Bayesian games: applications

In Bayesian Games we study interactions in which there may be some uncertainty about the characteristics of the other players (or the state of the nature).

We model the players' uncertainty introducing a set Ω of states of the nature.

States of Nature are a description of the player's relevant characteristics.

This leads to the definition of a Bayesian Game.

Definition. *A Bayesian game consists of:*

- *A finite set N of players.*
- *A finite set of state of nature* Ω *(for simplicity here).*
- *For each* $i \in N$*:*
  - *A set* A_i *of actions*
  - *A finite set of types* T_i *and a signal function* $\tau_i : \Omega \to T_i$
  - *A probability measure* p_i *over* Ω *with* $p_i(\tau_i^{-1}(t_i)) > 0$ *for all* $t_i \in T_i$
  - *A preference relation* $\succeq_i$ *over* $A \times \Omega$.

Note that a Bayesian game is a tuple:
$\langle N, \Omega, (A_i), (T_i), (\tau_i), (p_i)(u_i) \rangle$.

Often a Bayesian game is not defined in terms of Ω and the *signal function* τ_i, but directly in terms of types,

Sometimes it is described in terms of Ω and a signal structure expressed as a conditional distribution over types T_i.

In this case, we denote the conditional probability as $\tau_i(t_i;\omega)$.

In the definition, we allow for heterogeneous priors on Ω, often it is assumed that there is a common prior.

We define a Nash eq. of a Bayesian game in terms of the Nash eq. of an associated strategic game.

Definition. *A Nash equilibrium of a Bayesian game is the Nash of a strategic game defined as follows:*

- *The set of players is the set of pairs* (i, t_i) *for each* $i \in N$ *and* $t_i \in T_i$.
- *The set of actions of player* (i, t_i) *is* A_i.
- *The preferences of* $\succeq_{(i,t_i)}$:

$$a^* \succeq_{(i,t_i)} b^* \Leftrightarrow L_i(a^*, t_i) \succeq_i L_i(b^*, t_i)$$

where $L_i(a, t_i)$ *is a lottery that assigns probability* $\frac{p_i(\omega)}{p_i(\tau_i^{-1}(t_i))}$ *to*

$\left((a^*(j,\tau_j(\omega)))_{j\in N},\omega\right)$ *if* $\omega \in \tau_i^{-1}(t_i)$ *and* 0 *otherwise.*

We study today a few applications of Bayesian games.

Collective action

Consider a problem in which a group of n players can obtain a public good if at least one of them volunteers.

The public good generates a value $v < 1$ to each player.

Volunteering costs c_i to player i.

We assume $c_i \sim F(\bullet)$ with support $[0, \bar{c}]$.

We focus on symmetric equilibria.

In the formalism of above:

States: $(c_1, \ldots, c_n) \in \Omega$

Signals ($\tau_i : \Omega \to T_i$): $\tau_i(c_1, \ldots, c_n) = c_i$

Believes: $Pr(c_{-i}; c_i) = \prod_{j \neq i} f(c_j)$

Utility:

$$u(a,c) = \begin{cases} 0 & \text{no contribution} \\ v & \text{at least some } j \neq i \text{ contributes} \\ v - c_i & i \text{ contributes} \end{cases}$$

Assume probability of contribution for a player is σ.

Then the expected utility of contributing is:

$$v - c_i$$

The utility of not contributing is:

$$\Pr(\#volunteers \neq i \geq 1)v$$

So the net utility of contributing is:

$$\begin{aligned}\Delta u_i &= [1 - \Pr(\#volunteers \neq i = 1)]v - c_i \\ &= \Pr(\#volunteers \neq i = 0)v - c_i\end{aligned}$$

If $\Delta u_i \geq 0$, then $\Delta u_j > 0$ if $c_j < c_i$

So there is a c^* such that a player c_i volunteers if and only if $c_i \leq c^*$ (indifferent if $c_i \leq c^*$).

What is c^*?

$$c^* = \Pr(\#volunteers = 0)v$$

$$= [1 - F(c^*)]^{n-1}v$$

Note that the rhs is increasing in c^* and the lhs is decreasing; moreover $0 < [1 - F(0)]^{n-1}v = v$ and $1 > [1 - F(1)]^{n-1}v = 0$, so we have a unique equilibrium.

We can study what happens as $n \to \infty$.

Assume $c_n^* \to c_\infty^* > 0$, then...we have a contradiction (can you see it?)

So $c_n^* \to 0$

But what about the probability of success?

$$P = 1 - [1 - F(c_n^*)]^n = 1 - \left(\frac{c_n^*}{v}\right)^{\frac{n-1}{n}} \to 1$$

Juries

Consider a trial in which n jurors decide to acquit or convict a defendant

The defendant is guilty (G) with prior probability π, and innocent (I) otherwise

Each juror receives an informative signal $\{g, i\}$ with probability $p(g;G) = p > 1/2$, and $p(i;I) = q > 1/2$

The defendant is convicted if all jurors vote to convict.

We assume that voter's utility is:

$$\begin{cases} 0 & \text{if decision is correct} \\ -z & \text{if innocent is convicted} \\ -(1-z) & \text{if guilty defendant is acquitted} \end{cases}$$

z can be interpreted as the belief threshold on the guiltyness of the defendant, such that below it acquitting is optimal.

Indeed, if r is the probability a defendant is guilty:

Expected payoff if acquit$= -r(1-z) + (1-r) \bullet 0$

Expected payoff if convict$= r \bullet 0 - (1-r)z$

So:

$$Acquitt \succeq Convict \Leftrightarrow r \geq z$$

Jurors face two types of costly mistakes:

- convict an innocent;
- acquit a guilty defendant.

They need to assess the trade-off on the basis of all the information they have.

Why is this a strategic situation?

The outcome of the decision does not depend on a single decision-maker, but on the decision of all players.

The decision of a juror matters only in one event: that all other jurors vote to convict.

If the other jurors vote on the basis of their information, the very fact that a juror is pivotal has informational content!

A vote to acquit by i matters only if another jurors vote to convict.

Given this, should a juror reconsider the vote? This fact should be incorporated in the decision, but how?

We need to study this as a game.

The state here is: $\Omega(\{G,I\},(\{i,g\})^n)$

The payoffs in the game depend on the state and actions of all players:

$$u(a,\omega) = \begin{cases} 0 & a \neq (C,C..C) \text{ and } \omega_1 = I \\ 0 & a = (C,C..C) \text{ and } \omega_1 = G \\ -z & a = (C,C..C) \text{ and } \omega_1 = I \\ -(1-z) & a \neq (C,C..C) \text{ and } \omega_1 = G \end{cases}$$

Single decision maker

The simplest case is when $n = 1$, since we have no strategic interactions.

In this case the posterior after a i signal is:

$$\Pr(G;i) = \frac{\Pr(i;G)\Pr(G)}{\Pr(i;G)\Pr(G) + \Pr(i;I)(1-\Pr(G))}$$
$$= \frac{(1-p)\pi}{(1-p)\pi + q(1-\pi)}$$

The juror is willing to acquit following a innocent signal if:

$$z \geq \frac{(1-p)\pi}{(1-p)\pi + q(1-\pi)}$$

Similarly, s/he is willing to convict after a g signal if

$$z \leq \frac{(p)\pi}{p\pi+(1-q)(1-\pi)}.$$

A juror is willing to go with the signal if:

$$\frac{(1-p)\pi}{(1-p)\pi + q(1-\pi)} \leq z \leq \frac{(p)\pi}{p\pi + (1-q)(1-\pi)}$$

It is not always the case that the signal determines the decision because the signal may be weak when compared to the prior π, or the preferences z.

Two jurors

With 2 jurors, the vote of a juror matters only if the other voters to convict.

Can we have an equilibrium in which both jurors vote according to their signal, i.e. convict if g; acquit if i?

If this is the case, then the relevant posterior for i is

$$\begin{aligned}
\Pr(G;i,g) &= \frac{\Pr(i,g;G)\Pr(G)}{\Pr(i,g;G)\Pr(G)+\Pr(i,g;I)(1-\Pr(G))} \\
&= \frac{(1-p)p\pi}{(1-p)p\pi+q(1-q)(1-\pi)}
\end{aligned}$$

And a juror with a signal i votes to acquit only if:

$$z \geq \frac{(1-p)p\pi}{(1-p)p\pi + q(1-q)(1-\pi)}$$

From similar calculations, such an equilibrium exists only if:

$$\frac{(1-p)p\pi}{(1-p)p\pi + q(1-q)(1-\pi)} \leq z \leq \frac{(p)^2\pi}{p^2\pi + (1-q)^2(1-\pi)}$$

Note that:

$$\frac{(1-p)\pi}{(1-p)\pi + q(1-\pi)} = \frac{\pi}{\pi + \frac{q}{1-p}(1-\pi)}$$

and $$\frac{(1-p)p\pi}{(1-p)p\pi + q(1-q)(1-\pi)} = \frac{\pi}{\pi + \frac{q(1-q)}{p(1-p)}(1-\pi)}$$

But: $\frac{q}{1-p} > \frac{q(1-q)}{p(1-p)} \Leftrightarrow 1-q < p$, always true, so:

$$\frac{(1-p)p\pi}{(1-p)p\pi + q(1-q)(1-\pi)} > \frac{(1-p)\pi}{(1-p)\pi + q(1-\pi)}$$

This implies that the juror is more reluctant to vote to acquit.

Many jurors

In this case the relevant posterior is:

$$\Pr(G; i, (g)_{-i}) = \frac{\Pr(i, (g)_{-i}; G)\Pr(G)}{\Pr(i, (g)_{-i}; G)\Pr(G) + \Pr(i, (g)_{-i}; I)(1 - \Pr(G))}$$

$$= \frac{(1-p)p^{n-1}\pi}{(1-p)p^{n-1}\pi + q(1-q)^{n-1}(1-\pi)}$$

A juror with a i signal, acquits only if:

$$z \geq \frac{(1-p)p^{n-1}\pi}{(1-p)p^{n-1}\pi + q(1-q)^{n-1}(1-\pi)}$$

$$= \frac{\pi}{\pi + \frac{q}{(1-p)}(\frac{1-q}{p})^{n-1}(1-\pi)}$$

As $n \to \infty$, the right hand side converges to 1, so for no $z < 1$, the juror is willing to follow his own signal!

This means that for large n, we cannot have an equilibrium in which jurors follow their signals with probability 1.

Under some conditions there is an equilibrium in which the jurors partially use the infromation of the signals.

In tehse equilibria the jurors with a g signal vote to convict.

The juors with a i signal, vote to convict with probability σ and to acquit with probability $1-\sigma$

When σ is high, the infromative contant of the fact that $n-1$ other jurors vote to convict is sufficientlt small that it does not dominate the jurors' private signals.

Standard Auctions

Basic environment

A seller with an indivisible item for sale, zero cost.

I bidders/players: $i = 1, .., I$.

Each bidder i has private information $v_i \in V_i$.

Given the profile $v = (v_i, v_{-i})$.

Bidder i's valuation is $u_i(v_i; v_{-i})$ if he gets the item and zero otherwise.

The prior distribution over $V = \prod_i V_i$ is $F(v)$.

After knowing one's own v_i , bidder i forms the posterior distribution of others' valuation payoff as $F_i(v_{-i}; v_i)$.

All bidders and seller have quasi-linear, von-Neumann Morgenstern expected utility functions.

Pure strategy for bidder i is $s_i \in S_i = [0, \infty)$.

Winning probability is $P_i(s) = P_i(s_i, s_{-i})$.

Monetary payment $T_i(s_i, s_{-i})$

A model has private values if $u_i(v_i, v_{-i}) = u_i(v_i, v'_{-i})$ for any v_{-i}, v'_{-i}

A model has common values if it has not private values

A model has independent values if $f(v) = \prod f_i(v)$.

A model has symmetric values if $f_i(v) = f(v)$.

From now on we focus on the case with:

- independent, symmetric private values;
- risk neutral bidders, so:

$$\pi(v_i, s) = P_i(s_i, s_{-i})v_i - T_i(s_i, s_{-i})$$

Equilibrium concepts

Bidder's utility is:

$$\pi(s_i; v_i, s_{-i}) = v_i \bullet P(s_i; s_{-i}) - T(s_i; s_{-i})$$

A pure strategy is a map $s_i^* : V \to R^+$

A strategy profile $(s_1^*, \ldots, s_I^*)$ is an equilibrium in weakly undominates strategies if:

$$\pi(s_i^*(v_i); v_i, s_{-i}) \geq \pi(s_i; v_i, s_{-i}) \text{ for any } s_i,\ s_{-i}$$

A strategy profile $(s_1^*, \ldots, s_I^*)$ is a Bayesian equilibrium if:

$$E[\pi(s_i^*(v_i); s_{-i}^*(v_{-i})); v_i] \geq E[\pi(s_i; s_{-i}^*(v_{-i})); v_i] \text{ for any } s_i$$

where $s_{-i}^*(v_{-i})$ is the profile of strategies of other players:

$$s_1^*(v_1), \ldots, s_{i-1}^*(v_{i-1}), s_{i+1}^*(v_{i+1}) \ldots s_n^*(v_n)$$

Note that any WUS equilibrium is a BE:

$$\pi(s_i^*(v_i); s_{-i}) \geq \pi(s_i; s_{-i}) \text{ for any } s_i,\ s_{-i} \ \Rightarrow$$

$$E[\pi(s_i^*(v_i); s_{-i}^*(v_{-i})); v_i] \geq E[\pi(s_i; s_{-i}^*(v_{-i})); v_i] \text{ for any } s_i$$

The opposite is not in general true.

Types of Auctions

First price auction

- Bidders simultaneously submit bids;
- the highest bidder receives the good;
- winner pays the (highest) bid.

In this Auction:

$$P_i(v_i, v'_{-i}) = \begin{cases} 1 & s_i > s_j \ \forall i \neq j \\ 1/k & s_i \geq s_j \ \forall i \neq j \text{ and } i \text{ ties with } k-1 \text{ other bids} \\ 0 & else \end{cases}$$

$$T_i(v_i, v'_{-i}) = \begin{cases} s_i & i \text{ wins} \\ 0 & else \end{cases}$$

Second price Auction

In this auction:

- Bidders simultaneously submit bids;
- the highest bidder receives the good;
- winner pays the second highest bid.

This Auction format was invented by Vickery (1961).

Not common but very important.

In this Auction:

$$P_i(v_i, v'_{-i}) = \begin{cases} 1 & s_i > s_j \ \forall i \neq j \\ 1/k & s_i \geq s_j \ \forall i \neq j \text{ and } \\ & i \text{ ties with } k-1 \text{ other bids} \\ 0 & else \end{cases}$$

$$T_i(v_i, v'_{-i}) = \begin{cases} \max_{j \neq i} s_j & i \text{ wins} \\ 0 & else \end{cases}$$

English Auction

In this auction:

- Bidders bidders announce bids in successive rounds, each bid must be higher than the previous;
- bids stop when no bidder is willing to submit a higher bid;
- highest bidder wins.

Dutch Auction

In this auction:

- Auctioneer starts from a high price and continuously reduces the price until some bidder accepts the price;

- The bidder who stops the decline (the highest bidder) wins the object and pays the price.

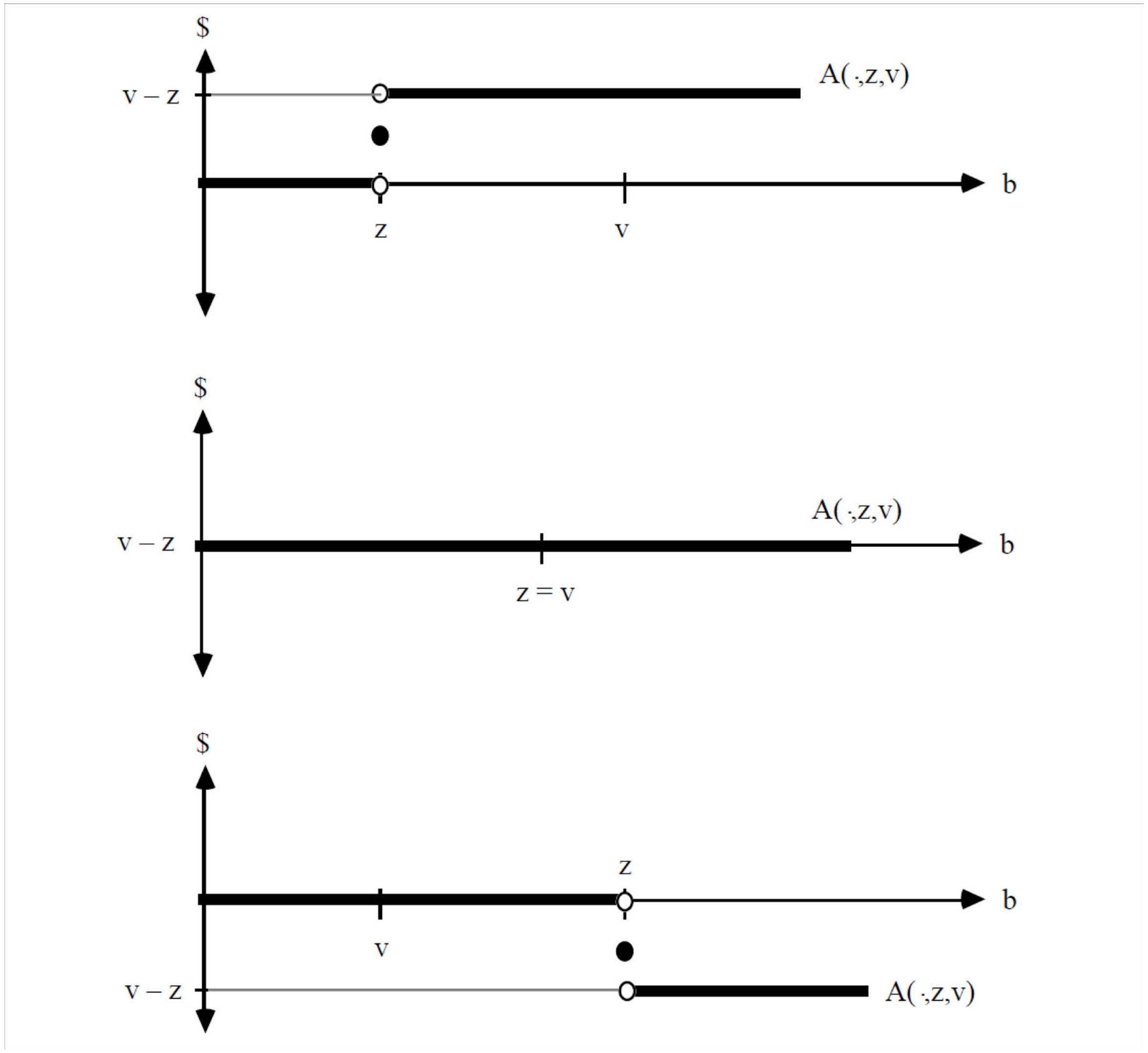
Second price auction

Equilibrium behavior

It is easy to see that $s_i = v_i$ is a (weakly) dominant strategy.

Take a generic profile v_{-i}, with maximum element z:

- If $z \geq v_i$, then the bidder's maximal payoff achievable with b is zero: it is achieved with $b = v_i$.
- If $z < v_i$, then the maximal achievable payoff is $v_i - z > 0$, achieved with $b = v_i$.


$
v – z
A(·,z,v)
b
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A(·,z,v)
v – z
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z
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v
v – z
A(·,z,v)

Given this equilibrium behavior a buyer wins with probability:

$$Q(v_i) = G(v_i) = \Pr(v_i \text{ is maximal}) = F(v_i)^{n-1}$$

The expected value of a bidder is:

$$\begin{aligned}\pi(v) &= \int_0^v (v-x)G'(x)dx \\ &= |(v-x)G(x)|_0^v + \int_0^v G(x)dx \\ &= \int_0^v G(x)dx\end{aligned}$$

The seller's expected revenues are:

$$R = E(v_{(2)})$$

where $v_{(2)}$ is the second highest realized value:

$$v_{(2)} = \max\{v_1, \ldots, v_I\} \backslash v_{(1)}$$

Note:

$$\Pr[v_{(2)} < v] = \Pr[v_{(1)} < v] + \Pr[v_{(2)} \le v < v_{(1)}]$$
$$= [F(v)]^n + n[F(v)]^{n-1}[1 - F(v)] = H(v)$$

So:

$$R = \int_0^1 v \bullet dH(v)$$

$$
\begin{aligned}
R &= \int_0^1 x \cdot dH(x) = \int_0^1 x \cdot d\Big[[F(x)]^n + n[F(x)]^{n-1}[1 - F(x)] \Big] \\
&= n \int_0^1 x \cdot [1 - F(x)] \cdot \Big[(n-1)[F(x)]^{n-2} f(x) \Big] dx \\
&= n \int_0^1 x \cdot [1 - F(x)] \cdot dG(x)
\end{aligned}
$$

$$
\begin{aligned}
R &= n\int_0^1 x \bullet [1 - F(x)] \bullet dG(x) \\
&= n \bullet |x \bullet [1 - F(x)] \bullet G(x)|_0^1 - n\int_0^1 [1 - F(x) - xf(x)]G(x)dx \\
&= n\int_0^1 \left[x - \frac{1 - F(x)}{f(x)} \right] G(x)dF(x)
\end{aligned}
$$

First Price Auction

Bidder's strategies are $b : [0, 1] \rightarrow R^+$

Bidder i's expected payoff is:

$$\pi_i(v_i, b) = (v_i - b)Q_i(b)$$

$Q_i(b)$ is the probability of winning in equilibrium with a bid b.

We study symmetric, pure strategy equilibria.

Proposition. *There is unique symmetric, pure strategy equilibrium in which:*

$$b^*(v) = \int_0^v x \frac{g(x)}{G(v)} dx$$

In a symmetric equilibrium bids are equal to the expectation of the maximum of his competitors' values conditional on that value being less than his own.

An implication of this is that bidders underbid their valuations.

Integrating by parts and noting that $G(x) = F(x)^{n-1}$ we have:

$$b^*(v) = v - \int_0^v \left[\frac{F(x)}{F(v)} \right]^{n-1} dx < v$$

Proof. Let us make two guesses about the equilibrium, we will then verify them.

- Bids are increasing in v.
- Bids are differentiable in v.

Given this we have that:

$$\begin{aligned} Q_i(b) &= \Pr(b \text{ is the highest bid}) \\ &= \Pr(v_i \text{ is the highest value}) \\ &= G(v_i) \end{aligned}$$

The bidder's objective is:

$$\pi_i(v_i, b) = (v_i - b)G([b^{-1}](b))$$

where $b([b^{-1}](b)) = b$.

The necessary foc is:

$$-G(v_i) + (v_i - b(v_i))g(v_i)[b^{-1}]'(b) = 0$$

From $b([b^{-1}](b)) = b$, $[b^{-1}]'(b(v)) = 1/b'(v)$

$$-G(v_i) + \frac{v_i - b(v_i)}{b'(v_i)} g(v_i) = 0$$

So we have:

$$v_i g(v_i) = b'(v_i)G(v_i) + b(v_i)g(v_i)$$

This is a differential equation in b and b'.

Rearranging we have:

$$v_i G'(v_i) = \frac{\partial(b(v_i)G(v_i))}{\partial v_i}$$

Integrating both sides:

$$b(v)G(v) = \int_0^v \left(\frac{\partial(b(x)G(x))}{\partial x} \right) dx = \int_0^v x g(x) dx$$

$$\Leftrightarrow b(v) = \int_0^v x \frac{g(x)}{G(v)} dx$$

Note that this bid function is non-decreasing in v.

The condition:

$$b^*(v) = \int_0^v x \frac{g(x)}{G(v)} dx$$

is necessary. Is it also sufficient?

We need to check the second order condition.

To this goal we will prove that:

$$\pi(v_i, b) = (v_i - b)Q(b)$$

is quasi concave:

- $\frac{\partial}{\partial b}\pi_i(v_i, b) > 0 \text{ for } b < b^*(v_i);$
- $\frac{\partial}{\partial b}\pi_i(v_i, b) < 0 \text{ for } b > b^*(v_i)$

The key to prove the result is to show that $\pi_{vb}(v_i, b) > 0$.

Note that:

$$\begin{aligned}
\pi_v(v_i, b) &= Q(b) - [Q(b) - (v_i - b)Q'(b)]b'(v) \\
&= G([b^*]^{-1}(b)) - b'(v)[Q(b) - (v_i - b)Q'(b)] \\
&= G([b^*]^{-1}(b))
\end{aligned}$$

and:

$$\pi_{vb}(v_i, b) = g([b^*]^{-1}(b)) \bullet \frac{\partial}{\partial b}[b^*]^{-1}(b) = \frac{g([b^*]^{-1}(b))}{[b^*]'(v_i)} > 0$$

Now take a bid $\widehat{b} < b^*(v_i)$ and let $\widehat{v} = [b^*]^{-1}(\widehat{b})$.

since $\widehat{b} < b^*(v_i)$ we have:

$$\widehat{v} = [b^*]^{-1}(\widehat{b}) < [b^*]^{-1}(b^*(v)) \quad = v$$

But then since $\pi_{bv}(v_i, b) > 0$ we have:

$$\pi_b(v_i, \widehat{b}) \geq \pi_b(\widehat{v}, \widehat{b})$$

Note that the fact that $\widehat{b} = b^*(\widehat{v})$ implies $\pi_b(\widehat{v}, \widehat{b}) = 0$, so:

$$\pi_b(v_i, \widehat{b}) \geq \pi_b(\widehat{v}, \widehat{b}) = 0$$

The argument that for $\widehat{b} > b^*(v)$ $\pi_b(v_i, \widehat{b}) \leq 0$ is similar.

Revenue Equivalence

As we said, integrating by parts the bids $b^*(v) = \int_0^v x \frac{g(x)}{G(v)} dx$ in the FPA and noting that $G(x) = F(x)^{n-1}$ we have:

$$b^*(v) = v - \int_0^v \left[\frac{F(x)}{F(v)} \right]^{n-1} dx$$

In equilibrium bidder's utilities are:

$$\begin{aligned} \pi(v) &= (v - b^*(v))G(v) \\ &= \int_0^v \left[\frac{G(x)}{G(v)} \cdot G(v) \right] dx = \int_0^v G(x) dx \end{aligned}$$

This is the same value we obtained in the second price auction.

With risk neutral bidders, bidders are indifferent between FPA and SPA

Consider the seller now:

$$R = E(b^*(v_{(1)})) = \int_0^1 \left(v - \int_0^v \frac{G(x)}{G(v)} dx \right) d[F(v)^n]$$

$$= \int_0^1 \left(v - \int_0^v \frac{G(x)}{G(v)} dx \right) nG(v)dF(v)$$

$$= n \int_0^1 \left(vG(v) - \int_0^v G(x)dx \right) dF(v)$$

$$= n \int_0^1 \left(\int_0^v xdG(x) \right) dF(v) = n \int_0^1 v[1 - F(v)]dG(v)$$

$$= |v[1 - F(v)]G(v)|_0^1 - n \int_0^1 [1 - F(v) - vf(v)]G(v)dv$$

$$= n \int_0^1 \left[x - \frac{1 - F(x)}{f(x)} \right] G(x)dF(x)$$

The bottom line is that:

$$R_{FPA} = n\int_0^1 \left[x - \frac{1 - F(x)}{f(x)} \right] G(x)dF(x) = R_{SPA}$$

The first price auction and the second price auction generate the same revenues.

As we will see this does not happen by chance by the fact that the two auctions implement the same allocation of the good.