Midterm
ECON 6140, Spring 2020

March 13

This is your midterm exam. You have 1 hour and 15 minutes (75 minutes) to answer these questions. The "relative prices" can be found next to each question for a total of 70 points and 10 "extra" bonus points. **If you get stuck at some point in the exam, keep moving and come back to those questions at the end.**
Read the instructions carefully before writing anything. **If you need to add any additional assumptions, please state them clearly.**

Workers' Health, Consumption and Growth

Time is continuous and indexed by t. There is a single good produced in the economy combining capital, $K(t)$, and labor, $L(t)$, through a constant returns to scale technology F. Labor efficiency is indexed by $Z(t)$ and grows exogenously at rate $\gamma_z > 0$.
The total population of the economy is denoted by $N(t)$ and population growth is denoted by $\gamma_n > 0$. The initial population of the economy is $N(0) > 0$. At each point in time, a constant fraction of the population, π, gets sick and **cannot** work.
Capital is accumulated with a linear technology that transforms consumption goods into capital as follows,

$$\dot{K}(t) = X(t) - \delta K(t)$$

The initial level of capital, K_0, is given.
At each time t, there is a continuum of measure N(t) of identical households with preferences,

$$\int_{t=0}^{\infty} \exp^{-\rho t} \frac{c(t)^{1-\sigma}}{1-\sigma}$$

1. (5pt) Describe the planner's problem associated to this economy.

   The planner solves,

$$\int_{t=0}^{\infty} \exp^{-\rho t} \frac{C(t)^{1-\sigma}}{1-\sigma}$$

subject to

$$C(t) + X(t) \leq F(K(t), Z(t)L(t))$$

$$\dot{K}(t) = X(t) - \delta K(t) \qquad L(t) = (1-\pi)N(t)$$

$$\dot{N}(t) = \gamma_n N(t) \qquad \dot{Z}(t) = \gamma_z Z(t)$$

where aggregate consumption is $C(t) = N(t)c(t)$

2. (10pt) Define a Balanced Growth Path and characterize the key dynamic equations describing the equilibrium growth rate of consumption, γ_c and capital γ_K.

   A Balanced growth path (BGP) is an allocation such that the growth rate of consumption, output, investment and capital is constant and possibly different among them.

   We can write the Hamiltonian associated to the planner's problem described above. Let $\lambda(t)$ be the co-state variable of the problem and compute the sufficient conditions for an optimum as follows

$$\lambda(t) = C(t)^{-\sigma}$$

$$\frac{\dot{C}(t)}{C(t)} = \frac{1}{\sigma}[F_1(K(t), Z(t)L(t)) - \delta - \rho] \tag{1}$$

$$C(t) + \dot{K}(t) + \delta K(t) = F(K(t), Z(t)L(t)) \tag{2}$$

$$\lim_{t\to\infty} \exp^{-\rho t} \lambda t K(t) = 0 \tag{TVC}$$

   The above three equations characterize the equilibrium growth rate of consumption (equation 1) and the growth rate of capital (equation 1). Using the assumption of CRS of F, we can rewrite equation 1 as

$$\gamma_C = \frac{\dot{C}(t)}{C(t)} = \frac{1}{\sigma}[F_1(1, \frac{Z(t)L(t)}{K(t)}) - \delta - \rho]$$

   So for a BGP to exists (γ_C constant) we need that the input ratio is constant, $\frac{Z(t)L(t)}{K(t)}$. Hence, the growth rate of capital

$$\gamma_K = \gamma_n + \gamma_z.$$

   Dividing through equation 2 by the stock of capital,

$$\frac{C(t)}{K(t)} + \gamma_K + \delta = F(1, \frac{Z(t)L(t)}{K(t)})$$

Hence, consumption needs to grow at the same rate as capital along the BGP. Finally, using the constant returns to scale assumption for output, we obtain that output also needs to grow at the same rate as capital, since $Y = F(1, \frac{Z(t)L(t)}{K(t)})K(t)$ and the first term in the LHS is a constant.

From now on, assume that the production technology is Cobb-Douglas,

$$Y(t) = K(t)^{\alpha}(Z(t)L(t))^{1-\alpha}$$

3. (10pt) Detrend the economy and characterize the steady state level of capital per worker (in efficiency units, $\frac{K(t)}{Z(t)L(t)}$).

   Explain how the level of capital per worker (in efficiency units) depends on (i) the population growth rate, and (ii) the probability of being sick.

   I redefine all variables in efficiency units of labor, $\zeta(t) = \frac{C(t)}{Z(t)L(t)}$, $\kappa(t) = \frac{K(t)}{Z(t)L(t)}$ and $\iota(t) = \frac{Y(t)}{Z(t)L(t)}$.

   You can solve for the optimality conditions of the problem again (or rewrite the stationary counterpart of 1 and 2 as follows,

$$\frac{\dot{\zeta}(t)}{\zeta(t)} = \frac{1}{\sigma}[\alpha(\kappa(t))^{\alpha-1} - \delta - \rho - \gamma_n - \gamma_z] \tag{3}$$

$$\zeta(t) + \dot{\kappa}(t) + (\delta + \gamma_n + \gamma_z)\kappa(t) = (\kappa(t))^{\alpha} \tag{4}$$

$$\lim_{t\to\infty} \exp^{-\rho t} \zeta(t)^{-\sigma}\kappa(t) = 0 \tag{TVC}$$

   The steady state capital per worker κ can be solved for using 3 when $\frac{\dot{\zeta}(t)}{\zeta(t)} = 0$ so that

$$\kappa^{\star} = \left(\frac{\alpha}{\delta + \rho + \gamma_n + \gamma_z}\right)^{\frac{1}{1-\alpha}}$$

   Hence, (i) s.s. capital per worker decreases in population growth and (ii) it is independent of the probability of being sick.

4. (10pt) Assume that the economy is transiting a BGP and unexpectedly the probability of being sick increases permanently, $\pi' > \pi$. Describe the qualitative features of

the transition dynamics to the new BGP (or the new steady state of the detrended economy). Particularly, describe the time path for aggregate consumption and capital.

As described before, nothing happens to the steady state of the economy (in efficiency units). But the **level** of capital is decreasing in the probability of being sick. This results follows from the fact that the RHS of the above condition is constant and independent of π and the definition of κ. Higher probability of being sick implies less workers, so that level of capital that can be sustained in steady state is lower.

The shock is analogous to moving the economy to a point in the saddle path where the level of capital in efficiency units is higher than its steady state. The optimal transition of the economy will generate decreasing levels of capital in efficiency units (and in levels), and decreasing levels of consumption (because the marginal utility of consumption is increasing as the economy approached the steady state from the "right".

5. (5pt) What happens to long term growth?

   Long term growth in this economy is exogenous, and nothing happens to it.

   Now let us decentralize the economy described above. Assume there is a representative firm that produces output using the technology described before, F to maximize profits taking the price of capital, $r(t)$ and labor, $w(t)$ as given.

   Assume that households maximize the utility and their preferences are as described above. Assume that households are endowed with a unit of time that they can supply for work. Households are heterogeneous in the probability of being sick π_i. Assume that the distribution of these probabilities is such that the average probability of being sick coincides with π, the probability we used when solving the planner's problem. Finally, assume that households can accumulate assets and that at the beginning of their life they are endowed with equal amount of assets, $a_{0i} = \frac{K(0)}{N(0)}$.

6. (5pt) Describe the problem of the households.

   The households solve the following problem,

$$E_t \int_{t=0}^{\infty} \exp^{-\rho t} \frac{c(t)^{1-\sigma}}{1-\sigma}$$

subject to

$$c(t) + \dot{a}(t) = w(t) + r(t)a(t) \qquad \text{with probability } (1-\pi_i)$$

$$c(t) + \dot{a}(t) = r(t)a(t) \qquad \text{with probability } (\pi_i)$$

$$\lim_{t\to\infty} \exp^{-\rho t} \lambda(t)a(t) = 0$$

with $a(0)$ given.

Note that when markets are complete, the allocation for the household is the same as if it was solving (without uncertainty)

$$\int_{t=0}^{\infty} \exp^{-\rho t} \frac{c(t)^{1-\sigma}}{1-\sigma}$$

subject to

$$c(t) + \dot{a}(t) = w(t)(1-\pi_i) + r(t)a(t)$$

$$\lim_{t\to\infty} \exp^{-\rho t} \lambda(t)a(t) = 0$$

with $a(0)$ given.

7. (5pt) Describe the market clearing conditions of the economy.

$$N(t)c_i(t) + \dot{K}(t) + \delta K(t) = F(K(t), Z(t)L(t))$$

$$L(t) = (1-\pi)N(t) = \int_1^{N(t)} (1-\pi_i)di$$

$$\int_1^{N(t)} a_{it}di = K(t)$$

8. (10pt) Explain what is the difference between a No-Ponzi Scheme Condition and a Transversality Condition.

   A No Ponzi Scheme Condition is a restriction on the problem that prevents agents from borrowing infinite amounts and making the economic problem trivial. The TVC is an optimality condition of the problem. That is, every infinite horizon problem has a TVC condition associated to the optimal path. The No Ponzi Scheme condition is not necessary and can be replaced with minimum borrowing constraints for example.

9. (10pt) Show that the dynamics of the aggregate economy is the same as the one characterized before. Explain what feature of the problem grants this result.

   The dynamics of the economy is the same because the only difference across households stems from their labor income. But the mean of the shock is the same in the decentralized economy than in the planner's allocation for the feasibility constraints of the problem (in the aggregate) are identical to the ones before. Given that preferences are identical, the **unique** solution to the problem has to be the one characterized before. The formal proofs follows the class notes replacing the labor income e_i for the probability of working $(1-\pi_i)$.

10. (**EXTRA**: 10pt) Assume the economy is transiting its BGP and there is a shock such that households are not allowed to borrow any more in the future, $a_{ti} \geq 0$. Go as far as you can showing what would happen with the consumption paths of households relative to the economy where borrowing is allowed. What happens with consumption in this economy as the probability of being sick increases?

    The consumption paths will be permanently lower without borrowing. Households will need to save to self finance periods where they cannot work because they are sick. The level of consumption is lower than the efficient level.

    When the probability of being sick increases consumption for all these households falls. Asset poor households are particularly worse off (in utility terms) because they are consuming relatively low levels (where the curvature of the utility function is higher, and therefore fluctuations in consumption are more costly).