Games and Nash

Strategic interactions

A strategic game is a model of interactive decision-making in which:

- Each decision-maker chooses a plan of action;
- choices are made simultaneously (or without knowledge of other players' plans);
- the plan is chosen to maximize the decision maker's utility, that depends on the action profile.

To study this type of situations, we need a formalism.

Definition. A strategic game consists of:

- A finite set N of players.
- for each player i a non-empty set of actions A_i. The set of actions is $A = \prod_{i \in N} A_i$.
- for each i, a utility function $u_i : A \to R$.

In short a game is a tuple: $\langle N, (A_i), (u_i) \rangle$.

This is an abstract description of a strategic interaction.

Alternatively, we could allow actions to affect consequences through some function $g : A \to C$ and define utilities over consequences.

For example firms may decide production plans $\mathbf{q} = (q_1, \ldots, q_n)$, then their utility may depend indirectly on quantities and directly on profits $\pi(\mathbf{q})$.

The consequences may depend on unknown variables, say a random variable ω in Ω.

It is easy to cover this case:

- either by allowing a consequence function that depends on ω: $g(a,\omega)$;
- or by introducing a fictitious player, Nature (more on this later).

With 2 (or 3) players we can represent the game in matrix form.

An example:

	L	R
T	a_1, a_2	b_1, b_2
B	c_1, c_2	d_1, d_2

Here:

- $N = \{r, c\}$
- $A_r = \{T, B\}, A_c = \{R, L\}$
- $u_r(T, L) = a_1, u_c(B, L) = c_2, \ldots$

Notes:

- Players do not need to choose actions simultaneously, what is important is that the players make decision independently without knowing the choice of the opponents.

- Rules of the game and preferences are common knowledge: Everybody knows; Everybody knows that everybody knows; Everybody knows that everybody knows that everybody knows; etc.

- Actions may be fairly complicated contingent plans: for example c may selects $\{R, L\}$; but r may be able to select a contingent plan: $A_r = \{RT, RB, LT, LB\}$. This is a contingent plan because it may depend on c's action.

- There is no contradiction with the previous point because the action in A_r is selected without knowing r's action.

Nash Equilibrium

A solution concept is a rule that assigns to each game $\langle N, A_i, u_i \rangle$ a prediction of an action profile.

Which actions will be played?

- Normative interpretation: How they game should be played.
- Positive interpretation: How they game is played

Definition. *A (pure) Nash Equilibrium of a strategic game* (N, A_i, u_i) *is a profile* $a^* = (a_1^*, \ldots, a_n^*) \in A$ *of actions such that for every* $i \in N$*:*

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*)$$

for all $a_i \in A_i$*.*

So in correspondence to a^*, no player has a strictly profitable deviation.

It is sometimes useful to define the best response correspondence:

$$B_i(a_{-i}) := \left\{a_i \in A_i : u_i(a_{-i}, a_i) \geq u_i(a_{-i}, a_i') \text{ for all } a_i' \in A_i\right\}$$

Given this a Nash equilibrium is given by a fixed point of B:

$$a_i^* \in B_i(a_{-i}^*)$$

Examples

Prisoner's Dilemma

	Don't Confess	Confess
Don't Confess	$\mathbf{3,3}$	$\mathbf{0,5}$
Confess	$\mathbf{5,0}$	$\mathbf{1,1}$

The game has a unique equilibrium: C, C.

Dove-Hawk

	Dove	Hawk
Dove	$3, 3$	$1, 4$
Hawk	$4, 1$	$0, 0$

In this game, two animals may behave aggressively, or cooperatively.

The game has two equilibria D, H and H, D.

Matching Pennies

	Head	Tail
Head	$1, -1$	$-1, 1$
Tail	$-1, 1$	$1, -1$

The game has no (pure) Nash equilibria.

Cournot Competition

This is more than an application, since Cournot stated his model before a Nash equilibrium was formalized as a general tool.

Two firms, 1 and 2, simultaneously choose output levels $q_i \in [0,\infty)$

The price is $p(q_1,q_2)$, assumed differentiable.

Profit is:

$$u_i(q_1,q_2) = q_i \bullet p(q_1,q_2) - c_i(q_i)$$

To find a Nash equilibrium, let's find the reaction functions: $r_i(q_{-i})$:

$$p(q_{-i} + r_i(q_{-i})) \;\; + p'(q_{-i} + r_i(q_{-i}))r_i(q_{-i}) - c_i'(r_i(q_{-i})) \;\; = \; 0$$

If we assume linear costs and demand:

$$c_i(q) \; = \; cq$$

$$p(q_1, q_2) \; = \; \max(0, 1 - q_1 - q_2)$$

Then in an interior solution we have:

$$r_i(q_{-i}) = \frac{1 - q_{-i} - c}{2}$$

A Nash equilibrium is given by the system:

$$r_1(q_2) = \frac{1 - q_2 - c}{2}, r_2(q_1) = \frac{1 - q_1 - c}{2}$$

So $q_1 = q_2 = (1 - c)/3$.

Mixed strategies

Given a profile a_{-i}, a best response for i is a set $B_i(a_{-i})$.

For a Nash equilibrium a^*, we require i to choose some $a_i^* \in B_i(a_{-i}^*)$.

But why?

For i it would be equally rational to select any distribution with positive probability in $B_i(a_{-i}^*)$.

Why should we allow this? and what are its implications?

Let us denote by $\Delta(A_i)$ the set of probability distributions over A_i.

An element of $\alpha_i \in \Delta(A_i)$ is denoted a mixed strategy of payer i.

A degenerate element of $\Delta(A_i)$ that puts probability 1 on an element is called a pure strategy of i.

The expected utility of a player given a profile $\alpha = (\alpha_1, \ldots, \alpha_n)$ is:

$$U_i(\alpha) = \sum_{a \in A} \left[\prod_{j \in N} \alpha_j(a_j) \right] u_i(a)$$

Given a game $\langle N, (A_i), (u_i) \rangle$, an *mixed extension* is a game $\langle N, (\Delta(A_i)), (U_i) \rangle$.

Definition. *A mixed strategy Nash equilibrium of a game* $\langle N, (A_i), (u_i) \rangle$ *is a Nash equilibrium of its mixed extension.*

Existence

Why introducing mixed equilibria?

Recall that a pure strategy Nash Equilibrium may fail to exist.

It turns out that a mixed equilibrium always exists under plausible assumptions.

To prove this result, we use a theorem called Kakutani's fixed point theorem.

Theorem. *Let X be a compact, convex subset of R^n and let $f : X \to X$ be a correspondence for which:*

- *for all $x \in X$ the set $f(x)$ is non empty and convex*
- *The graph of f is closed: for all sequences $\{x_n\}$, $\{y_n\}$ for which $y_n \in f(x_n)$ for all n, $x_n \to x$, and $y_n \to y$, we have $y \in f(x)$.*

Then there exists a x^* such that $x^* \in f(x^*)$.

Such a point is called a fixed point of f.

Why is this useful?

Recall that a definition of a Nash equilibrium a^* is that $a_i^* \in B_i(a_{-i}^*)$, or in vector form:

$$a^* = \begin{pmatrix} a_1^* \\ \dots \\ a_n^* \end{pmatrix} \in \begin{pmatrix} B_1(a_{-1}^*) \\ \dots \\ B_1(a_{-n}^*) \end{pmatrix} = B(a^*)$$

So we only need to check conditions under which Kakutani's theorem is applicable to B.

Let us start from a preliminary result.

Theorem. *A strategic game* $\langle N,(A_i),(u_i)\rangle$ *has a Nash equilibrium if for all* i*:*

- *the set* A_i *of actions for i is a non empty, compact, convex subset of a Euclidean space;*
- *The utility* u_i *is continuous and quasi concave on* A_i*.*

Proof. We proceed in steps:

- The set $B_i(a_{-i})$ is non empty since the expected utility is continuous and A_i is non-empty and compact.
- The set $B_i(a_{-i})$ is convex since u_i is quasi concave;
- B has a closed graph since (u_i) are continuous.

This is not yet our theorem.

We need to prove that any finite game $\langle N, (A_i), (u_i) \rangle$ has a mixed extension $\langle N, (\Delta(A_i)), (U_i) \rangle$ that satisfies the requirement of the existence result we have proven.

Theorem. *Every finite strategic game has a mixed strategy Nash Equilibrium.*

Proof. The set of mixed strategies is the set of real vectors $p = (p_1, \ldots, p_{m_i})$, with $m_i = |A_i|$ and $p_i \geq 0$, $\sum p_i = 1$.

This set is non-empty, convex and compact.

U_i are linear functions of α_i, so quasi concave and continuos.

The mixed extension satisfies the conditions for existence.

Do we need the assumptions?

Bayesian Games

We are often interested in interactions in which there may be some uncertainty about the characteristics of the other players (or the state of the nature).

To this goal we model the players' uncertainty introducing a set Ω of states of the nature.

States of Nature are a description of the player's relevant characteristics.

This leads to the definition of a Bayesian Game.

Definition. *A Bayesian game consists of:*

- *A finite set N of players.*
- *A finite set of state of nature Ω (for simplicity here).*
- *For each $i \in N$:*
  - *A set A_i of actions;*
  - *A finite set of types T_i and a signal function $\tau_i : \Omega \to T_i$;*
  - *A probability measure p_i over Ω with $p_i(\tau_i^{-1}(t_i)) > 0$ for all $t_i \in T_i$ (the prior belief).*
  - *A preference relation $\succeq_i$ over $A \times \Omega$.*

Note that a Bayesian game is a tuple:

$$\langle N, \Omega, (A_i), (T_i), (\tau_i), (p_i)(u_i) \rangle$$

Often a Bayesian game is not defined in terms of Ω and the *signal function* τ_i, but directly in terms of types,

Sometimes it is described in terms of Ω and a signal structure expressed as a conditional distribution over types T_i.

In the definition, we allow for heterogeneous priors on Ω, often it is assumed that there is a common prior.

We define a Nash eq. of a Bayesian game in terms of the Nash eq. of an associated strategic game.

Definition. *A Nash equilibrium of a Bayesian game is the Nash eq. of a strategic game defined as follows:*

- *The set of players is the set of pairs* (i, t_i) *for each* $i \in N$ *and* $t_i \in T_i$.
- *The set of actions of player* (i, t_i) *is* A_i.
- *The preferences of* $\succsim_{(i,t_i)}$ *:*

$$a^* \succsim_{(i,t_i)} b^* \Leftrightarrow L_i(a^*, t_i) \succsim_i L_i(b^*, t_i)$$

where $L_i(a, t_i)$ *is a lottery over* $A \times \Omega$ *that assigns probability* $\frac{p_i(\omega)}{p_i(\tau_i^{-1}(t_i))}$ *to* $\left((a^*(j, \tau_j(\omega)))_{j \in N}, \omega\right)$ *if* $\omega \in \tau_i^{-1}(t_i)$ *and* 0 *otherwise.*

An example: the volunteers' dilemma

$N = \{1, \ldots, n\}$

$T_i = c_i = [0, 1]$

Types are independent, $c_i \sim F(\bullet)$

So we have: $F_{-i}(c_{-i}) = \prod_{l \neq i} F(c_l)$

And preferences are:

$$U_i(a) = \begin{cases} v - c & \text{if } i \text{ volunteers} \\ v & \text{if } i \text{ does not volunteer, but someone else does} \\ 0 & \text{no player volunteers} \end{cases}$$

Relationship with our definition?

Ω is $[0,1]^n$

$\tau_i : \mathbf{c} \to c_i$

p_{-i} is the density associated to $F_{-i}(c_{-i})$

Let us compute the utility of choosing to volunteer (V) or not (NV):

$$EU_i(a;V) = [1 - P_{-i}]v + P_{-i}v - c$$

$$EU_i(a;NV) = P_{-i}v$$

So i volunteers if:

$$c \leq [1 - P_{-i}]v = c^*$$

This implies that an agent volunteers with probability:
$\sigma = F(c^*)$

And we have:

$$1 - P_{-i} = [1 - F(c^*)]^{n-1}$$

We can therefore characterize the equilibrium with:

$$c^* = [1 - F(c^*)]^{n-1}$$

Bayesian games can also describe situations in which uncertainty is about what other players know.

Consider a game with $N = \{1, 2\}$ and 3 states $\omega_1, \omega_2, \omega_3$.

For 1: $\tau_1(\omega_1) = \tau_1(\omega_2) = t_1'$, $\tau_1(\omega_3) = t_1''$.

$(b, \omega_j) \succ_1 (c, \omega_j)$ for $j = 1, 2$ but $(c, \omega_3) \succ_1 (b, \omega_3)$

For 2: $\tau_1(\omega_1) = t_2'$, $\tau_1(\omega_2) = \tau_1(\omega_3) = t_2''$

Here in state ω_1, 2 knows that 1 strictly prefers b to c.

In state ω_2, however:

- 2 doesn't distinguish between ω_2 and ω_3,
- so 2 does not know whether $(b, \omega_j) \succ_1 (c, \omega_j)$ or $(c, \omega_3) \succ_1 (b, \omega_3)$.

However in state ω_1, 1 does not know if 2 knows this fact, since 1 cannot distinguish between ω_1 and ω_2