On Two Expressions of the MLE for a Special Case of the Extended Growth Curve Models

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ABSTRACT

An algebraic proof is given on the equivalence between two solutions of MLEs for a special case of the extended growth curve models called the Banken model. One solution given by Verbyla and Venables is an iterative solution in the general case but reduces to a non-iterative one in the case of the Banken model. The other solution given by von Rosen is a closed-form solution specifically targeted at the Banken model. The proof has turned out to be quite challenging yet intriguing as it touches on many aspects of intricate matrix theory involving projection matrices.

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1 Introduction

Growth curve models (GCM; [1]) provide versatile techniques for analyzing patterns of change in repeated/longitudinal data and investigating how such patterns are related to subject characteristics. Let X denote an n by p data matrix, where n indicates the number of subjects, and p the number of measurement occasions. Let G (n by m) and H (p by q) represent design matrices for subjects and measurement occasions, respectively. The matrix G, for example, may be the matrix of dummy variables indicating treatment groups, and H the matrix of orthogonal polynomials of certain order to capture basic trends in repeated measurements. For simplicity, we assume that both G and H are columnwise nonsingular (without much loss of generality). The conventional growth curve model (hereafter, simply GCM) postulates

$$X = GAH' + E, \tag{1}$$

where A is the m by q matrix of regression coefficients, and E is the matrix of disturbance terms. Maximum likelihood estimates of regression parameters have been derived [2, 3, 4] under the distributional assumption that

$$\mathbf{e} \sim \mathcal{N}(\mathbf{0}, \Sigma \otimes I_n), \tag{2}$$

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where $\mathbf{e} = \mathrm{vec}(E)$ with an unknown covariance matrix Σ between the elements of a row vector of E, and $\otimes$ indicates a Kronecker product defined by $[\sigma_{ij}I]$, where $\Sigma = [\sigma_{ij}]$. See Lemma 7 in Section 3 for details.

The ordinary GCM described above has been generalized to the extended GCM (ExGCM) by including more than one structural term in the model. Let this model be written as

$$X = \sum_{j=1}^{J} G_j A_j H_j' + E, \tag{3}$$

where G_j, H_j, and A_j are analogous to G, H, and A in (1), and E is, as before, the matrix of disturbance terms. The general form of the extended GCM given above was first proposed by Verbyla and Venables [5], who also derived an iterative procedure for finding maximum likelihood estimates (MLE) of A_j's under a similar distributional assumption to (2). This is an alternating maximum likelihood (AML) algorithm for solving maximum likelihood equations, in which a subset of parameter estimates are updated alternately with other subsets of parameters temporarily fixed until convergence is attained.

In this paper, we primarily focus on a special case of (3) called the Banken model [6], in which it is assumed that

$$\mathrm{Sp}(G_j) \supset \mathrm{Sp}(G_{j+1}) \tag{4}$$

for $j = 1, \cdots, J-1$, where Sp indicates a range space. Under this assumption, a closed-form solution for the MLE of A_j's has been derived by von Rosen [6]. Verbyla and Venables' [5] iterative solution, on the other hand, converges in one step for the Banken model. These two solutions of the MLE for the Banken model will be described in Section 3. As will be seen, they appear completely different despite the fact that they are the same MLEs of the same model. This paper provides an explicit proof that they are indeed algebraically equivalent in the limited case of $J = 2$. This exercise turned out to be quite challenging yet intriguing from a linear algebra perspective. In particular, it touched on many aspects of intricate matrix theory involving projection matrices [7].

The rest of this paper is organized as follows. The next section (Section 2) provides some basic results of matrix algebra related to projection matrices useful in subsequent sections. Section 3 first reviews what is known about the MLE for the Banken model, and then gives a precise statement of the problem to be solved in this paper. Section 4 gives our main results, the proof of the equivalence of the two MLEs. The proof is presented in several steps. The final section concludes the paper.

2 Preliminaries

Let Z be an n by p matrix, where we assume $n \geq p$. Define the orthogonal projector P_Z onto $\mathrm{Sp}(Z)$ (the range space of Z) by

$$P_Z = Z(Z'Z)^{-}Z', \tag{5}$$

where $(Z'Z)^{-}$ indicates a generalized inverse of $Z'Z$. Let Q_Z denote the orthogonal complement of P_Z defined by

$$Q_Z = I - P_Z, \tag{6}$$

The matrix Q_Z is the orthogonal projector onto $\mathrm{Ker}(Z')$ (the null space of Z). We generalize the two projectors defined above by introducing a symmetric *nnd* (non-negative definite) matrix K of order n, such that

$$\mathrm{rank}(KZ) = \mathrm{rank}(Z). \tag{7}$$

The matrix K is often referred to as a metric matrix. Let

$$P_{Z/K} = Z(Z'KZ)^{-}Z'K. \tag{8}$$

This matrix represents the projector onto $\mathrm{Sp}(Z)$ along $\mathrm{Ker}(Z'K)$. (See, for example, [7, 17].) Its transpose, denoted by $P'_{Z/K}$, is the projector onto $\mathrm{Sp}(KZ)$ along $\mathrm{Ker}(Z')$. We also define the orthogonal complement of $P_{Z/K}$ with respect to the metric matrix K, namely

$$Q_{Z/K} = I - P_{Z/K}. \tag{9}$$

This matrix represents the projector onto $\mathrm{Ker}(Z'K)$ along $\mathrm{Sp}(Z)$. Its transpose $Q'_{Z/K}$ is the projector onto $\mathrm{Ker}(Z')$ along $\mathrm{Sp}(KZ)$. The four types of projectors defined above are the only types of projectors we use in this paper.

We begin with the following lemma giving the basic decompositions of P_Z when Z is partitioned into two column blocks, namely $Z = [M, N]$.

Lemma 1 *Let Z, M and N be matrices as introduced above. Then,*

$$\begin{aligned} P_Z &= P_M + P_{Q_M N} = P_N + P_{Q_N M}, && (10)\\ &= P_{M/Q_N} + P_{N/Q_M} \quad \textit{(if and only if } M \textit{ and } N \textit{ are disjoint).} && (11) \end{aligned}$$

For a proof, see e.g., Rao and Yanai [8]. Note that the decompositions (10) hold generally, while the decomposition (11) holds if and only if M and N are disjoint, i.e., $\mathrm{Sp}(M) \cap \mathrm{Sp}(N) = \{0\}$. The former are useful when we first fit one of two predictor sets (M or N), and then fit the other to residuals from the first in regression analysis. The latter is useful when we fit both predictor sets simultaneously. Note that

$$P_{Q_M N} = Q_M P_{N/Q_M} \;\text{ and }\; P_{Q_N M} = Q_N P_{M/Q_N}, \tag{12}$$

because both Q_M and Q_N are idempotent.

The following lemma generalizes Lemma 1 to the case of nonidentity metric K.

Lemma 2 *Let $Z = [M, N]$, and let K satisfy condition (7). Then,*

$$\begin{aligned} P_{Z/K} &= P_{M/K} + P_{Q_{M/K}N/K} = P_{N/K} + P_{Q_{N/K}M/K}, && (13)\\ &= P_{M/KQ_{N/K}} + P_{N/KQ_{M/K}} \quad \textit{(if and only if } M \textit{ and } N \textit{ are disjoint).} && (14) \end{aligned}$$

A proof is omitted. See, for example, Takane and Yanai [9].

The following two lemmas describing basic properties of projectors are easy to establish.

Lemma 3 *Let $Z = [M, N]$, where M and N are disjoint. Then,*

$$Q_Z = Q_M Q_{N/Q_M} = Q_N Q_{M/Q_N}. \tag{15}$$

Let K satisfy the condition stated in (7). Then,

$$Q_{Z/K} = Q_{M/K} Q_{N/KQ_{M/K}} = Q_{N/K} Q_{M/KQ_{N/K}}. \tag{16}$$

Proof. We first prove (15). By (10), $P_M P_{Q_M N} = O$, and (12), we have $Q_Z = I - P_M - P_{Q_M N} = I - P_M - P_{Q_M N} + P_M P_{Q_M N} = (I - P_M)(I - P_{Q_M N}) = Q_M Q_{Q_M N} = Q_M Q_{N/Q_M}$. Note that P_M and $P_{Q_M N}$ commute, so $Q_M Q_{Q_M N} = Q_{Q_M N} Q_M$. By symmetry, the role of M and N can be exchanged, and the second equality in (15) also holds. Identities in (16) can be proven along a similar line. □

Lemma 4 *Let Z, M, and N be as introduced in Lemma 3. Then,*

$$P_{M/Q_N} = P_M Q_{N/Q_M} = P_M - P_M P_{N/Q_M}. \tag{17}$$

Let K satisfy the condition stated in (7). Then,

$$P_{M/KQ_{N/K}} = P_{M/K} Q_{N/KQ_{M/K}} = P_{M/K} - P_{M/K} P_{N/KQ_{M/K}}. \tag{18}$$

Proof. By (11), (10), and (12), we have $P_{M/Q_N} = P_Z - P_{N/Q_M} = P_M + P_{Q_M N} - P_{N/Q_M} = P_M + Q_M P_{N/Q_M} - P_{N/Q_M} = P_M + P_{N/Q_M} - P_M P_{N/Q_M} - P_{N/Q_M} = P_M Q_{N/Q_M}$. Identities in (18) can be proven similarly. □

The following corollary, which we call Seber's [10, pp. 465-466] trick, follows immediately from Lemma 4, and is useful in the sequel.

Corollary 1 *Let $X = MA + NB + E^*$ be a regression model with multivariate criterion variables X, two disjoint sets of predictor variables M and N with matrices of regression coefficients A and B, and a matrix of disturbance terms E^*. Then, the OLS (ordinary least squares) estimates of MA and NB are given by*

$$N\hat{B} = P_{N/Q_M} X = P_N Q_{M/Q_N} X = P_N (I - P_{M/Q_N}) X = P_N (X - M\hat{A}), \tag{19}$$

and

$$M\hat{A} = P_M (X - N\hat{B}) = P_M (X - P_{N/Q_M} X) = P_M Q_{N/Q_M} X = P_{M/Q_N} X. \tag{20}$$

The two equations above could be written in a completely parallel manner because the two regression terms in the model play algebraically symmetric roles. However, they are deliberately arranged in different orders. This is because in statistics, the roles that the two regression terms play are often not symmetric (e.g., one is the predictor set of our interest and the other the set of covariates whose effects are to be eliminated), and some expressions in the two equations are more important than others in certain contexts.

Corollary 1 can be extended to accommodate a nonidentity metric matrix K. We state this extension for only (20). (The other can be easily deduced.) Let K satisfy (7). Then,

$$\begin{aligned} M\hat{A} &= P_{M/K}(X - N\hat{B}) = \\ &P_{M/K}(X - P_{N/KQ_{M/K}}X) = P_{M/K}Q_{N/KQ_{M/K}}X = P_{M/KQ_{N/K}}X. \end{aligned} \tag{21}$$

Note that in (21) $\hat{A} = (M'KQ_{N/K}M)^{-}M'KQ_{N/K}X$ and $\hat{B} = (N'KQ_{M/K}N)^{-}N'KQ_{M/K}X$.

The following lemma states which projector will win when two projectors with the same onto space but in different metrics are successively applied. These results can be readily verified [11, Appendix II], but explicit statements are still handy to have.

Lemma 5 *Let L denote another nnd matrix satisfying a similar condition to (7), i.e.,* $\mathrm{rank}(LZ) = \mathrm{rank}(Z)$. *Then,*

$$\begin{aligned} &\text{(a)}\ \ P_{Z/K}P_{Z/L} = P_{Z/L}, \quad \text{(b)}\ \ P'_{Z/K}P'_{Z/L} = P'_{Z/K}, \\ &\text{(c)}\ \ Q_{Z/K}Q_{Z/L} = Q_{Z/K}, \quad \text{(d)}\ \ Q'_{Z/K}Q'_{Z/L} = Q'_{Z/L}. \end{aligned} \tag{22}$$

Proof. Proposition (a) can be verified directly. The others follow immediately from (a). □

The following lemma, which we call Khatri's [2] lemma, will be repeatedly used in subsequent sections.

Lemma 6 *Let S be a pd (positive definite) matrix of order p, and let M $(p \times p_1)$ and N $(p \times p_2)$ be such that* $\mathrm{Sp}(N) = \mathrm{Ker}(M')$. Then,

$$P'_{M/S} = Q_{N/S^{-1}}, \tag{23}$$

where the matrices $P'_{M/S}$ and $Q_{N/S^{-1}}$ are the projector onto $\mathrm{Sp}(SM) = \mathrm{Ker}(N'S^{-1})$ along $\mathrm{Ker}(M') = \mathrm{Sp}(N)$.

For a proof, see Khatri [2]. Note that S and S^{-1} in (23) are exchangeable. We note in passing that this lemma has been generalized in several ways, to a *psd* (positive-semidefinite) S [12], to a nonsymmetric and singular S such that $\mathrm{Sp}(N) \subset \mathrm{Sp}(S)$ and $\mathrm{Sp}(M) \subset \mathrm{Sp}(S')$ [13], and to a rectangular S [14, Appendix].

3 A Statement of the Problem

In this section, we give a precise statement of the problem we aim to solve in this paper. We begin by stating Khatri's [2] solution (Lemma 7 below) for ordinary GCM, which is in fact a special case of the extended GCM (ExGCM) with $J = 1$. This solution serves as a building block for more general cases. We then discuss Verbyla and Venables' [5] solution and von Rosen's [6] solution for the Banken model focusing on $J = 2$. A full understanding of the two solutions is essential to motivate our main results to be reported in the next section.

Lemma 7 *Let $J = 1$ in model (3), and let X, G_1, H_1, and A_1 be as introduced in (3). Then, the MLEs of $G_1 A_1 H_1'$ and Σ under (2) are given by*

$$G_1 \hat{A}_1 H_1' = P_{G_1} X P'_{H_1/S_1^{-1}}, \tag{24}$$

and

$$n\hat{\Sigma} = S_1 + Q_{H_1/S_1^{-1}} X' P_{G_1} X Q'_{H_1/S_1^{-1}}, \tag{25}$$

where

$$S_1 = X' Q_{G_1} X. \tag{26}$$

Proof. A proof is given by Khatri [2; see also Seber [10, pp. 480-482]]. It is reproduced below for later references. Let $T = [T_1, T_2]$ be such that $H_1' T_1 = I$ and $\mathrm{Sp}(T_2) = \mathrm{Ker}(H_1')$. Setting $J = 1$ in (3), and postmultiplying by T, we obtain

$$Y = [Y_1, Y_2] = XT = [G_1 A_1, O] + E^*, \tag{27}$$

where $E^* = ET$. We use the information in the conditional expectation of Y_1 given Y_2 for estimating A_1 (covariance adjustment), namely

$$\mathrm{Ex}[Y_1|Y_2] = G_1 A_1 + Y_2 B_1, \tag{28}$$

where Ex indicates an expectation operation, and B_1 is the matrix of regression weights applied to Y_2. Estimates (conditional MLEs) of parameters in (28) can be obtained by the OLS estimation of regression parameters in

$$Y_1 = G_1 A_1 + Y_2 B_1 + \tilde{E}. \tag{29}$$

This leads to

$$\begin{aligned} G_1 \hat{A}_1 H_1' &= P_{G_1}(Y_1 - P_{Y_2/Q_{G_1}} Y_1) H_1' \quad \text{(by Seber's trick)} \\ &= P_{G_1} X Q_{T_2/S_1} T_1 H_1' \quad (\text{since } Y_1 = XT_1 \text{ and } Y_2 = XT_2) \\ &= P_{G_1} X P'_{H_1/S_1^{-1}} T_1 H_1' \quad \text{(by Khatri's lemma)} \\ &= P_{G_1} X P'_{H_1/S_1^{-1}} \end{aligned} \tag{30}$$

Since the marginal distribution of Y_2 is unrelated to A_1, this is also the unconditional MLE of $G_1 A_1 H_1'$. The MLE of Σ follows immediately from

$$n\hat{\Sigma} = (X - G_1 \hat{A}_1 H_1')'(X - G_1 \hat{A}_1 H_1'). \tag{31}$$

Note that

$$H_1' S_1^{-1} = H_1' (n\hat{\Sigma})^{-1}, \tag{32}$$

since $(n\hat{\Sigma})^{-1}$ can be written as $(n\hat{\Sigma})^{-1} = S_1^{-1} - S_1^{-1} Q_{H_1/S_1^{-1}} M$ for some M (see Lemma 2.1(i) of von Rosen [6], or A3.3 of Seber [10, Appendix]), and $H_1' S_1^{-1} Q_{H_1/S_1^{-1}} = O$, so that $P_{H_1/S_1^{-1}} = P_{H_1/\hat{\Sigma}^{-1}}$. □

Verbyla and Venables [5] derived an iterative solution for ExGCM, which reduced to a noniterative one for the Banken model. It uses a transformational approach similar to the case above for $J = 1$. For $J = 2$, let $T = [T_{11}, T_{12}, T_2]$ be such that $H_1'T_{11} = I$, $H_1'T_{12} = O$, $H_2'T_{11} = O$, $H_2'T_{12} = I$, and $\mathrm{Sp}(T_2) = \mathrm{Ker}(H')$, where $H = [H_1, H_2]$. Setting $J = 2$ in (3) and postmultiplying by T, we obtain

$$Y = [Y_{11}, Y_{12}, Y_2] = XT = [G_1A_1, G_2A_2, O] + E^*, \tag{33}$$

where $E^* = ET$. Using the same logic as above for $J = 1$, we obtain the MLEs of A_1 and A_2 by OLS estimations of

$$Y_{12} = G_2A_2 + Y_2B_2 + \tilde{E}_2, \tag{34}$$

and

$$Y_{11} = G_1A_1 + \hat{U}_{12}C_1 + Y_2B_1 + \tilde{E}_1, \tag{35}$$

where $\hat{U}_{12}$ is the matrix of residuals from (34), namely $\hat{U}_{12} = Y_{12} - G_2\hat{A}_2 - Y_2\hat{B}_2 = Q_{[G_2,Y_2]}Y_{12}$. One may rightfully wonder why (34) does not need a residual term like $\hat{U}_{21} = Q_{[G_1,Y_2]}Y_{11}$ analogous to $\hat{U}_{12}$ in (35). This is because the term $(\hat{U}_{21})$ is orthogonal to both of the two regression terms in (34), and consequently has no effect on the OLS estimates of A_2 and B_2 in (34), and can therefore be omitted.

The OLS estimation of (34) leads to

$$G_2\hat{A}_2H_2' = P_{G_2}(Y_{12} - P_{Y_2/Q_{G_2}}Y_{12})H_2', \tag{36}$$

using Seber's trick. The regression model in (35), on the other hand, can be rewritten as

$$Y_{11} = G_1A_1 + Q_{G_2/Q_{Y_2}}Y_{12}C + Y_2B_1^* + \tilde{E}_1 \tag{37}$$

by transferring the part of the effect of $Q_{[G_2,Y_2]}Y_{12}$ pertaining to Y_2 to the third term. More formally, from Lemma 3, we have $Q_{[G_2,Y_2]}Y_{12} = Q_{Y_2}Q_{G_2/Q_{Y_2}}Y_{12} = (Q_{G_2/Q_{Y_2}} - P_{Y_2}Q_{G_2/Q_{Y_2}})Y_{12} = Q_{G_2/Q_{Y_2}}Y_{12} - Y_2(Y_2'Y_2)^{-1}Y_2'Q_{G_2/Q_{Y_2}}Y_{12}$, so the $(Y_2'Y_2)^{-1}Y_2'Q_{G_2/Q_{Y_2}}Y_{12}$ part of the second term can be subtracted from B_1 in (35) to define $\hat{B}_1^*$ in (37). The OLS estimation of (37) leads to

$$G_1\hat{A}_1H_1' = P_{G_1}(Y_{11} - P_{[Q_{Y_2/Q_{G_2}}Y_{12},Y_2]/Q_{G_1}}Y_{11})H_1', \tag{38}$$

again using Seber's trick. Once $G_1\hat{A}_1H_1'$ and $G_2\hat{A}_2H_2'$ are obtained, the estimate of $n\hat{\Sigma}$ is obtained by

$$n\hat{\Sigma} = (X - G_1\hat{A}_1H_1' - G_2\hat{A}_2H_2')'(X - G_1\hat{A}_1H_1' - G_2\hat{A}_2H_2'). \tag{39}$$

von Rosen [6] derived an elegant solution for the MLEs of A_j's for the Banken model. Below we summarize his solution. Let

$$R_r = \prod_{k=0}^{r-1} F_k, \text{ where } F_0 = I \tag{40}$$

for $r = 1, \cdots, J+1$,

$$F_i = I - R_iH_i(H_i'R_i'S_i^{-1}R_iH_i)^{-1}H_i'R_i'S_i^{-1} \tag{41}$$

for $i = 1, \cdots, J$, and

$$S_i = \sum_{j=1}^{i} K_j, \tag{42}$$

where

$$K_j = R_j X' P_{G_{j-1}} Q_{G_j} P_{G_{j-1}} X R_j' \tag{43}$$

with $G_0 = I$. Then,

$$G_r \hat{A}_r H_r' = P_{G_r}(X - \sum_{i=r+1}^{J} G_i \hat{A}_i H_i') P'_{R_r H_r / S_r^{-1}}, \tag{44}$$

and

$$\begin{aligned} n\hat{\Sigma} &= (X - \sum_{i=1}^{J} G_i \hat{A}_i H_i')'(X - \sum_{i=1}^{J} G_i \hat{A}_i H_i') \\ &= S_J + R_{J+1} X' P_J X R_{J+1}'. \end{aligned} \tag{45}$$

For $J = 2$, we obtain

$$R_1 = F_0 = I, \tag{46}$$

$$F_1 = I - H_1 (H_1' S_1^{-1} H_1)^{-1} H_1' S_1^{-1} = Q_{H_1/S_1^{-1}}, \tag{47}$$

where S_1 is as defined in (26),

$$R_2 = F_1 F_0 = F_1 R_1 = Q_{H_1/S_1^{-1}}, \tag{48}$$

$$\begin{aligned} F_2 &= I - Q_{H_1/S_1^{-1}} H_2 (H_2' Q'_{H_1/S_1^{-1}} S_2^{-1} Q_{H_1/S_1^{-1}} H_2)^{-1} H_2' Q'_{H_1/S_1^{-1}} S_2^{-1} \\ &= Q_{Q_{H_1/S_1^{-1}} H_2 / S_2^{-1}}, \end{aligned} \tag{49}$$

where

$$S_2 = S_1 + Q_{H_1/S_1^{-1}} X' P_{G_1} Q_{G_2} P_{G_1} X Q'_{H_1/S_1^{-1}}, \tag{50}$$

and

$$R_3 = F_2 F_1 F_0 = F_2 R_2 = Q_{Q_{H_1/S_1^{-1}} H_2 / S_2^{-1}} Q_{H_1/S_1^{-1}} = Q_{H/S_2^{-1}}, \tag{51}$$

where, as before, $H = [H_1, H_2]$. Note that the last equality in (51) holds because of (16). Note also that in (49) $Q'_{H_1/S_1^{-1}} S_2^{-1} Q_{H_1/S_1^{-1}} = Q'_{H_1/S_1^{-1}} S_2^{-1} = S_2^{-1} Q_{H_1/S_1^{-1}}$. Finally,

$$S_3 = S_2 + Q_{H/S_2^{-1}} X' P_{G_2} X Q'_{H/S_2^{-1}}, \tag{52}$$

where we assumed $Q_{G_3} = I$. For reasons similar to (32), we have

$$H_1' S_1^{-1} = H_1' S_2^{-1} = H_1' S_3^{-1}, \tag{53}$$

and

$$H_2' S_2^{-1} = H_2' S_3^{-1}. \tag{54}$$

The MLE of $G_2A_2H_2'$, $G_1A_1H_1'$, and Σ can be explicitly written as

$$G_2\hat{A}_2H_2' = P_{G_2}XP'_{H_2/S_2^{-1}Q_{H_1/S_1^{-1}}}, \tag{55}$$

$$G_1\hat{A}_1H_1' = P_{G_1}(X - P_{G_2}XP'_{H_2/S_2^{-1}Q_{H_1/S_1^{-1}}})P'_{H_1/S_1^{-1}}, \tag{56}$$

and

$$n\hat{\Sigma} = S_3. \tag{57}$$

Considering (53), (54), and (57), S_1^{-1} and S_2^{-1} in (55) and (56) can all be replaced by $\hat{\Sigma}$.

Expressions of MLEs of $G_2A_2H_2'$ and $G_1A_1H_1'$ given in (36) and (38), and in (55) and (56) look completely different despite the fact that they are MLEs of the same parameters in the same model. In the following section we explicitly show that they are indeed equivalent expressions.

4 Main Results

We first present a lemma which essentially shows

$$P_{H_2/\tilde{S}^{-1}Q_{H_1/\tilde{S}^{-1}}} = P_{H_2/S_2^{-1}Q_{H_1/S_1^{-1}}} = P_{H_2/\hat{\Sigma}^{-1}Q_{H_1/\hat{\Sigma}^{-1}}}, \tag{58}$$

where

$$\tilde{S} = X'Q_{G_2}X, \tag{59}$$

and where S_1, S_2 and $\hat{\Sigma}$ are as defined in (26), (50), and (57). This identity plays a crucial role in both Theorems 1 and 2.

Lemma 8 *Let T_{12} and T_2 be as defined right above (33), and S_1, S_2, and $\tilde{S}$ be as defined in (26), (50), and (59). Then,*

$$\tilde{S}^{-1}Q_{H_1/\tilde{S}^{-1}} = S_2^{-1}Q_{H_1/S_1^{-1}}. \tag{60}$$

Proof. Let $T^* = [T_{12}, T_2]$. Then, the left-hand side of (60) is equal to $T^*(T^{*'}\tilde{S}T^*)^{-1}T^{*'}$, and the right-hand side is equal to $S_2^{-1}Q_{H_1/S_1^{-1}} = S_2^{-1}Q_{H_1/S_2^{-1}}$(by (53)) $= T^*(T^{*'}S_2T^*)^{-1}T^{*'}$, both by Khatri's lemma. So we are to show that

$$T^{*'}\tilde{S}T^* = T^{*'}S_2T^*. \tag{61}$$

The left-hand side of (61) is equal to $T^{*'}X'Q_{G_2}XT^* = T^{*'}X'(P_{G_1} + Q_{G_1})Q_{G_2}(P_{G_1} + Q_{G_1})XT^* = T^{*'}X'P_{G_1}Q_{G_2}P_{G_1}XT^* + T^{*'}S_1T^* = T^{*'}(S^* + S_1)T^*$, where $S^* = X'P_{G_1}Q_{G_2}P_{G_1} \times X$. The right-hand side of (61), on the other hand, is equal to $T^{*'}S_1T^* + T^{*'}Q_{H_1/S_1^{-1}}S^* \times Q'_{H_1/S_1^{-1}}T^* = T^{*'}(S_1 + S^*)T^*$, since $Q'_{H_1/S_1^{-1}}T^* = T^*$, concluding the proof. □

It is obvious that the first equality in (58) follows from Lemma 8. (The two metric matrices that define the first two projectors in (58) are identical.) The second equality in (58) follows from (57), (54), and (53). We now proceed to our main results.

Theorem 1 *The OLS estimate of $G_2A_2H_2'$ in (36) can be expressed as*

$$G_2\hat{A}_2H_2' = P_{G_2}XP'_{H_2/\tilde{S}^{-1}Q_{H_1/\tilde{S}^{-1}}}, \tag{62}$$

where

$$\begin{aligned} P_{H_2/\tilde{S}^{-1}Q_{H_1/\tilde{S}^{-1}}} &= H_2(H_2'\tilde{S}^{-1}Q_{H_1/\tilde{S}^{-1}}H_2)^{-1}H_2'\tilde{S}^{-1}Q_{H_1/\tilde{S}^{-1}} \\ &= H_2(H_2'S_2^{-1}Q_{H_1/S_1^{-1}}H_2)^{-1}H_2'S_2^{-1}Q_{H_1/S_1^{-1}} \quad \text{(by (58))} \\ &= P_{H_2/S_2^{-1}Q_{H_1/S_1^{-1}}}. \end{aligned} \tag{63}$$

Proof. We have

$$\begin{aligned} G_2\hat{A}_2H_2' &= P_{G_2}(Y_{12} - P_{Y_2/Q_{G_2}}Y_{12})H_2' \quad \text{(by Seber's trick)} \\ &= P_{G_2}XT_{12}H_2' - P_{G_2}XP_{T_2/\tilde{S}}T_{12}H_2' \\ &= P_{G_2}XT_{12}H_2' - P_{G_2}XQ'_{H/\tilde{S}^{-1}}T_{12}H_2' \quad \text{(by Khatri's lemma)} \\ &= P_{G_2}X\tilde{S}^{-1}H(H'\tilde{S}^{-1}H)^{-1}\begin{bmatrix} O \\ I \end{bmatrix}H_2' \\ &= P_{G_2}X\tilde{S}^{-1}Q_{H_1/\tilde{S}^{-1}}H_2(H_2'\tilde{S}^{-1}Q_{H_1/\tilde{S}^{-1}}H_2)^{-1}H_2' \\ &= P_{G_2}XP'_{H_2/\tilde{S}^{-1}Q_{H_1/\tilde{S}^{-1}}} = P_{G_2}XP'_{H_2/S_2^{-1}Q_{H_1/S_1^{-1}}} \quad \text{(by (58))} \end{aligned} \tag{64}$$

□

Theorem 2 *The OLS estimate of $G_1A_1H_1'$ in (38) can be expressed as*

$$\begin{aligned} G_1\hat{A}_1H_1' &= P_{G_1}(X - P_{G_2}XP'_{H_2/\tilde{S}^{-1}Q_{H_1/\tilde{S}^{-1}}})P'_{H_1/S_1^{-1}} \\ &= P_{G_1}(X - P_{G_2}XP'_{H_2/S_2^{-1}Q_{H_1/S_1^{-1}}})P'_{H_1/S_1^{-1}} \quad \text{(by (58))}. \end{aligned} \tag{65}$$

Proof. We note first that $G_1\hat{A}_1H_1'$ in (38) can be rewritten as

$$\begin{aligned} G_1\hat{A}_1H_1' &= P_{G_1}(I - P_{[Q_{G_2/Q_{Y_2}}Y_{12},Y_2]/Q_{G_1}})Y_{11}H_1' \quad \text{(by Seber's trick)} \\ &= P_{G_1}(I - P_{[Y_{12},Y_2]/Q_{G_1}})Y_{11}H_1' + P_{G_2/Q_{Y_2}}P_{[Y_{12},Y_2]/Q_{G_1}}Y_{11}H_1', \end{aligned} \tag{66}$$

because $P_{[Q_{G_2/Q_{Y_2}}Y_{12},Y_2]/Q_{G_1}}$ can be expanded as

$$\begin{aligned} &P_{[Q_{G_2/Q_{Y_2}}Y_{12},Y_2]/Q_{G_1}} \\ &\quad = [Q_{G_2/Q_{Y_2}}Y_{12}, Y_2]U^{-1}V \\ &\quad = [Y_{12}, Y_2]U^{-1}V - [P_{G_2/Q_{Y_2}}Y_{12}, O]U^{-1}V, \\ &\quad = [Y_{12}, Y_2]U^{-1}V - P_{G_2/Q_{Y_2}}[Y_{12}, Y_2]U^{-1}V, \end{aligned} \tag{67}$$

where

$$U = \begin{bmatrix} Y_{12}'Q_{G_1}Y_{12} & Y_{12}'Q_{G_1}Y_2 \\ Y_2'Q_{G_1}Y_2 & Y_2'Q_{G_1}Y_2 \end{bmatrix}, \tag{68}$$

and

$$V = \begin{bmatrix} Y_{12}'Q_{G_1} \\ Y_2'Q_{G_1} \end{bmatrix}, \tag{69}$$

and so

$$[Y_{12}, Y_2]U^{-1}V = P_{[Y_{12},Y_2]/Q_{G_1}}. \tag{70}$$

In the derivation above, it is crucial to notice that $Q'_{G_2/Q_{Y_2}}Q_{G_1} = Q_{G_1} = Q_{G_1}Q_{G_2/Q_{Y_2}}$, and that $P_{G_2/Q_{Y_2}}Y_2 = O$. The first term on the right hand side of (66) is equal to the OLS estimate of $G_1A_1H_1'$ when it is the only structural term in GCM, so it must be equal to $P_{G_1}XP'_{H_1/S_1^{-1}}$ according to Lemma 7. That is,

$$P_{G_1}(I - P_{[Y_{12},Y_2]/Q_{G_1}})Y_{11}H_1' = P_{G_1}XP'_{H_1/S_1^{-1}}. \tag{71}$$

The second term, on the other hand, can be expanded as follows:

$$\begin{aligned}
&P_{G_2/Q_{Y_2}}(Y_{11}H_1' - XP'_{H_1/S_1^{-1}}) \\
&\quad = (P_{G_2} - P_{G_2}P_{Y_2/Q_{G_2}})(Y_{11}H_1' - XP'_{H_1/S_1^{-1}}) \\
&\quad = P_{G_2}XT_{11}H_1' - P_{G_2}P_{Y_2/Q_{G_2}}XT_{11}H_1' - P_{G_2}XP'_{H_1/S_1^{-1}} \\
&\qquad\qquad\qquad\qquad + P_{G_2}P_{Y_2/Q_{G_2}}XP'_{H_1/S_1^{-1}},
\end{aligned} \tag{72}$$

the second term of which can be further expanded as

$$\begin{aligned}
-P_{G_2}P_{Y_2/Q_{G_2}}XT_{11}H_1' &= -P_{G_2}XP_{T_2/\tilde{S}}T_{11}H_1' \\
&= -P_{G_2}XQ'_{H/\tilde{S}^{-1}}T_{11}H_1' \quad \text{(by Khatri's lemma)} \\
&= -P_{G_2}XT_{11}H_1' + P_{G_2}XP'_{H_1/\tilde{S}^{-1}Q_{H_2/\tilde{S}^{-1}}},
\end{aligned} \tag{73}$$

so the first term in (72) and that in (73) cancel out. The fourth term of (72), on the other hand, can be expanded as

$$\begin{aligned}
P_{G_2}P_{Y_2/Q_{G_2}}XP'_{H_1/S_1^{-1}} &= P_{G_2}XP_{T_2/\tilde{S}}P'_{H_1/S_1^{-1}} \\
&= P_{G_2}XQ'_{H/\tilde{S}1}P'_{H_1/S_1^{-1}} \quad \text{(by Khatri's lemma)} \\
&= P_{G_2}XP'_{H_1/S_1^{-1}} - P_{G_2}XP'_{H/\tilde{S}^{-1}}P'_{H_1/S_1^{-1}},
\end{aligned} \tag{74}$$

so the third term in (72) and the first term in (74) cancel out. This implies that the second term in (66) is equal to

$$\begin{aligned}
&P_{G_2}XP'_{H_1/\tilde{S}^{-1}Q_{H_1/\tilde{S}^{-1}}} - P_{G_2}XP'_{H/\tilde{S}^{-1}}P'_{H_1/S_1^{-1}} \\
&\quad = P_{G_2}X(P'_{H_1/\tilde{S}^{-1}Q_{H_2/\tilde{S}^{-1}}} - P'_{H/\tilde{S}^{-1}})P'_{H_1/S_1^{-1}} \\
&\quad = -P_{G_2}XP'_{H_2/\tilde{S}^{-1}Q_{H_1/\tilde{S}^{-1}}}P'_{H_1/S_1^{-1}} \quad \text{(by (14))}.
\end{aligned} \tag{75}$$

The first equality in (75) holds because $P'_{H_1/\tilde{S}^{-1}Q_{H_2/\tilde{S}^{-1}}}P'_{H_1/S_1^{-1}} = P'_{H_1/\tilde{S}^{-1}Q_{H_2/\tilde{S}^{-1}}}$ by Lemma 5(b). Combining the expression above for the second term of (66) and that of the first term given in (71), we get the complete expression for $G_1\hat{A}_1H_1'$. This concludes the proof of Theorem 2. □

Theorems 1 and 2 together establish the equivalence of Verbyla-Venables' and von Rosen's solutions.

5 Discussion

Fujikoshi and his collaborator [15] also derived the MLE of parameters in ExGCM in closed form, but under the condition that

$$\mathrm{Sp}(H_j) \subset \mathrm{Sp}(H_{j+1}) \tag{76}$$

for $j = 1, \cdots, J-1$. This condition and (4) look different. However, it turns out that they are equivalent because the model under one condition could be reparameterized into the other, as is shown below for $J = 2$. Thus, there is no need for a separate treatment of this case.

Let the Banken model be written as

$$X = G_1A_1H_1' + G_2A_2H_2' + E^* \tag{77}$$

for $J = 2$. Let $\tilde{G}_1$ be such that $\mathrm{Sp}(\tilde{G}_1) \oplus \mathrm{Sp}(G_2) = \mathrm{Sp}(G_1)$. Then, (77) can be rewritten as

$$\begin{aligned} X &= [\tilde{G}_1, G_2]\begin{bmatrix} A_{11}^* \\ A_{21}^* \end{bmatrix} H_1' + G_2A_2H_2' + E^* \\ &= \tilde{G}_1A_{11}^*H_1' + G_2[A_{21}^*, A_2]\begin{bmatrix} H_1' \\ H_2' \end{bmatrix} + E^*. \end{aligned} \tag{78}$$

If we reset $[H_1, H_2]$ as H_2, it holds that $\mathrm{Sp}(H_2) \supset \mathrm{Sp}(H_1)$. Note that what is important is the subspaces spanned by $\tilde{G}_1$, G_2, H_1, and H_2, not the specific basis vectors spanning these subspaces.

Fujikoshi and Satoh [15] actually derived a third expression of the MLE but under (76) (which is equivalent to the Banken model as shown above) with $J = 2$, which can also be shown to be identical to von Rosen's solution. However, their solution does not seem to be readily extensible to $J > 2$ [16].

In this paper, we have shown the equivalence between two solutions of MLEs for the Banken model. In doing so, we mostly focus on the case of $J = 2$. Presumably, we should be able to show the equivalence for $J > 2$, following a similar line of proof.

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