DIXIT-STIGLITZ (CES) DEMAND SYSTEMS
LECTURE NOTES

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1. Demand

These notes derive product demand and price indices for CES aggregators of the Dixit-Stiglitz form. Monopolistically competitive markets are often modeled by using a utility function defined over a basket of differentiated goods of the form

$$U(C) = \frac{C^{1-\sigma}}{1-\sigma} \tag{1.1}$$

where C_t is CES aggregator over a continuum of goods

$$C \equiv \left(\int_0^1 C_i^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}} : \varepsilon > 1 \tag{1.2}$$

where $i \in (0,1)$. For a finite ε, goods are imperfect substitutes and firms have some market power over the pricing of goods.

Now, consider the problem of maximizing C subject to the budget constraint

$$\int_0^1 P_i C_i di \leq R \tag{1.3}$$

where R is the nominal budget constraint. Set up the lagrangian

$$\max_{C_i} \left(\int_0^1 C_i^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}} - \lambda \left(\int_0^1 P_i C_i di - R \right) \tag{1.4}$$

and take f.o.c.

$$\frac{\varepsilon}{\varepsilon-1} \frac{\varepsilon-1}{\varepsilon} \left(\int_0^1 C_i^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}-1} C_i^{\frac{\varepsilon-1}{\varepsilon}-1} = \lambda P_i \tag{1.5}$$

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This expression can be simplified to

$$\left(\int_0^1 C_i^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{1}{\varepsilon-1}} C_i^{-\frac{1}{\varepsilon}} = \lambda P_i \tag{1.6}$$

Now use that the definition of C (1.2) implies that

$$C^{\frac{\varepsilon-1}{\varepsilon}} \equiv \int_0^1 C_i^{\frac{\varepsilon-1}{\varepsilon}} di \tag{1.7}$$

so that (1.6) can be rewritten as

$$C^{\frac{1}{\varepsilon}} C_i^{-\frac{1}{\varepsilon}} = \lambda P_i \tag{1.8}$$

implying

$$C_i = (\lambda P_i)^{-\varepsilon} C. \tag{1.9}$$

We now have demand for good i as a function of the aggregate consumption good, the price of good i and the lagrange multiplier. We would like to be able to express demand as a function of the aggregate consumption good and the price of good i relative to a price index P such that

$$PC = \int_0^1 P_i C_i di \tag{1.10}$$

To get there, first multiply both sides of (1.8) with C_i

$$C^{\frac{1}{\varepsilon}} C_i^{1-\frac{1}{\varepsilon}} = \lambda P_i C_i \tag{1.11}$$

and integrate over i

$$C^{\frac{1}{\varepsilon}} \int_0^1 C_i^{1-\frac{1}{\varepsilon}} di = \lambda \int_0^1 P_i C_i di \tag{1.12}$$

so that

$$C^{\frac{1}{\varepsilon}+1-\frac{1}{\varepsilon}} = \lambda \int_0^1 P_i C_i di \tag{1.13}$$

which by the definition of P implies that $P = \lambda^{-1}$. Plugging into (1.9) gives the demand schedule for good i of the desired form

$$C_i = \left(\frac{P_i}{P}\right)^{-\varepsilon} C. \tag{1.14}$$

1.1. **Why is the derivative of $\partial C/\partial C_i$ not zero?** It is arbitrarily close to zero. When deriving the equations above we skipped one step. To do it properly, we need to consider the integral as the limit of the sum over n goods as n tend to infinity and with the weight of each good normalized by $1/n$. That is, the consumption aggregator C and the total expenditure R are defined as

$$\left(\int_0^1 C_i^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}} \equiv \left(\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} C_i^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{1.15}$$

and

$$\int_0^1 P_i C_i di \equiv \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} P_i C_i. \tag{1.16}$$

The first order condition then becomes

$$\frac{\varepsilon}{\varepsilon-1} \frac{\varepsilon-1}{\varepsilon} \left(\int_0^1 C_i^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}-1} \frac{1}{n} C_i^{\frac{\varepsilon-1}{\varepsilon}-1} = \frac{1}{n} \lambda P_i. \tag{1.17}$$

Since this expression holds for any n, we can multiply both sides with n to get the f.o.c. (1.5).

2. The price index P

To find the expression for the price index P, use the expression for demand for good C_i. The following manipulations

$$\begin{aligned}
C_i &= \left(P^{-1}P_i\right)^{-\varepsilon} C && (2.1)\\
\left(P^{-1}P_i\right)^{\varepsilon} &= CC_i^{-1} && (2.2)\\
P^{-1}P_i &= C^{\frac{1}{\varepsilon}}C_i^{-\frac{1}{\varepsilon}} && (2.3)\\
\left(P^{-1}P_i\right)^{1-\varepsilon} &= C^{\frac{1-\varepsilon}{\varepsilon}}C_i^{-\frac{1-\varepsilon}{\varepsilon}} && (2.4)\\
P^{\varepsilon-1}\int_0^1 P_i^{1-\varepsilon}di &= C^{\frac{1-\varepsilon}{\varepsilon}}\int_0^1 C_i^{-\frac{1-\varepsilon}{\varepsilon}} && (2.5)\\
P^{\varepsilon-1}\int_0^1 P_i^{1-\varepsilon}di &= C^{\frac{1-\varepsilon}{\varepsilon}}C^{-\frac{1-\varepsilon}{\varepsilon}} && (2.6)\\
&= 1 && (2.7)
\end{aligned}$$

then delivers the desired expression

$$P = \left(\int_0^1 P_i^{1-\varepsilon}di\right)^{\frac{1}{1-\varepsilon}}. \tag{2.8}$$

3. The optimal price in a static setting

Consider a firm producing the good i facing the demand curve (1.14) using production function

$$Y_i = AN_i^{(1-\alpha)}. \tag{3.1}$$

The maximization problem can then be written as

$$\max_{P_i} P_iY_i - \mathcal{C}\left(Y_i\right). \tag{3.2}$$

Use the product rule for differentiation

$$\left(f \cdot g\right)^{'} = f \cdot g' + g \cdot f'$$

to get the f.o.c.

$$Y_i + P_i \frac{\delta Y_i}{\delta P_i} - \frac{\delta \mathcal{C}}{\delta Y_i}\frac{\delta Y_i}{\delta P_i} = 0 \tag{3.3}$$

Denote the nominal marginal cost as $\psi_i \equiv \delta\mathcal{C}/\partial Y_i$. The following steps

$$\begin{aligned}
Y_i + P_i \frac{\delta Y_i}{\delta P_i} - \psi_i \frac{\delta Y_i}{\delta P_i} &= 0 && (3.4)\\
1 + \frac{P_i}{Y_i}\frac{\delta Y_i}{\delta P_i} - \psi_i \frac{1}{Y_i}\frac{\delta Y_i}{\delta P_i} &= 0 && (3.5)\\
1 - \varepsilon + \psi_i \frac{\varepsilon}{P_i} &= 0 && (3.6)\\
1 + \psi_i \frac{\varepsilon}{P_i} &= \varepsilon && (3.7)\\
P_i &= \frac{\varepsilon}{\varepsilon - 1}\psi_i && (3.8)
\end{aligned}$$

then shows that the optimal price is the markup $\mathcal{M} = \frac{\varepsilon}{\varepsilon-1}$ times marginal cost.