

ECON 6110
Problem Set 1

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Problem 1

1. **Proof.** Take some profile s that is a Nash equilibrium of the primal game. Note that the set of strategies available in the game in which strategies have been removed by iterated strict dominance is a subset of the set of strategies available in the primal game. Thus, it suffices to show that s is available in the game in which strategies have been removed by iterated strict dominance to show that s is a Nash equilibrium of that game. FSOC, assume not, meaning that there is some player i for which s_i is dominated, meaning that there is some (possibly mixed) strategy σ_i such that

$$U_i(\sigma_i, s_{-i}) > U_i(s_i, s_{-i})$$

Then we can immediately see that s is not a Nash equilibrium of the primal game, since player i would prefer σ_i to it. This is a contradiction of the assumption. \square

2. **Proof.** Proof by contrapositive. Take some profile s that is not a Nash equilibrium of the primal game. It remains to show that it is not a Nash equilibrium of the game in which strategies have been removed by iterated strict dominance. Since s is not a Nash equilibrium, using the negation of the Nash equilibrium definition there must exist some player i and some (possibly mixed) strategy σ'_i for which

$$U_i(\sigma'_i, s_{-i}) > U_i(s_i, s_{-i})$$

Thus, s_i is strictly dominated for player i , so it is not available in the game in which strategies have been removed by iterated strict dominance, so s is not an equilibrium of that game. \square

3. **Proof.** Take some profile s that is a Nash equilibrium of the game where strategies have been removed by iterated weak dominance. FSOC, assume that it is not a Nash equilibrium of the larger game, meaning that for some i there is some (possibly mixed) strategy σ_i that they prefer, such that

$$U_i(\sigma_i, s_{-i}) > U_i(s_i, s_{-i})$$

Then s_i is strictly dominated, meaning that it is weakly dominated, meaning that it would not be available in the game where strategies have been removed by iterated weak dominance. This is a contradiction of the fact that s is a Nash equilibrium of that game. \square

4. Consider the following example:

	L	R
U	1, 1	0, 0
D	1, 0	-1, 0

D is weakly dominated, but D, L is a Nash equilibrium.

Problem 2

1. There is one pure-strategy Nash equilibrium, (U, L) . There are a continuum of mixed strategy equilibria. Assume that column chooses L with probability p . Then we must have that

$$u(U, p) = 4p + 3(1 - p) = 4p + 0(1 - p) = u(D, p) \implies p = 1$$

Thus, if column chooses L all the time, row is indifferent. Assume that row chooses U with probability q . Column will be indifferent if and only if

$$u(R, q) = u(L, q) \implies q + (1 - q)6 = 5q \implies q = \frac{6}{10}$$

So there are a continuum of mixed strategy equilibria, where column plays L with probability 1 and row plays U with probability $q \in [0.6, 1]$.

2. There is only one pure-strategy Nash equilibrium, (U, R) . Observe that D is strictly dominated by U , and once D is eliminated L is strictly dominated by R . Thus, this is the only Nash equilibrium.
3. There are two pure-strategy Nash equilibria, (U, L) and (M, C) . There are also mixed strategy equilibria. Assume that column chooses L with probability p_1 , C with probability p_2 , and R with probability p_3 . The expected payoff for each pure strategy for row is

$$u(U, p) = 6p_1 + p_2 + 3p_3$$

$$u(M, p) = 2p_1 + 4p_2 + 4p_3$$

$$u(D, p) = 3p_1 + 2p_2 + 3p_3$$

Assume that row chooses U with probability q_1 , M with probability q_2 , and D with probability q_3 . The expected payoff for each pure strategy for column is

$$u(L, q) = 6q_1 + 2q_2 + 3q_3$$

$$u(C, q) = q_1 + 7q_2 + 3q_3$$

$$u(R, q) = 4q_1 + 5q_2 + 9q_3$$

By solving the systems $u(U, p) = u(M, p) = u(D, p)$ and $u(L, q) = u(C, q) = u(R, q)$ and verifying that they have no solutions (in the open unit interval), we can confirm that neither player is ever purely mixing. Next, we check whether they will ever mix between two strategies. Assume first that $p_1 = 0$. Then M strictly dominates either U or D . If $p_2 = 0$, then U strictly dominates D for any positive p_1 . Once D is eliminated, R is strictly dominated by a combination of L and C , so it will also happen that $p_3 = 0$, and we are back to pure. Thus, it must be the case that $p_3 = 0$, so column mixes between L and C . Similarly, assume first that $q_1 = 0$. Then L is strictly dominated, and once L is eliminated we saw above that $q_3 = 0$ also and we are back to pure. Next, assume that $q_2 = 0$. Then C is strictly dominated, and again we are back in the above case where $p_2 = 0$, so it would again happen that $q_3 = 0$. Thus, it must be the case that $q_3 = 0$, and row mixes between U and M .

Once we are in the world where row mixes between U and M and column mixes between L and C , say that column chooses L with probability p and row chooses U with probability q . It must hold that

$$u(U, p) = u(M, p) \implies 6p + (1 - p) = 2p + 4(1 - p) \implies p = \frac{3}{7}$$

and

$$u(L, q) = u(C, q) \implies 6q + 1 - q = 2q + 7(1 - q) \implies q = \frac{3}{5}$$

So there is one mixed Nash equilibrium,

$$\left(\frac{3}{5}U + \frac{2}{5}M, \frac{3}{7}L + \frac{4}{7}C \right)$$

Remark. There are a few way easier ways to do this. It would be significantly more straightforward to simply observe that D and R are strictly dominated by some mixed combination of U, M and L, C respectively, which follows directly from the only pure Nash equilibria being in those strategies. I'm choosing to leave my high effort version because of my intense moral fibre that does not allow me to take credit for Finn's observations, and also because I already did the work and I'm quite lazy. Hi Ruqing! Hope you're having a good day.

Problem 3

1. We will define the easiest possible probability space $\Omega = \{a, b, c, d\}$, with $\pi(a) = \pi(b) = \pi(c) = \frac{1}{3}$ and $\pi(d) = 0$. This state space coincides with the action space $\{(U, L), (U, R), (D, L), (D, R)\}$, which we can do because of the Proposition presented by Ruqing in her week 2 TA section, in which she showed it is without loss to do so. Intuitively, if the state space was larger than the action space, it would necessarily collapse down to the action space. If it were smaller, it could be made equivalent by putting 0 probability on all actions not represented in the state space.
2. Row has the partitions $P_1 = \{a\}$, $P_2 = \{b, c\}$, and $P_3 = \{d\}$. Column has the partitions $Q_1 = \{a, b\}$, $Q_2 = \{c\}$, and $Q_3 = \{d\}$. Observe that P_3 and Q_3 have probability 0 of occurring, so we will henceforth ignore them.
3. $\sigma(P_1) = D$, $\sigma(P_2) = U$, $\sigma(Q_1) = L$, $\sigma(Q_2) = R$. Using the probabilities defined above, we have that $\mathbb{P}(U, L) = 1/3$, $\mathbb{P}(U, R) = 1/3$, $\mathbb{P}(D, L) = 1/3$, and $\mathbb{P}(D, R) = 0$.
4. These strategies are a correlated equilibrium. Observe that if row sees P_2 , they evaluate the conditional probability $\mathbb{P}(b | P_2) = \mathbb{P}(c | P_2) = 1/2$, so their expected utility for playing U is $2 \cdot (1/2) + 0 = 1$, and their expected utility for playing D is $1 \cdot (1/2) + 0 = 1/2$, so they will adhere to the prescribed strategy and play U . If row sees P_1 , they will either 1 for playing D or 0 for playing U , so they will play D . Since the game is symmetric, the same holds for column – if they see Q_1 , they will perceive a higher expected utility from L than R , and if they see Q_2 they know they will attain 1 rather than 0. Thus, neither player will deviate and this is a correlated equilibrium.

Problem 4

1. Any policy $a_i \in (0, 1)$ is rationalizable. Take some $a_{-i} \in [0, 1]$. If $F(a_{-i}) < 1/2$, then $\exists \varepsilon > 0$ such that $F(a_{-i} + \varepsilon) = 1/2$, and thus any $a_i \in (a_{-i}, a_{-i} + 2\varepsilon)$ will attain 1, and is a best response. If $F(a_{-i}) > 1/2$, then $\exists \varepsilon > 0$ such that $F(a_{-i} - \varepsilon) = 1/2$, and thus any $a_i \in (a_{-i} - 2\varepsilon, a_{-i})$ will attain 1, and is a best response. Finally, if $F(a_{-i}) = 1/2$, then the only best response is $a_i = a_{-i}$. Thus, by perturbing a_{-i} , we can see that any a_i is the best response to some a_{-i} , with the only exceptions being $a_i \in \{0, 1\}$, which are strictly dominated by ε and $1 - \varepsilon$ for small ε .
2. Implicitly define the pure Nash equilibrium a^* such that $F(a^*) = 1/2$. Observe that this a^* will exist by the Intermediate Value Theorem. Then note that the payoff for a^* is

$$v_i(a_i^*, a_{-i}) = \begin{cases} 1 & \text{if } a_{-i} > a_i^* \\ 1/2 & \text{if } a_{-i} = a_i^* \\ 1 & \text{if } a_{-i} < a_i^* \end{cases}$$

Fix some $a_i \neq a_i^*$. For any $a_{-i} \neq a_i^*$, $v_i(a_i, a_{-i}) \leq 1$. Furthermore, if $a_{-i} = a_i^*$, then $v(a_i, a_{-i}) = 0$. Thus, a_i^* weakly dominates any other strategy. If $a_{-i} = a_i^*$, then $v_i(a_i^*, a_{-i}) = 1/2 > 0 = v_i(a_i', a_{-i})$ for any $a_i' \neq a_i^*$, so this is a pure-strategy Nash equilibrium.

3. Towards a contradiction, assume that there exists some other equilibrium a' where i plays $a_i' \neq a_i^*$. WLOG, assume that $a_i' > a_i^*$. If $a_{-i} = a^*$, then i attains 0 always and could improve to $1/2$ by switching to a_i^* . Since 0 and 1 are strictly dominated, in any Nash equilibrium $a_{-i}' \in (0, 1)$. If $a_{-i}' = a_i'$, then i could improve by switching to a_i^* and getting 1 instead of $1/2$. If $a_{-i}' < a_i'$, then i could improve by switching to a_i^* and getting 1 instead of 0. If $a_{-i}' > a_i'$, then $-i$ would improve by switching to a_i^* and attain 1 instead of 0. In all cases, this is not a Nash equilibrium because at least one player is incentivized to deviate. Note that this holds also with mixed strategies, considering expected payoff rather than actual payoff. Thus, there can be no other equilibria and a^* is the unique Nash equilibrium.

Problem 5

1. No. The strategy of voting for the opposing candidate is strictly dominated by not voting, since (i) in the case where the supporting candidate wins, they attain $1 - c$ instead of 1, (ii) in the case where the opposing candidate wins, they attain $-c$ instead of 0, and (iii) in the case where they are pivotal, they attain $-c$ instead of 1. In every case, they get strictly lower payoff.
2. We have to complete two tasks here: construct q such that every member of T_1 is indifferent between voting and not voting, and construct a condition under which every member of T_2 is content to abstain. If T_2 never votes, a single vote will win (and zero votes will lose). Take some i in T_1 . The expected payoff from not voting is 1 times the probability that somebody else votes, plus zero times the probability that nobody else votes. The expected payoff from voting is $1 - c$. We will set these equal to ensure a mixed strategy exists:

$$u_i(\text{abstain}) = 1 - (1 - q)^{M-1} = 1 - c = u_i(\text{vote}) \implies q = 1 - c^{1/(M-1)}$$

Next, we check that a member $j \in T_2$ is willing to abstain under these conditions. They attain 1 if nobody on T_1 votes, 0 otherwise. For voting, they attain $1 - c$ if exactly 0 or 1 members of T_1 vote, and $-c$ otherwise. For them to abstain, we need that

$$u_j(\text{abstain}) \geq u_j(\text{vote}) \implies (1 - q)^M \geq [(1 - q)^M + Mq(1 - q)^{M-1}](1 - c) - (1 - [(1 - q)^M + Mq(1 - q)^{M-1}])c$$

which simplifies to

$$(1 - q)^M \geq [(1 - q)^M + Mq(1 - q)^{M-1}] - c \implies c \geq Mq(1 - q)^{M-1}$$

Substituting in the formula for q from T_1 's condition, we get that this is equivalent to

$$c \geq M(1 - c^{1/(M-1)})c \implies c \geq \left(1 - \frac{1}{M}\right)^{M-1}$$

Thus, as long as c is large enough, there exists an equilibrium in which each member of T_1 votes with probability q and each member of T_2 abstains. Putting this into practice with $c = 1/2$ and $M = 25$, we first check:

$$\frac{1}{2} \geq \left(1 - \frac{1}{25}\right)^{24} \approx 0.375$$

So T_2 is willing to abstain. Then,

$$q = 1 - \left(\frac{1}{2}\right)^{1/24} \approx 1 - 0.972 = 0.028$$

so each member of T_1 votes approximately 2.8% of the time.

3. Fix some $k \in (0, \min\{M - 1, N\})$. Observe that if all members of T_1 except for i vote with probability q , the number of other votes is $X \sim \text{binom}(q, M - 1)$. We need i to be indifferent between voting and not voting. We can characterize their expected utility as follows:

$$\begin{aligned} u(\text{abstain}) &= \mathbb{P}\{X > k\} \cdot 1 + \mathbb{P}\{X \leq k\} \cdot 0 = \mathbb{P}\{X \geq k + 1\} \\ u(\text{vote}) &= \mathbb{P}\{X \geq k\} \cdot (1 - c) + \mathbb{P}\{X < k\} \cdot (-c) = \mathbb{P}\{X \geq k\} - c \end{aligned}$$

Setting them equal, we get

$$\mathbb{P}\{X \geq k + 1\} = \mathbb{P}\{X \geq k\} - c \implies c = \mathbb{P}\{X = k\} = \binom{M-1}{k} q^k (1 - q)^{M-1-k}$$

which defines q , if a solution to this equation in $(0, 1)$ exists. To check whether the members of T_2 will adhere to their strategy, we will check the voters (among the k) and the abstainers (among the $N - k$). Define the number of T_1 votes as $Y \sim \text{binom}(q, M)$. For a voter, we need that the utility of voting is greater than the utility of abstaining, so

$$\mathbb{P}\{Y \leq k\} \cdot (1 - c) + \mathbb{P}\{Y > k\} \cdot -c \geq \mathbb{P}\{Y \leq k - 1\}$$

which simplifies to

$$\mathbb{P}\{Y \leq k\} - c \geq \mathbb{P}\{Y \leq k - 1\} \implies c \leq \mathbb{P}\{Y = k\}$$

and since Y is binomially distributed, this becomes

$$c \leq \binom{M}{k} q^k (1 - q)^{M-k}$$

For an abstainer, we need that the utility of abstaining is greater than the utility of voting, so

$$\mathbb{P}\{Y \leq k\} \geq \mathbb{P}\{Y \leq k + 1\} \cdot (1 - c) + \mathbb{P}\{Y > k + 1\} \cdot (-c)$$

which simplifies to

$$c \geq \mathbb{P}\{Y = k + 1\} = \binom{M}{k+1} q^{k+1} (1 - q)^{M-1-k}$$

So the necessary and sufficient conditions for this equilibrium to exist are that:

$$c \in \left[\binom{M}{k+1} q^{k+1} (1 - q)^{M-1-k}, \binom{M}{k} q^k (1 - q)^{M-k} \right]$$

and that there exists $q \in (0, 1)$ that satisfies

$$c = \binom{M-1}{k} q^k (1 - q)^{M-1-k}$$

4. Fix some $c \in (0, 1)$ and some $M, N \geq 2$. Assume that k members of T_2 are voting. From the condition in part (3), there must exist q such that

$$c = \binom{M-1}{k} q^k (1 - q)^{M-1-k}$$

By continuity, this determines a unique q as long as

$$c \leq \max_{q \in (0, 1)} \left[\binom{M-1}{k} q^k (1 - q)^{M-1-k} \right]$$

Where the condition is met for any k since the Binomial pdf can attain any value in $(0, 1]$. Furthermore, assuming that the condition defining q holds, it implies that

$$c \leq \binom{M}{k} q^k (1 - q)^{M-k}$$

so the only condition left on the T_2 voters is that

$$c \geq \binom{M}{k+1} q^{k+1} (1 - q)^{M-1-k}$$

and combining with the condition defining q , this simplifies to

$$q \leq \frac{k+1}{M}$$

Thus, if $N \geq M - 1$, this condition can be satisfied trivially by choosing $k = M - 1$. It remains to show that this holds when $k = N < M - 1$. The T_1 condition admits a unique q , as always, and we need to check that $q \leq \frac{N+1}{M}$. We have that

$$c = \binom{M-1}{N} q^N (1-q)^{M-N-1}$$

So by the properties of binomial pdfs, the unique maximum of q here is $q = \frac{N}{M-1} \leq \frac{N+1}{M}$. Thus, the unique q will satisfy this condition, and there will always exist (at least one) equilibrium for any $c \in (0, 1)$ in which $k \in \{M-1, N\}$ members of T_2 vote and there exists a unique q such that each member of T_1 votes with probability q .