ECON6110: Problem Set 2

Spring 2025

This problem set is due on at 23:59 on February 28, 2025. Every student must write their own solution and submit it individually. Problem set submissions are submitted electronically and may be typed or handwritten. If handwritten, please ensure your work and scan are legible. **Illegible submissions will not be graded.**

Problem 1

Consider the following game between a union (player 1) and a monopolist (player 2). The union moves first and sets wage $w \geq 0$. The monopolistic firm, after observing the wage set by the union, chooses output $q \geq 0$. Assume that workers have opportunistic cost $l \in (0, 1)$ of their time, that one unit of labor is needed for each unit of output, and firms faces downward-sloping inverse demand $p(q) = 1 - q$. Overall:

$$v_1(w, q) = (w - l)q \quad \text{and} \quad v_2(w, q) = q((1 - q) - w).$$

Question 1: Find a subgame perfect equilibrium.

Question 2: Find a Nash equilibrium that is not subgame perfect.

Solution:

Question 1: We use backward induction. Consider the subgame played by the monopolist:

$$\max_{q \geq 0} v_2 = q((1 - q) - w)$$

Since it is a concave function, FOC would grantee a global maximum. Solving the FOC gives $q^* = \frac{1-w}{2}$. However, this optimum is subject to the nonnegativity constraint $q \geq 0$. So the firm will shut down if a wage dictates a negative quantity as the optimal. Taking together, the best response function of the monopolist to a wage w set by the union is:

$$BR_2(w) = \begin{cases} \frac{1-w}{2} & \text{if } 0 \leq w \leq 1 \\ 0 & \text{if } w \geq 1 \end{cases}$$

We fold back to stage 1 where the union sets the wage. Now the union's decision problem is:

$$\max_{w \geq 0} v_1 = \begin{cases} (w - l)\frac{1-w}{2} & \text{if } 0 \leq w \leq 1 \\ 0 & \text{if } w \geq 1 \end{cases}$$

Solving the first case gives an optimal wage $w^* = \frac{1}{2} + \frac{1}{2}l \in (\frac{1}{2}, 1)$, and the optimal value in this case is nonnegative ($w - l$ and $1 - w \geq 0$), so w^* is truly the best response to q^*. We know in class that backward induction gives the SPE. So the SPE in this case is $(w^* = \frac{1}{2} + \frac{1}{2}l, q^* = \frac{1-w^*}{2})$.

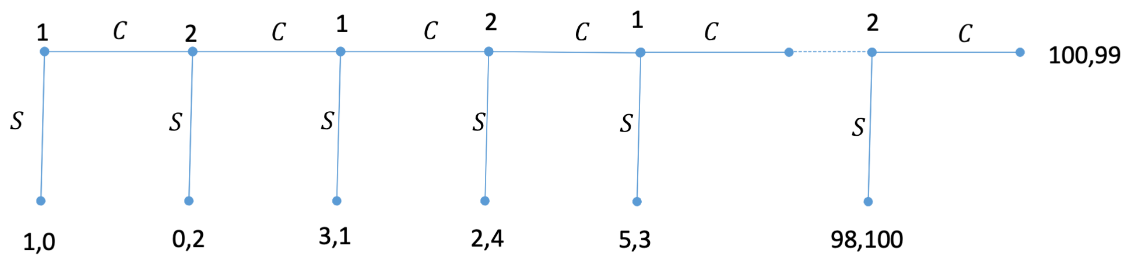
Question 2: Consider the followng strategy of the monopolist:

$$q(w) = \begin{cases} \frac{1-l}{2} & \text{if } w = l \\ 0 & \text{if } w \neq l \end{cases}$$

It is easy to verify that $(w = l, q = \frac{1-l}{2})$ is a Nash equilibrium, since both players are best responding to the other's strategy. But this is not subgame perfect because (intuitively) the equilibrium is maintained by a non-credible threat from the monopolist. Rigorously, we can verify that the subgame played by the monopolist does not attain a Nash equilibrium.

Problem 2

Consider the following Centipede game:



Question 1: Find a Nash equilibrium that is not a SPE, or prove that there are none.

Question 2: Is the subgame-perfect equilibrium outcome the unique Nash equilibrium outcome of the centipede game?

Solution:

Question 1: The centipede game does admit NE which are not subgame-perfect. Consider the following strategies: player 2 always exits, player 1 exits in the first information set, and thereafter continues. This is a NE. Given player 2's strategy, the BR of player 1 is to exit at the first information set. Given player 1's strategy, player 2 can not do better by changing her strategy.

Question 2: All NE have the same outcome. To see that, suppose there is a NE in which player 1 continues at the first move with some positive probability. This will be true only if player 2 is playing "continue" in the following move with a strictly positive probability. However, for player 2 to play continue with positive probability, it has to be the case that player 1 is playing "continue" at the third node with a strictly positive probability. The argument continues to the last move of the game, where again one need to assume positive probability (approaching one) of player 2 to continue. However, this does not constitute a NE, since player 2 has a profitable deviation (exiting) at the last node.

Problem 3

Suppose a parent and child play the following game, first analyzed by Gary Becker. First, the child takes an action, A; that produces income for the child, $I_C(A)$; and income for the parent, $I_P(A)$. (Think of $I_C(A)$ as the child's income net of any cost of the action A). Second, the parent observes the incomes $I_C(A)$ and $I_P(A)$ and then chooses a bequest, B; to leave to the child. The child's payoff is:

$$U(I_C(A) + B)$$

The parent's is:

$$V(I_P(A) - B) + kU(I_C(A) + B)$$

where $k > 0$ reflects the parent's concern for the child's well-being. Assume that: the action is a nonnegative number, $A > 0$; the income functions $I_C(A)$ and $I_P(A)$ are strictly concave and maximized at $A_C > 0$ and $A_P > 0$, respectively; the bequest B can be positive or negative; and the utility functions U and V are increasing and strictly concave. Prove the "Rotten Kid" Theorem (G. Becker): in the subgame-perfect equilibrium outcome, the child chooses the action that maximizes the family's aggregate income, $I_C(A) + I_P(A)$; even though only the parent's payoff exhibits altruism.

Solution:

By backward induction, in the second period the parent chooses B to solve:

$$\max_B V\left(I_P(A) - B\right) + kU(I_C(A) + B)$$

for any value of A. The parents reaction function $B^*(A)$ is implicitly defined by the following FOC:

$$-V'\left(I_P(A) - B^*(A)\right) + kU'(I_C(A) + B^*(A)) = 0$$

Since we are looking for a SPE (and not just a NE), this condition has to hold for any A. Using the implicit function theorem, we get:

$$-V''\left(I_P(A)-B^*(A)\right)\left[I_P'(A)-\frac{dB^*(A)}{dA}\right]$$
$$+kU''(I_C(A)+B^*(A))\left[I_C'(A)+\frac{dB^*(A)}{dA}\right]=0$$

So:

$$\frac{dB^*(A)}{dA}=-\frac{kU''\cdot I_C'(A)-V''\cdot I_P'(A)}{kU''+V''} \tag{1}$$

In the first period, the child, in a SPE, chooses A to solve:

$$\max_A U(I_C(A)+B^*(A))$$

The foc is:

$$U'(I_C(A)+B^*(A))\left[I_C'(A)+\frac{dB^*(A)}{dA}\right]=0$$
$$\Leftrightarrow I_C'(A)+\frac{dB^*(A)}{dA}=0$$

Using (1) in the previous equation:

$$I_C'(A)-\frac{kU''\cdot I_C'(A)-V''\cdot I_P'(A)}{kU''+V''}=0$$

which simplifies to:

$$\begin{aligned} V''\left[I_C'(A)+I_P'(A)\right] &= 0 \\ &\Leftrightarrow I_C'(A)+I_P'(A)=0. \end{aligned}$$

This is the necessary and sufficient condition for the maximization of $I_C(A)+I_P(A)$, since these functions are concave.

Problem 4

Two partners would like to complete a project. Each partner receives the payoff V when the project is completed but neither receives any payoff before completion. The cost remaining before the project can be completed is R. Neither partner can commit to making a future contribution towards completing the project, so they decide to play the following two-period game: In period one partner 1 chooses to contribute $c_1 \geq 0$ towards completion. If this contribution is sufficient to complete the project then the game ends and each partner receives V: If this contribution is not sufficient to complete the project (i.e., $c_1 < R$) then in period two partner 2 chooses to contribute $c_2 \geq 0$ towards completion. If the (undiscounted) sum of the two contributions is sufficient to complete the project then the game ends and each partner receives V: If this sum is not sufficient to complete the project then the game ends and both partners receive zero.

Each partner must generate the funds for a contribution by taking money away from other profitable activities. The optimal way to do this is to take money away from the least profitable alternative first. The resulting (opportunity) cost of a contribution is thus convex in the size of the contribution. Suppose that the cost of contribution c is c^2 for each partner. Assume that partner 1 discounts second-period benefits by the discount factor $\delta \in (0,1)$: Compute the unique backwards-induction outcome of this two-period contribution game for each triple of parameters $\{V, R, \delta\}$.

Solution:

We solve this game using backwards induction. Starting with the final stage, observing C_1 the second partner will set:

$$C_2 = \begin{cases} R - C_1 & \text{if } (R - C_1)^2 \leq V \text{ and } R - C_1 \geq 0 \\ 0 & \textit{else} \end{cases}$$

Thus, anticipating the reaction function of the second partner, the first one will choose:

$$C_1 = \begin{cases} 0 & R \leq \sqrt{V} \text{ and } \delta V \geq V - R^2 \\ R & R \leq \sqrt{V} \text{ and } \delta V < V - R^2 \\ 0 & R > \sqrt{V} \text{ and } \delta V - (R - \sqrt{V})^2 < 0 \\ R - \sqrt{V} & R > \sqrt{V} \text{ and } \delta V - (R - \sqrt{V})^2 \geq 0 \end{cases}$$

So the equilibrium strategy is:

$$(C_1, C_2) = \begin{cases} (0, R) & R \leq \sqrt{V} \text{ and } \delta V \geq V - R^2 \\ (R, 0) & R \leq \sqrt{V} \text{ and } \delta V < V - R^2 \\ (0, 0) & R > \sqrt{V} \text{ and } \delta V - (R - \sqrt{V})^2 < 0 \\ (R - \sqrt{V}, \sqrt{V}) & R > \sqrt{V} \text{ and } \delta V - (R - \sqrt{V})^2 \geq 0 \end{cases}$$

And the equilibrium payoffs are:

$$(U_1, U_2) = \begin{cases} (\delta V, V - R^2) & R \leq \sqrt{V} \text{ and } \delta V \geq V - R^2 \\ (V - R^2, V) & R \leq \sqrt{V} \text{ and } \delta V < V - R^2 \\ (0, 0) & R > \sqrt{V} \text{ and } \delta V - (R - \sqrt{V})^2 < 0 \\ (\delta V - (R - \sqrt{V})^2, 0) & R > \sqrt{V} \text{ and } \delta V - (R - \sqrt{V})^2 \geq 0 \end{cases}$$