H3 Mathematics 2025

Gerard Sayson

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Contents

Chapter 1

Mathematical Statements

1.1 Quantification

Mathematicians use many symbols to quantify objects in logical statements. Consider the statement

$$\forall x \exists y : x + y = 0$$

Here, $x, y \in \mathbb{Z} \setminus \{0\}$. What this statement is saying is that for all ($\forall$) x, there exists some (one or more) y such that $x+y = 0$.

If we write $\exists y$ alone, it means that there can exist *any number* of such y (including only one!). But in our above statement, only one such y can be the negative of some integer x. To emphasize this uniqueness of y, we further annotate $\exists!$.

1.1.1 For all ($\forall$)

The $\forall$ (read "for all") symbol is used to specify some arbitrary element (which can be anything!) in a set of objects. Consider the set of real numbers $\mathbb{R}$. If we want to state a logical statement $P(x)$ which applies to any x in $\mathbb{R}$, we say $\forall x \in \mathbb{R} : P(x)$.

One usually uses this to declare dummy variables which are used in proofs. Therefore, dummy variables are declared before existence clauses.

1.1.2 There exists ($\exists$ or $\exists!$)

There are two ways we can specify existence: do we know that there can only one such object ($\exists!$), or do we not know how many such objects can exist ($\exists$)?

Consider the statement: *For all integers n there exists another integer m such that $n + m$ is even.* We can write this using the symbols we have learnt, as

$$\forall n \in \mathbb{Z} \exists m \in \mathbb{Z} : n + m \text{ is even}$$

In the above statement, there exist many $m \in \mathbb{Z}$ which can make $n+m$ even for any $n \in \mathbb{Z}$. Take for example $n = 1$. Then when $m = 1$, $n + m = 2$ which is even. But $m = 3$ is possible, since $n + m = 4$ which is also even. Hence, m is not unique, and we can only write $\exists$ without the exclamation mark '!'.

Now, consider our earlier statement $\forall x \exists y : x+y = 0$ where $x, y \in \mathbb{Z}$. Here, y must be unique; each integer possesses a unique additive inverse. Hence, we can emphasize on this uniqueness by writing $\forall x \exists! y : x + y = 0$.

1.2 Definitions, propositions, theorems

Mathematicians use many terms to classify statements which tell us *what* something is, or whether it has been proved. The terminology in use can also tell us about its importance. In this section, we will discuss briefly the terms:

- Theorem
- Definition
- Proposition
- Collorary
- Lemma
- Conjecture

We introduce the first two terms by way of an example.

Definition 1.2.1. A prime number p is a positive integer which is divisible only by 1 and itself.

This leads to the following theorem.

Theorem 1.2.2 (Euclid)**.** *There are infinitely many prime numbers.*

Proof. Suppose that there exists a finite set of primes $\mathcal{P} = \{p_1, p_2, p_3, ..., p_n\}$, where p_k is the k-th prime number. Now consider $j = p_1 p_2 p_3 ... p_n + 1$. If j is prime and not in $\mathcal{P}$, there is a contradiction. Otherwise j is divisible by some prime number p_z. But that implies that p_z divides 1, another contradiction. Hence there are infinitely many prime numbers. $\square$

(How this proof is constructed will be seen in Chapter **??**.)

As seen above, theorems are proven purely by deductive reasoning, and they are based on other true statements.

Propositions are theorems that are less important; they are considered so trivial that it may be stated without any proof. A collorary is a proposition that is immediately implied by some theorem or other true statement, and a lemma is a proposition mainly suited in some proof. (Note that over time, lemmas may rise in importance to the level of theorems, but the term "lemma" remains in the name. An example is Bézout's lemma.)

Conjectures on the other hand are statements that are generally believed to be true, but lack proof. We introduce this concept by way of another example from [2].

n	$2^n - 1$	Is n prime?	Is $2^n - 1$ prime?
2	3	Yes	Yes
3	7	Yes	Yes
5	31	Yes	Yes
7	127	Yes	Yes
9	511	No	No

Table 1.1: Primes of the form $2^n - 1$.

In the above table, we notice a pattern: if n is prime, then $2^n - 1$ must be prime. Hence, we make a conjecture as follows.

Conjecture 1.2.3. *For any prime p, $2^p - 1$ is prime.*

Let's check the case $n = 11$ to make sure this holds.

$$2^{11} - 1 = 2047 = 23 \times 89$$

Unfortunately, our pattern does not hold. The existence of one counterexample immediately proves our claim false; this is a method of proof detailed in Chapter 2.

For the cases where $2^n - 1$ is prime, such numbers are called *Mersenne primes*. It is conjectured that there are infinitely many such primes.

1.3 Connectives and conditionals

Connectives can be thought of as 'conjunctions', just like in any language.

Suppose that P and Q are two statements. Then we write 'P and Q' as $P \wedge Q$' (the *conjunction* of P and Q), 'P or Q' as $P \vee Q$ (the *disjunction* of P and Q), and 'not P' as $\neg P$ (the *negation* of P).

Now, suppose that if P, then Q. We write this as $P \implies Q$ (P "implies" Q) where P is known as the *antecedent* and Q is known as the *consequent*. (Some authors may annotate $P \to Q$ instead.)

The statement $P \implies Q$ can also be thought of as "if P, then Q".

Example 1.3.1. Write, in logical form, the statement '*If a rose is given to Guy or Guy gets a fiancée, Guy will be happy*', if P stands for the statement '*A rose is given to Guy*', Q stands for the statement '*Guy gets a fiancée*' and R stands for the statement '*Guy will be happy*'. State the antecedent and the consequent.

Solution. $(P \vee Q) \implies R$. The antecedent is $(P \vee Q)$, and the consequent is R. △

Chapter 2

Mathematical Proof

Chapter 3

Problem-solving heuristics

Chapter 4

Introduction to limits

The concept of a limit is extremely fundamental to the understanding of calculus, namely differentiation and integration; their formal definitions are in terms of limits. Here, we will introduce limits for sequences and functions, their arithmetic operations, the squeeze theorem, L'Hopital's rule, and apply it to formally defining the derivative.

4.1 Limits of sequences and functions

4.1.1 Limits of sequences

Suppose we have a sequence

$$a_1, a_2, a_3, a_4, \ldots$$

If the term a_n converges to a fixed value L as n tends to infinity, then we write

$$\lim_{n \to \infty} a_n = L$$

This is read as "the *limit* of the sequence a_n as n tends to infinity is L".

This concept of a limit can be stated with the following definition:

Definition 4.1.1. If, for any arbitrarily small positive number ε, one can always find a term in a sequence a_n such that $|a_n - L| < \varepsilon$ if $n > N(\varepsilon)$, and $N(\varepsilon)$ is a function of ε.

A more advanced treatment of limits, and how the above definition can prove common limits, can be found in [1].

Take note that there are sequences which diverge (i.e. they do not converge). Consider the harmonic series, where the n-th term is

$$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

Although it increases slowly, it can be proven that

$$\lim_{n\to\infty} H_n = \infty$$

This proof is left as an exercise to the reader.

4.1.2 Limits of functions

Similar to the above interpretation of limits to sequences, suppose that we have a function $f(x)$. If $f(x)$ approaches L as x tends to c from both sides (i.e., from $-\infty$, and from ∞), then $\lim_{x\to c} f(x) = L$.

Example 4.1.2. Find the limit $\lim_{x\to\infty} \frac{1}{x}$.

Solution. The limit is

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

△

A function $f(x)$ is said to be continuous at $x = c$, if $\lim_{x\to c} = f(c)$.

4.2 Finding limits

The above sections now beg the question, "how do we find what a limit is?" In this section, we will discover the idea of a 'limit', how we can approximate it, and then how we can find the exact value of the limit.

Because some functions may not be defined at some particular value, for example $f(x) = \frac{\sin(x)}{x}$ is not defined at $x = 0$ due to division by zero being undefined. Instead of leaving it undefined, we can attempt to 'complete' the graph of $y = f(x)$, by trying to find some suitable value; this is where limits come in.

The process of 'completing' the graph of $y = f(x)$ at $x = c$ (where $f(c)$ does not exist) will definitely involve studying the *behavior* of the function around $x = c$. Even if $f(c)$ existed, we ignore that, and find our own 'completion' to the graph.

We will start with this idea of 'completing the graph' in the following section.

4.2.1 Evaluation by approximation

Consider the function $f(x) = \frac{\sin(x)}{x}$, where x is in degrees. If we graph it, we get:

In Figure 4.1, y is not defined at $x = 0$ since division by zero is undefined. However, y can be seen approaching 1 as $x \to 0$, **from both sides**. In limit notation, $\lim_{x\to 0} y = 1$.

(One-sided limits, where x approaches from only one side, will be covered later.)

Now, let us define a sequence where the n-th term

$$T_n = \frac{\sin(n)}{n}$$

for $n \in \mathbb{Z}^+$.

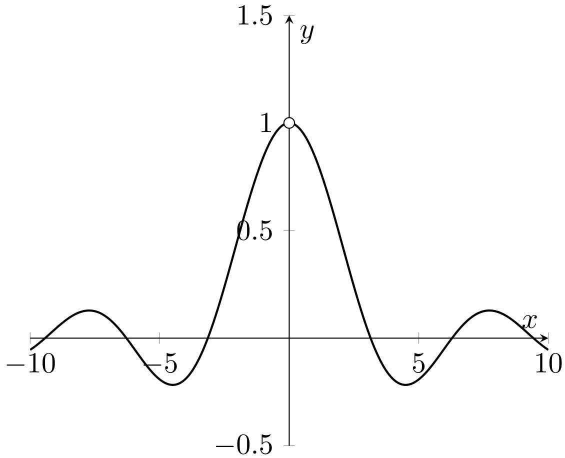

Figure 4.1: Plot of $y = \frac{\sin(x)}{x}$

As seen in Table 4.1, T_n quickly approaches 1 as n approaches zero. In notation,

$$T_n \to 1 \text{ as } n \to 0$$

In other words,

$$\lim_{n\to 0} T_n = \lim_{n\to 0} \frac{\sin(n)}{n} = 1$$

The reason that we can find the limit like so, is because $\frac{\sin(x)}{x}$ actually approaches **only one** value (from both sides), as seen in Figure 4.1.

Contrast this with the function $f(x) = \sin\left(\frac{1}{x}\right)$.

As seen in Figure 4.2, $y = f(x)$ oscillates wildly. It does not even approach a value as $f(x)$ gets smaller; we will leave the verification of this to the reader. Hence, the limit does not exist here, as $f(x)$ never converges to any particular value.

$-\lg(n)$	T_n (approx.)
1	0.99833416647
2	0.999983333417
3	0.99999983333
4	0.99999999833
5	0.99999999983

Table 4.1: Values of $-\lg(n)$ and T_n for $0 \leq n \leq 5$.

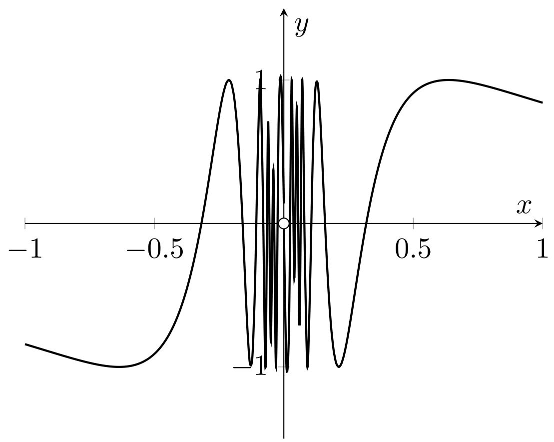

Figure 4.2: Plot of $y = f(x) = \sin\left(\frac{1}{x}\right)$

In fact, the limit does not even need to exist. Consider the sign function $\text{sgn}(x)$, where $\text{sgn}(x) = -1$ for $x < 0$, $\text{sgn}(x) = 1$ for $x > 0$ and $\text{sgn}(x) = 0$ for $x = 0$, and its graph in Figure 4.3.

We know that $\text{sgn}(x)$ is defined everywhere, including 0. However, limits study the behavior of functions *around* some point, not *at* that point.

If we look from the left of Figure 4.2, it seems like $\text{sgn}(x)$ is approaching -1 as x approaches 0 from $-\infty$. However, from the right, it seems like $\text{sgn}(x)$ is approaching 1 as x approaches 0 from $+\infty$ (the positive sign is explicitly used here to emphasize that we approach from the right).

Hence, because $\text{sgn}(x)$ does not settle on only one value when x approaches 0 from both sides (i.e., the values are different, as stated previously), the limit of $\text{sgn}(x)$ as $x \to 0$ **does not exist**.

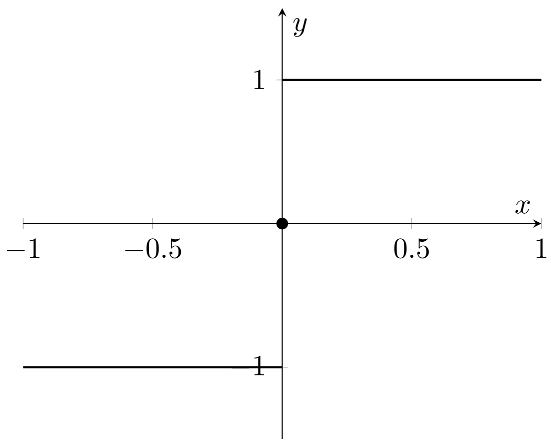

Figure 4.3: Plot of $y = \text{sgn}(x)$

4.3 Operations of limits

Suppose that $\lim_{x\to c} f(x) = A$ and $\lim_{x\to c} g(x) = B$ for the functions $f(x)$, $g(x)$, and A, B are constants. Then:

$$\lim_{x\to c}(f(x) \pm g(x)) = A \pm B,$$

$$\lim_{x\to c}(f(x)g(x)) = AB,$$

$$\lim_{x\to c}\left(\frac{f(x)}{g(x)}\right) = \frac{A}{B},\ B \neq 0,$$

$$\lim_{x\to c} kf(x) = k\lim_{x\to c} f(x) = kA,\ k \text{ is a constant.}$$

Example 4.3.1. Find the limit of ke^{-x}, as x tends to infinity.

Solution. The limit is

$$\lim_{x\to\infty} ke^{-x} = k \lim_{x\to-\infty} e^{-x} = k \times 0 = 0.$$

One can study the behavior of e^{-x} graphically. △

4.4 The squeeze theorem

The squeeze theorem is extremely useful in determining important limits; in fact, it is key to proving the derivative of $\sin(x)$ (with respect to x). Hence, we begin with a statement of the theorem.

Theorem 4.4.1 (Squeeze theorem)**.** *Suppose that we have three functions $f(x), g(x), h(x)$ such that $g(x) \leq f(x) \leq h(x)$. Then*

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L \implies \lim_{x\to c} f(x) = L$$

where c is a constant.

Proof. The proof can be found on Wikipedia. □

Now, we begin with some examples.

Example 4.4.2. Find the limit

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right).$$

Solution. We cannot apply the law

$$\lim_{x \to c}(f(x)g(x)) = AB$$

because $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist. However, since the range of $\sin(x)$ is $[-1, 1]$, we can establish the inequality

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1.$$

Multiplying both sides by x^2, we obtain

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2.$$

Evaluating the leftmost and rightmost limits by direct substitution,

$$\lim_{x \to 0} x^2 = \lim_{x \to 0}(-x^2) = 0 \implies \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$

by the squeeze theorem, and we are done. △

4.5 Differentiation from first principles

Differentiation in the 'A'-Level and 'O'-Level has, in the author's experience, only been taught in terms of memorizing formulae and identities. For example, one simply assumes that

$$\frac{d}{dx}\sin(x) = \cos(x).$$

The proof of this 'trivial' identity, is not 'trivial'; it has never ever been covered in any ordinary course. This is because the proof of this identity utilizes Theorem 4.4.1. In fact, most trigonometric identities stem from the aforementioned theorem.

Now, we must start with the very definition of what a derivative is. Instead of introducing the definition directly without further explanation, let us briefly derive the definition of the derivative ourselves.

Recall that the gradient of a straight line y passing through two points (x_1, y_1) and (x_2, y_2) is defined as $m = \frac{y_2 - y_1}{x_2 - x_1}$., where $x_2 > x_1$ and $x, y \in \mathbb{R}$. If $y = f(x)$, then $m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$.

Now, consider the difference between x_2 and x_1; let this difference be $\delta = x_2 - x_1$. Then $x_2 = x_1 + \delta$. Replacing all such occurences of x_2, one obtains

$$m = \frac{f(x_1 + \delta) - f(x_1)}{\delta}.$$

We have learnt that we can draw a tangent at a point in a graph to find the gradient at that point; this is exactly the idea we use here to formally define the derivative. As δ tends to 0, we will get a better approximation of the gradient at the point (x_1, y_1). Alas, using limits, we have the following definition:

Definition 4.5.1. The derivative of a function $f(x)$, with respect to x, is defined as

$$f'(x) = \lim_{\delta \to 0} \frac{f(x + \delta) - f(x)}{\delta}$$

When we apply this definition in finding a derivative, the process is dubbed as *differentiating from first principles.*

Example 4.5.2. Find the derivative of $f(x) = 3x^2 + 2x + 1$.

Solution. By differentiating from first principles, one obtains

$$
\begin{aligned}
f'(x) &= \lim_{\delta\to 0} \frac{f(x+\delta)-f(x)}{\delta} \\
&= \lim_{\delta\to 0} \frac{(3(x+\delta)^2+2(x+\delta)+1)-(3x^2+2x+1)}{\delta} \\
&= \lim_{\delta\to 0} \frac{3(x+\delta)^2+2x+2\delta+1-3x^2-2x-1}{\delta} \\
&= \lim_{\delta\to 0} \frac{3(x+\delta)^2+2\delta-3x^2}{\delta} \\
&= \lim_{\delta\to 0} \frac{3(x^2+2x\delta+\delta^2)+2\delta-3x^2}{\delta} \\
&= \lim_{\delta\to 0} \frac{3x^2+6x\delta+3\delta^2+2\delta-3x^2}{\delta} \\
&= \lim_{\delta\to 0} \frac{6x\delta+3\delta^2+2\delta}{\delta} \\
&= \lim_{\delta\to 0} 6x+3\delta+2 \\
&= 6x+3(0)+2 \\
&= 6x+2
\end{aligned}
$$

Indeed, this is what we expect since

$$\frac{d}{dx}(3x^2+2x+1) = 6x+2.$$

Hence we are done. $\triangle$

4.6 L'Hôpital's rule

L'Hôpital's rule is a theorem popularly used to evaluate limits of expressions which take, after direct substitution, $\frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$. If we assume that $f(x)$ and $g(x)$ are differentiable everywhere, and they are well defined everywhere, we state L'Hôpitals rule as follows:

Theorem 4.6.1 (L'Hôpital's rule)**.** *For any functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ differentiable everywhere and well-defined everywhere,*

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

if the following conditions hold:

- $\lim_{x\to c} f(x) = \lim_{x \to c} g(x) = 0$ *or* $\pm\infty$
- $g'(x) \neq 0$
- $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ *exists*

where $f(x), g(x)$ *are functions of* x.

4.7 Exercises

Exercise 4.7.1. approximate the limit of $\frac{1}{x}$ as x approaches 0 from both sides (i.e., approximate the left and right limits). Hence, explain why

$$lim_{x\to 0}\frac{1}{x}$$

does not exist.

Exercise 4.7.2. Prove that

$$\lim_{x\to 0}|f(x)| = 0 \implies \lim_{x\to 0} f(x) = 0$$

using the squeeze theorem. (Hint: $|-f(x)| = |f(x)|$. Form an inequality.)

Exercise 4.7.3. It is known that the sum of a geometric series up to n terms is given by

$$S_n = \sum_{k=1}^{n} ar^{k-1} = \frac{a(r^n - 1)}{r - 1}$$

If $|r| < 0$, take the limit of S_n as n approaches ∞.

References

[1] Sheng Gong and Youhong Gong. *Concise Calculus.* WORLD SCIENTIFIC, 2017.

[2] Daniel J. Velleman. *How to Prove It: A Structured Approach.* Cambridge University Press, 3 edition, 2019.