H3 Mathematics 2025

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Chapter 1

Mathematical Statements

1.1 Quantification

Mathematicians use many symbols to quantify objects in logical statements. Consider the statement

$$\forall x \exists y : x + y = 0$$

Here, $x, y \in \mathbb{Z} - \{0\}$. What this statement is saying is that for all ($\forall$) x, there exists some (one or more) y such that $x+y = 0$.

If we write $\exists y$ alone, it means that there can exist *any number* of such y (including only one!). But in our above statement, only one such y can be the negative of some integer x. To emphasize this uniqueness of y, we further annotate $\exists!$.

1.1.1 For all ($\forall$)

The $\forall$ (read ”for all”) symbol is used to specify some arbitrary element (which can be anything!) in a set of objects. Consider the set of real numbers $\mathbb{R}$. If we want to state a logical statement $P(x)$ which applies to any x in $\mathbb{R}$, we say $\forall x \in \mathbb{R} : P(x)$.

One usually uses this to declare dummy variables which are used in proofs. Therefore, dummy variables are declared before existence clauses.

1.1.2 There exists ($\exists$ or $\exists!$)

There are two ways we can specify existence: do we know that there can only one such object ($\exists!$), or do we not know how many such objects can exist ($\exists$)?

Consider the statement: *For all integers n there exists another integer m such that $n + m$ is even.* We can write this using the symbols we have learnt, as

$$\forall n \in \mathbb{Z} \exists m \in \mathbb{Z} : n + m \text{ is even}$$

In the above statement, there exist many $m \in \mathbb{Z}$ which can make $n+m$ even for any $n \in \mathbb{Z}$. Take for example $n = 1$. Then when $m = 1$, $n + m = 2$ which is even. But $m = 3$ is possible, since $n + m = 4$ which is also even. Hence, m is not unique, and we can only write $\exists$ without the exclamation mark '!'.

Now, consider our earlier statement $\forall x \exists y : x+y = 0$ where $x, y \in \mathbb{Z}$. Here, y must be unique; each integer possesses a unique additive inverse. Hence, we can emphasize on this uniqueness by writing $\forall x \exists! y : x + y = 0$.

1.2 Definitions, propositions, theorems

Mathematicians use many terms to classify statements which tell us *what* something is, or whether it has been proved. The terminology in use can also tell us about its importance. In this section, we will discuss briefly the terms:

- Theorem
- Definition
- Proposition
- Collorary
- Lemma
- Conjecture

We introduce the first two terms by way of an example.

Definition 1.2.1. A prime number p is a positive integer which is divisible only by 1 and itself.

This leads to the following theorem.

Theorem 1.2.2 (Euclid)**.** *There are infinitely many prime numbers.*

Proof. Suppose that there exists a finite set of primes $\mathcal{P} = \{p_1, p_2, p_3, ..., p_n\}$, where p_k is the k-th prime number. Now consider $j = p_1p_2p_3...p_n + 1$. If j is prime and not in $\mathcal{P}$, there is a contradiction. Otherwise j is divisible by some prime number p_z. But that implies that p_z divides 1, another contradiction. Hence there are infinitely many prime numbers. $\square$

(How this proof is constructed will be seen in Chapter 2.)

As seen above, theorems are proven purely by deductive reasoning, and they are based on other true statements.

Propositions are theorems that are less important; they are considered so trivial that it may be stated without any proof. A collorary is a proposition that is immediately implied by some theorem or other true statement, and a lemma is a proposition mainly suited in some proof. (Note that over time, lemmas may rise in importance to the level of theorems, but the term "lemma" remains in the name. An example is Bézout's lemma.)

Conjectures on the other hand are statements that are generally believed to be true, but lack proof. We introduce this concept by way of another example from [2].

n	$2^n - 1$	Is n prime?	Is $2^n - 1$ prime?
2	3	Yes	Yes
3	7	Yes	Yes
5	31	Yes	Yes
7	127	Yes	Yes
9	511	No	No

Table 1.1: Primes of the form $2^n - 1$.

In the above table, we notice a pattern: if n is prime, then $2^n - 1$ must be prime. Hence, we make a conjecture as follows.

Conjecture 1.2.3. *For any prime p, $2^p - 1$ is prime.*

Let's check the case $n = 11$ to make sure this holds.

$$2^{11} - 1 = 2047 = 23 \times 89$$

Unfortunately, our pattern does not hold. The existence of one counterexample immediately proves our claim false; this is a method of proof detailed in Chapter 2.

For the cases where $2^n - 1$ is prime, such numbers are called *Mersenne primes*. It is conjectured that there are infinitely many such primes.

1.3 Connectives and conditionals

Connectives can be thought of as 'conjunctions', just like in any language.

Suppose that P and Q are two statements. Then we write 'P and Q' as $P \wedge Q$' (the *conjunction* of P and Q), 'P or Q' as $P \vee Q$ (the *disjunction* of P and Q), and 'not P' as $\neg P$ (the *negation* of P).

Now, suppose that if P, then Q. We write this as $P \implies Q$ (P "implies" Q) where P is known as the *antecedent* and Q is known as the *consequent*. (Some authors may annotate $P \to Q$ instead.)

The statement $P \implies Q$ can also be thought of as "if P, then Q".

Example 1.3.1. Write, in logical form, the statement '*If a rose is given to Guy or Guy gets a fiancée, Guy will be happy*', if P stands for the statement '*A rose is given to Guy*', Q stands for the statement '*Guy gets a fiancée*' and R stands for the statement '*Guy will be happy*'. State the antecedent and the consequent.

Solution. $(P \vee Q) \implies R$. The antecedent is $(P \vee Q)$, and the consequent is R. △

Chapter 2

Mathematical Proof

Chapter 3

Problem-solving heuristics

Chapter 4

Introduction to limits

The concept of a limit is extremely fundamental to the understanding of calculus, namely differentiation and integration; their formal definitions are in terms of limits. Here, we will introduce limits for sequences and functions, their arithmetic operations, the squeeze theorem, L'Hopital's rule, and apply it to formally defining the derivative.

4.1 Limits of sequences and functions

4.1.1 Limits of sequences

Suppose we have a sequence

$$a_1, a_2, a_3, a_4, ...$$

If the term a_n converges to a fixed value L as n tends to infinity, then we write

$$\lim_{n\to\infty} a_n = L$$

This is read as "the *limit* of the sequence a_n as n tends to infinity is L".

This concept of a limit can be stated with the following definition:

Definition 4.1.1. If, for any arbitrarily small positive number ε, one can always find a term in a sequence a_n such that $|a_n - L| < \varepsilon$ if $n > N(\varepsilon)$, and $N(\varepsilon)$ is a function of ε.

A more advanced treatment of limits, and how the above definition can prove common limits, can be found in [1].

Take note that there are sequences which diverge (i.e. they do not converge). Consider the harmonic series, where the n-th term is

$$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

Although it increases slowly, it can be proven that

$$\lim_{n\to\infty} H_n = \infty$$

This proof is left as an exercise to the reader.

4.1.2 Limits of functions

Similar to the above interpretation of limits to sequences, suppose that we have a function $f(x)$. If $f(x)$ approaches L as x tends to c from both sides (i.e., from $-\infty$, and from ∞), then $\lim_{x\to c} f(x) = L$.

Example 4.1.2. Find the limit $\lim_{x\to\infty} \frac{1}{x}$.

Solution. The limit is

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

△

A function $f(x)$ is said to be continuous at $x = c$, if $\lim_{x\to c} = f(c)$.

4.2 Operations of limits

Suppose that $\lim_{x\to c} f(x) = A$ and $\lim_{x\to c} g(x) = B$ for the functions $f(x)$, $g(x)$, and A, B are constants. Then:

$$\lim_{x\to c}(f(x) \pm g(x)) = A \pm B,$$

$$\lim_{x\to c}(f(x)g(x)) = AB,$$

$$\lim_{x\to c}\left(\frac{f(x)}{g(x)}\right) = \frac{A}{B},\ B \neq 0,$$

$$\lim_{x\to c} kf(x) = k\lim_{x\to c} f(x) = kA,\ k \text{ is a constant.}$$

Example 4.2.1. Find the limit of ke^{-x}, as x tends to infinity.

Solution. The limit is

$$\lim_{x\to\infty} ke^{-x} = k\lim_{x\to-\infty} e^{-x} = k \times 0 = 0.$$

One can study the behavior of e^{-x} graphically. △

4.3 The squeeze theorem

The squeeze theorem is extremely useful in determining important limits; in fact, it is key to proving the derivative of $\sin(x)$ (with respect to x). Hence, we begin with a statement of the theorem.

Theorem 4.3.1 (Squeeze theorem)**.** *Suppose that we have three functions $f(x), g(x), h(x)$ such that $g(x) \leq f(x) \leq h(x)$. Then*

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L \implies \lim_{x\to c} f(x) = L$$

where c is a constant.

Proof. The proof can be found on Wikipedia. □

Now, we begin with some examples.

Example 4.3.2. Find the limit

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right).$$

Solution. We cannot apply the law

$$\lim_{x\to c}(f(x)g(x)) = AB$$

because $\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$ does not exist. However, since the range of $\sin(x)$ is $[-1,1]$, we can establish the inequality

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1.$$

Multiplying both sides by x^2, we obtain

$$-x^2 \leq x^2\sin\left(\frac{1}{x}\right) \leq x^2.$$

Evaluating the leftmost and rightmost limits by direct substitution,

$$\lim_{x\to 0} x^2 = \lim_{x\to 0}(-x^2) = 0 \implies \lim_{x\to 0} x^2\sin\left(\frac{1}{x}\right) = 0$$

by the squeeze theorem, and we are done. △

4.4 Differentiation from first principles

Differentiation in the 'A'-Level and 'O'-Level has, in the author's experience, only been taught in terms of memorizing formulae and identities. For example, one simply assumes that

$$\frac{d}{dx}\sin(x) = \cos(x).$$

The proof of this 'trivial' identity, is not 'trivial'; it has never ever been covered in any ordinary course. This is because the proof of this identity utilizes Theorem 4.3.1. In fact, most trigonometric identities stem from the aforementioned theorem.

Now, we must start with the very definition of what a derivative is. Instead of introducing the definition directly without further explanation, let us briefly derive the definition of the derivative ourselves.

Recall that the gradient of a straight line y passing through two points (x_1, y_1) and (x_2, y_2) is defined as $m = \frac{y_2 - y_1}{x_2 - x_1}$., where $x_2 > x_1$ and $x, y \in \mathbb{R}$. If $y = f(x)$, then $m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$.

Now, consider the difference between x_2 and x_1; let this difference be $\delta = x_2 - x_1$. Then $x_2 = x_1 + \delta$. Replacing all such occurences of x_2, one obtains

$$m = \frac{f(x_1 + \delta) - f(x_1)}{\delta}.$$

We have learnt that we can draw a tangent at a point in a graph to find the gradient at that point; this is exactly the idea we use here to formally define the derivative. As δ tends to 0, we will get a better approximation of the gradient at the point (x_1, y_1). Alas, using limits, we have the following definition:

Definition 4.4.1. The derivative of a function $f(x)$, with respect to x, is defined as

$$f'(x) = \lim_{\delta \to 0} \frac{f(x + \delta) - f(x)}{\delta}$$

When we apply this definition in finding a derivative, the process is dubbed as *differentiating from first principles.*

Example 4.4.2. Find the derivative of $f(x) = 3x^2 + 2x + 1$.

Solution. By differentiating from first principles, one obtains

$$\begin{aligned}
f'(x) &= \lim_{\delta\to 0} \frac{f(x+\delta)-f(x)}{\delta} \\
&= \lim_{\delta\to 0} \frac{(3(x+\delta)^2+2(x+\delta)+1)-(3x^2+2x+1)}{\delta} \\
&= \lim_{\delta\to 0} \frac{3(x+\delta)^2+2x+2\delta+1-3x^2-2x-1}{\delta} \\
&= \lim_{\delta\to 0} \frac{3(x+\delta)^2+2\delta-3x^2}{\delta} \\
&= \lim_{\delta\to 0} \frac{3(x^2+2x\delta+\delta^2)+2\delta-3x^2}{\delta} \\
&= \lim_{\delta\to 0} \frac{3x^2+6x\delta+3\delta^2+2\delta-3x^2}{\delta} \\
&= \lim_{\delta\to 0} \frac{6x\delta+3\delta^2+2\delta}{\delta} \\
&= \lim_{\delta\to 0} 6x+3\delta+2 \\
&= 6x+3(0)+2 \\
&= 6x+2
\end{aligned}$$

Indeed, this is what we expect since

$$\frac{d}{dx}(3x^2+2x+1) = 6x+2.$$

Hence we are done. $\triangle$

4.5 L'Hopital's rule

4.6 Exercises

Exercise 4.6.1. Prove that

$$\lim_{x \to 0} |f(x)| = 0 \implies \lim_{x \to 0} f(x) = 0$$

using the squeeze theorem. (Hint: $|-f(x)| = |f(x)|$.)

Exercise 4.6.2. It is known that the sum of a geometric series up to n terms is given by

$$S_n = \sum_{k=1}^{n} ar^{k-1} = \frac{a(r^n - 1)}{r - 1}$$

If $|r| < 0$, take the limit of S_n as n approaches ∞.

References

[1] Sheng Gong and Youhong Gong. *Concise Calculus*. WORLD SCIENTIFIC, 2017.

[2] Daniel J. Velleman. *How to Prove It: A Structured Approach*. Cambridge University Press, 3 edition, 2019.