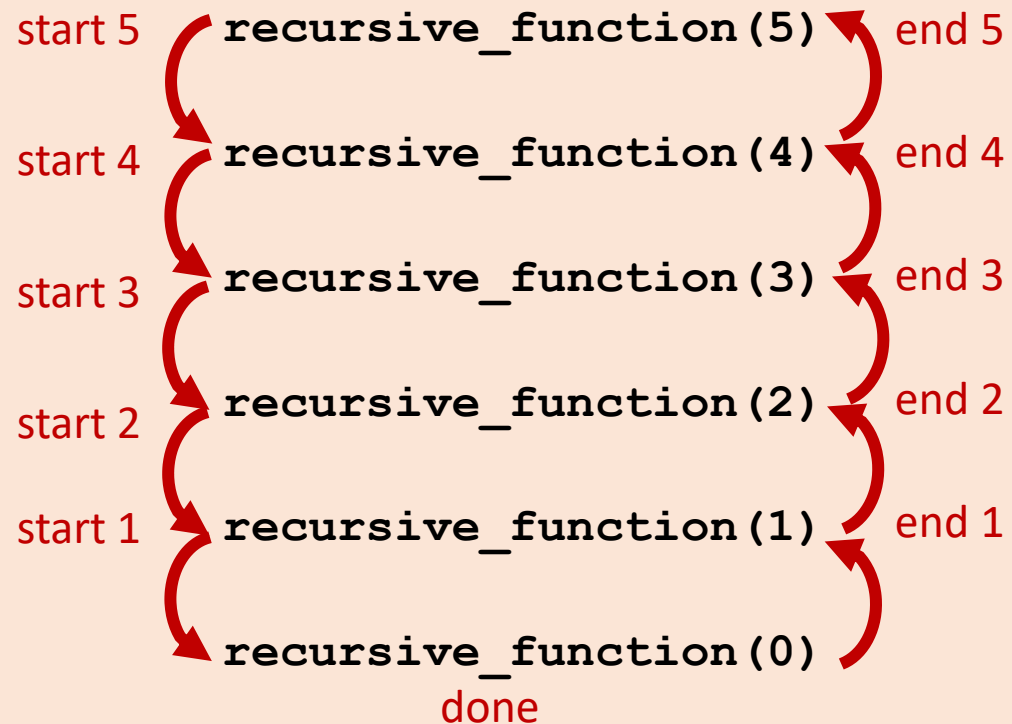


Recursion

- symbol table
- stack frames

Recursion

Recursive function
≡
"function that calls itself"

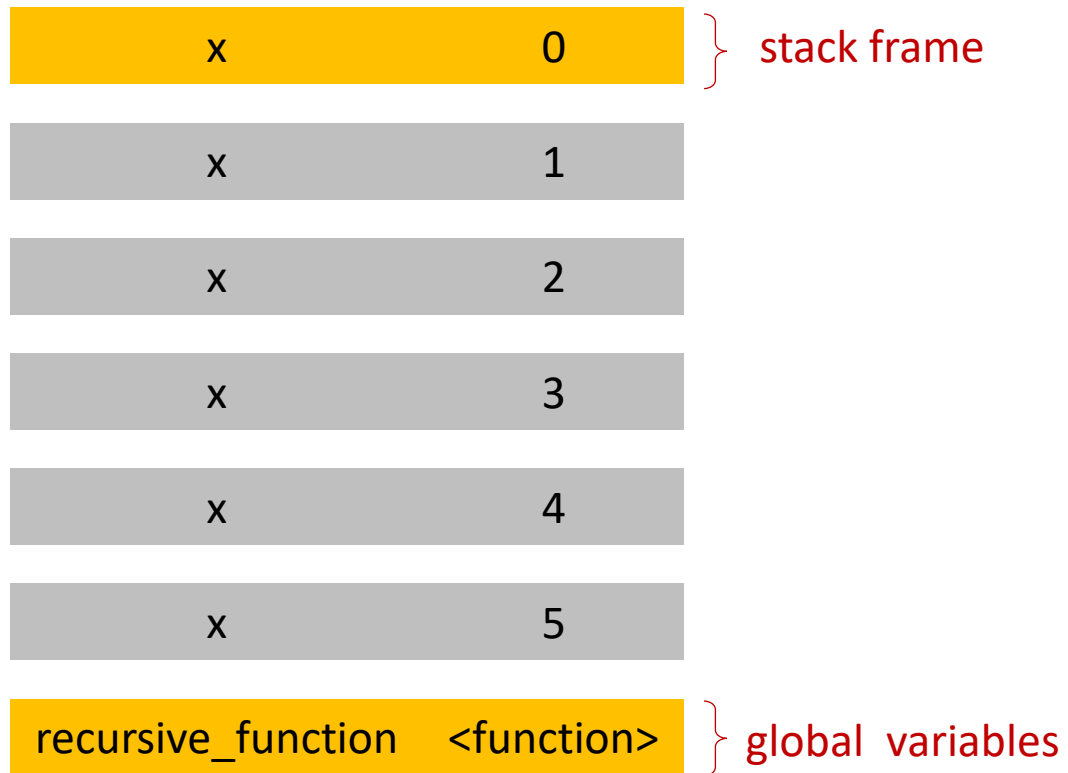


Python shell

```
> def recursive_function(x):
    if x > 0:
        print("start", x)
        recursive_function(x - 1)
        print("end", x)
    else:
        print("done")

> recursive_function(5)
| start 5
| start 4
| start 3
| start 2
| start 1
| done
| end 1
| end 2
| end 3
| end 4
| end 5
```

Recursion



Recursions stack when x = 0 is reached

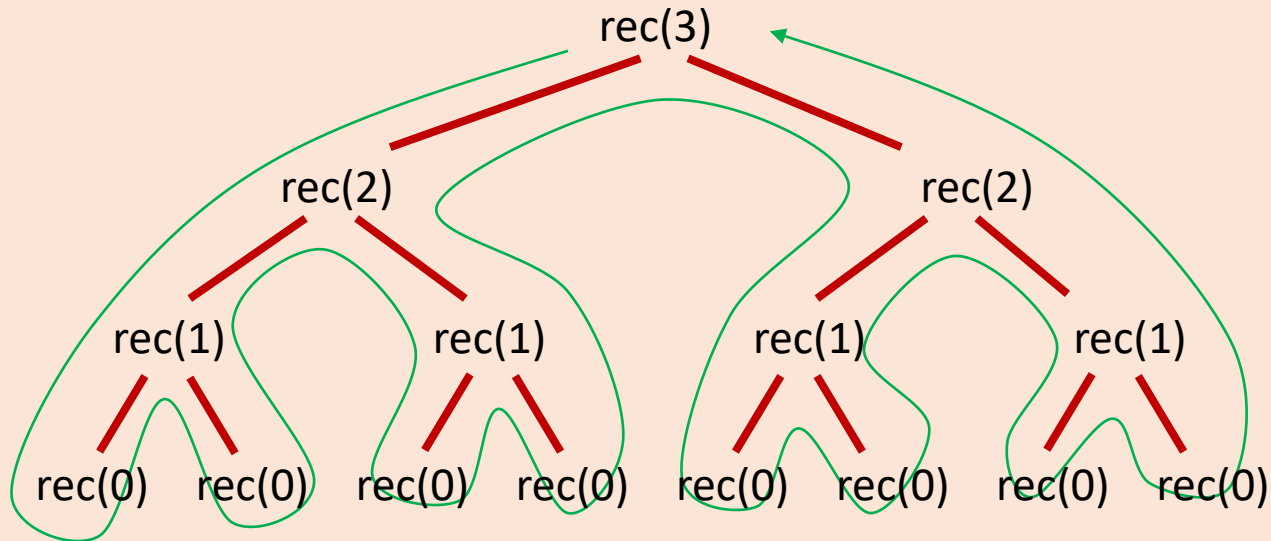
Python shell

```
> def recursive_function(x):  
    if x > 0:  
        print("start", x)  
        recursive_function(x - 1)  
    print("end", x)  
    else:  
        print("done")  
  
> recursive_function(5)  
| start 5  
| start 4  
| start 3  
| start 2  
| start 1  
| done  
| end 1  
| end 2  
| end 3  
| end 4  
| end 5
```

Python shell

```
> def rec(x):  
    if x > 0:  
        print("start", x)  
        rec(x - 1)  
        rec(x - 1)  
        print("end", x)  
    else:  
        print("done")
```

Recursion tree



Python shell

```
> rec(3)  
| start 3  
| start 2  
| start 1  
| done  
| done  
| end 1  
| start 1  
| done  
| done  
| end 1  
| end 2  
| start 2  
| start 1  
| done  
| done  
| end 1  
| end 2  
| end 3
```

Question – How many times does `rec(5)` print "done"?

Python shell

```
> def rec(x):  
    if x > 0:  
        print("start", x)  
        rec(x - 1)  
        rec(x - 1)  
        rec(x - 1)  
        print("end", x)  
    else:  
        print("done")
```

a) 3

b) 5

c) 15

d) 81

e) 125



f) $243 = 3^5$

g) Don't know

Factorial

$$n! = n \cdot \underbrace{(n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1}_{(n-1)!}$$

Observation
(recursive definition)

$$1! = 1$$

$$n! = n \cdot (n-1)!$$

factorial.py

```
def factorial(n):  
    if n <= 1:  
        return 1  
    return n * factorial(n - 1)
```

factorial.py

```
def factorial(n):  
    return n * factorial(n - 1) if n > 1 else 1
```

factorial_iterative.py

```
def factorial(n):  
    result = 1  
    for i in range(2, n + 1):  
        result *= i  
    return result
```

Binomial coefficient $\binom{n}{k}$

- $\binom{n}{k}$ = number of ways to pick k elements from a set of size n

- $$\binom{n}{k} = \begin{cases} 1 & \text{if } k = 0 \text{ or } k = n \\ \binom{n-1}{k} + \binom{n-1}{k-1} & \text{otherwise} \end{cases}$$

`binomial_recursive.py`

```
def binomial(n, k):  
    if k == 0 or k == n:  
        return 1  
    return binomial(n - 1, k) + binomial(n - 1, k - 1)
```

- Unfolding computation shows $\binom{n}{k}$ 1's are added → **slow**

Readable functions ? – return early / bail out fast

- Treat simple cases first and return
- Do not put `else` after `if` ending with `return`
- Avoid unnecessary nesting of code
- 1-liners are not always the most readable code

`binomial_return_early.py`

```
def binomial(n, k): # Ugly, nested indentations and redundant else
    if k == 0:
        return 1
    else:
        if k == n:
            return 1
        else:
            return binomial(n - 1, k) + binomial(n - 1, k - 1)

def binomial(n, k): # Treat each special case first and return
    if k == 0:
        return 1
    if k == n:
        return 1
    return binomial(n - 1, k) + binomial(n - 1, k - 1)

def binomial(n, k): # Several cases simultaneously - is test obvious?
    if k == 0 or k == n:
        return 1
    return binomial(n - 1, k) + binomial(n - 1, k - 1)

def binomial(n, k): # 1-liner, but is this the easiest to read?
    return binomial(n - 1, k) + binomial(n - 1, k - 1) if 0 < k < n else 1
```


Tracing the recursion

- At beginning of function call, **print** arguments
- Before returning, **print** return value
- Keep track of recursion depth in a argument to print **indentation**

binomial_trace.py

```
def binomial(n, k, indent=0):  
    print('    ' * indent + f'binomial({n}, {k})')  
    if k == 0 or k == n:  
        result = 1  
    else:  
        result = binomial(n - 1, k, indent=indent + 1) + \  
            binomial(n - 1, k - 1, indent=indent + 1)  
    print('    ' * indent + f'return {result}')  
    return result
```

Python shell

```
> binomial(4, 2)  
binomial(4, 2)  
    binomial(3, 2)  
        binomial(2, 2)  
        return 1  
        binomial(2, 1)  
            binomial(1, 1)  
            return 1  
            binomial(1, 0)  
            return 1  
        return 2  
    return 3  
    binomial(3, 1)  
        binomial(2, 1)  
            binomial(1, 1)  
            return 1  
            binomial(1, 0)  
            return 1  
        return 2  
        binomial(2, 0)  
        return 1  
    return 3  
return 6  
6
```

Binomial coefficient $\binom{n}{k}$

Observation $\binom{n}{k} = \frac{n!}{(n-k)! \cdot k!}$

```
binomial_factorial.py
```

```
def binomial(n, k):  
    return factorial(n) // factorial(k) // factorial(n - k)
```

- Unfolding computation shows $2n - 2$ multiplications and 2 divisions → **fast**
- Intermediate value $n!$ can have significantly more digits than result (**bad**)

Binomial coefficient $\binom{n}{k}$

Observation $\binom{n}{k} = \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{k \cdot (k-1) \cdot (k-2) \cdots 1} = \binom{n-1}{k-1} \cdot \frac{n}{k}$


`binomial_recursive_product.py`

```
def binomial(n, k):  
    if k == 0:  
        return 1  
    else:  
        return binomial(n - 1, k - 1) * n // k
```

- Unfolding computation shows k multiplications and divisions \rightarrow fast
- Multiplication with fractions $\geq 1 \rightarrow$ intermediate numbers limited size

Questions – Which correctly computes $\binom{n}{k}$?

Observation $\binom{n}{k} = \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{k \cdot (k-1) \cdot (k-2) \cdots 1}$

- a) binomial_A
-  b) binomial_B
- c) both
- d) none
- e) Don't know

binomial_iterative.py

```
def binomial_A(n, k):  
    result = 1  
    for i in range(k):  
        result = result * (n - i) // (k - i)  
    return result  
  
def binomial_B(n, k):  
    result = 1  
    for i in range(k) [::-1]:  
        result = result * (n - i) // (k - i)  
    return result
```

Python shell

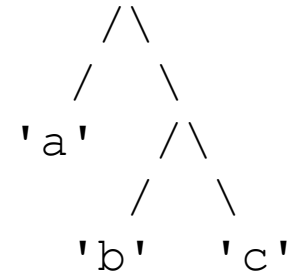
```
> binomial_A(5, 2)  
| 8  
> binomial_B(5, 2)  
| 10
```

Recursively print all leaves of a tree

- Assume a recursively nested tuple represents a tree with strings as leaves

Python shell

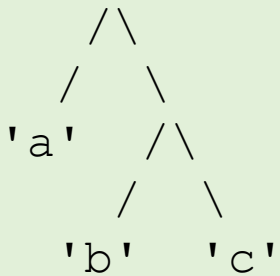
```
> def print_leaves(tree):  
    if isinstance(tree, str):  
        print("Leaf:", tree)  
    else:  
        for child in tree:  
            print_leaves(child)  
  
> print_leaves(('a', ('b', 'c')))  
| Leaf: a  
| Leaf: b  
| Leaf: c
```



Question – How many times is `print_leaves` function called in the example?

Python shell

```
> def print_leaves(tree):  
    if isinstance(tree, str):  
        print("Leaf:", tree)  
    else:  
        for child in tree:  
            print_leaves(child)  
  
> print_leaves(('a', ('b', 'c')))  
| Leaf: a  
| Leaf: b  
| Leaf: c
```



a) 3

b) 4

 c) 5

d) 6

e) Don't know

Python shell

```
> def collect_leaves_wrong(tree, leaves = set()):  
    if isinstance(tree, str):  
        leaves.add(tree)  
    else:  
        for child in tree:  
            collect_leaves_wrong(child, leaves)  
    return leaves
```



```
> def collect_leaves_right(tree, leaves = None):  
    if leaves == None:  
        leaves = set()  
    if isinstance(tree, str):  
        leaves.add(tree)  
    else:  
        for child in tree:  
            collect_leaves_right(child, leaves)  
    return leaves
```

```
> collect_leaves_wrong(('a', ('b', 'c')))  
| {'a', 'c', 'b'}  
> collect_leaves_wrong(('d', ('e', 'f')))  
| {'b', 'e', 'a', 'f', 'c', 'd'}
```

```
> collect_leaves_right(('a', ('b', 'c')))  
| {'b', 'a', 'c'}  
> collect_leaves_right(('d', ('e', 'f')))  
| {'f', 'd', 'e'}
```

Python shell

```
> def collect_leaves(tree):  
    leaves = set()  
  
    def traverse(tree):  
        nonlocal leaves # can be omitted  
        if isinstance(tree, str):  
            leaves.add(tree)  
        else:  
            for child in tree:  
                traverse(child)  
  
    traverse(tree)  
    return leaves  
  
> collect_leaves(('a', ('b', 'c')))  
| {'b', 'a', 'c'}  
> collect_leaves(('d', ('e', 'f')))  
| {'f', 'd', 'e'}
```


Maximum recursion depth ?

- Python's maximum allowed recursion depth can be increased by

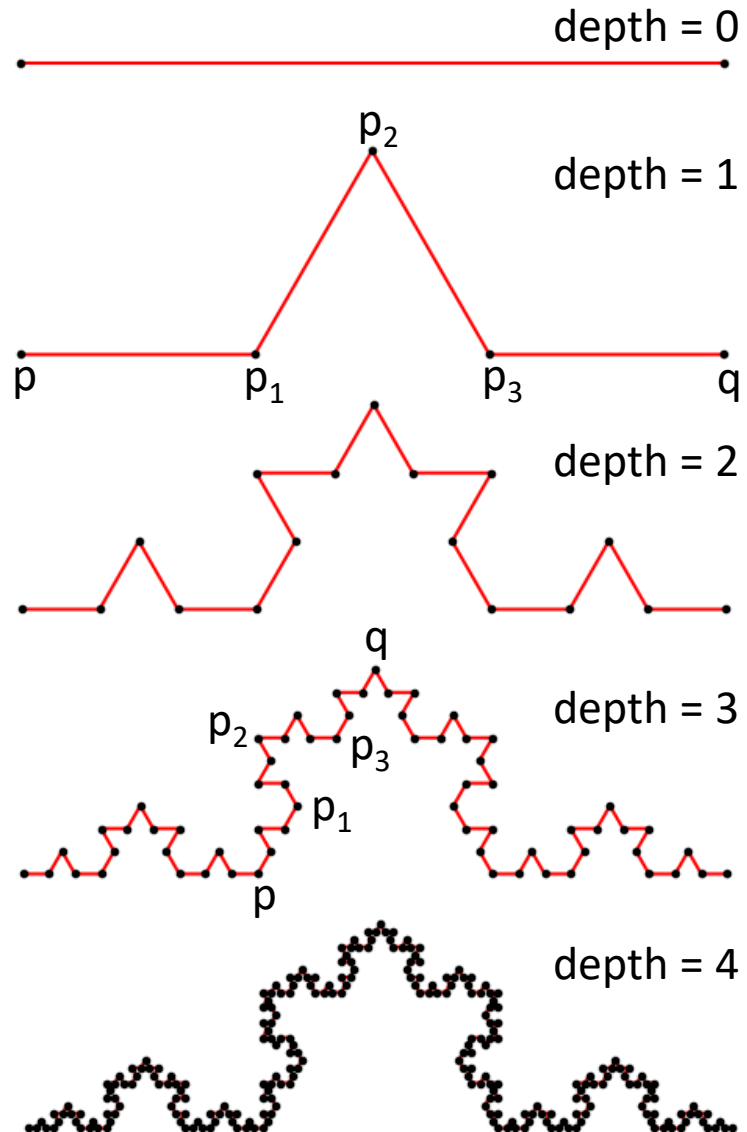
```
import sys
sys.setrecursionlimit(1500)
```

Python shell

```
> def f(x):
    print("#", x)
    f(x + 1)

> f(1)
| # 1
| # 2
| # 3
| ...
| # 975
| # 976
| # 977
| # 978
| RecursionError: maximum
  recursion depth exceeded
  while pickling an object
```

Koch Curves



koch_curve.py

```
import matplotlib.pyplot as plt
from math import sqrt

def koch(p, q, depth=3):
    if depth == 0:
        return [p, q]

    (px, py), (qx, qy) = p, q
    dx, dy = qx - px, qy - py
    h = 1 / sqrt(12)

    p1 = px + dx / 3, py + dy / 3
    p2 = px + dx / 2 - h * dy, py + dy / 2 + h * dx
    p3 = px + dx * 2 / 3, py + dy * 2 / 3

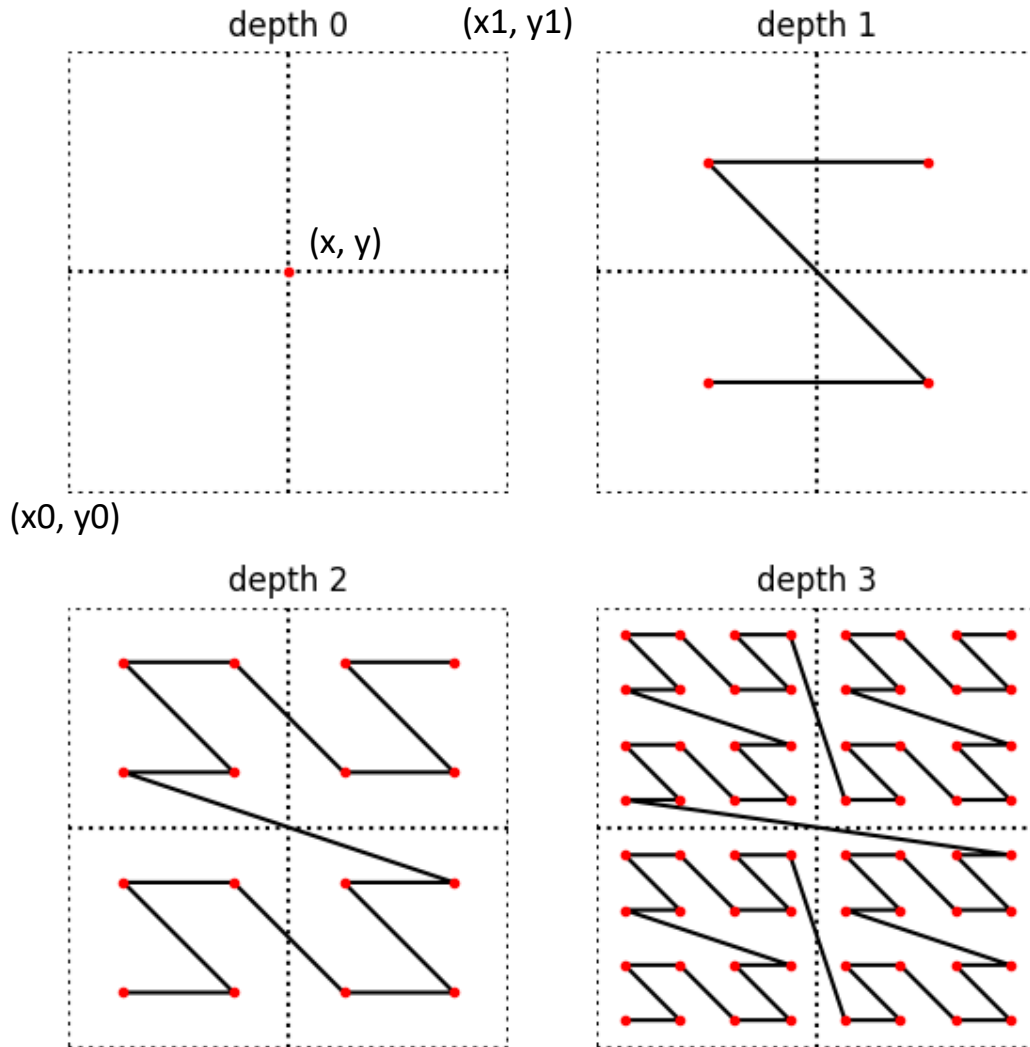
    return (koch(p, p1, depth - 1)[: -1]
            + koch(p1, p2, depth - 1)[: -1]
            + koch(p2, p3, depth - 1)[: -1]
            + koch(p3, q, depth - 1))

points = koch((0, 0), (1, 0), depth=3)
X, Y = zip(*points)

plt.subplot(aspect='equal')
plt.plot(X, Y, 'r-')
plt.plot(X, Y, 'k.')
plt.show()
```

remove last point
(equal to first point in
next recursive call)

Z-curves



z_curve.py

```
import matplotlib.pyplot as plt

def z_curve(depth, x0=0, y0=0, x1=1, y1=1):
    x, y = (x0 + x1) / 2, (y0 + y1) / 2
    if depth == 0:
        return [(x, y)]
    return [
        *z_curve(depth - 1, x0, y0, x, y),
        *z_curve(depth - 1, x, y0, x1, y),
        *z_curve(depth - 1, x0, y, x, y1),
        *z_curve(depth - 1, x, y, x1, y1)
    ]

for depth in range(4):
    X, Y = zip(*z_curve(depth))
    plt.subplot(2, 2, 1 + depth, aspect='equal')
    plt.title(f'depth {depth}')
    plt.axis('off')
    plt.axis([0, 1, 0, 1])
    plt.plot(
        [0,1,1,0,0], [0,0,1,1,0], 'k:', # dash box
        [0.5,0.5], [0,1], 'k:', # dash vertical
        [0,1], [0.5,0.5], 'k:', # dash horizontal
        X, Y, 'k-', # Z-curve
        X, Y, 'r.', # Z-curve points
    )
plt.show()
```