Macroeconomics ECON 6140 (Second Half)

Lecture 5
Solving Linear Rational Expectations Models

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How to solve a linear rational expectations model

There are at least three different ways of solving linear rational expectations models

1. Method of undetermined coefficients, can be very quick when feasible and illustrates the fixed point nature of the rational expectations solution.
2. Decouple the stable and unstable dynamics of the model and set the unstable part to zero.
3. Replacing expectations with linear projections onto observable variables

We will discuss methods 1 and 2 today.

Based on lecture notes posted on Canvas.

The 3 equation NK model

As a vehicle to demonstrate the different solution methods, we will use the basic New-Keynesian model

$$\begin{aligned}
\pi_t &= \beta E_t \pi_{t+1} + \kappa \widetilde{y}_t \\
\widetilde{y}_t &= E_t \widetilde{y}_{t+1} - \frac{1}{\sigma}\left(i_t - E_t \pi_{t+1} - r_t^n\right) \\
i_t &= \rho + \phi_\pi \pi_t + \phi_y \widehat{y}_t + v_t \\
r_t^n &= \rho - \sigma\left(1 - \rho_a\right)\psi_{ya} a_t + (1 - \rho_z) z_t
\end{aligned}$$

where $\pi_t, \widetilde{y}_t, \widehat{y}_t, i_t, r_t^n$ are inflation, output gap, output deviation from steady state, nominal interest rate and the natural rate of interest.

Method I:
Method of undetermined coefficients

Method I: Method of undetermined coefficients

Pros

- Method is quick when feasible
- Illustrates well the fixed point nature of rational expectations equilibria.

Cons

- Difficult to implement in larger models

Method of undetermined coefficients

Start by substituting in the interest rate equation into the IS equation to get

$$\widetilde{y}_t = E_t \widetilde{y}_{t+1} - \frac{1}{\sigma} \left(\rho + \phi_\pi \pi_t + \phi_y \widehat{y}_t + v_t - E_t \pi_{t+1} - r_t^n \right)$$

Define the composite shock u_t

$$\begin{aligned} u_t &\equiv r_t^n - \phi_y \widehat{y}_t^n - v_t \\ &= -\psi_{ya}(\phi_y + \sigma(1-\rho_a))a_t + (1-\rho_z)z_t - v_t \end{aligned}$$

and use that

$$\widehat{y}_t = \widetilde{y}_t + \widehat{y}_t^n, \quad \widehat{y}_t^n = \psi_{ya} a_t$$

to simplify the IS equation to

$$\widetilde{y}_t = E_t \widetilde{y}_{t+1} - \frac{1}{\sigma} \left(\phi_\pi \pi_t + \phi_y \widetilde{y}_t - E_t \pi_{t+1} + u_t \right)$$

Method of undetermined coefficients

By assuming that the composite shock

$$u_t = -\psi_{ya}(\phi_y + \sigma(1 - \rho_a))a_t + (1 - \rho_z)z_t - v_t$$

is an AR(1) process with persistence parameter ρ_u

$$u_t = \rho_u u_{t-1} + \eta_t$$

we can write all endogenous variables as functions only of u_t.

- By assuming common persistence of shocks, decomposition of u_t does not matter.

Solving model using method of undetermined coefficients

Conjecture that model can be put in the form

$$\begin{aligned}\pi_t &= \psi_\pi u_t \\ \widetilde{y}_t &= \psi_y u_t\end{aligned}$$

so that

$$\begin{aligned}E_t\pi_{t+1} &= \psi_\pi \rho_u u_t \\ E_t\widetilde{y}_{t+1} &= \psi_y \rho_u u_t\end{aligned}$$

Solving the model implies finding the coefficients ψ_π and ψ_y.

Solving model using method of undetermined coefficients

Substitute in the conjectured solution into the structural Phillips curve and IS equation

$$
\begin{aligned}
\psi_\pi u_t &= \beta \psi_\pi \rho_u u_t + \kappa \psi_y u_t \\
\psi_y u_t &= \psi_y \rho_u u_t - \frac{1}{\sigma}\left(\phi_\pi \psi_\pi u_t + \phi_y \psi_y u_t - \psi_\pi \rho_u u_t + u_t\right)
\end{aligned}
$$

Equating coefficients on the LHS and the RHS we get

$$
\begin{aligned}
\psi_\pi - \beta \psi_\pi \rho_u - \kappa \psi_y &= 0 \\
\psi_y - \psi_y \rho_u + \frac{1}{\sigma}\left(\phi_\pi \psi_\pi + \phi_y \psi_y - \psi_\pi \rho_u\right) &= -\frac{1}{\sigma}
\end{aligned}
$$

which is a system of linear equations in ψ_π and ψ_y

$$
\begin{bmatrix} (1-\beta\rho) & -\kappa \\ \left(\frac{1}{\sigma}\phi_\pi - \frac{1}{\sigma}\rho_u\right) & \left(1-\rho_u+\frac{1}{\sigma}\phi_y\right) \end{bmatrix} \begin{bmatrix} \psi_\pi \\ \psi_y \end{bmatrix} = \begin{bmatrix} 0 \\ -\sigma \end{bmatrix}
$$

Solving for ψ_π and ψ_y

Pre-multiplying both sides with the inverse of the coefficient matrix

$$\begin{aligned}
\begin{bmatrix} \psi_\pi \\ \psi_y \end{bmatrix} &= \begin{bmatrix} (1-\beta\rho) & -\kappa \\ \left(\frac{1}{\sigma}\phi_\pi - \frac{1}{\sigma}\rho_u\right) & \left(1-\rho_u+\frac{1}{\sigma}\phi_y\right) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -\frac{1}{\sigma} \end{bmatrix} \\
&= \Lambda \begin{bmatrix} \left(1-\rho_u+\frac{1}{\sigma}\phi_y\right) & \kappa \\ -\left(\frac{1}{\sigma}\phi_\pi - \frac{1}{\sigma}\rho_u\right) & (1-\beta\rho_u) \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{1}{\sigma} \end{bmatrix} \\
&= \Lambda \begin{bmatrix} -\kappa \\ (\beta\rho_u - 1) \end{bmatrix}
\end{aligned}$$

where Λ is the determinant of the matrix on the l.h.s. of (8)

$$\Lambda \equiv \frac{1}{(1-\beta\rho_\upsilon)\left[\sigma(1-\rho_u)+\phi_y\right]+\kappa(\phi_\pi-\rho_u)} > 0$$

so that

$$\begin{aligned}
\pi_t &= -\kappa\sigma\Lambda u_t \\
\widetilde{y}_t &= -\Lambda\sigma(1-\beta\rho_u)\,u_t
\end{aligned}$$

Method II:
Stable/unstable decoupling

Method II: Stable/unstable decoupling

Originally due to Blanchard and Kahn (1980)

- Computational aspects of the method has been further developed by others, for instance Klein (2000).
- The most accessible reference is probably Soderlind (1999).

The method has several advantages:

- Fast
- Provides conditions for when a solution exists
- Provides conditions for when the solution is unique.

The model in matrix form

Start by putting the model into matrix form

$$
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \frac{1}{\sigma} & 1 \end{bmatrix}
\begin{bmatrix} a_{t+1} \\ z_{t+1} \\ v_{t+1} \\ E_t \pi_{t+1} \\ E_t \widetilde{y}_{t+1} \end{bmatrix}
$$

$$
= \begin{bmatrix} \rho_a & 0 & 0 & 0 & 0 \\ 0 & \rho_z & 0 & 0 & 0 \\ 0 & 0 & \rho_v & 0 & 0 \\ 0 & 0 & 0 & 1 & -\kappa \\ \frac{1}{\sigma}\left(\phi_y \psi_{ya} + (1-\rho_a)\psi_{ya}\right) & -\frac{1}{\sigma}(1-\rho_z) & \frac{1}{\sigma} & \frac{1}{\sigma}\phi_\pi & \left(1+\frac{1}{\sigma}\phi_y\right) \end{bmatrix}
\begin{bmatrix} a_t \\ z_t \\ v_t \\ \pi_t \\ \widetilde{y}_t \end{bmatrix}
$$

$$
+ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{u}_{t+1}
$$

The model in matrix form

More compactly

$$A_0 \begin{bmatrix} x^1_{t+1} \\ E_t x^2_{t+1} \end{bmatrix} = A_1 \begin{bmatrix} x^1_t \\ x^2_t \end{bmatrix} + C_1 \mathbf{u}_{t+1}$$

where x^1_t is vector containing the pre-determined and/or exogenous variables and x^2_t a vector containing the forward looking ("jump") variables, i.e.

$$x^1_t \equiv \begin{bmatrix} a_t & z_t & v_t \end{bmatrix}', \quad x^2_t \equiv \begin{bmatrix} \pi_t & \widetilde{y}_t \end{bmatrix}'.$$

Stable/unstable decoupling

Pre-multiply both sides of

$$A_0 \begin{bmatrix} x^1_{t+1} \\ E_t x^2_{t+1} \end{bmatrix} = A_1 \begin{bmatrix} x^1_t \\ x^2_t \end{bmatrix} + C_1 \mathbf{u}_{t+1}$$

by A_0^{-1} to get

$$\begin{bmatrix} x^1_{t+1} \\ E_t x^2_{t+1} \end{bmatrix} = A \begin{bmatrix} x^1_t \\ x^2_t \end{bmatrix} + C \mathbf{u}_{t+1}$$

where $A = A_0^{-1} A_1$ and $C = A_0^{-1} C_1$.

For the model to have unique stable solution the number of stable eigenvalues of A must be equal to the number of exogenous (or pre-determined) variables.

Stable/unstable decoupling

Use a Schur decomposition to get

$$A = ZTZ^H$$

where T is upper block triangular

$$T = \begin{bmatrix} T_{11} & T_{12} \\ \mathbf{0} & T_{22} \end{bmatrix}$$

and Z is a unitary matrix so that $Z^H Z = ZZ^H = I$ ($\implies Z^H = Z^{-1}$).

- For any square matrix W, $W^{-1}AW$ is a so called similarity transformation of A.
- Similarity transformations do not change the eigenvalues of a matrix
- It is a property of the Shur decomposition that it is always possible to choose Z and T so that the unstable eigenvalues of A are shared with T_{22}

Stable/unstable decoupling

Define the auxiliary variables

$$\begin{bmatrix} \gamma_t \\ \delta_t \end{bmatrix} = Z^H \begin{bmatrix} x_t^1 \\ x_t^2 \end{bmatrix}$$

We can then rewrite the system as

$$Z^H \begin{bmatrix} x_{t+1}^1 \\ E_t x_{t+1}^2 \end{bmatrix} = Z^H Z T Z^H \begin{bmatrix} x_t^1 \\ x_t^2 \end{bmatrix}$$

or equivalently

$$E \begin{bmatrix} \gamma_{t+1} \\ \delta_{t+1} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ \mathbf{0} & T_{22} \end{bmatrix} \begin{bmatrix} \gamma_t \\ \delta_t \end{bmatrix}$$

since $Z^H Z = I$.

Stable/unstable decoupling

For this system to be stable, the auxiliary variables associated with the unstable roots in T_{22} must be zero for all t. (WHY?)

Imposing $\delta_t = 0 \forall t$ reduces the relevant state dynamics to

$$\gamma_t = T_{11}\gamma_{t-1}$$

To get back the original variables we simply use that

$$\begin{bmatrix} x_t^1 \\ x_t^2 \end{bmatrix} = \begin{bmatrix} Z_{11} \\ Z_{21} \end{bmatrix} \gamma_t$$

or

$$\begin{bmatrix} x_t^1 \\ x_t^2 \end{bmatrix} = \begin{bmatrix} Z_{11} \\ Z_{21} \end{bmatrix} Z_{11}^{-1} x_t^1$$

which is the solution to the model.

The solved model

The solved model is in the form

$$\begin{aligned} x_t^1 &= M x_{t-1}^1 + C\mathbf{u}_t \\ x_t^2 &= G x_t^1 \end{aligned}$$

where

$$\begin{aligned} M &= Z_{11} T_{11} Z_{11}^{-1} \\ &= \begin{bmatrix} \rho_a & 0 & 0 \\ 0 & \rho_z & 0 \\ 0 & 0 & \rho_u \end{bmatrix} \end{aligned}$$

and $G = Z_{21} Z_{11}^{-1}$.

Stable/unstable decoupling in Matlab

Matlab: Housekeeping

```matlab
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Housekeeping
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear var
clc
keep_plot=1;
if keep_plot ==0
    close all
end
shock=2;%1=prod, 2=demand, 3=monetary policy
```

Matlab: Assign parameter values

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Assign values to parameters
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
r_a=.9; %   Persistence of potential output
r_z=.9; %   Persistence of demand shocks
r_v=.9; %   Persistence of monetary policy shocks

b=.95;  %  Discount rate (beta)
sig = 1; % Curvature in consumption
varphi= 5 ; %curvature in labor supply
epsilon= 9 ; % CES aggregator elasticity

fi_pi=1.5 ; % Taylor rule parameter on inflation
fi_y=0.125 ; % Taylor rule parameter on output
theta=0.75 ; %Calvo parameter

alpha = .5; % Labor share in production function
lambda = ((1-theta)*(1-theta*b)/theta)* ( (1-alpha)/ (1-alpha+epsilon*alpha));
k = lambda * (sig + (varphi + (alpha + varphi)/(1-alpha)));
psi_ya = (1 + varphi)/(sig*(1-alpha) + varphi + alpha);
```

Matlab: Put model in matrix form

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Soderlind of stable/unstable decoupling
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
cutoff=.999999; %Define the cutoff for stable vs unstable eignevalues (should be just below

% Define model matrices
A0=[1,0,0,0,0;
    0,1,0,0,0;
    0,0,1,0,0;
    0,0,0,b,0;
    0,0,0,sig,1;];

A1=[r_a,0,0,0,0;
    0,r_z,0,0,0;
    0,0,r_v,0,0;
    0,0,0,1,-k;
    sig*(fi_y*psi_ya +sig*(1-r_a)*psi_ya),-sig*(1-r_z),sig,sig*fi_pi,1+sig*fi_y;];

C1= [eye(3);zeros(2,3);];

A=A0\A1;
C=A0\C1;
C=C(1:3,1:3);
```

Matlab: Stable/unstable decoupling

```
egen = abs(eig(A)) < cutoff;

n1=3; %Number of predetermined variables
n2=2; % Number of jump variables
n = n1 + n2; %Total number of variables

%MatLab, complex generalized Schur decomposition
[S,T,Qa,Z] = qz(eye(size(A)),A); %MatLab: I=Q'SZ' and A=Q'TZ'; Paul S:  I=QSZ'
[S,T,Qa,Z] = reorder(S,T,Qa,Z);   % reordering of generalized eigenvalues, T(i,
logcon = abs(diag(T)) <= (abs(diag(S))*cutoff);  %1 for stable eigenvalue

if sum(logcon) < n1
    warning('Too few stable roots: no stable solution');
    M = NaN; G = NaN;J0 = NaN;
    return;
elseif sum(logcon) > n1
    warning('Too many stable roots: inifite number of stable solutions');
    M = NaN; G = NaN;J0 = NaN;
    return;
end
```

Matlab: Stable/unstable decoupling, cont'd

```
Stt = S(1:n1,1:n1);
Zkt = Z(1:n1,1:n1);
Zlt = Z(n1+1:n,1:n1);
Ttt = T(1:n1,1:n1);

if cond(Zkt) > 1e+14
    warning('Zkt is singular: rank condition for solution not satisfied');
    M = NaN; G = NaN;J0 = NaN;
    return;
end

Zkt_1 = inv(Zkt);          %inverting
Stt_1 = inv(Stt);


M = real(Zkt*Stt_1*Ttt*Zkt_1);        %x1(t+1) = M*x1(t) + e(t+1)
G = real(Zlt*Zkt_1) ;               %x2(t) = G*x1(t)
```

Matlab: Find equilibrium objects as function of state

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Construct equilibrium functions for output, labor, interest rate
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
G_pi=G(1,:);
G_yg = G(2,:);
G_y = G(2,:) + [psi_ya ,0,0;];
G_n = (1/(1-alpha))*(G_y - [1,0,0;]);
G_i= fi_y*G_y + fi_pi*G_pi + [0,0,1;];
```

Impulse Response Functions

Matlab: Impulse Response Functions

```
periods = 50;
IRF = zeros(6,periods);
for t=1:periods
    IRF(1:5,t)=[G_pi;G_yg;G_y;G_n;G_i;]*M^(t-1)*C(:,shock);
    if t==1
        IRF(6,t)=G_pi*M^(t-1)*C(:,shock);
    else
        IRF(6,t)=IRF(6,t-1)+G_pi*M^(t-1)*C(:,shock);
    end
end
```

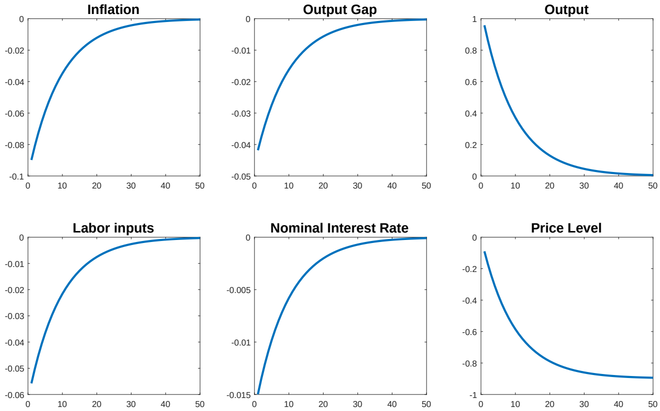
Matlab: Plot figures

```
figure(1)
subplot(2,3,1);
plot(IRF(1,:),'linewidth',2);title('Inflation','fontsize',16);
subplot(2,3,2);
plot(IRF(2,:),'linewidth',2);title('Output Gap','fontsize',16)
subplot(2,3,3);
plot(IRF(3,:),'linewidth',2);title('Output','fontsize',16)
subplot(2,3,4);
plot(IRF(4,:),'linewidth',2);title('Labor inputs','fontsize',16)
subplot(2,3,5);
plot(IRF(5,:),'linewidth',2);title('Nominal Interest Rate','fontsize',16)
subplot(2,3,6);
plot(IRF(6,:),'linewidth',2);title('Price Level','fontsize',16)
```

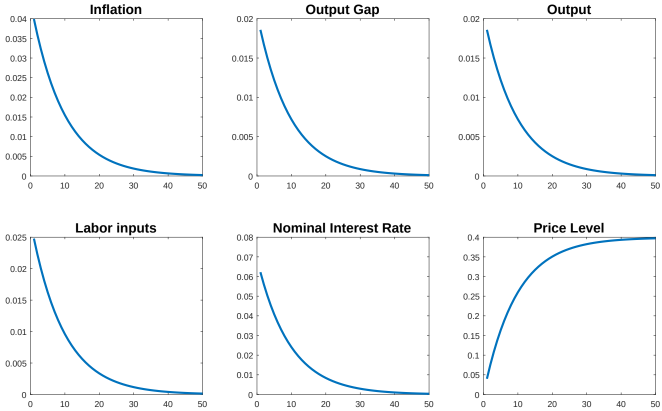
Baseline parameterization

- Households: $\sigma = 1; \varphi = 5, \beta = 0.95, \epsilon = 9, \rho_z = 0.9$
- Firms: $\alpha = 1/4, \theta = 3/4, \rho_a = 0.9$
- Policy rules: $\phi_\pi = 1.5, \phi_y = 0.125, \rho_v = 0.9$

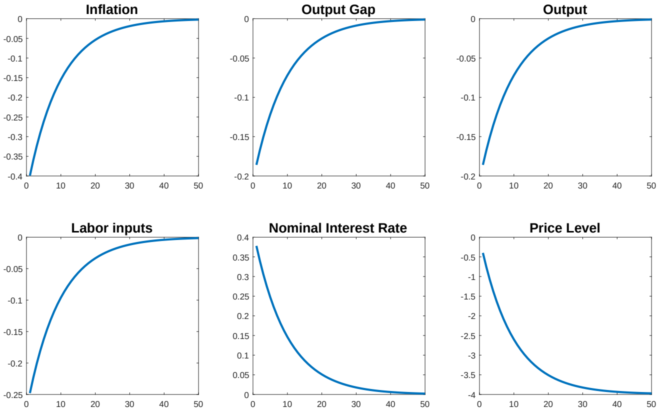
The response to productivity shocks



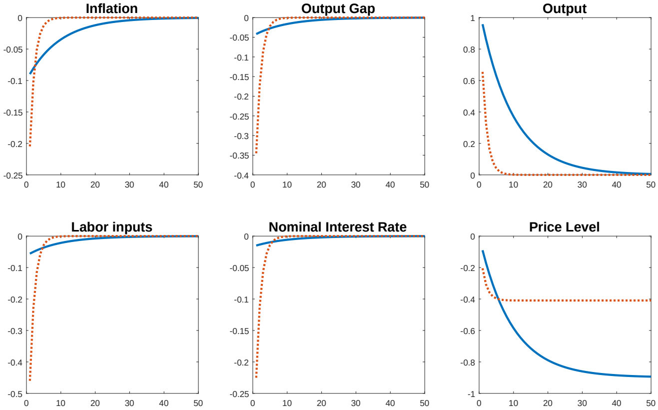
The response to demand shocks



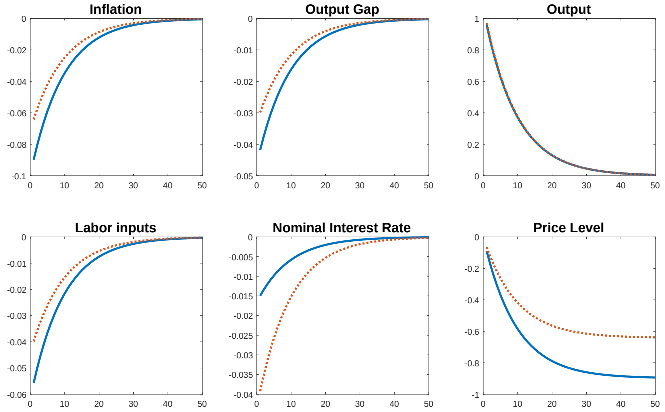
The response to monetary policy shocks



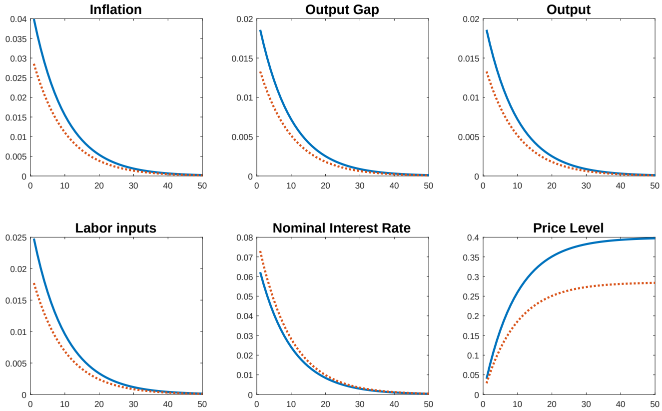
Persistence is exogenous: IRF to prod shock with $\rho_a = 0.5$



IRF to prod shocks: $\phi_\pi = 2.5$



IRF to demand shocks: $\phi_\pi = 2.5$



Summary

Linear(-ized) rational expectations models can be solved multiple ways

- "Rational" in this context means "model consistent"
- Equilibrium is a fixed point: Expectations are model consistent when they are optimal forecasts of endogenous variables that are functions of the very same expectations...

You should know how to

- Solve a linear RE model
- Simulate the solved model
- Compute impulse responses of all variables to all shocks