Imperfect Private monitoring

The problem with private monitoring

Consider the following simple two period game.

There are two players. In the first period, they play a prisoners' dilemma.

	Cooperate	Defect
Cooperate	$1, 1$	$-1, 2$
Defect	$2, -1$	$0, 0$

In periods 2, they play a coordination game:

	G	B
G	k, k	$0, 0$
B	$0, 0$	$1, 1$

with $k > 2$.

Perfect monitoring

With perfect monitoring, the following strategies will support first period cooperation:

Period 1 : Play C

Period 2 : Play G if first period outcome is (C, C), else play B.

These strategies lead to an equilibrium in the second period, regardless of the first period outcome.

In the first period, consider i's incentives:

Play $C \Rightarrow 1 + k$

Play $D \Rightarrow 2 + 1$

Since $k > 2$, it is optimal to follow the specified strategy strategy.

So we can sustain the efficient solution at $t = 1$.

Imperfect, private independent monitoring

Now suppose that first period actions (a_1, a_2) are not observed.

Rather, each player i observes a signal $y_i \in \{c, d\}$ about her opponent's action.

Suppose that:

$$\Pr(y_i = c|a_j) = \begin{cases} 1-\varepsilon & a_j = C \\ \varepsilon & a_j = D \end{cases}$$

If ε is small, monitoring is almost perfect.

However, we can verify that no pure-strategy equilibrium supports (C, C) in the first period.

Why?

Observe that in the second period, i will want to play G if and only if she assigns probability $1/(k+1)$ or greater to j playing G.

Consider strategies that call for each player to play C in the first period, and G in the second period if and only if $y_i = c$.

If i plays C in the first period, she assigns probability $1 - \varepsilon$ to j observing c, and hence to j playing G.

But then regardless of the signal she observes, she will want to play G, and so she won't follow the strategy.

Note that i and j's second period signals are **conditionally independent**.

So long as i **cooperates** in the first period, she will assign **high probability** to j observing **a good signal** regardless of her own signal.

Consequently, she prefers to just keep cooperating.

In short, the private monitoring means that:

- there is no way to coordinate on the punishment equilibrium in period two following a bad outcome;

- and on the good equilibrium otherwise.

Thus, there's no way to enforce cooperation in the first period.

Imperfect correlated monitoring

The coordination problem is reduced if the private signals are correlated rather than independent.

Let's consider the extreme case of perfect correlation.

Suppose for simplicity that $y_i = y_j = y \in \{c, d\}$, where:

$$\Pr(y_i = c, y_j = c | (a_1, a_2)) = \begin{cases} 1 - \varepsilon & a_1 = a_2 = C \\ \varepsilon & \text{else} \end{cases}$$

Consider strategies that call for each player to play C in the first period, and to play G in the second period if and only if they observe c.

This clearly gives an equilibrium in the second period, and checking first period incentives:

Play $C \Rightarrow\ 1 + (1 - \varepsilon)(k - 1) + 1$

Play $D \Rightarrow\ 2 + \varepsilon(k - 1) + 1$

It is optimal to follow the specified strategy if $k \geq 1 + 1/(1 - 2\varepsilon)$, which is ensured for ε small.

More generally, if y_i and y_j are highly, but not perfectly, correlated, it will be possible to support cooperation in the first period by coordinating on different second period play depending on the signals.

Because signals are correlated at $t = 2$, they "self enforce."

What we have now is really a game with imperfect public monitoring.

Mixed strategies

Even if the private signals are independent, the players may be able to correlate their beliefs by playing mixed strategies in the first period.

Returning to the independent signals set-up from above, consider the following strategies.

Period 1 : Play C, D with probabilities $\alpha, 1 - \alpha$

Period 2 : i plays G if and only if $a_i = C$ and $y_i = c$.

In the second period, i will want to play G if and only if she assigns probability $1/(k+1)$ or greater to j playing G.

She assigns this probability by conditioning on what she knows, her action a_i and her signal y_i:

$$\Pr(j \text{ will play } G|a_i, y_i) = Pr(y_j = c|a_i) \times Pr(a_j = C|y_i)$$

This because j plays G if $a_j = C$ and $y_j = c$ (if $a_j \neq C$, then it can't happen).

Bayes rule gives probabilities:

a_i, y_i	$\Pr(j \text{ will play } G \mid a_i, y_i)$
C, c	$(1-\varepsilon)\frac{(1-\varepsilon)\alpha}{(1-\varepsilon)\alpha+\varepsilon(1-\alpha)}$
C, d	$(1-\varepsilon)\frac{\varepsilon\alpha}{(1-\varepsilon)\alpha+\varepsilon(1-\alpha)}$
D, c	$\varepsilon\frac{(1-\varepsilon)\alpha}{(1-\varepsilon)\alpha+\varepsilon(1-\alpha)}$
D, d	$\varepsilon\frac{\varepsilon\alpha}{(1-\varepsilon)\alpha+\varepsilon(1-\alpha)}$

To see this note that:

$$\Pr(j \text{ playes } G; a_i, y_i) = \Pr(y_j = c; a_i) \bullet \Pr(a_j = C; y_i)$$

Consider $a_i, y_i = C, c$. We have $\Pr(y_j = c; a_i) = 1 - \varepsilon$; and:

$$\Pr(a_j = C; y_i) = \frac{\Pr(y_i; a_j = C)\Pr(a_j = C)}{\begin{bmatrix} \Pr(y_i; a_j = C)\Pr(a_j = C) \\ +\Pr(y_i; a_j = D)\Pr(a_j = D) \end{bmatrix}}$$

$$= \frac{(1-\varepsilon)\alpha}{(1-\varepsilon)\alpha + \varepsilon(1-\alpha)}$$

Consider $a_i, y_i = D, c$. We have: $\Pr(y_j = c; a_i) = \varepsilon$ and:

$$\Pr(a_j = C; y_i) = \frac{\Pr(y_i; a_j = C)\Pr(a_j = C)}{\left[\begin{array}{c}\Pr(y_i; a_j = C)\Pr(a_j = C) \\ +\Pr(y_i; a_j = D)\Pr(a_j = D)\end{array}\right]}$$

$$= \frac{\varepsilon\alpha}{(1-\varepsilon)\alpha + \varepsilon(1-\alpha)}$$

Aside for the (C, c) case, the probability that j will play G is of order ε.

So, for sufficiently small values of ε, i will be willing to following the prescribed strategy in the second period.

Now consider i's incentives in the **first period**. Her expected payoffs are:

Play

$$C \Rightarrow \alpha\left[\begin{array}{c} 1 + (1-\varepsilon)^2 k + (1-\varepsilon)\varepsilon \cdot 0 \\ +(1-\varepsilon)\varepsilon \cdot 0 + \varepsilon^2 \cdot 1 \end{array}\right]$$

$$+ (1-\alpha)[-1 + (1-\varepsilon) \cdot 1 + \varepsilon \cdot 0]$$

$$= \alpha\left[1 + (1-\varepsilon)^2 k + \varepsilon^2\right] - (1-\alpha) + (1-\alpha)(1-\varepsilon)$$

$$= \alpha(1+\varepsilon^2) - (1-\alpha) + (1-\varepsilon)^2[\alpha k + (1-\alpha)] \simeq \alpha(1+k)$$

Play

$$D \Rightarrow \alpha(2 + \varepsilon \bullet 0 + (1 - \varepsilon)) + (1 - \alpha)[0 + 1]$$
$$= 2\alpha + 1 - \varepsilon\alpha \simeq 2\alpha + 1$$

We can make her just indifferent between C and D by setting:

$$\alpha \to \frac{1}{(k-1)} \text{ as } \varepsilon \to 0$$

Note that a key to the mixed equilibrium is that *player i conditions his second period behavior on the result of his first period randomization*.

Because of this, player j's first period signal is informative about i's second period behavior.

This means that player j will want to condition his second period action on his first period signal, which means in turn that some incentive can be provided for i to cooperate in the first period.

Belief Free equilibria

The problem with imperfect private independent monitoring is that with pure strategies and signals with full support, there is no Bayesian updating: low signals are attributed to chance.

This would not happen if the players played mixed startegies, correlated with the state in which the players are (cooperation or punishment phase).

Ely and Valimaki [2002] provide an ingenuous construction to prove a folk theorem with private monitoring when there is almost-perfect (but not necessarily correlated) signals.

The idea is to construct equilibria in which the players are indifferent between C, D at every point in time.

Consider the PD:

	Cooperate	Defect
Cooperate	$1, 1$	$-1, 2$
Defect	$2, -1$	$0, 0$

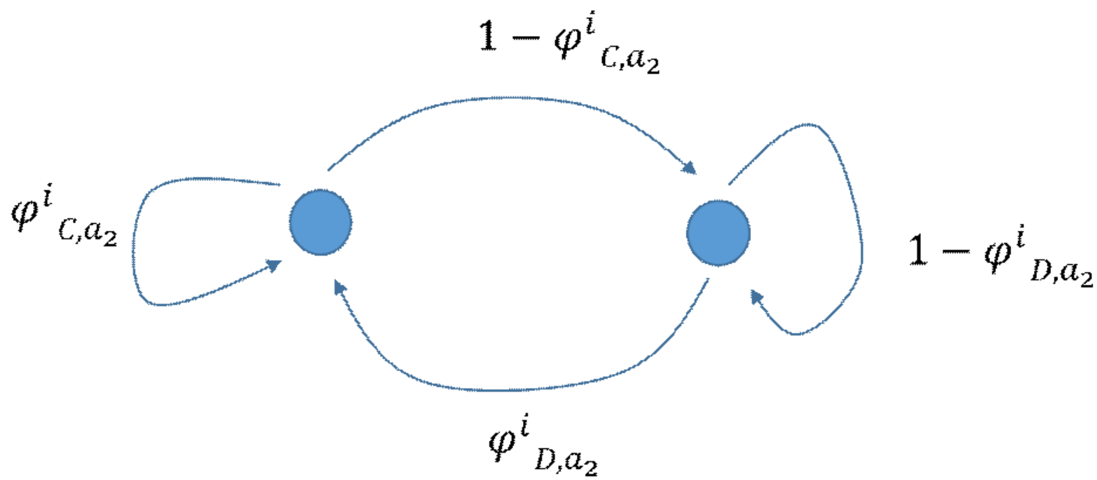
Now player i observes a signal $y_i = \{c, d\}$ correlated to j's action.

The signal is:

$$\Pr(y_i = c) = \begin{cases} 1-\varepsilon & a_j = C \\ \varepsilon & a_j = D \end{cases}$$

To get a sense of the construction, consider first the case with perfect monitoring.

Construct a state contingent mixed strategy in which $\varphi^i_{a_i,y_i}$ probability of selecting C in state a_i, y_i ("$= a_j$") (a_i is the action selected by i at $t-1$, y_i is the signal observed at t on the action chosen by j at $t-1$). So:



To see how this works, let us assume for simplicty here perfect monitoring, $\varepsilon = 0$.

We construct an equilibrium in which:

- when j selects C at t, i is indifferent between C and D and receives V_C^i;

- when j selects D at t, i is indifferent between C and D and receives V_D^i.

This equilibrium generates value functions:

$$V_C^i = (1-\delta) + \delta\left[\varphi^j_{C,C} V_C^i + (1-\varphi^j_{C,C}) V_D^i\right]$$

$$V_C^i = 2(1-\delta) + \delta\left[\varphi^j_{C,D} V_C^i + (1-\varphi^j_{C,D}) V_D^i\right]$$

$$V_D^i = -(1-\delta) + \delta\left[\varphi^j_{D,C} V_C^i + (1-\varphi^j_{D,C}) V_D^i\right]$$

$$V_D^i = 0 \bullet (1-\delta) + \delta\left[\varphi^j_{D,D} V_C^i + (1-\varphi^j_{D,D}) V_D^i\right]$$

Consider the first 2 lines:

- If i plays C, then j will be instate CC with probability $\varphi^{j}_{C,C}$;
- If i plays D, then j will be in CC with probability $\varphi^{j}_{C,D}$

It can be shown that any V_C^i V_D^i such that

$$1 \geq V_C^i \geq V_D^i \geq 0$$

can be achieved in equilibrium if δ is near 1.

This allows us to obtain any pair $(v_1, v_2) \in V = (0, 1]$.

To see this, suppose we want to achieve $V^i_C = 1$, $V^i_D = 1/2$ with $\delta = 0.9$.

Note that for δ near 1, the system above implies:

$$\delta\left(\varphi^j_{D,C} - \varphi^j_{D,D}\right)[V^i_C - V^i_D] = (1 - \delta)$$

$$\delta\left(\varphi^j_{C,C} - \varphi^j_{C,D}\right)[V^i_C - V^i_D] = (1 - \delta)$$

So we need: $\varphi^j_{C,C} - \varphi^j_{C,D} = \varphi^j_{D,C} - \varphi^j_{D,D} = \frac{2}{9}$

From the value functions we get:

$$1 = 0.1 + 0.9 \cdot \frac{1}{2} \cdot (1 + \varphi^j_{C,C})$$

$$\frac{1}{2} = 0.9 \cdot \frac{1}{2} \cdot (1 + \varphi^j_{D,D})$$

So:

$$\varphi^j_{C,C} = 1,\ \varphi^j_{C,D} = 7/9,\ \varphi^j_{D,C} = 3/9,\ \varphi^j_{D,D} = 1/9$$

This construction can be generalized to imperfect monitoring, i.e. $\varepsilon > 0$ (but small).

This gives a lot of equilibria, but it is still not sufficient for a Fork Theorem, which requires all feasible, individually rational payoffs.

To do this EV add a preliminary phase of finite length and support different payoffs in this preliminary phases with different continuation payoffs from the unit square.

This gives the folk theorem for the case where monitoring is nearly perfect.