Macroeconomics, PhD core

Lecture #11

Julieta Caunedo

Intro

- Non-trivial transition dynamics in a convex economy
- Recursive representation in continuous time
- Growth through variety innovation

Transition Dynamics

- The economies we studied so far have trivial dynamics in the sense that the economy is on the BGP from t=0.
- What if we depart from it? i.e. Jones & Manuelli (1990)

$$Y = F(K, L) = AK + BK^{\alpha} L^{1-\alpha}$$

so that

$$y = Ak + Bk^{\alpha} \qquad \text{and } \lim_{t\to\infty} f'(k) = A$$

Dynamic Equations

- $$\frac{\dot{k}}{k} = \frac{f(k)}{k} - \frac{c}{k} - (n + \delta)$$
  $$\frac{\dot{c}}{c} = \frac{1}{\theta}(A + B\alpha k^{\alpha-1} - (\rho + \delta))$$
  where θ is the elasticity of substitution in consumption.
- BGP growth rates
  $$\gamma^* = \frac{1}{\theta}(A - (\rho + \delta))$$
- Problem, there is no steady state.

Detrending

- Rewrite the dynamic variables in stationary terms, i.e. DETREND
- Which ones? Depends on the problem (here, use capital)

$$z = f(k)/k \qquad \text{and} \qquad \chi = c/k$$

- Lot's of algebra, and you can rewrite the system of dynamic equations as

$$\dot{z} = -(1-\alpha)(z-A)(z-\chi-n-\delta)$$

$$\dot{\chi} = \chi((\chi-\varphi) - \frac{\theta-\alpha 1}{\theta}(z-A))$$

where $\varphi \equiv (A-\delta)\frac{\theta-1}{\theta} + \frac{\rho}{\theta} - n$

- You can draw a Phase Diagram in (z,χ) space!

Hamilton-Jacobi-Bellman equations

$$V(k_0) = \max_{c(t)} \int_0^\infty e^{-\rho t} U(c(t)) dt$$

s.t.

$$\dot{k}(t) = F(k(t)) - \delta k(t) - c(t)$$

for $t \geq 0$, $k(0) = k_0$ given

- State, $x = k(t)$ and control, $u = c(t)$
- Let $h(x,u) = U(u)$ and $g(x,u) = F(x) - \delta x - u$

Hamilton-Jacobi-Bellman equations

- The value function of the generic optimal control problem satisfies the Hamilton-Jacobi-Bellman equation

$$\rho V(x) = \max_u h(x, u) + V'(x) g(x, u)$$

- with multiple states, $V'(x)$ is a vector of dimension m (# of states)
- This implies,

$$\rho V(k) = \max_c U(c) + V'(k)[F(k) - \delta k - c]$$

taking first order conditions

$$u'(c) = V'(k)$$

Derivation from discrete time

- Discount factor, $\beta(\Delta) = e^{-\rho\Delta}$ where Δ is the length of a period.
- Bellman equation

$$V(k_t) = \max_{c_t} \Delta U(c_t) + e^{-\rho\Delta} V(k_{t+\Delta})$$

s.t.

$$k_{t+\Delta} = \Delta[F(k_t) - \delta k_t - c_t] + k_t$$

- For small Δ, $e^{-\rho\Delta} \sim (1 - \rho\Delta)$

$$\rho\Delta V(k_t) = \max_{c_t} \Delta U(c_t) + (1 - \rho\Delta)(V(k_{t+\Delta}) - V(k_t))$$

$$\rho V(k_t) = \max_{c_t} U(c_t) + (1 - \rho\Delta)\left(\frac{V(k_{t+\Delta}) - V(k_t)}{k_{t+\Delta} - k_t} \frac{k_{t+\Delta} - k_t}{\Delta}\right)$$

Take the limit as $\Delta \to 0$

$$\rho V(k_t) = \max_{c} U(c_t) + V'(k_t)\dot{k}_t$$

Connection: Hamiltonian and HJB

- Hamiltonian

$$H(x, u, \lambda) \equiv h(x, u) + \lambda g(x, u)$$

- Bellman

$$\rho V(x) = \max_u h(x, u) + V'(x)g(x, u)$$

- Connection, $\lambda(t) = V'(x(t))$, co-state = shadow value

$$\rho V(x) = \max_{u \in U} H(x, u, V'(x))$$

- therefore the "hamilton" in the HJB

Innovation

- Expanding input varieties
- Greater variety of inputs increases the "division of labor" raising the productivity of final good firms
- Romer (1990)
    - Final goods — competitive
    - Intermediate goods — monopolistic competition
    - R&D — competitive

Innovation

- Final goods

$$Y(t) = L(t)^{1-\alpha} \left(\int_0^{A(t)} x_i(t)^{1-\mu} di \right)^{\frac{\alpha}{1-\mu}}$$

  where $1/\mu$ is the elasticity of substitution. We are interested in $\mu \in (0,1)$, i.e. some substitution.

- Intermediate goods

$$x_i(t) = a l_i(t)$$

- R&D, CRS

$$\dot{A}(t) = bX(t)$$

  where $X(t)$ are final goods devoted to R&D

Innovation

- Normalize population size to 1.
- Planner's problem

$$\max \int_0^\infty e^{-\rho t} \frac{c(t)^{1-\sigma}}{1-\sigma} dt$$

$$s.t. \qquad c(t) + X(t) = L(t)^{1-\alpha} \left(\int_0^{A(t)} x_i(t)^{1-\mu} di \right)^{\frac{\alpha}{1-\mu}}$$

$$x_i(t) = a l_i(t)$$

$$\dot{A}(t) = bX(t)$$

$$L(t) + \int_0^{A(t)} l_i(t) di = 1$$

Innovation

- Given that there is imperfect substitution, optimal planner strategy is

$$x_i(t) = x(t) \qquad \text{and} \qquad l_i(t) = l(t)$$

Hence

$$c(t) + X(t) = L(t)^{1-\alpha} \left(A(t) \left(a l(t) \right)^{1-\mu} \right)^{\frac{\alpha}{1-\mu}}$$

Feasibility

$$l(t) = \frac{1 - L(t)}{A(t)}$$

- Equilibrium Aggregate output

$$Y(t) = a^{\alpha} L(t)^{1-\alpha} \left(1 - L(t) \right)^{\alpha} A(t)^{\frac{\alpha\mu}{1-\mu}}$$

- Output Maximization implies, $L(t) = (1 - \alpha)$

Innovation

- Planner's problem

$$\max \int_0^\infty e^{-\rho t} \frac{c(t)^{1-\sigma}}{1-\sigma} dt$$

$$s.t. \qquad c(t) + \frac{\dot{A}(t)}{b} = CA(t)^{\frac{\alpha\mu}{1-\mu}}$$

where $C = a^\alpha (1-\alpha)^{1-\alpha} \alpha^\alpha$.

- If $\frac{\alpha\mu}{1-\mu} \in (0, 1)$ we have Ramsey-Caas-Koopmans
- Romer needs $\frac{\alpha\mu}{1-\mu} = 1$ so that we have an "AK" model

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma} [bC - \rho]$$

on BGP from the beginning!

Decentralized economy

- When a new idea is introduced, R&D sector charges $k(t)$
- The sector is competitive and techno CRS, so $\pi_{R\&D} = 0$.
- Assume $k(t)$ equals the PV of the profits of the monopolistic firms
- $p_i(t)$ be the price of intermediate goods
- $p(t)$ the price of final goods
- $w(t)$ the cost of labor

Decentralized economy

- Final good firms maximize profits. Optimality

$$w(t) = (1-\alpha)p(t)L(t)^{-\alpha}\left(\int_0^{A(t)} x_i(t)^{1-\mu}di\right)^{\frac{\alpha}{1-\mu}} \quad (L(t))$$

$$w(t) = (1-\alpha)p(t)\frac{Y(t)}{L(t)}$$

$$p_i(t) = \alpha p(t)L(t)^{1-\alpha}\left(\int_0^{A(t)} x_i(t)^{1-\mu}di\right)^{\frac{\alpha}{1-\mu}-1} x_i(t)^{-\mu} \quad (x_i(t))$$

$$x_i(t)^{\mu} = \alpha\frac{p(t)}{p_i(t)}\frac{Y(t)}{\left(\int_0^{A(t)} x_i(t)^{1-\mu}di\right)}$$

Decentralized economy

- Because markets are competitive, $\pi = 0$
- Use the zero profit condition in this market to solve for $x_i^*(t)$

$$x_i^*(t) = \alpha^{\frac{1}{\mu}} \left(\frac{p(t)}{p_i(t)} \right)^{\frac{1}{\mu}} Y(t)^{\frac{\mu+\alpha-1}{\alpha\mu}} L(t)^{\frac{(1-\mu)(1-\alpha)}{\alpha\mu}}$$

Decentralized economy

- Intermediate goods producer
  (they already paid the fixed cost)

$$\max_{p_i(t)} p_i(t)x_i(t) - w(t)\frac{x_i(t)}{a}$$

  because of monopolistic competition, $x_i(t)(p_i(t))$

- Optimality yields

$$p_i(t) = \frac{w(t)}{a(1-\mu)}$$

  Constant markup over marginal cost!

Decentralized economy vs. Planner's

- In equilibrium, all firms are identical, hence $p_i(t) = \widehat{p}(t)$
- You can show

$$L^{CE}(t) = \frac{1-\alpha}{1-\alpha\mu} > 1-\alpha$$

  so there is more labor in final goods than in the planner's allocation.
- $x_i(t)$ is relatively expensive, firms switch to L. Less investment in ideas
- Growth rates

$$\frac{\dot{c}^{CE}(t)}{c^{CE}(t)} = \frac{1}{\sigma}\left(b\alpha^{\alpha}\alpha\mu\frac{1-\alpha}{1-\alpha\mu} - \rho\right) < \frac{\dot{c}(t)}{c(t)}$$