Statistical Inference Notes

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4 April 2018

Variability

- Standard normals have a variance of 1; means of n standard normals have standard deviation $1/\sqrt{n}$

R code below for demonstration:

```
nosim <- 1000
n <- 10
sd(apply(matrix(rnorm(nosim * n), nosim), 1, mean))
```

```
## [1] 0.323238
```

```
1/sqrt(n)
```

```
## [1] 0.3162278
```

- Standard uniforms have variance 1/12; means of random samples of n uniforms have sd $\frac{1}{\sqrt{(12 \times n)}}$

R code below for demonstration:

```
nosim <- 1000
n <- 10
sd(apply(matrix(runif(nosim * n), nosim), 1, mean))
```

```
## [1] 0.08761867
```

```
1/sqrt(12 * n)
```

```
## [1] 0.09128709
```

- Poisson(4) have variance 4; means of random samples of n Poisson(4) have sd $\frac{2}{\sqrt{n}}$

R code below for demonstration:

```
nosim <- 1000
n <- 10
sd(apply(matrix(rpois(nosim * n, 4), nosim), 1, mean))
```

```
## [1] 0.6218639
```

```
2 / sqrt(n)
```

```
## [1] 0.6324555
```

- Fair coin flips have variance 0.25; means of random samples of n coin flips have sd $\frac{1}{2\sqrt{n}}$

R code below for demonstration:

```
nosim <- 1000
n <- 10
sd(apply(matrix(sample(0 : 1, nosim * n, replace = TRUE), nosim), 1, mean))
```

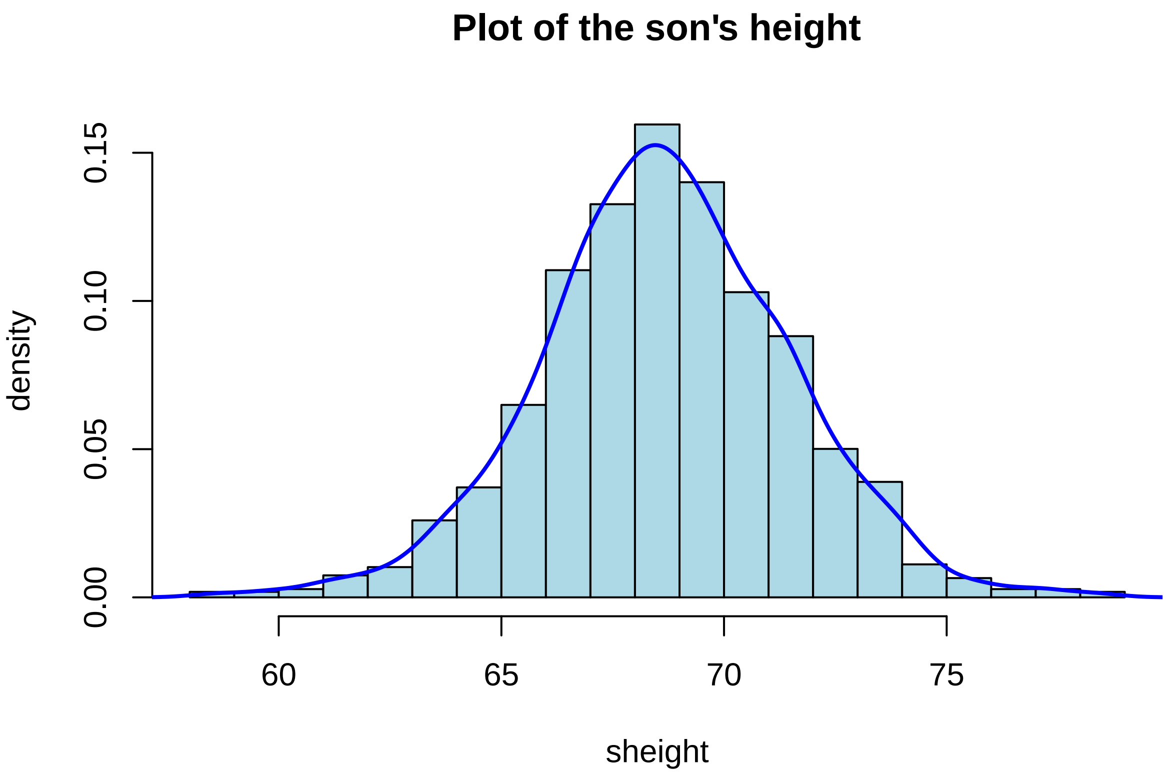
```
## [1] 0.1505728
```

```r
1 / (2 * sqrt(n))
```

```
## [1] 0.1581139
```

- Data example

```r
library(UsingR); data(father.son);
x <- father.son$sheight
n <-length(x)
hist(x, breaks = 20, prob = TRUE, col = "lightblue", main = "Plot of the son's height",
     ylab = "density", xlab = "sheight")
lines(density(x), col="blue", lwd=2)
```




```r
round(c(var(x), var(x) / n, sd(x), sd(x) / sqrt(n)),2)
```

```
## [1] 7.92 0.01 2.81 0.09
```

- Variance of X: var(x) = **7.92** (in sq inches)
- Standard deviation of X: sd(x) = **2.81** (in inches)
- Variablility in the average of 10 childrens heights: var(x)/n = **0.01**
- Variablility in the average of 10 childrens heights: sd(x)/sqrt(n) = **0.09** (standard error estimate)

Distributions

Bernoulli distribution

PMF $P(X = x) = p^x(1-p)^{1-x}$

The mean of a Bernoulli random variable is p and the variance is $p(1-p)$

It is typical to call $X = 1$ as a "success" and $X = 0$ as a "failure"

1. Approximately 68%, 95% and 99% of the normal density lies within 1, 2 and 3 standard deviations from the mean, respectively
2. -1.28, -1.645, -1.96 and -2.33 are the 10^{th}, 5^{th}, 2.5^{th} and 1^{st} percentiles of the standard normal distribution respectively
3. By symmetry, 1.28, 1.645, 1.96 and 2.33 are the 90^{th}, 95^{th}, 97.5^{th} and 99^{th} percentiles of the standard normal distribution respectively

Figure 1:

Binomial distribution

- The binomial random variables are obtained as the sum of iid (independed identically distributed) Bernoulli trials $X = \sum_{i=1}^{n} X_i$

Binomial mass function: $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$

$\binom{n}{x} = \frac{n!}{x!(n-x)!}$ and $\binom{n}{0} = \binom{n}{n} = 1$

- Suppose a friend has 8 children 7 of which are girls and none are twins
- If each gener has an independent 50% probability for each birth, what is the probability of getting 7 or more girls out of 8 births?

$\binom{8}{7} .5^7 (1-.5)^1 + \binom{8}{8} .5^8 (1-.5)^0 \approx 0.04$

```r
choose(8, 7) * 0.5^8 + choose(8, 8) * 0.5^8
```

```
## [1] 0.03515625
```

```r
pbinom(6, size = 8, prob = 0.5, lower.tail = FALSE)
```

```
## [1] 0.03515625
```

Normal distribution

A distribution is **normal/Gaussian** with a mean μ and a variance σ^2 if the variable density is $(2\pi\sigma^2)^{-\frac{1}{2}} e^{\frac{(-x-\mu)^2}{2\sigma^2}}$

If X is a random varabile this the density above then $E[X] = \mu$ and $Var(X) = \sigma^2$ written $X \sim N(\mu, \sigma^2)$

For the standard normal distribution $\mu = 0$ and $\sigma = 1$.

Within -1 and 1 standard deviation area = 68%

Within -2 and 2 standard deviations area = 95% with 2.5% in either tail (confidence levels)

Within -3 and 3 = 99%

The 10% and 90% quantiles in a standard normal distribution are at -1.28 and +1.28. For a non-standard normal they are at $\mu - 1.28 * \sigma$ and $\mu + 1.28 * \sigma$

In R you can use **qnorm()** example: *qnorm(.95, mean = mu, sd = sd)* or for a standard normal $\mu + \sigma * 1.645$

What is the probability that a $N(\mu, \sigma^2)$ random variable is larger than x? You could estimate how many standard deviations x is from the mean by using: $\frac{X-\mu}{\sigma}$

Example from the slides:

Assume that the number of daily ad clicks for a company is approximately normally distributed with a mean of 1020 and a standard deviation of 50. What is the probability of getting more than 1,160 clicks in a day? It is not very likely since 1,160 is 2.8 standard deviation from the mean.

```r
pnorm(1160, mean = 1020, sd = 50, lower.tail = FALSE)
```

```
## [1] 0.00255513
```

```r
pnorm(2.8, lower.tail = FALSE)
```

```
## [1] 0.00255513
```

Second example from the slides:

Assume that the number of daily ad clicks for a company is approximately normally distributed with a mean of 1020 and a standard deivation of 50. What number of daily ad clicks would represent the one where 75% of days have fewer clicks (assuming days are independent and identically distributed)?

```r
qnorm(0.75, mean = 1020, sd = 50)
```

```
## [1] 1053.724
```

Poisson distribution

This distribution is used to model counts.

The mass function is : $P(X = x; \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$ for $x = 0, 1, ...$

The mean is λ

The variance is λ

x ranges from 0 to ∞

Used for:

- Modeling count data
- Modeling event-time or survival data
- Modeling contingency tables
- Approximating binomials when n is large and p is small

Used to model rates:

$X \sim Poisson(\lambda t)$ where $\lambda = E[\frac{X}{t}]$ is the expected count per unit of time and t is the total monitoring time.

Example of modeling rates:

The number of people that show up at a bus stop is Poisson with a mean of 2.5 per hour.

If watching the bus stop for 4 hours, what is the probability that 3 or fewer people show up for the whole time?

```r
ppois(3, lambda = 2.5 * 4)
```

```
## [1] 0.01033605
```

Example of modeling a binomial when n is large and p is small:

- $X \sim Binomial(n, p)$
- $\lambda = np$
- n gets large
- p gets small

We flip a coin with success probability of 0.01 five hundred times.

What is the probability of 2 or fewer successes?

```
pbinom(2, size = 500, prob = 0.01)
```

```
## [1] 0.1233858
```

```
ppois(2, lambda = 500 * 0.01)
```

```
## [1] 0.124652
```

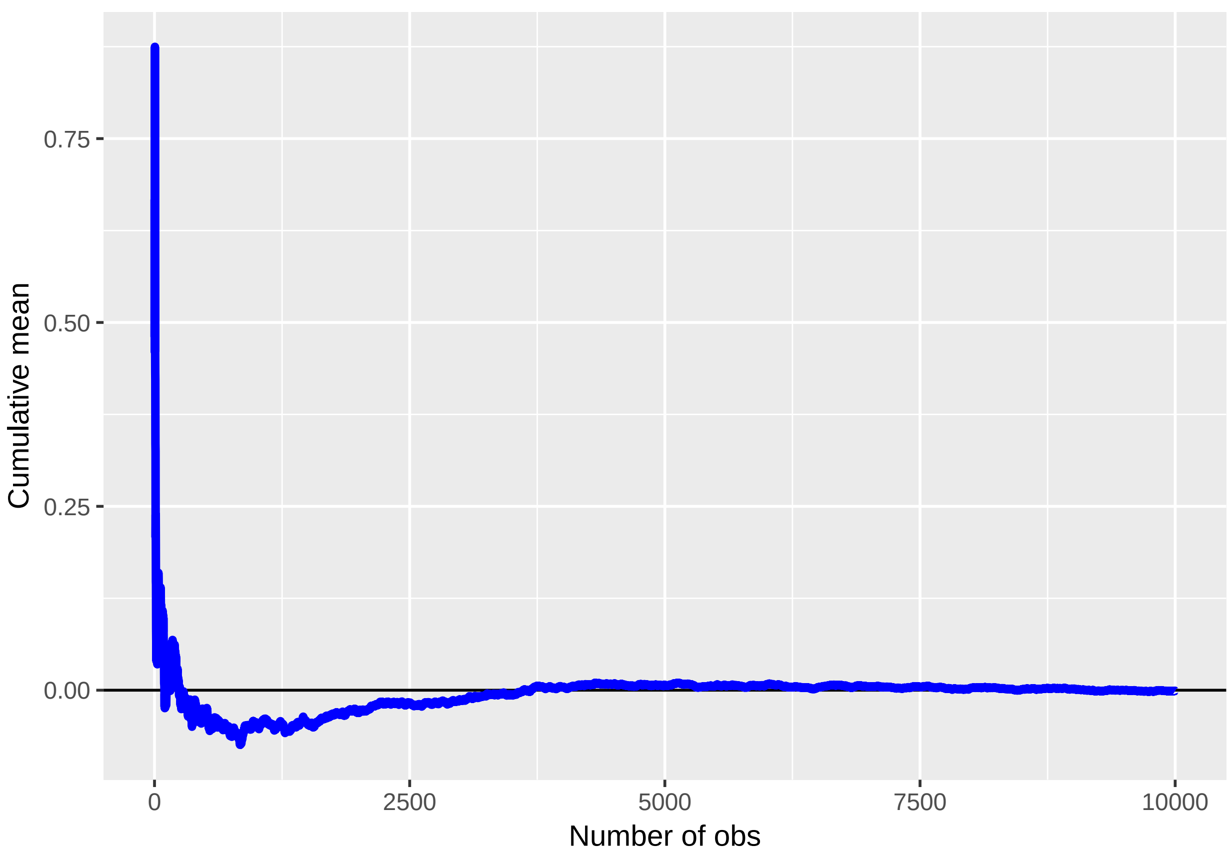
Asymptotics

Asymptotics is the term for the behavior of statistics as the sample size (or some other relevant quantity) limits to infinity (or some other relevant number)

The law of large numbers

- The average limits to what it is estimating, the population mean

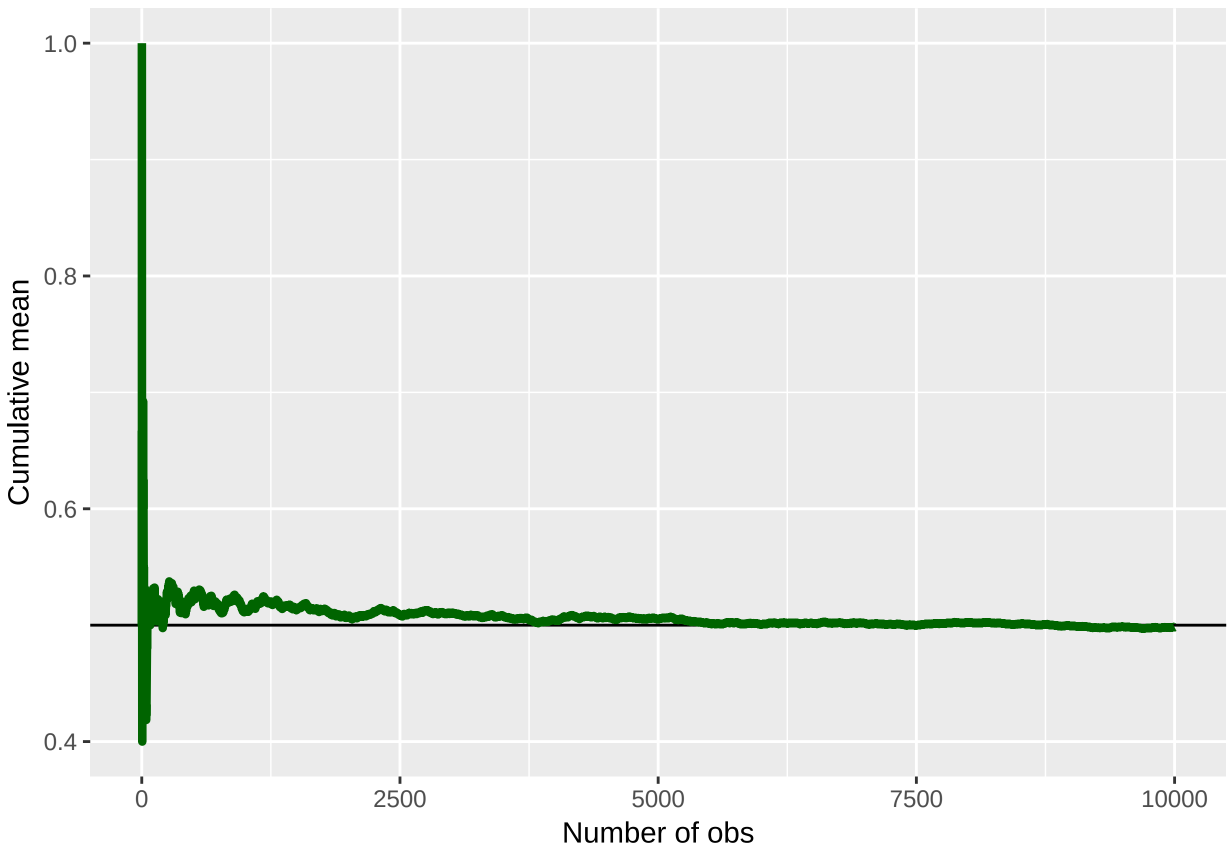
```
n <- 10000
means <- cumsum(rnorm(n))/(1:n)
library(ggplot2)
g <- ggplot(data.frame(x = 1:n, y = means), aes(x = x, y = y))
g <- g + geom_hline(yintercept = 0) + geom_line(color = "blue",size = 1.5)
g <- g + labs(x = "Number of obs", y = "Cumulative mean")
g
```



```r
rm(list=ls(all=TRUE))
```

Coin flip example:

```r
n <- 10000
means <- cumsum(sample(0:1, n, replace = TRUE))/(1:n)
g <- ggplot(data.frame(x = 1:n, y = means), aes(x = x, y = y))
g <- g + geom_hline(yintercept = 0.5) + geom_line(color = "darkgreen", size = 1.5)
g <- g + labs(x = "Number of obs", y = "Cumulative mean")
g
```




```r
rm(list=ls(all=TRUE))
```

An estimator is **consistent** if it converges to what you want to estimate.

The Law of Large Numbers says that the sample mean of an **iid** sample (independent and identically distributed) is consistent for the population mean. The sample standard deviation and variance of **iid** randoms variables are also consistent.

Central Limit Theorem

- One of the most important theorems in statistics
- States that the distribution of averages of **iid** variables (properly normalized) becomes that of a standard normal as the sample size increases
- $$\frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} = \frac{\text{Estimate} - \text{Mean of estimate}}{\text{Std. Err. of estimate}}$$

has a distribution like that of a standard normal for large n.

- $\bar{X}_n$ is approximately $N(\mu, \frac{\sigma^2}{n})$

Confidence intervals

$\bar{X}$ is approximately normal with mean μ and sd $\frac{\sigma}{\sqrt{n}}$. The probability that $\bar{X}$ is bigger than $\mu + \frac{2\sigma}{\sqrt{n}}$ or smaller than $\mu - \frac{2\sigma}{\sqrt{n}}$ is 5%. $\bar{X} \pm \frac{2\sigma}{\sqrt{n}}$ is called a 95% interval for μ.

Confidence interval for the average height of sons:

```
library(UsingR)
data(father.son)
x <- father.son$sheight
(mean(x) + c(-1, 1) * qnorm(0.975) * sd(x)/sqrt(length(x)))/12
```

```
## [1] 5.709670 5.737674
```

Wald confidence interval (Sample proportions)

Replace p with $\hat{p}$ in the standard error.

For 95% intervals $\hat{p} \pm \frac{1}{\sqrt{n}}$ is a quick CI estimate for p

Example:

In a random sample of 100 likely voters, 56 intend to vote for you. Can you relax?

Using the Wald confidence interval:

$\frac{1}{\sqrt{100}} = 0.1$ so $\hat{p} \pm 0.1 = 56 \pm 0.1 = (0.46, 0.66)$ for the 95% confidence interval. Keep campaigning!

Rough guidelines: 100 for 1 decimal place, 10,000 for 2, 1,000,000 for 3.

Binomial interval for the same problem, first manually then using **binom**:

```
0.56 + c(-1, 1) * qnorm(0.975) * sqrt(0.56 * 0.44/100)
```

```
## [1] 0.4627099 0.6572901
```

```
binom.test(56, 100)$conf.int
```

```
## [1] 0.4571875 0.6591640
## attr(,"conf.level")
## [1] 0.95
```

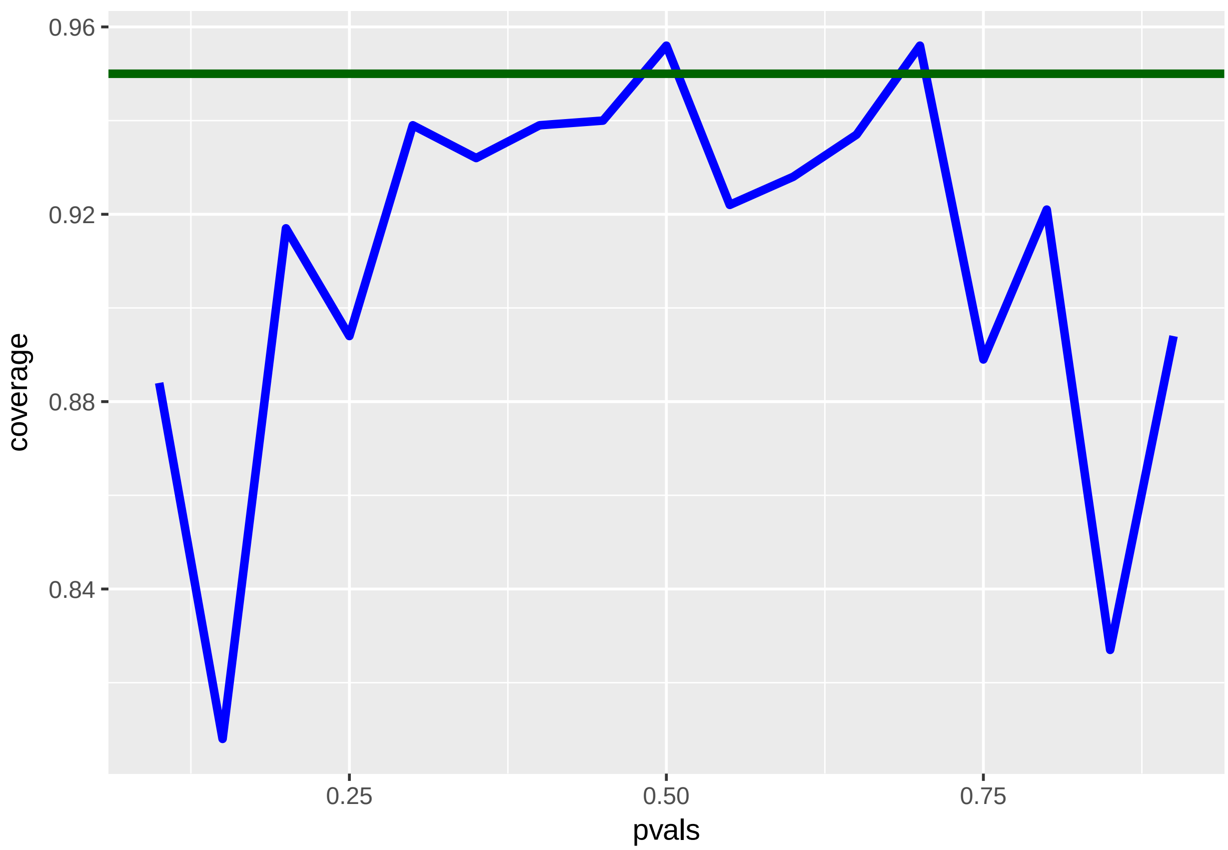
Simulation:

Flip a coin and calcualte the percentage of time the Wald interval covers the true coin probability used to generate the data.

```
# Twenty coin flips in each simulation
n <- 20
# True values varied between 0.1 and 0.9 step through by 0.05
pvals <- seq(0.1, 0.9, by = 0.05)
# 1000 simulations
nosim <- 1000

# Loop through and for each true success probability
coverage <- sapply(pvals, function(p) {
  # Generate a 1000 sets of 10 coin flips and take the
```

```r
  # sample proporation
  phats <- rbinom(nosim, prob = p, size = n)/n
  # Calculate lower limit
  ll <- phats - qnorm(0.975) * sqrt(phats * (1 - phats)/n)
  # Calculate uppper limit
  ul <- phats + qnorm(0.975) * sqrt(phats * (1 - phats)/n)
  # Calculate the proportion of time the cover the true value
  # of p that was used to simulate the data
  mean(ll < p & ul > p)
})
df <- data.frame(coverage,pvals)
ggplot(data = df, aes(x=pvals, y=coverage)) + geom_line(color = "blue", size=1.5) +
    geom_hline(yintercept = 0.95, color="darkgreen", size = 1.5)
```



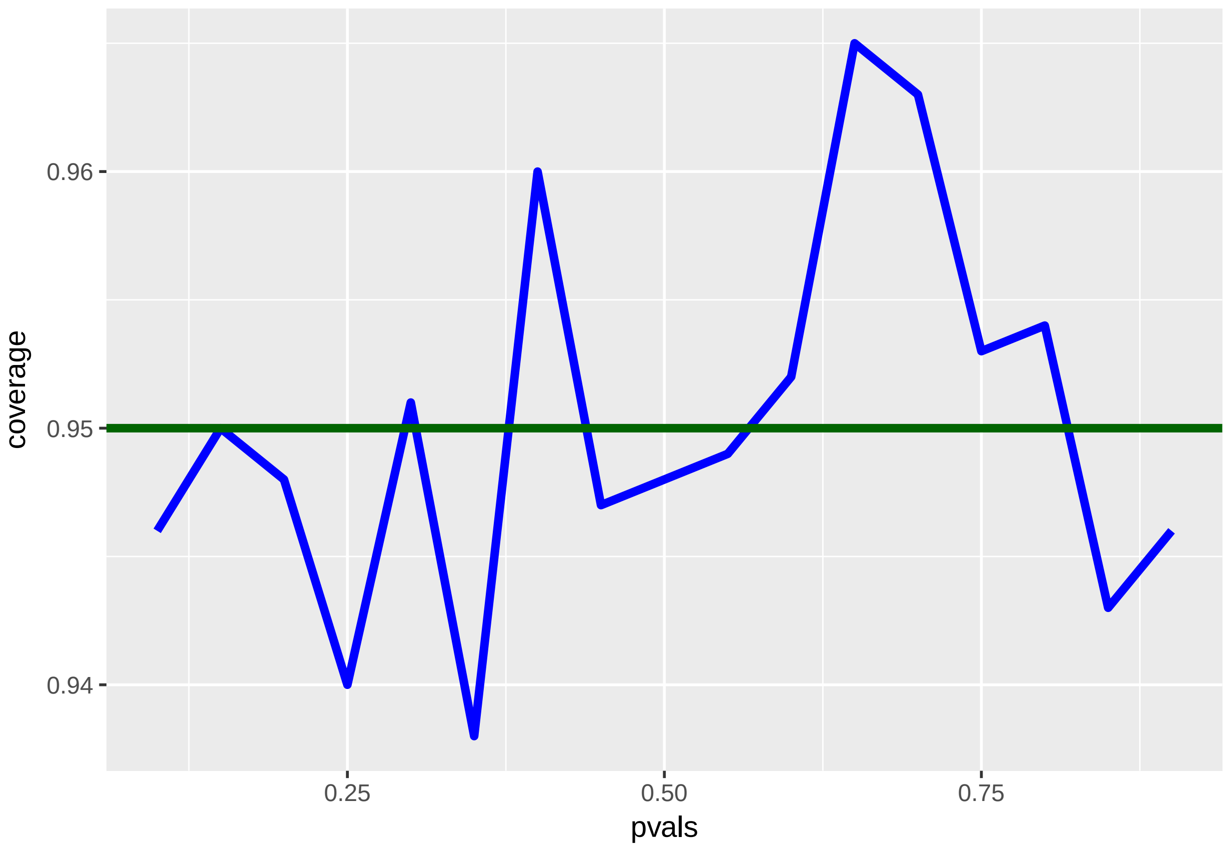
Better than 95% for 0.5, but it is very far off for smaller/larger values of n. In the case above n isn't large enough for the CLT to be applicable for many of the values of p.

Quick fix (Agresti/Coull interval) form the interval as: $\frac{X+2}{n+4}$ adding two successes and failures.

First show how the CLT improves with larger values of n.

```r
# Twenty coin flips in each simulation
n <- 10000
# True values varied between 0.1 and 0.9 step through by 0.05
pvals <- seq(0.1, 0.9, by = 0.05)
# 1000 simulations
nosim <- 1000
```

```
# Loop through and for each true success probability
coverage <- sapply(pvals, function(p) {
  # Generate a 1000 sets of 10 coin flips and take the
  # sample proporation
  phats <- rbinom(nosim, prob = p, size = n)/n
  # Calculate lower limit
  ll <- phats - qnorm(0.975) * sqrt(phats * (1 - phats)/n)
  # Calculate uppper limit
  ul <- phats + qnorm(0.975) * sqrt(phats * (1 - phats)/n)
  # Calculate the proportion of time the cover the true value
  # of p that was used to simulate the data
  mean(ll < p & ul > p)
})
df <- data.frame(coverage,pvals)
ggplot(data = df, aes(x=pvals, y=coverage)) + geom_line(color = "blue", size=1.5) +
    geom_hline(yintercept = 0.95, color="darkgreen", size = 1.5)
```

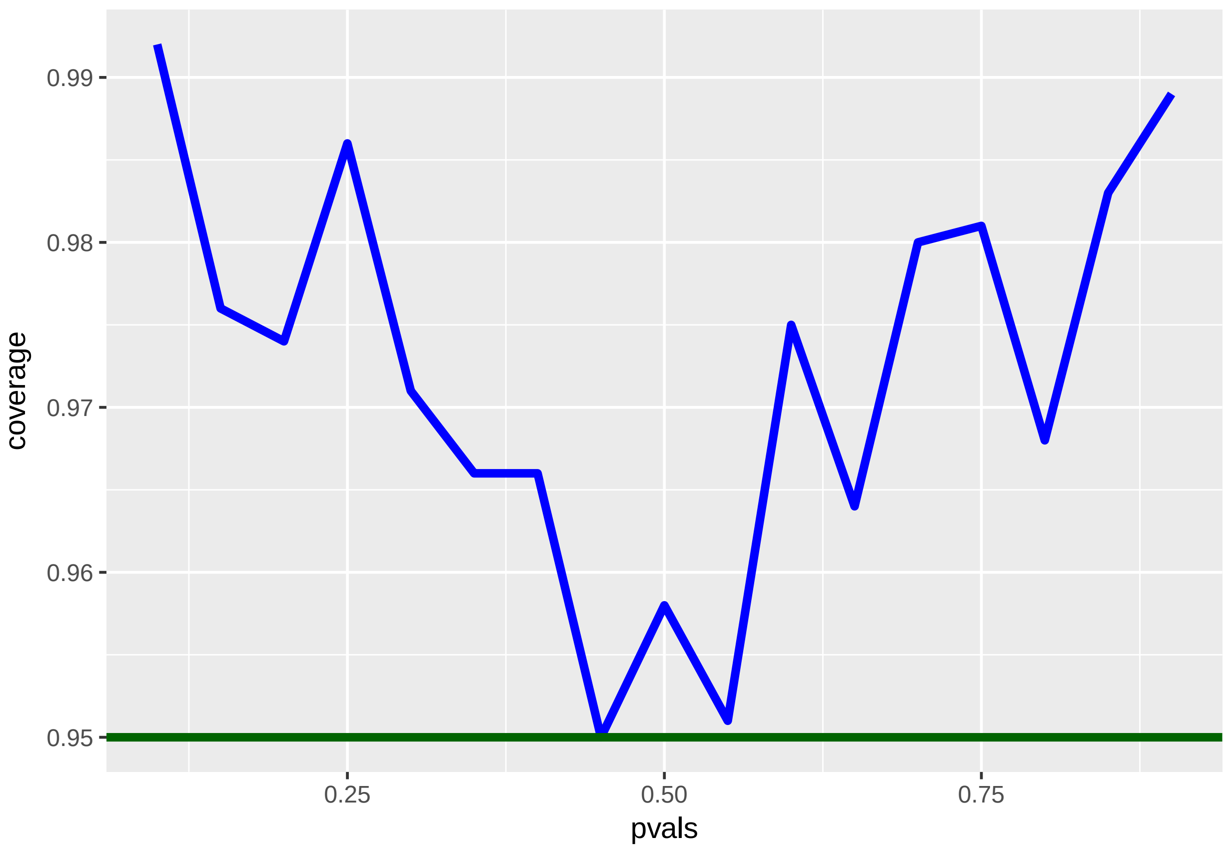


Now try the Agresti/Coull interval on the $n = 20$ simulation by adding two successes and two failures.

```
# Twenty coin flips in each simulation
n <- 20
# True values varied between 0.1 and 0.9 step through by 0.05
pvals <- seq(0.1, 0.9, by = 0.05)
# 1000 simulations
nosim <- 1000

# Loop through and for each true success probability
```

```r
coverage <- sapply(pvals, function(p) {
  # Generate a 1000 sets of 10 coin flips and take the
  # sample proportion
  phats <- (rbinom(nosim, prob = p, size = n) + 2)/(n + 4)
  # Calculate lower limit
  ll <- phats - qnorm(0.975) * sqrt(phats * (1 - phats)/n)
  # Calculate uppper limit
  ul <- phats + qnorm(0.975) * sqrt(phats * (1 - phats)/n)
  # Calculate the proportion of time the cover the true value
  # of p that was used to simulate the data
  mean(ll < p & ul > p)
})
df <- data.frame(coverage,pvals)
ggplot(data = df, aes(x=pvals, y=coverage)) + geom_line(color = "blue", size=1.5) +
    geom_hline(yintercept = 0.95, color="darkgreen", size = 1.5)
```



In general the add 2 successes and 2 failures should be used for small values of n rather than the Wald interval.

Poisson interval

A nuclear pump failed 5 times of of 94.32 days, give a 95% confidence interfal for the failure rate per day?

- $X \sim Poisson(\lambda t)$
- Estimate $\hat{\lambda} = \frac{X}{t}$
- $Var(\hat{\lambda}) = \frac{\lambda}{t}$
- $\frac{\hat{\lambda}}{t}$ is the variance estimate

Note: Failure rate is λ and number of days is t

R code for the calculation:

```
# Number of events
x <- 5
# Monitoring time
t <- 94.32
# Estimate of the rate
lambda <- x/t
# Confidence interval estimate
round(lambda + c(-1, 1) * qnorm(0.975) * sqrt(lambda/t), 3)
```

```
## [1] 0.007 0.099
```

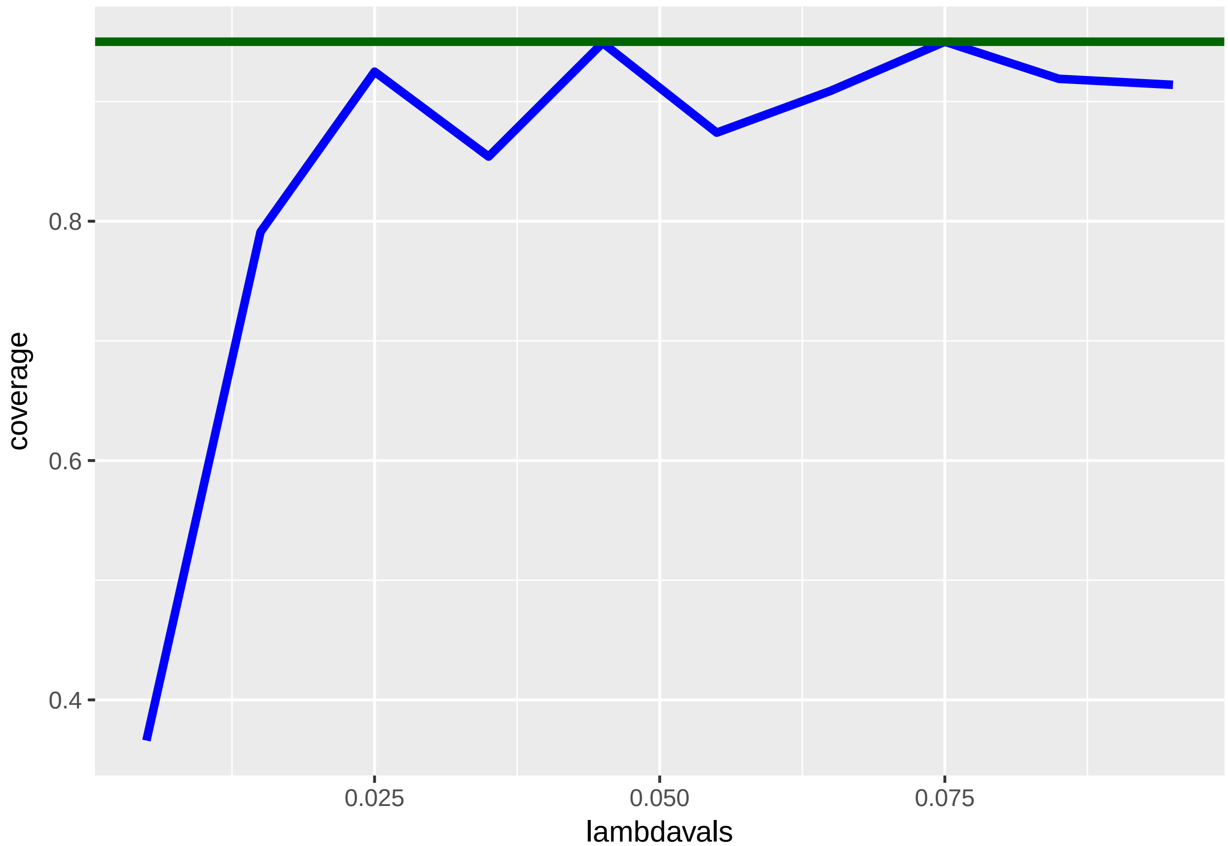
```
# Exact interval (could be wider than desired)
poisson.test(x, T = 94.32)$conf
```

```
## [1] 0.01721254 0.12371005
## attr(,"conf.level")
## [1] 0.95
```

Simulation of the coverage rate using Poisson:

```
lambdavals <- seq(0.005, 0.1, by = 0.01)
nosim <- 1000
t <- 100
coverage <- sapply(lambdavals, function(lambda) {
 lhats <- rpois(nosim, lambda = lambda * t)/t
 ll <- lhats - qnorm(0.975) * sqrt(lhats/t)
 ul <- lhats + qnorm(0.975) * sqrt(lhats/t)
 mean(ll < lambda & ul > lambda)
})

df <- data.frame(coverage,lambdavals)
ggplot(data = df, aes(x=lambdavals, y=coverage)) + geom_line(color = "blue", size=1.5) +
    geom_hline(yintercept = 0.95, color="darkgreen", size = 1.5)
```

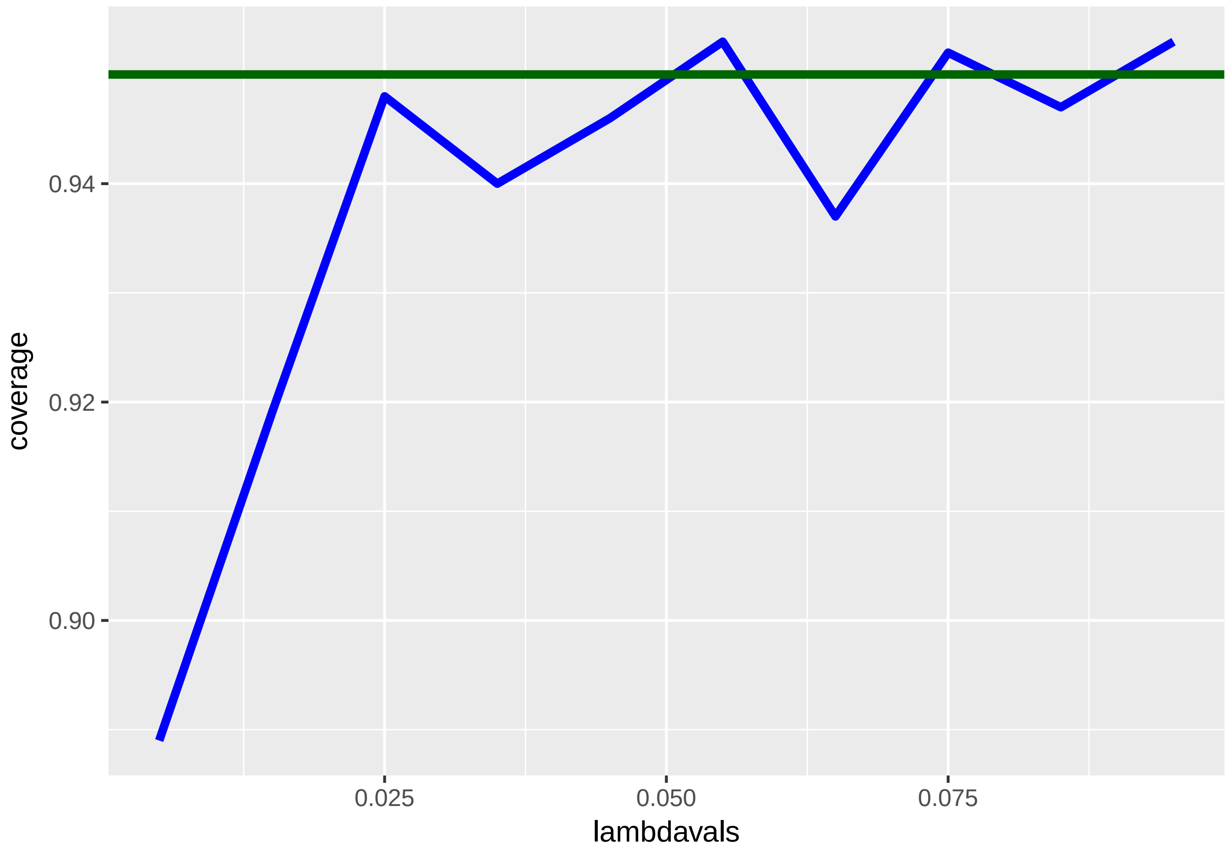


The result is really bad for small values of lambda.

Next increase t to 1000:

```
lambdavals <- seq(0.005, 0.1, by = 0.01)
nosim <- 1000
t <- 1000
coverage <- sapply(lambdavals, function(lambda) {
 lhats <- rpois(nosim, lambda = lambda * t)/t
 ll <- lhats - qnorm(0.975) * sqrt(lhats/t)
 ul <- lhats + qnorm(0.975) * sqrt(lhats/t)
 mean(ll < lambda & ul > lambda)
})

df <- data.frame(coverage,lambdavals)
ggplot(data = df, aes(x=lambdavals, y=coverage)) + geom_line(color = "blue", size=1.5) +
    geom_hline(yintercept = 0.95, color="darkgreen", size = 1.5)
```



Summary

- The **LLN** (Law of Large Numbers) states that averages of **iid** (Independent Identically Distributed) samples converge to the population means that they are estimating.
- The **CLT** (Central Limit Theorem) states that averages are approximately normal, with distributions
  - centered at the population mean
  - with standard deviation equal to the standard error of the mean
  - CLT gives no guarentee that n is large enough
- Taking the mean and adding and subtracting the relevant normal quantile times the **SE** (Standard Error) yields a confidence interval for the mean
  - Adding and subtracting 2 SEs works for 95% intervals
- Confidence intervals get wider as the coverage increases
- Confidence intervals get narrower with less variability or larege sample sizes
- The Poisson and binomial case have exact intervals that don't reqire the CLT
  - But a quick fix for small sample size binomial calculation is to add 2 successes and failures

Appendix

- mu μ is the mean
- sigma σ is the standard deviation
- lambda λ is the $E[\frac{X}{t}]$
- standard error of the sample means is the square root of its vairance
- variance of a sample mean is σ^2/n esitmated with s^2/n
- $s/\sqrt{n}$ standard error

- `rnorm(n,mean,sd)` generates n independent random normal samples with the specified mean and sd. Defaults are mean 0 and sd 1.
- `sd(apply(matrix(rnorm(10000),1000),1,mean))` returns the standard deviation of 1000 averages each of a sample of 10 random normals
- $2/\sqrt{n}$ is the sd of n Poisson(4)
- Averages of 10 Poisson(4) samples sd = 2/sqrt(10)
- `sd(apply(matrix(rpois(10000,4),1000),1,mean))` to check the above

Quiz

1. What is the variance of the distribution of the average an IID draw of n observations from a population with mean μ and variance σ^2?

Ans: $\frac{\sigma^2}{n}$

2. Suppose that diastolic blood pressures (DBPs) from men aged 35-44 are normally distributed with a mean of 80mmHg and a standard deviation of 10 mmHg. About what is the probability that a random 35-44 year old has a DBP less than 70 ?

Ans:

```r
pnorm(70, mean = 80, sd = 10)
```

```
## [1] 0.1586553
```

3. Brain volume for adult women is normally distributed with a mean of about 1,100 cc for women with a standard deviation of 75 cc. What brain volume represents the 95th percentile?

Ans:

```r
qnorm(0.95, mean = 1100, sd = 75)
```

```
## [1] 1223.364
```

4. Refer to the previous question. Brain volume for adult women is about 1,100 cc for women with a standard deviation of 75 cc. Consider the sample mean of 100 random adult women from this population. What is the 95th percentile of the distribution of that sample mean?

Ans:

```r
qnorm(0.95, mean = 1100, sd = 75/sqrt(100))
```

```
## [1] 1112.336
```

5. You flip a fair coin 5 times, about what's the probability of getting 4 or 5 heads?

Ans:

```r
# 3 for q since we want 4 or 5 heads, 5 flips, probability 0.5
pbinom(3, size = 5, prob = 0.5, lower.tail = FALSE)
```

```
## [1] 0.1875
```

6. The respiratory disturbance index (RDI), a measure of sleep disturbance, for a specific population has a mean of 15 (sleep events per hour) and a standard deviation of 10. They are not normally distributed. Give your best estimate of the probability that a sample mean RDI of 100 people is between 14 and 16 events per hour?

Ans:

```r
# Use pnorm, even tho the distribution isn't normal. Subtraction the probabilty of
# the result begin at 1 sd from 14 from the probability of it being 1 sd from 16
# should give the probability of it being between 14-16 given the mean is 15.
pnorm(16, mean = 15, sd = 1) - pnorm(14, mean = 15, sd = 1)
```

```
## [1] 0.6826895
```

7. Consider a standard uniform density. The mean for this density is .5 and the variance is 1 / 12. You sample 1,000 observations from this distribution and take the sample mean, what value would you expect it to be near?

Ans:

```r
mean(rnorm(1000, mean = 0.5, sd = sqrt(1/12)))
```

```
## [1] 0.4917305
```

8. The number of people showing up at a bus stop is assumed to be Poisson with a mean of 5 people per hour. You watch the bus stop for 3 hours. About what's the probability of viewing 10 or fewer people?

Ans:

```r
ppois(10, lambda = 5 * 3)
```

```
## [1] 0.1184644
```