Cart-Pole Derivation

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1 Equation of Motion

Let x denote the cart's position. Let θ denote the pole angle, measured c.c.w. and with $\theta = 0$ pointing downward.

Let M denote the cart's mass. Let m denote the pole's mass. Let P be the center of mass the pole, and let L denote the distance between P and the axle connecting the pole to the cart. Let I_{CM} denote the pole's moment of inertia about about an axis through its center of mass P and parallel to the axle.

We will take a Lagrangian approach. The kinetic energy will have three terms: one for the cart's linear motion, one for the linear motion of the pole's CoM, and one for the rotational motion of the pole about its CoM:

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\dot{\vec{\mathbf{r}}}_P^2 + \frac{1}{2}I_{\text{CM}}\dot{\theta}^2.$$

Here, v_P is the velocity of point P. Note that the angular velocity of the pole about its CoM is the same as its angular velocity about the axle, $\dot{\theta}$.

Now, we have

$$\begin{aligned} \vec{\mathbf{r}}_P &= \hat{\mathbf{x}}(x + L\sin(\theta)) - \hat{\mathbf{y}}\cos(\theta) \\ \implies \dot{\vec{\mathbf{r}}}_P &= \hat{\mathbf{x}}(\dot{x} + L\cos(\theta)\dot{\theta}) + \hat{\mathbf{y}}L\sin(\theta)\dot{\theta} \\ \implies \dot{\vec{\mathbf{r}}}_P^2 &= \dot{x}^2 + 2\dot{x}\dot{\theta}L\cos(\theta) + L^2\dot{\theta}^2 \\ \implies T &= \frac{1}{2}(M+m)\dot{x}^2 + mL\cos(\theta)\dot{x}\dot{\theta} + \frac{1}{2}(I_{\text{CM}} + mL^2)\dot{\theta}^2. \end{aligned}$$

We can then recognize $I \equiv I_{\text{CM}} + mL^2$ as the pole's moment of inertia about the axle (via the parallel axis theorem).

For the potential energy, we have

$$U = mgL(1 - \cos(\theta)),$$

if we choose $U(\theta = 0) = 0$. So

$$\mathcal{L} = T - U = \frac{1}{2}(M+m)\dot{x}^2 + mL\cos(\theta)\dot{x}\dot{\theta} + \frac{1}{2}I\dot{\theta}^2 + mgL(\cos(\theta) - 1).$$

With coordinates $\vec{\mathbf{q}} = (x, \theta)$, the Euler-Lagrange equation $(\mathrm{d}/\mathrm{d}t)\vec{\boldsymbol{\nabla}}_{\dot{\vec{\mathbf{q}}}}\mathcal{L} = \vec{\boldsymbol{\nabla}}_{\vec{\mathbf{q}}}\mathcal{L}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial x} \implies (M+m)\ddot{x} + mL(\cos(\theta)\ddot{\theta} - \sin(\theta)\dot{\theta}^2) = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial \mathcal{L}}{\partial \theta} \implies mL(\cos(\theta)\ddot{x} - \cancel{\sin(\theta)\dot{x}\dot{\theta}}) + I\ddot{\theta} = -\cancel{mL\sin(\theta)\dot{x}\dot{\theta}} - mgL\sin(\theta).$$

We must also account for the horizontal control force $\vec{\mathbf{F}} = \hat{\mathbf{x}}u$ applied to the cart, which makes the system non-conservative. To do this, we interpret $\partial\mathcal{L}/\partial\dot{x}$ as the generalized momentum for coordinate x. We may then simply add u as a term to $(\mathrm{d}/\mathrm{d}t)\,\partial\mathcal{L}/\partial\dot{x}$:

$$(M+m)\ddot{x} + mL(\cos(\theta)\ddot{\theta} - \sin(\theta)\dot{\theta}^2) = u.$$

Solving for $\ddot{x}$ and $\ddot{\theta}$, we have (in manipulator form)

$$\ddot{\vec{\mathbf{q}}} = \mathbf{H}^{-1}(\vec{\mathbf{q}})[\vec{\mathbf{U}} - \mathbf{C}(\vec{\mathbf{q}},\dot{\vec{\mathbf{q}}})\dot{\vec{\mathbf{q}}} - \vec{\mathbf{G}}(\vec{\mathbf{q}})]$$

$$\text{where} \qquad \mathbf{H}(\vec{\mathbf{q}}) \equiv \begin{bmatrix} M+m & mL\cos(\theta) \\ mL\cos(\theta) & I \end{bmatrix}, \quad \mathbf{C}(\vec{\mathbf{q}},\dot{\vec{\mathbf{q}}}) \equiv \begin{bmatrix} 0 & -mL\sin(\theta)\dot{\theta} \\ 0 & 0 \end{bmatrix},$$

$$\vec{\mathbf{G}}(\vec{\mathbf{q}}) \equiv \begin{bmatrix} 0 \\ mgL\sin(\theta) \end{bmatrix}, \quad \text{and} \qquad \vec{\mathbf{U}} \equiv \begin{bmatrix} u \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \frac{1}{(M+m)I - (mL\cos(\theta))^2} \begin{bmatrix} mL\sin(\theta)[mgL\cos(\theta) + I\dot{\theta}^2] + Iu \\ -mL[mL\sin(\theta)\cos(\theta)\dot{\theta}^2 + (M+m)g\sin(\theta) + u\cos(\theta)] \end{bmatrix}.$$

2 Linear-Quadratic Regulator for Balance Control

To develop a linear-quadratic regulator for balance control, we first need to linearize the equation of motion about the unstable equilibrium. We seek a linearized equation of the form

$$\dot{\vec{\mathbf{s}}} = \mathbf{A}(\vec{\mathbf{s}} - \vec{\mathbf{s}}^*) + \mathbf{B}(\vec{\mathbf{u}} - \vec{\mathbf{u}}^*), \quad \text{where} \quad \vec{\mathbf{s}} = \begin{bmatrix} \vec{\mathbf{q}} \\ \dot{\vec{\mathbf{q}}} \end{bmatrix} \equiv \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix}, \quad \vec{\mathbf{u}} = \begin{bmatrix} \vec{\mathbf{0}} \\ \vec{\mathbf{U}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ u \\ 0 \end{bmatrix}$$

is the control, $\vec{\mathbf{s}}^* = \begin{bmatrix} \text{any} & \pi & 0 & 0 \end{bmatrix}^T$ is the unstable equilibrium that we want to balance at, and $\vec{\mathbf{u}}^* = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$ indicates that no control should be applied when the system is exactly at the unstable equilibrium in phase space. The matrices $\mathbf{A}$ and $\mathbf{B}$ can be computed by writing our equation of motion in the form $\dot{\vec{\mathbf{s}}} = \vec{\mathbf{f}}(\vec{\mathbf{s}},\vec{\mathbf{u}})$ and linearizing about the unstable equilibrium via the evaluated Jacobians

$$\mathbf{A} = \left.\frac{\partial \dot{\vec{\mathbf{s}}}}{\partial \vec{\mathbf{s}}}\right|_{\vec{\mathbf{s}}^*,\vec{\mathbf{u}}^*} \quad \text{and} \quad \mathbf{B} = \left.\frac{\partial \dot{\vec{\mathbf{s}}}}{\partial \vec{\mathbf{u}}}\right|_{\vec{\mathbf{s}}^*,\vec{\mathbf{u}}^*}.$$

For clarity, since different authors define and notate Jacobians differently, we require

$$A_{ij} = \left.\frac{\partial \dot{s}_i}{\partial s_j}\right|_{\vec{\mathbf{s}}^*,\vec{\mathbf{u}}^*} \quad \text{and} \quad B_{ij} = \left.\frac{\partial \dot{s}_i}{\partial u_j}\right|_{\vec{\mathbf{s}}^*,\vec{\mathbf{u}}^*}.$$

Now, the equation of motion is (in block form)

$$\dot{\vec{\mathbf{s}}} = \begin{bmatrix} \dot{\vec{\mathbf{q}}} \\ \mathbf{H}^{-1}(\vec{\mathbf{q}})[\vec{\mathbf{U}} - \mathbf{C}(\vec{\mathbf{q}},\dot{\vec{\mathbf{q}}})\dot{\vec{\mathbf{q}}} - \vec{\mathbf{G}}(\vec{\mathbf{q}})] \end{bmatrix}$$

The top two components of $\dot{\vec{\mathbf{s}}}$ are trivial since they are already linear:

$$\begin{bmatrix} \dot{\vec{\mathbf{q}}} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\vec{\mathbf{s}}.$$

For the bottom two rows of $\mathbf{A}$ ($i \in \{3, 4\}$), we have

$$A_{ij} = \frac{\partial}{\partial s_j} \Big(\mathbf{H}^{-1}(\vec{\mathbf{q}})[\vec{\mathbf{U}} - \mathbf{C}(\vec{\mathbf{q}}, \dot{\vec{\mathbf{q}}})\dot{\vec{\mathbf{q}}} - \vec{\mathbf{G}}(\vec{\mathbf{q}})] \Big)_i \Big|_{\vec{\mathbf{s}}^*, \vec{\mathbf{u}}^*}.$$

This looks unpleasant, but it simplifies quite a bit:

- The term involving a derivative of $\mathbf{H}^{-1}$ vanishes since $[\mathbf{U} - \mathbf{C}\dot{\vec{\mathbf{q}}} - \vec{\mathbf{G}}] = \ddot{\vec{\mathbf{q}}} = \vec{\mathbf{0}}$ at the equilibrium.
- $\mathbf{U}$ has no s_j dependence.
- The term involving a derivative of $\mathbf{C}$ vanishes since $\dot{\vec{\mathbf{q}}} = 0$ at the equilibrium.
- The term involving a derivative of $\dot{\vec{\mathbf{q}}}$ vanishes since $\mathbf{C} = \mathbf{0}$ at the equilibrium.
- The term involving a derivative of $\vec{\mathbf{G}}$ vanishes if $j \in \{3, 4\}$ since $\vec{\mathbf{G}}$ has no dependence on the components of $\dot{\vec{\mathbf{q}}}$.

So (in block notation)

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{H}^{-1}\, \partial\vec{\mathbf{G}} \big/ \partial\vec{\mathbf{q}} & \mathbf{0} \end{bmatrix} \Bigg|_{\vec{\mathbf{s}}^*, \vec{\mathbf{u}}^*}.$$

For the bottom two rows of $\mathbf{B}$ ($i \in \{3, 4\}$), we have

$$B_{ij} = \frac{\partial}{\partial u_j} \Big(\mathbf{H}^{-1}(\vec{\mathbf{q}})[\vec{\mathbf{U}} - \mathbf{C}(\vec{\mathbf{q}}, \dot{\vec{\mathbf{q}}})\dot{\vec{\mathbf{q}}} - \vec{\mathbf{G}}(\vec{\mathbf{q}})] \Big)_i \Big|_{\vec{\mathbf{s}}^*, \vec{\mathbf{u}}^*}.$$

Since $\vec{\mathbf{u}} = \begin{bmatrix} 0 & 0 & u & 0 \end{bmatrix}^T$, only the $j = 3$ elements survive. But since $\mathbf{U} = \begin{bmatrix} u & 0 \end{bmatrix}^T$,

$$B_{(\text{bottom})3} = \mathbf{H}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Bigg|_{\vec{\mathbf{s}}^*, \vec{\mathbf{u}}^*}.$$

So (in block notation)

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (\mathbf{H}^{-1})_{11} & 0 \\ 0 & 0 & (\mathbf{H}^{-1})_{12} & 0 \end{bmatrix} \Bigg|_{\vec{\mathbf{s}}^*, \vec{\mathbf{u}}^*}.$$

Plugging in, we have

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \boldsymbol{\mathcal{A}} & \mathbf{0} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\mathcal{B}} \end{bmatrix}$$

$$\text{where} \qquad \boldsymbol{\mathcal{A}} \equiv \frac{mgL}{(M+m)I - (mL)^2} \begin{bmatrix} 0 & mL \\ 0 & M+m \end{bmatrix} \quad \text{and} \quad \boldsymbol{\mathcal{B}} \equiv \frac{1}{(M+m)I - (mL)^2} \begin{bmatrix} I & 0 \\ mL & 0 \end{bmatrix}.$$

Now, an infinite-horizon, continuous-time LQR minimizes the quadratic loss function

$$J = \int_0^\infty \mathrm{d}t \left[(\vec{\mathbf{s}} - \vec{\mathbf{s}}^*)^T \mathbf{Q} (\vec{\mathbf{s}} - \vec{\mathbf{s}}^*) + (\vec{\mathbf{u}} - \vec{\mathbf{u}}^*)^T \mathbf{R} (\vec{\mathbf{u}} - \vec{\mathbf{u}}^*) + 2(\vec{\mathbf{s}} - \vec{\mathbf{s}}^*)^T \mathbf{N} (\vec{\mathbf{u}} - \vec{\mathbf{u}}^*) \right]$$

via a linear feedback control law $\vec{\mathbf{u}} = -\mathbf{K}(\vec{\mathbf{s}} - \vec{\mathbf{s}}^*)$. The optimal feedback matrix $\mathbf{K}$ is given by some complicated linear algebra that we will not derive here, but I highly recommend reading through a derivation yourself, as there are some neat lemmas used.

Now, we need to pick some matrices $\mathbf{Q}$ $\mathbf{R}$, and $\mathbf{N}$ for our loss function. Note that due to the form of $\vec{\mathbf{u}}$, the only element of $\mathbf{R}$ that actually affects our loss function is R_{33}, so we only have one scalar to choose for $\mathbf{R}$. Similarly, due to the form of $\vec{\mathbf{u}}$, only $\mathbf{N}$ four components of $\mathbf{N}$ matter.

The effect of positive R_{33} is to penalize excessive control. Intuitively, we might expect such a penalty to push the controller to be more conservative and less prone to overshooting the equilibrium.

For $\mathbf{Q}$, a good choice is a diagonal matrix, so that our quadratic form in $\vec{\mathbf{s}} - \vec{\mathbf{s}}^*$ does not contain cross terms. In most cases a diagonal $\mathbf{Q}$ will be perfectly sufficient, though the relative weights may take some tuning. While we could also introduce nonzero weights to off-diagonal cross terms in the quadratic form, it turns out that in practice, doing so is typically unnecessary and serves only to complicate the process of tuning weights.

For getting a good controller, it is also perfectly sufficient to entirely neglect $\mathbf{N}$, giving no weights to cross terms between u and components of the state vector.