Extensive Games with Perfect Information

Definition

In many strategic situations:

- We have additional information other than $\langle N, (A_i), (u_i) \rangle$:
  ex.: order of play, information on other of payers' actions.
- This information may be useful to predict the outcome of the strategic interactions.

In these lectures:

- we study how to describe these environments;
- and introduce solution concepts that will exploit this information.

Key in this analysis is the introduction of an extensive game.

An extensive game is a detailed description of the sequential structure of a decision problem.

Two cases are relevant:

- Perfect information: each player when making a decision, is perfectly informed of all the events that have previously occurred. Here we can have environments in which:
  - players make decision one at a time;
  - or there are decision nodes in which more than one player makes a decision.
  - We will initially focus on the case with one decision at a time.
- Imperfect information, in which players are not perfectly informed about events that previously occurred.

We initially focus on Perfect information, with one decision at a time.

Definition. *An extensive game with perfect information* $\langle N, H, P, (u_i) \rangle$ *is the following:*

- *A set N of players;*
- *A set H of sequences (histories) with 3 properties:*
  - $\varnothing \in H$.
  - $(a^k)_{k=1,\ldots,K} \in H$, *where K may be infinite, then* $(a^k)_{k=1,\ldots,L} \in H$ *for* $L \leq K$.
  - *if an infinite sequence* $(a^k)_{k=1}^{\infty}$ *satisfies* $(a^k)_{k=1,\ldots,K} \in H$ *for every L, then* $(a^k)_{k=1}^{\infty} \in H$.

- *A function $P : h \to N$ that assigns to each non-terminal history a member of N.*

  A history is terminal if infinite or $\exists$ K such that $(a^k)_{k=1,\ldots,K} \in H$, but $(a^k)_{k=1,\ldots,K}, a^{K+1} \notin H$ for any a^{K+1}).

- *Preferences over terminal histories $u_i : Z \to R$.*

An extensive game in which H is finite is called a finite extensive game.

The interpretation of this construct is as follows:

- each history corresponds to a node;
- after each history h player $P(h)$ chooses an action in the set:

$$A(h) := \{a : (h, a) \in H\}.$$

- The empty history is the initial history;

An extensive game can be represented by a *tree*: a connected graph with no cycles.

In this interpretation:

- each node corresponds to an history;
- any pair of nodes that are connected corresponds to an action.

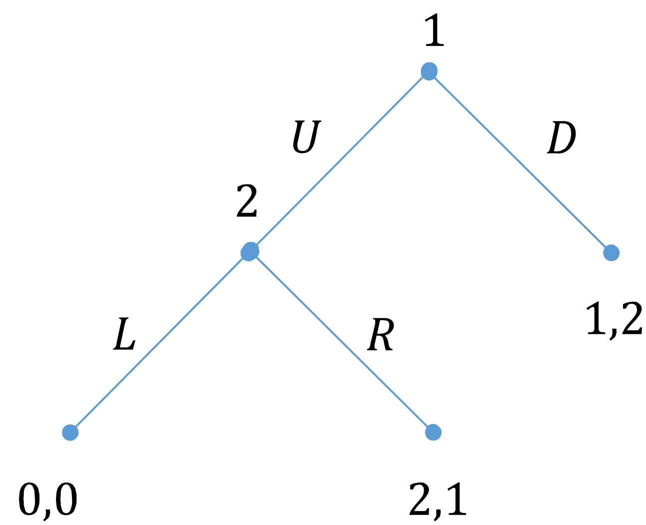
Each node has exactly one predecessor.

So a node is a complete description of all events that preceded it: not just a state, or complete physical situation.

Consider this simple extensive game $\langle N, H, P, (u_i) \rangle$:

- $N = 2$
- $H = \varnothing, U, D, UL, UR.$
- $P(\varnothing) = 1, P(\theta) = 2$ for $\theta = U, D$;
- $(U, L) \in Z, (UR) \in Z$,...else.
- $u_1(UR) = 2, u_2(UR) = 1$, etc.

The corresponding tree is:



Strategies

Definition. *A strategy of a player i in an extensive game with perfect information is a function:*

$$s_i(h) \to A(h)$$

for any $h \in H \backslash Z$ s.t. $P(h) = i$.

A strategy specifies an action for any node in which a player is asked to choose an action.

A strategy profile: $s = (s_1, \ldots, s_n)$.

Definition. *For each strategy profile an outcome* $O(s)$ *is the terminal node associated to the strategy profile.*

Note that for now we are considering pure strategies.

If we consider randomizations, then the outcome may be a distribution over terminal histories.

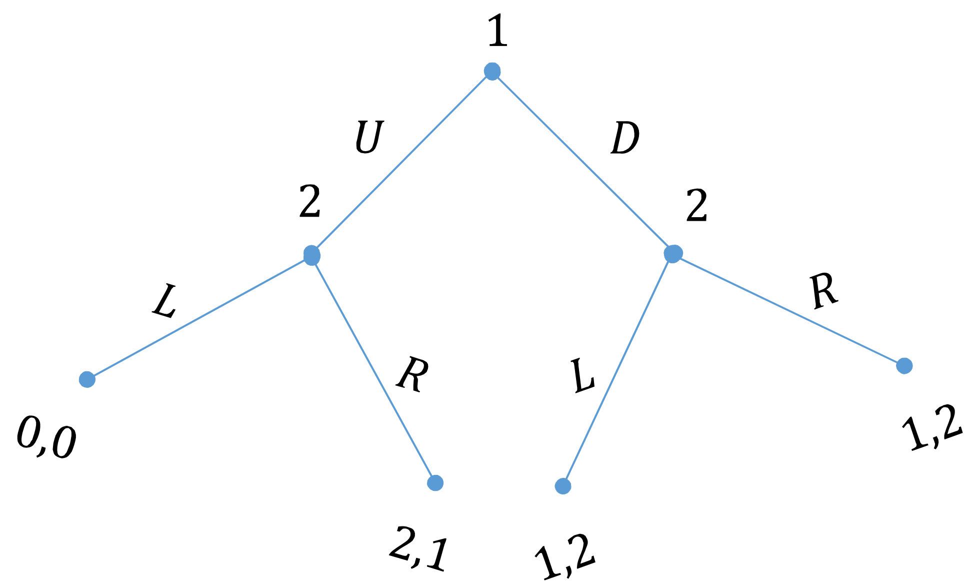
Nash equilibrium

Definition. *The strategic form of an extensive game with perfect information* $\langle N, H, P, (u_i)\rangle$ *is the strategic game* $\langle N, (S_i), (\widetilde{u}_i)\rangle$ *in which:*

- S_i *is the set of strategies in the extensive game;*
- $\widetilde{u}_i(s) = u_i(O(s))$.

Definition. *A Nash equilibrium of an extensive game is a Nash equilibrium of the associated strategic form game.*

Lets consider a slight modification of the previous example.



The strategic form game is:

	LL	LR	RL	RR
U	0,0	0, 0	2,1	2,1
D	1,2	1, 2	1,2	1,2

Here, say, LL stands for $s_2(U) = L, s_2(D) = L; LR$ for $s_2(U) = L, s_2(D) = R$.

Nash equilibria of the previous game are U, RL; U, RR; $D, LL; D, LR$.

We could simplify a little the representation as follows:

Definition. *Define two strategies* s_i *and* s_i' *to be equivalent if for each* s_{-i} *we have* $u_i(s_i, s_{-i}) = u_i(s_i', s_{-i})$.

Definition. *The reduced strategic form of an extensive game is the associated strategic form in which:*

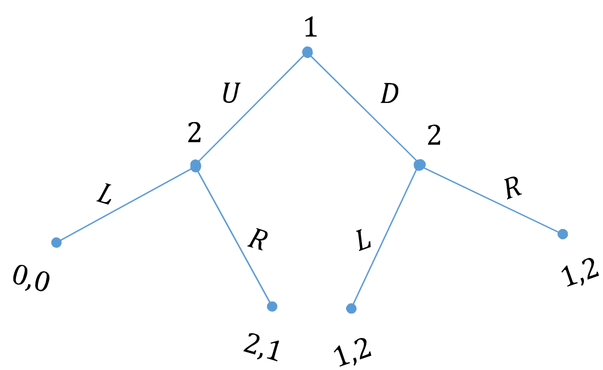
- *we include only one member for each set of equivalent strategies;*
- *and define preferences over this set of strategies.*

So we have the following:

	L	R
U	0,0	$2, 1$
D	1,2	$1, 2$

Problems with the Nash solution

Once player 1 chooses U, it is natural to assume that 2 selects R.



The equilibria $D, LL; D, LR$ exist based on the conjecture that if 1 selected U, then 2 selects L.

Subgame perfect equilibrium

Definition. *The subgame of the extensive game with perfect information* $\Gamma = \langle N, H, P, (u_i) \rangle$ *that follows from history* h *is the extensive form game* $\Gamma(h) = \langle N, H|_h, P|_h, (u_i)|_h \rangle$ *where:*

- $H|_h$ *is the set of sequences* h' *of actions for which* $(h, h') \in H$;
- $P|_h$ *is such that* $P|_h(h') = P(h, h')$ $(h, h') \in H$;
- and $u_i(h'; h) \geq u_i(h''; h)$ iff $u_i((h, h')) \geq u_i((h, h''))$.

Definition. *We define subgame perfect equilibrium to be a strategy profile* s^* *in* Γ *for which for any history* h *the strategy profile* $s^*|_h$ *is a (subgame perfect) Nash equilibrium of the subgame* $\Gamma(h)$*.*

Where $s^*|_h(h') = s^*(h, h')$.

Stackelberg

Two firms, 1 and 2, choose output levels $q_i \in [0, \infty)$.

Firm 1 moves first.

The price is $p(q_1, q_2)$.

Profit is:

$$u_i(q_1, q_2) = q_i \cdot p(q_1, q_2) - c_i(q_i)$$

To find a Nash equilibrium, let's find the reaction functions: $r_2(q_1)$:

$$p(q_1 + r_2(q_1)) \ + p'(q_1 + r_2(q_1))r_2(q_1) - c_1'(r_2(q_1)) \ = 0$$

If we assume linear costs and demand:

$$c_i(q) = cq$$

$$p(q_1, q_2) = \max(0, 1 - q)$$

Then in an interior solution we have:

$$r_i(q_{-i}) = \frac{1 - q_{-i} - c}{2}$$

A Nash equilibrium is given by the system:

$$q_2 = r_2(q_1) = \frac{1 - q_1 - c}{2}$$

$$q_1 = r_1(q_2) = \frac{1 - q_2 - c}{2}$$

Assume 1 chooses first, then 2.

Now firm 1 optimizes "knowing" firm's 2 reaction function.

$$q_1 \bullet \left[1 - q_1 - \frac{1 - q_1 - c}{2} \right] - cq_1$$

$$= q_1 \bullet \left[\frac{1 - q_1 + c}{2} \right] - cq_1$$

From the foc:

$$\frac{1}{2}[1-2q_1+c]=c$$

$$q_1=\frac{1-c}{2}$$

$$q_2=\frac{1-\frac{1-c}{2}-c}{2}=\frac{1-c}{4}$$

One-stage deviation principle

To verify that a strategy s^* is a subgame perfect equilibrium we need to check that for every $i \in N$, and every subgame $\Gamma(h)$, no strategy is a strictly positive deviation.

The following result simplifies the calculation, by reducing the *class of deviations* that need to be checked.

It states that a strategy profile is a SPE iff for each subgame the player who makes the **first move cannot** obtain a better outcome by changing only his **initial action**.

Theorem. *In a finite extensive game with observed actions, a strategy profile s is a SPE iff the one-stage deviation condition that no player i can gain by deviating from s in a single stage and conforming to s thereafter.*

Proof. We need to prove that s is a SPE iff there is no i and no $\widehat{s}_i$ that agrees with s_i except at a single t and h^t, and such that $\widehat{s}_i$ is a better response to s_{-i} then s_i conditionally on h^t.

The only if part is immediate. Lets us focus on the "if".

We prove the counterpositive: if it is not a SPE, then the one-stage deviation property is violated.

Assume s is not a SPE. Then there is a t and a h^t such that some i has a deviation $\widehat{s}_i$ in the subgame $\Gamma(h^t)$.

Let $\widehat{t}$ be the largest t' such that $\widehat{s}_i(h^{t'}) \neq s_i|_{h^t}(h^{t'})$.

Consider an alternative strategy $\widetilde{s}_i$ that agrees with $\widehat{s}_i$ for all $t < \widehat{t}$ and agrees with $s_i|_{h^t}$ from $\widehat{t}$ on.

Since from any $h^{\widehat{t}}$ it agrees with $s_i|_{h^t}$ except for the first move, *by the one stage deviation principle*, this change can only increase the utility of i at any $h^{\widehat{t}}$.

Of course if the One-stage DP fails, then we are done

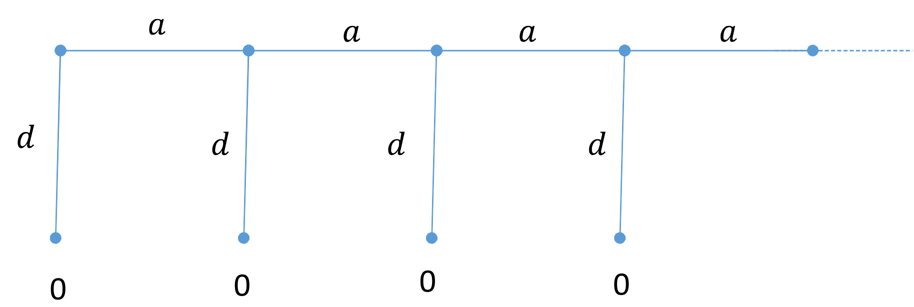
So $\widetilde{s}_i$ is as good a response as $\widehat{s}_i$ at h^t.

If $\widehat{t} = t + 1$, then $\widetilde{s}_i = s_i$, and we have a contradiction.

If $\widehat{t} > t + 1$, then iterate the procedure until we have a contradiction.

With no additional assumptions, this theorem fails to be true if we assume there is an infinite number of periods.

Consider this one player game in which the payoff of playing infinite a is 1:



The strategy d after every history satisfies One Stage DP, but not a SPE.

Definition. *A game is continuous at infinity if for each player i the utility function $u_i(h)$ satisfies:*

$$\sup_{h,\widehat{h} \text{ s.t. } h^t=\widehat{h}^t} \left| u_i(h) - u_i(\widehat{h}) \right| \to 0 \text{ as } t \to \infty$$

where $h, \widehat{h}$ are infinite histories, and $u_i(h), u_i(\widehat{h})$ their respective utilities.

This condition is satisfied if the utilities are equal to a discounted sum of per period payoffs $U_i^t(a^t)$ that are uniformly bounded.

Theorem. *In an infinite horizon extensive game with observed actions that is continuous at infinity, a profile* s *is a SPE iff there is no player* i *and strategy* $\hat{s}_i$ *that agrees with* s_i *except at a single* t *and* h^t *and that is a strictly superior deviation to* s_{-i}*.*

Existence of a SPE

Theorem. (Kuhn's Theorem) *Every finite extensive game with perfect information has a SPE*.

Proof. Start with a finite extensive game with perfect information Γ with subgames $(\Gamma(h))$, which are finite.

Define $l(\Gamma(h))$ the length of the maximal history of $\Gamma(h)$.

We now define $R(h)$ a function that associates a terminal history h' to each history h such that $h' \geq h$.

When $l(\Gamma(h)) = 0$, then we are in a terminal history and $R(h) = h$.

Assume we have defined $R(h)$ for all h such that $l(\Gamma(h)) \leq k$.

Consider a h' such that $l(\Gamma(h')) = k + 1$

We have $l(\Gamma(h', a)) \leq k$ for all $a \in A(h')$.

Let $s_i(h')$ be the such that $u_i(R(h', s_i(h'))) \geq u_i(R(h', a))$ for all $a \in A(h')$.

Define $R(h') = R(h', s_i(h'))$.

We have defined by induction $R(h)$ and a strategy profile s

that is SPE by the one-stage deviation principle.

The idea behind the proof is called backward induction.

The gist of it is just to solve the game starting from the end, with the simplest subgame.

Exogenous uncertainty

We might want to describe situations in which nature also moves.

This can easily be incorporated in the basic framework.

An extensive game with perfect information and chance moves is a tuple $\langle N, H, P, f_c, (\succeq_i)\rangle$, where now:

- P is a function from H to $N \cup \{c\}$, where c stands for chance.
- for each h such that $P(h) = c$, $f_c(\bullet; h)$ is a probability distribution over $A(h)$.
- $(\succeq_i)$ are preferences over lotteries over terminal nodes.

Simultaneous moves

Definition *An extensive game with perfect information and simultaneous moves is a tuple* $\langle N, H, P, (u_i)\rangle$ *such that:*

- N *is the set of players;*
- H *is a sequence of* $|P(h)|$ *dimensional vectors of actions;*
- P *identifies the set of players who chose after history* h;
- u_i *is just as before.*

A strategy is a function $s_i(h) \to A_i(h)$ for all $i \in P(h)$.

The definition of a subgame and SPE applies to this more general game.

Definition. *We define subgame perfect equilibrium to be a strategy profile s^* in Γ for which for any history h the strategy profile $s^*|_h$ is a (subgame perfect) Nash equilibrium of the subgame $\Gamma(h)$.*

When we represent a game with simultaneous moves, we don’t have perfect information.

If we want to use a game tree representation, we need to describe this information.

To this goal we use information sets:

- Partitions of the histories (nodes) with the interpretation that a player at a node nodes x in a partition $z(x)$ is unsure if the node is x or any other $x' \in z(x)$;

- The same player moves at x and x'

- We require that $x', x \in z(x)$ implies $A(x) = A(x')$.

Information sets can be used to describe information in a game tree.

The could also describe situations in which information is degraded, i.e. a player forgets information he once knew.

Games with perfect recall are games in which no player ever forgets information s/he once knew.

It can be shown, using the same logic as above, that the one-stage deviation principle holds in games with simultaneous moves.

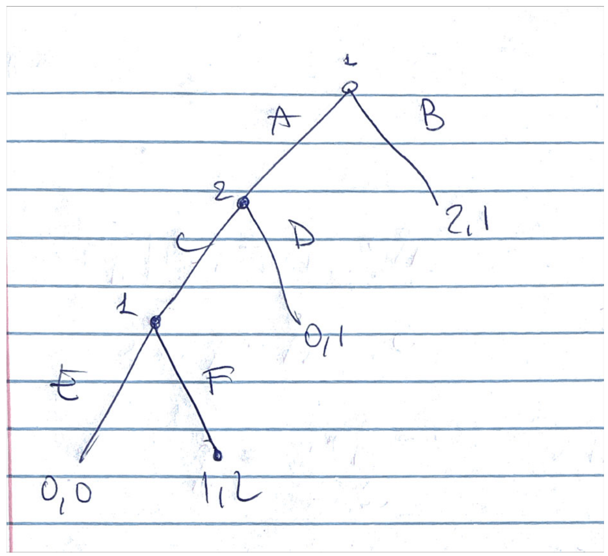
We, however, might not have a SPE (in pure strategies).

What is an example of this failure?

Interpretations and critiques

On the interpretation of strategies

Strategies specify actions after each node.



Interpretation of this?: $\sigma_1(\emptyset) = B; \sigma_1(A, C) = F$

History has zero probability given $1's$ strategy.

We are requiring a player to make up his/her mind even for events that are impossible given his/her strategy.

We do this because this forms a basis for the belief of 2.

Given the strategy, 2 selects C, since this leads to $1, 2$ rather than $0, 1$.

It is sensible to require 2 to assume 1 would choose F rather than E.

But 2 has evidence that 1 is not rational (else B would have been preferred).

Here rationality is assumed a priori with prior probability of 1.

The key assumption is that rationality remains the **guiding principle** no matter what is observed.

One way to rationalize this is to assume that any *deviation from rationality* is **more likely** to derive from some mistake.

Forward induction

When we end up in a history that has probability zero, we should question: what does this imply for the rationality of the other players?

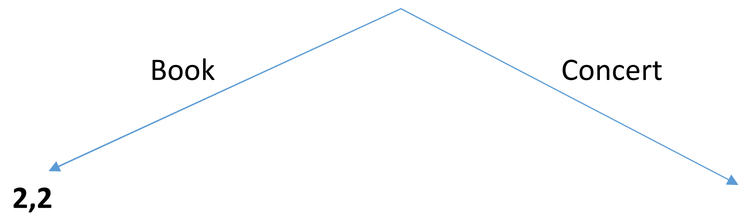
And then: what does this imply for how the player **will** play?

Perhaps the player is irrational, and this has implication on how to play.

More generally, **past actions** may be **informative** on how the opponents **will play** if there is "ambiguity" in the continuation subgame.

Iterated deletion of weakly dominated strategies may capture this type of reasoning.

Consider this game:



	Bach	**Stravinski**
Bach	$\mathbf{3, 1}$	$\mathbf{0, 0}$
Stravinski	$\mathbf{0, 0}$	$\mathbf{1, 3}$

The subgames are $(Book, S)$,S and $(Concert, B), B$.

But is the first equilibrium plausible?

- If 2 gets to play, she should realize that 1 could have obtained 2 with book, but instead choose Concert.
- This suggests he expects B, B.
- In this case B is optimal for 2.

Consider the strategic form of the game.

	B	S
Book	$2, 2$	$2, 2$
B	3,1	0,0
S	$0, 0$	$1, 3$

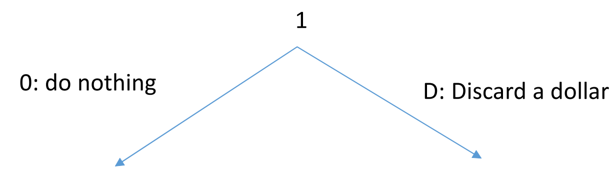
Note that Book strictly dominates S for 1

Given this B weakly dominates S for 2.

BB is the equilibrium.

Burning money

Consider now this game:



	B	S
B	3,1	0,0
S	0, 0	1, 3

	B	S
B	2,1	-1,0
S	-1, 0	0, 3

This is the BoS but now 1 can burn 1$ before playing.

2 observes 1.

Lets try to solve it by iterated deletion of weakly dominated strategies.

	BB	BS	SB	SS
0B	3,1	3,1	0, 0	0, 0
0S	0,0	0,0	1,3	1,3
DB	2,1	-1,0	2,1	-1,0
DS	-1,0	0,3	−1, 1	0, 3

	BB	BS	SB
0B	3,1	3,1	0, 0
0S	0,0	0,0	1,3
DB	2,1	-1,0	2,1

So the equilibrium is the preferred outcome by 1, and with no money burning!

What is the logic?

In the original game:

	B	S
B	**3,1**	**0,0**
S	**0, 0**	**1, 3**

A player can guarantee a payoff:

$$\pi = \min_{\alpha \in [0,1]} \max\{3\alpha, 1-\alpha\} = \frac{3}{4}$$

In the original game, this is not relevant.

But after burning 1$, the only way to achieve this is to play B.

So after burning 1$, 2 expects B and plys B.

This generates a payoff of 2 for 1.

With no burning, the only way to achieve this is to play B.

So now 2 expects B, and the equilibrium is BB!

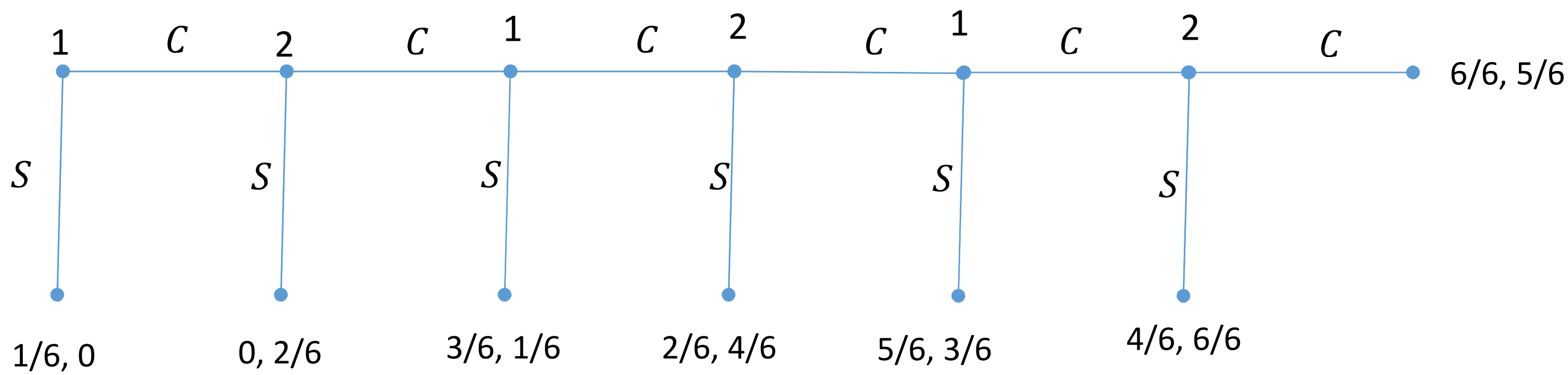
The centipede game

Two players are in a process that they can alternatively stop or continue.

At each t, each player prefers to stop the process than to continue and let the other player stop at $t+1$.

In the last period $t = T-1$ the player prefers to stop than to continue.

However the terminal history at T is better for both players than stopping at any period $t < T-1$.



There is a unique SPE in this game: $s_i(h^t) = S$.

Any pair of strategies in which player 1 chooses S in period 1 and 2 plays S in period 2 is a Nash equilibrium.

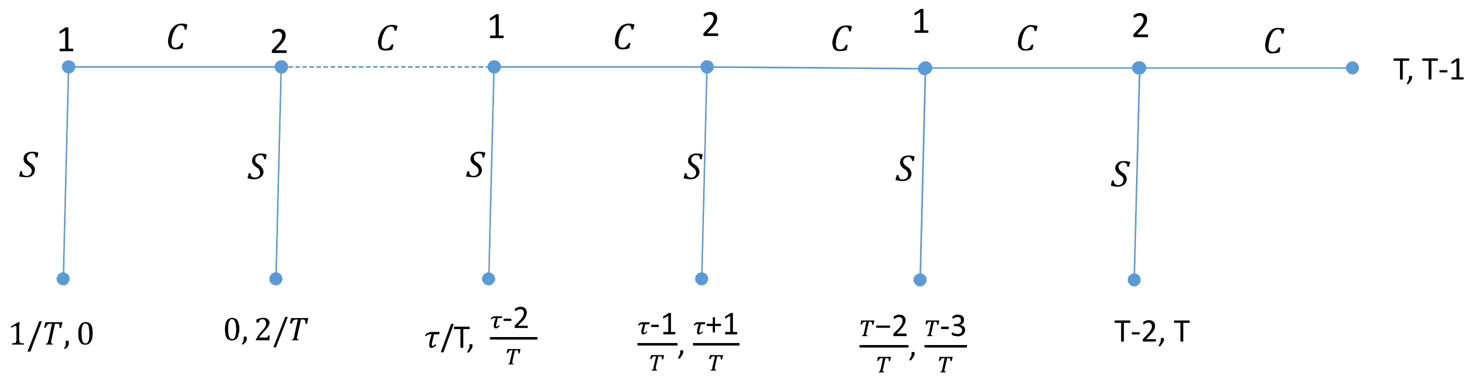
Are these predictions realistic?

A way to reconcile the observation is to note that cooperation is close to an equilibrium if the length of the game is sufficiently high.

Definition. *A profile* s^* *is an* ε*-Nash equilibrium if, for all players* i *and strategies* s_i*, we have:*

$$u_i(s^*) \geq u_i(s_i, s^*_{-i}) - \varepsilon$$

Consider now the game with T stages, so payoffs go up to $T, T-1$.



Normalize the payoffs dividing them by T (so imagine the horizon is the same but increments are smaller).

Is cooperation up to k (for some k) optimal if T is large enough?

No deviation is optimal for $T \geq k$, since the startegy reccomends to stop;

No deviation is also optimal at $t \leq k - 2$, since it is optimal to C if the other player does not stop at $t + 1$ (so stops at any $t' > t + 1$).

At $\tau = k - 1$, the net benefit of a deviation is $1/T < \varepsilon$ if T is large enough.