# ECON 6200: Section 2

1/31/25

## Interpreting the Linear Model

1) Best Linear Predictor
   - makes sense in very general cases
   - almost always interpreted as a projection coefficient under square loss

2) Causal Linear Model
   - underlying "true" model is linear
   - more restrictive, but gives you a causal interpretation

Note: All finite sample theory results are based on the causal interpretation of OLS

## Assumptions (Causal Model)

(1) Linearity $\qquad Y = X\beta + \varepsilon$

(2) Strong Exogeneity $\qquad E(\varepsilon | X) = 0$

↳ conditioning on <u>all</u> elements in $X$

(3) Rank Condition $\qquad \text{rank}(X) = K$ a.s.
where $X \in \mathbb{R}^{n \times k}$

↳ ie: $X'X$ nonsingular

↳ Identification assumption
If fails, then there exists a set of observationally equivalent "true" coefficient that form a linear subspace of $\mathbb{R}^k$ and all induce the same $\hat{Y}$

(4) Spherical Errors $\qquad E[\varepsilon\varepsilon' | X] = \sigma^2 I_n$

↳ combines homoscedasticity $E[\varepsilon_i^2 | X] = \sigma^2$
and no correlation between errors
$E[\varepsilon_i \varepsilon_j | X] = 0$, for $i \neq j$

Note: Identification (in simple terms) means that a parameter is uniquely determined by the distribution of the observed variables

# Finite Sample Theory

## Theorem

Under the assumptions (1) - (4), we have:

A) $\hat{\beta}$ is unbiased
↳ $E(\hat{\beta} \mid X) = \beta$

B) $var(\hat{\beta} \mid X) = \sigma^2 (X'X)^{-1}$

C) Gauss - Markov Thm
↳ $\hat{\beta}$ is BLUE

A) Prove $\hat{\beta}$ unbiased

Recall

$$\begin{aligned}
\hat{\beta} &= (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y} \\
&= (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'(\boldsymbol{X}\beta + \varepsilon) \\
&= \beta + \underbrace{(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\varepsilon}_{=\text{estimation error}} .
\end{aligned}$$

Thus,

$$E[\hat{\beta} \mid X] = \beta + E[(X'X)^{-1}X'\varepsilon \mid X]$$

$$= \beta + (X'X)^{-1}X' \underbrace{E[\varepsilon \mid X]}_{=0 \text{ by } ②}$$

$$= \beta \checkmark$$

B) Prove $var(\hat{\beta} \mid X) = \sigma^2 (X'X)^{-1}$

$$var(\hat{\beta} \mid X) = var(\underset{\text{constant}}{\beta} + (X'X)^{-1}X'\varepsilon \mid X)$$

$$= var((X'X)^{-1}X'\varepsilon \mid X)$$

$var(AY) = A\, var(Y) A'$

$$= (X'X)^{-1}X'\, var(\varepsilon \mid X)\, ((X'X)^{-1}X')'$$

② $E[\varepsilon \mid X] = 0$

$$= (X'X)^{-1}X'\sigma^2 I_n X(X'X)^{-1}$$

④ $E[\varepsilon\varepsilon' \mid X] = \sigma^2 I_n$

$Var(Y) = E[Y^2] - E[Y]^2$

$$= \sigma^2 (X'X)^{-1}\cancel{X'X(X'X)^{-1}}$$

$$= \sigma^2 (X'X)^{-1} \checkmark$$

c) Gauss Markov Thm

- OLS estimator $\hat{\beta}$ is "BLUE"
  - ↳ BLUE = Best Linear Unbiased Estimator
- Variance is the metric we use to assess "Best"
- Obviously, there exists better estimators with a smaller variance, but they may not <u>linear</u> or <u>unbiased</u>

**Theorem 4.4 Gauss-Markov**
In the homoskedastic linear regression model (Assumption 4.3) with i.i.d. sampling (Assumption 4.1), if $\widetilde{\beta}$ is a linear unbiased estimator of $\beta$ then

$$\operatorname{var}[\widetilde{\beta} \mid \boldsymbol{X}] \geq \sigma^2 (\boldsymbol{X}'\boldsymbol{X})^{-1}.$$

Let $\tilde{\beta}$ be a linear unbiased estimator that is <u>not</u> the OLS estimator. Let us represent $\tilde{\beta}$ as

$$\begin{aligned} \tilde{\beta} &= \beta_{OLS} + DY \\ &= CY \end{aligned}$$

since $\tilde{\beta}$ is linear in $Y$

where $D = C - (X'X)^{-1}X'$

$$\Rightarrow C = (X'X)^{-1}X' + D$$

Note: Intuitively, $\tilde{\beta} = \beta_{OLS} + \text{noise}$

[WTS: $\tilde{\beta}$ unbiased]

$$
\begin{aligned}
E[\tilde{\beta} \mid X] &= E[CY \mid X] \\
&= E[((X'X)^{-1}X' + D)Y \mid X] \\
&= E[(X'X)^{-1}X'Y \mid X] + E[DY \mid X] \qquad \text{Distribute} \\
&= \beta_{OLS} + E[D(X\beta_{OLS} + \varepsilon) \mid X] \\
&= \beta_{OLS} + E[DX\beta_{OLS} \mid X] + D\underbrace{E[\varepsilon \mid X]}_{=0 \text{ by } (2)}
\end{aligned}
$$

Note: D is a func of X

In order for $\tilde{\beta}$ to be unbiased, we must have

$$E[DX\beta_{OLS} \mid X] = 0 \quad \text{for any } \beta$$

$\Rightarrow$ Since $\beta$ is a constant and $D$ is a function of $X$, we must have $DX = 0$

[WTS: $var(\tilde{\beta} | X) \geq \sigma^2 (X'X)^{-1}$

$$var(\tilde{\beta} | X) = var\left( \left( (X'X)^{-1}X' + D \right) Y | X \right)$$

$$= var\left( \left( (X'X)^{-1}X' + D \right)(X\beta + \varepsilon) | X \right)$$

everything here is either a scalar or function of $X$

$$= var\left[ \underbrace{\left( (X'X)^{-1}X' + D \right) X\beta}_{\text{treat this as a constant when conditioning on } X} + \left( (X'X)^{-1}X' + D \right) \varepsilon | X \right]$$

$$= var\left[ \underbrace{\left( (X'X)^{-1}X' + D \right)}_{\text{func. of } X \Rightarrow \text{ treat as constant}} \varepsilon | X \right]$$

$$= \left( (X'X)^{-1}X' + D \right) var(\varepsilon | X) \left( (X'X)^{-1}X' + D \right)'$$

$$= \underbrace{\sigma^2}_{\text{by } 4} \left[ \left( (X'X)^{-1}X' + D \right) \left( (X'X)^{-1}X' + D \right)' \right]$$

Recall
$(A+B)' = A' + B'$

$$= \sigma^2 \left[ \left( (X'X)^{-1}X' + D \right) \left( X (X'X)^{-1} + D' \right) \right]$$

$$= \sigma^2 \left[ (X'X)^{-1} \cancel{X'X (X'X)^{-1}} + \cancel{(X'X)^{-1}} \underbrace{X'D'}_{(DX)' = 0} + \underbrace{DX}_{= 0} (X'X)^{-1} + DD' \right]$$

$$= \sigma^2 \left[ (X'X)^{-1} + \underbrace{DD'}_{\text{PSD (all entries } \geq 0)} \right]$$

$$\geq \sigma^2 (X'X)^{-1} = var(\hat{\beta} | X)$$

Note: This proof assumes homoscedasticity (i.e.: 4 $E[\varepsilon^2 | X] = \sigma^2 I_n$
2 $E[\varepsilon | X] = 0$ )

# Standard Errors

(*) What are standard errors?
- estimators of standard deviations
- quantifies the uncertainty or variability in an estimator

Ex: $SE(\hat{\beta})$ = uncertainty or variability in the estimate of the regression coefficient $\hat{\beta}$

If I repeatedly sampled from the population and re-estimated the regression model, how does my $\hat{\beta}$ change?

## Standard Errors vs Standard Deviation

| Standard Errors | Standard Deviation |
|---|---|
| - variability between multiple samples from the same population | - variability inside a sample |
| - how precise is our estimate? | - how spread out is my data? |

## ⊛ Estimating Variance

To estimate the variance, in class we first considered:

- plug-in estimator

$$\hat{\sigma}^2 \equiv \frac{1}{n}\hat{\varepsilon}'\hat{\varepsilon} = \frac{1}{n}\sum_{i=1}^{n}\hat{\varepsilon}_i^2 .$$

However, $\hat{\sigma}^2$ is biased

$$E[\hat{\sigma}^2 | X] = \frac{n-k}{n}\sigma^2 \neq \sigma^2$$

So now take this into account and create:

- df-adjusted estimator

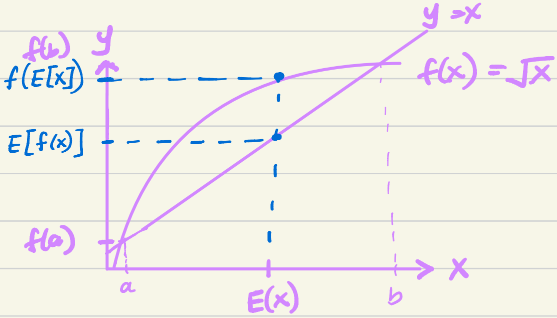
$$s^2 \equiv \frac{1}{n-K}\sum_{i=1}^{n}\hat{\varepsilon}_i^2 .$$

where $s^2$ is unbiased $E[s^2 | X] = \sigma^2$

[ Question: If we want to estimate the standard deviation, is $\sqrt{s^2}$ unbiased ? ]

Ans: No! Expectation is a linear operator. By Jensen's Inequality, we can see that $E[\sqrt{s^2}] \le \underbrace{\sqrt{E[s^2]}}_{\text{standard deviation}}$

Note: Since $\sqrt{(\cdot)}$ is a concave function, by Jensen's Inequality



Previously, we made no distributional assumptions on $\varepsilon$. However, if we impose a normality assumption on $\varepsilon$, we can derive an exact finite sample distribution and perform hypothesis testing

Let

$$\varepsilon | X \sim N(0, \sigma^2 I_n)$$

then we have

$$(\hat{\beta} - \beta) | X \sim N(0, \sigma^2 (X'X)^{-1})$$

Proof

Previously, we showed that $\hat{\beta} = \beta + (X'X)^{-1} X' \varepsilon$

$$E[\hat{\beta} - \beta | X] = E[\hat{\beta} | X] - \beta = 0$$

$$\begin{aligned} Var(\hat{\beta} - \beta | X) &= Var(\hat{\beta} | X) \\ &= \sigma^2 (X'X)^{-1} \end{aligned}$$

$$\Rightarrow \hat{\beta} - \beta | X \sim N(0, \sigma^2 (X'X)^{-1})$$

## Problem: MLE vs OLS Under Normality Assumption

$$Y = X\beta + \varepsilon$$

By OLS, our objective function is

$$\hat{\beta}_{OLS} = \arg\min_{b} (y - Xb)'(y - Xb)$$

$$\Rightarrow \hat{\beta}_{OLS} = (X'X)^{-1}(X'Y)$$

assuming $(X'X)$ invertible.

Given data $Y_1, \dots, Y_n | X$, show that

$$\hat{\beta}_{MLE} = \hat{\beta}_{OLS} \quad \text{when} \quad \varepsilon \sim N(0, \sigma^2 I_n)$$

If $Y = X\beta + \varepsilon$, then

$$Y_i \mid \mathbf{X}_i \sim N(X_i\beta, \sigma^2 I_n)$$

Since $Y_1, \dots, Y_n$ iid given $\mathbf{X}$

$$\mathcal{L}(\beta, \sigma^2) = \prod_{i=1}^{n} f(y_i \mid x_i, \beta, \sigma^2)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y_i - x_i\beta)^2}{2\sigma^2}\right)$$

Taking the log-likelihood

$$\ell(\beta, \sigma^2) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - x_i\beta)^2$$

$$\frac{\partial \ell(\beta, \sigma^2)}{\partial \beta} = -\frac{1}{\cancel{2}\sigma^2}\sum_{i}^{n} \cancel{2}(y_i - x_i\beta)(-x_i) = 0$$

$$\Rightarrow \sum_{i=1}^{n} y_i x_i - \beta \sum x_i^2 = 0$$

$$\hat{\beta}_{MLE} = \frac{\sum y_i x_i}{\sum x_i^2}$$

$$\frac{\partial \ell(\beta, \sigma^2)}{\partial \sigma^2} = -\frac{n}{2}\frac{1}{2\pi\sigma^2}(2\pi) + 2 \cdot \frac{1}{2(\sigma^2)^2}\sum(y_i - X_i\beta)^2 = 0$$

$$\Rightarrow \quad \hat{\sigma}^2 = \frac{1}{n}\sum_i^n (y_i - X_i\hat{\beta})^2$$

And check SOC!