ECON 6200 : Section 5

2/21/25

Agenda

- GMM Intuition
- Review Moments
- GMM Definition
- Estimation
  - ↳ Derivation
- Relationship w/ TSLS, IV, OLS
- GMM Asymptotics

GMM Intuition

GMM generalizes the classical method of moments by allowing for more equations than unknown parameters

Like method of moments, GMM starts by postulating moment conditions, which we assume to be true in the population
↳ ie: there exists population parameters that solve the system of moment equation

How is GMM more "general"?

1) Overidentification ($\ell \geq k$)
   ↳ # moment eq $\geq$ # of regressors
   ↳ cannot solve for a closed form solution <u>AND</u> meeting our sample moment conditions

   Solution: Minimize "distance" of sample moment conditions to 0

2) Heteroscedasticity
   ↳ Previously, OLS, IV, and (implicitly) TSLS assume homoscedasticity

Review: Method of Moments

- Random vectors D
- Parameter of interest $\theta \in \mathbb{R}^k$, for some $k < \infty$
- Known function $g(\cdot)$

We have a moment condition that looks like

$$E[g(D;\theta)] = 0$$

The method of moments estimator $\hat{\theta}$ can be constructed by solving the sample analog of the moment condition

$$\frac{1}{n}\sum_{i=1}^{n} g(D;\hat{\theta}) = 0$$

GMM Formal Definition

Given random vectors $Y \in \mathbb{R}$, $X \in \mathbb{R}^k$, $Z \in \mathbb{R}^\ell$, and moment conditions

$$E[g(Y, X, Z; \theta)] = 0$$

where $\theta \in \mathbb{R}^k$ and $g(\cdot)$ is a known smooth function mapping into $\mathbb{R}^\ell$, $\ell \geq k$.

Moreover, for now, we assume $g(\cdot)$ is <u>linear</u>

Hence, in the context of this class / overidentified IV, we can write the moment condition as

$$E[Z(Y - X'\beta)] = 0$$

Estimation

The GMM estimator is defined as

$$\hat{\theta}(W) = \underset{\theta}{\arg\min}\, J_n(\theta)$$

where $J_n(\theta) = n\, \bar{g}_n(\theta)' W \bar{g}_n(\theta)$, and $\bar{g}_n(\theta) = \frac{1}{n}\sum_{i=1}^{n} g(\cdot\,; \theta)$

Derivation of $\hat{\beta}_{GMM}$

Under the linear case

$$J_n(\beta) = n\left(Z'(Y-X\beta)\right)' W \left(Z'(Y-X\beta)\right)$$

$$= n\left(Z'Y - Z'X\beta\right)' W \left(Z'Y - Z'X\beta\right)$$

Some algebra:

$$\begin{aligned}
&(Y'Z - \beta'X'Z)\,W\,(Z'Y - Z'X\beta) \\
&= (Y'ZW - \beta'X'ZW)(Z'Y - Z'X\beta) \\
&= Y'ZWZ'Y - Y'ZWZ'X\beta - \beta'X'ZWZ'Y + \beta'X'ZWZ'X\beta
\end{aligned}$$

FOC

$[\beta]$ $$2X'ZWZ'X\hat{\beta} - 2X'ZWZ'Y = 0$$

$$\hat{\beta}_{GMM} = (X'ZWZ'X)^{-1}(X'ZWZ'Y)$$

Special Cases

- OLS estimator

  If $X = Z$

$$\hat{\beta} = (X'XWX'X)^{-1} X'XWX'Y$$
$$= (X'X)^{-1} W (X'X)^{-1} X'X\, W\, X'Y$$
$$= (X'X)^{-1} (XY)$$

- IV estimator

  If $(X'Z)$ square and invertible (ie: $l = k$)

$$\hat{\beta} = (Z'X)^{-1} W^{-1} (X'Z)^{-1} X'ZWZ'Y$$
$$= (Z'X)^{-1} Z'Y$$

- TSLS estimator

  If $W = (Z'Z)$

$$\hat{\beta}_{TSLS} = (X'Z(Z'Z)^{-1}Z'X)^{-1} X'Z(Z'Z)^{-1}Z'Y$$

vs.

$$\hat{\beta}_{GMM} = (X'ZWZ'X)^{-1} X'ZWZ'Y$$

GMM Asymptotics

GMM Assumptions

1. We observe i.i.d. realizations (Y_i, X_i, Z_i), $i = 1, \ldots, n$.
2. $\mathbb{E}(Z(Y - X'\beta)) = 0$.
3. $\mathbb{E}(|Y^4|) < \infty$,
4. $\mathbb{E}(\|X\|^4) < \infty$,
5. $\mathbb{E}(\|Z\|^4) < \infty$,
6. $\boldsymbol{Q} \equiv \mathbb{E}(ZX')$ has full rank k,
7. $\boldsymbol{W}$ is positive definite,
8. $\Omega \equiv \mathbb{E}(ZZ'\varepsilon^2)$ is positive definite.

Asymptotic Distribution

The following algebra generalizes previous algebra for OLS:

$$\begin{aligned}
\hat{\beta}_{GMM}(\boldsymbol{W}) &= \left(\boldsymbol{X}'\boldsymbol{Z}\boldsymbol{W}\boldsymbol{Z}'\boldsymbol{X}\right)^{-1}\boldsymbol{X}'\boldsymbol{Z}\boldsymbol{W}\boldsymbol{Z}'\boldsymbol{Y} \\
&= \beta + \left(\boldsymbol{X}'\boldsymbol{Z}\boldsymbol{W}\boldsymbol{Z}'\boldsymbol{X}\right)^{-1}\boldsymbol{X}'\boldsymbol{Z}\boldsymbol{W}\boldsymbol{Z}'\varepsilon \\
&= \beta + \left(\tfrac{1}{n}\boldsymbol{X}'\boldsymbol{Z}\boldsymbol{W}\tfrac{1}{n}\boldsymbol{Z}'\boldsymbol{X}\right)^{-1}\tfrac{1}{n}\boldsymbol{X}'\boldsymbol{Z}\boldsymbol{W}\tfrac{1}{n}\boldsymbol{Z}'\varepsilon \\
&= \beta + \left(\mathbb{E}(XZ')\boldsymbol{W}\mathbb{E}(ZX')\right)^{-1}\mathbb{E}(XZ')\boldsymbol{W}\tfrac{1}{n}\boldsymbol{Z}'\varepsilon + o_P(1) \\
&= \beta + \left(\boldsymbol{Q}'\boldsymbol{W}\boldsymbol{Q}\right)^{-1}\boldsymbol{Q}'\boldsymbol{W}\tfrac{1}{n}\boldsymbol{Z}'\varepsilon + o_P(1)
\end{aligned}$$

By WLLN, $\frac{1}{n}Z'\varepsilon \xrightarrow{P} E[Z'\varepsilon] = E[Z'(Y - X\beta)] = 0$

1) Consistency

$$\hat{\beta}_{GMM}(\boldsymbol{W}) - \beta \quad \xrightarrow{p} \quad 0,$$

2) Asymptotic Normality

$$\sqrt{n}(\hat{\beta}_{GMM}(\boldsymbol{W}) - \beta) \quad \xrightarrow{d} \quad N(0, (\boldsymbol{Q}'\boldsymbol{W}\boldsymbol{Q})^{-1}\boldsymbol{Q}'\boldsymbol{W}\Omega\boldsymbol{W}\boldsymbol{Q}(\boldsymbol{Q}'\boldsymbol{W}\boldsymbol{Q})^{-1}).$$

By CLT, $\sqrt{n}\left(\frac{1}{n}Z'\varepsilon\right) \xrightarrow{d} N(0, \Omega)$