Recursion and iteration

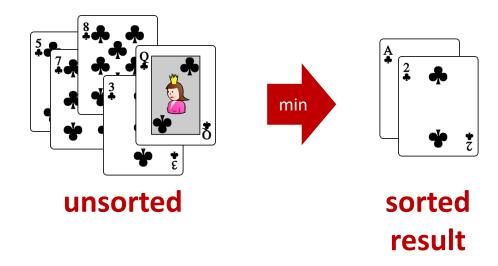
algorithm examples

Standard 52-card deck

	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
Clubs	*	2 4 • • • • •	3 4 4 5 5	44 4 4 4;	\$ * * * * * *	64 4 4 4 4 45	7.4.4. 4.4. 4.4.2	****	9.4.4. 4.4.4. 4.4.4.4.4.4.4.4.4.4.4.4.4.	10 * * * * * * * * * * * * * * * * * * *	i i	\$ 8	K X
Diamonds	* • • • • • • • • • • • • • • • • • • •	2	3 • • • • • • • • • • • • • • • • • • •	4 ♦ ♦	5 * * * * * * * * * *	♦♦♦♦♦	? ◆◆◆ ◆ ◆ ◆	8 4 4 4 8	9	10 • • • • • • • • • • • • • • • • • • •	i	Q &	K X
Hearts	•	2 ♥	3 • • • • • • • • • • • • • • • • • • •	4 V V A A †	\$ \\\\\\ \\\\\\\\\\\\\\\\\\\\\\\\\\\\\\	♥♥♥AABBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBB<	₹ ₩₩₩₩₩₩₩₩₩₩₩₩₩₩₩₩₩₩₩₩₩₩₩₩₩₩₩₩₩₩₩₩₩₩₩₩	8 W W W W W W W W W W W W W W W W W W W	\$\times \times \		i	\$	K X
Spades	•	2 4 • • • • • • • • • • • • • • • • • • •	3 4 4 5 5	4 4 4 4 • • • • •	5 4 4 4 5	6	7.4.4.4.4.4.4.4.4.4.4.4.4.4.4.4.4.4.4.4	**************************************	9.4.4.4.4.4.4.4.4.4.4.4.4.4.4.4.4.4.4.4		i i	\$	K X

Selection sort

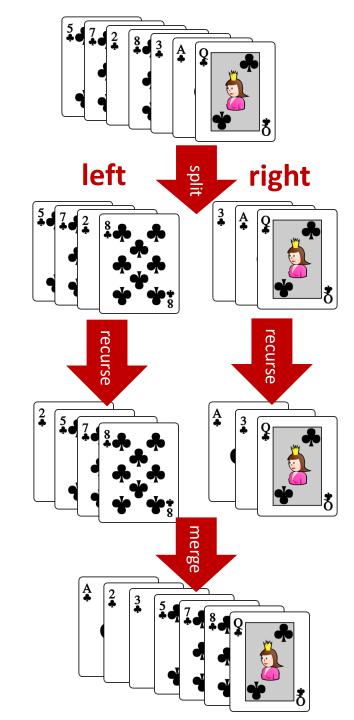
```
selection_sort.py
def selection sort(L):
    unsorted = L[:]
    result = []
    while unsorted:
        e = min(unsorted)
        unsorted.remove(e)
        result.append(e)
    return result
```



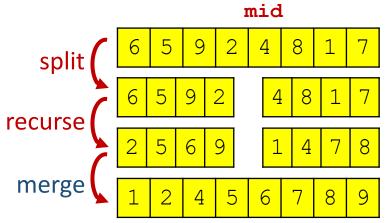
- min and .remove scan the remaining unsorted list for each element moved to result
- order |L|² comparisons

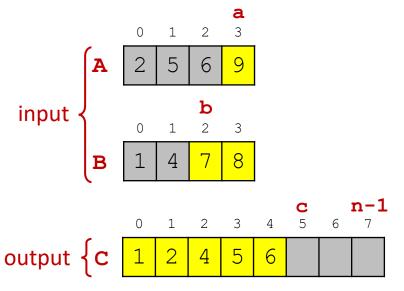
Sorting a pile of cards (Merge sort)

- If one card in pile, i.e. pile is sorted
- Otherwise
 - 1) Split pile into two piles, **left** and **right**, of approximately same size
 - 2) Sort left and right recursively (independently)
 - 3) Merge left and right (which are sorted)



```
merge sort.py
def merge_sort(L):
    n = len(L)
    if n \le 1:
        return L[:]
    mid = n // 2
    left, right = L[:mid], L[mid:]
    return merge (merge sort(left), merge sort(right))
def merge(A, B):
    n = len(A) + len(B)
    C = n * [None]
    a, b = 0, 0
    for c in range(n):
        if a < len(A) and (b == len(B) \text{ or } A[a] < B[b]):
            C[c] = A[a]
            a = a + 1
        else:
            C[c] = B[b]
            b = b + 1
    return C
```

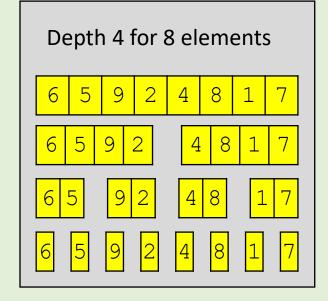




Question – Depth of recursion for 52 elements

- a) 1
- b) 2
- c) 3
- d) 4
- e) 5
- f) 6
- 😶 g) 7
 - h) 8
 - i) 9
 - j) 10
 - k) Don't know

Max recursive subproblem size $52 \rightarrow 26 \rightarrow 13 \rightarrow 7 \rightarrow 4 \rightarrow 2 \rightarrow 1$



Question – Order of comparisons by Merge sort?

```
    a) ~ n
    b) ~ n√n
    c) ~ n log<sub>2</sub> n
    d) ~ n<sup>2</sup>
    e) ~ n<sup>3</sup>
    f) Don't know
```

```
merge sort.py
def merge sort(L):
    n = len(L)
    if n <= 1:
        return L[:]
    else:
        mid = n // 2
        left, right = L[:mid], L[mid:]
        return merge (merge sort(left), merge sort(right))
def merge(A, B):
    n = len(A) + len(B)
    C = n * [None]
    a, b = 0, 0
    for c in range(n):
        if a < len(A) and (b == len(B) \text{ or } A[a] < B[b]):
            C[c] = A[a]
            a = a + 1
        else:
            C[c] = B[b]
            b = b + 1
    return C
```

Merge sort without recursion

- Start with piles of size one
- Repeatedly merge two smallest piles

```
merge sort.py
def merge sort iterative(L):
                                         insert at front of
    Q = [[x] \text{ for } x \text{ in } L]
    while len(Q) > 1;
         Q.insert(0, merge(Q.pop(), Q.pop()))
    return Q[0]
                                              deques are a
                                       generalization of lists
from collections import deque
                                       with efficient updates
def merge sort deque(L):
                                              at both ends
    Q = deque([[x] for x in L])
    while len(Q) > 1:
         Q.appendleft(merge(Q.pop(), Q.pop()))
    return Q[0]
```

```
merge\_sort\_iterative([7,1,9,3,-2,5])
```

Values of Q in while-loop

```
[[7], [1], [9], [3], [-2], [5]]

[[-2, 5], [7], [1], [9], [3]]

[[3, 9], [-2, 5], [7], [1]]

[[1, 7], [3, 9], [-2, 5]]

[[-2, 3, 5, 9], [1, 7]]

[[-2, 1, 3, 5, 7, 9]]
```

Note: Lists in Q appear in non-increasing length order, where longest $\leq 2 \cdot$ shortest

Question – Number of iterations of while-loop?

```
merge_sort_iterative([7, 1, 9, 3, -2, 5])
```

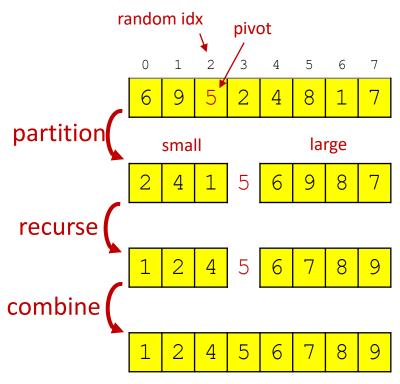
- a) 1
- b) 2
- c) 3
- d) 4
- <u>e</u>)
 - f) 6
 - g) 7
 - h) Don't know

```
merge_sort.py

def merge_sort_iterative(L):
    Q = [[x] for x in L]
    while len(Q) > 1:
        Q.insert(0, merge(Q.pop(), Q.pop()))
    return Q[0]
```

Quicksort (randomized)

```
quicksort.py
import random
def quicksort(L):
    if len(L) <= 1:
        return L[:]
    idx = random.randint(0, len(L) - 1)
    pivot = L[idx]
    other = L[:idx] + L[idx + 1:]
    small = [e for e in other if e < pivot]</pre>
    large = [e for e in other if e >= pivot]
    return quicksort(small) + [pivot] + quicksort(large)
```



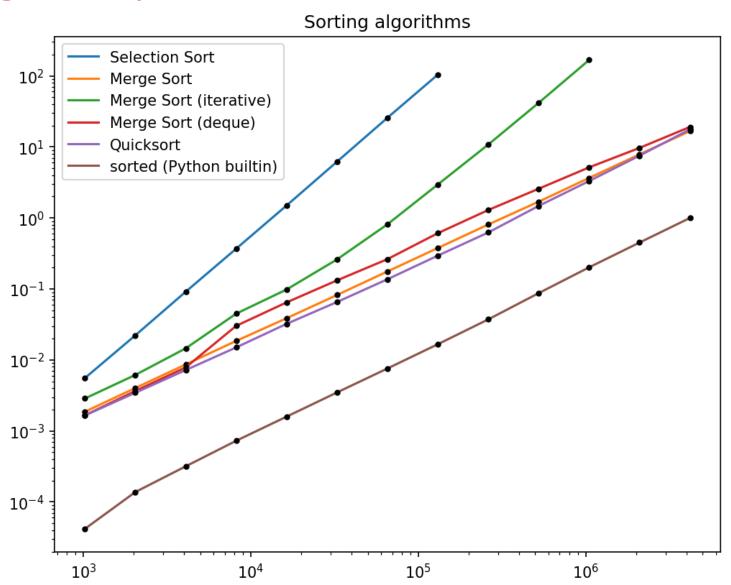
Sorting comparison (single run)

tuned merge-sort (Tim-sort) implementation in C



[L]	Selection sort	Merge sort Recursive	Merge sort Iterative	Merge sort Deque	Quicksort	sorted (Python builtin)
2 ¹⁰	0.006	0.002	0.003	0.002	0.002	0.00004
211	0.02	0.004	0.006	0.000	0.003	0.0001
212	0.09	0.008	0.01	0.008	0.007	0.0003
2 ¹³	0.37	0.02	0.04	0.03	0.02	0.0007
214	1.50	0.04	0.10	0.06	0.03	0.002
2 ¹⁵	6.19	0.08	0.26	0.13	0.07	0.003
2 ¹⁶	25.67	0.18	0.81	0.26	0.14	0.008
2 ¹⁷	104.20 × 4	0.38	2.96	0.61	0.29	0.02
2 ¹⁸		0.81	10.78	1.29	0.62	0.04
2 ¹⁹		1.69	41.71) x 4	2.58	1.48	0.09
2 ²⁰		3.65	167.31 ^{7 × 4}	5.15	3.30	0.20
2 ²¹		7.85		9.68	7.53	0.45
2 ²²		16.69 x 2		19.09 x 2	17.6 x 2	1.00 ² x 2

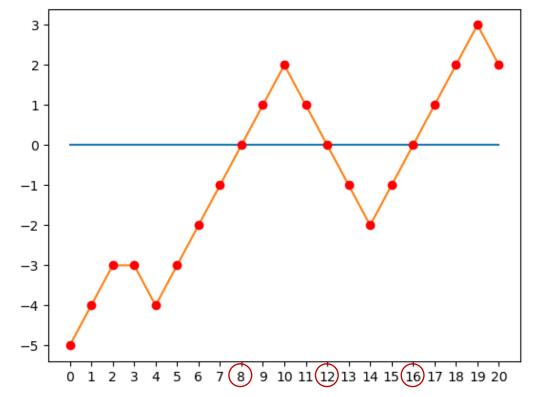
Sorting comparison



Find zero

• Given a list L of integers starting with a negative and ending with a positive integer, and where $|L[i+1] - L[i]| \le 1$, find the position of a zero in L.

L = [-5, -4, -3, -3, -4, -3, -2, -1, 0, 1, 2, 1, 0, -1, -2, -1, 0, 1, 2, 3, 2]



```
find zero.py
def find zero loop(L):
    i = 0
    while L[i] != 0:
        i += 1
    return i
def find zero enumerate(L):
    for i, e in enumerate(L):
        if e == 0:
            return i
def find zero index(L):
    return L.index(0)
     -1
     -2 -
     -3 -
```

0 1 2 3 4 5 6 7 (8) 9 10 11 (12) 13 14 15 (16) 17 18 19 20

```
def find zero binary search(L):
    low = 0
    high = len(L) - 1
    while True: # L[low] < 0 < L[high]
        mid = (low + high) // 2
        if L[mid] == 0:
            return mid
        elif L[mid] < 0:</pre>
            low = mid
        else:
            high = mid
def find zero recursive(L):
    def search(low, high):
        mid = (low + high) // 2
        if L[mid] == 0:
            return mid
        elif L[mid] < 0:</pre>
            return search(mid, high)
        else:
            return search (low, mid)
    return search(0, len(L) - 1)
```

find zero.py

```
def find zero loop(L):
    i = 0
    while L[i] != 0:
        i += 1
    return i
def find zero enumerate(L):
    for i, e in enumerate(L):
        if e == 0:
            return i
def find zero index(L):
    return L.index(0)
```

Function (L = 10 ⁶)	Time, sec
find_zero_loop	0.13
find_zero_enumerate	0.10
find_zero_index	0.015
find_zero_binary_search	0.000015
find_zero_recursive	0.000088

```
def find zero binary search(L):
    low = 0
    high = len(L) - 1
    while True: # L[low] < 0 < L[high]
        mid = (low + high) // 2
        if L[mid] == 0:
            return mid
       elif L[mid] < 0:</pre>
            low = mid
        else:
            high = mid
def find zero recursive(L):
    def search(low, high):
        mid = (low + high) // 2
        if L[mid] == 0:
            return mid
        elif L[mid] < 0:</pre>
            return search(mid, high)
        else:
            return search (low, mid)
    return search(0, len(L) - 1)
```

Greatest Common Divisor (GCD)

Notation

$$x\uparrow y$$
 denotes y is divisible by x, e.g. $3\uparrow 12$ i.e. $y = a \cdot x$ for some integer a

Definition

$$gcd(m, n) = max \{x \mid x \uparrow m \text{ and } x \uparrow n \}$$

Fact

if
$$x\uparrow y$$
 and $x\uparrow z$ then $x\uparrow (y+z)$ and $x\uparrow (y-z)$

Observation

(recursive definition)

$$\gcd(m, n) = \begin{cases} m & \text{if } m = n \\ \gcd(m, n - m) & \text{if } m < n \\ \gcd(m - n, n) & \text{if } m > n \end{cases}$$

gcd(90, 24)

m	n
90	24
66	24
42	24
18	24
18	6
12	6
6	6

Greatest Common Divisor (GCD)

```
gcd.py

def gcd(m, n):
    while n != 0:
        m, n = n, m % n
    return m
```

```
gcd_slow_recursive.py

def gcd(m, n):
    if m == n:
        return m
    elif m > n:
        return gcd(m - n, n)
    else:
        return gcd(m, n - m)
```

```
gcd_recursive.py

def gcd(m, n):
    if n == 0:
        return m
    else:
        return gcd(n, m % n)
```

```
gcd_recursive_one_line.py

def gcd(m, n):
    return m if n == 0 else gcd(n, m % n)
```

Permutations

Generate a list L of all permutations of a tuple

```
permutations.py

def permutations(L):
    if len(L) == 0:
        return [L[:]] # empty tuple (ensures same type as L)
    else:
        P = permutations(L[1:])
        return [p[:i] + L[:1] + p[i:] for p in P for i in range(len(L))]
```

An implementation of permutations exists in the itertools module

Maze solver

Input

- First line #rows and #columns
- Following #rows lines contain strings containing #column characters
- There are exactly one 'A' and one 'B'
- '.' are free cells and '#' are blocked cells

Output

 Print whether there is a path from 'A' to 'B' or not

maze input

```
########################
# . . . . . . # . . . . . # . . . . #
# • # # # • # # # • • • # • # • # • #
# . . . # . . . . . # . # . . . # . #
# . # . # # # . # . # . # . # # # . #
# . # . . . . . # . # . # . . . #
# • # # # # # # # # # # # . # • # • #
# . # . # . . . . . # . . . # . # . #
# . . . . . . . . # . . . . . # . #
# # # # # # # # # # # # # # B # # #
```

Maze solver (recursive)

```
maze solver.py
def explore(i, j):
                                                                                                                                                                                                                                                         def find(symbol):
                   global solution, visited
                                                                                                                                                                                                                                                                             for i, row in enumerate (maze):
                                                                                                                                                                                                                                                                                                j = row.find(symbol)
                                                                                                                                                                                                                                                                                               if i >= 0:
                   if (0 \le i \le n \text{ and } 0 \le j \le m \text{ and } 0 \le j \le
                                                                                                                                                                                                                                                                                                                   return (i, j)
                                      maze[i][j] != '#' and not visited[i][j]):
                                                                                                                                                                                                                                                         n, m = [int(x) for x in input().split()]
                                      visited[i][j] = True
                                                                                                                                                                                                                                                         maze = [input() for i in range(n)]
                                                                                                                                                    maze input
                                      if maze[i][j] == 'B':
                                                                                                                                                     11 19
                                                         solution = True
                                                                                                                                                                                                                                                         solution = False
                                                                                                                                                      visited = [m * [False] for i in range(n)]
                                                                                                                                                      # - - - - - # - - - - # - - - #
                                      explore(i - 1, j)
                                                                                                                                                      # . # # # . # # # . . . # . # . # . #
                                      explore(i + 1, j)
                                                                                                                                                                                                                                                         explore(*find('A'))
                                                                                                                                                      # . . . # . . . . . # . # . . . # . #
                                      explore(i, j - 1)
                                                                                                                                                      # . # . # # # . # . # . # . # # # . #
                                      explore(i, j + 1)
                                                                                                                                                                                                                                                         if solution:
                                                                                                                                                      # . # . . . . . # . # . # . . . #
                                                                                                                                                                                                                                                                           print('path from A to B exists')
                                                                                                                                                      else:
                                                                                                                                                      print('no path')
                                                                                                                                                      # . # . # # # # # . # # # # . # . #
                                                                                                                                                      # - - - - - # - - - - # - #
```

###############

Maze solver (iterative)

```
maze solver iterative.py
def explore(i, j):
                                                                                                                                                                                                                                                                                                                   def find(symbol):
                      global solution, visited
                                                                                                                                                                                                                                                                                                                                         for i, row in enumerate(maze):
                                                                                                                                                                                                                                                                                                                                                                j = row.find(symbol)
                     Q = [(i, j)] \# cells to visit
                                                                                                                                                                                                                                                                                                                                                               if j >= 0:
                                                                                                                                                                                                                                                                                                                                                                                     return (i, j)
                     while Q:
                                           i, j = Q.pop()
                                                                                                                                                                                                                                                                                                                  n, m = [int(x) for x in input().split()]
                                            if (0 \le i \le n \text{ and } 0 \le j \le m \text{ and } 0 \le j \le
                                                                                                                                                                                                                                                                                                                 maze = [input() for i in range(n)]
                                                                 maze[i][j] != '#' and not visited[i][j]):
                                                                                                                                                                                                                                                                                                                   solution = False
                                                                                                                                                                                                                                                                                                                   visited = [m*[False] for i in range(n)]
                                                                 visited[i][j] = True
                                                                  if maze[i][j] == 'B':
                                                                                                                                                                                                                                                                                                                   explore(*find('A'))
                                                                                        solution = True
                                                                                                                                                                                                                                                                                                                   if solution:
                                                                  Q.append((i - 1, j))
                                                                                                                                                                                                                                                                                                                                       print("path from A to B exists")
                                                                  Q.append((i + 1, j))
                                                                                                                                                                                                                                                                                                                   else:
                                                                  Q.append((i, j - 1))
                                                                                                                                                                                                                                                                                                                                       print("no path")
                                                                  Q.append((i, j + 1))
```