On reverse-order laws for least-squares g-inverses and minimum norm g-inverses of a matrix product

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Abstract. Let $A^{(1,3)}$ and $A^{(1,4)}$ denote a least-squares g-inverse and a minimum norm g-inverse of a matrix A, respectively. In this paper, we establish necessary and sufficient conditions for $\{B^{(1,3)}A^{(1,3)}\} \subseteq \{(AB)^{(1,3)}\}$ and $\{B^{(1,4)}A^{(1,4)}\} \subseteq \{(AB)^{(1,4)}\}$ to hold. We also show that the well-known reverse-order law $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ is equivalent to $\{B^{(1,3)}A^{(1,3)}\} \subseteq \{(AB)^{(1,3)}\}$ and $\{B^{(1,4)}A^{(1,4)}\} \subseteq \{(AB)^{(1,4)}\}$.

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1 Introduction

Let $\mathbb{C}^{m\times n}$ denote the set of all $m \times n$ matrices over the field of complex numbers. For a matrix $A \in \mathbb{C}^{m\times n}$, the Moore-Penrose inverse $A^{\dagger}$ is defined to be the unique solution of the following four Penrose equations

$$\text{(i)} \quad AXA = A, \quad \text{(ii)} \quad XAX = X, \quad \text{(iii)} \quad (AX)^* = AX, \quad \text{(iv)} \quad (XA)^* = XA,$$

where $(\cdot)^*$ denotes the conjugate transpose of a matrix. A matrix X is called a generalized inverse of A, denoted by A^-, if it satisfies (i), an outer inverse of A, denoted by $A^{(2)}$, if it satisfies (ii), a reflexive generalized inverse of A, denoted by $A^{(1,2)}$, if it satisfies both (i) and (ii), a $\{1, 3\}$-inverse of A, or least-squares g-inverse of A, denoted by $A^{(1,3)}$, if it satisfies both (i) and (iii), while the collection of all $A^{(1,3)}$ is denoted by $\{A^{(1,3)}\}$, a $\{1, 4\}$-inverse of A, or minimum norm g-inverse of A, denoted by $A^{(1,4)}$, if it satisfies both (i) and (iv), while the collection of all $A^{(1,4)}$ is denoted by $\{A^{(1,4)}\}$. $\{1, 2, 3\}$-inverse and $\{1, 2, 4\}$-inverse of A are defined similarly. General properties of the above generalized inverses can be found in [2, 4].

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As the names indicate, least-squares g-inverses and minimum norm g-inverses of a matrix arise from expressing least-squares solutions and minimum norm solutions of linear matrix equations. Suppose we are given a linear matrix equation $Ax = b$. The least-squares solution of $Ax = b$ is defined as $x = Gy$, which minimizes $\| b - Ax \|^2 = (b - Ax)^*(b - Ax)$. In this case, G is a least-squares g-inverse of A. If $Ax = b$ is consistent, the minimum norm solution of $Ax = b$ is defined as $x = Hy$, which minimizes $\|x\|^2 = x^*Ax$ subject to $Ax = b$. In this case, H is a minimum norm g-inverse of A. Least-squares solutions and minimum norm solutions of linear matrix equations, as well as least-squares g-inverses and minimum norm g-inverses of matrices have been extensively investigated, see, e.g., [7, 24]. Some recent work on minimum norm g-inverses of block matrices and its applications is given in [19].

In the theory of generalized inverses and its applications, a fundamental matrix operation is to find generalized inverses of matrix products. If A and B are two nonsingular matrices of the same size, the product AB is nonsingular, too, and the usual inverse of AB can be expressed as $(AB)^{-1} = B^{-1}A^{-1}$. This equality is called the reverse-order law in matrix theory, which can be used to simplify various matrix expressions that involve inverses of matrix products. However, this law cannot trivially be extended to generalized inverses of matrix products. Because a matrix has different types of g-inverses, reverse-order laws for the product of matrices have different forms, such as, $(AB)^- = B^-A^-$, $(AB)^{(1,2)} = B^{(1,2)}A^{(1,2)}$, $(AB)^\dagger = B^\dagger A^\dagger$, $(AB)^\dagger = B^-A^-$, $(AB)^{(1,3)} = B^{(1,3)}A^{(1,3)}$, $(AB)^{(1,4)} = B^{(1,4)}A^{(1,4)}$. Reverse-order laws for generalized inverses have been investigated in the literature since 1960s. This is due to the applications of these laws in simplifying matrix expressions involving generalized inverses. Some common methods for characterizing reverse-order laws for matrix products include definitions of g-inverses, range and null spaces of matrices, matrix rank equalities, as well as various matrix decompositions. A well-known result due to Greville [5] is

$$(AB)^\dagger = B^\dagger A^\dagger \ \Leftrightarrow \ \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B) \text{ and } \mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*), \tag{1.1}$$

where $\mathcal{R}(\cdot)$ denotes the column space of a matrix. The two range conditions in (1.1) are also equivalent to the two rank equalities $r[\, B, \ A^*AB \,] = r(B)$ and $r[\, A^*, \ BB^*A^* \,] = r(A)$, where $r(\cdot)$ denotes the rank of a matrix. Another well-known necessary and sufficient condition for $(AB)^\dagger = B^\dagger A^\dagger$ to hold is $\mathcal{R}(A^*ABB^*) = \mathcal{R}(BB^*A^*A)$, see [1]. A new investigation to the equivalence of $(AB)^\dagger = B^\dagger A^\dagger$ and other mixed-type reverse-order laws is given in Tian [16]. For the reverse-order law $(AB)^- = B^-A^-$, Werner [21, 22] showed the following

$$\{B^-A^-\} \subseteq \{(AB)^-\} \ \Leftrightarrow \ r(AB) = r(A) + r(B) - n.$$

In a recent paper [20], Wei and Guo considered the set inclusions

$$\{B^{(1,3)}A^{(1,3)}\} \subseteq \{(AB)^{(1,3)}\}, \tag{1.2}$$

$$\{B^{(1,4)}A^{(1,4)}\} \subseteq \{(AB)^{(1,4)}\}, \tag{1.3}$$

and gave some necessary and sufficient conditions for (1.2) and (1.3) to hold by applying the product singular value decomposition (P-SVD).

It has been noticed since 1970s that the rank of matrix provides a powerful method for investigating the relations between any two matrix expressions involving generalized inverses. In fact, any two matrices A and B of the same size are equal if and only if $r(A - B) = 0$, where $r(\cdot)$ denotes the rank of a matrix. If one can find some nontrivial formulas for the rank of $A - B$, then necessary and sufficient conditions for $A = B$ to hold can be derived from these rank formulas. Tian [14] recently establishes a set of rank equalities for matrix expressions consisting of $(AB)^{\dagger}$, $A^{\dagger}$ and $B^{\dagger}$.

Lemma 1.1. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Then,*

$$\begin{aligned} r[\,(AB)(AB)^{\dagger} - (AB)(B^{\dagger}A^{\dagger})\,] &= r(\, A^*AB - BB^{\dagger}A^*AB\,) \\ &= r[\, ABB^{\dagger} - (AB)(AB)^{\dagger}A\,] \\ &= \frac{1}{2}r(\, A^*ABB^{\dagger} - BB^{\dagger}A^*A\,) \\ &= r[\,(ABB^{\dagger})^{\dagger} - BB^{\dagger}A^{\dagger}\,] \\ &= r[\,B^{\dagger}(ABB^{\dagger})^{\dagger} - B^{\dagger}A^{\dagger}\,] \\ &= r[\,B,\ A^*AB\,] - r(B). \end{aligned} \tag{1.4}$$

Hence the following eight statements are equivalent:

(a) $AB(AB)^{\dagger} = ABB^{\dagger}A^{\dagger}$.
(b) $A^*AB = BB^{\dagger}A^*AB$.
(c) $ABB^{\dagger} = (AB)(AB)^{\dagger}A$.
(d) $A^*ABB^{\dagger} = BB^{\dagger}A^*A$.
(e) $(ABB^{\dagger})^{\dagger} = BB^{\dagger}A^{\dagger}$.
(f) $B^{\dagger}(ABB^{\dagger})^{\dagger} = B^{\dagger}A^{\dagger}$.
(g) $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$.
(h) $B^{\dagger}A^{\dagger} \subseteq \{(AB)^{(1,\,2,\,3)}\}$.

Lemma 1.2. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Then,*

$$\begin{aligned} r[\,(AB)^{\dagger}(AB) - B^{\dagger}A^{\dagger}(AB)\,] &= r(\, ABB^* - ABB^*A^{\dagger}A\,) \\ &= r[\, A^{\dagger}AB - B(AB)^{\dagger}(AB)\,] \\ &= \frac{1}{2}r(\, A^{\dagger}ABB^* - BB^*A^{\dagger}A\,) \\ &= r[\,(A^{\dagger}AB)^{\dagger} - B^{\dagger}A^{\dagger}A\,] \\ &= r[\,(A^{\dagger}AB)^{\dagger}A^{\dagger} - B^{\dagger}A^{\dagger}\,] \\ &= r\begin{bmatrix} A \\ ABB^* \end{bmatrix} - r(A). \end{aligned} \tag{1.5}$$

Hence, the following eight statements are equivalent:

(a) $(AB)^{\dagger}AB = B^{\dagger}A^{\dagger}AB$.
(b) $ABB^* = ABB^*A^{\dagger}A$.
(c) $A^{\dagger}AB = B(AB)^{\dagger}(AB)$.

(d) $A^{\dagger}ABB^* = BB^*A^{\dagger}A$.
(e) $(A^{\dagger}AB)^{\dagger} = B^{\dagger}A^{\dagger}A$.
(f) $(A^{\dagger}AB)^{\dagger}A^{\dagger} = B^{\dagger}A^{\dagger}$.
(g) $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$.
(h) $B^{\dagger}A^{\dagger} \subseteq \{(AB)^{(1,2,4)}\}$.

The combination of Lemmas 1.1(a)–(h) and 1.2(a)–(h) yields a group of necessary and sufficient conditions for $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ to hold. Most of these conditions can be found in the literature.

Just as in Lemmas 1.1 and 1.2, we are able to establish a group of rank formulas for $(AB)^{(1,3)} - B^{(1,3)}A^{(1,3)}$ and $(AB)^{(1,4)} - B^{(1,4)}A^{(1,4)}$, and then derive from them necessary and sufficient conditions for (1.2) and (1.3) to hold. In addition, we are also able to establish necessary and sufficient conditions for the following set inclusions

$$\{B^{(1,i)}A^{(1,i)}\} \subseteq \{(AB)^{(1)}\} \ \text{ for } i = 3,\ 4, \tag{1.6}$$

to hold by the matrix rank method.

A simple rank formula for the difference of two outer inverses of a matrix is needed in the sequel.

Lemma 1.3[11]**.** *Let X_1 and X_2 be a pair of outer inverses of a matrix A, that is, $X_1AX_1 = X_1$ and $X_2AX_2 = X_2$. Then,*

$$r(\, X_1 - X_2\,) = r\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + r[\, X_1,\ X_2\,] - r(X_1) - r(X_2). \tag{1.7}$$

In particular, if X_1 and X_2 are both idempotent, then (1.7) *holds.*

In addition, we need the following several rank formulas for partitioned matrices.

Lemma 1.4[6]**.** *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$ and $C \in \mathbb{C}^{l\times n}$. Then,*

$$r[\, A,\ B\,] = r(A) + r(\, B - AA^{\dagger}B\,), \tag{1.8}$$

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(\, A - AC^{\dagger}C\,), \tag{1.9}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r[\,(I - BB^{\dagger})A(I - C^{\dagger}C)\,]. \tag{1.10}$$

Lemma 1.5[12, 17]**.** *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$ and $C \in \mathbb{C}^{l\times n}$. Then,*

$$\max_X r(\, A - BXC\,) = \min\left\{ r[\, A,\ B\,],\ \ r\begin{bmatrix} A \\ C \end{bmatrix} \right\}, \tag{1.11}$$

$$\min_X r(\, A - BXC\,) = r[\, A,\ B\,] + r\begin{bmatrix} A \\ C \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \tag{1.12}$$

In particular, if A is square, then

$$\max_X r(A - BX) = \{ r[A,\ B],\quad n\}, \tag{1.13}$$

$$\min_X r(A - BX) = r[A,\ B] - r(B), \tag{1.14}$$

$$\max_X r(A - XC) = \min\left\{ r\begin{bmatrix} A \\ C \end{bmatrix},\quad m \right\}, \tag{1.15}$$

$$\min_X r(A - XC) = r\begin{bmatrix} A \\ C \end{bmatrix} - r(C). \tag{1.16}$$

The following formula is shown in Tian [10]

$$r(D - CA^{\dagger}B) = r\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix} - r(A). \tag{1.17}$$

2 The relations between $B^{(1,i)}A^{(1,i)}$ and $(AB)^{(1,i)}$

It is well known that $X \in \{A^{(1,3)}\}$ if and only if $AX = AA^{\dagger}$, and $X \in \{A^{(1,4)}\}$ if and only if $XA = A^{\dagger}A$, see e.g., [2, p. 44]. Hence, there are $A^{(1,3)}$ and $B^{(1,3)}$ such that $B^{(1,3)}A^{(1,3)} \in \{(AB)^{(1,3)}\}$ if and only if

$$\min_{A^{(1,3)},\, B^{(1,3)}} r[\, ABB^{(1,3)}A^{(1,3)} - AB(AB)^{\dagger}\,] = 0.$$

The set inclusion $\{B^{(1,3)}A^{(1,3)}\} \subseteq \{(AB)^{(1,3)}\}$ holds if and only if

$$\max_{A^{(1,3)},\, B^{(1,3)}} r[\, ABB^{(1,3)}A^{(1,3)} - AB(AB)^{\dagger}\,] = 0.$$

Similarly, there are $A^{(1,4)}$ and $B^{(1,4)}$ such that $B^{(1,4)}A^{(1,4)} \in \{(AB)^{(1,4)}\}$ if and only if

$$\min_{A^{(1,4)},\, B^{(1,4)}} r[\, B^{(1,4)}A^{(1,4)}AB - (AB)^{\dagger}AB\,] = 0.$$

The set inclusion $\{B^{(1,4)}A^{(1,4)}\} \subseteq \{(AB)^{(1,4)}\}$ holds if and only if

$$\max_{A^{(1,4)},\, B^{(1,4)}} r[\, B^{(1,4)}A^{(1,4)}AB - (AB)^{\dagger}AB\,] = 0.$$

It is well known that the general expressions of $A^{(1,3)}$ and $B^{(1,3)}$ can be written as

$$A^{(1,3)} = A^{\dagger} + (I_n - A^{\dagger}A)V_1, \qquad B^{(1,3)} = B^{\dagger} + (I_p - B^{\dagger}B)V_2, \tag{2.1}$$

$$A^{(1,4)} = A^{\dagger} + W_1(I_m - AA^{\dagger}), \qquad B^{(1,4)} = B^{\dagger} + W_2(I_n - BB^{\dagger}), \tag{2.2}$$

where the matrices V_1, V_2, W_1 and W_2 are arbitrary, see [2, p. 44]. Hence,

$$\begin{aligned} & ABB^{(1,3)}A^{(1,3)} - AB(AB)^{\dagger} \\ & = ABB^{\dagger}A^{\dagger} - AB(AB)^{\dagger} + ABB^{\dagger}(I_n - A^{\dagger}A)V_1, \end{aligned} \tag{2.3}$$

$$\begin{aligned} & B^{(1,4)}A^{(1,4)}AB - (AB)^{\dagger}AB \\ & = B^{\dagger}A^{\dagger}AB - (AB)^{\dagger}AB + W_2(I_n - BB^{\dagger})A^{\dagger}AB. \end{aligned} \tag{2.4}$$

Clearly, (2.3) and (2.4) are two linear matrix expressions with variable matrices V_1 and W_2.

Theorem 2.1. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Then,*

$$\min_{A^{(1,3)},\, B^{(1,3)}} r[\, ABB^{(1,3)}A^{(1,3)} - AB(AB)^{\dagger}\,] = r[\, B,\ A^{*}AB\,] - r[\, B,\ A^{*}\,] - r(AB) + r(A), \tag{2.5}$$

$$\max_{A^{(1,3)},\, B^{(1,3)}} r[\, ABB^{(1,3)}A^{(1,3)} - AB(AB)^{\dagger}\,] = r[\, B,\ A^{*}AB\,] - r(B). \tag{2.6}$$

Hence,

(a) *There are $A^{(1,3)}$ and $B^{(1,3)}$ such that $B^{(1,3)}A^{(1,3)} \in \{(AB)^{(1,3)}\}$ if and only if*
$$r[\, B,\ A^{*}AB\,] = r[\, B,\ A^{*}\,] + r(AB) - r(A).$$

(b) $\{B^{(1,3)}A^{(1,3)}\} \subseteq \{(AB)^{(1,3)}\}$ *holds if and only if* $\mathcal{R}(A^{*}AB) \subseteq \mathcal{R}(B)$.

Proof. Let $M = AB$ and $F_A = I_n - A^{\dagger}A$. Then we see by (2.3) and (1.14) that

$$\begin{aligned} &\min_{A^{(1,3)},\, B^{(1,3)}} r(\, MB^{(1,3)}A^{(1,3)} - MM^{\dagger}\,) \\ &= \min_{V_1} r(\, MB^{\dagger}A^{\dagger} - MM^{\dagger} + MB^{\dagger}F_AV_1\,) \\ &= r[\, MB^{\dagger}A^{\dagger} - MM^{\dagger},\ MB^{\dagger}F_A\,] - r(MB^{\dagger}F_A). \end{aligned} \tag{2.7}$$

Applying (1.9) and elementary block matrix operations, we find that

$$\begin{aligned} r[\, MB^{\dagger}A^{\dagger} - MM^{\dagger},\ MB^{\dagger}F_A\,] &= r\begin{bmatrix} MB^{\dagger}A^{\dagger} - MM^{\dagger} & MB^{\dagger} \\ 0 & A \end{bmatrix} - r(A) \\ &= r\begin{bmatrix} MM^{\dagger} & MB^{\dagger} \\ AA^{\dagger} & A \end{bmatrix} - r(A) \\ &= r\begin{bmatrix} 0 & MB^{\dagger} - MM^{\dagger}A \\ AA^{\dagger} & 0 \end{bmatrix} - r(A) \\ &= r(\, MB^{\dagger} - MM^{\dagger}A\,) \\ &= r[\, B,\ A^{*}AB\,] - r(B) \quad (\text{by } (1.4)). \end{aligned} \tag{2.8}$$

Also by (1.17) and elementary block matrix operations,

$$\begin{aligned} r(MB^{\dagger}F_A) &= r\begin{bmatrix} B^{*}BB^{*} & B^{*} - B^{*}A^{\dagger}A \\ ABB^{*} & 0 \end{bmatrix} - r(B) \\ &= r\begin{bmatrix} B^{*}A^{\dagger}ABB^{*} & B^{*} - B^{*}A^{\dagger}A \\ ABB^{*} & 0 \end{bmatrix} - r(B) \\ &= r\begin{bmatrix} 0 & B^{*} - B^{*}A^{\dagger}A \\ ABB^{*} & 0 \end{bmatrix} - r(B) \end{aligned}$$

$$
\begin{aligned}
&= r(AB) + r(\, B^* - B^*A^\dagger A\,) - r(B) \\
&= r(AB) + r\begin{bmatrix} B^* \\ A \end{bmatrix} - r(A) - r(B) \qquad \text{(by (1.9))} \\
&= r[\, A^*,\, B\,] + r(AB) - r(A) - r(B).
\end{aligned} \tag{2.9}
$$

Substituting (2.8) and (2.9) into (2.7) yields (2.5). Applying (1.13) to (2.3) gives

$$
\begin{aligned}
&\max_{A^{(1,3)},\ B^{(1,3)}} r(\, MB^{(1,3)}A^{(1,3)} - MM^\dagger\,) \\
&= \max_{V_1} r(\, MB^\dagger A^\dagger - MM^\dagger + MB^\dagger F_A V_1\,) \\
&= r[\, MB^\dagger A^\dagger - MM^\dagger,\ MB^\dagger F_A\,] \\
&= r[\, B,\ A^*AB\,] - r(B) \qquad \text{(by (2.8))}.
\end{aligned}
$$

Thus, we have (2.6). Results (a) and (b) of the theorem are direct consequences of (2.5) and (2.6). $\square$

By a similar argument, we can derive the following result from (2.4), (1.5) and (1.8)–(1.17).

Theorem 2.2. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Then,*

$$
\begin{aligned}
\min_{A^{(1,4)},\, B^{(1,4)}} r[\, B^{(1,4)}A^{(1,4)}AB - (AB)^\dagger AB\,] &= r\begin{bmatrix} A \\ ABB^* \end{bmatrix} - r\begin{bmatrix} A \\ B^* \end{bmatrix} \\
&\quad - r(AB) + r(B), \\
\max_{A^{(1,4)},\, B^{(1,4)}} r[\, B^{(1,4)}A^{(1,4)}AB - (AB)^\dagger AB\,] &= r\begin{bmatrix} A \\ ABB^* \end{bmatrix} - r(A).
\end{aligned}
$$

Hence,

(a) *There are $A^{(1,4)}$ and $B^{(1,4)}$ such that $B^{(1,4)}A^{(1,4)} \in \{(AB)^{(1,4)}\}$ if and only if*

$$
r\begin{bmatrix} A \\ ABB^* \end{bmatrix} = r\begin{bmatrix} A \\ B^* \end{bmatrix} - r(AB) + r(B).
$$

(b) $\{B^{(1,4)}A^{(1,4)}\} \subseteq \{(AB)^{(1,4)}\}$ *holds if and only if* $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$.

Combining Lemma 1.1(a)–(h) and Theorem 2.1(b), we see that the set inclusion in (1.1) holds if and only if AB satisfies any one of Lemma 1.1(a)–(h); combining Lemma 1.2(a)–(h) and Theorem 2.2(b), we see that the set inclusion (1.1) holds if and only if AB satisfies any one of Lemma 1.2(a)–(h). Combining (1.1) with Theorems 2.1(b) and 2.2(b), we see that

$$
(AB)^\dagger = B^\dagger A^\dagger \Leftrightarrow \{B^{(1,3)}A^{(1,3)}\} \subseteq \{(AB)^{(1,3)}\} \text{ and } \{B^{(1,4)}A^{(1,4)}\} \subseteq \{(AB)^{(1,4)}\}.
$$

The above results imply that the two set inclusions in (1.2) and (1.3) can be characterized by various conventional methods. The equivalence of $(AB)^\dagger = B^\dagger A^\dagger$ and other mixed-type reverse-order laws are given in [16].

3 The relationships between $B^{(1,i)}A^{(1,i)}$ and $(AB)^-$

In addition to (1.2) and (1.3), we are also able to investigate the relations between the reverse product $B^{(1,i)}A^{(1,i)}$ and $(AB)^-$ for $i = 3,\ 4$ by the matrix rank method. Obviously, there are $B^{(1,3)}$ and $A^{(1,3)}$ such that $B^{(1,i)}A^{(1,i)} \in \{(AB)^-\}$ if and only if

$$\min_{A^{(1,i)},\ B^{(1,i)}} r(\, AB - ABB^{(1,i)}A^{(1,i)}AB\,) = 0, \quad i = 3,\ 4.$$

The set inclusion $\{B^{(1,i)}A^{(1,i)}\} \subseteq \{(AB)^-\}$ holds if and only if

$$\max_{A^{(1,i)},\ B^{(1,i)}} r(\, AB - ABB^{(1,i)}A^{(1,i)}AB\,) = 0, \quad i = 3,\ 4.$$

From (2.3) and (2.4), we obtain that

$$\begin{aligned} & AB - ABB^{(1,3)}A^{(1,3)}AB \\ & = AB - ABB^{\dagger}A^{\dagger}AB - ABB^{\dagger}(\, I_n - A^{\dagger}A\,)V_1AB, \qquad (3.1) \\ & AB - ABB^{(1,4)}A^{(1,4)}AB \\ & = AB - ABB^{\dagger}A^{\dagger}AB - ABW_2(\, I_n - BB^{\dagger}\,)A^{\dagger}AB. \qquad (3.2) \end{aligned}$$

These are two linear matrix expressions.

Theorem 3.1. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Then,*

$$\min_{A^{(1,i)},\ B^{(1,i)}} r(\, AB - ABB^{(1,i)}A^{(1,i)}AB\,) = 0, \quad i = 3,\ 4, \qquad (3.3)$$

$$\begin{aligned} \max_{A^{(1,i)},\ B^{(1,i)}} r(\, AB - ABB^{(1,i)}A^{(1,i)}AB\,) & = r[\, A^*,\ B\,] + r(AB) \\ & \quad - r(A) - r(B),\ i = 3, 4. \qquad (3.4) \end{aligned}$$

Hence,

(a) *For any given $B^{(1,3)}$, there is $A^{(1,3)}$ such that $B^{(1,3)}A^{(1,3)} \in \{(AB)^-\}$.*

(b) *For any given $A^{(1,4)}$, there is $B^{(1,4)}$ such that $B^{(1,4)}A^{(1,4)} \in \{(AB)^-\}$.*

(c) *The following statements are equivalent:*

  (i) $\{B^{(1,3)}A^{(1,3)}\} \subseteq \{(AB)^-\}$.

  (ii) $\{B^{(1,4)}A^{(1,4)}\} \subseteq \{(AB)^-\}$.

  (iii) $B^{\dagger}A^{\dagger} \in \{(AB)^-\}$.

  (iv) $r[\, A^*,\ B\,] = r(A) + r(B) - r(AB)$.

Proof. Let $M = AB$ and $F_A = I_n - A^{\dagger}A$. Applying (1.12) and elementary block matrix operations to (3.1) yields

$$\min_{A^{(1,3)},\ B^{(1,3)}} r(\, M - MB^{(1,3)}A^{(1,3)}M\,)$$

$$
\begin{aligned}
&= \min_{V_1} r(\, M - MB^{\dagger}A^{\dagger}M - MB^{\dagger}F_AV_1M\,) \\
&= r[\, M - MB^{\dagger}A^{\dagger}M,\ MB^{\dagger}F_A\,] + r\begin{bmatrix} M - MB^{\dagger}A^{\dagger}M \\ M \end{bmatrix} \\
&\quad - r\begin{bmatrix} M - MB^{\dagger}A^{\dagger}M & MB^{\dagger}F_A \\ M & 0 \end{bmatrix} \\
&= r[\, 0,\ MB^{\dagger}F_A\,] + r(M) - r\begin{bmatrix} 0 & MB^{\dagger}F_A \\ M & 0 \end{bmatrix} = 0.
\end{aligned}
$$

Let $E_B = I_n - BB^{\dagger}$. Applying (1.12) and elementary block matrix operations to (3.2) yields

$$
\begin{aligned}
&\min_{A^{(1,4)},\ B^{(1,4)}} r(\, M - MB^{(1,4)}A^{(1,4)}M\,) \\
&= \min_{W_2} r(\, M - MB^{\dagger}A^{\dagger}M - MW_2E_BA^{\dagger}M\,) \\
&= r[\, M - MB^{\dagger}A^{\dagger}M,\ M\,] + r\begin{bmatrix} M - MB^{\dagger}A^{\dagger}M \\ E_BA^{\dagger}M \end{bmatrix} \\
&\quad - r\begin{bmatrix} M - MB^{\dagger}A^{\dagger}M & M \\ E_BA^{\dagger}M & 0 \end{bmatrix} \\
&= r(M) + r\begin{bmatrix} 0 \\ E_BA^{\dagger}M \end{bmatrix} - r\begin{bmatrix} 0 & M \\ E_BA^{\dagger}M & 0 \end{bmatrix} = 0.
\end{aligned}
$$

The above two results are summarized in (3.3). Applying (1.11) and elementary block matrix operations to (3.1) yields

$$
\begin{aligned}
&\max_{A^{(1,3)},\ B^{(1,3)}} r(\, M - MB^{(1,3)}A^{(1,3)}M\,) \\
&= \max_{V_1} r(\, M - MB^{\dagger}A^{\dagger}M - MB^{\dagger}F_AV_1M\,) \\
&= \min\left\{ r[\, M - MB^{\dagger}A^{\dagger}M,\ MB^{\dagger}F_A\,],\quad r\begin{bmatrix} M - MB^{\dagger}A^{\dagger}M \\ M \end{bmatrix} \right\} \\
&= \min\left\{ r[\, 0,\ MB^{\dagger}F_A\,],\quad r(M) \right\} \\
&= r(MB^{\dagger}F_A) = r[\, A^*,\ B\,] + r(AB) - r(A) - r(B). \quad \text{(by (2.9))}
\end{aligned}
$$

Applying (1.11) and elementary block matrix operations to (3.2) yields

$$
\begin{aligned}
&\max_{A^{(1,4)},\ B^{(1,4)}} r(\, M - MB^{(1,4)}A^{(1,4)}M\,) \\
&= \max_{W_2} r(\, M - MB^{\dagger}A^{\dagger}M - MW_2E_BA^{\dagger}M\,) \\
&= \min\left\{ r[\, M - MB^{\dagger}A^{\dagger}M,\ M\,],\quad r\begin{bmatrix} M - MB^{\dagger}A^{\dagger}M \\ E_BA^{\dagger}M \end{bmatrix} \right\} \\
&= \min\left\{ r(M),\quad r\begin{bmatrix} 0 \\ E_BA^{\dagger}M \end{bmatrix} \right\} \\
&= r(E_BA^{\dagger}M) \\
&= r[\, B,\ A^*\,] + r(AB) - r(A) - r(B) \quad \text{(by (2.9))}.
\end{aligned}
$$

Hence we have the two results (3.4). The result in (a) follows from (3.1) and (3.3). The result in (b) follows from (3.2) and (3.3). The equivalence of the four statements in (c) follows from (3.4). $\square$

In addition to the four equivalent conditions in Theorem 3.1(c), there are many other statements equivalent to these four conditions. Let $P_A = AA^{\dagger}$, which is the orthogonal projector onto the range of A. It was shown in [3] that

$$\begin{aligned} r(\, P_{A^*}P_B - P_BP_{A^*}\,) &= 2r(\, P_{A^*}P_B - P_{A^*}P_BP_{A^*}\,) \\ &= 2r(\, P_{A^*}P_B - P_BP_{A^*}P_B\,) \\ &= 2r[\, P_{A^*}P_B - (P_{A^*}P_B)^2\,]. \end{aligned}$$

Also note that

$$\begin{aligned} r[\, P_{A^*}P_B - (P_{A^*}P_B)^2\,] &= r(\, A^{\dagger}ABB^{\dagger} - A^{\dagger}ABB^{\dagger}A^{\dagger}ABB^{\dagger}\,) \\ &= r(\, AB - ABB^{\dagger}A^{\dagger}AB\,). \end{aligned}$$

Hence, each of the following conditions

(v) $P_{A^*}P_B = P_BP_{A^*}P_B$,

(vi) $P_{A^*}P_B = P_{A^*}P_BP_{A^*}$,

(vii) $P_{A^*}P_B = P_BP_{A^*}P_B$,

(vii) $P_{A^*}P_B = (P_{A^*}P_B)^2$,

is equivalent to Theorem 3.1(c).

4 The reverse-order laws $(AB)^{\dagger} = B^{(1,4)}A^{\dagger}$, $(AB)^{\dagger} = B^{\dagger}A^{(1,3)}$ and $(AB)^{\dagger} = B^{(1,4)}A^{(1,3)}$

In additions to the reverse-order law $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$, $(AB)^{\dagger}$ may be expressed as

$$(AB)^{\dagger} = B^{\dagger}A^{\dagger} + X \quad \text{or} \quad (AB)^{\dagger} = B^{\dagger}YA^{\dagger},$$

where X and Y are matrices consisting of A and B. Moreover, $(AB)^{\dagger}$ may be expressed as $(AB)^{\dagger} = B^{-}A^{-}$, etc. In this section, we investigate the relations between $(AB)^{\dagger}$ and the products $B^{\dagger}A^{(1,i)}$, $B^{(1,i)}A^{\dagger}$, $B^{(1,i)}A^{(1,j)}$ for $i,\ j = 3,\ 4$.

Theorem 4.1. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Then,*

$$\min_{A^{(1,3)}} r[\,(AB)^{\dagger} - B^{\dagger}A^{(1,3)}\,] = r[\,B,\ A^*AB\,] - r[\,B,\ A^*\,] - r(AB) + r(A), \tag{4.1}$$

$$\min_{B^{(1,4)}} r[\,(AB)^{\dagger} - B^{(1,4)}A^{\dagger}\,] = r\begin{bmatrix} A \\ ABB^* \end{bmatrix} - r\begin{bmatrix} A \\ B^* \end{bmatrix} - r(AB) + r(B). \tag{4.2}$$

Hence,

(a) *There is* $A^{(1,3)}$ *such that* $(AB)^{\dagger} = B^{\dagger}A^{(1,3)}$ *if and only if*

$$r[\, B, \ A^*AB \,] = r[\, B, \, A^* \,] + r(AB) - r(A).$$

(b) *There is* $B^{(1,4)}$ *such that* $(AB)^{\dagger} = B^{(1,4)}A^{\dagger}$ *if and only if*

$$r\begin{bmatrix} A \\ ABB^* \end{bmatrix} = r\begin{bmatrix} A \\ B^* \end{bmatrix} + r(AB) - r(B).$$

Proof. From (2.1) and (2.2), we obtain

$$(AB)^{\dagger} - B^{\dagger}A^{(1,3)} = (AB)^{\dagger} - B^{\dagger}A^{\dagger} - B^{\dagger}(\, I_n - A^{\dagger}A\,)V, \tag{4.3}$$

$$(AB)^{\dagger} - B^{(1,4)}A^{\dagger} = (AB)^{\dagger} - B^{\dagger}A^{\dagger} - W(\, I_n - BB^{\dagger}\,)A^{\dagger}, \tag{4.4}$$

where V and W are arbitrary. Applying (1.14) to (4.3) and then simplifying by elementary block matrix operations give

$$\begin{aligned}
&\min_{A^{(1,3)}} r[\,(AB)^{\dagger} - B^{\dagger}A^{(1,3)}\,] \\
&= \min_{V} r[\,(AB)^{\dagger} - B^{\dagger}A^{\dagger} - B^{\dagger}(\, I_n - A^{\dagger}A\,)V\,] \\
&= r[\,(AB)^{\dagger} - B^{\dagger}A^{\dagger}, \, B^{\dagger}(\, I_n - A^{\dagger}A\,)\,] - r[\, B^{\dagger}(\, I_n - A^{\dagger}A\,)\,] \\
&= r\begin{bmatrix} (AB)^{\dagger} - B^{\dagger}A^{\dagger} & B^{\dagger} \\ 0 & A \end{bmatrix} - r\begin{bmatrix} B^{\dagger} \\ A \end{bmatrix} \quad \text{(by (1.9))} \\
&= r\begin{bmatrix} (AB)^{\dagger} & B^{\dagger} \\ AA^{\dagger} & A \end{bmatrix} - r\begin{bmatrix} B^* \\ A \end{bmatrix} \\
&= r\begin{bmatrix} 0 & B^{\dagger} - (AB)^{\dagger}A \\ AA^{\dagger} & 0 \end{bmatrix} - r\begin{bmatrix} B^* \\ A \end{bmatrix} \\
&= r[\, B^{\dagger} - (AB)^{\dagger}A\,] + r(A) - r[\, A^*, \ B\,].
\end{aligned} \tag{4.5}$$

Note that $B^{\dagger}B[\, B^{\dagger} - (AB)^{\dagger}A\,] = B^{\dagger} - (AB)^{\dagger}A$. We see that

$$r[\, BB^{\dagger} - B(AB)^{\dagger}A\,] = r[\, B^{\dagger} - (AB)^{\dagger}A\,]. \tag{4.6}$$

Also note that both $BB^{\dagger}$ and $B(AB)^{\dagger}A$ are idempotent. We find by (1.7) that

$$\begin{aligned}
&r[\, BB^{\dagger} - B(AB)^{\dagger}A\,] \\
&= r\begin{bmatrix} BB^{\dagger} \\ B(AB)^{\dagger}A \end{bmatrix} + r[\, BB^{\dagger}, \, B(AB)^{\dagger}A\,] - r(BB^{\dagger}) - r[\, B(AB)^{\dagger}A\,] \\
&= r\begin{bmatrix} B^* \\ (AB)^*A \end{bmatrix} - r(AB) = r[\, A^*AB, \, B\,] - r(AB).
\end{aligned} \tag{4.7}$$

Substituting (4.7) into (4.6) and then substituting (4.6) into (4.5) give (4.1). Similarly we can show (4.2) by (4.4). $\square$

Theorem 4.2[23]. *Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{n\times p}$. *Then, there are* $A^{(1,2,3)}$ *and* $B^{(1,2,4)}$ *such that* $(AB)^{\dagger} = B^{(1,2,4)}A^{(1,2,3)}$.

Proof. Let $M = AB$, $F_A = I_n - A^{\dagger}A$ and $E_B = I_n - BB^{\dagger}$. Recall that the general expressions of $A^{(1,2,3)}$ and $B^{(1,2,4)}$ can be written as

$$A^{(1,2,3)} = A^{\dagger} + F_A V A A^{\dagger}, \quad B^{(1,2,4)} = B^{\dagger} + B^{\dagger}BWE_B,$$

where V and W are arbitrary; see [2, p. 46]. Hence, the rank of $M^{\dagger} - B^{(1,2,4)}A^{(1,2,3)}$ can be written as

$$\begin{aligned}
& r(M^{\dagger} - B^{(1,2,4)}A^{(1,2,3)}) \\
& = r[\, M^{\dagger} - (\, B^{\dagger} + B^{\dagger}BWE_B\,)(\, A^{\dagger} + F_AVAA^{\dagger}\,)\,] \\
& = r\begin{bmatrix} M^{\dagger} & B^{\dagger} + B^{\dagger}BWE_B \\ A^{\dagger} + F_AVAA^{\dagger} & I_n \end{bmatrix} - n \\
& = r\left(\begin{bmatrix} M^{\dagger} & B^{\dagger} \\ A^{\dagger} & I_n \end{bmatrix} + \begin{bmatrix} 0 \\ F_A \end{bmatrix} V[\, AA^{\dagger},\, 0\,] + \begin{bmatrix} B^{\dagger}B \\ 0 \end{bmatrix} W[\, 0,\, E_B\,] \right) - n. \quad (4.8)
\end{aligned}$$

Applying the following rank formula in Tian [13]

$$\min_{X_1,\, X_2} r(\, A - B_1X_1C_1 - B_2X_2C_2\,) = r\begin{bmatrix} A \\ C_1 \\ C_2 \end{bmatrix} + r[\, A,\, B_1,\, B_2\,] + \max\{s_1, \;\; s_2\},$$

where

$$s_1 = r\begin{bmatrix} A & B_1 \\ C_2 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 & B_2 \\ C_2 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix},$$

$$s_2 = r\begin{bmatrix} A & B_2 \\ C_1 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_2 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix},$$

to (4.8) gives

$$\begin{aligned}
& \min_{V,\, W} r[\, M^{\dagger} - (\, B^{\dagger} + B^{\dagger}BWE_B\,)(\, A^{\dagger} + F_AVAA^{\dagger}\,)\,] \\
& \quad = r\begin{bmatrix} M^{\dagger} & B^{\dagger} \\ A^{\dagger} & I_n \\ AA^{\dagger} & 0 \\ 0 & E_B \end{bmatrix} + r\begin{bmatrix} M^{\dagger} & B^{\dagger} & 0 & B^{\dagger}B \\ A^{\dagger} & I_n & F_A & 0 \end{bmatrix} - n + \max\{\, s_1,\; s_2\,\}, \quad (4.9)
\end{aligned}$$

where

$$s_1 = r\begin{bmatrix} M^{\dagger} & B^{\dagger} & 0 \\ A^{\dagger} & I_n & F_A \\ 0 & E_B & 0 \end{bmatrix} - r\begin{bmatrix} M^{\dagger} & B^{\dagger} & 0 & B^{\dagger}B \\ A^{\dagger} & I_n & F_A & 0 \\ 0 & E_B & 0 & 0 \end{bmatrix} - r\begin{bmatrix} M^{\dagger} & B^{\dagger} & 0 \\ A^{\dagger} & I_n & F_A \\ 0 & E_B & 0 \\ AA^{\dagger} & 0 & 0 \end{bmatrix},$$

$$s_2 = r\begin{bmatrix} M^{\dagger} & B^{\dagger} & B^{\dagger}B \\ A^{\dagger} & I_n & 0 \\ AA^{\dagger} & 0 & 0 \end{bmatrix} - r\begin{bmatrix} M^{\dagger} & B^{\dagger} & B^{\dagger}B & 0 \\ A^{\dagger} & I_n & 0 & F_A \\ AA^{\dagger} & 0 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} M^{\dagger} & B^{\dagger} & B^{\dagger}B \\ A^{\dagger} & I_n & 0 \\ AA^{\dagger} & 0 & 0 \\ 0 & E_B & 0 \end{bmatrix}.$$

Simplifying the block matrices in (4.9) by (1.8), (1.9), (1.10) and elementary block matrix operations gives

$$\begin{aligned}&\min_{V,\,W} r[\, M^{\dagger} - (\, B^{\dagger} + B^{\dagger}BWE_B\,)(\, A^{\dagger} + F_AVAA^{\dagger}\,)\,] \\ &= \max\{\, 0, \quad r(A) + r(B) - n - r(AB)\,\}. \end{aligned} \tag{4.10}$$

Also by the well-known Frobenius rank inequality $r(AB) \geq r(A)+r(B)-n$, the right-hand side of (4.10) becomes zero. Hence the result of the theorem follows. $\square$

Applying the results in Sections 2, 3 and 4 to A^2 leads to the following results.

Theorem 4.3. *Let $A \in \mathbb{C}^{m\times m}$. Then:*

(a) *There are $A_1^{(1,3)}, A_2^{(1,3)} \in \{A^{(1,3)}\}$ such that $A_1^{(1,3)}A_2^{(1,3)} \in \{(A^2)^{(1,3)}\}$ if and only if $r[\, A,\ A^*A^2\,] = r[\, A,\ A^*\,] + r(A^2) - r(A)$.*

(b) *$\{A^{(1,3)}A^{(1,3)}\} \subseteq \{(A^2)^{(1,3)}\}$ if and only if $\mathcal{R}(A^*A^2) \subseteq \mathcal{R}(A)$.*

(c) *There are $A_1^{(1,4)}, A_2^{(1,4)} \in \{A^{(1,4)}\}$ such that $A_1^{(1,4)}A_2^{(1,4)} \in \{(A^2)^{(1,4)}\}$ if and only if $r\begin{bmatrix} A \\ A^2A^* \end{bmatrix} = r[A,\ A^*\,] - r(A^2) + r(A)$.*

(d) *$\{A^{(1,4)}A^{(1,4)}\} \subseteq \{(A^2)^{(1,4)}\} \Leftrightarrow \mathcal{R}(AA^*A^*) \subseteq \mathcal{R}(A^*)$.*

(e) *For any given $A_1^{(1,3)} \in \{A^{(1,3)}\}$, there is $A_2^{(1,3)} \in \{A^{(1,3)}\}$ such that $A_1^{(1,3)}A_2^{(1,3)} \in \{(A^2)^{-}\}$.*

(f) *For any given $A_2^{(1,4)} \in \{A^{(1,4)}\}$, there is $A_1^{(1,4)} \in \{A^{(1,4)}\}$ such that $A_1^{(1,4)}A_2^{(1,4)} \in \{(A^2)^{-}\}$.*

(g) *The following statements are equivalent:*

   (1) $\{A^{(1,3)}A^{(1,3)}\} \subseteq \{(A^2)^{-}\}$.

   (2) $\{A^{(1,4)}A^{(1,4)}\} \subseteq \{(A^2)^{-}\}$.

   (3) $(A^{\dagger})^2 \in \{(A^2)^{-}\}$.

   (4) $r[\, A^*,\ A\,] = 2r(A) - r(A^2)$.

(h) *There is $A^{(1,3)}$ such that $(A^2)^{\dagger} = A^{\dagger}A^{(1,3)}$ if and only if*

$$r[\, A,\ A^*A^2\,] = r[\, A,\ A^*\,] + r(A^2) - r(A).$$

(i) *There is $A^{(1,4)}$ such that $(A^2)^{\dagger} = A^{(1,4)}A^{\dagger}$ if and only if*

$$r\begin{bmatrix} A \\ A^2A^* \end{bmatrix} = r[\, A,\ A^*\,] + r(A^2) - r(A).$$

(j) *There are $A^{(1,2,3)}$ and $A^{(1,2,4)}$ such that $(A^2)^{\dagger} = A^{(1,2,4)}A^{(1,2,3)}$.*

5 Conclusion remarks

In the theory of generalized inverses of matrices and applications, there are various matrix expressions involving generalized inverses of matrices, as well as equalities consisting of these matrix expressions. A general research topic on these matrix expressions is:

> Given two matrix expressions $p(A_1, \ldots, A_k)$ and $q(A_1, \ldots, A_k)$ of the same size consisting of matrices $A_1, \ldots, A_k$ and their generalized inverses, determine necessary and sufficient conditions for

$$\begin{aligned} p(A_1, \ldots, A_k) &= q(A_1, \ldots, A_k), \\ \{ p(A_1, \ldots, A_k) \} &\subseteq \{ q(A_1, \ldots, A_k) \}, \\ \{ p(A_1, \ldots, A_k) \} &= \{ q(A_1, \ldots, A_k) \} \end{aligned}$$

to hold. Theoretically, these problems can be solved by determining the maximal or minimal ranks of the corresponding matrix expressions. Many results on these problems have been derived by the matrix rank method for example, necessary and sufficient conditions for $(AB)^{\dagger} = B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger}$, $(AB)^{\dagger} = B^{*}(A^{*}ABB^{*})^{\dagger}A^{*}$, $(AB)^{\dagger} = B^{\dagger}A^{\dagger} - B^{\dagger}[\, (I_n - BB^{\dagger})(I_n - A^{\dagger}A)\,]^{\dagger}A^{\dagger}$, $(ABC)^{\dagger} = C^{\dagger}B^{\dagger}A^{\dagger}$, $(ABC)^{\dagger} = (BC)^{\dagger}B(AB)^{\dagger}$, $(A_1A_2 \cdots A_k)^{\dagger} = A_k^{\dagger} \cdots A_2^{\dagger}A_1^{\dagger}$, $A^kA^{\dagger} = A^{\dagger}A^k$, $A^*A^{\dagger} = A^{\dagger}A^*$, $AA^{\dagger}_{M,N} = A^{\dagger}_{M,N}A$, $AA^{-} = A^{-}A$, $A^kA^{-} = A^{-}A^k$, $AA^{-} = BB^{-}$ and so on, see [8–16]. As an extension of this paper, it may be of interest to investigate the reverse-order laws $(ABC)^{(1,i)} = C^{(1,i)}B^{(1,i)}A^{(1,i)}$ and $(ABC)^{\dagger} = C^{(1,i)}B^{(1,i)}A^{(1,i)}$ $(i = 3,\, 4)$ by the matrix rank method.

References

[1] Arghiriade, E.: Sur les matrices qui sont permutables avec leur inverse généralisée. Atti. Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Natur. Ser. VIII 35(1963), 244–251.

[2] Ben-Israel, A. and Greville, T.N.E.: Generalized inverses: theory and applications, Corrected reprint of the 1974 original, R. E. Krieger Publishing Company, Huntington, New York, 1980.

[3] Bérubé, J., Hartwig, R.E. and Styan, G.P.H.: On canonical correlations and the degrees of non-orthogonality in the three-way layout. In: Matusita, K., Puri, M.L. and Hayakawa, T. (eds.), Statistical Sciences and Data Analysis, Proceedings of the Third Pacific Area Statistical Conference. VSP International Science Publishers, Utrecht, The Netherlands, 1993, pp. 247–252.

[4] Campbell, S.L. and Meyer, C.D.: Generalized Inverses of Linear Transformations. Corrected reprint of the 1979 original. Dover Publications, Inc., New York, (1991).

[5] Greville, T.N.E.: Note on the generalized inverse of a matrix product. SIAM Rev., 8(1966), 518–521.

[6] Marsaglia, G. and Styan, G.P.H.: Equalities and inequalities for ranks of matrices. Linear and Multilinear Algebra, 2(1974), 269–292.

[7] Rao, C.R. and Mitra, S.K.: Generalized Inverse of Matrices and Its Applications. Wiley, New York, 1971.

[8] Tian, Y.: The Moore-Penrose inverse of a triple matrix product. Mathematics in Theory and Practice, No.1(1992), 64–70.

[9] Tian, Y.: Reverse order laws for the generalized inverses of multiple matrix products. Linear Algebra Appl., 211(1994), 85–100.

[10] Tian, Y.: How to characterize equalities for the Moore-Penrose inverse of a matrix. Kyungpook Math. J., 41(2001), 1–15.

[11] Tian, Y.: Rank equalities related to outer inverses of matrices and applications. Linear and Multilinear Algebra, 49(2002), 269–288.

[12] Tian, Y.: The maximal and minimal ranks of some expressions of generalized inverses of matrices. Southeast Asian Bull. Math., 25(2002), 745–755.

[13] Tian, Y.: Upper and lower bounds for ranks of matrix expressions using generalized inverses. Linear Algebra Appl., 355(2002), 187–214.

[14] Tian, Y.: Using rank formulas to characterize equalities for Moore-Penrose inverses of matrix products. Appl. Math. Comput., 147(2004), 581–600.

[15] Tian, Y.: On mixed-type reverse-order laws for the Moore-Penrose inverse of a matrix product. Internat. J. Math. Math. Sci., 58(2004), 3103–3116.

[16] Tian, Y.: The equivalence between $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ and other mixed-type reverse-order laws. Int. J. Math. Educ. Sci. Technol., in press.

[17] Tian, Y. and Cheng, S.: The maximal and minimal ranks of $A - BXC$ with applications. New York J. Math., 9(2003), 345–362.

[18] Tian, Y. and Styan, G.P.H.: Rank equalities for idempotent and involutory matrices. Linear Algebra Appl., 335(2001), 101–117.

[19] Udwadia, F.E. and Kalaba, R.E.: Sequential determination of the $\{1, 4\}$-inverse of a matrix. J. Optim. Theory Appl., 117(2003), 1–7.

[20] Wei, W. and Guo, W.: Reverse order laws for least squares g-inverses and minimum norm g-inverses of products of two matrices. Linear Algebra Appl., 342(2002), 117–132.

[21] Werner, H.J.: G-inverse of matrix products. In: Schach, S. and Trenkler, G. (eds.), Data Analysis and Statistical Inference. Eul Verlag, Bergisch-Gladbach, 1992, pp. 531–546.

[22] Werner, H.J.: When is $B^{-}A^{-}$ a generalized inverse of AB? Linear Algebra Appl., 210(1994), 255–263.

[23] Wibker, E.A., Howe, R.B. and Gilbert,J.D.: Explicit solution to the reverse order law $(AB)^{+} = B^{-}_{mr}A^{-}_{lr}$. Linear Algebra Appl., 25(1999), 107–114.

[24] Yanai. H: Some generalized forms of least squares g-inverse, minimum norm g-inverse, and Moore-Pernrose inverse matrices. Compt. Statist. Data Anal., 10(1990), 251–260.