ECON 6110, Microeconomic Theory III	**Section 4**
TA: Ruqing Xu	**February 16, 2024**

1 Supermodular game

Definition 1.1. $u_i(s_i, s_{-i})$ has increasing differences in (s_i, s_{-i}) if for all $(s_i, \tilde{s}_i)$ and $(s_{-i}, \tilde{s}_{-i})$ such that $s_i \geq \tilde{s}_i$ and $s_{-i} \geq \tilde{s}_{-i}$, we have:

$$u_i(s_i, s_{-i}) - u_i(\tilde{s}_i, s_{-i}) \geq u_i(s_i, \tilde{s}_{-i}) - u_i(\tilde{s}_i, \tilde{s}_{-i})$$

Definition 1.2. $u_i(s_i, s_{-i})$ is supermodular in s_i if for each s_{-i}:

$$u_i(s_i, s_{-i}) + u_i(\tilde{s}_i, s_{-i}) \leq u_i(s_i \wedge \tilde{s}_i, s_{-i}) + u_i(s_i \vee \tilde{s}_i, s_{-i})$$

Remark. Note that if S_i is linearly ordered (as $\mathbb{R}$), then u_i is trivially supermodular in s_i as the above inequality is vacuously satisfied as equality.

Definition 1.3. A (resp., strictly) supermodular game is a game in which for each i:

- S_i is a sublattice of R^{m_i}
- u_i has (resp., strictly) increasing differences in (s_i, s_{-i})
- u_i is (resp., strictly) supermodular in s_i

Remark. If every players' strategy is single-dimensional, the definition of supermodular game boils down to just increasing differences.

Theorem 1.1. *Let (S, u) be a supermodular game. Then:*

- *the set of strategies surviving iterated strict dominance has greatest and least elements* $\overline{a}, \underline{a}$.
- *and* $\overline{a}, \underline{a}$ *are both Nash equilibria.*

2 Exercise

ECON 6110: 2021 Prelim #1 Question #2

Two students are deciding how long to spend studying for 6110 on the night before the exam. Let e_i be the fraction of the available time student i devotes to studying with $0 \leq e_i \leq 1$. Assume that the students' payoffs are

$$\begin{aligned} v_1(e_1, e_2) &= \log(1 + 3e_1 - e_2) - e_1, \\ v_2(e_1, e_2) &= \log(1 + 3e_2 - e_1) - e_2. \end{aligned}$$

Note: Please ignore the two action profiles that render one of the value functions undefined :)

(a) Show that the game is supermodular.

(b) Find the set of rationalizable actions.

(c) Find the Nash equilibria.

Solution:

(a) Each player has one-dimensional strategy, so only the requirement on increasing difference has bite. Since the payoff functions for both players are $\mathcal{C}^2$, we can check for increasing difference with cross partials. Since

$$\frac{\partial^2}{\partial e_i \partial e_{-i}} v_i(e_i, e_{-i}) = \frac{3}{(1 + 3e_i - e_{-i})^2} \geq 0,$$

the game is indeed supermodular.

(b) The only rationalizable action is 1 for both players. To find the set of rationalizable strategies, we just need to perform iterated elimination of strictly dominated strategies (IESDS). This is equivalent to eliminating strategies that are never-best responses.

In particular, taking the first-order condition for each player (and noting that the objective function for each player is strictly concave for any action taken by the opponent), we have

$$\frac{\partial v_1}{\partial e_1} = \frac{3}{1 + 3e_1 - e_2} - 1 = 0$$

This can be re-written as

$$e_1^* = BR(e_2) = \frac{2 + e_2}{3}$$

Step 1: The original action set:

$$A_i^0 = [0, 1]$$

Step 2: only a_i that is a best response to some belief over player $-i$'s actions in Step 1 survives IESDS.

$$A_i^1 = [\frac{2}{3}, 1]$$

Step 3: only a_i that is a best response to some belief over player $-i$'s actions in Step 2 survives IESDS.

$$A_i^2 = [\frac{2}{3} + \frac{2}{9}, 1]$$

Generalizing to arbitrary t, we have

$$A_i^t = [\frac{2}{3} + \frac{2}{9} + ... + \frac{2}{3^t}, 1] = [2\sum_{k=1}^{t} \frac{1}{3^k}, 1]$$

The 1-truncated geometric series is

$$\sum_{k=1}^{\infty} \frac{1}{3^k} = \frac{1}{1 - \frac{1}{3}} - 1 = \frac{1}{2}$$

Hence we have

$$\lim_{t \to \infty} A_i^t = \{1\}$$

so that $e_i = 1$ is the only strategy that survives iterated deletion, and is thus the unique rationalizable strategy.

(c) Since the only rationalizable strategy is $(1, 1)$, which is indeed a Nash equilibrium, it is the game's unique Nash equilibrium.

3 Extensive game

Definition 3.1. A **multi-stage game with observed actions** consists of

(i) (Finite) set of players, $N = \{1, \dots, n\}$

(ii) A (possibly infinite) set of stages, $\{0, 1, \dots\}$

(iii) At stage 0:

  (a) An initial history $h^0 = \emptyset$

  (b) Set of feasible actions for each player i at h^0, $A_i(h^0)$

  (c) Set of action profiles played by players $a^0 \in \bigtimes_{i=1}^{n} A_i(h^0)$

(iv) At each stage $k > 0$

  (a) Set H^k of partial histories $h^k = (a^0, \dots, a^{k-1})$

  (b) Set of feasible actions for each player i at each h^k, $A_i(h^k)$

  (c) Set of action profiles played by players $a^k \in \bigtimes_{i=1}^{n} A_i(h^k)$ at each h^k

(v) Set Z of terminal histories $z = (a^0, a^1, \dots)$

(vi) Payoff function of player i, $v_i : Z \to \mathbb{R}$

Definition 3.2. A history $h \in H$ is a sequence of actions taken by the players $(a^k)_{k=1,\dots,K}$. The set of terminal histories is denoted Z.

Definition 3.3. A strategy of a player i in an extensive game with perfect information is a function

$$s_i(h) \to A(h)$$

for any $h \in H \setminus Z$ such that $P(h) = i$.

Remark. A strategy specifies an action for *each* (non-terminal) history in which a player is asked to choose an action, even for histories that, if the strategy is followed, are never reached.

Definition 3.4. Denote a strategy profile $s = (s_1, \ldots, s_n)$. For each strategy profile an outcome $O(s)$ is the terminal history associated with the strategy profile.

Definition 3.5. A strategy profile, $s = (s_1, \ldots, s_n)$ is a **Nash equilibrium** if for all players i and all deviations $\hat{s}_i$,

$$u_i(s_i, s_{-i}) \geq u_i(\hat{s}_i, s_{-i})$$

Definition 3.6. The **subgame** of the extensive game with perfect information $\Gamma =< N, H, P, (u_i) >$ that follows the history h is the extensive game $\Gamma(h) =< N, H \mid_h, P \mid_h, (u_i) \mid_h >$, where $H \mid_h, P \mid_h, (u_i) \mid_h$ are consistent with the original game starting at history h.

Definition 3.7. A strategy profile, s is a **subgame perfect equilibrium** in Γ if for any history h the strategy profile $s \mid_h$ is a Nash equilibrium of the subgame $\Gamma(h)$.

Definition 3.8. For fixed s_i and history h, a **one-stage deviation** is a strategy $\hat{s}_i$ in the subgame $\Gamma(h)$ that differs from $s_i \mid_h$ only in the action it prescribes after the initial history of $\Gamma(h)$.

Theorem 3.1 (One-stage deviation principle). *In a finite-horizon extensive game, a strategy profile s is an SPE if and only if for all players i, all histories $h \in H$, and one-stage deviations $\hat{s}_i$,*

$$u_i(s_i \mid_h, s_{-i} \mid_h) \geq u_i(\hat{s}_i, s_{-i} \mid_h)$$

Example 3.2 (Entry game).