On the necessary and sufficient condition for the extended Wedderburn-Guttman theorem

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ABSTRACT

Let A be a u by v matrix, and let M and N be u by p and v by q matrices, where p may not be equal to q or $\text{rank}(M'AN) < \min(p, q)$. Recently, Galantai [A. Galantai, A note on the generalized rank reduction, Acta Mathematica Hungarica 116(2007)239-246] presented what he claimed to be the necessary and sufficient condition for $\text{rank}(A - AN(M'AN)^- M'A) = \text{rank}(A) - \text{rank}(AN(M'AN)^- M'A)$ to hold. This rank subtractivity formula along with the condition under which it holds is called the extended Wedderburn-Guttman theorem. In this paper, we show that some of Galantai's assertions are incorrect.

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1 Introduction

Let A be a u by v matrix, and let M and N be u by p and v by q, where p and q are not necessarily equal to each other or $M'AN$ is not necessarily nonsingular. (Throughout this paper, matrices are all assumed to be real matrices.) Takane and Yanai [4] investigated a necessary and sufficient condition as well as several sufficient conditions for

$$\text{rank}(A - AN(M'AN)^- M'A) = \text{rank}(A) - \text{rank}(AN(M'AN)^- M'A) \tag{1}$$

to hold. This rank subtractivity formula along with the condition under which it holds is called the extended Wedderburn-Guttman theorem. Recently, Galantai [3] asserted that some of the sufficient conditions of Takane and Yanai [4] were also necessary. In this paper, we show that Galantai's assertion is incorrect.

2 A summary of the key results in Takane and Yanai [4, 5]

Let

$$B = N(M'AN)^- M', \tag{2}$$

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so that

$$AN(M'AN)^{-}M'A = ABA. \tag{3}$$

Table 1 summarizes the four conditions most pertinent to Galantai's assertions. Takane and Yanai [4] have shown that Condition (A) is a necessary and sufficient condition for (1) ([4], Theorem 2.1), and that (D) implies (B1) and (B2) (Lemma 2.4 and Corollary 2.2), either one of which in turn implies (A) (Theorem 2.2 and Corollary 2.1). In the following table, the four conditions are characterized by matrix equality and rank equality conditions, which are equivalent (see [4] for proofs):

Table 1: Four conditions surrounding the extended Wedderburn-Guttman theorem.

Condition	Matrix Equality	Rank Equality
(A)	$ABABA = ABA$	$\text{rank}(ABA) = \text{rank}(M'AN)$
(B1)	$(AB)^2 = AB$	$\text{rank}(AB) = \text{rank}(M'AN)$
(B2)	$(BA)^2 = BA$	$\text{rank}(BA) = \text{rank}(M'AN)$
(D)	$BAB = B$	$\text{rank}(B) = \text{rank}(M'AN)$

The above four conditions were further analyzed by Takane and Yanai [5] using the product singular value decomposition (PSVD) of the matrix triplets, A, M, and N. This analysis has revealed that the following four rank conditions are particularly important in characterizing the above conditions:

$$\begin{array}{llll} \text{(C1)} & s = 0, & \text{where} & s = \text{rank}(AN) - \text{rank}(M'AN), \\ \text{(C2)} & j = 0, & \text{where} & j = \text{rank}(M'A) - \text{rank}(M'AN), \\ \text{(G1)} & i = 0, & \text{where} & i = \text{rank}(M) - \text{rank}(M'A), \\ \text{(G2)} & t = 0, & \text{where} & t = \text{rank}(N) - \text{rank}(AN). \end{array} \tag{4}$$

Combinations of these rank conditions are called rank profiles.

There are two kinds of conditions to be distinguished under which the four conditions in Table 1 hold. One is the condition (called the rank profile conditions) under which the four conditions in Table 1 hold irrespective of the g-invese of $M'AN$ used. The other is the condition on the g-inverse of $M'AN$ under which the four conditions in Table 1 still hold despite the failure of the rank profile conditions. According to Theorem 1 in Takane and Yanai [5], Condition (A) holds if and only if (C1) or (C2) holds, or a special g-inverse of $M'AN$ is used for $(M'AN)^{-}$. Condition (B1) holds if and only if (C1), or (C2) and (G1) hold, or a special g-inverse of $M'AN$ is used for $(M'AN)^{-}$. Condition (B2) holds if and only if (C2), or (C1) and (G2) hold, or a special g-inverse of $M'AN$ is used for $(M'AN)^{-}$. Condition (D) holds if and only if (C1) and (G2), or (C2) and (G1) hold, or a special g-inverse of $M'AN$ is used for $(M'AN)^{-}$.

There are sixteen rank profiles that can be created by the combinations of the four rank conditions in (4). Table 2 summarizes those sixteen rank profiles. In the table, a 0 in a column under "Rank profile" means the corresponding s, t, j, or i is zero (i.e., the associated rank condition, (C1), (G2), (C2), or (G1) holds), and a 1 means they are nonzero (positive). For example, row 6 corresponds with the rank profile of $s = 0$, $t \neq 0$, $j = 0$, and $i \neq 0$. A "Y " in the table indicates that a particular condition in Table 1 is satisfied under the given rank profile, no matter which g-inverse of $M'AN$ is used. For example, Condition (A) has Y's in rows corresponding $s = 0$ (C1) and $j = 0$ (C2), meaning that Condition (A) holds if C1 or C2 holds irrespective of the g-inverse of $M'AN$ used. Lower cases letters in the table indicate conditions on the special g-inverse of $M'AN$ required (described below), which depend on both rank profiles and conditions in Table 1 to be satisfied.

Table 2: Rank conditions (A, B1, B2, and D) characterized by rank profiles, and conditions on $(M'AN)^-$.

	Rank Profile				Rank Condition			
No.	s	t	j	i	A	B1	B2	D
1	0	0	0	0	Y	Y	Y	Y
2	0	0	0	1	Y	Y	Y	Y
3	0	0	1	0	Y	Y	Y	Y
4	0	0	1	1	Y	Y	Y	Y
5	0	1	0	0	Y	Y	Y	Y
6	0	1	0	1	Y	Y	Y	d
7	0	1	1	0	Y	Y	c	c
8	0	1	1	1	Y	Y	c	cd
9	1	0	0	0	Y	Y	Y	Y
10	1	0	0	1	Y	b	Y	b
11	1	0	1	0	a	a	a	a
12	1	0	1	1	a	ab	a	ab
13	1	1	0	0	Y	Y	Y	Y
14	1	1	0	1	Y	b	Y	bd
15	1	1	1	0	a	a	ac	ac
16	1	1	1	1	a	ab	ac	abcd

What kind of special g-inverse is needed in which situations? To answer this question, we need to introduce a few more symbols and conditions. Let $h = \text{rank}(M'AN)$, $b = p - (h + j + i)$, and $e = q - (h + s + t)$, where p and q are the number of columns in M and N, respectively. Let $M'AN = U\Delta V'$ denote the complete SVD of $M'AN$, where Δ is partitioned into:

$$\Delta = \begin{array}{cc} \begin{bmatrix} S & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} h \\ j \\ i \\ b \end{array} \\ \begin{array}{cccc} h & s & t & e \end{array} & \end{array}. \tag{5}$$

(Symbols in the right and bottom margins indicate the height and width of blocks, respectively. We also temporarily assume that $h > 0$.) Then, $(M'AN)^-$ can generally be represented as

$$(M'AN)^- = V\Delta^- U', \tag{6}$$

where

$$\Delta^- = \begin{array}{c} \left[\begin{array}{cccc} S^{-1} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{41} & G_{42} & G_{43} & G_{44} \end{array} \right] \begin{array}{c} h \\ s \\ t \\ e \end{array} \\ \begin{array}{cccc} h & j & i & b \end{array} \end{array}, \tag{7}$$

and G's are arbitrary except for their size. Again, symbols in the margins indicate the size of the blocks.) Consider the following four conditions on the elements of Δ^-:

$$\begin{aligned} &\text{Condition (a)} \quad G_{22} = G_{21}SG_{12}, \\ &\text{Condition (b)} \quad G_{23} = G_{21}SG_{13}, \\ &\text{Condition (c)} \quad G_{32} = G_{31}SG_{12}, \\ &\text{Condition (d)} \quad G_{33} = G_{31}SG_{13}. \end{aligned} \tag{8}$$

(Matrix S is incorrectly specified as S^{-1} in C, KCK, $KCKCK$, KC, $(KC)^2$, CK, $(CK)^2$, CKC, Theorem 2 and Note 3 in Takane and Yanai [5].) The specific g-inverse needed can be characterzed by the combinations of the above four conditions. Note that for $(M'AN)^-$ to be a reflexive g-inverse of $M'AN$, the following condition must be satisfied:

$$\left[\begin{array}{c} G_{21} \\ G_{31} \\ G_{41} \end{array} \right] S \left[\begin{array}{ccc} G_{12} & G_{13} & G_{14} \end{array} \right] = \left[\begin{array}{ccc} G_{22} & G_{23} & G_{24} \\ G_{32} & G_{33} & G_{34} \\ G_{42} & G_{43} & G_{44} \end{array} \right]. \tag{9}$$

Note also that the four conditions in (8) are subsets of the conditions in (9). (When $h = 0$, Conditions (a), (b), (c), and (d) in (8) reduce to $G_{22} = 0$, $G_{23} = 0$, $G_{32} = 0$, and $G_{33} = 0$, respectively. Similarly, (9) reduces to the condition in which the matrix on the right hand side is a zero matrix.)

Lower case letters in Table 2 indicate the combinations of the conditions in (8) that are required of a special g-inverse of $M'AN$ to satisfy a particular condition in Table 1. For example, under the rank profile 12, both Conditions (a) and (b) are required to satisfy Condition (B1), whereas under the rank profile 16, all four conditions in (8) are required to sastisfy Condition (D).

Let $\{(M'AN)^-_A\}$ denote the set of g-inverses of $M'AN$ under which Condition (A) holds for each rank profile. Let $\{(M'AN)^-_{B1}\}$, $\{(M'AN)^-_{B2}\}$, and $\{(M'AN)^-_D\}$ be similarly defined. Then, for every rank profile,

$$\{(M'AN)^-\} \supset \{(M'AN)^-_A\} \supset \{(M'AN)^-_{B1}\} \supset \{(M'AN)^-_D\} \supset \{(M'AN)^-_r\},$$

and

$$\{(M'AN)^-\} \supset \{(M'AN)^-_A\} \supset \{(M'AN)^-_{B2}\} \supset \{(M'AN)^-_D\} \supset \{(M'AN)^-_r\},$$

where $\{(M'AN)^-\}$ denotes the set of all g-inverses, and $\{(M'AN)^-_r\}$ the set of reflexive g-inverses of $M'AN$.

The above analysis indicates that Condition (D) implies (B1) and (B2), either one of which in turn implies (A), but not vice versa.

3 The main assertions

Galantai [3] (see also Galantai [2], Theorem 9) uncritically took Cline and Funderlic's [1] following claim as the necessary and sufficient condition for (1):

Galantai's [3] Theorem 5. The equality $\mathrm{rank}(A-H) = \mathrm{rank}(A) - \mathrm{rank}(H)$ holds if and only if there is a matrix B such that $H = ABA$, and $BAB = B$.

This condition corresponds with Condition (D) in Table 1. The "only if" part of the theorem is incorrect. Although this is clear from the above analysis, we give a simple counter example. Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$, $M = \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 1 & 1 \end{bmatrix}$, and $N = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Then, Condition (A) holds ($\mathrm{rank}(A - ABA) = 1$, $\mathrm{rank}(A) = 2$, and $\mathrm{rank}(ABA) = 1$) irrespective of the g-inverse of $M'AN$ used, whereas $BAB = B$ does not hold unless a special g-inverse of $M'AN$ is used for $(M'AN)^-$. In this case, $\mathrm{rank}(M) = \mathrm{rank}(N) = \mathrm{rank}(AN) = 2$, and $\mathrm{rank}(M'A) = \mathrm{rank}(M'AN) = 1$, so that $j = 0$, $i \neq 0$, $s \neq 0$, and $t = 0$. This case corresponds with the rank profile 10 in Table 2. Conditions (A) and (B2) hold because $j = 0$ (C2), but (B1) or (D) do not (unless a special g-inverse of $M'AN$ is used) because $i \neq 0$ and $s \neq 0$.

Galantai [3] goes on to state the next proposition, which is also incorrect.

Galantai's [3] Proposition 6. The matrix $B = N(M'AN)^-M'$ is the solution of $BAB = B$ if and only if $(M'AN)^-$ is a reflexive g-inverse of $M'AN$.

Again, the "only if" part is incorrect. There are other ways by which $BAB = B$ holds: (x1) $j = 0$ and $i = 0$ (C2 and G1), (x2) $s = 0$ and $t = 0$ (C1 and G2), and (x3) a special non-reflexive g-inverse of $M'AN$ is used. In case of (x1) and (x2) $BAB = B$ holds irrespective of $(M'AN)^-$ used. Here is an example of (x1). ((x2) is similar.) Let A and N be the same as before, but $M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then, $BAB = B$, no matter which g-inverse of $M'AN$ is used. In this case, $\mathrm{rank}(N) = \mathrm{rank}(AN) = 2$, and $\mathrm{rank}(M) = \mathrm{rank}(M'A) = \mathrm{rank}(M'AN) = 1$, so that $j = 0$, $i = 0$, $s \neq 0$, $t = 0$. This case coresponds with the rank profile 9 in Table 2. Conditions (A) and (B2) hold because $j = 0$ (C2), and (B1) and (D) hold because $j = 0$ and $i = 0$ (C2 and G1). That is, all four conditions in Table 1 hold in this case.

Here is an example of (x3). Let

$$A = \begin{bmatrix} 1 & -3 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 3 & 1 & 0 \end{bmatrix}, \ M = \begin{bmatrix} 1 & -3 & -1 & 0 \\ 1 & -1 & 3 & 0 \\ 1 & 1 & -3 & 0 \\ 1 & 3 & 1 & 0 \end{bmatrix}, \text{and } N = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}.$$

Then, $\mathrm{rank}(A) = \mathrm{rank}(M) = \mathrm{rank}(N) = 3$, $\mathrm{rank}(M'A) = \mathrm{rank}(AN) = 2$, and $\mathrm{rank}(M'AN) = 1$, so that $j = 1$, $i = 1$, $s = 1$, $t = 1$, $h = 1$, $b = 1$, and $e = 1$. This case corresponds with the rank profile 16 in Table 2. We have

$$M'AN = \begin{array}{cc} \left[\begin{array}{cccc} 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] & \begin{array}{c} h \\ j \\ i \\ b \end{array} \\ \begin{array}{cccc} h & s & t & e \end{array} & \end{array},$$

where the symbols in the margin indicate the width of the corresponding rows and columns. Since $M'AN$ is diagonal in this case (both U and V in (6) are identity matrices in this case), its g-inverse can be obtained by

$$(M'AN)^- = \left[\begin{array}{cccc} 1/8 & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{array}\right], \tag{10}$$

where g's are arbitrary. A reflexive g-inverse requires:

$$8\left(\begin{array}{c} g_{21} \\ g_{31} \\ g_{41} \end{array}\right)\left(\begin{array}{ccc} g_{12} & g_{13} & g_{14} \end{array}\right) = \left[\begin{array}{ccc} g_{22} & g_{23} & g_{24} \\ g_{32} & g_{33} & g_{34} \\ g_{42} & g_{43} & g_{44} \end{array}\right]. \tag{11}$$

(The above condition is a special case of (9). In the event that $h = 0$, the above relation reduces to the condition in which the matrix on the right hand side is a zero matrix.) If (11) holds, all four conditions in Table 1 are satisfied. That is, a reflexive g-inverse of $M'AN$ is a sufficient condition for all of them.

Consider next the following four conditions. (a): $8g_{21}g_{12} = g_{22}$, (b): $8g_{21}g_{13} = g_{23}$, (c): $8g_{31}g_{12} = g_{32}$, and (d): $8g_{31}g_{13} = g_{33}$. These conditions are special cases of the four conditions in (8). These conditions are a subset of the nine conditions in (11). (Note that when $h = 0$, the above conditions degenerate into: (a): $g_{22} = 0$, (b): $g_{23} = 0$, (c): $g_{32} = 0$, and (d): $g_{33} = 0$.

Suppose that all the above four conditions are satisfied, but other entries (i.e., those in the last row and/or the last column) in (10) are arbitrary. Then, obviously such a $(M'AN)^-$ is not a reflexive g-inverse of $M'AN$ because not all nine conditions in (11) are satisfied. However, Conditions (A) through (D) are all satisfied. This indicates that whenever $b \neq 0$ and/or $e \neq 0$, it is possible to get a non-reflexive g-inverse of $M'AN$ that satisfies all of Conditions (A) through (D).

Suppose now that only (a), (b), and (c) hold, but other g's are arbitrary. Then, Conditions (A), (B1), and (B2) hold, but not (D). Suppose that only (a) and (b) hold. Then, (A) and (B1) hold, but neither (B2) nor (D) hold. Suppose that only (a) and (c) hold. Then, (A) and (B2) hold, but neither (B1) nor (D) hold. Suppose that only (a) holds. Then, only (A) holds, but not (B1), (B2) or (D).

Theorem 7 and Corollary 8 of Galantai [3] are based on his Theorem 5 and Proposition 6, and consequently both are incorrect.

Galantai's [3] Theorem 7. The rank subtractivity condition (1) holds if and only if $(M'AN)^-$

is a reflexive g-inverse of $M'AN$.

The first example given above already serves as a counter example to this assertion. It clearly shows that there are situations in which (1) holds irrespective of the choice of g-inverse of $M'AN$. The example for (x3) above also indicates that in the event that neither (C1) nor (C2) holds, there are certain non-reflexive g-inverses of $M'AN$ that satisfy (1).

Galantai's [3] Corollary 8. Conditions (A), (B1), (B2), and (D) are all equivalent.

It is abundantly clear by now that this assertion is incorrect.

References

[1] R. E. Cline, R. E. Funderlic, The rank of a difference of matrices and associated generalized inverses, Linear Algebra and Its Applications, 24 (1979) 185-215.

[2] A. Galantai, Rank reduction: Theory and applications, International Journal of Mathematics, Game Theory and Algebra, 13 (2003) 173-189.

[3] A. Galantai, A note on the generalized rank reduction, Acta Mathematica Hungarica, 116 (2007) 239-246.

[4] Y. Takane, H. Yanai, On the Wedderburn-Guttman Theorem, Linear Algebra and Its Applications, 410 (2005) 267-278.

[5] Y. Takane, H. Yanai, Alternative characterizations of the extended Wedderburn-Guttman theorem, Linear Algebra and Its Applications, 422 (2007) 701-711.