

Multi-dimensional data

- NumPy
- matrix multiplication, @
- `numpy.linalg.solve`, `numpy.polyfit`

Array programming with NumPy

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- NumPy is a Python package for dealing with multi-dimensional data

www.numpy.org

docs.scipy.org/doc/numpy/user/quickstart.html

pylab ?

- Guttag [2nd edition] uses pylab in the examples, but...

*“**pylab** is a convenience module that bulk imports **matplotlib.pyplot** (for plotting) and **numpy** (for mathematics and working with arrays) in a single name space. Although many examples use pylab, **it is no longer recommended.**”*

NumPy arrays (example)

Python shell

```
> range(0, 1, .3)
| TypeError: 'float' object cannot be
| interpreted as an integer
> [1 + i / 4 for i in range(5)]
| [1.0, 1.25, 1.5, 1.75, 2.0]
```

} python only supports ranges of int

} generate 5 uniform values in range [1,2]

Python shell

```
> import numpy as np
> np.arange(0, 1, 0.3)
| array([0. , 0.3, 0.6, 0.9])
> type(np.arange(0, 1, 0.3))
| <class 'numpy.ndarray'>
> help(numpy.ndarray)
| +2000 lines of text
> np.linspace(1, 2, 5)
| array([1. , 1.25, 1.5 , 1.75, 2.  ])
```

} numpy can generate ranges with float

} returns a "NumPy array" (not a list)

"arange" \equiv "array range" and generates the array explicitly

} generate n uniformly spaced values

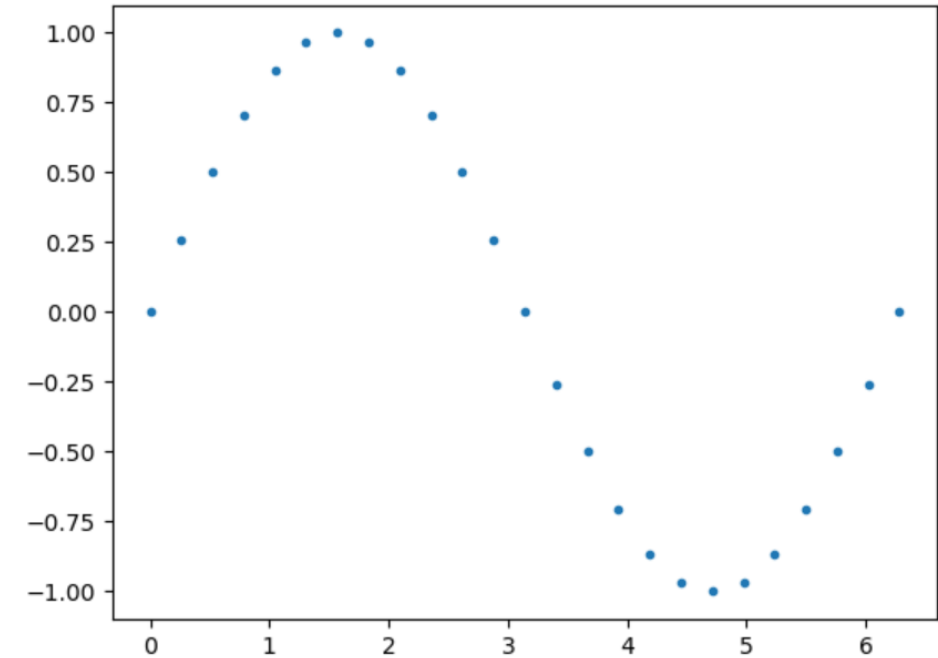
Plotting a function (example)

sin.py

```
import matplotlib.pyplot as plt
import math
n = 25
x = [2 * math.pi * i / (n - 1) for i in range(n)]
y = [math.sin(v) for v in x]
plt.plot(x, y, '.')
plt.show()
```

sin_numpy.py

```
import matplotlib.pyplot as plt
import numpy as np
x = np.linspace(0, 2 * np.pi, 25)
y = np.sin(x)
plt.plot(x, y, '.')
plt.show()
```

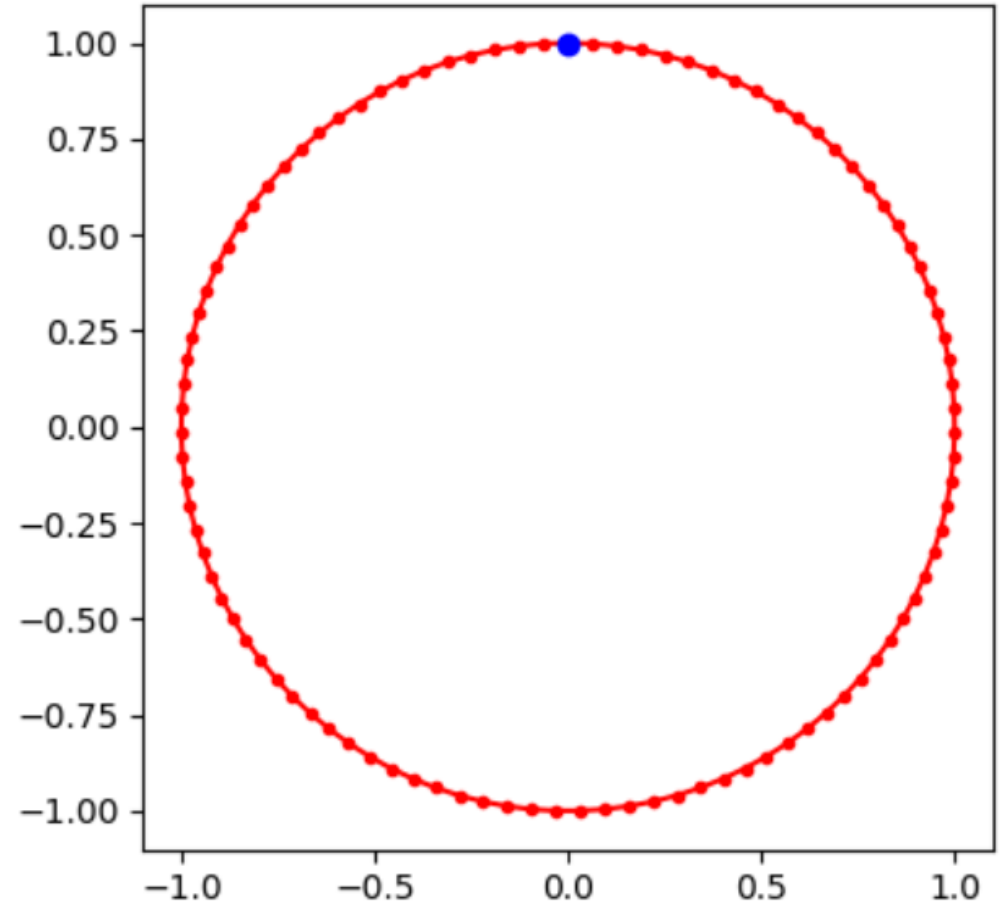


- `np.sin` applies the `sin` function *to each* element of `x`
- `pyplot` accepts NumPy arrays
- `math.pi == np.pi $\approx \frac{22}{7}$`

A circle

circle.py

```
import matplotlib.pyplot as plt
import numpy as np
a = np.linspace(0, 2 * np.pi, 100)
x = np.sin(a)
y = np.cos(a)
plt.plot(x, y, 'r.-')
plt.plot(x[0], y[0], 'bo')
plt.show()
```



Two half circles

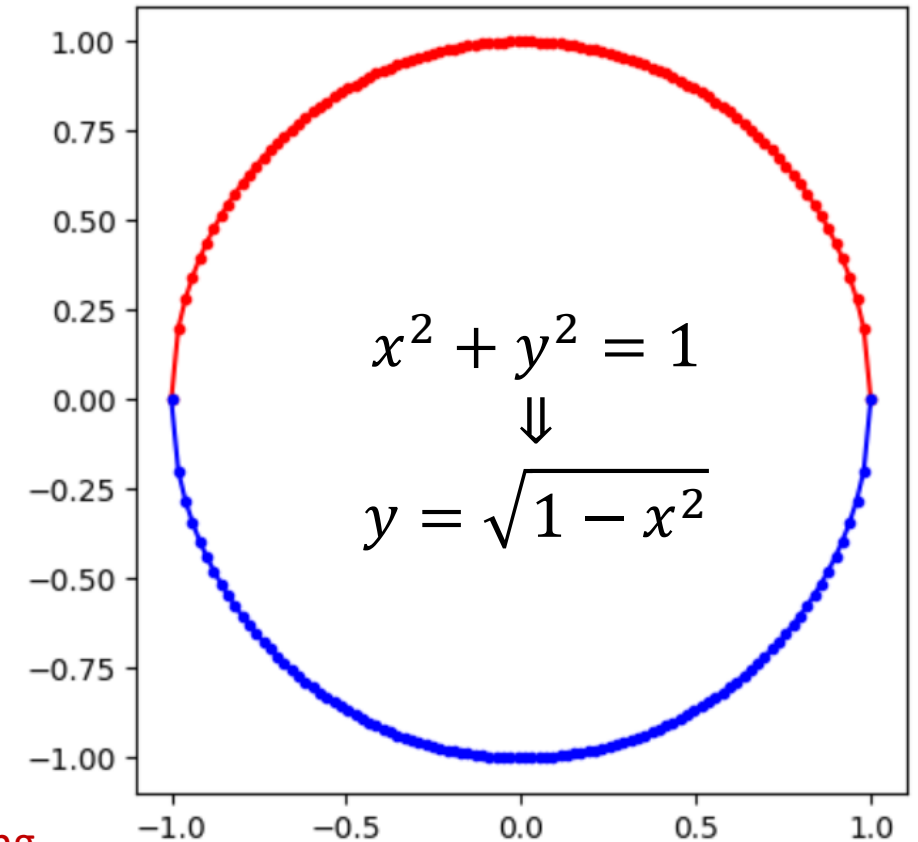
half_circles.py

```
import matplotlib.pyplot as plt
import numpy as np

x = np.linspace(-1, 1, 100)
plt.plot(x, np.sqrt(1 - x ** 2), 'r.-')
plt.plot(x, -np.sqrt(1 - x ** 2), 'b.-')
plt.show()
```

compact expression computing
something quite complicated

- `x` is a NumPy array
- `**` NumPy method `__pow__` squaring each element in `x`
- `binary` – NumPy method `__rsub__` that for each element `e` in `x` computes `1 - e`
- `np.sqrt` NumPy method computing the square root of each element in `x`
- `unary` – NumPy method `__neg__` that negates each element in `x`



Creating one-dimensional NumPy arrays

Python shell

```
> np.array([1, 2, 3])
| array([1, 2, 3])
> np.array((1, 2, 3))
| array([1, 2, 3])
> np.array(range(1, 4))
| array([1, 2, 3])
> np.arange(1., 4., 1.)
| array([1., 2., 3.])
> np.linspace(1, 3, 3)
| array([1., 2., 3.])
> np.zeros(3)
| array([0., 0., 0.])
> np.ones(3)
| array([1., 1., 1.])
> np.full(3, 7)
| array([7, 7, 7])
> np.random.random(3)
| array([0.73761651,
0.60607355, 0.3614118 ])
```

```
> np.array([1, 2, 3]).dtype # type of all values
| dtype('int32')
> np.arange(3, dtype='float')
| array([0., 1., 2.])
> np.arange(3, dtype='int16') # 16 bit integers
| array([0, 1, 2], dtype=int16)
> np.arange(3, dtype='int32') # 32 bit integers
| array([0, 1, 2])
> 1000 ** np.arange(5)
| array([1, 1000, 1000000, 10000000000,
-727379968], dtype=int32) # OOPS.. overflow
> 1000 ** np.arange(5, dtype='O')
| array([1, 1000, 1000000, 10000000000,
10000000000000], dtype=object) # Python integer
> np.arange(3, dtype='complex')
| array([0.+0.j, 1.+0.j, 2.+0.j])
```



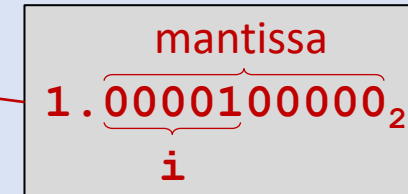
Elements of a NumPy array are not arbitrary precision integers by default – you can select between +25 number representations



Mantissa size in various numpy floats

Python shell

```
> for data_type in ['half', 'float', 'single', 'double', 'longdouble', 'float32', 'float64']:
    x = np.array([1], dtype=data_type)
    for i in range(100):
        if x == x + (x / 2) ** i:
            break
    print(data_type, i - 1, 'bits mantissa')
half 10 bits mantissa
float 52 bits mantissa
single 23 bits mantissa
double 52 bits mantissa
longdouble 52 bits mantissa
float32 23 bits mantissa      # platform independent
float64 52 bits mantissa     # platform independent
```



Creating multi-dimensional NumPy arrays

Python shell

```
> np.array([[1, 2, 3], [4, 5, 6]])  
| array([[1, 2, 3],  
|        [4, 5, 6]])  
> np.arange(1, 7).reshape(2, 3)  
| array([[1, 2, 3],  
|        [4, 5, 6]])  
> x = np.arange(12).reshape(2, 2, 3)  
> x  
| array([[[ 0,  1,  2],  
|         [ 3,  4,  5]],  
|        [[ 6,  7,  8],  
|         [ 9, 10, 11]])]  
> numpy.zeros((2, 5), dtype='int32')  
| array([[0, 0, 0, 0, 0],  
|        [0, 0, 0, 0, 0]])
```

```
> x.size  
| 12  
> x.ndim  
| 3  
> x.shape  
| (2, 2, 3)  
> x.dtype  
| dtype('int32')  
> x.flatten()  
| array([0,1,2,3,4,5,6,7,8,9,10,11])  
> list(x.flat) # flat is an iterator  
| [0,1,2,3,4,5,6,7,8,9,10,11]  
> np.eye(3) # diagonal 2D array  
| array([[1., 0., 0.],  
|        [0., 1., 0.],  
|        [0., 0., 1.]])
```

View vs Copy

- Numpy is optimized to handle big multi-dimensional arrays
- To *avoid copying* data, **views** allows one to look at the underlying data in different ways (data can be shared by multiple views)
- `reshape`, `ravel` and slicing are examples creating views
- `flatten` and `ravel` both turn multiple dimensional arrays into one dimensional arrays but `flatten` creates an explicit copy whereas `ravel` creates a space efficient view

Python shell

```
> x = np.arange(6)
> y = x.reshape(2, 3) # view
> y[0][0] = 42        # updates x
> x
| array([42, 1, 2, 3, 4, 5])
> y
| array([[42, 1, 2],
|        [ 3, 4, 5]])
> z = y.flatten()      # copy
> z[5] = 0
> z
| array([42, 1, 2, 3, 4, 0])
> x
| array([42, 1, 2, 3, 4, 5])
> w = y.ravel()         # view
> w[5] = -1
> w
| array([42, 1, 2, 3, 4, -1])
> x
| array([42, 1, 2, 3, 4, -1])
```

NumPy operations

Python shell

```
> x = numpy.arange(3)
> x
| array([0, 1, 2])
> x + x # elementwise addition
| array([0, 2, 4])
> 1 + x # add integer to each element
| array([1, 2, 3])
> x * x # elementwise multiplication
| array([0, 1, 4])
> np.dot(x, x) # dot product
| 5
> np.cross([1, 2, 3], [3, 2, 1]) # cross product
| array([-4, 8, -4])
```

```
> a = np.arange(6).reshape(2,3)
> a
| array([[0, 1, 2],
|        [3, 4, 5]])
> a.T # matrix transposition, view
| array([[0, 3],
|        [1, 4],
|        [2, 5]])
> a @ a.T # matrix multiplication
| array([[ 5, 14],
|        [14, 50]])
> a += 1
> a
| array([[1, 2, 3],
|        [4, 5, 6]])
```

Universal functions (apply to each entry)

Python shell

```
> x = np.array([[1, 2], [3, 4]])
> np.sin(x) # also: cos, exp, sqrt, log, ceil, floor, abs
| array([[ 0.84147098,  0.90929743],
|         [ 0.14112001, -0.7568025 ]])
> np.sign(np.sin(x))
| array([[ 1.,  1.],
|         [ 1., -1.]])
> np.mod(np.arange(10), 3) # same as: np.arange(10) % 3
| array([0, 1, 2, 0, 1, 2, 0, 1, 2, 0], dtype=int32)
```

Axis

Python shell

```
> x = np.arange(1, 7).reshape(2, 3)
> x
| array([[1, 2, 3],
|        [4, 5, 6]])
> x.sum()    # = x.sum(axis=(0, 1))
| 21
> x.sum(axis=0)
| array([5, 7, 9])
> x.sum(axis=1)
| array([6, 15])
> x.min()    # = x.min(axis=(0, 1))
| 1
```

Python shell

```
> x.min(axis=0)
| array([1, 2, 3])
> x.min(axis=1)
| array([1, 4])
> x.cumsum()
| array([ 1,  3,  6, 10, 15, 21], dtype=int32)
> x.cumsum(axis=0)
| array([[1, 2, 3],
|        [5, 7, 9]], dtype=int32)
> x.cumsum(axis=1)
| array([[ 1,  3,  6],
|        [ 4,  9, 15]], dtype=int32)
```

Slicing

Python shell

```
> x = numpy.arange(20).reshape(4,5)
> x
| array([[ 0,  1,  2,  3,  4],
|        [ 5,  6,  7,  8,  9],
|        [10, 11, 12, 13, 14],
|        [15, 16, 17, 18, 19]])
> x[2, 3] # = x[(2, 3)]
| 13
> x[1:4:2, 2:4:1] # rows 1 and 3, and columns 2 and 3, view
| array([[ 7,  8],
|        [17, 18]])
> x[:, 3]
| array([ 3,  8, 13, 18])
> x[..., 3] # ... is placeholder for ':' for all missing dimensions
| array([ 3,  8, 13, 18])
> type(...)
| <class 'ellipsis'>
```

Broadcasting (stretching arrays to get same size)

- Numpy tries to apply *broadcasting*, if array shapes do not match, i.e. adds missing leading dimensions and repeats a dimension with only one element:

```
[[1],[2]] + [10,20]    column + row vector
≡ [[1],[2]] + [[10,20]] both ndim = 2
≡ [[1,1],[2,2]] + [[10,20]]
≡ [[1,1],[2,2]] + [[10,20],[10,20]]
≡ [[11,21],[12,22]]
```

- To prevent unexpected broadcasting, add an assertion to your program:



```
assert x.shape == y.shape
```

Python shell

```
> x = np.array([[1, 2, 3], [4, 5, 6]])
> y = np.array([1, 2, 3]) # one row
> z = np.array([[1], [2]]) # one column
> x + 3 # add 3 to each entry
| array([[4, 5, 6],
|        [7, 8, 9]])
> x + y # add y to each row
| array([[2, 4, 6],
|        [5, 7, 9]])
> x + z # add z to each column
| array([[2, 3, 4],
|        [6, 7, 8]])
> y + z # 2 rows with y + 3 columns with z
| array([[2, 3, 4],
|        [3, 4, 5]])
> z == z.T # [[1,1],[2,2]] == [[1,2],[1,2]]
| array([[ True, False],
|        [False,  True]])
```



Masking

Python shell


```
> x = np.arange(1, 11).reshape(2, 5)
> x
| array([[ 1,  2,  3,  4,  5],
|        [ 6,  7,  8,  9, 10]])
> x % 3
| array([[1, 2, 0, 1, 2],
|        [0, 1, 2, 0, 1]], dtype=int32)
> x % 3 == 0
| array([[False, False,  True, False, False],
|        [ True, False, False,  True, False]])
> x[x % 3 == 0] # use Boolean matrix to select entries
| array([3, 6, 9])
> x[:, x.sum(axis=0) % 3 == 0] # columns with sum divisible by 3
| array([[ 2,  5],
|        [ 7, 10]])
```

Numpy is fast... but be aware of dtype

Python shell



```
> sum([x**2 for x in range(1000000)])
| 333332833333500000
> (np.arange(1000000)**2).sum()
| 584144992 # wrong since overflow when default dtype='int32'
> (np.arange(1000000, dtype='int64')**2).sum()
| 333332833333500000 # 64 bit integers do not overflow
> import timeit from timeit
> timeit('sum([x**2 for x in range(1000000)])', number=1)
| 0.5614346340007614
> timeit('(np.arange(1000000)**2).sum()', setup='import numpy as np', number=1)
| 0.014362967000124627 # ridiculous fast but also wrong result...
> timeit('(np.arange(1000000, dtype='int64')**2).sum()',
| setup='import numpy as np', number=1)
| 0.048017077999247704 # fast and correct
> np.iinfo(np.int32).min
| -2147483648
> np.iinfo(np.int32).max
| 2147483647
```



numpy.int32 – 32 bit signed two's-complement integers

32 bits $b_{31}b_{30}b_{29}b_{28}\cdots b_2b_1b_0$
represent the value

$$-b_{31} \cdot 2^{31} + \sum_{i=0}^{30} b_i \cdot 2^i$$

10000000000000000000000000000000	-2147483648
100000000000000000000000000000001	-2147483647
⋮	
11111111111111111111111111111110	-2
11111111111111111111111111111111	-1
00000000000000000000000000000000	0
000000000000000000000000000000001	1
⋮	
01111111111111111111111111111110	2147483646
01111111111111111111111111111111	2147483647

Note: There is one more negative number than positive

Python shell

```
> np.int32(- 2 ** 31)
| -2147483648
> np.int32(- 2 ** 31) + 1
| -2147483647
> np.int32(- 2 ** 31) - 1
| 2147483647
> np.int32(2 ** 31)
| OverflowError: Python int too
  large to convert to C long
> np.int32(2 ** 31 - 1)
| 2147483647
> np.int32(2 ** 31 - 1) + 1
| -2147483648
> abs(np.int32(-2147483647))
| 2147483647
> abs(np.int32(-2147483648))
| -2147483648
```



numpy.org/doc/stable/user/basics.types.html

en.wikipedia.org/wiki/Two's_complement

Linear algebra

Python shell

```
> x = np.arange(1, 5, dtype=float).reshape(2, 2)
> x
| array([[1., 2.],
|        [3., 4.]])
> x.T # matrix transpose
| array([[1., 3.],
|        [2., 4.]])
> np.linalg.det(x) # matrix determinant
| -2.0000000000000004
> np.linalg.inv(x) # matrix inverse
| array([[-2. ,  1. ],
|        [ 1.5, -0.5]])
> np.linalg.eig(x) # eigenvalues and eigenvectors
| (array([-0.37228132,  5.37228132]),
|   array([[-0.82456484, -0.41597356], [0.56576746, -0.90937671]]))
> y = np.array([[5.], [7.]])
> np.linalg.solve(x, y) # solve linear matrix equations
| array([[-3.],
|        [ 4.]]) # z1
|               # z2
```



numpy.matrix

"It is no longer recommended to use this class, even for linear algebra. Instead use regular arrays. The class may be removed in the future."

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

Singular value decomposition, `np.linalg.svd`

$$\begin{matrix} \boxed{M} \\ m \times n \end{matrix} = \begin{matrix} \boxed{U} \\ m \times m \end{matrix} \begin{matrix} \boxed{S} \\ m \times n \end{matrix} \begin{matrix} \boxed{V} \\ n \times n \end{matrix}$$

The diagram illustrates the SVD decomposition of matrix M into three matrices: U , S , and V . Matrix U is $m \times m$ and contains r non-zero singular values. Matrix S is $m \times n$ and contains r non-zero singular values. Matrix V is $n \times n$ and contains r non-zero singular values. The rank r is indicated by red boxes and labels in the diagram.

- U and V unitary matrix ($UU^T = I$)
- S diagonal matrix, decreasing singular values

`np.ndarray shape=(520, 800) dtype=uint8 min=0 max=252`

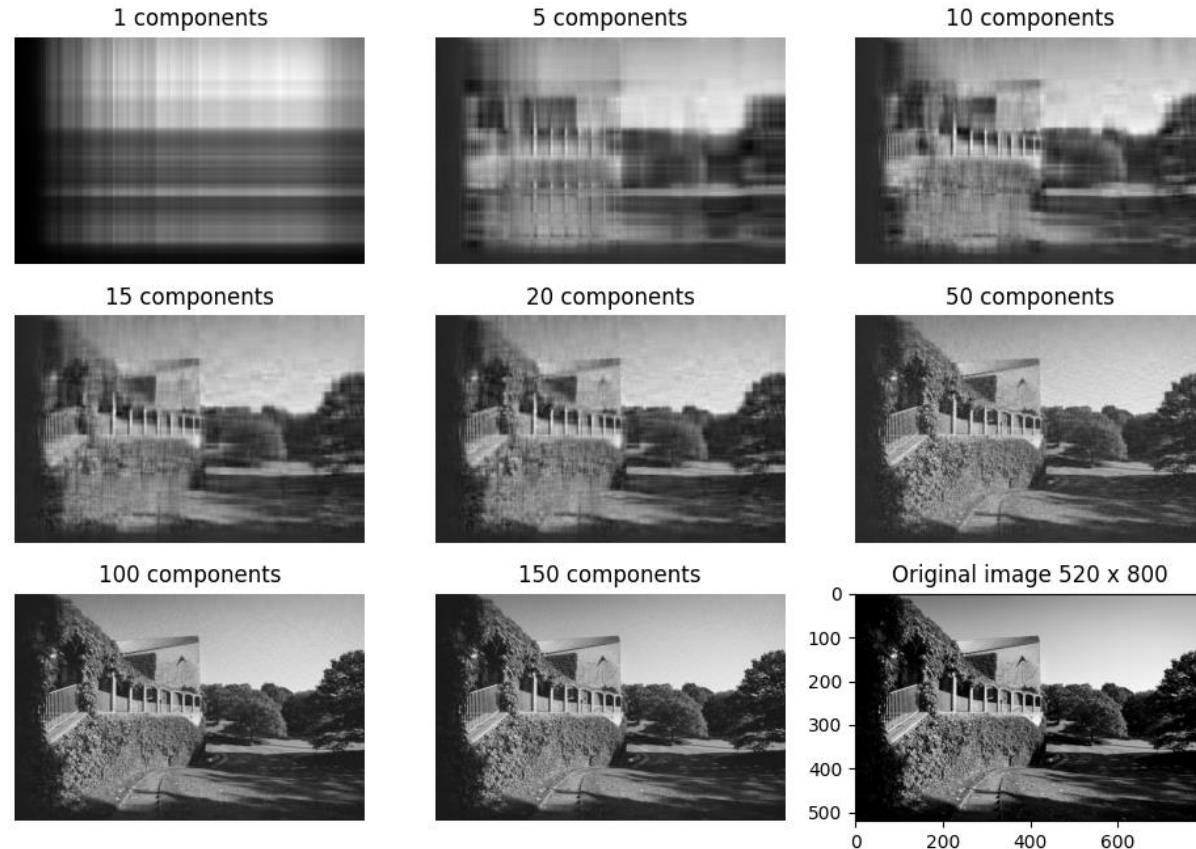
image-reconstruction.py

```
import numpy as np
import cv2 # Computer Vision, opencv.org
import matplotlib.pyplot as plt

color = cv2.imread('university.jpg') # color image
gray = cv2.cvtColor(color, cv2.COLOR_BGR2GRAY) # convert to gray

u, s, v = np.linalg.svd(gray) # Calculating the SVD

for i, r in enumerate([1, 5, 10, 15, 20, 50, 100, 150], start=1):
    rank_r = u[:, :r] @ np.diag(s[:r]) @ v[:, r, :]
    plt.subplot(3, 3, i)
    plt.imshow(rank_r, cmap='gray', vmin=0, vmax=255)
    plt.title(f'{r} components')
    plt.axis('off')
plt.subplot(3, 3, 9)
plt.imshow(gray, cmap='gray')
plt.title(f'Original image {gray.shape[0]} x {gray.shape[1]}')
plt.show()
```



... and in color

image-reconstruction-color.py

```
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.image import imread

color = imread('university.jpg')
color = color / 255 # convert integers 0..255 to floats 0..1

plt.subplot(4, 2, 8)
plt.imshow(color)
plt.axis('off')
plt.title(f'Original')

height, width, colors = color.shape

u, s, v = np.linalg.svd(color.reshape((height, width * colors)),
                        full_matrices=False)

for i, r in enumerate([1, 2, 5, 10, 25, 50, 125], start=1):
    rank_r = u[:, :r] @ np.diag(s[:r]) @ v[:, r, :]
    plt.subplot(4, 2, i)
    plt.imshow(rank_r.reshape((height, width, colors)))
    plt.title(f'{r} components')
    plt.axis('off')
plt.show()
```

shape=(520, 800, 3)
3 = (red, blue, green)

shape=(520, 2400)

v shape=(520, 2400)
instead of (2400, 2400)

shape=(520, 800, 3)

1 components



2 components



5 components



10 components



25 components



50 components



125 components



Original



... and in color (stacked)

image-reconstruction-color-stacked.py

```
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.image import imread

color = imread('university.jpg')
color = color / 255 # convert integers 0..255 to floats 0..1

plt.subplot(4, 2, 8)
plt.imshow(color)
plt.axis('off')
plt.title(f'Original')

u, s, v = np.linalg.svd(color.transpose(2, 0, 1), full_matrices=False)
print(f'{u.shape=} {s.shape=} {v.shape=}')

for i, r in enumerate([1, 2, 5, 10, 25, 50, 125], start=1):
    rank_r = (u[:, :, :r] * s[:, None, :r]) @ v[:, :r, :]
    plt.subplot(4, 2, i)
    plt.imshow(rank_r.transpose(1, 2, 0))
    plt.title(f'{r} components')
    plt.axis('off')
plt.show()
```

Annotations:

- `color = imread('university.jpg')` → `shape = (520, 800, 3)`
- `color.transpose(2, 0, 1)` → `shape = (3, 520, 800)`
- `rank_r = (u[:, :, :r] * s[:, None, :r]) @ v[:, :r, :]` → `element-wise multiplication (*), broadcasting (None), stacked for each color (:)`
- `rank_r.transpose(1, 2, 0)` → `shape = (520, 800, 3)`

Python shell

```
| u.shape=(3, 520, 520) s.shape=(3, 520) v.shape=(3, 520, 800)
```

1 components



2 components



5 components



10 components



25 components



50 components



125 components



Original



matplotlib.image.imread



cv2.imread



cv2.imread corrected



color-image.py

```
import matplotlib.pyplot as plt
import matplotlib.image
import cv2

img1 = matplotlib.image.imread('university.jpg')
img2 = cv2.imread('university.jpg') # cv2 uses BGR instead of RGB
img3 = img2[:, :, ::-1]           # change color order BGR to RGB
Images = [(img1, 'matplotlib.image.imread'), (img2, 'cv2.imread'), (img3, 'cv2.imread corrected')]

for i, (img, title) in enumerate(images, start=1):
    plt.subplot(1, 3, i)
    plt.imshow(img)
    plt.axis('off')
    plt.title(title)

plt.show()
```

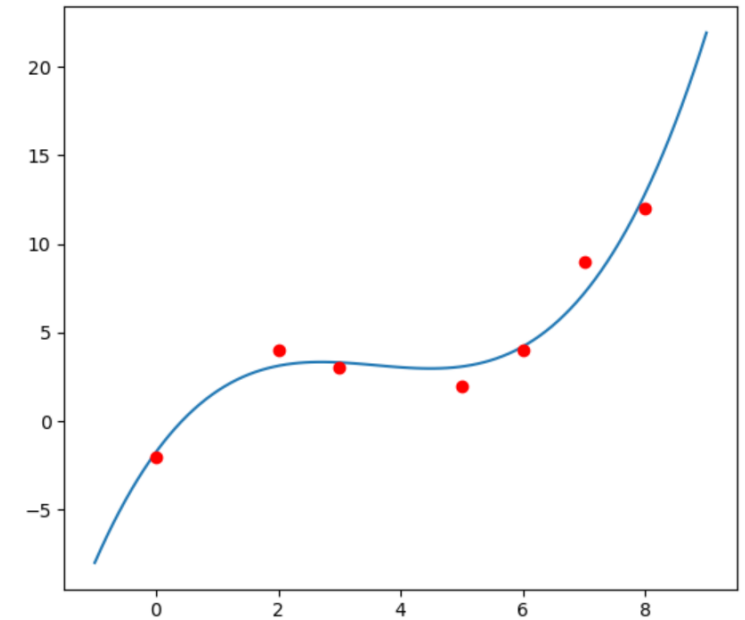


numpy.polyfit

- Given n points with $(x_0, y_0), \dots, (x_{n-1}, y_{n-1})$
- Find polynomial p of degree d that minimizes

$$\sum_{i=0}^{n-1} (y_i - p(x_i))^2$$

- know as least squares fit / linear regression / polynomial regression



fit.py

```
import matplotlib.pyplot as plt
import numpy as np

x = [0, 2, 3, 5, 6, 7, 8]
y = [-2, 4, 3, 2, 4, 9, 12]

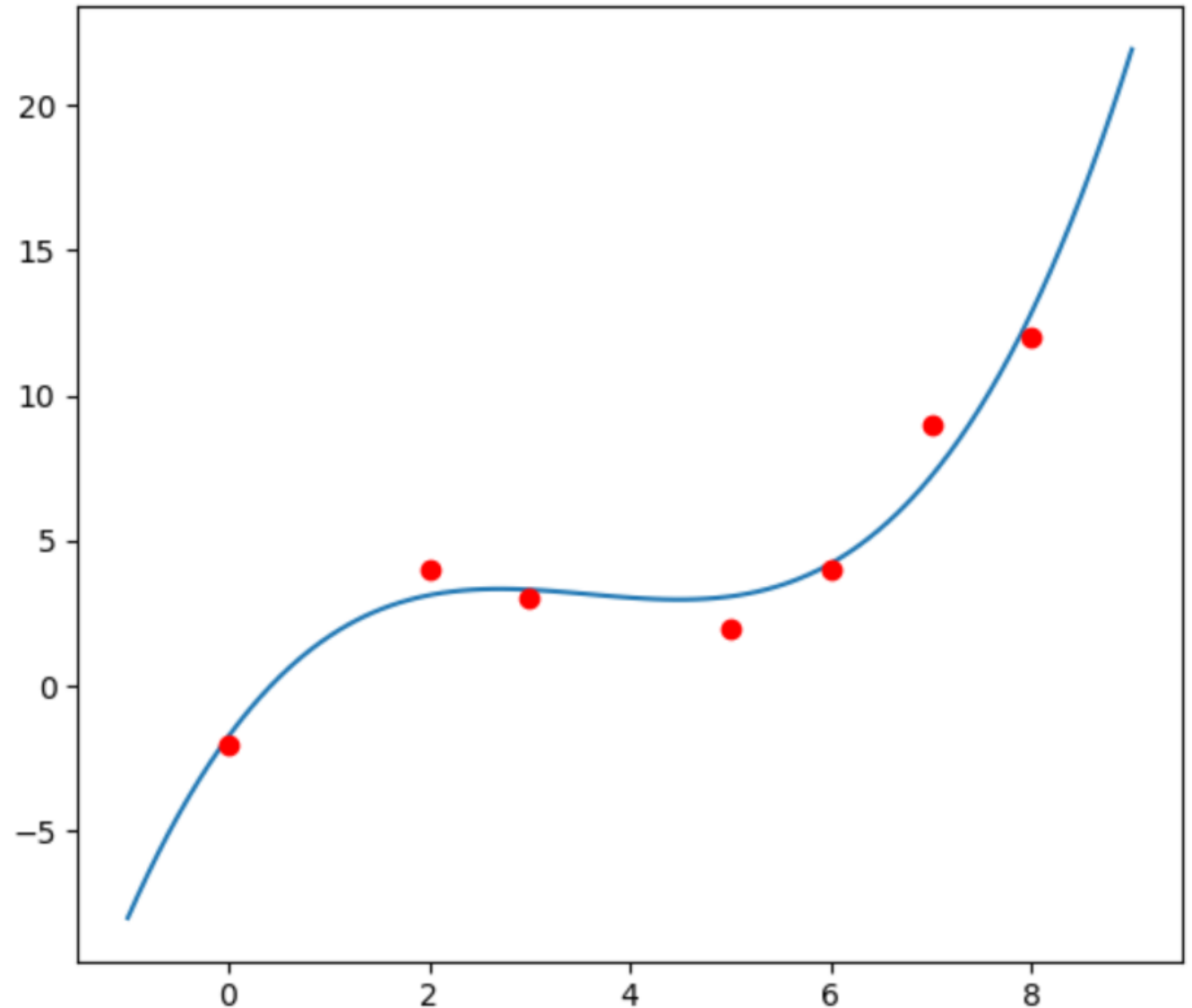
coefficients = np.polyfit(x, y, 3)

fx = np.linspace(-1, 9, 100)
fy = np.polyval(coefficients, fx)

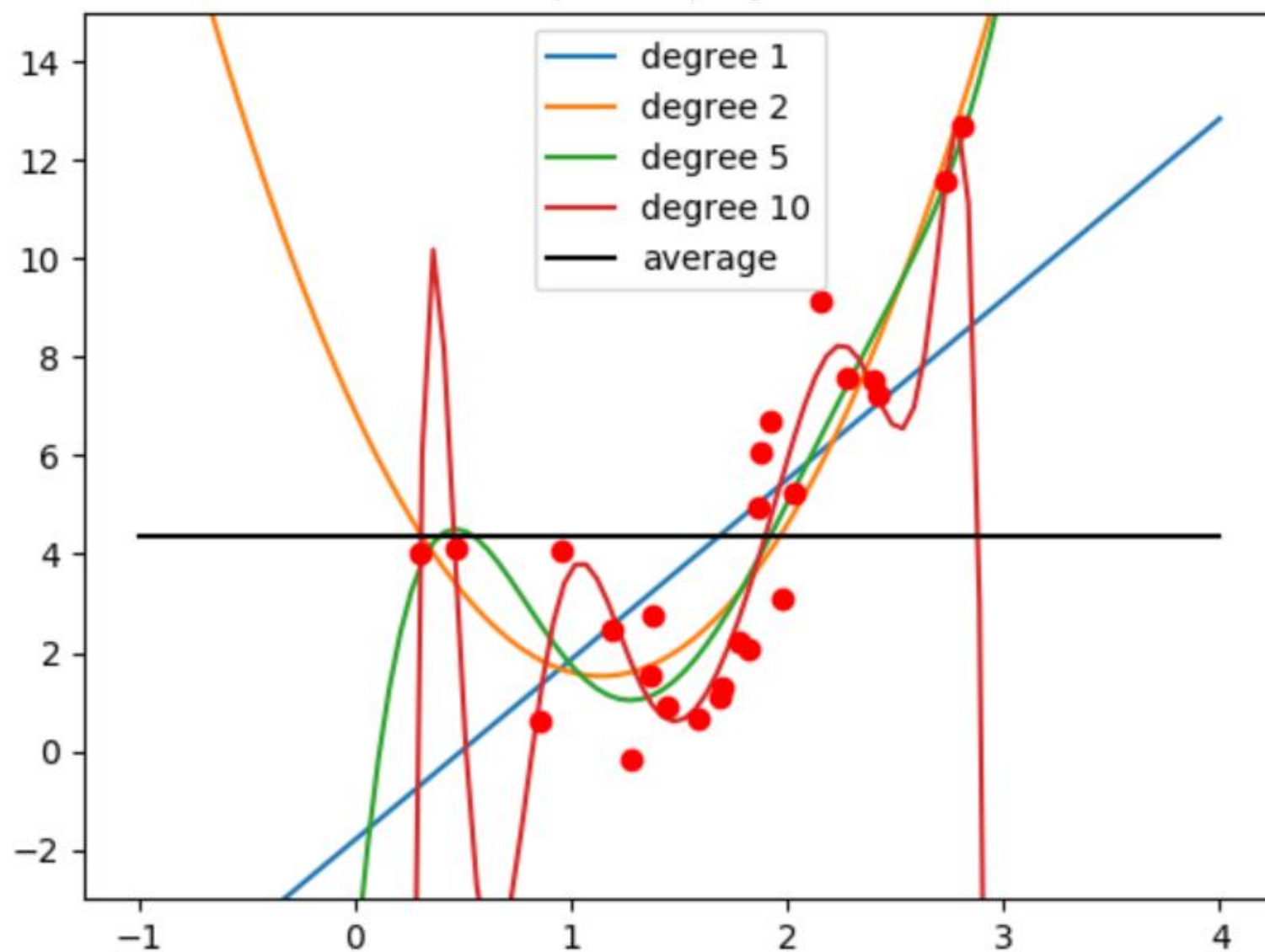
plt.plot(fx, fy, '-')
plt.plot(x, y, 'ro')

plt.show()
```

degree
↓



Least squares polynomial fit



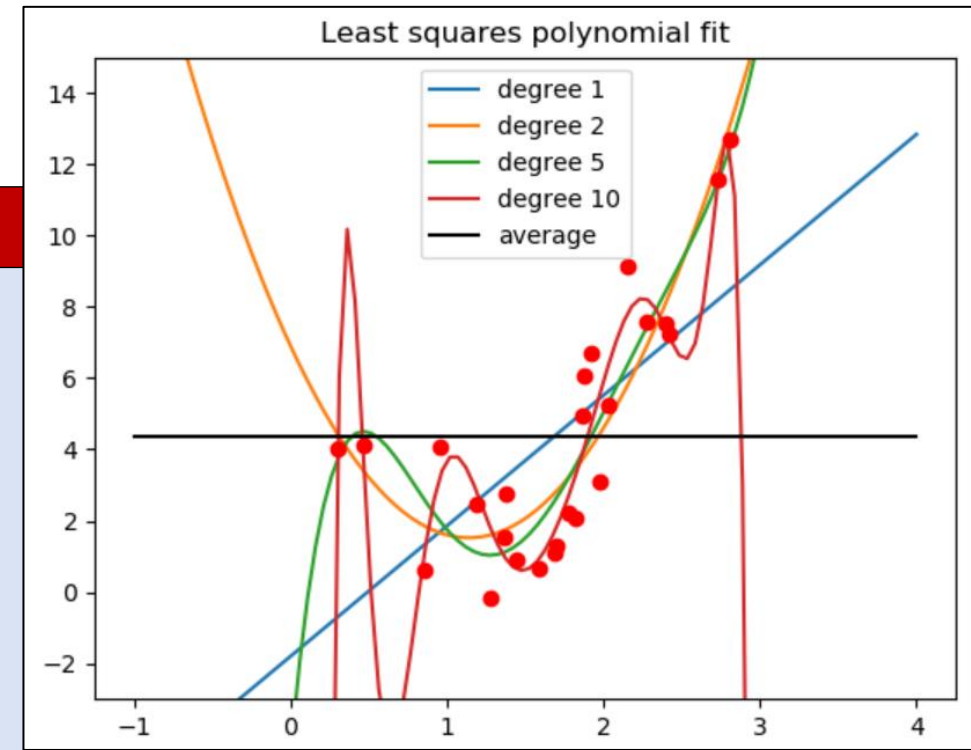
polyfit.py

```
import matplotlib.pyplot as plt
import numpy as np

x = 3 * np.random.random(25)
noise = np.random.random(x.size) ** 2
y = 5 * x ** 2 - 12 * x + 7 + 5 * noise

for degree in [1, 2, 5, 10]:
    coefficients = np.polyfit(x, y, degree)
    fx = np.linspace(-1, 4, 100)
    fy = np.polyval(coefficients, fx)
    plt.plot(fx, fy, '-', label=f'degree {degree}')

avg = np.average(y)
plt.plot(x, y, 'ro')
plt.plot([-1, 4], [avg, avg], 'k-', label='average')
plt.ylim(-3, 15)
plt.title('Least squares polynomial fit')
plt.legend()
plt.show()
```

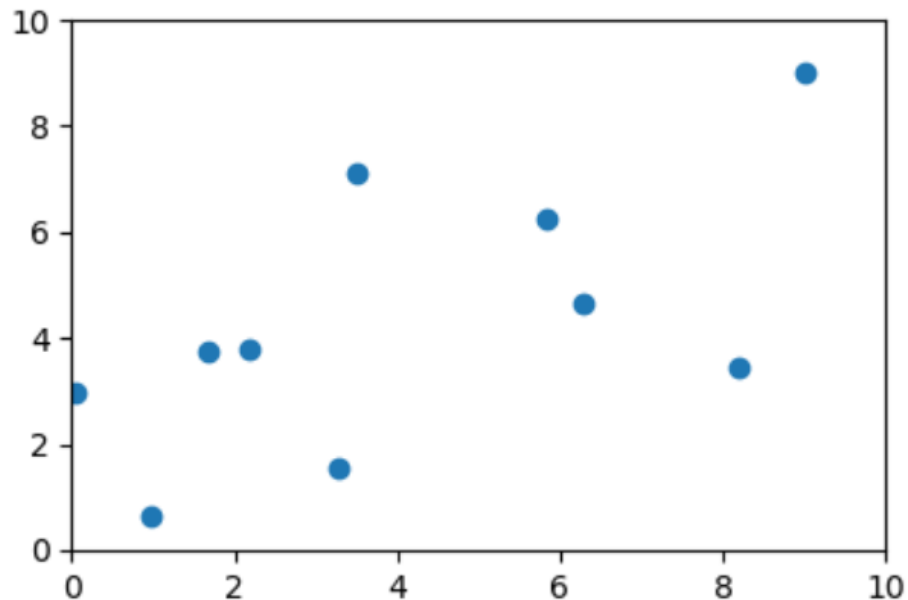


Animating bouncing balls

- matplotlib figures can be animated using

`matplotlib.animation.FuncAnimation`

that as arguments takes the figure to be updated/redrawn, a function to call for each update, and an interval in milliseconds between updates



`balls.py`

```
import matplotlib.pyplot as plt
from matplotlib.animation import FuncAnimation
from numpy import zeros, maximum, minimum
from numpy.random import random

g = 0.01
N = 10
x, y = 10.0 * random(N), 1.0 + 9.0 * random(N)
dx, dy = random(N) / 5, zeros(N)

fig = plt.figure()
plt.xlim(0, 10)
plt.ylim(0, 10)
balls, = plt.plot(x, y, 'o') # returns Line2D obj

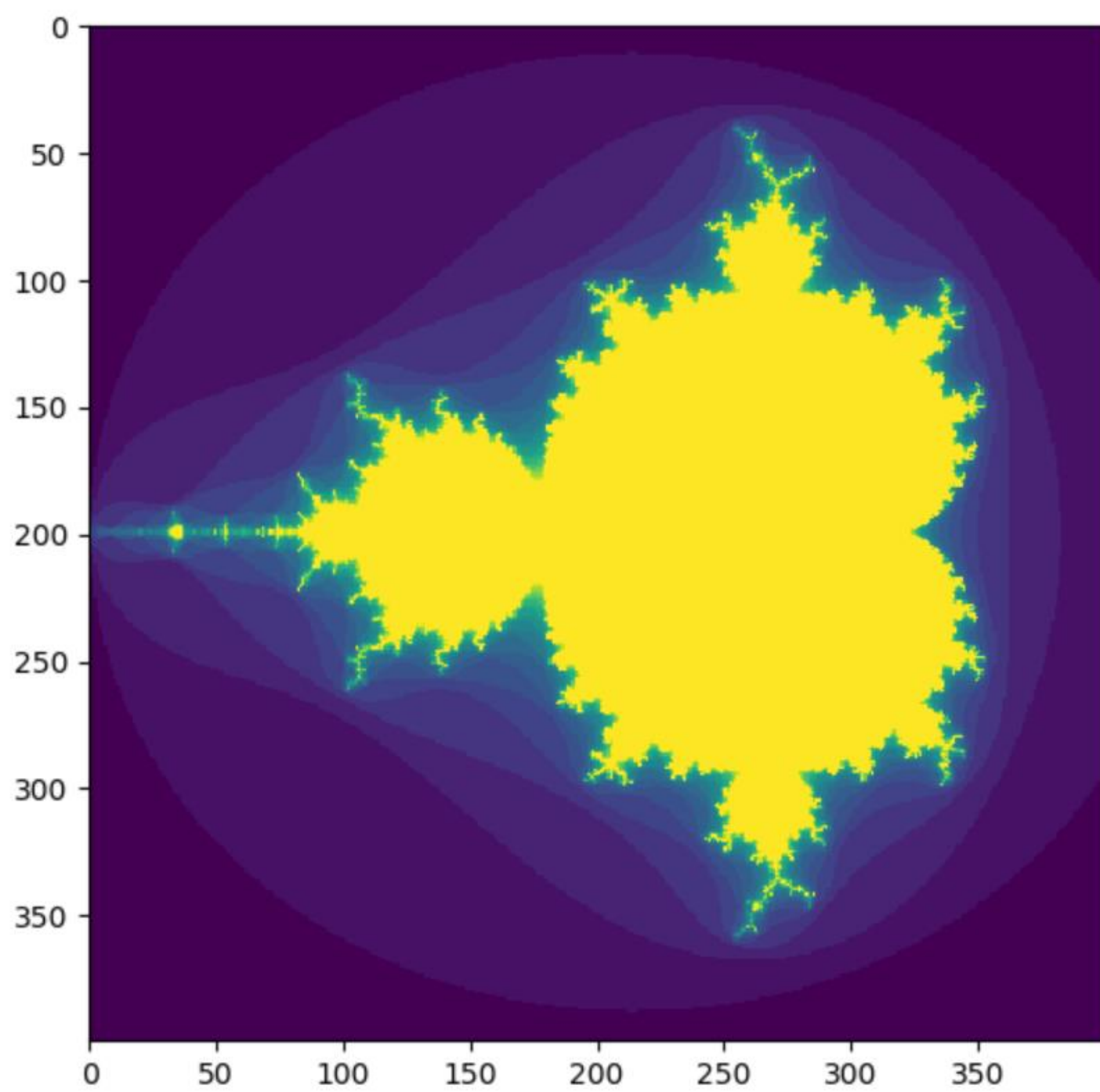
def move(frame):
    global x, y, dx, dy

    x += dx
    bounce = (x > 10.0) | (x < 0.0) # numpy mask
    dx[bounce] = -dx[bounce]
    x = minimum(10.0, maximum(0.0, x))

    y += dy
    bounce = y < 0.0 # numpy mask
    y[bounce] -= dy[bounce]
    dy[bounce] = -dy[bounce]
    dy -= g

    balls.set_data(x, y) # update positions

# removing 'ani =' causes program to fail...
ani = FuncAnimation(fig, move, interval=25)
plt.show()
```



mandelbrot.py

```
import numpy as np
import matplotlib.pyplot as plt

def mandelbrot(h, w, maxit=20):
    '''Returns an image of the Mandelbrot fractal of size (h, w).'''
    x = np.linspace(-2.0, 0.8, w).reshape(1, w) # row vector
    y = np.linspace(-1.4, 1.4, h).reshape(h, 1) # column vector
    c = x + y * 1j # broadcast & complex
    z = c
    divtime = np.full(z.shape, maxit, dtype=int) # all values = maxit
    for i in range(maxit):
        z = z * z + c # elementwise
        diverge = z * np.conj(z) > 4 # who is diverging
        div_now = diverge & (divtime == maxit) # who is diverging now
        divtime[div_now] = i # note when
        z[diverge] = 0 # limit divergence
    return divtime # (avoids overflows)

plt.imshow(mandelbrot(400, 400))
plt.show()
```

