Credit Markets and Economic Growth, continued

Consider an economy where individuals can either supply their labor to firms or operate an individual-specific technology. Individuals in this economy are heterogeneous in terms of their wealth and entrepreneurial ability. However, everyone has the same labor productivity and behaves competitively.

Output is produced using labor (l), capital (k), and entrepreneurial input (e_i):

$$f(e_i, k, l) = e_i^v l^{(1-v)(1-\alpha)} k^{\alpha(1-v)}$$

Individuals' preference is given by

$$\int_0^\infty e^{-\rho t} \frac{c_i(t)^{1-\sigma}}{1-\sigma} dt$$

To simplify, assume that the entrepreneurial ability can either be $e_L = 0$ or $e_H > 0$. Note that an individual's ability does not change over time. The measure of the high-ability individuals is normalized to one, and that of the low-ability ones is π. Assume that π is large enough that high-ability types always choose to operate their technology. Further assume that all the initial capital stock in the economy is owned by the individuals with high ability and is equally distributed among them.

Assume that the capital rental market is shut down, i.e. to accumulate capital you depend on your own savings.

1. Can you compute the competitive equilibrium by solving the planner problem as before (when we had perfect credit markets)? Why or why not?

Since the markets are not complete, we can no longer apply the First Welfare Theorem, and the competitive equilibrium may differ from the solution to the social planner's problem.

2. Consider the entrepreneur's static profit maximization. Taking the individual capital stock and the market wage as given, derive the demand for labor of an individual entrepreneur.

The entrepreneur's static profit maximization problem is:

$$\max_{l(t)} \pi_H(t) \equiv e_H^v l(t)^{(1-v)(1-\alpha)} k(t)^{\alpha(1-v)} - w(t)l(t).$$

FOC:

$$(1-v)(1-\alpha)e_H^v l(t)^{(1-v)(1-\alpha)-1}k(t)^{\alpha(1-v)} = w(t).$$

Hence, the demand for labor of an individual entrepreneur is

$$l_H^{d*}(t) = \Big(\frac{w(t)}{(1-v)(1-\alpha)e_H^v k(t)^{\alpha(1-v)}}\Big)^{\frac{1}{(1-v)(1-\alpha)-1}}$$

3. Now compute its profits given k_t and w_t.

Substituting our answer from Part 2 into the profit expression:

$$\pi_H^*(t) = e_H^v l_H^{d*}(t)^{(1-v)(1-\alpha)}k(t)^{\alpha(1-v)} - w(t)l_H^{d*}(t) =$$

$$= e_H^v\Big(\frac{w(t)}{(1-v)(1-\alpha)e_H^v k(t)^{\alpha(1-v)}}\Big)^{\frac{(1-v)(1-\alpha)}{(1-v)(1-\alpha)-1}}k(t)^{\alpha(1-v)} - \frac{w(t)^{\frac{(1-v)(1-\alpha)}{(1-v)(1-\alpha)-1}}}{\Big((1-v)(1-\alpha)e_H^v k(t)^{\alpha(1-v)}\Big)^{\frac{1}{(1-v)(1-\alpha)-1}}} =$$

$$= w(t)^{\frac{(1-v)(1-\alpha)}{(1-v)(1-\alpha)-1}}\Big[(1-v)(1-\alpha)\,e_H^v\,k(t)^{\alpha(1-v)}\Big]^{\frac{-1}{(1-v)(1-\alpha)-1}}\frac{1-(1-v)(1-\alpha)}{(1-v)(1-\alpha)} =$$

$$= w(t)^{\frac{(1-v)(1-\alpha)}{\alpha v-\alpha-v}}\Big[(1-v)(1-\alpha)\,e_H^v\,k(t)^{\alpha(1-v)}\Big]^{\frac{-1}{\alpha v-\alpha-v}}\frac{\alpha+v-\alpha v}{(1-v)(1-\alpha)}$$

4. Express the wage that clears the labor market as a function of K_t. Note that $k_t = K_t$ in equilibrium, because all high-ability entrepreneurs are identical.

Since, in equilibrium, all high-ability individuals operate their technology and accumulate capital and all low-ability individuals simply supply their labor, the market-clearing conditions are:

$$l_H^d(t) = l_L^s(t) = \pi$$

$$k(t) = K(t)$$

In Part 2, we obtained

$$w(t) = (1-v)(1-\alpha)e_H^v l^{d*}(t)^{(1-v)(1-\alpha)-1}k(t)^{\alpha(1-v)}.$$

Using market clearing

$$w(t) = (1-v)(1-\alpha)e_H^v \pi^{(1-v)(1-\alpha)-1}K(t)^{\alpha(1-v)}.$$

5. Show that the profit of an entrepreneur can be expressed as $AK_t k_t^{\phi}$ where

$$AK_t = (\alpha + v - \alpha v)e^v \pi^{(1-v)(1-\alpha)} K_t^{-\frac{\alpha(1-\alpha)(1-v)^2}{\alpha+v-\alpha v}}$$

and

$$\phi = \frac{\alpha(1-v)}{\alpha + v - \alpha v}$$

Combining our answers to Parts 3 and 4

$$\pi_H^*(t) = w(t)^{\frac{(1-v)(1-\alpha)}{\alpha v - \alpha - v}} \left[(1-v)(1-\alpha)\, e_H^v\, k(t)^{\alpha(1-v)}\right]^{\frac{-1}{\alpha v - \alpha - v}} \frac{\alpha + v - \alpha v}{(1-v)(1-\alpha)} =$$

$$= \left((1-v)(1-\alpha)e_H^v \pi^{(1-v)(1-\alpha)-1} K(t)^{\alpha(1-v)}\right)^{\frac{(1-v)(1-\alpha)}{\alpha v - \alpha - v}} \left[(1-v)(1-\alpha)\, e_H^v\, k(t)^{\alpha(1-v)}\right]^{\frac{-1}{\alpha v - \alpha - v}} \frac{\alpha + v - \alpha v}{(1-v)(1-\alpha)} =$$

$$= (\alpha + v - \alpha v)e_H^v K(t)^{\alpha(1-v)\frac{(1-v)(1-\alpha)}{\alpha v - \alpha - v}} k(t)^{\alpha(1-v)\frac{-1}{\alpha v - \alpha - v}} =$$

$$= (\alpha + v - \alpha v)e_H^v K(t)^{-\frac{\alpha(1-\alpha)(1-v)^2}{\alpha + v - \alpha v}} k(t)^{\frac{\alpha(1-v)}{\alpha + v - \alpha v}}.$$

6. The budget constraint of an individual entrepreneur is

$$\dot{k}(t) = AK_t k_t^{\phi} - \delta k_t - c_t$$

where $c(t)$ is the consumption of a high-ability individual. Write down the Hamiltonian and derive a pair of ODEs in $c(t)$ and $k(t)$.

$$\max_{c(t),k(t)} \int_0^{\infty} e^{-\rho t} \frac{c_H(t)^{1-\sigma}}{1-\sigma} dt$$

s.t.

$$\dot{k}(t) = AK(t)k(t)^{\phi} - \delta k(t) - c(t)$$

$$\mathcal{H}(t, c_H(t), k(t), \lambda(t)) = e^{-\rho t} \frac{c_H(t)^{1-\sigma}}{1-\sigma} + \lambda(t)[AK(t)k(t)^{\phi} - \delta k(t) - c(t)]$$

$$\frac{\partial \mathcal{H}}{\partial c_H(t)} : e^{-\rho t} c_H(t)^{-\sigma} = \lambda(t)$$

$$\frac{\partial \mathcal{H}}{\partial k(t)} : \lambda(t)[\phi AK(t)k(t)^{\phi-1} - \delta] = -\dot{\lambda}(t)$$

Hence, the system of ODE in $c_H(t)$ and $k(t)$ is

$$\frac{\dot{c}_H(t)}{c_H(t)} = \frac{1}{\sigma}[\phi AK_t k_t^{\phi-1} - \delta - \rho],$$
$$\dot{k}(t) = AK_t k_t^{\phi} - \delta k(t) - c(t).$$

7. In equilibrium, $k_t = K_t$. Derive a pair of ODEs in $c(t)$ and $K(t)$.

From part (6), given that in equilibrium $k(t) = K(t)$, the system of ODE becomes

$$\frac{\dot{c}_H(t)}{c_H(t)} = \frac{1}{\sigma}[\phi AK(t)^{\phi} - \delta - \rho],$$

$$\dot{K}(t) = AK_t^{\phi+1} - \delta K(t) - c_H(t),$$

or equivalently,

$$\frac{\dot{c}_H(t)}{c_H(t)} = \frac{1}{\sigma}[\phi AK(t)^{\frac{\alpha(1-v)}{\alpha+v-\alpha v}} - \delta - \rho],$$
$$\dot{K}(t) = AK_t^{\frac{\alpha(1-v)}{\alpha+v-\alpha v}+1} - \delta K(t) - c_H(t).$$

8. If we assume that initial capital stock belongs to the low-ability agents, how would the dynamics of aggregate variables change?

Since the rental markets are shut down, low-ability agents cannot rent their capital to high-ability agents. Nor can they use it themselves as their ability is too low to operate the technology. Hence, nothing is produced in this economy and we have a degenerate equilibrium with zero production and consumption.

Extra points. Go back to the economy with perfect credit markets (previous HW).

1. Let $y(t) = [C(t), K(t)]$. Linearize the system that describes the dynamics.

The equilibrium dynamics are given by the system

$$\begin{aligned}
&\frac{\dot{C}(t)}{C(t)} = \frac{1}{\sigma}\Big(\alpha(1-v)e_H^v\,\pi^{(1-v)(1-\alpha)}\,K(t)^{\alpha(1-v)-1} - \delta - \rho\Big),\\
&\dot{K}(t) = e_H^v\,\pi^{(1-v)(1-\alpha)}\,K(t)^{\alpha(1-v)} - \delta K(t) - C(t),\\
&\lim_{t\to\infty}\lambda(t)K(t) = 0.
\end{aligned}$$

Define

$$A = e_H^v\,\pi^{(1-v)(1-\alpha)}.$$

Then the system can be rewritten as

$$\begin{aligned}
\dot{C}(t) &= \frac{C(t)}{\sigma}\left(\alpha(1-v)A\,K(t)^{\alpha(1-v)-1} - \delta - \rho\right),\\
\dot{K}(t) &= A\,K(t)^{\alpha(1-v)} - \delta K(t) - C(t).
\end{aligned}$$

An equilibrium (C^*, K^*) satisfies:

$$\alpha(1-v)A\,(K^*)^{\alpha(1-v)-1} = \delta+\rho \quad \text{and} \quad C^* = A\,(K^*)^{\alpha(1-v)} - \delta K^*.$$

Define

$$F(C,K) = \frac{C}{\sigma}\Big(\alpha(1-v)A\,K^{\alpha(1-v)-1} - \delta - \rho\Big), \quad G(C,K) = A\,K^{\alpha(1-v)} - \delta K - C.$$

Let $y(t) = \begin{pmatrix} C(t) \\ K(t) \end{pmatrix}$. The Jacobian matrix J is

$$J = \begin{pmatrix} F_C & F_K \\ G_C & G_K \end{pmatrix}_{(C^*,K^*)}.$$

For $F(C,K)$:

$$F_C(C,K) = \frac{1}{\sigma}\Big(\alpha(1-v)A\,K^{\alpha(1-v)-1} - \delta - \rho\Big),$$

so that at (C^*, K^*) (using $\alpha(1-v)A\,(K^*)^{\alpha(1-v)-1} = \delta + \rho$),

$$F_C(C^*, K^*) = 0.$$

Next,

$$F_K(C, K) = \frac{C}{\sigma} \cdot \frac{d}{dK}\Big[\alpha(1-v)A\,K^{\alpha(1-v)-1}\Big] = \frac{C}{\sigma}\,\alpha(1-v)A\,(\alpha(1-v)-1)K^{\alpha(1-v)-2}.$$

Thus,

$$F_K(C^*, K^*) = \frac{C^*}{\sigma}\,\alpha(1-v)A\,(\alpha(1-v)-1)(K^*)^{\alpha(1-v)-2}.$$

For $G(C, K)$:

$$G_C(C, K) = -1, \quad G_K(C, K) = A\,\alpha(1-v)K^{\alpha(1-v)-1} - \delta.$$

At (C^*, K^*) we have

$$G_K(C^*, K^*) = \alpha(1-v)A\,(K^*)^{\alpha(1-v)-1} - \delta = (\delta + \rho) - \delta = \rho.$$

Define

$$b = \frac{C^*}{\sigma}\,\alpha(1-v)A\,(\alpha(1-v)-1)(K^*)^{\alpha(1-v)-2}.$$

Then,

$$J = \begin{pmatrix} 0 & b \\ -1 & \rho \end{pmatrix}.$$

The linearized system is

$$\frac{d}{dt}\begin{pmatrix} C - C^* \\ K - K^* \end{pmatrix} = \begin{pmatrix} 0 & b \\ -1 & \rho \end{pmatrix}\begin{pmatrix} C - C^* \\ K - K^* \end{pmatrix}.$$

2. Describe the system in matrix form. Go as far as you can characterizing local stability of the system (i.e. what is the sign of the eigenvalues for alternative parameters?).

The characteristic equation is

$$\det\begin{pmatrix} -\lambda & b \\ -1 & \rho - \lambda \end{pmatrix} = \lambda^2 - \rho\lambda + b = 0.$$

Note that the sign of b depends on $\alpha(1-v)(\alpha(1-v)-1)$:

- If $\alpha(1-v) < 1$, then $\alpha(1-v) - 1 < 0$ so that $b < 0$. The characteristic equation has one positive and one negative real eigenvalue, and hence the equilibrium is a saddle point.

- If $\alpha(1-v) > 1$, then $b > 0$ (with $\rho > 0$) leading to both eigenvalues having positive real parts, and hence instability.

- The case $\alpha(1-v) = 1$ implies $b = 0$ and a degenerate linearization.

Optimal R&D in an AK world

Consider a closed economy populated by a continuum of households of measure 1 with homogeneous CES preferences,

$$U \equiv \int_0^\infty e^{-\rho t} \frac{c(t)^{1-\sigma}}{1-\sigma} dt$$

defined over a final good, Y. Final goods are produced using labor $L(t)$ and intermediate inputs, $x_i(t)$ with a constant returns to scale technology

$$Y(t) = L(t)^{1-\alpha} \left(\int_0^{A(t)} x_i(t)^{1-\mu} di \right)^{\frac{\alpha}{1-\mu}}$$

where $1/\mu$ is the elasticity of substitution across intermediate goods. We are interested in $\mu \in (0, 1)$, i.e. some substitution; and non-trivial factor shares, $\alpha \in (0, 1)$. At each point in time, there exists a continuum of intermediate goods, x_i, with endogenous support $i \in [0, A(t)]$. Intermediate goods are produced with a linear technology in labor

$$x_i(t) = a l_i(t)$$

for $a > 0$. Finally, the support of available intermediate goods for production evolves according to

$$\dot{A}(t) = bX(t)$$

where $X(t)$ are final goods devoted to R&D and $b > 0$.

1. Describe the planner's problem in this economy and characterize the optimal input and labor allocation.

The social planner maximizes utility subject to the constraints above:

$$\begin{aligned}
\max_{\{c(t), x_i(t), l_i(t), L(t), X(t), A(t)\}} \quad & \int_0^\infty e^{-\rho t} \frac{c(t)^{1-\sigma}}{1-\sigma} dt, \\
\text{s.t.} \quad & c(t) + X(t) = L(t)^{1-\alpha} \left(\int_0^{A(t)} x_i(t)^{1-\mu} di \right)^{\frac{\alpha}{1-\mu}}, \\
& \dot{A}(t) = bX(t), \quad A(0) = A_0, \\
& L(t) + \int_0^{A(t)} l_i(t) di = 1.
\end{aligned}$$

Given the symmetry of the problem and imperfect substitutability, i.e., using the fact that $\mu \in (0, 1)$, Jensen's inequality implies

$$x_i^*(t) = x(t), \quad l_i^*(t) = l(t).$$

To show this formally, note that each period, the planner aims to choose a combination of intermediate inputs that will maximize their combined output, given the planner's choice of the total labor allocated to intermediate goods production. Thus, the static problem is

$$\max_{l_i,\ \forall i} \int_0^{A(t)} (al_i)^{1-\mu} di$$

s.t.

$$\int_0^{A(t)} l(i)di = C,$$

where C is a constant representing the total labor allocated to intermediate goods production in a given period. The Lagrangian is

$$\mathcal{L} = \int_0^{A(t)} al_i^{1-\mu} di + \lambda[C - \int_0^{A(t)} l(i)di]$$

FOCs give us:

$$a^{1-\mu} l_i^{-\mu} = \lambda,$$
$$a^{1-\mu} l_j^{-\mu} = \lambda,$$

which imply that $l_i = l_j,\ \forall i, j$. Hence, $x_i = x_j,\ \forall i, j$.

Therefore,

$$\int_0^{A(t)} l_i(t)di = A(t)l(t) \quad \Longrightarrow \quad L(t) = 1 - A(t)l(t).$$

Since $x(t) = al(t)$

$$\int_0^{A(t)} x_i(t)^{1-\mu} di = A(t)x(t)^{1-\mu} = A(t)(al(t))^{1-\mu}.$$

Therefore, we can rewrite the planner's problem as

$$\max_{\{c(t),l(t),L(t),A(t)\}} \int_0^\infty e^{-\rho t}\frac{c(t)^{1-\sigma}}{1-\sigma}dt,$$
$$\text{s.t. } \dot{A}(t) = b\Big[a^\alpha L(t)^{1-\alpha}\, A(t)^{\frac{\alpha}{1-\mu}}\, l(t)^\alpha - c(t)\Big], \quad A(0) = A_0,$$
$$L(t) + \int_0^{A(t)} l_i(t)di = 1.$$

The current-value Hamiltonian

$$\mathcal{H} = \frac{c(t)^{1-\sigma}}{1-\sigma} + \lambda(t)\, b\Big[a^\alpha L(t)^{1-\alpha}\, A(t)^{\frac{\alpha}{1-\mu}}\, l(t)^\alpha - c(t)\Big] + \mu(t)\Big[1 - L(t) - A(t)l(t)\Big].$$

The first order conditions with respect to $L(t)$ and $l(t)$ (for now, ignoring other FOCs, transversality conditions and the costate dynamics for brevity) yield:

$$\frac{\partial\mathcal{H}}{\partial L(t)}: \quad \lambda(t)\, b\, a^\alpha(1-\alpha)L(t)^{-\alpha}\, A(t)^{\frac{\alpha}{1-\mu}}\, l(t)^\alpha = \mu(t),$$
$$\frac{\partial\mathcal{H}}{\partial l(t)}: \quad \lambda(t)\, b\, a^\alpha\alpha\, L(t)^{1-\alpha}\, A(t)^{\frac{\alpha}{1-\mu}}\, l(t)^{\alpha-1} = \mu(t)\, A(t),$$

Taking the ratio of the conditions with respect to $L(t)$ and $l(t)$ gives:

$$\frac{(1-\alpha)}{\alpha}\frac{l(t)}{L(t)} = \frac{1}{A(t)} \implies l(t) = \frac{\alpha}{1-\alpha}\frac{L(t)}{A(t)}.$$

Substituting this result into the labor resource constraint,

$$L(t) + A(t)l(t) = L(t) + A(t)\left[\frac{\alpha}{1-\alpha}\frac{L(t)}{A(t)}\right] = L(t)\left(1 + \frac{\alpha}{1-\alpha}\right) = \frac{L(t)}{1-\alpha} = 1,$$

implies

$$L^*(t) = 1-\alpha.$$

Remark: Alternatively, instead of taking the FOCs of the Hamiltonian, you could solve a static problem of output maximization w.r.t. $L(t)$. It would give the same result.

It then follows that

$$l^*(t) = \frac{\alpha}{A(t)}.$$

Since $x(t) = al(t)$

$$x^*(t) = a\frac{\alpha}{A(t)}.$$

2. Replace the optimal factor allocations into the problem. Does this problem have a recursive representation? If so describe it using the Hamilton-Jacobi-Belman equation*, if not, explain what feature prevents recursivity. Finally, show under which conditions the economy reduces to an AK framework.

With the optimal labor allocations determined, the production function becomes:

$$\begin{aligned} Y(t) &= a^{\alpha}(1-\alpha)^{1-\alpha}\, A(t)^{\frac{\alpha}{1-\mu}} \left(\frac{\alpha}{A(t)}\right)^{\alpha} = \\ &= a^{\alpha}(1-\alpha)^{1-\alpha}\alpha^{\alpha}\, A(t)^{\frac{\alpha}{1-\mu}-\alpha} = \\ &= a^{\alpha}(1-\alpha)^{1-\alpha}\alpha^{\alpha}\, A(t)^{\frac{\alpha\mu}{1-\mu}}. \end{aligned}$$

Therefore, we can rewrite the planner's problem as

$$\max_{\{c(t),l(t),L(t),A(t)\}} \int_0^{\infty} e^{-\rho t}\frac{c(t)^{1-\sigma}}{1-\sigma}dt,$$
$$\text{s.t. } \dot{A}(t) = b\Big[a^{\alpha}(1-\alpha)^{1-\alpha}\alpha^{\alpha}\, A(t)^{\frac{\alpha\mu}{1-\mu}} - c(t)\Big], \quad A(0) = A_0.$$

The corresponding HJB equation is defined by

$$\rho V(A) = \max_{c\geq 0}\Big\{\frac{c^{1-\sigma}}{1-\sigma} + V'(A)b\Big[a^{\alpha}(1-\alpha)^{1-\alpha}\alpha^{\alpha}\, A^{\frac{\alpha\mu}{1-\mu}} - c\Big]\Big\},$$

with $A(0) = A_0$ given.

For this economy to reduce to AK, we need a constant marginal product of A, i.e., we need $\frac{\alpha\mu}{1-\mu} = 1$.

3. Characterize the growth rate of available intermediate inputs along a BGP, g_A. What is the role of R&D investment for long-run growth?

$$\max_{\{c(t),l(t),L(t),A(t)\}} \int_0^{\infty} e^{-\rho t}\frac{c(t)^{1-\sigma}}{1-\sigma}dt,$$
$$\text{s.t. } \dot{A}(t) = b\Big[a^{\alpha}(1-\alpha)^{1-\alpha}\alpha^{\alpha}\, A(t)^{\frac{\alpha\mu}{1-\mu}} - c(t)\Big], \quad A(0) = A_0.$$

The current-value Hamiltonian

$$\mathcal{H} = \frac{c(t)^{1-\sigma}}{1-\sigma} + \lambda(t)\, b\Big[a^{\alpha}(1-\alpha)^{1-\alpha}\alpha^{\alpha}\, A(t)^{\frac{\alpha\mu}{1-\mu}} - c(t)\Big].$$

$$\frac{\partial \mathcal{H}}{\partial c(t)}: \quad c(t)^{-\sigma} = \lambda(t)\, b,$$
$$\frac{\partial \mathcal{H}}{\partial A(t)}: \quad -\lambda(t) b\, a^{\alpha}(1-\alpha)^{1-\alpha}\alpha^{\alpha} A(t)^{\frac{\alpha\mu}{1-\mu}-1} = \dot{\lambda}(t) - \rho\lambda(t)$$

By rearranging the FOCs

$$\gamma_c = \frac{\dot{c}(t)}{c(t)} = -\frac{1}{\sigma}\frac{\dot{\lambda}(t)}{\lambda(t)},$$

$$-\frac{\dot{\lambda}(t)}{\lambda(t)} = b a^{\alpha}(1-\alpha)^{1-\alpha}\alpha^{\alpha} A(t)^{\frac{\alpha\mu}{1-\mu}-1} - \rho.$$

This implies

$$\gamma_c = \frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}\Big[b a^{\alpha}(1-\alpha)^{1-\alpha}\alpha^{\alpha} A(t)^{\frac{\alpha\mu}{1-\mu}-1} - \rho\Big].$$

Hence, to obtain a BGP, we must ensure $A(t)^{\frac{\alpha\mu}{1-\mu}-1}$ is constant over time.

In the AK world, $\frac{\alpha\mu}{1-\mu} = 1$, so

$$\gamma_c = \frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}\Big[b a^{\alpha}(1-\alpha)^{1-\alpha}\alpha^{\alpha} - \rho\Big],$$

i.e., the economy is on the BGP from the beginning.

Recall the feasibility constraint:

$$c(t) + \frac{\dot{A}(t)}{b} = a^{\alpha}(1-\alpha)^{1-\alpha}\alpha^{\alpha}\, A(t)^{\frac{\alpha\mu}{1-\mu}}.$$

Dividing it by $A(t)$ gives us

$$\frac{c(t)}{A(t)} + \frac{\dot{A}(t)}{bA(t)} = a^{\alpha}(1-\alpha)^{1-\alpha}\alpha^{\alpha}\, A(t)^{\frac{\alpha\mu}{1-\mu}-1},$$

$$\frac{c(t)}{A(t)} + \frac{\gamma_a}{b} = a^{\alpha}(1-\alpha)^{1-\alpha}\alpha^{\alpha}\, A(t)^{\frac{\alpha\mu}{1-\mu}-1}.$$

Along the BGP, γ_A is constant, the RHS is constant, so $c(t)$ and $A(t)$ must grow at the same rate, i.e., in the AK-world the growth rates are $\gamma_a = \gamma_c = \frac{1}{\sigma}\Big[b a^{\alpha}(1-\alpha)^{1-\alpha}\alpha^{\alpha} - \rho\Big]$. If we do not make such AK-assumption, the model reduces to the RCK setting, so the economy converges to a no-growth steady state.

To decentralize the economy, assume that intermediate input producers compete monopolistically and charge $p_i(t)$ for each variety. In addition, assume that the final good sector as well as the production of new varieties of intermediate goods are competitive. The price of the final good is $p(t)$ while the cost of labor is $w(t)$. Whenever a new variety is available in the market, R&D firms charge $k(t)$ units for the innovation and an intermediate good producer can start producing such a variety. Because of free-entry into the R&D sector equilibrium profits are zero, so that in equilibrium, the return to innovation for the R&D firm, $k(t)$, equals the present discounted value of the profits of the intermediate good producer that manufactures such a variety.

4. What is the optimal pricing schedule for intermediate good producers? What features of the problem warrant this result?

First, we need to solve the problem of the final good producer to find their optimal demand for an intermediate good:

$$\max_{x_i(t),\, \forall i, L(t)} p(t)L(t)^{1-\alpha}\left(\int_0^{A(t)} x_i(t)^{1-\mu}di\right)^{\frac{\alpha}{1-\mu}} - w(t)L(t) - \int_0^{A(t)} p_i(t)x_i(t)di$$

FOCs:

$$L(t): \quad w(t) = (1-\alpha)p(t)L(t)^{-\alpha}\left(\int_0^{A(t)} x_i(t)^{1-\mu}di\right)^{\frac{\alpha}{1-\mu}} = \frac{(1-\alpha)p(t)Y(t)}{L(t)}$$

$$x_i(t): \quad p_i(t) = \alpha p(t)L(t)^{1-\alpha}\left(\int_0^{A(t)} x_i(t)^{1-\mu}di\right)^{\frac{\alpha}{1-\mu}-1} x_i(t)^{-\mu}$$

$$\implies x_i^*(t)^{\mu} = \alpha\frac{p(t)}{p_i(t)}L(t)^{1-\alpha}\left(\int_0^{A(t)} x_i(t)^{1-\mu}di\right)^{\frac{\alpha}{1-\mu}-1}$$

Important step: recall $Y(t) = L(t)^{1-\alpha}\left(\int_0^{A(t)} x_i(t)^{1-\mu}di\right)^{\frac{\alpha}{1-\mu}}$. Hence,

$$x_i^*(t)^{\mu} = \alpha\frac{p(t)}{p_i(t)}\frac{Y(t)}{\int_0^{A(t)} x_i(t)^{1-\mu}di} = \alpha\frac{p(t)}{p_i(t)}\frac{Y(t)}{\left[\frac{Y(t)}{L(t)^{1-\alpha}}\right]^{\frac{1-\mu}{\alpha}}}$$

Each intermediate good producer takes $x_i(t)$ as given and solves the following problem:

$$\max_{p_i(t)} p_i(t)x_i(t) - l_i(t)w(t) = x_i(t)\left[p_i(t) - \frac{1}{a}w(t)\right]$$

FOC:

$$x_i(t) - \frac{\partial x_i(t)}{\partial p_i(t)}\Big[p_i(t) - \frac{1}{a}w(t)\Big] = 0$$

From the final producer's problem:

$$x_i^*(t)^\mu = \alpha \frac{p(t)}{p_i(t)} \frac{Y(t)}{\left[\frac{Y(t)}{L(t)^{1-\alpha}}\right]^{\frac{1-\mu}{\alpha}}} \implies \frac{\partial x_i(t)}{\partial p_i(t)} = \frac{1}{\mu}\frac{1}{p_i(t)}x_i(t).$$

Hence, the FOC becomes

$$x_i(t) - \frac{1}{\mu}\frac{1}{p_i(t)}x_i(t)\Big[p_i(t) - \frac{1}{a}w(t)\Big] = 0$$

$$\implies p_i^*(t) = \frac{w(t)}{a(1-\mu)},$$

which is the same for all firms, implying $p_i^*(t) = p_j^*(t) = \hat{p}(t), \quad x_i^*(t) = x_j^*(t) = x^*(t)$.

This is a standard result for monopolistic competition: firms charge a constant markup $\frac{1}{1-\mu}$ on the marginal cost.

5. How does R&D investment along the BGP compares to the efficient level under the assumptions that warrant an AK-specification?

Part 1. Household problem:

The household problem is simply

$$\max_{c(t)} \int_0^\infty e^{-\rho t}\frac{c(t)^{1-\sigma}}{1-\sigma}dt$$

s.t.

$$\int_0^\infty p(t)c(t) = \int_0^\infty w(t)dt + k(0)A(0),$$

where $A(0) = A_0$ and $k(0) = k_0$ is given.

FOC:

$$c(t)^{-\sigma} = \lambda p(t),$$

which implies

$$\gamma_c = \frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}\Big[-\frac{\dot{p}(t)}{p(t)} - \rho\Big] = \frac{1}{\sigma}\Big[-\gamma_p - \rho\Big].$$

Remark: Note that the household does not accumulate capital or assets, so there is no state variable in this problem.

Now we need to find γ_p and the growth rates of the other key variables.

Part 2. Optimal allocation of labor:

Since demand for all intermediate goods is the same, the final good's production function becomes

$$Y(t) = L(t)^{1-\alpha}(A(t)^{\frac{1}{1-\mu}}x(t))^{\alpha}$$

From the final producer's FOCs:

$$\alpha p(t)Y(t) = A(t)x(t)\hat{p}(t),$$

$$w(t)L(t) = (1-\alpha)p(t)Y(t).$$

This implies

$$\frac{A(t)x(t)\hat{p}(t)}{w(t)L(t)} = \frac{\alpha}{1-\alpha}.$$

Substituting $\hat{p}(t) = \frac{w(t)}{a(1-\mu)}$

$$\frac{A(t)x(t)}{a(1-\mu)L(t)} = \frac{\alpha}{1-\alpha} \implies A(t)x(t) = \frac{\alpha}{1-\alpha}a(1-\mu)L(t).$$

From the labor market clearing condition,

$$L(t) + A(t)\frac{x(t)}{a} = 1$$

implies

$$L(t) = 1 - A(t)\frac{x(t)}{a} = 1 - \frac{\alpha}{1-\alpha}a(1-\mu)L(t)\frac{1}{a} \implies L(t) = \frac{1-\alpha}{1-\alpha\mu},$$

i.e. $L(t) = L$ is constant, i.e., $\gamma_L = 0$.

Part 3. Idea producer problem: The idea producer maximizes profit

$$\max \int_0^{\infty} \Big[k(t)\dot{A}(t) - p(t)X(t)\Big]dt$$

s.t. $\dot{A}(t) = bX(t)$.

Zero profit condition implies

$$k(t) = \frac{p(t)}{b}.$$

Note that $\dot{A}(t) = bX(t)$ implies that along the BGP $\gamma_A = \gamma_X$.

Part 4. Zero profit and free entry:

Now we need to use the zero profit condition. Recall the problem says:

*Whenever a new variety is available in the market, R&D firms charge $k(t)$ units for the innovation and an intermediate good producer can start producing such a variety. Because of **free-entry** into the R&D sector **equilibrium profits are zero**, so that in equilibrium, the return to innovation for the R&D firm, $k(t)$, equals the present discounted value of the profits of the intermediate good producer that manufactures such a variety.*

Hence, for all intermediate producers

$$k(t) = \int_t^{\infty} \pi_i(\tau) d\tau.$$

Since all intermediate good producers choose the same prices and face the same demand, we can rewrite it as

$$k(t) = \int_t^{\infty} \pi(\tau) d\tau.$$

For each intermediate good producer, by substituting $w(t)$ and then $x(t)\hat{p}(t)$ from the final producer's FOC

$$\pi(t) = x(t)\left[\hat{p}(t) - \frac{1}{a}w(t)\right] = x(t)\hat{p}(t)\mu = \mu\alpha\frac{p(t)Y(t)}{A(t)}.$$

Hence, zero profit and free entry conditions imply

$$k(t) = \int_t^{\infty} \mu\alpha\frac{p(\tau)Y(\tau)}{A(\tau)} d\tau.$$

Part 5. Existence of BGP:

1) The resource constraint $Y(t) = X(t) + c(t)$ implies

$$\gamma_Y = \gamma_c = \gamma_X.$$

From Part 3, we also know that $\gamma_A = \gamma_X$. Hence,

$$\gamma_Y = \gamma_A = \gamma_c = \gamma_X.$$

2) We have shown that $\gamma_L = 0$. Then, using labor market clearing $L = 1 - A(t)\frac{x(t)}{a}$, we must have $\gamma_A = -\gamma_x$.

Recall the production function:

$$Y(t) = L(t)^{1-\alpha}(A(t)^{\frac{1}{1-\mu}}x(t))^{\alpha}$$

Since we have already shown that $\gamma_L = 0$, the production growth rate is

$$\gamma_Y = \alpha\gamma_x + \frac{\alpha}{1-\mu}\gamma_A = \frac{\alpha\mu}{1-\mu}\gamma_A.$$

(1) and (2) imply:

$$\gamma_Y = \gamma_A,$$
$$\gamma_Y = \frac{\alpha\mu}{1-\mu}\gamma_A.$$

Hence, there are two possibilities:

- $\gamma_Y = \gamma_A = 0$, and again the model boils down to the RCK case with no-growth steady state.
- $\frac{\alpha\mu}{1-\mu} = 1$, and the **BGP exists**.

From now on, we will be considering the second case.

6. **Combining Parts 3 and 4**:

$$\frac{p(t)}{b} = \int_t^{\infty} \mu\alpha\frac{p(\tau)Y(\tau)}{A(\tau)}d\tau.$$

From $\gamma_Y = \gamma_A$ it follows that $\frac{Y(t)}{A(t)} = \frac{Y(0)}{A(0)}$ - constant.

Now we need to find this ratio. From the definition of the production function at the initial period

$$Y(0) = L(0)^{1-\alpha}x(0)^{\alpha}A(0)^{\frac{\alpha}{1-\mu}}$$

Hence

$$\frac{Y(0)}{A(0)} = L^{1-\alpha}x(0)^{\alpha}A(0)^{\frac{\alpha}{1-\mu}-1} =$$
$$= L(0)^{1-\alpha}x(0)^{\alpha}A(0)^{\alpha},$$

where I used $\frac{\alpha\mu}{1-\mu} = 1$.

Therefore,

$$\frac{Y(0)}{A(0)} = L^{1-\alpha}(a(1-L))^{\alpha} = \frac{1-\alpha}{1-\alpha\mu}a^{\alpha}.$$

Now we can substitute this ratio into the free entry condition:

$$\frac{p(t)}{b} = \mu\alpha\frac{1-\alpha}{1-\alpha\mu}a^{\alpha}\int_t^{\infty} p(\tau)d\tau.$$

Note that for $\tau \geq t$ $p(\tau) = p(t)e^{\gamma_p(\tau-t)}$, so the expression above becomes

$$\frac{p(t)}{b} = \mu\alpha\frac{1-\alpha}{1-\alpha\mu}a^{\alpha}\int_t^{\infty} p(t)e^{\gamma_p(\tau-t)}d\tau$$

By integrating and dividing both sides of the expression by $p(t)$, we obtain

$$1 = -b\mu\alpha\frac{1-\alpha}{1-\alpha\mu}a^{\alpha}\frac{1}{\gamma_p}.$$

Hence,

$$\gamma_p = -b\mu\alpha\frac{1-\alpha}{1-\alpha\mu}a^{\alpha}.$$

Part 7. (Finally) Combining everything, we have:

$$\gamma_Y = \gamma_A = \gamma_X = \gamma_c = \frac{1}{\sigma}\Big[-\gamma_p - \rho\Big] =$$
$$= \frac{1}{\sigma}\Big[b\mu\alpha\frac{1-\alpha}{1-\alpha\mu}a^{\alpha} - \rho\Big].$$

Note that since $\mu\alpha < 1$, these growth rates of consumption and R&D investments are lower than the ones we obtained in the SPP.

6. Would a subsidy to R&D investment restore efficiency in the market? If yes, explain how. If

not, describe an alternative policy that would restore efficiency.

We have shown that the decentralized version of the model exhibits inefficiency due to monopolistic competition in the intermediate goods market. Market power leads to the underproduction of intermediate goods, which slows the growth of production and consumption. By incentivizing investments in research on new varieties, a subsidy to R&D investment may lead to higher intermediate input production and generate a Pareto improvement, restoring market efficiency.