ECON 6110, Microeconomic Theory III	**Section 5**
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1 Extensive Game (Cont.)

Definition 1.1. A history $h \in H$ is a sequence of actions taken by the players $(a^k)_{k=1,\dots,K}$. The set of terminal histories is denoted Z.

Definition 1.2. A strategy of a player i in an extensive game with perfect information is a function

$$s_i(h) \to A(h)$$

for any $h \in H \setminus Z$ such that $P(h) = i$.

Remark. A strategy specifies an action for *each* (non-terminal) history in which a player is asked to choose an action, even for histories that, if the strategy is followed, are never reached.

Definition 1.3. Denote a strategy profile $s = (s_1, \dots, s_n)$. For each strategy profile an outcome $O(s)$ is the terminal history associated with the strategy profile.

Definition 1.4. A strategy profile, $s = (s_1, \dots, s_n)$ is a **Nash equilibrium** if for all players i and all deviations $\hat{s}_i$,

$$u_i(s_i, s_{-i}) \geq u_i(\hat{s}_i, s_{-i})$$

where $u_i(s) = u_i(O(s))$.

Definition 1.5. The **subgame** of the extensive game with perfect information $\Gamma = \langle N, H, P, (u_i) \rangle$ that follows the history h is the extensive game $\Gamma(h) = \langle N, H|_h, P|_h, (u_i)|_h \rangle$, where $H|_h, P|_h, (u_i)|_h$ are consistent with the original game starting at history h.

Definition 1.6. A strategy profile, s is a **subgame perfect equilibrium** in Γ if for any history h the strategy profile $s|_h$ is a Nash equilibrium of the subgame $\Gamma(h)$.

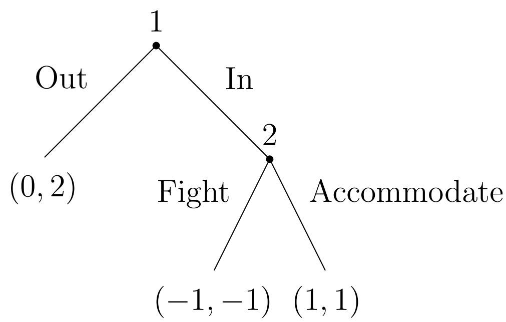
Definition 1.7. For fixed s_i and history h, a **one-stage deviation** is a strategy $\hat{s}_i$ in the subgame $\Gamma(h)$ that differs from $s_i|_h$ only in the action it prescribes after the initial history of $\Gamma(h)$.

Theorem 1.1 (One-stage deviation principle)**.** *In a finite-horizon extensive game or infinite horizon games continuous at infinity, a strategy profile s is an SPE if and only if for all players i, all histories $h \in H$, and one-stage deviations $\hat{s}_i$,*

$$u_i(s_i|_h, s_{-i}|_h) \geq u_i(\hat{s}_i, s_{-i}|_h)$$

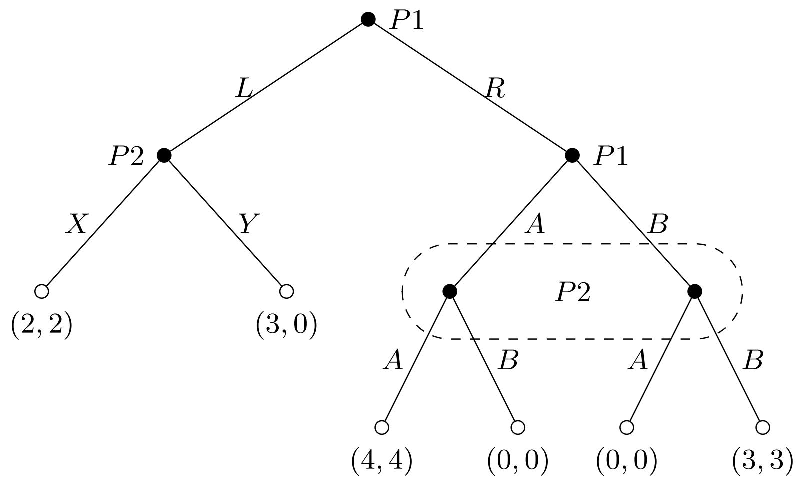
Theorem 1.2 (Kuhn's)**.** *SPE for finite extensive games can be found by Backward induction.*

Example 1.3 (Entry game)*.*



2 Microeconomic Theory Qualification Exam, 2018 Retake: Question III

Part III (20 Points)
Consider the following dynamic game in extensive form:



(a) (3 points) List all pure strategies that each player has.

(b) (3 points) How many subgames are there? Please describe them.

(c) (9 points) Find all (pure or mixed) subgame perfect equilibria.

(d) (5 points) Find a Nash equilibrium that is not subgame perfect.

Proof. **(a)** A pure strategy specifies what each player does at all of their respective decision nodes. Thus, the pure strategies are

- Player 1: $\{L, R\} \times \{A, B\}$
- Player 2: $\{X, Y\} \times \{A, B\}$.

(b) There are three subgames:

- Subgame 1: The coordination game that occurs after Player 1 plays R
- Subgame 2: Player 2's decision problem that occurs after Player 1 plays L.

- Subgame 3: The entire game.

(c) For a SPNE, a Nash must be played in each proper subgame:

- Subgame 1: The three NE are $(A, A); (B, B)$; and $([3/7]A + [4/7]B, [3/7]A + [4/7]B)$.
- Subgame 2: The Unique NE is (Idle, X)

Thus, if Player 1 plays L, she is guaranteed a payoff of 2, but if she plays R, she can either receive a payoff of $4, 3$, or $12/7$ in the case of the interior subgame NE. Thus, the SPNE are as follows:

- {(R, A); (X, A)}
- {(R, B); (X, B)}
- {(L, [3/7]A+[4/7]B); (X, [3/7]A+[4/7]B)}

(d) The strategy profile

$$\{(R, A); (Y, A)\}$$

is a NE, but it is not subgame perfect as Player 2 is not playing a best response in Subgame 2.

□

3 Ultimatum Game

Consider the following two stage game, about how two players split a pile of 100 gold coins. The action (strategy) set of player 1 is given by $S_1 = \{0, \cdots, 100\}$, with choice i meaning that player 1 proposes to keep i of the gold coins. Player 2 learns the choice of player one, and then takes one of two actions in response: 1 (accept) or 0 (reject). If player two plays accept, the payoff vector is $(i, 100 - i)$. If player two plays reject, the payoff vector is $(0, 0)$.

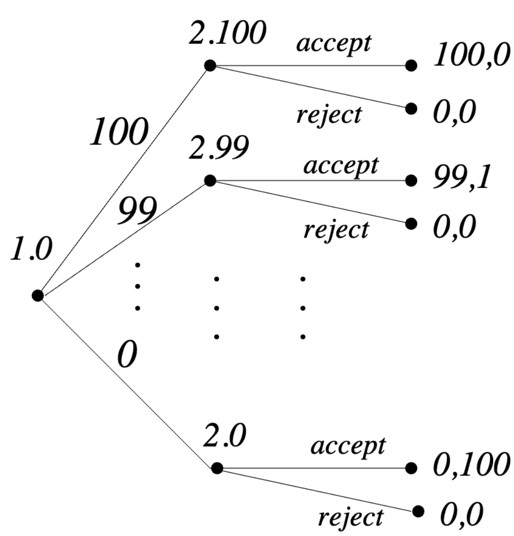
(a) Describe the extensive form version of the game using a game tree.

(b) Describe the normal form of the game. It suffices to specify the strategy spaces and payoff functions. (Hint: Player 2 has 2^{101} pure strategies.)

(c) Identify a Nash equilibrium of the normal form game with payoff vector $(50, 50)$.

(d) Identify the subgame perfect equilibria of the extensive form game. (Hint: There are two of them.)

(e) Do the subgame perfect equilibria change if player 1's strategy space is now continuous, i.e., $S_1 = [0, 100]$?



Proof. (a)

(b) The strategy spaces are $S_1 = \{0, \cdots, 100\}$ and $S_2 = \{0, 1\}^{101}$. A strategy of player 2 can be represented as a vector $d = (d(i) : 0 \leq i \leq 100)$. $d(i) = 1$ if player 2 accepts when player 1 plays i, and $d(i) = 0$ if player 2 rejects when player 1 plays i.

The payoff functions are given by

$$u_1(i, d) = i \cdot d(i)$$
$$u_2(i, d) = (100 - i) \cdot d(i)$$

(c) Denote a cutoff strategy of player 2 d_k as $d(i) = 1, \forall i \leq k$ and $d(i) = 0 \, \forall i > k$. The strategy profile $(50, d_{50})$ is a Nash equilibrium.

(d) To find SPE we use backward induction. For all subgame $i \leq 99$, the subgame Nash is accept. Only for the last subgame, both accept and reject are Nash. Suppose player 2 always accept, then for player 1 the best response is 100. Suppose player 2 reject only if $i = 100$, then for player 1 the best response is 99. Then, $(99, d_{99})$ and $(100, d_{100})$ are subgame perfect equilibria.

(e) Now, only $(100, d_{100})$ is subgame perfect equilibrium. If player 2 rejects at $i = 100$, then no strategy of player 1 is Nash.

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