On consistency, natural restrictions and estimability under classical and extended growth curve models

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Abstract. For the general growth curve model $\{\mathbf{Y},\, \mathbf{X}_1\mathbf{\Theta}\mathbf{X}_2,\, \sigma^2(\mathbf{\Sigma}_2\otimes\mathbf{\Sigma}_1)\}$ with $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ of arbitrary ranks, we study the following three problems:

(I) Necessary and sufficient conditions for the model to be consistent.

(II) Natural restrictions to the parameter matrix $\mathbf{\Theta}$.

(III) Unbiasedness of linear estimators and estimability of parameter matrices.

Some generalizations of this work to the extended growth curve model $\{\mathbf{Y},\, \sum_{i=1}^{t}\mathbf{X}_{1i}\mathbf{\Theta}_i\mathbf{X}_{2i},\, \sigma^2(\mathbf{\Sigma}_2\otimes\mathbf{\Sigma}_1)\}$ are also given.

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1 Introduction

Throughout this paper, $\mathbb{R}^{m\times n}$ stands for the collection of all $m\times n$ real matrices. The symbols $\mathbf{A}'$, $\mathrm{tr}(\mathbf{A})$, $r(\mathbf{A})$ and $\mathscr{R}(\mathbf{A})$ stand for the transpose, the trace, the rank and the range (column space) of a matrix $\mathbf{A}\in\mathbb{R}^{m\times n}$ respectively. The Kronecker product of any two matrices $\mathbf{A}\in\mathbb{R}^{m\times n}$ and $\mathbf{B}\in\mathbb{R}^{p\times q}$ is defined to be $\mathbf{A}\otimes\mathbf{B}=(a_{ij}\mathbf{B})$. The vec operation of the matrix $\mathbf{A}=[\,\mathbf{a}_1,\ldots,\mathbf{a}_n\,]$ is defined to be $\mathrm{vec}(\mathbf{A})=[\,\mathbf{a}_1',\ldots,\mathbf{a}_n'\,]'$. A well-known formula for the vec operation is $\mathrm{vec}(\mathbf{AXB})=(\mathbf{B}'\otimes\mathbf{A})\mathrm{vec}(\mathbf{X})$.

Suppose we are given a growth curve model

$$\mathbf{Y}=\mathbf{X}_1\mathbf{\Theta}\mathbf{X}_2+\boldsymbol{\varepsilon},\quad E(\boldsymbol{\varepsilon})=\mathbf{0},\quad Cov[\mathrm{vec}(\boldsymbol{\varepsilon})]=\sigma^2(\mathbf{\Sigma}_2\otimes\mathbf{\Sigma}_1),\tag{1.1}$$

or in the compact form

$$\mathscr{M}=\{\mathbf{Y},\,\mathbf{X}_1\mathbf{\Theta}\mathbf{X}_2,\,\sigma^2(\mathbf{\Sigma}_2\otimes\mathbf{\Sigma}_1)\},\tag{1.2}$$

where

$\mathbf{Y}=(y_{ij})\in\mathbb{R}^{n\times m}$ is an observable random matrix (a longitudinal data set),
$\mathbf{X}_1=(x_{1ij})\in\mathbb{R}^{n\times p}$ and $\mathbf{X}_2=(x_{2ij})\in\mathbb{R}^{q\times m}$ are two known model matrices of arbitrary ranks,
$\mathbf{\Theta}=(\theta_{ij})\in\mathbb{R}^{p\times q}$ is a matrix of unknown parameters to be estimated,
$\mathbf{\Sigma}_1=(\sigma_{1ij})\in\mathbb{R}^{n\times n}$ and $\mathbf{\Sigma}_2=(\sigma_{2ij})\in\mathbb{R}^{m\times m}$ are two known nonnegative definite matrices,
σ^2 is a positive unknown scalar.

If one of $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ is a singular matrix, (1.1) is also said to be a singular growth curve model.

Through the Kronecker product and the vec operation of matrices, the model in (1.1) can alternatively be written as

$$\mathrm{vec}(\mathbf{Y})=(\mathbf{X}_2'\otimes\mathbf{X}_1)\mathrm{vec}(\mathbf{\Theta})+\mathrm{vec}(\boldsymbol{\varepsilon}),\quad E[\mathrm{vec}(\boldsymbol{\varepsilon})]=\mathbf{0},\quad Cov[\mathrm{vec}(\boldsymbol{\varepsilon})]=\sigma^2(\mathbf{\Sigma}_2\otimes\mathbf{\Sigma}_1).\tag{1.3}$$

Because (1.3) is a linear model, many results on linear models can be extended to (1.1). Note, however, that both $\mathrm{vec}(\mathbf{Y})$ and $\mathrm{vec}(\mathbf{\Theta})$ in (1.3) are just column vectors, many algebraic properties and restrictions related to the structures of the two matrices $\mathbf{Y}$ and $\mathbf{\Theta}$ in (1.1), such as their ranks, ranges, singularity, symmetry, partitioned representations, can hardly be demonstrated in the expressions of $\mathrm{vec}(\mathbf{Y})$ and $\mathrm{vec}(\mathbf{\Theta})$. Conversely, not all estimators of $\mathrm{vec}(\mathbf{\Theta})$ and $(\mathbf{X}_1'\otimes\mathbf{X}_2)\mathrm{vec}(\mathbf{\Theta})$ under (1.3) can be written in the forms of $\mathbf{\Theta}$ and $\mathbf{X}_1\mathbf{\Theta}\mathbf{X}_2$. That is to say, some problems on the model in (1.1) can be studied through (1.3), others can only be done from the original model in (1.1).

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The growth curve model is an extension of multivariate linear models, which was originally proposed by Potthoff and Roy (1964), and subsequently was studied by many authors, such as Khatri (1966), Pan and Fang (2002), Rao (1965, 1966), Seber (1985), von Rosen (1989, 1991), and Woolson and Leeper (1980) among many others. Because the four matrices $\mathbf{X}_1$ and $\mathbf{X}_2$, $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ can be of arbitrary rank, some essential relations between $\mathbf{Y}$ and the four matrices need clarifications. In this paper, we investigate the following three basic problems on the general growth curve model in (1.1):

(I) Necessary and sufficient conditions for the model to be consistent.

(II) Natural restrictions to the parameter matrix $\mathbf{\Theta}$.

(III) Unbiasedness of linear estimators and estimability of parametric functions.

If the matrices $\mathbf{X}_1$ and $\mathbf{X}_2$, $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ in (1.1) are of full rank, the above three problems reduce to trivial cases. Theoretically, the above three problems can easily be solved through the transformed model in (1.3). However, the challenging part is how to reasonably convert the results derived from vec operations in (1.3) to those that can be expressed in the forms of given matrices in (1.1). In this paper, we derive some satisfactory solutions to these problems through Kronecker products, vec operations, generalized inverses of matrices, matrix equations and the matrix rank method.

The Moore-Penrose inverse of $\mathbf{A} \in \mathbb{R}^{m\times n}$, denoted by $\mathbf{A}^+$, is defined to be the unique solution $\mathbf{G}$ to the four matrix equations

$$\text{(i) } \mathbf{AGA} = \mathbf{A}, \quad \text{(ii) } \mathbf{GAG} = \mathbf{G}, \quad \text{(iii) } (\mathbf{AG})' = \mathbf{AG}, \quad \text{(iv) } (\mathbf{GA})' = \mathbf{GA}.$$

A matrix $\mathbf{G} \in \mathbb{R}^{n\times m}$ is called a generalized inverse (g-inverse) of $\mathbf{A}$, denoted by $\mathbf{G} = \mathbf{A}^-$, if it satisfies (i). Further, $\mathbf{P_A} = \mathbf{AA}^+$, $\mathbf{E_A} = \mathbf{I}_m - \mathbf{AA}^+$ and $\mathbf{F_A} = \mathbf{I}_n - \mathbf{A}^+\mathbf{A}$ stand for the three orthogonal projectors induced from $\mathbf{A}$. One of the most important applications of Moore-Penrose inverses of matrices is to derive closed-form formulas for ranks of partitioned matrices, as well as general solutions of matrix equations; see Lemmas 1.1, 1.2, 1.6 and 1.7 below.

Some well-known rank formulas for partitioned matrices due to Marsaglia and Styan (1974) are given below, which can be used to simplify various matrix expressions involving Moore-Penrose inverses.

Lemma 1.1 *Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{m\times k}$ *and* $\mathbf{C} \in \mathbb{R}^{l\times n}$. *Then*

$$r[\,\mathbf{A},\,\mathbf{B}\,] = r(\mathbf{A}) + r[\,(\,\mathbf{I}_m - \mathbf{AA}^+\,)\mathbf{B}\,] = r(\mathbf{B}) + r[\,(\,\mathbf{I}_m - \mathbf{BB}^+\,)\mathbf{A}\,], \tag{1.4}$$

$$r\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} = r(\mathbf{A}) + r[\,\mathbf{C}(\,\mathbf{I}_n - \mathbf{A}^+\mathbf{A}\,)\,] = r(\mathbf{C}) + r[\,\mathbf{A}(\,\mathbf{I}_n - \mathbf{C}^+\mathbf{C}\,)\,], \tag{1.5}$$

$$r\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} = r(\mathbf{B}) + r(\mathbf{C}) + r[\,(\,\mathbf{I}_m - \mathbf{BB}^+\,)\mathbf{A}(\,\mathbf{I}_n - \mathbf{C}^+\mathbf{C}\,)\,]. \tag{1.6}$$

The following two lemmas are well known; see Penrose (1955).

Lemma 1.2 *Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{p\times q}$ *and* $\mathbf{C} \in \mathbb{R}^{m\times q}$. *Then the following statements are equivalent:*

(a) *The matrix equation* $\mathbf{AZB} = \mathbf{C}$ *is solvable for* $\mathbf{Z}$.

(b) *The matrix equation* $(\mathbf{B}' \otimes \mathbf{A})\mathrm{vec}(\mathbf{Z}) = \mathrm{vec}(\mathbf{C})$ *is solvable for* $\mathrm{vec}(\mathbf{Z})$.

(c) $\mathscr{R}(\mathbf{C}) \subseteq \mathscr{R}(\mathbf{A})$ *and* $\mathscr{R}(\mathbf{C}') \subseteq \mathscr{R}(\mathbf{B}')$.

(d) $\mathbf{AA}^+\mathbf{CB}^+\mathbf{B} = \mathbf{C}$.

(e) $\mathscr{R}[\mathrm{vec}(\mathbf{C})] \subseteq \mathscr{R}(\mathbf{B}' \otimes \mathbf{A})$.

In this case, the general solution to the equation can be written in the following parametric form

$$\mathbf{Z} = \mathbf{A}^+\mathbf{CB}^+ + \mathbf{F_A}\mathbf{U}_1 + \mathbf{U}_2\mathbf{E_B},$$

where $\mathbf{U}_1, \mathbf{U}_2 \in \mathbb{R}^{n\times p}$ *are arbitrary.*

Lemma 1.3 *Suppose* $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{B}_1$ *and* $\mathbf{B}_2$ *are nonzero matrices. Then the equality* $\mathbf{A}_1\mathbf{Z}_1\mathbf{B}_1 = \mathbf{A}_2\mathbf{Z}_2\mathbf{B}_2$ *holds for any two matrices* $\mathbf{Z}_1$ *and* $\mathbf{Z}_2$ *if and only if* $\mathbf{A}_1 = \lambda\mathbf{A}_2$ *and* $\mathbf{B}_1 = \mu\mathbf{B}_2$ *with* $\lambda\mu = 1$.

It is obvious that the equality $\mathbf{AZB} = \mathbf{0}$ holds for any matrix $\mathbf{Z}$ if and only if $\mathbf{A} = \mathbf{0}$ and $\mathbf{B} = \mathbf{0}$. However, the equality $\mathbf{L}_1\mathbf{Y}\mathbf{L}_2 = \mathbf{0}$ holds with probability 1 under (1.1) does not necessarily imply $\mathbf{L}_1 = \mathbf{0}$ or $\mathbf{L}_2 = \mathbf{0}$. Instead, we have the following result.

Theorem 1.4 *Let the growth curve model be as given in* (1.1), *and let* $\mathbf{L}_1$ *and* $\mathbf{L}_2$ *be two given matrices of appropriate sizes.*

(a) *Under the condition that* $\mathbf{\Theta}$ *is a free matrix, the equality* $\mathbf{L}_1\mathbf{Y}\mathbf{L}_2 = \mathbf{0}$ *holds with probability 1 if one of the four conditions* (i) $\mathbf{L}_1\mathbf{X}_1 = \mathbf{0}$ *and* $\mathbf{L}_1\mathbf{\Sigma}_1 = \mathbf{0}$, (ii) $\mathbf{L}_1\mathbf{X}_1 = \mathbf{0}$ *and* $\mathbf{\Sigma}_2\mathbf{L}_2 = \mathbf{0}$, (iii) $\mathbf{X}_2\mathbf{L}_2 = \mathbf{0}$ *and* $\mathbf{L}_1\mathbf{\Sigma}_1 = \mathbf{0}$, *and* (iv) $\mathbf{X}_2\mathbf{L}_2 = \mathbf{0}$ *and* $\mathbf{\Sigma}_2\mathbf{L}_2 = \mathbf{0}$ *holds.*

(b) *Under the conditions that both* $\mathbf{\Sigma}_1$ *and* $\mathbf{\Sigma}_2$ *are positive definite,* $\mathbf{L}_1\mathbf{Y}\mathbf{L}_2 = \mathbf{0}$ *holds with probability* 1 *if and only if* $\mathbf{L}_1 = \mathbf{0}$ *or* $\mathbf{L}_2 = \mathbf{0}$.

Proof It is obvious that $\mathbf{L}_1\mathbf{Y}\mathbf{L}_2 = \mathbf{0}$ if and only if $(\mathbf{L}_2'\otimes\mathbf{L}_1)\mathrm{vec}(\mathbf{Y}) = \mathbf{0}$. Also note that $(\mathbf{L}_2'\otimes\mathbf{L}_1)\mathrm{vec}(\mathbf{Y}) = \mathbf{0}$ holds with probability 1 if and only if the following two equalities

$$E[(\mathbf{L}_2'\otimes\mathbf{L}_1)\mathrm{vec}(\mathbf{Y})] = (\mathbf{L}_2'\mathbf{X}_2'\otimes\mathbf{L}_1\mathbf{X}_1)\mathrm{vec}(\mathbf{\Theta}) = \mathbf{0}, \tag{1.7}$$

$$Cov[(\mathbf{L}_2'\otimes\mathbf{L}_1)\mathrm{vec}(\mathbf{Y})] = (\mathbf{L}_2'\otimes\mathbf{L}_1)(\mathbf{\Sigma}_2\otimes\mathbf{\Sigma}_1)(\mathbf{L}_2\otimes\mathbf{L}_1') = \mathbf{0} \tag{1.8}$$

hold, that is,

$$\mathbf{L}_1\mathbf{X}_1\mathbf{\Theta}\mathbf{X}_2\mathbf{L}_2 = \mathbf{0} \ \text{ and } \ \mathbf{L}_2'\mathbf{\Sigma}_2\otimes\mathbf{L}_1\mathbf{\Sigma}_1 = \mathbf{0}. \tag{1.9}$$

If $\mathbf{\Theta}$ is a free matrix, then (1.9) is equivalent to $\mathbf{L}_1\mathbf{X}_1 = \mathbf{0}$ and $\mathbf{L}_1\mathbf{\Sigma}_1 = \mathbf{0}$, or $\mathbf{L}_1\mathbf{X}_1 = \mathbf{0}$ and $\mathbf{\Sigma}_2\mathbf{L}_2 = \mathbf{0}$, or $\mathbf{X}_2\mathbf{L}_2 = \mathbf{0}$ and $\mathbf{L}_1\mathbf{\Sigma}_1 = \mathbf{0}$, or $\mathbf{X}_2\mathbf{L}_2 = \mathbf{0}$ and $\mathbf{\Sigma}_2\mathbf{L}_2 = \mathbf{0}$. Hence we have the conclusion in (a). Under the conditions that both $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ are positive definite, the second equality in (1.9) is equivalent to $\mathbf{L}_1 = \mathbf{0}$ or $\mathbf{L}_2 = \mathbf{0}$, so that (b) holds. □

In order to characterize relations between two linear estimators under (1.1), we need the following result.

Theorem 1.5 *Suppose* $\mathbf{L}_{11}$, $\mathbf{L}_{12}$, $\mathbf{L}_{21}$ *and* $\mathbf{L}_{22}$ *are nonzero matrices. Then the equality* $\mathbf{L}_{11}\mathbf{Y}\mathbf{L}_{12} = \mathbf{L}_{21}\mathbf{Y}\mathbf{L}_{22}$ *holds with probability* 1 *under* (1.1) *if*

$$\mathbf{L}_{11}[\,\mathbf{X}_1,\,\mathbf{\Sigma}_1\,] = \mathbf{L}_{21}[\,\lambda_1\mathbf{X}_1,\,\mu_1\mathbf{\Sigma}_1\,], \quad \begin{bmatrix}\mathbf{X}_2\\ \mathbf{\Sigma}_2\end{bmatrix}\mathbf{L}_{12} = \begin{bmatrix}\lambda_2\mathbf{X}_2\\ \mu_2\mathbf{\Sigma}_2\end{bmatrix}\mathbf{L}_{22}, \quad \lambda_1\lambda_2 = 1 \ \ \textit{and} \ \ \mu_1\mu_2 = 1. \tag{1.10}$$

Proof Note that $\mathbf{L}_{11}\mathbf{Y}\mathbf{L}_{12} = \mathbf{L}_{21}\mathbf{Y}\mathbf{L}_{22}$ is equivalent to

$$(\mathbf{L}_{12}'\otimes\mathbf{L}_{11})\mathrm{vec}(\mathbf{Y}) = (\mathbf{L}_{22}'\otimes\mathbf{L}_{21})\mathrm{vec}(\mathbf{Y}). \tag{1.11}$$

Also note that

$$E[\,(\mathbf{L}_{12}'\otimes\mathbf{L}_{11})\mathrm{vec}(\mathbf{Y}) - (\mathbf{L}_{22}'\otimes\mathbf{L}_{21})\mathrm{vec}(\mathbf{Y})\,] = (\,\mathbf{L}_{12}'\mathbf{X}_2'\otimes\mathbf{L}_{11}\mathbf{X}_1 - \mathbf{L}_{22}'\mathbf{X}_2'\otimes\mathbf{L}_{21}\mathbf{X}_1\,)\mathrm{vec}(\mathbf{\Theta}),$$

and

$$\begin{aligned} &Cov[\,(\mathbf{L}_{12}'\otimes\mathbf{L}_{11})\mathrm{vec}(\mathbf{Y}) - (\mathbf{L}_{22}'\otimes\mathbf{L}_{21})\mathrm{vec}(\mathbf{Y})\,] \\ &= \sigma^2(\,\mathbf{L}_{12}'\otimes\mathbf{L}_{11} - \mathbf{L}_{22}'\otimes\mathbf{L}_{21}\,)(\mathbf{\Sigma}_2\otimes\mathbf{\Sigma}_1)(\,\mathbf{L}_{12}'\otimes\mathbf{L}_{11} - \mathbf{L}_{22}'\otimes\mathbf{L}_{21}\,)'. \end{aligned}$$

Hence (1.11) holds with probability 1 if

$$\mathbf{L}_{12}'\mathbf{X}_2'\otimes\mathbf{L}_{11}\mathbf{X}_1 = \mathbf{L}_{22}'\mathbf{X}_2'\otimes\mathbf{L}_{21}\mathbf{X}_1 \ \text{ and } \ \mathbf{L}_{12}'\mathbf{\Sigma}_2\otimes\mathbf{L}_{11}\mathbf{\Sigma}_1 = \mathbf{L}_{22}'\mathbf{\Sigma}_2\otimes\mathbf{L}_{21}\mathbf{\Sigma}_1,$$

which, by Lemma 1.3, is equivalent to

$$\mathbf{L}_{11}\mathbf{X}_1 = \lambda_1\mathbf{L}_{21}\mathbf{X}_1, \quad \mathbf{X}_2\mathbf{L}_{12} = \lambda_2\mathbf{X}_2\mathbf{L}_{22}, \quad \lambda_1\lambda_2 = 1,$$

$$\mathbf{L}_{11}\mathbf{\Sigma}_1 = \mu_1\mathbf{L}_{21}\mathbf{\Sigma}_1, \quad \mathbf{\Sigma}_2\mathbf{L}_{12} = \mu_2\mathbf{\Sigma}_2\mathbf{L}_{22}, \quad \mu_1\mu_2 = 1,$$

as required for (1.10). □

The following result was given by Mitra (1984).

Lemma 1.6 *The pair of matrix equations* $\mathbf{A}_1\mathbf{Z} = \mathbf{B}_1$ *and* $\mathbf{Z}\mathbf{A}_2 = \mathbf{B}_2$ *have a common solution for* $\mathbf{Z}$ *if and only if* $\mathscr{R}(\mathbf{B}_1) \subseteq \mathscr{R}(\mathbf{A}_1)$, $\mathscr{R}(\mathbf{B}_2') \subseteq \mathscr{R}(\mathbf{A}_2')$ *and* $\mathbf{A}_1\mathbf{B}_2 = \mathbf{B}_1\mathbf{A}_2$. *In such a case, the general common solution to* $\mathbf{A}_1\mathbf{Z} = \mathbf{B}_1$ *and* $\mathbf{Z}\mathbf{A}_2 = \mathbf{B}_2$ *can be written in the following parametric form*

$$\mathbf{Z} = \mathbf{A}_1^{+}\mathbf{B}_1 + \mathbf{B}_2\mathbf{A}_2^{+} - \mathbf{A}_1^{+}\mathbf{A}_1\mathbf{B}_2\mathbf{A}_2^{+} + \mathbf{F}_{\mathbf{A}_1}\mathbf{U}\mathbf{E}_{\mathbf{A}_2}, \tag{1.12}$$

where the matrix $\mathbf{U}$ *is arbitrary.*

Lemma 1.7 *Let* $\mathbf{A}_i \in \mathbb{R}^{m_i\times n_i}$, $\mathbf{B}_i \in \mathbf{C}^{m_i\times p}$ *and* $\mathbf{C}_i \in \mathbb{R}^{q\times n_i}$ *be given for* $i = 1, 2$. *Then,*

(a) (Özgüler and Akar, 1991) *The pair of matrix equations*

$$\mathbf{A}_1\mathbf{Z}\mathbf{B}_1 = \mathbf{C}_1 \quad \textit{and} \quad \mathbf{A}_2\mathbf{Z}\mathbf{B}_2 = \mathbf{C}_2$$

have a common solution if and only if each of the two matrix equations is consistent, i.e., $\mathscr{R}(\mathbf{C}_i) \subseteq \mathscr{R}(\mathbf{A}_i)$ *and* $\mathscr{R}(\mathbf{C}_i') \subseteq \mathscr{R}(\mathbf{B}_i')$ *for* $i = 1, 2$, *and*

$$r\begin{bmatrix} \mathbf{C}_1 & \mathbf{0} & \mathbf{A}_1 \\ \mathbf{0} & -\mathbf{C}_2 & \mathbf{A}_2 \\ \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{0} \end{bmatrix} = r\begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix} + r[\,\mathbf{B}_1,\, \mathbf{B}_2\,].$$

(b) (Tian, 2000) *Under* (a), *the general common solution of the pair of equations can be written in the parametric form*

$$\mathbf{Z} = \mathbf{Z}_0 + \mathbf{F}_{\mathbf{A}}\mathbf{U}_1 + \mathbf{U}_2\mathbf{E}_{\mathbf{B}} + \mathbf{F}_{\mathbf{A}_1}\mathbf{U}_3\mathbf{E}_{\mathbf{B}_2} + \mathbf{F}_{\mathbf{A}_2}\mathbf{U}_4\mathbf{E}_{\mathbf{B}_1},$$

where $\mathbf{Z}_0$ *is a special common solution to the pair of equations,* $\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix}$, $\mathbf{B} = [\,\mathbf{B}_1,\, \mathbf{B}_2\,]$, *and* $\mathbf{U}_1, \ldots, \mathbf{U}_4 \in \mathbb{R}^{p\times q}$ *are arbitrary.*

The following result was shown by Özgüler (1991).

Lemma 1.8 *The matrix equation* $\mathbf{A}_1\mathbf{Z}_1\mathbf{B}_1 + \mathbf{A}_2\mathbf{Z}_2\mathbf{B}_2 = \mathbf{C}$ *is solvable for* $\mathbf{Z}_1$ *and* $\mathbf{Z}_2$ *if and only if*

$$r[\,\mathbf{C},\, \mathbf{A}_1,\, \mathbf{A}_2\,] = r[\,\mathbf{A}_1,\ \mathbf{A}_2\,], \quad r\begin{bmatrix} \mathbf{C} \\ \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} = r\begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix},$$

$$r\begin{bmatrix} \mathbf{C} & \mathbf{A}_1 \\ \mathbf{B}_2 & \mathbf{0} \end{bmatrix} = r(\mathbf{A}_1) + r(\mathbf{B}_2), \quad r\begin{bmatrix} \mathbf{C} & \mathbf{A}_2 \\ \mathbf{B}_1 & \mathbf{0} \end{bmatrix} = r(\mathbf{A}_2) + r(\mathbf{B}_1).$$

2 Consistency of a singular growth curve model

Let $\mathbf{M} = [\,\mathbf{X}_2' \otimes \mathbf{X}_1,\, \mathbf{\Sigma}_2 \otimes \mathbf{\Sigma}_1\,]$. Under the assumptions in (1.3), it is easy to obtain

$$E[\mathbf{M}\mathbf{M}^{+}\mathrm{vec}(\mathbf{Y}) - \mathrm{vec}(\mathbf{Y})] = \mathbf{M}\mathbf{M}^{+}(\mathbf{X}_2' \otimes \mathbf{X}_1)\mathrm{vec}(\mathbf{\Theta}) - (\mathbf{X}_2' \otimes \mathbf{X}_1)\mathrm{vec}(\mathbf{\Theta}) = \mathbf{0},$$
$$Cov[\mathbf{M}\mathbf{M}^{+}\mathrm{vec}(\mathbf{Y}) - \mathrm{vec}(\mathbf{Y})] = (\mathbf{M}\mathbf{M}^{+} - \mathbf{I})(\mathbf{\Sigma}_2 \otimes \mathbf{\Sigma}_1)(\mathbf{M}\mathbf{M}^{+} - \mathbf{I})' = \mathbf{0}.$$

Both equalities imply that

$$\mathrm{vec}(\mathbf{Y}) \in \mathscr{R}[\,\mathbf{X}_2' \otimes \mathbf{X}_1,\, \mathbf{\Sigma}_2 \otimes \mathbf{\Sigma}_1\,] \tag{2.1}$$

holds with probability 1. In view of this, we have the following definition.

Definition 2.1 The growth curve model in (1.1) is said to be *consistent* if (2.1) holds with probability 1.

The consistency condition in (2.1) is defined through the transformed model in (1.3). We next show how to equivalently write (2.1) in the forms associated with (1.1).

Theorem 2.2 *The following statements are equivalent*:

(a) *The growth curve model in* (1.1) *is consistent*, *i.e.*, (2.1) *holds with probability* 1.

(b) *The following matrix equation*

$$\mathbf{X}_1\mathbf{V}\mathbf{X}_2 + \mathbf{\Sigma}_1\mathbf{W}\mathbf{\Sigma}_2 = \mathbf{Y} \tag{2.2}$$

is solvable for $\mathbf{V}$ *and* $\mathbf{W}$ *with probability* 1.

(c) *The following four rank equalities*

$$r[\,\mathbf{X}_1,\, \mathbf{\Sigma}_1,\, \mathbf{Y}\,] = r[\,\mathbf{X}_1,\, \mathbf{\Sigma}_1\,], \quad r\begin{bmatrix} \mathbf{X}_2 \\ \mathbf{\Sigma}_2 \\ \mathbf{Y} \end{bmatrix} = r\begin{bmatrix} \mathbf{X}_2 \\ \mathbf{\Sigma}_2 \end{bmatrix}, \tag{2.3}$$

$$r\begin{bmatrix} \mathbf{Y} & \mathbf{X}_1 \\ \mathbf{\Sigma}_2 & \mathbf{0} \end{bmatrix} = r(\mathbf{X}_1) + r(\mathbf{\Sigma}_2), \quad r\begin{bmatrix} \mathbf{Y} & \mathbf{\Sigma}_1 \\ \mathbf{X}_2 & \mathbf{0} \end{bmatrix} = r(\mathbf{X}_2) + r(\mathbf{\Sigma}_1) \tag{2.4}$$

hold with probability 1.

(d) *The following four equalities*

$$[\,\mathbf{X}_1,\, \mathbf{\Sigma}_1\,][\,\mathbf{X}_1,\, \mathbf{\Sigma}_1\,]^{+}\mathbf{Y} = \mathbf{Y}, \quad \mathbf{Y}\begin{bmatrix} \mathbf{X}_2 \\ \mathbf{\Sigma}_2 \end{bmatrix}^{+}\begin{bmatrix} \mathbf{X}_2 \\ \mathbf{\Sigma}_2 \end{bmatrix} = \mathbf{Y}, \tag{2.5}$$

$$(\mathbf{I} - \mathbf{X}_1\mathbf{X}_1^{+}\,)\mathbf{Y}(\mathbf{I} - \mathbf{\Sigma}_2^{+}\mathbf{\Sigma}_2\,) = \mathbf{0}, \quad (\mathbf{I} - \mathbf{\Sigma}_1\mathbf{\Sigma}_1^{+}\,)\mathbf{Y}(\mathbf{I} - \mathbf{X}_2^{+}\mathbf{X}_2\,) = \mathbf{0} \tag{2.6}$$

hold with probability 1.

Proof Note that the equation in (2.2) is equivalent to

$$(\mathbf{X}_2' \otimes \mathbf{X}_1)\mathrm{vec}(\mathbf{V}) + (\mathbf{\Sigma}_2 \otimes \mathbf{\Sigma}_1)\mathrm{vec}(\mathbf{W}) = \mathrm{vec}(\mathbf{Y}).$$

It is obvious that this equation is solvable for $\mathrm{vec}(\mathbf{V})$ and $\mathrm{vec}(\mathbf{W})$ if and only if (2.1) holds, so that (a) and (b) are equivalent. The equivalence of (b) and (c) follows from Lemma 1.8. Applying (1.4), (1.5) and (1.6) to the four rank equalities in (b) gives the equivalence of (c) and (d). □

More equivalent statements for the consistency of (1.1) are given in Section 3.

3 Natural restrictions under a singular growth curve model

The concept of natural restrictions to the unknown parameter vector in a singular linear model was proposed by Rao (1973a, p. 279; 1976, p. 1033). Many discussions on natural restrictions to singular linear models can be found in the literature, see, e.g., Baksalary, Rao and Markiewicz (1992), Groß (2004), Harville (1981), Haupt and Oberhofer (2002), McCulloch and Searle (1995), Puntanen and Scott (1996). A recent paper by Tian, Beisiegel, Dagenais and Haines (2008) gave a new investigation to this problem by the matrix rank method. In this section, we introduce the concept of natural restrictions to the unknown parameter matrix in a general growth curve model, and consider some problems related to the natural restrictions.

Theorem 3.1 *The parameter matrix* $\mathbf{\Theta}$ *in the growth curve model* (1.1) *satisfies the following matrix equation*

$$\mathbf{Y} - \mathbf{P}_{\mathbf{\Sigma}_1}\mathbf{Y}\mathbf{P}_{\mathbf{\Sigma}_2} = \mathbf{X}_1\mathbf{\Theta}\mathbf{X}_2 - \mathbf{P}_{\mathbf{\Sigma}_1}\mathbf{X}_1\mathbf{\Theta}\mathbf{X}_2\mathbf{P}_{\mathbf{\Sigma}_2} \tag{3.1}$$

with probability 1.

Proof Pre-multiplying both sides of (1.3) by $\mathbf{E}_{(\mathbf{\Sigma}_2\otimes\mathbf{\Sigma}_1)}$ gives the following transformed model

$$\mathbf{E}_{(\mathbf{\Sigma}_2\otimes\mathbf{\Sigma}_1)}\mathrm{vec}(\mathbf{Y}) = \mathbf{E}_{(\mathbf{\Sigma}_2\otimes\mathbf{\Sigma}_1)}(\mathbf{X}_2'\otimes\mathbf{X}_1)\mathrm{vec}(\mathbf{\Theta}) + \mathbf{E}_{(\mathbf{\Sigma}_2\otimes\mathbf{\Sigma}_1)}\mathrm{vec}(\boldsymbol{\varepsilon}). \tag{3.2}$$

It is easy to verify that

$$E[\mathbf{E}_{(\mathbf{\Sigma}_2\otimes\mathbf{\Sigma}_1)}\mathrm{vec}(\boldsymbol{\varepsilon})] = \mathbf{0} \ \text{ and } \ Cov[\mathbf{E}_{(\mathbf{\Sigma}_2\otimes\mathbf{\Sigma}_1)}\mathrm{vec}(\boldsymbol{\varepsilon})] = \mathbf{0}.$$

These two equalities imply that the equality

$$\mathbf{E}_{(\mathbf{\Sigma}_2\otimes\mathbf{\Sigma}_1)}\mathrm{vec}(\mathbf{Y}) = \mathbf{E}_{(\mathbf{\Sigma}_2\otimes\mathbf{\Sigma}_1)}(\mathbf{X}_2'\otimes\mathbf{X}_1)\mathrm{vec}(\mathbf{\Theta}) \tag{3.3}$$

holds with probability 1, that is, (3.1) holds with probability with 1. □

The equation in (3.1) is called the *natural restriction* to the parameter matrix $\mathbf{\Theta}$ in (1.1). It is easy to verify that the equation in (3.3) is solvable for $\mathrm{vec}(\mathbf{\Theta})$ if and only if

$$r[\,\mathbf{E}_{(\mathbf{\Sigma}_2\otimes\mathbf{\Sigma}_1)}\mathrm{vec}(\mathbf{Y}),\, \mathbf{E}_{(\mathbf{\Sigma}_2\otimes\mathbf{\Sigma}_1)}(\mathbf{X}_2'\otimes\mathbf{X}_1)\,] = r[\mathbf{E}_{(\mathbf{\Sigma}_2\otimes\mathbf{\Sigma}_1)}(\mathbf{X}_2'\otimes\mathbf{X}_1)]. \tag{3.4}$$

Applying (1.4) to both sides of (3.4) gives

$$r[\,\mathrm{vec}(\mathbf{Y}),\, \mathbf{X}_2'\otimes\mathbf{X}_1,\, \mathbf{\Sigma}_2\otimes\mathbf{\Sigma}_1\,] = r[\,\mathbf{X}_2'\otimes\mathbf{X}_1,\, \mathbf{\Sigma}_2\otimes\mathbf{\Sigma}_1\,],$$

which is obviously equivalent to (2.1). Combining Theorem 3.1 with Definition 2.1 yields the following result.

Theorem 3.2 *The matrix equation in* (3.1) *is solvable for* $\mathbf{\Theta}$ *with probability* 1 *if and only if the model in* (1.1) *is consistent.*

Another set of natural restrictions on the parameter matrix $\mathbf{\Theta}$ in (1.1) is give below.

Theorem 3.3 *The parameter matrix* $\mathbf{\Theta}$ *in the growth curve model* (1.1) *satisfies the pair of matrix equations*

$$\mathbf{E}_{\mathbf{\Sigma}_1}\mathbf{Y} = \mathbf{E}_{\mathbf{\Sigma}_1}\mathbf{X}_1\mathbf{\Theta}\mathbf{X}_2 \ \ \textit{and} \ \ \mathbf{Y}\mathbf{E}_{\mathbf{\Sigma}_2} = \mathbf{X}_1\mathbf{\Theta}\mathbf{X}_2\mathbf{E}_{\mathbf{\Sigma}_2} \tag{3.5}$$

with probability 1.

Proof Pre- and post-multiplying both sides of (1.1) by $\mathbf{E}_{\mathbf{\Sigma}_1}$ and $\mathbf{E}_{\mathbf{\Sigma}_2}$, respectively, gives the following two transformed models

$$\mathbf{E}_{\mathbf{\Sigma}_1}\mathbf{Y} = \mathbf{E}_{\mathbf{\Sigma}_1}\mathbf{X}_1\mathbf{\Theta}\mathbf{X}_2 + \mathbf{E}_{\mathbf{\Sigma}_1}\boldsymbol{\varepsilon} \quad \text{and} \quad \mathbf{Y}\mathbf{E}_{\mathbf{\Sigma}_2} = \mathbf{X}_1\mathbf{\Theta}\mathbf{X}_2\mathbf{E}_{\mathbf{\Sigma}_2} + \boldsymbol{\varepsilon}\mathbf{E}_{\mathbf{\Sigma}_2}.$$

It is easy to verify that

$$E(\mathbf{E}_{\mathbf{\Sigma}_1}\boldsymbol{\varepsilon}) = \mathbf{0}, \ \ E(\boldsymbol{\varepsilon}\mathbf{E}_{\mathbf{\Sigma}_2}) = \mathbf{0}, \ \ Cov[(\mathbf{I}\otimes\mathbf{E}_{\mathbf{\Sigma}_1})\mathrm{vec}(\boldsymbol{\varepsilon})] = \mathbf{0}, \ \ Cov[(\mathbf{E}_{\mathbf{\Sigma}_2}\otimes\mathbf{I})\mathrm{vec}(\boldsymbol{\varepsilon})] = \mathbf{0}.$$

These four equalities imply that the pair of equalities in (3.5) hold with probability 1. □

Concerning solvability conditions and general common solutions to the pair of matrix equations in (3.5), we have the following result.

Theorem 3.4

(a) *The first matrix equation in* (3.5) *is solvable for* $\mathbf{\Theta}$ *if and only if* $\mathscr{R}(\mathbf{E}_{\mathbf{\Sigma}_1}\mathbf{Y}) \subseteq \mathscr{R}(\mathbf{E}_{\mathbf{\Sigma}_1}\mathbf{X}_1)$ *and* $\mathscr{R}[(\mathbf{E}_{\mathbf{\Sigma}_1}\mathbf{Y})'] \subseteq \mathscr{R}(\mathbf{X}_2')$, *that is,*

$$r[\,\mathbf{X}_1,\, \mathbf{\Sigma}_1, \mathbf{Y}\,] = r[\,\mathbf{X}_1,\, \mathbf{\Sigma}_1\,] \ \ \textit{and} \ \ r\begin{bmatrix} \mathbf{Y} & \mathbf{\Sigma}_1 \\ \mathbf{X}_2 & \mathbf{0} \end{bmatrix} = r(\mathbf{\Sigma}_1) + r(\mathbf{X}_2). \tag{3.6}$$

(b) *The second matrix equation in* (3.5) *is solvable for* $\mathbf{\Theta}$ *if and only if* $\mathscr{R}(\mathbf{Y}\mathbf{E}_{\mathbf{\Sigma}_2}) \subseteq \mathscr{R}(\mathbf{X}_1)$ *and* $\mathscr{R}[(\mathbf{Y}\mathbf{E}_{\mathbf{\Sigma}_2})'] \subseteq \mathscr{R}[(\mathbf{X}_2\mathbf{E}_{\mathbf{\Sigma}_2})']$, *that is,*

$$r\begin{bmatrix} \mathbf{\Sigma}_2 \\ \mathbf{X}_2 \\ \mathbf{Y} \end{bmatrix} = r\begin{bmatrix} \mathbf{\Sigma}_2 \\ \mathbf{X}_2 \end{bmatrix} \quad \textit{and} \quad r\begin{bmatrix} \mathbf{Y} & \mathbf{X}_1 \\ \mathbf{\Sigma}_2 & \mathbf{0} \end{bmatrix} = r(\mathbf{X}_1) + r(\mathbf{\Sigma}_2). \tag{3.7}$$

(c) *The pair of matrix equations in* (3.5) *have a common solution for* $\mathbf{\Theta}$ *if and only if both* (3.6) *and* (3.7) *hold.*

(d) *Under both* (3.6) *and* (3.7), *the general solution of the pair of the matrix equations in* (3.5) *can be expressed in the following parametric form*

$$\mathbf{\Theta} = \mathbf{\Theta}_0 + \mathbf{F}_{\mathbf{X}_1}\mathbf{U}_1 + \mathbf{U}_2\mathbf{E}_{\mathbf{X}_2} + [\,\mathbf{I} - (\mathbf{E}_{\mathbf{\Sigma}_1}\mathbf{X}_1)^{+}(\mathbf{E}_{\mathbf{\Sigma}_1}\mathbf{X}_1)\,]\mathbf{U}_3[\,\mathbf{I} - (\mathbf{X}_2\mathbf{E}_{\mathbf{\Sigma}_2})(\mathbf{X}_2\mathbf{E}_{\mathbf{\Sigma}_2})^{+}\,], \tag{3.8}$$

where $\mathbf{\Theta}_0$ *is a special solution to the pair of equations in* (3.5), *and* $\mathbf{U}_1$, $\mathbf{U}_2$ *and* $\mathbf{U}_3$ *are three arbitrary matrices.*

Proof It follows from Lemma 1.7. $\square$

Combining Theorems 2.2, 3.2 and 3.4 leads to the following result.

Theorem 3.5 *The following statements are equivalent*:

(a) *The matrix equation in* (3.1) *is solvable for* $\mathbf{\Theta}$ *with probability* 1.

(b) *The pair of the matrix equations in* (3.5) *have a common solution for* $\mathbf{\Theta}$ *with probability* 1.

(c) *The model in* (1.1) *is consistent.*

Theorem 3.5 indicates that if the model in (1.1) is consistent, then the unknown parameter matrix $\mathbf{\Theta}$ in (1.1) satisfies the natural restrictions in (3.1) or (3.5) with probability 1. In this case, combining (1.1) with (3.1) gives the following restricted growth curve model

$$\begin{cases} \mathbf{Y} = \mathbf{X}_1\mathbf{\Theta}\mathbf{X}_2 + \boldsymbol{\varepsilon}, \quad E(\boldsymbol{\varepsilon}) = \mathbf{0}, \ Cov[\mathrm{vec}(\boldsymbol{\varepsilon})] = \sigma^2(\mathbf{\Sigma}_2 \otimes \mathbf{\Sigma}_1), \\ \mathbf{Y} - \mathbf{P}_{\mathbf{\Sigma}_1}\mathbf{Y}\mathbf{P}_{\mathbf{\Sigma}_2} = \mathbf{X}_1\mathbf{\Theta}\mathbf{X}_2 - \mathbf{P}_{\mathbf{\Sigma}_1}\mathbf{X}_1\mathbf{\Theta}\mathbf{X}_2\mathbf{P}_{\mathbf{\Sigma}_2} \ \text{ with probability 1.} \end{cases}$$

It should be pointed out that not all estimators under (1.1) satisfy the natural restriction in (3.1) or (3.5). Recall that the ordinary least-squares estimator (OLSE) of $\mathbf{\Theta}$ in (1.1), denoted by OLSE($\mathbf{\Theta}$), is defined to be

$$\hat{\mathbf{\Theta}} = \operatorname*{argmin}_{\mathbf{\Theta}} \mathrm{tr}[\,(\,\mathbf{Y} - \mathbf{X}_1\mathbf{\Theta}\mathbf{X}_2\,)'(\,\mathbf{Y} - \mathbf{X}_1\mathbf{\Theta}\mathbf{X}_2\,)\,], \tag{3.9}$$

and the OLSE of $\mathbf{X}_1\mathbf{\Theta}\mathbf{X}_2$ in (1.1) is defined to be OLSE($\mathbf{X}_1\mathbf{\Theta}\mathbf{X}_2$) $= \mathbf{X}_1\hat{\mathbf{\Theta}}\mathbf{X}_2$. As is well known, the normal equation corresponding to (3.9) is given by

$$\mathbf{X}_1'\mathbf{X}_1\mathbf{\Theta}\mathbf{X}_2\mathbf{X}_2' = \mathbf{X}_1'\mathbf{Y}\mathbf{X}_2'.$$

This equation is always consistent, and the general expression of $\mathbf{\Theta}$ can be written in the following parametric form

$$\mathrm{OLSE}(\mathbf{\Theta}) = \mathbf{X}_1^{+}\mathbf{Y}\mathbf{X}_2^{+} + \mathbf{F}_{\mathbf{X}_1}\mathbf{U}_1 + \mathbf{U}_2\mathbf{E}_{\mathbf{X}_2},$$

where $\mathbf{U}_1$, $\mathbf{U}_2 \in \mathbb{R}^{p\times q}$ are arbitrary. Correspondingly, the OLSE of $\mathbf{X}_1\mathbf{\Theta}\mathbf{X}_2$ in (1.1) is given by

$$\mathrm{OLSE}(\mathbf{X}_1\mathbf{\Theta}\mathbf{X}_2) = \mathbf{X}_1\mathbf{X}_1^{+}\mathbf{Y}\mathbf{X}_2^{+}\mathbf{X}_2 = \mathbf{P}_{\mathbf{X}_1}\mathbf{Y}\mathbf{P}_{\mathbf{X}_2'}. \tag{3.10}$$

Theorem 3.6 *The following statements are equivalent*:

(a) *The* OLSE($\mathbf{X}_1\mathbf{\Theta}\mathbf{X}_2$) *in* (3.10) *satisfies the natural restriction in* (3.5) *with probability* 1.

(b) *There exists a matrix* $\mathbf{Z}$ *such that* $\mathbf{Y}$ *can be decomposed as*

$$\mathbf{Y} = \mathrm{OLSE}(\mathbf{X}_1\mathbf{\Theta}\mathbf{X}_2) + \mathbf{\Sigma}_1\mathbf{Z}\mathbf{\Sigma}_2 \tag{3.11}$$

with probability 1.

(c) *One of the following three conditions holds*

(i) $\mathbf{\Sigma}_1\mathbf{X}_1 = \mathbf{0}$,

(ii) $\mathbf{X}_2\mathbf{\Sigma}_2 = \mathbf{0}$,

(iii) $\mathbf{P}_{\mathbf{X}_1}\mathbf{P}_{\mathbf{\Sigma}_1} = \mathbf{P}_{\mathbf{\Sigma}_1}\mathbf{P}_{\mathbf{X}_1}$ *and* $\mathbf{P}_{\mathbf{X}_2'}\mathbf{P}_{\mathbf{\Sigma}_2} = \mathbf{P}_{\mathbf{\Sigma}_2}\mathbf{P}_{\mathbf{X}_2'}$.

Proof Substituting (3.10) into the two equations in (3.5) gives

$$\mathbf{E}_{\mathbf{\Sigma}_1}\mathbf{Y} = \mathbf{E}_{\mathbf{\Sigma}_1}\mathbf{P}_{\mathbf{X}_1}\mathbf{Y}\mathbf{P}_{\mathbf{X}_2'} \quad \text{and} \quad \mathbf{Y}\mathbf{E}_{\mathbf{\Sigma}_2} = \mathbf{P}_{\mathbf{X}_1}\mathbf{Y}\mathbf{P}_{\mathbf{X}_2'}\mathbf{E}_{\mathbf{\Sigma}_2}. \tag{3.12}$$

It is easy to verify by definition that the first equality in (3.12) holds with probability 1 under (1.1) if and only if $\mathbf{X}_2\mathbf{\Sigma}_2 = \mathbf{0}$ or $\mathbf{P}_{\mathbf{X}_1}\mathbf{P}_{\mathbf{\Sigma}_1} = \mathbf{P}_{\mathbf{\Sigma}_1}\mathbf{P}_{\mathbf{X}_1}$, and the second equality in (3.12) holds with probability 1 under (1.1) if and only if $\mathbf{\Sigma}_1\mathbf{X}_1 = \mathbf{0}$ or $\mathbf{P}_{\mathbf{X}_2'}\mathbf{P}_{\mathbf{\Sigma}_2} = \mathbf{P}_{\mathbf{\Sigma}_2}\mathbf{P}_{\mathbf{X}_2'}$, establishing the equivalence of (a) and (c). Equation (3.11) can be written as

$$\mathbf{Y} = \mathbf{P}_{\mathbf{X}_1}\mathbf{Y}\mathbf{P}_{\mathbf{X}_2'} + \mathbf{\Sigma}_1\mathbf{Z}\mathbf{\Sigma}_2. \tag{3.13}$$

From Lemma 1.2, the equation is solvable for $\mathbf{Z}$ if and only if (3.12) holds. Hence (a) and (b) are equivalent. $\square$

4 Estimability of parametric functions under the growth curve model

Definition 4.1 Suppose a matrix $\mathbf{K} \in \mathbb{R}^{k\times pq}$ is given. Then the linear transformation $\mathbf{K}\mathrm{vec}(\mathbf{\Theta})$ under (1.1) is said to be estimable if there exists a matrix $\mathbf{L} \in \mathbb{R}^{k\times nm}$ so that $E[\mathbf{L}\mathrm{vec}(\mathbf{Y})] = \mathbf{K}\mathrm{vec}(\mathbf{\Theta})$ under (1.1).

The following result follows from Alalouf and Styan (1979).

Lemma 4.2 *A linear transformation* $\mathbf{K}\mathrm{vec}(\mathbf{\Theta})$ *is estimable under* (1.1) *if and only if* $\mathscr{R}(\mathbf{K}') \subseteq \mathscr{R}(\mathbf{X}_2 \otimes \mathbf{X}_1')$.

Because $\mathbf{\Theta}$ in (1.1) is a parameter matrix, linear transformations of $\mathbf{\Theta}$ may have various forms. A general linear transformation of $\mathbf{\Theta}$ has the following form

$$\mathbf{K}_{11}\mathbf{\Theta}\mathbf{K}_{21} + \cdots + \mathbf{K}_{1k}\mathbf{\Theta}\mathbf{K}_{2k}, \tag{4.1}$$

where $\mathbf{K}_{11}, \ldots, \mathbf{K}_{1k} \in \mathbb{R}^{s\times p}$ and $\mathbf{K}_{21}, \ldots, \mathbf{K}_{2k} \in \mathbb{R}^{q\times t}$ are given. In fact, any linear combination of the entries in $\mathbf{\Theta}$ can be written in the form of (4.1). For example,

(i) the trace of the square matrix $\mathbf{\Theta}$ in (1.1) can be written as $\mathbf{e}_1'\mathbf{\Theta}\mathbf{e}_1 + \cdots + \mathbf{e}_p'\mathbf{\Theta}\mathbf{e}_p$, where $\mathbf{e}_i' = [\,0, \ldots, 1, \ldots, 0\,]$, $i = 1, \ldots, p$;

(ii) the parametric function $\mathrm{tr}(\mathbf{K}\mathbf{\Theta})$ can also be written as in (4.1);

(iii) the submatrix $\mathbf{\Theta}_{11}$ in $\mathbf{\Theta} = \begin{bmatrix} \mathbf{\Theta}_{11} & \mathbf{\Theta}_{12} \\ \mathbf{\Theta}_{21} & \mathbf{\Theta}_{22} \end{bmatrix}$ can be written as $\mathbf{\Theta}_{11} = \mathbf{K}_1\mathbf{\Theta}\mathbf{K}_2$;

(iv) the diagonal block matrix $\begin{bmatrix} \mathbf{\Theta}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Theta}_{22} \end{bmatrix}$ can be written as $\mathbf{K}_{11}\mathbf{\Theta}\mathbf{K}_{21} + \mathbf{K}_{12}\mathbf{\Theta}\mathbf{K}_{22}$.

Because $\mathbf{Y}$ in (1.1) is a matrix, a general expression of linear estimators under (1.1) can be written in the following two-sided form

$$\mathbf{L}_{11}\mathbf{Y}\mathbf{L}_{21} + \cdots + \mathbf{L}_{1l}\mathbf{Y}\mathbf{L}_{2l} + \mathbf{L}_{l+1}, \tag{4.2}$$

where $\mathbf{L}_{11}, \ldots, \mathbf{L}_{1l}$, $\mathbf{L}_{21}, \ldots, \mathbf{L}_{2l}$ and $\mathbf{L}_{l+1}$ are given matrices. In fact, any linear combination of the entries in $\mathbf{Y}$ can be written in the form of (4.2). The estimability of the parameter matrix in (4.1) and the unbiasedness of the linear estimator in (4.2) are defined below.

Definition 4.3 Let $\mathbf{K}_{11}, \ldots, \mathbf{K}_{1k} \in \mathbb{R}^{s\times p}$ and $\mathbf{K}_{21}, \ldots, \mathbf{K}_{2k} \in \mathbb{R}^{q\times t}$ be two sets of given matrices. Then the linear transformation $\mathbf{K}_{11}\mathbf{\Theta}\mathbf{K}_{21} + \cdots + \mathbf{K}_{1k}\mathbf{\Theta}\mathbf{K}_{2k}$ of the parameter matrix Θ is said to be *estimable* under (1.1) if there exist $\mathbf{L}_{11}, \ldots, \mathbf{L}_{1l} \in \mathbb{R}^{s\times m}$, $\mathbf{L}_{21}, \ldots, \mathbf{L}_{2l} \in \mathbb{R}^{n\times t}$ and $\mathbf{L}_{l+1} \in \mathbb{R}^{s\times t}$ such that

$$E(\mathbf{L}_{11}\mathbf{Y}\mathbf{L}_{21} + \cdots + \mathbf{L}_{1l}\mathbf{Y}\mathbf{L}_{2l} + \mathbf{L}_{l+1}) = \mathbf{K}_{11}\mathbf{\Theta}\mathbf{K}_{21} + \cdots + \mathbf{K}_{1k}\mathbf{\Theta}\mathbf{K}_{2k},$$

i.e.,

$$\mathbf{L}_{11}\mathbf{X}_1\mathbf{\Theta}\mathbf{X}_2\mathbf{L}_{21} + \cdots + \mathbf{L}_{1l}\mathbf{X}_1\mathbf{\Theta}\mathbf{X}_2\mathbf{L}_{2l} + \mathbf{L}_{l+1} = \mathbf{K}_{11}\mathbf{\Theta}\mathbf{K}_{21} + \cdots + \mathbf{K}_{1k}\mathbf{\Theta}\mathbf{K}_{2k}. \tag{4.3}$$

In this case, the linear estimator $\mathbf{L}_{11}\mathbf{Y}\mathbf{L}_{21} + \cdots + \mathbf{L}_{1l}\mathbf{Y}\mathbf{L}_{2l} + \mathbf{L}_{l+1}$ is said to be *unbiased* for the parametric transformation $\mathbf{K}_{11}\mathbf{\Theta}\mathbf{K}_{21} + \cdots + \mathbf{K}_{1k}\mathbf{\Theta}\mathbf{K}_{2k}$.

Two basic problems associated with estimability of linear transformations of the parameter matrix $\mathbf{\Theta}$ in (4.2) are:

(I) Necessary and sufficient conditions for $\mathbf{K}_{11}\mathbf{\Theta}\mathbf{K}_{21} + \cdots + \mathbf{K}_{1k}\mathbf{\Theta}\mathbf{K}_{2k}$ to be estimable.

(II) General expressions of $\mathbf{L}_{11}, \ldots, \mathbf{L}_{1l}$, $\mathbf{L}_{21}, \ldots, \mathbf{L}_{2l}$ and $\mathbf{L}_{l+1}$ satisfying (4.3) if $\mathbf{K}_{11}\mathbf{\Theta}\mathbf{K}_{21} + \cdots + \mathbf{K}_{1k}\mathbf{\Theta}\mathbf{K}_{2k}$ is estimable.

Through the vec operation and Kronecker product of matrices, (4.3) is equivalent to

$$\begin{aligned}&[(\mathbf{X}_2\mathbf{L}_{21})' \otimes (\mathbf{L}_{11}\mathbf{X}_1)]\mathrm{vec}(\mathbf{\Theta}) + \cdots + [(\mathbf{X}_2\mathbf{L}_{2l})' \otimes (\mathbf{L}_{1l}\mathbf{X}_1)]\mathrm{vec}(\mathbf{\Theta}) + \mathrm{vec}(\mathbf{L}_{l+1}) \\ &\quad = (\mathbf{K}_{21}' \otimes \mathbf{K}_{11})\mathrm{vec}(\mathbf{\Theta}) + \cdots + (\mathbf{K}_{2k}' \otimes \mathbf{K}_{1k})\mathrm{vec}(\mathbf{\Theta}).\end{aligned} \tag{4.4}$$

If $\mathbf{\Theta}$ in (4.4) is taken as a free matrix, in other words, (3.1) is not taken into account in estimating (1.1), than we obtain the following result by comparing both sides of (4.4).

Theorem 4.4 *If $\mathbf{\Theta}$ is a free matrix, then* (4.4) *is equivalent to*

$$(\mathbf{X}_2\mathbf{L}_{21})' \otimes (\mathbf{L}_{11}\mathbf{X}_1) + \cdots + (\mathbf{X}_2\mathbf{L}_{2l})' \otimes (\mathbf{L}_{1l}\mathbf{X}_1) = \mathbf{K}_{21}' \otimes \mathbf{K}_{11} + \cdots + \mathbf{K}_{2k}' \otimes \mathbf{K}_{1k} \text{ and } \mathbf{L}_{l+1} = \mathbf{0}. \tag{4.5}$$

In particular, if both $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ in (1.1) *are positive definite, then* (4.5) *holds.*

Observe that the first equality in (4.5) is a quadratic matrix equation for the unknown matrices $\mathbf{L}_{11}, \ldots, \mathbf{L}_{1l}$, $\mathbf{L}_{21}, \ldots, \mathbf{L}_{2l}$. It seems impossible to solve this matrix equation for $l > 1$ or $k > 1$ by current tools in matrix theory. However, if $l = k = 1$, we have the following result.

Theorem 4.5 *Let $\mathbf{K}_1 \in \mathbb{R}^{s\times p}$ and $\mathbf{K}_2 \in \mathbb{R}^{q\times t}$ be two given matrices. and assume that both $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ in* (1.1) *are positive definite. Then,*

(a) *There exist two matrices $\mathbf{L}_1$ and $\mathbf{L}_2$ such that $E(\mathbf{L}_1\mathbf{Y}\mathbf{L}_2) = \mathbf{K}_1\mathbf{\Theta}\mathbf{K}_2$, that is,*

$$\mathbf{L}_1\mathbf{X}_1\mathbf{\Theta}\mathbf{X}_2\mathbf{L}_2 = \mathbf{K}_1\mathbf{\Theta}\mathbf{K}_2 \tag{4.6}$$

holds under (1.1) *if and only if*

$$\mathbf{L}_1\mathbf{X}_1 = \lambda_1\mathbf{K}_1, \quad \mathbf{X}_2\mathbf{L}_2 = \lambda_2\mathbf{K}_2, \quad \lambda_1\lambda_2 = 1. \tag{4.7}$$

(b) $\mathbf{K}_1\mathbf{\Theta}\mathbf{K}_2$ *is estimable if and only if* $\mathscr{R}(\mathbf{K}_1') \subseteq \mathscr{R}(\mathbf{X}_1')$ *and* $\mathscr{R}(\mathbf{K}_2) \subseteq \mathscr{R}(\mathbf{X}_2)$. *In this case, the general expressions of* $\mathbf{L}_1$ *and* $\mathbf{L}_2$ *satisfying* (4.7) *are given by*

$$\mathbf{L}_1 = \lambda_1\mathbf{K}_1\mathbf{X}_1^{+} + \mathbf{U}_1(\,\mathbf{I} - \mathbf{X}_1\mathbf{X}_1^{+}\,), \quad \mathbf{L}_2 = \lambda_2\mathbf{X}_2^{+}\mathbf{K}_2 + (\,\mathbf{I} - \mathbf{X}_2^{+}\mathbf{X}_2\,)\mathbf{U}_2, \quad \lambda_1\lambda_2 = 1,$$

where $\mathbf{U}_1$ *and* $\mathbf{U}_2$ *are arbitrary matrices.*

As shown in Section 3, the parameter matrix $\mathbf{\Theta}$ in (1.1) satisfies the natural restriction in (3.1) or (3.5) if the Kronecker product $\mathbf{\Sigma}_2 \otimes \mathbf{\Sigma}_1$ in (1.1) is singular. In this case, (4.4) is not necessarily equivalent to (4.5). In order to obtain necessary and sufficient conditions for $\mathbf{K}_{11}\mathbf{\Theta}\mathbf{K}_{21} + \cdots + \mathbf{K}_{1k}\mathbf{\Theta}\mathbf{K}_{2k}$ to be estimable under (1.1), we have to substitute (3.8) into (4.3), and then derive estimability conditions from the new matrix equality. Needless to say, it is impossible to solve this problem at current time. However for the simplest parametric function $\mathrm{tr}(\mathbf{K}\mathbf{\Theta})$ under (1.1), we are able to derive its estimability conditions.

Theorem 4.6 *Let* $\mathbf{K} \in \mathbb{R}^{q\times p}$ *be given. Then the following statements are equivalent*:

(a) $\mathrm{tr}(\mathbf{K}\mathbf{\Theta})$ *is estimable under* (1.1).

(b) $\mathrm{vec}'(\mathbf{K}')\mathrm{vec}(\mathbf{\Theta})$ *is estimable under* (1.3).

(c) *The matrix equation* $\mathbf{X}_2\mathbf{L}\mathbf{X}_1 = \mathbf{K}$ *is solvable.*

(d) $\mathscr{R}(\mathbf{K}) \subseteq \mathscr{R}(\mathbf{X}_2)$ *and* $\mathscr{R}(\mathbf{K}') \subseteq \mathscr{R}(\mathbf{X}_1')$.

Proof Note that $\mathrm{tr}(\mathbf{K}\mathbf{\Theta})$ can be written as

$$\mathrm{tr}(\mathbf{K}\mathbf{\Theta}) = \mathrm{vec}'(\mathbf{K}')\mathrm{vec}(\mathbf{\Theta}).$$

Hence (a) and (b) are equivalent. It follows from Lemma 4.2 that (b) holds if and only if $\mathscr{R}[\mathrm{vec}(\mathbf{K}')] \subseteq \mathscr{R}(\mathbf{X}_2 \otimes \mathbf{X}_1')$, which is equivalent to (c) and (d) by Lemma 1.2. $\square$

In statistical applications, parameters in regression models often satisfy some restrictions. For instance, assume that the unknown parameter matrix $\mathbf{\Theta}$ in (1.1) satisfies a pair of consistent linear matrix equations

$$\mathbf{A}_1\mathbf{\Theta} = \mathbf{B}_1, \quad \mathbf{\Theta}\mathbf{A}_2 = \mathbf{B}_2, \tag{4.8}$$

where $\mathbf{A}_1 \in \mathbb{R}^{n_1\times p}$, $\mathbf{A}_2 \in \mathbb{R}^{q\times m_1}$ and $\mathbf{B}_1 \in \mathbb{R}^{n_1\times q}$ and $\mathbf{B}_2 \in \mathbb{R}^{p\times m_1}$ are four known matrices. In such a case, the model in (1.1) together with (4.8) can be written as

$$\mathscr{M}_r = \{\mathbf{Y},\, \mathbf{X}_1\mathbf{\Theta}\mathbf{X}_2 \,|\, \mathbf{A}_1\mathbf{\Theta} = \mathbf{B}_1,\, \mathbf{\Theta}\mathbf{A}_2 = \mathbf{B}_2,\, \sigma^2(\mathbf{\Sigma}_2 \otimes \mathbf{\Sigma}_1)\}, \tag{4.9}$$

and is called a restricted growth curve model. Usually, (4.9) can be transformed into an unrestricted growth curve model by substituting the general solution of (4.8) into (1.1).

Because there are two restrictions to the parameter matrix $\mathbf{\Theta}$ in (4.9), the estimability of the parametric function $\mathrm{tr}(\mathbf{K}\mathbf{\Theta})$ under (4.9) is different from that of $\mathrm{tr}(\mathbf{K}\mathbf{\Theta})$ under (1.1). Concerning the estimability of the parametric function $\mathrm{tr}(\mathbf{K}\mathbf{\Theta})$ under (4.9), we have the following result.

Theorem 4.7 *Suppose that both* $\mathbf{\Sigma}_1$ *and* $\mathbf{\Sigma}_2$ *in* (4.9) *are positive definite, and let* $\mathbf{K} \in \mathbb{R}^{q\times p}$ *be given. Then the following statements are equivalent*:

(a) *There exist a matrix* $\mathbf{L} \in \mathbb{R}^{m\times n}$ *and a scalar* c *such that* $\mathrm{tr}(\mathbf{L}\mathbf{Y}) + c$ *is an unbiased estimator of* $\mathrm{tr}(\mathbf{K}\mathbf{\Theta})$ *under* (4.9), *i.e.*, $\mathrm{tr}(\mathbf{K}\mathbf{\Theta})$ *is estimable under* (4.9).

(b) *There exists a matrix* $\mathbf{L} \in \mathbb{R}^{m\times n}$ *such that* $\mathbf{E}_{\mathbf{A}_2}\mathbf{X}_2\mathbf{L}\mathbf{X}_1\mathbf{F}_{\mathbf{A}_1} = \mathbf{E}_{\mathbf{A}_2}\mathbf{K}\mathbf{F}_{\mathbf{A}_1}$.

(c) $r\begin{bmatrix} \mathbf{K} & \mathbf{X}_2 & \mathbf{A}_2 \\ \mathbf{A}_1 & \mathbf{0} & \mathbf{0} \end{bmatrix} = r[\,\mathbf{X}_2,\, \mathbf{A}_2\,] + r(\mathbf{A}_1)$ *and* $r\begin{bmatrix} \mathbf{K} & \mathbf{A}_2 \\ \mathbf{X}_1 & \mathbf{0} \\ \mathbf{A}_1 & \mathbf{0} \end{bmatrix} = r\begin{bmatrix} \mathbf{X}_1 \\ \mathbf{A}_1 \end{bmatrix} + r(\mathbf{A}_2)$.

Proof Under the assumption that both $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ in (4.9) are positive definite, the natural restriction in (3.1) to $\mathbf{\Theta}$ vanishes. In this case, substituting the general solution (1.12) of the equations in (4.8) into (1.1) gives the following reparameterized model

$$\mathbf{Y} - \mathbf{X}_1\mathbf{A}_1^{+}\mathbf{B}_1\mathbf{X}_2 - \mathbf{X}_1\mathbf{B}_2\mathbf{A}_2^{+}\mathbf{X}_2 + \mathbf{X}_1\mathbf{A}_1^{+}\mathbf{A}_1\mathbf{B}_2\mathbf{A}_2^{+}\mathbf{X}_2 = \mathbf{X}_1\mathbf{F}_{\mathbf{A}_1}\mathbf{U}\mathbf{E}_{\mathbf{A}_2}\mathbf{X}_2 + \boldsymbol{\varepsilon},$$

where $\mathbf{U}$ is a new free parameter matrix. Correspondingly, both $E[\mathrm{tr}(\mathbf{LY})+c]$ and $\mathrm{tr}(\mathbf{K\Theta})$ can be written as

$$\begin{aligned} &E[\mathrm{tr}(\mathbf{LY}) + c] \\ &= \mathrm{tr}(\,\mathbf{L}\mathbf{X}_1\mathbf{A}_1^{+}\mathbf{B}_1\mathbf{X}_2 + \mathbf{L}\mathbf{X}_1\mathbf{B}_2\mathbf{A}_2^{+}\mathbf{X}_2 - \mathbf{L}\mathbf{X}_1\mathbf{A}_1^{+}\mathbf{A}_1\mathbf{B}_2\mathbf{A}_2^{+}\mathbf{X}_2\,) + \ \mathrm{tr}(\mathbf{L}\mathbf{X}_1\mathbf{F}_{\mathbf{A}_1}\mathbf{U}\mathbf{E}_{\mathbf{A}_2}\mathbf{X}_2\,) + c \\ &= \mathrm{tr}(\,\mathbf{L}\mathbf{X}_1\mathbf{A}_1^{+}\mathbf{B}_1\mathbf{X}_2 + \mathbf{L}\mathbf{X}_1\mathbf{B}_2\mathbf{A}_2^{+}\mathbf{X}_2 - \mathbf{L}\mathbf{X}_1\mathbf{A}_1^{+}\mathbf{A}_1\mathbf{B}_2\mathbf{A}_2^{+}\mathbf{X}_2\,) + \mathrm{tr}(\mathbf{E}_{\mathbf{A}_2}\mathbf{X}_2\mathbf{L}\mathbf{X}_1\mathbf{F}_{\mathbf{A}_1}\mathbf{U}) + c, \\ \mathrm{tr}(\mathbf{K\Theta}) &= \mathrm{tr}(\mathbf{K}\mathbf{A}_1^{+}\mathbf{B}_1 + \mathbf{K}\mathbf{B}_2\mathbf{A}_2^{+} - \mathbf{K}\mathbf{A}_1^{+}\mathbf{A}_1\mathbf{B}_2\mathbf{A}_2^{+} + \mathbf{K}\mathbf{F}_{\mathbf{A}_1}\mathbf{U}\mathbf{E}_{\mathbf{A}_2}) \\ &= \mathrm{tr}(\mathbf{K}\mathbf{A}_1^{+}\mathbf{B}_1 + \mathbf{K}\mathbf{B}_2\mathbf{A}_2^{+} - \mathbf{K}\mathbf{A}_1^{+}\mathbf{A}_1\mathbf{B}_2\mathbf{A}_2^{+}\,) + \mathrm{tr}(\mathbf{E}_{\mathbf{A}_2}\mathbf{K}\mathbf{F}_{\mathbf{A}_1}\mathbf{U}). \end{aligned}$$

Note that $\mathbf{U}$ is free, so that $E[\mathrm{tr}(\mathbf{LY}) + c] = \mathrm{tr}(\mathbf{K\Theta})$ is equivalent to

$$\begin{aligned} &\mathrm{tr}(\,\mathbf{L}\mathbf{X}_1\mathbf{A}_1^{+}\mathbf{B}_1\mathbf{X}_2 + \mathbf{L}\mathbf{X}_1\mathbf{B}_2\mathbf{A}_2^{+}\mathbf{X}_2 - \mathbf{L}\mathbf{X}_1\mathbf{A}_1^{+}\mathbf{A}_1\mathbf{B}_2\mathbf{A}_2^{+}\mathbf{X}_2\,) + c \\ &= \mathrm{tr}(\mathbf{K}\mathbf{A}_1^{+}\mathbf{B}_1 + \mathbf{K}\mathbf{B}_2\mathbf{A}_2^{+} - \mathbf{K}\mathbf{A}_1^{+}\mathbf{A}_1\mathbf{B}_2\mathbf{A}_2^{+}\,), \end{aligned} \tag{4.10}$$

and

$$\mathbf{E}_{\mathbf{A}_2}\mathbf{X}_2\mathbf{L}\mathbf{X}_1\mathbf{F}_{\mathbf{A}_1} = \mathbf{E}_{\mathbf{A}_2}\mathbf{K}\mathbf{F}_{\mathbf{A}_1}. \tag{4.11}$$

By Lemma 1.2, the matrix equation in (4.11) is solvable for $\mathbf{L}$ if and only if

$$\mathscr{R}(\mathbf{E}_{\mathbf{A}_2}\mathbf{K}\mathbf{F}_{\mathbf{A}_1}) \subseteq \mathscr{R}(\mathbf{E}_{\mathbf{A}_2}\mathbf{X}_2), \quad \mathscr{R}[(\mathbf{E}_{\mathbf{A}_2}\mathbf{K}\mathbf{F}_{\mathbf{A}_1})'] \subseteq \mathscr{R}[(\mathbf{X}_1\mathbf{F}_{\mathbf{A}_1})'].$$

The two conditions are equivalent to (c) by Lemma 1.1. Solving $\mathbf{L}$ from (4.11) and substituting the solution into (4.10) yields the matrix $\mathbf{L}$ and the scalar c such that $E[\mathrm{tr}(\mathbf{LY}) + c] = \mathrm{tr}(\mathbf{K\Theta})$ holds. □

Assume now that the unknown parameter matrix $\mathbf{\Theta}$ in (1.1) satisfies a consistent linear matrix equation

$$\mathbf{A}_1\mathbf{\Theta}\mathbf{A}_2 = \mathbf{B}, \tag{4.12}$$

where $\mathbf{A}_1 \in \mathbb{R}^{n_1\times p}$, $\mathbf{A}_2 \in \mathbb{R}^{q\times m_1}$ and $\mathbf{B} \in \mathbb{R}^{n_1\times m_1}$ are three known matrices. In such a case, the model in (1.1) together with (4.12) is written as

$$\{\mathbf{Y},\ \mathbf{X}_1\mathbf{\Theta}\mathbf{X}_2 \mid \mathbf{A}_1\mathbf{\Theta}\mathbf{A}_2 = \mathbf{B},\ \sigma^2(\mathbf{\Sigma}_2 \otimes \mathbf{\Sigma}_1)\}. \tag{4.13}$$

Concerning the estimability of the parametric function $\mathrm{tr}(\mathbf{K\Theta})$ under (4.13), we have the following result.

Theorem 4.8 *Suppose that both $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ in (4.13) are positive definite, and let $\mathbf{K} \in \mathbb{R}^{q\times p}$ be given. Then the following statements are equivalent*:

(a) *There exist a matrix $\mathbf{L} \in \mathbb{R}^{m\times n}$ and a scalar c such that $\mathrm{tr}(\mathbf{LY}) + c$ is an unbiased estimator of $\mathrm{tr}(\mathbf{K\Theta})$ under (4.13), i.e., $\mathrm{tr}(\mathbf{K\Theta})$ is estimable under (4.13).*

(b) *There exists a matrix $\mathbf{L} \in \mathbb{R}^{m\times n}$ such that $\mathbf{X}_2\mathbf{L}\mathbf{X}_1\mathbf{F}_{\mathbf{A}_1} = \mathbf{K}\mathbf{F}_{\mathbf{A}_1}$ and $\mathbf{E}_{\mathbf{A}_2}\mathbf{X}_2\mathbf{L}\mathbf{X}_1 = \mathbf{E}_{\mathbf{A}_2}\mathbf{K}$.*

(c) $\mathscr{R}(\mathbf{K}) \subseteq \mathscr{R}[\,\mathbf{X}_2,\ \mathbf{A}_2\,]$, $\mathscr{R}(\mathbf{K}') \subseteq \mathscr{R}[\,\mathbf{X}_1',\ \mathbf{A}_1'\,]$ *and*

$$r\begin{bmatrix} \mathbf{K} & \mathbf{X}_2 \\ \mathbf{A}_1 & \mathbf{0} \end{bmatrix} = r(\mathbf{X}_2) + r(\mathbf{A}_1), \quad r\begin{bmatrix} \mathbf{K} & \mathbf{A}_2 \\ \mathbf{X}_1 & \mathbf{0} \end{bmatrix} = r(\mathbf{X}_1) + r(\mathbf{A}_2).$$

(d) *The matrix equation $\mathbf{X}_2\mathbf{V}\mathbf{X}_1 + \mathbf{A}_2\mathbf{W}\mathbf{A}_1 = \mathbf{K}$ is solvable for $\mathbf{V}$ and $\mathbf{W}$.*

Proof From Lemma 1.2, the matrix equation in (4.12) is consistent if and only if $\mathbf{A}_1\mathbf{A}_1^{+}\mathbf{B}\mathbf{A}_2^{+}\mathbf{A}_2 = \mathbf{B}$. In this case, the general solution of the equation can be written as $\mathbf{\Theta} = \mathbf{A}_1^{+}\mathbf{B}\mathbf{A}_2^{+} + \mathbf{F}_{\mathbf{A}_1}\mathbf{U}_1 + \mathbf{U}_2\mathbf{E}_{\mathbf{A}_2}$, where $\mathbf{U}_1$ and $\mathbf{U}_2$ are arbitrary. Substituting the general solution into (1.1) gives the following reparameterized model

$$\mathbf{Y} - \mathbf{X}_1\mathbf{A}_1^{+}\mathbf{B}\mathbf{A}_2^{+}\mathbf{X}_2 = \mathbf{X}_1\mathbf{F}_{\mathbf{A}_1}\mathbf{U}_1\mathbf{X}_2 + \mathbf{X}_1\mathbf{U}_2\mathbf{E}_{\mathbf{A}_2}\mathbf{X}_2 + \boldsymbol{\varepsilon}.$$

In this case, $E[\,\mathrm{tr}(\mathbf{LY}) + c\,]$ and $\mathrm{tr}(\mathbf{K\Theta})$ can be written as

$$\begin{aligned} E[\mathrm{tr}(\mathbf{LY}) + c] &= \mathrm{tr}(\,\mathbf{L}\mathbf{X}_1\mathbf{A}_1^{+}\mathbf{B}\mathbf{A}_2^{+}\mathbf{X}_2 + \mathbf{L}\mathbf{X}_1\mathbf{F}_{\mathbf{A}_1}\mathbf{U}_1\mathbf{X}_2 + \mathbf{L}\mathbf{X}_1\mathbf{U}_2\mathbf{E}_{\mathbf{A}_2}\mathbf{X}_2) + c \\ &= \mathrm{tr}(\,\mathbf{X}_2\mathbf{L}\mathbf{X}_1\mathbf{A}_1^{+}\mathbf{B}\mathbf{A}_2^{+} + \mathbf{X}_2\mathbf{L}\mathbf{X}_1\mathbf{F}_{\mathbf{A}_1}\mathbf{U}_1 + \mathbf{E}_{\mathbf{A}_2}\mathbf{X}_2\mathbf{L}\mathbf{X}_1\mathbf{U}_2) + c \end{aligned}$$

and

$$\begin{aligned} \mathrm{tr}(\mathbf{K\Theta}) &= \mathrm{tr}(\mathbf{K}\mathbf{A}_1^{+}\mathbf{B}\mathbf{A}_2^{+} + \mathbf{K}\mathbf{F}_{\mathbf{A}_1}\mathbf{U}_1 + \mathbf{K}\mathbf{U}_2\mathbf{E}_{\mathbf{A}_2}) \\ &= \mathrm{tr}(\mathbf{K}\mathbf{A}_1^{+}\mathbf{B}\mathbf{A}_2^{+} + \mathbf{K}\mathbf{F}_{\mathbf{A}_1}\mathbf{U}_1 + \mathbf{E}_{\mathbf{A}_2}\mathbf{K}\mathbf{U}_2). \end{aligned}$$

So that $E[\,\mathrm{tr}(\mathbf{LY}) + c\,] = \mathrm{tr}(\mathbf{K\Theta})$ is equivalent to

$$\mathrm{tr}(\,\mathbf{X}_2\mathbf{L}\mathbf{X}_1\mathbf{A}_1^{+}\mathbf{B}\mathbf{A}_2^{+}) + c = \mathrm{tr}(\mathbf{K}\mathbf{A}_1^{+}\mathbf{B}\mathbf{A}_2^{+}), \tag{4.14}$$

$$\mathbf{X}_2\mathbf{L}\mathbf{X}_1\mathbf{F}_{\mathbf{A}_1} = \mathbf{K}\mathbf{F}_{\mathbf{A}_1}, \quad \mathbf{E}_{\mathbf{A}_2}\mathbf{X}_2\mathbf{L}\mathbf{X}_1 = \mathbf{E}_{\mathbf{A}_2}\mathbf{K}. \tag{4.15}$$

By Lemma 1.7, the pair of matrix equations in (4.15) have a common solution for $\mathbf{L}$ if and only if

$$\begin{aligned} &\mathscr{R}(\mathbf{K}\mathbf{F}_{\mathbf{A}_1}) \subseteq \mathscr{R}(\mathbf{X}_2), \quad \mathscr{R}[(\mathbf{K}\mathbf{F}_{\mathbf{A}_1})'] \subseteq \mathscr{R}[(\mathbf{X}_1\mathbf{F}_{\mathbf{A}_1})'], \\ &\mathscr{R}(\mathbf{E}_{\mathbf{A}_2}\mathbf{K}) \subseteq \mathscr{R}(\mathbf{E}_{\mathbf{A}_2}\mathbf{X}_2), \quad \mathscr{R}[(\mathbf{E}_{\mathbf{A}_2}\mathbf{K})'] \subseteq \mathscr{R}(\mathbf{X}_1'), \end{aligned}$$

and

$$r\begin{bmatrix} \mathbf{K}\mathbf{F}_{\mathbf{A}_1} & \mathbf{0} & \mathbf{X}_2 \\ \mathbf{0} & -\mathbf{E}_{\mathbf{A}_2}\mathbf{K} & \mathbf{E}_{\mathbf{A}_2}\mathbf{X}_2 \\ \mathbf{X}_1\mathbf{F}_{\mathbf{A}_1} & \mathbf{X}_1 & \mathbf{0} \end{bmatrix} = r\begin{bmatrix} \mathbf{X}_2 \\ \mathbf{E}_{\mathbf{A}_2}\mathbf{X}_2 \end{bmatrix} + r[\,\mathbf{X}_1\mathbf{F}_{\mathbf{A}_1},\, \mathbf{X}_1\,].$$

The results are further equivalent to those in (c). The equivalence of (c) and (d) is established by Lemma 1.8. $\square$

In what follows, we give the definition of the best linear unbiased estimator (BLUE) of parametric functions under (1.1).

Definition 4.9 A linear estimator $\mathbf{G}_{11}\mathbf{Y}\mathbf{G}_{21} + \cdots + \mathbf{G}_{1l}\mathbf{Y}\mathbf{G}_{2l}$ is said to be the BLUE of parametric functions $\mathbf{K}_{11}\mathbf{\Theta}\mathbf{K}_{21} + \cdots + \mathbf{K}_{1k}\mathbf{\Theta}\mathbf{K}_{2k}$ under (1.1) if $E(\,\mathbf{G}_{11}\mathbf{Y}\mathbf{G}_{21} + \cdots + \mathbf{G}_{1l}\mathbf{Y}\mathbf{G}_{2l}\,) = \mathbf{K}_{11}\mathbf{\Theta}\mathbf{K}_{21} + \cdots + \mathbf{K}_{1k}\mathbf{\Theta}\mathbf{K}_{2k}$ and $Cov[\mathbf{L}\mathrm{vec}(\mathbf{Y})] - Cov[(\mathbf{G}_{21}' \otimes \mathbf{G}_{11} + \cdots + \mathbf{G}_{2l}' \otimes \mathbf{G}_{1l})\mathrm{vec}(\mathbf{Y})]$ is nonnegative definite for any other unbiased estimator $\mathbf{L}\mathrm{vec}(\mathbf{Y})$ of $\mathbf{K}_{11}\mathbf{\Theta}\mathbf{K}_{21} + \cdots + \mathbf{K}_{1k}\mathbf{\Theta}\mathbf{K}_{2k}$.

Note the parametric functions $\mathbf{K}_{11}\mathbf{\Theta}\mathbf{K}_{21} + \cdots + \mathbf{K}_{1k}\mathbf{\Theta}\mathbf{K}_{2k}$ under (1.1) can alternatively be expressed as $\mathbf{K}\mathrm{vec}(\Theta)$. In such a case, the BLUE of $\mathbf{K}\mathrm{vec}(\Theta)$ can also be defined through (1.3).

Definition 4.10 A linear estimator $\mathbf{G}\mathrm{vec}(\mathbf{Y})$ is said to be the BLUE of the parametric functions $\mathbf{K}\mathrm{vec}(\Theta)$ under (1.3), denoted by $\mathrm{BLUE}_{\mathscr{M}_v}[\mathbf{K}\mathrm{vec}(\Theta)]$, if $\mathbf{G}\mathrm{vec}(\mathbf{Y})$ is unbiased for $\mathbf{K}\mathrm{vec}(\Theta)$, i.e., $E[\mathbf{G}\mathrm{vec}(\mathbf{Y})] = \mathbf{K}\mathrm{vec}(\Theta)$, and $Cov[\mathbf{G}_1\mathrm{vec}(\mathbf{Y})] - Cov[\mathbf{G}\mathrm{vec}(\mathbf{Y})]$ is nonnegative definite for any other unbiased estimator $\mathbf{G}_1\mathrm{vec}(\mathbf{Y})$ of $\mathbf{K}\mathrm{vec}(\Theta)$.

Through generalized inverses of matrices, the general expression of the BLUE of $\mathbf{K}\mathrm{vec}(\mathbf{\Theta})$ under (1.3) can be written in some closed form. The following result follows from Rao (1973b, p. 282).

Lemma 4.11 *Assume that* $\mathbf{K}\mathrm{vec}(\mathbf{\Theta})$ *is estimable under* (1.3). *Then the general expression of* $\mathrm{BLUE}[\mathbf{K}\mathrm{vec}(\mathbf{\Theta})]$ *under* (1.3) *is given by*

$$\mathrm{BLUE}[\,\mathbf{K}\mathrm{vec}(\mathbf{\Theta})\,] = \mathbf{G}\mathrm{vec}(\mathbf{Y}), \tag{4.16}$$

where $\mathbf{G}$ *is the general expression of the following matrix equation*

$$\mathbf{G}[\,\mathbf{X}_2' \otimes \mathbf{X}_1,\, (\mathbf{\Sigma}_2 \otimes \mathbf{\Sigma}_1)\mathbf{E}_{\mathbf{X}_2' \otimes \mathbf{X}_1}\,] = [\,\mathbf{K},\, \mathbf{0}\,]. \tag{4.17}$$

This equation is always consistent, and the general solution of (4.17) *can be expressed as*

$$\mathbf{G} = [\,\mathbf{K},\, \mathbf{0}\,][\,\mathbf{X}_2' \otimes \mathbf{X}_1,\, (\mathbf{\Sigma}_2 \otimes \mathbf{\Sigma}_1)\mathbf{E}_{\mathbf{X}_2'\otimes\mathbf{X}_1}\,]^+ + \mathbf{U}\mathbf{E}_{[\,\mathbf{X}_2'\otimes\mathbf{X}_1,\, \mathbf{\Sigma}_2\otimes\mathbf{\Sigma}_1\,]}, \tag{4.18}$$

where $\mathbf{U} \in \mathbb{R}^{k\times pq}$ *is arbitrary.*

It is obvious that if $\mathbf{G}_{11}\mathbf{Y}\mathbf{G}_{21} + \cdots + \mathbf{G}_{1l}\mathbf{Y}\mathbf{G}_{2l}$ is a BLUE of $\mathbf{K}_{11}\mathbf{\Theta}\mathbf{K}_{21} + \cdots + \mathbf{K}_{1k}\mathbf{\Theta}\mathbf{K}_{2k}$ under (1.1), then $(\mathbf{G}_{21}' \otimes \mathbf{G}_{11} + \cdots + \mathbf{G}_{2l}' \otimes \mathbf{G}_{1l})\mathrm{vec}(\mathbf{Y})$ is a BLUE of $[\,\mathbf{K}_{21}' \otimes \mathbf{K}_{11} + \cdots + \mathbf{K}_{2k}' \otimes \mathbf{K}_{1k}\,]\mathrm{vec}(\mathbf{\Theta})$ under (1.3). Some special forms of BLUEs of $\mathbf{X}_1\mathbf{\Theta}\mathbf{X}_2$ in (1.1) with the form $\mathbf{G}_1\mathbf{Y}\mathbf{G}_2$ were given by Zhang and Zhu (2000).

Many problems on BLUEs of parametric functions under (1.1), such as their existence, general expressions, uniqueness, rank, range, additive and block decompositions are yet to be investigated.

5 Extensions to the extended growth curve model

A useful extension of the model in (1.1) is given by

$$\mathbf{Y} = \sum_{i=1}^{t} \mathbf{X}_{1i}\mathbf{\Theta}_i\mathbf{X}_{2i} + \boldsymbol{\varepsilon}, \quad E(\boldsymbol{\varepsilon}) = \mathbf{0}, \quad Cov[\mathrm{vec}(\boldsymbol{\varepsilon})] = \sigma^2(\mathbf{\Sigma}_2 \otimes \mathbf{\Sigma}_1), \tag{5.1}$$

where $\mathbf{Y} \in \mathbb{R}^{n\times m}$ is an observable random matrix, $\mathbf{X}_{1i} \in \mathbb{R}^{n\times p_i}$ and $\mathbf{X}_{2i} \in \mathbb{R}^{q_i\times m}$ are known matrices of arbitrary rank, $\mathbf{\Theta}_i \in \mathbb{R}^{p_i\times q_i}$ are matrices of unknown parameters, $\mathbf{\Sigma}_1 \in \mathbb{R}^{n\times n}$ and $\mathbf{\Sigma}_2 \in \mathbb{R}^{m\times m}$ are two known nonnegative definite matrices of arbitrary rank, and σ^2 is a positive unknown scalar. This mode is called the extended growth curve model in the literature. This kind of models have many applications in statistics and other disciplines; see, e.g., Kollo and von Rosen (2005), Seid Hamid and von Rosen (2006), Srivastava (2002), Takane and Hunter (2001), Takane, Kiers and de Leeuw (1995), Takane and Shibayama (1991), von Rosen (1989, 1991), and Verbyla and Venables (1988), among others.

The model in (5.1) can also be written as

$$\begin{cases} \mathbf{Y} = [\,\mathbf{X}_{11}, \ldots, \mathbf{X}_{1t}\,] \begin{bmatrix} \mathbf{\Theta}_1 & & \\ & \ddots & \\ & & \mathbf{\Theta}_t \end{bmatrix} \begin{bmatrix} \mathbf{X}_{21} \\ \vdots \\ \mathbf{X}_{2t} \end{bmatrix} + \boldsymbol{\varepsilon}, \\ E(\boldsymbol{\varepsilon}) = \mathbf{0},\ Cov[\mathrm{vec}(\boldsymbol{\varepsilon})] = \sigma^2(\mathbf{\Sigma}_2 \otimes \mathbf{\Sigma}_1). \end{cases} \tag{5.2}$$

Through the Kronecker product and the vec operation of matrices, (5.1) can alternatively be written as

$$\mathrm{vec}(\mathbf{Y}) = \mathbf{X}\boldsymbol{\theta} + \mathrm{vec}(\boldsymbol{\varepsilon}),\ E[\mathrm{vec}(\boldsymbol{\varepsilon})] = \mathbf{0},\ Cov[\mathrm{vec}(\boldsymbol{\varepsilon})] = \sigma^2(\mathbf{\Sigma}_2 \otimes \mathbf{\Sigma}_1), \tag{5.3}$$

where $\mathbf{X} = [\,\mathbf{X}_{21}' \otimes \mathbf{X}_{11}, \ldots, \mathbf{X}_{2t}' \otimes \mathbf{X}_{1t}\,]$ and $\boldsymbol{\theta} = [\,\mathrm{vec}'(\mathbf{\Theta}_1), \ldots, \mathrm{vec}'(\mathbf{\Theta}_t)\,]'$. It is straightforward to extend most of the results in the previous sections to the model in (5.1).

5.1 Consistency of the extended growth curve model

Definition 5.1 The extended growth curve model in (5.1) is said to be *consistent* if

$$\mathrm{vec}(\mathbf{Y}) \in [\,\mathbf{X}_{21}' \otimes \mathbf{X}_{11}, \ldots, \mathbf{X}_{2t}' \otimes \mathbf{X}_{1t},\, \mathbf{\Sigma}_2 \otimes \mathbf{\Sigma}_1\,] \tag{5.4}$$

holds with probability 1.

Theorem 5.2 *The following statements are equivalent*:

(a) *The extended growth curve model in* (5.1) *is consistent.*

(b) *The following matrix equation*

$$\mathbf{X}_{11}\mathbf{V}_1\mathbf{X}_{21} + \cdots + \mathbf{X}_{1t}\mathbf{V}_t\mathbf{X}_{2t} + \mathbf{\Sigma}_1\mathbf{W}\mathbf{\Sigma}_2 = \mathbf{Y} \tag{5.5}$$

is solvable for $\mathbf{V}_1, \ldots, \mathbf{V}_t$ *and* $\mathbf{W}$ *with probability* 1.

Proof The equivalence of (a) and (b) follows directly from the Kronecker product and the vec operation of matrices. □

For $t = 2$, the solvability condition of (5.5) can also be characterized by rank equalities and generalized inverses of matrices, for more details on this topic, see Tian (2000).

Definition 5.3 (a) A linear estimator $\mathbf{G}\mathrm{vec}(\mathbf{Y})$ is said to be the BLUE of parametric functions $\mathbf{K}\boldsymbol{\theta}$ under the transformed model in (5.3), denoted by $\mathrm{BLUE}(\mathbf{X}\boldsymbol{\theta})$, if $E[\mathbf{G}\mathrm{vec}(\mathbf{Y})] = \mathbf{X}\boldsymbol{\theta}$, and $Cov[\mathbf{L}\mathrm{vec}(\mathbf{Y})] - Cov[\mathbf{G}\mathrm{vec}(\mathbf{Y})]$ is nonnegative definite for any other unbiased estimator $\mathbf{L}\mathrm{vec}(\mathbf{Y})$ of $\mathbf{X}\boldsymbol{\theta}$.

(b) A linear estimator $\sum_{i=1}^{l} \mathbf{G}_{1i}\mathbf{Y}\mathbf{G}_{2i}$ is said to be the BLUE of $\sum_{i=1}^{t} \mathbf{K}_{1i}\boldsymbol{\Theta}_i\mathbf{K}_{2i}$ under (5.1) if $\sum_{i=1}^{l} \mathbf{G}_{1i}\mathbf{Y}\mathbf{G}_{2i}$ is unbiased for $\sum_{i=1}^{t} \mathbf{K}_{1i}\boldsymbol{\Theta}_i\mathbf{K}_{2i}$, and $Cov[\mathbf{L}\mathrm{vec}(\mathbf{Y})] - Cov[(\sum_{i=1}^{l} \mathbf{G}_{2i}' \otimes \mathbf{G}_{1i})\mathrm{vec}(\mathbf{Y})]$ is nonnegative definite for any other unbiased estimator $\mathbf{L}\mathrm{vec}(\mathbf{Y})$ of $\sum_{i=1}^{t} \mathbf{K}_{1i}\boldsymbol{\Theta}_i\mathbf{K}_{2i}$ under (5.1).

It is more challenging to study existence, general expressions, uniqueness, ranks, range, decompositions of BLUEs under (5.1).

5.2 Natural restrictions under the extended growth curve model

Theorem 5.4

(a) *The parameter matrices* $\boldsymbol{\Theta}_1, \ldots, \boldsymbol{\Theta}_t$ *in the growth curve model* (5.1) *satisfy the following matrix equation*

$$\mathbf{Y} - \mathbf{P}_{\boldsymbol{\Sigma}_1}\mathbf{Y}\mathbf{P}_{\boldsymbol{\Sigma}_2} = \sum_{i=1}^{t} \mathbf{X}_{1i}\boldsymbol{\Theta}_i\mathbf{X}_{2i} - \sum_{i=1}^{t} \mathbf{P}_{\boldsymbol{\Sigma}_1}\mathbf{X}_{1i}\boldsymbol{\Theta}_i\mathbf{X}_{2i}\mathbf{P}_{\boldsymbol{\Sigma}_2} \tag{5.6}$$

with probability 1.

(b) *The parameter matrices* $\boldsymbol{\Theta}_1, \ldots, \boldsymbol{\Theta}_t$ *in the growth curve model* (5.1) *satisfy the following pair of matrix equations*

$$\mathbf{E}_{\boldsymbol{\Sigma}_1}\mathbf{Y} = \sum_{i=1}^{t} \mathbf{E}_{\boldsymbol{\Sigma}_1}\mathbf{X}_{1i}\boldsymbol{\Theta}_i\mathbf{X}_{2i} \quad \textit{and} \quad \mathbf{Y}\mathbf{E}_{\boldsymbol{\Sigma}_2} = \sum_{i=1}^{t} \mathbf{X}_{1i}\boldsymbol{\Theta}_i\mathbf{X}_{2i}\mathbf{E}_{\boldsymbol{\Sigma}_2} \tag{5.7}$$

with probability 1.

(c) *The following statements are equivalent*:

(i) *The matrix equation in* (5.6) *is solvable for* $\boldsymbol{\Theta}_1, \ldots, \boldsymbol{\Theta}_t$ *with probability* 1.

(ii) *The pair of matrix equations in* (5.7) *have common solutions for* $\boldsymbol{\Theta}_1, \ldots, \boldsymbol{\Theta}_t$ *with probability* 1.

(iii) *The model in* (5.1) *is consistent.*

Proof It follows from (5.3), (3.2), and the Kronecker product and the vec operation of matrices. □

5.3 Estimability of parameter matrices under the extended growth curve model

Because there are t parameter matrices $\boldsymbol{\Theta}_1, \ldots, \boldsymbol{\Theta}_t$ in (5.1), a general linear matrix consisting of $\boldsymbol{\Theta}_1, \ldots, \boldsymbol{\Theta}_t$ may be written as

$$\sum_{i=1}^{k_1} \mathbf{K}_{1i1}\boldsymbol{\Theta}_1\mathbf{K}_{1i2} + \cdots + \sum_{i=1}^{i_k} \mathbf{K}_{ti1}\boldsymbol{\Theta}_t\mathbf{K}_{ti2}. \tag{5.8}$$

A special form of (5.8) is $\sum_{i=1}^{t} \mathrm{tr}(\mathbf{K}_i\boldsymbol{\Theta}_i)$. General expression of linear estimators under (5.1) can be written as (4.2).

As pointed out in Section 4, it is quite difficult to derive necessary and sufficient conditions for the unbiasedness of general linear estimators and the estimability of general parametric matrices. However, for the parametric function $\sum_{i=1}^{t} \mathrm{tr}(\mathbf{K}_i\boldsymbol{\Theta}_i)$ we have the following result.

Theorem 5.5 *Suppose that both* $\mathbf{\Sigma}_1$ *and* $\mathbf{\Sigma}_2$ *in* (5.1) *are positive definite, and let* $\mathbf{K}_1 \in \mathbb{R}^{q_1 \times p_1}, \ldots, \mathbf{K}_t \in \mathbb{R}^{q_t \times p_t}$ *be given. Then the following statements are equivalent*:

(a) *There exists a matrix* $\mathbf{L} \in \mathbb{R}^{m \times n}$ *such that* $\mathrm{tr}(\mathbf{LY})$ *is an unbiased estimator of* $\sum_{i=1}^{t} \mathrm{tr}(\mathbf{K}_i \mathbf{\Theta}_i)$ *under* (5.1), *i.e.*, $\sum_{i=1}^{t} \mathrm{tr}(\mathbf{K}_i \mathbf{\Theta}_i)$ *is estimable under* (5.1).

(b) *There exists a matrix* $\mathbf{L} \in \mathbb{R}^{m \times n}$ *such that* $\mathbf{X}_{21}\mathbf{L}\mathbf{X}_{11} = \mathbf{K}_1, \ldots, \mathbf{X}_{2t}\mathbf{L}\mathbf{X}_{1t} = \mathbf{K}_t$.

Proof Note that

$$E[\mathrm{tr}(\mathbf{LY})] = \mathrm{tr}[\mathbf{L}E(\mathbf{Y})] = \mathrm{tr}\left(\mathbf{L}\sum_{i=1}^{t} \mathbf{X}_{1i}\mathbf{\Theta}_i\mathbf{X}_{2i}\right) = \mathrm{tr}\left(\sum_{i=1}^{t} \mathbf{X}_{2i}\mathbf{L}\mathbf{X}_{1i}\mathbf{\Theta}_i\right).$$

Hence $E[\mathrm{tr}(\mathbf{LY})] = \sum_{i=1}^{t} \mathrm{tr}(\mathbf{K}_i \mathbf{\Theta}_i)$ is equivalent to

$$\mathrm{tr}\left(\sum_{i=1}^{t} \mathbf{X}_{2i}\mathbf{L}\mathbf{X}_{1i}\mathbf{\Theta}_i\right) = \sum_{i=1}^{t} \mathrm{tr}(\mathbf{K}_i \mathbf{\Theta}_i). \tag{5.9}$$

Since $\mathbf{\Theta}_1, \ldots, \mathbf{\Theta}_t$ are free parameter matrices under the given conditions, (5.9) is equivalent to $\mathbf{X}_{2i}\mathbf{L}\mathbf{X}_{1i} = \mathbf{K}_i$, $i = 1, \ldots t$. Hence (a) and (b) are equivalent. □

A special case of the model in (5.1) is

$$\mathbf{Y} = \mathbf{X}_{11}\mathbf{\Theta}_1\mathbf{X}_{21} + \mathbf{X}_{21}\mathbf{\Theta}_2\mathbf{X}_{22} + \boldsymbol{\varepsilon}, \quad E(\boldsymbol{\varepsilon}) = \mathbf{0}, \quad Cov[\mathrm{vec}(\boldsymbol{\varepsilon})] = \sigma^2(\mathbf{\Sigma}_2 \otimes \mathbf{\Sigma}_1). \tag{5.10}$$

Applying Theorem 5.5 to this model leads to the following result.

Corollary 5.6 *Suppose that both* $\mathbf{\Sigma}_1$ *and* $\mathbf{\Sigma}_2$ *in* (5.10) *are positive definite, and let* $\mathbf{K}_1 \in \mathbb{R}^{q_1 \times p_1}$ *and* $\mathbf{K}_2 \in \mathbb{R}^{q_2 \times p_2}$ *be given. Then the following statements are equivalent*:

(a) *There exists a matrix* $\mathbf{L} \in \mathbb{R}^{m \times n}$ *such that* $\mathrm{tr}(\mathbf{LY})$ *is an unbiased estimator of* $\mathrm{tr}(\mathbf{K}_1\mathbf{\Theta}_1 + \mathbf{K}_2\mathbf{\Theta}_2)$ *under* (5.10), *i.e.*, $\mathrm{tr}(\mathbf{K}_1\mathbf{\Theta}_1 + \mathbf{K}_2\mathbf{\Theta}_2)$ *is estimable.*

(b) *There exists a matrix* $\mathbf{L} \in \mathbb{R}^{m \times n}$ *such that* $\mathbf{X}_{21}\mathbf{L}\mathbf{X}_{11} = \mathbf{K}_1$ *and* $\mathbf{X}_{22}\mathbf{L}\mathbf{X}_{12} = \mathbf{K}_2$.

(c) $\mathscr{R}(\mathbf{K}_i) \subseteq \mathscr{R}(\mathbf{X}_{2i})$, $\mathscr{R}(\mathbf{K}_i') \subseteq \mathscr{R}(\mathbf{X}_{1i}')$, $i = 1, 2$, *and*

$$r\begin{bmatrix} \mathbf{K}_1 & \mathbf{0} & \mathbf{X}_{21} \\ \mathbf{0} & -\mathbf{K}_2 & \mathbf{X}_{22} \\ \mathbf{X}_{11} & \mathbf{X}_{12} & \mathbf{0} \end{bmatrix} = r\begin{bmatrix} \mathbf{X}_{21} \\ \mathbf{X}_{22} \end{bmatrix} + r[\,\mathbf{X}_{11},\, \mathbf{X}_{12}\,].$$

Proof The equivalence of (b) and (c) follows from Lemma 1.7. □

6 Concluding remarks

We have derived a variety of results on consistency, natural restrictions and estimability under classical and extended growth curve models. These results can be used to further investigate various problems associated with the growth curve models. In particular, they can be used to study various algebraic and statistical properties of estimators under the growth curve models. Some recent results on the weighted least-squares estimators of the parameter matrix $\mathbf{\Theta}$, the mean matrix $\mathbf{X}_1\mathbf{\Theta}\mathbf{X}_2$ and the linear transformation $\mathbf{K}_1\mathbf{\Theta}\mathbf{K}_2$ under the general growth curve model in (1.1) can be found in Tian and Takane (2007).

Finally, we propose an open problem on the estimability of the linear transformation $\mathbf{K}_1\mathbf{\Theta}\mathbf{K}_2$: In order for a given linear transformation $\mathbf{K}_1\mathbf{\Theta}\mathbf{K}_2$ to be estimable under (1.1) what are the possible restrictions that are need to impose to the parameter matrix $\mathbf{\Theta}$ in (1.1)?

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