

ECON 6140
Problem Set 2

Gabe Sekeres

February 10, 2025

Taxation and human capital

1. The household takes prices, wages, rental prices, interest rates, and taxes as given, and maximizes their utility by choosing a stream of consumption, total labor, productive labor, capital investment, next-period capital, next-period human capital, and bond investment. They solve the problem

$$\max_{\{c_t, n_t, n_{m,t}, x_t, k_{t+1}, h_{t+1}, b_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_t)$$

subject to, for $t = 0, 1, \dots$,

$$\begin{aligned} (1 + \tau_t^c)c_t + p_{k,t}x_t + b_{t+1} &\leq (1 - \tau_t^n)w_t(h_t n_{m,t}) - \tau_t^w + r_t k_t - \tau_t^k(r_t - \delta_k)k_t + (1 + (1 - \tau_t^b)r_t^b)b_t \\ k_{t+1} &\leq x_t + (1 - \delta_k)k_t \\ h_{t+1} &= B(n_t - n_{m,t})^\gamma h_t^{1-\gamma} + (1 - \delta_h)h_t \\ 0 &= \lim_{T \rightarrow \infty} \beta^T u'_c(c_T, 1 - n_T)b_{T+1} \\ 0 &\leq (c_t, x_t, k_{t+1}) \\ n_t &\in [0, 1] \end{aligned}$$

where τ_t^c is the consumption tax, τ_t^n is the labor tax, τ_t^k is the capital rent tax, τ_t^b is the capital gains tax, and τ_t^w is the (non-distortionary) lump-sum tax on income. All other variables are as defined, either in the problem or the notes. The price of capital investment, $p_{k,t}$, will end up being 1 in equilibrium, as money can be freely allocated between investment and consumption. For simplicity, we work in terms of $n_{m,t}$ and n_t , endogenizing the conditions that $n_{m,t} + n_{h,t} = n_t$ and that $l_t + n_t = 1$.

The firm solves the simple, unconstrained problem

$$\max_{k_t, z_t} F(k_t, z_t) - r_t k_t - w_t z_t \equiv \max_{k_t, n_{m,t}} F(k_t, h_t n_{m,t}) - r_t k_t - w_t (h_t n_{m,t})$$

for each $t = 0, 1, \dots$, where the second problem endogenizes the productive effect of human capital.

2. A recursive competitive equilibrium is a stream of prices $\{w_t^*, r_t^*, (r_t^b)^*\}_{t=0}^\infty$, a stream of allocations $\{c_t^*, n_t^*, n_{t,m}^*, x_t^*, k_{t+1}^*, h_{t+1}^*\}_{t=0}^\infty$, and a sequence of bond holdings $\{b_{t+1}^*\}$ such that
 - (a) Given prices, the allocations and bonds satisfy the household's maximization problem
 - (b) Given prices, the allocations satisfy the firm's maximization problem
 - (c) Markets clear (meaning that the supply and demand of labor and capital are the same in the household and firm's problems respectively)
 - (d) $b_0^* = b_0 = 0$, $k_0^* = k_0 > 0$, $p_{k,0}^* = p_{k,t}^* = 1$, and $h_0^* = h_0$ are each given

Note that this equilibrium implicitly defines the level of investment in human capital under the model primitives, so the equilibrium formulation is identical to the class formulation, with the addition of $n_{t,m}$ and h_{t+1} . Formally, the stream $\{h_t^*\}_{t=0}^\infty$ is characterized precisely by h_0 (given) and condition (a), that the equilibrium allocations are optimal for the household. Additionally, we take the problem parameters (including, importantly, the stream of taxes!) as exogenous.

3. The household's Lagrangian is

$$\begin{aligned} \mathcal{L} = \sum_{t=0}^{\infty} \beta^t & \left\{ u(c_t, 1 - n_t) + \right. \\ & + \lambda_t \left[(1 - \tau_t^n) w_t (h_t n_{m,t}) - \tau_t^w + r_t k_t - \tau_t^k (r_t - \delta_k) k_t + (1 + (1 - \tau_t^b) r_t^b) b_t \right. \\ & \left. - (1 + \tau_t^c) c_t - p_{k,t} x_t - b_{t+1} \right] \\ & + \theta_t [x_t + (1 - \delta_k) k_t - k_{t+1}] \\ & \left. + \mu_t \left[B(n_t - n_{m,t})^\gamma h_t^{1-\gamma} + (1 - \delta_h) h_t - h_{t+1} \right] \right\} \end{aligned}$$

where the non-negativity constraints and the constraint $n_t \leq 1$ can be eliminated because of the Inada conditions on utility. This admits the first order conditions

$$\begin{aligned} 0 &= u'_c(c_t, 1 - n_t) - \lambda_t(1 + \tau_t^c) & (c_t) \\ 0 &= -u'_n(c_t, 1 - n_t) + \mu_t B \gamma (n_t - n_{m,t})^{\gamma-1} h_t^{1-\gamma} & (n_t) \\ 0 &= \lambda_t(1 - \tau_t^n) w_t h_t - \mu_t B \gamma (n_t - n_{m,t})^{\gamma-1} h_t^{1-\gamma} & (n_{m,t}) \\ 0 &= -\lambda_t p_{k,t} + \theta_t & (x_t) \\ 0 &= -\theta_t + \beta [\lambda_{t+1}(r_{t+1} - \tau_{t+1}^k (r_{t+1} - \delta_k)) + \theta_{t+1}(1 - \delta_k)] & (k_{t+1}) \\ 0 &= -\mu_t + \beta \left[\lambda_{t+1}(1 - \tau_{t+1}^n) w_{t+1} n_{m,t+1} + \mu_{t+1} \left(B(1 - \gamma) \left(\frac{n_{t+1} - n_{m,t+1}}{h_{t+1}} \right)^\gamma + (1 - \delta_h) \right) \right] & (h_{t+1}) \\ 0 &= -\lambda_t + \beta \lambda_{t+1} (1 + (1 - \tau_{t+1}^b) r_{t+1}^b) & (b_{t+1}) \\ 0 &= \lim_{T \rightarrow \infty} \beta^T \lambda_T k_{t+1} & (\text{TVC}) \\ 0 &= \lim_{T \rightarrow \infty} \beta^T \lambda_T b_{t+1} & (\text{No Ponzi}) \end{aligned}$$

for $t = 0, 1, \dots$. The first order conditions for the firm are, as always,

$$F_k(k_t, h_t n_{m,t}) = r_t \quad \text{and} \quad F_n(k_t, h_t n_{m,t}) = w_t h_t$$

with these conditions, we can characterize the steady state, assuming that the exogenous streams of taxes τ , bond returns r^b , and government spending g are non-stochastic. We have that

$$p_k[1 - \beta(1 - \delta_k)] = \beta[(1 - \tau^k) F_k(k^*, n_m^* h^*) + \tau^k \delta_k] \quad (k)$$

$$1 = \beta[1 + \rho] \quad (b)$$

$$u'_n(c^*, 1 - n^*) = u'_c(c^*, 1 - n^*) \frac{1 - \tau^n}{1 + \tau^c} F_n(k^*, h^* n_m^*) \quad (n_m)$$

$$(1 - \beta(1 - \delta_h)) \frac{1}{B\gamma} \left(\frac{n^* - n_m^*}{h^*} \right)^{1-\gamma} = \beta \left(1 + \frac{1 - \gamma}{\gamma} \frac{n^* - n_m^*}{h^*} \right) \quad (h)$$

$$x^* = \delta_k k^* \quad (x)$$

$$h^* = \left(\frac{B}{\delta_h} \right)^{\frac{1}{\gamma}} (n^* - n_m^*) \quad (n)$$

$$c^* + \delta_k k^* + g = F(k^*, h^* n_m^*) \quad (c)$$

where, as in the notes, we define $\rho = (1 + \tau^b)r^b$. We can unpack condition (n_m) to observe the effects of taxes on the steady-state level of human capital, recalling that $F_n(\cdot) = wh^*$, and observing that no taxes appear explicitly in conditions (h) and (n) , the other conditions that concern human capital. We have that

$$u'_n(\cdot) = u'_c(\cdot) \frac{1 - \tau^n}{1 + \tau^c} wh^* \implies h^* = \frac{u'_n(\cdot)}{u'_c(\cdot)} \frac{1}{w} \frac{1 + \tau^c}{1 - \tau^n}$$

So we can directly see that h^* is increasing in both τ^c and τ^n . Both of these conditions make sense – as consumption and productive labor get more expensive, the household is incentivized to divert energy towards human capital. Additionally, h^* is not dependent on τ^x and τ^k , as those two taxes affect only the ratio of investment between bonds and future capital, and it is not dependent on τ^w as that is a lump-sum tax and therefore non-distortionary.

4. Describe the changes in the following taxes on the stock of capital, human capital, capital output ratios, and labor:

- (a) Capital gains (in our terms, τ^b). Changes in the capital gains tax τ^b will have no effect on any variables in the steady state. To see why, note that the only place that the capital gains tax appears in the steady state equations is in equation (b) , as a part of ρ . Since this condition must hold in steady state, any change in τ^b will be compensated by a corresponding change in r^b , like we saw in class. Formally, we have that

$$\frac{\partial \rho}{\partial \tau^b} = \frac{\partial}{\partial \tau^b} [(1 + \tau^b)r^b] = \frac{\partial}{\partial \tau^b} \underbrace{\left[\frac{1}{\beta} - 1 \right]}_{\text{constant}} = 0$$

Since ρ is necessarily a constant, it will not change when τ^b changes. Thus, a change in τ^b will have no effect in equilibrium. We saw in class the same thing, where changes in the capital gains tax are compensated by changes in the risk-free return rate.

- (b) Labor income (in our terms, τ^n). Note first that τ^n appears only in the steady state condition n_m . Holding all else constant, and rearranging, we can see that this condition becomes

$$u'_c(c^*, 1 - n^*) = \frac{1}{1 - \tau^n} [(1 + \tau^c)F_n(\cdot)u'_n(c^*, 1 - n^*)]$$

so $u'_c(\cdot)$ is increasing in τ^n . Since u is increasing and concave in consumption, this means that c^* is decreasing in τ^n , which implies that x^* is decreasing in τ^n , since the two move together. Thus, since $x^* = \delta_k k^*$, k^* is also decreasing in τ^n . Human capital h^* and labor n^* are not changing in τ^n . To see why, note that both are pinned down by the steady state equations (n_m) , (n) , and (h) . We saw earlier that the change in τ^n is compensated by a change in $u'_c(\cdot)$, so the other choices in (n_m) are not affected. Other than that, τ^n does not appear in the other steady state equations characterizing h^* and n^* , so they do not change when τ^n changes. For an intuition, observe that changing the bond ratio will change how investment is divided between consumption, capital, and bonds, but will not change how much the household chooses to work (or how they divide that work into productive labor and human capital investment labor). Finally, the capital-output ratio $k^*/F(k^*, h^*, n_m^*)$ will depend on whether the production function faces increasing, decreasing, or constant returns to scale. We know that h^* and n_m^* do not change, and we know that k^* decreases when τ^n increases. Thus, an increase in τ^n will lead to an increase in capital-output ratio if there are increasing returns to scale, a decrease if there are decreasing returns to scale, and no change if there are constant returns to scale. The same holds in the opposite direction if τ^n decreases.

5. If the labor income tax τ^n changes by 10%, we could replicate the changes described in (4b) by changing τ^c such that the ratio $\frac{1 - \tau^n}{1 + \tau^c}$ is the same as when τ^n changed. To see why this replicates the change precisely, note that the only place that either τ^n or τ^c appear in the steady state is in that precise ratio.

As long as the ratio remains constant, there will be no effect on the steady state values. Specifically, a decrease in τ^n could be replicated by a decrease in τ^c . Intuitively, it must be the case that the marginal utility of leisure and the marginal utility of consumption are in ratio of the respective taxes, so we could replicate the same choices of leisure and consumption by either increasing the tax on working or increasing the tax on consumption – either way, the household will move to increase leisure. The same holds, of course, for decreases.

A primer to OLG

1. Each generation t solves the problem

$$\max_{s_t} \ln(c_t^t) + \beta \ln(c_{t+1}^t)$$

subject to

$$\begin{aligned} c_t^t &\leq w_t - \delta_{y,t} - s_t \\ c_{t+1}^t &\leq (1 + r_{t+1})s_t - \delta_{o,t+1} \end{aligned}$$

(which will, of course, each hold with equality). The Lagrangian, substituting the constraints, is

$$\mathcal{L} = \ln(w_t - \delta_{y,t} - s_t) + \beta \ln((1 + r_{t+1})s_t - \delta_{o,t+1})$$

which admits the first order condition

$$\frac{\partial \mathcal{L}}{\partial s_t} = -\frac{1}{w_t - \delta_{y,t} - s_t} + \frac{\beta(1 + r_{t+1})}{(1 + r_{t+1})s_t - \delta_{o,t+1}} = 0$$

implying that

$$s_t^* = \frac{\beta}{1 + \beta}(w_t - \delta_{y,t}) + \frac{1}{1 + \beta} \frac{\delta_{o,t+1}}{1 + r_{t+1}}$$

Substituting back to get optimal consumption, we get that

$$\begin{aligned} (c_t^t)^* &= w_t - \delta_{y,t} - \left(\frac{\beta}{1 + \beta}(w_t - \delta_{y,t}) + \frac{1}{1 + \beta} \frac{\delta_{o,t+1}}{1 + r_{t+1}} \right) \\ &= \frac{1}{1 + \beta} \left(w_t - \delta_{y,t} - \frac{\delta_{o,t+1}}{1 + r_{t+1}} \right) \\ (c_{t+1}^t)^* &= (1 + r_{t+1}) \left(\frac{\beta}{1 + \beta}(w_t - \delta_{y,t}) + \frac{1}{1 + \beta} \frac{\delta_{o,t+1}}{1 + r_{t+1}} \right) - \delta_{o,t+1} \\ &= \frac{\beta}{1 + \beta} ((1 + r_{t+1})(w_t - \delta_{y,t}) - \delta_{o,t+1}) \end{aligned}$$

For the initial old generation, we have that they supply no labor and receive no wage, so they have no real choice problem – they consume their endowment, along with any tax imposed by the government. Formally,

$$c_0^0 = (1 + r_0)A_0 - \delta_{o,0}$$

Note that the government's budget constraint and spending do not factor in here at all – the households take them as exogenous, and optimize savings relative to the taxes they see. For consumption to be positive, it's necessary that the taxes are not too high. Specifically, we need that $\delta_{y,t} < w_t$, and that $\delta_{o,t+1}$ is not too large. It would suffice to impose that $\delta_{o,t+1} \leq 0$, so the government is on net making transfers to the old generation, as we would expect from actual policy.

2. In the steady state with Cobb-Douglas production technology and constant labor supply of 1, we can precisely characterize the wage and interest rate:

$$w = \frac{\partial F(k, 1)}{\partial n} = (1 - \alpha)k^\alpha \quad \text{and} \quad r = \frac{\partial F(k, 1)}{\partial k} - 1 = \alpha k^{\alpha-1} - 1$$

From part 1, we have that the steady state optimal savings rate is

$$s = \frac{\beta}{1 + \beta}(w - \delta_y) + \frac{1}{1 + \beta} \frac{\delta_o}{1 + r}$$

Additionally, the government must fund in each period g and the interest on their debt rd , so their budget constraint is

$$g + rd = \delta_y + \delta_o$$

Finally, the market for assets must clear, meaning that the young must save precisely enough to buy the capital and debt from the old. We must have that

$$s = k + d$$

Along with the exogenous parameters, these conditions precisely characterize the steady state.

3. Observe that the condition $s = k + d$ characterizes precisely where the savings of the economy go. If $d > 0$, then some savings must go to debt rather than capital. As d increases, r also increases, assuming that $\alpha \in (0, 1)$. This would necessarily lower wages w and consumption c in the steady state, at all levels of d . Thus, since the economy starts with $d_0 = 0$, optimally the government will hold no debt. Relating to what we talked about in class, we can see that this economy does *not* face dynamic inefficiencies, since $r > 0$ and there is no population growth. In that case, it cannot be that $k^* > k^{\text{gold}}$, so there are no gains to reducing capital. Thus, the government should hold no debt and finance their spending purely through (non-distortionary) taxes.
4. I solved for the steady state in Julia, making some additional assumptions for tractability. Along with assuming that $d = 0$, I also imposed that $\delta_o = 0$, so there are no taxes (or transfers) on the old. Since taxes are linear in the government's budget constraint and the households take r as given, this does not impose any behavioral constraints on the problem, and ensures a single (interior) solution to the government's optimal policy, rather than the continuum when both are free variables. I used the **NLsolve** root-finding algorithm, and the entirety of the code used for this problem is below.¹ I got the steady state levels:

$$\begin{aligned} k &= 0.16576237075421454 \\ F(k, 1) &= 0.5832379634630681 \\ w &= 0.40826657442414765 \\ r &= 0.0555555416033509 \\ g &= 0.058323796346306814 \\ \delta_y &= 0.058323796346306814 \\ \delta_o &= 0 \end{aligned}$$

Checking optimal savings policy to confirm, I got that it equalled the steady-state capital, which would be a confirmation that this algorithm worked. From these, optimal consumption levels are

$$c_y = 0.18418040951465306 \quad ; \quad c_0 = 0.17497138672616994$$

¹This is a slow algorithm, and it could be improved if I used Newton-Raphson since the policy function is smooth. However, I am lazy and don't want to write an entire gradient descent algorithm. This is fast enough for this problem, it runs entirely in 0.25 seconds.

5. The immediate effect of the government/output ratio changing to 0.15 will be that g increases, leading to an increase in δ_y , as it is directly in proportion to the government policy. We can directly calculate the new steady state and compare:

Variable	Old Steady State	New Steady State	Change
k	0.1658	0.1464	↓
$F(k, 1)$	0.5832	0.5619	↓
w	0.4083	0.3933	↓
r	0.0556	0.1515	↑
g	0.0583	0.0843	↑
δ_y	0.0583	0.0843	↑
δ_o	0	0	=
c_y	0.1842	0.1627	↓
c_o	0.1750	0.1686	↓

As we can see, the shift decreases steady-state consumption, output, capital, and wages, and increases the interest rate, government spending, and taxes on the young.

6. If taxes are now labor-income taxes rather than lump-sum, the generation's problem becomes

$$\max_{s_t} \ln(c_t^t) + \beta \ln(c_{t+1}^t)$$

subject to

$$\begin{aligned} c_t^t &\leq w_t(1 - \tau_t) - s_t \\ c_{t+1}^t &\leq s_t(1 + r_{t+1}) \end{aligned}$$

The Lagrangian is now

$$\mathcal{L} = \ln(w_t(1 - \tau_t) - s_t) + \beta \ln(s_t(1 + r_{t+1}))$$

which admits the first order condition

$$\frac{\partial \mathcal{L}}{\partial s_t} = -\frac{1}{w_t(1 - \tau_t) - s_t} + \frac{\beta}{s_t} = 0 \implies s_t^* = \frac{\beta}{1 + \beta} w_t(1 - \tau_t)$$

and substituting back to get optimal consumption, we get that

$$\begin{aligned} (c_t^t)^* &= \frac{1}{1 + \beta} w_t(1 - \tau_t) \\ (c_{t+1}^t)^* &= \frac{\beta}{1 + \beta} w_t(1 - \tau_t)(1 + r_{t+1}) \end{aligned}$$

In the steady state, the wage and interest rate remain the same, as the technology has not changed:

$$w = \frac{\partial F(k, 1)}{\partial n} = (1 - \alpha)k^\alpha \quad \text{and} \quad r = \frac{\partial F(k, 1)}{\partial k} - 1 = \alpha k^{\alpha-1} - 1$$

The steady state optimal savings rate is simply

$$s = \frac{\beta}{1 + \beta} w(1 - \tau)$$

and the government's budget constraint is now

$$g + rd = w\tau \implies \tau = \frac{g}{w}$$

Finally, the market for assets must again clear so we need

$$s = k + d$$

Computationally, we have that using the parameters from above

$$\begin{aligned} k &= 0.16576237075421468 \\ F(k, 1) &= 0.5832379634630682 \\ w &= 0.40826657442414765 \\ r &= 0.0555555416033502 \\ g &= 0.058323796346306814 \\ \tau &= 0.14285714285714288 \end{aligned}$$

Checking optimal savings policy to confirm, I got that it equalled the steady-state capital, which would be a confirmation that this algorithm worked. From these, optimal consumption levels are

$$c_y = 0.18418040951465306 \quad ; \quad c_o = 0.17497138672616994$$

So nothing has changed from the steady state above, except for the tax levels!

7. If the government-output ratio increases to 0.15, we can calculate the new steady state and compare, like above:

Variable	Old Steady State	New Steady State	Change	Same?
k	0.1658	0.1464	↓	✓
$F(k, 1)$	0.5832	0.5619	↓	✓
w	0.4083	0.3933	↓	✓
r	0.0556	0.1515	↑	✓
g	0.0583	0.0843	↑	✓
τ	0.1429	0.2142	↑	NA
c_y	0.1842	0.1627	↓	✓
c_o	0.1750	0.1686	↓	✓

So everything is the same as above! This is a strange result, but comes from the fact that in the lump-sum model, taxes were only on the young, and in this model taxes are only on those making income – basically, the young. While we are rescaling the taxes, an increase in them will affect the generations in the exact same way.

Code. I wrote the code in Julia, and it is included here:

```
using NLSolve
```

```
@time begin
```

```
# Calculate excess savings
```

```
function excess_savings(k; beta, alpha, g_ratio)
```

```
    # Penalize non-positive capital
```

```
    if k <= 0
```

```
        return 1e8
```

```
    end
```

```
# Calculate savings
```

```
    s = (beta / (1 + beta)) * ((1 - alpha) * k ^ alpha - g_ratio * k ^ alpha)
```

```

    # Calculate excess savings
    return s - k
end

# Solve for k using a root-finding algorithm
function find_steady_state(; guess = 0.5, beta = 0.9, alpha = 0.3, g_ratio =
    0.1)
    f!(F, x) = (F[1] = excess_savings(x[1]; beta=beta, alpha=alpha, g_ratio=
        g_ratio))
    sol = nlsolve(f!, [guess])
    return sol.zero[1]
end

# Set model parameters
beta = 0.9
alpha = 0.3
g_ratio = 0.1

# Calculate steady state capital
k_ss = find_steady_state(guess = 0.1, beta = beta, alpha = alpha, g_ratio =
    g_ratio)

println("Steady-state capital K* = $k_ss")

# Calculate other steady state variables
Y_ss = k_ss ^ alpha
w_ss = (1 - alpha) * Y_ss
r_ss = alpha * k_ss ^ (alpha - 1)
g_ss = g_ratio * Y_ss
delta_y_ss = g_ss

println("Steady-state output Y* = $Y_ss")
println("Steady-state wage w* = $w_ss")
println("Steady-state interest rate r* = $r_ss")
println("Steady-state government spending g* = $g_ss")
println("Steady-state taxes delta_y* = $delta_y_ss")

# Household saving s_star should equal K_star
s_ss = (beta/(1+beta)) * (w_ss - delta_y_ss)
println("Check s* = $s_ss, which should equal K*.")

cy_ss = (1 / (1 + beta)) * (w_ss - delta_y_ss)
println("Optimal Consumption Cy* = $cy_ss")

co_ss = (beta / (1 + beta)) * (r_ss) * (w_ss - delta_y_ss)
println("Optimal Consumption Co* = $co_ss")

# Calculate steady state after shock to g_ratio
g_ratio_new = 0.15

```



```

k_ss_new = find_steady_state(guess = 0.1, beta = beta, alpha = alpha, g_ratio
    = g_ratio_new)

println("Steady-state capital K* = $k_ss_new")

# Calculate other steady state variables after shock
Y_ss_new = k_ss_new ^ alpha
w_ss_new = (1 - alpha) * Y_ss_new
r_ss_new = alpha * k_ss_new ^ (alpha - 1)
g_ss_new = g_ratio_new * Y_ss_new
delta_y_ss_new = g_ss_new

println("Steady-state output Y* = $Y_ss_new")
println("Steady-state wage w* = $w_ss_new")
println("Steady-state interest rate r* = $r_ss_new")
println("Steady-state government spending g* = $g_ss_new")
println("Steady-state taxes delta_y* = $delta_y_ss_new")

# Household saving s_star should equal K_star
s_ss_new = (beta/(1+beta)) * (w_ss_new - delta_y_ss_new)
println("Check s* = $s_ss_new, which should equal K*.")

cy_ss_new = (1 / (1 + beta)) * (w_ss_new - delta_y_ss_new)
println("Optimal Consumption Cy* = $cy_ss_new")

co_ss_new = (beta / (1 + beta)) * (r_ss_new) * (w_ss_new - delta_y_ss_new)
println("Optimal Consumption Co* = $co_ss_new")


# Repeat but with income taxes rather than lump sum
# Calculate excess savings
function excess_savings_income(k; beta, alpha, g_ratio)
    # Penalize non-positive capital
    if k <= 0
        return 1e8
    end

    # Calculate savings
    s = (beta / (1 + beta)) * ((1 - alpha) * k ^ alpha * (1 - g_ratio / (1 -
        alpha)))

    # Calculate excess savings
    return s - k
end

```

```

# Solve for k using a root-finding algorithm
function find_steady_state_income(; guess = 0.5, beta = 0.9, alpha = 0.3,
    g_ratio = 0.1)
    f!(F, x) = (F[1] = excess_savings_income(x[1]; beta=beta, alpha=alpha,
        g_ratio=g_ratio))
    sol = nlsolve(f!, [guess])
    return sol.zero[1]
end

# Calculate steady state capital
k_ss_income = find_steady_state_income(guess = 0.1, beta = beta, alpha = alpha
    , g_ratio = g_ratio)

println("Income Tax Steady-state capital K* = $k_ss_income")

# Calculate other steady state variables
Y_ss_income = k_ss_income ^ alpha
w_ss_income = (1 - alpha) * Y_ss_income
r_ss_income = alpha * k_ss_income ^ (alpha - 1)
g_ss_income = g_ratio * Y_ss_income
tau_ss_income = g_ss_income / w_ss_income

println("Income Tax Steady-state output Y* = $Y_ss_income")
println("Income Tax Steady-state wage w* = $w_ss_income")
println("Income Tax Steady-state interest rate r* = $r_ss_income")
println("Income Tax Steady-state government spending g* = $g_ss_income")
println("Income Tax Steady-state taxes tau* = $tau_ss_income")

# Household saving s_star should equal K_star
s_ss_income = (beta/(1+beta)) * w_ss_income * (1 - tau_ss_income)
println("Check s* = $s_ss_income, which should equal K*.")

cy_ss_income = (1 / (1 + beta)) * w_ss_income * (1 - tau_ss_income)
println("Optimal Consumption Cy* = $cy_ss_income")

co_ss_income = (beta / (1 + beta)) * (r_ss_income) * w_ss_income * (1 -
    tau_ss_income)
println("Optimal Consumption Co* = $co_ss_income")

# Calculate steady state after shock to g_ratio
g_ratio_new = 0.15
k_ss_income_new = find_steady_state_income(guess = 0.1, beta = beta, alpha =
    alpha, g_ratio = g_ratio_new)

println("Income Tax Steady-state capital K* = $k_ss_income_new")

# Calculate other steady state variables
Y_ss_income_new = k_ss_income_new ^ alpha
w_ss_income_new = (1 - alpha) * Y_ss_income_new
r_ss_income_new = alpha * k_ss_income_new ^ (alpha - 1)

```

```

g_ss_income_new = g_ratio_new * Y_ss_income_new
tau_ss_income_new = g_ss_income_new / w_ss_income_new

println("Income Tax Steady-state output Y* = $Y_ss_income_new")
println("Income Tax Steady-state wage w* = $w_ss_income_new")
println("Income Tax Steady-state interest rate r* = $r_ss_income_new")
println("Income Tax Steady-state government spending g* = $g_ss_income_new")
println("Income Tax Steady-state taxes tau* = $tau_ss_income_new")

# Household saving s_star should equal K_star
s_ss_income_new = (beta/(1+beta)) * w_ss_income_new * (1 - tau_ss_income_new)
println("Check s* = $s_ss_income_new, which should equal K*.")

cy_ss_income_new = (1 / (1 + beta)) * w_ss_income_new * (1 - tau_ss_income_new)
println("Optimal Consumption Cy* = $cy_ss_income_new")

co_ss_income_new = (beta / (1 + beta)) * (r_ss_income_new) * w_ss_income_new *
    (1 - tau_ss_income_new)
println("Optimal Consumption Co* = $co_ss_income_new")

end

```