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TA, Physics 211
4/14/2017

An Angular Momentum Nonconservation Problem

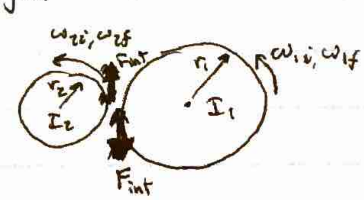
Problem. Two round objects, one with radius r_1, moment of inertia I_1, and angular velocity ω_{1i} and the other analogously with r_2, I_2 and ω_{2i}, each mounted at their geometric center through parallel axes, are brought into contact. What are their final angular velocities after the "angular collision"?

Solution. After the collision, the surface speeds of the two objects will be the same, $-r_1\omega_{f1} = r_2\omega_{f2}$. This has the two unknowns we want, so we need another equation.

Define the angular analog of impulse via the following manipulation. Given a time-dependent net torque $\tau(t)$ on an object with (constant) moment of inertia I, we know the response angular acceleration will be $\alpha(t)$. Integrating both sides from t_1 to t_2,

$$\begin{aligned}\int_{t_1}^{t_2} \tau_{net}(t)\,dt &= \int_{t_1}^{t_2} I\alpha(t)\,dt \\ &= I\left[\omega(t)\right]_{t_1}^{t_2} \\ &= I(\omega_2-\omega_1) \\ &= \Delta L.\end{aligned}$$

Applying this to the current problem, and considering the two objects separately, we find that each has a single torque acting on it due to the interaction $F_{int}(t)$ between the two (where the origin of each rotation is the given rotation axis). Then



$$\begin{aligned}I_1(\omega_{1f}-\omega_{1i}) &= \int_{t_i}^{t_f} \tau_{1\,net}(t)\,dt \\ &= \int_{t_i}^{t_f} r_1 F_{int}(t)\,dt \\ &= r_1\int_{t_i}^{t_f} F_{int}(t)\,dt\end{aligned}$$

$$\begin{aligned}I_2(\omega_{2f}-\omega_{2i}) &= \int_{t_i}^{t_f} \tau_{2\,net}(t)\,dt \\ &= r_2\int_{t_i}^{t_f} F_{int}(t)\,dt\end{aligned}$$

Hence $\frac{I_1}{r_1}(\omega_{1f}-\omega_{1i}) = \frac{I_2}{r_2}(\omega_{2f}-\omega_{2i})$. Using $-r_1\omega_{1f} = r_2\omega_{2f}$ to solve for ω_{1f}, then proceeding to solve for ω_{2f}, yields

$$\begin{cases}\omega_{1f} = \dfrac{\frac{I_1}{r_1}\omega_{1i} - \frac{I_2}{r_2}\omega_{2i}}{\frac{I_1}{r_1} + \frac{I_2 r_1}{r_2^2}} \\[2ex] \omega_{2f} = \dfrac{\frac{I_2}{r_2}\omega_{2i} - \frac{I_1}{r_1}\omega_{1i}}{\frac{I_2}{r_2} + \frac{I_1 r_2}{r_1^2}}\end{cases}$$