ECON 6110, Microeconomic Theory III	Section 13
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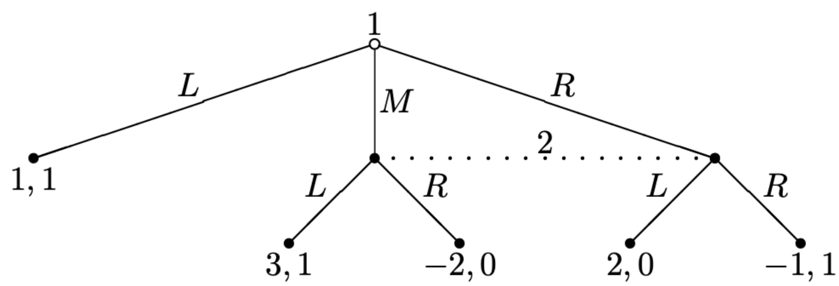
1 Sequential Equilibrium (SE)

Definition 1.1. An assessment (β, μ) is **consistent** if there is a sequence $((\beta^n, \mu^n))_{n=1}^{\infty}$ of assessments that converges to (β, μ) in Euclidian space and that

1. Each strategy profile β^n is completely mixed,
2. Each belief system μ^n is derived from β^n using Bayes' rule.

Definition 1.2. An assessment (β, μ) is a **sequential equilibrium** of a finite extensive game with perfect recall if it is sequentially rational and consistent.

Example 1.1.



Find the set of sequential equilibria of the game.

Solution: Since SE is a refinement of PBE, we can find all PBEs first, and then check if any of them still holds to be a SE.

You should find a PBE as follows:

$$\beta = \left(\begin{array}{ccc|cc} L & M & R & L & R \\ 0 & 1 & 0 & 1 & 0 \end{array}\right) \quad \text{and} \quad \mu = \begin{pmatrix} M & R \\ 1 & 0 \end{pmatrix}$$

Consider the sequence of totally mixed strategies:

$$\beta^n = \left(\begin{array}{ccc|cc} L & M & R & L & R \\ 1/n & 1-2/n & 1/n & 1-1/n & 1/n \end{array}\right) \quad \text{and} \quad \mu^n = \begin{pmatrix} M & R \\ \frac{1-2/n}{1-1/n} & \frac{1/n}{1-1/n} \end{pmatrix}$$

We can verify that since $\frac{1-2/n}{1-1/n} = 2 - \frac{1}{1-1/n} \to 1$, (β, μ) is consistent so it is also SE.

You should find another class of PBEs $(1 > \delta > 0)$:

$$\beta = \left(\begin{array}{ccc|cc} L & M & R & L & R \\ 1 & 0 & 0 & \delta & 1-\delta \end{array} \right) \quad \text{and} \quad \mu = \begin{pmatrix} M & R \\ 1/2 & 1/2 \end{pmatrix}$$

Consider the sequence of totally mixed strategies:

$$\beta^n = \left(\begin{array}{ccc|cc} L & M & R & L & R \\ 1-2/n & 1/n & 1/n & \delta & 1-\delta \end{array} \right) \quad \text{and} \quad \mu^n = \begin{pmatrix} M & R \\ 1/2 & 1/2 \end{pmatrix}$$

We can verify that since $\beta^n \to \beta$ and $\mu^n = \mu$, so (β, μ) is consistent so it is also SE.

You should find yet another class of PBEs $(p \leq 1/2)$:

$$\beta = \left(\begin{array}{ccc|cc} L & M & R & L & R \\ 1 & 0 & 0 & 0 & 1 \end{array} \right) \quad \text{and} \quad \mu = \begin{pmatrix} M & R \\ p & 1-p \end{pmatrix}$$

Consider the sequence of totally mixed strategies:

$$\beta^n = \left(\begin{array}{ccc|cc} L & M & R & L & R \\ 1-1/n & p/n & (1-p)/n & 1/n & 1-1/n \end{array} \right) \quad \text{and} \quad \mu^n = \begin{pmatrix} M & R \\ p & 1-p \end{pmatrix}$$

We can verify that since $\beta^n \to \beta$ and $\mu^n = \mu$, so (β, μ) is consistent so it is also SE.

2 Trembling Hand Perfect Equilibrium

Sequential equilibrium restricts the player to hold "reasonable" off-path beliefs.

Trembling Hand Perfect Equilibria take a different route: each player allows the other players to make uncorrelated mistakes (their hands may tremble) that lead to off-path events. Moreover, each player's equilibrium strategy must be **robust** to such small mistakes of the other players. The essence of this concept is captured in static games.

Definition 2.1. A **trembling hand perfect equilibrium** of a finite strategic game is a mixed strategy profile σ with the property that there exists a sequence $(\sigma^k)_{k=0}^{\infty}$ of completely mixed strategy profiles that converges to σ such that for each player i the strategy σ_i is a best response to σ_{-i}^k for all values of k.

Remark. Trembling hand perfect equilibrium requires that σ_i is a best response to σ_{-i}^k, but this implies σ_i is a best response to σ_{-i}. This is because expected utility is linear in σ_{-i}^k, so

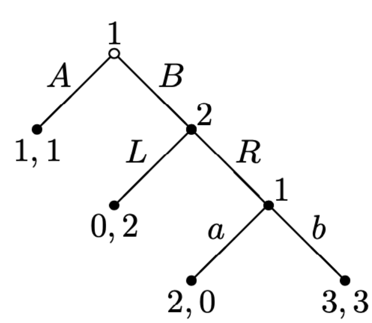
$$\mathbb{E}[u_i(\sigma_i, \sigma_{-i}^k)] - \mathbb{E}[u_i(\sigma_i', \sigma_{-i}^k)] \geq 0 \quad \forall \sigma_i', k$$

implies

$$\mathbb{E}[u_i(\sigma_i, \sigma_{-i})] - \mathbb{E}[u_i(\sigma_i', \sigma_{-i})] \geq 0$$

Thus, trembling hand perfect equilibrium $\subseteq$ Nash equilibrium.

Remark. When applying the solution concept in extensive games, we need to also allow the possibility a player's past and future selves to make mistakes. We study the trembling hand perfect equilibria of the **agent strategic form of the game**, in which there is one agent for each information set belonging to the same player.



In the agent strategic form of this game, $((A, a), L)$ is not a trembling hand perfect equilibrium since for any pair of completely mixed strategies of player 1's first agent and player 2, the unique best response of player 1's second agent is the pure strategy b.

Remark (Comparison with sequential equilibrium).

Proposition 2.1. *For every trembling hand perfect equilibrium β of a finite extensive game with perfect recall there is a belief system μ such that (β,μ) is a sequential equilibrium of the game.*

3 Knowledge and Equilibrium

Definition 3.1. A model of knowledge consists of:

1. Set of states Ω. Only one of the state can be true (but unobservable).

2. An **information function** for each agent h such that $h(\omega) \subset \Omega$. When a true state $\omega \in \Omega$ occurs, any state in $h(\omega)$ is deemed possible by the agent.

3. A subset of states $E \subset \Omega$ is an **event**. If $h(\omega) \subset E$, then we say in state ω, the agent **knows** E. We define the agent's **knowledge function** K by:

$$K(E) = \{\omega \in \Omega : h(\omega) \subset E\}.$$

   $K(E)$ is the set of states where the agent knows E.

Definition 3.2. An information function is **partitional** if there is some partition of Ω such that for any $\omega \in \Omega$, $h(\omega)$ is the element of the partition that contains ω.

Example 3.1. Suppose $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and that the agent's partition is $\{\{\omega_1, \omega_2\}, \{\omega_3\}, \{\omega_4\}\}$. Then $h(\omega_3) = \{\omega_3\}$, while $h(\omega_1) = \{\omega_1, \omega_2\}$.

Suppose $E = \{\omega_3\}$. Then $K(E) = \{\omega_3\}$. Similarly, $K(\{\omega_3, \omega_4\}) = \{\omega_3, \omega_4\}$ and $K(\{\omega_1, \omega_3\}) = \{\omega_3\}$.

3.1 Common Knowledge

Suppose there are I agents with partitional information functions $h_1, \ldots, h_I$ and associated knowledge functions $K_1, \ldots, K_I$.

Definition 3.3. We say that an event $E \subset \Omega$ is **mutual knowledge** in state ω if it is known to all agent. In math, let

$$K_1(E) \cap K_2(E) \cap \cdots \cap K_I(E) \equiv K^1(E)$$

Then $\omega \in K^1(E)$.

Definition 3.4. An event F is **self-evident** if for all $\omega \in F$ and $i = 1, \ldots, I$, we have $h_i(\omega) \subset F$.

Remark. A "self-evident" event is one whose occurrence is immediately and transparently known to each agent, purely by virtue of their information partitions. Formally, an event F is self-evident for agent i if, whenever the true state ω lies in F, the entire information cell $h_i(\omega)$ is contained in F, which implies $K_i(F) = F$.

Definition 3.5. The following two definitions of **common knowledge** is equivalent:

1. An event $E \subset \Omega$ is common knowledge in state ω if $\omega \in K^1(E) \cap K^1K^1(E) \cap \cdots$. (An event is "common knowledge" if is it mutual knowledge and each individual knows that all other individuals know it, each individual knows that all other individuals know that all the individuals know it, and so on.)

2. An event $E \subset \Omega$ is common knowledge in state $\omega \in \Omega$ if there is a self-evident event F for which $\omega \in F \subset E$.

Example 3.2. In a two-player example, an event $E \subseteq \Omega$ is common knowledge between 1 and 2 in the state $\omega \in \Omega$ if ω is a member of every set in the infinite sequence $K_1(E),\ K_2(E),\ K_1(K_2(E)),$ $K_2(K_1(E)), \ldots$.

3.2 Epistemic Foundations for Equilibrium

Fix a game $G = (I, \{S_i\}, \{u_i\})$. Let Ω be a set of states. Each state is a complete description of each player's knowledge, action and belief. Formally, each state $\omega \in \Omega$ specifies for each i,

- $h_i(\omega) \subset \Omega$, i's knowledge in state ω.
- $s_i(\omega) \in S_i$, i's pure strategy in state ω.
- $\mu_i(\omega) \in \Delta(S_{-i})$, i's belief about the actions of others (note that i may believe other players actions are correlated).

We assume that among the players, it is common knowledge that the game being played is G.

Proposition 3.3. *Suppose that in state $\omega \in \Omega$, each player i:*

1. *knows the others' actions: $h_i(\omega) \subset \{\omega' \in \Omega : s_{-i}(\omega') = s_{-i}(\omega)\}$.*

2. *has a belief consistent with this knowledge: supp$(\mu_i(\omega)) \subset \{s_{-i}(\omega') \in S_{-i} : \omega' \in h_i(\omega)\}$*

3. is rational: $s_i(\omega)$ *is a best response to* $\mu_i(\omega)$*,*

Then $s(\omega)$ *is a pure strategy Nash equilibrium of* G*.*

Proof. By *(iii)*, $s_i(\omega)$ is a best response for i to his belief, which by *(ii)* and *(i)* assigns probability one to the profile $s_{-i}(\omega)$. □

Remark. Previously, we defined Nash equilibrium as "mutual best responses." Here, we are able to arrive at the same place by using the common knowledge assumption of rationality.

With a bit more work, we can also provide an epistemic characterization of mixed Nash equilibrium, correlated equilibrium, and rationalizability.

Example 3.4. It seems natural to extend this reasoning to dynamic games.

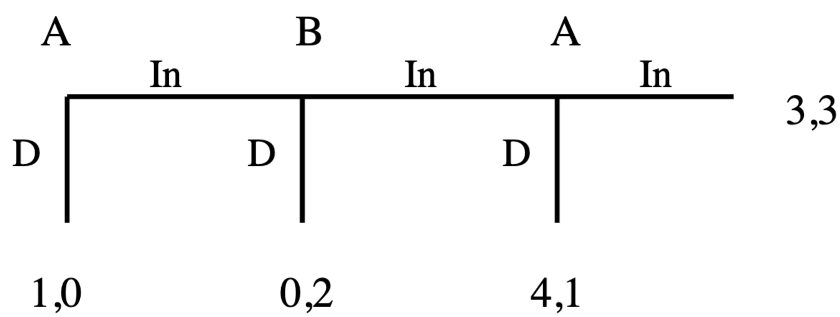


Figure 1: The Centepede Game

Consider the following argument for why common knowledge of rationality should imply the backward induction solution that A play D immediately:

If A is rational, A will play D at the last node; if B knows A is rational, then B knows this; if B herself is rational, she must then play D at the second to last node; if A knows B is rational, and that B knows that A is rational, then A knows that B will play D at the second to last node; thus, if A is rational, A must play D immediately.

The subtlety, however, is *what will happen if* A plays In? At that point, how will B assess A's rationality?