Optimal Contracts: general model

Finite types

Model

Let us now extend the model in two directions.

A monopolists sells a good q at a price T.

The good is produced at a cost $C(q)$

First, we assume that there are n types: $\theta_1, \theta_2, \ldots, \theta_n$, with $\theta_i < \theta_j$ for $i < j$

We assume that the probability of type i is p_i, with distribution $P_i = \sum_j^i p_j$

Second we adopt a more general utility function: $u(q,\theta) - T$

We also assume the so called single crossing condition. For q, q', θ, θ' such that $q < q'$ and $\theta < \theta'$

$$u(q',\theta) - u(q,\theta) > 0 \rightarrow u(q',\theta') - u(q,\theta') > 0$$
$$u(q',\theta) - u(q,\theta) \geq 0 \rightarrow u(q',\theta') - u(q,\theta') \geq 0$$

Note that when $u(q,\theta)$ is differentiable in q, θ, then single crossing is the same thing as assuming $u_{q\theta}(q,\theta) \geq 0$.

If $u_{q\theta}(q,\theta) > 0$, then we have:

$$0 < \int_{q}^{q'} \int_{\theta}^{\theta'} u_{q\theta}(q,\theta) d\theta dq = \int_{q}^{q'} [u_q(q,\theta') - u_q(q,\theta')] dq$$

$$[u(q',\theta') - u(q',\theta')] - [u(q,\theta') - u(q,\theta')]$$

$$\rightarrow u(q',\theta') - u(q,\theta') > u(q',\theta) - u(q,\theta) > 0$$

The reverse is proven similarly.

This condition has a nice interpretation in the state space q, T.

The slope of an indifference curve is:

$$\frac{dT}{dq}(q,\theta) = u_q(q,\theta)$$

if we have single crossing, then the slope is monotonically non-decreasing in θ.

Finally, we assume that the cost function is convex in q:
$C(q)$ with $C'(q) > 0$ and $C''(q) > 0$

The optimal contract

We can still apply the revelation principle in this more general environment.

The seller's problem is:

$$\max_{T_i, x_i} \sum_i p_i (T_i - c(q_i))$$

$$s.t. \quad \begin{array}{c} u(q_i, \theta_i) - T_i \geq u(q_j, \theta_i) - T_j \text{ for all } i, j \text{ IC(i,j)} \\ u(q_i, \theta_i) - T_i \geq 0 \text{ for all } i \text{ IR(i)} \end{array}$$

What makes this problem complex is the fact that we have a lot of constraints: $n(n-1) + n$.

We start from a result that simplifies the analysis to $2(n-1)+n$ constraints only.

Proposition. Assuming the single crossing condition, IC is satisfied if and only if the local incentive constraints are satisfied, that is:

$$u(q_i,\theta_i) - T_i \geq u(q_{i-1},\theta_i) - T_{i-1} \text{ for all } i > 1 \text{ (DLIC)}$$

$$u(q_i,\theta_i) - T_i \geq u(q_{i+1},\theta_i) - T_{i+1} \text{ for all } i < n \text{ (ULIC)}$$

Note first that these conditions are necessary, as they are a subset of the general conditions.

We therefore focus on the proof that they are sufficient.

First we note that the local IC constraints imply a monotonic allocation rule, that is $q_i > q_j$ for $i > j$.

To see this note that LICs imply:

$$u(q_i,\theta_i) - u(q_{i-1},\theta_i) \geq T_i - T_{i-1} \geq u(q_i,\theta_{i-1}) - u(q_{i-1},\theta_{i-1})$$

So we have:

$$u(q_i,\theta_i) - u(q_i,\theta_{i-1}) \geq u(q_{i-1},\theta_i) - u(q_{i-1},\theta_{i-1})$$

This is possible only if $q_i \geq q_{i-1}$. But then we have:

$$q_i - q_j = q_i - q_{i-1} + q_{i-1} - q_{i-2} +..-q_{j+1} + q_{j+1} + q_j > 0$$

Consider DLIC for i and $i-1$:

$$u(q_i,\theta_i) - u(q_{i-1},\theta_i) \geq T_i - T_{i-1}$$
$$u(q_{i-1},\theta_{i-1}) - u(q_{i-2},\theta_{i-1}) \geq T_{i-1} - T_{i-2}$$

Lets sum these two inequalities:

$$u(q_i,\theta_i) - u(q_{i-1},\theta_i) + u(q_{i-1},\theta_{i-1}) - u(q_{i-2},\theta_{i-1}) \geq T_i - T_{i-2}$$

Note that single crossing implies that:

$$\begin{aligned}
& u(q_i,\theta_i) - u(q_{i-1},\theta_i) + u(q_{i-1},\theta_{i-1}) - u(q_{i-2},\theta_{i-1}) \\
\leq\; & u(q_i,\theta_i) - u(q_{i-1},\theta_i) + u(q_{i-1},\theta_i) - u(q_{i-2},\theta_i) \\
=\; & u(q_i,\theta_i) - u(q_{i-2},\theta_i)
\end{aligned}$$

So we have that $DLIC(i,i-1)$ and $DLIC(i-1,i-2)$ implies

$IC(i, i-2)$:

$$u(q_i, \theta_i) - u(q_{i-2}, \theta_i) \geq T_i - T_{i-2} \quad (IC(i, i-2))$$

We can now complete the proof proceeding inductively.

We conclude that the $DLIC$s imply the DICs.

A similar argument can be used to show that $ULIC$s imply the UICs

Lets consider the following relaxed problem:

$$\max_{T_i, x_i} \sum_i p_i(T_i - c(q_i))$$

$$s.t. \quad \begin{aligned} u(q_i, \theta_i) - T_i &\geq u(q_{i-1}, \theta_i) - T_{i-1} \text{ for all } i > 1 \ (DLIC) \\ u(q_1, \theta_1) - T_1 &\geq 0 \\ q_i &> q_j \text{ for } i > j \end{aligned}$$

Why is this a relaxed program? We have eliminated the $ULIC$ and the $IR(j)$ $j > 1$ constraints.

Note that we have added the monotonicity constraint, that is implied by the LICs.

However, we have seen that monotonicity is an implication of the other constraints.

So the feasibility set in the relaxed problem is a superset of the feasibility in the unrelaxed.

To analyze the relaxed program, we make two observations.

First, $DLIC(i)$ must be binding.

Assume not, then there must be a i and a $\varepsilon > 0$ such that:

$$u(q_i, \theta_i) - T_i - u(q_{i-1}, \theta_i) + T_{i-1} > \varepsilon$$

So we can raise T_j for all $j \geq i$ without violating any constraint and increasing the value function, a contradiction.

The solution of the relaxed program satisfies all the constraints of the unrelaxed program so must be a solution of the unrelaxed program.

To see this note that the binding $DLIC(i+1)$ implies:

$$u(q_{i+1},\theta_{i+1}) - u(q_i,\theta_{i+1}) = T_{i+1} - T_i$$

Single crossing and monotonicity imply then:

$$u(q_{i+1},\theta_i) - u(q_i,\theta_i) \leq T_{i+1} - T_i$$

and so:

$$u(q_i,\theta_i) - T_i \geq u(q_{i+1},\theta_i) - T_i$$

That is $ULIC(i)$. Proceeding in this way we can show that all LIC are satisfied.

To study the optimal contract we can therefore focus on the relaxed program.

Characterization of the optimal contract

We now have to solve:

$$\max_{T_i, x_i} \sum_i p_i (T_i - c(q_i))$$

$$s.t. \quad \begin{aligned} u(q_i,\theta_i) - T_i &= u(q_{i-1},\theta_i) - T_{i-1} \text{ for all } i > 1 \text{ (DLIC)} \\ u(q_1,\theta_1) &= T_1 \\ q_i &> q_j \text{ for } i > j \end{aligned}$$

We can write

$$\begin{aligned} T_i - c(q_i) &= u(q_i,\theta_i) - c(q_i) - (u(q_i,\theta_i) - T_i) \\ &= S(q_i,\theta_i) - U(\theta_i) \end{aligned}$$

where $S(q_i,\theta_i)$ is the surplus generated and $U(\theta_i) = u(q_i,\theta_i) - T_i$ is the indirect utility of agent i.

Moreover, using the binding DLICs, we have:

$$
\begin{aligned}
U(\theta_i) &= u(q_i,\theta_i) - T_i = u(q_{i-1},\theta_i) - T_{i-1} \\
&= U(\theta_{i-1}) + \left[u(q_{i-1},\theta_i) - u(q_{i-1},\theta_{i-1})\right]
\end{aligned}
$$

So we can get rid of transfers and solve:

$$\max_{U_i, q_i} \sum_i p_i(S(q_i, \theta_i) - U(\theta_i))$$

$$U(\theta_i) = U(\theta_{i-1}) + [u(q_{i-1}, \theta_i) - u(q_{i-1}, \theta_{i-1})] \text{ for all } i > 1 \text{ (DLIC)}$$

$s.t.$

$$U(\theta_1) = 0$$

$$q_i > q_j \text{ for } i > j$$

Let us denote:

$$\Phi(q_{i-1}, \theta_i, \theta_{i-1}) = u(q_{i-1}, \theta_i) - u(q_{i-1}, \theta_{i-1})$$

The indirect utility a type i is equal to the utility of type $i-1$ plus a rent due to the informational advantage equal to $\Phi(q_{i-1}, \theta_i, \theta_{i-1})$.

Solving for U recursively we have:

$$U(\theta_1) = 0$$

$$U(\theta_2) = U(\theta_1) + \Phi(q_1, \theta_2, \theta_1)$$

$$\text{so } U(\theta_2) = \Phi(q_1, \theta_2, \theta_1)$$

In general:

$$U(\theta_i) = \sum_{j=2}^{i} \Phi(q_{j-1}, \theta_i, \theta_{i-1})$$

One way to solve the problem is to insert the formula for $U(\theta_i)$ in the objective function to obtain:

$$\max_{U_i, q_i} \sum_i p_i \Big(S(q_i, \theta_i) - \sum_{j=2}^{i} \Phi(q_{j-1}, \theta_i, \theta_{i-1}) \Big)$$

$$s.t. \quad q_i > q_j \text{ for } i > j$$

(with the convention that $\Phi(q_0, \theta_i, \theta_0) = 0$)

This problem can be rewritten as:

$$\max_{U_i,q_i} \sum_i (p_i S(q_i,\theta_i) - (1-P_i)\Phi(q_i,\theta_{i+1},\theta_i))$$

$$s.t. \quad q_i > q_j \text{ for } i > j$$

where we use the convention that $\Phi(q_n,\theta_{n+1},\theta_n) = 0$.

It is generally assumed that monotonicity constraint can be ignored, we will verify in a second when this is ok.

It is also generally assumed that the objective function is quasi concave.

Then the optimum is characterized by:

$$S'(q_i,\theta_i) - \frac{1-P_i}{p_i}\Phi'(q_i,\theta_{i+1},\theta_i) = 0$$

$$\Rightarrow u_q(q_i,\theta_i) = C'(q) + \frac{1-P_i}{p_i}[u_q(q_i,\theta_{i+1}) - u_q(q_i,\theta_i)]$$

for all $i < n$, and:

$$S'(q_i,\theta_i) = 0$$

$$\Rightarrow u_q(q_i,\theta_i) = C'(q) \text{ for all } i = n$$

Lets assume $u(q,\theta) = q\theta$ and $C(q) = q^2/2$, we have:

$$\theta_i - q_i - \frac{1 - P_i}{p_i}[\theta_{i+1} - \theta_i] = 0$$
$$\Rightarrow q_i = \theta_i - \frac{1 - P_i}{p_i}[\theta_{i+1} - \theta_i] \; i < n$$

and

$$q_n = \theta_n.$$

Note:

- We have no distortion at the top, but now this phenomenon concerns only a type with probability p_n;

- All types below the highest are distorted below, that is they are sold less than efficient quantities.

- **We still need to check monotonicity**. Assume $\theta_{i+1} - \theta_i = \Delta\theta$, then we have:

$$q_i = \theta_i - \frac{1 - P_i}{p_i}\Delta\theta$$

so a sufficient condition for monotonicity is that $\frac{1-P_i}{p_i}$ is non increasing in i.

Continuous types

Let us now assume that we have a continuum of types $\theta \in [0, 1]$ (without loss of generality)

The distribution of types is F.

The utility function is $u(q,\theta)$ with $u_\theta(q,\theta) > 0$, $u_{\theta q}(q,\theta) > 0$ or alternatively $u_\theta(q,\theta) < 0$, $u_{\theta q}(q,\theta) < 0$

A direct mechanism is now a function $h(\theta) = (q(\theta), T(\theta))$

A direct mechanism is incentive compatible if:

$$u(q(\theta), \theta) - T(\theta) \geq u(q(\theta'), \theta) - T(\theta') \text{ for all } \theta, \theta'$$

The optimal contract can now be written as:

$$\max_{T,q} \int T(\theta) - C(q(\theta))dF(\theta)$$

$$s.t. \quad \begin{array}{c} u(q(\theta),\theta) - T(\theta) \geq u(q(\theta'),\theta) - T(\theta') \text{ for all } \theta,\theta' \\ u(q(\theta),\theta) - T(\theta) \geq 0 \text{ for all } \theta \end{array}$$

We first study the constraint set, then the solution of this problem.

Implementability

A direct mechanism $h = (q, T)$ is compact valued if

$$\left\{(q,T) \text{ s.t. } \exists\theta' \text{ such that } q,T = (q(\theta'), T(\theta'))\right\}$$

is compact.

We now show that if $u_{\theta q}(q,\theta) > 0$ and a direct mechanism $h(\theta)$ is compact valued then $h(\theta)$ is incentive compatible if and only if:

$$U(\theta'') - U(\theta') = \int_{\theta'}^{\theta''} u_\theta(q(x),x)dx \text{ for any } \theta'',\theta' \text{ s.t. } \theta' < \theta''$$

and $q(\theta)$ is non decreasing

Necessity

$IC(\theta';\theta)$ constraint implies:

$$
\begin{aligned}
U(\theta) &= u(q(\theta),\theta) - T(\theta) \geq u(q(\theta'),\theta) - T(\theta') \\
&= U(\theta') + [u(q(\theta'),\theta) - u(q(\theta'),\theta')]
\end{aligned}
$$

Or:

$$U(\theta) - U(\theta') \geq [u(q(\theta'),\theta) - u(q(\theta'),\theta')]$$

Similarly $IC(\theta;\theta')$ implies:

$$U(\theta') - U(\theta) \geq [u(q(\theta),\theta') - u(q(\theta),\theta)]$$

$$\Rightarrow U(\theta) - U(\theta') \leq [u(q(\theta),\theta) - u(q(\theta),\theta')]$$

We have:

$$u(q(\theta'),\theta) - u(q(\theta'),\theta') \leq U(\theta) - U(\theta') \leq u(q(\theta),\theta) - u(q(\theta),\theta')$$

The single crossing condition implies that $q(\theta) \geq q(\theta')$ for $\theta \geq \theta'$.

Moreover we have:

$$\frac{u(q(\theta'),\theta) - u(q(\theta'),\theta')}{\theta - \theta'} \leq \frac{U(\theta) - U(\theta')}{\theta - \theta'} \leq \frac{u(q(\theta),\theta) - u(q(\theta),\theta')}{\theta - \theta'}$$

Taking the limit as $\theta - \theta' \to 0$, we have:

$$U'(\theta) = u_\theta(q(\theta),\theta)$$

at all points of continuity of $q(\theta)$.

Now observe that:

- given that h is compact valued;
- u is continuous.

Then $U(\theta)$ must be continuous by the theorem of the maximum since:

$$U(\theta) = \max_{\theta' \in [0,1]} \{u(q(\theta'), \theta) - T(\theta')\}$$

Since $U(\theta)$:

- is continuous over a compact set;
- with bounded derivative (at all point of existence).

Then the fundamental theorem of calculus implies that it is integrable.

Sufficiency

Assume:

$$U(\theta'') - U(\theta') = \int_{\theta'}^{\theta''} u_\theta(q(x),x)dx \text{ for any } \theta'',\theta' \text{ s.t. } \theta' < \theta''$$

and $q(\theta)$ is non decreasing

If the mechanism is not IC then there must be a θ and a θ' such that

$$\begin{aligned} U(\theta') + u(q(\theta'),\theta) - u(q(\theta'),\theta') &= u(q(\theta'),\theta) - T(\theta') \\ &\geq u(q(\theta),\theta) - T(\theta) = U(\theta) \end{aligned}$$

and the reverse.

So we can write:

$$\begin{aligned} u(q(\theta'),\theta) - u(q(\theta'),\theta') &> U(\theta) - U(\theta') \\ &= u(q(\theta),\theta) - u(q(\theta'),\theta') \\ &= \int_{\theta'}^{\theta} u_\theta(q(x),x)dx \end{aligned}$$

Or:

$$\int_{\theta'}^{\theta} u_\theta(q(\theta'),x)dx > \int_{\theta'}^{\theta} u_\theta(q(x),x)dx$$

That is:

$$\int_{\theta'}^{\theta} [u_\theta(q(\theta'), x) - u_\theta(q(x), x)]dx > 0$$

But using the monotonicity of $q(x)$, we have:

$$u_\theta(q(\theta'), x) - u_\theta(q(x), x) \leq u_\theta(q(\theta'), x) - u_\theta(q(\theta'), x) = 0$$

a contradiction.

Solving the seller's problem

It follows that the optimal contract is:

$$\max_{T,q} \int [T(\theta) - C(q(\theta))] dF(\theta)$$

$$s.t. \begin{cases} U(\theta) = \int_0^\theta u_\theta(q(x), x) dx \\ q(\theta) \text{ non decreasing} \\ u(q(0), 0) - T(0) = 0 \end{cases}$$

Note that $T(\theta) - C(q(\theta)) = S(q(\theta), \theta) - U(\theta)$.

So we can write it as:

$$\max_{U,q} \int [S(q(\theta), \theta) - U(\theta)] dF(\theta)$$

$$s.t. \quad U(\theta) = \int_0^\theta u_\theta(q(x), x) dx$$

$q(\theta)$ non decreasing and $U(0) = 0$

We can substitute the first constraint in the profit function.

We obtain:

$$\pi(q) = \max_{U,q} \int [S(q(\theta),\theta) - U(\theta)]dF(\theta)$$

$$= \max_{U,q} \int \left[S(q(\theta),\theta) - \int_0^\theta u_\theta(q(x),x)dx \right] f(\theta)d\theta$$

Remember that by integration by parts we have:

$$\int_a^b kz' dx = [kz]_a^b - \int_a^b k'z dx$$

Let us apply this to:

$$-\int_0^1 \left[\int_0^\theta u_\theta(q(x), x) dx \right] f(\theta) d\theta$$

Letting

$$z = -[1 - F(\theta)] \text{ so } z' = F'(\theta) = f(\theta)$$

$$\text{and } k = \int_0^\theta u_\theta(q(x),x)dx \text{ so } k' = u_\theta(q(x),x).$$

We have:

$$\begin{aligned}
EU(\theta) &= \int_0^1 U(\theta)dF(\theta) \\
&= \int_0^1 \int_0^\theta u_\theta(q(x),x)dx \bullet F'(\theta)d\theta \\
&= -[U(\theta)[1 - F(\theta)]_0^1 + \int_0^1 u_\theta(q(x),x) \bullet [1 - F(\theta)]d\theta \\
&= U(0) + E\left[u_\theta(q(\theta),\theta)\frac{1 - F(\theta)}{f(\theta)} \right]
\end{aligned}$$

So the problem becomes:

$$\max_{q} \int \left[S(q(\theta),\theta) - u_{\theta}(q(\theta),\theta)\frac{1 - F(\theta)}{f(\theta)} - U(0) \right] dF(\theta)$$

$s.t.\ q(\theta)$ non decreasing and $U(0) = 0$

This problem is not necessarily concave and does not necessarily an interior solution.

In the following we assume that:

$$\Phi(q,\theta) = S(q,\theta) - u_\theta(q,\theta)\frac{1 - F(\theta)}{f(\theta)}$$

is quasiconcave in q and has a unique interior maximum.

Sufficient conditions for quasi concavity are:

$S(q,\theta)$, typically uncontroversial

$u_\theta(q,\theta)$ not too concave

The focs are:

$$S'(q(\theta),\theta) - u_\theta(q(\theta),\theta)\frac{1 - F(\theta)}{f(\theta)} = 0$$

Assume that $u(q,\theta) = q\theta$ and $C(q) = \frac{q^2}{2}$. Then we have:

$$S'(q(\theta),\theta) - u_\theta(q(\theta),\theta)\frac{1 - F(\theta)}{f(\theta)} = \theta - q(\theta) - \frac{1 - F(\theta)}{f(\theta)}q(\theta)$$

Note that under these assumptions $\Phi(q,\theta)$ is concave and has a unique interior maximum:

$$q(\theta) = \theta - \frac{1 - F(\theta)}{f(\theta)}$$

To prove that this is a solution we need to verify monotonicity.

A necessary and sufficient condition for monotonicity of the solution of the relaxed problem is that $\Phi_{q\theta}(q,\theta) \geq 0$ for all q,θ.

To see this differentiate the foc and obtain:

$$\begin{aligned}
\Phi_{qq}(q,\theta)dq + \Phi_{q\theta}(q,\theta)d\theta &= 0 \\
\rightarrow \frac{dq}{d\theta} &= -\frac{\Phi_{q\theta}(q,\theta)}{\Phi_{qq}(q,\theta)}
\end{aligned}$$

A sufficient condition for this is that $u_{q\theta} \geq 0$ and $u_{q\theta\theta}(q,\theta) \leq 0$ and that types satisfy the monotone hazard rate condition, that is: $\frac{f}{1-F}$ non decreasing.

In the example seen above we have $u = \theta q$, $u_{q\theta} = 1$, $u_{q\theta\theta} = 0$ so the MHRC alone is sufficient.

What have we learned?

There is a trade off between efficiency and incentives:

$$S(q(\theta),\theta) - \frac{1-F(\theta)}{f(\theta)} u_\theta(q(\theta),\theta)$$

This leads to quantities that are distorted, lower than efficient.

The previous formulation of surplus is very similar to the formulation with discrete types:

$$S(q_i,\theta_i) - \frac{1-P_i}{p_i}[u(q_{i-1},\theta_i) - u(q_i,\theta_i)]$$

We still have no distortion at the top, but now this concerns a measure zero of types (only the highest type).