Sequential equilibrium

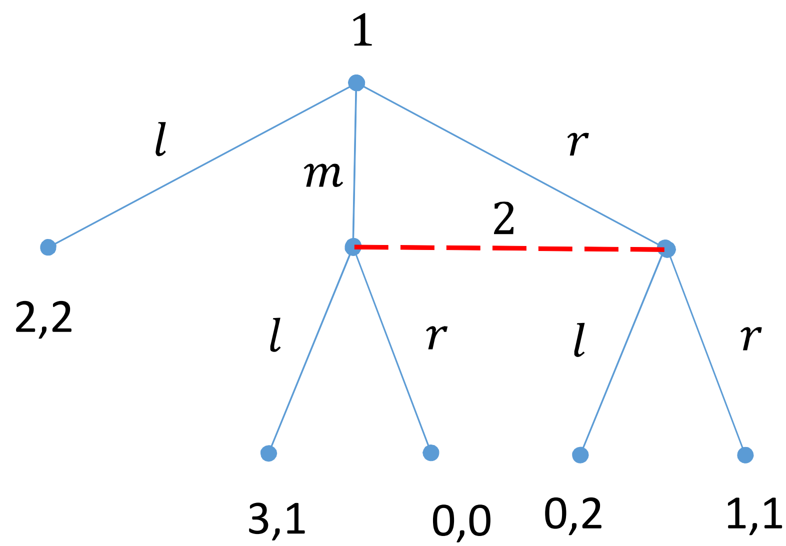
The problem

When we studied extensive games with perfect information, we found that dynamic structure helps refining equilibria.

In a **SPE**, a strategy profile is a **Nash equilibrium at any history** after which a player is asked to take an action.

The natural extension is to require that a strategy profile is a **Nash equilibrium at any information set** at which a player is asked to take an action.

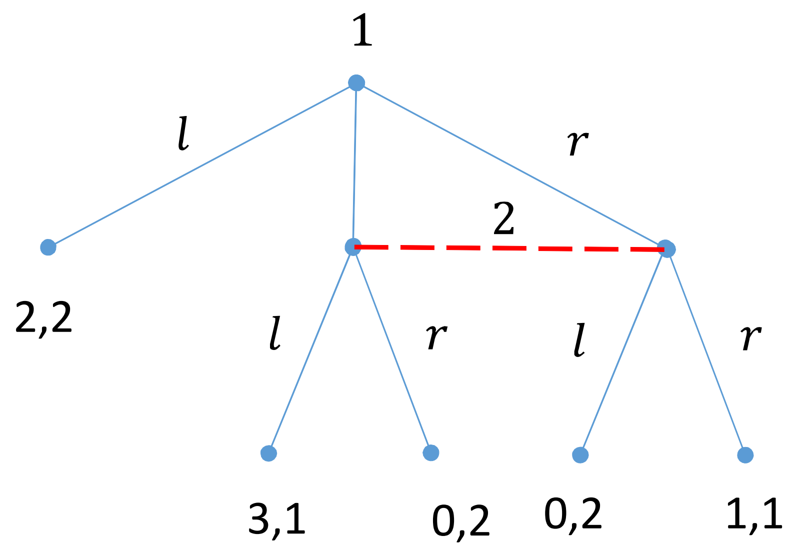
Sometimes this works with no complications. For example:



Here (l,r) is a Nash equilibrium, but it can be refined.

Indeed, 2's decision is suboptimal at I_i for any belief.

Consider now:



Here, 2 optimal action depends on her beliefs at I_2.

R is optimal if at least probability $1/2$ is on m; otherwise it is L.

Beliefs are not pinned down by strategies.

The key idea we will explore is to extend the definition of equilibrium beyond strategies, to *include* **beliefs** *as* ***integral part*** *of the equilibrium definition*.

We define an **assessment** to be a strategy profile and a belief system.

Two requirements:

- *Sequential rationality*: actions are optimal given beliefs.
- *Consistency* of beliefs with strategies and structure of the game.

Consistency require different desiderata:

- **Consistency with strategies**. Beliefs are obtained by Bayes' rule whenever possible.
- **Structural Consistency**. Even at information sets in which Bayes' rule has no bite, beliefs are derived by *some strategy profile* using Bayes' rule.
- **Common beliefs**. Players have the same beliefs after unexpected events.

Sequential equilibrium

The key concept is:

Definition. *An assessment in an extensive game is a pair* (β, μ), *where:*

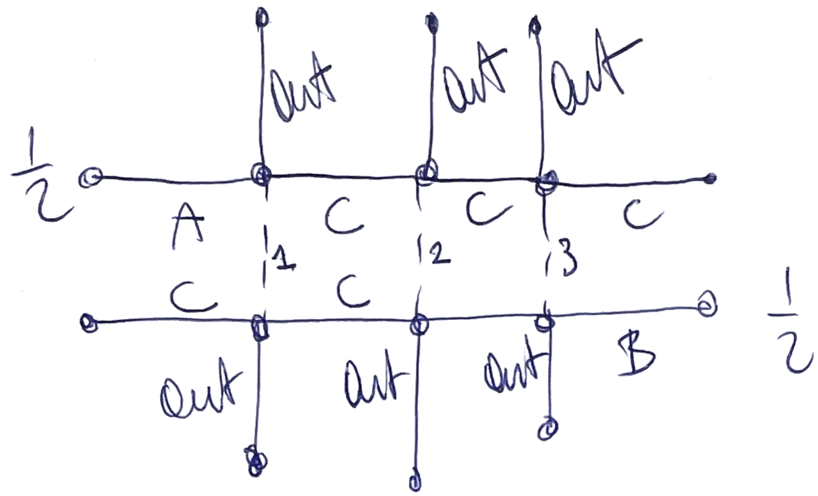
- *β is a profile of behavioral strategies;*
- *and μ is a function that assigns to every information set a probability measure on the set of histories in the information set.*

Given (β, μ) we can define a distribution on outcomes $O((\beta, \mu); I_i)$:

Definition. *The assessment* (β, μ) *is sequentially rational if for every player* $i \in N$ *and every information set* I_i *we have:*

$$O((\beta, \mu); I_i) \succeq_i O((\beta_i', \beta_{-i}, \mu); I_i)$$

Note that this definition may generate some paradoxical results.



If $\beta_1 = \beta_3 = out$, player's 2 information set is never reached.

Assume that $\mu_2(A, C; I_2) > 0$, $\mu_2(B, C; I_2) > 0$.

Then 2 evaluates the distribution over terminal histories with the posterior and the strategies.

Player 2 evaluate the future, i.e. $O(\beta, \mu; I)$, assuming that β_{-2} is used; but forms a belief assuming startegies different than β_{-2}.

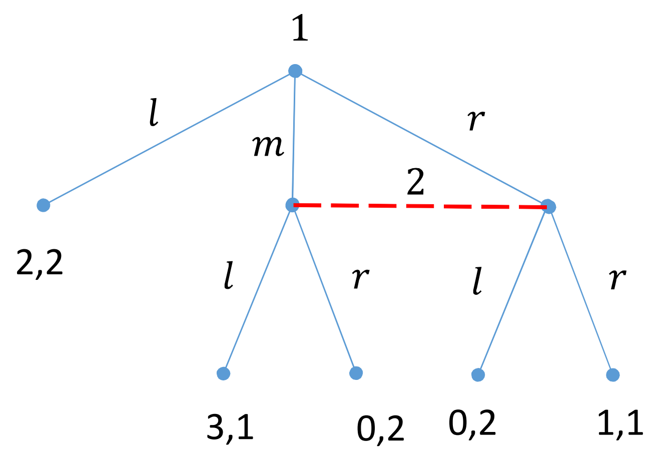
Definition. *An assessment* (β, μ) *is consistent if:*

- *there is a sequence* (β^n, μ^n) *of assessments that converges to* (β, μ) *in Euclidian space;*
- *each strategy profile* β^n *is completely mixed;*
- *each belief system* μ^n *is derived from* β^n *using Bayes' rule.*

The idea behind this requirement is that the probability of events conditional on zero-probability events must approximate probabilities that are *derived* from *strategies that assign positive probability to every action*.

Definition. *An assessment is a sequential equilibrium of a finite extensive game with perfect recall if it is sequentially rational and consistent.*

Example 1. Consider again:



An assessment in which $\beta_1(l) = 1$, $\beta_2(r) = 1$ and $\mu_2(\{m,r\})(m) = \alpha \geq 1/2$ is a sequential equilibrium.

Consider:

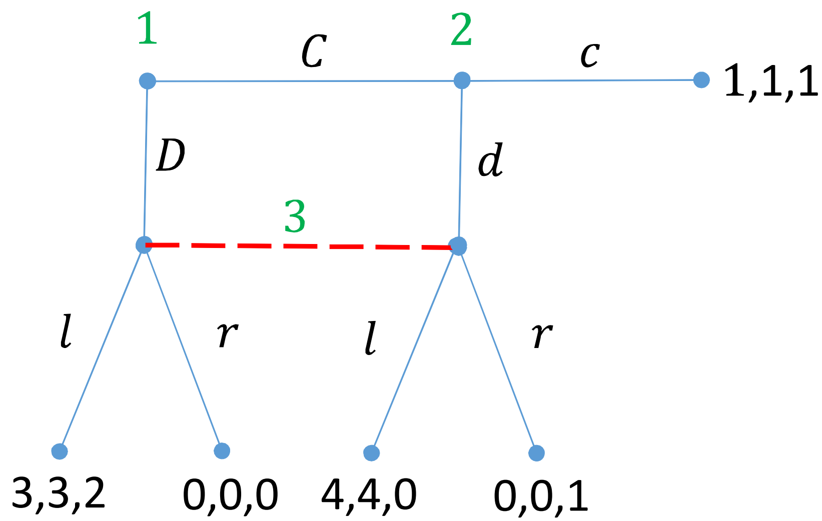
$$\beta_1^n = \left(1 - \frac{1}{n}, \frac{\alpha}{n}, (1-\alpha)\frac{1}{n}\right),$$

$$\beta_2^n = \left(1 - \frac{1}{n}, \frac{1}{n}\right)$$

and:

$$\mu(\{m,r\})(m) = \frac{\alpha/n}{\alpha/n + (1-\alpha)/n} = \alpha$$

Another example (Selten's horse):



Here we have two types of Nash equilibria:

- $\beta_1(\emptyset)(D) = 1,\ \frac{1}{3} \leq \beta_2(C)(c) \leq 1,\ \beta_3(I_3)(l) = 1$
- $\beta_1(\emptyset)(C) = 1,\ \beta_2(C)(c) = 1,\ \beta_3(I_3)(r) \geq 3/4.$

However, the first class is not sequentially rational:

- For any strategy $\beta_1(\emptyset)(D) < 1 \simeq 1$ $\beta_3(C)(l) \simeq 1$, the sequentially rational choice for 3 is l.

- But then 2 would choose d.

The second class is sequentially rational:

$$\beta_1^n(\emptyset)(C) = 1 - \frac{1}{n},$$

$$\beta_2^n(C) = 1 - \frac{2}{n},$$

$$\beta_3^n(I)(r) = \beta_3(I)(r) - \frac{1}{n}.$$

Perfect Bayesian equilibrium

In a large class of interesting games, we have more structure, that can be exploited to simplify the equilibrium analysis.

In Bayesian extensive games with observed actions, players observe all actions in the game.

All uncertainty is about the type of the players.

Definition. A Bayesian extensive game with observable actions is a tuple $\langle \Gamma, (\Theta_i), (p_i), (u_i) \rangle$, where:

- $\Gamma = \langle N, H, P \rangle$ is an extensive game form with perfect information and (potentially) simultaneous moves.
- Θ_i is the finite set of types with $\Theta = \times_i \Theta_i$.
- p_i is a probability measure on Θ_i, with $p_i(\theta_i) > 0$ being the probability that player i is of type θ_i, and p_i are assumed to be stochastically independent.
- $u_i : \Theta \times Z \to R$ is a von Neumann-Morgenstern utility function. Specifically we denote $u_i(\theta, h)$ to be the expected utility of i when the profile of types is θ and the termina history is h.

The definition above is closely connected to the definition of an extensive game with imperfect information (which is a generalization):

$$\langle N, H', P, f_c, (\Im_i), (\succeq_i)\rangle$$

Note that in the definition above, we introduce nature selecting types, at time 0.

To reconduct this game to the previous analysis, we can define an extensive game with $H' = \{\varnothing\} \cup \{\Theta \times H\}$ with $P(\varnothing) = c$.

And define the informations sets $I_i(\theta, h)$ to be:

$$I_i(\theta, h) := \{((\theta_i, \theta_{-i}), h) : \theta_{-i} \in \Theta_{-i}\}$$

for $i \in P(h) = P(I_i(\theta, h))$.

In this context an assessment is a pair:

$$\sigma = (\sigma_i(\theta_i)(h))_i, \mu = (\mu_i(h))_i,$$

where:

- $\sigma_i(\theta_i)(h)$ is a probability distribution over $A_i(h)$ if $i \in P(h)$.
- $\mu_i(h)$ is a probability distribution over Θ_i, conditioning on h (so for players $-i$).

Given σ, μ, we can define:

$$O(\sigma_{-i}, \sigma_i, \mu_{-i}; h)$$

to be the probability measure on the set of terminal histories Z of Γ, given that i plays σ_i, the other players play σ_{-i}, and the beliefs on the other players' types is μ_{-i}.

Definition. *For a Bayesian extensive game with observed actions, σ, μ is a Perfect Bayesian equilibrium if:*

- **Sequential rationality**.
  $O(\sigma_{-i}, \sigma_i(\theta_i), \mu_{-i}; h) \succeq_{\theta_i} O(\sigma_{-i}, s_i, \mu_{-i}; h)$ *for any other strategy of* $i \in P(h)$ *in* Γ.

- **Correct initial beliefs**: $\mu_i(\emptyset) = p_i$.

- **Action determined beliefs**. *If* $i \notin P(h)$, then $\mu_i(h,a) = \mu_i(h)$. If $i \in P(h)$, $a, a' \in A(h)$ with $a_i = a_i'$ then $\mu_i(h,a) = \mu_i(h,a')$.

- ...and...

- **Bayesian updating**. *If* $i \in P(h)$ *and* a_i *is in the support of* $\sigma_i(\theta_i)(h)$ *for some* θ_i *in the support of* $\mu_i(h)$, *then for any* $\theta_i' \in \Theta_i$:

$$\mu_i(h,a)(\theta_i') = \frac{\sigma_i(\theta_i')(h)(a_i) \cdot \mu_i(h)(\theta_i')}{\sum_{\theta_i \in \Theta_i} \sigma_i(\theta_i)(h)(a_i) \cdot \mu_i(h)(\theta_i)}.$$

An example

Consider a game in which nature first chooses one of two types of player 1: (t_1, t_2). Each type is chosen with equal probability.

Player 1 observes her type and decides whether to choose L or R.

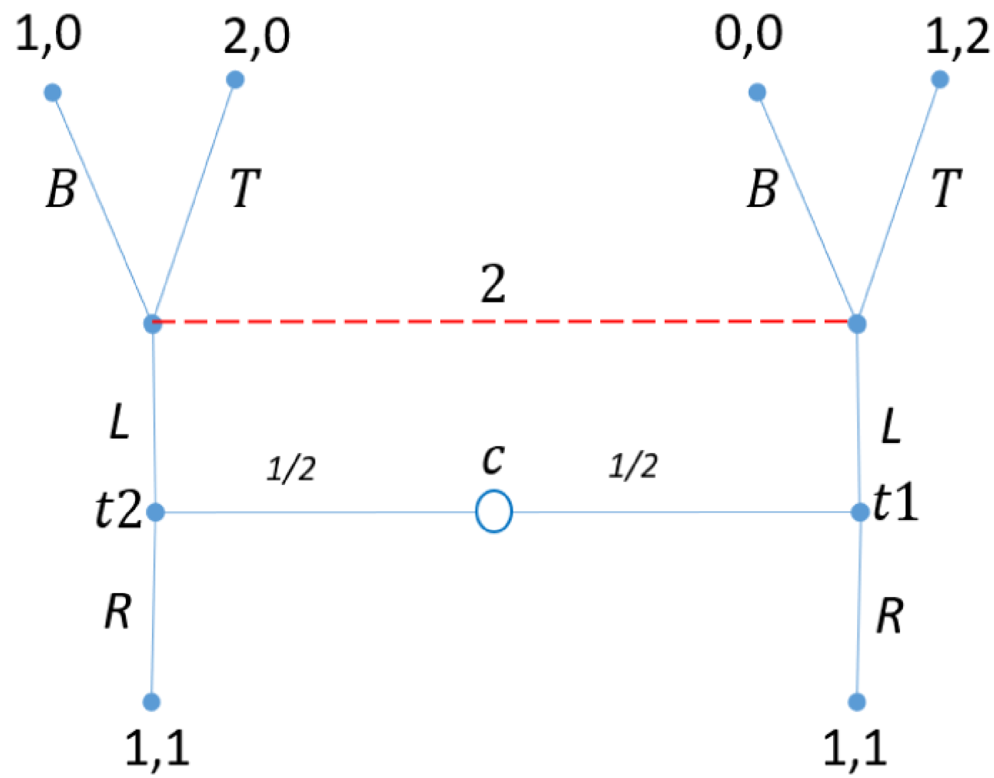
If player 1 chooses R, the game ends and the payoffs are $(1,1)$.

However, if player 1 chooses L, player 2 has to choose between T and B (without knowing the type of player 1).

The payoff after history $h = (t_1, LT), (t_1, LB)$ are, respectively $(1, 2)$ and $(0, 0)$.

The payoff after history $h = (t_2, LT), (t_2, LB)$ are, respectively $(2, 0)$ and $(1, 0)$.

Represent the game in extensive form in a labelled figure.



Find all Perfect Bayesian equilibria, in pure and mixed strategies. (Recall to specify the beliefs and explain how you determine them!).

Since both player’s actions are binary, it can be represented with the probability that each player puts on their first action. Let $\sigma_2 = \Pr(T)$ and $\sigma_1^t = \Pr(L), \forall t \in \{1, 2\}$.

We procede by explusion: 1. $\sigma_2 = 1$, 2. $\sigma_2 = 0$, and 3. $\sigma_2 \in (0, 1)$.

Step 1. Assume that player 2 is playing pure strategy T (i.e. so $\sigma_2 = 1$).

Note that this is optimal for player 2, no matter what beliefs she holds.

For player 1, any $\sigma_1^1 \in [0,1]$ and $\sigma_1^2 = 1$ is a best response to $\sigma_2 = 1$.

It follows that there exist a continuum of equilibiria on the following form:

$$\sigma_1 = (\sigma_1^1, \sigma_1^2) \ s.t. \ \sigma_1^1 \in [0,1] \text{ and } \sigma_1^2 = 1$$

$$\sigma_2 = 1 \text{ and } \mu = \frac{\sigma_1^1}{\sigma_1^1 + \sigma_1^2}$$

Step 2. Assume that player 2 is playing pure strategy B (i.e. assume that $\sigma_2 = 0$).

Note that this is an optimal strategy for player 2 if $\mu = 0$.

Then, note that the best response of player 1 to $\sigma_2 = 0$ is any strategy $\sigma_1^1 = 0$ and $\sigma_1^2 \in [0, 1]$.

If $\sigma_1^2 > 0$, beliefs are determined by Bayes' rule, and it must always be that $\mu = 0$.

If $\sigma_1^2 = 0$, we are "free" to assign beliefs.

In particular, for beliefs $\mu = 0$, the strategy-beliefs pair constitutes an equilibrium.

Hence, we have a continuum of equilibria on the following form:

$$\sigma_1 = (\sigma_1^1, \sigma_1^2) \ s.t. \ \sigma_1^2 \in [0,1] \text{ and } \sigma_1^1 = 0$$

$$\sigma_2 = 0 \text{ and } \mu = 0$$

Step 3. Assume that player 2 is playing a fully mixed strategy [i.e. assume that $\sigma_2 \in (0,1)$].

In any such equilibrium, it would have to be the case that type 1 plays R, while type 2 plays L.

Given this strategy of player 1, player 2 is indifferent between T and B, so any fully mixed strategy is in fact a best response.

Hence, we have a continuum of equilibria on the following form:

$$\sigma_1 = (\sigma_1^1, \sigma_1^2) \ s.t. \ \sigma_1^1 = 0 \text{ and } \sigma_1^2 = 1$$

$$\sigma_2 \in (0, 1) \text{ and } \mu = 0$$