ECON 6110, Microeconomic Theory III	**Section 4**
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1 Supermodular game

Definition 1.1. $u_i(s_i, s_{-i})$ has increasing differences in (s_i, s_{-i}) if for all $(s_i, \tilde{s}_i)$ and $(s_{-i}, \tilde{s}_{-i})$ such that $s_i \geq \tilde{s}_i$ and $s_{-i} \geq \tilde{s}_{-i}$, we have:

$$u_i(s_i, s_{-i}) - u_i(\tilde{s}_i, s_{-i}) \geq u_i(s_i, \tilde{s}_{-i}) - u_i(\tilde{s}_i, \tilde{s}_{-i})$$

Definition 1.2. $u_i(s_i, s_{-i})$ is supermodular in s_i if for each s_{-i}:

$$u_i(s_i, s_{-i}) + u_i(\tilde{s}_i, s_{-i}) \leq u_i(s_i \wedge \tilde{s}_i, s_{-i}) + u_i(s_i \vee \tilde{s}_i, s_{-i})$$

Remark. Note that if S_i is linearly ordered (as $\mathbb{R}$), then u_i is trivially supermodular in s_i as the above inequality is vacuously satisfied as equality.

Definition 1.3. A (resp., strictly) supermodular game is a game in which for each i:

- S_i is a sublattice of R^{m_i}
- u_i has (resp., strictly) increasing differences in (s_i, s_{-i})
- u_i is (resp., strictly) supermodular in s_i

Remark. If every players' strategy is single-dimensional, the definition of supermodular game boils down to just increasing differences.

Theorem 1.1. *Let (S, u) be a supermodular game. Then:*

- *the set of strategies surviving iterated strict dominance has greatest and least elements* $\overline{a}, \underline{a}$.
- *and* $\overline{a}, \underline{a}$ *are both Nash equilibria.*

2 Exercise

ECON 6110: 2021 Prelim #1 Question #2

Two students are deciding how long to spend studying for 6110 on the night before the exam. Let e_i be the fraction of the available time student i devotes to studying with $0 \leq e_i \leq 1$. Assume that the students' payoffs are

$$\begin{aligned} v_1(e_1, e_2) &= \log(1 + 3e_1 - e_2) - e_1, \\ v_2(e_1, e_2) &= \log(1 + 3e_2 - e_1) - e_2. \end{aligned}$$

Note: Please ignore the two action profiles that render one of the value functions undefined :)

(a) Show that the game is supermodular.

(b) Find the set of rationalizable actions.

(c) Find the Nash equilibria.

3 Extensive game

Definition 3.1. A **multi-stage game with observed actions** consists of

(i) (Finite) set of players, $N = \{1, \dots, n\}$

(ii) A (possibly infinite) set of stages, $\{0, 1, \dots\}$

(iii) At stage 0:

- (a) An initial history $h^0 = \emptyset$
- (b) Set of feasible actions for each player i at h^0, $A_i(h^0)$
- (c) Set of action profiles played by players $a^0 \in \bigtimes_{i=1}^n A_i(h^0)$

(iv) At each stage $k > 0$

- (a) Set H^k of partial histories $h^k = (a^0, \dots, a^{k-1})$
- (b) Set of feasible actions for each player i at each h^k, $A_i(h^k)$
- (c) Set of action profiles played by players $a^k \in \bigtimes_{i=1}^n A_i(h^k)$ at each h^k

(v) Set Z of terminal histories $z = (a^0, a^1, \dots)$

(vi) Payoff function of player i, $v_i : Z \to \mathbb{R}$

Definition 3.2. A history $h \in H$ is a sequence of actions taken by the players $(a^k)_{k=1,\dots,K}$. The set of terminal histories is denoted Z.

Definition 3.3. A strategy of a player i in an extensive game with perfect information is a function

$$s_i(h) \to A(h)$$

for any $h \in H \setminus Z$ such that $P(h) = i$.

Remark. A strategy specifies an action for *each* (non-terminal) history in which a player is asked to choose an action, even for histories that, if the strategy is followed, are never reached.

Definition 3.4. Denote a strategy profile $s = (s_1, \ldots, s_n)$. For each strategy profile an outcome $O(s)$ is the terminal history associated with the strategy profile.

Definition 3.5. A strategy profile, $s = (s_1, \ldots, s_n)$ is a **Nash equilibrium** if for all players i and all deviations $\hat{s}_i$,

$$u_i(s_i, s_{-i}) \geq u_i(\hat{s}_i, s_{-i})$$

Definition 3.6. The **subgame** of the extensive game with perfect information $\Gamma =< N, H, P, (u_i) >$ that follows the history h is the extensive game $\Gamma(h) =< N, H \mid_h, P \mid_h, (u_i) \mid_h>$, where $H \mid_h, P \mid_h, (u_i) \mid_h$ are consistent with the original game starting at history h.

Definition 3.7. A strategy profile, s is a **subgame perfect equilibrium** in Γ if for any history h the strategy profile $s \mid_h$ is a Nash equilibrium of the subgame $\Gamma(h)$.

Definition 3.8. For fixed s_i and history h, a **one-stage deviation** is a strategy $\hat{s}_i$ in the subgame $\Gamma(h)$ that differs from $s_i \mid_h$ only in the action it prescribes after the initial history of $\Gamma(h)$.

Theorem 3.1 (One-stage deviation principle)**.** *In a finite-horizon extensive game, a strategy profile s is an SPE if and only if for all players i, all histories $h \in H$, and one-stage deviations $\hat{s}_i$,*

$$u_i(s_i \mid_h, s_{-i} \mid_h) \geq u_i(\hat{s}_i, s_{-i} \mid_h)$$

Example 3.2 (Entry game).