsf{v}_{s_2}) \le \mathsf{d}(u_{t+h}, \mathsf{v}_{s_1}) + \mathsf{d}(\mathsf{v}_{s_1}, \mathsf{v}_{s_2}), \qquad \mathsf{d}(u_t, \mathsf{v}_{s_2}) = \mathsf{d}(u_t, \mathsf{v}_{s_1}) + \mathsf{d}(\mathsf{v}_{s_1}, \mathsf{v}_{s_2}),$$

whence

$$\frac{\mathsf{d}(u_{t+h}, \mathsf{v}_{s_2}) - \mathsf{d}(u_t, \mathsf{v}_{s_2})}{h} \le \frac{\mathsf{d}(u_{t+h}, \mathsf{v}_{s_1}) - \mathsf{d}(u_t, \mathsf{v}_{s_1})}{h}. \tag{3.58}$$

The thesis follows by passing to the limit in (3.58) as $h \downarrow 0$. $\quad \square$

3.4. Examples

We already considered the case of λ-convex functionals in Hilbert spaces in the Introduction; many examples of applications can be found in [15,18–20,83]. A discussion concerning the smooth Riemannian case is contained in [89, Proposition 23.1].

3.4.1. Hadamard NPC spaces, CAT(k) *spaces and convexity along generalized geodesics*

One of the nicest metric setting where general existence of EVI$_\lambda$-gradient flows can be proved is provided by the class of Hadamard *non positively curved* (NPC) metric spaces, which we introduce here by one of their equivalent characterizations [13, Section 1.2]. A Hadamard (or CAT(0)) space is a geodesic space (or set, recall Definition 2.3) such that the map $x \mapsto \frac{1}{2}\mathsf{d}^2(y,x)$ is strongly 1-convex for every $y \in X$; that is, for every $\mathsf{x} \in \mathrm{Geo}[\mathsf{x}_0 \to \mathsf{x}_1]$ and every $y \in X$ we have

$$\mathsf{d}^2(y,\mathsf{x}_t) \le (1-t)\mathsf{d}^2(y,\mathsf{x}_0) + t\mathsf{d}^2(y,\mathsf{x}_1) - t(1-t)\mathsf{d}^2(\mathsf{x}_0,\mathsf{x}_1). \tag{3.59}$$

When X is a smooth Riemannian manifold, the above condition is equivalent to requiring the non-positivity of its sectional curvature.

If ϕ is geodesically convex in a Hadamard space, MAYER [63] and JOST [52] proved the convergence of the Minimizing Movement scheme (recall (1.4a,b) or (1.7); see also Section 5) to a contraction semigroup $(\mathsf{S}_t)_{t \ge 0}$ in $\overline{\mathrm{Dom}(\phi)}$, using in particular the fact that the map $x \mapsto \mathsf{J}_\tau[x]$ is a contraction; they also provide nice applications to the harmonic map flow in Hadamard spaces. The generation result has been extended to arbitrary CAT(k) spaces, $k \in \mathbb{R}$, by [73]. In this framework, the link between the limit contraction semigroup (obtained by the convergence of the variational scheme) with the EVI$_\lambda$-formulation has been clarified by [5], with two further important developments: the optimal error estimate of order 1 (as in Hilbert spaces, [14,71])

$$\mathsf{d}(\mathsf{S}_t(u_0), \mathsf{J}^n_{t/n}(u_0)) \le \frac{t}{n\sqrt{2}} |\partial\phi|(u_0) \tag{3.60}$$

and a considerably weaker assumption on the system (X, d, ϕ). It is in fact sufficient that for every triple of points $\mathsf{x}_0, \mathsf{x}_1, y \in \mathrm{Dom}(\phi)$ there exists a curve $\mathsf{x} : [0,1] \to X$ connecting x_0 to x_1 (and possibly depending on y) along which (3.59) holds and ϕ satisfies the λ-convexity inequality (2.27). This condition clearly covers the case of Hadamard NPC spaces but also allows for important applications to the Kantorovich-Rubinstein-Wasserstein space $(\mathscr{P}_2(\mathbb{R}^d), \mathsf{d}_W)$, which is not an NPC space if $d \ge 2$: a large class of interesting functionals in $\mathscr{P}_2(\mathbb{R}^d)$ are in fact *convex along generalized geodesics*, we refer to [5, Section 11.2] for more details.

We can recap the above discussion in the following result:

Theorem 3.14 *(Existence of* EVI$_\lambda$*-flows in* CAT *spaces). Let* (X, d, ϕ) *be a metric-functional system, for which* (X, d) *is a complete geodesic metric space. Let us assume that at least one of the following two conditions holds:*

1. *(X, d) is a* CAT(k) *space for some $k \in \mathbb{R}$ and ϕ is λ-convex;*
2. *for every $\mathsf{x}_0, \mathsf{x}_1, y \in \mathrm{Dom}(\phi)$ there exists a curve $\mathsf{x} : [0,1] \to \mathrm{Dom}(\phi)$ connecting x_0 to x_1 such that (3.59) and (2.27) hold.*

Then the functional ϕ generates an EVI$_\lambda$*-flow in* $\overline{\mathrm{Dom}(\phi)}$*, according to Definition 3.1.*

3.4.2. Alexandrov spaces

A second important class of metric spaces where λ-convex functionals generate an EVI_λ-flow is provided by Alexandrov spaces with curvature bounded from below [22, 23,49,79,84]: they can be considered as the natural non-smooth metric version of Riemannian manifolds with sectional curvature bounded from below. Referring to the above quoted papers for the general definition, which is based on triangle comparison with the reference 2-dimensional model of constant curvature [1], here we limit ourselves to recalling one of the equivalent characterizations of positively curved (PC) geodesic spaces: the squared distance function $x \mapsto \frac{1}{2}\mathsf{d}^2(y,x)$ is -1-concave along geodesics; more explicitly, for every $y \in X$ and $\mathsf{x} \in \text{Geo}[\mathsf{x}_0 \to \mathsf{x}_1]$ we have

$$\mathsf{d}^2(y,\mathsf{x}_t) \geq (1-t)\mathsf{d}^2(y,\mathsf{x}_0) + t\mathsf{d}^2(y,\mathsf{x}_1) - t(1-t)\mathsf{d}^2(\mathsf{x}_0,\mathsf{x}_1).$$

Existence of gradient flows for λ-convex functionals in complete m-dimensional (in particular locally compact) Alexandrov spaces has been proved by [77]. In [70] we will generalize this result to cover arbitrary Alexandrov spaces and even more general cases, characterized by a new geometric property, weaker than the Alexandrov one, based on a local angle condition between geodesics and on the semi-concavity of the squared distance [35,82] (see also [59] and [72] for a different viewpoint). Such an approach has also the advantage to be stable with respect to the Wasserstein construction, in the sense that the above property is shared by the Wasserstein space $(\mathscr{P}_2(X), \mathsf{d}_W)$. Note that the space $(\mathscr{P}_2(X), \mathsf{d}_W)$ constructed on an Alexandrov space (X, d) is not finite dimensional and fails to be an Alexandrov space if (X, d) is not positively curved (see [5, Chapter 12.3], [86, Proposition 2.10]).

Referring to [70] for a more detailed discussion, here we limit ourselves to stating the Alexandrov case:

Theorem 3.15 *(Existence of EVI_λ-flows in Alexandrov spaces with lower curvature bounds). Let (X, d, ϕ) be a metric-functional system, for which (X, d) is a complete geodesic metric space satisfying a uniform lower curvature bound in the sense of Alexandrov and ϕ is λ-convex. Then the functional ϕ generates an EVI_λ-flow in $\overline{\text{Dom}(\phi)}$, according to Definition 3.1.*

3.4.3. The Kantorovich-Rubinstein-Wasserstein space $(\mathscr{P}_2(X), \mathsf{d}_W)$

One of the main recent motivations to study gradient flows in metric spaces comes from the remarkable OTTO's interpretation of the heat and Fokker-Planck flows [51] and of a large class of nonlinear diffusion equations [75] as gradient flows of suitable entropy functionals in the Kantorovich-Rubinstein-Wasserstein (KRW) space of probability measures $(\mathscr{P}_2(X), \mathsf{d}_W)$.

We assume for simplicity that (X, d) is complete and separable. We denote by $\mathscr{P}(X)$ the space of Borel probability measures on X and by $\mathscr{P}_2(X)$ the subset of those $\mu \in \mathscr{P}(X)$ with finite quadratic moment:

$$\int_X \mathsf{d}^2(x,o)\,\mathrm{d}\mu(x) < \infty \quad \text{for some (and thus any) } o \in X.$$

The KRW-distance between two measures $\mu_1, \mu_2 \in \mathscr{P}_2(X)$ can be defined as

$$\mathsf{d}_W^2(\mu_1, \mu_2) := \min\left\{\int_{X \times X} \mathsf{d}^2(x_1, x_2)\,\mathrm{d}\boldsymbol{\mu}(x_1, x_2) : \boldsymbol{\mu} \in \mathscr{P}(X \times X), \quad \pi_\sharp^i \boldsymbol{\mu} = \mu_i\right\},$$

where $\pi^i : X \times X \to X$, $\pi^i(x_1, x_2) := x_i$, are the Cartesian projections and for a Borel map $\boldsymbol{t} : X \to Y$ between metric spaces, $\boldsymbol{t}_\sharp : \mathscr{P}(X) \to \mathscr{P}(Y)$ denotes the push-forward operation defined by

$$\boldsymbol{t}_\sharp \mu(B) := \mu(\boldsymbol{t}^{-1}(B)) \quad \text{for every Borel subset } B \subset Y \text{ and every } \mu \in \mathscr{P}(X).$$

It turns out (see e.g. [5,89]) that $(\mathscr{P}_2(X), \mathsf{d}_W)$ is complete and separable, and it is also a geodesic (resp. length) space provided (X, d) is geodesic (resp. length).

If $X = \mathbb{R}^d$ with the usual Euclidean distance, by Otto's [75] and subdifferential [5] calculus, one can see that the metric gradient flow of the entropy functional (also called *Csiszàr f-divergence*)

$$\mathscr{F}(\mu) := \int_X F\left(\frac{\mathrm{d}\mu}{\mathrm{d}\mathfrak{m}}\right)\mathrm{d}\mathfrak{m}, \tag{3.61}$$

where $F : [0, +\infty) \to [0, \infty)$ is a smooth, convex and superlinear function and $\mathfrak{m} := \mathrm{e}^{-V}\mathscr{L}^d$ is a Borel measure induced by a smooth potential $V : X \to \mathbb{R}$ (for simplicity), provides a solution to the diffusion equation

$$\partial_t \mu = \nabla \cdot (\mu \nabla F'(\varrho)) = \nabla \cdot \left(\mathrm{e}^{-V}\nabla(\varrho F'(\varrho) - F(\varrho))\right) \quad \text{in } (0, \infty) \times \mathbb{R}^d, \quad \varrho = \frac{\mathrm{d}\mu}{\mathrm{d}\mathfrak{m}}, \tag{3.62}$$

e.g. in the weak sense

$$\int_0^\infty \int_{\mathbb{R}^d} \partial_t \zeta(t, x)\,\mathrm{d}\mu_t(x)\,\mathrm{d}t = \int_0^\infty \int_{\mathbb{R}^d} \nabla \zeta(t, x) \cdot \nabla F'(\varrho_t(x))\,\mathrm{d}\mu_t(x)\,\mathrm{d}t$$

for every $\zeta \in \mathrm{C}_c^\infty((0, \infty) \times \mathbb{R}^d)$. Among the class of entropy functionals (3.61), the logarithmic entropy one

$$\mathscr{E}(\mu) := \int_X E\left(\frac{\mathrm{d}\mu}{\mathrm{d}\mathfrak{m}}\right)\mathrm{d}\mathfrak{m}, \quad E(\varrho) := \varrho \log \varrho, \tag{3.63}$$

plays a distinguished role, since one easily sees that in the case of the functional $\mathscr{E}$ the density $\sigma = \mathrm{e}^{-V}\varrho$ of μ w.r.t. $\mathscr{L}^d$ solves the Fokker-Planck equation

$$\partial_t \sigma = \Delta \sigma + \nabla \cdot (\sigma \nabla V) \quad \text{in } (0, \infty) \times \mathbb{R}^d, \tag{3.64}$$

which reduces to the heat equation when V is constant (so that $\mathfrak{m} = \mathscr{L}^d$).

It has been a striking achievement of McCann [64] to show that if F satisfies the condition (clearly fulfilled by E)

$$s \mapsto \mathrm{e}^s F(\mathrm{e}^{-s}) \quad \text{is convex and nonincreasing in } (0, \infty), \tag{3.65}$$

and $\mathfrak{m}$ is log-concave (equivalently, V is convex in $\mathbb{R}^d$), then the functional $\mathscr{F}$ of (3.61) is convex in $(\mathscr{P}_2(X), \mathsf{d}_W)$; it turns out that it is also convex along generalized geodesics [5, Chapter 9], according to the condition stated in Theorem 3.14 (a more refined characterization of the λ-convexity of $\mathscr{F}$, involving the dimension d and the first and second differential of V, is also available, see [89]).

This remarkable property has been used in [5] to show that the gradient flow of $\mathscr{F}$ in $(\mathscr{P}_2(X), \mathsf{d}_W)$ (initially obtained as a limit of the Minimizing Movement scheme) is in fact an EVI_0-flow in the sense of Definition 3.1. Such a characterization, further extended to λ-convex potentials V and to functionals including interaction energies, has been extremely useful in the study of the asymptotic behavior [24] and in the analysis of more complex flows, see e.g. [17,53,62].

Starting from further geometric investigations of the link between lower Ricci curvature bounds and the geodesic convexity of the entropy functionals (3.61) when X is a complete Riemannian manifold endowed with the Riemannian distance d [29,30,76,85], an intensive effort has been devoted to identify the Wasserstein gradient flow of the logarithmic entropy with the heat flow in X or, more generally, with the $L^2(X, \mathfrak{m})$-gradient flow of the Dirichlet form

$$\mathscr{D}(u) := \frac{1}{2} \int\limits_X |\nabla u(x)|^2 \, \mathrm{d}\mathfrak{m}(x)$$

associated with the Sobolev space $W^{1,2}(X, \mathsf{d}, \mathfrak{m})$. The case of smooth Riemannian manifolds has been fully analyzed by Erbar [41] and Villani [89, Chapter 23]; the case where X is a Hilbert space and $\mathfrak{m}$ is a log-concave measure has been studied in [6], whereas Alexandrov spaces have been considered in [47,72]. We can summarize part of the above discussion in the following result (which, however, is far from being exhaustive):

Theorem 3.16 *(Heat flow as EVI_λ-flow in KRW spaces). Let (X, d) be a complete, separable geodesic metric space endowed with a Borel measure $\mathfrak{m}$, $\lambda \in \mathbb{R}$ and let us consider the metric-functional system $\mathbb{X} = (\mathscr{P}_2(X), \mathsf{d}_W, \mathscr{F})$, where $\mathscr{F}$ is the entropy functional (3.61) satisfying McCann's condition (3.65). If one of the following assumptions holds*

1. *(X, d) is a d-dimensional smooth Riemannian manifold with Ricci curvature $\geq \kappa_1$, $\mathfrak{m} := \mathrm{e}^{-V} \mathrm{Vol}$ where $V : X \to \mathbb{R}$ is a geodesically κ_2-convex potential, $\lambda := \kappa_1 + \kappa_2$;*
2. *(X, d) is a Hilbert space and $\mathfrak{m}$ is a log-concave measure, $\lambda := 0$;*

3. (X, d) *is a compact d-dimensional Alexandrov space with curvature* $\geq \kappa$ *and* $\mathfrak{m} = \mathscr{H}^d$
   is the Hausdorff d-dimensional measure, $\lambda := \kappa$,

then $\mathscr{F}$ *generates an* EVI_λ*-flow in* $(\mathscr{P}_2(X), \mathsf{d}_W)$.

As we already observed, when X is not positively curved, $(\mathscr{P}_2(X), \mathsf{d}_W)$ is not an Alexandrov space so that Theorem 3.15 cannot be directly applied.

3.4.4. $\mathrm{RCD}(K, \infty)$ *metric-measure spaces*

The equivalence between a lower Ricci curvature bound $\mathrm{Ric} \geq K$ and the K-convexity of the logarithmic entropy functional (3.63) in Riemannian manifolds endowed with the Riemannian volume measure $\mathfrak{m}$ motivated the deep investigations of LOTT-VILLANI [58] and STURM [86,87] to solve the problem of finding synthetic notions of lower Ricci curvature bounds for general metric-measure spaces $(X, \mathsf{d}, \mathfrak{m})$, a structure formed by a complete, separable, and length metric space (X, d) and a Borel measure $\mathfrak{m}$ satisfying the growth condition

$$\mathfrak{m}(\mathrm{B}(o, r)) \leq A\mathrm{e}^{Br^2} \text{ for some } o \in X \text{ and constants } A, B \geq 0,$$

where $\mathrm{B}(o, r) := \{x \in X : \mathsf{d}(x, o) < r\}$. According to their definition, such a structure satisfies the *Curvature-Dimension* $\mathrm{CD}(K, \infty)$ condition if the logarithmic entropy functional $\mathscr{E}$ (3.63) is K-convex in $(\mathscr{P}_2(X), \mathsf{d}_W)$.

In order to capture a Riemannian-like structure, related to the linearity of the heat flow in $(X, \mathsf{d}, \mathfrak{m})$, the Lott-Sturm-Villani condition has been reinforced in [8] (see also [7]) by asking that the logarithmic Entropy function $\mathscr{E}$ generates an EVI_K-flow in $(\mathscr{P}_2(X), \mathsf{d}_W)$. This condition precisely characterizes the class of $\mathrm{RCD}(K, \infty)$ metric-measure spaces:

Definition 3.17 *(Metric-measure spaces with Ricci curvature bounded from below).* A complete, separable metric-measure space $(X, \mathsf{d}, \mathfrak{m})$ satisfies the $\mathrm{RCD}(K, \infty)$ condition if the logarithmic entropy function $\mathscr{E}$ of (3.63) generates an EVI_K-flow in $(\mathscr{P}_2(X), \mathsf{d}_W)$.

It is a remarkable fact that in this case the gradient flow of $\mathscr{E}$ is a semigroup of operators in $\mathscr{P}_2(X)$ satisfying $\mathsf{S}_t(\alpha\mu + (1 - \alpha)\sigma) = \alpha\mathsf{S}_t(\mu) + (1 - \alpha)\mathsf{S}_t(\sigma)$ for every $\alpha \in (0, 1)$, a property encoded by the logarithmic structure of $\mathscr{E}$ and the variational formulation of the EVI_K-flow [82].

The $\mathrm{RCD}(K, \infty)$-condition shows the relevance of the synthetic notion of EVI_K-flows; its equivalence with the celebrated Bakry-Émery curvature-dimension condition has been proved in [9] and its N-dimensional version has been introduced and deeply studied by ERBAR-KUWADA-STURM in [45] (by using a refined notion of the EVI_λ-flow) and in [12] (by studying the gradient flows generated by other entropy functionals). We refer to the survey [11] for a brief introduction to this theory and to its further developments.

We conclude this discussion by observing that the existence of the EVI_K flow of the entropy functional $\mathscr{E}$ at the level of measures $(\mathscr{P}_2(X), \mathsf{d}_W)$ implies relevant generation

properties also for the original space (X, d). In fact, STURM [88] proved that for locally compact metric-measure spaces satisfying the $\mathrm{RCD}(K, \infty)$-condition every continuous λ-convex functional $\phi : X \to \mathbb{R}$ generates an EVI_λ-flow. As a byproduct of our analysis of the variational convergence of EVI_λ-flows in [69] we will show that this property actually holds in arbitrary $\mathrm{RCD}(K, \infty)$ spaces, thus obtaining the following result.

Theorem 3.18 *(Existence of* EVI_λ*-flows in* RCD *spaces). Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *metric measure space and let* $\phi : X \to (-\infty, +\infty]$ *be a continuous λ-convex functional with proper domain. Then ϕ generates an* EVI_λ*-flow in* $\overline{\mathrm{Dom}(\phi)}$*, according to Definition 3.1.*

4. Energy-dissipation inequality, curves of maximal slope and EVI

There exists a weaker notion of gradient flow for the system (X, d, ϕ), which is strictly related to the energy identity (3.17). Referring to [5, Chapters 2, 3] for more details (see also [3]), we provide the following definition, which is again independent of λ.

Definition 4.1 *(EDI and curves of maximal slope).* Let (X, d, ϕ) be a metric-functional system as in (3.1) and $v \in \mathrm{AC}^2_{\mathrm{loc}}([0, +\infty); X)$. We say that v satisfies the *Energy-Dissipation Inequality (EDI$_0$)* starting from $t = 0$ if

$$v_0 \in \mathrm{Dom}(\phi), \quad \int\limits_0^t \left(\frac{1}{2} |\dot{v}_r|^2 + \frac{1}{2} |\partial \phi|^2(v_r) \right) \mathrm{d}r + \phi(v_t) \le \phi(v_0) \quad \text{for every } t > 0. \quad (\mathrm{EDI}_0)$$

Moreover, we say that v is a *curve of maximal slope* if the map $t \mapsto \phi(v_t)$ is locally absolutely continuous in $[0, +\infty)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} \phi(v_t) \le -\frac{1}{2} |\dot{v}_t|^2 - \frac{1}{2} |\partial \phi|^2(v_t) \quad \text{for } \mathscr{L}^1\text{-a.e. } t > 0. \quad (4.1)$$

Note that every curve of maximal slope for the system (X, d, ϕ) satisfies (EDI_0): it is sufficient to integrate (4.1) in the interval $[0, t]$. In fact (4.1) is much stronger, since it also yields the identity

$$\frac{\mathrm{d}}{\mathrm{d}t} \phi(v_t) = -|\partial \phi|^2(v_t) = -|\dot{v}_t|^2 \quad \text{for } \mathscr{L}^1\text{-a.e. } t > 0. \quad (4.2)$$

Indeed, (2.17) and the differentiability of $t \mapsto \phi(v_t)$ and $t \mapsto v_t$ (in the metric sense (2.2)) for $\mathscr{L}^1$-a.e. $t > 0$ entail

$$\frac{\mathrm{d}}{\mathrm{d}t} \phi(v_t) \ge -|\partial \phi|(v_t) \, |\dot{v}_t| \quad \text{for } \mathscr{L}^1\text{-a.e. } t > 0,$$

which combined with (4.1) yields $(|\partial \phi|(v_t) - |\dot{v}_t|)^2 = 0$. In turn, (4.2) is equivalent to the Energy-Dissipation Equality

$$\int\limits_s^t \left(\frac{1}{2}|\dot{v}_r|^2 + \frac{1}{2}|\partial\phi|^2(v_r)\right) \mathrm{d}r + \phi(v_t) = \phi(v_s) \quad \text{for every } 0 \le s \le t. \tag{EDE}$$

When ϕ admits an EVI$_\lambda$-gradient flow according to Definition 3.1, the two notions in fact coincide. We have already seen in Theorem 3.5 (identity (3.17)) that every solution of EVI$_\lambda(X, \mathsf{d}, \phi)$ (starting from $u_0 \in \mathrm{Dom}(\phi)$) is a curve of maximal slope and therefore satisfies the Energy-Dissipation Equality (EDE). The converse implication, only upon requiring the Energy-Dissipation Inequality (EDI$_0$), is stated in the next theorem.

Theorem 4.2 (*Curves of maximal slope and* EVI$_\lambda$-*gradient flow*). *Let v be a curve that satisfies the Energy-Dissipation Inequality* (EDI$_0$). *Assume in addition that the* EVI$_\lambda$-*gradient flow* S_t *of ϕ exists (for some $\lambda \in \mathbb{R}$) in a set $D \subset \overline{\mathrm{Dom}(\phi)}$ that contains the image of v. Then $v_t = \mathsf{S}_t(v_0)$ for every $t \ge 0$.*

Proof. We can suppose with no loss of generality that $\lambda \le 0$. The map $t \mapsto |\partial\phi|^2(v_t)$ is Lebesgue measurable (recall (2.20)) and belongs to $L^1((0,T))$ for every $T > 0$ in view of (EDI$_0$). By applying Lemma 4.4 below, we can therefore find $\bar{t} \in (0,1)$, a sequence $\tau_k \downarrow 0$ and points $t_k := \tau_k \bar{t}$ such that, upon letting $t_k^n := t_k + n\tau_k$, $v_k^n := v_{t_k^n}$ and choosing $N_k \in \mathbb{N}$ so that $\tau_k(N_k + 2) \le T < \tau_k(N_k + 3)$, there holds

$$C_k := \sum_{n=0}^{N_k} \left| \tau_k |\partial\phi|^2(v_k^n) - \int\limits_{t_k^n}^{t_k^{n+1}} |\partial\phi|^2(v_r) \, \mathrm{d}r \right|, \qquad \lim_{k\to\infty} C_k = 0. \tag{4.3}$$

Note that eventually $v_k^n \in \mathrm{Dom}(|\partial\phi|)$ for all n as above. Now let us set $u_k^n := \mathsf{S}_{n\tau_k}[v_k^0]$ and $w_k^{n+1} := \mathsf{S}_{\tau_k}[v_k^n]$; in order to estimate $\mathsf{d}(v_k^n, u_k^n)$ we start from the basic recurrence relations

$$\mathrm{e}^{\lambda t_k^n}\mathsf{d}(v_k^n, u_k^n) \le \mathrm{e}^{\lambda t_k^n}\Big(\mathsf{d}(v_k^n, w_k^n) + \mathsf{d}(w_k^n, u_k^n)\Big) \overset{(3.54)}{\le} \mathrm{e}^{\lambda t_k^n}\mathsf{d}(v_k^n, w_k^n) + \mathrm{e}^{\lambda t_k^{n-1}}\mathsf{d}(v_k^{n-1}, u_k^{n-1}),$$

so that a telescopic summation plus the inequality $\lambda t_k^n \le \lambda\tau_k$ (for $n \ge 1$) yield

$$\sup_{1 \le n \le N_k} \mathrm{e}^{\lambda t_k^n}\mathsf{d}(v_k^n, u_k^n) \le \sum_{n=1}^{N_k} \mathrm{e}^{\lambda\tau_k}\mathsf{d}(v_k^n, w_k^n). \tag{4.4}$$

It is therefore crucial to estimate the terms $\mathsf{d}(v_k^n, w_k^n)$. If we apply (3.15) with $u_t = w_k^n(t) = \mathsf{S}_t[v_k^{n-1}]$, $v = v_k^n$ and $t = \tau_k$, we obtain:

$$\frac{\mathrm{e}^{2\lambda\tau_k}}{2}\mathsf{d}^2(w_k^n, v_k^n) - \frac{1}{2}\mathsf{d}^2(v_k^{n-1}, v_k^n) \le \mathsf{E}_{2\lambda}(\tau_k)\big(\phi(v_k^n) - \phi(v_k^{n-1})\big) + \frac{\tau_k^2}{2}|\partial\phi|^2(v_k^{n-1}); \tag{4.5}$$

on the other hand, by the definition of metric slope and Hölder's inequality we infer that

$$\frac{1}{2}\mathsf{d}^2(v_k^{n-1}, v_k^n) \le \frac{\tau_k}{2} \int_{t_k^{n-1}}^{t_k^n} |\dot{v}_r|^2 \, \mathrm{d}r = \tau_k \, I_k^n + \tau_k \phi(v_k^{n-1}) - \tau_k \phi(v_k^n) - \frac{\tau_k}{2} \int_{t_k^{n-1}}^{t_k^n} |\partial\phi|^2(v_r) \, \mathrm{d}r,$$

$$(4.6)$$

where

$$I_k^n := \phi(v_k^n) - \phi(v_k^{n-1}) + \frac{1}{2} \int_{t_k^{n-1}}^{t_k^n} \left(|\dot{v}_r|^2 + |\partial\phi|^2(v_r) \right) \mathrm{d}r.$$

Summing up (4.5) and (4.6), after a multiplication by a factor 2 we end up with

$$\mathrm{e}^{2\lambda\tau_k}\mathsf{d}^2(w_k^n, v_k^n) \le \tau_k\big(a_k^n + c_k^n + I_k^n\big),$$

$$a_k^n := 2\big(1 - \tau_k^{-1}\mathsf{E}_{2\lambda}(\tau_k)\big)\big(\phi(v_k^{n-1}) - \phi(v_k^n)\big), \qquad c_k^n := \tau_k|\partial\phi|^2(v_k^{n-1}) - \int_{t_k^{n-1}}^{t_k^n} |\partial\phi|^2(v_r) \, \mathrm{d}r,$$

so that by (4.4) there follows

$$\sup_{1 \le n \le N_k} \mathrm{e}^{\lambda t_k^n}\mathsf{d}(v_k^n, u_k^n) \le \sqrt{T} \left(\sum_{n=1}^{N_k} a_k^n + c_k^n + I_k^n \right)^{\frac{1}{2}}. \tag{4.7}$$

An elementary numerical inequality yields

$$2\big(1 - \tau_k^{-1}\mathsf{E}_{2\lambda}(\tau_k)\big) \le 2|\lambda|\tau_k;$$

thus, letting $T_k := t_k^{N_k}$, a telescopic summation and $\phi(v_{t_k}) \le \phi(v_0)$ ensure that

$$\sum_{n=1}^{N_k} a_k^n \le 2|\lambda|\tau_k \left| \sum_{n=1}^{N_k} \big(\phi(v_k^{n-1}) - \phi(v_k^n)\big) \right| \le 2|\lambda|\tau_k\big(\phi(v_0) - \phi(v_{T_k})\big), \qquad \sum_{n=1}^{N_k} c_k^n \le C_k.$$

Furthermore, another telescopic summation entails

$$\sum_{n=1}^{N_k} I_k^n = \phi(v_{T_k}) - \phi(v_{t_k}) + \frac{1}{2} \int_{t_k}^{T_k} \left(|\dot{v}_r|^2 + |\partial\phi|^2(v_r) \right) \mathrm{d}r \overset{(\mathrm{EDI}_0)}{\le} \phi(v_0) - \phi(v_{t_k}).$$

Recalling (4.3), (4.7), the lower semicontinuity of ϕ and the fact that $\phi(v_T) > -\infty$, we finally obtain

$$\lim_{k \to \infty} \sup_{1 \le n \le N_k} \mathrm{e}^{\lambda t_k^n}\mathsf{d}(v_k^n, u_k^n) = 0,$$

which shows that $v_t = \mathsf{S}_t(v_0)$ for every $t \geq 0$ upon noticing the joint continuity of the map $(t, u) \mapsto \mathsf{S}_t(u)$, consequence of (3.54) and continuity w.r.t. t. $\square$

Remark 4.3 *(When $v_0 \notin \mathrm{Dom}(\phi)$).* The case of a curve v_t such that $v_0 \notin \mathrm{Dom}(\phi)$ can be dealt with similarly. Indeed, more in general, we can say that a curve $v \in \mathrm{AC}^2_{\mathrm{loc}}((0, +\infty); X) \cap C^0([0, +\infty); X)$ satisfies the Energy-Dissipation Inequality if

$$v_t \in \mathrm{Dom}(\phi), \qquad \liminf_{s \downarrow 0} \left[\int_s^t \left(\frac{1}{2}|\dot{v}_r|^2 + \frac{1}{2}|\partial \phi|^2(v_r) \right) \mathrm{d}r + \phi(v_t) - \phi(v_s) \right] \leq 0$$

for every $t > 0$. $\hspace{10cm}$ (EDI$_s$)

Note that (EDI$_s$) is consistent with (EDI$_0$), in the sense that if $v_0 \in \mathrm{Dom}(\phi)$ then it is implied by (EDI$_0$) (and it is equivalent to the latter provided $t \mapsto \phi(v_t)$ is continuous at $t = 0$). It is then straightforward to check that one can repeat the proof of Theorem 4.2 up to replacing the initial time 0 with s, eventually letting $s \downarrow 0$.

Lemma 4.4. *Let $g \in L^1((0, T))$ for some $T > 0$. There exists a sequence $\tau_k \downarrow 0$ and a set of points $\bar{t} \in (0, 1)$ with full $\mathscr{L}^1$-measure such that, by choosing $t_k^n := \tau_k(\bar{t} + n)$ and $N_k \in \mathbb{N}$ so that $T \in [\tau_k(N_k + 2), \tau_k(N_k + 3))$, there holds*

$$\lim_{k \to \infty} \sum_{n=0}^{N_k} \left| \tau_k g(t_k^n) - \int_{t_k^n}^{t_k^{n+1}} g(r) \, \mathrm{d}r \right| = 0.$$

Proof. First of all, let us trivially extend g by 0 outside the interval $(0, T)$ (for notational convenience we do not relabel such an extension). It is well known that the bounded map

$$y \in [0, +\infty) \mapsto I(y) := \int_0^T |g(x) - g(x + y)| \, \mathrm{d}x$$

is continuous; hence a further integration with respect to y and Fubini's Theorem yield

$$\limsup_{\tau \downarrow 0} \int_0^T \left| g(x) - \int_0^1 g(x + \tau y) \, \mathrm{d}y \right| \mathrm{d}x \leq \limsup_{\tau \downarrow 0} \int_0^T \int_0^1 \left| g(x) - g(x + \tau y) \right| \mathrm{d}y \, \mathrm{d}x$$

$$= \limsup_{\tau \downarrow 0} \int_0^1 I(\tau y) \, \mathrm{d}y = 0.$$

If we denote by $N(\tau)$ the unique integer such that $T \in [\tau(N(\tau) + 2), \tau(N(\tau) + 3))$, it follows that

$$\lim_{\tau \downarrow 0} \tau \sum_{n=0}^{N(\tau)} \int_0^1 \left| g(\tau(x+n)) - \int_0^1 g(\tau(x+y+n))\, \mathrm{d}y \right| \mathrm{d}x = 0;$$

in particular, there exists a vanishing subsequence τ_k such that the integrand

$$\sum_{n=0}^{N(\tau_k)} \tau_k \left| g(\tau_k(x+n)) - \int_0^1 g(\tau_k(x+y+n))\, \mathrm{d}y \right|$$

converges to 0 for $\mathscr{L}^1$-a.e. $x \in (0,1)$. $\square$

5. "Ekeland relaxation" of the Minimizing Movement method and uniform error estimates

A general variational method to approximate gradient flows (and often prove their existence) for the system (X, d, ϕ) is provided by the so called *Minimizing Movement* variational scheme. In its original formulation (see e.g. [36]), the method consists in finding a discrete approximation $\overline{U}_\tau$ of the continuous gradient flow u by solving a recursive variational scheme. In fact $\overline{U}_\tau$ is a piecewise constant function on the partition $\mathcal{P}_\tau := \{0, \tau, 2\tau, \cdots, n\tau, \cdots\}$ induced by the time step $\tau > 0$; in each interval $((n-1)\tau, n\tau]$ of the partition, $\overline{U}_\tau$ takes the value U_τ^n which minimizes the functional

$$U \mapsto \frac{1}{2\tau}\mathsf{d}^2(U, U_\tau^{n-1}) + \phi(U). \tag{5.1}$$

In order to carry out the iteration, one has to assign the starting value $U_\tau^0 = \overline{U}_\tau(0)$, which is supposed to be a suitable approximation of u_0.

The existence of a minimizing sequence $\{U_\tau^n\}_{n \in \mathbb{N}}$ is usually obtained by invoking the direct method of the Calculus of Variations, thus requiring that the functional (5.1) has compact sublevels with respect to some Hausdorff topology σ on X (see e.g. the setting of [5, Section 2.1]). Another possibility, still considered in [5], is to suppose that the functional (5.1) satisfies a strong convexity assumption.

Here we try to avoid these restrictions by applying Ekeland's Variational Principle to the functional (5.1), as we did in the Definition 2.9 of the Moreau-Yosida-Ekeland resolvent. This approach only requires the completeness of the sublevels of ϕ; we will show that the sole existence of an EVI_λ-gradient flow for ϕ provides an explicit error estimate between the solution to (EVI_λ) and any discrete approximation obtained by the variational method, so that we can drop any compactness assumptions.

Definition 5.1 *(The Ekeland relaxation of the Minimizing Movement scheme).* Let us consider a time step $\tau > 0$, a relaxation parameter $\eta \geq 0$, and a discrete initial datum $U_{\tau,\eta}^0 \in \mathrm{Dom}(\phi)$. A (τ, η)-discrete Minimizing Movement starting from $U_{\tau,\eta}^0$ is any sequence $(U_{\tau,\eta}^n)_{n \in \mathbb{N}}$ in $\mathrm{Dom}(\phi)$ s.t.

$$U_{\tau,\eta}^n \in \mathrm{J}_{\tau,\eta}[U_{\tau,\eta}^{n-1}] \quad \text{for every } n \in \mathbb{N},$$

i.e. $U_{\tau,\eta}^n$ is a solution of the family of variational problems

$$\frac{1}{2\tau}\mathsf{d}^2(U_{\tau,\eta}^n, U_{\tau,\eta}^{n-1}) + \phi(U_{\tau,\eta}^n)$$
$$\leq \frac{1}{2\tau}\mathsf{d}^2(V, U_{\tau,\eta}^{n-1}) + \phi(V) + \frac{\eta}{2}\,\mathsf{d}(U_{\tau,\eta}^n, U_{\tau,\eta}^{n-1})\,\mathsf{d}(V, U_{\tau,\eta}^n) \quad \text{for every } V \in \mathrm{Dom}(\phi), \tag{5.2a}$$

satisfying the further condition

$$\frac{1}{2\tau}\mathsf{d}^2(U_{\tau,\eta}^n, U_{\tau,\eta}^{n-1}) + \phi(U_{\tau,\eta}^n) \leq \phi(U_{\tau,\eta}^{n-1}), \tag{5.2b}$$

for every $n \in \mathbb{N}$. We denote by $\mathsf{MM}_{\tau,\eta}(U_{\tau,\eta}^0) \subset X^{\mathbb{N}}$ the corresponding collection of all the discrete minimizing movements that start from $U_{\tau,\eta}^0$.

Note that if we pick $\eta = 0$ in the previous definition then we have the usual minimizing movements: in this case, (5.2b) is a direct consequence of (5.2a), since it can easily be obtained by taking $V := U_\tau^{n-1}$. The particular scaling choice of the parameter η in (5.2a) is motivated by the simpler form of the next estimates, where η can be considered as a mild perturbation of the parameter λ and therefore it does not affect the stability on finite intervals and the order of convergence of the method. In particular, η can be taken fixed and positive as τ vanishes.

Theorem 2.10 ensures that if ϕ has complete sublevels and it is quadratically bounded from below according to (2.13), then a (τ, η)-discrete Minimizing Movement $(U_{\tau,\eta}^n)_{n \in \mathbb{N}} \in \mathsf{MM}_{\tau,\eta}(U_{\tau,\eta}^0)$ always exists for every $\tau \in (0, \tau_o)$, $\eta > 0$, and $U_{\tau,\eta}^0 \in \mathrm{Dom}(\phi)$, where τ_o is given by (2.16). The piecewise constant interpolant $\overline{U}_{\tau,\eta}$ is defined as (for any sequence actually, not necessarily a minimizing movement)

$$\overline{U}_{\tau,\eta}(t) := U_{\tau,\eta}^n \quad \text{if } t \in \big((n-1)\tau, n\tau\big], \qquad \overline{U}_{\tau,\eta}(0) := U_{\tau,\eta}^0. \tag{5.3}$$

We can now provide the definition of (continuous) Minimizing Movement.

Definition 5.2 *(Minimizing Movements).* We say that a curve $u : [0, +\infty) \to X$ is a Minimizing Movement and belongs to $\mathsf{MM}(X, \mathsf{d}, \phi; u_0)$ if $u(0) = u_0$ and there exist $\eta \geq 0$ and piecewise constant curves $\overline{U}_{\tau,\eta}$ associated with sequences $(U_{\tau,\eta}^n)_{n \in \mathbb{N}} \in \mathsf{MM}_{\tau,\eta}(u_0)$ (for sufficiently small $\tau > 0$) such that

$$\lim_{\tau \downarrow 0} \overline{U}_{\tau,\eta}(t) = u(t) \quad \text{for every } t \geq 0.$$

We say that u is a Generalized Minimizing Movement in $\mathsf{GMM}(X, \mathsf{d}, \phi; u_0)$ if $u(0) = u_0$ and there exist $\eta \geq 0$, a vanishing sequence $k \mapsto \tau(k) \downarrow 0$ and piecewise constant curves $\overline{U}_k = \overline{U}_{\tau(k),\eta}$ associated with sequences $(U_{\tau(k),\eta}^n)_{n \in \mathbb{N}} \in \mathsf{MM}_{\tau(k),\eta}(u_0)$ such that

$$\lim_{k \to \infty} \overline{U}_k(t) = u(t) \quad \text{for every } t \geq 0.$$

As is typical in numerical analysis of differential equations, uniform error estimates will result from the combination of the discrete uniform stability (as $\tau \downarrow 0$) of the method and a local error estimate, an approach known in the literature as *Lady Windermere's Fan* (see e.g. [50, Page 39]). The purpose of the next two lemmas is to somehow reproduce such a strategy in our setting. First of all, we will show that the discrete Minimizing Movements satisfy an approximate version of the Energy-Dissipation Inequality; from the latter, stability follows (see Proposition 5.8).

Lemma 5.3. *Let the sequence $(U_{\tau,\eta}^n)_{n \in \mathbb{N}} \in \mathsf{MM}_{\tau,\eta}(U_{\tau,\eta}^0)$ be a solution of the Minimizing Movement scheme of Definition 5.1, for some $\tau > 0$, $\eta \geq 0$. Then for every $n \in \mathbb{N}$*

$$(1 - \tfrac{1}{2}\eta\tau)|\partial\phi|(U_{\tau,\eta}^n) \leq \frac{\mathsf{d}(U_{\tau,\eta}^n, U_{\tau,\eta}^{n-1})}{\tau}. \tag{5.4a}$$

Moreover, if ϕ is approximately λ-convex then

$$(1 + \tfrac{1}{2}(\lambda - \eta)\tau)\frac{\mathsf{d}^2(U_{\tau,\eta}^n, U_{\tau,\eta}^{n-1})}{\tau} \leq \phi(U_{\tau,\eta}^{n-1}) - \phi(U_{\tau,\eta}^n), \tag{5.4b}$$

$$(1 + (\lambda - \eta)\tau)\frac{\mathsf{d}(U_{\tau,\eta}^n, U_{\tau,\eta}^{n-1})}{\tau} \leq |\partial\phi|(U_{\tau,\eta}^{n-1}), \qquad (1 - \lambda'\tau)|\partial\phi|(U_{\tau,\eta}^n) \leq |\partial\phi|(U_{\tau,\eta}^{n-1}), \tag{5.4c}$$

where $\lambda' \geq \eta(1 + \tfrac{1}{2}\tau\lambda^+) - \lambda$.

Note that when $\eta = 0$ we can choose $\lambda' = -\lambda$ in the above estimates. On the other hand, when $\lambda > 0$ we can always pick η sufficiently small (independently of τ ranging in a bounded interval) so that $\lambda - \eta \geq 0$ and $\lambda' = 0$.

Proof. Inequality (5.4a) follows directly from (2.24). In order to show (5.4b), for every $\vartheta, \varepsilon \in (0,1)$ we take a (ϑ, ε)-intermediate point $\mathsf{U}_{\vartheta,\varepsilon}^{n-1,n}$ between $U_{\tau,\eta}^{n-1}$ and $U_{\tau,\eta}^n$ such that

$$\phi(\mathsf{U}_{\vartheta,\varepsilon}^{n-1,n}) \leq (1-\vartheta)\phi(U_{\tau,\eta}^{n-1}) + \vartheta\phi(U_{\tau,\eta}^n) - \frac{\lambda - \varepsilon}{2}\vartheta(1-\vartheta)\mathsf{d}^2(U_{\tau,\eta}^{n-1}, U_{\tau,\eta}^n). \tag{5.5}$$

Now note that by picking $V = \mathsf{U}_{\vartheta,\varepsilon}^{n-1,n}$ in (5.2a), setting $\tilde{\eta} := \frac{\eta}{2}\mathsf{d}(U_{\tau,\eta}^n, U_{\tau,\eta}^{n-1})$, exploiting (5.5) and

$$\mathsf{d}(\mathsf{U}_{\vartheta,\varepsilon}^{n-1,n}, U_{\tau,\eta}^{n-1}) \leq \vartheta(1+\varepsilon)\mathsf{d}(U_{\tau,\eta}^{n-1}, U_{\tau,\eta}^n),$$
$$\mathsf{d}(\mathsf{U}_{\vartheta,\varepsilon}^{n-1,n}, U_{\tau,\eta}^n) \leq (1-\vartheta)(1+\varepsilon)\mathsf{d}(U_{\tau,\eta}^{n-1}, U_{\tau,\eta}^n),$$

we end up with

$$\frac{1}{2\tau}\mathsf{d}^2(U_{\tau,\eta}^n, U_{\tau,\eta}^{n-1}) + \phi(U_{\tau,\eta}^n)$$

$$\leq \frac{1}{2\tau}\mathsf{d}^2(\mathsf{U}_{\vartheta,\varepsilon}^{n-1,n}, U_{\tau,\eta}^{n-1}) + \phi(\mathsf{U}_{\vartheta,\varepsilon}^{n-1,n}) + \tilde{\eta}\mathsf{d}(\mathsf{U}_{\vartheta,\varepsilon}^{n-1,n}, U_{\tau,\eta}^n)$$

$$\leq \frac{\vartheta^2(1+\varepsilon)^2}{2\tau}\mathsf{d}^2(U_{\tau,\eta}^{n-1}, U_{\tau,\eta}^{n-1}) + (1-\vartheta)\phi(U_{\tau,\eta}^{n-1}) + \vartheta\phi(U_{\tau,\eta}^n) \tag{5.6a}$$

$$- \frac{\lambda-\varepsilon}{2}\vartheta(1-\vartheta)\mathsf{d}^2(U_{\tau,\eta}^{n-1}, U_{\tau,\eta}^n) + \tilde{\eta}(1-\vartheta)(1+\varepsilon)\mathsf{d}(U_{\tau,\eta}^{n-1}, U_{\tau,\eta}^n). \tag{5.6b}$$

First one lets $\varepsilon \downarrow 0$ and then observes that the corresponding second-order polynomial in $\vartheta \in [0, 1]$ defined by the right-hand side (5.6a)–(5.6b) evaluated at $\varepsilon = 0$ attains a minimum at $\vartheta = 1$; inequality (5.4b) then follows by taking the (left) derivative with respect to ϑ at $\vartheta = 1$.

Thanks to Proposition 2.16, we know in particular that $|\partial\phi|(U_{\tau,\eta}^{n-1}) = \mathfrak{L}_\lambda[\phi](U_{\tau,\eta}^{n-1})$, whence

$$\phi(U_{\tau,\eta}^{n-1}) - \phi(U_{\tau,\eta}^n) \leq |\partial\phi|(U_{\tau,\eta}^{n-1})\mathsf{d}(U_{\tau,\eta}^{n-1}, U_{\tau,\eta}^n) - \frac{\lambda}{2}\mathsf{d}^2(U_{\tau,\eta}^{n-1}, U_{\tau,\eta}^n)$$

and therefore, by exploiting (5.4b),

$$\left(1 + (\lambda - \tfrac{1}{2}\eta)\tau\right)\frac{\mathsf{d}^2(U_{\tau,\eta}^n, U_{\tau,\eta}^{n-1})}{\tau} \leq |\partial\phi|(U_{\tau,\eta}^{n-1})\mathsf{d}(U_{\tau,\eta}^{n-1}, U_{\tau,\eta}^n); \tag{5.7}$$

we thus get the first inequality of (5.4c) since $\eta \geq 0$. The second inequality is implied by (5.4a) and (5.7), which yield

$$\left(1 + (\lambda - \tfrac{1}{2}\eta)\tau\right)\left(1 - \tfrac{1}{2}\eta\tau\right)|\partial\phi|(U_{\tau,\eta}^n) \leq |\partial\phi|(U_{\tau,\eta}^{n-1}), \tag{5.8}$$

upon observing that $\lambda - \eta - \frac{1}{2}\eta(\lambda - \frac{1}{2}\eta)\tau \geq -\lambda'$. Actually (5.8) may not hold if the left-hand sides of both (5.4a) and (5.7) are negative; however, the second inequality of (5.4c) is trivially true. $\square$

The second step consists in providing the fundamental *local error estimate* between (τ, η)-minimizing movements and the EVI_λ-gradient flow, just upon assuming a relaxed version of (5.4a) and (5.4b). Here the refined local asymptotic expansion (3.15) turns out to be the crucial ingredient of the estimate. From now on, for simplicity we will only consider the case $\lambda \leq 0$: this is not restrictive, since a solution of EVI_λ for $\lambda > 0$ also solves EVI_0.

Lemma 5.4. *Let $u \in \mathrm{Lip}([0, +\infty); \mathrm{Dom}(\phi))$ be a solution of the Evolution Variational Inequality $\mathrm{EVI}_\lambda(X, \mathsf{d}, \phi)$, $\lambda \leq 0$, with $u_0 \in \mathrm{Dom}(|\partial\phi|)$ and let $U \in \mathrm{Dom}(|\partial\phi|)$ satisfy the estimates*

$$\tau(1 - \tfrac{1}{2}\eta\tau)^2|\partial\phi|^2(U) \leq \frac{\mathsf{d}^2(U, u_0)}{\tau} + \varepsilon, \qquad (1 - \tfrac{1}{2}\beta\tau)\frac{\mathsf{d}^2(U, u_0)}{\tau} \leq \phi(u_0) - \phi(U) + \varepsilon \tag{5.9}$$

for some $\tau > 0$, $\varepsilon \geq 0$, $\beta, \eta \in [0, 2/\tau)$. If $(\eta + \beta - 2\lambda)\tau < 1$, then for every $\alpha \leq 0$ complying with $2\alpha \leq \tau^{-1} \log(1 - (\eta + \beta - 2\lambda)\tau)$ there holds

$$\mathrm{e}^{2\lambda\tau}\mathsf{d}^2(U, u_\tau) \leq \tau^2\Big(|\partial\phi|^2(u_0) - \mathrm{e}^{2\alpha\tau}|\partial\phi|^2(U)\Big) + 3\varepsilon\tau. \qquad (5.10)$$

Before proceeding to prove the lemma, let us point out that estimate (5.10) looks particularly simple in the case $\lambda = \eta = \beta = 0$:

$$\mathsf{d}^2(U, u_\tau) \leq \tau^2\Big(|\partial\phi|^2(u_0) - |\partial\phi|^2(U)\Big) + 3\varepsilon\tau.$$

Proof. The local expansion (3.15) with $v = U$ yields

$$\frac{\mathrm{e}^{2\lambda\tau}}{2}\mathsf{d}^2(u_\tau, U) - \frac{1}{2}\mathsf{d}^2(u_0, U) + \mathsf{E}_{2\lambda}(\tau)\Big(\phi(u_0) - \phi(U)\Big) \leq \frac{\tau^2}{2}|\partial\phi|^2(u_0). \qquad (5.11)$$

Multiplying the second inequality of (5.9) by $2\mathsf{E}_{2\lambda}(\tau)$ and summing it to (5.11) we obtain

$$\mathrm{e}^{2\lambda\tau}\mathsf{d}^2(u_\tau, U) \leq \tau^2|\partial\phi|^2(u_0) - c_1\mathsf{d}^2(u_0, U) + 2\mathsf{E}_{2\lambda}(\tau)\,\varepsilon \qquad (5.12)$$

where $c_1 := 2\tau^{-1}(1 - \frac{1}{2}\beta\tau)\mathsf{E}_{2\lambda}(\tau) - 1$. Using the elementary inequalities

$$\tau(1 + \lambda\tau) \leq \mathsf{E}_{2\lambda}(\tau) \leq \tau, \quad 1 \geq c_1 \geq 1 + (2\lambda - \beta)\tau,$$

and the first inequality of (5.9), which yields

$$\mathsf{d}^2(U, u_0) \geq (1 - \eta\tau)\tau^2|\partial\phi|^2(U) - \varepsilon\tau,$$

from (5.12) we end up with (5.10). $\quad\square$

5.1. Error estimates and convergence for initial data in $\mathrm{Dom}(|\partial\phi|)$

In the next result we establish uniform error estimates between ε-relaxed (τ, η)-minimizing movements (in the sense that they are assumed to satisfy only (5.9) at each discrete time step) and the gradient flow. Here we consider the case of *regular* discrete initial data, namely we suppose that $U^0_{\tau,\eta} \in \mathrm{Dom}(|\partial\phi|)$.

For convenience, let us introduce the following notations:

$$\gamma := 2\eta - 3\lambda, \qquad t_\tau := \min\{k\tau : k \in \mathbb{N}, \ k\tau \geq t\} \quad \text{for every } t \geq 0. \qquad (5.13)$$

Theorem 5.5 *(Uniform error estimate for regular data). Let the following assumptions hold:*

1. the EVI_λ-gradient flow ($\lambda \leq 0$) S_t of ϕ exists in $D \subset \overline{\mathrm{Dom}(\phi)}$;

2. *for some $\tau > 0$, $\eta \geq 0$ and $\varepsilon \geq 0$ such that $4\gamma\tau \leq 1$, the sequence $(U_{\tau,\eta}^n)_{n \in \mathbb{N}} \subset D \cap \mathrm{Dom}(|\partial\phi|)$ satisfies for all $n \in \mathbb{N}$*

$$
\tau(1 - \tfrac{1}{2}\eta\tau)^2 |\partial\phi|^2(U_{\tau,\eta}^n) \leq \frac{\mathsf{d}^2(U_{\tau,\eta}^n, U_{\tau,\eta}^{n-1})}{\tau} + \varepsilon,
$$
$$
\left(1 - \tfrac{\eta - \lambda}{2}\tau\right) \frac{\mathsf{d}^2(U_{\tau,\eta}^n, U_{\tau,\eta}^{n-1})}{\tau} \leq \phi(U_{\tau,\eta}^{n-1}) - \phi(U_{\tau,\eta}^n) + \varepsilon.
$$

(5.14)

Then, for all $u_0 \in D$, the following estimate holds for every $t > 0$:

$$
\mathsf{d}(\mathsf{S}_t(u_0), \overline{U}_{\tau,\eta}(t)) \leq \mathrm{e}^{-\lambda t} \mathsf{d}(u_0, U_{\tau,\eta}^0) + \left(\sqrt{\tau t_\tau} + t_\tau - t\right) \mathrm{e}^{\gamma t_\tau} |\partial\phi|(U_{\tau,\eta}^0) + 2\sqrt{\frac{\varepsilon}{\tau} t_\tau \, \mathsf{E}_{2\gamma}(t_\tau)}.
$$

(5.15)

Proof. Thanks to (5.14), by applying Lemma 5.4 we obtain the validity of estimate (5.10) with u_0 replaced by $U_{\tau,\eta}^{n-1}$, U replaced by $U_{\tau,\eta}^n$, u_τ replaced by $\mathsf{S}_\tau(U_{\tau,\eta}^{n-1})$ (for all $n \in \mathbb{N}$) and $\beta = \eta - \lambda$. Let us set $\gamma_\tau := -\tau^{-1} \log(1 - \gamma\tau)$. Upon multiplying (5.10) by $\mathrm{e}^{2\alpha(n-1)\tau}$ and noticing that there holds $2\alpha \leq -\gamma_\tau \leq -\gamma \leq 2\lambda$, we end up with

$$
\mathrm{e}^{2\alpha n\tau} \mathsf{d}^2(U_{\tau,\eta}^n, \mathsf{S}_\tau(U_{\tau,\eta}^{n-1})) \leq \tau^2 (A^n + \varepsilon B^n),
$$

where $\quad A^n := \mathrm{e}^{2\alpha(n-1)\tau} |\partial\phi|^2(U_{\tau,\eta}^{n-1}) - \mathrm{e}^{2\alpha n\tau} |\partial\phi|^2(U_{\tau,\eta}^n) \quad$ and $\quad B^n := \dfrac{3\mathrm{e}^{2\alpha(n-1)\tau}}{\tau}$.

On the other hand, since $\alpha \leq \lambda \leq 0$, if we set $E^n := \mathrm{e}^{\alpha n\tau} \mathsf{d}(U_{\tau,\eta}^n, \mathsf{S}_{n\tau}(U_{\tau,\eta}^0))$ we obtain:

$$
\begin{aligned}
E^n &\leq \mathrm{e}^{\alpha n\tau} \mathsf{d}(U_{\tau,\eta}^n, \mathsf{S}_\tau(U_{\tau,\eta}^{n-1})) + \mathrm{e}^{\alpha n\tau} \mathsf{d}(\mathsf{S}_\tau(U_{\tau,\eta}^{n-1}), \mathsf{S}_\tau(\mathsf{S}_{(n-1)\tau}(U_{\tau,\eta}^0))) \\
&\leq \mathrm{e}^{\alpha n\tau} \mathsf{d}(U_{\tau,\eta}^n, \mathsf{S}_\tau(U_{\tau,\eta}^{n-1})) + \mathrm{e}^{\alpha(n-1)\tau} \mathsf{d}(U_{\tau,\eta}^{n-1}, \mathsf{S}_{(n-1)\tau}(U_{\tau,\eta}^0)) \\
&\leq \mathrm{e}^{\alpha n\tau} \mathsf{d}(U_{\tau,\eta}^n, \mathsf{S}_\tau(U_{\tau,\eta}^{n-1})) + E^{n-1},
\end{aligned}
$$

so that if $m \in \mathbb{N}$ is such that $t \in ((m-1)\tau, m\tau]$, or equivalently $t_\tau = m\tau$, there holds (note that $E^0 = 0$)

$$
\begin{aligned}
E^m &\leq \sum_{n=1}^m \mathrm{e}^{\alpha n\tau} \mathsf{d}(U_{\tau,\eta}^n, \mathsf{S}_\tau(U_{\tau,\eta}^{n-1})) \leq \tau \sum_{n=1}^m (A^n + \varepsilon B^n)^{1/2} \\
&\leq \tau \sum_{n=1}^m (A^n)^{1/2} + \tau\sqrt{\varepsilon} \sum_{n=1}^m (B^n)^{1/2} \\
&\leq \tau\sqrt{m} \left(\sum_{n=1}^m A^n\right)^{1/2} + \sqrt{3m\varepsilon\tau} \left(\sum_{n=1}^m \mathrm{e}^{2\alpha(n-1)\tau}\right)^{1/2} \\
&\leq \sqrt{\tau t_\tau} |\partial\phi|(U_{\tau,\eta}^0) + \sqrt{3\varepsilon t_\tau \frac{1 - \mathrm{e}^{2\alpha t_\tau}}{1 - \mathrm{e}^{2\alpha\tau}}},
\end{aligned}
$$

whence, recalling (3.10) and (3.15),

$$
\begin{aligned}
\mathsf{d}(\overline{U}_{\tau,\eta}(t), \mathsf{S}_t(u_0)) & \\
&\leq \mathsf{d}(\mathsf{S}_t(U^0_{\tau,\eta}), \mathsf{S}_t(u_0)) + \mathrm{e}^{-\lambda t}\mathsf{d}(\mathsf{S}_{m\tau-t}(U^0_{\tau,\eta}), U^0_{\tau,\eta}) + \mathsf{d}(U^m_{\tau,\eta}, \mathsf{S}_{m\tau}(U^0_{\tau,\eta})) \\
&\leq \mathrm{e}^{-\lambda t}\mathsf{d}(u_0, U^0_{\tau,\eta}) + (t_\tau - t)\,\mathrm{e}^{-\lambda t_\tau}|\partial\phi|(U^0_{\tau,\eta}) + \mathrm{e}^{-\alpha t_\tau} E^m \\
&\leq \mathrm{e}^{-\lambda t}\mathsf{d}(u_0, U^0_{\tau,\eta}) + \left(\sqrt{\tau t_\tau} + t_\tau - t\right)\mathrm{e}^{-\alpha t_\tau}|\partial\phi|(U^0_{\tau,\eta}) + \sqrt{3\varepsilon t_\tau\frac{\mathrm{e}^{-2\alpha t_\tau}-1}{1-\mathrm{e}^{2\alpha\tau}}},
\end{aligned}
$$

namely (5.15) upon choosing $2\alpha = -\gamma_\tau$, exploiting the inequality

$$
\frac{1-\mathrm{e}^{2\alpha\tau}}{2\alpha\tau} = \frac{1}{\tau}\int_0^\tau \mathrm{e}^{2\alpha r}\,\mathrm{d}r \geq \mathrm{e}^{2\alpha\tau} = \mathrm{e}^{-\tau\gamma_\tau} \geq (1-\gamma\tau) \geq 3/4
$$

and observing that (we use (5.29) below)

$$
\gamma_\tau = \log\left(1-\gamma\tau\right)^{-\frac{1}{\tau}} \leq \log\left[\mathrm{e}^\gamma(1-\gamma\tau)^{-\gamma}\right] \leq 2\gamma
$$

since $1-\gamma\tau \geq \mathrm{e}^{-1}$. □

Let us make explicit two important consequences of the previous result, keeping in mind the notations (5.13) for γ and t_τ.

Corollary 5.6 *(Error estimate for the Minimizing Movement scheme). Let us suppose that* $\mathrm{Dom}(\phi)$ *is an* approximate length subset *of* X, *and the* EVI_λ-*gradient flow* ($\lambda \leq 0$) S_t *of* ϕ *exists in* $\overline{\mathrm{Dom}(\phi)}$. *If, for some* $\tau > 0$ *and* $\eta \geq 0$ *satisfying* $4\gamma\tau \leq 1$, *the sequence* $(U^n_{\tau,\eta})_{n\in\mathbb{N}} \subset \mathrm{Dom}(\phi)$ *is a* (τ,η)-*discrete Minimizing Movement, according to Definition 5.1, then for all* $u_0 \in \overline{\mathrm{Dom}(\phi)}$ *the following estimate holds:*

$$
\mathsf{d}(\mathsf{S}_t(u_0), \overline{U}_{\tau,\eta}(t)) \leq \mathrm{e}^{-\lambda t}\mathsf{d}(u_0, U^0_{\tau,\eta}) + \left(\sqrt{\tau t_\tau} + t_\tau - t\right)\mathrm{e}^{\gamma t_\tau}|\partial\phi|(U^0_{\tau,\eta}) \quad \textit{for every } t > 0.
\tag{5.16}
$$

Proof. Since $\mathrm{Dom}(\phi)$ is an approximate length subset, we know by Theorem 3.10 that ϕ is approximately λ-convex. The sequence $(U^n_{\tau,\eta})_{n\in\mathbb{N}}$ thus satisfies the *a priori* estimates of Lemma 5.3, so that we can apply Theorem 5.5 with $\varepsilon = 0$. □

In order to appreciate the strength of (5.16), let us consider the case where $\lambda = 0$ and $\eta = 0$, so that the sequence $n \mapsto U^n_\tau = U^n_{\tau,0}$ is in fact a solution of the usual Minimizing Movement scheme. For a fixed final time t, we choose a uniform partition of step size $\tau = t/n$ and as initial datum $U^0_\tau = u_0 \in \mathrm{Dom}(|\partial\phi|)$, obtaining

$$
\mathsf{d}(\mathsf{S}_t(u_0), U^n_\tau) = \mathsf{d}(\mathsf{S}_t(u_0), \overline{U}_\tau(t)) \leq \frac{t}{\sqrt{n}}|\partial\phi|(u_0),
$$

which reproduces (with a better constant) the celebrated Crandall-Liggett estimate for the generation of contraction semigroups in Banach spaces governed by m-accretive operators.

As a second consequence, we are able to compare the EVI_λ-formulation with the notion of Minimizing Movements recalled in Definition 5.2.

Corollary 5.7 *(Existence of Minimizing Movements). Let us suppose that ϕ has complete sublevels,* $\text{Dom}(\phi)$ *is an approximate length subset of X, and the* EVI_λ*-gradient flow* ($\lambda \leq 0$) S_t *of ϕ exists in* $\overline{\text{Dom}(\phi)}$*. Then for every $u_0 \in \text{Dom}(|\partial\phi|)$ the sets* $\mathsf{GMM}(X, \mathsf{d}, \phi; u_0)$ *and* $\mathsf{MM}(X, \mathsf{d}, \phi; u_0)$ *coincide and contain as a unique element the curve* $(\mathsf{S}_t(u_0))_{t \geq 0}$*.*

Proof. As in the previous corollary, we know that ϕ is approximately λ-convex; since ϕ has complete sublevels, it is quadratically bounded from below by Theorem 2.17 and the set of (τ, η)-Minimizing Movements is surely not empty thanks to Theorem 2.10, at east if $\eta > 0$ and τ is sufficiently small. Estimate (5.16) then shows that $\lim_{\tau\downarrow 0} \mathsf{d}(\overline{U}_{\tau,\eta}(t), \mathsf{S}_t(u_0)) = 0$ for every $t \geq 0$, so that the curve $t \mapsto \mathsf{S}_t(u_0)$ is the unique element of $\mathsf{MM}(X, \mathsf{d}, \phi; u_0)$ and $\mathsf{GMM}(X, \mathsf{d}, \phi; u_0)$. $\square$

We point out that one could also drop the length assumption on $\text{Dom}(\phi)$ in the previous Corollaries 5.6 and 5.7 (provided $u_0 \in \overline{\text{Dom}(\phi)}^{\mathsf{d}_\ell}$), by replacing the minimizing movements generated by d with the corresponding ones generated by d_ℓ in $X = \overline{\text{Dom}(\phi)}^{\mathsf{d}_\ell}$: it is enough to apply Theorems 3.9 and 3.10. Note that the sublevels of ϕ stay complete also w.r.t. d_ℓ if they were w.r.t. d.

5.2. Error estimates and convergence for initial data in $\text{Dom}(\phi)$

The error estimates we obtained in the previous Theorem 5.5 involve the slope of the initial datum. One can expect that for less regular initial data, namely belonging only to $\text{Dom}(\phi)$, a lower-order error estimate should still be available. Usually, such weaker estimates can be derived by combining the stronger ones with suitable contraction properties of the discrete scheme, through an interpolation technique. In our situation, however, only the continuous semigroup exhibits contraction properties of the type of (3.10), but we do not know if the Minimizing Movement scheme shares an analogous good dependence on perturbations of the initial condition. In this regard, we are only able to prove estimate (5.18) below.

To overcome this difficulty, we adopt a different approach, which will allow us to prove Theorem 5.9: because calculations are much more complicated with respect to Section 5.1, here we do not aim at finding optimal constants. First of all, we establish the following refined version of Lemma 5.3, where the rate of approximation depends on the *Moreau-Yosida* regularization (2.15) of ϕ and the times at which the minimizing movements are considered are not necessarily consecutive (compare with the *a priori* estimates and asymptotic expansions of Theorem 3.5, see also Remark 3.7).

Proposition 5.8 *(Refined continuous stability estimates). Let the assumptions of Lemma 5.3 be fulfilled with $\lambda \leq 0$ and $\beta := \lambda' = \eta - \lambda$. Suppose in addition that ϕ is quadratically bounded from below for all $\kappa_o > \beta$. If $4\beta\tau \leq 1$ then for every $t, s > 0$ the following estimates hold:*

$$\frac{1}{2}\mathsf{d}^2(\overline{U}_{\tau,\eta}(t), U^0_{\tau,\eta}) \leq \mathrm{e}^{2\beta t_{\tau,\beta}} \, \mathsf{E}_{-\beta}(t_{\tau,\beta}) \left[\phi(U^0_{\tau,\eta}) - \phi_{\mathsf{E}_{-\beta}(t_{\tau,\beta})}(U^0_{\tau,\eta})\right], \qquad (5.17)$$

$$\frac{1}{2}\mathsf{d}^2(\overline{U}^s_{\tau,\eta}(t), \overline{U}_{\tau,\eta}(t)) \leq \mathrm{e}^{2\beta(t_{\tau,\beta}+s_{\tau,\beta})+\eta t_{\tau,\beta}} \, \mathsf{E}_{-\beta}(s_{\tau,\beta}) \left[\phi(U^0_{\tau,\eta}) - \phi_{\mathsf{E}_{-\beta}(s_{\tau,\beta})}(U^0_{\tau,\eta})\right], \qquad (5.18)$$

$$\frac{1}{2}|\partial\phi|^2(\overline{U}_{\tau,\eta}(t)) \leq (1 + 2\eta\tau) \frac{\mathrm{e}^{2\beta t_{\tau,\beta}}}{\mathsf{E}_{-\beta}(t_\tau)} \left[\phi(U^0_{\tau,\eta}) - \phi_{\mathsf{E}_{-\beta}(t_{\tau,\beta})}(U^0_{\tau,\eta})\right], \qquad (5.19)$$

where by $\overline{U}^s_{\tau,\eta}(\cdot)$ we denote the piecewise constant interpolant of the (τ, η)-discrete Minimizing Movement $\left(U^{n+h}_{\tau,\eta}\right)_{n\in\mathbb{N}}$ starting from $\overline{U}_{\tau,\eta}(s)$ (let $h := s_\tau/\tau$) and for every $t > 0$ we set

$$t_{\tau,\beta} := (1 + 4\beta\tau)t_\tau.$$

Proof. We can suppose with no loss of generality that $\beta > 0$: the estimates corresponding to $\beta = 0$ just follow by letting $\beta \to 0$ (i.e. $\eta, \lambda \to 0$) in (5.17)–(5.19). To begin with, for all $k \in \mathbb{N}$ let us write

$$-(1 - \beta\tau)^k \phi(U^k_{\tau,\eta}) + (1 - \beta\tau)^{k-1} \phi(U^{k-1}_{\tau,\eta})$$
$$= \frac{\beta\tau}{1 - \beta\tau}(1 - \beta\tau)^k \phi(U^k_{\tau,\eta}) + (1 - \beta\tau)^{k-1}\left[\phi(U^{k-1}_{\tau,\eta}) - \phi(U^k_{\tau,\eta})\right]. \qquad (5.20)$$

Now note that estimate (5.4b) entails

$$(1 - \beta\tau)^{k-1}\left[\phi(U^{k-1}_{\tau,\eta}) - \phi(U^k_{\tau,\eta})\right]$$
$$\geq \frac{1}{2\tau}(1 - \tfrac{1}{2}\beta\tau)(1 - \beta\tau)^{k-1}\mathsf{d}^2(U^k_{\tau,\eta}, U^{k-1}_{\tau,\eta}) + \frac{1}{2}(1 - \beta\tau)^{k-1}\left[\phi(U^{k-1}_{\tau,\eta}) - \phi(U^k_{\tau,\eta})\right], \qquad (5.21)$$

so that by combining (5.20), (5.21) and summing up we obtain for all $n \in \mathbb{N}$

$$\phi(U^0_{\tau,\eta}) - (1 - \beta\tau)^n \phi(U^n_{\tau,\eta})$$
$$\geq \frac{\beta\tau}{1 - \beta\tau} \sum_{k=1}^n (1 - \beta\tau)^k \phi(U^k_{\tau,\eta})$$
$$+ \frac{1}{2\tau}(1 - \tfrac{1}{2}\beta\tau) \sum_{k=1}^n (1 - \beta\tau)^{k-1}\mathsf{d}^2(U^k_{\tau,\eta}, U^{k-1}_{\tau,\eta})$$
$$+ \frac{1}{2} \sum_{k=1}^n (1 - \beta\tau)^{k-1}\left[\phi(U^{k-1}_{\tau,\eta}) - \phi(U^k_{\tau,\eta})\right]. \qquad (5.22)$$

It is then direct to check that there holds (discrete integration by parts)

$$\sum_{k=1}^{n}(1-\beta\tau)^{k-1}\big[\phi(U_{\tau,\eta}^{k-1})-\phi(U_{\tau,\eta}^{k})\big]$$

$$= \phi(U_{\tau,\eta}^{0}) - (1-\beta\tau)^{n}\phi(U_{\tau,\eta}^{n}) - \frac{\beta\tau}{1-\beta\tau}\sum_{k=1}^{n}(1-\beta\tau)^{k}\phi(U_{\tau,\eta}^{k}). \qquad (5.23)$$

Moreover,

$$\mathsf{d}^{2}(U_{\tau,\eta}^{n},U_{\tau,\eta}^{0}) \le \left(\sum_{k=1}^{n}\mathsf{d}(U_{\tau,\eta}^{k},U_{\tau,\eta}^{k-1})\right)^{2}$$

$$\le \frac{1-\beta\tau}{\beta\tau}\big[(1-\beta\tau)^{-n}-1\big]\sum_{k=1}^{n}(1-\beta\tau)^{k-1}\mathsf{d}^{2}(U_{\tau,\eta}^{k},U_{\tau,\eta}^{k-1}). \qquad (5.24)$$

Hence (5.22)–(5.24) yield

$$(1-\beta\tau)^{n}\left[\frac{\beta}{2[1-(1-\beta\tau)^{n}]}\mathsf{d}^{2}(U_{\tau,\eta}^{n},U_{\tau,\eta}^{0})+\phi(U_{\tau,\eta}^{n})\right]$$

$$-\frac{(1-\beta\tau)^{n}}{2}\phi(U_{\tau,\eta}^{n})+\frac{\beta\tau}{2(1-\beta\tau)}\sum_{k=1}^{n}(1-\beta\tau)^{k}\phi(U_{\tau,\eta}^{k}) \le \frac{1}{2}\phi(U_{\tau,\eta}^{0}); \qquad (5.25)$$

recalling the definition of Moreau-Yosida approximation and setting

$$\Phi_{n} := -\sum_{k=1}^{n}(1-\beta\tau)^{k}\phi(U_{\tau,\eta}^{k}), \quad \Phi_{0} := 0,$$

from (5.25) we can therefore deduce

$$\Phi_{n}-\Phi_{n-1}-\frac{\beta\tau}{1-\beta\tau}\Phi_{n} \le \phi(U_{\tau,\eta}^{0})-2(1-\beta\tau)^{n}\phi_{\frac{1-(1-\beta\tau)^{n}}{\beta}}(U_{\tau,\eta}^{0}) \quad \text{for every } n \in \mathbb{N}\setminus\{0\}; \qquad (5.26)$$

by iterating (5.26) and using the fact that $\tau \mapsto \phi_{\tau}(U_{\tau,\eta}^{0})$ is nonincreasing, we end up with

$$\frac{\beta\tau}{1-\beta\tau}\Phi_{n} \le \big[(1-\beta\tau)^{n}-1\big]\phi(U_{\tau,\eta}^{0})$$

$$+\left[\left(\frac{1-\beta\tau}{1-2\beta\tau}\right)^{n}-(1-\beta\tau)^{n}\right]\left[\phi(U_{\tau,\eta}^{0})-\phi_{\frac{1-(1-\beta\tau)^{n}}{\beta}}(U_{\tau,\eta}^{0})\right]. \qquad (5.27)$$

We point out that the r.h.s. of (5.27) is finite thanks to the assumptions on $\beta\tau$ and ϕ. Now we reconsider (5.22) and still apply (5.4b) to the last term in the r.h.s. (together with (5.24)), to get

$$\frac{\beta(1-\beta\tau)^n}{2[1-(1-\beta\tau)^n]} \mathsf{d}^2(U_{\tau,\eta}^n, U_{\tau,\eta}^0)$$

$$\leq \phi(U_{\tau,\eta}^0) - (1-\beta\tau)^n \left[\frac{\beta}{2[1-(1-\beta\tau)^n]} \mathsf{d}^2(U_{\tau,\eta}^n, U_{\tau,\eta}^0) + \phi(U_{\tau,\eta}^n) \right] + \frac{\beta\tau}{1-\beta\tau} \Phi_n;$$

estimate (5.27) plus once again the definition Moreau-Yosida approximation then yield

$$\frac{1}{2}\mathsf{d}^2(U_{\tau,\eta}^n, U_{\tau,\eta}^0) \leq \frac{1-(1-\beta\tau)^n}{\beta(1-2\beta\tau)^n} \left[\phi(U_{\tau,\eta}^0) - \phi_{\frac{1-(1-\beta\tau)^n}{\beta}}(U_{\tau,\eta}^0) \right]. \tag{5.28}$$

The validity of (5.17) is ensured upon recalling (5.13), the definition (5.3) of piecewise constant interpolant $\overline{U}_{\tau,\eta}(t)$ and the elementary inequality

$$(1-x)^{-1} \leq \mathrm{e}^{\frac{x}{1-x}} \quad \text{for every } x \in (0,1), \tag{5.29}$$

which implies

$$(1-2\beta\tau)^{-n} \leq \mathrm{e}^{2\beta(1+4\beta\tau)t_\tau} \quad \text{and} \quad (1-\beta\tau)^{-n} \leq \mathrm{e}^{\beta(1+\frac{4}{3}\beta\tau)t_\tau} \qquad \text{provided } 4\beta\tau \leq 1. \tag{5.30}$$

In order to prove (5.18), let $n, h \in \mathbb{N}$ be such that $t \in ((n-1)\tau, n\tau]$ and $s \in ((h-1)\tau, h\tau]$, namely $n = t_\tau/\tau$ and $h = s_\tau/\tau$. Similarly to (5.24), we have:

$$\mathsf{d}^2(U_{\tau,\eta}^{n+h}, U_{\tau,\eta}^n) \leq \frac{1-\beta\tau}{\beta\tau} \left[(1-\beta\tau)^{-h} - 1 \right] \sum_{k=1}^{h} (1-\beta\tau)^{k-1} \mathsf{d}^2(U_{\tau,\eta}^{k+n}, U_{\tau,\eta}^{k+n-1}).$$

On the other hand, (5.4a) and the left-hand inequality in (5.4c) yield (up to an iteration)

$$\mathsf{d}(U_{\tau,\eta}^{k+n}, U_{\tau,\eta}^{k+n-1}) \leq (1-\beta\tau)^{-n}(1-\tfrac{1}{2}\eta\tau)^{-n} \mathsf{d}(U_{\tau,\eta}^k, U_{\tau,\eta}^{k-1}),$$

whence

$$\mathsf{d}^2(U_{\tau,\eta}^{n+h}, U_{\tau,\eta}^n) \leq \frac{1-\beta\tau}{\beta\tau} \frac{(1-\beta\tau)^{-h} - 1}{(1-\beta\tau)^{2n}(1-\tfrac{1}{2}\eta\tau)^{2n}} \sum_{k=1}^{h} (1-\beta\tau)^{k-1} \mathsf{d}^2(U_{\tau,\eta}^k, U_{\tau,\eta}^{k-1}). \tag{5.31}$$

Now we observe that, by the above method of proof, estimate (5.28) still holds if we replace $\mathsf{d}^2(U_{\tau,\eta}^n, U_{\tau,\eta}^0)$ with the r.h.s. of (5.24), so that this refined information plus (5.31) entail

$$\frac{1}{2}\mathsf{d}^2(U_{\tau,\eta}^{n+h}, U_{\tau,\eta}^n) \leq \frac{1-(1-\beta\tau)^h}{\beta(1-2\beta\tau)^h(1-\beta\tau)^{2n}(1-\tfrac{1}{2}\eta\tau)^{2n}} \left[\phi(U_{\tau,\eta}^0) - \phi_{\frac{1-(1-\beta\tau)^h}{\beta}}(U_{\tau,\eta}^0) \right]. \tag{5.32}$$

Estimate (5.18) is therefore a consequence of (5.32) up to exploiting again (5.29) as we did in (5.30) (now with $x = \beta\tau$, $x = 2\beta\tau$ and $x = \eta\tau/2$), noting that $U_{\tau,\eta}^n = \overline{U}_{\tau,\eta}(t)$ and $U_{\tau,\eta}^{n+h} = \overline{U}_{\tau,\eta}^s(t)$ by the definition of piecewise constant interpolant.

We are then left with establishing (5.19). To this aim, first of all note that by virtue of the right-hand inequality in (5.4c) we know that the map $n \mapsto (1 - \beta\tau)^n |\partial\phi|(U_{\tau,\eta}^n)$ is not increasing. Furthermore, (5.4a) and (5.4b) yield

$$\tau(1 - \tfrac{1}{2}\eta\tau)^2(1 - \tfrac{1}{2}\beta\tau)|\partial\phi|^2(U_{\tau,\eta}^n) \le \phi(U_{\tau,\eta}^{n-1}) - \phi(U_{\tau,\eta}^n); \tag{5.33}$$

if we plug (5.33) into (5.22) and proceed exactly as above to estimate the remaining terms, we obtain:

$$
\begin{aligned}
&\frac{(1 - \beta\tau)^n[1 - (1 - \beta\tau)^n]}{2\beta}|\partial\phi|^2(U_{\tau,\eta}^n) \\
&= \frac{\tau}{2}\sum_{k=1}^{n}(1 - \beta\tau)^{-k}\left[(1 - \beta\tau)^n|\partial\phi|(U_{\tau,\eta}^n)\right]^2 \\
&\le \frac{\tau}{2}(1 - \tfrac{1}{2}\beta\tau)\sum_{k=1}^{n}(1 - \beta\tau)^{k-1}|\partial\phi|^2(U_{\tau,\eta}^k) \\
&\le (1 - \tfrac{1}{2}\eta\tau)^{-2}\left(\frac{1 - \beta\tau}{1 - 2\beta\tau}\right)^n\left[\phi(U_{\tau,\eta}^0) - \phi_{\frac{1-(1-\beta\tau)^n}{\beta}}(U_{\tau,\eta}^0)\right].
\end{aligned}
\tag{5.34}
$$

Estimate (5.19) finally follows from (5.34), by recalling (5.13) along with the definition of piecewise constant interpolant and using (5.30), the inequality $(1 - x) \le e^{-x}$, valid for all $x \in (0, 1)$, applied to $x = \beta\tau$ and the elementary estimate

$$(1 - \tfrac{1}{2}\eta\tau)^{-2} \le 1 + 2\eta\tau \qquad \text{ensured by } 4\eta\tau \le 4\beta\tau \le 1. \quad \square$$

We are now able to establish the analogue of Theorem 5.5 for initial (discrete) data that merely belong to $\mathrm{Dom}(\phi)$. For simplicity, here we treat the case $\varepsilon = 0$ only.

Theorem 5.9 (Uniform error estimate for data in $\mathrm{Dom}(\phi)$). *Let the following assumptions hold:*

1. *the EVI_λ-gradient flow ($\lambda \le 0$) S_t of ϕ exists in $D \subset \overline{\mathrm{Dom}(\phi)}$;*
2. *for some $\tau \in (0, 1)$ and $\eta \ge 0$ such that $4\gamma\tau \le 1$ (let γ be as in (5.13)), the sequence $(U_{\tau,\eta}^n)_{n\in\mathbb{N}} \subset D \cap \mathrm{Dom}(\phi)$ is a (τ, η)-discrete Minimizing Movement, according to Definition 5.1;*
3. *ϕ is approximately λ-convex in D.*

Then, for all $u_0 \in D$, the following estimate holds for every $t > 0$:

$$\mathsf{d}(\mathsf{S}_t(u_0), \overline{U}_{\tau,\eta}(t)) \leq \mathrm{e}^{-\lambda t} \mathsf{d}(u_0, U^0_{\tau,\eta}) + 10 \sqrt[4]{\tau t_\tau}\, \mathrm{e}^{2\gamma t_\tau} \sqrt{\phi(U^0_{\tau,\eta}) - \phi_{\mathsf{E}_{\lambda-\eta}(3\sqrt{\tau t_\tau})}(U^0_{\tau,\eta})}.$$

(5.35)

Note that once condition 1. is satisfied, then by virtue of Theorem 3.10 condition 3. is also satisfied as long as D is an approximate length subset.

Proof. In order to establish (5.35), the idea is to start from the trivial inequality

$$\mathsf{d}(\mathsf{S}_t(U^0_{\tau,\eta}), \overline{U}_{\tau,\eta}(t)) \leq \mathsf{d}(\mathsf{S}_t(U^0_{\tau,\eta}), \overline{U}^s_{\tau,\eta}(t)) + \mathsf{d}(\overline{U}^s_{\tau,\eta}(t), \overline{U}_{\tau,\eta}(t)),$$

(5.36)

valid for all $s > 0$. If we apply (5.15) with $\overline{U}_{\tau,\eta}$ replaced by $\overline{U}^s_{\tau,\eta}$, $u_0 = U^0_{\tau,\eta}$ and $\varepsilon = 0$, we obtain:

$$\mathsf{d}(\mathsf{S}_t(U^0_{\tau,\eta}), \overline{U}^s_{\tau,\eta}(t)) \leq \mathrm{e}^{-\lambda t} \mathsf{d}(U^0_{\tau,\eta}, \overline{U}_{\tau,\eta}(s)) + \left(\sqrt{\tau t_\tau} + t_\tau - t\right) \mathrm{e}^{\gamma t_\tau} |\partial\phi|(\overline{U}_{\tau,\eta}(s))$$

for every $t > 0$.

(5.37)

Hence by exploiting (3.10), (5.36), (5.37) and (5.17)–(5.19), we deduce the estimate

$$\mathsf{d}(\mathsf{S}_t(u_0), \overline{U}_{\tau,\eta}(t))$$
$$\leq \mathrm{e}^{-\lambda t} \mathsf{d}(u_0, U^0_{\tau,\eta}) + \mathrm{e}^{\beta(t_{\tau,\beta}+s_{\tau,\beta})+\frac{\eta}{2}t_{\tau,\beta}} \sqrt{2\,\mathsf{E}_{-\beta}(s_{\tau,\beta}) \left[\phi(U^0_{\tau,\eta}) - \phi_{\mathsf{E}_{-\beta}(s_{\tau,\beta})}(U^0_{\tau,\eta})\right]}$$
$$+ \mathrm{e}^{-\lambda t + \beta s_{\tau,\beta}} \sqrt{2\,\mathsf{E}_{-\beta}(s_{\tau,\beta}) \left[\phi(U^0_{\tau,\eta}) - \phi_{\mathsf{E}_{-\beta}(s_{\tau,\beta})}(U^0_{\tau,\eta})\right]}$$
$$+ \left(\sqrt{\tau t_\tau} + t_\tau - t\right) \mathrm{e}^{\gamma t_\tau + \beta s_{\tau,\beta}} \sqrt{\frac{2 + 4\eta\tau}{\mathsf{E}_{-\beta}(s_\tau)} \left[\phi(U^0_{\tau,\eta}) - \phi_{\mathsf{E}_{-\beta}(s_{\tau,\beta})}(U^0_{\tau,\eta})\right]}.$$

(5.38)

Now we make the choice $s = \sqrt{\tau t_\tau}$ and try to simplify (5.38) as much as possible by means of the following elementary inequalities, valid under the running assumptions on the parameters τ, η and λ:

$$s_{\tau,\beta} \leq 3\sqrt{\tau t_\tau}, \quad \mathsf{E}_{-\beta}(s_{\tau,\beta}) \leq 3\sqrt{\tau t_\tau}, \quad \mathsf{E}_{-\beta}(s_\tau) \geq \sqrt{\tau t_\tau}\, \mathrm{e}^{-\frac{\beta}{2}t_\tau - \frac{1}{16}},$$
$$\sqrt{\tau t_\tau} + t_\tau - t \leq 2\sqrt{\tau t_\tau},$$

(5.39)

and

$$2 + 4\eta\tau \leq \frac{5}{2}, \quad \mathrm{e}^{\beta(t_{\tau,\beta}+s_{\tau,\beta})+\frac{\eta}{2}t_{\tau,\beta}} \leq \mathrm{e}^{2\gamma t_\tau + \frac{3}{16}}, \quad \mathrm{e}^{-\lambda t + \beta s_{\tau,\beta}} \leq \mathrm{e}^{\gamma t_\tau + \frac{3}{16}},$$
$$\mathrm{e}^{\gamma t_\tau + \beta s_{\tau,\beta}} \leq \mathrm{e}^{\gamma t_\tau + \frac{3}{2}\beta t_\tau + \frac{3}{16}}.$$

(5.40)

The validity of (5.35) is then ensured by (5.38)–(5.40) up to some further trivial numerical inequalities. Note that the right-hand side is surely finite: indeed, by virtue of

Theorem 3.5 we know that ϕ is quadratically bounded from below for all $\kappa_o > -\lambda$, and in the case where $\lambda < 0$ there holds $\mathsf{E}_{-\beta}(3\sqrt{\tau t_\tau}) < 1/\beta \le -1/\lambda$. $\square$

Let us point out that in the simplified case $\lambda = \eta = 0$ with $\inf_X \phi > -\infty$, if we set $n \mapsto U_\tau^n = U_{\tau,0}^n$, choose a uniform partition of step size $\tau = t/n$ and an initial datum $U_\tau^0 = u_0 \in \mathrm{Dom}(\phi)$, we obtain the following uniform error estimate of order $1/4$:

$$\mathsf{d}(\mathsf{S}_t(u_0), U_\tau^n) = \mathsf{d}(\mathsf{S}_t(u_0), \overline{U}_\tau(t)) \le 10\, \frac{\sqrt{t}}{\sqrt[4]{n}}\, \sqrt{\phi(u_0) - \inf_X \phi}.$$

Moreover, if $U_{\tau,\eta}^0 \in \mathrm{Dom}(|\partial\phi|)$ the last term in (5.35) involving the Moreau-Yosida regularization is of order $\sqrt{\tau}$ (recall (3.50)), so that the latter reproduces the same convergence rate as (5.16), up to different multiplicative constants (in order to end up with a much more readable estimate we have given up "optimal" constants in (5.35)).

Appendix A. Right Dini derivatives

For every real function $\zeta : [a, b] \to \mathbb{R}$ and $t \in [a, b]$ we consider the lower and upper right *Dini derivatives*

$$\frac{\mathrm{d}}{\mathrm{d}t_+}\, \zeta(t) := \liminf_{h \downarrow 0} \frac{\zeta(t+h) - \zeta(t)}{h}, \quad \frac{\mathrm{d}^+}{\mathrm{d}t}\, \zeta(t) := \limsup_{h \downarrow 0} \frac{\zeta(t+h) - \zeta(t)}{h}.$$

In the beginning of Section 3 we take advantage of the following basic lemma (see e.g. [46]).

Lemma A.1. *Let $\zeta, \eta : (a, b) \to \mathbb{R}$ be lower semicontinuous functions. If*

$$\frac{\mathrm{d}}{\mathrm{d}t_+}\, \zeta(t) + \eta(t) \le 0 \quad \text{for every } t \in (a, b), \tag{A.1}$$

then η is locally integrable in (a, b) and for every $a_0 \in (a, b)$ the function

$$\tilde{\zeta}(t) := \zeta(t) + \int_{a_0}^t \eta(s)\, \mathrm{d}s \quad \text{is nonincreasing and right continuous in } (a, b).$$

In particular, (A.1) is equivalent to

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\, \zeta(t) + \eta(t) \le 0 \quad \text{for every } t \in (a, b)$$

and implies the distributional inequality $\frac{\mathrm{d}}{\mathrm{d}t}\zeta + \eta \le 0$, i.e.

$$\zeta, \eta \in L^1_{\mathrm{loc}}((a,b)) \quad and \quad \int_a^b (-\zeta \psi' + \eta \psi)\, \mathrm{d}t \le 0 \quad for\ every\ nonnegative\ \psi \in C^\infty_{\mathrm{c}}((a,b)),$$

$$(\mathrm{A.2})$$

which is in turn equivalent to (A.1) *under the additional assumption that* ζ *is right continuous.*

Proof. Let us first consider the case $\eta = 0$. If $a < t_0 < t_0 + \tau < b$ existed with $\delta := \tau^{-1}\big(\zeta(t_0 + \tau) - \zeta(t_0)\big) > 0$, then a minimum point $\bar{t} \in [t_0, t_0 + \tau)$ of $t \mapsto \zeta_\delta(t) := \zeta(t) - \zeta(t_0) - \delta(t - t_0)$ would satisfy

$$\liminf_{h \downarrow 0} \frac{\zeta_\delta(\bar{t} + h) - \zeta_\delta(\bar{t})}{h} = \frac{\mathrm{d}}{\mathrm{d}t+} \zeta(\bar{t}) - \delta \ge 0, \qquad \text{which contradicts (A.1)}.$$

The right continuity is then a trivial consequence of the lower semicontinuity. In the general case, since η is lower semicontinuous it admits a minimum $m(c,d)$ in every interval $[c,d] \subset (a,b)$. It follows that the function $\zeta_{c,d}(t) := \zeta(t) + m(c,d)t$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t+} \zeta_{c,d}(t) \le -(\eta(t) - m(c,d)) \le 0 \quad \text{for every } t \in (c,d),$$

whence $\zeta_{c,d}$ is nonincreasing in (c,d) thanks to the first part of the proof. Thus $\zeta_{c,d}$ is right continuous in (c,d), is differentiable $\mathscr{L}^1$-a.e. in (c,d), its pointwise derivative $\dot{\zeta}_{c,d}$ coincides with $\frac{\mathrm{d}}{\mathrm{d}t+}$ up to a $\mathscr{L}^1$-negligible set and $\dot{\zeta}_{c,d} \in L^1_{\mathrm{loc}}((c,d))$; since $\eta \le -\frac{\mathrm{d}}{\mathrm{d}t+}\zeta_{c,d} + m(c,d)$ and η is locally bounded from below, it follows that also $\eta \in L^1_{\mathrm{loc}}((c,d))$. All of the just proved properties being independent of $a < c < d < b$, we have established that $\eta \in L^1_{\mathrm{loc}}((a,b))$ and that ζ is right continuous.

We can finally introduce the primitive function $H(t) := \int_{a_0}^t \eta(s)\, \mathrm{d}s$. Because $\frac{\mathrm{d}}{\mathrm{d}t}\zeta_{c,d} \le \dot{\zeta}_{c,d}$ in $\mathscr{D}'((c,d))$, there holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\zeta} = \frac{\mathrm{d}}{\mathrm{d}t}(\zeta + H) = \frac{\mathrm{d}}{\mathrm{d}t}(\zeta_{c,d} + H - m(c,d)t) \le \dot{\zeta}_{c,d} + \eta - m(c,d) = \frac{\mathrm{d}}{\mathrm{d}t+}\zeta + \eta \le 0,$$

still in the sense of distributions in (c,d). On the other hand, since $a < c < d < b$ are arbitrary and $\tilde{\zeta} = \zeta + H$ is right continuous, it follows that it is nonincreasing in (a,b). The last assertions concerning (A.2) are direct consequences of the previous results. $\quad\square$

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