On $\mathbf{V}$-orthogonal projectors associated with a semi-norm

Short Title: $\mathbf{V}$-orthogonal projectors

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Abstract. For any $n \times p$ matrix $\mathbf{X}$ and $n \times n$ nonnegative definite matrix $\mathbf{V}$, the matrix $\mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V}$ is called a $\mathbf{V}$-orthogonal projector with respect to the semi-norm $\|\cdot\|_{\mathbf{V}}$, where $(\cdot)^{+}$ denotes the Moore-Penrose inverse of a matrix. Various new properties of the $\mathbf{V}$-orthogonal projector were derived under the condition that $\mathrm{rank}(\mathbf{VX}) = \mathrm{rank}(\mathbf{X})$, including its rank, complement, equivalent expressions, conditions for additive decomposability, equivalence conditions between two ($\mathbf{V}$-)orthogonal projectors, etc.

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1 Introduction

Throughout this paper, $\mathbb{R}^{m\times n}$ stands for the collections of all $m \times n$ real matrices. The symbols $\mathbf{A}'$, $r(\mathbf{A})$, $\mathscr{R}(\mathbf{A})$ and $\mathscr{N}(\mathbf{A})$ stand for the transpose, the rank, the range (column space) and the kernel (null space) of a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, respectively; $\mathscr{R}^{\perp}(\mathbf{A})$ stands for the orthogonal complement of $\mathscr{R}(\mathbf{A})$. The Moore-Penrose inverse of $\mathbf{A}$, denoted by $\mathbf{A}^{+}$, is defined to be the unique solution $\mathbf{G}$ to the four matrix equations

$$\text{(i) } \mathbf{AGA} = \mathbf{A}, \quad \text{(ii) } \mathbf{GAG} = \mathbf{G}, \quad \text{(iii) } (\mathbf{AG})' = \mathbf{AG}, \quad \text{(iv) } (\mathbf{GA})' = \mathbf{GA}.$$

A matrix $\mathbf{G}$ is called a generalized inverse (g-inverse) of $\mathbf{A}$, denoted by $\mathbf{A}^{-}$, if it satisfies (i), an outer inverse of $\mathbf{A}$ if it satisfies (ii). Further, let $\mathbf{P}_{\mathbf{A}}$, $\mathbf{F}_{\mathbf{A}}$ and $\mathbf{E}_{\mathbf{A}}$ stand for the three orthogonal projectors $\mathbf{P}_{\mathbf{A}} = \mathbf{A}\mathbf{A}^{+}$, $\mathbf{E}_{\mathbf{A}} = \mathbf{I}_m - \mathbf{P}_{\mathbf{A}} = \mathbf{I}_n - \mathbf{A}\mathbf{A}^{+}$ and $\mathbf{F}_{\mathbf{A}} = \mathbf{I}_n - \mathbf{P}_{\mathbf{A}'} = \mathbf{I}_n - \mathbf{A}^{+}\mathbf{A}$.

Let $\mathbf{V} \in \mathbb{R}^{n\times n}$ be a nonnegative definite (nnd) matrix, i.e., $\mathbf{V}$ can be expressed as $\mathbf{V} = \mathbf{ZZ}'$ for some $\mathbf{Z}$. The seminorm of a vector $\mathbf{x} \in \mathbb{R}^{m\times 1}$ induced by $\mathbf{V}$ is defined by $\|\mathbf{x}\|_{\mathbf{V}} = (\mathbf{x}'\mathbf{V}\mathbf{x})^{1/2}$.

Suppose we are given a general linear model

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \quad E(\boldsymbol{\varepsilon}) = \mathbf{0}, \quad Cov(\boldsymbol{\varepsilon}) = \sigma^2\boldsymbol{\Sigma}, \tag{1}$$

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or in the triplet form

$$\mathscr{M} = \{\mathbf{y},\, \mathbf{X}\boldsymbol{\beta},\, \sigma^2\boldsymbol{\Sigma}\}, \tag{2}$$

where $\mathbf{X} \in \mathbb{R}^{n\times p}$ is a known matrix, $\mathbf{y} \in \mathbb{R}^{n\times 1}$ is an observable random vector, $\boldsymbol{\varepsilon} \in \mathbb{R}^{n\times 1}$ is a random error vector, $\boldsymbol{\beta} \in \mathbb{R}^{n\times 1}$ is a vector of unknown parameters, σ^2 is a positive unknown parameter, $\boldsymbol{\Sigma} \in \mathbb{R}^{n\times n}$ is a known nonnegative definite matrix.

As is well known, for a given nnd matrix $\mathbf{V}$, the WLSE of $\boldsymbol{\beta}$ under (2), denoted by $\mathrm{WLSE}_{\mathscr{M}}(\boldsymbol{\beta})$, is defined to be

$$\tilde{\boldsymbol{\beta}} = \underset{\boldsymbol{\beta}}{\operatorname{argmin}}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'\mathbf{V}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}). \tag{3}$$

The WLSE of $\mathbf{X}\boldsymbol{\beta}$ under (2), denoted by $\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta})$, is defined to be $\mathbf{X}\tilde{\boldsymbol{\beta}}$. The normal equation associated with (3) is given by $\mathbf{X}'\mathbf{V}\mathbf{X}\boldsymbol{\beta} = \mathbf{X}'\mathbf{V}\mathbf{y}$. This equation is always consistent. Solving this equation gives the following well-known result.

Lemma 1 *The general expression of the WLSE of $\boldsymbol{\beta}$ under* (2) *is given by*

$$\tilde{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V}\mathbf{y} + [\,\mathbf{I} - (\mathbf{V}\mathbf{X})^{+}(\mathbf{V}\mathbf{X})\,]\mathbf{u} = (\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V}\mathbf{y} + \mathbf{F}_{\mathbf{V}\mathbf{X}}\mathbf{u}, \tag{4}$$

where $\mathbf{u} \in \mathbb{R}^{p\times 1}$ is arbitrary.

For $\mathbf{y} \neq \mathbf{0}$, let $\mathbf{u} = \mathbf{U}\mathbf{y}$ in (4), where $\mathbf{U} \in \mathbb{R}^{p\times n}$ is arbitrary. Then the WLSEs of $\boldsymbol{\beta}$ and $\mathbf{X}\boldsymbol{\beta}$ under (2) can be written in the following homogeneous forms

$$\mathrm{WLSE}_{\mathscr{M}}(\boldsymbol{\beta}) = [\,(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V} + \mathbf{F}_{\mathbf{V}\mathbf{X}}\mathbf{U}\,]\mathbf{y}, \tag{5}$$

$$\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta}) = [\,\mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V} + \mathbf{X}\mathbf{F}_{\mathbf{V}\mathbf{X}}\mathbf{U}\,]\mathbf{y}. \tag{6}$$

Further, let $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ denote the matrix pre-multiplied to $\mathbf{y}$ in (6):

$$\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V} + \mathbf{X}\mathbf{F}_{\mathbf{V}\mathbf{X}}\mathbf{U}, \tag{7}$$

which is called the projector into $\mathscr{R}(\mathbf{X})$ with respect to the semi-norm $\|\cdot\|_{\mathbf{V}}$, (see Mitra and Rao (1974)). Because there is an arbitrary matrix $\mathbf{U}$ in (7), it is possible to take the $\mathbf{U}$ such that $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ has some special forms, for example,

$$\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{-}\mathbf{X}'\mathbf{V}, \tag{8}$$

$$\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V}, \tag{9}$$

$$\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{X}\mathbf{X}^{-} + \mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{-}\mathbf{X}'\mathbf{V}(\,\mathbf{I} - \mathbf{X}\mathbf{X}^{-}\,), \tag{10}$$

$$\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{X}\mathbf{X}^{+} + \mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V}(\,\mathbf{I} - \mathbf{X}\mathbf{X}^{+}\,). \tag{11}$$

It can be seen from (7) that $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ is not necessarily unique. In addition, $\mathbf{P}_{\mathbf{X}:\mathbf{V}}^2 = \mathbf{P}_{\mathbf{X}:\mathbf{V}}$ and $\mathbf{P}_{\mathbf{X}:\mathbf{V}}\mathbf{X} = \mathbf{X}$ do not necessarily hold for a given $\mathbf{U}$ in (7). Using the notation in (7), the expectation and the covariance matrix of $\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta})$ are given by

$$E[\,\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta})\,] = \mathbf{P}_{\mathbf{X}:\mathbf{V}}\mathbf{X}\boldsymbol{\beta} \quad \text{and} \quad Cov[\,\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta})\,] = \sigma^2\mathbf{P}_{\mathbf{X}:\mathbf{V}}\boldsymbol{\Sigma}\mathbf{P}_{\mathbf{X}:\mathbf{V}}'. \tag{12}$$

Eqs. (6) and (12) indicate that the algebraic and statistical properties of the WLSE of $\mathbf{X}\boldsymbol{\beta}$ under (2) are mainly determined by the projector $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ in (7). Hence it is quite essential to know various properties of the projector when applying $\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta})$ in statistical practice, for example, the rank, range, trace, norm, uniqueness, idempotency, symmetry, decompositions of the projector, as well as various equalities involving projectors. The projector $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ in (7) and its special cases have widely been investigated in the literature since 1970s, see, e.g., Harville (1997), Mitra and Rao (1974), Rao (1974), Rao and Mitra (1971a, 1971b), Takane and Yanai (1999), Tian and Takane (2007b). In Tian and Takane (2007b), a variety of new properties were derived on projectors associated with WLSEs by making use of the matrix rank method. As further extensions of Tian and Takane (2007b), of particular interest in the present paper are various properties of $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ when it is unique.

Some rank formulas for partitioned matrices due to Marsaglia and Styan (1974, Theorem 19) are given below, which can be used to simplify various matrix expressions involving generalized inverses.

Lemma 2 *Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{m\times k}$ *and* $\mathbf{C} \in \mathbb{R}^{l\times n}$. *Then*

$$r[\,\mathbf{A},\, \mathbf{B}\,] = r(\mathbf{A}) + r[\,(\,\mathbf{I}_m - \mathbf{A}\mathbf{A}^-\,)\mathbf{B}\,] = r(\mathbf{B}) + r[\,(\,\mathbf{I}_m - \mathbf{B}\mathbf{B}^-\,)\mathbf{A}\,], \tag{13}$$

$$r\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} = r(\mathbf{A}) + r[\,\mathbf{C}(\,\mathbf{I}_n - \mathbf{A}^-\mathbf{A}\,)\,] = r(\mathbf{C}) + r[\,\mathbf{A}(\,\mathbf{I}_n - \mathbf{C}^-\mathbf{C}\,)\,], \tag{14}$$

where the ranks do not depend on the particular choice of $\mathbf{A}^-$, $\mathbf{B}^-$ *and* $\mathbf{C}^-$.

The results in the following lemma are given in Tian and Styan (2001, Theorem 2.1 and Corollary 2.3).

Lemma 3 *Any pair of idempotent matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *of the order* m *satisfy the following two rank formulas*

$$r(\,\mathbf{A} - \mathbf{B}\,) = r\begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix} + r[\,\mathbf{A},\, \mathbf{B}\,] - r(\mathbf{A}) - r(\mathbf{B}), \tag{15}$$

$$r(\,\mathbf{I}_m - \mathbf{A} - \mathbf{B}\,) = m + r(\mathbf{A}\mathbf{B}) + r(\mathbf{B}\mathbf{A}) - r(\mathbf{A}) - r(\mathbf{B}). \tag{16}$$

Hence,

$$\mathbf{A} = \mathbf{B} \Leftrightarrow \mathscr{R}(\mathbf{A}) = \mathscr{R}(\mathbf{B}) \text{ and } \mathscr{R}(\mathbf{A}') = \mathscr{R}(\mathbf{B}'), \tag{17}$$

$$\mathbf{A} + \mathbf{B} = \mathbf{I}_m \Leftrightarrow r(\mathbf{A}) + r(\mathbf{B}) = m + r(\mathbf{A}\mathbf{B}) + r(\mathbf{B}\mathbf{A}). \tag{18}$$

The result in the following lemma is shown in Tian and Takane (2007a, Lemma 1.3).

Lemma 4 *Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$, *and let* $\mathbf{G}_1,\, \mathbf{G}_2,\, \mathbf{G}_3 \in \mathbb{R}^{n\times m}$ *be three outer inverses of* $\mathbf{A}$, *i.e.,* $\mathbf{G}_i\mathbf{A}\mathbf{G}_i = \mathbf{G}_i$, $i = 1, 2, 3$. *Also suppose* $\mathscr{R}(\mathbf{G}_i) \subseteq \mathscr{R}(\mathbf{G}_1)$ *and* $\mathscr{R}(\mathbf{G}_i') \subseteq \mathscr{R}(\mathbf{G}_1')$, $i = 2, 3$. *Then*

$$r(\,\mathbf{G}_1 - \mathbf{G}_2 - \mathbf{G}_3\,) = r(\mathbf{G}_1) - r(\mathbf{G}_2) - r(\mathbf{G}_3) + r(\mathbf{G}_2\mathbf{A}\mathbf{G}_3) + r(\mathbf{G}_3\mathbf{A}\mathbf{G}_2). \tag{19}$$

The following result is given in Tian (2004, Corollary 2).

Lemma 5 *Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$ *and* $\mathbf{B} \in \mathbb{R}^{m\times k}$. *Then*

$$(\mathbf{P_A P_B})^+ = \mathbf{P_B P_A} - \mathbf{P_B}(\mathbf{E_B E_A})^+\mathbf{P_A}. \tag{20}$$

We also use the following well-known results on Moore-Penrose inverses, ranges and ranks of matrices:

$$\mathbf{A} = \mathbf{AA}'(\mathbf{A}^+)' = (\mathbf{A}^+)'\mathbf{A}'\mathbf{A}, \quad (\mathbf{A}^+)^+ = \mathbf{A}, \quad (\mathbf{A}^+)' = (\mathbf{A}')^+, \tag{21}$$

$$\mathbf{A}^+ = (\mathbf{A}'\mathbf{A})^+\mathbf{A}' = \mathbf{A}'(\mathbf{AA}')^+, \tag{22}$$

$$\mathscr{R}(\mathbf{B}) \subseteq \mathscr{R}(\mathbf{A}) \Leftrightarrow r[\,\mathbf{A},\, \mathbf{B}\,] = r(\mathbf{A}) \Leftrightarrow \mathbf{AA}^+\mathbf{B} = \mathbf{B}, \tag{23}$$

$$\mathscr{R}(\mathbf{B}) \subseteq \mathscr{R}(\mathbf{A}) \text{ and } r(\mathbf{B}) = r(\mathbf{A}) \Leftrightarrow \mathscr{R}(\mathbf{B}) = \mathscr{R}(\mathbf{A}), \tag{24}$$

$$\mathscr{R}(\mathbf{A}) = \mathscr{R}(\mathbf{AA}') = \mathscr{R}(\mathbf{AA}^+) = \mathscr{R}[\,(\mathbf{A}^+)'\,], \tag{25}$$

$$\mathscr{R}(\mathbf{A}') = \mathscr{R}(\mathbf{A}'\mathbf{A}) = \mathscr{R}(\mathbf{A}^+\mathbf{A}) = \mathscr{R}(\mathbf{A}^+), \tag{26}$$

$$\mathscr{R}(\mathbf{AB}^+\mathbf{B}) = \mathscr{R}(\mathbf{AB}^+) = \mathscr{R}(\mathbf{AB}'), \tag{27}$$

$$\mathscr{R}(\mathbf{A}_1) = \mathscr{R}(\mathbf{A}_2) \text{ and } \mathscr{R}(\mathbf{B}_1) = \mathscr{R}(\mathbf{B}_2) \Rightarrow r[\,\mathbf{A}_1,\, \mathbf{A}_2\,] = r[\,\mathbf{B}_1,\, \mathbf{B}_2\,]. \tag{28}$$

Moreover, if $\mathbf{V}$ is nnd, then

$$\mathbf{VV}^+ = \mathbf{V}^+\mathbf{V},\ \mathscr{R}(\mathbf{V}) = \mathscr{R}(\mathbf{V}^{1/2}) = \mathscr{R}(\mathbf{V}^+),\ \mathscr{R}(\mathbf{A}'\mathbf{V}) = \mathscr{R}(\mathbf{A}'\mathbf{V}^{1/2}) = \mathscr{R}(\mathbf{A}'\mathbf{V}^+), \tag{29}$$

where $\mathbf{V}^{1/2}$ is the nnd square root of $\mathbf{V}$, see, e.g., Ben-Israel and Greville (2003), and Rao and Mitra (1971).

2 Properties of $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ when it is unique

We first give a series of equivalent statements on the uniqueness of the projector $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ in (7).

Theorem 6 *Let* $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ *be as given in* (7). *Then the following statements are equivalent*:

(a) $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ *is unique.*

(b) $\mathscr{R}(\mathbf{X}'\mathbf{V}) = \mathscr{R}(\mathbf{X}')$.

(c) $r(\mathbf{VX}) = r(\mathbf{X})$.

(d) $\mathscr{R}(\mathbf{X}) \cap \mathscr{R}(\mathbf{E_V}) = \{\mathbf{0}\}$.

(e) $\mathscr{R}(\mathbf{X}) \cap \mathscr{R}(\mathbf{E_{VX}}) = \{\mathbf{0}\}$.

(f) $r[\,\mathbf{E_X},\ \mathbf{V}\,] = n$.

(g) $\mathbf{E_X} + \mathbf{V}$ *is positive definite (pd).*

In this case, the unique $\mathbf{V}$*-orthogonal projector can be written as*

$$\mathbf{P_{X:V}} = \mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V}. \tag{30}$$

Proof From (7), the projector $\mathbf{P_{X:V}}$ is unique if and only if $\mathbf{X}(\mathbf{VX})^{+}(\mathbf{VX}) = \mathbf{X}$, which, by (23) is equivalent to $\mathscr{R}(\mathbf{X}'\mathbf{V}) = \mathscr{R}(\mathbf{X}')$, and to $r(\mathbf{VX}) = r(\mathbf{X})$. It is easy to find from (13) that

$$\begin{aligned}
r[\,\mathbf{X},\ \mathbf{E_V}\,] &= r(\mathbf{P_V X}) + r(\mathbf{E_V}) = r(\mathbf{VX}) + r(\mathbf{E_V}) = r(\mathbf{VX}) + n - r(\mathbf{V}),\\
r[\,\mathbf{X},\ \mathbf{E_{VX}}\,] &= r(\mathbf{P_{VX} X}) + r(\mathbf{E_{VX}}) = r(\mathbf{VX}) + r(\mathbf{E_{VX}}) = n,\\
r[\,\mathbf{E_X},\ \mathbf{V}\,] &= r(\mathbf{E_X}) + r(\mathbf{P_X V}) = r(\mathbf{E_X}) + r(\mathbf{VX}) = n - r(\mathbf{X}) + r(\mathbf{VX}),\\
r(\,\mathbf{E_X} + \mathbf{V}\,) &= r[\,\mathbf{E_X},\ \mathbf{V}\,] = r(\mathbf{E_X}) + r(\mathbf{VX}) = n - r(\mathbf{X}) + r(\mathbf{VX})
\end{aligned}$$

hold. Hence,

$$\begin{aligned}
\mathscr{R}(\mathbf{X}) \cap \mathscr{R}(\mathbf{E_V}) = \{\mathbf{0}\} &\Leftrightarrow r[\,\mathbf{X},\ \mathbf{E_V}\,] = r(\mathbf{X}) + r(\mathbf{E_V}) \Leftrightarrow r(\mathbf{VX}) = r(\mathbf{X}),\\
\mathscr{R}(\mathbf{X}) \cap \mathscr{R}(\mathbf{E_{VX}}) = \{\mathbf{0}\} &\Leftrightarrow r[\,\mathbf{X},\ \mathbf{E_{VX}}\,] = r(\mathbf{X}) + r(\mathbf{E_{VX}}) \Leftrightarrow r(\mathbf{VX}) = r(\mathbf{X}),\\
r[\,\mathbf{E_X},\ \mathbf{V}\,] = n &\Leftrightarrow r(\mathbf{VX}) = r(\mathbf{X}),\\
\mathbf{E_X} + \mathbf{V} \text{ is pd} &\Leftrightarrow r[\,\mathbf{E_X},\ \mathbf{V}\,] = n,
\end{aligned}$$

establishing the equivalence of (c), (d), (e), (f) and (g). $\square$

If the projector $\mathbf{P_{X:V}}$ in (7) is unique, it is often called the $\mathbf{V}$-orthogonal projector onto $\mathscr{R}(\mathbf{X})$. The following theorem gives a variety of properties of the $\mathbf{V}$-orthogonal projector $\mathbf{P_{X:V}}$ in (30), some of which were given in the literature, see, e.g., Harville (1997), and Mitra and Rao (1974).

Theorem 7 *Let* $\mathbf{P_{X:V}}$ *be as given in* (30) *and suppose* $r(\mathbf{VX}) = r(\mathbf{X})$. *Then*:

(a) $\mathbf{P_{X:V}}\mathbf{Z} = \mathbf{Z}$ *holds for any* $\mathbf{Z}$ *with* $\mathscr{R}(\mathbf{Z}) \subseteq \mathscr{R}(\mathbf{X})$.

(b) $\mathbf{P}^2_{\mathbf{X:V}} = \mathbf{P_{X:V}}$.

(c) $\mathscr{R}(\mathbf{P_{X:V}}) = \mathscr{R}(\mathbf{X})$ *and* $\mathscr{R}(\mathbf{P}'_{\mathbf{X:V}}) = \mathscr{R}(\mathbf{VX})$.

(d) $\mathscr{N}(\mathbf{P_{X:V}}) = \mathscr{N}(\mathbf{X}'\mathbf{V})$ *and* $\mathscr{N}(\mathbf{P}'_{\mathbf{X:V}}) = \mathscr{N}(\mathbf{X}')$.

(e) $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ *can be written in the following nine forms*

$$\begin{aligned}\mathbf{P}_{\mathbf{X}:\mathbf{V}} &= \mathbf{X}(\mathbf{V}^{1/2}\mathbf{X})^{+}\mathbf{V}^{1/2} && (31)\\ &= \mathbf{X}\mathbf{X}'\mathbf{V}(\mathbf{X}'\mathbf{V}\mathbf{X}\mathbf{X}'\mathbf{V})^{+}\mathbf{X}'\mathbf{V} && (32)\\ &= \mathbf{X}(\mathbf{X}\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}\mathbf{X}'\mathbf{V} && (33)\\ &= \mathbf{P}_{\mathbf{X}}\mathbf{V}(\mathbf{X}'\mathbf{V}\mathbf{P}_{\mathbf{X}}\mathbf{V})^{+}\mathbf{X}'\mathbf{V} && (34)\\ &= \mathbf{X}(\mathbf{P}_{\mathbf{X}}\mathbf{V}\mathbf{X})^{+}\mathbf{P}_{\mathbf{X}}\mathbf{V} && (35)\\ &= (\mathbf{P}_{\mathbf{V}\mathbf{X}}\mathbf{P}_{\mathbf{X}})^{+} && (36)\\ &= \mathbf{P}_{\mathbf{X}}\mathbf{P}_{\mathbf{V}\mathbf{X}} - \mathbf{P}_{\mathbf{X}}(\mathbf{E}_{\mathbf{X}}\mathbf{E}_{\mathbf{V}\mathbf{X}})^{+}\mathbf{P}_{\mathbf{V}\mathbf{X}} && (37)\\ &= \mathbf{X}\mathbf{X}'(\,\mathbf{X}\mathbf{X}' + \mathbf{E}_{\mathbf{V}\mathbf{X}}\,)^{-1} && (38)\\ &= \mathbf{P}_{\mathbf{X}}(\,\mathbf{P}_{\mathbf{X}} + \mathbf{E}_{\mathbf{V}\mathbf{X}}\,)^{-1}, && (39)\end{aligned}$$

where $\mathbf{V}^{1/2}$ *is the nonnegative definite square root of* $\mathbf{V}$.

(f) $\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{P}_{\mathbf{X}:\mathbf{V}}\mathbf{V}^{+}\mathbf{P}'_{\mathbf{X}:\mathbf{V}}\mathbf{V}$.

(g) $\mathbf{P}_{\mathbf{X}}\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{P}_{\mathbf{X}:\mathbf{V}}$, $\mathbf{P}_{\mathbf{X}:\mathbf{V}}\mathbf{P}_{\mathbf{X}} = \mathbf{P}_{\mathbf{X}}$ *and* $(\,\mathbf{P}_{\mathbf{X}} - \mathbf{P}_{\mathbf{X}:\mathbf{V}}\,)^2 = \mathbf{0}$.

(h) $\mathbf{P}_{\mathbf{X}\mathbf{X}':\mathbf{V}}$, $\mathbf{P}_{\mathbf{P}_{\mathbf{X}}:\mathbf{V}}$, $\mathbf{P}_{\mathbf{P}_{\mathbf{X}:\mathbf{V}}:\mathbf{V}}$, $\mathbf{P}_{\mathbf{X}:(\lambda\mathbf{E}_{\mathbf{X}}+\mathbf{V})}$ *and* $\mathbf{P}_{\mathbf{X}:\mathbf{P}_{\mathbf{V}\mathbf{X}}}$ *are unique, and*

$$\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{P}_{\mathbf{X}\mathbf{X}':\mathbf{V}} = \mathbf{P}_{\mathbf{P}_{\mathbf{X}}:\mathbf{V}} = \mathbf{P}_{\mathbf{P}_{\mathbf{X}:\mathbf{V}}:\mathbf{V}} = \mathbf{P}_{\mathbf{X}:(\lambda\mathbf{E}_{\mathbf{X}}+\mathbf{V})} = \mathbf{P}_{\mathbf{X}:\mathbf{P}_{\mathbf{V}\mathbf{X}}},$$

where λ *is any real number.*

(i) $\mathbf{P}'_{\mathbf{X}:\mathbf{V}} = \mathbf{P}_{\mathbf{V}\mathbf{X}:(\mathbf{E}_{\mathbf{X}}+\mathbf{V})^{-1}}$.

(j) $\mathbf{P}_{\mathbf{X}:\mathbf{V}} + \mathbf{P}'_{\mathbf{E}_{\mathbf{X}}:(\mathbf{E}_{\mathbf{X}}+\mathbf{V})^{-1}} = \mathbf{I}_n$.

(k) $\mathbf{P}_{\mathbf{X}:\mathbf{V}} + \mathbf{P}_{(\mathbf{E}_{\mathbf{X}}+\mathbf{V})^{-1}\mathbf{E}_{\mathbf{X}}:(\mathbf{E}_{\mathbf{X}}+\mathbf{V})} = \mathbf{I}_n$.

(l) $\mathbf{P}_{\mathbf{V}\mathbf{X}:\mathbf{V}^{+}}$ *is unique and* $\mathbf{P}_{\mathbf{V}\mathbf{X}:\mathbf{V}^{+}} = \mathbf{V}\mathbf{P}_{\mathbf{X}:\mathbf{V}}\mathbf{V}^{+}$.

Proof Parts (a) and (b) are obvious. Parts (c) and (d) are derived by (27). Applying (22) to $(\mathbf{V}^{1/2}\mathbf{X})^{+}$ gives

$$\mathbf{X}(\mathbf{V}^{1/2}\mathbf{X})^{+}\mathbf{V}^{1/2} = \mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V},$$

as required for (31). Denote the eight matrices in (32)–(39) by

$$\begin{aligned}\mathbf{Z}_1 &= \mathbf{X}\mathbf{X}'\mathbf{V}(\mathbf{X}'\mathbf{V}\mathbf{X}\mathbf{X}'\mathbf{V})^{+}\mathbf{X}'\mathbf{V},\\ \mathbf{Z}_2 &= \mathbf{X}(\mathbf{X}\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}\mathbf{X}'\mathbf{V},\\ \mathbf{Z}_3 &= \mathbf{P}_{\mathbf{X}}\mathbf{V}(\mathbf{X}'\mathbf{V}\mathbf{P}_{\mathbf{X}}\mathbf{V})^{+}\mathbf{X}'\mathbf{V},\\ \mathbf{Z}_4 &= \mathbf{X}(\mathbf{P}_{\mathbf{X}}\mathbf{V}\mathbf{X})^{+}\mathbf{P}_{\mathbf{X}}\mathbf{V},\\ \mathbf{Z}_5 &= (\mathbf{P}_{\mathbf{V}\mathbf{X}}\mathbf{P}_{\mathbf{X}})^{+},\\ \mathbf{Z}_6 &= \mathbf{P}_{\mathbf{X}}\mathbf{P}_{\mathbf{V}\mathbf{X}} - \mathbf{P}_{\mathbf{X}}(\mathbf{E}_{\mathbf{X}}\mathbf{E}_{\mathbf{V}\mathbf{X}})^{+}\mathbf{P}_{\mathbf{V}\mathbf{X}},\\ \mathbf{Z}_7 &= \mathbf{X}\mathbf{X}'(\,\mathbf{X}\mathbf{X}' + \mathbf{E}_{\mathbf{V}\mathbf{X}}\,)^{-1},\\ \mathbf{Z}_8 &= \mathbf{P}_{\mathbf{X}}(\,\mathbf{P}_{\mathbf{X}} + \mathbf{E}_{\mathbf{V}\mathbf{X}}\,)^{-1}.\end{aligned}$$

Then it is easy to verify that $\mathbf{Z}_1, \dots, \mathbf{Z}_5$ are all idempotent and that

$$\mathscr{R}(\mathbf{Z}_1) = \mathscr{R}(\mathbf{Z}_2) = \mathscr{R}(\mathbf{Z}_3) = \mathscr{R}(\mathbf{Z}_4) = \mathscr{R}(\mathbf{Z}_5) = \mathscr{R}(\mathbf{X}) = \mathscr{R}(\mathbf{P}_{\mathbf{X}:\mathbf{V}}),$$

$$\mathscr{R}(\mathbf{Z}_1') = \mathscr{R}(\mathbf{Z}_2') = \mathscr{R}(\mathbf{Z}_3') = \mathscr{R}(\mathbf{Z}_4') = \mathscr{R}(\mathbf{Z}_5') = \mathscr{R}(\mathbf{V}\mathbf{X}) = \mathscr{R}(\mathbf{P}_{\mathbf{X}:\mathbf{V}}').$$

In these cases, applying (17) to $\mathbf{P}_{\mathbf{X}:\mathbf{V}}, \mathbf{Z}_1, \dots, \mathbf{Z}_5$ yields $\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{Z}_1 = \dots = \mathbf{Z}_5$. The equality $\mathbf{Z}_5 = \mathbf{Z}_6$ follows from (20). By (13),

$$r(\,\mathbf{X}\mathbf{X}' + \mathbf{E}_{\mathbf{V}\mathbf{X}}\,) = r[\,\mathbf{X},\, \mathbf{E}_{\mathbf{V}\mathbf{X}}\,] = r(\mathbf{P}_{\mathbf{V}\mathbf{X}}\mathbf{X}) + r(\mathbf{E}_{\mathbf{V}\mathbf{X}}) = r(\mathbf{V}\mathbf{X}) + r(\mathbf{E}_{\mathbf{V}\mathbf{X}}) = n,$$

$$r(\,\mathbf{P}_{\mathbf{X}} + \mathbf{E}_{\mathbf{V}\mathbf{X}}\,) = r[\,\mathbf{P}_{\mathbf{X}},\, \mathbf{E}_{\mathbf{V}\mathbf{X}}\,] = r(\mathbf{P}_{\mathbf{V}\mathbf{X}}\mathbf{P}_{\mathbf{X}}) + r(\mathbf{E}_{\mathbf{V}\mathbf{X}}) = r(\mathbf{V}\mathbf{X}) + r(\mathbf{E}_{\mathbf{V}\mathbf{X}}) = n.$$

Hence both $\mathbf{X}\mathbf{X}' + \mathbf{E}_{\mathbf{V}\mathbf{X}}$ and $\mathbf{P}_{\mathbf{X}} + \mathbf{E}_{\mathbf{V}\mathbf{X}}$ are nonsingular. From (a) and (30), we obtain that $\mathbf{P}_{\mathbf{X}:\mathbf{V}}\mathbf{X}\mathbf{X}' = \mathbf{X}\mathbf{X}'$, $\mathbf{P}_{\mathbf{X}:\mathbf{V}}\mathbf{P}_{\mathbf{X}} = \mathbf{P}_{\mathbf{X}}$ and $\mathbf{P}_{\mathbf{X}:\mathbf{V}}(\mathbf{V}\mathbf{X})(\mathbf{V}\mathbf{X})^{+} = \mathbf{P}_{\mathbf{X}:\mathbf{V}}$, so that

$$\begin{aligned}
r(\,\mathbf{P}_{\mathbf{X}:\mathbf{V}} - \mathbf{Z}_7\,) &= r[\,\mathbf{P}_{\mathbf{X}:\mathbf{V}} - \mathbf{X}\mathbf{X}'(\,\mathbf{X}\mathbf{X}' + \mathbf{E}_{\mathbf{V}\mathbf{X}}\,)^{-1}\,] \\
&= r[\,\mathbf{P}_{\mathbf{X}:\mathbf{V}}(\,\mathbf{X}\mathbf{X}' + \mathbf{E}_{\mathbf{V}\mathbf{X}}\,) - \mathbf{X}\mathbf{X}'\,] \\
&= r[\,\mathbf{P}_{\mathbf{X}:\mathbf{V}} - \mathbf{P}_{\mathbf{X}:\mathbf{V}}(\mathbf{V}\mathbf{X})(\mathbf{V}\mathbf{X})^{+}\,] \\
&= 0, \\
r(\,\mathbf{P}_{\mathbf{X}:\mathbf{V}} - \mathbf{Z}_8\,) &= r[\,\mathbf{P}_{\mathbf{X}:\mathbf{V}} - \mathbf{P}_{\mathbf{X}}(\,\mathbf{P}_{\mathbf{X}} + \mathbf{E}_{\mathbf{V}\mathbf{X}}\,)^{-1}\,] \\
&= r[\,\mathbf{P}_{\mathbf{X}:\mathbf{V}}(\,\mathbf{P}_{\mathbf{X}} + \mathbf{E}_{\mathbf{V}\mathbf{X}}\,) - \mathbf{P}_{\mathbf{X}}\,] \\
&= r[\,\mathbf{P}_{\mathbf{X}:\mathbf{V}} - \mathbf{P}_{\mathbf{X}:\mathbf{V}}(\mathbf{V}\mathbf{X})(\mathbf{V}\mathbf{X})^{+}\,] \\
&= 0.
\end{aligned}$$

Hence $\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{Z}_7 = \mathbf{Z}_8$, establishing (e). The result in (f) is straightforward from (30). The first two equalities in (g) are derived from (a). The third equality in (g) follows from (b) and the first two equalities in (f). If $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ is unique, then the five projectors $\mathbf{P}_{\mathbf{X}\mathbf{X}':\mathbf{V}}$, $\mathbf{P}_{\mathbf{P}_{\mathbf{X}}:\mathbf{V}}$, $\mathbf{P}_{\mathbf{X}:(\lambda\mathbf{E}_{\mathbf{X}}+\mathbf{V})}$, $\mathbf{P}_{\mathbf{P}_{\mathbf{X}:\mathbf{V}}:\mathbf{P}_{\mathbf{V}}}$ and $\mathbf{P}_{\mathbf{X}:\mathbf{P}_{\mathbf{V}\mathbf{X}}}$ are also unique by Theorem 6. In these cases, it is easy to verify that

$$\mathscr{R}(\mathbf{P}_{\mathbf{X}\mathbf{X}':\mathbf{V}}) = \mathscr{R}(\mathbf{P}_{\mathbf{P}_{\mathbf{X}}:\mathbf{V}}) = \mathscr{R}(\mathbf{P}_{\mathbf{P}_{\mathbf{X}:\mathbf{V}}:\mathbf{V}}) = \mathscr{R}(\mathbf{P}_{\mathbf{X}:(\lambda\mathbf{E}_{\mathbf{X}}+\mathbf{V})}) = \mathscr{R}(\mathbf{P}_{\mathbf{X}:\mathbf{P}_{\mathbf{V}\mathbf{X}}}) = \mathscr{R}(\mathbf{X}),$$

$$\mathscr{R}(\mathbf{P}_{\mathbf{X}\mathbf{X}':\mathbf{V}}') = \mathscr{R}(\mathbf{P}_{\mathbf{P}_{\mathbf{X}}:\mathbf{V}}') = \mathscr{R}(\mathbf{P}_{\mathbf{P}_{\mathbf{X}:\mathbf{V}}:\mathbf{V}}') = \mathscr{R}(\mathbf{P}_{\mathbf{X}:(\lambda\mathbf{E}_{\mathbf{X}}+\mathbf{V})}') = \mathscr{R}(\mathbf{P}_{\mathbf{X}:\mathbf{P}_{\mathbf{V}\mathbf{X}}}') = \mathscr{R}(\mathbf{V}\mathbf{X}).$$

Hence the equalities in (h) hold by (17). If $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ is unique, then $\mathbf{E}_{\mathbf{X}} + \mathbf{V}$ is pd by Theorem 6(g), so that the projector $\mathbf{P}_{\mathbf{V}\mathbf{X}:(\mathbf{E}_{\mathbf{X}}+\mathbf{V})^{-1}}$ is unique, too. In this case,

$$r[\,(\mathbf{E}_{\mathbf{X}} + \mathbf{V})^{-1}\mathbf{V}\mathbf{X},\, \mathbf{X}\,] = r[\,\mathbf{V}\mathbf{X},\, (\mathbf{E}_{\mathbf{X}} + \mathbf{V})\mathbf{X}\,] = r[\,\mathbf{V}\mathbf{X},\, \mathbf{V}\mathbf{X}\,] = r(\mathbf{V}\mathbf{X}) = r(\mathbf{X}). \quad (40)$$

This implies $\mathscr{R}[\,(\mathbf{E}_{\mathbf{X}} + \mathbf{V})^{-1}\mathbf{V}\mathbf{X}\,] = \mathscr{R}(\mathbf{X})$ by (24). In these cases, we derive from (b) that

$$\begin{aligned}
r(\mathbf{P}_{\mathbf{V}\mathbf{X}:(\mathbf{E}_{\mathbf{X}}+\mathbf{V})^{-1}}) &= r(\mathbf{V}\mathbf{X}) = r(\mathbf{X}), \\
\mathscr{R}(\mathbf{P}_{\mathbf{V}\mathbf{X}:(\mathbf{E}_{\mathbf{X}}+\mathbf{V})^{-1}}) &= \mathscr{R}(\mathbf{V}\mathbf{X}), \\
\mathscr{R}(\mathbf{P}_{\mathbf{V}\mathbf{X}:(\mathbf{E}_{\mathbf{X}}+\mathbf{V})^{-1}}') &= \mathscr{R}[\,(\mathbf{E}_{\mathbf{X}} + \mathbf{V})^{-1}\mathbf{V}\mathbf{X}\,] = \mathscr{R}(\mathbf{X}).
\end{aligned}$$

Hence applying (17) to the two idempotent matrices $\mathbf{P}'_{\mathbf{X}:\mathbf{V}}$ and $\mathbf{P}_{\mathbf{VX}:(\mathbf{E_X}+\mathbf{V})^{-1}}$ leads to (i). Since both $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ and $\mathbf{P}_{\mathbf{E_X}:(\mathbf{E_X}+\mathbf{V})^{-1}}$ are unique, we obtain from (c) that

$$r(\mathbf{P}_{\mathbf{X}:\mathbf{V}}) = r(\mathbf{X}) \quad \text{and} \quad r(\mathbf{P}_{\mathbf{E_X}:(\mathbf{E_X}+\mathbf{V})^{-1}}) = r(\mathbf{E_X}) = n - r(\mathbf{X}).$$

Thus

$$r(\mathbf{P}_{\mathbf{X}:\mathbf{V}}) + r(\mathbf{P}_{\mathbf{E_X}:(\mathbf{E_X}+\mathbf{V})^{-1}}) = n. \tag{41}$$

On the other hand, (40) is equivalent to $\mathbf{E_X}(\mathbf{E_X}+\mathbf{V})^{-1}\mathbf{VX} = \mathbf{0}$ by (13). In such a case, it is easy to verify that

$$\mathbf{P}_{\mathbf{X}:\mathbf{V}}\mathbf{P}'_{\mathbf{E_X}:(\mathbf{E_X}+\mathbf{V})^{-1}} = \mathbf{0} \quad \text{and} \quad \mathbf{P}'_{\mathbf{E_X}:(\mathbf{E_X}+\mathbf{V})^{-1}}\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{0}. \tag{42}$$

Applying (18) to (41) and (42) results in (j). Similarly, we can show that

$$r(\mathbf{P}_{\mathbf{X}:\mathbf{V}}) + r(\mathbf{P}_{(\mathbf{E_X}+\mathbf{V})^{-1}\mathbf{E_X}:(\mathbf{E_X}+\mathbf{V})}) = n,$$

$$\mathbf{P}_{\mathbf{X}:\mathbf{V}}\mathbf{P}_{(\mathbf{E_X}+\mathbf{V})^{-1}\mathbf{E_X}:(\mathbf{E_X}+\mathbf{V})} = \mathbf{P}_{(\mathbf{E_X}+\mathbf{V})^{-1}\mathbf{E_X}:(\mathbf{E_X}+\mathbf{V})}\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{0}.$$

Hence we obtain (k) from (18). Since $r(\mathbf{V}^+\mathbf{VX}) = r(\mathbf{VX})$, the projector $\mathbf{P}_{\mathbf{VX}:\mathbf{V}^+}$ is unique by Theorem 6(c). In such a case, we obtain from (30) that

$$\mathbf{P}_{\mathbf{VX}:\mathbf{V}^+} = \mathbf{VX}(\mathbf{X}'\mathbf{VV}^+\mathbf{VX})^+\mathbf{X}'\mathbf{VV}^+ = \mathbf{VX}(\mathbf{X}'\mathbf{VX})^+\mathbf{X}'\mathbf{VV}^+ = \mathbf{VP}_{\mathbf{X}:\mathbf{V}}\mathbf{V}^+,$$

as required for (l). $\square$

In the remaining part of this section, we give some rank equalities for two projectors and then use them to characterize relations between two projectors. The results obtained can be used to investigate relationships between two estimators under two linear models.

Theorem 8 *Let* $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ *be as given in* (30) *and suppose* $r(\mathbf{VX}) = r(\mathbf{X})$. *Then*

$$r(\mathbf{P}_{\mathbf{X}:\mathbf{V}} - \mathbf{P}'_{\mathbf{X}:\mathbf{V}}) = 2r[\mathbf{X},\ \mathbf{VX}] - 2r(\mathbf{X}), \tag{43}$$
$$r(\mathbf{P}_{\mathbf{X}:\mathbf{V}} - \mathbf{P}_{\mathbf{X}}) = r[\mathbf{X},\ \mathbf{VX}] - r(\mathbf{X}), \tag{44}$$
$$r(\mathbf{P}_{\mathbf{VX}} - \mathbf{P}_{\mathbf{X}}) = 2r[\mathbf{X},\ \mathbf{VX}] - 2r(\mathbf{X}). \tag{45}$$

Hence, the following statements are equivalent:

(a) $\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{P}'_{\mathbf{X}:\mathbf{V}}$.

(b) $\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{P}_{\mathbf{X}}$, *i.e.*, $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ *is the orthogonal projector onto* $\mathscr{R}(\mathbf{X})$.

(c) $\mathbf{P}_{\mathbf{VX}} = \mathbf{P}_{\mathbf{X}}$, *i.e.*, $\mathbf{P}_{\mathbf{VX}}$ *is the orthogonal projector onto* $\mathscr{R}(\mathbf{X})$.

(d) $\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{P}_{\mathbf{VX}:(\mathbf{E_X}+\mathbf{V})^{-1}}$.

(e) $\mathbf{P}_{\mathbf{E_X}:(\mathbf{E_X}+\mathbf{V})^{-1}}$ *is the orthogonal projector onto* $\mathscr{R}^{\perp}(\mathbf{X})$.

(f) $\mathbf{P}_{(\mathbf{E_X}+\mathbf{V})^{-1}\mathbf{E_X}:(\mathbf{E_X}+\mathbf{V})}$ *is the orthogonal projector onto* $\mathscr{R}^{\perp}(\mathbf{X})$.

(g) $\mathscr{R}(\mathbf{VX}) \subseteq \mathscr{R}(\mathbf{X})$.

Proof Since both $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ and $\mathbf{P}'_{\mathbf{X}:\mathbf{V}}$ are idempotent, we derive from (15) that

$$\begin{aligned} r(\,\mathbf{P}_{\mathbf{X}:\mathbf{V}} - \mathbf{P}'_{\mathbf{X}:\mathbf{V}}\,) &= 2r[\,\mathbf{P}_{\mathbf{X}:\mathbf{V}},\, \mathbf{P}'_{\mathbf{X}:\mathbf{V}}\,] - 2r(\mathbf{P}_{\mathbf{X}:\mathbf{V}}) \\ &= 2r[\,\mathbf{X},\, \mathbf{VX}\,] - 2r(\mathbf{X}) \quad \text{(by (28) and Theorem 7(c))}, \end{aligned}$$

establishing (43). Also by (15), we find that

$$\begin{aligned} r(\,\mathbf{P}_{\mathbf{X}:\mathbf{V}} - \mathbf{P}_{\mathbf{X}}\,) &= r[\,\mathbf{P}'_{\mathbf{X}:\mathbf{V}},\, \mathbf{P}_{\mathbf{X}}\,] + r[\,\mathbf{P}_{\mathbf{X}:\mathbf{V}},\, \mathbf{P}_{\mathbf{X}}\,] - r(\mathbf{P}_{\mathbf{X}:\mathbf{V}}) - r(\mathbf{X}) \\ &= r[\,\mathbf{VX},\, \mathbf{X}\,] - r(\mathbf{X}), \end{aligned}$$

and

$$\begin{aligned} r(\,\mathbf{P}_{\mathbf{VX}} - \mathbf{P}_{\mathbf{X}}\,) &= 2r[\,\mathbf{P}_{\mathbf{VX}},\, \mathbf{P}_{\mathbf{X}}\,] - r(\mathbf{P}_{\mathbf{VX}}) - r(\mathbf{X}) \\ &= 2r[\,\mathbf{VX},\, \mathbf{X}\,] - 2r(\mathbf{X}), \end{aligned}$$

establishing (44) and (45). The equivalence of (a), (b), (c) and (g) follows from (43), (44) and (45). The equivalence of (a), (d), (e) and (f) follows from Theorem 7(i), (j) and (k). □

Harville (1997, Sec. 14.2e) investigated relationships between two projectors and gave some necessary and sufficient conditions for $\mathbf{P}_{\mathbf{X}:\mathbf{V}_1} = \mathbf{P}_{\mathbf{X}:\mathbf{V}_2}$ to hold when $\mathbf{V}_1$ and $\mathbf{V}_2$ are pd. A general result on the equality $\mathbf{P}_{\mathbf{X}_1:\mathbf{V}_1} = \mathbf{P}_{\mathbf{X}_2:\mathbf{V}_2}$ is given below.

Theorem 9 *Let* $\mathbf{X}_1 \in \mathbb{R}^{n\times p}$ *and* $\mathbf{X}_2 \in \mathbb{R}^{n\times k}$, *let* $\mathbf{V}_1,\, \mathbf{V}_2 \in \mathbb{R}^{n\times n}$ *be nnd with* $r(\mathbf{V}_1\mathbf{X}_1) = r(\mathbf{X}_1)$ *and* $r(\mathbf{V}_2\mathbf{X}_2) = r(\mathbf{X}_2)$. *Then*

$$r(\,\mathbf{P}_{\mathbf{X}_1:\mathbf{V}_1} - \mathbf{P}_{\mathbf{X}_2:\mathbf{V}_2}\,) = r[\,\mathbf{V}_1\mathbf{X}_1,\, \mathbf{V}_2\mathbf{X}_2\,] + r[\,\mathbf{X}_1,\, \mathbf{X}_2\,] - r(\mathbf{X}_1) - r(\mathbf{X}_2). \tag{46}$$

Hence, the following statements are equivalent:

(a) $\mathbf{P}_{\mathbf{X}_1:\mathbf{V}_1} = \mathbf{P}_{\mathbf{X}_2:\mathbf{V}_2}$.

(b) $\mathscr{R}(\mathbf{X}_1) = \mathscr{R}(\mathbf{X}_2)$ *and* $\mathscr{R}(\mathbf{V}_1\mathbf{X}_1) = \mathscr{R}(\mathbf{V}_2\mathbf{X}_2)$.

Proof Since both $\mathbf{P}_{\mathbf{X}_1:\mathbf{V}_1}$ and $\mathbf{P}_{\mathbf{X}_2:\mathbf{V}_2}$ are idempotent, we obtain by (15), (28) and Theorem 7(c) that

$$\begin{aligned} & r(\,\mathbf{P}_{\mathbf{X}_1:\mathbf{V}_1} - \mathbf{P}_{\mathbf{X}_2:\mathbf{V}_2}\,) \\ &= r[\,\mathbf{P}'_{\mathbf{X}_1:\mathbf{V}_1},\, \mathbf{P}'_{\mathbf{X}_2:\mathbf{V}_2}\,] + r[\,\mathbf{P}_{\mathbf{X}_1:\mathbf{V}_1},\, \mathbf{P}_{\mathbf{X}_2:\mathbf{V}_2}\,] - r(\mathbf{P}_{\mathbf{X}_1:\mathbf{V}_1}) - r(\mathbf{P}_{\mathbf{X}_2:\mathbf{V}_2}) \\ &= r[\,\mathbf{V}_1\mathbf{X}_1,\, \mathbf{V}_2\mathbf{X}_2\,] + r[\,\mathbf{X}_1,\, \mathbf{X}_2\,] - r(\mathbf{X}_1) - r(\mathbf{X}_2), \end{aligned}$$

establishing (46). Also note that

$$r[\,\mathbf{V}_1\mathbf{X}_1,\, \mathbf{V}_2\mathbf{X}_2\,] \geq r(\mathbf{V}_1\mathbf{X}_1) = r(\mathbf{X}_1), \quad r[\,\mathbf{V}_1\mathbf{X}_1,\, \mathbf{V}_2\mathbf{X}_2\,] \geq r(\mathbf{V}_2\mathbf{X}_2) = r(\mathbf{X}_2),$$

$$r[\,\mathbf{X}_1,\, \mathbf{X}_2\,] \geq r(\mathbf{X}_1), \quad r[\,\mathbf{X}_1,\, \mathbf{X}_2\,] \geq r(\mathbf{X}_2).$$

Thus it follows from (46) that $\mathbf{P}_{\mathbf{X}_1:\mathbf{V}_1} = \mathbf{P}_{\mathbf{X}_2:\mathbf{V}_2}$ holds if and only if

$$r[\,\mathbf{V}_1\mathbf{X}_1,\ \mathbf{V}_2\mathbf{X}_2\,] = r(\mathbf{V}_1\mathbf{X}_1) = r(\mathbf{V}_2\mathbf{X}_2), \quad r[\,\mathbf{X}_1,\ \mathbf{X}_2\,] = r(\mathbf{X}_1) = r(\mathbf{X}_2),$$

both of which are equivalent to the two range equalities in (b) by (23) and (24). □

For more relations between $\mathbf{P}_{\mathbf{X}_1:\mathbf{V}_1}$ and $\mathbf{P}_{\mathbf{X}_2:\mathbf{V}_2}$ without the assumptions $r(\mathbf{V}_1\mathbf{X}_1) = r(\mathbf{X}_1)$ and $r(\mathbf{V}_2\mathbf{X}_2) = r(\mathbf{X}_2)$, see Tian and Takane (2007b).

Two results on sum decompositions of $\mathbf{V}$-orthogonal projectors are given below.

Theorem 10 *Let $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ be as given in (30) with $r(\mathbf{V}\mathbf{X}) = r(\mathbf{X})$, and partition $\mathbf{X}$ as $\mathbf{X} = [\,\mathbf{X}_1,\ \mathbf{X}_2\,]$. Then:*

(a) *Both $\mathbf{P}_{\mathbf{X}_1:\mathbf{V}}$ and $\mathbf{P}_{\mathbf{X}_2:\mathbf{V}}$ are unique.*

(b) $\mathbf{P}_{\mathbf{X}:\mathbf{V}}\mathbf{P}_{\mathbf{X}_i:\mathbf{V}} = \mathbf{P}_{\mathbf{X}_i:\mathbf{V}}$, $i = 1, 2$.

(c) $r(\,\mathbf{P}_{\mathbf{X}:\mathbf{V}} - \mathbf{P}_{\mathbf{X}_i:\mathbf{V}}\,) = r(\mathbf{X}) - r(\mathbf{X}_i)$, $i = 1, 2$.

(d) $r(\,\mathbf{P}_{\mathbf{X}:\mathbf{V}} - \mathbf{P}_{\mathbf{X}_1:\mathbf{V}} - \mathbf{P}_{\mathbf{X}_2:\mathbf{V}}\,) = r(\mathbf{X}) + 2r(\mathbf{X}_1'\mathbf{V}\mathbf{X}_2) - r(\mathbf{X}_1) - r(\mathbf{X}_2)$.

(e) $\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{P}_{\mathbf{X}_1:\mathbf{V}}$ *holds if and only if* $\mathscr{R}(\mathbf{X}_2) \subseteq \mathscr{R}(\mathbf{X}_1)$; $\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{P}_{\mathbf{X}_2:\mathbf{V}}$ *holds if and only if* $\mathscr{R}(\mathbf{X}_1) \subseteq \mathscr{R}(\mathbf{X}_2)$.

(f) $\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{P}_{\mathbf{X}_1:\mathbf{V}} + \mathbf{P}_{\mathbf{X}_2:\mathbf{V}}$ *holds if and only if* $\mathbf{X}_1'\mathbf{V}\mathbf{X}_2 = \mathbf{0}$.

Proof Note that $\mathscr{R}(\mathbf{X}'\mathbf{V}) = \mathscr{R}(\mathbf{X}')$ obviously implies $\mathscr{R}(\mathbf{X}_1'\mathbf{V}) = \mathscr{R}(\mathbf{X}_1')$ and $\mathscr{R}(\mathbf{X}_2'\mathbf{V}) = \mathscr{R}(\mathbf{X}_2')$. Hence both $\mathbf{P}_{\mathbf{X}_1:\mathbf{V}}$ and $\mathbf{P}_{\mathbf{X}_2:\mathbf{V}}$ are unique, too, by Theorem 6(b). Also note from Theorem 7(c) that

$$\mathscr{R}(\mathbf{P}_{\mathbf{X}:\mathbf{V}}) = \mathscr{R}(\mathbf{X}), \quad \mathscr{R}(\mathbf{P}_{\mathbf{X}_1:\mathbf{V}}) = \mathscr{R}(\mathbf{X}_1), \quad \mathscr{R}(\mathbf{P}_{\mathbf{X}_2:\mathbf{V}}) = \mathscr{R}(\mathbf{X}_2),$$

and that

$$\mathscr{R}(\mathbf{X}_1) \subseteq \mathscr{R}(\mathbf{X}) \ \text{ and } \ \mathscr{R}(\mathbf{X}_2) \subseteq \mathscr{R}(\mathbf{X}).$$

Hence the results in (b) follow from Theorem 7(a). Since $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$, $\mathbf{P}_{\mathbf{X}_1:\mathbf{V}}$ and $\mathbf{P}_{\mathbf{X}_2:\mathbf{V}}$ are idempotent when they are unique, we obtain from (15) and (28) that

$$\begin{aligned} r(\,\mathbf{P}_{\mathbf{X}:\mathbf{V}} - \mathbf{P}_{\mathbf{X}_1:\mathbf{V}}\,) &= r\begin{bmatrix} \mathbf{P}_{\mathbf{X}:\mathbf{V}} \\ \mathbf{P}_{\mathbf{X}_1:\mathbf{V}} \end{bmatrix} + r[\,\mathbf{P}_{\mathbf{X}:\mathbf{V}},\ \mathbf{P}_{\mathbf{X}_1:\mathbf{V}}\,] - r(\mathbf{P}_{\mathbf{X}:\mathbf{V}}) - r(\mathbf{P}_{\mathbf{X}_1:\mathbf{V}}) \\ &= r\begin{bmatrix} \mathbf{X}'\mathbf{V} \\ \mathbf{X}_1'\mathbf{V} \end{bmatrix} + r[\,\mathbf{X},\ \mathbf{X}_1\,] - r(\mathbf{X}) - r(\mathbf{X}_1) \\ &= r(\mathbf{X}) - r(\mathbf{X}_1), \end{aligned}$$

establishing the rank equality in (c) for $i = 1$. The equality in (c) for $i = 2$ can be shown similarly. The rank equality in (d) is derived from (19), the details are omitted. The results in (e) and (f) follow directly from (c) and (d). □

For further results on relations between $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ in (7) and $\mathbf{P}_{\mathbf{X}_1:\mathbf{V}} + \mathbf{P}_{\mathbf{X}_2:\mathbf{V}}$ and their applications to estimations under (2), see Tian and Takane (2007a).

Theorem 11 *Let* $\mathbf{X}$, $\mathbf{Y} \in \mathbb{R}^{n\times p}$, *let* $\mathbf{V} \in \mathbb{R}^{n\times n}$ *be nnd, and suppose* $r(\,\mathbf{VX}+\mathbf{VY}\,) = r(\,\mathbf{X}+\mathbf{Y}\,)$, $r(\mathbf{VX}) = r(\mathbf{X})$ *and* $r(\mathbf{VY}) = r(\mathbf{Y})$. *If* $\mathbf{X}'\mathbf{VY} = \mathbf{0}$ *and* $\mathbf{XY}' = \mathbf{0}$, *then* $\mathbf{P}_{(\mathbf{X}+\mathbf{Y}):\mathbf{V}} = \mathbf{P}_{\mathbf{X}:\mathbf{V}} + \mathbf{P}_{\mathbf{Y}:\mathbf{V}}$.

Proof It is well known that if $\mathbf{X}'\mathbf{Y} = \mathbf{0}$ and $\mathbf{XY}' = \mathbf{0}$, which are equivalent to $\mathbf{X}^+\mathbf{Y} = \mathbf{Y}^+\mathbf{X} = \mathbf{0}$ and $\mathbf{YX}^+ = \mathbf{XY}^+ = \mathbf{0}$, then $(\,\mathbf{X}+\mathbf{Y}\,)^+ = \mathbf{X}^+ + \mathbf{Y}^+$. Hence if $\mathbf{X}'\mathbf{VY} = \mathbf{0}$ and $\mathbf{XY}' = \mathbf{0}$, then

$$(\,\mathbf{V}^{1/2}\mathbf{X} + \mathbf{V}^{1/2}\mathbf{Y}\,)^+ = (\mathbf{V}^{1/2}\mathbf{X})^+ + (\mathbf{V}^{1/2}\mathbf{Y})^+.$$

Applying this equality and (29) to $\mathbf{P}_{(\mathbf{X}+\mathbf{Y}):\mathbf{V}}$ leads to

$$\begin{aligned}
\mathbf{P}_{(\mathbf{X}+\mathbf{Y}):\mathbf{V}} &= (\mathbf{X}+\mathbf{Y})[\,\mathbf{V}^{1/2}(\mathbf{X}+\mathbf{Y})\,]^+\mathbf{V}^{1/2}\\
&= (\mathbf{X}+\mathbf{Y})[\,(\mathbf{V}^{1/2}\mathbf{X})^+ + (\mathbf{V}^{1/2}\mathbf{Y})^+\,]\mathbf{V}^{1/2}\\
&= \mathbf{X}(\mathbf{V}^{1/2}\mathbf{X})^+\mathbf{V}^{1/2} + \mathbf{Y}(\mathbf{V}^{1/2}\mathbf{Y})^+\mathbf{V}^{1/2}\\
&= \mathbf{P}_{\mathbf{X}:\mathbf{V}} + \mathbf{P}_{\mathbf{Y}:\mathbf{V}},
\end{aligned}$$

as required. $\square$

3 Properties of $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ when $\mathbf{V}$ is positive definite

In statistical practice, the weight matrix $\mathbf{V}$ is often assumed to be positive definite. In this case, the rank equality $r(\mathbf{VX}) = r(\mathbf{X})$ is satisfied, and the projector $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ in (7) is unique and possesses all the properties presented in the previous section. Because the inverse of $\mathbf{V}^{1/2}$ exists, the $\mathbf{V}$-orthogonal projector $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ in (30) can be written as

$$\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{X}(\mathbf{V}^{1/2}\mathbf{X})^+\mathbf{V}^{1/2} = \mathbf{V}^{-1/2}(\mathbf{V}^{1/2}\mathbf{X})(\mathbf{V}^{1/2}\mathbf{X})^+\mathbf{V}^{1/2} = \mathbf{V}^{-1/2}\mathbf{P}_{\mathbf{V}^{1/2}\mathbf{X}}\mathbf{V}^{1/2}. \quad (47)$$

This result indicates that $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ is similar to the orthogonal projector $\mathbf{P}_{\mathbf{V}^{1/2}\mathbf{X}}$.

In addition, the $\mathbf{V}$-orthogonal projector in (47) can also be expressed through the weighted Moore-Penrose inverse of $\mathbf{X}$. Recall that the weighted Moore-Penrose inverse of a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ with respect to two pd matrices $\mathbf{M} \in \mathbb{R}^{m\times m}$ and $\mathbf{N} \in \mathbb{R}^{n\times n}$ is defined to be the unique solution $\mathbf{G}$ satisfying the following four matrix equations

$$\text{(i) } \mathbf{AGA} = \mathbf{A}, \quad \text{(ii) } \mathbf{GAG} = \mathbf{G}, \quad \text{(iii) } (\mathbf{MAG})' = \mathbf{MAG}, \quad \text{(iv) } (\mathbf{NGA})' = \mathbf{NGA},$$

and is denoted by $\mathbf{G} = \mathbf{A}^+_{\mathbf{M},\mathbf{N}}$. It is well known that the weighted Moore-Penrose inverse $\mathbf{A}^+_{\mathbf{M},\mathbf{N}}$ of $\mathbf{A} \in \mathbb{R}^{m\times n}$ can be rewritten as

$$\mathbf{A}^+_{\mathbf{M},\mathbf{N}} = \mathbf{N}^{-1/2}(\mathbf{M}^{1/2}\mathbf{A}\mathbf{N}^{-1/2})^+\mathbf{M}^{1/2}, \quad (48)$$

where $\mathbf{M}^{1/2}$ and $\mathbf{N}^{1/2}$ are the pd square roots of $\mathbf{M}$ and $\mathbf{N}$, respectively: see, e.g., Ben-Israel and Greville (2003).

The following result reveals relations between $\mathbf{V}$-orthogonal projectors and weighted Moore-Penrose inverses of matrices.

Theorem 12 *Let $\mathbf{X} \in \mathbb{R}^{n\times p}$, and let $\mathbf{M} \in \mathbb{R}^{n\times n}$ and $\mathbf{N} \in \mathbb{R}^{p\times p}$ be pd. Then*

$$\mathbf{P}_{\mathbf{X}:\mathbf{M}} = \mathbf{X}\mathbf{X}^{+}_{\mathbf{M},\mathbf{I}_p}, \quad \mathbf{P}'_{\mathbf{X}':\mathbf{N}^{-1}} = \mathbf{X}^{+}_{\mathbf{I}_n,\mathbf{N}}\mathbf{X}, \quad \mathbf{X}^{+}_{\mathbf{M},\mathbf{N}} = \mathbf{P}'_{\mathbf{X}':\mathbf{N}^{-1}}\mathbf{X}^{-}\mathbf{P}_{\mathbf{X}:\mathbf{M}}. \tag{49}$$

Proof Applying (30) and (48) to $\mathbf{X}\mathbf{X}^{+}_{\mathbf{M},\mathbf{I}_p}$ and $\mathbf{X}^{+}_{\mathbf{I}_n,\mathbf{N}}\mathbf{X}$ gives

$$\mathbf{X}\mathbf{X}^{+}_{\mathbf{M},\mathbf{I}_p} = \mathbf{X}(\mathbf{M}^{\frac{1}{2}}\mathbf{X})^{+}\mathbf{M}^{\frac{1}{2}} = \mathbf{P}_{\mathbf{X}:\mathbf{M}} \ \text{ and } \ \mathbf{X}^{+}_{\mathbf{I}_n,\mathbf{N}}\mathbf{X} = \mathbf{N}^{-1/2}(\mathbf{X}\mathbf{N}^{-1/2})^{+}\mathbf{X} = \mathbf{P}'_{\mathbf{X}':\mathbf{N}^{-1}}.$$

Also by definition and (48)

$$\begin{aligned}\mathbf{X}^{+}_{\mathbf{M},\mathbf{N}} = \mathbf{X}^{+}_{\mathbf{M},\mathbf{N}}\mathbf{X}\mathbf{X}^{+}_{\mathbf{M},\mathbf{N}} &= (\mathbf{X}^{+}_{\mathbf{M},\mathbf{N}}\mathbf{X})\mathbf{X}^{-}(\mathbf{X}\mathbf{X}^{+}_{\mathbf{M},\mathbf{N}})\\ &= (\mathbf{X}^{+}_{\mathbf{I}_n,\mathbf{N}}\mathbf{X})\mathbf{X}^{-}(\mathbf{X}\mathbf{X}^{+}_{\mathbf{M},\mathbf{I}_p})\\ &= \mathbf{P}'_{\mathbf{X}':\mathbf{N}^{-1}}\mathbf{X}^{-}\mathbf{P}_{\mathbf{X}:\mathbf{M}}.\end{aligned}$$

Hence, the three equalities in (49) hold. □

Further properties of $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ with $\mathbf{V}$ pd are given below.

Theorem 13 *Let $\mathbf{X} \in \mathbb{R}^{n\times p}$, and let $\mathbf{V} \in \mathbb{R}^{n\times n}$ be pd. Then:*

(a) $\mathbf{P}_{\mathbf{X}:\mathbf{V}} + \mathbf{P}'_{\mathbf{E}_\mathbf{X}:\mathbf{V}^{-1}} = \mathbf{I}_n$.

(b) $\mathbf{P}_{\mathbf{X}:\mathbf{V}} + \mathbf{P}_{\mathbf{V}^{-1}\mathbf{E}_\mathbf{X}:\mathbf{V}} = \mathbf{I}_n$, *i.e.*, $\mathbf{I}_n - \mathbf{P}_{\mathbf{X}:\mathbf{V}}$ *is the* $\mathbf{V}$*-orthogonal projector onto* $\mathscr{R}(\mathbf{V}^{-1}\mathbf{E}_\mathbf{X})$.

(c) $\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{X}\mathbf{X}'(\,\mathbf{X}\mathbf{X}' + \mathbf{V}^{-1}\mathbf{E}_\mathbf{X}\mathbf{V}^{-1}\,)^{-1} = \mathbf{X}\mathbf{X}'\mathbf{V}(\,\mathbf{V}\mathbf{X}\mathbf{X}'\mathbf{V} + \mathbf{E}_\mathbf{X}\,)^{-1}\mathbf{V}$.

(d) $\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{P}_\mathbf{X}(\,\mathbf{P}_\mathbf{X} + \mathbf{V}^{-1}\mathbf{E}_\mathbf{X}\mathbf{V}^{-1}\,)^{-1} = \mathbf{P}_\mathbf{X}\mathbf{V}(\,\mathbf{V}\mathbf{P}_\mathbf{X}\mathbf{V} + \mathbf{E}_\mathbf{X}\,)^{-1}\mathbf{V}$.

(e) $\mathbf{P}_{\mathbf{V}\mathbf{X}:\mathbf{V}^{-1}} = \mathbf{V}\mathbf{P}_{\mathbf{X}:\mathbf{V}}\mathbf{V}^{-1} = \mathbf{P}'_{\mathbf{X}:\mathbf{V}}$.

(f) $\mathbf{P}_{\mathbf{V}^{-1}\mathbf{X}:\mathbf{V}} = \mathbf{V}^{-1}\mathbf{P}_{\mathbf{X}:\mathbf{V}^{-1}}\mathbf{V} = \mathbf{P}'_{\mathbf{X}:\mathbf{V}^{-1}}$.

Theorem 13(a) is called Khatri's lemma (1966). Its generalization has been given by Yanai and Takane (1992).

Proof It is easy to verify that

$$r(\mathbf{P}_{\mathbf{X}:\mathbf{V}}) + r(\mathbf{P}'_{\mathbf{E}_\mathbf{X}:\mathbf{V}^{-1}}) = n, \quad r(\mathbf{P}_{\mathbf{X}:\mathbf{V}}) + r(\mathbf{P}_{\mathbf{V}^{-1}\mathbf{E}_\mathbf{X}:\mathbf{V}}) = n,$$

$$\mathbf{P}_{\mathbf{X}:\mathbf{V}}\mathbf{P}'_{\mathbf{E}_\mathbf{X}:\mathbf{V}^{-1}} = \mathbf{P}'_{\mathbf{E}_\mathbf{X}:\mathbf{V}^{-1}}\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{0}, \quad \mathbf{P}_{\mathbf{X}:\mathbf{V}}\mathbf{P}_{\mathbf{V}^{-1}\mathbf{E}_\mathbf{X}:\mathbf{V}} = \mathbf{P}_{\mathbf{V}^{-1}\mathbf{E}_\mathbf{X}:\mathbf{V}}\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{0}.$$

Hence (a) and (b) follow from (18). It can be seen from (13) that

$$\begin{aligned} r[\,\mathbf{X},\, \mathbf{V}^{-1}\mathbf{E}_\mathbf{X}\,] &= r(\mathbf{X}) + r(\mathbf{E}_\mathbf{X}\mathbf{V}^{-1}\mathbf{E}_\mathbf{X}) = r(\mathbf{X}) + r(\mathbf{E}_\mathbf{X}) = n,\\ r[\,\mathbf{P}_\mathbf{X},\, \mathbf{V}^{-1}\mathbf{E}_\mathbf{X}\,] &= r(\mathbf{P}_\mathbf{X}) + r(\mathbf{E}_\mathbf{X}\mathbf{V}^{-1}\mathbf{E}_\mathbf{X}) = r(\mathbf{X}) + r(\mathbf{E}_\mathbf{X}) = n.\end{aligned}$$

Hence the following four matrices

$$\begin{aligned} \big[\,\mathbf{X},\, \mathbf{V}^{-1}\mathbf{E}_\mathbf{X}\,\big]\,\big[\,\mathbf{X},\, \mathbf{V}^{-1}\mathbf{E}_\mathbf{X}\,\big]' &= \mathbf{X}\mathbf{X}' + \mathbf{V}^{-1}\mathbf{E}_\mathbf{X}\mathbf{V}^{-1},\\ [\,\mathbf{V}\mathbf{X},\, \mathbf{E}_\mathbf{X}\,]\,[\,\mathbf{V}\mathbf{X},\, \mathbf{E}_\mathbf{X}\,]' &= \mathbf{V}\mathbf{X}\mathbf{X}'\mathbf{V} + \mathbf{E}_\mathbf{X}\mathbf{V},\\ \big[\,\mathbf{P}_\mathbf{X},\, \mathbf{V}^{-1}\mathbf{E}_\mathbf{X}\,\big]\,\big[\,\mathbf{P}_\mathbf{X},\, \mathbf{V}^{-1}\mathbf{E}_\mathbf{X}\,\big]' &= \mathbf{P}_\mathbf{X} + \mathbf{V}^{-1}\mathbf{E}_\mathbf{X}\mathbf{V}^{-1},\\ [\,\mathbf{V}\mathbf{P}_\mathbf{X},\, \mathbf{E}_\mathbf{X}\,]\,[\,\mathbf{V}\mathbf{P}_\mathbf{X},\, \mathbf{E}_\mathbf{X}\,]' &= \mathbf{V}\mathbf{P}_\mathbf{X}\mathbf{V} + \mathbf{E}_\mathbf{X}\end{aligned}$$

are nonsingular. In these cases,

$$\begin{aligned}
r[\,\mathbf{P}_{\mathbf{X}:\mathbf{V}} - \mathbf{X}\mathbf{X}'(\,\mathbf{X}\mathbf{X}' + \mathbf{V}^{-1}\mathbf{E}_{\mathbf{X}}\mathbf{V}^{-1}\,)^{-1}\,] &= r[\,\mathbf{P}_{\mathbf{X}:\mathbf{V}}(\,\mathbf{X}\mathbf{X}' + \mathbf{V}^{-1}\mathbf{E}_{\mathbf{X}}\mathbf{V}^{-1}\,) - \mathbf{X}\mathbf{X}'\,]\\
&= r[\,\mathbf{P}_{\mathbf{X}:\mathbf{V}}\mathbf{X}\mathbf{X}' + \mathbf{P}_{\mathbf{X}:\mathbf{V}}\mathbf{X}\mathbf{V}^{-1}\mathbf{E}_{\mathbf{X}}\mathbf{V}^{-1} - \mathbf{X}\mathbf{X}'\,]\\
&= r(\,\mathbf{X}\mathbf{X}' - \mathbf{X}\mathbf{X}'\,) = 0,\\
r[\,\mathbf{P}_{\mathbf{X}:\mathbf{V}} - \mathbf{P}_{\mathbf{X}}(\,\mathbf{P}_{\mathbf{X}} + \mathbf{V}^{-1}\mathbf{E}_{\mathbf{X}}\mathbf{V}^{-1}\,)^{-1}\,] &= r[\,\mathbf{P}_{\mathbf{X}:\mathbf{V}}(\,\mathbf{P}_{\mathbf{X}} + \mathbf{V}^{-1}\mathbf{E}_{\mathbf{X}}\mathbf{V}^{-1}\,) - \mathbf{P}_{\mathbf{X}}\,]\\
&= r[\,\mathbf{P}_{\mathbf{X}:\mathbf{V}}\mathbf{P}_{\mathbf{X}} + \mathbf{P}_{\mathbf{X}:\mathbf{V}}\mathbf{X}\mathbf{V}^{-1}\mathbf{E}_{\mathbf{X}}\mathbf{V}^{-1} - \mathbf{P}_{\mathbf{X}}\,]\\
&= r(\,\mathbf{P}_{\mathbf{X}} - \mathbf{P}_{\mathbf{X}}\,) = 0,
\end{aligned}$$

so that

$$\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{X}\mathbf{X}'(\,\mathbf{X}\mathbf{X}' + \mathbf{V}^{-1}\mathbf{E}_{\mathbf{X}}\mathbf{V}^{-1}\,)^{-1} = \mathbf{P}_{\mathbf{X}}(\,\mathbf{P}_{\mathbf{X}} + \mathbf{V}^{-1}\mathbf{E}_{\mathbf{X}}\mathbf{V}^{-1}\,)^{-1}. \tag{50}$$

Substituting the following two equalities

$$\begin{aligned}
(\,\mathbf{X}\mathbf{X}' + \mathbf{V}^{-1}\mathbf{E}_{\mathbf{X}}\mathbf{V}^{-1}\,)^{-1} &= \mathbf{V}(\,\mathbf{V}\mathbf{X}\mathbf{X}'\mathbf{V} + \mathbf{E}_{\mathbf{X}}\,)^{-1}\mathbf{V},\\
(\,\mathbf{P}_{\mathbf{X}} + \mathbf{V}^{-1}\mathbf{E}_{\mathbf{X}}\mathbf{V}^{-1}\,)^{-1} &= \mathbf{V}(\,\mathbf{V}\mathbf{P}_{\mathbf{X}}\mathbf{V} + \mathbf{E}_{\mathbf{X}}\,)^{-1}\mathbf{V}
\end{aligned}$$

into (50) gives the equalities in (c) and (d). By (30),

$$\begin{aligned}
\mathbf{P}_{\mathbf{V}\mathbf{X}:\mathbf{V}^{-1}} &= \mathbf{V}\mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{V}^{-1}\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V}\mathbf{V}^{-1} = \mathbf{V}\mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V}\mathbf{V}^{-1} = \mathbf{V}\mathbf{P}_{\mathbf{X}:\mathbf{V}}\mathbf{V}^{-1},\\
\mathbf{P}_{\mathbf{V}\mathbf{X}:\mathbf{V}^{-1}} &= \mathbf{V}\mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}' = \mathbf{P}'_{\mathbf{X}:\mathbf{V}},
\end{aligned}$$

as required for (e). Replacing $\mathbf{V}$ with $\mathbf{V}^{-1}$ in (e) leads to (f). □

Theorem 14 *Let* $\mathbf{X}_1 \in \mathbb{R}^{n\times p}$ *and* $\mathbf{X}_2 \in \mathbb{R}^{n\times k}$, *and let* $\mathbf{V}_1$, $\mathbf{V}_2 \in \mathbb{R}^{n\times n}$ *be pd. Then the following statements are equivalent*:

(a) $\mathbf{P}_{\mathbf{X}_1:\mathbf{V}_1} = \mathbf{P}_{\mathbf{X}_2:\mathbf{V}_2}$.

(b) $\mathbf{P}_{\mathbf{E}_{\mathbf{X}_1}:\mathbf{V}_1^{-1}} = \mathbf{P}_{\mathbf{E}_{\mathbf{X}_2}:\mathbf{V}_2^{-1}}$.

(c) $\mathbf{P}_{\mathbf{V}_1^{-1}\mathbf{E}_{\mathbf{X}_1}:\mathbf{V}_1} = \mathbf{P}_{\mathbf{V}_2^{-1}\mathbf{E}_{\mathbf{X}_2}:\mathbf{V}_2}$.

(d) $\mathbf{P}_{\mathbf{V}_1\mathbf{X}_1:\mathbf{V}_1^{-1}} = \mathbf{P}_{\mathbf{V}_2\mathbf{X}_2:\mathbf{V}_2^{-1}}$.

(e) $\mathscr{R}(\mathbf{X}_1) = \mathscr{R}(\mathbf{X}_2)$ *and* $\mathscr{R}(\mathbf{V}_1\mathbf{X}_1) = \mathscr{R}(\mathbf{V}_2\mathbf{X}_2)$.

(f) $\mathscr{R}(\mathbf{E}_{\mathbf{X}_1}) = \mathscr{R}(\mathbf{E}_{\mathbf{X}_2})$ *and* $\mathscr{R}(\mathbf{V}_1^{-1}\mathbf{E}_{\mathbf{X}_1}) = \mathscr{R}(\mathbf{V}_2^{-1}\mathbf{E}_{\mathbf{X}_2})$.

Proof It follows from Theorem 8(a) and (b) and Theorem 13(a), (b) and (e). □

When $\mathbf{X}_1 = \mathbf{X}_2$, statements (a) and (e) above are equivalent to Theorem 14.12.18 of Harville (1997).

Theorem 15 *Let* $\mathbf{X} \in \mathbb{R}^{n\times p}$ *and let* $\mathbf{V} \in \mathbb{R}^{n\times n}$ *be pd. Then the following statements are equivalent*:

(a) $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ *is the orthogonal projector onto* $\mathscr{R}(\mathbf{X})$.

(b) $\mathbf{P}_{\mathbf{VX}}$ *is the orthogonal projector onto* $\mathscr{R}(\mathbf{X})$.

(c) $\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{P}_{\mathbf{VX}:\mathbf{V}^{-1}}$.

(d) $\mathbf{P}_{\mathbf{E}_{\mathbf{X}}:\mathbf{V}^{-1}}$ *is the orthogonal projector onto* $\mathscr{R}^{\perp}(\mathbf{X})$.

(e) $\mathbf{P}_{\mathbf{V}^{-1}\mathbf{E}_{\mathbf{X}}:\mathbf{V}}$ *is the orthogonal projector onto* $\mathscr{R}^{\perp}(\mathbf{X})$.

(f) $\mathscr{R}(\mathbf{VX}) \subseteq \mathscr{R}(\mathbf{X})$.

Proof It follows from Theorem 8(b), (c) and (g), and Theorem 13(a), (b) and (e). □

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