Notable dynamic models

Rubinstein's bargaining model

In Rubinstein (1982), two player must split a pie of size 1.

They alternate in making offers.

In period $t = 0, 2, 4, \ldots$:

- Player 1 proposes a sharing $(x, 1 - x)$, where x is the share allocated to 1.
- If player 2 accepts, the payoffs are $x, 1 - x$.

- If player 2 rejects, then we move to $t + 1$ and payoffs are discounted by $\delta < 1$.

In period $t = 1, 3, 5, \ldots$:

- Player 2 proposes a sharing $(x, 1 - x)$, where x is the share allocated to 1.
- If player 1 accepts, the payoffs are $x, 1 - x$
- If player 1 rejects, then we move to $t + 1$ and payoffs are discounted by $\delta < 1$.

Note that here we have periods ($t = 1, 2, 3$) and stages $t = 1.1, 1.2$ etc.

Rubinstein's model is a little more general, but the essence of the model is here.

It is relatively easy to see that this game has a plethora of Nash equilibria.

For example, for any $x \in [0,1]$, we have a Nash equilibrium supporting the immediate allocation $(x, 1-x)$.

For this equilibrium consider the strategies:

- 1 always demands 1 and refuses anything less;
- 2 demand 0 and accepts anything.

Given 2’s strategy, it is optimal to ask 1 by 1; similarly 2’s strategy is optimal since s7he does note expect anything in the following periods.

This Nash equilibrium, however, is not SPE.

If player 2 rejects the offer, s/he gets a chance to make an offer.

2 may then offer $x \in (\delta, 1)$.

It is rational for 1 to accept, since:

$$u_1(x) = x > \delta = \delta u_1(1)$$

Note that in no subgame equilibrium, 1 can get more than $u_1(1)$ at $t+1$.

So what can be a SPE? Is the game indeterminate, or we have a precise prediction?

Here is a SPE:

- Player i demands a share:

$$x_i^i = \frac{1-\delta_j}{1-\delta_i\delta_j}$$

when s/he makes an offer.

- Player i demands:

$$x_i^j = \frac{\delta_i(1-\delta_j)}{1-\delta_i\delta_j}$$

when s/he does not make an offer

To check that this is a SPE, let us use the one-step deviation principle.

Can i profitably deviate when proposing?

Any proposal must satisfy:

$$1-\widetilde{x}_i^i \geq x_j^i = \frac{\delta_j(1-\delta_i)}{1-\delta_i\delta_j} = 1-\frac{1-\delta_j}{1-\delta_i\delta_j} = 1-x_i^i$$

$$\Leftrightarrow \widetilde{x}_i^i \leq x_i^i$$

Since i knows that x_i^i is accepted, it must also be that $\widetilde{x}_i^i \geq x_i^i$.

Similarly, 2 refuses if:

$$\widetilde{x}_j^i < \delta_j x_j^j = \frac{\delta_j(1-\delta_i)}{1-\delta_i\delta_j} = x_j^i$$

and is willing to accept otherwise.

Are there other SPE?

The key result in Rubinstein (1982) is that this is the unique SPE.

To see this lets us define $\bar{v}_i$ and $\underline{v}_i$ to be player i's supremum and infimum payoffs in the set of possible payoffs in a SPE.

We must have:

$$\underline{v}_1 \geq 1 - \delta_2 \bar{v}_2$$

Since 2 would always accept anything larger than $\delta_2 \bar{v}_2$.

Similarly, we must have $\underline{v}_2 \geq 1 - \delta_1 \bar{v}_1$.

Moreover, we must have:

$$\bar{v}_1 \leq \max\left\{1 - \delta_2 \underline{v}_2, (\delta_1)^2 \bar{v}_1\right\}$$

The first inequality $\bar{v}_1 \leq 1 - \delta_2 \underline{v}_2$ means that 2 rejects anything that gives her less than $\delta_2 \underline{v}_2$, implying $1 - x_1^1 \geq \delta_2 \underline{v}_2$.

the second follows from the fact that 1 can go for rejected offer and wait one turn.

But:

$$\max\left\{1-\delta_2\underline{v}_2,(\delta_1)^2\bar{v}_1\right\} = 1-\delta_2\underline{v}_2$$

Since else we would have $1-\delta_2\underline{v}_2 < (\delta_1)^2\bar{v}_1$, so $\bar{v}_1 < (\delta_1)^2\bar{v}_1$, a contradiction.

So

$$\bar{v}_i \le 1-\delta_j\underline{v}_j$$

Combining the inequalities we have:

$$\underline{v}_i \geq 1 - \delta_j \bar{v}_j \geq 1 - \delta_j(1 - \delta_i \underline{v}_i)$$

$$\Leftrightarrow \underline{v}_i \geq \frac{1 - \delta_j}{1 - \delta_i \delta_j}$$

Similarly, we have:

$$\bar{v}_i \leq 1 - \delta_j \underline{v}_j \leq 1 - \delta_j (1 - \delta_i \bar{v}_i)$$

$$\Leftrightarrow \bar{v}_i \leq \frac{1 - \delta_j}{1 - \delta_i \delta_j}$$

We conclude that:

$$v_i = \underline{v}_j = \bar{v}_i = \frac{1 - \delta_j}{1 - \delta_i \delta_j}$$

This says that the values are uniquely defined.

It is easy to see that this implies that the strategies must also be uniquely defined.

Given v_j, i must offer $x_j^i \geq 1 - \delta_j v_j$

If $x_j^i = 1 - \delta_j v_j$, j is willing to accept, but can we have an equilibrium in which j randomizes?

No, in this case no offer for i would be optimal and we can't have a SPE.

So $x_j^i = 1 - \delta_j v_j$ and j accepts with probability 1 in all equilibria.

What determines the share of the pie in the asymmetric case?

Let us assume $\delta_i = \exp(-r_i \Delta)$.

For $\Delta \to 0$, we have $\delta_i \to 1 - r_i \Delta$, implying:

$$v_i = \frac{r_j}{r_i + r_j}$$

An interesting case is the symmetric case with $\delta_j = \delta_i = \delta$.

In this case, we have

$$v_i = v_j = \frac{1-\delta}{1-\delta^2} = \frac{1}{1+\delta} > \frac{1}{2}$$

This implies that proposer gets more than 50%

However as $\delta \to 1$, $v_i \to v_j \to \frac{1}{2}$.

Repeated games: **a first look**

Often a game is played repeatedly over time.

In this case:

- the game that is played repeatedly is called the *stage game*;
- and the overall game is called a *repeated game*.

Even when this is done in finite horizons or the game has a unique equilibrium, this may lead to a larger set of equilibria.

Repetitions allow the players to condition their actions on the actions taken by the players in previous periods.

Even if the past actions are payoff irrelevant (i.e. they do not directly affect the payoffs, conditioning on past actions makes the strategies "interactive" and thus more powerful.

Equilibria may be associated to payoffs that are higher or lower for all players that the payoff in the unique equilibrium of the stage game.

The repeated prisoner's dilemma

Consider this stage game:

	Cooperate	**Defect**
Cooperate	1.1	$-1, 2$
Defect	$2, -1$	$0, 0$

It is easy to see that defect is the unique equilibrium.

Consider the repeated version of this game in which the strategy is a function of the sequence of past actions a^t: $\sigma_i(a^t)$.

In this case, if the game is repeated for T periods, we can write the payoff as:

$$U_i = \frac{1-\delta}{1-\delta^{T+1}} \sum_{t=0}^{T} u_i(\sigma(a^t))$$

where the first term allows to express the payoff as "average discounted payoffs".

As $T \to \infty$, we have: $U_i = (1-\delta)\sum_{t=0}^{\infty} u_i(\sigma(a^t))$.

We claim that in this game, as $T \to \infty$ when $\delta \geq 1/2$ there is a subgame perfect equilibrium in which cooperation is chosen in all periods.

Consider the following "grimm trigger" strategies:

- **State I**: Play C in every periods unless some player plays D. If some player plays D, then move to State II.

- **State II**. Play D forever.

To check that this is a subgame perfect equilibrium, we use the "one-stage deviation principle".

There are two cases to consider:

- Suppose we are in state I. Then payoffs are:
  - Play C: $(1-\delta)\sum_{t=0}^{\infty}\delta^t = 1$
  - Play D: $(1-\delta)[2+\delta \bullet 0+\delta^2 \bullet 0+\ldots] = 2(1-\delta)$

  It follows that C is weakly preferred to D if $\delta \geq 1/2$

- Suppose we are in State II. Note that D is a Nash equilibrium of the stage game and strategies are not interactive anymore (i.e. they never look at past actions anymore). So D is a subgame perfect equilibrium.

Note that $(0,0)$ is the minimax of the original game.

Now, we can obtain a payoff that is strictly higher 1,1.

A general result that we will prove is the *folk theorem*: any feasible payoff above the minimax payoff is the payoff of some subgame perfect equilibrium for a sufficiently large δ.

Carrot-stick strategies

In repeated games, the payers may end up in SPE in which they obtain payoffs lower than the payoff in the unique equilibrium of the stage game.

Consider this game:

	A	B	C
A	2,2	2, 1	0,0
B	1,2	1,1	-1,0
C	0, 0	0, −1	-1,-1

This game has unique equilibrium (A, A) with payoff $(2, 2)$.

Consider this subgame perfect equilibrium:

- **State I**: Play *B* in every periods unless some player plays something else ("deviates"). If some player deviates, then move to State II.

- **State II**. Play *C*. If all plays play *C*, return to State I. If some players does not play *C*, stay in State I.

To check this is a SPE, again there are two cases:

- Suppose we are in State I. Then payoffs are:
  - Play B: $(1-\delta)\sum_{t=0}^{\infty}\delta^t = 1$
  - Best deviation A:
    $(1-\delta)[2+\delta \bullet -1+\delta^2+\delta^3+\ldots] = 1+(1-\delta)(1-2\delta)$

  It follows that B is weakly preferred to A (an thus anything else) if $\delta \geq 1/2$

- Suppose we are in State II. Then payoffs are:
  - Play C: $(1-\delta)[-1+\delta+\delta^2+\delta^3+\ldots] = 1-2(1-\delta)$
  - Best deviation A:
    $(1-\delta)[0+\delta\bullet -1+\delta^2+\delta^3+\ldots] = 1+(1-\delta)(1+2\delta)$

  so C is weakly preferred to A (an thus anything else) if again $\delta \geq 1/2$

Finitely repeated games

In the first example (repeated prisoner’s dilemma), a finitely repeated version has the same equilibria as a static, because the stage game has a unique equilibrium.

If the stage game had multiple equilibria, then the repeated game would also have multiple equilibria.

But the interesting observation is that repetition allow to leverage on multiplicity to generate even more equilibria.

Consider this game:

	A	B	C
A	0,0	3,4	6,0
B	4,3	0,0	0,0
C	0,6	0,0	5,5

This game has 2 pure equilibria: $(B,A),(A,B)$

And a mixed eq.: $3/7\ A$, $4/7\ B$ and $4/7\ A$, $3/7\ B$.

Note that all payoffs fall short of the maximal: $(5, 5)$.

In the twice repeated version we have an equilibrium in which the maximal payoff $(5, 5)$ is achieved.

The strategies are:

- Play (C, C) at $t = 1$.
- If at $t = 1$ (C, C) is played, then play (B, A) at $t = 2$, else play the mixed equilibrium.

In equilibrium the payoffs are $(5,5)+\delta(4,3)$

If a player deviates, the gain is at most 1 at $t = 1$, the loss is at minimum $\delta(3 - 12/7)$.

So the strategies are a SPE if $\delta(3 - 12/7) \geq 1 \Leftrightarrow \delta \geq \frac{7}{9}$.

A simple war of attrition

Two animals are fighting for a prize with value v.

The fighting cost is 1 per period.

if an animal stops fighting at t, the opponent wins v, there is no fighting in that period and the game stops.

There is a per period discount factor δ.

The payoff of quitting (alone) at $\hat{t}$ is:

$$L(\hat{t}) = -\left(1 + \delta + \delta^2 + \ldots + \delta^{\hat{t}-1}\right)$$

The payoff for the winner is:

$$\begin{aligned} W(\hat{t}) &= -\left(1 + \delta + \delta^2 + \ldots + \delta^{\hat{t}-1}\right) + \delta^{\hat{t}} v \\ &= L(\hat{t}) + \delta^{\hat{t}} v \end{aligned}$$

If both animals quit together, then we assume the payoff is $L(\hat{t})$ for both.

As for the bargaining game, here we have several Nash equilibria.

For example:

- Player i never stops,
- Player j stops as soon as possible.

Is this subgame perfect? Yes

Is the game indeterminate? It is natural to look at a symmetric equilibrium, and this is unique.

A symmetric equilibrium looks like: "if we are still playing the game, stop with probability p."

what is the equilibrium p?

It easy to see that p must be interior, i.e. $p \in (0,1)$.

In equilibrium we must have:

$$L(t) = pW(t) + (1-p)L(t+1)$$

$$\Leftrightarrow L(t) - L(t+1) = p[W(t) - L(t+1)]$$

The left hand side of the first equation is the payoff to stopping; the payoff on the RHS is the payoff of continuing.

Note that:

$$L(t) - L(t+1) = \delta^t$$

$$W(t) - L(t+1) = \delta^t + \delta^t v$$

So we must have:

$$1 = p(1+v)$$

$$\Leftrightarrow p^* = \frac{1}{1+v}$$

Note that both the symmetric an asymmetric equilibria that we have seen are stationary.

A stationary Nash equilibrium is always a Nash equilibrium, since subgames are all strategically equivalent.