Macroeconomics, PhD core

Lecture #4-5

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Lecture Road Map

- The Overlapping Generations Model

The OLG

- Besides the one sector neoclassical growth model, the OLG model is the second major workhorse of **modern** macroeconomics.
  Allais, '47; Samuelson, '58; Diamond, '65

- Shortcomings of the infinitely lived agents model: individuals do not live forever?
  an altruistic bequest motive makes individuals that live for a finite number of period maximize the utility of an entire dynasty.

- The real deal?

  We want models where agents have interesting life-cycle:
  born, education, labor income, plan for retirement, partner up, have children, retire, die.

- Why? Integrate micro and macro data $\rightarrow$ **modern macro**

The OLG
Basic Set up

- Time is discrete t=1,2,3,.... and the economy (but not people) lives forever.
- Single non-storable consumption good in each period.
- A new generation is born in each period,
  index generations by year born.
- People live for two periods and then die.

The OLG

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  What happens to population in this economy?
- Alternative: do not keep track of age distributions, i.e. people die stochastically (Blanchard, '85).

The OLG

Endowments & consumption

- Generation t's endowment of the consumption good in period 1 and 2 of life, (e_t^t, e_{t+1}^t).

- Generation t's consumption in period 1 and 2 of life, (c_t^t, c_{t+1}^t).

- At each point in time there are two generations alive,
  - One **old** generation, with endowment and consumption (e_t^{t-1}, c_t^{t-1}).
  - One **young** generation, with endowment and consumption (e_t^t, c_t^t).
- At time zero, there is one old generation, (e_1^0, c_1^0).
- Exponential population growth, $L_0 = 1$.

$$L_t = (1+n)^t L_0$$

Timing

Generation/Time	1	2	....	t	t+1
0	(c_1^0, e_1^0)				
1	(c_1^1, e_1^1)	(c_2^1, e_2^1)			
2					
.					
.					
t-1				(c_t^{t-1}, e_t^{t-1})	
t				(c_t^t, e_t^t)	(c_{t+1}^t, e_{t+1}^t)
t+1					$(c_{t+1}^{t+1}, e_{t+1}^{t+1})$

Production Technology

- Assume that the only endowment is labor (time)
- 1 unit supplied inelastically when young in return for w_t.

- CRS technology for production

$$Y_t = F(K_t, L_t),$$

competitive factor markets.

- Capital: assume $\delta = 1$
$k \equiv \frac{K}{L}$, $f(k) \equiv F(k, 1)$, and the gross return on saving (rental rate of capital)

$$1 + r_t = R_t = f'(k_t) \tag{1}$$

- Wage rate

$$w_t = f(k_t) - k_t f'(k_t) \tag{2}$$

Consumption-savings decisions

Savings of a generation

$$\max_{c_t^t, c_{t+1}^t, s_t} = u(c_t^t) + \beta u(c_{t+1}^t)$$

subject to

$$c_t^t + s_t \leq w_t$$

$$c_{t+1}^t \leq R_{t+1} s_t$$

no altruism, no bequest?

- Old agents rent their savings to firms as capital.
- With U strictirly increasing + inada, constraint w/equality
- Non-negativity constraints?

Consumption-savings decisions

Optimality

- Euler equation

$$u'(c_t^t) = \beta R_{t+1} u'(c_{t+1}^t) \tag{3}$$

  Problem of individuals is concave, so Euler is sufficient.

- Obtain a savings function s: $\mathbb{R}^2 \to \mathbb{R}$ with

$$s_t = s(w_t, R_{t+1}) \tag{4}$$

  s increasing in w and increasing or decreasing in R.

- Aggregate savings

$$S_t = L_t s_t$$

- With full depreciation, capital stock

$$K_{t+1} = L_t s(w_t, R_{t+1}) \tag{5}$$

Competitive Equilibrium

Definition

A competitive equilibrium is a sequence of aggregate capital stocks, individual consumption and factor prices, $\{K_t, (c_t^t, c_{t+1}^t), R_t, w_t\}_{t=0}^{\infty}$, s.t. the factor prices sequence satisfies 1 and 2, individual consumption decisions are given by 3 and 7, and the aggregate capital stocks follows, 5.

- Steady state is defined as usual such that $k \equiv \frac{K}{L}$ constant.
- Equilibrium characterization requires normalizing by the size of the population $L_{t+1} = (1+n)L_t$

Equilibrium characterization

$$k_{t+1} = \frac{K_{t+1}}{L_{t+1}} = \frac{L_t}{L_{t+1}} s(w_t, R_{t+1})$$

Using 1 and 2

$$k_{t+1} = \frac{s(f(k_t) - k_t f'(k_t), f'(k_{t+1}))}{1+n} \tag{6}$$

- Steady state is a solution s.t. $k_{t+1} = k_t = k^\star$.
- Since s(.) can take any form, in pple multiple steady states are possible, as well as complicated dynamics.

Special Case

- CRRA Utility functions

$$U_t = \frac{(c_t^t)^{1-\theta} - 1}{1-\theta} + \beta \left(\frac{(c_{t+1}^t)^{1-\theta} - 1}{1-\theta} \right)$$

for $\theta > 0$ & $\beta \in (0, 1)$.

- Cobb-Douglas technology

$$f(k) = k^{\alpha}$$

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- Key outcome: Euler equation

$$\frac{c_{t+1}^t}{c_t^t} = (\beta R_{t+1})^{\frac{1}{\theta}}$$

Special Case

- Key outcome: Euler equation

$$\frac{c_{t+1}^t}{c_t^t} = (\beta R_{t+1})^{\frac{1}{\theta}}$$

- Rewritten in terms of savings rates..

$$s_t^\theta \beta R_{t+1}^{1-\theta} = (w_t - s_t)^{-\theta}$$

$$s_t = \frac{w_t}{\phi_{t+1}} \tag{7}$$

for $\phi_{t+1} \equiv \left[1 + \beta^{-1/\theta} R_{t+1}^{-\frac{1-\theta}{\theta}}\right] > 1$ so that $s_t \leq w_t$

Comparative Statics

Savings

- Comparative statics to wages

$$s_w = \frac{\partial s}{\partial w} = \frac{1}{\phi_{t+1}} \in (0,1)$$

- Comparative statics to the interest rate

$$s_R = \frac{\partial s}{\partial R} = \frac{1-\theta}{\theta}(\beta R_{t+1})^{-\frac{1}{\theta}} \frac{s_t}{\phi_{t+1}}$$

sign depends on intertemporal elasticity of substitution.

$s_R > 0$ if $\theta > 1$

$s_R < 0$ if $\theta < 1$

$s_R = 0$ if $\theta = 1$ (log-preferences/cobb-douglas)

Characterization

- Equations 7 and 6 imply

$$k_{t+1} = \frac{s_t}{1+n} = \frac{w_t}{(1+n)\phi_{t+1}}$$

- Using the expression for wages,

$$k_{t+1} = \frac{f(k_t) - k_t f'(k_t)}{(1+n)\left[1 + \beta^{-\frac{1}{\theta}} f'(k_{t+1})^{-\frac{1-\theta}{\theta}}\right]}$$

- Therefore the steady-state is implicitly defined by

$$k^\star = \frac{f(k^\star) - k^\star f'(k^\star)}{(1+n)\left[1 + \beta^{-\frac{1}{\theta}} f'(k^\star)^{-\frac{1-\theta}{\theta}}\right]}$$

Characterization Cobb-Douglas Techno

- Therefore the steady-state is implicitly defined by

$$k^\star = \frac{(1-\alpha)k^{\star\alpha}}{(1+n)\left[1+\beta^{-\frac{1}{\theta}}(\alpha k^{\star\alpha-1})^{-\frac{1-\theta}{\theta}}\right]} \tag{8}$$

- One can alternatively solve for the interest rate $R_t = \alpha k^{\star\alpha-1}$

$$(1+n)\left[1+\beta^{-\frac{1}{\theta}}R^{\star-\frac{1-\theta}{\theta}}\right] = \frac{1-\alpha}{\alpha}R^\star$$

- Dynamics are given by the difference equation

$$k_{t+1} = \frac{(1-\alpha)k_t^\alpha}{(1+n)\left[1+\beta^{-\frac{1}{\theta}}(\alpha k_{t+1}^{\alpha-1})^{-\frac{1-\theta}{\theta}}\right]}$$

Characterization

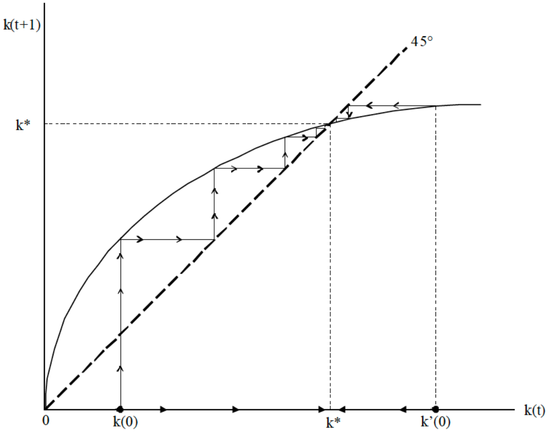
Proposition

In the OLG model w/generations that live for two periods, Cobb-Douglas technology, and CRRA preferences, there exist a steady state, $k^\star$, characterized by 8. The steady state is unique and stable for all $k(0) > 0$.

- In this well behaved case, equilibrium dynamics $\approx$ Solow model.
- Even with CRRA and Cobb-Douglas the model gets messy.
- Canonical model: log-preferences, $\theta = 1$

Characterization

Capital dynamics



Planners problem

- Planner solves

$$\sum_{t=0}^{\infty} \beta_s^t U_t \equiv \sum_{t=0}^{\infty} \beta_s^t \left(u(c_t^t) + \beta u(c_{t+1}^t) \right)$$

subject to

$$F(K_t, L_t) = K_{t+1} + L_t c_t^t + L_{t-1} c_t^{t-1}$$

where β_s is the planner's discount factor across generations.

- Very common issue... value the unborn?
Measure Growth when Life is Worth Living
`https://web.stanford.edu/~chadj/popwelfare.pdf`

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subject to

$$f(k_t) = (1+n)k_{t+1} + c_t^t + \frac{c_t^{t-1}}{(1+n)}$$

dividing by L_t

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Planners problem

Optimality

- Euler equation (FONC)

$$u'(c_t^t) = \beta f'(k_{t+1}) u'(c_{t+1}^t)$$

- Since $f'(k_{t+1}) = R_{t+1}$, the intertemporal consumption decisions identical to individual's.
- No "distortion" in consumption allocation over time.
- What about across generations?

  Not clear: planner weights different generations.

Planners problem

Overaccumulation ?

- Steady state national income accounts:

$$f(k^\star) - (1+n)k^\star = \overbrace{c_1^\star + \frac{c_2^\star}{1+n}}^{c^\star}$$

  where (c_1, c_2) are consumption when young and old, respectively.
- Think about maximizing overall consumption

$$\frac{\partial c^\star}{\partial k^\star} = f'(k^\star) - (1+n)$$

- steady-state capital that maximizes consumption, k^{gold}

$$f'(k^{\text{gold}}) = (1+n)$$

- If $k^\star > k^{\text{gold}}$, then $\frac{\partial c^\star}{\partial k^\star} < 0$, i.e. lower savings increases consumption for everyone!

Planners problem

Over-accumulation?

Definition

The economy is dynamically inefficient if in involves over-accumulation, i.e. $k^\star > k^{\text{gold}}$.

- An alternative to this condition is

$$R^\star < (1+n) \qquad \text{or} \qquad r^\star < n$$

- Transversality condition in a standard one sector growth model requires $r > (g+n)$.
- In an OLG this transversality condition is not there (agents live for two periods and solve finite problems).

Intuition

Over-accumulation?

- Individuals born at time t face prices determined by the stock of capital chosen by the previous generation.
- Pecuniary externality: actions of previous generation affect current on.
- Pecuniary externalities typically do not matter for welfare ¿¿ second order.

  ...but here a infinite stream of newborn agents are affected.
- These pecuniary externalities can be exploited (we will see this in the application).

Over-accumulation

Proposition

In the baseline OLG, the competitive equilibrium is not necessarily Pareto optimal. Whenever $r^\star < n$ *the economy is dynamically inefficient. Hence, it is possible to reduce the capital stock in the steady state and increase consumption of all generations.*

Over-accumulation

Proposition

In the baseline OLG, the competitive equilibrium is not necessarily Pareto optimal. Whenever $r^\star < n$ the economy is dynamically inefficient. Hence, it is possible to reduce the capital stock in the steady state and increase consumption of all generations.

Proof. Consider change in next period's capital stock $-\Delta k < 0$ and then move towards s.s.

- Lower savings first period

$$\Delta c_T = (1+n)\Delta k > 0$$

- Since $k^\star > k^{\text{gold}}$, for small Δk,

$$\Delta c_t = -(f'(k^\star - \Delta k) - (1+n))\Delta k \text{ for } t > T$$

$$f'(k^\star - \Delta k) - (1+n) < 0 \rightarrow \Delta c_t > 0 \text{ for } t > T$$

dealing with dynamic inefficiencies?

- Fully-funded system: young make contributions to the Social Security System and contributions are paid back in their old age.
- Unfunded system (pay as you go): transfers from the young go directly to the current old.
- Pay-as-you-go discourages aggregate savings. $\rightarrow$ may lead to Pareto improvement.

Social Security

dealing with dynamic inefficiencies?

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Fully Funded Social Security

- Individual's problem

$$\max_{c_t^t, c_{t+1}^t, s_t} u(c_t^t) + \beta u(c_{t+1}^t)$$

subject to

$$c_t^t + s_t + d_t \leq w_t$$

$$c_{t+1}^t \leq R_{t+1}(s_t + d_t)$$

- Gvn'ment raises d_t from young, invest in the capital stock, and pays $R_{t+1} d_t$ when old.
- Market clearing for capital requires

$$s_t + d_t = (1 + n) k_{t+1}$$

Fully Funded Social Security

characterization

- no longer the HH chooses $s_t > 0$ necessarily.
- if s_t is unconstrainted, then given a sequence $\{d_t\}_{t=0}^{\infty}$ (feasible), the set of CE without social security is the set of CE with social security if $s_t > 0$.
- if one imposes $s_t \geq 0$, i.e. no borrowing, then a sequence $\{d_t\}_{t=0}^{\infty}$ (feasible) is a CE if the equilibrium savings is such that $s_t > 0$ for all t.
- Implications
  - There can't be a Pareto improvement if we impose $s_t \geq 0$

Unfunded Social Security

- Individual's problem

$$\max_{c_t^t, c_{t+1}^t, s_t} u(c_t^t) + \beta u(c_{t+1}^t)$$

subject to

$$c_t^t + s_t + d_t \leq w_t$$

$$c_{t+1}^t \leq R_{t+1} s_t + (1+n) d_{t+1}$$

- Gvn'ment raises d_t from young, and distributes to the current old with a transfer $b_t = (1+n)d_t$
- Rate of return on social security is $1+n$ rather than $R_{t+1} - 1$ (pure transfer)
- Only s_t goes to capital accumulation.

UnFunded Social Security

characterization

- Unfunded Social Security reduces capital accumulation, negative consequences on growth and welfare?
- If the economy is dynamically inefficient this may be good!

  ... But much of the evidence show that capital accumulation in poorer societies is suboptimally low.
- Social Security transfers resources from future generations to initial old generation.

  ... with no dynamic inefficiency any transfer of resources would make some future generation worse off!