CHESS AROUND A BLACK HOLE

Course notes
Winter 2013

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Dear reader,

You probably think you know what gravity is. I'm here to break it to you gently that, you are, unfortunately, sadly mistaken. Nobody knows what gravity *really* is—but physicists probably know better than most (by which I mean they have *precisely stated* ideas that they believe for reasons you would probably find convincing). Popular science often tries to dig inside physicists' heads to help other people understand what's so compelling about, say, Einstein's theory of gravity, but unless you can show me someone who demonstrates a profound understanding of relativity based on reading those articles, I maintain that they do a pretty poor job of it. But it's a starting point, and that's most likely the extent of your knowledge right now. So keep a mental inventory of the things you believe now, but be prepared to step beyond the limits of your intuition and revise that list as you come to understand gravity better. Most of all, relish those special wtf moments, when all the pieces fit together and make sense in a totally unexpected way—in my experience, they become the most enjoyable moments of learning relativity, but more importantly the stark paradigm shift helps reveal how much of our thinking could be imprecise and totally inaccurate. But that's a story we'll resume later.

Before we dive into gravity, I wanted to acquaint you with the format of the course. I'll do minimal lecturing; most of the learning will take place when we're doing challenge problems or going over them as a class. Right now you're reading these notes, but when you join back in with the class this will make more sense: spend no more than the allotted time on each problem, even if you don't finish. The goal is for you to think about the problem on your own and get an intuition for what would otherwise be a highly nonintuitive situation; at that point, you'll probably learn more quickly from seeing my answer and other students' answers than from continuing to work on your own.

Another note: once you join the class, you'll be working in a group. You should prioritize helping your group members over working ahead on your own—first of all, you'll probably learn something from hearing what confuses your group members, and second of all I'll be pacing the class for the slowest people instead of the fastest. Likewise, if you're having difficulty, please ask your group for help so that they can catch you up.

Final note: Please send me any feedback you think would help improve something I'm doing badly or reinforce something I'm doing well! This is your class, so make it a good one.

Thanks for the privilege,

Gabriel

SESSION 1:
A Bishop on Battle Mountain

You probably know how to play chess… on a regular chessboard. (If not, check out `chessusa.com/CHESS_RULES.html`). But what if I gave you a different chessboard?

Without further ado…

Challenge Problem 1: How do you play chess on this chessboard? How do the following pieces move?

- Rook
- Bishop
- Knight
- Pawn

Spend no more than 10 minutes on this question. You'll find my answers on the next page.

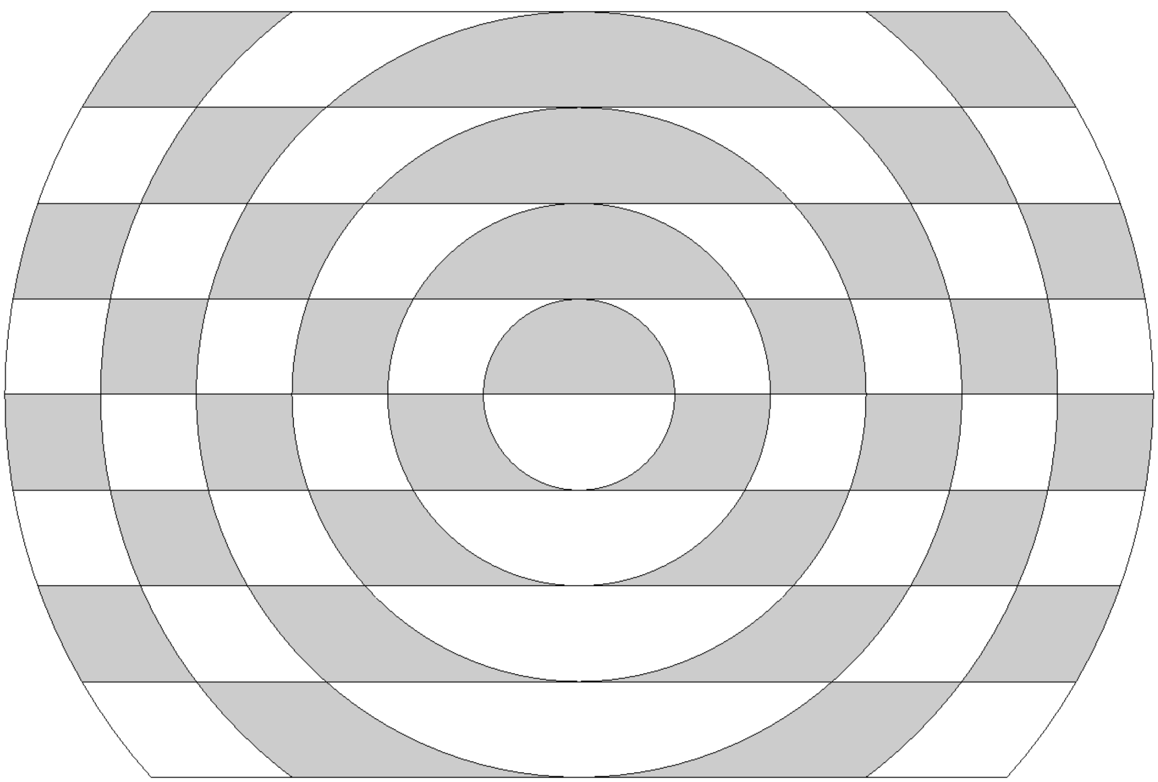


Figure 1: Singularity chessboard. Draw here.

My Answer: Check out the figures below. Here's my reasoning.

For the rook, we're used to thinking about it as moving along horizontal or vertical lines that go all the way across the board. But here's another way of stating that rule: "Each time a rook enters a square, the rook may either stop in that square or continue through the side opposite the one it entered."

(You'll notice one of these versions makes a statement pertaining to the whole chessboard or movement, while the other makes a statement about each individual step in a movement. We call the former a *global* rule and the latter a *local* one. For reasons that will soon become apparent, relativity relies largely on local rules.)

The only trick here is that it's much more difficult to find the opposite side on this chessboard than it was in a regular chessboard. But here's how it's the same: *every* square has four sides and four corners; opposite sides are the ones that don't touch. This works completely, as long as you legislate that there is a vertex between the two middle squares. I'm not saying it's the only right answer—but it works. This is how I got the rook's path, and I used similar rule switches to arrive at the paths of the other pieces. See if you can figure out what they are.

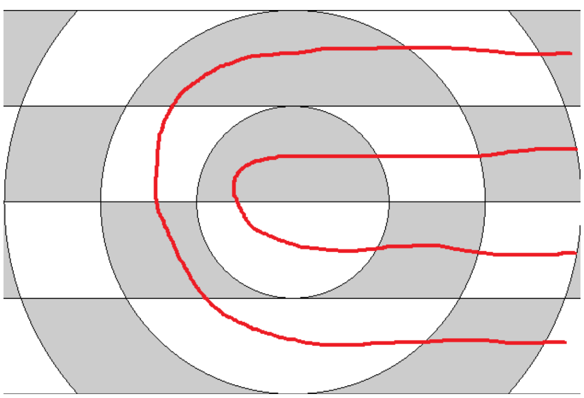


Figure 2. The rook

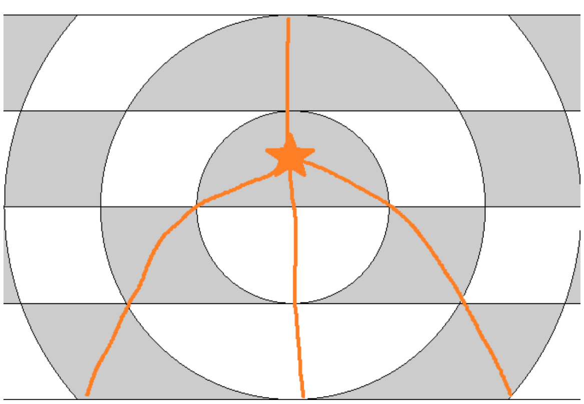


Figure 3. The bishop

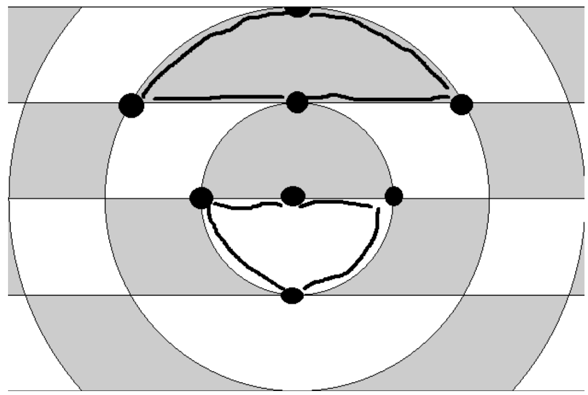


Figure 4. The sides and vertices of a "square"

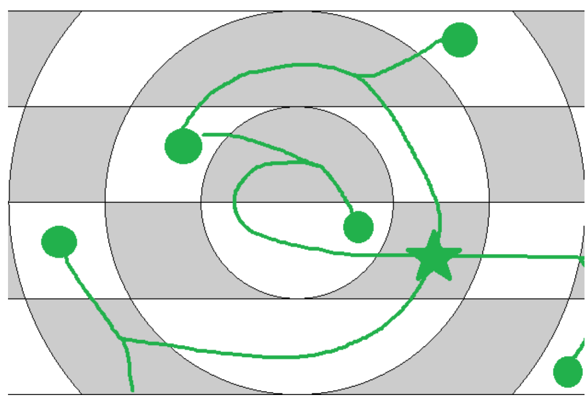


Figure 5. The knight. I used "2 forward, 1 left/right."

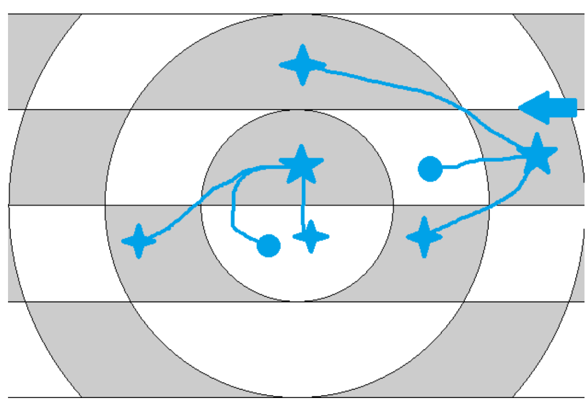


Figure 6. The pawn. I assumed pawns attack diagonally forward, instead of e.g. 1 right, 1 forward.

There are a number of counterintuitive results that we get when we use these rules.

- The rook's path is just weird.
- If we allow the bishop to move through the vertex at the center of the board, then we have to allow the bishop to change colors. It's no longer true that both a bishop moves from one corner to the opposite corner *and* a bishop stays on its original color; we have to give up the latter in order to use the former.
- Whereas in the regular chessboard the knight could never move next to itself, under these rules it certainly can.
- The pawn can now attack a square it can move into. This is if you assume a pawn attacks diagonally forward. You get a different answer if you assume pawns attack 1 square to the right/left of the square in front of them—but then a pawn would be able to attack its own square.
- The overall trajectory of a pawn is, as usual, the same as that of a rook—which means a pawn can never get queened, unless you legislate that it has to return to the side it started from.

You may be noticing that a lot of this weirdness occurs mostly at the origin. As you go far away from it, it looks and behaves more and more like a regular chessboard. As you'll soon see, this is because the center is a *singularity*—stuff breaks down there.

(You may have heard this term associated with black holes, and that is indeed what this chessboard is supposed to represent. However, as we'll see in a few weeks, this isn't really how black holes work.)

Differential Geometry, i.e. The Pythagorean Theorem But Not

In order to understand better where this chessboard came from and why it works the way it does, we'll now be learning the intuition behind a field of math called differential geometry.

But first, I want to make sure you're acquainted with a commonly used explanation for Einstein's theory of gravity, which physicists usually call *general relativity*. The idea is that space and time are joined together, and they're just a stretched rubber sheet hanging in midair or something. Now normally if you rolled a golf ball on the sheet, it would roll in a straight line. However, when you put a bowling ball on the sheet, it creates a dent in the sheet, such that the path of the ball is curved around the dent. See the picture below.

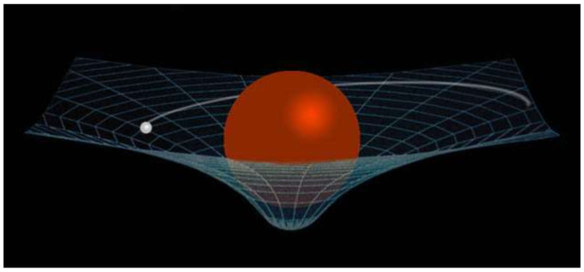


Figure 7. The ball-on-a-rubber-sheet metaphor

This metaphor does a pretty good job of introducing something called a *manifold*—essentially a smooth surface that may be curvy or flat. In addition, the path of the ball suggests that when the manifold is not flat, the path that the ball travels doesn't look straight anymore—but no path along it looks straight anyway, so how do we really know what "straight" is? Manifolds and straight lines on manifolds are a big part of differential geometry, and they play an especially big role in general relativity. We'll resume this discussion soon, after seeing why this class isn't about the rubber sheet metaphor.

I just mentioned some of the good things about the rubber sheet metaphor; now we'll look at some of the problems. Here's one: what happens if you turn the whole setup upside down?

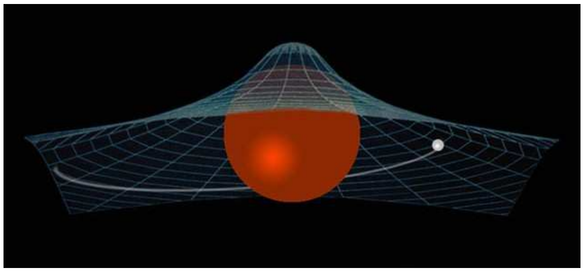


Figure 8. Upside down

Now what? Well obviously if gravity is still pulling the balls down, they'll simply fall out of the sheet—which is nonsense, if it's supposed to describe our universe. Planets don't fall out of existence, as far as we know. One solution to this is to say that gravity (or whatever is pulling the balls downwards in the first picture) always pulls in the same direction compared to the orientation of the sheet, but it turns out there are situations in which it becomes impossible to say which direction that is. What if your rubber sheet were the surface of a balloon? Then which way are the balls pulled?

Another intuitive problem with this metaphor is that we're using a gravitational force to explain a theory of gravity. It seems like circular logic, but on its own this isn't too big of a problem—just call the force something else. But it brings up another, subtler problem. Consider again a rubber sheet distorted like that in Fig. 8. If indeed the golf ball (representing a planet, satellite, or other real object in the universe) is "inside" the rubber sheet (representing the universe), then it shouldn't matter which side the golf ball is on—it should roll the same either way. But imagine putting a golf ball on the *top* side of the sheet to the left, instead of the bottom (as illustrated). It's like rolling a ball towards a hill; instead of being *attracted* to where the bowling ball is, it's *repelled*. The path of the planet doesn't curve *towards* the sun; it curves *away*. That's simply not what happens in our universe.

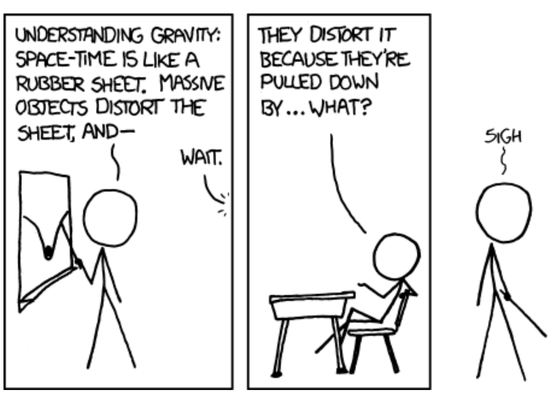


Figure 9. [http://xkcd.com/895 Part 1](http://xkcd.com/895)

Don't worry if you didn't follow some of that. My goal was to unsettle you about the way general relativity is usually explained. That way, you'll be ready when I give you a totally different paradigm, and hopefully you'll be thinking critically about the strengths and shortcomings of the new model. Next week: spacetime is a manifold.

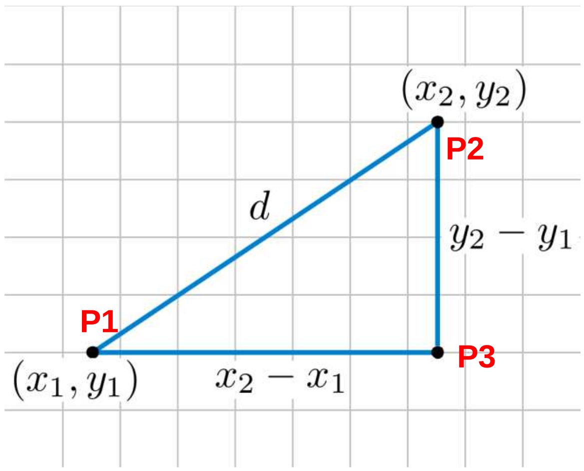
Manifolds, Metrics, and Geodesics, Oh My!

Do you remember the strengths of the rubber sheet model? That it makes you think about these strange things called manifolds, and makes you ask what "straight" means on a manifold? Now we'll develop some intuition and a tiny bit of math to understand all this.

The purpose of this section is to understand how to find a *geodesic* on a given manifold, a fancy word that means "the straightest line we could find on this manifold." To get there, we'll be using a mathematical object called a *metric*. As mathematical objects go, metrics are pretty friendly. In fact, you already know one, although you know it by a different name: the Pythagorean Theorem.

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$$

What does the Pythagorean Theorem do? If you want to know the length of the hypotenuse of a right triangle, just give it the lengths of the other two sides and it spits out the answer.



Now let's consider a different situation: you're given two points P1 and P2 on a grid, and you want to find the distance between them. How do you do it? Simply add in a point P3 to make a right triangle, and then you can use the Pythagorean Theorem. So the Pythagorean Theorem tells you the distance between any two points.

It turns out this is exactly what a metric does: point to any spot on the manifold, and the metric will tell you the distance between any two points in the neighborhood of that spot. If you looked at any random metric, it's totally possible that the distances in the nearby spot A differ from the distances nearby spot B; the Pythagorean Theorem is a special case in which distances are the same everywhere.

In particular, we say the Pythagorean Theorem is the metric for a flat space. If I gave you any other surface (i.e. manifold), like a sphere, you'd expect to find a totally different metric to describe distances on it. And here it is, for a sphere:

$$(\Delta s)^2 = R^2(\Delta\theta)^2 + R^2 \sin^2\theta(\Delta\phi)^2$$

…That's quite a bit more complicated. (Note: R is the radius of the sphere, θ is the latitude, and φ is the longitude.) Applying what I said before about metrics, we can just pick a spot at a certain latitude and longitude, and this metric will tell us how to find distances between any two points nearby that spot. Put another way, there are two steps we need to take to find the distance between two nearby points:

1. Find the *approximate* coordinates (θ, φ) of the points. This is our "spot on the manifold." Plug in these values in for all the occurrences of θ and φ that *are not preceded by a capital delta*. We'll fill those in during the next step. In the sphere metric, we just plug in our coordinate θ into $\sin^2\theta$ and leave everything else alone. (Assume we've already plugged in the radius R of the sphere.)
2. Find the difference between the θ coordinates of the two original points, and call it $\Delta\theta$. Likewise, find the difference between the φ coordinates of the two points, and call it $\Delta\varphi$. Plug in $\Delta\theta$ and $\Delta\varphi$, take a square root, and you've found your distance!

I want to emphasize that the two points need to be very close in order for the distance to be accurate (infinitesimally, if you want perfect accuracy), because as the points get farther apart, the coordinates of the spot become a worse approximation.

(In the challenge problem that follows, feel free to attempt to use the metric in the detailed manner described here. However, since it gets tricky really quickly, you may not get far.)

Plotting a Metric

For this class, I've had to develop a way of plotting a metric on paper, because we'll be using these plots extensively. I'm still refining it, but here's what I've come up with so far.

On a square grid, notice that as long as you're traveling from one vertex, along a gridline, to the next vertex, you always know exactly how long the path is—no calculation needed. Just count the number of times you went from one vertex to the next. That's because regardless of your location, moving one unit purely in the *x* direction corresponds to moving one unit of distance.

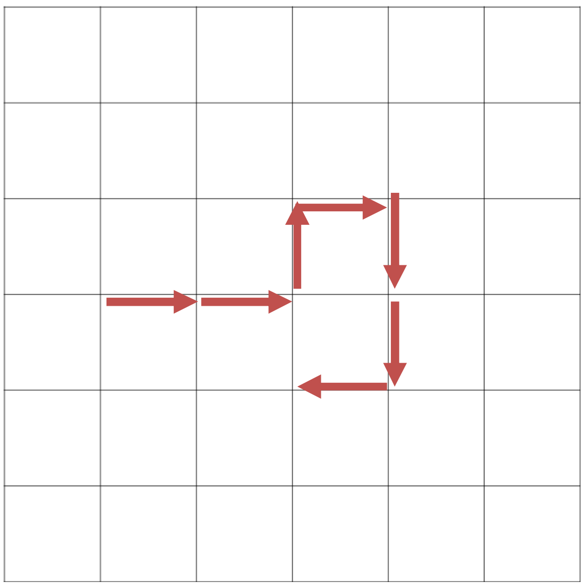


Figure 10. This path has length 7 units.

Finding a path with an exact length in a different metric is pretty similar—*but you don't travel on gridlines*! Instead, always move horizontally and vertically. For example, take the following metric:

$$(\Delta s)^2 = (\Delta x)^2 + x^2(\Delta y)^2$$

Now if you take any two points that have the same *y* coordinate, then Δy is zero, so the distance is just their separation in *x*. Likewise, if you take any two points that have the same *x* coordinate, then Δx is zero, so the distance is just *x* times their separation in *y*. Now if *x* is horizontal and *y* vertical, then clearly we can say exactly how far we've traveled along a path consisting purely of horizontal and vertical segments.

That's far from obvious from the way I've graphed the metric:

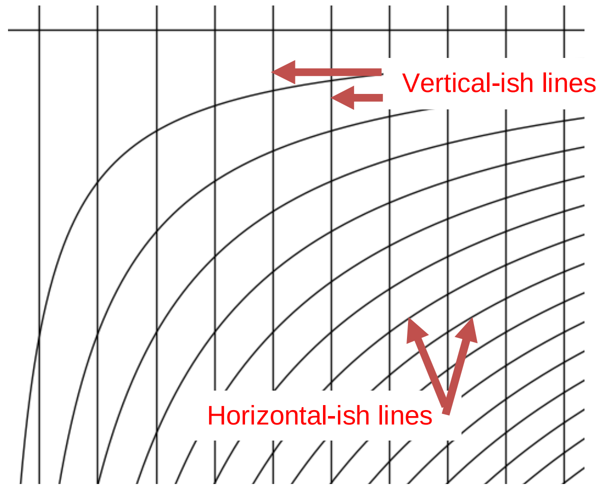


Figure 11. The same metric. The left-hand side is $x = 0$.

The key to making the connection is the following: *The distance along a vertical path is the number of horizontal-ish lines it crosses*. Likewise, *the distance along a horizontal path is the number of vertical-ish lines it crosses*. More accurately, at a particular spot on the manifold, one unit of vertical distance is illustrated as the separation between two horizontal-ish lines.

I don't want to give too much away now, but here are two features that might make it clear why this is a correct way of graphing the metric.

1. If you're traveling vertically and you want to move one unit of distance, you have to travel a much larger Δy when *x* is close to 0 than when *x* is big. You'll notice that this lines up with the metric.
2. If you're traveling horizontally, one unit of distance is the same Δx everywhere. But notice that you're always traveling perpendicular to the vertical lines, *not parallel to the horizontal-ish lines*, since the horizontal-ish lines only tell you about distance in the vertical direction.

Finally, an example of a valid path:

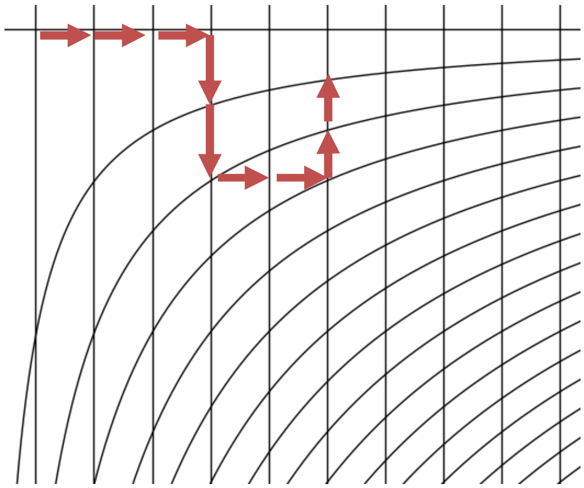


Figure 12. A path in this metric that we know how to estimate. It has length approximately 9 ("approximately" because the last horizontal segment might not end exactly on the vertex).

This section has been all about how to *use* a metric so that you can start to *understand* them and then, starting next week, learn why we care. Now that you have an idea about how to use the metric, we can start talking about how to find a straight line.

What Is a Straight Line?

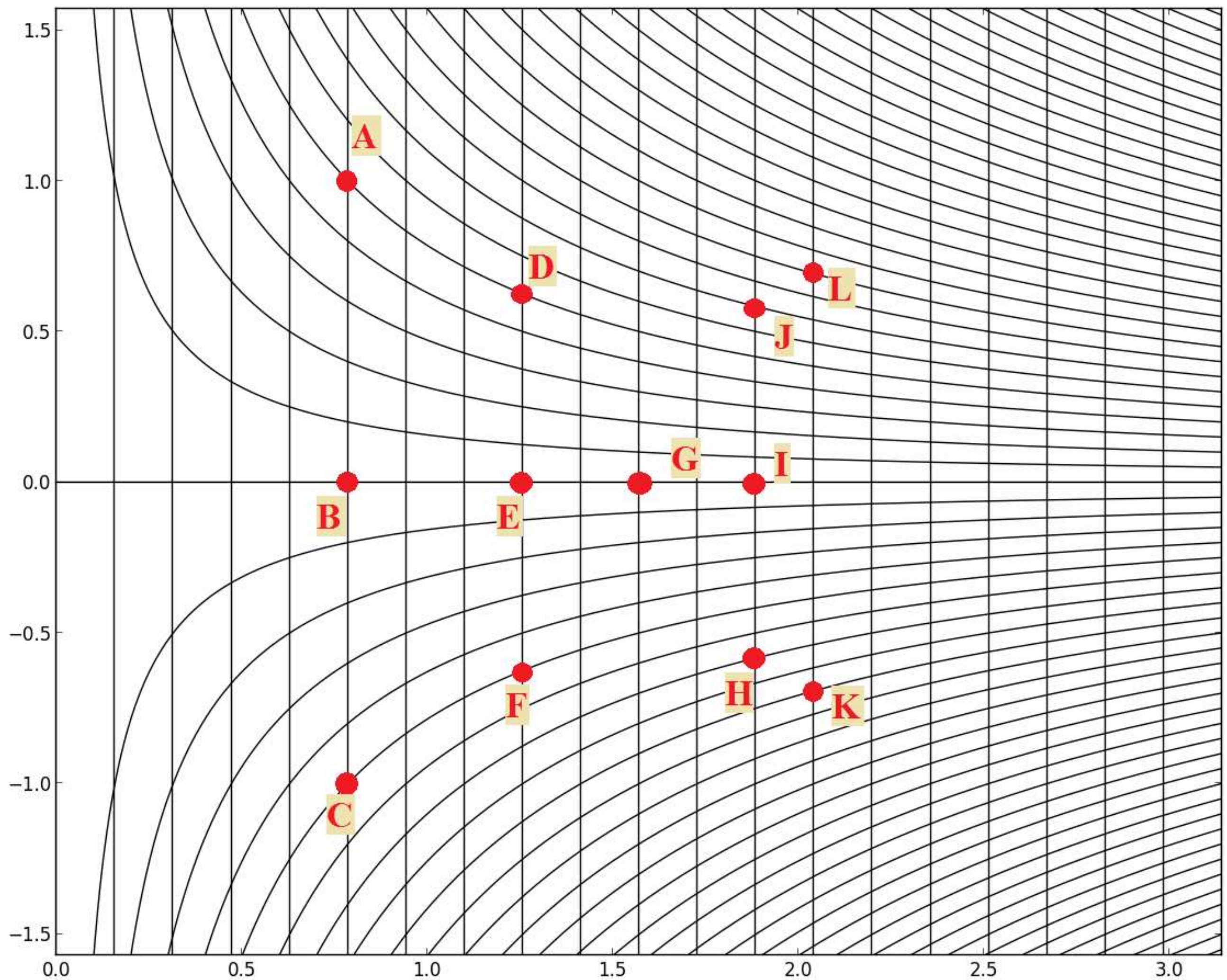
You've probably heard a definition of "straight line" at some point, and you've probably heard that it's the path of shortest distance between any two points. Well, you heard right! There are other definitions, but for our purposes they turn out to be equivalent. For this class, we'll be defining a straight line on a manifold (recall that I called it a *geodesic*) as *the path of shortest distance between any two points it touches*. That is, if I take a pen and draw some curvy path on a manifold, how can you tell if it's a geodesic? Take any two points on it; if the shortest path between them is *not* along the path, then it's *not* a geodesic. If you keep trying it with new pairs of points over and over again and *never* find a pair that tell you it's *not* a geodesic… then it is a geodesic.

Equipped with nothing but that, here's your next challenge!

Challenge Problem 2: Turn to the plot on the next page. Find the shortest path using just horizontal and vertical segments between each of these pairs of points:

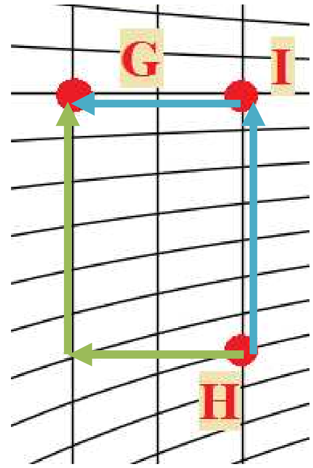
- B and I
- C and A
- F and D
- H and G
- H and J

Furthermore, see if you can figure out what geodesic connects each pair of points. Spend no more than 10 minutes on this problem. Try to spend a little time on all five. You'll find my answer on the page after.

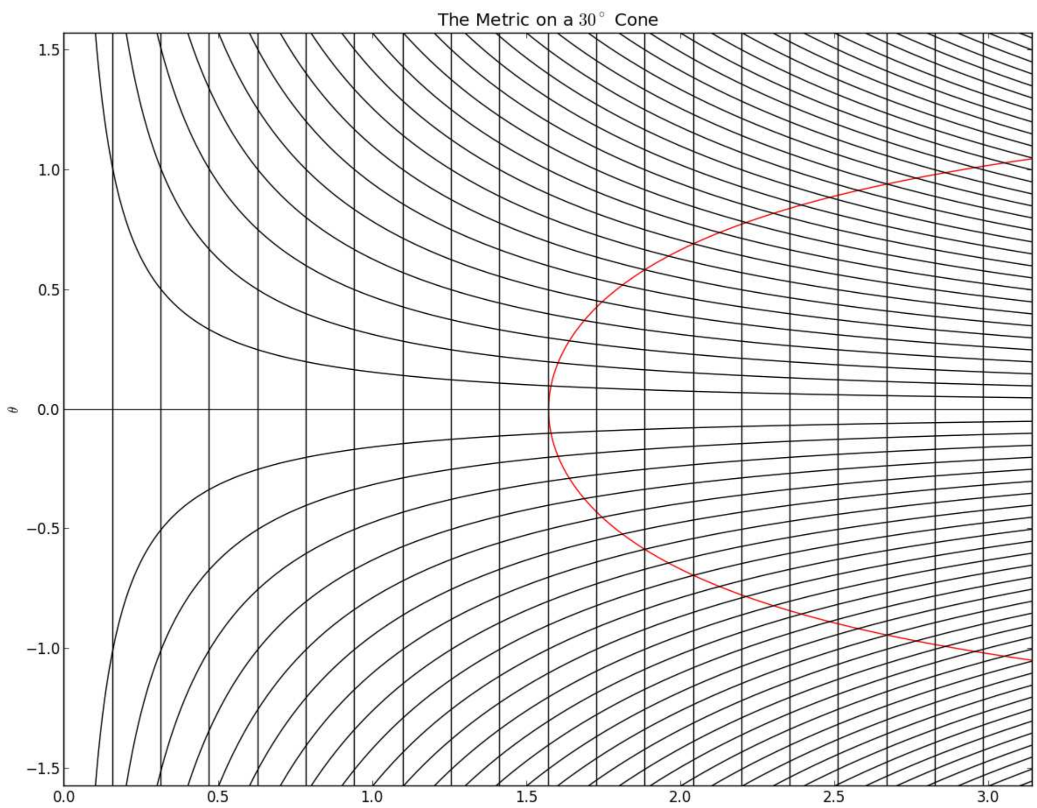

A
B
C
D
E
F
G
H
I
J
K
L
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0.0
0.5
1.0
1.5
2.0
2.5
3.0

My answer: I'll just skip right to the last two. First, consider any rectangle in a square grid. Note that if we start at one corner and want to get to the opposite corner, the two paths around the perimeter are the same length. Furthermore, we already know that the shortest path (i.e. the straight line between them) is simply the diagonal.

Now consider the following two paths around a rectangle from H to G. Note that the green path is about 1 unit shorter than the blue path. Since it's shorter to go around to the left, we might expect that the shorter path goes somewhat to the left of the diagonal.



In fact, that's true; the geodesic looks like this:



And now we know that the geodesic connecting points H and J is in fact not a straight line, as you may have found, but the geodesic pictured above.

Next week, we connect the dots: This metric is simply the metric on a cone, and it looks straight when you unfold the cone. Finally, when you look at the cone from above, you see the singularity chessboard.