Macroeconomics, PhD core

Lecture #8

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Bewley Economies

- Computation incomplete markets economy
  - Collocation (today)
  - Endogenous grid method (an alternative in Carroll, 2006)

- Uses of compecon

  `https://pfackler.wordpress.ncsu.edu/compecon/154-2/`

  Based on R. Guntin's notes... based on Gianluca V. and Virgiliu M. at NYU

Application: Incomplete Markets Model

We want to solve

$$V\left(a,\epsilon\right)=\max_{a^{'},\tilde{c}\geq 0} u\left(\tilde{c}\right)+\beta\mathbb{E}\left[V\left(a^{'},\epsilon^{'}\right)\right]$$

subject to

$$\begin{aligned}\tilde{c}+a^{'} &= w\epsilon+\left(1+r\right)a\\ a^{'} &\geq \underline{a}\\ \epsilon^{'} &= \rho\epsilon+\varepsilon\\ \varepsilon &\sim^{iid}\end{aligned}$$

- solve using collocation method $\rightarrow V(a,\epsilon)\approx B(a,\epsilon)c$
  where B are basis functions and c are "collocation" coefficients

Function approximation

- Objective is to find $\hat{f}$ such that it minimizes

$$\left\| f - \hat{f} \right\|_\infty = \sup_{x \in D} \mid f(x) - \hat{f}(x) \mid$$

- Function interpolation
  - $\hat{f}(x, c)$ linear combination of polynomials with coefficients c
  - choose c to minimize $\mid f - \hat{f} \mid$ at finite number of nodes
  - Key: choice of polynomial and nodes

Choice of polynomial and nodes

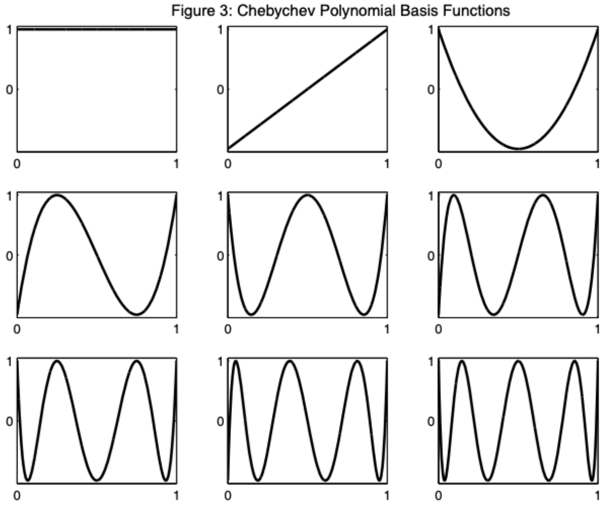
Choose basis functions $b_1(x), \dots, b_n(x)$

- $b_n(x) = x^n$ typically bad idea
- Chebyshev orthogonal polynomials
- Splines k-th order polynomials spliced together (linear popular choice)

Choose nodes $x = x_1, \dots, x_n$

- Equidistant typically bad
- Chebyshev nodes (roots of n-th degree Chebyshev polynomial) optimal
- use function `funnode`

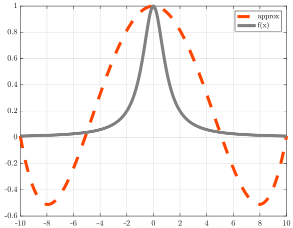
Chebyshev basis functions: b_i



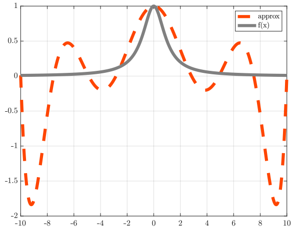
Approximating Function Example: $1/(1+x^2)$

- Chebyshev basis and equidistant nodes

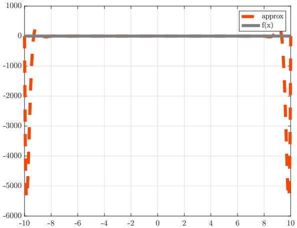
n = 5



n = 9



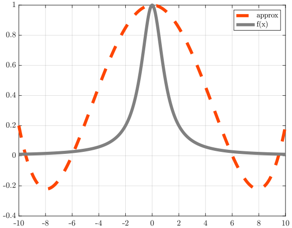
n = 30



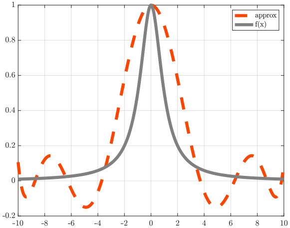
Approximating Function Example: $1/(1+x^2)$

- Chebyshev basis and Chebyshev nodes

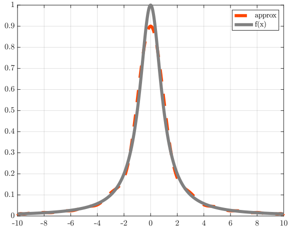
n = 5



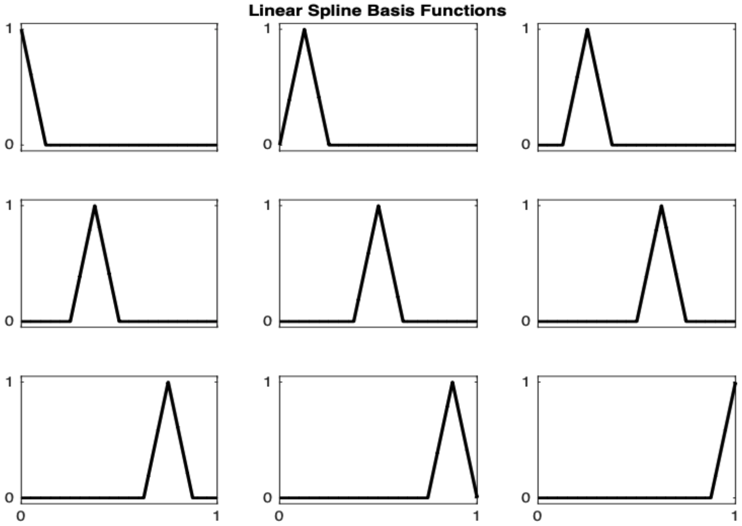
n = 9



n = 30



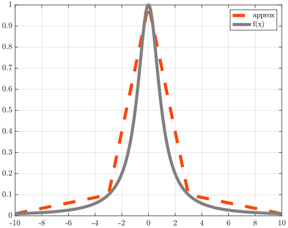
Splines basis functions: b_i



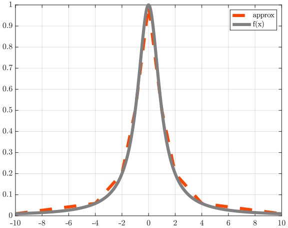
Approximating Function Example: $1/(1+x^2)$

- Splines (linear) basis and n break points

n = 5



n = 9



- Splines can be non-linear (e.g., cubic)

Computation

1. Choose n order of polynomial
2. Choose $m \geq n$ nodes
3. Evaluate at nodes $B(x)$ where

$$B(x) = \begin{bmatrix} b_1(x_1) & b_2(x_1) & \dots & b_n(x_1) \\ b_1(x_2) & b_2(x_2) & \dots & b_n(x_2) \\ \vdots & \vdots & \dots & \vdots \\ b_1(x_m) & b_2(x_m) & \dots & b_n(x_m) \end{bmatrix}$$

4. Find c using $F = Bc$ then compute $c = B \backslash F$ (F also evaluated at x)

Example: Code using Splines

- Matlab code

```
a = -10;
b = 10;
xnode = (a:.001:b)';

break_points = [-10; -3; 0; 3 ; 10];
fspace = fundef({'spli', break_points,0,1}); % linear splines
s = funnode(fspace); % nodes
Phi = funbas(fspace, s); % evaluate n by n matrix of n basis functions at n points in s

F = 1./(1+s.^ 2);
c = Phi \ F;

Phinode = funbas(fspace, xnode); % basis functions at xnode
plot(xnode,Phinode*c) % plot approximation
```

More than 1 dimension

- Goal: approximate $f(x, y)$
- Generate x and y nodes
- Basis functions b_i^x and b_i^x
- Approximate

$$F = \sum_i^{n_x} \sum_j^{n_y} b_i^x(x)\, b_i^y(y)\, c_{ij}$$

- find c with $c = B \backslash F$ where $B = B_d \otimes B_{d-1} \ldots \otimes B_1$

Example: $1/(1+x^2y^2)$

- Matlab code

```
n=[7,7]; % nodes in each dimension
a=[-10,-10]; b=[10,10]; % bounds in each dimension

fspace=fundefn('cheb',n,a,b) % define func. approx. space
xnode=funnode(fspace); % default nodes
xnode=gridmake(xnode); % Cartesian product of univariate nodes
Phi=funbas(fspace,xnode,[0, 0]); % basis function at Cartesian product

F=1./(1+xnode(:,1).^2.*xnode(:,2).^2); %eval. function at nodes
c = Phi \ F;
```

Application: Incomplete Markets Model

We want to solve

$$V(a,\epsilon) = \max_{a^{'},\tilde{c}\geq 0} u(\tilde{c}) + \beta\mathbb{E}\left[V\left(a^{'},\epsilon^{'}\right)\right]$$

subject to

$$\begin{aligned}\tilde{c} + a^{'} &= w\epsilon + (1+r)\,a \\ a^{'} &\geq \underline{a} \\ \epsilon^{'} &= \rho\epsilon + \varepsilon \\ \varepsilon &\sim^{iid}\end{aligned}$$

- Discretize shock process
- solve using collocation method $\rightarrow V(a,\epsilon) \approx B(a,\epsilon)c$

Markov Transition Probability: Discretization

- Discretize process: $e' = \rho e + \epsilon$
- Commonly used Tauchen, Rouwenhorst
- Using Compecon: example, assume $\sim^{\text{iid}} N(0, \sigma_e^2)$
  `https://pfackler.wordpress.ncsu.edu/compecon/154-2/`

  1. [epse, we] = qnwnorm(n_points, 0, sde^2); %discretize normal distribution
  2. fspace = fundef({'spli', egrid, 0, 3}); s = funnode(fspace); Ns = size(s, 1);
  3. Compute e'

     for j = 1 : numel(epse)

     eprime = rhoe*s + p.epse(j);
     eprime = max(min(eprime, p.emax), p.emin); % Replace for bounds

     end

  or quantecon
  `https://python.quantecon.org/finite_markov.html`

Application: Incomplete Markets Model

Broad steps

1. Parameters and grid
2. Define space (e.g., B and B^E)
   - Notice that B^E can computed using B and **!**

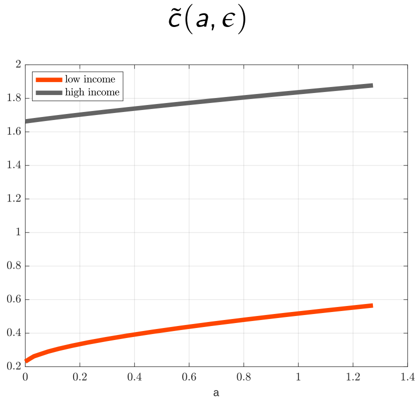
$$\mathbb{E}\left[V\left(a,\epsilon^{'}\right)\right] = B\left(a,\epsilon\right)c_E = \sum_{\omega_i}\omega_i V\left(a,\rho\epsilon+\varepsilon_i\right) = \sum_{\omega_i}\omega_i B\left(a,\rho\epsilon+\varepsilon_i\right)c$$

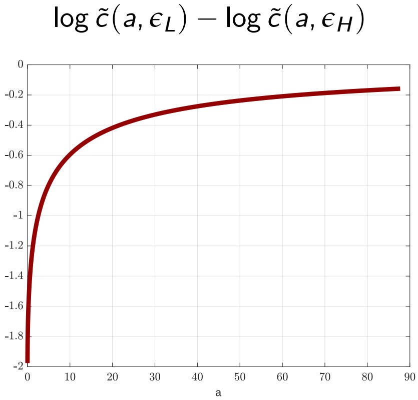
$$\rightarrow c_E = \underbrace{B\left(a,\epsilon\right)^{-1}\sum_{\omega_i}\omega_i B\left(a,\rho\epsilon+\varepsilon_i\right)}_{B^E}c$$

3. Guess c_0 and solve non-linear equation until convergence

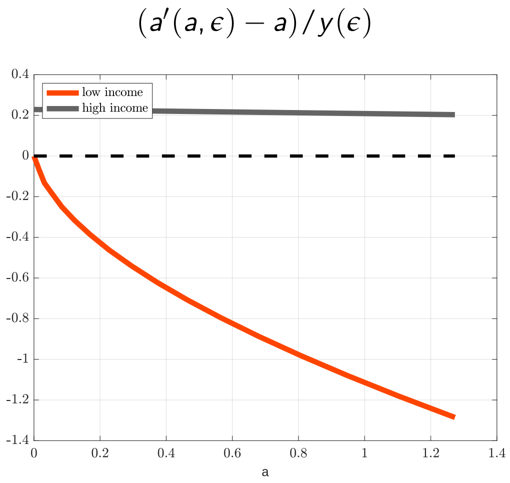
   $B\left(a,\epsilon\right)c_{j+1} = \max_{a^{'}\geq 0} u\left(w\epsilon+\left(1+r\right)a-a^{'}\right)+\left(a^{'},\epsilon\right)B^E c_j$

Consumption $c(a, \epsilon)$





Savings rate $(a'(a,\epsilon) - a)/y(\epsilon)$



Ergodic Distribution

- Aggregate assets are $A = \sum_{\epsilon} a \times \phi(a, \epsilon)$
- Assume $A = K$ in equilibrium then find r such that market clearing holds
- To aggregate we need to compute the Transition Probability Matrix (TPM) P using the policy functions
- Compute ergodic distribution $\phi(a, \epsilon)$ such that

$$\phi(a, \epsilon)P = \phi(a, \epsilon)$$

is a fixed point. We can solve it by computing the TPM and finding the eigenvector

Detour on Markov Processes

- A stochastic process is a sequence of random vectors. We typically use time to index it.
- A stochastic process has the *Markov property* if for all t and $k \geq 1$,

$$Prob(x_{t+1}|x_t, \dots x_{t-k}) = Prob(x_{t+1}|x_t)$$

- Assuming the Markov Property we can characterize a process by a *Markov chain*.
  - π_0 initial distribution,
  - P transition matrix,

$$P_{ij} = Prob(x_{t+1} = e_j | x_t = e_i). \qquad \sum_j P_i j = 1$$

  - vector of zeros except 1 in entry i, e_i.

- Probability of landing in state j in two periods

$$P_{ij}^{(2)} = Prob(x_{t+2} = e_j | x_t = e_i) = \sum_{h=1}^{n} Prob(x_{t+2} = e_j | x_{t+1} = e_h) Prob(x_{t+1} = e_h | x_t =$$

- Unconditional probabilities are therefore determined by

$$\pi_1' = Prob(x_1) = \pi_0' P$$

$$\pi_2' = Prob(x_2) = \pi_0' P^2$$

Detour on Markov Processes

- A stationary distribution satisfies

$$\pi' = \pi' P \quad \rightarrow \quad \pi'(I - P) = 0 \quad (I - P')\pi = 0$$

  Does this distribution exist?

  at least one unit eigenvalue & at least one eigenvector π normalized so that $\sum_i \pi_i = 1$.

- Multiplicity is possible.
- convergence? for given π_0

$$\lim_{t \to \infty} \pi_t = \pi_\infty$$

  If yes and π_∞ independent of π_0 then the process is assymptotically stationary with unique distribution.

  π_∞ is the assymptotic (or invariant) distribution of P.

Other Applications, for your future :)

- Compecon is very flexible and has very handy computational tools
- Discrete and continuous choice (e.g., problems with occupational choice and consumption-savings)
- Heterogeneous firms models

Panel Simulation

- We can also simulate many households
- For example: a standard simulated panel will follow this steps
  - select a very long period of time and N number of agents
  - make many draws of ϵ
  - use policy function to compute the consumption and assets of each agents
  - for initial conditions to remain irrelevant remove the first periods of the simulation
  - simulation can be "on the grid" or use a continuous process and interpolate the policy functions
- We can study micro behavior of consumption and income jointly, check if the distribution convergence to the one we compute inverting the TPM, etc

Aggregate Shocks

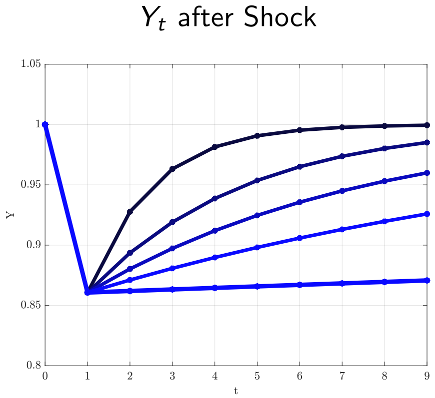
- Assume shock unexpected ("MIT" shock)
- Shock $\epsilon_0^Y < 0$ proportional shock with persistence $\rho^Y < 1$
- Computation
  - assume r fixed so no need to solve for prices along transition path
  - for t large enough we assume $Y_t = Y_\infty = Y$
  - steps: (i) policy functions along transition path backward iteration; (ii) simulate forward the distribution and compute aggregates
- Study cross-sectional responses to aggregate shock

Solving Transition Path

- Simplifications:
  1. unexpected and perfect foresight about aggregate shock
  2. r exogenous and endowment economy
- Backward computation of $\{c_t(a,\epsilon), a'_t(a,\epsilon)\}$
  - at $t \geq T$ and $t < 0$ policy functions $\{c_t(a,\epsilon), a'_t(a,\epsilon)\} = \{c(a,\epsilon), a'(a,\epsilon)\}$
  - for $t \in [0, T)$ we solve $\{c_t(a,\epsilon), a'_t(a,\epsilon)\}$ given $\{c_{t+1}(a,\epsilon), a'_{t+1}(a,\epsilon)\}$
    - same algorithm but without checking for convergence!
  - using $\{c_t(a,\epsilon), a'_t(a,\epsilon)\}$ compute forward the $\{\phi_t(a,\epsilon)\}$ (and aggregates)
    - at T we may not be in the steady state distribution
- Very simple to extend to a permanent shock where, e.g., $Y_\infty \neq Y$

Aggregate Shocks

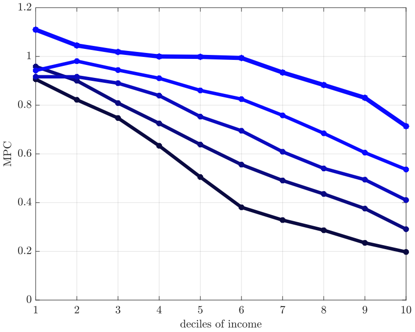
- $\rho^Y = [0.5; 0.75; 0.85; .92; .99]$
- Assume $\nu = 0$



Heterogeneous Responses

- Across deciles (repeated cross-section moment)
- Two moments: MPC $\frac{\bar{c}(Ye^{\epsilon_0^Y},\epsilon_i)-\bar{c}(Y,\epsilon_i)}{Ye^{\epsilon_0^Y}-Y}$ and elasticity $\frac{\ln\bar{c}(Ye^{\epsilon_0^Y},\epsilon_i)-\ln\bar{c}(Y,\epsilon_i)}{\epsilon_0^Y}$

MPC



Elasticity C-to-Y