Let P, Q be 2 point clouds with confidence w_i's in space $\mathbb{R}^d$ i.e. p_i, q_i are d dimension points

to find R, t s.t

$$\sum_{i=1}^{n} w_i \| (R p_i + t) - q_i \|^2$$

is min. & $R^T R = I$
rotational matrix

i.e.

$$(R, t) = \underset{R, t}{\operatorname{argmin}} \sum_{i=1}^{n} w_i \| (R p_i + t) - q_i \|^2$$

Objective func^n:

$$F(R, t) = \sum_{i=1}^{n} w_i \| (R p_i \;\; t) - q_i \|^2$$

Differentiating w.r.t t & = 0
for min

$$\frac{\partial F}{\partial t} = \sum_{i=1}^{n} 2 w_i \, (R p_i + t - q_i)$$

$$= 0$$

$$\Rightarrow R \sum_{i=1}^{n} w_i p_i - \sum_{i=1}^{n} w_i q_i + t = 0$$

as $(\sum w_i = 1)$

let $p_0 = \dfrac{\sum_{i=1}^{n} w_i p_i}{\sum_{i=1}^{n} w_i}$ $q_0 = \dfrac{\sum_{i=1}^{n} w_i q_i}{\sum_{i=1}^{n} w_i}$

$\Rightarrow t = q_0 - R p_0$

$\Rightarrow$ Optimal translation maps weighted centroid P to weighted centroid of Q

plug t to $F(R, t)$

$$F(R) = \sum_{i=1}^{n} w_i \| (R p_i + t) - q_i \|^2$$

$$= \sum_{i=1}^{n} w_i \| R p_i + q_0 - R p_0 - q_i \|^2$$

$$= \sum_{i=1}^{n} w_i \| R(p_i - p_0) - (q_i - q_0) \|^2$$

assume $x_i = p_i - p_0$

$y_i = q_i - q_0$

$$F(R) = \sum_{i=1}^{n} w_i \| R x_i - y_i \|^2$$

$$\|Rx_i - y_i\|^2 = (Rx_i - y_i)^T (Rx_i - y_i)$$

$$= (x_i^T R^T - y_i^T)(Rx_i - y_i)$$

$$= x_i^T \overbrace{R^T R}^{I} x_i - x_i^T R^T y_i - y_i^T R x_i + y_i^T y_i$$

$$= x_i^T x_i - x_i^T R^T y_i - y_i^T R x_i + y_i^T y_i$$

Shape of $x_i^T R^T y_i$

$x_i \to d\times 1$

$R \to d\times d$

$y_i = d\times 1$

$$\underbrace{(1\times d)(d\times d)}\underbrace{(d\times 1)}$$

$$\to (1\times 1)$$

$$\therefore (x_i^T R^T y_i)^T = x_i^T R^T y_i$$

$$y_i^T R x_i = x_i^T R^T y_i$$

$$\Rightarrow \|Rx_i - y_i\|^2 = x_i^T x_i + y_i^T y_i - 2 y_i^T R x_i$$

$$R = \arg\min_R w_i \|Rx_i - y_i\|^2$$

$$= \arg\min_R w_i \left(x_i^T x_i + y_i^T y_i - 2 y_i^T R x_i \right)$$

$$= \arg\min_R w_i (-2 y_i^T R x_i)$$

$$= \arg\max_R (w_i y_i^T R x_i)$$

let $W = \begin{bmatrix} w_1 & 0 & 0 & \cdots \\ 0 & w_2 & 0 & \cdots \\ 0 & 0 & w_3 & \cdots \\ & & & \ddots \\ & & & & w_n \end{bmatrix}$

$Y = \begin{bmatrix} | & | & | & & | \\ y_1 & y_2 & y_3 & \cdots & y_n \\ | & | & | & & | \end{bmatrix}$

$X = \begin{bmatrix} | & | & | & & | \\ x_1 & x_2 & x_3 & \cdots & x_n \\ | & | & | & & | \end{bmatrix}$

$W Y^T R X$

$\rightarrow$

$= \begin{bmatrix} w_1 & 0 & \cdots & 0 \\ 0 & w_2 & \cdots & 0 \\ 0 & 0 & w_3 \cdots & 0 \\ & & \ddots & \\ & \cdots & & w_n \end{bmatrix} \begin{bmatrix} - y_1^T - \\ - y_2^T - \\ \vdots \\ - y_n^T - \end{bmatrix} \begin{bmatrix} R \end{bmatrix} \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix}$

$= \begin{bmatrix} - w_1 y_1^T - \\ - w_2 y_2^T - \\ - w_3 y_3^T - \\ \vdots \\ - w_n y_n^T - \end{bmatrix} \begin{bmatrix} | & | & | & & | \\ Rx_1 & Rx_2 & Rx_3 & \cdots & Rx_n \\ | & | & | & & | \end{bmatrix}$

$$= \begin{bmatrix} w_1 y_1^T R x_1 & & & & * \\ & w_2 y_2^T R x_2 & & & \\ & & w_3 y_3^T R x_3 & & \\ & & & \ddots & \\ & * & & & w_n y_n^T R x_n \end{bmatrix}$$

$$\therefore tr(W Y^T R X) = \sum_{i=1}^{n} w_i y_i^T R x_i$$

$$R = \underset{R}{\operatorname{argmax}}\ tr(W Y^T R X)$$

$$= \underset{R}{\operatorname{argmax}}\ tr((W Y^T)(R X))$$

$$= \underset{R}{\operatorname{argmax}}\ tr((R X)(W Y^T))$$

Let's denote $d \times d$ co-variance matrix $X W Y^T \to S$

S.V.D of $S = U \Sigma V^T$

$$tr(R X W Y^T) = tr(R S)$$

$$= tr(R U \Sigma V^T)$$

$$= tr(\Sigma V^T R U)$$

Let $M = V^T R U$

$= tr(\Sigma M)$

$\therefore R = \underset{R}{\operatorname{argmax}}\ tr(\Sigma M)$

as V, R, U are orthogonal matrices

$\therefore$ $M = V^T R U$ is also orthogonal matrix

$\therefore$ its column are orthonormal vectors

$$M = \begin{bmatrix} | & | & | & & | \\ m_1 & m_2 & m_3 & \cdots & m_n \\ | & | & | & & | \end{bmatrix}$$

$m_j^T m_j = 1$

$\Rightarrow \sum_{i=1}^{d} m_{ij}^2 = 1 \Rightarrow m_{ij}^2 \leq 1 \Rightarrow |m_{ij}| \leq 1$

Σ is a diagonal matrix with diagonal entries as $\sigma_1, \sigma_2 \ldots \sigma_n > 0$

$$tr(\Sigma M) = \begin{pmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \sigma_3 & & \\ & & & \ddots & \\ & & & & \sigma_n \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1d} \\ \cdots & m_{22} & & \vdots \\ & & \ddots & \vdots \\ \cdot & \cdot & \cdot & m_{nn} \end{pmatrix}$$

$$= \sum_{i=1}^{d} \sigma_i m_{ii} \leq \sum_{i=1}^{n} \sigma_i$$

$\therefore$ $m_{ii} = 1 \quad \forall i$

and all other elements are 0

$M = I$

$V^T R U = I$

$\Rightarrow R = V U^T$ //

$\therefore$ optimal $R = VU^T$

$$t = q_0 - Rp_0$$

$$p_0 = \frac{\sum_{i=1}^{n} w_i p_i}{\sum w_i} \qquad q_0 = \frac{\sum_{i=1}^{n} w_i q_i}{\sum_{i=1}^{n} w_i}$$

Note: if P & Q have reflection transformation ($\det(VU^T) = -1$) no R (rotational matrix) can give exact transformation

in that case we take

$$R = V \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & -1 \end{bmatrix} U^T$$