Alternative Characterizations of the Extended Wedderburn-Guttman Theorem

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ABSTRACT

Let A be a u by v matrix of rank a, and let M and N be u by p and v by q matrices, respectively, where p is not necessarily equal to q or $\mathrm{rank}(M'AN) < \min(p, q)$. Takane and Yanai (Linear Algebra and Its Applications 410 (2005) 267-278) investigated the conditions under which $\mathrm{rank}(A - AN(M'AN)^{-}M'A) = \mathrm{rank}(A) - \mathrm{rank}(AN(M'AN)^{-}M'A)$. This is called the extended Wedderburn-Guttman theorem. In this paper, we give alternative characterizations of these conditions using the product singular value decomposition (PSVD) of matrix triplets.

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1 Introduction

Let A be a u by v matrix of rank a, and let M and N be u by g and v by g matrices, respectively, such that $M'AN$ is nonsingular. Then, $\mathrm{rank}(A - AN(M'AN)^{-1}M'A) = a - g$, where $g = \mathrm{rank}(AN(M'AN)^{-1}M'A) = \mathrm{rank}(M'AN)$. This is called the Wedderburn-Guttman theorem (Guttman, 1944, 1957; Wedderburn, 1934; see also Egerváry, 1960). The theorem is used extensively in numerical linear algebra (Chu, Funderlic, and Golub, 1995; Galántai, 2003) as a rank reduction method, and in psychometrics (e.g., Guttman, 1952; Takane and Hunter, 2001), and statistics (e.g., Rao, 1964) as a means of extracting components which are known linear combinations of observed variables. Recently, Hubert, Heiser, and Meulman (2000) reviewed the history behind the theorem. See also Groß and Tian (2006) who investigated various invariance properties of a triple matrix product of the

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form $AB^{-}C$ over the choice of a g-inverse B^{-} of B.

Takane and Yanai (2005) recently extended this theorem to the situation in which the regular inverse of $M'AN$ does not exist. Let M and N be u by p and v by q matrices, respectively, where p is not necessarily equal to q or $\mathrm{rank}(M'AN) < \min(p, q)$. Then, under a certain minimal rank additivity condition,

$$\mathrm{rank}(A - AN(M'AN)^{-}M'A) = \mathrm{rank}(A) - \mathrm{rank}(AN(M'AN)^{-}M'A), \tag{1}$$

where $(M'AN)^{-1}$ in the original formula was replaced by a g-inverse $(M'AN)^{-}$. Let

$$g = \mathrm{rank}(AN(M'AN)^{-}M'A), \tag{2}$$

and

$$h = \mathrm{rank}(M'AN). \tag{3}$$

It was also shown that under the same condition $g = h$, although in general $g \geq h$. Takane and Yanai (2005) also investigated other conditions, starting from the minimal condition referred to above, and gradually introducing stronger conditions. In this paper we give alternative characterizations of these conditions using the product singular value decomposition (PSVD; e.g., Zha, 1991) of matrix triplets.

Let

$$B = N(M'AN)^{-}M', \tag{4}$$

so that

$$AN(M'AN)^{-}M'A = ABA. \tag{5}$$

The conditions investigated by Takane and Yanai (2005) are summarized in Table 1. In the table, these conditions are characterized in two ways, one in terms of matrix equalities and the other in terms of rank equalities. For reference, matrix and rank equalities that hold unconditionally are listed at the top of the table. The minimal (necessary and sufficient, *ns*) condition, referred to above, for (1) to hold is labelled as Condition A, which is characterized by $ABABA = ABA$ or equivalently $\mathrm{rank}(ABA) = \mathrm{rank}(M'AN)$ (i.e., $g = h$). This condition is weaker than AB being idempotent (Condition B1) and BA being idempotent (Condition B2). Conditions B1 and B2 are in turn weaker than C1 and C2, respectively. The latter conditions are interesting because either one of them is *ns* for the uniqueness of $\mathrm{rank}(ABA)$ over the choice of a g-inverse of $M'AN$. (Note that this means that there are cases in which (1) holds even if $\mathrm{rank}(ABA)$ is not unique.) Conditions G1 and G2 (and G) were not explicitly discussed by Takane and Yanai (2005). These conditions are useful, combined with C2 and C1, respectively, in defining E1 and E2, which are the assumptions often made in statistical contexts (e.g., Takane and Hunter, 2001). Condition G is defined as both Conditions G1 and G2 being true. Condition D is of interest because prior to Takane and Yanai (2005), this was believed to be the *ns* condition for (1) (e.g., Cline and Funderlic

Table 1: Conditions surrounding the extended Wedderburn-Guttman theorem.

Condition	Matrix Equality	Rank Equality
No Condition	$\begin{cases} ABABAN = ABAN \\ M'ABABA = M'ABA \end{cases}$	$\begin{cases} \text{rank}(ABAN) = \text{rank}(M'AN) \\ \text{rank}(M'ABA) = \text{rank}(M'AN) \end{cases}$
A	$ABABA = ABA$	$\text{rank}(ABA) = \text{rank}(M'AN)$
B	B1 and B2	
B1	$(AB)^2 = AB$	$\text{rank}(AB) = \text{rank}(M'AN)$
B2	$(BA)^2 = BA$	$\text{rank}(BA) = \text{rank}(M'AN)$
C	C1 and C2	
C1	$ABAN = AN$	$\text{rank}(AN) = \text{rank}(M'AN)$
C2	$M'ABA = M'A$	$\text{rank}(M'A) = \text{rank}(M'AN)$
G	G1 and G2	
G1	$M'A(M'A)^-M' = M'$	$\text{rank}(M) = \text{rank}(M'A)$
G2	$N(AN)^-AN = N$	$\text{rank}(N) = \text{rank}(AN)$
D	$BAB = B$	$\text{rank}(B) = \text{rank}(M'AN)$
E	E1 and E2	
E1	$M'AB = M'$	$\text{rank}(M) = \text{rank}(M'AN)$
E2	$BAN = N$	$\text{rank}(N) = \text{rank}(M'AN)$
F	$ABA = A$	$\text{rank}(A) = \text{rank}(M'AN)$

(1979)). Obviously, this condition is sufficient but not necessary for (1). Condition F makes the residual matrix equal to zero.

Roughly speaking, conditions listed toward the bottom of the table represent stronger conditions. The implication relationships are not of strict order, however. More precise relationships are depicted in Figure 1. In the figure a condition at the origin of an arrow indicates a stronger condition than the one at the terminus of the arrow. A formal proof of these relationships are given in Takane and Yanai (2005). It is important to note, however, that there are more than one way in which each of these conditions can occur. For example, there are several different ways in which Condition A occurs. It occurs when certain rank conditions are satisfied among the three matrices involved (A, M and N) irrespective of the g-inverse $(M'AN)^-$ used. It also occurs under the use of a specific g-inverse of $M'AN$ regardless of the rank conditions. In this paper, we examine various ways in which the conditions described in Table 1 occur, using PSVD. The implication relations among various conditions are also clarified by the PSVD representation.

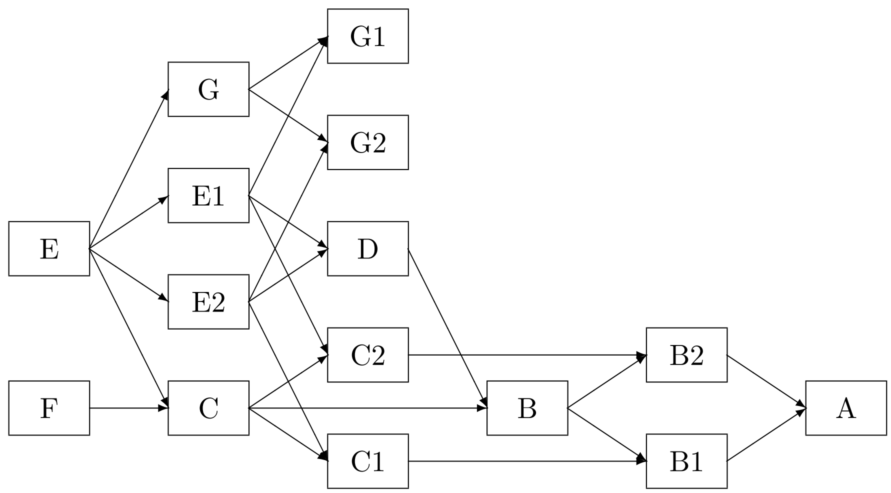

Figure 1: The relationship among the conditions.

2 Representation by the Product SVD

Transforming the matrices into quasi-diagonal ("canonical") form has proven to be useful in establishing many matrix results in a variety of statistical contexts. We first present an explicit representation of $(M'AN)^-$ by the product singular value decomposition (PSVD) of matrix triplets (Bojanczyk, Ewerbring, Luk, and van Dooren, 1991; Fernando and Hammarling, 1988; Zha, 1991), and then give expressions of ABA, $ABABA$, etc. in terms of PSVD. Let

$$j = \text{rank}(M'A) - h, \tag{6}$$

$$i = \text{rank}(M) - \text{rank}(M'A), \tag{7}$$

$$s = \text{rank}(AN) - h, \tag{8}$$

$$t = \text{rank}(N) - \text{rank}(AN), \tag{9}$$

and

$$k = a - \text{rank}(M'A) - \text{rank}(AN) + h. \tag{10}$$

We immediately note that $\text{rank}(M) = h+i+j$, $\text{rank}(N) = h+s+t$, $\text{rank}(A) = h+j+s+k$, $\text{rank}(M'A) = h + j$, and $\text{rank}(AN) = h + s$. Note also that Condition C1 is equivalent to $s = 0$, Condition C2 to $j = 0$, Condition G1 to $i = 0$, Condition G2 to $t = 0$, Condition E1 to $i = 0$ and $j = 0$, Condition E2 to $s = 0$ and $t = 0$, and Condition F to $j = 0, s = 0$, and $k = 0$. (These observations are summarized in Table 2 shown below.)

PSVD of matrix triplets was initially developed as a way of obtaining the ordinary SVD of a product of the matrix triplets without actually computing the product. This is done by separately decomposing the three matrices in a special way. When the decomposed matrices are put together in multiplicative form, the SVD of the product of the original matrices is obtained. In PSVD, matrices M', A, and N are expressed as

$$M' = UDX_1^{-1}, \tag{11}$$

$$A = X_1 K X_2, \tag{12}$$

and

$$N = X_2^{-1} J V', \tag{13}$$

where $X_1(u \times u)$ and $X_2(v \times v)$ are nonsingular, $U(p \times p)$ and $V(q \times q)$ are orthogonal, and

$$D_{p\times u} = \begin{bmatrix} S_{h\times h} & 0_{h\times j} & 0_{h\times i} & 0_{h\times k} & 0_{h\times s} & 0_{h\times c} \\ 0_{j\times h} & I_{j\times j} & 0_{j\times i} & 0_{j\times k} & 0_{j\times s} & 0_{j\times c} \\ 0_{i\times h} & 0_{i\times j} & I_{i\times i} & 0_{i\times k} & 0_{i\times s} & 0_{i\times c} \\ 0_{b\times h} & 0_{b\times j} & 0_{b\times i} & 0_{b\times k} & 0_{b\times s} & 0_{b\times c} \end{bmatrix},$$

$$K_{u\times v} = \begin{bmatrix} I_{h\times h} & 0_{h\times j} & 0_{h\times k} & 0_{h\times s} & 0_{h\times t} & 0_{h\times d} \\ 0_{j\times h} & I_{j\times j} & 0_{j\times k} & 0_{j\times s} & 0_{j\times t} & 0_{j\times d} \\ 0_{i\times h} & 0_{i\times j} & 0_{i\times k} & 0_{i\times s} & 0_{i\times t} & 0_{i\times d} \\ 0_{k\times h} & 0_{k\times j} & I_{k\times k} & 0_{k\times s} & 0_{k\times t} & 0_{k\times d} \\ 0_{s\times h} & 0_{s\times j} & 0_{s\times k} & I_{s\times s} & 0_{s\times t} & 0_{s\times d} \\ 0_{c\times h} & 0_{c\times j} & 0_{c\times k} & 0_{c\times s} & 0_{c\times t} & 0_{c\times d} \end{bmatrix},$$

and

$$J_{v\times q} = \begin{bmatrix} I_{h\times h} & 0_{h\times s} & 0_{h\times t} & 0_{h\times e} \\ 0_{j\times h} & 0_{j\times s} & 0_{j\times t} & 0_{j\times e} \\ 0_{k\times h} & 0_{k\times s} & 0_{k\times t} & 0_{k\times e} \\ 0_{s\times h} & I_{s\times s} & 0_{s\times t} & 0_{s\times e} \\ 0_{t\times h} & 0_{t\times s} & I_{t\times t} & 0_{t\times e} \\ 0_{d\times h} & 0_{d\times s} & 0_{d\times t} & 0_{d\times e} \end{bmatrix}.$$

with $S_{h\times h}$ being a diagonal matrix of order h with the nonzero singular values of $M'AN$ as its diagonal entries. I's and 0's are identity matrices and zero matrices of appropriate sizes, and $b = p - (h + j + i)$, $c = u - (h + j + i + k + s)$, $d = v - (h + j + k + s + t)$ and $e = q - (h + s + t)$. Note that some of the row and/or column blocks in the above matrices and those given below may be null (order 0) (i.e., it may be that $j = 0$, $s = 0$, $i = 0$, $t = 0$, $k = 0$, or $h = 0$).

It follows that

$$(M'AN)_{p\times q} = U(DKJ)V', \qquad \text{(This is the complete SVD of } M'AN.\text{)} \tag{14}$$

and

$$(M'AN)^{-}_{q\times p} = V(DKJ)^{-}U', \tag{15}$$

where

$$(DKJ)_{p\times q} = \begin{bmatrix} S_{h\times h} & 0_{h\times s} & 0_{h\times t} & 0_{h\times e} \\ 0_{j\times h} & 0_{j\times s} & 0_{j\times t} & 0_{j\times e} \\ 0_{i\times h} & 0_{i\times s} & 0_{i\times t} & 0_{i\times e} \\ 0_{b\times h} & 0_{b\times s} & 0_{b\times t} & 0_{b\times e} \end{bmatrix},$$

and

$$(DKJ)^{-}_{q\times p} = \begin{bmatrix} S^{-1}_{h\times h} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{41} & G_{42} & G_{43} & G_{44} \end{bmatrix},$$

where G's are arbitrary. Define $C = J(DKJ)^{-}D$. Then,

$$C_{v\times u} = \begin{bmatrix} I_{h\times h} & G_{12} & G_{13} & 0_{h\times k} & 0_{h\times s} & 0_{h\times c} \\ 0_{j\times h} & 0_{j\times j} & 0_{j\times i} & 0_{j\times k} & 0_{j\times s} & 0_{j\times c} \\ 0_{k\times h} & 0_{k\times j} & 0_{k\times i} & 0_{k\times k} & 0_{k\times s} & 0_{k\times c} \\ G_{21}S & G_{22} & G_{23} & 0_{s\times k} & 0_{s\times s} & 0_{s\times c} \\ G_{31}S & G_{32} & G_{33} & 0_{t\times k} & 0_{t\times s} & 0_{t\times c} \\ 0_{d\times h} & 0_{d\times j} & 0_{d\times i} & 0_{d\times k} & 0_{d\times s} & 0_{d\times c} \end{bmatrix}.$$

$$(ABA)_{u\times v} = X_1KCKX_2, \tag{16}$$

$$(ABABA)_{u\times v} = X_1KCKCKX_2, \tag{17}$$

$$(AB)_{u\times u} = X_1KCX_1^{-1}, \tag{18}$$

$$(AB)^2_{u\times u} = X_1(KC)^2X_1^{-1}, \tag{19}$$

$$(BA)_{v\times v} = X_2^{-1}CKX_2, \tag{20}$$

$$(BA)^2_{v\times v} = X_2^{-1}(CK)^2X_2, \tag{21}$$

$$B_{v\times u} = X_2^{-1}CX_1^{-1}, \tag{22}$$

and

$$(BAB)_{v\times u} = X_2^{-1}CKCX_1^{-1}, \tag{23}$$

where

$$(KCK)_{u\times v} = \begin{bmatrix} I_{h\times h} & G_{12} & 0_{h\times k} & 0_{h\times s} & 0_{h\times t} & 0_{h\times d} \\ 0_{j\times h} & 0_{j\times j} & 0_{j\times k} & 0_{j\times s} & 0_{j\times t} & 0_{j\times d} \\ 0_{i\times h} & 0_{i\times j} & 0_{i\times k} & 0_{i\times s} & 0_{i\times t} & 0_{i\times d} \\ 0_{k\times h} & 0_{k\times j} & 0_{k\times k} & 0_{k\times s} & 0_{k\times t} & 0_{k\times d} \\ G_{21}S & G_{22} & 0_{s\times k} & 0_{s\times s} & 0_{s\times t} & 0_{s\times d} \\ 0_{c\times h} & 0_{c\times j} & 0_{c\times k} & 0_{c\times s} & 0_{c\times t} & 0_{c\times d} \end{bmatrix},$$

$$(KCKCK)_{u\times v} = \begin{bmatrix} I_{h\times h} & G_{12} & 0_{h\times k} & 0_{h\times s} & 0_{h\times t} & 0_{h\times d} \\ 0_{j\times h} & 0_{j\times j} & 0_{j\times k} & 0_{j\times s} & 0_{j\times t} & 0_{j\times d} \\ 0_{i\times h} & 0_{i\times j} & 0_{i\times k} & 0_{i\times s} & 0_{i\times t} & 0_{i\times d} \\ 0_{k\times h} & 0_{k\times j} & 0_{k\times k} & 0_{k\times s} & 0_{k\times t} & 0_{k\times d} \\ G_{21}S & G_{21}SG_{12} & 0_{s\times k} & 0_{s\times s} & 0_{s\times t} & 0_{s\times d} \\ 0_{c\times h} & 0_{c\times j} & 0_{c\times k} & 0_{c\times s} & 0_{c\times t} & 0_{c\times d} \end{bmatrix},$$

$$(KC)_{u\times u} = \begin{bmatrix} I_{h\times h} & G_{12} & G_{13} & 0_{h\times k} & 0_{h\times s} & 0_{h\times c} \\ 0_{j\times h} & 0_{j\times j} & 0_{j\times i} & 0_{j\times k} & 0_{j\times s} & 0_{j\times c} \\ 0_{i\times h} & 0_{i\times j} & 0_{i\times i} & 0_{i\times k} & 0_{i\times s} & 0_{i\times c} \\ 0_{k\times h} & 0_{k\times j} & 0_{k\times i} & 0_{k\times k} & 0_{k\times s} & 0_{k\times c} \\ G_{21}S & G_{22} & G_{23} & 0_{s\times k} & 0_{s\times s} & 0_{s\times c} \\ 0_{c\times h} & 0_{c\times j} & 0_{c\times i} & 0_{c\times k} & 0_{c\times s} & 0_{c\times c} \end{bmatrix},$$

$$(KC)^2_{u\times u} = \begin{bmatrix} I_{h\times h} & G_{12} & G_{13} & 0_{h\times k} & 0_{h\times s} & 0_{h\times c} \\ 0_{j\times h} & 0_{j\times j} & 0_{j\times i} & 0_{j\times k} & 0_{j\times s} & 0_{j\times c} \\ 0_{i\times h} & 0_{i\times j} & 0_{i\times i} & 0_{i\times k} & 0_{i\times s} & 0_{i\times c} \\ 0_{k\times h} & 0_{k\times j} & 0_{k\times i} & 0_{k\times k} & 0_{k\times s} & 0_{k\times c} \\ G_{21}S & G_{21}SG_{12} & G_{21}SG_{13} & 0_{s\times k} & 0_{s\times s} & 0_{s\times c} \\ 0_{c\times h} & 0_{c\times j} & 0_{c\times i} & 0_{c\times k} & 0_{c\times s} & 0_{c\times c} \end{bmatrix},$$

$$(CK)_{v\times v} = \begin{bmatrix} I_{h\times h} & G_{12} & 0_{h\times k} & 0_{h\times s} & 0_{h\times t} & 0_{h\times d} \\ 0_{j\times h} & 0_{j\times j} & 0_{j\times k} & 0_{j\times s} & 0_{j\times t} & 0_{j\times d} \\ 0_{k\times h} & 0_{k\times j} & 0_{k\times k} & 0_{k\times s} & 0_{k\times t} & 0_{k\times d} \\ G_{21}S & G_{22} & 0_{s\times k} & 0_{s\times s} & 0_{s\times t} & 0_{s\times d} \\ G_{31}S & G_{32} & 0_{t\times k} & 0_{t\times s} & 0_{t\times t} & 0_{t\times d} \\ 0_{d\times h} & 0_{d\times j} & 0_{d\times k} & 0_{d\times s} & 0_{d\times t} & 0_{d\times d} \end{bmatrix},$$

$$
(CK)^2_{v\times v} = \begin{bmatrix} I_{h\times h} & G_{12} & 0_{h\times k} & 0_{h\times s} & 0_{h\times t} & 0_{h\times d} \\ 0_{j\times h} & 0_{j\times j} & 0_{j\times k} & 0_{j\times s} & 0_{j\times t} & 0_{j\times d} \\ 0_{k\times h} & 0_{k\times j} & 0_{k\times k} & 0_{k\times s} & 0_{k\times t} & 0_{k\times d} \\ G_{21}S & G_{21}SG_{12} & 0_{s\times k} & 0_{s\times s} & 0_{s\times t} & 0_{s\times d} \\ G_{31}S^{-1} & G_{31}S^{-1}G_{12} & 0_{t\times k} & 0_{t\times s} & 0_{t\times t} & 0_{t\times d} \\ 0_{d\times h} & 0_{d\times j} & 0_{d\times k} & 0_{d\times s} & 0_{d\times t} & 0_{d\times d} \end{bmatrix},
$$

and

$$
(CKC)_{v\times u} = \begin{bmatrix} I_{h\times h} & G_{12} & G_{13} & 0_{h\times k} & 0_{h\times s} & 0_{h\times c} \\ 0_{j\times h} & 0_{j\times j} & 0_{j\times i} & 0_{j\times k} & 0_{j\times s} & 0_{j\times c} \\ 0_{k\times h} & 0_{k\times j} & 0_{k\times i} & 0_{k\times k} & 0_{k\times s} & 0_{k\times c} \\ G_{21}S & G_{21}SG_{12} & G_{21}SG_{13} & 0_{s\times k} & 0_{s\times s} & 0_{s\times c} \\ G_{31}S & G_{31}SG_{12} & G_{31}SG_{13} & 0_{t\times k} & 0_{t\times s} & 0_{t\times c} \\ 0_{d\times h} & 0_{d\times j} & 0_{d\times i} & 0_{d\times k} & 0_{d\times s} & 0_{d\times c} \end{bmatrix}.
$$

We now give two theorems based on the above representations. (In what follows, we use the expression "irrespective of $(M'AN)^-$" to mean "irrespective of the choice of $(M'AN)^-$.)

Theorem 1.

(A) Condition A holds irrespective of $(M'AN)^-$ if and only if $s = 0$, or $j = 0$.

(B1) Condition B1 holds irrespective of $(M'AN)^-$ if and only if $s = 0$, or $j = 0$ and $i = 0$.

(B2) Condition B2 holds irrespective of $(M'AN)^-$ if and only if $j = 0$, or $s = 0$ and $t = 0$.

(D) Condition D holds irrespective of $(M'AN)^-$ if and only if $s = 0$ and $t = 0$, or $j = 0$ and $i = 0$.

Proof of Theorem 1. (A) $ABABA = ABA$ if and only if $KCKCK = KCK$, which holds irrespective of $(M'AN)^-$ if and only if there are no s blocks, or there are no j blocks in KCK and $KCKCK$. (B1) Similarly, $(AB)^2 = AB$ if and only if $(KC)^2 = KC$, which holds irrespective of $(M'AN)^-$ if and only if there are no s blocks, or no j and i blocks in KC and $(KC)^2$. (B2) $(BA)^2 = BA$ if and only if $(CK)^2 = CK$, which holds irrespective of $(M'AN)^-$ if and only if there are no j blocks, or no s and t blocks in CK and $(CK)^2$. (D) $BAB = B$ if and only if $CKC = C$, which holds irrespective of $(M'AN)^-$ if and only if there are no j and i blocks, or no s and t blocks in C and CKC. QED.

The comparison between the two relevant matrices in each of the four conditions (A, B1, B2, and D) above also reveals that we can always make these four conditions hold by special choice of $(M'AN)^-$ regardless of the rank conditions described in Theorem 1.

Theorem 2.

Consider the following four conditions: (a) $G_{22} = G_{21}SG_{12}$, (b) $G_{23} = G_{21}SG_{13}$, (c) $G_{32} = G_{31}SG_{12}$, and (d) $G_{33} = G_{31}SG_{13}$. Then,

(A) Condition A holds if Condition (a) holds (i.e., if $(M'AN)^-$ that satisfies (a) is chosen).

(B1) Condition B1 holds if Conditions (a) and (b) hold.

(B2) Condition B2 holds if Conditions (a) and (c) hold.

(D) Condition D holds if all four conditions ((a), (b), (c), and (d)) hold.

Proof of Theorem 2. (A) Matrices KCK and $KCKCK$ can always be made identical by choosing a $(M'AN)^-$ in which Condition (a) holds. (B1) Similarly, matrices KC and $(KC)^2$ can be made identical by choosing a $(M'AN)^2$ in which Conditions (a) and (b) hold. (B2) Matrices CK and $(CK)^2$ can be made identical by choosing a $(M'AN)^-$ in which Conditions (a) and (c) hold. (D) Matrices C and CKC can be made identical by choosing a $(M'AN)^-$ in which all of the four conditions ((a), (b), (c), and (d)) hold. QED.

Note 1. In Theorem 2(A), Condition (a) is *ns* for Condition A when $s \neq 0$ and $j \neq 0$. In Theorem 2(B1), both Conditions (a) and (b) are necessary only when neither $j = 0$ nor $i = 0$. When $i = 0$ but $j \neq 0$, only (a) is necessary, and when $j = 0$ but $i \neq 0$, only (b) is necessary. Similarly in Theorem 2(B2), both Conditions (a) and (c) are necessary only when neither $s = 0$ nor $t = 0$. When $t = 0$ but $s \neq 0$, only (a) is necessary, and when $s = 0$ but $t \neq 0$, only (c) is necessary. The situation is much more complicated in Theorem 2(D). All of the four conditions ((a), (b), (c), and (d)) are necessary when none of j, i, s, and t are zero. When only i is 0 (the others are nonzero), only (a) and (c) are necessary, when only j is 0, only (b) and (d) are necessary, when only t is 0, only (a) and (b) are necessary, and when only s is 0, only (c) and (d) are necessary. Furthermore, when only t and i are 0 (s and j are nonzero), only (a) is necessary, when only t and j are 0, only (b) is necessary, when only s and i are 0, only (c) is necessary, and when only s and j are 0, only (d) is necessary. These are summarized in Table 3 below.

Note 2. The four conditions in Theorem 2 partially "degenerate" when $h = 0$: $G_{22} = 0$ in Condition (a) when neither $s = 0$ nor $j = 0$ (G_{22} exists only when neither $s = 0$ nor $j = 0$), $G_{23} = 0$ in Condition (b) when neither $s = 0$ nor $i = 0$, $G_{32} = 0$ in Condition (c) when neither $t = 0$ nor $j = 0$, and $G_{33} = 0$ in Condition (d) when neither $t = 0$ nor $i = 0$.

Note 3. We also see that $g = h + \mathrm{rank}(G_{22} - G_{21}SG_{12})$, so that in general $g \geq h$ as has been alluded to earlier, and $g = h$ if and only if either $s = 0$, $j = 0$, or $\mathrm{rank}(G_{22} - G_{21}SG_{12}) = 0$.

Observations in Theorems 1 and 2 as well as our earlier observations are summarized in the following table.

Table 2: Summary of the new characterizations of the conditions described in Table 1.

Condition	Conditions on rank or a g-inverse	Remark
A	$\{s=0\}\cup\{j=0\}$, or a special $(M'AN)^-$	C1 | C2 | a special $(M'AN)^-$
B	$(\{s=0\}\cup(\{j=0\}\cap\{i=0\}))\cap$ $(\{j=0\}\cup(\{s=0\}\cap\{t=0\}))$, or a special $(M'AN)^-$	B1 & B2
B1	$\{s=0\}\cup(\{j=0\}\cap\{i=0\})$, or a special $(M'AN)^-$	C1 | E1 | a special $(M'AN)^-$
B2	$\{j=0\}\cup(\{s=0\}\cap\{t=0\})$, or a special $(M'AN)^-$	C2 | E2 | a special $(M'AN)^-$
C	$\{s=0\}\cap\{j=0\}$	C1 & C2
C1	$s=0$	
C2	$j=0$	
G	$\{i=0\}\cap\{t=0\}$	G1 & G2
G1	$i=0$	
G2	$t=0$	
D	$(\{j=0\}\cap\{i=0\})\cup(\{s=0\}\cap\{t=0\})$, or a special $(M'AN)^-$	E1 | E2 | a special $(M'AN)^-$
E	$\{j=0\}\cap\{i=0\}\cap\{s=0\}\cap\{t=0\}$	E1 & E2
E1	$\{j=0\}\cap\{i=0\}$	G1 & C2
E2	$\{s=0\}\cap\{t=0\}$	G2 & C1
F	$\{j=0\}\cap\{s=0\}\cap\{k=0\}$	C & $\{k=0\}$

"&" indicates a logical "and", and "\|" a logical "or".
Special g-inverses $(M'AN)^-$ required are given in Table 3.

Table 3 presents another way of looking at the two theorems. In this table, rank identifiability conditions are characterized in terms of sets of rank profiles and sets of conditions on $(M'AN)^-$ that they should satisfy. Rank profiles are defined by combinations of four rank conditions: (1) $s=0$ or $s\neq 0$ (In the table, "$\neq 0$" is indicated by "$=1$".), (2) $t=0$ or $t\neq 0$, (3) $j=0$ or $j\neq 0$, and (4) $i=0$ or $i\neq 0$. Symbol "Y" under a particular rank identifiability condition indicates that this condition is satisfied (irrespective of conditions on $(M'AN)^-$) for a particular rank profile (corresponding to a row of the table). The rank profiles for which a particular rank identifiability condition is marked by Y may be called the Y-profiles associated with the condition. For non-Y-profiles, some conditions on $(M'AN)^-$ are required to satisfy rank identifiability conditions. These are indicated by strings of up to four lower case letters, each letter signifying a condition. For example, ab under Condition B1 and rank profile 12 indicates that both Conditions (a) and (b) are necessary to satisfy

Table 3: Rank identifiability conditions characterized by rank profiles, and conditions on $(M'AN)^{-}$.

	Rank Profile				Rank Identifiability Condition													
No.	s	t	j	i	A	B1	B2	B	C1	C2	C	G1	G2	G	D	E1	E2	E
1	0	0	0	0	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
2	0	0	0	1	Y	Y	Y	Y	Y	Y	Y		Y		Y		Y	
3	0	0	1	0	Y	Y	Y	Y	Y			Y	Y	Y	Y		Y	
4	0	0	1	1	Y	Y	Y	Y	Y				Y		Y		Y	
5	0	1	0	0	Y	Y	Y	Y	Y	Y	Y	Y			Y	Y		
6	0	1	0	1	Y	Y	Y	Y	Y	Y	Y				d			
7	0	1	1	0	Y	Y	c	c	Y			Y			c			
8	0	1	1	1	Y	Y	c	c	Y						cd			
9	1	0	0	0	Y	Y	Y	Y		Y		Y	Y	Y	Y	Y		
10	1	0	0	1	Y	b	Y	b		Y			Y		b			
11	1	0	1	0	a	a	a	a				Y	Y	Y	a			
12	1	0	1	1	a	ab	a	ab					Y		ab			
13	1	1	0	0	Y	Y	Y	Y		Y		Y			Y	Y		
14	1	1	0	1	Y	b	Y	b		Y					bd			
15	1	1	1	0	a	a	ac	ac				Y			ac			
16	1	1	1	1	a	ab	ac	abc							abcd			

0 under the rank profile means s, t, j, and i are zero, 1 means they are nonzero, e.g., Row 6 corresponds to the rank profile of $s = 0$, $t \neq 0$, $j = 0$, and $i \neq 0$.
"Y" indicates that a particular rank identifiability condition is satisfied under a particular rank profile, and the lower case letters indicate conditions required of special $(M'AN)^{-}$ to satisfy a particular rank identifiability condition.

Condition B1 under rank profile 12. A blank entry in the table indicates an empty set. (No conditions on $(M'AN)^-$ will satisfy particular rank identifiability conditions.)

A rank identifiability condition is said to be a special case of another when Y-profiles and a set of conditions on $(M'AN)^-$ associated with the former is a subset of those associated with the latter. For example, Y-profiles of E1 is a subset of those for C2. Thus, Condition E1 is a special case of C2. Similarly, E1 is a special case of G1, C1 is a special case of B1, C2 is a special case of B2, and both Conditions B1 and B2 are special cases of Condition A. (Y-profiles of B1 is a subset of those of A, and conditions on $(M'AN)^-$ for B1 in particular rank profiles are subsets of those for A under the same rank profiles. Conditions (ab) (both Conditions (a) and (b)) is a subset of Condition (a). Y-profiles constitute a universal condition for the choice of $(M'AN)^-$, since they impose no conditions on $(M'AN)^-$. The case of B2 is similar.) Figure 1 was actually constructed from these observations.

Unions and intersections of rank identifiability conditions may be defined by the same operations on Y-profiles and conditions on $(M'AN)^-$ associated with the rank identifiability conditions. For example, the intersection of C1 and C2 (Condition C) is characterized by rank profiles 1, 2, 5 and 6 (which is indeed the intersection of the Y-profiles associated with C1 and C2). Similarly, the union of C1 and C2 (C1 or C2) is characterized by profiles 1 through 10, 13, and 14. The intersection of Y-profiles of C2 and G1 is equal to the Y-profiles of E1. Thus, E1 is equal to the intersection of C2 and G1. The intersection B1 and B2 (Condition B) is characterized by the intersection of Y-profiles, and the intersection of conditions on $(M'AN)^-$ associated with these conditions. These come down to rank profiles 1 through 6, 9, and 13, and Condition (c) under rank profiles 7 and 8, Condition (b) under profiles 10 and 14, Condition (a) under profile 11, Condition (ab) under profile 12, Condition (ac) under profile 15, and Condition (abc) for profile 16. (Note that the intersection of Conditions (ab) and (a), for example, is equal to Condition (ab).) The union of B1 and B2 (B1 or B2) is characterized by profiles 1 through 10, 13, and 14, and Condition (a). These profiles and condition are equal to those of Condition A. Thus, Condition A and Condition B1 or B2 are equivalent.

Recently, Tian and Styan (2004, Corollary 2.3) have shown that $\text{rank}(A - ABA) = \text{rank}(A) - \text{rank}(M'AN)$ holds unconditionally (whereas $\text{rank}(ABA)$ is not necessarily equal to $\text{rank}(M'AN)$ in general, and $\text{rank}(A - ABA) = \text{rank}(A) - \text{rank}(ABA)$ as well as $\text{rank}(ABA) = \text{rank}(M'AN)$ require a condition). This somewhat counter-intuitive result can easily be shown using the PSVD framework. Note that $A - ABA = X_1(K - KCK)X_2$, and that $\text{rank}(A - ABA) = \text{rank}(K - KCK)$, where

$$
(K - KCK)_{u\times v} = \begin{bmatrix} 0_{h\times h} & -G_{12} & 0_{h\times k} & 0_{h\times s} & 0_{h\times t} & 0_{h\times d} \\ 0_{j\times h} & I_{j\times j} & 0_{j\times k} & 0_{j\times s} & 0_{j\times t} & 0_{j\times d} \\ 0_{i\times h} & 0_{i\times j} & 0_{i\times k} & 0_{i\times s} & 0_{i\times t} & 0_{i\times d} \\ 0_{k\times h} & 0_{k\times j} & I_{k\times k} & 0_{k\times s} & 0_{k\times t} & 0_{k\times d} \\ -G_{21}S & -G_{22} & 0_{s\times k} & I_{s\times s} & 0_{s\times t} & 0_{s\times d} \\ 0_{c\times h} & 0_{c\times j} & 0_{c\times k} & 0_{c\times s} & 0_{c\times t} & 0_{c\times d} \end{bmatrix}.
$$

The rank of this matrix is equal to $j+k+s = a-h$ regardless of G_{12}, G_{21}, and G_{22}. Ouellette (1981) has pointed out that if either Condition C1 or C2 holds, $\mathrm{rank}(A - ABA) = a - h$. However, it is clear from the above exercise (as well as from Tian and Styan (2004)) that no conditions are necessary for $\mathrm{rank}(A - ABA) = a - h$. Rather, Condition C1 or C2 is equivalent to the condition under which Condition A holds irrespective of conditions on $(M'AN)^-$.

Although PSVD was originally developed for computational purposes, this paper demonstrates that it is also useful as a tool for mathematical proof. This is no different from the ordinary SVD.

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