Remark: Suggested solutions for Problem 1 are based on answers provided by previous TAs.

Skilled-biased technical change

... Historical evidence suggests that complementarity between skilled labor and capital has characterized technological developments throughout the entire 20th century. (Goldin and Katz, 1998)

Consider an economy with two types of workers, skilled, s, and unskilled, u. All workers have identical preferences given by

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \quad \beta \in (0,1)$$

where u is twice differentiable and concave, and satisfies the Inada conditions in leisure, l and consumption, c.

The technology for production is

$$f(z, k_t, n_t) = z(\lambda(\mu(k_t)^\rho + (1-\mu)(n_s)^\rho)^{\frac{\sigma}{\rho}} + (1-\lambda)n_u^\sigma)^{\frac{1}{\sigma}}$$

where n_s is the amount of skilled labor and n_u is the amount of unskilled labor.

Feasibility dictates

$$\begin{aligned} c_t + x_{kt} &\leq f(z, k_t, n_t) \quad k_0 > 0 \\ k_{t+1} &\leq (1-\delta_k)k_t + x_{kt} \\ n_t + l_t &\leq 1 \\ n_s + n_u &= n_t \end{aligned}$$

where z is the overall productivity of the economy.

1. Show that this production technology displays a constant elasticity of substitution between capital and skilled labor. How does the elasticity depend on the parameter $\rho \in (0,1)$, and μ?

The formula for elasticity of substitution between capital and skilled labor is:

$$\epsilon_{k,n_s} = -\frac{dlog(\frac{k}{n_s})}{dlog(\frac{f_k}{f_{n_s}})}$$

Since the technology for production is:

$$f(z,k_t,n_t) = z(\lambda(\mu(k_t)^\rho + (1-\mu)(n_s)^\rho)^{\frac{\sigma}{\rho}} + (1-\lambda)n_u^\sigma)^{\frac{1}{\sigma}}$$

For simplicity, define

$$A \equiv \lambda(\mu(k_t)^\rho + (1-\mu)(n_s)^\rho)^{\frac{\sigma}{\rho}} + (1-\lambda)n_u^\sigma$$
$$B \equiv \mu(k_t)^\rho + (1-\mu)(n_s)^\rho$$

Therefore

$$f_k = \frac{\partial f(z,k_t,n_t)}{\partial k_t} = \frac{1}{\sigma} z A^{\frac{1}{\sigma}-1}\frac{\sigma}{\rho}\lambda B^{\frac{\sigma}{\rho}-1}\rho\mu k_t^{\rho-1}$$
$$f_{n_s} = \frac{\partial f(z,k_t,n_t)}{\partial n_s} = \frac{1}{\sigma} z A^{\frac{1}{\sigma}-1}\frac{\sigma}{\rho}\lambda B^{\frac{\sigma}{\rho}-1}\rho(1-\mu) {n_s}^{\rho-1}$$

$$\frac{f_k}{f_{n_s}} = \frac{\mu}{1-\mu}\left(\frac{k_t}{n_s}\right)^{\rho-1}$$
$$log\left(\frac{f_k}{f_{n_s}}\right) = log\left(\frac{\mu}{1-\mu}\right) + (\rho-1)log\left(\frac{k_t}{n_s}\right)$$
$$\epsilon_{k,n_s} = -\frac{dlog(\frac{k}{n_s})}{dlog(\frac{f_k}{f_{n_s}})} = \frac{1}{1-\rho}$$

which is positive for $\rho \in (-\infty, 1)$.

Now we can evaluate how ϵ_{k,n_s} depends on ρ and μ:

- $\frac{\partial \epsilon_{k,n_s}}{\partial \rho} = \left(\frac{1}{1-\rho}\right)^2 > 0$ elasticity of substitution between capital and skilled labor increases with ρ.
- $\frac{\partial \epsilon_{k,n_s}}{\partial \mu} = 0$ elasticity of substitution between capital and labor does not depend on μ.

2. Go as far as you can providing conditions that guarantee existence of a steady state.

First we state the conditions that guarantee an interior solution:

- Inada conditions for utility function

- Inada conditions for production function
  $\lim_{i\to 0} f_i(.) = \infty$, $\lim_{i\to\infty} f_i(.) = 0$, for $i = k, n_s, n_u$
- No free lunch $f(0) = 0$
- $k_0 > 0$

The Social Planner's Problem

$$\begin{aligned} \max_{c_t, x_{kt}, l_t, k_{t+1}} & \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \\ \text{s.t.} \quad c_t + x_{kt} &\leq f(z, k_t, n_t) \\ k_{t+1} &\leq (1-\delta_k)k_t + x_{kt} \\ n_t + l_t &\leq 1 \\ (c_t, x_{kt}, l_t, k_{t+1}) &\geq (0,0,0,0) \end{aligned}$$

Since the supply of skilled workers n_s and the supply of unskilled workers n_u are fixed and the utility function is assumed to be strictly increasing in leisure l_t, then in the equilibrium we have $l_t = 1 - n_t$ for all t.

The Lagrangian Equation

$$\begin{aligned} \mathcal{L}_{\mathcal{SPP}} = & \sum_{t=0}^{\infty} \beta^t \{u(c_t, l_t) + \theta_{1t}[f(z, k_t, n_t) - c_t - x_{kt}] + \theta_{2t}[(1-\delta_k)k_t + x_{kt} - k_{t+1}] \\ & + \eta_{1t}c_t + \eta_{2t}x_{kt} + \eta_{3t}k_{t+1}\} \end{aligned}$$

FOCs

$$\begin{aligned} [c_t] &\quad u_c(c_t, l_t) - \theta_{1t} + \eta_{1t} = 0 \\ [x_{kt}] &\quad -\theta_{1t} + \theta_{2t} + \eta_{2t} = 0 \\ [k_{t+1}] &\quad -\theta_{2t} + \eta_{3t} + \beta(\theta_{1t+1} f_k(z, k_{t+1}, n_{t+1}) + \theta_{2t+1}(1-\delta_k)) = 0 \end{aligned}$$

Complementary Slackness Conditions

$$
\begin{aligned}
&[CS_1] \quad \theta_{1t}[f(z,k_t,n_t) - c_t - x_{kt}] = 0 \\
&[CS_2] \quad \theta_{2t}[(1-\delta_k)k_t + x_{kt} - k_{t+1}] = 0 \\
&[CS_3] \quad \eta_{1t}c_t = 0 \\
&[CS_4] \quad \eta_{2t}x_{kt} = 0 \\
&[CS_5] \quad \eta_{3t}k_{t+1} = 0
\end{aligned}
$$

For interior solutions, we have $(c_t, x_{kt}, l_t, k_{t+1}) > (0,0,0,0)$ for all t. Hence complementary slackness conditions imply $\eta_{1t} = \eta_{2t} = \eta_{3t} = 0$ for all t.

Combining FOCs, we have

$$
[\text{Euler Equation}] \quad u_c(c_t, l_t) = \beta u_c(c_{t+1}, l_{t+1})(f_k(z, k_{t+1}, n_{t+1}) + 1 - \delta_k)
$$

Transversality Condition

$$
[TVC] \quad \lim_{T\to\infty} \beta^T u_c(c_T, l_T)k_{T+1} = 0
$$

To guarantee the existence and uniqueness of the steady state, we can assume $f(z,k_t,n_t)$ is strictly increasing and strictly concave in k_t. Assume $f_k(z,k_t,n_t)$ is continuous and it satisfies:

$$
\begin{aligned}
\lim_{k_t\to 0} f_k(z,k_t,n_t) > \frac{1}{\beta} + 1 - \delta_k \\
\lim_{k_t\to \infty} f_k(z,k_t,n_t) < \frac{1}{\beta} + 1 - \delta_k
\end{aligned}
$$

Define $L_t = \beta[f_k(z,k_t,n_t) + (1-\delta_k)]$. Then L_t has the property

$$
\lim_{k_t\to 0} L_t > 1 \qquad \lim_{k_t\to\infty} L_t < 1
$$

Therefore it follow that in the steady state

$$
\begin{aligned}
\beta[f_k(z^*,k^*,n^*) + (1-\delta_k)] = 1 \\
c^* + \delta_k k^* = f(z^*,k^*,n^*)
\end{aligned}
$$

By the assumptions about $f_k(z,k_t,n_t)$ we know that there exists $k^* > 0$. We need to check if $c^* > 0$. We can check if $f_k(z,k_t,n_t) < \delta_k$ at the point $\hat{k}$ such that $f(z,\hat{k},n) = \delta_k\hat{k}$ (i.e., $\hat{c} = 0$). Then we should have that for any $k < \hat{k}$, $f(z,k_t,n) > \delta_k k$ (i.e., $c > 0$) . Note that from the Euler Equation, in the steady state $f_k(z^*,k^*,n^*) = \frac{1}{\beta} + 1 - \delta_k > \delta_k$ for $\beta \in (0,1)$. Therefore $k^* < \hat{k}$ and

$c^* > 0$.

3. Go as far as you can characterizing the skill premium $\left(\frac{w_s}{w_u}\right)$. Assume that $\sigma > \rho$ estimated in Krusell et.al., Econometrica, 2000.

Profit Maximization Problem

$$\max_{k_t, n_{st}, n_{ut}} f(z, k_t, n_{st}, n_{ut}) - r_t k_t - w_s n_{st} - w_u n_{ut}$$

FOCs

$$\begin{aligned}
[n_{st}] \quad w_{st} = f_{n_s} &= \frac{1}{\sigma} z A^{\frac{1}{\sigma}-1} \frac{\sigma}{\rho} \lambda B^{\frac{\sigma}{\rho}-1} \rho (1-\mu) n_{st}^{\rho-1} \\
[n_{ut}] \quad w_{ut} = f_{u_s} &= \frac{1}{\sigma} z A^{\frac{1}{\sigma}-1} (1-\lambda) \sigma n_{ut}^{\sigma-1}
\end{aligned}$$

Skill premum is given by:

$$\begin{aligned}
\frac{w_{st}}{w_{ut}} &= \frac{\lambda(1-\mu) B^{\frac{\sigma}{\rho}-1} n_{st}^{\rho-1}}{(1-\lambda) n_{ut}^{\sigma-1}} \\
&= \frac{\lambda(1-\mu)(\mu(k_t)^\rho + (1-\mu)(n_{st})^\rho)^{\frac{\sigma}{\rho}-1} n_{st}^{\rho-1}}{(1-\lambda) n_{ut}^{\sigma-1}} \\
&= \frac{\lambda(1-\mu)(n_{st}^\rho(\mu(\frac{k_t}{n_{st}})^\rho + (1-\mu)))^{\frac{\sigma}{\rho}-1} n_{st}^{\rho-1}}{(1-\lambda) n_{ut}^{\sigma-1}} \\
&= \frac{\lambda(1-\mu)(\mu(\frac{k_t}{n_{st}})^\rho + (1-\mu))^{\frac{\sigma}{\rho}-1} n_{st}^{\sigma-\rho} n_{st}^{\rho-1}}{(1-\lambda) n_{ut}^{\sigma-1}} \\
&= \frac{\lambda(1-\mu)}{1-\lambda} \left(\mu \left(\frac{k_t}{n_{st}}\right)^\rho + (1-\mu)\right)^{\frac{\sigma-\rho}{\rho}} \left(\frac{n_{st}}{n_{ut}}\right)^{\sigma-1} \\
&= \frac{\lambda(1-\mu)}{1-\lambda} \left(\mu \left(\frac{k_t}{n_{st}}\right)^\rho + (1-\mu)\right)^{\frac{\sigma-\rho}{\rho}} \left(\frac{n_{ut}}{n_{st}}\right)^{1-\sigma}
\end{aligned}$$

In equilibrium, $n_{st} \equiv n_s$ and $n_{ut} \equiv n_u$. Taking log on both sides

$$log\left(\frac{w_{st}}{w_{ut}}\right) = log\left(\frac{\lambda(1-\mu)}{1-\lambda}\right) + \frac{\sigma-\rho}{\rho} log\left(\mu \left(\frac{k_t}{n_s}\right)^\rho + (1-\mu)\right) + (1-\sigma) log\left(\frac{n_u}{n_s}\right)$$

Since we have $1 > \sigma > \rho$, we get that skill premium is increasing in $\frac{k_t}{n_s}$ and $\frac{n_u}{n_s}$.

4. Assume that countries differ in the value of productivity z. If every country is in steady state,

what does the model imply for the cross country differences in

(a) the capital skills ratio k/n_s

(b) output per worker

(c) the skill-premium

(a) Denote $\frac{k}{n_s}$ as k_s and $\frac{n_u}{n_s}$ as n_{us}. From the Euler Equation in steady state

$$1 = \beta[f_k(z^*, k^*, n^*) + (1 - \delta_k)]$$
$$\frac{1}{\beta} - 1 + \delta_k = zA^{\frac{1}{\sigma}-1}\lambda B^{\frac{\sigma}{\rho}-1}\mu(k^*)^{\rho-1}$$
$$\frac{1}{\beta} - 1 + \delta_k = z\lambda\mu[\lambda(\mu(k^*)^\rho + (1-\mu)(n_s^*)^\rho)^{\frac{\sigma}{\rho}} + (1-\lambda)(n_u^*)^\sigma]^{\frac{1}{\sigma}-1}[\mu(k^*)^\rho + (1-\mu)(n_s^*)^\rho]^{\frac{\sigma}{\rho}-1}(k^*)^{\rho-1}$$
$$\frac{1}{\beta} - 1 + \delta_k = z\lambda\mu[\lambda(\mu(k_s^*)^\rho + (1-\mu))^{\frac{\sigma}{\rho}} + (1-\lambda)(n_{us}^*)^\sigma]^{\frac{1}{\sigma}-1}[\mu(k_s^*)^\rho + (1-\mu)]^{\frac{\sigma}{\rho}-1}(k_s^*)^{\rho-1}$$

Take log on both two sides

$$log\Big(\frac{1}{\beta} - 1 + \delta_k\Big) = logz + log\lambda\mu + \frac{1-\sigma}{\sigma}log\Big(\lambda(\mu(k_s^*)^\rho + (1-\mu))^{\frac{\sigma}{\rho}} + (1-\lambda)(n_{us}^*)^\sigma\Big)$$
$$\frac{\sigma-\rho}{\rho}log\Big(\mu(k_s^*)^\rho + (1-\mu)\Big) + (\rho-1)logk_s^*$$

Differentiate w.r.t z :

$$0 = \frac{1}{z} + \frac{1-\sigma}{\sigma}\frac{\lambda\sigma\left(\mu\left(k_s^*\right)^\rho + (1-\mu)\right)^{\frac{\sigma}{\rho}-1}\mu(k_s^*)^{\rho-1}\frac{\partial k_s^*}{\partial z}}{\lambda\left(\mu\left(k_s^*\right)^\rho + (1-\mu)\right)^{\frac{\sigma}{\rho}} + (1-\lambda)\left(n_{us}^*\right)^\sigma}$$
$$+ \frac{\sigma-\rho}{\rho}\frac{\rho\mu(k_s^*)^{\rho-1}\frac{\partial k_s^*}{\partial z}}{\mu\left(k_s^*\right)^\rho + (1-\mu)} + \frac{(\rho-1)}{k_s^*}\frac{\partial k_s^*}{\partial z}$$

Then we have

$$-\frac{1}{z} = \underbrace{\left(\frac{(1-\sigma)(\lambda\left(\mu\left(k_s^*\right)^\rho + (1-\mu)\right)^{\frac{\sigma}{\rho}-1}\mu(k_s^*)^{\rho-1})}{\lambda\left(\mu\left(k_s^*\right)^\rho + (1-\mu)\right)^{\frac{\sigma}{\rho}} + (1-\lambda)\left(n_{us}^*\right)^\sigma} + \frac{(\sigma-\rho)\mu(k_s^*)^{\rho-1}}{\mu\left(k_s^*\right)^\rho + (1-\mu)} + \frac{(\rho-1)}{k_s^*}\right)}_{P}\frac{\partial k_s^*}{\partial z}$$

Therefore

$$\frac{\partial k_s^*}{\partial z} = -\frac{1}{zP}$$

To determine the sign of P, we can use another piece of information that we have, that is, f is strictly concave in k, so we must have $f_{kk} < 0$

$$f_k = z\lambda\mu[\lambda(\mu(k_s^*)^\rho + (1-\mu))^{\frac{\sigma}{\rho}} + (1-\lambda)(n_{us}^*)^\sigma]^{\frac{1}{\sigma}-1}[\mu(k_s^*)^\rho + (1-\mu)]^{\frac{\sigma}{\rho}-1}(k_s^*)^{\rho-1}$$

$$\begin{aligned} log f_k &= log z + log\lambda\mu + \frac{1-\sigma}{\sigma} log\Big(\lambda(\mu(k_s^*)^\rho + (1-\mu))^{\frac{\sigma}{\rho}} + (1-\lambda)(n_{us}^*)^\sigma\Big) \\ &+ \frac{\sigma-\rho}{\rho} log\Big(\mu(k_s^*)^\rho + (1-\mu)\Big) + (\rho-1) log k_s^* \end{aligned}$$

Differentiate w.r.t k_t and evaluate at k_{ss}

$$\frac{f_{kk}}{f_k} = \underbrace{\left(\frac{(1-\sigma)(\lambda\left(\mu\left(k_s^*\right)^\rho + (1-\mu)\right)^{\frac{\sigma}{\rho}-1}\mu(k_s^*)^{\rho-1})}{\lambda\left(\mu\left(k_s^*\right)^\rho + (1-\mu)\right)^{\frac{\sigma}{\rho}} + (1-\lambda)\left(n_{us}^*\right)^\sigma} + \frac{(\sigma-\rho)\mu(k_s^*)^{\rho-1}}{\mu\left(k_s^*\right)^\rho + (1-\mu)} + \frac{(\rho-1)}{k_s^*}\right)}_{P}\frac{1}{n_s} < 0$$

by the assumption of strict concavity of f in k. Hence we can conclude that we must have $P < 0$. This implies that $\frac{\partial k_s^*}{\partial z} > 0$. Hence, a country with higher productivity has higher capital skill ratio.

(b) output per worker

Total Output

$$Y = f(z, k_t, n_t) = z(\lambda(\mu(k_t)^\rho + (1-\mu)(n_s)^\rho)^{\frac{\sigma}{\rho}} + (1-\lambda)n_u^\sigma)^{\frac{1}{\sigma}}$$

Steady State

$$Y^* = f(z, k^*, n^*) = z(\lambda(\mu(k_s^*)^\rho + (1-\mu))^{\frac{\sigma}{\rho}} + (1-\lambda)(n_{us}^*)^\sigma)^{\frac{1}{\sigma}} n_s^*$$

Output per worker

$$y^* = \frac{Y^*}{n^*} = z(\lambda(\mu(k_s^*)^\rho + (1-\mu))^{\frac{\sigma}{\rho}} + (1-\lambda)(n_{us}^*)^\sigma)^{\frac{1}{\sigma}} \frac{n_s^*}{n^*}$$

Note that $\frac{n^*}{n_s^*} = 1 + n_{us}^*$. Take log on both sides

$$log y^* = log z + \frac{1}{\sigma} log(\lambda(\mu(k_s^*)^\rho + (1-\mu))^{\frac{\sigma}{\rho}} + (1-\lambda)(n_{us}^*)^\sigma) - log(1 + n_{us}^*)$$

Differentiate w.r.t z :

$$\frac{1}{y^*}\frac{\partial y^*}{\partial z} = \frac{1}{z} + \frac{1}{\sigma}\frac{\lambda\sigma\left(\mu\left(k_s^*\right)^\rho + (1-\mu)\right)^{\frac{\sigma}{\rho}-1}\mu(k_s^*)^{\rho-1}\frac{\partial k_s^*}{\partial z}}{\lambda\left(\mu\left(k_s^*\right)^\rho + (1-\mu)\right)^{\frac{\sigma}{\rho}} + (1-\lambda)\left(n_{us}^*\right)^\sigma}$$

Then we have
$$\frac{\partial y^*}{\partial z} = \frac{y^*}{z} + \underbrace{\frac{\lambda y^* \left(\mu \left(k_s^*\right)^{\rho} + (1-\mu)\right)^{\frac{\sigma}{\rho}-1} \mu (k_s^*)^{\rho-1}}{\lambda \left(\mu \left(k_s^*\right)^{\rho} + (1-\mu)\right)^{\frac{\sigma}{\rho}} + (1-\lambda)\left(n_{us}^*\right)^{\sigma}}}_{Q>0} \frac{\partial k_s^*}{\partial z}$$

Since we know that $Q > 0$ and $\frac{\partial k_s^*}{\partial z} > 0$, then we must have $\frac{\partial y^*}{\partial z} > 0$. A country has higher output per worker if it has higher productivity.

(c) Recall the equation of skill premium

$$log\Big(\frac{w_s^*}{w_u^*}\Big) = log\Big(\frac{\lambda(1-\mu)}{1-\lambda}\Big) + \frac{\sigma-\rho}{\rho} log(\mu(k_s^*)^{\rho} + (1-\mu)) + (1-\sigma) log\Big(\frac{n_u}{n_s}\Big)$$

Let the steady-state level of skill premium be $\pi^* \equiv \frac{w_s^*}{w_u^*}$

Differentiate w.r.t z:
$$\frac{1}{\pi^*}\frac{\partial \pi^*}{\partial z} = \frac{\sigma-\rho}{\rho}\frac{\mu\rho(k_s^*)^{\rho-1}}{\mu(k_s^*)^{\rho} + (1-\mu)}\frac{\partial k_s^*}{\partial z}$$

Therefore
$$\frac{\partial \pi^*}{\partial z} = \underbrace{\frac{\pi^*(\sigma-\rho)\mu(k_s^*)^{\rho-1}}{\mu(k_s^*)^{\rho} + (1-\mu)}}_{R>0}\frac{\partial k_s^*}{\partial z}$$

Again, as $\frac{\partial k_s^*}{\partial z} > 0$, we can conclude that $\frac{\partial \pi^*}{\partial z} > 0$. For a country with higher productivity it will have a higher skill premium.

5. Assume that countries differ in the relative supply of skill to unskill workers, $\frac{n_u}{n_s}$. If every country is in steady state, what does the model imply for the cross country differences in

(a) the capital skills ratio k/n_s

(b) output per worker

(c) the skill-premium

(a) Recall the Euler Equation

$$log\Big(\frac{1}{\beta}-1+\delta_k\Big) = logz + log\lambda\mu + \frac{1-\sigma}{\sigma}log\Big(\lambda(\mu(k_s^*)^\rho+(1-\mu))^{\frac{\sigma}{\rho}}+(1-\lambda)(n_{us}^*)^\sigma\Big)$$
$$\frac{\sigma-\rho}{\rho}log\Big(\mu(k_s^*)^\rho+(1-\mu)\Big)+(\rho-1)logk_s^*$$

Differentiate w.r.t n_{us}^* :

$$0=\frac{1-\sigma}{\sigma}\frac{\lambda\sigma\left(\mu\left(k_s^*\right)^\rho+(1-\mu)\right)^{\frac{\sigma}{\rho}-1}\mu(k_s^*)^{\rho-1}\frac{\partial k_s^*}{\partial n_{us}^*}+(1-\lambda)\sigma(n_{us}^*)^{\sigma-1}}{\lambda\left(\mu\left(k_s^*\right)^\rho+(1-\mu)\right)^{\frac{\sigma}{\rho}}+(1-\lambda)\left(n_{us}^*\right)^\sigma}$$
$$+\frac{\sigma-\rho}{\rho}\frac{\rho\mu(k_s^*)^{\rho-1}\frac{\partial k_s^*}{\partial n_{us}^*}}{\mu\left(k_s^*\right)^\rho+(1-\mu)}+\frac{(\rho-1)}{k_s^*}\frac{\partial k_s^*}{\partial n_{us}^*}$$

Then we have

$$\frac{(\sigma-1)(1-\lambda)(n_{us}^*)^{\sigma-1}}{\lambda\left(\mu\left(k_s^*\right)^\rho+(1-\mu)\right)^{\frac{\sigma}{\rho}}+(1-\lambda)\left(n_{us}^*\right)^\sigma}$$
$$=\underbrace{\left(\frac{(1-\sigma)(\lambda\left(\mu\left(k_s^*\right)^\rho+(1-\mu)\right)^{\frac{\sigma}{\rho}-1}\mu(k_s^*)^{\rho-1})}{\lambda\left(\mu\left(k_s^*\right)^\rho+(1-\mu)\right)^{\frac{\sigma}{\rho}}+(1-\lambda)\left(n_{us}^*\right)^\sigma}+\frac{(\sigma-\rho)\mu(k_s^*)^{\rho-1}}{\mu\left(k_s^*\right)^\rho+(1-\mu)}+\frac{(\rho-1)}{k_s^*}\right)}_{P}\frac{\partial k_s^*}{\partial n_{us}^*}$$

Again, as $P<0$, we can conclude that $\frac{\partial k_s^*}{\partial n_{us}^*}>0$. If a country has high unskilled-skilled ratio, then it has high capital skills ratio.

(b) Recall the equation about the output per worker

$$logy^*=logz+\frac{1}{\sigma}log(\lambda(\mu(k_s^*)^\rho+(1-\mu))^{\frac{\sigma}{\rho}}+(1-\lambda)(n_{us}^*)^\sigma)-log(1+n_{us}^*)$$

Differentiate w.r.t n_{us}^* :

$$\frac{1}{y^*}\frac{\partial y^*}{\partial n_{us}^*}=\frac{1}{\sigma}\frac{\lambda\sigma\left(\mu\left(k_s^*\right)^\rho+(1-\mu)\right)^{\frac{\sigma}{\rho}-1}\mu(k_s^*)^{\rho-1}\frac{\partial k_s^*}{\partial n_{us}^*}+(1-\lambda)\sigma(n_{us}^*)^{\sigma-1}}{\lambda\left(\mu\left(k_s^*\right)^\rho+(1-\mu)\right)^{\frac{\sigma}{\rho}}+(1-\lambda)\left(n_{us}^*\right)^\sigma}-\frac{1}{1+n_{us}^*}$$
$$\frac{\partial y^*}{\partial n_{us}^*}=\underbrace{\frac{\lambda y^*\left(\mu\left(k_s^*\right)^\rho+(1-\mu)\right)^{\frac{\sigma}{\rho}-1}\mu(k_s^*)^{\rho-1}}{\lambda\left(\mu\left(k_s^*\right)^\rho+(1-\mu)\right)^{\frac{\sigma}{\rho}}+(1-\lambda)\left(n_{us}^*\right)^\sigma}}_{Q>0}\frac{\partial k_s^*}{\partial n_{us}^*}+\frac{(1-\lambda)y^*(n_{us}^*)^{\sigma-1}}{\lambda\left(\mu\left(k_s^*\right)^\rho+(1-\mu)\right)^{\frac{\sigma}{\rho}}+(1-\lambda)\left(n_{us}^*\right)^\sigma}-\frac{1}{1+n_{us}^*}$$

Hence it is unclear whether a country with high unskilled-skilled ratio has a high output per worker or not.

(c) Recall the equation of skill premium

$$log(\pi^*) = log\Big(\frac{\lambda(1-\mu)}{1-\lambda}\Big) + \frac{\sigma-\rho}{\rho}log(\mu(k_s^*)^\rho + (1-\mu)) + (1-\sigma)log(n_{us}^*)$$

Differentiate w.r.t n_{us}^* :

$$\frac{1}{\pi^*}\frac{\partial \pi^*}{\partial n_{us}^*} = \frac{\sigma-\rho}{\rho}\frac{\mu\rho(k_s^*)^{\rho-1}}{\mu(k_s^*)^\rho + (1-\mu)}\frac{\partial k_s^*}{\partial n_{us}^*} + \frac{1-\sigma}{n_{us}^*}$$

Therefore

$$\frac{\partial \pi^*}{\partial n_{us}^*} = \underbrace{\frac{(\sigma-\rho)\mu\pi^*(k_s^*)^{\rho-1}}{\mu(k_s^*)^\rho + (1-\mu)}}_{R>0}\frac{\partial k_s^*}{\partial n_{us}^*} + \frac{(1-\sigma)\pi^*}{n_{us}^*}$$

Since we assume $1 > \sigma$ and we have concluded that $\frac{\partial k_s^*}{\partial n_{us}^*} > 0$, we have $\frac{\partial \pi^*}{\partial n_{us}^*} > 0$. If a country has high unskilled-to-skilled ratio, then this country has high risk premium.

6. Can you describe what's the incidence of skill-biased technical change on the skill-premium as function of the characteristics of f? HINT: think about different parameterizations.

We can evaluate how the skill premium responds to the change in λ. If λ increases, then the marginal product of skilled workers will increase while the marginal product of unskilled workers will decrease. Hence skill-biased technical change involves the increase in λ. Skill premium in the steady state is given by:

$$\pi^* = \frac{\lambda(1-\mu)}{1-\lambda}(\mu(k_s^*)^\rho + (1-\mu))^{\frac{\sigma-\rho}{\rho}}(n_{us}^*)^{1-\sigma}$$

Firstly, we need to know how k_s^* responds to an increase in λ. Recall the Euler Equation

$$\begin{aligned} log\Big(\frac{1}{\beta} - 1 + \delta_k\Big) =& logz + log\lambda\mu + \frac{1-\sigma}{\sigma}log\Big(\lambda(\mu(k_s^*)^\rho + (1-\mu))^{\frac{\sigma}{\rho}} + (1-\lambda)(n_{us}^*)^\sigma\Big) \\ & \frac{\sigma-\rho}{\rho}log\Big(\mu(k_s^*)^\rho + (1-\mu)\Big) + (\rho-1)logk_s^* \end{aligned}$$

Differentiate w.r.t λ :

$$\begin{aligned} 0 =& \frac{1}{\lambda} + \frac{1-\sigma}{\sigma}\frac{(\mu(k_s^*)^\rho + (1-\mu))^{\frac{\sigma}{\rho}} - (n_{us}^*)^\sigma + \lambda\sigma\left(\mu\left(k_s^*\right)^\rho + (1-\mu)\right)^{\frac{\sigma}{\rho}-1}\mu(k_s^*)^{\rho-1}\frac{\partial k_s^*}{\partial\lambda}}{\lambda\left(\mu\left(k_s^*\right)^\rho + (1-\mu)\right)^{\frac{\sigma}{\rho}} + (1-\lambda)\left(n_{us}^*\right)^\sigma} \\ &+ \frac{\sigma-\rho}{\rho}\frac{\rho\mu(k_s^*)^{\rho-1}\frac{\partial k_s^*}{\partial\lambda}}{\mu\left(k_s^*\right)^\rho + (1-\mu)} + \frac{(\rho-1)}{k_s^*}\frac{\partial k_s^*}{\partial\lambda} \end{aligned}$$

Rearrange and we get

$$-\frac{1}{\lambda}+\frac{\sigma-1}{\sigma}\frac{(\mu(k_s^*)^\rho+(1-\mu))^{\frac{\sigma}{\rho}}-(n_{us}^*)^\sigma}{\lambda\left(\mu\left(k_s^*\right)^\rho+(1-\mu)\right)^{\frac{\sigma}{\rho}}+(1-\lambda)\left(n_{us}^*\right)^\sigma}=P\frac{\partial k_s^*}{\partial\lambda}$$

However, since we are not able to determine the sign of $(\mu(k_s^*)^\rho+(1-\mu))^{\frac{\sigma}{\rho}}-(n_{us}^*)^\sigma$, we cannot determine the sign of $\frac{\partial k_s^*}{\partial\lambda}$. Suppose we assume $\frac{\partial k_s^*}{\partial\lambda}>0$. The equation of skill premium shows that if k_s^* is kept fixed, then π^* increases if λ increases. Hence with the assumption $\frac{\partial k_s^*}{\partial\lambda}>0$, we should get $\frac{\partial\pi^*}{\partial\lambda}>0$. That is, higher λ leads to higher skill premium.

Transition paths in the one sector growth model

Consider the one sector growth model as described in the class notes (lecture 1). Suppose that the economy is initiated with 85% of the steady state level of capital, $k_0 = 0.85k^*$, and that there is no government expenditure.

1. Implement the shooting algorithm described in lecture 2 and compute the saddle path of the economy. Assume the following values as benchmark:

- $\Delta t = 1$.
- $N = 600$.
- A production function of the form, zk^{α} where $A = 1$ and $\alpha = 0.33$.
- $\beta = 0.98$.
- $\delta = 0.10$.
- CES preferences, $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ with $\sigma = 0.95$.

2. Plot the phase diagram and the saddle path of the benchmark economy in the (k, c) space.

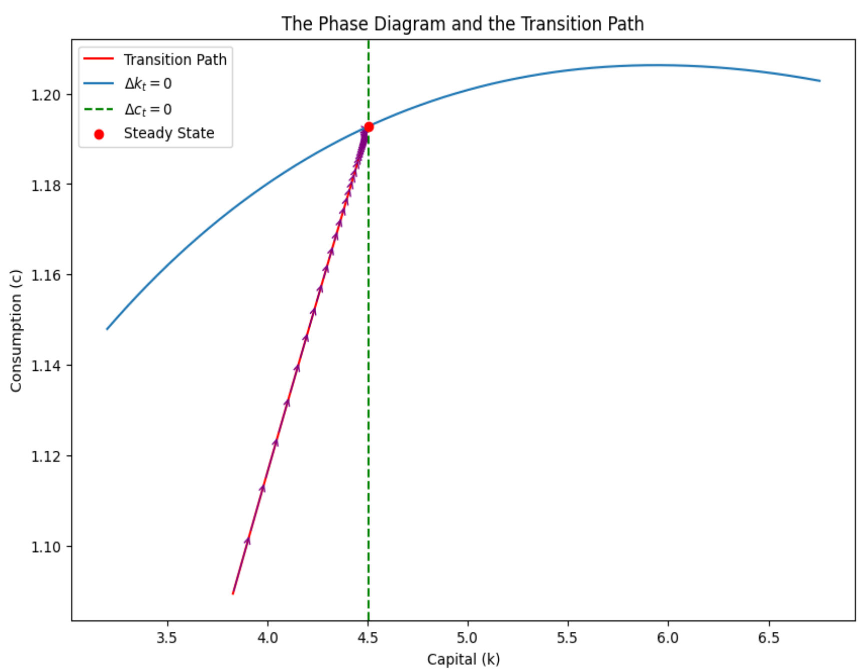

Figure 1: Phase diagram (baseline case).

Remark: In Figure 1, I plotted the transition path. Note the difference:

- The transition path is the actual trajectory the economy takes, given a specific initial condition, as it converges to the steady state.
- The saddle path is the unique stable trajectory that leads the economy to the steady state from any initial k.

Thus, in Figure 1, I plotted only the portion of the saddle path that corresponds to the transition from $k_0 = 0.85k^*$ to the steady state. If you plotted the entire saddle path, that is also correct.

3. What happens with the transition dynamic when agents become more impatient? How does the speed of convergence to the steady state change?

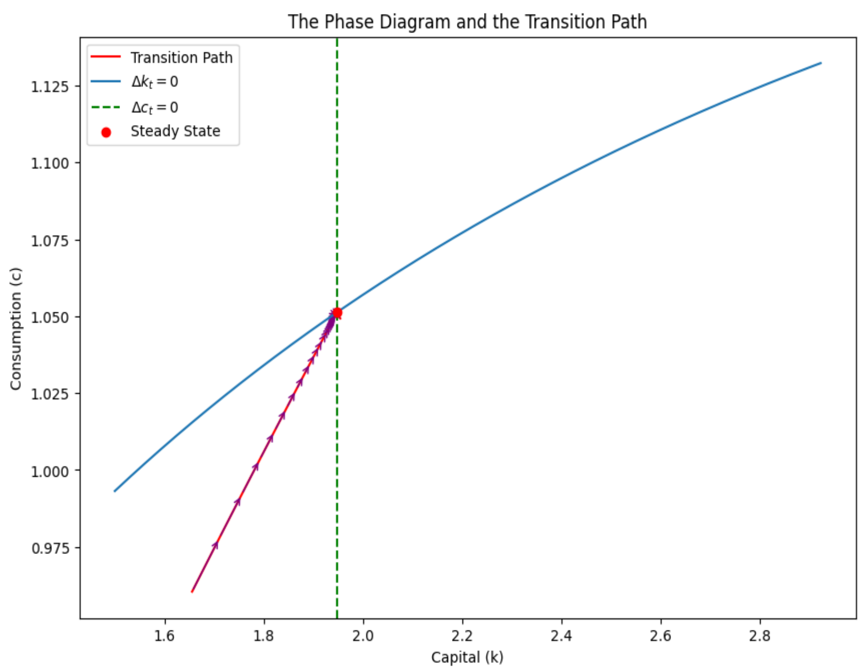

Figure 2: Phase diagram ($\beta = 0.9$)

As agents become more impatient, i.e., when β decreases, they discount future utility more and prioritize consumption over savings. This leads to lower capital accumulation and causes both the steady-state levels of capital and consumption to decrease. The economy converges to the new steady state faster (with this paramerization) because the distance between the initial and steady state is shorter, as illustrated in the phase diagram in Figure 2.

4. Write the one sector model in recursive form. Solve for the value function of the representative household via value function iteration. Compute and plot the optimal accumulation policy as well as the residuals of the Euler equation.

Recursive form:

$$V(k) = \max_{k'} u(f(k) + (1-\delta)k - k') + \beta V(k')$$

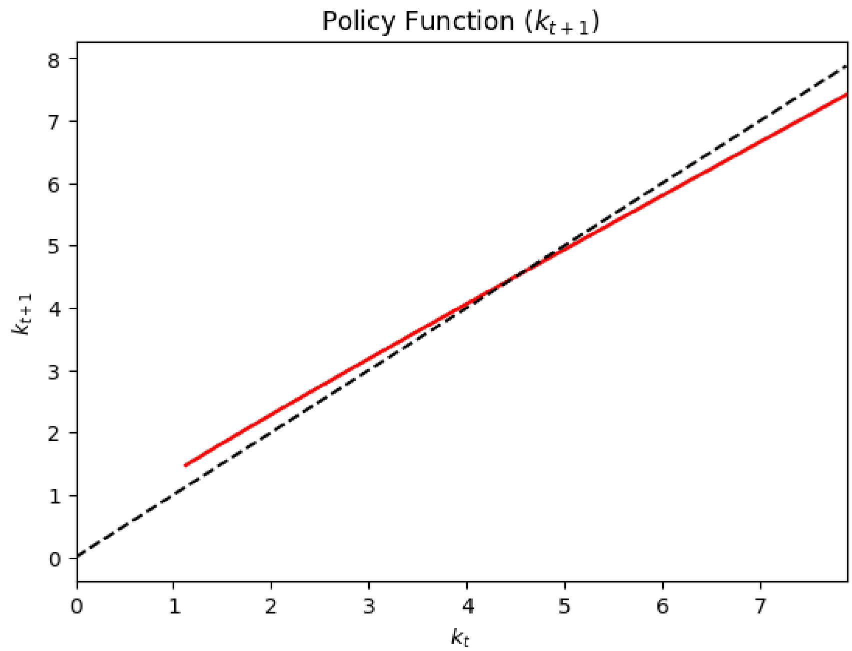

Figure 3: Optimal accumulation policy

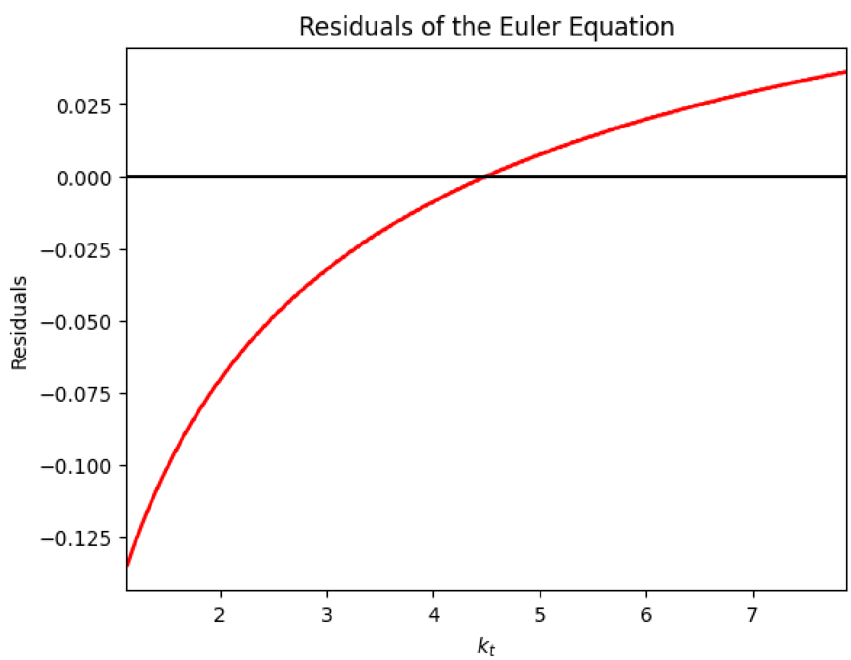

Figure 4: The residuals of the Euler equation

5. Using the policy function from the previous question, describe the transition path of capital when you start with $k_0 = 0.85k^*$. Compare this path to the one obtained in question (1).

Both algorithms converge to the same transition path of capital. However, small discrepancies in the initial periods are expected due to numerical factors such as grid density, tolerance levels, and interpolation errors. That is, the paths should be qualitatively identical, provided that both algorithms are correctly implemented. Any observed deviations are primarily numerical rather than substantive.