About TA sections:

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Section time and location: 8:40am - 9:55am Rockefeller Hall 132

Office hours: Tuesday 4:30-5:30 pm in Uris Hall 451; other times available by appointment (just send me an email).

Our plan for today:[1]

[1]The notes for this section are **heavily** based on Manuelli's notes and Julieta's slides. For more details, you can refer to those directly, or feel free to discuss them with me during my office hours.

1 Constrained Optimization: Review

1.1 Kuhn-Tucker Theorem

Consider the following problem:

$$\max\ f_0(x) \tag{*}$$

subject to

$$x \in S; \quad f_1(x) \geq 0,\ f_2(x) \geq 0, \ldots, f_m(x) \geq 0.$$

Assume that f_i, $i = 0, 1, \ldots, m$, are differentiable, concave and real-valued functions defined on a convex domain $S \subset \mathbb{R}^n$.

Slater Condition: $\exists \hat{x} \in$ interior of S such that $\forall i$, $f_i(\hat{x}) > 0$ for $i = 1, 2, \ldots, m$.

Associated with (*), we can define a function L (sometimes called the Lagrangian), $L : S \times \mathbb{R}^n_+ \to \mathbb{R}$, by

$$L(x, \lambda) = f_0(x) + \sum_{i=1}^{n} \lambda_i f_i(x).$$

Note that for a given λ, $L(x, \cdot)$ is concave, while for a given x, $L(\cdot, \lambda)$ is convex. A function with these properties is sometimes called a *saddle function*.

A point (x^*, λ^*) is a **saddle-point** of $L(x, \lambda)$ if

$$L(x^*, \lambda) \leq L(x^*, \lambda^*) \leq L(x, \lambda^*), \quad \text{for all } (x, \lambda) \in S \times \mathbb{R}^n_+.$$

If x^* is in the interior of S, (x^*, λ^*) is a saddle-point iff

$$\text{(C.1)} \quad Df_0(x^*) + \sum_{i=1}^{m} \lambda_i Df_i(x^*) = 0.$$

$$\text{(C.2)} \quad f_i(x^*) \geq 0,\ \lambda_i^* \geq 0,\ i = 1, 2, \ldots, m.$$

$$\text{(C.3)} \quad \sum_{i=1}^{m} \lambda_i^* f_i(x^*) = 0.$$

(A version of) Kuhn-Tucker Theorem: Assume $f_0, f_1, \ldots, f_m$ are concave, continuous functions from S (convex) $\subset \mathbb{R}^n$ into $\mathbb{R}$. Let the problem (*) and the function $L(x, \lambda)$ be as described above. Then

(i) If $(x^*, \lambda^*) \in S \times \mathbb{R}^m_+$ is a saddle-point of $L(x, \lambda)$, then x^* solves (*).

(ii) Assume that the Slater condition holds. Then if $x^* \in S$ is a solution to (*), there exists a $\lambda^* \in \mathbb{R}^m_+$, such that (x^*, λ^*) is a saddle-point of $L(x, \lambda)$.

1.2 The Transversality Condition (TVC)

Remark: We talked about this concept in the previous semester. While the logic remains the same, the notations and components of the model have changed slightly. Let's review what we (hopefully!) already know :)

The Transversality Condition (TVC) is a key **optimality** condition in infinite-horizon problems. It acts as a first-order necessary condition "at infinity", ensuring that the derived optimality conditions are sufficient under certain assumptions (e.g., concavity). Without the TVC, the first-order conditions derived from the Lagrangian are not sufficient to guarantee optimality.

Intuition: The TVC prevents inefficient accumulation of resources (e.g., capital), ensuring that the planner does not overinvest in assets that yield no utility or output in the long run. In economic terms, it ensures that the marginal value of capital or other state variables asymptotically vanishes.

Recall the algorithm we discussed last semester:

1. Derive the optimality conditions for the finite problem and take the limit as $T \to \infty$.

2. Use other optimality conditions to rearrange terms and derive the TVC in its desired form.

While this approach is rather "informal", it works well to provide intuition about how to derive the TVC. In more formal treatments, the TVC often emerges naturally as a boundary condition when solving the Hamiltonian or Bellman equation, ensuring consistency in the infinite-horizon optimization framework.

2 One sector Growth Model in Discrete Time: Review

2.1 Social Planner's Problem

Consider the problem of a planner who wants to maximize the utility of the representative agent (dynasty) subject to the feasibility constraints and the exogenously given sequence of government spending $\{g_t\}$. This problem is given by

$$\max_{\{c_t\},\{x_t\},\{k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

subject to

$$c_t + x_t + g_t \leq f(k_t)$$
$$k_{t+1} \leq (1-\delta)k_t + x_t$$
$$(c_t, x_t, k_{t+1}) \geq (0,0,0),$$
$$k_0 > 0, \text{given},$$

where c_t is household's consumption, k_t is capital stock, x_t is capital investment, g_t is government spending.

Definitions from the lecture notes:

An *allocation* is a set of sequences $\{\{c_t\}_{t=0}^{\infty}, \{x_t\}_{t=0}^{\infty}, \{k_{t+1}\}_{t=0}^{\infty}\}$.

An *allocation* is *feasible* if it satisfies

$$c_t + x_t + g_t \leq f(k_t)$$
$$k_{t+1} \leq (1-\delta)k_t + x_t$$
$$(c_t, x_t, k_{t+1}) \geq (0,0,0).$$

The **solution to the planner's problem** is a *feasible allocation* that has the property that it *maximizes the utility of the representative household.*

Assumptions:

- $u(c_t)$ is strictly increasing, strictly concave and differentiable,
- $f(k_t)$ is strictly increasing, strictly concave and differentiable, with $f(0) = 0$ (no free lunch condition),

- $\lim_{k_t \to 0} f'(k) > \frac{1}{\beta} - 1 + \delta$ to guarantee that that the solution is interior and that there is a steady state,
- $\lim_{k_t \to \infty} f'(k) < \frac{1}{\beta} - 1 + \delta$ to ensure that utility maximization is well defined and that the economy attains a steady state.

2.2 The Lagrangian

$$\mathcal{L}(c, x, k, \lambda, \theta, \gamma) = \sum_{t=0}^{\infty} \beta^t \Big\{ u(c_t) + \lambda_t \big[f(k_t) - c_t - x_t - g_t \big] + \theta_t \big[(1-\delta)k_t + x_t - k_{t+1} \big] + \gamma_{1t} c_t + \gamma_{2t} x_t + \gamma_{3t} k_{t+1} \Big\}$$

Optimality (first-order *necessary*) conditions:

$$u'(c_t) - \lambda_t + \gamma_{1t} = 0 \tag{c_t}$$

$$-\lambda_t + \theta_t + \gamma_{2t} = 0 \tag{x_t}$$

$$-\theta_t + \beta\theta_{t+1}(1-\delta) + \beta\lambda_{t+1} f'(k_{t+1}) + \gamma_{3t} = 0 \tag{k_{t+1}}$$

$$\lim_{T \to \infty} \beta^T \lambda_T k_{T+1} = 0 \tag{TVC}$$

along with non-negativity constraints and complementary slackness.

If we impose additional assumptions, for instance, the Inada conditions, we can guarantee an interior solution. This will simplify the system of optimality conditions described above substantially: if we know that $(c_t, x_t, k_{t+1}) > (0, 0, 0), \forall t$, i.e., strictly positive in the interior, we can use complementary slackness to set all γ's to zero.

Assuming that the sequence of $\{g_t\}_{t=0}^{\infty}$ is given, the ***interior*** solution for the planner's problem is fully described by:

$$u'(c_t) = \beta u'(c_{t+1})[1 - \delta + f'(k_{t+1})] \qquad \text{(Euler equation)}$$

$$c_t + x_t + g_t = f(k_t) \qquad \text{(Feasibility constraint)}$$

$$k_{t+1} = (1-\delta)k_t + x_t \qquad \text{(Law of motion for capital)}$$

$$(c_t, x_t, k_{t+1}) > (0, 0, 0) \qquad \text{(Interiority constraints)}$$

$$\lim_{T \to \infty} \beta^T u(c_T) k_{T+1} = 0 \qquad \text{(Transversality condition)}$$

2.3 Steady State

Characterization:

A **steady state** is a feasible allocation that is such that $\forall t$, $c_t = c$, $x_t = x$, $k_{t+1} = k$.

By substituting steady-state values into the system of equations above, we see that a steady state (if one exists) satisfies the following conditions:

$$u'(c^*) = \lambda^*,$$

$$1 = \beta[1 - \delta + f'(k^*)],$$

$$c^* + x^* + g = f(k^*),$$

$$k^* = (1 - \delta)k^* + x^*.$$

These conditions can be simplified to

$$1 = \beta[1 - \delta + f'(k^*)] \qquad \text{(Euler equation)}$$

$$f(k^*) = c^* + \delta k^* + g \qquad \text{(Feasibility)}$$

where we substituted $x^* = \delta k^*$ from the law of motion for capital.

Existence:

A steady state exists if 1) $\exists k^*$ s.t. $1 = \beta[1 - \delta + f'(k^*)]$ and 2) $c(k^*) \geq 0$.

The existence of k^* s.t. $1 = \beta[1 - \delta + f'(k^*)]$ is guaranteed by continuity of $f'(k)$ and the assumptions we made earlier:

$$\lim_{k_t \to 0} f'(k) > \frac{1}{\beta} - 1 + \delta,$$

$$\lim_{k_t \to \infty} f'(k) < \frac{1}{\beta} - 1 + \delta.$$

From the feasibility constraint, we see that

$$c^* = f(k^*) - \delta k^* - g_t$$

which means that $c(k^*) \geq 0$ iff $f(k^*) - \delta k^* - g_t \geq 0$ which depends on the value of g_t, as discussed in the lecture.

Uniqueness:

A sufficient (but not necessary) condition for uniqueness is that $f'(k)$ is strictly decreasing, or equivalently, $f''(k) < 0$. This ensures that $\beta[1 - \delta + f'(k^*)]$ crosses 1 only once, given the earlier assumptions:

$$\lim_{k_t \to 0} f'(k) > \frac{1}{\beta} - 1 + \delta,$$

and

$$\lim_{k_t \to \infty} f'(k) < \frac{1}{\beta} - 1 + \delta.$$

Since we assumed that $f(\cdot)$ is strictly concave, this condition is automatically satisfied.

2.4 Comparative Statics

Comparative statics involves analyzing how changes in model parameters affect the equilibrium outcomes of the system. For example, in the model above, the steady state is defined by the following key equations:

$$1 = \beta[1 - \delta + f'(k^*)] \qquad \text{(Euler equation)}$$

$$f(k^*) = c^* + \delta k^* + g \qquad \text{(Feasibility)}$$

Let us assume that $f(k) = zk^\alpha$. By substituting this functional form into the equations above and rearranging terms, we obtain:

$$1 = \beta[1 - \delta + z\alpha k^{*\alpha - 1}]$$

$$c^* = zk^{*\alpha} - \delta k^* - g$$

We might be interested in answering the following questions: what happens to the steady-state level of consumption in response to a positive productivity shock (i.e., if z increases)? Or, alternatively, what is the difference in consumption levels between two countries with different values of total factor productivity (TFP)? To address these questions, we need to understand how c^* depends on z, i.e., we want to know the sign of $\partial c^*/\partial z$. This can be derived directly from the equations above:

$$1 = \beta[1 - \delta + z\alpha k^{*\alpha - 1}] \implies k^* = \sqrt[1-\alpha]{\frac{z\alpha}{1/\beta - 1 + \delta}}$$

$$\implies c^* = z^{\frac{1}{1-\alpha}}\left[\left(\frac{\alpha}{1/\beta - 1 + \delta}\right)^{\frac{\alpha}{1-\alpha}} - \delta\left(\frac{\alpha}{1/\beta - 1 + \delta}\right)^{\frac{1}{1-\alpha}}\right] - g$$

Do we know the sign of $\partial c^*/\partial z$ yet?