Some properties of projectors associated with the WLSE under a general linear model

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Abstract

Projectors associated with a particular estimator in a general linear model play an important role in characterizing statistical properties of the estimator. A variety of new properties were derived on projectors associated with the weighted least-squares estimator (WLSE). These properties include maximal and minimal possible ranks, rank invariance, uniqueness, idempotency, and other equalities involving the projectors. Applications of these properties were also suggested. Proofs of the main theorems demonstrate how to use the matrix rank method for deriving various equalities involving the projectors under the general linear model.

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1 Introduction and Preliminary Results

Throughout this paper, $\mathbb{R}^{m\times n}$ stands for the collection of all $m \times n$ matrices. $\mathbf{A}'$, $r(\mathbf{A})$ and $\mathscr{R}(\mathbf{A})$ stand for the transpose, the rank and the range (column space) of a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, respectively. The Moore-Penrose inverse of $\mathbf{A}$, denoted by $\mathbf{A}^+$, is defined to be the unique solution $\mathbf{G}$ to the four matrix equations

$$\text{(i) } \mathbf{AGA} = \mathbf{A}, \quad \text{(ii) } \mathbf{GAG} = \mathbf{G}, \quad \text{(iii) } (\mathbf{AG})' = \mathbf{AG}, \quad \text{(iv) } (\mathbf{GA})' = \mathbf{GA}.$$

A matrix $\mathbf{G}$ is called a generalized inverse (g-inverse) of $\mathbf{A}$, denoted by $\mathbf{A}^-$, if it satisfies (i), while the collection of all g-inverses of $\mathbf{A}$ is denoted by $\{\mathbf{A}^-\}$. Further, let $\mathbf{P_A}$, $\mathbf{E_A}$ and $\mathbf{F_A}$ stand for the three orthogonal projectors $\mathbf{P_A} = \mathbf{AA}^+$, $\mathbf{E_A} = \mathbf{I}_m - \mathbf{AA}^+$ and $\mathbf{F_A} = \mathbf{I}_n - \mathbf{A}^+\mathbf{A}$. Let $\mathbf{V} \in \mathbb{R}^{n\times n}$ be a nonnegative definite matrix, i.e., $\mathbf{V}$ can be written as $\mathbf{V} = \mathbf{ZZ}'$ for some matrix $\mathbf{Z}$. The seminorm of a vector $\mathbf{x} \in \mathbb{R}^{n\times 1}$ induced by $\mathbf{V}$ is defined by $\|\mathbf{x}\|_{\mathbf{V}} = (\mathbf{x}'\mathbf{V}\mathbf{x})^{1/2}$.

Suppose we are given a general linear model

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \quad E(\boldsymbol{\varepsilon}) = \mathbf{0}, \quad Cov(\boldsymbol{\varepsilon}) = \sigma^2\boldsymbol{\Sigma}, \tag{1.1}$$

or in the triplet form

$$\mathscr{M} = \{\mathbf{y},\, \mathbf{X}\boldsymbol{\beta},\, \sigma^2\boldsymbol{\Sigma}\}, \tag{1.2}$$

where $\mathbf{X} \in \mathbb{R}^{n\times p}$ is a known matrix of arbitrary rank, $\mathbf{y} \in \mathbb{R}^{n\times 1}$ is an observable random vector, $\boldsymbol{\beta} \in \mathbb{R}^{p\times 1}$ is a vector of unknown parameters to be estimated, $\boldsymbol{\Sigma} \in \mathbb{R}^{n\times n}$ is a known nonnegative definite matrix of arbitrary rank, and σ^2 is a positive unknown parameter. If $\boldsymbol{\Sigma}$ is a singular matrix, (1.1) is also said to be a singular linear model.

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The weighted least-squares estimator (WLSE) is often used to estimate unknown parameters in a general linear model. The WLSE of the parameter vector $\boldsymbol{\beta}$ under the model $\mathscr{M}$ in (1.2), denoted by $\mathrm{WLSE}_{\mathscr{M}}(\boldsymbol{\beta})$, is defined to be

$$\hat{\boldsymbol{\beta}} = \underset{\boldsymbol{\beta}}{\operatorname{argmin}}(\,\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\,)'\mathbf{V}(\,\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\,). \tag{1.3}$$

The WLSE of $\mathbf{X}\boldsymbol{\beta}$ under (1.2) is defined to be $\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta}) = \mathbf{X}\mathrm{WLSE}_{\mathscr{M}}(\boldsymbol{\beta})$.

The following well-known result on the general solution of a linear matrix equation (see, e.g., Penrose [8], and Rao and Mitra [11, Theorem 2.3.1(b)]) can be used to derive general expressions of the WLSEs of $\boldsymbol{\beta}$ and $\mathbf{X}\boldsymbol{\beta}$ under $\mathscr{M}$.

Lemma 1.1 *A linear matrix equation $\mathbf{A}\boldsymbol{\beta} = \mathbf{y}$ is solvable for $\boldsymbol{\beta}$ if and only if $\mathbf{A}\mathbf{A}^{-}\boldsymbol{\beta} = \mathbf{y}$. In this case, the general solution of $\mathbf{A}\boldsymbol{\beta} = \mathbf{y}$ can be written as*

$$\boldsymbol{\beta} = \mathbf{A}^{-}\mathbf{y} + (\,\mathbf{I} - \mathbf{A}^{-}\mathbf{A}\,)\mathbf{u}, \tag{1.4}$$

where $\mathbf{u}$ is an arbitrary vector. In particular, the solution to $\mathbf{A}\boldsymbol{\beta} = \mathbf{y}$ is unique if and only if $\mathbf{A}$ has full column rank.

Through generalized inverses of matrices and (1.4), the general expressions of the WLSEs of $\boldsymbol{\beta}$ and $\mathbf{X}\boldsymbol{\beta}$ under $\mathscr{M}$ are given in the following lemma.

Lemma 1.2 *The normal equation associated with* (1.3) *is given by $\mathbf{X}'\mathbf{V}\mathbf{X}\boldsymbol{\beta} = \mathbf{X}'\mathbf{V}\mathbf{y}$. This equation is always consistent and the general expression of the WLSE of $\boldsymbol{\beta}$ under* (1.2), *denoted by* $\mathrm{WLSE}_{\mathscr{M}}(\boldsymbol{\beta})$, *is given by $\tilde{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{V}\mathbf{X})^{-}\mathbf{X}'\mathbf{V} + [\,\mathbf{I} - (\mathbf{X}'\mathbf{V}\mathbf{X})^{-}(\mathbf{X}'\mathbf{V}\mathbf{X})\,]\mathbf{u}$, where $\mathbf{u} \in \mathbb{R}^{p\times 1}$ is arbitrary. Let $\mathbf{u} = \mathbf{U}\mathbf{y}$ for $\mathbf{y} \neq \mathbf{0}$, where $\mathbf{U} \in \mathbb{R}^{p\times n}$ is arbitrary. Then the WLSEs of $\boldsymbol{\beta}$ and $\mathbf{X}\boldsymbol{\beta}$ under* (1.2) *can be written in the following homogeneous forms*

$$\mathrm{WLSE}_{\mathscr{M}}(\boldsymbol{\beta}) = \{(\mathbf{X}'\mathbf{V}\mathbf{X})^{-}\mathbf{X}'\mathbf{V} + [\,\mathbf{I} - (\mathbf{X}'\mathbf{V}\mathbf{X})^{-}(\mathbf{X}'\mathbf{V}\mathbf{X})\,]\mathbf{U}\}\mathbf{y}, \tag{1.5}$$

$$\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta}) = \{\mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{-}\mathbf{X}'\mathbf{V} + [\,\mathbf{X} - \mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{-}(\mathbf{X}'\mathbf{V}\mathbf{X})\,]\mathbf{U}\}\mathbf{y}. \tag{1.6}$$

In what follows, let $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ denote the matrix pre-multiplied to $\mathbf{y}$ in (1.6):

$$\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{-}\mathbf{X}'\mathbf{V} + [\,\mathbf{X} - \mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{-}(\mathbf{X}'\mathbf{V}\mathbf{X})\,]\mathbf{U}, \tag{1.7}$$

where $\mathbf{U} \in \mathbb{R}^{p\times n}$ is arbitrary, which is called the projector into $\mathscr{R}(\mathbf{X})$ with respect to the seminorm $\|\cdot\|_{\mathbf{V}}$, see Rao and Mitra [10, 11, Notes 3.2.5 and 3.2.7], and Mitra and Rao [7].

The expectation and the covariance matrix of $\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta})$ in (1.6) are given by

$$E[\,\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta})\,] = \mathbf{P}_{\mathbf{X}:\mathbf{V}}\mathbf{X}\boldsymbol{\beta}, \quad Cov[\,\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta})\,] = \sigma^2\mathbf{P}_{\mathbf{X}:\mathbf{V}}\boldsymbol{\Sigma}\mathbf{P}'_{\mathbf{X}:\mathbf{V}}. \tag{1.8}$$

These two results indicate that the projector $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ plays a key role in characterizing statistical properties of the WLSEs of $\mathbf{X}\boldsymbol{\beta}$ under $\mathscr{M}$.

It can be seen from (1.6) and (1.8) that the algebraic and statistical properties of the WLSE of $\mathbf{X}\boldsymbol{\beta}$ under (1.2) are primarily determined by the projector $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ in (1.7). Hence it is essential to investigate various properties of the projector, for example, its rank, range, trace, norm, uniqueness, idempotency, symmetry, decompositions, equalities, as well as relations between projectors of WLSEs under different models. Because there is an arbitrary matrix $\mathbf{U}$ in (1.7), it is possible to take the $\mathbf{U}$ such that $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ has some special forms, for example,

$$\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{-}\mathbf{X}'\mathbf{V}, \tag{1.9}$$

$$\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V}, \tag{1.10}$$

$$\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{X}\mathbf{X}^{-} + \mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{-}\mathbf{X}'\mathbf{V}(\,\mathbf{I} - \mathbf{X}\mathbf{X}^{-}\,), \tag{1.11}$$

$$\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{X}\mathbf{X}^{+} + \mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V}(\,\mathbf{I} - \mathbf{X}\mathbf{X}^{+}\,). \tag{1.12}$$

These special forms provide $\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta})$ with various prescribed properties. Indeed, the projector $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ in (1.7) and its special cases have been widely investigated in the literature, see, e.g., [1, 2, 7, 9, 10, 12, 13, 14, 17] and [11, Sections 5.1 and 5.2].

Because the projector $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ in (1.7) is a matrix pencil consisting of generalized inverses and an arbitrary matrix, we shall use the following rank formulas for partitioned matrices due to Marsaglia and Styan [5] to simplify various matrix operations related to $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$.

Lemma 1.3 *Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{m\times k}$ *and* $\mathbf{C} \in \mathbb{R}^{l\times n}$. *Then*

$$r[\,\mathbf{A},\,\mathbf{B}\,] = r(\mathbf{A}) + r[\,(\,\mathbf{I}_m - \mathbf{A}\mathbf{A}^-\,)\mathbf{B}\,] = r(\mathbf{B}) + r[\,(\,\mathbf{I}_m - \mathbf{B}\mathbf{B}^-\,)\mathbf{A}\,], \tag{1.13}$$

$$r\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} = r(\mathbf{A}) + r[\,\mathbf{C}(\,\mathbf{I}_n - \mathbf{A}^-\mathbf{A}\,)\,] = r(\mathbf{C}) + r[\,\mathbf{A}(\,\mathbf{I}_n - \mathbf{C}^-\mathbf{C}\,)\,]. \tag{1.14}$$

The following results are shown in [15, 16].

Lemma 1.4 *Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{m\times k}$ *and* $\mathbf{C} \in \mathbb{R}^{l\times n}$. *Then*

$$\max_{\mathbf{Y}\in\mathbb{R}^{k\times n}} r(\,\mathbf{A} - \mathbf{B}\mathbf{Y}\,) = \min\{\,r[\,\mathbf{A},\,\mathbf{B}\,],\quad n\,\}, \tag{1.15}$$

$$\min_{\mathbf{Y}\in\mathbb{R}^{k\times n}} r(\,\mathbf{A} - \mathbf{B}\mathbf{Y}\,) = r[\,\mathbf{A},\,\mathbf{B}\,] - r(\mathbf{B}), \tag{1.16}$$

$$\max_{\mathbf{Y}\in\mathbb{R}^{k\times l}} r(\,\mathbf{A} - \mathbf{B}\mathbf{Y}\mathbf{C}\,) = \min\left\{ r[\,\mathbf{A},\,\mathbf{B}\,],\quad r\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} \right\}. \tag{1.17}$$

We also use the following simple results (see [3, 4, 11, Lemma 2.2.4 and and Section 3.3]) on the Moore-Penrose inverse, the range and the rank of matrices:

$$\mathbf{A} = \mathbf{A}\mathbf{A}'(\mathbf{A}^+)' = (\mathbf{A}^+)'\mathbf{A}'\mathbf{A}, \quad (\mathbf{A}^+)^+ = \mathbf{A}, \quad (\mathbf{A}^+)' = (\mathbf{A}')^+, \tag{1.18}$$

$$\mathscr{R}(\mathbf{B}) \subseteq \mathscr{R}(\mathbf{A}) \Leftrightarrow r[\,\mathbf{A},\,\mathbf{B}\,] = r(\mathbf{A}) \Leftrightarrow \mathbf{A}\mathbf{A}^+\mathbf{B} = \mathbf{B}, \tag{1.19}$$

$$\mathscr{R}(\mathbf{A}) \subseteq \mathscr{R}(\mathbf{B}) \text{ and } r(\mathbf{A}) = r(\mathbf{B}) \Leftrightarrow \mathscr{R}(\mathbf{A}) = \mathscr{R}(\mathbf{B}) \Leftrightarrow \mathbf{A}\mathbf{A}^+ = \mathbf{B}\mathbf{B}^+, \tag{1.20}$$

$$\mathscr{R}(\mathbf{A}\mathbf{B}^+\mathbf{B}) = \mathscr{R}(\mathbf{A}\mathbf{B}^+) = \mathscr{R}(\mathbf{A}\mathbf{B}'). \tag{1.21}$$

Moreover, if $\mathbf{V}$ is nnd, then

$$\mathbf{V}\mathbf{V}^+ = \mathbf{V}^+\mathbf{V}, \ \mathscr{R}(\mathbf{V}) = \mathscr{R}(\mathbf{V}^{1/2}) = \mathscr{R}(\mathbf{V}^+), \ \ \mathscr{R}(\mathbf{X}'\mathbf{V}\mathbf{X}) = \mathscr{R}(\mathbf{X}'\mathbf{V}) \ \text{ for an nnd } \mathbf{V}, \tag{1.22}$$

where $\mathbf{V}^{1/2}$ is square root of $\mathbf{V}$.

2 Properties of the projector $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$

It can been seen from (1.6) the WLSE of $\mathbf{X}\boldsymbol{\beta}$ is a projection of $\mathbf{y}$ into $\mathscr{R}(\mathbf{X})$ through the linear transformation $\mathbf{P}_{\mathbf{X}:\mathbf{V}}\mathbf{y}$. Hence, algebraic properties of $\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta})$ are completely determined by the projector $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$. In this section, we derive a variety of properties of $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$, and give some applications to $\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta})$.

Let a g-inverse of $\mathbf{X}'\mathbf{V}\mathbf{X}$ in (1.7) be the Moore-Penrose inverse of $\mathbf{X}'\mathbf{V}\mathbf{X}$. Then (1.7) reduces to

$$\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^+\mathbf{X}'\mathbf{V} + \mathbf{X}\mathbf{F}_{\mathbf{V}\mathbf{X}}\mathbf{U}, \tag{2.1}$$

where $\mathbf{U} \in \mathbb{R}^{p\times n}$ is arbitrary. Because the projector $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ in (2.1) is a linear matrix expression with respect to $\mathbf{U}$, we can use the rank formulas in Lemmas 1.3 and 1.4 to derive properties of $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$. In this section, we investigate the following problems on the projector in (2.1):

(a) The maximal and minimal possible ranks of $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ (Theorem 2.1).

(b) Rank invariance of $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ (Theorem 2.1).

(c) Uniqueness of $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ (Theorem 2.1).

(d) Necessary and sufficient conditions for $\mathbf{Z} \in \{\mathbf{P}_{\mathbf{X}:\mathbf{V}}\}$ to hold (Theorem 2.1).

(e) Necessary and sufficient conditions for $\mathbf{P}_{\mathbf{X}:\mathbf{V}}\mathbf{X} = \mathbf{X}$ to hold (Theorem 2.2).

(f) Various equations satisfied by $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ (Theorem 2.4).

(g) Necessary and sufficient conditions for $\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{P}_{\mathbf{X}_0:\mathbf{V}_0}$ to hold (Theorems 2.5).

(h) Necessary and sufficient conditions for $\mathbf{P}_{\mathbf{X}:\mathbf{V}} + \mathbf{P}_{\mathbf{X}_0:\mathbf{V}} = \mathbf{I}_n$ and $\mathbf{P}_{\mathbf{X}:\mathbf{V}} + \mathbf{P}_{\mathbf{X}:\mathbf{V}_0} = \mathbf{I}_n$ to hold (Theorems 2.10).

Theorem 2.1 *Let $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ be as given in* (2.1), *and let $\mathbf{Z} \in \mathbb{R}^{n\times n}$ be given. Then*

$$\min_{\mathbf{P}_{\mathbf{X}:\mathbf{V}}} r(\,\mathbf{Z} - \mathbf{P}_{\mathbf{X}:\mathbf{V}}\,) = r\begin{bmatrix} \mathbf{E}_{\mathbf{X}}\mathbf{Z} \\ \mathbf{X}'\mathbf{V}\mathbf{Z} - \mathbf{X}'\mathbf{V} \end{bmatrix}, \tag{2.2}$$

$$\max_{\mathbf{P}_{\mathbf{X}:\mathbf{V}}} r(\mathbf{P}_{\mathbf{X}:\mathbf{V}}) = r(\mathbf{X}), \tag{2.3}$$

$$\min_{\mathbf{P}_{\mathbf{X}:\mathbf{V}}} r(\mathbf{P}_{\mathbf{X}:\mathbf{V}}) = r(\mathbf{V}\mathbf{X}). \tag{2.4}$$

Hence,

(a) $\mathbf{Z} \in \{\mathbf{P}_{\mathbf{X}:\mathbf{V}}\}$ *if and only if* $\mathscr{R}(\mathbf{Z}) \subseteq \mathscr{R}(\mathbf{X})$ *and* $\mathbf{X}'\mathbf{V}\mathbf{Z} = \mathbf{X}'\mathbf{V}$ [9].

(b) *The rank of* $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ *is invariant if and only if* $r(\mathbf{V}\mathbf{X}) = r(\mathbf{X})$, *i.e.,* $\mathscr{R}(\mathbf{X}'\mathbf{V}) = \mathscr{R}(\mathbf{X}')$.

(c) $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ *is unique if and only if* $r(\mathbf{V}\mathbf{X}) = r(\mathbf{X})$, *in which case,* $\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V}$.

Proof From (2.1), $\mathbf{Z} - \mathbf{P}_{\mathbf{X}:\mathbf{V}}$ can be written as

$$\mathbf{Z} - \mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{Z} - \mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V} - \mathbf{X}\mathbf{F}_{\mathbf{V}\mathbf{X}}\mathbf{U},$$

where $\mathbf{U}$ is arbitrary. Applying (1.16) to this expression gives

$$\begin{aligned}\min_{\mathbf{P}_{\mathbf{X}:\mathbf{V}}} r(\,\mathbf{Z} - \mathbf{P}_{\mathbf{X}:\mathbf{V}}\,) &= \min_{\mathbf{U}} r[\,\mathbf{Z} - \mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V} - \mathbf{X}\mathbf{F}_{\mathbf{V}\mathbf{X}}\mathbf{U}\,] \\ &= r[\,\mathbf{Z} - \mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V},\ \mathbf{X}\mathbf{F}_{\mathbf{V}\mathbf{X}}\,] - r(\mathbf{X}\mathbf{F}_{\mathbf{V}\mathbf{X}}).\end{aligned}$$

Applying (1.13) and (1.14) and simplifying by elementary block matrix operations (EBMOs), we obtain

$$\begin{aligned}& r[\,\mathbf{Z} - \mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V},\ \mathbf{X}\mathbf{F}_{\mathbf{V}\mathbf{X}}\,] \\ &= r\begin{bmatrix} \mathbf{Z} - \mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V} & \mathbf{X} \\ \mathbf{0} & \mathbf{V}\mathbf{X} \end{bmatrix} - r(\mathbf{V}\mathbf{X}) \quad \text{(by (1.14))} \\ &= r\begin{bmatrix} \mathbf{Z} & \mathbf{X} \\ -\mathbf{V}\mathbf{Z} + \mathbf{V}\mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V} & \mathbf{0} \end{bmatrix} - r(\mathbf{V}\mathbf{X}) \\ &= r\begin{bmatrix} \mathbf{E}_{\mathbf{X}}\mathbf{Z} \\ \mathbf{V}\mathbf{Z} - \mathbf{V}\mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V} \end{bmatrix} + r(\mathbf{X}) - r(\mathbf{V}\mathbf{X}) \quad \text{(by (1.13))}\end{aligned}$$

and

$$r(\mathbf{X}\mathbf{F}_{\mathbf{V}\mathbf{X}}) = r\begin{bmatrix} \mathbf{X} \\ \mathbf{V}\mathbf{X} \end{bmatrix} - r(\mathbf{V}\mathbf{X}) = r(\mathbf{X}) - r(\mathbf{V}\mathbf{X}) \quad \text{(by (1.14))}.$$

Thus

$$\min_{\mathbf{P}_{\mathbf{X}:\mathbf{V}}} r(\,\mathbf{Z} - \mathbf{P}_{\mathbf{X}:\mathbf{V}}\,) = r\begin{bmatrix} \mathbf{E}_{\mathbf{X}}\mathbf{Z} \\ \mathbf{V}\mathbf{Z} - \mathbf{V}\mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V} \end{bmatrix}.$$

It is easy to verify that

$$\begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{X}' \end{bmatrix} \begin{bmatrix} \mathbf{E}_{\mathbf{X}}\mathbf{Z} \\ \mathbf{VZ} - \mathbf{VX}(\mathbf{X}'\mathbf{VX})^{+}\mathbf{X}'\mathbf{V} \end{bmatrix} = \begin{bmatrix} \mathbf{E}_{\mathbf{X}}\mathbf{Z} \\ \mathbf{X}'\mathbf{VZ} - (\mathbf{X}'\mathbf{VX})(\mathbf{X}'\mathbf{VX})^{+}\mathbf{X}'\mathbf{V} \end{bmatrix} = \begin{bmatrix} \mathbf{E}_{\mathbf{X}}\mathbf{Z} \\ \mathbf{X}'\mathbf{VZ} - \mathbf{X}'\mathbf{V} \end{bmatrix}, \tag{2.5}$$

and

$$\begin{aligned} &\begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{V} - \mathbf{VX}(\mathbf{X}'\mathbf{VX})^{+}\mathbf{X}'\mathbf{V} & \mathbf{VX}(\mathbf{X}'\mathbf{VX})^{+} \end{bmatrix} \begin{bmatrix} \mathbf{E}_{\mathbf{X}}\mathbf{Z} \\ \mathbf{X}'\mathbf{VZ} - \mathbf{X}'\mathbf{V} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{E}_{\mathbf{X}}\mathbf{Z} \\ [\,\mathbf{V} - \mathbf{VX}(\mathbf{X}'\mathbf{VX})^{+}\mathbf{X}'\mathbf{V}\,]\,\mathbf{E}_{\mathbf{X}}\mathbf{Z} + \mathbf{VX}(\mathbf{X}'\mathbf{VX})^{+}\mathbf{X}'\mathbf{VZ} - \mathbf{VX}(\mathbf{X}'\mathbf{VX})^{+}\mathbf{X}'\mathbf{V} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{E}_{\mathbf{X}}\mathbf{Z} \\ \mathbf{VZ} - \mathbf{VX}(\mathbf{X}'\mathbf{VX})^{+}\mathbf{X}'\mathbf{V} \end{bmatrix}. \end{aligned} \tag{2.6}$$

Equalities (2.5) and (2.6) imply that

$$r\begin{bmatrix} \mathbf{E}_{\mathbf{X}}\mathbf{Z} \\ \mathbf{VZ} - \mathbf{VX}(\mathbf{X}'\mathbf{VX})^{+}\mathbf{X}'\mathbf{V} \end{bmatrix} = r\begin{bmatrix} \mathbf{E}_{\mathbf{X}}\mathbf{Z} \\ \mathbf{X}'\mathbf{VZ} - \mathbf{X}'\mathbf{V} \end{bmatrix},$$

as required for (2.2). Applying (1.14) and (1.15) to (2.1) gives

$$\begin{aligned} \max_{\mathbf{P}_{\mathbf{X}:\mathbf{V}}} r(\mathbf{P}_{\mathbf{X}:\mathbf{V}}) &= \max_{\mathbf{U}} r[\,\mathbf{X}(\mathbf{X}'\mathbf{VX})^{+}\mathbf{X}'\mathbf{V} + \mathbf{XF}_{\mathbf{VX}}\mathbf{U}\,] \\ &= r[\,\mathbf{X}(\mathbf{X}'\mathbf{VX})^{+}\mathbf{X}'\mathbf{V},\ \mathbf{XF}_{\mathbf{VX}}\,] \\ &= r\begin{bmatrix} \mathbf{X}(\mathbf{X}'\mathbf{VX})^{+}\mathbf{X}'\mathbf{V} & \mathbf{X} \\ \mathbf{0} & \mathbf{VX} \end{bmatrix} - r(\mathbf{VX}) \\ &= r\begin{bmatrix} \mathbf{0} & \mathbf{X} \\ \mathbf{VX}(\mathbf{X}'\mathbf{VX})^{+}\mathbf{X}'\mathbf{V} & \mathbf{0} \end{bmatrix} - r(\mathbf{VX}) \\ &= r(\mathbf{X}) + r[\,\mathbf{VX}(\mathbf{X}'\mathbf{VX})^{+}\mathbf{X}'\mathbf{V}\,] - r(\mathbf{VX}) \\ &= r(\mathbf{X}), \end{aligned}$$

establishing (2.3). Letting $\mathbf{Z} = \mathbf{0}$ in (2.2) results in (2.4). Let the right-hand side of (2.2) be zero, we see that $\mathbf{Z} \in \{\mathbf{P}_{\mathbf{X}:\mathbf{V}}\}$ if and only if $\mathbf{X}'\mathbf{V} = \mathbf{X}'\mathbf{VZ}$ and $\mathbf{E}_{\mathbf{X}}\mathbf{Z} = \mathbf{0}$. The second equality $\mathbf{E}_{\mathbf{X}}\mathbf{Z} = \mathbf{0}$ is equivalent to $\mathscr{R}(\mathbf{Z}) \subseteq \mathscr{R}(\mathbf{X})$ by (1.19). Hence we have (a). Result (b) follows from (2.3) and (2.4). If $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ is unique, then the rank of $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ is invariant, too. Thus $r(\mathbf{VX}) = r(\mathbf{X})$. Conversely, it follows from (1.19) and (1.20) that

$$r(\mathbf{VX}) = r(\mathbf{X}) \Leftrightarrow \mathscr{R}(\mathbf{X}'\mathbf{V}) = \mathscr{R}(\mathbf{X}') \Leftrightarrow (\mathbf{X}'\mathbf{V})(\mathbf{X}'\mathbf{V})^{+}\mathbf{X}' = \mathbf{X}' \Leftrightarrow \mathbf{X}(\mathbf{VX})^{+}(\mathbf{VX}) = \mathbf{X}.$$

Thus $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ is unique by (2.1). $\square$

Theorem 2.1(a) gives a characterization of $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$, that is, $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ is in fact a solution of the matrix equation $\mathbf{X}'\mathbf{VZ} = \mathbf{X}'\mathbf{V}$ under the restriction $\mathscr{R}(\mathbf{Z}) \subseteq \mathscr{R}(\mathbf{X})$.

Theorem 2.2 *Let $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ be as given in* (1.7), *and let $\mathbf{U} = \mathbf{X}^{-}$ in* (1.7). *Then:*

(a) *The projector*

$$\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{XX}^{-} + \mathbf{X}(\mathbf{X}'\mathbf{VX})^{-}\mathbf{X}'\mathbf{V}(\,\mathbf{I}_n - \mathbf{XX}^{-}\,) \tag{2.7}$$

is idempotent and $\mathbf{P}_{\mathbf{X}:\mathbf{V}}\mathbf{X} = \mathbf{X}$ for any $\mathbf{X}^{-}$.

(b) *For any $\mathbf{X}^{-}$ and $(\mathbf{X}'\mathbf{VX})^{-}$, the following WLSE*

$$\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta}) = [\,\mathbf{XX}^{-} + \mathbf{X}(\mathbf{X}'\mathbf{VX})^{-}\mathbf{X}'\mathbf{V}(\,\mathbf{I}_n - \mathbf{XX}^{-}\,)\,]\mathbf{y} \tag{2.8}$$

is unbiased for $\mathbf{X}\boldsymbol{\beta}$ under (1.2).

Proof For the projector $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ in (2.7), the two equalities $\mathbf{P}^2_{\mathbf{X}:\mathbf{V}} = \mathbf{P}_{\mathbf{X}:\mathbf{V}}$ and $\mathbf{P}_{\mathbf{X}:\mathbf{V}}\mathbf{X} = \mathbf{X}$ are derived from $(\mathbf{X}\mathbf{X}^-)^2 = \mathbf{X}\mathbf{X}^-$, $\mathbf{X}\mathbf{X}^-\mathbf{X} = \mathbf{X}$ and $(\mathbf{I}_n - \mathbf{X}\mathbf{X}^-)\mathbf{X} = \mathbf{0}$. In this case,

$$\begin{aligned}&E\{[\,\mathbf{X}\mathbf{X}^- + \mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^-\mathbf{X}'\mathbf{V}(\,\mathbf{I}_n - \mathbf{X}\mathbf{X}^-\,)\,]\}\mathbf{y}\\ &= [\,\mathbf{X}\mathbf{X}^- + \mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^-\mathbf{X}'\mathbf{V}(\,\mathbf{I}_n - \mathbf{X}\mathbf{X}^-\,)\,]\mathbf{X}\boldsymbol{\beta} = \mathbf{X}\boldsymbol{\beta}.\end{aligned}$$

Hence the WLSE in (2.8) is unbiased for $\mathbf{X}\boldsymbol{\beta}$ in (1.2). $\square$

Theorem 2.2(b) indicates that there always exists an unbiased WLSE of $\mathbf{X}\boldsymbol{\beta}$ under $\mathscr{M}$ irrespective of the weight matrix $\mathbf{V}$ in (1.3). This somewhat unexpected result is truly significant since the prevailing belief has been contrary to this assertion. In the literature on projectors and WLSEs, the WLSE of $\mathbf{X}\boldsymbol{\beta}$ under $\mathscr{M}$ is often taken as $\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta}) = \mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^-\mathbf{X}'\mathbf{V}\mathbf{y}$, see, e.g., [2, 6, 9]. In this case, it is impossible to take $(\mathbf{X}'\mathbf{V}\mathbf{X})^-$ such that $\mathrm{WLSE}_{\mathscr{M}}(\boldsymbol{\beta})$ is unbiased for $\mathbf{X}\boldsymbol{\beta}$ under $\mathscr{M}$ when $r(\mathbf{V}\mathbf{X}) < r(\mathbf{X})$.

Corollary 2.3 *Let* $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ *be as given in* (2.1). *Then*

$$\min_{\mathbf{P}_{\mathbf{X}:\mathbf{V}}} r(\,\mathbf{P}_{\mathbf{X}} - \mathbf{P}_{\mathbf{X}:\mathbf{V}}\,) = r(\,\mathbf{P}_{\mathbf{X}}\mathbf{V}\mathbf{P}_{\mathbf{X}} - \mathbf{P}_{\mathbf{X}}\mathbf{V}\,) = \frac{1}{2}r(\,\mathbf{P}_{\mathbf{X}}\mathbf{V} - \mathbf{V}\mathbf{P}_{\mathbf{X}}\,).$$

Hence, $\mathbf{P}_{\mathbf{X}} \in \{\mathbf{P}_{\mathbf{X}:\mathbf{V}}\}$ *if and only if* $\mathbf{P}_{\mathbf{X}}\mathbf{V} = \mathbf{V}\mathbf{P}_{\mathbf{X}}$.

Further interesting properties of $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ are given in the following theorem.

Theorem 2.4 *Let* $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ *be as given in* (2.1). *Then*:

(a) $\mathscr{R}(\mathbf{P}_{\mathbf{X}:\mathbf{V}}) \subseteq \mathscr{R}(\mathbf{X})$.

(b) $\mathbf{V}\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ *can uniquely be written as* $\mathbf{V}\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{V}\mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^+\mathbf{X}'\mathbf{V} = \mathbf{V}^{1/2}\mathbf{P}_{\mathbf{V}^{1/2}\mathbf{X}}\mathbf{V}^{1/2}$ *with* $\mathscr{R}(\mathbf{V}\mathbf{P}_{\mathbf{X}:\mathbf{V}}) = \mathscr{R}(\mathbf{V}\mathbf{X})$ *and* $r(\mathbf{V}\mathbf{P}_{\mathbf{X}:\mathbf{V}}) = r(\mathbf{V}\mathbf{X})$.

(c) $(\mathbf{P}_{\mathbf{X}:\mathbf{V}})'\mathbf{V}\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{V}\mathbf{P}_{\mathbf{X}:\mathbf{V}}$.

(d) $(\mathbf{V}\mathbf{P}_{\mathbf{X}:\mathbf{V}})' = \mathbf{V}\mathbf{P}_{\mathbf{X}:\mathbf{V}}$, *i.e.,* $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ *is right-symmetric with respect to* $\mathbf{V}$.

(e) $\mathbf{V}\mathbf{P}^2_{\mathbf{X}:\mathbf{V}} = \mathbf{V}\mathbf{P}_{\mathbf{X}:\mathbf{V}}$, *i.e.,* $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ *is right-idempotent with respect to* $\mathbf{V}$.

(f) $\mathbf{V}\mathbf{P}_{\mathbf{X}:\mathbf{V}}\mathbf{X} = \mathbf{V}\mathbf{X}$.

(g) $\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{P}_{\mathbf{X}:(\mathbf{V}+\lambda\mathbf{E}_{\mathbf{X}})}$, *where* λ *is any real number.*

(h) $\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{P}_{\mathbf{X}:(\mathbf{V}+\mathbf{E}_{\mathbf{X}}\mathbf{Z}\mathbf{E}_{\mathbf{X}})}$, *where* $\mathbf{Z}$ *is any symmetric matrix.*

(i) $\mathbf{Z}_1\mathbf{Z}_2 \in \{\mathbf{P}_{\mathbf{X}:\mathbf{V}}\}$ *and* $\lambda\mathbf{Z}_1 + (1-\lambda)\mathbf{Z}_2 \in \{\mathbf{P}_{\mathbf{X}:\mathbf{V}}\}$ *hold for any* $\mathbf{Z}_1$, $\mathbf{Z}_2 \in \{\mathbf{P}_{\mathbf{X}:\mathbf{V}}\}$, *where* λ *is any real number.*

Proof The range inclusion in (a) is obvious from (2.1). Result (b) also follows from (2.1). Results (c), (d), (e) and (f) are derived from (b). Results (g) and (h) follow from the expression in (2.1). Verification of (i) is straightforward. $\square$

Suppose the model matrix $\mathbf{X}$ in (1.2) is misspecified as $\mathbf{X}_0 \in \mathbb{R}^{n\times p}$, and the weight matrix $\mathbf{V}$ in (1.3) is alternatively taken as $\mathbf{V}_0$. In these cases, we have the misspecified linear model

$$\mathscr{M}_0 = \{\mathbf{y},\, \mathbf{X}_0\boldsymbol{\beta},\, \sigma^2\boldsymbol{\Sigma}\}. \tag{2.9}$$

Correspondingly, the WLSE of $\mathbf{X}_0\boldsymbol{\beta}$ under $\mathscr{M}_0$ is

$$\mathrm{WLSE}_{\mathscr{M}_0}(\mathbf{X}_0\boldsymbol{\beta}) = \mathbf{P}_{\mathbf{X}_0:\mathbf{V}_0}\mathbf{y}, \tag{2.10}$$

where

$$\mathbf{P}_{\mathbf{X}_0:\mathbf{V}_0} = \mathbf{X}_0(\mathbf{X}_0'\mathbf{V}_0\mathbf{X}_0)^+\mathbf{X}_0'\mathbf{V}_0 + \mathbf{X}_0\mathbf{F}_{\mathbf{V}_0\mathbf{X}_0}\mathbf{U}_0, \tag{2.11}$$

and $\mathbf{U}_0 \in \mathbb{R}^{p\times n}$ is arbitrary. Concerning relationships between $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ and $\mathbf{P}_{\mathbf{X}_0:\mathbf{V}_0}$ in (2.1) and (2.11), we have the following result.

Theorem 2.5 *Let* $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ *and* $\mathbf{P}_{\mathbf{X}_0:\mathbf{V}_0}$ *be as given in* (2.1) *and* (2.11). *Then*

$$\min_{\mathbf{P}_{\mathbf{X}:\mathbf{V}},\,\mathbf{P}_{\mathbf{X}_0:\mathbf{V}_0}} r(\,\mathbf{P}_{\mathbf{X}:\mathbf{V}} - \mathbf{P}_{\mathbf{X}_0:\mathbf{V}_0}\,) = r[\,\mathbf{VX},\, \mathbf{V}_0\mathbf{X}_0\,] + r[\,\mathbf{X},\, \mathbf{X}_0\,] - r(\mathbf{N}), \tag{2.12}$$

$$\max_{\mathbf{P}_{\mathbf{X}_0:\mathbf{V}_0}} \min_{\mathbf{P}_{\mathbf{X}:\mathbf{V}}} r(\,\mathbf{P}_{\mathbf{X}:\mathbf{V}} - \mathbf{P}_{\mathbf{X}_0:\mathbf{V}_0}\,) = r[\,\mathbf{VX},\, \mathbf{V}_0\mathbf{X}_0\,] + r[\,\mathbf{X},\, \mathbf{X}_0\,] - r(\mathbf{V}_0\mathbf{X}_0) - r(\mathbf{X}), \tag{2.13}$$

where $\mathbf{N} = \begin{bmatrix} \mathbf{X} & \mathbf{X}_0 \\ \mathbf{VX} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_0\mathbf{X}_0 \end{bmatrix}$. *Hence,*

(a) *There exist* $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ *and* $\mathbf{P}_{\mathbf{X}_0:\mathbf{V}_0}$ *such that* $\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \mathbf{P}_{\mathbf{X}_0:\mathbf{V}_0}$ *if and only if* $r[\,\mathbf{VX},\, \mathbf{V}_0\mathbf{X}_0\,] + r[\,\mathbf{X},\, \mathbf{X}_0\,] = r(\mathbf{N})$.

(b) *The set inclusion* $\{\mathbf{P}_{\mathbf{X}_0:\mathbf{V}_0}\} \subseteq \{\mathbf{P}_{\mathbf{X}:\mathbf{V}}\}$ *holds if and only if* $\mathscr{R}(\mathbf{VX}) \subseteq \mathscr{R}(\mathbf{V}_0\mathbf{X}_0)$ *and* $\mathscr{R}(\mathbf{X}_0) \subseteq \mathscr{R}(\mathbf{X})$.

(c) *The set equality* $\{\mathbf{P}_{\mathbf{X}:\mathbf{V}}\} = \{\mathbf{P}_{\mathbf{X}_0:\mathbf{V}_0}\}$ *holds if and only if* $\mathscr{R}(\mathbf{X}) = \mathscr{R}(\mathbf{X}_0)$ *and* $\mathscr{R}(\mathbf{VX}) = \mathscr{R}(\mathbf{V}_0\mathbf{X}_0)$.

Proof From (2.1) and (2.11), the difference $\mathbf{P}_{\mathbf{X}:\mathbf{V}} - \mathbf{P}_{\mathbf{X}_0:\mathbf{V}_0}$ can be written as

$$\mathbf{P}_{\mathbf{X}:\mathbf{V}} - \mathbf{P}_{\mathbf{X}_0:\mathbf{V}_0} = \mathbf{G} + \mathbf{X}\mathbf{F}_{\mathbf{VX}}\mathbf{U} - \mathbf{X}_0\mathbf{F}_{\mathbf{V}_0\mathbf{X}_0}\mathbf{U}_0 = \mathbf{G} + [\,\mathbf{X}\mathbf{F}_{\mathbf{VX}},\, \mathbf{X}_0\mathbf{F}_{\mathbf{V}_0\mathbf{X}_0}\,]\begin{bmatrix} \mathbf{U} \\ -\mathbf{U}_0 \end{bmatrix}, \tag{2.14}$$

where $\mathbf{G} = \mathbf{X}(\mathbf{X}'\mathbf{VX})^{+}\mathbf{X}'\mathbf{V} - \mathbf{X}_0(\mathbf{X}_0'\mathbf{V}_0\mathbf{X}_0)^{+}\mathbf{X}_0'\mathbf{V}_0$, and $\mathbf{U}$ and $\mathbf{U}_0$ are arbitrary. Applying (1.16) to (2.14) gives

$$\begin{aligned}
\min_{\mathbf{P}_{\mathbf{X}:\mathbf{V}},\,\mathbf{P}_{\mathbf{X}_0:\mathbf{V}_0}} r(\,\mathbf{P}_{\mathbf{X}:\mathbf{V}} - \mathbf{P}_{\mathbf{X}_0:\mathbf{V}_0}\,) &= \min_{\mathbf{U},\,\mathbf{U}_0} r\left(\mathbf{G} + [\,\mathbf{X}\mathbf{F}_{\mathbf{VX}},\, \mathbf{X}_0\mathbf{F}_{\mathbf{V}_0\mathbf{X}_0}\,]\begin{bmatrix} \mathbf{U} \\ -\mathbf{U}_0 \end{bmatrix}\right) \\
&= r[\,\mathbf{G},\, \mathbf{X}\mathbf{F}_{\mathbf{VX}},\, \mathbf{X}_0\mathbf{F}_{\mathbf{V}_0\mathbf{X}_0}\,] - r[\,\mathbf{X}\mathbf{F}_{\mathbf{VX}},\, \mathbf{X}_0\mathbf{F}_{\mathbf{V}_0\mathbf{X}_0}\,].
\end{aligned} \tag{2.15}$$

Applying (1.14) and simplifying by EBMOs, we obtain

$$\begin{aligned}
& r[\,\mathbf{G},\, \mathbf{X}\mathbf{F}_{\mathbf{VX}},\, \mathbf{X}_0\mathbf{F}_{\mathbf{V}_0\mathbf{X}_0}\,] \\
&= r\begin{bmatrix} \mathbf{X}(\mathbf{X}'\mathbf{VX})^{+}\mathbf{X}'\mathbf{V} - \mathbf{X}_0(\mathbf{X}_0'\mathbf{V}_0\mathbf{X}_0)^{+}\mathbf{X}_0'\mathbf{V}_0 & \mathbf{X} & \mathbf{X}_0 \\ \mathbf{0} & \mathbf{VX} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{V}_0\mathbf{X}_0 \end{bmatrix} - r(\mathbf{VX}) - r(\mathbf{VX}_0) \\
&= r\begin{bmatrix} \mathbf{0} & \mathbf{X} & \mathbf{X}_0 \\ -\mathbf{VX}(\mathbf{X}'\mathbf{VX})^{+}\mathbf{X}'\mathbf{V} & \mathbf{VX} & \mathbf{0} \\ \mathbf{V}_0\mathbf{X}_0(\mathbf{X}_0'\mathbf{V}_0\mathbf{X}_0)^{+}\mathbf{X}_0'\mathbf{V}_0 & \mathbf{0} & \mathbf{V}_0\mathbf{X}_0 \end{bmatrix} - r(\mathbf{VX}) - r(\mathbf{V}_0\mathbf{X}_0) \\
&= r\begin{bmatrix} \mathbf{0} & \mathbf{X} & \mathbf{X}_0 \\ -\mathbf{VX}(\mathbf{X}'\mathbf{VX})^{+}\mathbf{X}'\mathbf{V} & \mathbf{0} & \mathbf{0} \\ \mathbf{V}_0\mathbf{X}_0(\mathbf{X}_0'\mathbf{V}_0\mathbf{X}_0)^{+}\mathbf{X}_0'\mathbf{V}_0 & \mathbf{V}_0\mathbf{X}_0(\mathbf{X}_0'\mathbf{V}_0\mathbf{X}_0)^{+}\mathbf{X}_0'\mathbf{V}_0\mathbf{X} & \mathbf{V}_0\mathbf{X}_0 \end{bmatrix} - r(\mathbf{VX}) - r(\mathbf{V}_0\mathbf{X}_0) \\
&= r\begin{bmatrix} \mathbf{0} & \mathbf{X} & \mathbf{X}_0 \\ -\mathbf{VX}(\mathbf{X}'\mathbf{VX})^{+}\mathbf{X}'\mathbf{V} & \mathbf{0} & \mathbf{0} \\ \mathbf{V}_0\mathbf{X}_0(\mathbf{X}_0'\mathbf{V}_0\mathbf{X}_0)^{+}\mathbf{X}_0'\mathbf{V}_0 & \mathbf{0} & \mathbf{0} \end{bmatrix} - r(\mathbf{VX}) - r(\mathbf{V}_0\mathbf{X}_0) \\
&= r\begin{bmatrix} \mathbf{VX}(\mathbf{X}'\mathbf{VX})^{+}\mathbf{X}'\mathbf{V} \\ \mathbf{VX}_0(\mathbf{X}_0'\mathbf{V}_0\mathbf{X}_0)^{+}\mathbf{X}_0'\mathbf{V}_0 \end{bmatrix} + r[\,\mathbf{X},\, \mathbf{X}_0\,] - r(\mathbf{VX}) - r(\mathbf{V}_0\mathbf{X}_0) \\
&= r\begin{bmatrix} \mathbf{X}'\mathbf{V} \\ \mathbf{X}_0'\mathbf{V}_0 \end{bmatrix} + r[\,\mathbf{X},\, \mathbf{X}_0\,] - r(\mathbf{VX}) - r(\mathbf{V}_0\mathbf{X}_0),
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
& r[\,\mathbf{X}\mathbf{F}_{\mathbf{VX}},\, \mathbf{X}_0\mathbf{F}_{\mathbf{V}_0\mathbf{X}_0}\,] \\
&= r\begin{bmatrix} \mathbf{X} & \mathbf{X}_0 \\ \mathbf{VX} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_0\mathbf{X}_0 \end{bmatrix} - r(\mathbf{VX}) - r(\mathbf{V}_0\mathbf{X}_0) = r(\mathbf{N}) - r(\mathbf{VX}) - r(\mathbf{V}_0\mathbf{X}_0).
\end{aligned} \tag{2.17}$$

Substituting (2.16) and (2.17) into (2.15) yields (2.12).

By (1.16),

$$\begin{aligned}
&\min_{\mathbf{P}_{\mathbf{X}:\mathbf{V}}} r(\,\mathbf{P}_{\mathbf{X}:\mathbf{V}} - \mathbf{P}_{\mathbf{X}_0:\mathbf{V}_0}\,)\\
&= \min_{\mathbf{U}} r[\,-\mathbf{P}_{\mathbf{X}_0:\mathbf{V}_0} + \mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V} - \mathbf{X}\mathbf{F}_{\mathbf{V}\mathbf{X}}\mathbf{U}\,]\\
&= r[\,-\mathbf{P}_{\mathbf{X}_0:\mathbf{V}_0} + \mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V},\ \mathbf{X}\mathbf{F}_{\mathbf{V}\mathbf{X}}\,] - r(\mathbf{X}\mathbf{F}_{\mathbf{V}\mathbf{X}})\\
&= r[\,-\mathbf{P}_{\mathbf{X}_0:\mathbf{V}_0} + \mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V},\ \mathbf{X}\mathbf{F}_{\mathbf{V}\mathbf{X}}\,] - r(\mathbf{X}) + r(\mathbf{V}\mathbf{X}) \quad \text{(by (1.14))}.
\end{aligned} \tag{2.18}$$

By (1.17),

$$\begin{aligned}
&\max_{\mathbf{P}_{\mathbf{X}_0:\mathbf{V}_0}} r[\,-\mathbf{P}_{\mathbf{X}_0:\mathbf{V}_0} + \mathbf{X}(\mathbf{X}'\mathbf{V}\mathbf{X})^{+}\mathbf{X}'\mathbf{V},\ \mathbf{X}\mathbf{F}_{\mathbf{V}\mathbf{X}}\,]\\
&= \max_{\mathbf{U}_0} r[\,\mathbf{G} - \mathbf{X}_0\mathbf{F}_{\mathbf{V}_0\mathbf{X}_0}\mathbf{U}_0,\ \mathbf{X}\mathbf{F}_{\mathbf{V}\mathbf{X}}\,]\\
&= \max_{\mathbf{U}_0} r\{\,[\,\mathbf{G},\ \mathbf{X}\mathbf{F}_{\mathbf{V}\mathbf{X}}\,] - \mathbf{X}_0\mathbf{F}_{\mathbf{V}_0\mathbf{X}_0}\mathbf{U}_0[\,\mathbf{I}_n,\ \mathbf{0}\,]\,\}\\
&= \min\left\{ r[\,\mathbf{G},\ \mathbf{X}_0\mathbf{F}_{\mathbf{V}_0\mathbf{X}_0},\ \mathbf{X}\mathbf{F}_{\mathbf{V}\mathbf{X}}\,],\quad r\begin{bmatrix} -\mathbf{G} & \mathbf{X}\mathbf{F}_{\mathbf{V}\mathbf{X}} \\ \mathbf{I}_n & \mathbf{0} \end{bmatrix} \right\}\\
&= \min\{\, r[\,\mathbf{G},\ \mathbf{X}_0\mathbf{F}_{\mathbf{V}_0\mathbf{X}_0},\ \mathbf{X}\mathbf{F}_{\mathbf{V}\mathbf{X}}\,],\quad n + r(\mathbf{X}\mathbf{F}_{\mathbf{V}\mathbf{X}})\}\\
&= r[\,\mathbf{G},\ \mathbf{X}_0\mathbf{F}_{\mathbf{V}_0\mathbf{X}_0},\ \mathbf{X}\mathbf{F}_{\mathbf{V}\mathbf{X}}\,]\\
&= r[\,\mathbf{V}\mathbf{X},\ \mathbf{V}_0\mathbf{X}_0\,] + r[\,\mathbf{X},\ \mathbf{X}_0\,] - r(\mathbf{V}\mathbf{X}) - r(\mathbf{V}_0\mathbf{X}_0).
\end{aligned} \tag{2.19}$$

Substituting (2.19) into (2.18) yields (2.13). □

Applying the results in Theorem 2.5 to the corresponding WLSEs of $\mathbf{X}\boldsymbol{\beta}$ under $\mathscr{M}$, we obtain the following results.

Corollary 2.6 *Let* $\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta})$ *and* $\mathrm{WLSE}_{\mathscr{M}_0}(\mathbf{X}_0\boldsymbol{\beta})$ *be as given in* (1.6) *and* (2.10).

(a) *If* $r[\,\mathbf{V}\mathbf{X},\ \mathbf{V}_0\mathbf{X}_0\,] + r[\,\mathbf{X},\ \mathbf{X}_0\,] = r(\mathbf{N})$, *then there exist* $\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta})$ *and* $\mathrm{WLSE}_{\mathscr{M}_0}(\mathbf{X}_0\boldsymbol{\beta})$ *such that* $\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta}) = \mathrm{WLSE}_{\mathscr{M}_0}(\mathbf{X}_0\boldsymbol{\beta})$.

(b) *If* $\mathscr{R}(\mathbf{V}\mathbf{X}) \subseteq \mathscr{R}(\mathbf{V}_0\mathbf{X}_0)$ *and* $\mathscr{R}(\mathbf{X}_0) \subseteq \mathscr{R}(\mathbf{X})$, *then* $\{\mathrm{WLSE}_{\mathscr{M}_0}(\mathbf{X}_0\boldsymbol{\beta})\} \subseteq \{\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta})\}$.

(c) *If* $\mathscr{R}(\mathbf{X}) = \mathscr{R}(\mathbf{X}_0)$ *and* $\mathscr{R}(\mathbf{V}\mathbf{X}) = \mathscr{R}(\mathbf{V}_0\mathbf{X}_0)$, *then* $\{\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta})\} = \{\mathrm{WLSE}_{\mathscr{M}_0}(\mathbf{X}_0\boldsymbol{\beta})\}$.

The following two corollaries are obtained by letting $\mathbf{X} = \mathbf{X}_0$ (but not $\mathbf{V} = \mathbf{V}_0$), and $\mathbf{V} = \mathbf{V}_0$ (but not $\mathbf{X} = \mathbf{X}_0$) in Corollary 2.6.

Corollary 2.7 *Let* $\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta})$ *be as given in* (1.6) *and let*

$$\mathrm{WLSE}_{\mathscr{M}_0}(\mathbf{X}_0\boldsymbol{\beta}) = [\,\mathbf{X}_0(\mathbf{X}_0'\mathbf{V}\mathbf{X}_0)^{+}\mathbf{X}_0'\mathbf{V} + \mathbf{X}_0\mathbf{F}_{\mathbf{V}\mathbf{X}_0}\mathbf{U}_0\,]\mathbf{y}.$$

(a) *If* $r[\,\mathbf{V}\mathbf{X},\ \mathbf{V}\mathbf{X}_0\,] + r[\,\mathbf{X},\ \mathbf{X}_0\,] = r(\mathbf{N})$, *then there exist* $\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta})$ *and* $\mathrm{WLSE}_{\mathscr{M}_0}(\mathbf{X}_0\boldsymbol{\beta})$ *such that* $\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta}) = \mathrm{WLSE}_{\mathscr{M}_0}(\mathbf{X}_0\boldsymbol{\beta})$.

(b) *If* $\mathscr{R}(\mathbf{V}\mathbf{X}) \subseteq \mathscr{R}(\mathbf{V}\mathbf{X}_0)$ *and* $\mathscr{R}(\mathbf{X}_0) \subseteq \mathscr{R}(\mathbf{X})$, *then* $\{\mathrm{WLSE}_{\mathscr{M}_0}(\mathbf{X}_0\boldsymbol{\beta})\} \subseteq \{\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta})\}$.

(c) *If* $\mathscr{R}(\mathbf{X}) = \mathscr{R}(\mathbf{X}_0)$, *then* $\{\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta})\} = \{\mathrm{WLSE}_{\mathscr{M}_0}(\mathbf{X}_0\boldsymbol{\beta})\}$.

Corollary 2.8 *Let* $\mathrm{WLSE}_{\mathscr{M},\mathbf{V}}(\mathbf{X}\boldsymbol{\beta})$ *be as given in* (1.6) *and let*

$$\mathrm{WLSE}_{\mathscr{M},\mathbf{V}_0}(\mathbf{X}\boldsymbol{\beta}) = [\mathbf{X}(\mathbf{X}'\mathbf{V}_0\mathbf{X})^{+}\mathbf{X}'\mathbf{V}_0 + \mathbf{X}\mathbf{F}_{\mathbf{V}_0\mathbf{X}}\mathbf{U}_0]\mathbf{y}.$$

(a) *If* $r[\,\mathbf{V}\mathbf{X},\ \mathbf{V}_0\mathbf{X}\,] = r\begin{bmatrix} \mathbf{V}\mathbf{X} \\ \mathbf{V}_0\mathbf{X} \end{bmatrix}$, *then then there exist* $\mathrm{WLSE}_{\mathscr{M},\mathbf{V}}(\mathbf{X}\boldsymbol{\beta})$ *and* $\mathrm{WLSE}_{\mathscr{M},\mathbf{V}_0}(\mathbf{X}\boldsymbol{\beta})$ *such that* $\mathrm{WLSE}_{\mathscr{M},\mathbf{V}}(\mathbf{X}\boldsymbol{\beta}) = \mathrm{WLSE}_{\mathscr{M},\mathbf{V}_0}(\mathbf{X}\boldsymbol{\beta})$.

(b) *If* $\mathscr{R}(\mathbf{VX}) \subseteq \mathscr{R}(\mathbf{V}_0\mathbf{X})$, *then* $\{\mathrm{WLSE}_{\mathscr{M},\mathbf{V}_0}(\mathbf{X}\boldsymbol{\beta})\} \subseteq \{\mathrm{WLSE}_{\mathscr{M},\mathbf{V}}(\mathbf{X}\boldsymbol{\beta})\}$.

(c) *If* $\mathscr{R}(\mathbf{VX}) = \mathscr{R}(\mathbf{V}_0\mathbf{X})$, *then* $\{\mathrm{WLSE}_{\mathscr{M},\mathbf{V}_0}(\mathbf{X}\boldsymbol{\beta})\} = \{\mathrm{WLSE}_{\mathscr{M},\mathbf{V}}(\mathbf{X}\boldsymbol{\beta})\}$.

In statistical practice, the weight matrices $\mathbf{V}$ and $\mathbf{V}_0$ in (1.6) and (2.10) are often taken as some matrices related to the covariance matrix $\mathbf{\Sigma}$ and its misspecified form $\mathbf{\Sigma}_0$. Assume now both $\mathbf{\Sigma}$ and $\mathbf{\Sigma}_0$ are positive definite and $r(\mathbf{X}) = r(\mathbf{X}_0) = p$ in (1.2) and (2.9), and let $\mathbf{V} = \mathbf{\Sigma}^{-1}$ and $\mathbf{V}_0 = \mathbf{\Sigma}_0^{-1}$ in (1.6) and (2.10). Then we obtain the following two WLSEs

$$\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta}) = \mathbf{X}(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{y}, \tag{2.20}$$

$$\mathrm{WLSE}_{\mathscr{M}_0}(\mathbf{X}_0\boldsymbol{\beta}) = \mathbf{X}_0(\mathbf{X}_0'\mathbf{\Sigma}_0^{-1}\mathbf{X}_0)^{-1}\mathbf{X}_0'\mathbf{\Sigma}_0^{-1}\mathbf{y}. \tag{2.21}$$

Necessary and sufficient conditions for the equality $\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta}) = \mathrm{WLSE}_{\mathscr{M}_0}(\mathbf{X}_0\boldsymbol{\beta})$ to hold are given below.

Corollary 2.9 *Let* $\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta})$ *and* $\mathrm{WLSE}_{\mathscr{M}_0}(\mathbf{X}_0\boldsymbol{\beta})$ *be as given in* (2.20) *and* (2.21). *Then* $\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta}) = \mathrm{WLSE}_{\mathscr{M}_0}(\mathbf{X}_0\boldsymbol{\beta})$ *holds if and only if* $\mathscr{R}(\mathbf{X}) = \mathscr{R}(\mathbf{X}_0)$ *and* $\mathscr{R}(\mathbf{VX}) = \mathscr{R}(\mathbf{V}_0\mathbf{X}_0)$.

Some previous work on equalities for estimations with an incorrect dispersion matrix can be found in Mitra and Moore [6].

Note from (1.6) that the residual vector with respect to $\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta})$ is

$$\mathbf{e} = \mathbf{y} - \mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta}) = (\,\mathbf{I}_n - \mathbf{P}_{\mathbf{X}:\mathbf{V}}\,)\mathbf{y}.$$

In the remaining of this section, we gives some properties of the differences $\mathbf{I}_n - \mathbf{P}_{\mathbf{X}:\mathbf{V}}$ and $\mathbf{I}_n - \mathbf{P}_{\mathbf{X}:\mathbf{V}} - \mathbf{P}_{\mathbf{X}_0:\mathbf{V}}$.

Theorem 2.10 *Let* $\mathbf{X} \in \mathbb{R}^{n\times p}$, $\mathbf{X}_0 \in \mathbb{R}^{n\times q}$, *and let* $\mathbf{V} \in \mathbb{R}^{n\times n}$ *be nnd. Then*

$$\min_{\mathbf{P}_{\mathbf{X}:\mathbf{V}}} r(\,\mathbf{I}_n - \mathbf{P}_{\mathbf{X}:\mathbf{V}}\,) = n - r(\mathbf{X}), \tag{2.22}$$

$$\min_{\mathbf{P}_{\mathbf{X}:\mathbf{V}},\,\mathbf{P}_{\mathbf{X}_0:\mathbf{V}}} r(\,\mathbf{I}_n - \mathbf{P}_{\mathbf{X}:\mathbf{V}} - \mathbf{P}_{\mathbf{X}_0:\mathbf{V}}\,) = n + 2r(\mathbf{X}'\mathbf{V}\mathbf{X}_0) - r(\mathbf{N}), \tag{2.23}$$

where $\mathbf{N} = \begin{bmatrix} \mathbf{X} & \mathbf{X}_0 \\ \mathbf{VX} & \mathbf{0} \\ \mathbf{0} & \mathbf{VX}_0 \end{bmatrix}$. *Hence, there exist* $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ *and* $\mathbf{P}_{\mathbf{X}_0:\mathbf{V}}$ *such that* $\mathbf{P}_{\mathbf{X}:\mathbf{V}} + \mathbf{P}_{\mathbf{X}_0:\mathbf{V}} = \mathbf{I}_n$ *if and only if* $r(\mathbf{N}) = n + 2r(\mathbf{X}'\mathbf{V}\mathbf{X}_0)$.

Eq. (2.22) is a direct consequence of (2.2). The proof of (2.23) is similar to that of Theorem 2.5, and therefore is omitted. Letting $\mathbf{X}_0 = \mathbf{E}_{\mathbf{X}}$ and $\mathbf{X}_0 = \mathbf{V}^{+}\mathbf{E}_{\mathbf{X}}$ in (2.23), respectively, gives the following corollary.

Corollary 2.11 *Let* $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ *be as given in* (2.1). *Then:*

(a) *There exist* $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ *and* $\mathbf{P}_{\mathbf{E}_{\mathbf{X}}:\mathbf{V}}$ *such that* $\mathbf{P}_{\mathbf{X}:\mathbf{V}} + \mathbf{P}_{\mathbf{E}_{\mathbf{X}}:\mathbf{V}} = \mathbf{I}_n$ *if and only if* $\mathbf{P}_{\mathbf{X}}\mathbf{V} = \mathbf{V}\mathbf{P}_{\mathbf{X}}$.

(b) *There exist* $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ *and* $\mathbf{P}_{\mathbf{V}^{+}\mathbf{E}_{\mathbf{X}}:\mathbf{V}}$ *such that* $\mathbf{P}_{\mathbf{X}:\mathbf{V}} + \mathbf{P}_{\mathbf{V}^{+}\mathbf{E}_{\mathbf{X}}:\mathbf{V}} = \mathbf{I}_n$ *if and only if*

$$r[\,\mathbf{X},\,\mathbf{V}\,] = r(\mathbf{X}) + r(\mathbf{V}) - r(\mathbf{VX}) \quad \textit{and} \quad r(\mathbf{VX}) = r(\mathbf{V}) + r(\mathbf{X}) - n.$$

In particular, if $\mathbf{V}$ *is pd, then* $\mathbf{P}_{\mathbf{X}:\mathbf{V}} + \mathbf{P}_{\mathbf{V}^{-1}\mathbf{E}_{\mathbf{X}}:\mathbf{V}} = \mathbf{I}_n$ *holds.*

3 Conclusion remarks

We derived a number of new properties of projectors associated with the WLSE of $\mathbf{X}\boldsymbol{\beta}$ under (1.2) and discussed their statistical implications. These properties may be used in the investigation of various other problems associated with the WLSEs. For example:

(I) Assume that the weight matrix $\mathbf{V}$ in (1.2) has the diagonal block form $\mathbf{V} = \begin{bmatrix} \mathbf{V}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_2 \end{bmatrix}$, where $\mathbf{V}_i \in \mathbb{R}^{n_i \times n_i}$ with $n_1 + n_2 = n$, $i = 1, 2$, and partition the model matrix $\mathbf{X}$ in (1.3) as $\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}$, where $\mathbf{X}_i \in \mathbb{R}^{n_i \times p}$, $i = 1, 2$. Then derive necessary and sufficient conditions for $\mathbf{P}_{\mathbf{X}:\mathbf{V}} = \begin{bmatrix} \mathbf{P}_{\mathbf{X}_1:\mathbf{V}_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_{\mathbf{X}_2:\mathbf{V}_2} \end{bmatrix}$ to hold, as well as $\mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta}) = \begin{bmatrix} \mathbf{P}_{\mathbf{X}_1:\mathbf{V}_1}\mathbf{y}_1 \\ \mathbf{P}_{\mathbf{X}_2:\mathbf{V}_2}\mathbf{y}_2 \end{bmatrix}$ to hold, where $\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}$.

(II) Partition the model matrix $\mathbf{X}$ in (1.2) as $\mathbf{X} = [\,\mathbf{X}_1,\, \mathbf{X}_2\,]$, where $\mathbf{X}_i \in \mathbb{R}^{n \times p_i}$ with $p_1 + p_2 = p$, $i = 1, 2$. Then derive necessary and sufficient conditions for

$$\begin{aligned} \mathbf{P}_{\mathbf{X}:\mathbf{V}} &= \mathbf{P}_{\mathbf{X}_1:\mathbf{V}_1} + \mathbf{P}_{\mathbf{X}_2:\mathbf{V}_2}, \\ \mathbf{P}_{\mathbf{X}:\mathbf{V}} &= \mathbf{P}_{\mathbf{X}_1:\mathbf{V}} + \mathbf{P}_{\mathbf{X}_2:\mathbf{V}} - \mathbf{P}_{\mathbf{X}_1:\mathbf{V}_1}\mathbf{P}_{\mathbf{X}_2:\mathbf{V}_2}, \\ \mathbf{P}_{\mathbf{X}:\mathbf{V}} &= \mathbf{P}_{(\mathbf{E}_{\mathbf{X}_2}\mathbf{X}_1):\mathbf{V}_1} + \mathbf{P}_{(\mathbf{E}_{\mathbf{X}_1}\mathbf{X}_2):\mathbf{V}_2} \end{aligned}$$

as well as

$$\begin{aligned} \mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta}) &= \mathbf{P}_{\mathbf{X}_1:\mathbf{V}_1}\mathbf{y} + \mathbf{P}_{\mathbf{X}_2:\mathbf{V}_2}\mathbf{y}, \\ \mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta}) &= \mathbf{P}_{\mathbf{X}_1:\mathbf{V}}\mathbf{y} + \mathbf{P}_{\mathbf{X}_2:\mathbf{V}}\mathbf{y} - \mathbf{P}_{\mathbf{X}_1:\mathbf{V}_1}\mathbf{P}_{\mathbf{X}_2:\mathbf{V}_2}\mathbf{y}, \\ \mathrm{WLSE}_{\mathscr{M}}(\mathbf{X}\boldsymbol{\beta}) &= \mathbf{P}_{(\mathbf{E}_{\mathbf{X}_2}\mathbf{X}_1):\mathbf{V}_1}\mathbf{y} + \mathbf{P}_{(\mathbf{E}_{\mathbf{X}_1}\mathbf{X}_2):\mathbf{V}_2}\mathbf{y} \end{aligned}$$

to hold, where $\mathbf{V}$, $\mathbf{V}_1$ and $\mathbf{V}_2$ are some weight matrices.

(III) Take the weight matrix $\mathbf{V}$ in (1.6) as $\mathbf{V} = \boldsymbol{\Sigma}^-$ or $\mathbf{V} = (\mathbf{XTX}' + \boldsymbol{\Sigma})^-$, where $\mathbf{T}$ is a symmetric matrix such that $r(\,\mathbf{XTX}' + \boldsymbol{\Sigma}\,) = r[\,\mathbf{X},\, \boldsymbol{\Sigma}\,]$. Then derive algebraic and statistical properties of the $\mathbf{P}_{\mathbf{X}:\mathbf{V}}$ and the corresponding WLSE along with the lines in Section 2.

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