About TA sections:

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In these notes:

1 Classical Monetary Model (CMM): Solving the Model

1.1 Setup

Household Problem The representative household maximizes its expected discounted utility subject to the budget constraint and the solvency constraint:

$$\max_{C_t,B_t,N_t} E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t)$$

subject to

$$P_t C_t + Q_t B_t \leq B_{t-1} + W_t N_t + D_t$$

$$\lim_{T\to\infty} E_t\{\Lambda_{t,T} \frac{B_T}{P_T}\} \geq 0$$

where $\Lambda_{t,T} = \beta^{T-t} \frac{U_{c,T}}{U_{c,t}}$ is the stochastic discount factor.

The household chooses how much to consume C_t, how much to work N_t, and how much to invest in bonds B_t.

Firm Problem In each period, firms maximize their profits:

$$\max_{N_t} P_t Y_t - W_t N_t$$

where $Y_t = A_t N_t^{1-\alpha}$ describes the production function and A_t follows the AR(1) process: $a_t = \rho_a a_{t-1} + \varepsilon_t^a$, where $a_t \equiv \log(A_t)$.

Firms choose how much labor to hire in each period.

1.2 Equilibrium

Solving the household's problem The Lagrangian of the household problem:

$$\mathcal{L}(C_t, N_t, B_t, \lambda_t) = E_0 \sum_{t=0}^{\infty} \beta^t [U(C_t, N_t) + \lambda_t (B_{t-1} + W_t N_t + D_t - P_t C_t - Q_t B_t)]$$

FOCs:

$$\frac{\partial \mathcal{L}}{\partial C_t} = \beta^t (U_{c,t} - \lambda_t P_t) = 0$$

$$\frac{\partial \mathcal{L}}{\partial N_t} = \beta^t (U_{n,t} + \lambda_t W_t) = 0$$

$$\frac{\partial \mathcal{L}}{\partial B_t} = \beta^{t+1} E_t \lambda_{t+1} - \beta^t \lambda_t Q_t = 0$$

Combine first two FOCs:

$$\lambda_t = \frac{U_{c,t}}{P_t} = -\frac{U_{n,t}}{W_t} \implies U_{c,t}\frac{W_t}{P_t} = -U_{n,t}.$$

Intuition: This means that households equate the marginal utility loss from working an additional unit of labor to the marginal utility gain from consuming the real wage earned.

From the third FOC, substituting λ from the expression above:

$$Q_t = \beta E_t \frac{\lambda_{t+1}}{\lambda_t} = \beta E_t \left[\frac{U_{c,t+1}}{U_{c,t}} \frac{P_t}{P_{t+1}} \right],$$

or equivalently,

$$U_{c,t} = \beta Q_t^{-1} E_t \left[U_{c,t+1} \frac{P_t}{P_{t+1}} \right].$$

Intuition: The household chooses consumption today to balance the marginal utility of consuming now versus the expected discounted utility of consuming in the next period, while accounting for corresponding price levels.

If we are given the following functional forms

$$U(C_t, N_t) = \begin{cases} \frac{C_t^{1-\sigma}-1}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi}, \sigma \neq 1, \\ \log C_t - \frac{N_t^{1+\varphi}}{1+\varphi}, \sigma = 1, \end{cases} \implies U_{c,t} = C_t^{-\sigma}, U_{n,t} = -N_t^{\varphi},$$

The optimality conditions become:

Labor supply decision:

$$\frac{U_{n,t}}{U_{c,t}} = -\frac{W_t}{P_t} \implies \frac{N_t^{\varphi}}{C_t^{-\sigma}} = \frac{W_t}{P_t}$$

Euler equation:

$$U_{c,t} = \beta Q_t^{-1} E_t \left[U_{c,t+1} \frac{P_t}{P_{t+1}} \right] \implies C_t^{-\sigma} = \beta Q_t^{-1} E_t \left[C_{t+1}^{-\sigma} \frac{P_t}{P_{t+1}} \right]$$

Solving the firm's problem Denote the profit function Π. FOC:

$$\frac{\partial \Pi_t(N_t)}{\partial N_t} = P_t(1-\alpha)A_t N_t^{-\alpha} - W_t = 0$$

$$\implies \frac{W_t}{P_t} = (1-\alpha)A_t N_t^{-\alpha}.$$

Intuition: This should look familiar. Since, in the CMM, firms are perfectly competitive, they hire labor until its marginal product equals its cost to a firm.

Market Clearing Conditions

Goods market: $Y_t = C_t, \forall t$.

Labor market: $N_t^s = N_t^d, \forall t$.

1.3 Log-linearized Version of Equilibrium

Notations: we will use familiar notations as $x_t \equiv \log(X_t)$.

Gather the system of equilibrium condition in linearized version:

1. Household's optimality conditions:

$$\varphi n_t + \sigma c_t = w_t - p_t, \tag{1}$$

$$c_t = E_t\{c_{t+1}\} - \frac{1}{\sigma}(i_t - \rho - E_t\{\pi_{t+1}\}), \tag{2}$$

where lowercase letters stand for logarithms of corresponding variables, $\rho = -\log \beta$, $i_t = -\log Q_t$, $\pi_{t+1} = \log P_{t+1} - \log P_t$.

2. Firms' optimality condition - labor demand:

$$w_t - p_t = log(1-\alpha) + a_t - \alpha n_t. \tag{3}$$

3. Market clearing:

$$y_t = c_t. \tag{4}$$

To solve the model, we will proceed with the following steps:

1. Log-linearize the production function:

$$y_t = a_t + (1-\alpha)n_t$$

2. Combine the equation above with the market clearing condition:

$$c_t = a_t + (1-\alpha)n_t$$

3. Combine (1) and (3):

$$\varphi n_t + \sigma c_t = log(1-\alpha) + a_t - \alpha n_t$$

4. Substitute c_t from Step 2:

$$\varphi n_t + \sigma(a_t + (1-\alpha)n_t) = log(1-\alpha) + a_t - \alpha n_t$$

5. Simplify and find n_t:

$$n_t = \frac{log(1-\alpha) + (1-\sigma)a_t}{\varphi + \alpha + \sigma(1-\alpha)} \implies$$

$$\implies n_t = \psi_{na}a_t + \psi_n$$

where $\psi_{na} = \frac{1-\sigma}{\varphi+\alpha+\sigma(1-\alpha)}$, $\psi_n = \frac{log(1-\alpha)}{\varphi+\alpha+\sigma(1-\alpha)}$.

6. Find y_t:

$$y_t = a_t + (1-\alpha)(\psi_{na}a_t + \psi_n)$$

$$\implies y_t = a_t(1 + (1-\alpha)\psi_{na})a_t + (1-\alpha)n_t$$

$$\implies y_t = \psi_{ya}a_t + \psi_y$$

where $\psi_{ya} = \frac{1+\varphi}{\varphi+\alpha+\sigma(1-\alpha)}$, $\psi_y = (1-\alpha)\psi_n$.

7. Find ω_t:

$$\omega_t = w_t - p_t = log(1-\alpha) + a_t - \alpha n_t \implies$$

$$\implies \omega_t = log(1-\alpha) + a_t - \alpha(\psi_{na}a_t + \psi_n) \implies$$

$$\omega_t = log(1-\alpha) - \alpha\psi_n + (1-\alpha\psi_{na})a_t \implies$$

$$\implies \omega_t = \psi_{\omega a}a_t + \psi_\omega$$

where $\psi_{\omega a} = \frac{\varphi+\sigma}{\varphi+\alpha+\sigma(1-\alpha)}$, $\psi_\omega = \frac{(\varphi+\sigma(1-\alpha))log(1-\alpha)}{\varphi+\alpha+\sigma(1-\alpha)}$.

Intuition: Why do we need all of this? Macroeconomists often aim to understand how the variables in their models react to shocks of various types. To do this, they analyze impulse response functions. Recall that an impulse response is typically defined as the expected difference in a variable, conditional on a shock happening versus not happening.

For now, a_t is the only exogenous variable in our system, and any exogenous changes to this variable will propagate to other variables within the system. This is what we refer to as a "***productivity shock***" - unexpected changes in a_t.

Remark: The coefficients ψ_{xa} represent the contemporaneous responses to a productivity shock. It is good practice to examine their signs to understand the direction of variable movements and to interpret the underlying economic mechanisms driving these responses (within this model).

2 Useful Math for Moment Conditions

Recall that the evolution of a_t is described by an AR(1) process:

$$a_t = \rho a_{t-1} + \varepsilon_t,$$

where ρ is a persistence parameter, and we typically assume that $\rho_a \in (0, 1)$ to have a stationary process. Shock terms ε_t are i.i.d. (hence, do not exhibit serial correlation) with zero mean and finite variance.

Remark 1: More generally, we may allow $\rho_a \in [0, 1]$:

- If $\rho_a = 0$, the process becomes a white noise process.
- If $\rho_a = 1$, the process becomes a random walk (unit root), which is non-stationary.

Remark 2: In many applications, we assume $\varepsilon_t \overset{\text{i.i.d.}}{\sim} N(0, \sigma_\varepsilon^2)$. This allows for simple analytical derivation of distributional properties, but normality is not necessary for stationarity.

How do we find the variance of a_t? Using our assumptions above,

$$var(a_t) = var(\rho a_{t-1} + \varepsilon_t) = \rho^2 var(a_{t-1}) + \sigma_\varepsilon^2.$$

Note that $var(a_{t-1}) = var(a_t)$ because we assume that the process is stationary, so the variance does not change over time. Then, we can rearrange and get

$$var(a_t) = \frac{\sigma_\varepsilon^2}{1 - \rho^2}.$$

What if we want to compute $var(n_t)$?

Recall

$$n_t = \psi_{na} a_t + \psi_n.$$

Hence, we can write

$$var(n_t) = var(\psi_{na} a_t + \psi_n) = \psi_{na}^2 var(a_t) = \psi_{na}^2 \frac{\sigma_\varepsilon^2}{1 - \rho^2}.$$

3 CES Aggregation

3.1 Demand for Variety of Goods

Intuition: In the Classical Monetary Model (CMM), there was only one consumption good. In contrast, the basic New Keynesian model features household preferences defined over a variety of goods (known as "love of variety"). This is the main difference between the two models we have studied so far.

In this new formulation, we have

$$U(C) = \frac{C^{1-\sigma}}{1-\sigma}$$

where C_t is a CES aggregator over a continuum of goods

$$C \equiv \left(\int_0^1 C_i^{\frac{\varepsilon-1}{\varepsilon}} \, di \right)^{\frac{\varepsilon}{\varepsilon-1}} \quad : \quad \varepsilon > 1$$

When we solve the households problem as above, we still find the optimality condition for consumption but in aggregate, i.e., total of all the goods consumed. Now we want to see how this total consumption is allocated across all the goods available in the market.

When we solve the household's problem as above, we still find the optimality condition for consumption, but this time in aggregate, i.e., the total of all goods consumed. Next, we examine how this total consumption is allocated across the variety of goods available in the market.

When choosing their demand for a variety of goods, the household aims to maximize their total consumption (as more consumption is always better because the utility function is strictly increasing) subject to the budget constraint. In other words, given a certain amount of money, the household aims to choose a consumption basket that maximizes total consumption, and hence, utility. We can formalize this problem as follows:

$$\max_{C_i, i\in(0,1)} C \equiv \left(\int_0^1 C_i^{\frac{\varepsilon-1}{\varepsilon}} \, di \right)^{\frac{\varepsilon}{\varepsilon-1}} \quad : \quad \varepsilon > 1$$

$$\int_0^1 P_i C_i \, di \leq R$$

where R is the total amount spent on consumption.

$$L = \left(\int_0^1 C_i^{\frac{\varepsilon-1}{\varepsilon}} \, di \right)^{\frac{\varepsilon}{\varepsilon-1}} + \lambda \left(R - \int_0^1 P_i C_i \, di \right)$$

FOC:

$$\frac{\varepsilon}{\varepsilon-1}\frac{\varepsilon-1}{\varepsilon}\left(\int_0^1 C_i^{\frac{\varepsilon-1}{\varepsilon}}\,di\right)^{\frac{1}{\varepsilon-1}} C_i^{-\frac{1}{\varepsilon}} = \lambda P_i.$$

Note that, by definition, $\left(\int_0^1 C_i^{\frac{\varepsilon-1}{\varepsilon}}\,di\right)^{\frac{1}{\varepsilon-1}} = C^{\frac{1}{\varepsilon}}$. Hence, we can simplify the expression above as

$$C^{\frac{1}{\varepsilon}} C_i^{-\frac{1}{\varepsilon}} = \lambda P_i.$$

Next, multiply both sides by C_i

$$C^{\frac{1}{\varepsilon}} C_i^{1-\frac{1}{\varepsilon}} = \lambda P_i C_i$$

and integrate over i

$$C^{\frac{1}{\varepsilon}} \int_0^1 C_i^{\frac{\varepsilon-1}{\varepsilon}}\,di = \lambda \int_0^1 P_i C_i\,di.$$

Substituting $\int_0^1 C_i^{\frac{\varepsilon-1}{\varepsilon}}\,di = C^{\frac{\varepsilon-1}{\varepsilon}}$, we get

$$C^{\frac{1}{\varepsilon}} C^{\frac{\varepsilon-1}{\varepsilon}} = \lambda PC \iff P = \frac{1}{\lambda}$$

(as C on both sides cancels out).

Now we can substitute $P = \frac{1}{\lambda}$ into the FOC w.r.t. C_i

$$C^{\frac{1}{\varepsilon}} C_i^{-\frac{1}{\varepsilon}} = \frac{1}{P} P_i$$

which gives us

$$C_i = \left(\frac{P_i}{P}\right)^{-\varepsilon} C.$$

Intuition: Note that in CES aggregation ε represents the elasticity of substitution between different goods. As $\varepsilon \to \infty$, goods become perfect substitute and the economy converges to perfect competition.

Important remark: "Love of variety" introduces inefficiency into the model, as it grants firms monopolistic power, leading them to charge higher prices (with a markup) compared to a perfectly competitive market with identical goods.

3.2 Price Index

Note that we can use the same optimality condition to derive the expression for the price index:

Move one step back to

$$C^{\frac{1}{\varepsilon}} C_i^{-\frac{1}{\varepsilon}} = \frac{1}{P} P_i$$

$$\iff C^{\frac{1-\varepsilon}{\varepsilon}} C_i^{\frac{\varepsilon-1}{\varepsilon}} = P^{\varepsilon-1} P_i^{1-\varepsilon}.$$

Integrate over i

$$C^{\frac{1-\varepsilon}{\varepsilon}} \int_0^1 C_i^{\frac{\varepsilon-1}{\varepsilon}} \, di = P^{\varepsilon-1} \int_0^1 P_i^{1-\varepsilon} \, di.$$

Again, substituting $\int_0^1 C_i^{\frac{\varepsilon-1}{\varepsilon}} \, di = C^{\frac{\varepsilon-1}{\varepsilon}}$, we get

$$C^{\frac{1-\varepsilon}{\varepsilon}} C^{\frac{\varepsilon-1}{\varepsilon}} = P^{\varepsilon-1} \int_0^1 P_i^{1-\varepsilon} \, di.$$

The LHS equals 1, hence, the price index is simply

$$P = \left(\int_0^1 P_i^{1-\varepsilon} \, di \right)^{\frac{1}{1-\varepsilon}}.$$

3.3 Optimal Price Setting

Monopolistically competing firms choose prices to maximize their profits:

$$\max_{P_i} P_i Y_i - C(Y_i), \text{ where } C(\cdot) \text{ represents nominal cost.}$$

FOC:

$$Y_i + P_i \frac{\partial Y_i}{P_i} - \frac{\partial C(Y_i)}{\partial Y_i} \frac{\partial Y_i}{\partial P_i} = 0.$$

Denote marginal cost as $\Psi_i \equiv \frac{\partial C(Y_i)}{\partial Y_i}$ and use the fact that, through market clearing, $Y_i = \left(\frac{P_i}{P}\right)^{-\varepsilon} Y$. Dividing the expression above by Y_i, we get

$$1 + \frac{P_i}{Y_i} \frac{\partial Y_i}{P_i} - \Psi_i \frac{\partial Y_i}{\partial P_i} \frac{1}{Y} = 0$$

$$\implies 1 - \varepsilon + \Psi_i \frac{\varepsilon}{P_i} = 0.$$

Hence, the optimal price is

$$P_i = \frac{\varepsilon}{\varepsilon - 1} \Psi_i,$$

i.e., firms charge a constant markup $\mathcal{M} \equiv \frac{\varepsilon}{\varepsilon-1}$ over the nominal marginal cost.

4 Optional: Exam Practice

Try to answer the following questions:

- What are the decisions taken by the household? What are the optimality conditions determining these decisions?
- What are the decisions taken by firms? Characterize the optimality conditions for these decisions.
- Solve for equilibrium output and labor as functions of technology a_t (*Remark*: could also be y_t, n_t, or any other variable; idea is the same). How does the labor response to a shock in a_t depend on model parameters?
- Derive the optimal demand for good i as a function of aggregate output and the relative price of good i.
- What is the optimal price set by firm i when the prices are flexible (note: we have only considered flexible prices so far)? In what parameter limit does the model converge to a perfectly competitive economy?

***Remark*:** All answers can be found in the current set of notes. However, it is a good exercise to work through the derivations yourself. If you have any questions, I am always happy to help!