Artificial Intelligence, Algorithmic Pricing, and Collusion

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Background

Motivation

- Algorithms becoming increasingly prevalent in practice
    - German gasoline markets (Assad et al. 2024)
    - Smartphone price discrimination (Kehoe et al. 2020)
- Regulatory questions:
    - How do algorithms get to collusive prices?
    - Can they do so in the absence of active principals?
    - Is algorithmic collusion visibly different than tacit collusion?
- Massive lack of theoretical guarantees for this (see Banchio and Mantegazza (2023); Possnig (2024); Lamba and Zhuk (2025))

Q-Learning Algorithms

- We're familiar with Q-learners
- Specifically, they learn *slowly*. This example has a massive state space and is trying to learn the opponent's policy as part of the state
- Our biggest criticisms are related to this. Specifically:
  - What is the loss in the learning phase?
  - How sensitive are these results to the initialization of the Q-matrix?

Model

Environment

Canonical oligopoly pricing game, with n firms / products and an outside good, where in each period t the demand for good i is

$$q_{i,t} = \frac{e^{\frac{a_i - p_{i,t}}{\mu}}}{\sum_{j=1}^{n} e^{\frac{a_j - p_{j,t}}{\mu}} + e^{\frac{a_0}{\mu}}}$$

where a_i is an index of quality, μ is in index of differentiation, and a_0 is an outside good. Firms choose $p_{i,t}$, and we have exogenous marginal costs c_i. The stage problem is:

$$\max_{p_{i,t}} q_{i,t}(p_{i,t}) \cdot p_{i,t} - q_{i,t}(p_{i,t}) \cdot c_{i,t}$$

This is quasiconcave but does not in general have a nice closed form solution.

Simplified Stage Environment

Assume $n = 2$, $c_i = 1$, $a_i = 2$, $a_0 = 0$, and $\mu = \frac{1}{4}$. Then the stage game reduces to

$$\max_{p_i} \frac{(p_i - 1)e^{8-4p_i}}{e^{8-4p_i} + e^{8-4p_j} + 1}$$

This is strictly concave, and we have that it admits Nash prices

$$p_i^N = p_j^N \approx 1.473$$

and monopoly prices are obtained from setting $n = 1$, where we attain

$$p^M = \frac{5}{4} - \frac{1}{4} W_n(2e^3) \approx 1.925$$

Simplified Stage Environment

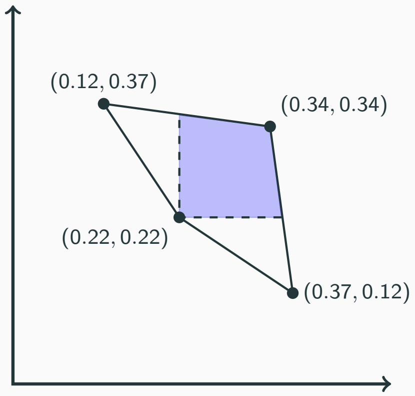
We basically have an extension of a Prisoner's Dilemma:

	N	M
N	$(0.22, 0.22)$	$(0.37, 0.12)$
M	$(0.12, 0.37)$	$(0.34, 0.34)$

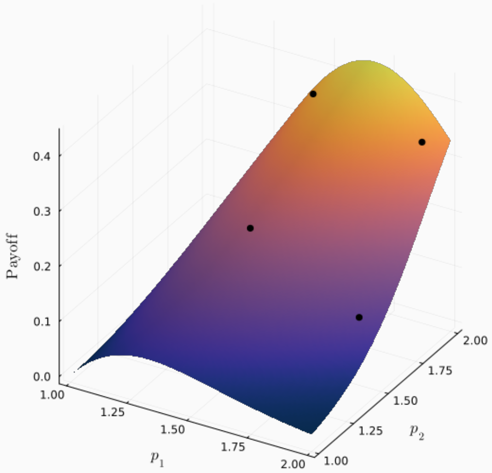
Since all of the involved functions are continuous and concave, this extends fairly nicely.

So we're making our Q-learners play a repeated Prisoner's Dilemma, and the strategies they learn *should* be similar to canonical repeated PD strategies.

Folk Theorem



Continuous Stage Payoffs



A Question

Why use this sigmoid demand function instead of exogenously imposing a reasonable range for prices and using *e.g.* linear demand?

We don't understand what the gain from this functional form is, and the fact that it doesn't in general have closed-form solutions is an annoyance.

Learning Theory

Learning in Repeated Games

Essentially, take the opponent's previous actions to be the state, along with whatever game parameters you need. Two issues:

1. The state space is increasing as the game continues.
   Solution: Bounded memory.
2. The optimization problem is non-stationary if the opponent(s) change strategy over time.
   No official solutions here, this is why we don't have theoretical guarantees[1]

[1] I'm fairly sure there should be something here. At least in probability. I'm confused why nobody has proved that yet - Gabe

Learning in Repeated Games

The Q-learners solve

$$Q(s,a) = \mathbb{E}(\pi \mid s,a) + \delta \mathbb{E}\left[\max_{a' \in A} Q(s',a') \mid s,a\right]$$

where $a \in A$ is the action (from the rules of the game), $s \in S$ is the state (defined as all player actions in the last k periods, where k is the memory). Once we discretize, $Q \in \mathbb{R}^{|A| \times |S|} = \mathbb{R}^{|A| \times |A|^{nk}}$

For simplicity, $k = 1$. Results robust to higher k.

Parameterization

Work in the simplified game as above, with $\delta = 0.95$ and $|A| = m = 15$. Discretize the price grid over

$$\left[p^N - 0.1(p^M - p^N), p^M + 0.1(p^M - p^N)\right]$$

Set the initial matrix to the discounted payoff if the other player randomized over all actions:

$$Q_{i,0}(s, a_i) = \frac{\sum_{a_{-i} \in A^{n-1}} \pi_i(a_i, a_{-i})}{(1-\delta)|A|^{n-1}}$$

Draw the initial state s_0 randomly as well.

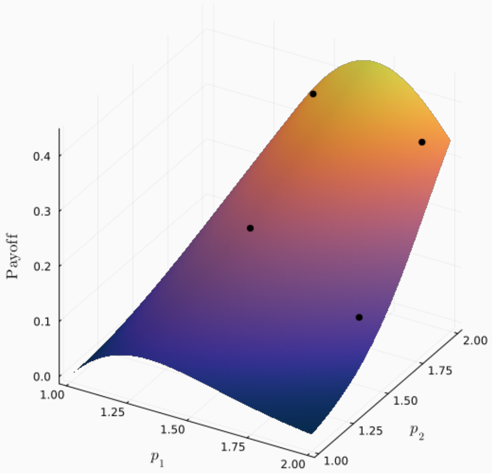
A Difference

Their definition of Q_0 incorporates the fact that the game is infinite, and defines as if we are taking the discounted sum of payoffs forever, assuming that the other player uniformly randomizes.

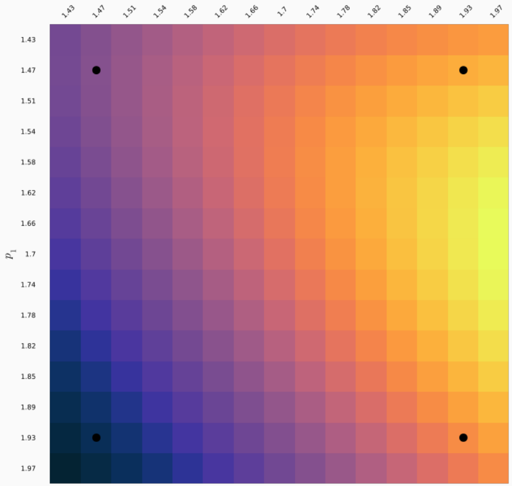
That's not how we've thought about this previously, because we're incorporating a meaningful discount factor for the first time. However, it means that Q_0 has a different qualitative meaning than we're used to. This is confusing and we don't particularly well understand the effects.

(We test both, and show some results later)

Continuous Stage Payoffs



Discretized Stage Payoffs



Parameterization

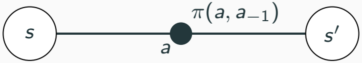
We will use ε-greedy learners, with $\varepsilon_t = \exp(-\beta t)$. The learning parameter α will be tested over a grid of 100 points in $[0.025, 0.25]$, and several different values of β are also tested.

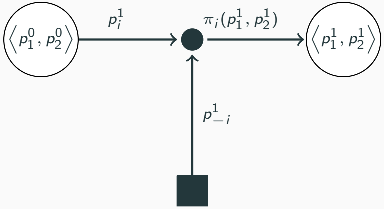
They define ν to be the number of times a certain cell is visited in expectation under a certain β. For our purposes, β will be tested over a grid of 100 points in $(0, 2 \cdot 10^{-5})$.

Recall from earlier:

$$Q(s,a) \leftarrow Q(s,a) + \alpha \left[\pi(s', a') + \delta \max_{a' \in A} Q(s', a') - Q(s,a) \right]$$

Learning Visualization




$\left\langle p_1^0, p_2^0 \right\rangle$
p_i^1
$\pi_i(p_1^1, p_2^1)$
$\left\langle p_1^1, p_2^1 \right\rangle$
p_{-i}^1

Remark

Prior to this, we've generally dealt with RL algorithms that are trying to learn payoffs in a static or stochastic game. These algorithms are (theoretically) learning strategies for the infinitely repeated game.

Thinking about the bounded memory, that's no longer such an innocuous assumption. For example: these algorithms will be able to learn tit-for-tat or Grim Trigger, but not trigger strategies with n periods of punishment for $n > k$.

Code

Python vs. Julia pt. N

- The authors wrote the code in Fortran, which neither of us know because we were both born in 2001
- Finn rewrote it in Python, and we'll present that code because it's the most similar to what we've seen previously
- Gabe refactored it into Julia and parallelized it, which increased the speed by a massive amount (2hrs $\rightarrow$ 5.5mins)
- The results below are from Julia, because it's more robust

High-level structure

- Exact same as when we defined Q-learning previously, except that we now update two Q-matrices instead of one
- Rough pseudocode:
  1. Define Q_0 for each player, taking the expected value if other player chooses randomly
  2. Define action choice function (as always, ε-greedy)
  3. Iteratively learn, updating using the rule:

$$Q(s,a) = \mathbb{E}(\pi \mid s,a) + \delta \mathbb{E}\left[\max_{a' \in A} Q(s',a') \mid s,a\right]$$

  4. End when $100,000$ periods of convergence, defined as either
     (i) $s_t = (a_t^1, a_t^2) = (a_{t+1}^1, a_{t+1}^2) = s_{t+1}$
     (ii) $s_t = s_{t-2}$ and $s_{t+1} = s_{t-1}$
  5. Return average per-firm profit once converged, compare to Nash / Monopoly profit

Changes

We had to make some small changes to make the logic work. The main ones are:

- Added an arbitrary cutoff for saying convergence failed (10m)
- Allowed cycles of 2 to be defined as convergence
- Ran many fewer times (25 experiments, across a 15×15 grid)

These were mainly runtime issues. If we had access to a supercomputing cluster, we could do a significantly more robust replication (*hint hint*)

Parameterization

Exact same as in the paper, except we used 15 realizations of α and β, on the intervals $[0.1, 0.2]$ and $[5 \cdot 10^{-6}, 1.5 \cdot 10^{-5}]$ respectively. Specifically, we have:

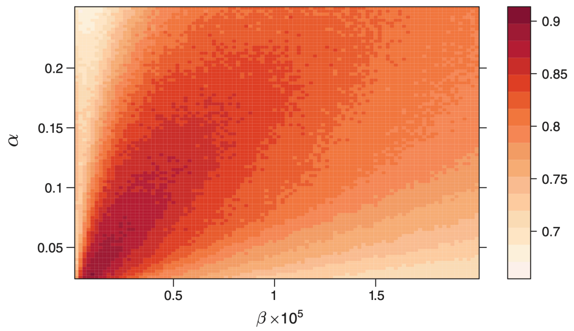


Figure 1. Average Profit Gain Δ for a Grid of Values of α and β

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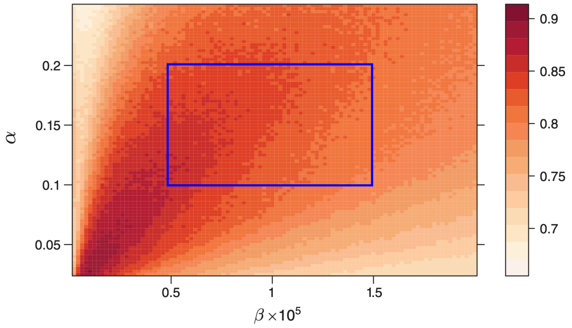
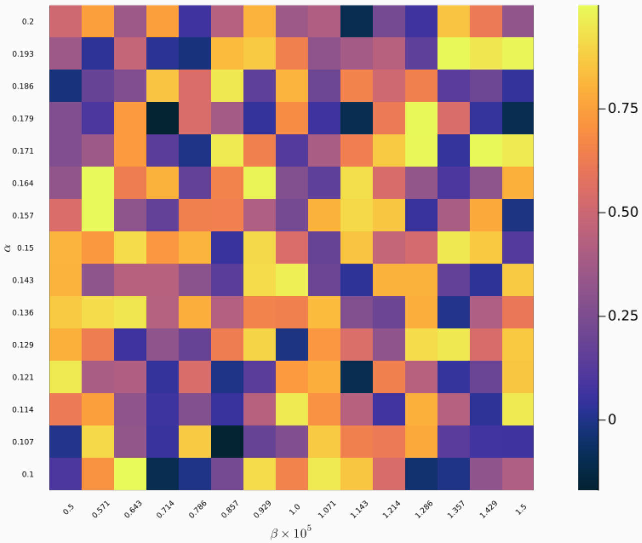


Figure 1. Average Profit Gain Δ for a Grid of Values of α and β

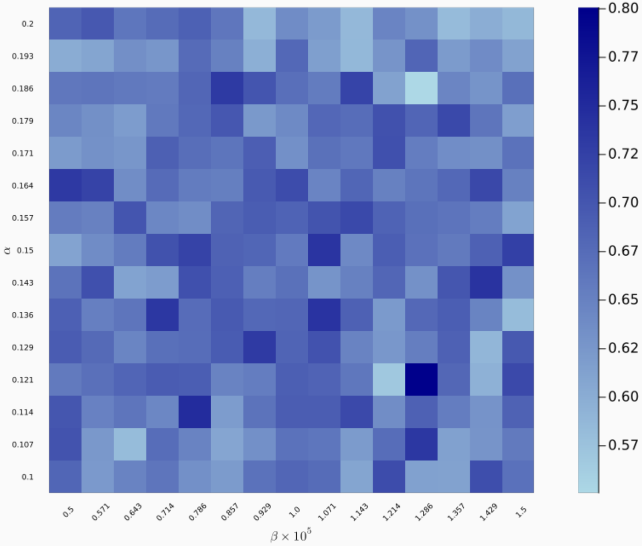
Results

One Run

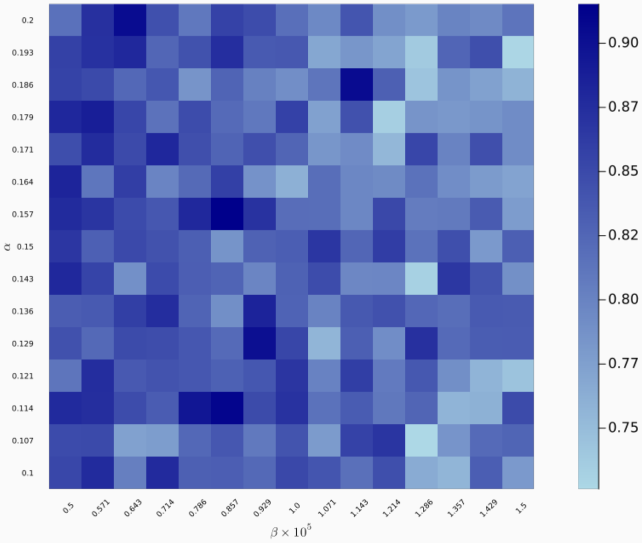


0.2
0.193
0.186
0.179
0.171
0.164
0.157
α 0.15
0.143
0.136
0.129
0.121
0.114
0.107
0.1
0.5
0.571
0.643
0.714
0.786
0.857
0.929
1.0
1.071
1.143
1.214
1.286
1.357
1.429
1.5
$\beta \times 10^5$
0.75
0.50
0.25
0

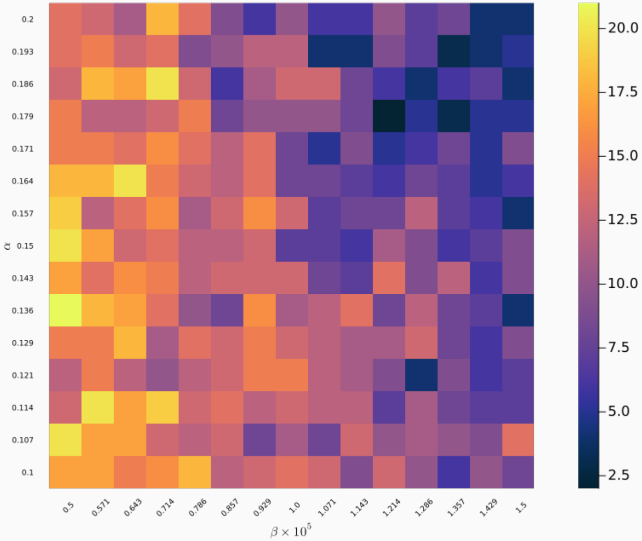
25 Runs, no $(1-\delta)$



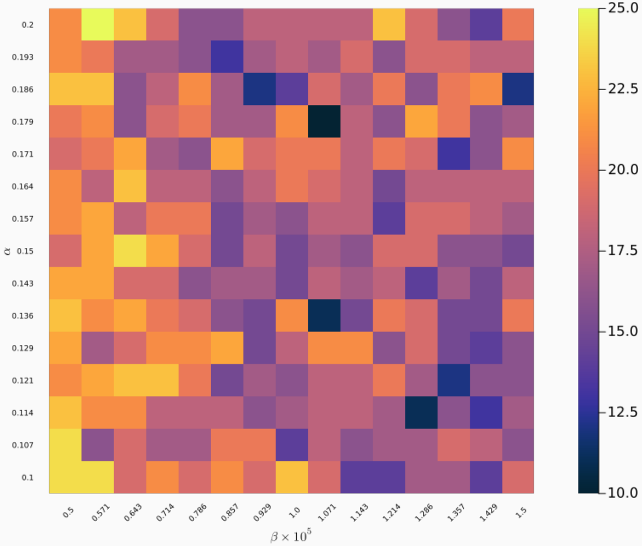
25 Runs, with $(1 - \delta)$



Convergence Rates, no $(1 - \delta)$



Convergence Rates, with $(1 - \delta)$


α
$\beta \times 10^5$