# Artificial Intelligence, Algorithmic Pricing, and Collusion

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# Background

# Motivation

- Algorithms becoming increasingly prevalent in practice
  - German gasoline markets (Assad et al. 2024)
  - Smartphone price discrimination (Kehoe et al. 2020)
- Regulatory questions:
  - How do algorithms get to collusive prices?
  - Can they do so in the absence of active principals?
  - Is algorithmic collusion visibly different than tacit collusion?
- Massive lack of theoretical guarantees for this (see Banchio and Mantegazza (2023); Possnig (2024); Lamba and Zhuk (2025))

## Q-Learning Algorithms

- We're familiar with Q-learners
- Specifically, they learn *slowly*. This example has a massive state space and is trying to learn the opponent's policy as part of the state
- Our biggest criticisms are related to this. Specifically:
    - What is the loss in the learning phase?
    - How sensitive are these results to the initialization of the Q-matrix?

# Model

# Environment

Canonical oligopoly pricing game, with $n$ firms / products and an outside good, where in each period $t$ the demand for good $i$ is

$$q_{i,t} = \frac{e^{\frac{a_i - p_{i,t}}{\mu}}}{\sum_{j=1}^{n} e^{\frac{a_j - p_{j,t}}{\mu}} + e^{\frac{a_0}{\mu}}}$$

where $a_i$ is an index of quality, $\mu$ is in index of differentiation, and $a_0$ is an outside good. Firms choose $p_{i,t}$, and we have exogenous marginal costs $c_i$. The stage problem is:

$$\max_{p_{i,t}} q_{i,t}(p_{i,t}) \cdot p_{i,t} - q_{i,t}(p_{i,t}) \cdot c_{i,t}$$

This is quasiconcave but does not in general have a nice closed form solution.

# Simplified Stage Environment

Assume $n = 2$, $c_i = 1$, $a_i = 2$, $a_0 = 0$, and $\mu = \frac{1}{4}$. Then the stage game reduces to

$$\max_{p_i} \frac{(p_i - 1)e^{8-4p_i}}{e^{8-4p_i} + e^{8-4p_j} + 1}$$

This is strictly concave, and we have that it admits Nash prices

$$p_i^N = p_j^N \approx 1.473$$

and monopoly prices are obtained from setting $n = 1$, where we attain

$$p^M = \frac{5}{4} - \frac{1}{4} W_n(2e^3) \approx 1.925$$

# Simplified Stage Environment

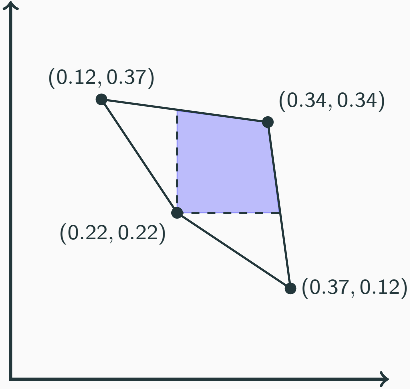
We basically have an extension of a Prisoner's Dilemma:

| | $N$ | $M$ |
|---|---|---|
| $N$ | $(0.22, 0.22)$ | $(0.37, 0.12)$ |
| $M$ | $(0.12, 0.37)$ | $(0.34, 0.34)$ |

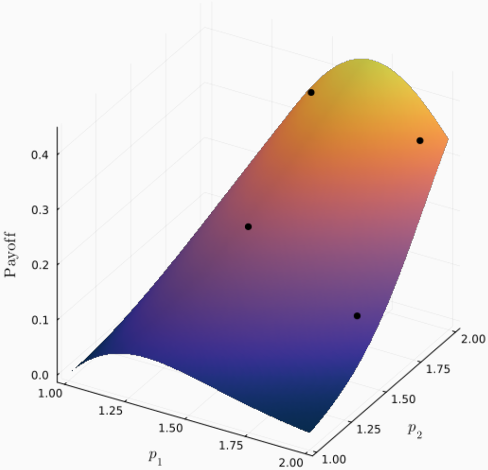
Since all of the involved functions are continuous and concave, this extends fairly nicely.

So we're making our Q-learners play a repeated Prisoner's Dilemma, and the strategies they learn *should* be similar to canonical repeated PD strategies.

# Folk Theorem



# Continuous Stage Payoffs



# A Question

Why use this sigmoid demand function instead of exogenously imposing a reasonable range for prices and using *e.g.* linear demand?

We don't understand what the gain from this functional form is, and the fact that it doesn't in general have closed-form solutions is an annoyance.

# Learning Theory

# Learning in Repeated Games

Essentially, take the opponent's previous actions to be the state, along with whatever game parameters you need. Two issues:

1. The state space is increasing as the game continues.
   Solution: Bounded memory.
2. The optimization problem is non-stationary if the opponent(s) change strategy over time.
   No official solutions here, this is why we don't have theoretical guarantees[1]

[1] I'm fairly sure there should be something here. At least in probability. I'm confused why nobody has proved that yet - Gabe

The Q-learners solve

$$Q(s, a) = \mathbb{E}(\pi \mid s, a) + \delta \mathbb{E}\left[\max_{a' \in A} Q(s', a') \mid s, a\right]$$

where $a \in A$ is the action (from the rules of the game), $s \in S$ is the state (defined as all player actions in the last $k$ periods, where $k$ is the memory). Once we discretize, $Q \in \mathbb{R}^{|A| \times |S|} = \mathbb{R}^{|A| \times |A|^{nk}}$

For simplicity, $k = 1$. Results robust to higher $k$.

# Parameterization

Work in the simplified game as above, with $\delta = 0.95$ and $|A| = m = 15$. Discretize the price grid over

$$\left[p^N - 0.1(p^M - p^N), p^M + 0.1(p^M - p^N)\right]$$

Set the initial matrix to the discounted payoff if the other player randomized over all actions:

$$Q_{i,0}(s, a_i) = \frac{\sum_{a_{-i} \in A^{n-1}} \pi_i(a_i, a_{-i})}{(1-\delta)|A|^{n-1}}$$

Draw the initial state $s_0$ randomly as well.

# An Issue

If you initialize $Q_0$ precisely the way that they say they do, stuff breaks immediately. Specifically, we saw one of two patterns: either for all parameters convergence was to the same pair ($\approx 1.58$), which is not the collusive equilibrium, or convergence was to a clearly suboptimal pair (*e.g.* 1.8926 and 1.4277).

This was fixed entirely by removing $(1-\delta)$ from the denominator.

$$\underbrace{\frac{\sum_{a_{-i} \in A^{n-1}} \pi_i(a_i, a_{-i})}{(1-\delta)|A|^{n-1}}}_{\text{Bad}} \qquad \underbrace{\frac{\sum_{a_{-i} \in A^{n-1}} \pi_i(a_i, a_{-i})}{|A|^{n-1}}}_{\text{Good}}$$

# Why does this happen?

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# Some Theories

- The obvious note is that this is the discounted payoff until infinity
- The issue is that this is not how learning works. We incorporate $\delta$ by updating one period at a time, not sending time far into the future:

$$Q(s, a) = \mathbb{E}(\pi \mid s, a) + \delta \mathbb{E}\left[\max_{a' \in A} Q(s', a') \mid s, a\right]$$

- By including the tail, we massively overestimate the value to each action
- $\varepsilon$ degenerating quickly explains convergence to a random suboptimal pair

## Some More Theories

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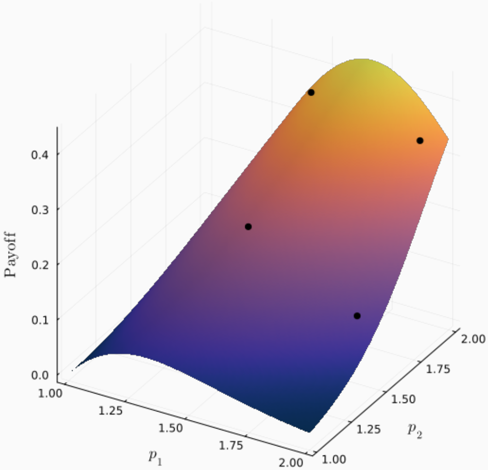
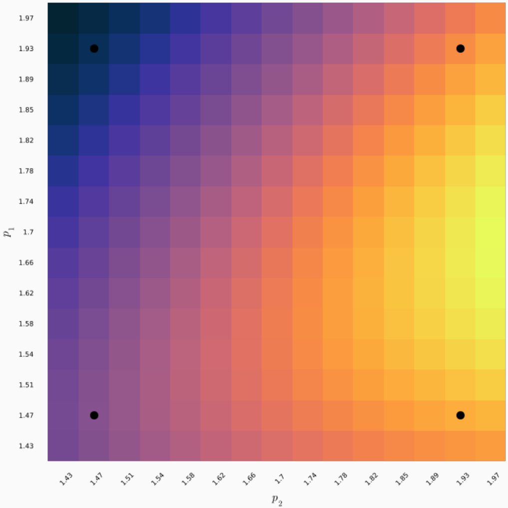
One of the algo just beat the other one to death and force them to play Nash

# Continuous Stage Payoffs



# Discretized Stage Payoffs



# Parameterization

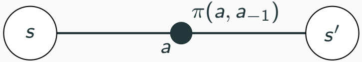
We will use $\varepsilon$-greedy learners, with $\varepsilon_t = \exp(-\beta t)$. The learning parameter $\alpha$ will be tested over a grid of 100 points in $[0.025, 0.25]$, and several different values of $\beta$ are also tested.

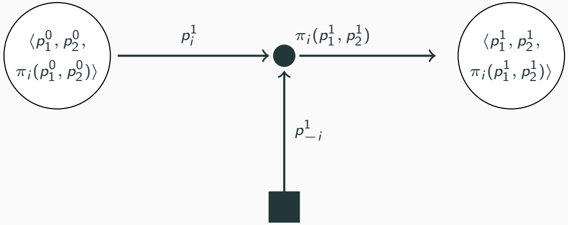
They define $\nu$ to be the number of times a certain cell is visited in expectation under a certain $\beta$. For our purposes, $\beta$ will be tested over a grid of 100 points in $(0, 2 \cdot 10^{-5})$.

Recall from earlier:

$$Q(s,a) \leftarrow Q(s,a) + \alpha \left[ \pi(s',a') + \delta \max_{a' \in A} Q(s',a') - Q(s,a) \right]$$

## Learning Visualization



## Remark

Prior to this, we've generally dealt with RL algorithms that are trying to learn payoffs in a static or stochastic game. These algorithms are (theoretically) learning strategies for the infinitely repeated game.

Thinking about the bounded memory, that's no longer such an innocuous assumption. For example: these algorithms will be able to learn tit-for-tat or Grim Trigger, but not trigger strategies with $n$ periods of punishment for $n > k$.

# Code

# Results