Unit 3 Homework

Answer Key

1. Processing Pasta

A certain manufacturing process creates pieces of pasta that vary by length. Suppose that the length of a particular piece, L, is a continuous random variable with the following probability density function.

$$f(l) = \begin{cases} 0, l \leq 0 \\ \frac{l}{2}, 0 < l \leq 2 \\ 0, 2 < l \end{cases}$$

(a) Write down a complete expression for the cumulative probability function of L.

$F(l) = \int_{\bar{l}=-\infty}^{l} f(\bar{l})d\bar{l}$

Notice that we needed a new variable to integrate over. The integrand is only positive between 0 and 2, so we already know F will be zero below this interval and 1 above this interval.

When $0 < l < 2$, we have

$F(l) = \int_{\bar{l}=-\infty}^{0} 0d\bar{l} + \int_{\bar{l}=0}^{l} \frac{\bar{l}}{2}d\bar{l} = 0 + \frac{\bar{l}^2}{4}\bigg|_0^l = \frac{l^2}{4}$

Putting these together, we have the complete expression,

$$F(l) = \begin{cases} 0, l \leq 0 \\ \frac{l^2}{4}, 0 < l < 2 \\ 1, 2 \leq l \end{cases}$$

(b) Using the definition of expectation for a continuous random variable, compute the expected length of the pasta, E(L).

Although some people will skip this step, I strongly recommend that you start your expression for expectation properly, by integrating all the way from $-\infty$ to ∞.

$$\begin{aligned} E(L) = \int_{-\infty}^{\infty} l \cdot f(l)dl &= \int_{-\infty}^{0} l \cdot 0dl + \int_{0}^{2} l \cdot \frac{l}{2}dl + \int_{2}^{\infty} l \cdot 0dl \\ &= 0 + \int_{0}^{2} l \cdot \frac{l}{2}dl + 0 = \int_{0}^{2} \frac{l^2}{2}dl = \frac{l^3}{6} = \frac{8}{6} \end{aligned}$$

2. The Warranty is Worth It

Suppose the life span of a particular (shoddy) server is a continuous random variable, T, with a uniform probability distribution between 0 and 1 year. The server comes with a contract that guarantees you money if the server lasts less than 1 year. In particular, if the server lasts t years, the manufacturer will pay you $g(t)$ = \$100$(1-t)^{\frac{1}{2}}$. Let $X = g(T)$ be the random variable representing the payout from the contract.

(a) Compute the expected payout from the contract, E(X) = E(g(T)), using the expression for the expectation of a function of a random variable.

$$E(X) = E(g(T)) = \int g(t)f(t)dt = \int 100(1-t)\frac{1}{2}dt$$

Let v = 1 - t and dv = -dt. Then

$$E(X) = \int -100v^{\frac{1}{2}}dv = -\frac{200}{3}(1-t)^{\frac{3}{2}}\Big|_{t=0}^{t=1} = \frac{200}{3} = \$66.67$$

3. (Lecture)#Fail

Suppose the length of Paul Laskowski's lecture in minutes is a continuous random variable C, with pmf $f(t) = e^{-t}$ for $t > 0$. This is an example of an exponential random variable, and it has some special properties. For example, suppose you have already sat through t minutes of the lecture, and are interested in whether the lecture is about to end immediately. In statistics, this can be represented by something called the *hazard rate*:

$$h(t) = \frac{f(t)}{1 - F(t)}$$

To understand the hazard rate, think of the numerator as the probability the lecture ends between time t and time $t + dt$. The denominator is just the probability the lecture does not end before time t. So you can think of the fraction as the conditional probability that the lecture ends between t and $t + dt$ given that it did not end before t.

Compute the hazard rate for C.

Solution:

The interpretation of the denominator means that $F(t)$ is the CDF for $f(t)$, so:

$$F(t) = \int_{-\infty}^{t} f(t)dt = \int_{-\infty}^{0} f(t)dt + \int_{0}^{t} f(t)dt = 0 + \int_{0}^{t} e^{-t}dt = -e^{-t}|_0^t = (-e^{-t}) - (-1) = 1 - e^{-t}$$

Plugging this into the hazard rate formula, we get $h(t) = \frac{e^{-t}}{1-(1-e^{-t})} = \frac{e^{-t}}{e^{-t}} = 1$, for $t > 0$.

For the given p.m.f., this hazard rate is constant with respect to time:

$$h(C) = 1$$

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4. Optional Advanced Exercise: Characterizing a Function of a Random Variable

Let X be a continuous random variable with probability density function $f(x)$, and let h be an invertible function where h^{-1} is differentiable. Recall that $Y = h(X)$ is itself a continuous random variable. Prove that the probability density function of Y is

$$g(y) = f(h^{-1}(y)) \cdot \left|\frac{d}{dy}h^{-1}(y)\right|$$

Solution:

Let us define the CDF of $f(x)$ to be $F(x) = P(X \leq x)$, and the CDF of $g(y)$ to be $G(y) = P(Y \leq y)$. Applying the function h, we get: $G(y) = P(h(X) \leq y)$. At this point, the inverse of h can be applied to show the relationship between $F(x)$ and $G(x)$; however, this depends on whether h^{-1} is an increasing or decreasing function.

Assuming h^{-1} is an increasing function:

$$G(y) = P(h(X) \leq y) = P(X \leq h^{-1}(y)) = F(h^{-1}(y))$$

Now we take the derivative of each side with respect to y:

$$\frac{d}{dy}(G(y)) = \frac{d}{dy}(F(h^{-1}(y)))$$

$$g(y) = f(h^{-1}(y))\frac{d}{dy}(h^{-1}(y))$$

Note that $\frac{d}{dy}(h^{-1}(y))$ is positive since we have assumed that h^{-1} is an increasing function.

Now, the same exercise can be repeated under the assumption that h^{-1} is a decreasing function:

$$G(y) = P(h(X) \leq y) = P(X \geq h^{-1}(y)) = 1 - F(h^{-1}(y))$$

Now we take the derivative of each side with respect to y:

$$\frac{d}{dy}(G(y)) = \frac{d}{dy}(1 - F(h^{-1}(y)))$$

$$g(y) = -f(h^{-1}(y))\frac{d}{dy}(h^{-1}(y))$$

Note that in this case, $\frac{d}{dy}(h^{-1}(y))$ is now negative since we have assumed that h^{-1} is a decreasing function, and $g(y)$ is still positive.

Since in either case $g(y)$ is positive, the p.d.f. of Y can be simplified to:

$$g(y) = f(h^{-1}(y)) \cdot \left|\frac{d}{dy}h^{-1}(y)\right|$$