Midterm

Macroeconomics II

6140

March 2025

*You have 90 minutes to answer these questions. The "relative prices" can be found next to each question, for a total of 100 points. If you need additional assumptions to solve any of these problems please state them clearly. I will only answer clarifying questions for the first 15 minutes. Please **write clearly**, if we can't read your answers we can't grant you points. If you get stuck at any point, move to the next question. Partial credit will be given for correct intuition.*

NAME:

CORNELL ID:

A Malthusian economy

Let's study a version of Diamond's (1965) OLG model. Capital, labor, and land are combined to produce output.

The production function is as follows:

$$Y_{Mt} = A_{Mt} K_{Mt}^{\phi} N_{Mt}^{\mu} L_{Mt}^{1-\phi-\mu}$$

where K_{Mt}, N_{Mt}, A_{Mt} denotes capital, labor and total factor productivity in sector $M. L_{Mt}$ is land which is in fixed supply, with a quantity of 1.

Output can be used for consumption or investment in capital. Capital depreciates fully at the end of each period, hence, the aggregate resource constraint is

$$C_t + K_{t+1} = Y_{Mt}$$

Households live for two periods and have preferences that depend on consumption in each period of life. In particular, a young household born in period t has preferences summarized by the following utility function:

$$U(c_{1t}, c_{2,t+1}) = \ln(c_{1t}) + \beta \ln(c_{2t+1})$$

where c_1 is consumption when young and c_2 consumption when old.

The number of households born in period t is denoted by N_t, where $N_{t+1} = g(c_{1t})N_t$ for some function g(.) that is increasing in leaving standards, as in Malthus (1798); continuous and differentiable.[1]

The initial old in the economy are endowed with $k_{t_0} \equiv \frac{K_{t_0}}{N_{t_0-1}}$ units of capital and $l_{t_0} = L_{t_0} = \frac{1}{N_{t_0-1}}$ units of land. Old agents rent capital and land to firms at rental rates r_K and r_L, respectively. At the end of the period, they sell their land to the young at price q_{t+1} in units of final good. Each young are endowed with one unit of labor and the labor income is used to finance consumption, the purchase of capital and land. The return on these assets will finance their consumption when they are old, i.e.

$$c_{2,t+1} = r_{K,t+1} k_{t+1} + (r_{L,t+1} + q_{t+1})\, l_{t+1}$$

[1] We measure it from the consumption of the young.

1. (15pt) Describe the problem of the household as well as the problem of the representative firm.

   The problem of a household born at t is

$$\max_{c_{1t},c_{2t+1},k_{t+1},l_{t+1}} U(c_{1t},c_{2,t+1}) = \ln(c_{1t}) + \beta\ln(c_{2t+1})$$

   subject to

$$c_{1t} + k_{t+1} + q_t l_{t+1} = w_t$$

$$c_{2,t+1} = r_{K,t+1}k_{t+1} + (r_{L,t+1} + q_{t+1})\, l_{t+1}$$

   Problem of the firm in sector j at each t

$$\max_{N_j,K_j,L_j} Y_j - wN_j - r_k K_j - r_L L_j$$

2. (10pt) Can households be borrowing constrained in this economy? Explain.

   Households cannot be borrowing constrained in this economy because they only earn income in their first period of their life and they derive utility from consumption when old. So they always want to save for the future.

3. (10pt) Define a competitive equilibrium in this economy. Be explicit about the market clearing conditions and be careful with the size of the population and factor supplies.

   Given $N_{t_0}, k_{t_o}, l_{t_o}$, *a competitive equilibrium in this economy are sequences for* $t \geq t_0$ *of prices* $q_t, w_t, r_{Kt}.r_{Lt}$; *firm allocations,* $q_t, w_t, r_{Kt}.r_{Lt}$; *and household allocations,* $c_{1t}, c_{2t+1}, k_{t+1}, l_{t+1}$ *such that the following holds:*

   (a) Households maximize utility subject to their budget constraints.

   (b) Firm maximize profits

   (c) Markets Clear:

$$K_{Mt} = k_t N_{t-1}$$

$$N_{Mt} = N_t$$

$$N_{t-1} l_t = 1$$

$$Y_{Mt} = N_t c_{1t} + N_{t-1}c_{2t} + N_t k_{t+1}$$

(d) The law of motion for population is

$$N_{t+1} = g(c_{1t})N_t$$

4. (5pt) Characterize the optimal marginal rate of substitution between consumption when young and old for the households in this economy.

The optimality condition for the household requires that the marginal rate of substitution equalizes the ratio of the shadow prices associate to the budget constraints when young and old.

$$\frac{1}{c_{1t}} = \lambda_{1t} \qquad \frac{\beta}{c_{2t+1}} = \lambda_{2t+1}$$

$$\frac{c_{2t+1}}{c_{1t}} = \beta\frac{\lambda_{1t}}{\lambda_{2t+1}}$$

Using the optimality condition for capital, the relative shadow values equalize the return on capital r_{Kt+1}.

$$\frac{c_{2t+1}}{c_{1t}} = \beta r_{Kt+1}$$

5. (5pt) Characterize the no-arbitrage condition that pins down the price of land in this economy. Explain the intuition behind this condition. *The no-arbitrage condition for land simply stems from the households optimality condition*

$$\frac{c_{2t+1}}{c_{1t}} = \beta r_{Kt+1} = \beta\frac{r_{Lt+1} + q_{t+1}}{q_t}$$

In other words, the return to land equalizes the return to capital. The reason for this is that both assets allow to transfer resources into the future so if agents hold both, they shall between indifferent between them. The return on land corresponds to the rental rate plus the revaluation of the asset through q.

The planner's problem that discounts time with $\nu \in (0,1)$ is

$$\max_{c_{1t},c_{2t+1},k_{t+1},l_{t+1}} \sum_{t=t_0}^{\infty} \nu^t \left(\ln(c_{1t}) + \beta\ln(c_{2,t+1})\right)$$

subject to

$$K_{Mt} = k_t N_{t-1}$$

$$N_{Mt} = N_t$$

$$N_{t-1}l_t = 1$$

$$Y_{Mt} = N_t c_{1t} + N_{t-1} c_{2t} + N_t k_{t+1}$$

$$N_{t+1} = g(c_{1t}) N_t$$

6. (10pt) What is the transversality condition, and is the transversality condition part of the solution to this problem? Yes/No and why?

   The TVC is an optimality condition for the optimal level of capital accumulation in an infinite horizon problem. We do not impose the TVC in this problem because agents live for two periods making their savings decisions bounded. The TVC is however part of the solution to the planner's problem because the planner's solves the infinite horizon control problem.

7. (15pt) Describe what happens to land holdings as population grows in this economy.

   Is this economy consistent with a steady state where consumption is positive and constant across generations? Yes/No why?

   [As population grows in this economy, land-holdings per capita fall sustainedly. The marginal product of labor is falling sustainedly as population grows (since $\phi + \mu < 1$), and therefore aggregate consumption is also falling, $A_M K_{Mt}^{\phi} N_{Mt}^{\mu-1}$. Consumption when young is $\frac{w_t}{1-\beta}$. In other words, the only steady state that the Malthusian economy admits is one where in consumption goes to zero, or a state of immiseration

From now on, assume that there is no land in the economy, and that the technology for production is constant returns to scale in capital and labor, $\mu + \phi = 1$. Further assume that population growth is exogenous at rate n

8. (20pt) Describe the planner's problem in per-capita terms and characterize the steady state levels of (aggregate) consumption and capital.

$$\max_{c_{1t}, c_{2t+1}, k_{t+1}, l_{t+1}} \sum_{t=t_0^\infty} \nu^t \left(\ln(c_{1t}) + \beta \ln(c_{2,t+1}) \right)$$

subject to

$$A_M \left(\frac{k_t}{(1+n)}\right)^\phi = c_{1t} + \frac{c_{2t}}{(1+n)} + k_{t+1}$$

$$\frac{N_{t+1}}{N_t} = (1+n)$$

where $\frac{K_{Mt}}{N_t} = \frac{k_t}{(1+n))}$.

Let aggregate consumption be $C_t = c_{1t} + \frac{c_{2t}}{(1+n)}$, which in the steady state is simply the solution to

$$A_M \left(\frac{k^\star}{(1+n)}\right)^\phi = C^\star + k^\star$$

The level of capital in steady state can be solved from the Euler equation, i.e

$$1 = \nu \left(\phi(1+n) A_M \left(\frac{k^\star}{(1+n)}\right)^{\phi-1} \right)$$

$$\frac{(1+n)^{\frac{\phi}{\phi-1}}}{(\nu \phi A_M)^{\frac{1}{\phi-1}}} = k^\star$$

9. (10pt) Describe how the steady state level of capital varies with TFP in the economy, as well as with population growth.

Capital in steady state increases with productivity as the marginal product of capital raises with it. Capital per capita falls with population growth.