About TA sections:

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Section time and location: 8:40am - 9:55am Rockefeller Hall 132

Office hours: Tuesday 4:30-5:30 pm in Uris Hall 451; other times available by appointment (just send me an email).

Our plan for today:[1]

[1] Materials adapted from notes provided by a previous Teaching Assistant, Zhuoheng Xu.

1 Dynamic Optimization in Continuous Time: Finite-Horizon

1.1 Notations

In continuous time, our variables depend on time and we can differentiate them with respect to time. Denote $x(t)$ as a variable x that depends on time t. Then, we can introduce the following notation:

$$\dot{x}(t) \equiv \frac{dx(t)}{dt} = \lim_{\Delta_t \to 0} \frac{x(t+\Delta_t) - x(t)}{\Delta_t}.$$

The growth rate of variable x, denoted γ_x, is then given by

$$\gamma_x \equiv \frac{\dot{x}(t)}{x(t)}.$$

If x grows at a constant rate (i.e., exponentially) and we know the initial level of x ($x(0) = x_0$), we can solve this as a simple differential equation and get a particular solution:

$$x(t) = x_0 e^{\gamma_x t}.$$

A discrete time analog is given by

$$\gamma_x = \frac{x_{t+1} - x_t}{x_t},$$

and

$$x_t = x_0(1+\gamma_x)^t$$

(as an approximation holds for small γ_x).

1.2 Basic Setup

Consider the following finite-horizon continuous time problem

$$\begin{aligned}
&\max_{x(t),y(t)} W(x(t), y(t)) \equiv \int_0^{t_1} f(t, x(t), y(t))dt \\
&\text{s.t.} \quad \dot{x}_i = g_i(t, x(t), y(t)) \quad \text{for } i = 1, 2, ...n \text{ and for all } t \\
&\qquad x \in \mathcal{X} \quad \text{for all } t \\
&\qquad y \in \mathcal{Y} \quad \text{for all } t \\
&\qquad x_i(0) = x_{i0}, \quad x_{i0} \text{ - given}, \quad \text{for } i = 1, 2, ...n
\end{aligned}$$

where

- $f(t, x(t), y(t))$ is the objective function, assumed to be twice continuously differentiable.
- $x(t)$ is an n-dimensional vector of **state variables** governed by law of motion $\dot{x}_i \equiv \frac{\partial x_i(t)}{\partial t} = g_i(t, x(t), y(t))$, with g_i assumed to be twice continuously differentiable. A state variable describes the state of the system at each point in time.
- $y(t)$ is an m-dimensional vector of **control variables** which belongs to the set $\mathcal{Y} \in \mathbb{R}^m$ nonempty and convex.

We are interested in finding an optimal solution $(\hat{x}(t), \hat{y}(t))$.

1.3 Variational Arguments

For simplicity, suppose both the state and control variables are one-dimensional. Let $(\hat{x}(t), \hat{y}(t))$ be the optimal solution. Moreover, we assume that $\hat{y}(\cdot)$ is continuous on $[0, t_1]$, $\hat{x}(t) \in \mathrm{Int}\mathcal{X}$ and $\hat{y}(t) \in \mathrm{Int}\mathcal{Y}$ for all t.

Now consider a **variation** of the function $\hat{y}(t)$, which is defined by

$$y(t, \varepsilon) \equiv \hat{y}(t) + \varepsilon\eta(t)$$

Interpretation: Given a perturbation function $\eta(t)$, the control variable $y(t, \varepsilon)$ represents a variation of $\hat{y}(t)$ where ε determines the perturbation's magnitude.

Note that for any given function $\eta(\cdot)$, there exists some $\varepsilon'_\eta > 0$ such that

$$y(t, \varepsilon) \equiv \hat{y}(t) + \varepsilon\eta(t) \in \mathrm{Int}\mathcal{Y}$$

for all $t \in [0, t_1]$ and for all $\varepsilon \in [-\varepsilon'_\eta, \varepsilon'_\eta]$.

Define the path of the state variable corresponding to the path of control $y(t, \varepsilon)$ as $x(t, \varepsilon)$

$$\dot{x}(t, \varepsilon) = g(t, x(t, \varepsilon), y(t, \varepsilon))$$

for all $t \in [0, t_1]$ with $x(0, \varepsilon) = x_0$. Then for $\varepsilon \in [-\varepsilon_\eta, \varepsilon_\eta] \subset [-\varepsilon'_\eta, \varepsilon'_\eta]$ for some $\varepsilon_\eta \leq \varepsilon'_\eta$, we also have $x(t, \varepsilon) \in \mathcal{X}$.

The condition can be rewritten as

$$g(t, x(t, \varepsilon), y(t, \varepsilon)) - \dot{x}(t, \varepsilon) = 0$$

Thus for any $\lambda : [0, t_1] \to \mathbb{R}$ we have

$$\int_0^{t_1} \lambda(t)[g(t, x(t, \varepsilon), y(t, \varepsilon)) - \dot{x}(t, \varepsilon)]dt = 0$$

The function $\lambda(\cdot)$ is assumed to be continuously differentiable and is referred to as the **costate variable**. Analogous to Lagrange multipliers in static optimization, it represents the shadow price of the state variable.

Define

$$\mathcal{W}(\varepsilon) \equiv W(x(t, \varepsilon), y(t, \varepsilon)) = \int_0^{t_1} f(t, x(t, \varepsilon), y(t, \varepsilon))dt$$

Combining two equations above we have

$$\mathcal{W}(\varepsilon) = \int_0^{t_1} f(t, x(t, \varepsilon), y(t, \varepsilon)) + \lambda(t)[g(t, x(t, \varepsilon), y(t, \varepsilon)) - \dot{x}(t, \varepsilon)]dt$$

Integration by parts

$$\int_0^{t_1} \lambda(t)\dot{x}(t, \varepsilon)dt = \lambda(t_1)x(t_1, \varepsilon) - \lambda(0)x_0 - \int_0^{t_1} \dot{\lambda}(t)x(t, \varepsilon)dt$$

Substitute back to the equation above

$$\mathcal{W}(\varepsilon) = \int_0^{t_1} [f(t, x(t, \varepsilon), y(t, \varepsilon)) + \lambda(t)g(t, x(t, \varepsilon), y(t, \varepsilon)) + \dot{\lambda}(t)x(t, \varepsilon)]dt - \lambda(t_1)x(t_1, \varepsilon) + \lambda(0)x_0$$

Since $\hat{y}(t)$ is defined as the optimal control variable, then we must have

$$\mathcal{W}(\varepsilon) \leq \mathcal{W}(0) \quad \text{for all } \varepsilon \in [-\varepsilon_\eta, \varepsilon_\eta]$$

Apply Leibniz's Rule and evaluate at $\varepsilon = 0$

$$\begin{aligned}\mathcal{W}'(0) = &\int_0^{t_1} [f_x(t, \hat{x}(t), \hat{y}(t)) + \lambda(t)g_x(t, \hat{x}(t), \hat{y}(t)) + \dot{\lambda}(t)]x_\varepsilon(t, 0)dt \\ &+ \int_0^{t_1} [f_y(t, \hat{x}(t), \hat{y}(t)) + \lambda(t)g_y(t, \hat{x}(t), \hat{y}(t))]\eta(t, 0)dt - \lambda(t_1)x_\varepsilon(t_1, 0)\end{aligned}$$

where x_ε and y_ε are the partial derivatives of x and y with respect to ε, respectively. The partial derivatives of f and g are denoted by f_x and f_y and so on.

The optimality implies

$$\mathcal{W}'(0) = 0 \quad \text{for all } \eta(t)$$

Since the equation above holds for any continuously differentiable $\lambda(t)$ function, then consider the function $\lambda(t)$ that is a solution to the differential equation

$$-\dot{\lambda}(t) = f_x(t, \hat{x}(t), \hat{y}(t)) + \lambda(t) g_x(t, \hat{x}(t), \hat{y}(t))$$

with the boundary condition

$$\lambda(t_1) = 0$$

These conditions imply

$$\int_0^{t_1} [f_x(t, \hat{x}(t), \hat{y}(t)) + \lambda(t) g_x(t, \hat{x}(t), \hat{y}(t)) + \dot{\lambda}(t)] x_\varepsilon(t, 0) dt - \lambda(t_1) x_\varepsilon(t_1, 0) = 0$$

Combining this result with the optimality condition we must have

$$\int_0^{t_1} [f_y(t, \hat{x}(t), \hat{y}(t)) + \lambda(t) g_y(t, \hat{x}(t), \hat{y}(t))] \eta(t, 0) dt = 0$$

Since the function $\eta(t)$ is arbitrary, we need to have

$$f_y(t, \hat{x}(t), \hat{y}(t)) + \lambda(t) g_y(t, \hat{x}(t), \hat{y}(t)) = 0 \quad \text{for all } t \in [0, t_1]$$

So far, we are able to conclude the necessary conditions of the maximum.

Theorem (Necessary Conditions) Consider the maximization problem described in the Basic Setup section with the additional assumption that $x(t)$ and $y(t)$ are one-dimensional. Suppose that this problem has an interior continuous solution $(\hat{x}(t), \hat{y}(t)) \in \text{Int}\mathcal{X} \times \text{Int}\mathcal{Y}$. Then there exists a continuously differentiable costate function $\lambda(\cdot)$ defined on $t \in [0, t_1]$ such that

$$\begin{gathered}
\dot{x} = g(t, x(t), y(t)) \quad \text{for all } t \in [0, t_1] \\
-\dot{\lambda}(t) = f_x(t, \hat{x}(t), \hat{y}(t)) + \lambda(t) g_x(t, \hat{x}(t), \hat{y}(t)) \quad \text{for all } t \in [0, t_1] \\
f_y(t, \hat{x}(t), \hat{y}(t)) + \lambda(t) g_y(t, \hat{x}(t), \hat{y}(t)) = 0 \quad \text{for all } t \in [0, t_1] \\
\lambda(t_1) = 0
\end{gathered}$$

Remark: The condition that $\lambda(t_1) = 0$ is the transversality condition of continuous-time optimization problems. Intuitively, there is no value to having more (or less) x after the planning horizon.

Remark: In some cases, there is a terminal condition $x(t_1) = x_1$. In these cases, we still have a similar set of necessary conditions, and the only difference is we will drop the condition $\lambda(t_1) = 0$ and add the condition $x(t_1) = x_1$.

1.4 The Maximum Principle of Pontryagin

The theorem presented in the previous section can be generalized into a multivariate problem, where the state and control variables are vectors. Define the Present Value Hamiltonian

$$H(t, x(t), y(t), \lambda(t)) \equiv f(t, x(t), y(t)) + \sum_{i=1}^{n} \lambda_i(t) g_i(t, x(t), y(t))$$

Note that since f and g are assumed to be continuously differentiable, the Hamiltonian H is also continuously differentiable in all its arguments.

Theorem (The Maximum Principle): Consider the maximization problem described in the Basic Setup section. Suppose that this problem has an interior continuous solution $\hat{y}(t) \in \text{Int}\mathcal{Y}$ with corresponding path of state variable $\hat{x}(t)$. Then there exists a continuously differentiable functions $\lambda(t) = (\lambda_1(t), ..., \lambda_n(t))'$ such that the optimal control $\hat{y}(t)$ and the corresponding path of the state variable $\hat{x}(t)$ satisfy the following necessary conditions:

$$\begin{gathered}
\frac{\partial H(\cdot)}{\partial y_j} = 0 \quad \text{for } j = 1, 2, ..., m \\
\frac{\partial H(\cdot)}{\partial x_i} = -\dot{\lambda}_i(t) \quad \text{for } i = 1, 2, ..., n \\
\frac{\partial H(\cdot)}{\partial \lambda_i} = \dot{\hat{x}}_i(t) \quad \text{for } i = 1, 2, ..., n \\
\lambda_i(t_1) \geq 0 \quad \lambda_i(t_1)\hat{x}_i(t_1) = 0 \quad \text{for } i = 1, 2, ..., n \quad \text{[TVC]}
\end{gathered}$$

with the Hamiltonian

$$H(t, x(t), y(t), \lambda(t)) = f(t, x(t), y(t)) + \sum_{i=1}^{n} \lambda_i(t) g_i(t, x(t), y(t))$$

Moreover, the Hamiltonian $H(t, x(t), y(t), \lambda(t))$ also satisfies the Maximum Principle that

$$H(t, \hat{x}(t), \hat{y}(t), \lambda(t)) \geq H(t, \hat{x}(t), y, \lambda(t))$$

for all $y \in \mathcal{Y}$ and for all $t \in [0, t_1]$.

The theorem states **necessary** conditions for a continuous interior solution but does not guarantee neither that such solution exists not that it is a global maximum.

For these conditions to be **sufficient**, we need the Hamiltonian $H(t, x(t), y(t), \lambda(t))$ to be jointly concave in (x, y) (see Mangasarian's sufficiency conditions). Moreover, if $H(t, x(t), y(t), \lambda(t))$ is strictly concave, then the solution that satisfies conditions above is unique.

2 Dynamic Optimization in Continuous Time: Discounted Infinite-Horizon

2.1 Basic Setup

Most economic problems, including almost all growth models, are more naturally formulated as infinite-horizon problems. Furthermore, in most growth models, future utility is discounted exponentially at a constant rate ρ to account for time preference. Taking this into account, the utility maximization problem can then be reformulated as:

$$\begin{aligned}
&\max_{x(t),y(t)} \int_0^\infty e^{-\rho t} f(t, x(t), y(t))dt \\
\text{s.t.} \quad & \dot{x}_i = g_i(t, x(t), y(t)) \quad \text{for } i = 1, 2, ...n \text{ and for all } t \\
& x \in \mathcal{X} \subseteq \mathbb{R}^n \quad \text{for all } t \\
& y \in \mathcal{Y} \subseteq \mathbb{R}^m \quad \text{for all } t \\
& x_i(0) = x_{i0} \quad \text{for } i = 1, 2, ...n
\end{aligned}$$

There are two common formulations: the **present-value Hamiltonian** and the **current-value Hamiltonian**. The difference between them lies in the treatment of the costate variable and discounting.

The **present-value Hamiltonian** in this case would be

$$H(t, x(t), y(t), \lambda(t)) \equiv e^{-\rho t} f(t, x(t), y(t)) + \sum_{i=1}^{n} \lambda_i(t) g_i(t, x(t), y(t))$$

The costate equation follows:

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x}.$$

The **current-value Hamiltonian** is

$$H_c(t, x(t), y(t), \tilde{\lambda}(t)) \equiv f(t, x(t), y(t)) + \sum_{i=1}^{n} \tilde{\lambda}_i(t) g_i(t, x(t), y(t))$$

The costate equation in this formulation is:

$$\dot{\tilde{\lambda}}(t) = -\frac{\partial \tilde{H}}{\partial x} + \rho \tilde{\lambda}(t).$$

Both formulations are mathematically equivalent and interchangeable via the transformation $\tilde{\lambda}(t) = e^{-\rho t}\lambda(t)$. The current-value Hamiltonian $H_c \equiv e^{\rho t}H$, with H being the present-value Hamiltonian. Therefore, the problem could perfectly be solved using the present-value Hamiltonian and in fact it would lead to the same solution.

2.2 Solving Current Value Hamiltonian

We can use the Maximum Principle on its infinite-horizon version to solve the optimal control problem. Define the Current Value Hamiltonian

$$H_c(t, x(t), y(t), \mu(t)) \equiv f(t, x(t), y(t)) + \sum_{i=1}^{n} \mu_i(t) g_i(t, x(t), y(t))$$

with $\mu_i(t) \equiv e^{\rho t}\lambda_i(t)$

Necessary conditions for a maximum:

- $\frac{\partial H_c(\cdot)}{\partial y_j} = 0 \quad \text{for } j = 1, 2, ..., m$
- $\frac{\partial H_c(\cdot)}{\partial x_i} = -\dot{\mu}_i(t) + \rho\mu_i(t) \quad \text{for } i = 1, 2, ..., n$
- $\frac{\partial H_c(\cdot)}{\partial \mu_i} = \dot{\hat{x}}_i(t) \quad \text{for } i = 1, 2, ..., n$
- $e^{-\rho t}\mu_i(T) \geq 0, \lim_{T\to\infty} e^{-\rho T}\mu_i(T)\hat{x}_i(T) = 0 \quad \text{for } i = 1, 2, ..., n \quad$ [TVC]

We still need concavity of H in y and x for these conditions to be sufficient.

3 Application: One Sector Growth Model

Consider a neoclassical economy with a representative household with preferences given by

$$\int_0^\infty e^{-\rho t}\frac{c(t)^{1-\theta}}{1-\theta}dt$$

The agent supplies labor inelastically and receives wage income $w(t)$.

The flow budget constraint is

$$c(t)+\dot{a}(t)=r(t)a(t)+w(t)$$

Plus assume we know the initial level of assets and impose no Ponzi condition to prevent infinite borrowing.

For now, our goal is to find the growth rate of consumption.

Present-value Hamiltonian:

$$H(t,c(t),a(t),\lambda(t))=e^{-\rho t}\frac{c(t)^{1-\theta}}{1-\theta}+\lambda(t)[r(t)a(t)+w(t)-c(t)]$$

Maximum Principle:

- $[c(t)]\quad \frac{\partial H(\cdot)}{\partial c(t)}=e^{-\rho t}c(t)^{-\theta}-\lambda(t)=0$
- $[a(t)]\quad \frac{\partial H(\cdot)}{\partial a(t)}=\lambda(t)r(t)=-\dot{\lambda}(t)$
- $[\lambda(t)]\quad \frac{\partial H(\cdot)}{\partial \lambda(t)}=r(t)a(t)+w(t)-c(t)=\dot{a}(t)$
- $[TVC]\ \lim_{t\to\infty}\lambda(t)a(t)=0$

The transversality condition becomes

$$\lim_{t\to\infty}a(t)e^{-r(t)t}=0$$

Consider the FOC about $c(t)$

$$\begin{aligned}\dot{\lambda}(t) \equiv \frac{\partial \lambda(t)}{\partial t} &= (-\rho)e^{-\rho t}c(t)^{-\theta} + (-\theta)e^{-\rho t}c(t)^{-\theta-1}\frac{\partial c(t)}{\partial t} \\ &= (-\rho)e^{-\rho t}c(t)^{-\theta} + (-\theta)e^{-\rho t}c(t)^{-\theta-1}\dot{c}(t) \\ &= e^{-\rho t}c(t)^{-\theta}[-\rho - \theta c^{-1}(t)\dot{c}(t)]\end{aligned}$$

Consider the FOC about $a(t)$

$$\begin{aligned}\lambda(t)r(t) &= -\dot{\lambda}(t) \\ e^{-\rho t}c(t)^{-\theta}r(t) &= e^{-\rho t}c(t)^{-\theta}[\rho + \theta c^{-1}(t)\dot{c}(t)] \\ r(t) &= \rho + \theta\frac{\dot{c}(t)}{c(t)} \\ \frac{\dot{c}(t)}{c(t)} &= \frac{r(t)-\rho}{\theta}\end{aligned}$$

The growth rate of consumption, which is defined as $\gamma_C = \frac{\dot{c}(t)}{c(t)}$, is equal to $\frac{r(t)-\rho}{\theta}$.

Does this remind you the Euler equation we have seen so many times in discrete time?

Remark: Alternatively, you can take natural logs of the FOC for $c(t)$, then differentiate w.r.t. t, and then substitute $\dot{\lambda}(t)/\lambda(t)$ from the FOC for $a(t)$. You will obtain the same expression for γ_C (you can check yourself). I would do it this way, but you can choose whatever you find more intuitive. In any case, it is useful to know the formula:

$$\frac{\dot{x}(t)}{x(t)} = \frac{\partial log[X(t)]}{\partial t}.$$