More on generalized inverses of partitioned matrices with Banachiewicz-Schur forms

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Abstract. Necessary and sufficient conditions are derived for a 2-by-2 partitioned matrix to have {1}-, {1,2}-, {1,3}-, {1,4}-inverses and the Moore-Penrose inverse with Banachiewicz-Schur forms. As applications, the Banachiewicz-Schur forms of {1}-, {1,2}-, {1,3}-, {1,4}-inverses and the Moore-Penrose inverse of a 2-by-2 partitioned Hermitian matrix are also given.

Keywords: Banachiewicz-Schur form; partitioned matrix; generalized inverse; Moore-Penrose inverse; Hermitian matrix; Schur complement; matrix rank method

Mathematics Subject Classifications: 15A03; 15A09

1 Introduction

Throughout this paper, $\mathbb{C}^{m\times n}$ stands for the set of all $m \times n$ matrices over the field of complex numbers. The symbols A^*, $r(A)$ and $\mathscr{R}(A)$ stand for the conjugate transpose, the rank and the range (column space) of a matrix $A \in \mathbb{C}^{m\times n}$, respectively; $[\,A,\, B\,]$ denotes a row block matrix consisting of A and B. The Moore-Penrose inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^{\dagger}$, is defined to be the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the following four matrix equations

$$(1)\;\; AXA = A \;\;(2)\;\; XAX = X, \;\;(3)\;\; (AX)^* = AX, \;\;(4)\;\; (XA)^* = XA.$$

Further, let $E_A = I_m - AA^{\dagger}$ and $F_A = I_n - A^{\dagger}A$ stand for the two orthogonal projectors. A matrix X is called an $\{i, \dots, j\}$-inverse of A, denoted by $A^{(i,\dots,j)}$, if it satisfies the $i, \dots, j$th equations. The collection of all $\{i, \dots, j\}$-inverses of A is denoted by $\{A^{(i,\dots,j)}\}$. Some frequently used generalized inverses of A are $A^{(1)}$, $A^{(1,2)}$, $A^{(1,3)}$ and $A^{(1,4)}$.

Let M be a 2×2 block matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{1.1}$$

where $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$ and $D \in \mathbb{C}^{l\times k}$. If A in (1.1) is square and nonsingular, then M can be decomposed as

$$M = \begin{bmatrix} I_m & 0 \\ CA^{-1} & I_l \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I_m & A^{-1}B \\ 0 & I_l \end{bmatrix}. \tag{1.2}$$

This decomposition is often called Aitken block-diagonalization formula in the literature, see Puntanen and Styan [9]. Moreover, if both M and A are nonsingular, then the Schur complement $S = D - CA^{-1}B$ is nonsingular too, and the inverse of M can be written in the following form

$$\begin{aligned} M^{-1} &= \begin{bmatrix} I_m & -A^{-1}B \\ 0 & I_l \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} I_m & 0 \\ -CA^{-1} & I_l \end{bmatrix} \\ &= \begin{bmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{bmatrix}. \end{aligned} \tag{1.3}$$

This well-known formula is called the Banachiewicz inversion formula for the inverse of a nonsingular matrix in the literature, see Puntanen and Styan [9], and can be found in most linear algebra books. The two formulas in (1.2) and (1.3) and their consequences are widely used in manipulating partitioned matrices and their operations. When both A and M in (1.1) are singular, the

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two formulas in (1.2) and (1.3) can be extended to generalized inverses of matrices. A reasonable extension of (1.3) with generalized inverses of submatrices in (1.1) is given by

$$
\begin{aligned}
N(A^{(i,\dots,j)},\, S^{(i,\dots,j)}) &= \begin{bmatrix} I_n & -A^{(i,\dots,j)}B \\ 0 & I_k \end{bmatrix} \begin{bmatrix} A^{(i,\dots,j)} & 0 \\ 0 & S^{(i,\dots,j)} \end{bmatrix} \begin{bmatrix} I_m & 0 \\ -CA^{(i,\dots,j)} & I_l \end{bmatrix} \\
&= \begin{bmatrix} A^{(i,\dots,j)} + A^{(i,\dots,j)}BS^{(i,\dots,j)}CA^{(i,\dots,j)} & -A^{(i,\dots,j)}BS^{(i,\dots,j)} \\ -S^{(i,\dots,j)}CA^{(i,\dots,j)} & S^{(i,\dots,j)} \end{bmatrix}, \qquad (1.4)
\end{aligned}
$$

where $S = D - CA^{(i,\dots,j)}B$. Eq. (1.4) is called the Banachiewicz-Schur form induced from M, see Baksalary and Styan [1]. It can be seen from (1.4) that the matrix $N(A^{(i,\dots,j)},\, S^{(i,\dots,j)})$ varies over the choice of $A^{(i,\dots,j)}$ and $S^{(i,\dots,j)}$. Let $\{N(A^{(i,\dots,j)},\, S^{(i,\dots,j)})\}$ denote the collection of all $N(A^{(i,\dots,j)},\, S^{(i,\dots,j)})$.

Although the right-hand side of (1.4) is obtained by replacing inverses with generalized inverses, it is not necessarily an $\{i, \dots, j\}$-inverse of M. In this case, it is of interest to investigate relations between generalized inverses of M in (1.1) and the matrix $N(A^{(i,\dots,j)},\, S^{(i,\dots,j)})$ in (1.4), in particular, to derive necessary and sufficient conditions for $N(A^{(i,\dots,j)},\, S^{(i,\dots,j)})$ to be generalized inverses of M in (1.1). some authors have investigated the relations between $M^{(i,\dots,j)}$ and $N(A^{(i,\dots,j)},\, S^{(i,\dots,j)})$ for some special choices of $\{i, \dots, j\}$. A well-known result asserts that

$$
M^{\dagger} = N(A^{\dagger},\, S^{\dagger}) \Leftrightarrow \mathscr{R}(B) \subseteq \mathscr{R}(A),\ \mathscr{R}(C^*) \subseteq \mathscr{R}(A^*),\ \mathscr{R}(C) \subseteq \mathscr{R}(S) \text{ and } \mathscr{R}(B^*) \subseteq \mathscr{R}(S^*), \quad (1.5)
$$

where $S = D - CA^{\dagger}B$; see, e.g., [1]. Other results can be found , e.g., in [2, 4, 5, 8, 15, 16]. In a recent paper [16], we considered relations between $M^{(1)}$ and $N(A^{(1)},\, S^{(1)})$ through the matrix rank method and obtain the following two rank formulas

$$
\begin{aligned}
&\min_{A^{(1)},\, M^{(1)}} r[\, M^{(1)} - N(A^{(1)},\, S^{(1)})\,] \\
&= \max\left\{ r(M) - r(A) - r[\, C,\, D\,],\ \ r(M) - r(A) - r\begin{bmatrix} B \\ D \end{bmatrix},\ \ 0 \right\}, \qquad (1.6) \\
&\max_{A^{(1)}} \min_{M^{(1)}} r[\, M^{(1)} - N(A^{(1)},\, S^{(1)})\,] \\
&= r\begin{bmatrix} A & 0 \\ 0 & B \\ C & D \end{bmatrix} + r\begin{bmatrix} A & 0 & B \\ 0 & C & D \end{bmatrix} - r\begin{bmatrix} B \\ D \end{bmatrix} - r[\, C,\, D\,] - 2r(A). \qquad (1.7)
\end{aligned}
$$

By setting the right-hand sides of the two rank equalities to zero, we obtain that

(a) There exist $A^{(1)}$ and $S^{(1)}$ such that $N(A^{(1)},\, S^{(1)})$ is a $\{1\}$-inverse of M if and only if

$$
r(M) \leqslant \min\left\{ r(A) + r[\, C,\, D\,],\ \ r(A) + r\begin{bmatrix} B \\ D \end{bmatrix} \right\} \qquad (1.8)
$$

holds.

(b) The set inclusion $\{N(A^{(1)},\, S^{(1)})\} \subseteq \{M^{(1)}\}$ holds if and only if

$$
r\begin{bmatrix} A & 0 & B \\ 0 & C & D \end{bmatrix} = r(A) + r[\, C,\, D\,] \ \text{ and } \ r\begin{bmatrix} A & 0 \\ 0 & B \\ C & D \end{bmatrix} = r(A) + r\begin{bmatrix} B \\ D \end{bmatrix}, \qquad (1.9)
$$

hold, or equivalently,

$$
\mathscr{R}\begin{bmatrix} 0 \\ (E_A B)^* \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} C^* \\ D^* \end{bmatrix} \ \text{ and } \ \mathscr{R}\begin{bmatrix} 0 \\ CF_A \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} B \\ D \end{bmatrix}
$$

hold.

As an extension of the previous investigation, we derive in this paper some rank formulas for the difference

$$M^{(i,\dots,j)} - N(A^{(i,\dots,j)},\, S^{(i,\dots,j)}) \tag{1.10}$$

for {1,2}-, {1,3}-, {1,4}-inverses and the Moore-Penroses of matrices, and use the rank formulas to characterize the following relations

$$\{N(A^{(i,\dots,j)},\, S^{(i,\dots,j)})\} \cap \{M^{(i,\dots,j)}\} \neq \emptyset, \tag{1.11}$$

$$\{N(A^{(i,\dots,j)},\, S^{(i,\dots,j)})\} \subseteq \{M^{(i,\dots,j)}\}. \tag{1.12}$$

In order to establish rank equalities associated with (1.10), we need a variety of rank formulas for partitioned matrices and generalized Schur complements.

Lemma 1.1 ([7]) *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$ *and* $D \in \mathbb{C}^{l\times k}$. *Then*

$$r[\, A,\, B\,] = r(A) + r(B - AA^{(1)}B) = r(B) + r(A - BB^{(1)}A), \tag{1.13}$$

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(\, C - CA^{(1)}A\,) = r(C) + r(\, A - AC^{(1)}C\,), \tag{1.14}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r[\,(I_m - BB^{(1)})A(I_n - C^{(1)}C)\,], \tag{1.15}$$

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) + r\begin{bmatrix} 0 & B - AA^{(1)}B \\ C - CA^{(1)}A & D - CA^{(1)}B \end{bmatrix}. \tag{1.16}$$

Lemma 1.2 ([12, 14]) *Let* M *be as given in* (1.1). *Then*

$$\min_{A^{(1)}} r(\, D - CA^{(1)}B\,) = r(A) + r[\, C,\, D\,] + r\begin{bmatrix} B \\ D \end{bmatrix} + r\begin{bmatrix} A & B \\ C & D \end{bmatrix} - r\begin{bmatrix} A & 0 & B \\ 0 & C & D \end{bmatrix} - r\begin{bmatrix} A & 0 \\ 0 & B \\ C & D \end{bmatrix}, \tag{1.17}$$

$$\max_{A^{(1)}} r(\, D - CA^{(1)}B\,) = \min\left\{ r[\, C,\, D\,],\ \ r\begin{bmatrix} B \\ D \end{bmatrix},\ \ r\begin{bmatrix} A & B \\ C & D \end{bmatrix} - r(A) \right\}, \tag{1.18}$$

$$\min_{A^{(1,2)}} r(\, D - CA^{(1,2)}B\,) = r\begin{bmatrix} B \\ D \end{bmatrix} + r[\, C,\ D\,] + r(A) + \max\{\, r_1,\ r_2\,\}, \tag{1.19}$$

$$\max_{A^{(1,2)}} r(\, D - CA^{(1,2)}B\,) = \min\left\{ r(A) + r(D),\ r[\, C,\, D\,],\ r\begin{bmatrix} B \\ D \end{bmatrix},\ r\begin{bmatrix} A & B \\ C & D \end{bmatrix} - r(A) \right\}, \tag{1.20}$$

$$\min_{A^{(1,3)}} r(\, D - CA^{(1,3)}B\,) = r\begin{bmatrix} A^*A & A^*B \\ C & D \end{bmatrix} + r\begin{bmatrix} B \\ D \end{bmatrix} - r\begin{bmatrix} A & 0 \\ 0 & B \\ C & D \end{bmatrix}, \tag{1.21}$$

$$\max_{A^{(1,3)}} r(\, D - CA^{(1,3)}B\,) = \min\left\{ r\begin{bmatrix} A^*A & A^*B \\ C & D \end{bmatrix} - r(A),\ \ \ r\begin{bmatrix} B \\ D \end{bmatrix} \right\}, \tag{1.22}$$

$$\min_{A^{(1,4)}} r(\, D - CA^{(1,4)}B\,) = r[\, C,\ D\,] + r\begin{bmatrix} AA^* & B \\ CA^* & D \end{bmatrix} - r\begin{bmatrix} A & 0 & B \\ 0 & C & D \end{bmatrix}, \tag{1.23}$$

$$\max_{A^{(1,4)}} r(\, D - CA^{(1,4)}B\,) = \min\left\{ r[\, C,\ D\,],\ r\begin{bmatrix} AA^* & B \\ CA^* & D \end{bmatrix} - r(A) \right\}, \tag{1.24}$$

$$r(\, D - CA^{\dagger}B\,) = r\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix} - r(A), \tag{1.25}$$

where

$$r_1 = r\begin{bmatrix} A & B \\ C & D \end{bmatrix} - r\begin{bmatrix} A & 0 & B \\ 0 & C & D \end{bmatrix} - r\begin{bmatrix} A & 0 \\ 0 & B \\ C & D \end{bmatrix}, \quad r_2 = r(D) - r\begin{bmatrix} A & 0 \\ C & D \end{bmatrix} - r\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}.$$

The following lemma is derived from Lemma 1.2 by setting B and C to identity matrices in (1.17), (1.19), (1.21) and (1.23).

Lemma 1.3 *Let $A \in \mathbb{C}^{m\times n}$ and $D \in \mathbb{C}^{n\times m}$. Then*

$$\min_{A^{(1)}} r(\, A^{(1)} - D\,) = r(\, A - ADA\,), \tag{1.26}$$

$$\min_{A^{(1,2)}} r(\, A^{(1,2)} - D\,) = \max\{\, r(\, A - ADA\,),\ r(D) + r(A) - r(DA) - r(AD)\,\}, \tag{1.27}$$

$$\min_{A^{(1,3)}} r(\, A^{(1,3)} - D\,) = r(\, A^*AD - A^*\,), \tag{1.28}$$

$$\min_{A^{(1,4)}} r(\, A^{(1,4)} - D\,) = r(\, DAA^* - A^*\,). \tag{1.29}$$

2 Generalized inverses of partitioned matrices with Banachiewicz-Schur forms

We first give two rank formulas for the difference $M^{(1,2)} - N(A^{(1,2)},\, S^{(1,2)})$.

Theorem 2.1 *Let M and $N(A^{(1,2)},\, S^{(1,2)})$ be as given in* (1.1) *and* (1.4), *respectively, where $S = D - CA^{(1,2)}B$. Then*

$$\min_{A^{(1,2)},\, M^{(1,2)}} r[\, M^{(1,2)} - N(A^{(1,2)},\, S^{(1,2)})\,] = \max\{r_1,\ \ r_2,\ \ r_3,\ \ 0\}, \tag{2.1}$$

$$\max_{A^{(1,2)}} \min_{M^{(1,2)}} r[\, M^{(1,2)} - N(A^{(1,2)},\, S^{(1,2)})\,] = \min\{\, s_1,\ \ s_2\,\}, \tag{2.2}$$

where

$$\begin{aligned}
r_1 &= r(M) - 2r(A) - r(D),\\
r_2 &= r(M) - r(A) - r[\, C,\, D\,],\\
r_3 &= r(M) - r(A) - r\begin{bmatrix} B \\ D \end{bmatrix},\\
s_1 &= r\begin{bmatrix} A & 0 \\ 0 & B \\ C & D \end{bmatrix} + r\begin{bmatrix} A & 0 & B \\ 0 & C & D \end{bmatrix} - r\begin{bmatrix} B \\ D \end{bmatrix} - r[\, C,\ D\,] - 2r(A),\\
s_2 &= r\begin{bmatrix} A & 0 \\ C & D \end{bmatrix} + r\begin{bmatrix} A & B \\ 0 & D \end{bmatrix} + r(M) - 2r(A) - r(D) - r[\, C,\ D\,] - r\begin{bmatrix} B \\ D \end{bmatrix}.
\end{aligned}$$

Hence,

(a) [16] *There exist $A^{(1,2)}$ and $S^{(1,2)}$ such that $N(A^{(1,2)},\, S^{(1,2)})$ is a $\{1,2\}$-inverse of M if and only if*

$$r(M) \leqslant \min\left\{ 2r(A) + r(D),\ \ r(A) + r\begin{bmatrix} B \\ D \end{bmatrix},\ \ r(A) + r[\, C,\, D\,] \right\}. \tag{2.3}$$

(b) *The set inclusion $\{N(A^{(1,2)},\, S^{(1,2)})\} \subseteq \{M^{(1,2)}\}$ holds if and only if* (1.9) *holds, or*

$$r(M) = r\begin{bmatrix} B \\ D \end{bmatrix} + r[\, C,\, D\,] - r(D) \ \text{and}\ r\begin{bmatrix} A & 0 \\ C & D \end{bmatrix} = r\begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = r(A) + r(D) \tag{2.4}$$

hold.

Proof It was shown in [16] that

$$\min_{M^{(1,2)}} r[\, M^{(1,2)} - N(A^{(1,2)},\, S^{(1,2)})\,] = r(M) - r(A) - r(\, D - CA^{(1,2)}B\,), \tag{2.5}$$

so that

$$\min_{A^{(1,2)},M^{(1,2)}} r[\,M^{(1,2)} - N(A^{(1,2)},\, S^{(1,2)})\,] = r(M) - r(A) - \max_{A^{(1,2)}} r(\,D - CA^{(1,2)}B\,), \tag{2.6}$$

$$\max_{A^{(1,2)}} \min_{M^{(1,2)}} r[\,M^{(1,2)} - N(A^{(1,2)},\, S^{(1,2)})\,] = r(M) - r(A) - \min_{A^{(1,2)}} r(\,D - CA^{(1,2)}B\,). \tag{2.7}$$

Substituting (1.19) and (1.20) into (2.6) and (2.7) gives (2.1) and (2.2). Setting the right-hand side of (2.1) to zero leads to $r_1 \le 0$, $r_2 \le 0$ and $r_3 \le 0$, that is, (2.3) holds. Setting the right-hand side of (2.2) to zero leads to $s_1 \le 0$ or $s_2 \le 0$. It can be seen from (1.7) that $s_1 \le 0$ is equivalent to (1.9). Note that s_2 in (2.2) can be rewritten as a sum of three parts

$$\begin{aligned} s_2 = {} & \left(r\begin{bmatrix} A & 0 \\ C & D \end{bmatrix} - r(A) - r(D) \right) + \left(r\begin{bmatrix} A & B \\ 0 & D \end{bmatrix} - r(A) - r(D) \right) \\ & + \left(r(M) + r(D) - r[\,C,\, D\,] - r\begin{bmatrix} B \\ D \end{bmatrix} \right), \end{aligned}$$

where each part is nonnegative. In this case, Setting $s_2 = 0$ leads to (2.4). $\square$

Theorem 2.2 *Let M and $N(A^{(1,3)},\, S^{(1,3)})$ be as given in* (1.1) *and* (1.4). *Then*

$$\min_{A^{(1,3)},\, M^{(1,3)}} r[\,M^{(1,3)} - N(A^{(1,3)},\, S^{(1,3)})\,] = \max\{\, r_1, \quad r_2\,\}, \tag{2.8}$$

$$\max_{A^{(1,3)}} \min_{M^{(1,3)}} r[\,M^{(1,3)} - N(A^{(1,3)},\, S^{(1,3)})\,] = r_1 + r_3, \tag{2.9}$$

where

$$r_1 = r[\,A,\, B\,] + r[\,C,\, D\,] - r\begin{bmatrix} A^*A & A^*B \\ C & D \end{bmatrix},$$

$$r_2 = r[\,A,\, B\,] + r[\,C,\, D\,] - r(A) - r\begin{bmatrix} B \\ D \end{bmatrix},$$

$$r_3 = r\begin{bmatrix} A & 0 \\ 0 & B \\ C & D \end{bmatrix} - r\begin{bmatrix} B \\ D \end{bmatrix} - r(A).$$

Hence,

(a) [16] *There exist $A^{(1,3)}$ and $S^{(1,3)}$ such that $N(A^{(1,3)},\, S^{(1,3)})$ is a $\{1,3\}$-inverse of M if and only if*

$$\mathscr{R}(B) \subseteq \mathscr{R}(A), \quad \mathscr{R}\begin{bmatrix} A^*A \\ B^*A \end{bmatrix} \cap \mathscr{R}\begin{bmatrix} C^* \\ D^* \end{bmatrix} = \{0\} \quad \text{and} \quad r[\,C,\, D\,] \leqslant r\begin{bmatrix} B \\ D \end{bmatrix}$$

hold.

(b) *The set inclusion $\{N(A^{(1,3)},\, S^{(1,3)})\} \subseteq \{M^{(1,3)}\}$ holds if and only if*

$$\mathscr{R}(B) \subseteq \mathscr{R}(A), \quad \mathscr{R}\begin{bmatrix} A^*A \\ B^*A \end{bmatrix} \cap \mathscr{R}\begin{bmatrix} C^* \\ D^* \end{bmatrix} = \{0\} \quad \text{and} \quad \mathscr{R}(CF_A) \subseteq \mathscr{R}(DF_B)$$

hold.

Proof The following formula

$$\min_{M^{(1,3)}} r[\,M^{(1,3)} - N(A^{(1,3)},\, S^{(1,3)})\,] = r[\,A,\, B\,] + r[\,C,\, D\,] - r(A) - r(\,D - CA^{(1,3)}B\,) \tag{2.10}$$

was shown in [16]. Substituting (1.21) and (1.22) into (2.10) gives (2.8) and (2.9). Setting the right-hand side of (2.8) to zero, we see that there exist $A^{(1,3)}$ and $S^{(1,3)}$ such that $N(A^{(1,3)},\, S^{(1,3)}) \in \{M^{(1,3)}\}$ if and only if

$$r[\,A,\, B\,] + r[\,C,\, D\,] = r\begin{bmatrix} A^*A & A^*B \\ C & D \end{bmatrix} \quad \text{and} \quad r[\,A,\, B\,] + r[\,C,\, D\,] \leqslant r(A) + r\begin{bmatrix} B \\ D \end{bmatrix}. \tag{2.11}$$

Note that

$$r[A,\, B] + r[C,\, D] \geqslant r(A) + r[C,\, D] \geqslant r\begin{bmatrix} A^*A & A^*B \\ C & D \end{bmatrix}.$$

Hence, the first equality in (2.11) is equivalent to

$$\mathscr{R}(B) \subseteq \mathscr{R}(A) \ \text{ and } \ r\begin{bmatrix} A^*A & A^*B \\ C & D \end{bmatrix} = r[AA^*,\, A^*B] + r[C,\, D],$$

and the second inequality in (2.11) is equivalent to $r[C,\, D] \leqslant r\begin{bmatrix} B \\ D \end{bmatrix}$. Setting the right-hand side of (2.9) to zero and noting that the two terms on the right-hand side of (2.9) are nonnegative, we obtain (b). $\square$

The following theorem can be shown similarly.

Theorem 2.3 *Let M and $N(A^{(1,4)},\, S^{(1,4)})$ be as given in* (1.1) *and* (1.4). *Then*

$$\min_{A^{(1,4)},\, M^{(1,4)}} r[M^{(1,4)} - N(A^{(1,4)},\, S^{(1,4)})] = \max\{r_1, \ \ r_2\},$$

$$\max_{A^{(1,4)}} \min_{M^{(1,4)}} r[M^{(1,4)} - N(A^{(1,4)},\, S^{(1,4)})] = r_1 + r_3,$$

where

$$r_1 = r\begin{bmatrix} A \\ C \end{bmatrix} + r\begin{bmatrix} B \\ D \end{bmatrix} - r\begin{bmatrix} AA^* & B \\ CA^* & D \end{bmatrix},$$

$$r_2 = r\begin{bmatrix} A \\ C \end{bmatrix} + r\begin{bmatrix} B \\ D \end{bmatrix} - r(A) - r[C,\, D],$$

$$r_3 = r\begin{bmatrix} A & 0 & B \\ 0 & C & D \end{bmatrix} - r[C,\, D] - r(A).$$

Hence,

(a) [16] *There exist $A^{(1,4)}$ and $S^{(1,4)}$ such that $N(A^{(1,4)},\, S^{(1,4)})$ is a $\{1,4\}$-inverse of M if and only if*

$$\mathscr{R}(C^*) \subseteq \mathscr{R}(A^*), \quad \mathscr{R}\begin{bmatrix} AA^* \\ CA^* \end{bmatrix} \cap \mathscr{R}\begin{bmatrix} B \\ D \end{bmatrix} = \{0\} \ \ \textit{and} \ \ r\begin{bmatrix} B \\ D \end{bmatrix} \leqslant r[C,\, D]$$

hold.

(b) *The set inclusion $\{N(A^{(1,4)},\, S^{(1,4)})\} \subseteq \{M^{(1,4)}\}$ holds if and only if*

$$\mathscr{R}(C^*) \subseteq \mathscr{R}(A^*), \quad \mathscr{R}\begin{bmatrix} AA^* \\ CA^* \end{bmatrix} \cap \mathscr{R}\begin{bmatrix} B \\ D \end{bmatrix} = \{0\} \ \ \textit{and} \ \ \mathscr{R}(B^*E_A) \subseteq \mathscr{R}(D^*E_C)$$

hold.

A special case of (1.4) corresponding to the Moore-Penrose inverse is given by

$$N(A^\dagger,\, S^\dagger) = \begin{bmatrix} A^\dagger + A^\dagger B S^\dagger C A^\dagger & -A^\dagger B S^\dagger \\ -S^\dagger C A^\dagger & S^\dagger \end{bmatrix}, \tag{2.12}$$

where $S = D - CA^\dagger B$. The relations between $N(A^\dagger,\, S^\dagger)$ and $\{i, \ldots, j\}$-inverse of M are given in the following theorems.

Theorem 2.4 *Let M and $N(A^\dagger,\, S^\dagger)$ be as given in* (1.1) *and* (2.12), *respectively. Then*

$$\min_{M^{(1)}} r[M^{(1)} - N(A^\dagger,\, S^\dagger)] = \min_{M^{(1,2)}} r[M^{(1,2)} - N(A^\dagger,\, S^\dagger)] = r(M) - r\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix}. \tag{2.13}$$

Hence, the following statements are equivalent:

(a) $N(A^\dagger,\, S^\dagger)$ *is a* $\{1\}$*-inverse of* M.

(b) $N(A^\dagger,\, S^\dagger)$ *is a* $\{1,2\}$*-inverse of* M.

(c) $r(M) = r(A) + r(\, D - CA^\dagger B\,)$.

(d) $r(M) = r\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix}$.

(e) $\mathscr{R}\begin{bmatrix} AA^* & B \\ CA^* & D \end{bmatrix} = \mathscr{R}(M)$ *and* $\mathscr{R}\begin{bmatrix} A^*A & C^* \\ B^*A & D^* \end{bmatrix} = \mathscr{R}(M^*)$.

Proof It follows from (1.26) and (1.27) that

$$\min_{M^{(1)}} r[\, M^{(1)} - N(A^\dagger,\, S^\dagger)\,] = r[\, M - MN(A^\dagger,\, S^\dagger)M\,], \tag{2.14}$$

$$\begin{aligned}\min_{M^{(1,2)}} r[\, M^{(1,2)} - N(A^\dagger,\, S^\dagger)\,] = \max\{\, & r[\, M - MN(A^\dagger,\, S^\dagger)M\,], \;\; r[N(A^\dagger,\, S^\dagger)] + r(M) \\ & - r[MN(A^\dagger,\, S^\dagger)] - r[N(A^\dagger,\, S^\dagger)M]\,\}. \end{aligned}\tag{2.15}$$

It is also easy to verify that

$$r[MN(A^\dagger,\, S^\dagger)] = r[N(A^\dagger,\, S^\dagger)M] = r(A) + r(S),$$
$$r[\, M - MN(A^\dagger,\, S^\dagger)M\,] = r(M) - r(A) - r(S),$$

where $S = D - CA^\dagger B$. Substituting these two equalities and (1.25) into (2.14) and (2.15) gives (2.13). The equivalence of (a), (b), (c) and (d) follows from (2.13). Recall a simple fact that

$$r(PAQ) = r(A) \Leftrightarrow \mathscr{R}(A^*P^*) = \mathscr{R}(A^*) \text{ and } \mathscr{R}(AQ) = \mathscr{R}(A).$$

Applying this result to (d) gives the equivalence of (d) and (e). $\square$

Theorem 2.5 *Let* M *and* $N(A^\dagger,\, S^\dagger)$ *be as given in* (1.1) *and* (2.12), *respectively. Then*

$$\min_{M^{(1,3)}} r[\, M^{(1,3)} - N(A^\dagger,\, S^\dagger)\,] = r[\, A,\, B\,] + r[\, C,\, D\,] - r\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix}. \tag{2.16}$$

Hence, the following statements are equivalent:

(a) $N(A^\dagger,\, S^\dagger)$ *is a* $\{1,3\}$*-inverse of* M.

(b) $r\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix} = r[\, A,\, B\,] + r[\, C,\, D\,]$.

(c) $r\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix} = r[\, A^*AA^*,\, A^*B\,] + r[\, CA^*,\, D\,]$, $r[\, C,\, D\,] = r[\, CA^*,\, D\,]$ *and* $r[\, A,\, B\,] = r[\, A^*AA^*,\, A^*B\,]$.

(d) $\mathscr{R}\begin{bmatrix} AA^*A \\ B^*A \end{bmatrix} \cap \mathscr{R}\begin{bmatrix} AC^* \\ D^* \end{bmatrix} = \{0\}$, $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ *and* $\mathscr{R}[\, C,\, D\,] = \mathscr{R}[\, CA^*,\, D\,]$.

Proof It follows from (1.28) that

$$\min_{M^{(1,3)}} r[\, M^{(1,3)} - N(A^\dagger,\, S^\dagger)\,] = r[\, M^*MN(A^\dagger,\, S^\dagger) - M^*\,]. \tag{2.17}$$

It is easy to verify that

$$MN(A^\dagger,\, S^\dagger) = \begin{bmatrix} AA^\dagger & E_ABS^\dagger \\ CA^\dagger & SS^\dagger \end{bmatrix}\begin{bmatrix} I_m & 0 \\ -CA^\dagger & I_l \end{bmatrix},$$

$$\begin{aligned} M^*MN(A^\dagger,\, S^\dagger) - M^* &= -M^*[\, I_{m+l} - MN(A^\dagger,\, S^\dagger)\,] \\ &= -M^*\begin{bmatrix} E_A + E_ABS^\dagger CA^\dagger & -E_ABS^\dagger \\ -E_SCA^\dagger & E_S \end{bmatrix}. \end{aligned}$$

Recall that elementary block matrix operations (EBMOs) don't change the rank of a matrix. Hence we can derive by elementary block matrix operations that

$$\begin{aligned} r[\, M^*MN(A^\dagger,\, S^\dagger) - M^* \,] &= r\left(M^* \begin{bmatrix} E_A + E_ABS^\dagger CA^\dagger & -E_ABS^\dagger \\ -E_SCA^\dagger & E_S \end{bmatrix} \right) \\ &= r\left(M^* \begin{bmatrix} E_A & 0 \\ 0 & E_S \end{bmatrix} \right) \\ &= r\begin{bmatrix} E_AA & E_AB \\ E_SC & E_SD \end{bmatrix} \\ &= r\begin{bmatrix} A & 0 & A & B \\ 0 & S & C & D \end{bmatrix} - r(A) - r(S) \quad \text{(by (1.13))} \\ &= r\begin{bmatrix} A & 0 & 0 & B \\ 0 & D & C & 0 \end{bmatrix} - r(A) - r(S) \\ &= r[\, A,\, B\,] + r[\, C,\, D\,] - r(A) - r(\, D - CA^\dagger B\,). \end{aligned}$$

Thus we have (2.16) by (1.25). Also note that

$$r\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix} \leqslant r[\, A^*AA^*,\, A^*B\,] + r[\, CA^*,\, D\,] \leqslant r[\, A,\, B\,] + r[\, C,\, D\,].$$

Applying this inequality to (b) leads to the equivalence of (b) and (c). The equivalence of (c) and (d) is obvious. $\square$

The following result can be shown similarly.

Theorem 2.6 *Let M and $N(A^\dagger,\, S^\dagger)$ be as given in* (1.1) *and* (2.12), *respectively. Then*

$$\min_{M^{(1,4)}} r[\, M^{(1,4)} - N(A^\dagger,\, S^\dagger)\,] = r\begin{bmatrix} A \\ C \end{bmatrix} + r\begin{bmatrix} B \\ D \end{bmatrix} - r\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix}.$$

Hence, the following statements are equivalent:

(a) $N(A^\dagger,\, S^\dagger)$ *is a* $\{1, 4\}$*-inverse of* M.

(b) $r\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix} = r\begin{bmatrix} A \\ C \end{bmatrix} + r\begin{bmatrix} B \\ D \end{bmatrix}$.

(c) $r\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix} = r\begin{bmatrix} A^*AA^* \\ CA^* \end{bmatrix} + r\begin{bmatrix} A^*B \\ D \end{bmatrix}$, $r\begin{bmatrix} A^*AA^* \\ CA^* \end{bmatrix} = r\begin{bmatrix} A \\ C \end{bmatrix}$ *and* $r\begin{bmatrix} A^*B \\ D \end{bmatrix} = r\begin{bmatrix} B \\ D \end{bmatrix}$.

(d) $\mathscr{R}\begin{bmatrix} A^*AA^* \\ CA^* \end{bmatrix} \cap \mathscr{R}\begin{bmatrix} A^*B \\ D \end{bmatrix} = \{0\}$, $\mathscr{R}[\, B^*A,\, D^*\,] = \mathscr{R}[\, B^*,\, D^*\,]$ *and* $\mathscr{R}(C^*) \subseteq \mathscr{R}(A^*)$.

3 Generalized inverses of partitioned Hermitian matrices

Let M be an Hermitian matrix, and partition M as

$$M = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}, \tag{3.1}$$

where $A = A^* \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{m\times n}$ and $D = D^* \in \mathbb{C}^{n\times n}$. The Banachiewicz-Schur form induced from M is

$$N(A^{(i,\dots,j)},\, S^{(i,\dots,j)}) = \begin{bmatrix} A^{(i,\dots,j)} + A^{(i,\dots,j)}BS^{(i,\dots,j)}B^*A^{(i,\dots,j)} & -A^{(i,\dots,j)}BS^{(i,\dots,j)} \\ -S^{(i,\dots,j)}B^*A^{(i,\dots,j)} & S^{(i,\dots,j)} \end{bmatrix}, \tag{3.2}$$

where $S = D - B^*A^{(i,\dots,j)}B$. Applying the results in Section 2 to (3.1) and (3.2) gives the following results.

Theorem 3.1 *Let M and $N(A^{(1)},\, S^{(1)})$ be as given in* (3.1) *and* (3.2). *Then*

$$\min_{A^{(1)},\, M^{(1)}} r[\, M^{(1)} - N(A^{(1)},\, S^{(1)})\,] = \max\{r(M) - r(A) - r[\, B^*,\, D\,],\ \ 0\},$$

$$\max_{A^{(1)}}\min_{M^{(1)}} r[\, M^{(1)} - N(A^{(1)},\, S^{(1)})\,] = 2r\begin{bmatrix} A & 0 & B \\ 0 & B^* & D \end{bmatrix} - 2r(A) - 2r[\, B^*,\, D\,].$$

Hence,

(a) *There exist $A^{(1)}$ and $S^{(1)}$ such that $N(A^{(1)},\, S^{(1)})$ is a $\{1\}$-inverse of M if and only if*

$$r(M) \leqslant r(A) + r[\, B^*,\, D\,].$$

(b) *The set inclusion $\{N(A^{(1)},\, S^{(1)})\} \subseteq \{M^{(1)}\}$ holds if and only if*

$$r\begin{bmatrix} A & 0 & B \\ 0 & B^* & D \end{bmatrix} = r(A) + r[\, B^*,\, D\,].$$

Proof It follows from (1.6) and (1.7) by setting $C = B^*$. □

Theorem 3.2 *Let M and $N(A^{(1,2)},\, S^{(1,2)})$ be as given in* (3.1) *and* (3.2), *where $S = D - B^*A^{(1,2)}B$. Then*

$$\min_{A^{(1,2)},\, M^{(1,2)}} r[\, M^{(1,2)} - N(A^{(1,2)},\, S^{(1,2)})\,] = \max\{r_1,\ \ r_2,\ \ 0\},$$

$$\max_{A^{(1,2)}}\min_{M^{(1,2)}} r[\, M^{(1,2)} - N(A^{(1,2)},\, S^{(1,2)})\,] = \min\{\, s_1,\ \ s_2\,\},$$

where

$$r_1 = r(M) - 2r(A) - r(D),$$
$$r_2 = r(M) - r(A) - r[\, B^*,\, D\,],$$
$$s_1 = 2r\begin{bmatrix} A & 0 & B \\ 0 & B^* & D \end{bmatrix} - 2r[\, B^*,\ D\,] - 2r(A),$$
$$s_2 = 2r\begin{bmatrix} A & B \\ 0 & D \end{bmatrix} + r(M) - 2r(A) - r(D) - 2r[\, B^*,\ D\,].$$

Hence,

(a) *There exist $A^{(1,2)}$ and $S^{(1,2)}$ such that $N(A^{(1,2)},\, S^{(1,2)})$ is a $\{1,2\}$-inverse of M if and only if*

$$r(M) \leqslant \min\{\, 2r(A) + r(D),\ \ r(A) + r[\, B^*,\, D\,]\,\}. \tag{3.3}$$

(b) *The set inclusion $\{N(A^{(1,2)},\, S^{(1,2)})\} \subseteq \{M^{(1,2)}\}$ holds if and only if* (3.3) *holds, or*

$$r(M) = 2r\begin{bmatrix} B \\ D \end{bmatrix} - r(D) \text{ and } r\begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = r(A) + r(D).$$

Proof It follows from Theorem 2.1 by setting $C = B^*$. □

Theorem 3.3 *Let M and $N(A^{(1,3)},\, S^{(1,3)})$ be as given in* (3.1) *and* (3.2). *Then*

$$\min_{A^{(1,3)},\, M^{(1,3)}} r[\, M^{(1,3)} - N(A^{(1,3)},\, S^{(1,3)})\,] = \max\{\, r_1,\ \ r_2\,\},$$

$$\max_{A^{(1,3)}}\min_{M^{(1,3)}} r[\, M^{(1,3)} - N(A^{(1,3)},\, S^{(1,3)})\,] = r_1 + r_3,$$

where

$$r_1 = r[\,A,\,B\,] + r[\,B^*,\,D\,] - r\begin{bmatrix} A^2 & AB \\ B^* & D \end{bmatrix},$$
$$r_2 = r[\,A,\,B\,] - r(A),$$
$$r_3 = r\begin{bmatrix} A & 0 \\ 0 & B \\ B^* & D \end{bmatrix} - r\begin{bmatrix} B \\ D \end{bmatrix} - r(A).$$

Hence,

(a) *There exist $A^{(1,3)}$ and $S^{(1,3)}$ such that $N(A^{(1,3)},\,S^{(1,3)})$ is $\{1,3\}$-inverse of M if and only if*

$$\mathscr{R}(B) \subseteq \mathscr{R}(A) \ \text{and}\ \mathscr{R}\begin{bmatrix} A^2 \\ B^*A \end{bmatrix} \cap \mathscr{R}\begin{bmatrix} B \\ D \end{bmatrix} = \{0\}.$$

(b) *The set inclusion $\{N(A^{(1,3)},\,S^{(1,3)})\} \subseteq \{M^{(1,3)}\}$ holds if and only if*

$$\mathscr{R}(B) \subseteq \mathscr{R}(A) \ \ \text{and} \ \ \mathscr{R}\begin{bmatrix} A^2 \\ B^*A \end{bmatrix} \cap \mathscr{R}\begin{bmatrix} B \\ D \end{bmatrix} = \{0\}.$$

Proof It follows from Theorem 2.2 by setting $C = B^*$. $\square$

A special case of (3.2) corresponding to the Moore-Penrose inverse is

$$N(A^{\dagger},\,S^{\dagger}) = \begin{bmatrix} A^{\dagger} + A^{\dagger}BS^{\dagger}B^*A^{\dagger} & -A^{\dagger}BS^{\dagger} \\ -S^{\dagger}B^*A^{\dagger} & S^{\dagger} \end{bmatrix}, \tag{3.4}$$

where $S = D - B^*A^{\dagger}B$. The relations between $N(A^{\dagger},\,S^{\dagger})$ and $\{i,\ldots,j\}$-inverse of M are given in the following theorems.

Theorem 3.4 *Let M and $N(A^{\dagger},\,S^{\dagger})$ be as given in* (3.1) *and* (3.4), *respectively. Then*

$$\min_{M^{(1)}} r[\,M^{(1)} - N(A^{\dagger},\,S^{\dagger})\,] = \min_{M^{(1,2)}} r[\,M^{(1,2)} - N(A^{\dagger},\,S^{\dagger})\,] = r(M) - r\begin{bmatrix} A^3 & AB \\ B^*A & D \end{bmatrix}.$$

Hence, the following statements are equivalent:

(a) $N(A^{\dagger},\,S^{\dagger})$ *is a $\{1\}$-inverse of M.*

(b) $N(A^{\dagger},\,S^{\dagger})$ *is a $\{1,2\}$-inverse of M.*

(c) $r(M) = r(A) + r(\,D - B^*A^{\dagger}B\,)$.

(d) $r(M) = r\begin{bmatrix} A^3 & AB \\ B^*A & D \end{bmatrix}$.

(e) $\mathscr{R}\begin{bmatrix} A^2 & B \\ B^*A & D \end{bmatrix} = \mathscr{R}(M)$.

Proof It follows from Theorem 2.4 by setting $C = B^*$. $\square$

Theorem 3.5 *Let M and $N(A^{\dagger},\,S^{\dagger})$ be as given in* (3.1) *and* (3.4), *respectively. Then*

$$\min_{M^{(1,3)}} r[\,M^{(1,3)} - N(A^{\dagger},\,S^{\dagger})\,] = \min_{M^{(1,4)}} r[\,M^{(1,4)} - N(A^{\dagger},\,S^{\dagger})\,] = r\begin{bmatrix} A \\ B^* \end{bmatrix} + r\begin{bmatrix} B \\ D \end{bmatrix} - r\begin{bmatrix} A^3 & AB \\ B^*A & D \end{bmatrix}.$$

Hence, the following statements are equivalent:

(a) $N(A^{\dagger},\,S^{\dagger})$ *is a $\{1,3\}$-inverse of M.*

(b) $N(A^{\dagger},\, S^{\dagger})$ *is a* $\{1,4\}$*-inverse of* M.

(c) $r\begin{bmatrix} A^3 & AB \\ B^*A & D \end{bmatrix} = r\begin{bmatrix} A \\ B^* \end{bmatrix} + r\begin{bmatrix} B \\ D \end{bmatrix}$.

(d) $\mathscr{R}\begin{bmatrix} A^3 \\ B^*A \end{bmatrix} \cap \mathscr{R}\begin{bmatrix} AB \\ D \end{bmatrix} = \{0\}$, $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ *and* $\mathscr{R}[\, B^*,\, D\,] = \mathscr{R}[\, B^*A,\, D\,]$.

Proof It follows from Theorems 2.5 and 2.6 by setting $C = B^*$. $\square$

Assume the Hermitian matrix in (3.1) is nonnegative definite, that is, there exists a matrix U such that $M = UU^*$. In this case, $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ and $\mathscr{R}(B^*) \subseteq \mathscr{R}(D)$ and $S = D - B^*A^{(1)}B = D - B^*A^{\dagger}B$ for any $A^{(1)}$ hold. Relations between $\{1\}$- and $\{1,2\}$-inverses of M and $N(A^{(1)},\, S^{(1)})$ and $N(A^{(1,2)},\, S^{(1,2)})$ were considered by Rhode [11]. Applying Theorems 3.1, 3.2, 3.3 and (1.5) to the nonnegative Hermitian matrix in (3.1) gives the following result.

Theorem 3.6 *Let* M *and* $N(A^{(1)},\, S^{(1)})$ *be as given in* (3.1) *and* (3.2), *and assume* M *is nonnegative definite. Also let* $S = D - B^*A^{\dagger}B$. *Then,*

(a) [11] *The set inclusion* $\{N(A^{(1)},\, S^{(1)})\} \subseteq \{M^{(1)}\}$ *always holds.*

(b) [11] *The set inclusion* $\{N(A^{(1,2)},\, S^{(1,2)})\} \subseteq \{M^{(1,2)}\}$ *always holds.*

(c) *The following statements are equivalent*:

  (i) *The set inclusion* $\{N(A^{(1,3)},\, S^{(1,3)})\} \subseteq \{M^{(1,3)}\}$ *holds.*

  (ii) *The set inclusion* $\{N(A^{(1,4)},\, S^{(1,4)})\} \subseteq \{M^{(1,4)}\}$ *holds.*

  (iii) $M^{\dagger} = N(A^{\dagger},\, S^{\dagger})$.

  (iv) $\mathscr{R}\begin{bmatrix} A \\ B^* \end{bmatrix} \cap \mathscr{R}\begin{bmatrix} B \\ D \end{bmatrix} = \{0\}$.

An $\{i, \ldots, j\}$-inverse of a square matrix A is said to be an Hermitian $\{i, \ldots, j\}$-inverse of A and is denoted by $A_h^{(i,\ldots,j)}$, if it Hermitian. It is easy to verify that any Hermitian matrix A always has an Hermitian $\{i, \ldots, j\}$-inverse for any give set $\{i, \ldots, j\}$. Further, the Hermitian Banachiewicz-Schur form induced from the Hermitian matrix M in (3.1) is defined to be

$$N(A_h^{(i,\ldots,j)},\, S_h^{(i,\ldots,j)}) = \begin{bmatrix} A_h^{(i,\ldots,j)} + A_h^{(i,\ldots,j)}BS_h^{(i,\ldots,j)}B^*A_h^{(i,\ldots,j)} & -A_h^{(i,\ldots,j)}BS_h^{(i,\ldots,j)} \\ -S_h^{(i,\ldots,j)}B^*A_h^{(i,\ldots,j)} & S_h^{(i,\ldots,j)} \end{bmatrix}, \tag{3.5}$$

where $S = D - B^*A_h^{(i,\ldots,j)}B$. In order to characterize relations between M and $N(A_h^{(i,\ldots,j)},\, S_h^{(i,\ldots,j)})$, we need to know the extremal ranks of $D - B^*A_h^{(i,\ldots,j)}B$ with respect to $A_h^{(i,\ldots,j)}$, which now are open problems.

4 Generalized inverses of a bordered Hermitian matrix

Setting $D = 0$ in (3.1) gives

$$M = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \tag{4.1}$$

where $A \in \mathbb{C}^{m \times m}$ is nonnegative definite and $B \in \mathbb{C}^{m \times n}$. This matrix occurs widely in various problems in matrix theory, in particular, in regression analysis. The Banachiewicz-Schur form induced from M is

$$N(A^{(i,\ldots,j)},\, S^{(i,\ldots,j)}) = \begin{bmatrix} A^{(i,\ldots,j)} + A^{(i,\ldots,j)}BS^{(i,\ldots,j)}B^*A^{(i,\ldots,j)} & -A^{(i,\ldots,j)}BS^{(i,\ldots,j)} \\ -S^{(i,\ldots,j)}B^*A^{(i,\ldots,j)} & S^{(i,\ldots,j)} \end{bmatrix}, \tag{4.2}$$

where $S = -B^*A^{(i,\ldots,j)}B$. Applying the results in Section 2 to (4.1) and (4.2) gives us the following results.

Theorem 4.1 *Let M and $N(A^{(1)}, S^{(1)})$ be as given in* (4.1) *and* (4.2). *Then the following statements are equivalent*:

(a) *There exist $A^{(1)}$ and $S^{(1)}$ such that $N(A^{(1)}, S^{(1)}) \in \{M^{(1)}\}$.*

(b) *The set inclusion $\{N(A^{(1)}, S^{(1)})\} \subseteq \{M^{(1)}\}$ holds.*

(c) *There exist $A^{(1,3)}$ and $S^{(1,3)}$ such that $N(A^{(1,3)}, S^{(1,3)}) \in \{M^{(1,3)}\}$.*

(d) *The set inclusion $\{N(A^{(1,3)}, S^{(1,3)})\} \subseteq \{M^{(1,3)}\}$ holds.*

(e) *There exist $A^{(1,4)}$ and $S^{(1,4)}$ such that $N(A^{(1,4)}, S^{(1,4)}) \in \{M^{(1,4)}\}$.*

(f) *The set inclusion $\{N(A^{(1,4)}, S^{(1,4)})\} \subseteq \{M^{(1,4)}\}$ holds.*

(g) $M^{\dagger} = N(A^{\dagger}, S^{\dagger})$.

(h) $\mathscr{R}(B) \subseteq \mathscr{R}(A)$.

Theorem 4.2 *Let M and $N(A^{(1)}, S^{(1)})$ be as given in* (4.1) *and* (4.2). *Then the following statements are equivalent*:

(a) *There exist $A^{(1,2)}$ and $S^{(1,2)}$ such that $N(A^{(1,2)}, S^{(1,2)}) \in \{M^{(1,2)}\}$.*

(b) *The set inclusion $\{N(A^{(1,2)}, S^{(1,2)})\} \subseteq \{M^{(1,2)}\}$ holds.*

(c) $r[\, A,\, B\,] \leqslant 2r(A) - r(B)$ *or* $\mathscr{R}(B) \subseteq \mathscr{R}(A)$.

5 Concluding remarks

If D in (1.1) is square and nonsingular, then the matrix in (1.1) can also be decomposed as

$$M = \begin{bmatrix} I_m & BD^{-1} \\ 0 & I_l \end{bmatrix} \begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I_m & 0 \\ D^{-1}C & I_l \end{bmatrix}. \tag{5.1}$$

If both M and D are nonsingular in (1.1), then the Schur complement $T = A - BD^{-1}C$ is nonsingular, too, and the inverse of M can also be written as

$$M^{-1} = \begin{bmatrix} T^{-1} & -T^{-1}BD^{-1} \\ -D^{-1}CT^{-1} & D^{-1} + D^{-1}CT^{-1}BD^{-1} \end{bmatrix}. \tag{5.2}$$

By symmetry, another type of Banachiewicz-Schur form induced from M is given by

$$K(D^{(i,\dots,j)}, T^{(i,\dots,j)}) = \begin{bmatrix} T^{(i,\dots,j)} & -T^{(i,\dots,j)}BD^{(i,\dots,j)} \\ -D^{(i,\dots,j)}CT^{(i,\dots,j)} & D^{(i,\dots,j)} + D^{(i,\dots,j)}CT^{(i,\dots,j)}BD^{(i,\dots,j)} \end{bmatrix}, \tag{5.3}$$

where $T = A - BD^{(i,\dots,j)}C$. Although (1.3) and (5.2) are identical, (1.4) and (5.3) are not necessarily the same. Applying the results in the previous sections to (5.3), we can derive various conclusions on relations between M and $K(D^{(i,\dots,j)}, T^{(i,\dots,j)})$. Furthermore, it is of interest to give necessary and sufficient conditions for the following equality

$$N(A^{(i,\dots,j)}, S^{(i,\dots,j)}) = K(D^{(i,\dots,j)}, T^{(i,\dots,j)}) \tag{5.4}$$

to hold, or equalities of submatrices in them to hold. All the results in this paper on partitioned matrices and Banachiewicz-Schur forms induced from the matrices can be used to further study various problems related to block matrices and their generalized inverses.

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References

[1] J.K. Baksalary and G.P.H. Styan. Generalized inverses of partitioned matrices in Banachiewicz-Schur form. *Linear Algebra Appl.* 354(2002) 41–47.

[2] A. Ben-Israel. A note on partitioned matrices and equations. *SIAM Rev.* 11(1969), 247–250.

[3] A. Ben-Israel and T.N.E. Greville. *Generalized Inverses: Theory and Applications*, 2nd Ed., Springer-Verlag, New York, 2003.

[4] P. Bhimasankaram. On generalized inverses of partitioned matrices. *Sankhyā Ser. A* 33(1971), 311–314.

[5] F. Burns, D. Carlson, E. Haynsworth and T. Markham. Generalized inverse formulas using the Schur complement. *SIAM. J. Appl. Math.* 26(1974), 254–259.

[6] S.L. Campbell and C.D. Meyer. *Generalized Inverses of Linear Transformations*, Corrected reprint of the 1979 original, Dover Publications, Inc., New York, 1991.

[7] G. Marsaglia and G.P.H. Styan. Equalities and inequalities for ranks of matrices. *Linear and Multilinear Algebra* 2(1974), 269–292.

[8] G. Marsaglia and G.P.H. Styan. Rank conditions for generalized inverses of partitioned matrices. *Sankhyā Ser. A* 36(1974), 437–442.

[9] S. Puntanen and G.P.H. Styan. Schur complements in statistics and probabiliy. In: *The Schur Complement and Its Applications* (F. Zhang ed.), Springer, pp. 163–226, 2005.

[10] C.R. Rao and S.K. Mitra. *Generalized Inverse of Matrices and Its Applications.* Wiley, New York, 1971.

[11] C.A. Rohde. Generalized inverses of partitioned matrices. *SIAM J. Appl. Math.* 13(1965), 1033–1035.

[12] Y. Tian. Upper and lower bounds for ranks of matrix expressions using generalized inverses. *Linear Algebra Appl.* 355(2002), 187–214.

[13] Y. Tian. Rank equalities for block matrices and their Moore-Penrose inverses. *Houston J. Math.* 30(2004), 483–510.

[14] Y. Tian. More on maximal and minimal ranks of Schur complements with applications. *Appl. Math. Comput.* 152(2004), 675–692.

[15] Y. Tian. Eight expressions for generalized inverses of a bordered matrix. *Linear and Multilinear Algebra*, in press.

[16] Y. Tian and Y. Takane. Schur complements and Banachiewicz-Schur forms. *Electron. J. Linear Algebra* 13(2005), 405–418.