Midterm Exam
ECON 6140
Fall 2014

This is your midterm exam. You have 1 hour and 45 minutes to answer these questions. The "relative prices" can be found next to each question, for a total of 150 points. The first and last sections are worth 60 points, the second section is worth, 30 points. If you get stuck at some point in the exam, keep moving and come back to those questions at the end.
I will answer questions for the 15 minutes only. Hence, read the instructions carefully before writing anything. If you need additional assumptions to solve any of the questions please state them clearly.

Investment Specific Technology

Consider an economy populated by a large number of identical infinitely lived households. Suppose that preferences are defined over consumption c_t and leisure l_t as

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t) + v(l_t)$$

Assume that u is strictly concave, increasing in both arguments, and twice differentiable. Suppose that the time endowment is 1, and denote n_t the labor supply.

$$n_t + l_t \leq 1$$

There are two sectors in the economy, one produces consumption goods and the other one produces capital goods. Allocations are feasible if

$$c_t \leq F(n_t^c, k_t^c) \qquad x_t \leq G(n_t^x, k_t^x) \tag{1}$$

$$k_{t+1} \leq (1-\delta)k_t + x_t \qquad k_0 > 0 \tag{2}$$

$$n_t^c + n_t^x \leq n_t \qquad \text{and} \qquad k_t^c + k_t^x \leq k_t \tag{3}$$

The functions F and G are assumed increasing in each argument, concave, and twice differentiable, and such that the marginal product of capital converges to zero as the allocation of capital to any of the sectors goes to infinity for any given value of labor, n.

1. [5pt] Describe the conditions that define an interior solution to the planner's problem.

   Std. concacity of F and G and concavity of U plus the Inada conditions.

2. [10pt]Does the Second Welfare Theorem hold? Argue which condition(s) is/or is not satisfied. If it does, go as far as you can characterizing the equilibrium price of capital for any decentralization.

   2WT hold, because strict concavity of F, G, U and v assure this is a standard convex problem.

   Apologies for the bad wording in the question, I meant characterize the price of investment goods relative to consumption.

   The cost of capital is the marginal product of capital in the **consumption sector** $F_2(n_{t+1}^c, k_{t+1}^c)$.

   The relative price of investment to consumption goods is more cumbersome. If the household buys consumption and investment goods, its budget constraint is

$$c_t + p_t^x(k_{t+1} - (1-\delta)k_t) \le r_t k_t + w_t n_t$$

   From the optimality conditions you get

$$p_t^x = \frac{u'(c_{t+1})}{u'(c_t)}\beta\left[r_{t+1} + p_{t+1}^x(1-\delta)\right]$$

   Call the lagrange multiplier associated to the technology in investment goods in the planner's problem μ_t, and the one associated to feasibility in capital, κ_t. From the optimality conditions of the planner you get

$$\mu_t = \beta\left[\kappa_{t+1} + (1-\delta)\mu_{t+1}\right]$$

   From the optimality condition of k_t^x, $\mu_{t+1}G_2(n_{t+1}^x, k_{t+1}^x) = \kappa_{t+1} = u'(c_{t+1})F_2(n_{t+1}^c, k_{t+1}^c)$. Hence,

$$\frac{F_2^t/G_2^t}{F_2^{t+1}/G_2^{t+1}} = \frac{u'(c_{t+1})}{u'(c_t)}\beta\left[G_2(n_{t+1}^x, k_{t+1}^x) + (1-\delta)\right]$$

   and $p_t^x = F_2^t/G_2^t$.

   Is this decentralization unique? To think..

3. [5pt] Find conditions for existence of a steady state.

   this is the standard in the notes, continuity and restrictions on beta and delta.

4. [5pt] Describe the differential equations that dictate the dynamic of aggregate capital, labor, consumption

   $\frac{F_2^t/G_2^t}{F_2^{t+1}/G_2^{t+1}} = \frac{u'(c_{t+1})}{u'(c_t)}\beta\left[G_2(n_{t+1}^x, k_{t+1}^x) + (1-\delta)\right]$

   $k_{t+1} - (1-\delta)k_t = G(n_t^x, k_t^x)$

   and the condition for labor is a static one,

   $v'(1-n_t) = \nu_t$ where ν_t is the lagrange multiplies associated to the feasibility constraint in labor, $F_1^{-1}(n_t^c, k_t^c) + G_1^{-1}(n_t^x, k_t^x) = n_t$

5. [10pt] Describe the allocation of labor and capital across sectors. Does it change in time? How? Why?

   From the optimality condition and assuming production functions are HOD1 you get

$$\frac{F_2(1,\frac{k_t^c}{n_t^c})}{F_1(1,\frac{k_t^c}{n_t^c})} = \frac{G_2(1,\frac{k_t-k_t^c}{n_t-n_t^c})}{G_1(1,\frac{k_t-k_t^c}{n_t-n_t^c})}$$

   which characterizes equilibrium capital labor ratios along the transition. In steady state, you can use the euler equation to pin it down,

$$1 = \beta\left[G_2(1,\frac{k_{ss}-k^c}{n_{ss}-n^c}) + (1-\delta)\right]$$

   Along the transition capital labor ratios will change. Assume that capital is relatively scarce initially $k_0 < k_{ss}$. We proved, that along the transition capital will increase. How capital labor ratios will move in each sector will depend on which sector is more capital intensive and which one is more productive.

   EXERCISE: assume technologies are Hicks neutral, and the shares of capital and labor are the same in F and G. In this case you can characterize the path.

6. Assume that F and G are HOD1 in capital and labor. Suppose there is an economy where the technology for producing investment goods is $bG(n,k)$ with $b > 1$ (a better financial system for example). What does the model say about the long run (i.e. steady state) differences across economies in

   (a) [5pt] the capital labor ratio k/n
   Use

$$1 = \beta\left[bG_2(1,\frac{k_{ss}-k^c}{n_{ss}-n^c}) + (1-\delta)\right]$$

   so that the capital labor ratio in the investment sector is higher in this economy.
   Use

$$\frac{F_2(1,\frac{k_t^c}{n_t^c})}{F_1(1,\frac{k_t^c}{n_t^c})} = \frac{G_2(1,\frac{k_t-k_t^c}{n_t-n_t^c})}{G_1(1,\frac{k_t-k_t^c}{n_t-n_t^c})}$$

   to show that the relative capital labor ratios across sectors, are independent of b. Hence, k/n must be higher in the economy with $b > 1$

   (b) [5pt] the allocation of labor across sectors
   Assume marginal product of labor is increasing in capital (Cobb-Douglas for example). Then, there will be more labor in sector G in the economy with $b > 1$

   (c) [5pt] output per worker
   Higher. Use the relationship in capital labor ratios across sectors.

   (d) [10pt] the price of capital
   The cost of capital, $r_t = F_2(1,\frac{k_{t+1}^c}{n_{t+1}^c})$ is lower in the economy with $b > 1$ (look at the argument for $6.a.$)
   The price of investment goods relative to consumption $p_t^x = F_2^t/G_2^t$ is **constant** given the form of the "productivity" improvement ("hicks neutral") (again (look at the argument for $6.a.$))

Taking the problem to the computer

1. [10pt] Describe the dynamic program that you would need to solve to characterize equilibrium allocation in the economy described in question 1.

$$V(k) = \max_{c,n,n^x,n^c,k,k^x,k^c} u(c) + v(1-n) + \beta V(k\prime)$$

subject to

$$c \le F(n^c, k^c) \qquad k' \le G(n^x, k^x) + (1-\delta)k$$

$$k_0 > 0$$

$$n^c + n^x \le n \quad \text{and} \quad k^c + k^x \le k$$

1. [5pt] Can you use some of the results of your analysis in question 1.5 to simplify the program? If yes, explain which ones. Otherwise, move on.

   relationship between capital labor ratios.

2. [15pt] Describe the algorithm that you will need to implement to solve it, i.e. a list of the actions that your computer will have to complete.

   [HINT: you can use some of the features of the contraction mapping theorem to design the algorithm]

Corruption and Growth

Consider an economy populated by a continuum of measure one of identical individuals. Individuals' preferences are given by

$$\int_0^\infty e^{-\rho t} \ln(c_i(t))dt$$

where $\rho > 0$. Each period, households can consume or save on an asset $a(t)$, with return $r(t) - \delta$ each period. Notice that $r(t)$ is the after tax return on capital. There is no population growth in the economy.
Output is produced using capital from the private sector (K), and infrastructure provided by the government, (I). To finance such expenses, the government levies a tax on capital earnings (assets of the households), $\tau \in (0,1)$.
There is a representative firm in the economy,operating

$$Y(t) = AK(t) + F(K(t), I(t))$$

where F is HOD1 in capital and infrastructure, strictly concave, and increasing in each argument. $F(K(t), I(t)) = [K(t)^\rho + I(t)^\rho]^{\frac{1}{\rho}}$ with $\frac{1}{1-\rho}$ being the elasticity of substitution between capital and infrastructure, and $\rho \in (0, \infty)$.
The technology for generating infrastructure is such that

$$\dot{I}(t) = \varphi(t) + (1-\delta_I)I(t)$$

where $\varphi(t)$ are the revenues of the government from tax collection.

1. Competitive Equilibrium

   (a) [5pt] Define the problem of a firm.

$$\pi(t) = \max_{K(t)} AK(t) + F(K(t), I(t)) - \chi(t)K(t)$$

   Note that there will be positive profits of the firm, as there are DRS in capital, $K(t)$

   (b) [5pt] Define the problem of the households.

$$\max_{c_i(t),\dot{a}(t)} \int_0^\infty e^{-\widehat{\rho}t} \ln(c_i(t))dt$$

$$c_i(t) + \dot{a}(t) = (r(t) - \delta)a(t) + \pi(t) \quad \text{and } a(0)\text{given}$$

   (c) [5pt] Describe the budget constraint of the government.

$$\tau(t)\chi(t)a(t) = \varphi(t)$$

   (d) [5pt] Define a competitive equilibrium

   The only tricky part is to realize that $K(t) = a(t)$ (assumming you already figured that profits are positive)

   Note that infraestructure investment is financed in "goods", because it depends on the return (in goods) of the available capital.

   The feasibility condition in goods is

$$c(t) + \dot{K}(t) + \delta K(t) + \varphi(t) = AK(t) + F(K(t), I(t))$$

$$c(t) + \dot{K}(t) + \delta K(t) + \dot{I}(t) + \widehat{\delta}_I I(t) = AK(t) + F(K(t), I(t))$$

   (see below for $\widehat{\delta}_I = -(1- \delta_I))$

2. [10pt] Provide conditions for the existence of a Balanced Growth Path (BGP)

   For the BGP to exist you need τ constant. We also need $\delta_I > 1$, (this is because I wrote the dynamic in an odd way.. sorry, the proper way would have been $\dot{I}(t) = \varphi(t) - \widehat{\delta}_I I(t)$, for $\widehat{\delta}_I = -(1- \delta_I))$

   Finally, we need $\delta = \widehat{\delta}_I$

3. [5pt] Go as far as you can characterizing the BGP.

   The consumption side of the economy is standard with constant elasticity preferences $(\sigma = 1)$

$$\frac{\dot{c}(t)}{c(t)} = [r(t) - \delta - \widehat{\rho}]$$

$r(t) = \chi(t)(1-\tau(t))$ and $\chi(t) = A + \left[1 + \frac{I(t)}{K(t)}^{\rho}\right]^{\frac{1}{\rho}-1}$.

$$\frac{\dot{I}(t)}{I(t)} = \tau(t)\frac{\chi(t)a(t)}{I(t)} - \widehat{\delta}_I$$

so that for infraestructure to grow at a constant rate we need capital and investment to be growing a constant rate. Note that $\tau(t) = \tau$, why? the rate belongs to the interval $[0,1]$ so it cannot be growing/shrinking indefinitely.

From the production side,

$$\frac{Y}{K} = A + \left[1 + \frac{I(t)}{K(t)}^{\rho}\right]^{\frac{1}{\rho}}$$

so that we need $I(t)$ and $K(t)$ growing at the same rate again. From the budget constraint of the households

$$\frac{c_i(t)}{a(t)} + \frac{\dot{a}(t)}{a(t)} = ((A + \left[1 + \frac{I(t)}{K(t)}^{\rho}\right]^{\frac{1}{\rho}-1})(1-\tau) - \delta)$$

Hence capital and consumption grow at the same rate.

If $\widehat{\delta}_I > \delta$,for consumption and infraestructure to grow at the same rate we need higher investment in infraestructure. But this is proportional to the stock of capital (due to the tax as the MPK is constant if $I(t)$ and $K(t)$ grow at the same rate), hence it cannot grow faster than the stock of capital, which will yield a contradiction.

4. [10pt] Suppose that a fraction $(1-\theta)$ of revenues goes to corruption, so

$$\dot{I}(t) = \theta\varphi(t) - \widehat{\delta}_I I(t)$$

Go as far as you can characterizing the impact of corruption on the growth rate of consumption per capita.

From the dynamic of infraestructure

$$\frac{\dot{I}(t)}{I(t)} = \theta\tau\frac{\chi(t)a(t)}{I(t)} - \widehat{\delta}_I$$

hence, its growth rate is lower. From the production function we know that infraestructure and capital should grow at the same rate. Hence, from the equation describing the dynamics, the growth rate of consumption per capita is the same as before. Of course LEVELS! are very different. In particular the ratio of infraestructure to capital (level) is lower.

5. [5pt] Is the equilibrium Pareto efficient? Explain.

Set up a planner's problem to solve for it

$$\max_{c_i(t),\dot{I}(t),\dot{\widehat{K}}(t)} \int_0^{\infty} e^{-\widehat{\rho}t}\ln(c_i(t))dt$$

subject to

$$c_i(t) + \dot{K}(t) + \dot{I}(t) = AK(t) + F(K(t), I(t)) - \delta K(t) - \widehat{\delta}_I \; I(t)$$

From here you can see that the market allocation will be efficient.

What is going on? There are no externalities here. Infraestructure is financed in "goods", and each input is priced at its marginal cost.

6. [10pt] Suppose that the government wants to maximize the proceeds from corruption, $(1-\theta)\varphi(t)$. Goes as far as you can characterizing the tax policy that would achieve that.

   Set up the problem of a government,

$$max_\tau (1-\theta)\tau\chi(t)a(t)$$

   subject to

$$\chi(t) = A + \left[1 + \frac{I(t)}{K(t)}^{\rho}\right]^{\frac{1}{\rho}-1}$$

$$a(t) = K(t)$$

$$\frac{c_i(t)}{a(t)} + \frac{\dot{a}(t)}{a(t)} = ((A + \left[1 + \frac{I(t)}{K(t)}^{\rho}\right]^{\frac{1}{\rho}-1})(1-\tau) - \delta)$$

$$\frac{\dot{c}(t)}{c(t)} = [\chi(t)(1-\tau) - \delta - \widehat{\rho}]$$

$$\frac{\dot{I}(t)}{I(t)} = \tau(t)\frac{\chi(t)a(t)}{I(t)} - \widehat{\delta}_I$$

   The government does not want to charge "too" much because then households do not save, there is less capital and tax revenues fall. There will be a sort of Laffer curve.