Mixed Nash Equilibria

Computing mixed equilibria

The "Battle of the Sexes" example

	Theater	Music
Theater	$2, 1$	$0, 0$
Music	$0, 0$	$1, 2$

There are two pure strategy equilibria: TT and MM. What about mixed strategy equilibria?

Assume 2 chooses T with probability $\alpha_2(T)$, in short α_2.

Then 1's expected utility of choosing T and M are:

$$U_1(T, \alpha_2) = 2\alpha_2 + 0 \cdot (1 - \alpha_2) = 2\alpha_2$$

$$U_1(M, \alpha_2) = 0 \cdot \alpha_2 + 1 \cdot (1 - \alpha_2) = 1 - \alpha_2$$

Player 1 prefers T if:

$$2\alpha_2 \geq 1 - \alpha_2 \Leftrightarrow \alpha_2 \geq 1/3$$

and strictly prefers M otherwise.

Similarly, assume 1 chooses T with probability α_1.

Then 2's expected utility of choosing T and M are:

$$U_2(T, \alpha_1) = \alpha_1 + 0 \bullet (1 - \alpha_1) = \alpha_1$$

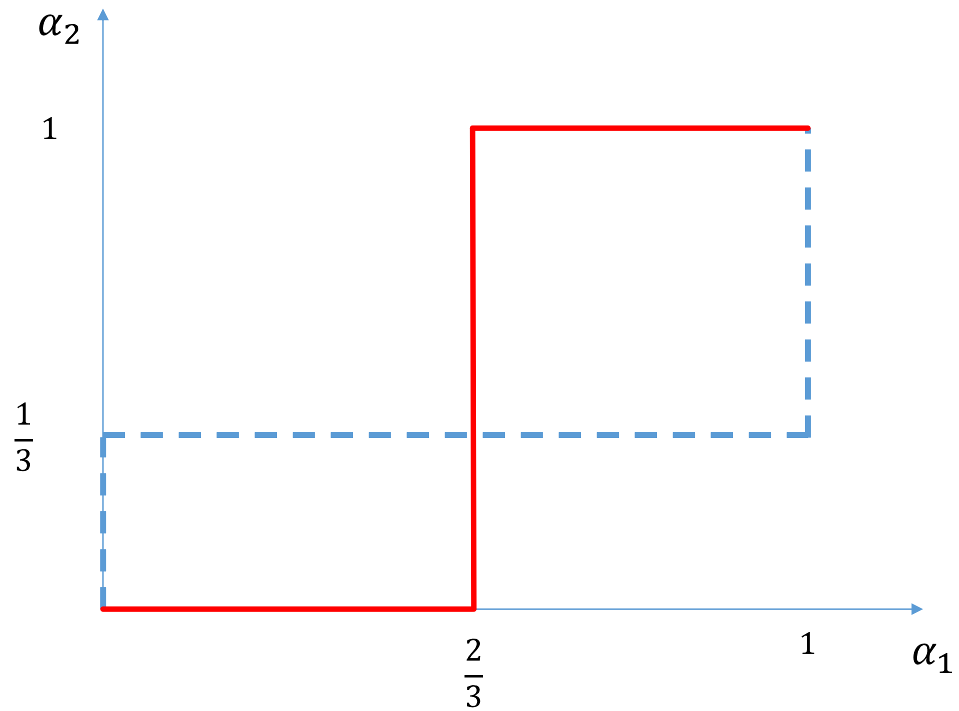
$$U_1(M, \alpha_1) = 0 \bullet \alpha_1 + 2 \bullet (1 - \alpha_1) = 2(1 - \alpha_1)$$

Player 1 prefers T if:

$$\alpha_1 \geq 2 - 2\alpha_1 \Leftrightarrow \alpha_1 \geq 2/3$$

and strictly prefers M otherwise.

Graphically:



So now the Nash equilibrium is $2/3, 1/3$, the probability that they both go to the theater is now $2/9$.

Comparative statics

What happens to the mixed Nash eq. when we increase the payoff of 1 for theater?

	Theater	Music
Theater	$\mathbf{5, 1}$	$\mathbf{0, 0}$
Music	$\mathbf{0, 0}$	$\mathbf{1, 2}$

Player 1's expected utility of choosing T and M are:

$$U_1(T, \alpha_2) = 5\alpha_2 + 0 \cdot (1 - \alpha_2) = 5\alpha_2$$

$$U_1(M, \alpha_2) = 0 \cdot \alpha_2 + 1 \cdot (1 - \alpha_2) = 1 - \alpha_2$$

Player 1 prefers T if:

$$5\alpha_2 \geq 1 - \alpha_2 \Leftrightarrow \alpha_2 \geq 1/6$$

and strictly prefers M otherwise.

Then 2's expected utility of choosing T and M are:

$$U_2(T, \alpha_1) = \alpha_1 + 0 \bullet (1 - \alpha_1) = \alpha_1$$

$$U_1(M, \alpha_1) = 0 \bullet \alpha_1 + 2 \bullet (1 - \alpha_1) = 2(1 - \alpha_1)$$

Player 1 prefers T if:

$$\alpha_1 \geq 2 - 2\alpha_1 \Leftrightarrow \alpha_1 \geq 2/3$$

and strictly prefers M otherwise.

So now the Nash equilibrium is $2/3, 1/6$, the probability that they both go to the theater is now $1/9$, lower than before the change.

Example 2: **Rock**, **Paper**, **Scissor**.

	Rock	**Paper**	**Scissor**
Rock	**0,0**	$-1, 1$	**1,-1**
Paper	**1,-1**	**0,0**	**-1,1**
Scissor	$-1, 1$	$1, -1$	**0,0**

Lets find all the equilibria.

It is easy to see that we cannot have a pure strategy equilibrium.

Nor that it is possible that a player plays an action with probability one.

We have two possibilities:

- A player mixes on fewer than 3 actions;
- Totally mixed strategies from both players.

We can exclude the first option.

	Rock	Paper	Scissor
Rock	0,0	$-1, 1$	1,-1
Paper	1,-1	0,0	-1,1
Scissor	$-1, 1$	$1, -1$	0,0

Assume w.l.g. that 1 mixes between R and S.

Then 2 can select rock and guarantee a strictly positive payoff.

This implies that 1 gets a strictly negative payoff.

This is a contradiction, since 1 could imitate 2’s strategy.

So we have a totally mixed equilibrium.

Lets show that there is a unique mixed strategy equilibrium.

2's strategy: $\sigma_R, \sigma_P, \sigma_S = 1 - \sigma_R - \sigma_P$

	Rock	Paper	Scissor
Rock	0,0	$-1, 1$	1,-1
Paper	1,-1	0,0	-1,1
Scissor	$-1, 1$	$1, -1$	0,0

For 1 to mix, we must have:

$$u_R = -\sigma_P + (1 - \sigma_R - \sigma_P)$$

$$u_P = \sigma_R - (1 - \sigma_R - \sigma_P)$$

$u_S = -\sigma_R + \sigma_P$ and $u_R = u_P, u_R = u_S$

Note: 2 variables and 2 independent equations, so (generically) a unique equilibrium.

Specifically

$$
\begin{aligned}
-\sigma_P + (1 - \sigma_R - \sigma_P) &= \sigma_R - (1 - \sigma_R - \sigma_P) \\
&\Leftrightarrow 2 - 3(\sigma_R + \sigma_P) = 0 \\
&\Leftrightarrow \sigma_R + \sigma_P = 2/3
\end{aligned}
$$

and:

$$
\begin{aligned}
-\sigma_P + (1 - \sigma_R - \sigma_P) &= -\sigma_R + \sigma_P \\
&\Leftrightarrow 1 - \sigma_R = 2\sigma_P \\
&\Leftrightarrow \sigma_P = 1 - (\sigma_R + \sigma_P) = 1/3 \\
&\Leftrightarrow \sigma_R = \sigma_S = 1/3
\end{aligned}
$$

A variant.

	Rock	Paper	Scissor
Rock	0,0	$-1, 1$	5,-5
Paper	1,-1	0,0	-1,1
Scissor	$-5, 5$	$1, -1$	0,0

Correlated Equilibrium

We have seen all the equilibria of the BoS game:

	Theater	**Music**
Theater	$2, 1$	$0, 0$
Music	$0, 0$	$1, 2$

There are 2 pure startegy equilibria and a mixed equilibrium.

There are other outcomes that may be rationalized:

- Assume with probability 1/2 it rains and 1/2 it is sunny;
- Suppose that the players have a convention that they go to T when raining and to M when it is sunny,
- So we would observe outcomes on the diagonal, each with probability 1/2.

But it could be even more complicated.

Suppose there are 3 states: $\{x, y, z\}$ each with probability $4/10, 2/10, 4/10$.

Suppose that player 1 observes $\{x\}$ or $\{y, z\}$, and 2 observes $\{x, y\}$ or $\{z\}$

Assume that 1 believes that 2 plays T if $\{x, y\}$ and M if $\{z\}$; and 2 believes that 1 plays T if $\{x\}$ and M if $\{y, z\}$.

Then for 1 the strategy is optimal if

$$U_1(T, \{x\}) = 2 \geq 0 = U_1(M, \{x\})$$

$$U_1(T, \{y, z\}) = \frac{1}{3} \cdot 2 \leq \frac{2}{3} \cdot 1 = U_1(M, \{y, z\})$$

so OK; for 2 the strategy is also optimal.

The probability of TT now is $\frac{4}{10} = 2/5 > 1/3$.

Definition. *A correlated equilibrium of a strategic game* $\langle N, (A_i), (u_i) \rangle$ *consists of:*

- *A finite probability space* Ω, π;
- *For each* $i \in N$*:*
    - *A partition* $\widetilde{P}_i$ *of* Ω *(player i's information partition);*
    - *A function* $\sigma_i : \Omega \to A_i$*, with* $\sigma_i(\omega) = \sigma_i(\omega')$ *if* $\omega, \omega' \in P_i$ *for some* $P_i \in \widetilde{P}_i$;
- Such that...

- *Such that or every i and every function* $\xi_i : \Omega \to A_i$ *with* $\xi_i(\omega) = \xi_i(\omega')$ *if* $\omega, \omega' \in P_i$ *for some* $P_i \in \widetilde{P}_i$ *we have:*

$$\sum_{\omega\in\Omega} \pi(\omega) \cdot u_i(\sigma_{-i}(\omega), \sigma_i(\omega))$$
$$\geq \sum_{\omega\in\Omega} \pi(\omega) \cdot u_i(\sigma_{-i}(\omega), \xi_i(\omega))$$

Notes

The probability space and the partitions are endogenous, part of the equilibrium definition.

A Nash equilibrium is a correlated equilibrium, but the opposite is not true, so the set of correlated equilibria is larger than the set of mixed equilibria.

Any convex combination of correlated equilibrium payoffs profile is a correlated equilibrium payoff profile of some correlated equilibrium.

Idea: first run a public randomization that identifies which equilibrium to play, then play that equilibrium.

In general we might not know what kind of correlation device is available to a group of players.

Studying the set of correlated equilibria can give us a sense of what outcome we might expect, and what outcomes are not to be expected.

In general we can assume without loss of generality that the state space coincides with the action space.

Theorem. *Let* $G = \langle N, (A_i), (u_i) \rangle$ *be a finite strategic game.*

Every probability distribution over outcomes that can be obtained in a correlated equilibrium of G *can be obtained in a correlated equilibrium in which:*

- *the set of states is* A*,*
- *and for each* $i \in N$ *player* i*'s information partition consists of all sets of the form* $\{a \ s.t. a_i = b_i\}$ *for some* $b_i \in A_i$*.*

Evolutionary Equilibrium

A variant of the concept of Nash equilibrium has been used to study the evolution of organisms (or other entities).

An organism has a possible range of actions B

Each organism is programmed to choose an action $b \in B$.

Organisms are paired in anonymous way to play a game.

If an organism chooses b and faces a distribution of actions β, the utility is the expected value of $u(b, b')$, where b' is distributed according to β.

As in a 2 player symmetric game: $u_1(b, b') = u(b, b')$ and $u_2(b, b') = u(b', b)$.

The utility reached in expectation determines the fitness of a type b.

Which steady state should we expect?

The notion of equilibrium here is designed to capture the steady state in a dynamical system in which all organisms take the equilibrium action and no mutant can invade the population with an alternative action.

For b^* *to be an Evolutionary Stable Strategy (ESS) we require:*

$$(1-\epsilon)u(b,b^*) + \epsilon u(b,b) < (1-\epsilon)u(b^*,b^*) + \epsilon u(b^*,b)$$

for all ϵ *sufficiently small and any* $b \in B$*.*

The lhs is the expected utility of a deviator b; the rhs is the expected utility of a non deviator.

If this condition is not satisfied, b has a superior fit than b^*.

The requirement above inspires the definition:

Definition. *A b^* is said to be an Evolutionary Stable Strategy (ESS)* iff:

- b^* is a Nash equilibrium of the symmetric game $\langle\{1,2\},(B,B),(u_i)\rangle$ with $u_1(a,b) = u_2(b,a)$;
- For all $b \neq b^*$ either $u(b,b^*) < u(b^*,b^*)$ (i.e. the equilibrium is strict), or $u(b,b^*) = u(b^*,b^*)$ and $u(b,b) < u(b^*,b)$.

Note that B may be the set of mixed strategies.

A game may have no ESS, consider:

	U	D
U	1, 1	1, 1
D	1, 1	1, 1

Every two-player symmetric strategic game in which each player with $|A_i| = 2$ and generic payoffs has a ESS.

In general, we have:

	U	D
U	w, w	x, y
D	y,x	z, z

w.l.g we can assume $w < y$ and $z < x$, else we have a strict pure strategy equilibrium.

We must have a totally mixed equilibrium:

$$
\begin{aligned}
w\alpha_U + x(1-\alpha_U) &= y\alpha_U + z(1-\alpha_U) \\
\Leftrightarrow \alpha_U &= \frac{(z-x)}{w+z-(x+y)}
\end{aligned}
$$

To verify that this mixed equilibrium is ESS, we need to show that for any α:

$$u(\alpha,\alpha) - u(\alpha_U,\alpha) < 0$$

$$\Leftrightarrow (\alpha - \alpha_U)[\alpha w + (1-\alpha)x] - (\alpha - \alpha_U)[\alpha y + (1-\alpha)z] < 0$$

$$\Leftrightarrow (\alpha - \alpha_U)[\alpha(w - y + z - x) + x - z] < 0$$

$$\Leftrightarrow (w - y + z - x)(\alpha - \alpha_U)\left[\alpha - \frac{(z-x)}{(w-y+z-x)}\right] < 0$$

$$\Leftrightarrow (w - y + z - x)(\alpha - \alpha_U)^2 < 0 \text{ a.v.}$$