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REGULARITY AND PERTURBATION RESULTS FOR MIXED SECOND ORDER ELLIPTIC PROBLEMS

Giuseppe Savaré

Istituto di Analisi Numerica del C.N.R.
Via Abbiategrasso 209, 27100 Pavia, Italy

Abstract. – We study a mixed boundary value problem for elliptic second order equations obtaining optimal regularity results under weak assumptions on the data. We also consider the dependence of the solution with respect to perturbations of the boundary sets carrying the Dirichlet and the Neumann conditions.

0. - Introduction.

Let Ω be a uniform $C^{1,1}$ open set of $\mathbf{R}^N$ with boundary $\Gamma = \partial\Omega$; we choose $C^{1,1}$ submanifolds (with boundary) $\Gamma_0 \subset \partial\Omega$ and $\Gamma_1 = \Gamma \setminus \overline{\Gamma_0}$, assuming that $\overline{\Gamma_0} \cap \overline{\Gamma_1}$ is nonempty. We shall study the mixed boundary value problem

$$(EP) \qquad \begin{cases} Au(x) = f(x) & \text{in } \Omega, \\ u(x) = g_0(x) & \text{on } \Gamma_0, \\ \dfrac{\partial u(x)}{\partial \nu_A} = g_1(x) & \text{on } \Gamma_1, \end{cases}$$

where A is a uniform elliptic second order operator with variable coefficients of the type

$$(0.1)\qquad Au = -\sum_{i,j} \frac{\partial}{\partial x_i}\Big(a^{ij}(x)\frac{\partial u}{\partial x_j}\Big) + \sum_i b^i(x)\frac{\partial u}{\partial x_i} + c(x)u$$

and $\nu_A = \nu_A(x)$ is the related conormal vector to Γ.

Problems of this kind have been deeply studied for long time from many points of view, starting from V. Volterra (1883): detailed and comparative references of the papers published before the Fifties are given in [21, 23].

It was immediately well known the fact that the solution of (EP) is not smooth in general, no matter how the data are regular. This singular behavior occurs "near" the points of $\overline{\Gamma_0} \cap \overline{\Gamma_1}$ and the typical counterexample is given by the harmonic function

$$(0.2)\qquad u(x,y) = \mathrm{Im}\big(x + i\,y\big)^{1/2}, \qquad x \in \mathbf{R},\ y > 0,$$

which satisfies a homogeneous Dirichlet (Neumann) condition on the positive (negative) real line.

In order to recover more regular solutions, one has to impose some compatibility conditions between the data, the elliptic operator and the boundary [13, 16, 9, 5, 2, 18], or to "measure" the regularity in weighted function spaces [22, 31, 17]; in these cases Γ_0 and Γ_1 can meet at a non-zero angle, whose amplitude affects the regularity of the solution. More generally, Fredholm-type properties are subtly investigated [31, 16, 9, 26] with fine structure analyses in the case of domains with corners (in particular polygonal or polyhedral domains: see e.g. [4, 9] and the references quoted therein).

On the other hand, it is interesting to know what is the maximal regularity of the solution allowed by the data without assuming any particular compatibility on them and choosing norms independent of the particular choice of Γ_0 in Γ; these informations turn out to be essential when we want to study the dependence of the solution u from perturbations of Γ_0, Γ_1.

Hölder continuity for variational solutions of (EP) follows from the general results of Stampacchia [29] and a precise estimate of the order of this regularity is given by Shamir when the coefficients of A are smooth: considering for simplicity the homogeneous case ($g_0, g_1 \equiv 0$), [28] shows

$$(0.3) \qquad f \in L^p(\Omega) \quad \Rightarrow \quad u \in W^{s,p}(\Omega), \qquad \forall\, s < 1/2 + 2/p,$$

which in particular implies $u \in C^{1/2-\varepsilon}(\Omega)$, $\forall\, \varepsilon > 0$, if f is essentially bounded.

When Γ_0 and Γ_1 do not meet tangentially and the supremum of their dihedral angles is strictly less than π, the situation is better, as showed by Dauge in the case of the Laplace operator on curvilinear polyhedra (cf. [9, Remark 23.6] and [10, 11]); however, the gain of regularity vanishes as the angles between Γ_0 and Γ_1 approach π.

We just consider this limit case, corresponding to the smoothness properties on Ω and Γ_0 we initially assumed; more precisely, we are concerned with the variational formulation of (EP) in $H^1(\Omega)$,[1] for which (0.3) becomes

$$(0.4) \quad f \in L^2(\Omega) \;\Rightarrow\; u \in H^{3/2-\varepsilon}(\Omega),\ \forall\, \varepsilon > 0, \quad \text{but } u \notin H^{3/2}(\Omega) \text{ in general.}$$

We refine this result in two directions; first of all, assuming that the coefficients of A are bounded and the principal ones Lipschitz, we shall see that u belongs in fact to the Besov space [2]

$$(0.5) \qquad u \in B^{3/2}_{2\,\infty}(\Omega) = \Big(H^1(\Omega), H^2(\Omega)\Big)_{1/2,\infty},$$

and this is the optimal regularity result, at least in the family of real interpolation spaces obtained from the $H^s(\Omega)$: we recall that $B^{3/2}_{2\,\infty}(\Omega) \subset H^{3/2-\varepsilon}(\Omega)$, $\forall\, \varepsilon > 0$, and the same example (0.2) shows $u \notin B^{3/2}_{2\,q}(\Omega)$, $\forall\, q < \infty$.

We observe that in the planar case ($N = 2$) we obtain an improvement of (0.3) for $p > 2$, too; in fact the Sobolev-Besov imbedding Theorem gives the optimal $u \in B^{1/2+2/p}_{p\,\infty}(\Omega)$ and, for $p = \infty$,

$$(0.6) \qquad u \in C^{1/2}(\Omega) = B^{1/2}_{\infty\,\infty}(\Omega),$$

reaching the maximal Hölder regularity allowed by (0.2).

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[1] To fix our ideas, we also assume for the moment that the bilinear form associated to A is coercive in $H^1(\Omega)$.

[2] We use the real interpolation functor $(\cdot,\cdot)_{\theta,p}$; see [6, 7].

The other feature we exploit is an estimation of the $B^{3/2}_{2\,\infty}(\Omega)$-norm of u in term of the data, which is well adapted to get interpolation results. We recall that $B^{3/2}_{2\,\infty}$-intermediate regularity is equivalent to a suitable bound of the L^2-modulus of continuity of the gradient of u; more precisely we shall prove

$$\text{(0.7)}\quad \int_{\Omega_{|\eta|}} |\nabla u(x+\eta)-\nabla u(x)|^2\,dx \leq C\,|\eta|\,\|f\|_{L^2(\Omega)}\|u\|_{H^1(\Omega)}, \qquad \forall\,\eta \in \mathbf{R}^n,$$

where $\Omega_{|\eta|} = \{x \in \Omega : d(x, \partial\Omega) > |\eta|\}$ and the constant C does not depend on u and f. This corresponds to the interpolation inequality [3]

$$\text{(0.8)}\qquad \|u\|^2_{B^{3/2}_{2\,\infty}(\Omega)} \leq C\,\|u\|_{H^1(\Omega)}\|f\|_{L^2(\Omega)},$$

which allows us to obtain the maximal (0.5) with weaker assumptions on the data and to prove new results in intermediate classes of spaces under the threshold given by $H^{3/2}(\Omega)$, avoiding in this way the loss of regularity due to the mixed boundary conditions.

For instance, for $-1/2 < \theta < 1/2$ we prove the optimal

$$f \in L^2(\Omega),\quad g_0 \in H^{1/2+\theta}(\Gamma_0),\quad g_1 \in H^{-1/2+\theta}(\Gamma_1) \quad\Rightarrow\quad u \in H^{1+\theta}(\Omega),$$

extending the results of [28] and [26]. Differently from these papers, where a singular integral technique is developed, our simple arguments rely on the difference quotient technique of Nirenberg [25] for variational problems in the Hilbertian setting, adapted to this intermediate situations via the interpolation theory.

Finally, we apply (0.8) to study the dependence of u from perturbation of Γ_0. Taking fixed f and the boundary conditions for simplicity, we find that the map

$$\text{(0.9)}\qquad \Gamma_0 \;\mapsto u \;\in H^1(\Omega)$$

[3] In the non homogeneous case, this estimate is substituted by

$$\|u\|^2_{B^{3/2}_{2\,\infty}(\Omega)} \leq C\,\|u\|_{H^1(\Omega)}\Big[\|f\|_{L^2(\Omega)} + \|g_0\|_{H^{3/2}(\Gamma_0)} + \|g_1\|_{H^{1/2}(\Gamma_1)}\Big].$$

is 1/2-Hölder continuous with respect to the Hausdorff distance between subsets of $\partial\Omega$: if Γ_0' is another $C^{1,1}$ submanifold of $\partial\Omega$ and u' is the related solution of (EP) we have

$$\|u-u'\|_{H^1(\Omega)} \le C\|f\|_{L^2(\Omega)}\, d(\Gamma_0,\Gamma_0')^{1/2}, \tag{0.10}$$

where the constant C depends only on the $C^{1,1}$ character of Γ_0 and Γ_0'. Taking account of example (0.2), it is easy to see that this Hölder exponent is the best possible.

Formulae (0.8) and (0.10) are also crucial for our approach to parabolic problems related to (EP) (see [27]) and to its "Minimizing Movements" formulation proposed by [14] in the general framework of the theory suggested by De Giorgi in [12].

The plan of this paper is the following: first we make precise our results in the next section; the second one contains the proofs of the regularity results and the last one is devoted to study the boundary perturbations.

1. - Main results.

We rewrite (EP) in the usual variational form; to this end we consider the continuous bilinear form on $V = H^1(\Omega)$ [4]

$$a(u,v)=\int_\Omega \Big\{ \sum_{i,j} a^{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}+\sum_i b^i\frac{\partial u}{\partial x_i}v + c\,u\,v\Big\}\,dx, \tag{1.1}$$

whose principal part we suppose symmetric, i.e. $a^{ij}=a^{ji}$; as usual we assume that there exist constants $\alpha,\beta,\gamma>0$ such that, for a.e. $x\in\Omega$,

$$\sum_{i,j} a^{ij}(x)\xi_i\xi_j \ge \alpha|\xi|^2, \quad \forall\,\xi\in\mathbf{R}^N, \tag{1.2}$$

$$\sum_{ij}|a^{ij}(x)|+\sum_i|b^i(x)|+|c(x)|\le\beta, \qquad \sum_{i,j,k}\left|\frac{\partial a^{ij}(x)}{\partial x_k}\right|\le\gamma. \tag{1.3}$$

[4] It is the usual Hilbert space of square summable functions with their gradient; we recall that the restriction operator $u\mapsto u_{|_\Gamma}$ defined on $C^\infty(\overline{\Omega})$-functions can be uniquely extended by density to a continuous and left invertible linear operator from $H^1(\Omega)$ to $H^{1/2}(\Gamma)$; for the function spaces on Γ we refer to [19, 16].

We also consider the closed subspace of V

$$V_0 = H^1_{\Gamma_0}(\Omega) = \big\{u \in H^1(\Omega) : \ u_{|_{\Gamma_0}} \equiv 0 \text{ in the sence of traces}\big\}. \tag{1.4}$$

To each couple of $f \in L^2(\Omega)$ and $g_1 \in \big(H^{1/2}_{00}(\Gamma_1)\big)'$ [5] we associate the linear functional on V_0, $L = L_{f,g_1}$:

$$v \in V_0 \mapsto {}_{V_0'}\langle L, v\rangle_{V_0} = \int_\Omega f\, v\, dx + {}_{\left(H^{1/2}_{00}(\Gamma_1)\right)'}\Big\langle g_1, v_{|_{\Gamma_1}}\Big\rangle_{H^{1/2}_{00}(\Gamma_1)} \tag{1.5}$$

and we will study

Problem 1. *Find $u \in V$ such that*

$$(EP') \qquad \begin{cases} a(u,v) = {}_{V_0'}\langle L, v\rangle_{V_0}, & \forall\, v \in V_0, \\ u - \tilde g_0 \in V_0, \end{cases}$$

where $\tilde g_0 \in V$ is a lifting of the Dirichlet datum $g_0 \in H^{1/2}(\Gamma_0)$:

$$\tilde g_0 \in H^1(\Omega) : \quad \tilde g_{0|_{\Gamma_0}} = g_0. \tag{1.6}$$

It is well known that Problem 1 admits a unique solution u if, for instance, $a(\cdot,\cdot)$ is coercive on V_0. In the case of Dirichlet (or Neumann) conditions we get $u \in H^2(\Omega)$ from $g_0 \in H^{3/2}(\Gamma)$ (or $g_1 \in H^{1/2}(\Gamma)$); in our mixed case we obtain

Theorem 1. *Let $u \in V$ satisfy (EP') with*

$$f \in L^2(\Omega), \quad g_0 \in H^{3/2}(\Gamma_0), \quad g_1 \in H^{1/2}(\Gamma_1); \tag{1.7}$$

[5] f could be chosen in bigger spaces: see [19] and the next Theorem 2; $H^{1/2}_{00}(\Gamma_1)$ is the Hilbert space of the $H^{1/2}(\Gamma_1)$-functions whose trivial extension outside Γ_1 belongs to $H^{1/2}(\Gamma)$. By using the real interpolation, we also have [19]

$$H^{1/2}_{00}(\Gamma_1) = \big(L^2(\Gamma_1), H^1_0(\Gamma_1)\big)_{1/2,2}.$$

Observe that $u \in H^1_{\Gamma_0}(\Omega) \Rightarrow u_{|_{\Gamma_1}} \in H^{1/2}_{00}(\Gamma_1)$.

then $u \in B^{3/2}_{2\,\infty}(\Omega)$ *and there exists a constant* $C > 0$*, depending only on* α, β, γ *and* $\delta = \delta_{\Omega,\Gamma_0}$ *such that* [6]

(1.8)

$$\|u\|^2_{B^{3/2}_{2\,\infty}(\Omega)} \le C\,\|u\|_{H^1(\Omega)}\Big\{\|f\|_{L^2(\Omega)}+\|g_0\|_{H^{3/2}(\Gamma_0)}+\|g_1\|_{H^{1/2}(\Gamma_1)}+\|u\|_{H^1(\Omega)}\Big\}.$$

1.1 Remark. Let us consider the linear map

$$\tag{1.9} \begin{cases} \mathcal{T}:\ (f,g) \in L^2(\Omega) \times H^{1/2}(\Gamma) \mapsto\ L_{f,g} \in V', \\ {}_{V'}\langle \mathcal{T}(f,g), v\rangle_{V} = \displaystyle\int_\Omega f(x)v(x)\,dx + \int_\Gamma g(x)\,v(x)\,d\mathcal{H}^{N-1}. \end{cases}$$

$\mathcal{T}$ is obviously continuous and injective, so we call W its range endowed with the topology induced by $\mathcal{T}$. Setting

$$\tag{1.10} \mathcal{D}(a;V_0,W) = \Big\{u \in V_0 : \exists\, w \in W \quad a(u,v) = {}_{V'}\langle w, v\rangle_{V}, \quad \forall\, v \in V_0\Big\}$$

with the natural norm, [7] formula (1.8) says that [6, Thm. 3.5.2(b)]

$$\tag{1.11} \Big(\mathcal{D}(a;V_0,W), V\Big)_{1/2,1} \subset B^{3/2}_{2\,\infty}(\Omega)$$

with continuous inclusion. □

Thanks to this last intepolation result, it is not surprising that we can obtain optimal regularity results under intermediate conditions on the data and find a new proof for a result of [28].

[6] We denote with δ_{Ω,Γ_0} a measure of the $C^{1,1}$-properties of the boundary; see the next section for a precise definition. Observe that, in the coercive case, we can drop the last addendum in the brackets.

[7] that is

$$\|u\|^2_{\mathcal{D}(a;V_0,W)} = \|u\|^2_V + \inf\big\{\|w\|^2_W :\ w \text{ satisfies } (1.10)\big\};$$

observe that, being $u_{|_{\Gamma_0}} \equiv 0$, this norm is only affected by the restriction of g to Γ_1.

Let us introduce the normal space of distributions [8]

$$X^{1/2}(\Omega) = \overline{H_0^1(\Omega)}^{B^{1/2}_{2\,\infty}(\Omega)} = \Big(L^2(\Omega), H^1(\Omega)\Big)^{\circ}_{1/2,\infty} \tag{1.12}$$

with dual $X^{-1/2}(\Omega)$; observe that [6, Rem. p. 55]

$$X^{-1/2}(\Omega) = \Big(L^2(\Omega), \big(H^1(\Omega)\big)'\Big)_{1/2,1}, \qquad H^{-\theta}(\Omega) \subset X^{-1/2}(\Omega), \quad \forall\, \theta < 1/2.$$

Since $X^{-1/2} \subset V'$, if f belongs to this space (EP') is meaningfull, provided that we replace the integral in Ω with the standard duality pairing. We have

Theorem 2. *Let $u \in V$ solve (EP') with $f \in X^{-1/2}(\Omega)$; then, $\forall\, s \in]0, 1/2[$*

$$g_0 \in H^{1/2+s}(\Gamma_0),\ g_1 \in H^{-1/2+s}(\Gamma_1) \quad \Rightarrow \quad u \in H^{1+s}(\Omega), \tag{1.13}$$

and [9]

$$g_0 \in B^1_{2\,1}(\Gamma_0),\ g_1 \in B^0_{2\,1}(\Gamma_1) \quad \Rightarrow \quad u \in B^{3/2}_{2\,\infty}(\Omega). \tag{1.14}$$

We are also able to consider (1.13) for negative s. The question of the existence of the solution will be reduced to the variational case, assuming that the bilinear form a is coercive on V_0, but different hypotheses are possible. [10]

[8] The precise definition of the "$\circ$" interpolation spaces would be [6, Thm. 3.4.2(c)]

$$X^{1/2}(\Omega) = \overline{H^1(\Omega)}^{B^{1/2}_{2\,\infty}(\Omega)};$$

on the other hand it easy to see that in this formula we can substitute $H^1(\Omega)$ with $H^{1/2}(\Omega)$ and then with $H^1_0(\Omega)$ by the well known density of this last space in $H^{1/2}(\Omega)$.

[9] We set

$$B^0_{2\,1}(\Gamma_1) = \Big(H^{-\sigma}(\Gamma_1), H^{\sigma}(\Gamma_1)\Big)_{1/2,1}, \quad B^1_{2\,1}(\Gamma_0) = \Big(H^{1-\sigma}(\Gamma_0), H^{1+\sigma}(\Gamma_0)\Big)_{1/2,1}$$

for a σ in $]0, 1/2[$; the Reiteration Theorem ensures that this definition does not depend on σ.

[10] For example, if Ω is bounded, $c(x) \geq 0$ will be sufficient; see [15, pp. 215-216].

Theorem 3. *Let us assume that the bilinear form a is coercive on V_0 and the coefficients $b^i(x)$ are Lipschitz, and let r be in* $]0,1/2[$ *. Then for any choice of*

$$f \in X^{-1/2}(\Omega), \quad g_0 \in H^{1/2-r}(\Gamma_0), \quad g_1 \in H^{-1/2-r}(\Gamma_1),$$

there exists a unique solution $u \in H^{1-r}(\Omega)$ of (EP).

1.2 Remark. Instead of $X^{-1/2}$ we could consider the $\Xi^\theta(\Omega)$-family of spaces introduced by [19, Ch.2, 6.3] for $\theta \le 0$; in (1.13) we can choose $f \in \Xi^{-1+s}(\Omega)$ and in the previous theorem $f \in \Xi^{-1-r}(\Omega)$. □

1.3 Remark. Let us set, following [19, Ch.2, 6.4]

$$D_A^{1+\theta}(\Omega) = \Big\{u \in H^{1+\theta}(\Omega) : Au \in \Xi^{-1+\theta}(\Omega)\Big\}, \quad -1/2 < \theta < 1/2. \tag{1.15}$$

We know [19, Ch. 2, Thm. 6.5] that the linear operator

$$\mathcal{E}u = \Big(Au, u|_{\Gamma_0}, \frac{\partial u}{\partial \nu_A}|_{\Gamma_1}\Big)$$

is well defined and continuous on $D_A^{1+\theta}(\Omega)$ with values in

$$\Xi^{-1+\theta}(\Omega) \times H^{1/2+\theta}(\Gamma_0) \times H^{-1/2+\theta}(\Gamma_1).$$

From the previous theorems we can also say that $\mathcal{E}$ is a surjective isomorphism between these spaces if it is an isomorphism for $\theta = 0$, for example in the coercive case. When we substitute $D_A^{1+\theta}(\Omega)$ with the smaller

$$D^{1+\theta}(A; L^2(\Omega)) = \Big\{u \in H^{1+\theta}(\Omega) : Au \in L^2(\Omega)\Big\}, \quad -1/2 < \theta < 1/2, \tag{1.16}$$

the analogous result was proved in [28] in the particular case of the half space $\Omega = \mathbf{R}^N_+ = \big\{(x_1, x_2, \dots, x_N) \in \mathbf{R}^N : \ x_N > 0\big\}$. Observe that when θ is negative, the result for general open sets Ω can not be easy deduced from the $\mathbf{R}^N_+$-one, since $D^{1+\theta}(A; L^2(\Omega))$ does not have the usual localization property (see [26], where a weaker Fredholm-type property is proved in this case.) □

Let us now consider another $C^{1,1}$ submanifold Γ_0' and let us assign a new set of data $L' = L_{f', g_1'}$ and g_0' with the associated $\tilde g_0'$; we also assume that the bilinear form a is coercive in $H^1(\Omega)$ so that Problem 1 is uniquely solvable. Finally we denote by u, u' the solutions relative to $L, \tilde g_0$ and $L', \tilde g_0'$ respectively.

We have

Theorem 4. *Let us assume that the bilinear form a is coercive on $H^1(\Omega)$ and let us choose $L, L' \in W$ and $\tilde{g}_0, \tilde{g}_0' \in H^2(\Omega)$, denoting by S the sum of their norms $\|L\|_W + \|L'\|_W + \|\tilde{g}_0\|_{H^2(\Omega)} + \|\tilde{g}_0'\|_{H^2(\Omega)}$; there exist positive constants C_0, C_1 depending only on δ_{Ω,Γ_0} and $\delta_{\Omega,\Gamma_0'}$ (see the following Definition 2.4) such that, if $d(\Gamma_0, \Gamma_0') \le C_1$ then*

$$\|u - u'\|_{H^1(\Omega)} \le C_0 \Big\{ \|L - L'\|_{V'} + \|\tilde{g}_0 - \tilde{g}_0'\|_{H^1(\Omega)} + S\, d(\Gamma_0, \Gamma_0')^{1/2} \Big\} \tag{1.17}$$

and, if $b^i(x)$ are Lipschitz,

$$\|u - u'\|_{L^2(\Omega)} \le C_0 \Big\{ \|L - L'\|_{V'} + \|\tilde{g}_0 - \tilde{g}_0'\|_{H^1(\Omega)} + S\, d(\Gamma_0, \Gamma_0') \Big\}. \tag{1.18}$$

1.4 Remark. Starting from this result it would not be difficult to show that we could compare g_0, g_0' and g_1, g_1' in $\Gamma_0 \cap \Gamma_0'$ and $\Gamma_1 \cap \Gamma_1'$ respectively instead of consider their extensions to Ω and Γ respectively. □

1.5 Remark. Let us see that 1/2 is the best exponent in (1.17); it is not difficult to modify the function u of (0.2) (e.g. by multiplying it with a suitable cut-off function) in order to obtain a function $\tilde{u}$ equal to u near $(0,0)$, satisfying the same boundary conditions and the Laplace equation

$$-\Delta \tilde{u} = f \quad \text{in } \mathbf{R}^2_+, \qquad f \in C_0^\infty(\mathbf{R}^2_+).$$

At this point we choose for a positive h

$$f'(x,y) = f(x+h, y), \qquad \Gamma_0' =]-h, +\infty[\,,$$

so that $\tilde{u}(x+h, y)$ is the new solution relative to f', Γ_0', and to the homogeneous boundary conditions. If (1.17) held with $\lambda \in [0,1]$ instead of 1/2, we could get

$$\|\tilde{u}(x+h,y) - \tilde{u}(x,y)\|_{H^1(\mathbf{R}^2_+)} \le C h^\lambda, \quad h > 0,$$

and, applying a similar version of Lemma 2.2 of the next section, we could control $\tilde{u}$ also in the y-direction, obtaining

$$\tilde{u} \in B^{1+\lambda}_{2\,\infty}(\mathbf{R}^2_+) \subset C^\lambda(\mathbf{R}^2_+),$$

which is possible only if $\lambda \le 1/2$. $\square$

The previous theorem is a consequence of the "one side" following version, which is interesting by itself; first we define the excess

$$e(\Gamma_0, \Gamma_0') = \sup_{x \in \Gamma_0} d(x, \Gamma_0'), \tag{1.19}$$

observing that

$$d(\Gamma_0, \Gamma_0') = e(\Gamma_0, \Gamma_0') + e(\Gamma_0', \Gamma_0).$$

Theorem 5. *Let $u \in V_0 = H^1_{\Gamma_0}(\Omega)$ be the solution of Problem 1 with respect to $L_{f,g} \in W$ and $g_0 = 0$, and let us consider a function $v \in H^1_{\Gamma_0'}(\Omega) \cap B^{3/2}_{2\,\infty}(\Omega)$; there exist constants $C_0, C_1 > 0$ depending only on the $C^{1,1}$ character of Γ_0 and Γ_0' (see Definition 2.4) such that, if $e(\Gamma_0, \Gamma_0') \le C_1$, then*

$$a(u, v) - {}_{V'}\langle L, v\rangle_V \le C_0 \, \|L\|_W \, \|v\|_{B^{3/2}_{2\,\infty}(\Omega)} \, e(\Gamma_0, \Gamma_0'). \tag{1.20}$$

As an example of the applications of this formula we consider the following mixed parabolic problem. For a fixed positive number $T > 0$ we choose a uniform family of $C^{1,1}$ submanifolds (with boundary) $\Gamma_0^t \subset \partial\Omega$, t varying in $[0, T]$; Σ_0 will be the subset of $\partial\Omega \times]0, T[$ covered by this family, that is

$$\Sigma_0 = \bigcup_{t \in]0,T[} \Gamma_0^t \times \{t\}, \qquad \Sigma_1 = \big(\partial\Omega \times]0, T[\big) \setminus \overline{\Sigma}_0,$$

and we want to study the mixed boundary value Cauchy problem

$$(PP) \qquad \begin{cases} \dfrac{\partial u(x,t)}{\partial t} + Au(x,t) = f(x,t) & \text{in } \Omega \times]0, T[, \\ u(x,t) = 0 & \text{on } \Sigma_0, \\ \dfrac{\partial u(x,t)}{\partial \nu_A} = 0 & \text{on } \Sigma_1, \\ u(x,0) = u_0(x) & \text{on } \Omega. \end{cases}$$

We shall assume that the mapping $t \mapsto \Gamma_0^t$ is Lipschitz with respect to the Hausdorff distance [11] and we apply the abstract results of [27] via our elliptic estimates; we find that for every choice of

$$f \in L^2(\Omega \times]0, T[), \qquad u_0 \in H^1(\Omega) \text{ with } u_0 = 0 \text{ on } \Gamma_0^0,$$

[11] but this condition could be relaxed; see [27].

there exists a unique solution of (PP), continuous with values in $H^1(\Omega)$ and satisfying the natural parabolic regularity

$$\frac{\partial u}{\partial t},\ Au\ \in L^2(\Omega\times]0,T[) \quad \square$$

2. - Proof of Theorems 1, 2, and 3.

First we consider Problem 1 with $g_0, \tilde{g}_0 = 0$, denoting it by **Problem** 1_0; the general one will be recovered by the standard trace results (see [19, 16]) at the end of this section. Since we are interested to the regularity of the solution, we can also assume that

$$b^i(x) \equiv 0, \qquad c(x) \equiv \alpha \quad \text{in } \Omega, \tag{2.1}$$

so that the bilinear form $a(\cdot,\cdot)$ is coercive in $H^1(\Omega)$:

$$a(v,v) \geq \alpha \|v\|^2_{H^1(\Omega)}, \qquad \forall\, v \in H^1(\Omega). \tag{2.2}$$

The first step is to consider particular but important cases for Ω and Γ_0. Obviously, when $\Omega \equiv \mathbf{R}^N$ or $\Omega = \mathbf{R}^N_+ = \big\{(x_1, x_2, \ldots, x_N) :\ x_N > 0\big\}$ with Γ_0 or Γ_1 empty, Theorem 1 follows from the classical H^2-estimates for u and the interpolation inequality [12]

$$\|u\|^2_{B^{3/2}_{2\,\infty}(\Omega)} \leq c\|u\|_{H^2(\Omega)}\|u\|_{H^1(\Omega)}. \tag{2.3}$$

Therefore we shall consider the case $\Omega = \mathbf{R}^N_+$ with

$$\Gamma_0 = \mathbf{R}^{N-1}_+ \times \{0\} = \big\{(x',0) : x' = (x_1,\ldots,x_{N-1}),\ x_{N-1} > 0\big\}. \tag{2.4}$$

For a given vector $\eta \in \mathbf{R}^N$ we denote by τ_η the translation

$$\tau_\eta(x) = x + \eta, \quad \tau_\eta f\,(x) = f(x+\eta). \tag{2.5}$$

$\mathbf{a}(x)$ is the symmetric matrix of the coefficients $\{a^{ij}(x)\}_{i,j=1,\ldots,N}$; by our hypotheses, there exist constants $\beta, \gamma \geq 0$ such that

$$\|\mathbf{a}(x)\| \leq \beta; \qquad \|\mathbf{a}(x+\eta) - \mathbf{a}(x)\| \leq \gamma\,|\eta|, \qquad \forall\, x,\, \eta \in \overline{\mathbf{R}^N_+}. \tag{2.6}$$

We have:

[12] From now on, we shall denote by c all the "universal" constants which are independent of our problem and by C those which are independent of the data f, g_0, g_1 and Γ_0. We only note that the cases $\Omega = \mathbf{R}^N$ or $\Omega = \mathbf{R}^N_+$ with Γ_0 or Γ_1 empty can be easily recovered by the next lemma.

2.1 Lemma. *Let* u *be the solution of Problem* 1_0 *with* Ω, Γ_0 *given by* (2.4), $\mathbf{a}$ *satisfying* (2.6) *and* $L \in W$*; then there exists a constant* $C_{\alpha,\gamma} > 0$ *depending only on* α, γ *such that*

$$\|u - \tau_\eta\, u\|_V^2 \le C_{\alpha,\gamma}\, |\eta|\, \|u\|_V\, \|L\|_W, \qquad \forall\, \eta \in \overline{\Gamma_0}. \tag{2.7}$$

Proof. Since we only consider translations which are *tangential* to $\partial\mathbf{R}^N_+$ and with nonnegative last component, we have obviously

$$\tau_\eta(V_0) \subset V_0, \tag{2.8}$$

so that we can choose $v = u - \tau_\eta u$ in (EP') obtaining ($\eta = (\eta', 0)$, $\eta' \in \mathbf{R}^{N-1}_+$)

$$a(u, u-\tau_\eta u) = \int_{\mathbf{R}^N_+} f\big(u(x)-u(x+\eta)\big)\, dx + \int_{\mathbf{R}^{N-1}_-} g(x')\big(u(x')-u(x'+\eta')\big)\, dx'.$$

Now we use the symmetry of a and the coercivity with respect to the $H^1(\mathbf{R}^N_+)$-norm to manipulate the lefthand member:

$$\begin{aligned}
&2\, a(u, u - \tau_\eta\, u) = a(u - \tau_\eta\, u, u - \tau_\eta\, u) + a(u,u) - a(\tau_\eta\, u, \tau_\eta\, u) \ge \\
&\alpha \|u - \tau_\eta\, u\|^2_{H^1(\mathbf{R}^N_+)} + \alpha \int_{\mathbf{R}^N_+} |u(x)|^2\, dx - \alpha \int_{\mathbf{R}^N_+} |u(x+\eta)|^2\, dx + \\
&\quad + \int_{\mathbf{R}^N_+} \Big(\mathbf{a}(x)\nabla u(x), \nabla u(x)\Big)\, dx - \int_{\mathbf{R}^N_+} \Big(\mathbf{a}(x)\nabla u(x+\eta), \nabla u(x+\eta)\Big)\, dx = \\
&\alpha \|u - \tau_\eta\, u\|^2_{H^1(\mathbf{R}^N_+)} + \int_{\mathbf{R}^N_+} \Big(\big[\mathbf{a}(x+\eta) - \mathbf{a}(x)\big]\nabla u(x+\eta), \nabla u(x+\eta)\Big)\, dx + \\
&+ \int_{\mathbf{R}^N_+} \Big(\mathbf{a}(x)\nabla u(x), \nabla u(x)\Big)\, dx - \int_{\mathbf{R}^N_+} \Big(\mathbf{a}(x+\eta)\nabla u(x+\eta), \nabla u(x+\eta)\Big)\, dx.
\end{aligned}$$

Since

$$\int_{\mathbf{R}^N_+} \Big(\mathbf{a}(x)\nabla u(x), \nabla u(x)\Big)\, dx = \int_{\mathbf{R}^N_+} \Big(\mathbf{a}(x+\eta)\nabla u(x+\eta), \nabla u(x+\eta)\Big)\, dx,$$

we obtain

$$2a(u, u - \tau_\eta\, u) \ge \alpha \|u - \tau_\eta\, u\|^2_{H^1(\mathbf{R}^N_+)} - \gamma\, |\eta|\, \|\nabla u\|^2_{L^2(\mathbf{R}^N_+)}. \tag{2.9}$$

On the other hand we have

$$\int_{\mathbf{R}^N_+} f(x)\big(u(x)-u(x+\eta)\big)\,dx \le |\eta|\,\|f\|_{L^2(\mathbf{R}^N_+)}\|\nabla u\|_{L^2(\mathbf{R}^N_+)},$$

and, recalling that $u\equiv 0$ on $\mathbf{R}^{N-1}_+$,

$$\int_{\mathbf{R}^{N-1}_-} g(x')\big(u(x')-u(x'+\eta')\big)\,dx' = \int_{\mathbf{R}^{N-1}_-} \big(g(x')-g(x'-\eta')\big)u(x')\,dx' \le$$
$$\|g-\tau_{-\eta}[g]\|_{\left(H^{1/2}_{00}(\mathbf{R}^{N-1}_-)\right)'}\|u\|_{H^{1/2}_{00}(\mathbf{R}^{N-1}_-)} \le |\eta|\,\|g\|_{H^{1/2}(\mathbf{R}^{N-1}_-)}\|u\|_{H^1(\mathbf{R}^N_+)}.$$

Combining these results and recalling that by (2.2)

$$\|u\|_{H^1(\mathbf{R}^N_+)} \le \frac{1}{\alpha}\|L\|_{V'} \le \frac{c}{\alpha}\|L\|_W,$$

we find (2.7). ∎

The interest of the previous estimate lies in the characterization of the Besov spaces via the translations (see [6, 7]); in particular a function $u\in H^1(\mathbf{R}^N_+)$ belongs to $B^{3/2}_{2\,\infty}(\mathbf{R}^N_+)$ if there exists a constant $B=B_u$ independent of η such that

$$\|\tau_\eta u-u\|^2_{H^1(\mathbf{R}^N_+)} \le B\,|\eta|, \qquad \forall\,\eta\in\overline{\mathbf{R}^N_+}. \tag{2.10}$$

Actually, it is sufficient to control the only translations in the (positive) directions of the canonical basis $\{\mathbf{e}_j\}_{j=1,\dots,N}$ of $\mathbf{R}^N$, so that an equivalent norm for $B^{3/2}_{2\,\infty}(\mathbf{R}^N_+)$ can be obtained by adding to the $H^1(\mathbf{R}^N_+)$-one the following seminorm

$$[u]^2_{B^{3/2}_{2\,\infty}(\mathbf{R}^N_+)} = \sum_{j=1}^{N}\sup_{t>0}\frac{\|u-\tau_{t\mathbf{e}_j}u\|^2_{H^1(\mathbf{R}^N_+)}}{t}. \tag{2.11}$$

Lemma 2.1 only shows that u belongs to

$$D'_{3/2} = \Big\{u\in H^1(\mathbf{R}^N_+):\ [u]^2_{D'_{3/2}} = \sum_{j=1}^{N-1}\sup_{t>0}\frac{\|u-\tau_{t\mathbf{e}_j}u\|^2_{H^1(\mathbf{R}^N_+)}}{t} < +\infty\Big\}, \tag{2.12}$$

with

$$[u]^2_{D'_{3/2}} \le C_{\alpha,\gamma}\|u\|_V\|L\|_W.$$

So we are lacking the control on the $\mathbf{e}_N$-direction but we can recover it by adapting the classic Nirenberg technique to our intermediate situation via an interpolation theorem of Baiocchi [3].

2.2 Lemma. *Let $u \in D'_{3/2}$ be such that $Au \in L^2(\Omega)$. Then $u \in B^{3/2}_{2\,\infty}(\mathbf{R}^N_+)$ with*

$$\|u\|^2_{B^{3/2}_{2\,\infty}(\mathbf{R}^N_+)} \leq C_{\alpha,\beta,\gamma}\Big[\|u\|^2_{D'_{3/2}} + \|u\|_{H^1(\mathbf{R}^N_+)}\|Au\|_{L^2(\mathbf{R}^N_+)}\Big]. \tag{2.13}$$

Proof. Let us introduce the space "of order 2" correponding to $D'_{3/2}$:

$$D'_2 = \Big\{u \in H^1(\mathbf{R}^N_+):\ [u]^2_{D'_2} = \sum_{j=1}^{N-1} \sup_{t>0} \frac{\|u - \tau_{t\mathbf{e}_j}u\|^2_{H^1(\mathbf{R}^N_+)}}{t^2} < +\infty\Big\}; \tag{2.14}$$

u belongs to D'_2 if and only if

$$\frac{\partial u}{\partial x_j} \in H^1(\mathbf{R}^N_+), \quad \text{for } j = 1, \ldots, N-1.$$

By the classic Nirenberg argument, if we know $u \in D'_2$ and $Au \in L^2(\mathbf{R}^N_+)$ then u is in $H^2(\mathbf{R}^N_+)$ with

$$\|u\|_{H^2(\mathbf{R}^N_+)} \leq C_{\alpha,\beta,\gamma}\Big[\|u\|_{D'_2} + \|Au\|_{L^2(\mathbf{R}^N_+)}\Big]. \tag{2.15}$$

In this way $H^2(\mathbf{R}^N_+)$ is isomorphic to the space $\{u \in D'_2 : Au \in L^2(\mathbf{R}^N_+)\}$ with its natural topology (see [3]).

On the other side, $H^1(\mathbf{R}^N_+)$ does not change if we add to it the condition $Au \in H^{-1}(\mathbf{R}^N_+)$, that is

$$H^1(\mathbf{R}^N_+) = \big\{u \in H^1(\mathbf{R}^N_+):\ Au \in H^{-1}(\mathbf{R}^N_+)\big\}$$ [13].

Then we can apply the "interpolation of operators" result [3, Thm. 2.1] [14] obtaining

$$\big(H^1(\mathbf{R}^N_+), H^2(\mathbf{R}^N_+)\big)_{1/2,\infty} = \Big\{u \in \big(H^1(\mathbf{R}^N_+), D'_2\big)_{1/2,\infty} : Au \in \big(H^{-1}(\mathbf{R}^N_+), L^2(\mathbf{R}^N_+)\big)_{1/2,\infty}\Big\}.$$

[13] Here (and in the following formulae) "=" stands by "with equivalent norm", as it is usual in interpolation theory.

[14] It requires the existence of a right inverse A^{-1} of A, bounded from $H^{-1}(\mathbf{R}^N_+)$ and $L^2(\mathbf{R}^N_+)$ to $H^1(\mathbf{R}^N_+)$ and D'_2 respectively, such that $A^{-1} \circ A$ maps D'_2 into itself; for example, we can consider the inversion of A with homogeneous Dirichlet condition on the whole boundary $\partial\mathbf{R}^N_+$.

We already noticed that

$$B^{3/2}_{2\,\infty}(\mathbf{R}^N_+) = \big(H^1(\mathbf{R}^N_+), H^2(\mathbf{R}^N_+)\big)_{1/2,\infty}$$

and consequently

(2.16)

$$\|u\|_{B^{3/2}_{2\,\infty}(\mathbf{R}^N_+)} \le C_{\alpha,\beta,\gamma}\Big\{\|u\|_{\big((H^1(\mathbf{R}^N_+),D'_2\big)_{1/2,\infty}} + \|Au\|_{\big(H^{-1}(\mathbf{R}^N_+),L^2(\mathbf{R}^N_+)\big)_{1/2,\infty}}\Big\}.$$

Now observe that the standard results on semigroup theory (see [7, 6]) give

$$\big(H^1(\mathbf{R}^N_+), D'_2\big)_{1/2,\infty} = D'_{3/2}$$

and the usual interpolation inequality gives

$$\|Au\|^2_{\big(H^{-1}(\mathbf{R}^N_+),L^2(\mathbf{R}^N_+)\big)_{1/2,\infty}} \le C_\beta \|Au\|_{L^2(\mathbf{R}^N_+)}\|u\|_{H^1(\mathbf{R}^N_+)};$$

substituting in (2.16) we get (2.13). ▪

2.3 Corollary. *Let* u *be the solution of Problem* 1_0 *with* Ω, Γ_0 *given by* (2.4), $\mathbf{a}$ *satisfying* (2.6) *and* $L \in W$*; then there exists a constant* $C = C_{\alpha,\beta,\gamma} > 0$ *depending only on* α, β, γ *such that*

$$\|u\|^2_{B^{3/2}_{2\,\infty}(\mathbf{R}^N_+)} \le C\,\|u\|_{H^1(\mathbf{R}^N_+)}\|L\|_W. \tag{2.17}$$

Proof. In our case we have $Au = f \in L^2(\mathbf{R}^N_+)$ and

$$\|u\|^2_{D'_{3/2}} \le C\|u\|_{H^1(\mathbf{R}^N_+)}\|L\|_W \ ;$$

by (2.13) we conclude. ▪

We come now to the proof of Theorem 1 for a generic $C^{1,1}$ open set Ω; since the argument is very classic, we only go into the details of some less obvious points, due to the possible *unboundedness* of $\partial\Omega$. In view of the next applications we are also interested to control exactly the dependence of the various constants on the regularity of the boundaries.

First of all we make precise our regularity hypothesis on Ω and Γ_0 (see [30, 1])

2.4 Definition. *We say that a couple (Ω,Γ_0) with $\Gamma_0\subset\partial\Omega$ is of class $C^{1,1}$ if there exist an $\varepsilon>0$, an integer ℓ, an $M>0$ and a (possible finite) sequence $U_1,\dots,U_k,\dots$ of open sets of $\mathbf{R}^N$ so that*

1. *if $x\in\partial\Omega$ then $\overline{B_\varepsilon(x)}\subset U_k$ for some k;*
2. *no point of $\mathbf{R}^N$ is contained in more than ℓ of the U_k's;*
3. *there exist $C^{1,1}$ diffeomorphisms*

$$\begin{cases} \Phi_k : U_k \mapsto V_k = B_1(x_k)\subset\mathbf{R}^N, \quad \Psi_k=\Phi_k^{-1},\\ \|\Phi_k\|_{C^{1,1}(U_k)}\le M, \quad \|\Psi_k\|_{C^{1,1}(V_k)}\le M, \end{cases}$$

with $B_1(x_k)=\{x\in\mathbf{R}^N : |x-x_k|<1\}$, $x_k\in\partial\mathbf{R}^N_+$, and

$$\begin{cases} \Phi_k(U_k\cap\Omega)=V_k\cap\mathbf{R}^N_+,\\ \Phi_k(U_k\cap\partial\Omega)=V_k\cap\partial\mathbf{R}^N_+,\\ \Phi_k(U_k\cap\Gamma_0)=V_k\cap\mathbf{R}^{N-1}_+\times\{0\},\\ \Phi_k(U_k\cap\partial\Gamma_0)=V_k\cap\mathbf{R}^{N-2}\times\{(0,0)\}. \end{cases} \tag{2.18}$$

We set

$$\delta=\delta_{\Omega,\Gamma_0}=\inf\{\ell+M+\varepsilon^{-1},\ \text{such that } 1,2,3 \text{ hold}\},$$

and $U_0=\{x\in\Omega:\ \overline{B_\varepsilon(x)}\subset\Omega\}$; clearly $\{U_k\}_{k\ge0}$ is an open cover of $\overline{\Omega}$. □

Correspondingly, we can find a C^∞ partition of unity $\{\theta_k(x)\}_{k\ge0}$ subordinate to the open cover $\{U_k\}_{k\ge0}$ such that

$$\begin{cases} \text{each } \theta_k(x) \text{ is supported in } U_k \text{ and valued in } [0,1],\\ \theta_k(x)\ge 1/(\ell+1) \text{ if } B_\varepsilon(x)\subset U_k,\\ \displaystyle\sum_k\big[\theta_k(x)\big]^2=1, \quad \forall\, x\in\overline{\Omega},\\ \|\theta_k\|_{W^{2,\infty}(\mathbf{R}^N)}\le C(\delta_{\Omega,\Gamma_0}), \quad \forall\, k. \end{cases}$$

Given a function u on Ω we set

$$u_k(x)=u(x)\theta_k(x), \qquad \tilde u_k(y)=\begin{cases} u_k(\Psi_k(y)) & \text{if } y\in\mathbf{R}^N_+\cap V_k,\\ 0 & \text{if } y\in\mathbf{R}^N_+\setminus V_k, \end{cases}$$

and we can recover u from the u_k's by $u(x)=\sum_k\theta_k(x)\,u_k(x)$, where for each x the sum involves al most $\ell+1$ non vanishing terms. We have the following well known facts (see [1, 30, 19, 16]).

2.5 Lemma. *For every* $s \in [0,2]$ *we have*

$$u \in H^s(\Omega) \;\Leftrightarrow u_k \in H^s(\Omega) \text{ and } \sum_k \|u_k\|^2_{H^s(\Omega)} < +\infty, \tag{2.19}$$

and there exist constants C_0, C_1 *only depending on* δ_{Ω,Γ_0} *such that*

$$\|u\|^2_{H^s(\Omega)} \le C_0 \sum_k \|u_k\|^2_{H^s(\Omega)} \le C_1 \|u\|^2_{H^s(\Omega)}.$$

The same relations hold for the H^s *spaces defined on* Γ, Γ_1. $\square$

2.6 Remark. For a generic Besov space we have to be more careful; nevertheless one of the implications of (2.19) still holds in the case we are interested in:

$$u_k \in B^{3/2}_{2\,\infty}(\Omega) \text{ and } \sum_k \|u_k\|^2_{B^{3/2}_{2\,\infty}(\Omega)} < +\infty \quad \Rightarrow \quad u \in B^{3/2}_{2\,\infty}(\Omega),$$

with the corresponding bound

$$\|u\|^2_{B^{3/2}_{2\,\infty}(\Omega)} \le C \sum_k \|u_k\|^2_{B^{3/2}_{2\,\infty}(\Omega)}. \tag{2.20}$$

This fact follows from the inclusions [15]

$$l^2\big(B^{3/2}_{2\,\infty}(\Omega)\big) = l^2\big((L^2(\Omega), H^2(\Omega))_{3/4,\infty}\big) \subset \Big(l^2\big(L^2(\Omega)\big), l^2\big(H^2(\Omega)\big)\Big)_{3/4,\infty}.$$

Now, the previous lemma says that the mapping $\{u_k\}_{k\in\mathbf{N}} \mapsto \sum_k \theta_k u_k$ is bounded from $l^2(H^s(\Omega))$ to $H^s(\Omega)$ and consequently from $l^2\big(B^{3/2}_{2\,\infty}(\Omega)\big)$ to $B^{3/2}_{2\,\infty}(\Omega)$ thanks to this last inclusion. $\square$

[15] see [8]; for a Banach space $\mathcal{B}$, $l^2(\mathcal{B})$ is the Banach space of the $\mathcal{B}$-valued sequences $\{b_k\}_{k\in\mathbf{N}}$ such that

$$\sum_k \|b_k\|^2_{\mathcal{B}} < +\infty.$$

2.7 Lemma. *Let $s \in [0,2]$; then*

$$u_k \in H^s(\Omega) \;\Leftrightarrow\; \tilde{u}_k \in H^s(\mathbf{R}^N_+)$$

and there exist constant C_0, C_1 depending only on δ_{Ω,Γ_0} such that

$$\|u_k\|_{H^s(\Omega)} \le C_0 \|\tilde{u}_k\|_{H^s(\mathbf{R}^N_+)} \le C_1 \|u\|_{H^s(\Omega)}.$$

The analogous result holds for the Besov spaces. □

Now we denote by Γ_{0k} the set

$$\Gamma_{0k} = \begin{cases} \partial \mathbf{R}^N_+ & \text{if } U_k \cap \Gamma_1 = \emptyset \\ \emptyset & \text{if } U_k \cap \Gamma_0 = \emptyset \\ \mathbf{R}^{N-1}_+ \times \{0\} & \text{if } U_k \cap \partial\Gamma_0 \neq \emptyset \end{cases} \tag{2.21}$$

and consequently $\Gamma_{1k} = \partial \mathbf{R}^N_+ \setminus \overline{\Gamma_{0k}}$.

It is not difficult to check that

$$u \in H^1_{\Gamma_0}(\Omega) \;\Rightarrow\; u_k \in H^1_{\Gamma_{0k}}(\mathbf{R}^N_+),$$

that is, following the order of (2.21), $H^1_0(\mathbf{R}^N_+)$, $H^1(\mathbf{R}^N_+)$ and $H^1_{\mathbf{R}^{N-1}_+}(\mathbf{R}^N_+)$ respectively. We have

2.8 Lemma. *Let $u \in H^1_{\Gamma_0}(\Omega)$ be the solution of (EP') with $L \in W$ and $\mathbf{a}$ satisfying*

$$\|\mathbf{a}(x)\| \le \beta, \qquad \left\| \frac{\partial \mathbf{a}(x)}{\partial x_j} \right\| \le \gamma, \qquad \forall\, x \in \Omega; \tag{2.22}$$

then there exist functions

$$\tilde{f}_k \in L^2(\mathbf{R}^N_+), \quad \tilde{g}_k \in H^{1/2}(\Gamma_{1k}), \quad \tilde{\mathbf{a}}_k \in W^{1,\infty}(\mathbf{R}^N_+; \mathbf{S}^{N\times N}),$$

such that $\tilde{u}_k \in H^1_{\Gamma_{0k}}(\mathbf{R}^N_+)$ satisfies

$$\int_{\mathbf{R}^N_+} \big(\tilde{\mathbf{a}}_k \nabla \tilde{u}_k, \nabla \tilde{v}\big) + \tilde{\alpha}\, \tilde{u}\tilde{v}\, dx = \int_{\mathbf{R}^N_+} \tilde{f}_k(x)\tilde{v}(x)\, dx + \int_{\Gamma_{1k}} \tilde{g}_k(x')\tilde{v}(x')\, dx'$$

for every choice of $\tilde{v}$ in $H^1_{\Gamma_{0k}}(\mathbf{R}^N_+)$. Moreover we have the bounds

$$\begin{cases} \displaystyle\sum_k \|\tilde{f}_k\|^2_{L^2(\mathbf{R}^N_+)} \le \tilde{C}\Big[\|f\|^2_{L^2(\Omega)} + \|u\|^2_{H^1(\Omega)}\Big], \\ \displaystyle\sum_k \|\tilde{g}_k\|^2_{H^{1/2}(\Gamma_{1k})} \le \tilde{C}\Big[\|g\|^2_{H^{1/2}(\Gamma_1)} + \|u\|^2_{H^1(\Omega)}\Big], \end{cases}$$

$$\begin{cases} \|\tilde{\mathbf{a}}_k(x)\| \le \tilde{\beta}, \quad \|\tilde{\mathbf{a}}_k(x+\eta) - \tilde{\mathbf{a}}_k(x)\| \le \tilde{\gamma}\,|\eta|, \\ \big(\tilde{\mathbf{a}}_k(x)\xi, \xi\big) \ge \tilde{\alpha}\,|\xi|^2, \end{cases} \qquad \forall \xi \in \mathbf{R}^N, \quad \forall\, x \in \mathbf{R}^N_+,$$

with $\tilde{C}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ *depending only on* α, β, γ *and* δ_{Ω,Γ_0}. □

We can now conclude the *proof of Theorem* 1; applying the estimates in $\mathbf{R}^N_+$ we get for every k

$$\|\tilde{u}_k\|^2_{B^{3/2}_{2\,\infty}(\mathbf{R}^N_+)} \le C \|\tilde{u}_k\|_{H^1(\mathbf{R}^N_+)} \Big[\|\tilde{f}_k\|_{L^2(\mathbf{R}^N_+)} + \|\tilde{g}_k\|_{H^{1/2}(\mathbf{R}^{N-1}_-)} \Big].$$

Lemma 2.7 allows us to suppress the ˜ over u_k, f_k, g_k, substituting $\mathbf{R}^N_+$ with Ω and $\mathbf{R}^{N-1}_-$ with Γ_1; summing up and taking into account (2.20) we get

$$\|u\|^2_{B^{3/2}_{2\,\infty}(\Omega)} \le C \left\{ \sum_k \|u_k\|^2_{H^1(\Omega)} \right\}^{1/2} \left\{ \sum_k \|f_k\|^2_{L^2(\Omega)} + \|g_k\|^2_{H^{1/2}(\Gamma_1)} \right\}^{1/2} \le$$
$$\le C \|u\|_{H^1(\Omega)} \Big\{ \|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\Gamma_1)} + \|u\|_{H^1(\Omega)} \Big\}.$$

Finally the $H^1(\Omega)$ norm of u between the brackets can be bounded by the other two terms. ■

We have to consider now the case $g_0 \ne 0$; we introduce a right inverse linear operator $\mathcal{R}$ of the trace map (see [19, 16]):

$$\mathcal{R} : H^{1/2}(\Gamma_0) \mapsto H^1(\Omega), \qquad \big(\mathcal{R}u\big)_{|\Gamma_0} = u,$$

such that

$$\mathcal{R} \text{ is bounded from } H^{3/2}(\Gamma_0) \text{ to } H^2(\Omega),$$

and we set $\tilde{g}_0 = \mathcal{R} g_0$ in Problem 1. We know that u is a solution of that problem if $\tilde{u} = u - \mathcal{R} g_0 \in V_0$ solves it with respect to the data

$$\tilde{f} = f - A\tilde{g}_0, \quad \tilde{g}_1 = g_1 - \frac{\partial \tilde{g}_0}{\partial \nu_A},$$

with

$$\|\tilde{f}\|_{L^2(\Omega)} + \|\tilde{g}_1\|_{H^{1/2}(\Gamma_1)} \le \|f\|_{L^2(\Omega)} + \|g_1\|_{H^{1/2}(\Gamma_1)} + C_{\beta,\gamma,\delta} \|g_0\|_{H^{3/2}(\Gamma_0)}.$$

By the previous result we get

$$\|\tilde{u}\|^2_{B^{3/2}_{2\,\infty}(\Omega)} \le C \|\tilde{u}\|_{H^1(\Omega)} \Big[\|f\|_{L^2(\Omega)} + \|g_1\|_{H^{1/2}(\Gamma_1)} + \|g_0\|_{H^{3/2}(\Gamma_0)} \Big],$$

whereas

$$\|\tilde u\|_{H^1(\Omega)} \le \|u\|_{H^1(\Omega)} + \|\mathcal{R}(u_{|_{\Gamma_0}})\|_{H^1(\Omega)} \le C\|u\|_{H^1(\Omega)},$$

and

$$\|u\|^2_{B^{3/2}_{2\,\infty}(\Omega)} \le 2\Big(\|\tilde u\|^2_{B^{3/2}_{2\,\infty}(\Omega)} + \|\mathcal{R}g_0\|^2_{B^{3/2}_{2\,\infty}(\Omega)}\Big) \le$$
$$2\Big(\|\tilde u\|^2_{B^{3/2}_{2\,\infty}(\Omega)} + C\|\mathcal{R}g_0\|_{H^2(\Omega)}\|\mathcal{R}(u_{|_{\Gamma_0}})\|_{H^1(\Omega)}\Big) \le$$
$$2\Big(\|\tilde u\|^2_{B^{3/2}_{2\,\infty}(\Omega)} + C\|g_0\|_{H^{3/2}(\Gamma_0)}\|u\|_{H^1(\Omega)}\Big)$$

so that (1.8) holds. ∎

The *proof of Theorem 2* follows now by interpolation. The linear map

$$\mathcal{G} : (f, g_0, g_1) \mapsto u$$

defined by Problem 1 is bounded

$$\text{from } \; B_0 = \big(H^1(\Omega)\big)' \times H^{1/2}(\Gamma_0) \times \big(H^{1/2}_{00}(\Gamma_1)\big)' \; \text{ to } \; H^1(\Omega)$$

and

$$\text{from } \; B_1 = L^2(\Omega) \times H^{3/2}(\Gamma_0) \times H^{1/2}(\Gamma_1) \; \text{ to } \; B^{3/2}_{2\,\infty}(\Omega),$$

with

$$\|\mathcal{G}(f, g_0, g_1)\|^2_{B^{3/2}_{2\,\infty}(\Omega)} \le C\|(f, g_0, g_1)\|_{B_1}\|(f, g_0, g_1)\|_{B_0}.$$

We deduce (by the same arguments of [6, 3.5(b), p. 49] that $\mathcal{G}$ is bounded from $\big(B_0, B_1\big)_{1/2,1}$ with values in the same $B^{3/2}_{2\,\infty}(\Omega)$; that is, for

$$\begin{cases} f \in X^{-1/2}(\Omega) = \Big(\big(H^1(\Omega)\big)', L^2(\Omega)\Big)_{1/2,1}, \\ g_0 \in B^1_{2\,1}(\Gamma_1) = \Big(H^{1/2}(\Gamma_0), H^{3/2}(\Gamma_0)\Big)_{1/2,1}, \\ g_1 \in B^0_{2\,1}(\Gamma_1) = \Big(\big(H^{1/2}_{00}(\Gamma_1)\big)', H^{1/2}(\Gamma_1)\Big)_{1/2,1}, \end{cases}$$

we have $u \in B^{3/2}_{2\,\infty}(\Omega)$. By the Reiteration Theorem we get also the optimal (1.13). An analogous argument cover the case of $\Xi^\theta(\Omega)$-spaces. ∎

Finally we come to the *proof of Theorem 3*. The method here is the standard transposition technique developed by [19], and we limit ourselves to consider the case $g_0 \equiv 0$ as before. We start with a simple estimate about the adjoint form of Problem 1_0:

2.9 Lemma. *Let $u \in V_0$ be the solution of*

$$(2.23) \qquad a(v,u) = {}_{V_0'}\langle L, v\rangle_{V_0}, \qquad \forall\, v \in V_0,$$

with $L \in \big(H^{1-s}_{\Gamma_0}(\Omega)\big)' \subset V_0'$, $s \in [0,1/2[$; then u belongs to $H^{1+s}_{\Gamma_0}(\Omega)$ with

$$(2.24) \qquad \|u\|_{H^{1+s}(\Omega)} \le C\|L\|_{\left(H^{1-s}_{\Gamma_0}(\Omega)\right)'}.$$

Proof. It is sufficient to show that

$$(2.25) \quad L \in L^2(\Omega) \quad \Rightarrow \quad u \in B^{3/2}_{2\,\infty}(\Omega), \qquad \|u\|_{B^{3/2}_{2\,\infty}(\Omega)} \le C\|u\|_V\|L\|_{L^2(\Omega)},$$

and then to reply the previous interpolation argument.

In order to prove (2.25), we define

$$a_0(u,v) = \int_\Omega \sum_i b^i(x)\frac{\partial u(x)}{\partial x_i}v(x)\,dx, \qquad a_1(u,v) = a(u,v) - a_0(u,v),$$

and we observe that u solves

$$a_1(u,v) = (L,v)_H - a_0(v,u), \qquad \forall\, v \in V_0,$$

where a_1 satisfies the hypotheses of Theorem 1. It remains to show that the linear functional

$$L_u : v \in V \mapsto {}_{V'}\langle L_u, v\rangle_V = -a_0(v,u)$$

belongs to W with norm bounded by $\|u\|_V$. Denoting by $\boldsymbol{b}(x)$ the vector field with components $b^i(x)$, we have

$$-a_0(v,u) = \int_\Omega \operatorname{div}\big(u\boldsymbol{b}\big)v\,dx - \int_\Gamma uv\big(\boldsymbol{b},\nu\big)\,d\mathcal{H}^{N-1}$$

and

$$\Big\|\operatorname{div}\big(u\boldsymbol{b}\big)\Big\|_{L^2(\Omega)} \le C\|u\|_{H^1(\Omega)}, \quad \big\|(\boldsymbol{b},\nu)u\big\|_{H^{1/2}(\Gamma)} \le C\|u\|_{H^1(\Gamma)} \quad \blacksquare$$

The other estimate we need to complete the proof of Theorem 3 is given by

2.10 Lemma. *Let $u \in V_0$ be the solution of Problem 1 with respect to $L \in V_0'$; for any $r \in [0, 1/2[$ there exists a constant $C > 0$ independent of L such that*

$$\|u\|_{H^{1-r}(\Omega)} \leq C\|L\|_{(H^{1+r}_{\Gamma_0}(\Omega))'}. \tag{2.26}$$

Proof. Choose $z \in \big(H^{1-r}(\Omega)\big)' \subset V_0'$ and let w_z be the solution of (2.23) with respect to z; we have:

$$w_z \in H^{1+r}_{\Gamma_0}(\Omega), \qquad \|w_z\|_{H^{1+r}(\Omega)} \leq C\,\|z\|_{(H^{1-r}(\Omega))'},$$

and consequently

$$\begin{aligned} {}_{V_0}\langle u, z\rangle_{V_0'} &= a(u, w_z) = {}_{V_0'}\langle L, w_z\rangle_{V_0} \leq \\ &\leq C\|L\|_{\left(H^{1+r}_{\Gamma_0}(\Omega)\right)'}\|w_z\|_{H^{1+r}(\Omega)} \leq C'\|L\|_{(H^{1+r}_{\Gamma_0}(\Omega))'}\|z\|_{(H^{1-r}(\Omega))'}. \end{aligned}$$

Since z is arbitrary, we get (2.26). ∎

By a density argument, it is easy to show that, for every $L \in \big(H^{1+r}_{\Gamma_0}(\Omega)\big)'$ there exists a unique solution $u \in H^{1-r}_{\Gamma_0}(\Omega)$ such that

$$a(u, v) = {}_{(H^{1+r}_{\Gamma_0}(\Omega))'}\langle L, v\rangle_{H^{1+r}_{\Gamma_0}(\Omega)}, \qquad \forall v \in H^{1+r}_{\Gamma_0}(\Omega).$$

Choosing now L of the form $L_{f,g}$ with

$$f \in L^2(\Omega), \qquad g \in H^{-1/2-r}(\Gamma_1), \text{ [16]}$$

we obtain $u \in D^{1-r}(A; L^2(\Omega))$ satisfying (EP) by the trace properties of this space. ∎

3. - Proof of Theorems 4 and 5.

Let Γ_0, Γ_0' be two $C^{1,1}$ submanifolds (with boundary) of $\Gamma = \partial\Omega$; we denote by δ, δ' respectively their $C^{1,1}$ bounds $\delta_{\Omega,\Gamma_0}, \delta_{\Omega,\Gamma_0'}$ introduced in Definition 2.4 and by $\sigma(\Gamma_0, \Gamma_0')$ the quantity

$$\sigma(\Gamma_0, \Gamma_0') = \sup_{x \in \Gamma_0 \setminus \Gamma_0'} d(x, \partial\Gamma_0') + \sup_{x \in \Gamma_0 \setminus \Gamma_0'} d(x, \partial\Gamma_0). \text{ [17]} \tag{3.1}$$

[16] but also $f \in X^{-1/2}(\Omega)$ or $f \in \Xi^{-1-r}(\Omega)$ are admissible; note that the trace on Γ of a $H^{1+r}_{\Gamma_0}(\Omega)$-function belongs to $H^{1/2+r}_0(\Gamma_1)$, so that the choice of g is justified.

[17] When $\Gamma \setminus \Gamma_0'$ is empty, we define 0 this number; of course, $\partial\Gamma_0$ and $\partial\Gamma_0'$ are the boundaries in the relative topology of Γ.

We shall see at the end of this section how $\sigma(\Gamma_0, \Gamma_0')$ depends on its first term $\sup_{x\in\Gamma_0} d(x, \Gamma_0')$. Our aim is now to prove the following

3.1 Proposition. *Let $u \in V_0 = H^1_{\Gamma_0}(\Omega)$ be the solution of Problem 1_0 with respect to $L_{f,g} \in W$ and $v \in H^1_{\Gamma_0'}(\Omega) \cap B^{3/2}_{2\,\infty}(\Omega)$; there exists a constant $C > 0$ depending only on $\alpha, \beta, \gamma, \delta$ and δ' such that*

$$a(u, v) - {}_{V'}\langle L, v\rangle_V \leq C\, \|L\|_W \|v\|_{B^{3/2}_{2\,\infty}(\Omega)} \sigma(\Gamma_0, \Gamma_0'). \tag{3.2}$$

We divide the *proof* in some steps, beginning with a simple extension of a well known property of the conormal map

$$u \mapsto \partial u/\partial \nu_A; \tag{3.3}$$

this is well defined and bounded from $H^2(\Omega)$ to $H^{1/2}(\Gamma)$ and can be continuosly extended by density to a linear continuous operator from (see (1.16) and [16])

$$D^1(A; L^2(\Omega)) \text{ to } H^{-1/2}(\Gamma), \tag{3.4}$$

so that the Green formula

$$a(u, v) = \int_\Omega (Au) v\, dx + {}_{H^{-1/2}(\Gamma)}\left\langle \frac{\partial u}{\partial \nu_A}, v \right\rangle_{H^{1/2}(\Gamma)}$$

still holds for $u \in D^1(A; L^2(\Omega)),\ v \in H^1(\Omega)$. We have:

3.2 Lemma. *There exists a constant $C > 0$ such that*

$$\left\| \frac{\partial u}{\partial \nu_A} \right\|_{B^0_{2\,\infty}(\Gamma)} \leq C \big[\|u\|_{B^{3/2}_{2\,\infty}(\Omega)} + \|Au\|_{L^2(\Omega)} \big]. \tag{3.5}$$

Proof. By interpolation we find that (3.3) is bounded from

$$\big(H^2(\Omega), D^1(A; L^2(\Omega))\big)_{1/2,\infty} \text{ to } \big(H^{1/2}(\Gamma), H^{-1/2}(\Gamma)\big)_{1/2,\infty} = B^0_{2\,\infty}(\Gamma) \tag{3.6}$$

and by standard results [3, (3,16)] we find

$$\big(H^2(\Omega), D^1(A; L^2(\Omega))\big)_{1/2,\infty} = D^{3/2}_{2\,\infty}(A; L^2(\Omega)) = \big\{ u \in B^{3/2}_{2\,\infty}(\Omega) : Au \in L^2(\Omega) \big\} \quad \blacksquare$$

3.3 Corollary. *Let* u, v *as in Proposition* 3.1 *with* $L = L_{f,g} \in W$, g *being defined on all* Γ *and belonging to* $H^{1/2}(\Gamma)$*; then*

$$a(u,v) - {}_{V'}\langle L, v\rangle_V = {}_{H^{-1/2}(\Gamma)}\left\langle \frac{\partial u}{\partial \nu_A} - g, v \right\rangle_{H^{1/2}(\Gamma)}, \tag{3.7}$$

with

$$\begin{cases} \tilde{u} = \dfrac{\partial u}{\partial \nu_A} - g \in B^0_{2\,\infty}(\Gamma), & \|\tilde{u}\|_{B^0_{2\,\infty}(\Gamma)} \le C\|L_{f,g}\|_W, \\ \tilde{v} = v_{|\Gamma} \in B^1_{2\,\infty}(\Gamma), & \|\tilde{v}\|_{B^1_{2\,\infty}(\Gamma)} \le C\|v\|_{B^{3/2}_{2\,\infty}(\Omega)}, \end{cases} \tag{3.8}$$

and

$$\mathrm{supp}(\tilde{u}) \subset \overline{\Gamma_0}, \qquad \mathrm{supp}(\tilde{v}) \subset \Gamma \setminus \Gamma_0' \quad \blacksquare \tag{3.9}$$

We have now to estimate the right hand side of (3.7) by using (3.8) and (3.9). The result we need is precisely the following

3.4 Proposition. *If* $\tilde{u} \in B^0_{2\infty}(\Gamma), \tilde{v} \in B^1_{2\,\infty}(\Gamma)$ *satisfy* (3.9)*, there exists a constant* $C = C(\delta, \delta')$ *such that*

$${}_{H^{-1/2}(\Gamma)}\langle \tilde{u}, \tilde{v}\rangle_{H^{1/2}(\Gamma)} \le C\|\tilde{u}\|_{B^0_{2\infty}(\Gamma)}\|\tilde{v}\|_{B^1_{2\,\infty}(\Gamma)}\,\sigma(\Gamma_0, \Gamma_0'). \tag{3.10}$$

This estimate requires a rather involved argument since $\tilde{u}$ does not belong in general to $L^2(\Gamma)$; we state whitout proof some more obvious facts which follow by standard estimates in $\mathbf{R}^{N-1}_+$ and the usual technique of partition of unity.

3.5 Lemma. *Let the support of* $w \in H^2(\Gamma)$ *be contained in* $\overline{\Gamma_0}$ *and let* G *be a measurable subset of* Γ_0*; then there exists a constant* $C = C(\delta)$ *such that*

$$\|w_{|G}\|_{L^2(G)} \le C\|w\|_{H^2(\Gamma)} \sup_{x \in G} d^2(x, \partial\Gamma_0) \quad \blacksquare$$

3.6 Corollary. *Let $\tilde{u}, \tilde{v}$ be $L^2(\Gamma)$-functions satisfying* (3.9)*; there exists a constant $C = C(\delta, \delta')$ such that*

$$\int_\Gamma \tilde{u}(x)\tilde{v}(x)\, d\mathcal{H}^{N-1} \le \begin{cases} C\|\tilde{u}\|_{L^2(\Gamma)}\|\tilde{v}\|_{H^2(\Gamma)}\, \sigma^2(\Gamma_0, \Gamma_0') & \text{if } \tilde{v} \in H^2(\Gamma), \\ C\|\tilde{u}\|_{H^2(\Gamma)}\|\tilde{v}\|_{L^2(\Gamma)}\, \sigma^2(\Gamma_0, \Gamma_0') & \text{if } \tilde{u} \in H^2(\Gamma). \end{cases}$$

Proof. We observe that

$$\int_\Gamma \tilde{u}(x)\tilde{v}(x)\, d\mathcal{H}^{N-1} = \int_{\Gamma_0 \setminus \Gamma_0'} \tilde{u}(x)\tilde{v}(x)\, d\mathcal{H}^{N-1} \le \|\tilde{u}\|_{L^2(\Gamma_0 \setminus \Gamma_0')}\|\tilde{v}\|_{L^2(\Gamma_0 \setminus \Gamma_0')}$$

and we apply the previous lemma. ∎

The next property is the last information we need

3.7 Lemma. *Let Γ_0 be a $C^{1,1}$ submanifold with boundary of Γ and $s \in [-2, 2]$; there exists a linear continuous projection operator*

$$\mathcal{P} = \mathcal{P}_{\Gamma_0} : H^s(\Gamma) \to H^s(\Gamma), \qquad \mathcal{P} \circ \mathcal{P} = \mathcal{P},$$

which satisfies the following properties:

(3.11)

$$\forall\, u \in H^s(\Gamma) : \ \|\mathcal{P}u\|_{H^s(\Gamma)} \le C(\delta)\|u\|_{H^s(\Gamma)}, \quad \operatorname{supp} u \subset \Gamma_0 \ \Leftrightarrow \ \mathcal{P}u = u.$$

Proof. We know [19, Lemma 12.2] that there exists a linear extension operator

$$\mathcal{P}^* : H^s(\Gamma \setminus \Gamma_0) \ \to \ H^s(\Gamma)$$

with

$$\|\mathcal{P}^* u\|_{H^s(\Gamma)} \le C(\delta)\|u\|_{H^s(\Gamma_1)}, \qquad \big(\mathcal{P}^* u\big)_{|\Gamma \setminus \Gamma_0} = u.$$

We set $\mathcal{P}u = u - \mathcal{P}^*(u_{|\Gamma_1})$. ∎

Now we prove Proposition 3.4 in a more general abstract form.

Let $H^2 \subset H \subset (H^2)' = H^{-2}$ be a (separable) Hilbert triple and let us set as usual

$$H^{s,p} = \Big(H^{-2}, H^2\Big)_{(2+s)/4,p}, \quad s \in]-2, 2[\,, \quad p \in [1, \infty].$$

Let $\mathcal{P}_0, \mathcal{P}_1$ be two linear projection operators with

$$\mathcal{P}_i \in \mathcal{L}(H^2) \cap \mathcal{L}(H^{-2}), \quad \|\mathcal{P}_i u\|_{H^{s,p}} \le P\|u\|_{H^{s,p}}, \quad \mathcal{P}_i \circ \mathcal{P}_i = \mathcal{P}_i,$$

and let us denote by K_i their images in H^{-2}

$$K_i = \{u \in H^{-2} : \mathcal{P}_i u = u\}, \qquad H^{s,p}_{\mathcal{P}_i} = \big\{x \in H^{s,p} : \mathcal{P}x = x\big\} = H^{s,p} \cap K_i.$$

3.8 Proposition. *Assume that there exists a constant σ such that for every couple $u \in H_{\mathcal{P}_0}, v \in H_{\mathcal{P}_1}$*

$$(u,v)_H \le \begin{cases} \sigma^2 \|u\|_H \|v\|_{H^2} & \text{if} \quad v \in H^2, \\ \sigma^2 \|u\|_{H^2} \|v\|_H & \text{if} \quad u \in H^2. \end{cases} \tag{3.12}$$

Then there exists a constant C depending on P such that, for every couple $u \in H^{0,\infty}_{\mathcal{P}_0}$, $v \in H^{\theta,\infty}_{\mathcal{P}_1}$

$${}_{H^{-\theta,1}}\langle u, v\rangle_{H^{\theta,\infty}} \le C\sigma^\theta \|u\|_{H^{0,\infty}} \|v\|_{H^{\theta,\infty}}, \quad \theta \in \,]0,2[\,.$$

Proof. First of all we observe that

$$\forall\, u \in H^{s,p}, \quad \|\mathcal{P}_i u\|_{\left(H^{-2}_{\mathcal{P}_i}, H^2_{\mathcal{P}_i}\right)_{(2+s)/4,p}} \le P\|u\|_{H^{s,p}};$$

in particular, if $u \in K_i$ we have

$$\|u\|_{\left(H^{-2}_{\mathcal{P}_i}, H^2_{\mathcal{P}_i}\right)_{(2+s)/4,p}} \le P\|u\|_{H^{s,p}}. \tag{3.13}$$

Interpolating between the two formulae of (3.12) and applying this last one we get

$$(u,v)_H \le cP^2\sigma^2 \|u\|_{H^{2-\theta,1}} \|v\|_{H^{\theta,\infty}}, \quad \forall\, u \in K_0,\ v \in K_1,\ \theta \in \,]0,2[\,. \tag{3.14}$$

Since the scalar product of H can be extended to the duality pairing between $H^{-\theta,1}$ and $H^{\theta,\infty}$ we find

$${}_{H^{-\theta,1}}\langle u, v\rangle_{H^{\theta,\infty}} \le \|u\|_{H^{-\theta,1}} \|v\|_{H^{\theta,\infty}}.$$

By interpolation with (3.14) we obtain

$$ {}_{H^{-\theta,1}}\langle u,v\rangle_{H^{\theta,\infty}} \le c(P^2\sigma^2)^{\theta/2}\|u\|_{\left(H^{2-\theta,1}_{\mathcal{P}_0},H^{-\theta,1}\right)_{1-\theta/2,\infty}}\|v\|_{H^{\theta,\infty}}. $$

By the Reiteration Theorem and (3.13) we conclude

$$ {}_{H^{-\theta,1}}\langle u,v\rangle_{H^{\theta,\infty}} \le cP^{1+\theta}\sigma^{\theta}\|u\|_{H^{0,\infty}}\|v\|_{H^{\theta,\infty}}, \qquad \forall\, u\in K_0,\ v\in K_1 \quad \blacksquare $$

Now we choose

$$ H=L^2(\Gamma), \quad H^2=H^2(\Gamma); \qquad \mathcal{P}_0=\mathcal{P}_{\Gamma_0}, \quad \mathcal{P}_1=\mathcal{P}_{\Gamma_1'} $$

and we get Proposition 3.4.

3.9 Corollary. *Let $u\in H^1_{\Gamma_0}(\Omega), u'\in H^1_{\Gamma_0'}(\Omega)$ be the two solutions of Problem 1_0 with respect to the data $L,L'\in W$; then there exists a constant C depending only on $\alpha,\beta,\gamma,\delta$ and δ' such that*

(3.15)

$$ \|u-u'\|_V \le C\left\{\|L-L'\|_{V'}+(\|L\|_W+\|L'\|_W)\Big[\sigma(\Gamma_0,\Gamma_0')+\sigma(\Gamma_0',\Gamma_0)\Big]^{1/2}\right\} $$

and, if $b^i(x)$ are Lipschitz,

(3.16)

$$ \|u-u'\|_H \le C\left\{\|L-L'\|_{V'}+(\|L\|_W+\|L'\|_W)\Big[\sigma(\Gamma_0,\Gamma_0')+\sigma(\Gamma_0',\Gamma_0)\Big]\right\}. $$

Proof. We have

$$ a(u,u-u') \le {}_{V'}\langle L,u-u'\rangle_V + C\|L\|_W\|L'\|_W\sigma(\Gamma_0,\Gamma_0'). $$

Changing the role of u and u' we obtain the symmetric inequality; summing up, we get (3.15).

To prove (3.16) we assume for simplicity $b^i(x)\equiv 0$ (otherwise we can use the transposition technique as in Lemma 2.9) so that $a(\cdot,\cdot)$ is a symmetric form. For a fixed $z\in H$, we consider the solution $w_z\in H^1_{\Gamma_0}(\Omega)$ of Problem 1_0 with respect to z, i.e.

$$ a(w_z,v)=(z,v)_H, \qquad \forall\, v\in H^1_{\Gamma_0}(\Omega). $$

We have

$$
\begin{aligned}
&(u-u',z)_H = (u,z)_H - (u',z)_H = a(w_z,u) - (u',z)_H = \\
&a(w_z,u-u') + a(w_z,u') - (u',z)_H \leq \\
&a(w_z,u-u') + C\|z\|_H \, \|L'\|_W \, \sigma(\Gamma_0,\Gamma_0').
\end{aligned}
$$

Now the first addendum of the last term becomes

$$
\begin{aligned}
a(w_z,u-u') = {}& {}_{V'}\langle L-L',w_z\rangle_V + {}_{V'}\langle L',w_z\rangle_V - a(u',w_z) \leq \\
& {}_{V'}\langle L-L',w_z\rangle_V + C\|L'\|_W \, \|z\|_H \, \sigma(\Gamma_0',\Gamma_0).
\end{aligned}
$$

Choosing $z = u-u'$, since $\|w_z\|_V \leq C\|z\|_H$, we get (3.16). ∎

Theorems 4 and 5 follow by the following geometrical lemma, [18] which gives a bound of $\sigma(\Gamma_0,\Gamma_0')$ in terms of $e(\Gamma_0,\Gamma_0')$.

3.10 Lemma. *There exists a constant $C = C(\delta') \geq 1$, such that from*

$$
e(\Gamma_0,\Gamma_0') = \sup_{x\in\Gamma_0} d(x,\Gamma_0') \leq C^{-1}
$$

it follows

$$
\sigma(\Gamma_0,\Gamma_0') \leq Ce(\Gamma_0,\Gamma_0'). \quad ∎
$$

[18] whose proof we omit for brevity; we limit to say that $4(\delta')^5$ is an admissible choice for the constant $C(\delta')$.

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