Problem Set on Convex Sets

Let's recall some definitions.

Definition 1. *A* ***closed half-space*** *in* $\mathbf{R^n}$ *is a set of points* $[p \geq \alpha] = \{x : p \cdot x \geq \alpha\}$ *or* $[p \leq \alpha] = \{x : p \cdot x \leq \alpha\}$ *for some* $p \in \mathbf{R^n}$ *and* $\alpha \in \mathbf{R}$.

A **half-space** is just the set of points one side or the other of a hyperplane $[p = \alpha] = \{x : p \cdot x = \alpha\}$. If the half-space includes the hyperplane, it is closed. If it contains no points in the hpyerplane, it is open.

Definition 2. *A hyperplane* $[p = \alpha]$ ***separates*** *convex sets* A *and* B *iff for all* $x \in A$ *and* $y \in B$, $p \cdot x \leq \alpha$ *and* $p \cdot y \geq \alpha$ *(or vice versa). That is,* $A \subset [p \leq \alpha]$ *and* $B \subset [p \geq \alpha]$ *(or vice versa). The separation is* ***proper*** *iff there is some* $x \in A$ *and* $y \in B$ *for which* $p \cdot x \neq p \cdot y$.

Proper just means that the two convex sets do not lie in the hyperplane.

Definition 3. *Two convex sets* A *and* B *are* ***strongly separated*** *by* p *iff there is an* $\epsilon > 0$ *and* α *such that* $A \subset [p \leq \alpha]$ *and* $B \subset [p \geq \alpha + \epsilon]$ *(or vice versa).*

The key result for us is

Theorem 1 (Strong Separating Hyperplane Theorem)**.** *If* K *and* C *are non-empty disjoint convex subsets of* $\mathbf{R^n}$ *with* K *compact and* C *closed, then there is a* $p \neq 0$ *which strongly separates* K *and* C.

Strong separation means that there is a separating hyperplane containing neither set.

There are two ways to describe a closed convex set C. The **primal** description of C is the list of elements in C. The **dual** description of C is the set of closed half-spaces containing C. Proving this, which we briefly discussed in class, is problem 1 below.

1. Use the strong separating hyperplane theorem to show that if C is a closed convex set, then C is the intersection of the half-spaces containing it.

2. For $p \in \mathbf{R^n}$, define $e_C(p) = \inf\{p \cdot x, x \in C\}$. $e_C(p)$ is called the **concave support function** of C.

   (a) Show that $e(p)$ is concave.

   (b) Show that $e(p)$ is homogeneous of degree 1.

   (c) What does it mean if $e(p) = -\infty$? You can explain with a picture.

   (d) Show that $[p \geq \alpha]$ ($[-p \leq -\alpha]$) contains C iff $\alpha \leq e(p)$.

3. Show that a function $f : \mathbf{R^n} \to \mathbf{R}$ is concave iff $\{(x, y) \in \mathbf{R^n} \times \mathbf{R} : y \leq f(x)\}$ is convex. This set is called the **subgraph** or **hypograph** of f.

4. Give an example of two closed convex sets that cannot be strongly separated.

5. Prove *Gordan's Lemma*: Either $Ax = 0, x > 0$ has a solution or $yA \ll 0$ has a solution. Hint: This follows from Farkas' lemma.

For notation in problem 5 (and throughout the course), $x \geq 0$ means $x \in \mathbf{R^n_+}$. $x > 0$ means that $x \geq 0$ and $x \neq 0$. Such vectors are called *semi-positive.* $x \gg 0$ means that each component of x is strictly positive. Similarly for the less-than relationships.