Notes on Discrete Time Economic Models: The Growth Model

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Preface

These notes were originally developed as part of the first year Macroeconomic Theory course at the University of Wisconsin. They have benefitted from suggestions by (so far) one cohort of students at Washington University. Several generations of sufferers/students deserve credit for finding numerous errors, and pointing alternative ways of discussing and presenting the material. They deserve recognition. The last iteration has benefitted from detailed comments by Nick Mader.

Introduction

The purpose of these notes is to briefly present some results from dynamic optimization as they apply to standard exogenous growth models. In addition, they also describe some basic techniques that can be used to study the dynamics of these models. The main application is the standard growth model. This basic model is used to study a variety of issues, ranging from population growth, the impact of new technologies on income distribution, and the effects of alternative tax and spending policies.

The notes are designed so that the exercises are an integral part of them (even though no answers are supplied at this time). It is suggested that the reader should try to work through as many of these exercises as possible.

Chapter 1

Basic Results in Dynamic Optimization

This section presents –with no proofs– some elementary results in dynamic optimization. The objective is to provide a brief overview of the basic mathematical techniques necessary to analyze dynamic models. Even though these tools will suffice to follow the arguments in these notes, they are **not** a good substitute for the formal study of dynamic optimization.

The basic idea of this section is that the analysis of dynamic problems can be successfully approached using a simple extension of the standard Kuhn-Tucker theorem. We first introduce some basic results in convex optimization, including the Kuhn-Tucker theorem; then we apply them to a simple two period economy. Finally we extend —using heuristic arguments— the conditions for dynamic optimization to the infinite dimensional case.

1.1 Concavity and Optimization

A function $f : \Re^n \to \Re$ is concave if $\forall \theta \in [0,1]$ and $x_1, x_2 \in \Re^n$

$$f(\theta x_1 + (1-\theta)x_2) \geq \theta f(x_1) + (1-\theta) f(x_2).$$

If the previous condition holds with strict inequality $\forall \theta \in (0,1)$ and $x_1 \neq x_2$, then f is strictly concave. A function f is convex iff $-f$ is concave.

Why does concavity matter? The answer is simple: if a differentiable function is concave its maximum can be characterized by finding a point $x \in \Re^n$ where $Df(x) = 0$. (Note that $Df(x) \in \Re^n$ and it is a vector of partial derivatives of the function f.) Thus if f is differentiable and concave and x maximizes f on $\Re^n$ then $Df(x) = 0$. (It turns out that this is an if and only if result that we will formally state in Theorem 3.)

Remark 1 (Properties of Concave Functions) *a) f and g concave* $\to af(x) + bg(x)$ *concave* $(a, b) \geq (0, 0)$.

b) f and g convex $\rightarrow af(x)+bg(x)$ convex $(a,b)\geq(0,0)$.
c) f concave and F concave and increasing $\rightarrow U(x)=F(f(x))$ concave.
d) f convex and F convex and increasing $\rightarrow U(x)=F(f(x))$ convex.
e) f and g concave $\rightarrow h(x)=min\{f(x),g(x)\}$ concave.
f) f and g convex $\rightarrow h(x)=max\{f(x),g(x)\}$ convex.

Theorem 2 *Let S be an open convex subset of $\Re^n$, and suppose that $f:S\rightarrow\Re$ has continuous partial derivatives. Then:*
a) f is concave on $S \leftrightarrow \forall(x^0,x)\in S\times S, f(x)-f(x^0)\leq\sum_i^n \frac{\partial f(x^0)}{\partial x_i}(x_i-x_i^0)$.
b) f is strictly concave $\leftrightarrow$ the above inequality is strict for all $x\neq x^0$.
c) The corresponding result for convex functions is attained reversing the inequality signs.

Define a stationary point x_0 as any point such that $Df(x^0)=0$.

Theorem 3 *Suppose that f has continuous partial derivatives in a convex set S in $\Re^n$, and let x_0 be an interior point of S.*
a) If f is concave, then x^0 is a global maximum of f in $S \leftrightarrow x^0$ is a stationary point of f.
b) If f is convex, then x^0 is a global minimum of f in $S \leftrightarrow x^0$ is a stationary point of f.

Thus, the theorem says that in the case of concave (convex) functions first order conditions are necessary and sufficient for an extremum.

1.2 Optimization Subject to Constraints

Consider the following problem:

$$\max f_0(x) \tag{P}$$

subject to

$$x\in S; f_1(x)\geq 0, f_2(x)\geq 0,..., f_m(x)\geq 0.$$

Assume that $f_i, i=0,1,...,m$ are differentiable, concave and real-valued functions defined on a convex domain $S\subset\Re^n$.

Condition 4 (Slater) *$\exists\hat{x}\in$ interior of S such that $\forall i$ $f_i(\hat{x})>0$ for $i=1,2,..,m$.*

Associated with (P) we can define a function L (sometimes called the Lagrangean), $L:S\times\Re^n_+\rightarrow\Re$ by,

$$L(x,\lambda)=f_0(x)+\sum_{i=1}^n\lambda_i f_i(x).$$

Note that for a given λ, $L(x,.)$ is concave, while for a given x, $L(.,\lambda)$ is convex. A function with these properties it is sometimes called a *saddle function*.

Definition 5 (Saddle Point) *A point* (x^*, λ^*) *is a saddle-point of* $L(x, \lambda)$ *if* $L(x, \lambda^*) \leq L(x^*, \lambda^*) \leq L(x^*, \lambda)$, *for all* $(x, \lambda) \in S \times \Re^n_+$.

This condition just says that for $\lambda = \lambda^*$, x^* maximizes $L(x, \lambda)$ over x, and for $x = x^*$, λ^* minimizes $L(x, \lambda)$ over λ. How can we characterize a saddle-point? It turns out that if x^* is in the interior of S, (x^*, λ^*) is a saddle-point iff

$(C.1)$ $Df_0(x^*) + \sum_{i=1}^m \lambda_i Df_i(x^*) = 0$.

$(C.2)$ $f_i(x^*) \geq 0, \lambda_i^* \geq 0, i = 1, 2, ..., m$.

$(C.3)$ $\sum_{i=1}^m \lambda_i^* f_i(x^*) = 0$.

Note that $(C.1)$ is equivalent to the statement x^* maximizes $L(x, \lambda^*)$. Condition $(C.2)$ guarantees that the constraints are satisfied, while $(C.3)$ implies that λ^* minimizes $L(x^*, .)$. It follows that –given $(C.2)$– condition $(C.3)$ is equivalent to,

$(C.3')$ $\lambda_i^* f_i(x^*) = 0, i = 1, 2, ..., m$.

Why do we care about saddle-points? It turns out that they are useful in characterizing (and sometimes helping to find) extrema. The following is a well-known theorem that provides an essential result in constrained optimization.

Theorem 6 (Kuhn-Tucker) *Assume* $f_0, f_1, ..., f_m$ *are concave, continuous functions from* S *(convex)* $\subset \Re^n$ *into* $\Re$. *Let the problem (P) and the function* $L(x, \lambda)$ *be as described above. Then*

i) If $(x^*, \lambda^*) \in S \times \Re^m$ *is a saddle-point of* $L(x, \lambda)$, *then* x^* *solves (P).*

ii) Assume that the Slater condition holds. Then if $x^* \in S$ *is a solution to (P), there exists a* $\lambda^* \in \Re^m$, *such that* (x^*, λ^*) *is a saddle-point of* $L(x, \lambda)$.

The vector λ is sometimes called the vector of *Lagrange multipliers* and, in economic applications, it is often referred to as *shadow prices*.

In some cases, it is useful to have a partial characterization of solutions even when the functions f_i are not concave. There is a version of the Kuhn-Tucker theorem that applies to this case. However, it assumes differentiability. To state the theorem, we need to define the notion of an *effective* constraint.

Definition 7 (Effective Constraint) *An inequality constraint* $f_i(x) \geq 0$ *is said to be effective at a certain point* x^* *if we have* $f_i(x^*) = 0$, *i.e. if the constraint holds with equality at* x^*

Theorem 8 (Kuhn-Tucker II) *Let* $f_0 : S \to \Re$ *be a* C^1 *function on a certain open set* $S \subseteq \Re^n$, *and let* $f_i : \Re^n \to \Re$, $i = 1, ..., m$, *be* C^1 *functions. Suppose that* x^* *is a local maximum of* f_0 *on the set*

$$D = S \cap \{x \in \Re^n \mid f_i(x) \geq 0, \ i = 1, ..., m\}.$$

Let $E \subseteq \{1, ..., m\}$ denote the set of effective constraints at x^*. Suppose that the derivatives $\{Df_i(x^*) \mid i \in E\}$ form an independent set of vectors. Then there exist $\lambda_i^* \in \Re,\ i = 1, 2, ..., m,$ such that

$$\begin{aligned} \lambda_i^* &\geq 0,\ i = 1, 2, ..., m, \\ \lambda_i^* f_i(x^*) &= 0,\ i = 1, 2, ..., m, \\ Df_0(x^*) + \sum_{i=1}^{m} \lambda f_i(x^*) &= 0 \end{aligned}$$

Note: This theorem only gives necessary conditions for a local maximum.

1.3 A Two Period Model

This section develops a very simple dynamic model that will be used to illustrate the power of the basic results in the previous section. In addition, the simple setting will demonstrate the connection between equilibrium problems –which are typically fairly complicated– and programming problems (like (P)) which can be "solved" using the results of section 1.2.

We consider an economy that is populated by a large number of identical households. Each household derives utility from a single consumption good in each period. Preferences are represented by a utility function given by,

$$U(c_1, c_2) = u(c_1) + \beta u(c_2)$$

where u is assumed increasing and concave. Although not necessary at this point, we will assume that the discount factor, β, is between zero and one. Intuitively, this captures the idea that one unit of instantaneous utility tomorrow is less valuable than a unit of utility today. Sometimes this property is described as "positive time preference." It turns out that it is both reasonable and necessary for technical reasons when we deal with more general models. Thus, we will always use discounted preferences. (However, note that the analysis in this section goes through independently of the value of β.) Each household is endowed with $e > 0$ units of consumption in their first period of life.

The technology in this economy is represented by a production function $f : \Re_+ \to \Re$. If k units of consumption are used in the first period in this technology, the economy produces $f(k)$ units of consumption in the second period. Assume that f is continuous and concave.

First, we will consider the problem faced by a planner who chooses investment to maximize the utility of the representative agent. The planner's problem is,

$$\max u(c_1) + \beta u(c_2) \tag{P.1}$$

subject to,

$$\begin{aligned} e - c_1 - k &\geq 0 \\ f(k) - c_2 &\geq 0 \\ (c_1, c_2, k) &\geq 0. \end{aligned}$$

It turns out that there is more than one way of mapping this problem into the "general" form of problem (P). Here is one: let $x = (x_1, x_2, x_3) = (c_1, c_2, k)$. Let $S \equiv \Re^3_+ \subset \Re^3$. Define,

$$\begin{aligned} f_0(x) &= u(x_1) + \beta u(x_2) \\ f_1(x) &= e - x_1 - x_3 \\ f_2(x) &= f(x_3) - x_2. \end{aligned}$$

In addition, there are some non-negativity constraints (on consumption and investment) that could be handled in the form of trivial f_i functions. However, it is more common in economics to handle non-negativity constraints in an implicit way. In this application –but not later– we will ignore this issue and assume that all equilibrium and optimal quantities satisfy the appropriate non-negativity constraints.

We are now ready to define the Lagrangean for this problem. It is given by,

$$L_1(x, \lambda) = u(x_1) + \beta u(x_2) + \lambda_1(e - x_1 - x_3) + \lambda_2(f(x_3) - x_2)$$

Exercise 9 *Verify that the planner's problem (P.1) satisfies the conditions for part i) of the Kuhn-Tucker Tucker Theorem. Can you think of an economic condition that guarantees that the Slater condition is satisfied?*

As interesting as "planner" problems are –especially from a normative point of view– we are also interested in how other institutions perform in terms of resource allocation. In this course we will spend most of our time studying markets and, in particular, competitive markets. To imagine this economy in a market environment we can consider that the technology f is operated by a firm that buys time one goods –let's call them capital or investment– and uses them to produce time two goods. If goods prices are denoted p_t, the firm's problem is,

$$\max_{k \geq 0} p_2 f(k) - p_1 k. \tag{PF(p)}$$

Let $\Pi(p)$ be the value of profits (i.e. the maximized value of $p_2 f(k) - p_1 k$) when the price vector is given by $p = (p_1, p_2)$. With some abuse of notation we will use Π (without the argument) to denote the equilibrium value of profits, that is, is just $\Pi(p)$ evaluated at the equilibrium p.

The representative consumer solves,

$$\max u(c_1) + \beta u(c_2) \tag{PC(Π, p)}$$

subject to,

$$\begin{aligned} \Pi + p_1(e - c_1) - p_2 c_2 &\geq 0 \\ (c_1, c_2) &\geq 0. \end{aligned}$$

We now define a competitive equilibrium for this economy.[1]

[1] Here, to simplify notation, we are assuming that each consumers 'owns' one firm. In general, the budget constraint of each household should depedn on the share of total profits in the economy owned by that household.

Definition 10 (Competitive Equilibrium) *A competitive equilibrium (CE) is a pair* $(\hat{x}, \hat{p})$ *that satisfies,*

i) $\hat{x}_3$ *solves (PF(p)),* *[profit maximization].*

ii) $(\hat{x}_1, \hat{x}_2)$ *solve* $(PC(\Pi, p))$ *[utility maximization].*

iii) $e \geq \hat{x}_1 + \hat{x}_3, f(\hat{x}_3) \geq \hat{x}_2$ *[market clearing].*

Note that condition *iii*) is a little bit "too general" in our case, since we already imposed that $f(\hat{x}_3) = \hat{x}_2$ when we formulated the firm's problem. In general, we should specify the firm's problem allowing for free disposal and, in this case, it would possible for some output to be thrown away in equilibrium.

Why do we study the planner's problem if we are interested in competitive equilibria? It turns out that the solution to (P.1) –the planner's problem– *is* the competitive equilibrium allocation. Thus, solving (relatively easy) programming problems is one way of characterizing (relatively hard) equilibrium problems. More formally we have,

Theorem 11 *Assume that* u *and* f *are continuous, strictly increasing and concave functions. Suppose that* $e > 0$ *and that* $f(0) = 0$.

a) If $(\hat{x}, \hat{p})$ *is a competitive equilibrium, then* $(\hat{x}, \gamma\hat{p})$ *is a saddle point of the Lagrangean* L_1 *for some nonnegative scale parameter* γ.

b) Conversely, if $(\hat{x}, \hat{p})$ *is a saddle point of* L_1*, then* $(\hat{x}, \hat{p})$ *is a competitive equilibrium.*

Proof. First we prove part *a*) [CE → Planner]. Given that $(\hat{x}, \hat{p})$ is a CE, then $(\hat{x}_1, \hat{x}_2)$ trivially is a saddle point of the following Lagrangean,

$$L_C(x, \gamma) = u(x_1) + \beta u(x_2) + \gamma[\hat{\Pi} + p_1(e - x_1) - p_2 x_2]$$

as it can be verified that the Slater condition holds since prices are non-zero and $\hat{\Pi} > 0$. (One can formally show that equilibrium prices are non-zero –in fact strictly positive– but the argument is already long even assuming that they are non-zero.) Similarly, $\hat{x}_3$ is a saddle point of L_F, where

$$L_F(x, \gamma) = \gamma[\hat{p}_2 f(x_3) - \hat{p}_1 x_3]$$

for any γ. Next recall that,

$$L_1(x, \lambda) = u(x_1) + \beta u(x_2) + \lambda_1(e - x_1 - x_3) + \lambda_2(f(x_3) - x_2)$$

and that,

$$\begin{aligned} L_C(x, \gamma) + L_F(x, \lambda) &= u(x_1) + \beta u(x_2) + \gamma[\hat{\Pi} + \hat{p}_1(e - x_1) - \hat{p}_2 x_2] \\ &\quad + \gamma[\hat{p}_2 f(x_3) - \hat{p}_1 x_3] \\ &= u(x_1) + \beta u(x_2) + \gamma\hat{\Pi} + \gamma\hat{p}_1[e - x_1 - x_3] \\ &\quad + \gamma\hat{p}_2[f(x_3) - x_2] \\ &= L_1(x, \gamma\hat{p}) + \gamma\hat{\Pi}. \end{aligned}$$

It follows that if $\hat{x}$ maximizes $L_C(x,\gamma)$ and $L_F(x,\lambda)$ given that $(\hat{p},\gamma)$ are taken as given, it also maximizes $L_1(x,\gamma\hat{p})$ given $\gamma\hat{p}$. Thus, we have proved that if $(\hat{x},\hat{p})$ is a CE, then $\hat{x}$ has "a chance" of being a saddle point of L_1. To complete the argument we need to show that at $x=\hat{x}$, $\gamma\hat{p}$ minimizes $L_1(\hat{x},\lambda)$ over λ. In section 1.2 we showed that this is equivalent to,

$$\lambda_1(e-\hat{x}_1-\hat{x}_3)+\lambda_2(f(\hat{x}_3)-\hat{x}_2)=0. \tag{*}$$

However, given that γ is the Lagrange multiplier of the consumer problem it is also a saddle point of the consumer's Lagrangean. Thus, it satisfies,

$$\gamma[\hat{\Pi}+\hat{p}_1(e-\hat{x}_1)-\hat{p}_2\hat{x}_2]=0.$$

But, $\hat{\Pi}=\hat{p}_2 f(\hat{x}_3)-\hat{p}_1\hat{x}_3$. Thus, substituting the equilibrium value of profits, the consumer's complementary slackness condition is,

$$\gamma[\hat{p}_2 f(\hat{x}_3)-\hat{p}_1\hat{x}_3+\hat{p}_1(e-\hat{x}_1)-\hat{p}_2\hat{x}_2]=0$$

or

$$\gamma\hat{p}_1[e-\hat{x}_1-\hat{x}_3]+\gamma\hat{p}_2[f(\hat{x}_3)-\hat{x}_2]=0.$$

Thus, by choosing $\lambda_1=\gamma\hat{p}_1$ and $\lambda_2=\gamma\hat{p}_2$ we find a vector of Lagrange multipliers that minimizes $L_1(\hat{x},\lambda)$ since it satisfies (*). To sum up, we showed that if $(\hat{x},\hat{p})$ is a CE, then $(\hat{x},\gamma\hat{p})$ is a saddle point of $L_1(x,\lambda)$. This, in turn, implies that $\hat{x}$ solves (P.1).

b) [Planner's $\rightarrow$ CE] Let $(\hat{x},\hat{p})$ be a saddle point of $L_1(x,\lambda)$. Since the constraints must be satisfied, we have that,

$$\begin{aligned} e-\hat{x}_1-\hat{x}_3 &\geq 0 \\ f(\hat{x}_3)-\hat{x}_2 &\geq 0. \end{aligned}$$

This implies that condition *iii*) of the definition of a competitive equilibrium [*market clearing*] is automatically satisfied. Thus, we only need to show that conditions *i*) and *ii*) of the definition of competitive equilibrium hold. By the Kuhn-Tucker theorem it suffices to show that $\exists\gamma$ such that $(\hat{x},\gamma)$ is a saddle point of $L_C(x,\gamma)$ and that $\hat{x}$ maximizes $L_F(x,\lambda)$.

To show this, first rewrite $L_1(x,\hat{p})$ as,

$$\begin{aligned} L_1(x,\hat{p}) &= u(x_1)+\beta u(x_2)+\hat{p}_1(e-x_1-x_3)+\hat{p}_2(f(x_3)-x_2) \\ &= u(x_1)+\beta u(x_2)+[\hat{p}_1(e-x_1)-\hat{p}_2x_2]+[\hat{p}_2f(x_3)-\hat{p}_1x_3]. \end{aligned}$$

Adding and subtracting an arbitrary Π, we obtain,

$$\begin{aligned} L_1(x,\hat{p}) &= u(x_1)+\beta u(x_2)+\Pi+[\hat{p}_1(e-x_1)-\hat{p}_2x_2]+[\hat{p}_2f(x_3)-\hat{p}_1x_3-\Pi] \\ L_1(x,\hat{p}) &= G_1(x_1,x_2,\hat{p},\Pi)+G_2(x_3,\hat{p},\Pi), \end{aligned}$$

where

$$\begin{aligned} G_1(x_1,x_2,\hat{p},\Pi) &= u(x_1)+\beta u(x_2)+\Pi+[\hat{p}_1(e-x_1)-\hat{p}_2x_2] \\ G_2(x_3,\hat{p},\Pi) &= \hat{p}_2f(x_3)-\hat{p}_1x_3-\Pi. \end{aligned}$$

Note that G_1 does not depend on x_3 and G_2 does not depend on (x_1, x_2). Thus, if $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ maximizes $G_1(x_1, x_2, \hat{p}, \Pi) + G_2(x_3, \hat{p}, \Pi)$, then it must be the case that $(\hat{x}_1, \hat{x}_2)$ maximizes G_1 and $\hat{x}_3$ maximizes G_2. We write this last implication as,

$$\hat{x}_3 \qquad \textit{maximizes} \qquad G_2(x_3, \hat{p}, \Pi).$$

This, however, is condition $i)$ of the definition of equilibrium. Thus, all that remains to do is to show that condition $ii)$ is satisfied. Recall that $ii)$ says that $(\hat{x}_1, \hat{x}_2)$ must solve $PC(\hat{\Pi}, \hat{p})$. A sufficient condition for this (again see the Kuhn-Tucker theorem) is that $(\hat{x}_1, \hat{x}_2, 1)$ be a saddle point of the associated Lagrangean $L_C(x, \lambda)$. It is easy to check that $L_C(x, 1) = G_1(x_1, x_2, \hat{p}, \hat{\Pi})$. Thus, the previous argument shows that $(\hat{x}_1, \hat{x}_2)$ maximizes $L_C(\cdot, 1)$. Next we need to show that 1 minimizes $L_C(x, \cdot)$. This is equivalent to,

$$\hat{\Pi} + [\hat{p}_1(e - x_1) - \hat{p}_2 x_2] = 0$$

or

$$\begin{aligned} \hat{p}_2 f(\hat{x}_3) - \hat{p}_1\hat{x}_3 + \hat{p}_1(e - \hat{x}_1) - \hat{p}_2\hat{x}_2 &= 0, \\ \hat{p}_1[e - \hat{x}_1 - \hat{x}_3] + \hat{p}_2[f(\hat{x}_3) - \hat{x}_2] &= 0. \end{aligned}$$

But since $(\hat{x}, \hat{p})$ is a saddle point of $L_1(x, \lambda)$ this last expression is zero. This completes the proof. ■

[Geometric interpretation missing]

1.4 An Extension of the Kuhn-Tucker Theorem

The previous section illustrated how optimization can be a very valuable tool in economic analysis. In this section we describe a multiperiod optimization problem (with a finite horizon) and derive the Kuhn-Tucker conditions. We then give a *heuristic* argument to derive the solution to an infinite horizon optimization problem.

Consider the following problem,

$$\max \sum_{t=0}^{T} \beta^t v(s_t) \tag{DP$_T$}$$

subject to

$$\begin{aligned} h(m_t, s_t) &\geq m_{t+1} && t = 0, 1, ..., T, \\ m_t &\geq 0 && t = 1, 2, ..., T+1, \\ s_t &\geq 0 && t = 0, 1, ..., T, \\ m_0 &> 0 && \textit{given}. \end{aligned}$$

In terms of the Kuhn-Tucker Theorem, let x be a vector in $\Re^{2T+2}$ which includes the $T+1$ values of s_t, and the $T+1$ values of m_{t+1}, $t = 0, 1, ..., T$. (Recall that m_0 is fixed.) Let $\mathbf{s}(\mathbf{m})$ stand for the $T+1$ vectors $\{s_t\}_{t=0}^T$ $(\{m_{t+1}\}_{t=0}^T)$. Assume that the functions v and h are concave and continuously differentiable.

Exercise 12 *Verify that for all T finite, (DP_T) satisfies the conditions for part i) of the Kuhn-Tucker Theorem. Make assumptions that will guarantee that the Slater condition is satisfied.*

Let the Lagrangean for (DP_T) be given by,

$$L_T(s, m, \lambda) = \sum_{t=0}^{T} \beta^t \{v(s_t) + \lambda_{1t}[h(m_t, s_t) - m_{t+1}] + \lambda_{2t} m_{t+1} + \lambda_{3t} s_t\}.$$

Note that in order to perfectly match this Lagrangean with our previous formulation the Lagrange multipliers are of the form $\beta^t \lambda_{it}$. Of course, with a finite "horizon" or dimension it does not matter if we multiply the "true" Lagrange multiplier by β^{-t}. It turns out that when we move to an infinite horizon the discounted formulation that we are using in this section is easier to deal with. However, the reader can check that *nothing* depends on whether the Lagrange multipliers are discounted or not.

To find a saddle point we need to maximize $L_T(s, m, \lambda)$ with respect to (s, m) and minimize it with respect to λ. The first order conditions are,

$$\begin{array}{rcll}
s_t & : & v'(s_t) + \lambda_{1t} h_s(m_t, s_t) + \lambda_{3t} = 0, & t = 0, 1, ..., T, \\
m_{t+1} & : & -\lambda_{1t} + \beta \lambda_{1t+1} h_m(m_{t+1}, s_{t+1}) + \lambda_{2t} = 0, & t = 0, 1, ..., T-1, \\
m_{T+1} & : & -\lambda_{1T} + \lambda_{2T} = 0, & \\
\lambda_{1t} & : & \lambda_{1t}[h(m_t, s_t) - m_{t+1}] = 0, & t = 0, 1, ..., T, \\
\lambda_{2t} & : & \lambda_{2t} m_{t+1} = 0, & t = 0, 1, ..., T, \\
\lambda_{3t} & : & \lambda_{3t} s_t = 0, & t = 0, 1, ..., T.
\end{array}$$

Note that the only "different" first order condition corresponds to m_{T+1}. In this example m_t is a *state variable* (and s_t is a *control variable*), and it is typical of these problems that the choice of the last state is different. The reason for this is simple: under standard economic conditions we can get s_t and m_t $t = 0, 1, ..., T$ to be interior and, hence, the non-negativity constraints do not bind. However, it is often the case that the non-negativity constraint is binding for m_{T+1}. Thus, the constraint $\lambda_{2T} m_{T+1} = 0$ typically bites, and there is no way of getting rid of it by making "reasonable" assumptions. One needs to verify that it is satisfied.

If we could find a solution to the previous system of first order conditions we would have found a solution to our problem. In these notes there will be plenty of opportunity to study solutions to these problems that are of economic interest. So, for now, we ignore the (interesting) issues of interpretation and we move on with the analysis of the formal arguments.

From an economic point of view it is often the case that a "natural" assumption is that horizons are infinite. Thus, one may be interested in the solution to problems like (DPT) when $T = \infty$. It turns out that dealing with infinite dimensional spaces (something like $\Re^\infty$ with some more structure or topology) requires substantially different arguments to prove something like the Kuhn-Tucker theorem. Fortunately, the outcome is very similar. Thus, in these notes we simply present a set of first order conditions that "works" without attempting to explain why.

What would be the problem if we just took $T = \infty$ in the previous system of first order conditions? It turns out that the "difficult" condition is $\lambda_{2T} m_{T+1} = 0$. Thus, in this course we are going to do the following in the case of a reasonable interior solution (the extension to corner solutions is immediate):

a) Impose the first order conditions on s_t and m_{t+1}. In terms of the previous example this corresponds to,

$$\begin{aligned} s_t &: \quad v'(s_t) + \lambda_{1t} h_s(m_t, s_t) = 0, && t = 0, 1, ..., \infty, \\ m_{t+1} &: \quad -\lambda_{1t} + \beta \lambda_{1t+1} h_m(m_{t+1}, s_{t+1}) = 0, && t = 0, 1, ..., \infty. \end{aligned}$$

[Note that we have set the $\lambda_{2t} = \lambda_{3t} = 0$ as s_t and m_{t+1} are strictly positive or interior.]

b) Replace the Lagrange multiplier in the condition $\lambda_{2T} m_{T+1} = 0$ with a "different" Lagrange multiplier with a more "fundamental" interpretation. In this setting, "fundamental" means **not associated to nonnegativity constraint**s. Thus in our example, since $-\lambda_{1T} + \lambda_{2T} = 0$, it follows that $\lambda_{2T} m_{T+1} = 0$ can be written as

$$\lambda_{1T} m_{T+1} = 0.$$

c) Bring back discounting into the previous equation and take the limit when T goes to ∞. Thus, the previous equation is replaced by,

$$\lim_{T \to \infty} \beta^T \lambda_{1T} m_{T+1} = 0,$$

or,

$$\lim_{T \to \infty} \beta^T [-\frac{v'(s_T)}{h_s(m_T, s_T)}] m_{T+1} = 0.$$

Thus, for the problem (DP_∞) the first order conditions for a (interior) solution are

$$\begin{aligned} s_t &: \quad v'(s_t) + \lambda_{1t} h_s(m_t, s_t) = 0, && t = 0, 1, ..., \infty, \\ m_{t+1} &: \quad -\lambda_{1t} + \beta \lambda_{1t+1} h_m(m_{t+1}, s_{t+1}) = 0, && t = 0, 1, ..., \infty, \\ \lambda_{1t} &: \quad h(m_t, s_t) = m_{t+1}, && t = 0, 1, ..., \infty, \\ TVC &: \quad \lim_{T \to \infty} \beta^T [-\frac{v'(s_T)}{h_s(m_T, s_T)}] m_{T+1} = 0. \end{aligned}$$

Equations that contain terms in t and $t+1$, like

$$-\lambda_{1t} + \beta \lambda_{1t+1} h_m(m_{t+1}, s_{t+1}) = 0$$

or, equivalently,

$$\frac{v'(s_t)}{h_s(m_t, s_t)} = \beta \frac{v'(s_{t+1})}{h_s(m_{t+1}, s_{t+1})},$$

are sometimes called *Euler Equations (EE)*, while the condition $\lim_{T\to\infty}\beta^T\lambda_{1T}m_{T+1} = 0$ is often referred to as the *Transversality Condition (TVC)*.

While the Euler equation does not present any problems of interpretation in most economic applications, the TVC is often a source of confusion. Unfortunately, since it plays no role when T is finite, intuition about its role relies on intuition about infinite dimensional spaces, which can only be "acquired" in upper level math courses. Thus, in these notes we will use this cookbook approach with the understanding that in order to do research in dynamic economics it is necessary to acquire the formal tools.

Exercise 13 *Consider a consumer who faces the following intertemporal maximization problem,*

$$\max c_1 + \beta c_2$$

subject to,

$$\begin{aligned} c_1 + s &\leq e \\ c_2 &\leq Rs, \end{aligned}$$

where the maximization is over $(c_1, c_2) \geq 0$ *and* s*. Assume that* $e \geq 0$*. The economic interpretation is simple:* c_1 *and* c_2 *correspond to consumption when young and old,* s *denotes saving (or borrowing if negative), and* R *is the gross interest rate.*

i) Describe the solution to this problem as carefully as you can. In doing so be specific about whether the Kuhn Tucker theorem applies. To characterize your solution note that you have to consider several cases. In particular, be specific about the solution when $e = 0$*, and* $e > 0$*. In addition, consider how the solution changes depending on whether* $\beta^{-1} = R$*,* $\beta^{-1} > R$*, or* $\beta^{-1} < R$*.*

ii) Describe a T *period version of this problem. Consider the case* $e > 0$ *(there is an endowment only in the first period). Could you determine the optimal solution for the cases* $\beta^{-1} > R$ *and* $\beta^{-1} < R$ *without going through the "full" derivation? Explain your argument.*

Exercise 14 *Consider the following problem,*

$$\max x$$

subject to

$$f_1(x) = -x^2 \geq 0.$$

Check if the conditions required to apply the Kuhn Tucker theorem are satisfied, and display the solution to this problem. If some conditions are not satisfied please indicate the nature of the problem.

Exercise 15 (Optimal and Equilibrium Allocations) *Consider a simple two period economy in which the representative agent is endowed with* e *units of output in the first period and nothing in the second. This agent has preferences defined over first and second period consumption given by,*

$$U = u(c_1) + \beta u(c_2),$$

where the function u is assumed twice differentiable and concave, and $0 < \beta < 1$. *The available technology satisfies,*

$$\begin{aligned} c_1 + k &\leq e \\ c_2 &\leq f(k) \end{aligned}$$

with $f'(k) > 0$, *and* $f''(k) > 0$. *That is, the technology displays increasing returns to scale.*

i) Go as far as you can describing the solution to the planner's problem. Please be explicit if different "types" of solution exist.

ii) Is it possible to decentralize the solution you found in i) as a competitive equilibrium? If not, explain what goes wrong.

iii) Discuss the following assertion in the context of the model of this exercise: It is impossible for the equilibrium allocation –even if one exists– to coincide with the planner's allocation.

Exercise 16 (Choice of Technology and Productivity) *One view of the development process is that production begins with technologies that require high labor input per unit of output, but little capital per unit of output; in other words, "traditional" technologies have low capital-labor ratios, while "industrial" technologies have high capital-labor ratios. In this exercise you study how productivity affects the choice of technology. Consider a two period economy. In period one the planner chooses the technology that will be used in period 2. No consumption or investment take place in period one. The two technologies available to country i are:*

$$y = f^1(n,k) = a_i \min(\lambda_1 n, \gamma_1 k),$$

and

$$y = f^2(n,k) = a_i \min(\lambda_2 n, \gamma_2 k)$$

where n and k are labor and capital inputs and a_i *is a known, country-specific, productivity parameter. Assume that,*

$$\lambda_1 < \lambda_2, \qquad \gamma_1 > \gamma_2, \qquad \text{and } a_i > 1/\gamma_2.$$

These parameter assumptions make technology one the "labor intensive" or "traditional" technology while technology two is the "capital intensive" or "industrial" technology. In addition, they guarantee that all countries are sufficiently productive.

In period two each economy has an endowment of labor equal to n, and a stock of a resource, e, that can be either directly consumed or transformed, one-for-one, into capital. This capital can be used immediately in either the f^1 *or the* f^2 *technologies to produce output.*

Suppose that all choices are made by a planner, and the objective is to maximize period two output. This, of course, is the sum of output produced using either technology (f^1 *or* f^2*), plus the unused part of the stock of the resource.*

i) Assume that,

$$\lambda_2/\gamma_2 < 1 \qquad \text{and} \qquad \lambda_2 n/\gamma_2 < e.$$

Under what parameter restrictions is technology two chosen in period one. Show your reasoning.

ii) How does maximized second period output depend on a_i? Interpret the results.

iii) What does this model say about observed differences across countries in the productivity of resources, e, and in labor productivity? (Here productivity is defined as total output divided the quantity of the resource.)

iv) Suppose now that, somehow, you introduce markets with individuals having linear preferences over second period consumption. Do you expect the competitive equilibrium allocation to coincide with the planner's allocation? Why? Why not?

Exercise 17 (Capacity Choice) *A firm has a production possibilities set given by*

$$F(x,k) = \begin{cases} Ax^{\alpha} & if\ x \leq k \\ Ak^{\alpha} & if\ x \geq k \end{cases} \qquad 0 < \alpha < 1.$$

In this context k is interpreted as the capacity of the firm. Let $v(k;w,A)$ be the value of a firm that has (fixed for now) capacity k. If we assume that the firm maximizes profits and the price of x is w, then

$$v(k;w,A) = \max_{x} Ax^{\alpha} - wx.$$

i) Check if this problem satisfies the conditions of the Kuhn-Tucker Theorem.

ii) Go as far as you can describing the solution to the firm's problem when capacity (k) is fixed.

iii) Go as far as you describing the impact of an increase in w on the value of the firm when capacity (k) is fixed.

iv) Assume now that a firm that has capacity equal to k can choose to expand its plant. The cost per unit of extra capacity is q. Thus, if the firm wants purchases z additional units, the cost is qz. Unfortunately (for this firm) there is a 'minimum size' extra capacity. Formally, the addition to capacity has to satisfy $z \geq \bar{z}$. Go as far as you can describing the optimal decision to expand capacity. What happens to the market value of the firm defined as $V(A,w,q) = v(k+z;w,A) - qz$ relative to the pre-expansion value?

v) Suppose that firms differ in terms of their productivity levels, A. Discuss the following claim: If two firms that have (initially) the same capacity, say k, are observed to invest exactly the same amount in capacity expansion, optimal investment theory implies that their market value —defined, as before, as $V(A,w,q)$— must be the same.

Exercise 18 *Consider the problem*

$$max_{x_1,x_2}\ [ax_1^{-\rho} + (1-a)x_2^{-\rho}]^{-1/\rho}, \qquad \rho > -1,\ 0 \leq a \leq 1,$$

subject to,

$$\begin{aligned} 2x_1 + x_2 &\leq 10, \\ x_1, x_2 &\geq 0. \end{aligned}$$

i) Go as far as you can describing the solution (s) to this problem.

ii) Does the qualitative nature of the solution depends on the value of a and ρ?

Exercise 19 (Redistribution) *Consider an individual i who has preferences over consumption and leisure given by*

$$u^i(c, \ell) = \ln(c) + \alpha^i \ln(\ell).$$

Given a wage rate, w^i, and an income transfer, y^i, the individual solves the following utility maximization problem,

$$v^i(w^i, y^i) \equiv \max u^i(c, \ell)$$

subject to

$$\begin{aligned} c &\leq w^i n + y^i, \\ n + \ell &\leq 1. \end{aligned}$$

Consider next a problem of a planner that wants to maximize a weighted average of the utilities of two individuals. More precisely, the planner can choose income transfers, (y^1, y^2) to maximize

$$\eta v^1(w^1, y^1) + (1 - \eta) v^2(w^2, y^2),$$

subject to

$$y^1 + y^2 \leq y,$$

and η an arbitrary share parameter that lies in the interval [0,1].

i) Go as far as you can characterizing the solution to the individual problem. Does the Kuhn-Tucker theorem apply?

ii) Go as far as you can characterizing the solution to the planner's problem. Does the Kuhn-Tucker theorem apply?

iii) What are the characteristics of the solution to the planner's problem? In particular discuss how transfers vary with η, w^i and α^i. Provide as much intuition as you can for your findings.

Chapter 2

The Basic Growth Model: Steady States

This chapter applies the results on dynamic optimization developed in the previous chapter to the analysis of the standard growth model.

2.1 The Basic Growth Model

Consider a representative agent one sector growth model. Preferences of the representative family are given by,

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

where the discount factor β is strictly between 0 and 1, and the function u is assumed strictly concave, increasing and differentiable.

This economy produces only one good that can be used for consumption or investment. Given an initial amount of capital k_t, output must satisfy,

$$c_t + x_t + g_t \leq f(k_t)$$

where x_t is investment, g_t is government spending, and $f(k_t)$ is an increasing, strictly concave production function, such that $f(0) = 0$. In addition, we assume that $\lim_{k \to 0} f'(k) > \beta^{-1} - (1 - \delta)$, and $\lim_{k \to \infty} f'(k) < \beta^{-1} - (1 - \delta)$. The first condition will guarantee that the solution is interior and that there is a steady state, while the second condition will ensure both that utility maximization is well defined, and that the economy attains a steady state.

Investment today is used to increase the future capital stock. The law of motion for capital is,

$$k_{t+1} \leq (1 - \delta)k_t + x_t.$$

It is assumed that all quantities must be nonnegative.

2.2 The Planner's Problem

Consider the problem of a planner who wants to maximize the utility of the representative agent subject to the feasibility constraints and the exogenously given sequence of government spending $\{g_t\}$. This problem is given by,

$$\max_{\{c_t\},\{x_t\},\{k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{P.1}$$

subject to

$$\begin{aligned} c_t + x_t + g_t &\leq f(k_t) \\ k_{t+1} &\leq (1-\delta)k_t + x_t \\ (c_t, x_t, k_{t+1}) &\geq (0,0,0), \end{aligned}$$

and the initial condition $k_0 > 0$, given. We will use the following definition:

Definition 20 (Allocation) *An allocation is a set of sequences* $[\{c_t\}_{t=0}^{\infty}, \{x_t\}_{t=0}^{\infty}, \{k_{t+1}\}_{t=0}^{\infty}]$.

Definition 21 (Feasible Allocation) *An allocation is feasible if it satisfies*

$$\begin{aligned} c_t + x_t + g_t &\leq f(k_t) \\ k_{t+1} &\leq (1-\delta)k_t + x_t \\ (c_t, x_t, k_{t+1}) &\geq (0,0,0). \end{aligned}$$

The solution to the planner's problem is a feasible allocation that has the property that it maximizes the utility of the representative household.

To describe the solution of the planner's problem we will use the appropriate version of the Kuhn Tucker Theorem, which was developed in 1. To this end, we first form the Lagrangean

$$\begin{aligned} L(\mathbf{x}, \mathbf{c}, \mathbf{k}, \boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\gamma}) &= \sum_{t=0}^{\infty} \beta^t \{u(c_t) + \lambda_t [f(k_t) - c_t - x_t - g_t] \\ &\quad + \theta_t [1-\delta)k_t + x_t - k_{t+1}] + \gamma_{1t} c_t + \gamma_{2t} x_t + \gamma_{3t} k_{t+1}\} \end{aligned}$$

where boldface letters indicate vectors (which in this case are infinite dimensional vectors). Note the "standard" Lagrange multipliers are given by the Lagrange multipliers that we use times the discount factor (e.g. the "standard" Lagrange multiplier associated with the feasibility constraint at time t is given by$\beta^t \lambda^t$). The first order conditions are,

$$\begin{aligned} c_t &: \ u'(c_t) - \lambda_t + \gamma_{1t} = 0 && \text{(2.1a)} \\ x_t &: \ -\lambda_t + \theta_t + \gamma_{2t} = 0 && \text{(2.1b)} \\ k_{t+1} &: \ -\theta_t + \beta\theta_{t+1}(1-\delta) + \beta\lambda_{t+1} f'(k_{t+1}) + \gamma_{3t} = 0 && \text{(2.1c)} \end{aligned}$$

In addition, we need to consider the transversality condition or "interiority condition at infinity." The approximate rule that we use is as follows:

a) Truncate the problem (say at $t = T$);

b) Consider the complementary slackness condition (by this we mean the condition that has the form of a variable multiplied by a Lagrange multiplier) corresponding to the "last" stock (in this case k_{T+1}). In this example (more precisely, for a finite T version of the example) it is given by, $\beta^T \gamma_{3T} k_{T+1}$;

c) Consider what other implications –or forms of writing this constraint– are available (in our case, for the truncated problem, there is one condition given by, $-\theta_T + \gamma_{3T} = 0$);

d) Find an expression for the complementary slackness condition in terms of "other" (i.e. no nonnegativity) Lagrange multipliers, and require that any candidate solution satisfy this condition. In this example a candidate solution must satisfy $\lim_{T\to\infty} \beta^T \theta_T k_{T+1}$.

If the solution is interior, it satisfies $\gamma_{it} = 0$ for all (i, t) (of course, this does not include the TVC), and $\lambda_t = \theta_t$, for all t. We can then rewrite the system of first order conditions as,

$$
\begin{aligned}
c_t &: \quad u'(c_t) = \lambda_t && \text{(2.2a)}\\
k_{t+1} &: \quad \lambda_t = \beta \lambda_{t+1}[(1-\delta) + f'(k_{t+1})] && \text{(2.2b)}\\
TVC &: \quad \lim_{T\to\infty} \beta^T \lambda_T k_{T+1} = 0 && \text{(2.2c)}\\
&: \quad c_t + x_t + g_t \le f(k_t) && \text{(2.2d)}\\
&: \quad k_{t+1} \le (1-\delta)k_t + x_t && \text{(2.2e)}
\end{aligned}
$$

The economic interpretation of these conditions is rather simple: (2.2a) simply defines the Lagrange multiplier as the marginal utility of consumption (a measure of how much another unit of consumption is worth at time t); (2.2b) equates the marginal cost –measured in utility units and given by λ_t– of giving up a unit of consumption today and turning it into capital, with the benefits of such a plan accruing tomorrow. Specifically, the benefits are the extra "total" output tomorrow (this is given by $1 - \delta + f\prime(k_{t+1})$) multiplied by the value of such output tomorrow (the price is just the discounted value of marginal gain in utility) given by $\beta \lambda_{t+1}$. This equation –which summarizes the intertemporal trade offs faced by the planner– is called the *Euler equation* (EE). Finally, the TVC (2.2c) simply says that the limiting value of the capital stock from the perspective of time zero (this is given by the price at time zero in utility terms —$\beta^T \lambda_T$— times the quantity k_{T+1}) converges to 0. Of course, (2.2d) and (2.2e) are just the feasibility constraints.

Exercise 22 *Describe in this economy the "optimal" level of government spending. To be precise, if the planner had been allowed to choose the sequence $\{g_t\}$, what would the chosen sequence look like?*

Exercise 23 *Modify the technology in the economy of problem (P.1) of this section as follows: Assume that government spending is productive and the production technology is given by,*

$$c_t + x_t + g_t \leq f(k_t, g_t)$$

where the function f is assumed to be strictly concave, increasing in each argument, twice differentiable, and such that the partial derivatives with respect to both arguments converge to zero as the quantity of them grows without bound.

Describe a set of equations that characterize an interior solution to the planner's problem when the planner can choose the sequence of government spending.

Exercise 24 *Consider an economy in which there are two capital stocks, physical and human capital, k and h , respectively. The technology is similar to that in problem (P.1) of this section except for the function f which is now dependent on both k and h. Specifically, assume that the feasibility constraints are given by,*

$$\begin{aligned}
c_t + x_{kt} + x_{ht} + g_t &\leq f(k_t, h_t) \\
k_{t+1} &\leq (1-\delta_k)k_t + x_{kt} \\
h_{t+1} &\leq (1-\delta_h)h_t + x_{ht} \\
(c_t, x_{kt}, x_{ht}, k_{t+1}, h_{t+1}) &\geq (0,0,0,0,0)
\end{aligned}$$

with k_0 and h_0 given. The function f is strictly increasing, concave, twice differentiable, and such that the marginal product of both forms of capital converge to zero as the stocks go to infinity.

i) Go as far as you can describing an interior solution for the planner's problem in this economy.

ii) Assume that the production function is of the form $f(k,h) = zk^{\alpha}h^{\epsilon}$, with $0 < \alpha, \epsilon < 1$, and $\alpha + \epsilon < 1$, and that the depreciation factor is the same for both stocks (i.e. $\delta_k = \delta_h$). What does this model imply about the optimal combination of physical and human capital for poor and rich countries? Note*: In this setting, define a country as "rich" if it has more k and more h than another country.*

Exercise 25 *Consider the basic economy described in (P.1) of this section with one modification. Instead of assuming that the labor supply is fixed at one, we include leisure in the utility function as well as in the production function. To simplify, we consider the total supply of time to be one. With this modification, preferences and technology are given by,*

$$\begin{aligned}
U &= \sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_t) \\
c_t + x_{kt} + g_t &\leq zf(k_t, n_t) \\
k_{t+1} &\leq (1-\delta_k)k_t + x_{kt}.
\end{aligned}$$

In this setting, n_t is the number of hours worked by the representative household at time t. The rest of the time (given by $1 - n_t$) is consumed as leisure. The functions u and f are assumed to be strictly increasing in each argument, concave, and twice differentiable. In addition, f is such that the marginal product of capital converges to zero as the stock goes to infinity for any given value of labor, n.

Go as far as you can describing an interior solution for the planner's problem in this economy.

2.3 Steady State Analysis

Consider first the question of what determines the level of the capital stock –or more precisely the stock of capital per worker– in an economy in which the planner chooses not to grow. For this to be the case, let's consider a stationary environment, i.e. an environment in which the level of government consumption is unchanged over time. Formally, we assume $g_t = g$, for all t. We then have

Definition 26 (Steady State) *A steady state is a feasible allocation that is such that $\forall t$, $c_t = c, x_t = x, k_{t+1} = k$.*

Since any optimal allocation (if interior) must satisfy the system (2.2), it follows that a steady state —if one exists— is such that the following hold

$$\begin{aligned} u'(c^*) &= \lambda^*, \\ 1 &= \beta[1 - \delta + f'(k^*)], \\ c^* + x^* + g &= f(k^*), \\ k^* &= (1-\delta)k^* + x^*. \end{aligned}$$

These conditions can be simplified to,

$$1 = \beta[1 - \delta + f'(k^*)] \tag{2.3a}$$

$$f(k^*) = c^* + \delta k^* + g \tag{2.3b}$$

where, in deriving (2.3b), we have used the fact that steady state investment is just enough to cover depreciation of the capital stock.

The first question to ask is: How do we know that a steady state exists? Note that from the system (2.3) it suffices to show that there is a capital stock k^* that satisfies (2.3a) *and* that at that capital stock ($k = k^*$), the level of consumption c^* is nonnegative. Let's concentrate on the first condition. Consider the right hand side of (2.3a). Because f is assumed twice differentiable, f' is a continuous function. Thus, it suffices to show that for small k, $\beta[1 - \delta + f'(k)] > 1$, and that for large k, $\beta[1 - \delta + f'(k)] < 1$. If this is the case, then there must be at least one k such that $\beta[1 - \delta + f'(k)] = 1$, and this k is our candidate k^* (*Question*: Where in this argument was the continuity of f' used?).

How do we know that the right hand side of (2.3a) is "high" for small $k's$ and "low" for large $k's$? It is enough to impose a condition that says that the marginal product of capital is high when the stock of capital per worker is low, and low when the stock of capital per worker is high. In fact, we have already made that assumption in this chapter. Specifically, we assumed that that $\lim_{k\to 0} f'(k) > \beta^{-1} - (1-\delta)$, and $\lim_{k\to\infty} f'(k) < \beta^{-1} - (1-\delta)$ which, in turn, implies the desired bounds. Thus we know that there is at least one k^* satisfying (2.3a). In Figure 1 we display a candidate $\beta[1-\delta+f'(k)]$ function.

Is the steady state unique? This amounts to determining whether more than one value of k can satisfy (2.3a). Recall that the argument we used to establish existence was to argue that the function $\beta[1-\delta+f'(k)]$ is greater than one for small $k's$, and less than one for high $k's$. If this function is strictly decreasing then it can cross one at just one value of k. Thus, a sufficient condition (but not necessary) for uniqueness is that the function $\beta[1-\delta+f'(k)]$ be a strictly decreasing function of k. This is equivalent to $f''(k)$ negative, which we already assumed when we stated that the function f is strictly concave. Thus, we have shown that a solution k^* to (2.3a)exists and is unique. (See Figure 2.1.)

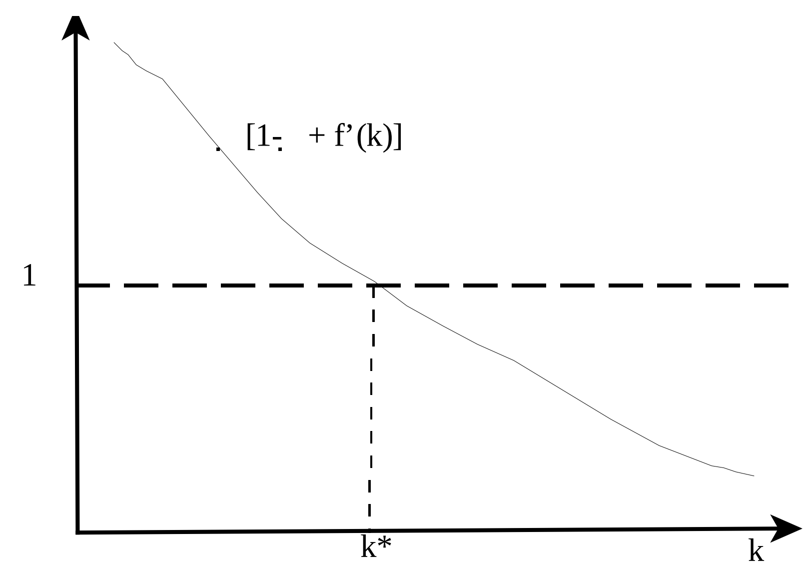


Figure 2.1: Existence and Uniqueness of the Steady State Capital per Worker

Are we done? Not quite. We need to show that (2.3b) implies that c^* is nonnegative. Consider first the case in which $g = 0$. In this case, it suffices to show that $f(k^*) - \delta k^* > 0$, or that $f(k^*) > \delta k^*$. To show that the function $f(k)$ is "above" the function δk, it is sufficient to show that $f'(k)$ is less than δ at the point of intersection. Why? Because at the point at which they intersect, the function $f(k)$ has to come from "above" (recall that it is a strictly concave function), and, hence, its derivative must be less than δ. Consequently, for all points at which $f'(k)$ is greater than δ, $f(k) > \delta k$. Is this satisfied at $k = k^*$? From the condition determining k^*, we have that

$$f'(k^*) = \beta^{-1} - (1 - \delta) > \delta$$

since $\beta^{-1} > 1$.

Therefore, for the case $g = 0$, we have a well defined steady state. What about for other values of g? Since the existence condition is satisfied as a strict inequality at $g = 0$, it will also hold for "small" values of g. Thus, this argument shows that if g is not too large, a steady state exists. On the other hand we can show that if g is sufficiently high no steady state exists. To see this pick g so that $g > f(k^*)$. In this case it is impossible to find a nonnegative c^*. Thus, a steady state will exist only for some (not too large) values of g. From now on (unless we explicitly say so) we will assume that g is "small enough" so that a steady state exists.

Now that we have already established existence and uniqueness, we can concentrate on the properties of the steady state. There are several things to note.

- First, the stock of capital per worker is independent of both the form of the instantaneous utility function (the function u) and the level of government spending. The intuition for this is as follows: the "efficient" amount of capital is such that the discounted value of one additional unit (given by $\beta[1-\delta+f'(k^*)]$) equals the cost, which is one. The reason for ignoring the "curvature" of u is that, in the steady state, consumption is constant and, hence, the marginal utility of consumption today is the same as that of consumption tomorrow. Since it is only the ratio of marginal utilities that matter (in micro jargon, it is the marginal rate of substitution between present and future consumption that matters), curvature plays no role at the steady state.

- Second, permanent changes in government spending have no impact upon aggregate output and capital, and they result in a one for one decrease in consumption. Thus, this model does not deliver the "standard" (undergraduate textbook) result that increases in government spending "crowd out" private investment. As before, the reason is that an increase in g does not affect the marginal rate of substitution between present and future consumption, nor does it affect the marginal rate of transformation between current and future output $(1 - \delta + f'(k^*))$. Consequently, the investment decision is unaffected, and increases in g have "pure income" effects; they reduce consumption.

- Third, more "patient" economies (i.e. those with higher discount factors (higher β)) will have higher steady state capital stocks. This follows from (2.3a) and the observation that f is a strictly concave function and, hence, $f'(k)$ is a decreasing function of k. It can also be shown –using the same argument we used in the case of the discount factor– that increases in the depreciation rate (increases in δ) result in a decrease in the steady state capital stock.

- Finally, to consider the impact of better technologies, let's modify the production function to allow for different technological levels. Specifically, let the production function be given by $zf(k)$, where z is interpreted as the "technology level." (Note that we assumed $z = 1$ in our analysis so far). In this setting it is possible to show that increases in technology (higher z) increase the steady state capital stock.

To sum up the results of this section, the simple model suggest that, if we restrict ourselves to steady states, to account for the differences in output per person or consumption per person across countries, the key factors to look at are differences in discounting (one element of preferences), depreciation factors or technological levels (elements of technologies). Of course, more complicated models (and some of them are presented in the exercises) suggest a different set of factors that can account for the differences in per capita income.

Exercise 27 *Assume that $g_t = g$, and g "small." Show that:*

i) In the "basic" economy (that described in (P.1)), an increase in the depreciation rate decreases steady state consumption.

ii) In the economy with a production function given by $zf(k)$, show that higher levels of technology (higher $z\prime s$) result in higher steady state consumption.

iii) Assume that $zf(k)$ is given by, zk^{α}, and that $\alpha = 0.33$. Describe what range of values of the technology level z would be required to make the model consistent with differences in per capita income across countries that are as high as 30 to 1. (Note: the level of z for the "poorest" country can be chosen to be equal to one.)

Exercise 28 *Consider the economy of Exercise* **??**.

i) Describe the steady state for the "general" specification of this economy. If necessary make assumptions to guarantee that such a steady state exists.

ii) Go as far as you can describing how the steady state levels of capital per worker and government spending per worker change as a function of the discount factor.

iii) Assume that –as in Exercise 27– the technology level can vary. More precisely, assume that the production function is given by $f(k, g, z) = zk^{\alpha}g^{\eta}$, where $0 < \alpha, \eta < 1$ and $\alpha + \eta < 1$. Go as far as you can describing how the investment/GDP ratio and the government spending/GDP ratio vary with the technology level z.

Exercise 29 *Consider the economy described in Exercise* **??***, and assume that $g_t = g$, for all t.*

i) Describe the steady state of this system for the "general" specification of this economy.

ii) Assume that $\delta_h = \delta_k$ and that the technology is given by $f(k, h) = zk^{\alpha}h^{\epsilon}$, with $0 < \alpha, \epsilon < 1$, and $\alpha + \epsilon < 1$. Go as far as you can tracing the impact upon *the steady state mix of physical and human capital (the ratio h/k) of changes in z and g. Does this model suggest that high technology countries have higher human/physical capital ratios? How about high government spending ratios?*

iii) Consider the case described in ii), except that we assume that the changes in technology level affect only human capital. That is, if the production function is written as $F(k,h,z)$, then there is some f satisfying $F(k,h,z) = f(k,zh)$. Could you find "nice" production functions (i.e. concave and increasing) such that: a) Changes in z do not affect the steady state ratio of h to k;b) Changes in z do affect the steady state ratio of h to k.

Exercise 30 *Consider the basic economy described in Exercise* **??** *Thus, preferences and technology are*

$$\begin{aligned} U &= \sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_t) \\ c_t + x_t + g_t &\leq f(k_t, n_t) \\ k_{t+1} &\leq (1-\delta_k)k_t + x_{kt}. \end{aligned}$$

In this setting, n_t is the number of hours worked by the representative household at time t. The rest of the time (given by $1-n_t$) is consumed as leisure. The functions u and f are assumed to be strictly increasing in each argument, concave, and twice differentiable. In addition, f is such that the marginal product of capital converges to zero as the stock goes to infinity for any given value of labor, n. Assume $g_t = g$ small.

i) Describe the steady state of this economy. If necessary make additional assumptions to guarantee that it exists and is unique. If you make additional assumptions, go as far as you can giving an economic interpretation of them.

ii) What is the impact (if any) upon the steady state level of employment of an increase in productivity. Are there cases in which "high technology" countries (high z) have lower level of employment (more leisure)? Are these assumptions reasonable from an economic point of view?

iii) Consider next an increase in g. Are there conditions under which an increase in g will result in an increase in the steady state k/n ratio? How about an increase in the steady state level of output per capita? Go as far as you can giving an economic interpretation of these conditions.

Exercise 31 (Vintage Capital and Cycles) *Consider a standard one sector optimal growth model with only one difference: If k_{t+1} new units of capital are built at time t, these units remain fully productive (i.e. they do not depreciate) until time $t+2$, at which point they disappear. Thus, the technology is given by,*

$$c_t + k_{t+1} \leq z f(k_t + k_{t-1})$$

i) Formulate the optimal growth problem.

ii) Show that, under standard conditions, a steady state exists and is unique.

iii) A researcher claims that with the unusual depreciation pattern, it is possible that the economy displays cycles. By this he means that, instead of a steady *state, the economy will converge to a period two sequence like* $(c_o, c_e, c_o, c_e, c_o, c_e, ...)$ *and* $(k_o, k_e, k_o, k_e, k_o, k_e, ...)$*, where* $c_o(k_o)$ *corresponds to consumption (investment) in odd periods, and* $c_e(k_e)$ *indicates consumption (investment) in even periods. Go as far as you can determining whether this can happen. If it is possible, try to provide an example.*

Exercise 32 (Excess Capacity) *In the standard growth model there is no room for varying the rate of utilization of capital. In this problem you explore how the nature of the solution is changed when variable rates of capital utilization are allowed. As in the standard model there is a representative agent with preferences given by*

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t) \qquad 0 < \beta < 1,$$

where u *is strictly increasing, concave, and twice differentiable. Output depends on the actual number of machines used at time* t, κ_t. *Thus, the aggregate resource constraint is,*

$$c_t + x_t \leq z f(\kappa_t)$$

where the function f *is strictly increasing, concave, and twice differentiable. In addition,* f *is such that the marginal product of capital converges to zero as the stock goes to infinity.*

Capital that is not used does not depreciate. Thus, capital accumulation satisfies the following,

$$k_{t+1} \leq (1-\delta)\kappa_t + (k_t - \kappa_t) + x_t$$

where we require that the number of machines used, κ_t*, is no greater than the number of machines available,* k_t*, or* $k_t \geq \kappa_t$. *This specification captures the idea that if some machines are not used,* $k_t - \kappa_t$ *in our notation, they do not depreciate.*

i) Describe the planner's problem and analyze, as thoroughly as you can, the first order conditions. Discuss your results.

ii) Describe the steady state of this economy. If necessary, make additional assumptions to guarantee that it exists and is unique. If you make additional assumptions, go as far as you can giving an economic interpretation of them.

iii) What is the optimal level of capacity utilization in this economy in the steady state?

iv) Is this model consistent with the view that cross country differences in output per capita are associated with differences in capacity utilization?

Exercise 33 (Government Spending and Labor Supply) *Consider an economy populated by a large number of identical agents. Assume that the instantaneous utility function depends on consumption,* c*, and leisure,* ℓ. *More precisely, preferences are given by,*

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t, \ell_t), \qquad 0 < \beta < 1.$$

Assume that the utility function u is such that consumption and leisure are both normal goods. In addition, assume that u is strictly increasing in each argument, concave, and twice differentiable. We normalize the time endowment to one. Thus, labor supply is n_t, with

$$n_t + \ell_t \leq 1.$$

In this economy there is a government that consumes g_t units of good in period t. Feasible allocations satisfy

$$\begin{aligned} c_t + x_{kt} + g_t &\leq zf(k_t, n_t), \qquad k_0 > 0, \\ k_{t+1} &\leq (1 - \delta_k)k_t + x_{kt}. \end{aligned}$$

The function f is strictly increasing in each argument, concave, and twice differentiable, and such that the marginal product of capital converges to zero as the stock goes to infinity for any given value of labor, n.

1. *Describe the conditions that define an interior solution to the planner's problem.*

2. *Assume that $g_t = g$ (which you may assume is small). Find conditions such that this economy has a steady state.*

3. *Assume that f is homogeneous of degree one. Suppose that different economies differ in terms of the size of the government (g). What does the model say about the long run (i.e. in the steady state) differences across economies in*

   (a) *The capital-labor ratio ($\kappa \equiv k/n$),*

   (b) *Per capita labor supply (n),*

   (c) *Output per worker.*

***Note**: For the purposes of this exercise you may use the following characterization of normality. Let $x(p_1, p_2, y)$ and $q(p_1, p_2, y)$ be the solutions to the following utility maximization problem*

$$\max u(x, q)$$

subject to

$$p_1 x + p_2 q \leq y.$$

Then, if $y' > y$, $x(p_1, p_2, y') > x(p_1, p_2, y)$, and $q(p_1, p_2, y') > q(p_1, p_2, y)$.

Problem 34 (Technological Change and Labor Supply) *Consider an economy in which the representative agent has preferences given by*

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t, \ell_t), \qquad 0 < \beta < 1,$$

where

$$u(c,\ell) = \frac{[\eta c^{-\mu} + (1-\eta)\ell^{-\mu}]^{-(1-\theta)/\mu}}{1-\theta}.$$

This function displays a constant elasticity of substitution between consumption (c) and leisure (ℓ). The parameter μ is restricted to the interval (-1,∞). Let $\sigma \equiv 1/(1+\mu)$ be the elasticity of substitution between consumption and leisure. It follows that σ ranges from 0 (perfect complements) to infinity (perfect substitutes). In general, we say that if $\sigma > 1$, consumption and leisure are substitutes, while if $\sigma < 1$, they are complements. (Pick your favorite name for the case $\sigma = 1$.) The parameter θ is strictly positive, and the share parameter, η, is restricted to lie between 0 and 1. The technology is given by

$$\begin{aligned} c_t + x_{kt} &\leq f(k_t, zn_t), \qquad k_0 > 0, \\ k_{t+1} &\leq (1-\delta_k)k_t + x_{kt}. \end{aligned}$$

In this setting, z is interpreted as labor specific technological change; i.e. high values of z correspond to higher productivity of labor. Assume that the function f is strictly increasing in each argument, homogeneous of degree one, concave, and twice differentiable, and such that the marginal product of capital converges to zero as the stock goes to infinity for any given value of labor, n. We normalize the time endowment to one. Thus, labor supply is n_t, with

$$n_t + \ell_t \leq 1.$$

1. *Go as far as you can providing conditions that guarantee existence of a steady state.*

2. *Assume that countries differ in the value of their productivity (z). If every country is in the steady state what does the model imply for the cross country differences in*

   (a) *Capital-labor ratios ($\kappa \equiv k/n$),*

   (b) *Per capita labor supply (n)*

   (c) *Output per worker.*

***Note**: If your answers depend on the value of σ, describe all the possible outcomes.*

Chapter 3

The Basic Growth Model: Dynamics

In this section we go beyond the steady state and, for the simple one sector growth model, we discuss the solution path for arbitrary initial levels of the capital stock, k_0. We also discuss the dynamic properties of more complicated economies that start with initial levels of capital "near" the steady state, k^*.

3.1 Global Dynamics

In chapter 2 we described the steady state of the system. However, the notion of a "stationary point" or steady state would not be of great interest unless we had some reason to believe that economies will converge to their steady states. In this section we show that this is indeed the case for the simple growth model. Specifically, we show that the dynamical system given by the system of equations (2.2) will converge to the steady state described in chapter 2. Moreover, we will show that this is the *only* possible long run behavior. Since we have already argued that the level of government spending does not affect the steady state of the economy, we will consider the simplest possible case and set for now $g_t = 0$, for all t.

Recall the dynamic behavior of the growth model is summarized by,

$$
\begin{aligned}
c_t &: \quad u'(c_t) = \lambda_t && (3.1a)\\
k_{t+1} &: \quad \lambda_t = \beta\lambda_{t+1}[(1-\delta) + f'(k_{t+1})] && (3.1b)\\
TVC &: \quad \lim_{T\to\infty} \beta^T \lambda_T k_{T+1} && (3.1c)\\
&: \quad c_t + x_t + g_t \le f(k_t) && (3.1d)\\
&: \quad k_{t+1} \le (1-\delta)k_t + x_t. && (3.1e)
\end{aligned}
$$

Let (3.1a) implicitly define the function (here we omit the time subscripts when there is no risk of confusion) $c(\lambda)$ by,

$$u'(c(\lambda)) \equiv \lambda \tag{3.2}$$

Given our assumptions on u this function is well defined if the range of $u'(c)$ is the nonnegative numbers. To guarantee that this is the case assume $\lim_{c \to 0} u'(c) = \infty$, and $\lim_{c \to \infty} u'(c) = 0$. It then follows that the function $c(\lambda)$ is well defined, continuous (and differentiable if u is twice continuously differentiable) and such that $\lim_{\lambda \to 0} c(\lambda) = \infty$, and $\lim_{\lambda \to \infty} c(\lambda) = 0$. In addition, $c(\lambda)$ is a decreasing function of λ.

In an interior solution –which we assume from now on in this section– equations (3.1d) and (3.1e) can be combined to yield,

$$k_{t+1} = (1-\delta)k_t + f(k_t) - c_t.$$

We can then completely summarize the dynamic behavior of the economy by the following pair of nonlinear difference equations,

$$\begin{aligned} k_{t+1} &= (1-\delta)k_t + f(k_t) - c(\lambda_t) \\ \lambda_t &= \beta\lambda_{t+1}[(1-\delta) + f'(k_{t+1})] \end{aligned}$$

The second of these equations is **not** in the form we typically associate with systems of difference equations. The "standard" form is one in which all variables dated $t+1$ appear on one side of the equations, while all variables dated t appear on the other side. While the first of the two preceding equations is in that form, the second is not. However, using the first equation into the second and rearranging we get,

$$k_{t+1} = (1-\delta)k_t + f(k_t) - c(\lambda_t) \tag{3.3a}$$

$$\lambda_{t+1} = \lambda_t\{\beta[(1-\delta) + f'((1-\delta)k_t + f(k_t) - c(\lambda_t))]\}^{-1} \tag{3.3b}$$

This system of two equations gives the values of (k_{t+1}, λ_{t+1}) given values of (k_t, λ_t). Thus, if we know (k_0, λ_0) we could "iterate" and study the behavior of (k_t, λ_t) as t goes to infinity. This strategy –which is what is normally done in the standard mathematical treatment of difference equations– is not available to us. The reason is simple: even though we **do** have an initial condition for k_0 (we assume that $k_0 > 0$ is given), we **do not** have an initial condition for λ_0. In fact, inspection of (3.2) shows that knowledge of λ_0 is equivalent to knowledge of c_0, and c_0 is an endogenous variable. The planner picks c_0 and, hence, it is not a "valid" initial condition. Since it is really necessary to have two conditions to solve the system (3.3) we will have to resort to some other *boundary* condition. It turns out that the role of the additional boundary condition will be provided by the transversality condition.

The first step in the analysis of the dynamics is to have a representation of the steady state in terms of the system (3.3). To this end, we will describe two loci. The first gives the combinations of (k, λ) consistent with $k_{t+1} = k_t = k$ in equation (3.3a). The second will give the pairs (k, λ) that guarantee $\lambda_{t+1} = \lambda_t = \lambda$ in equation (3.3b).

The first locus consists of the pairs (k, λ) that satisfy,

$$k = f(k) + (1 - \delta)k - c(\lambda)$$

or,

$$c(\lambda) = f(k) - \delta k.$$

We can think of this relationship defining a curve in (k, λ) space or, alternatively, a function $\lambda_1(k)$, satisfying $c(\lambda_1(k)) \equiv f(k) - \delta k$. The implicit function theorem (see any good Advanced Calculus book for a formal statement) can be used to show that this function is differentiable, and the derivative is given by,

$$c'(\lambda_1(k)) \frac{\partial \lambda_1(k)}{\partial k} \equiv f'(k) - \delta.$$

Let k_a be such that $f'(k_a) = \delta$ (*Question*: Do we know such a k_a exists? Is it unique?), then, we know that

$$\frac{\partial \lambda_1(k)}{\partial k} \lesseqgtr 0 \leftrightarrow k \lesseqgtr k_a$$

In Figure 3.1 we display a candidate $\lambda_1(k)$ function. Note that it must go to infinity both at $k = 0$ and at $k = k_m$, where $f(k_m) = \delta k_m$.

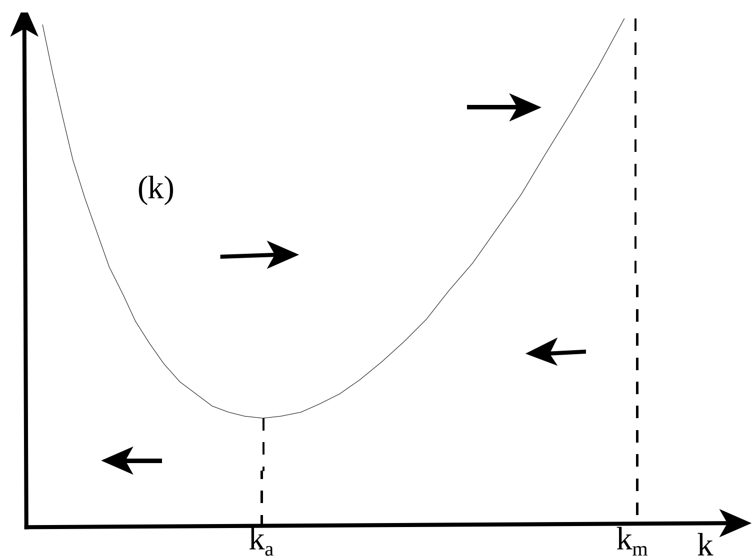


Figure 3.1: The $\lambda_1(k)$ Function

What happens if for a given k, λ is such that $\lambda > \lambda_1(k)$? In this case $c(\lambda)$ is "too low" to maintain $k_{t+1} = k$ and, hence, k_{t+1} is greater than k. Thus, for combinations (k, λ) above $\lambda_1(k)$ we use an arrow pointing to the right to indicate that, in that region, the capital stock is growing ($k_{t+1} > k$). Of course, for pairs (k, λ) below $\lambda_1(k)$, capital moves in the opposite direction.

Consider next the other locus. Imposing that $\lambda_{t+1} = \lambda_t$ in (3.3b), shows that the pairs (k, λ) that satisfy this condition are those that guarantee

$$k_{t+1} = (1-\delta)k_t + f(k_t) - c(\lambda_t) = k^*,$$

where k^* is the steady state we computed in Chapter 2 (recall that it solves $\beta[1-\delta+f'(k^*)] = 1$; see (2.3a)), and λ is any nonnegative number. Thus, we can describe these pairs by a function $\lambda_2(k)$, that is such for any (feasible) k, if $\lambda = \lambda_2(k)$, $\lambda_{t+1} = \lambda$. What does this function look like? For each k, it is given by the value of λ satisfying,

$$c(\lambda) = f(k) - \delta k + k - k^*$$

There are several things to note about this function.

First, if $k = k^*$ then $\lambda_2(k) = \lambda_1(k)$. Thus, at $k = k^*$ both functions $\lambda_1(k)$ and $\lambda_2(k)$ coincide (they intersect each other at that point).

Second, this function is not defined if k is too low. Actually we can set it equal to infinity if k is less than the value k_b, where k_b satisfies,

$$(1-\delta)k_b + f(k_b) = k^*$$

Note that at k_b, we must have $\lambda_2(k_b) = \infty$.

Third, the function $\lambda_2(k)$ is differentiable (wherever it is finite) and the derivative is given by,

$$\frac{\partial \lambda_2(k)}{\partial k} = \frac{f'(k) + 1 - \delta}{c'(\lambda)} < 0$$

We show one such function in Figure 3.2.

What do we know about the behavior of λ for pairs (k, λ) that are either above or below the locus $\lambda_2(k)$ in Figure (3.2)? Suppose that $\lambda < \lambda_2(k)$. This means that $c(\lambda) > c(\lambda_2(k))$, and hence, that $k_{t+1} < k^*$ (Recall that picking $\lambda = \lambda_2(k)$ guarantees that $k_{t+1} = k^*$). This, in turn, means that $\beta[f'(k_{t+1}) + 1 - \delta] > 1$ and, from (3.3b), that $\lambda_{t+1} < \lambda_t$. Hence, for λ below the function $\lambda_2(k)$ the next value of λ is decreasing and to indicate this we draw a downward arrow.

A similar argument can be used to show that if λ is chosen "too high" (i.e. above the locus $\lambda_2(k)$), the effect is that next period's λ will be even higher. Thus, the arrows point up for λ above $\lambda_2(k)$.

There is a third locus that is of interest, and it is given by the requirement that investment be nonnegative. From (3.3a) this requires that the feasible (k, λ) pairs satisfy,

$$f(k) - c(\lambda) \geq 0 \tag{3.4}$$

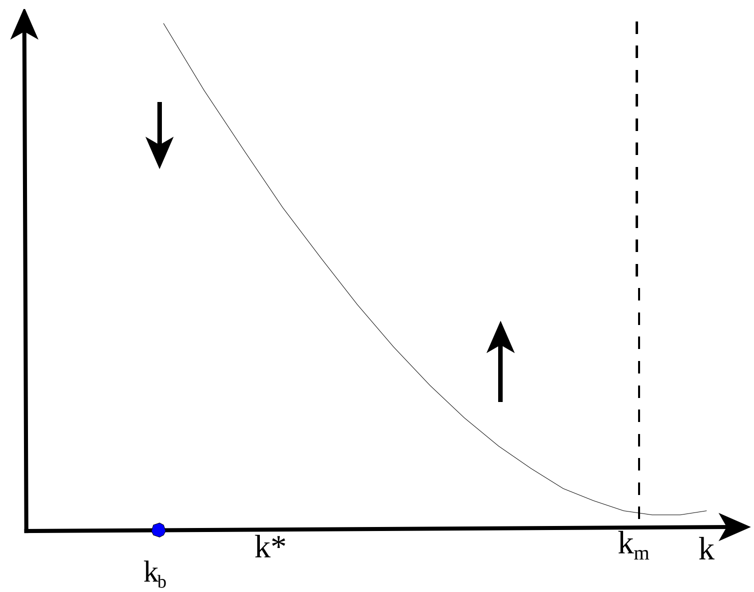


Figure 3.2: The $\lambda_2(k)$ Function

It turns out that this locus does not describe one the difference equations that govern the dynamic behavior of the system (as the other two did) and, hence, there are no "arrows" associated with it. It simply describes a nonnegative condition that any candidate solution must satisfy.

We can now put together all three loci to understand the dynamic behavior of the system. Figure (3.3) displays the relevant curves as well as the arrows that indicate the direction of movement.

Before we start using this representation of the dynamical system it is important that we show that the $\lambda_1(k)$ and $\lambda_2(k)$ loci are as shown. It turns out that the critical property is that the $\lambda_2(k)$ locus is above the $\lambda_1(k)$ locus for values of k lower than k^*, and it is below for values of k greater than k^* (recall that we have already established that they intersect at k^*). Consider any value of k less than k^* (we only care about values of $k > k_b$ for otherwise there is nothing to prove as $\lambda_2(k)$ is infinite). It follows that, by definition of $\lambda_2(k)$) we have,

$$c(\lambda_2(k)) \equiv f(k) - \delta k + k - k^*,$$

or,

$$c(\lambda_2(k)) \equiv c(\lambda_1(k)) + k - k^*$$

Since $k < k^*$, and $c'(\lambda) < 0$ it must be that $\lambda_2(k) > \lambda_1(k)$. A symmetric argument proves that $\lambda_2(k)<\lambda_1(k)$ whenever $k > k^*$.

The two loci and the nonnegativity constraint are displayed in Figure 3.3.

Before we get into a simple (but somewhat tedious) argument, let's remember what we are trying to prove. The conjecture is that all solutions to the planner's problem converge to the steady state. We will do this by arguing that paths that do not converge cannot be solutions.

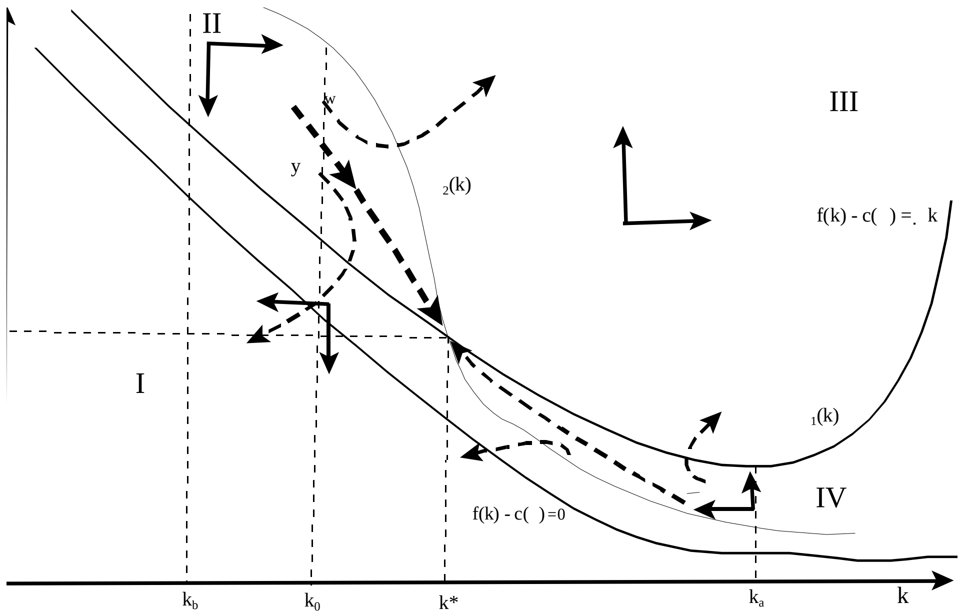


Figure 3.3: Dynamics of the One Sector Growth Model

Let's ignore for now the heavy dashed line with arrows pointing to the pair (k^*, λ^*). In Figure 3.3 the (k, λ) plane has been divided into four regions as determined by the first two loci that we analyzed (although we draw the locus that imposes nonnegative investment, it is not used to determine the four regions), and for each of them we use arrows to indicate the direction in which (k, λ) are moving. Consider first region I. As we argued before, in this region the capital stock is decreasing (it is below the locus that keeps the capital stock constant) and the Lagrange multiplier is decreasing as well (it is below the locus that keeps it constant $-\lambda_2(k)-$ and it results in decrease in λ_t). Thus, the pair of arrows pointing to the *southwest* indicates that if the system is ever in this region (that is, if the current "position" as described by a pair (k, λ) is in region I) the next pair will have both lower k and lower λ (i.e. it will be located *southwest* of the initial vector). The arrows corresponding to the other regions are interpreted similarly.

Consider now what happens when the initial capital stock is k_0 and we allow the planner to choose any λ_0. If the planner chooses a "high" λ it could end up in a trajectory like w in Figure 3.3. This trajectory moves *southeast* (λ decreases and k increases) for some time; at some point it hits the locus that keeps λ constant and —to the right of this point— both λ and k increase. Thus, given k_0, if the planner chooses a trajectory like w it will eventually enter region III. If on the other hand the planner chooses a "low" initial λ, it may end up in a trajectory like y. This trajectory stays in region II for a while (moving *southwest*) but eventually it must enter region I. Following this kind of argument it is possible to show –but it is hard– that regions I and III (this is the difficult one) are *absorbent* in the sense that once a pair (k, λ) is in one of those regions, all future values of (k, λ) induced by the system of difference equations (3.3) will remain in that region. Similarly —except for possibly one path— regions II and IV are *transient* in the sense that even if the initial condition is such that (k, λ) is in one of those regions, the dynamics induced by (3.3) will make the system enter either region I or III .

Finally, since all the functions are continuous functions there are infinitely many paths and, in fact, they fill the whole space. Thus, if for some initial value of λ —like the one inducing the path w— the process moves into region III, while for some other initial value of λ —like that inducing the path denoted y in Figure 3.3— the process moves to region I, there must be some initial condition —a value of λ— that is such that the ensuing path converges to the point (k^*, λ^*). Note that the pair (k^*, λ^*) lies at the intersection of the two loci and hence it has the property that, once the system **is** there, neither k nor λ moves.

An argument similar to the one above shows that –starting from region IV– there must be at least one path that converges to the stationary point (k^*, λ^*).

We now want to argue that *all* solutions to the planner's problem (Problem (P.1)) will converge to (k^*, λ^*). To do this we will show that unless the planner chooses the "right" initial condition (i.e. a value of λ_0 on the heavy dashed line converging to (k^*, λ^*)), the resulting candidate sequence violates some of the first order conditions of the planner's maximization problem (i.e. the conditions that describe a saddle point of the appropriate Lagrangean). We do this in two steps. First, we show that no solution can enter region I; second we also show that no solution can enter region III. Since we have already argued that regions II and IV are transient —except for paths that converge to (k^*, λ^*)— all that is left to prove is that there is only one such path.

Claim 35 *No solution to the planner's problem —infinite sequences satisfying the initial condition k_0 given, (3.3), and the TVC— can possibly enter region I.*

Proof. To see this consider a path that enters region I. Since it moves in the *southwest* direction, it will eventually (in finite time) cross the "nonnegativity locus." This, of course, violates the requirement that $x_t \geq 0$. [*Note*: It is possible to show that even if investment is not required to be nonnegative, no solution will ever enter in region I. The argument is that any path in region I will reach a level of zero capital —it will hit the y-axis— in finite time. At this point, current and all future levels of consumption must be zero (recall that $f(0) = 0$). With enough curvature in the utility function —formally, if $\lim_{c \to 0} u'(c) = \infty$— this cannot be a solution since another path that keeps the stock of capital (and consumption) bounded away from zero dominates.] ■

Claim 36 *No solution to the planner's problem —infinite sequences satisfying* *the initial condition k_0 given, (3.3), and the TVC— can possibly enter region III.*

Proof. Our strategy is to show that if it does, the resulting sequence of $k's$ and $\lambda's$ is such that it violates the transversality condition. Assume that at some point t, the system is in region III. Thus, we start with a pair (k_t, λ_t) in region III. To trace the evolution of the dynamical system it is useful to rewrite (3.3b) as,

$$\beta^{t+1}\lambda_{t+1} = \beta^t \lambda_t [1 - \delta + f'(k_{t+1})]^{-1}.$$

However, in region III the sequence $\{k_t\}$ is increasing. Thus, $\{k_{t+j}\}_{j=0}^{\infty}$ is an increasing sequence and as such, it either converges to a point or goes to infinity. As we argued before, there is a maximum sustainable capital stock, k_m, and it must be the case that $\lim_{j\to\infty} k_{t+j} = k_m$. Thus, for any (small) ϵ, there exists a j^* such that if $j \geq j^*$ then $\mid k_{t+j} - k_m \mid \leq \epsilon$. Since we have shown that $1 - \delta + f'(k_m) < 1$, it is possible to pick ϵ so that $1 - \delta + f'(k_{t+j}) \leq 1 - a < 1$ for all $j \geq j^*$.

It follows that, for $j \geq j^*$,

$$\beta^{t+j+1}\lambda_{t+j+1} \geq \beta^{t+j}\lambda_{t+j}[1 - \delta + f'(k_{t+j+1})]^{-1} = \beta^{t+j}\lambda_{t+j}\frac{1}{1-a}.$$

Thus, the sequence $\{\beta^{t+j}\lambda_{t+j}\}$ is increasing and has a rate of growth bounded away from one. This implies that $\lim_{j\to\infty} \beta^{t+j}\lambda_{t+j} = \infty$. Since $\lim_{j\to\infty} k_{t+j} = k_m$, it follows that the TVC (3.1c) is violated. This would complete the proof if it were true that once the solution enters region III it never leaves it. This is clearly the case if we are on the downward sloping part of $\lambda_1(k)$. However, we still need to show that a solution sequence that starts in region III "next" to the boundary in the upward sloping part $\lambda_1(k)$ will not "escape" into region IV, only to return —after some time— to region III. If this were the case our arguments —that relied on the solution moving *northeast* as soon as it hits region III— would not be valid. We now show that a solution that is in region III will not "escape" to region IV.

The argument is somewhat tedious and here it is a "sketch" of the proof: Suppose that there is an "arrow," a path, going from III to IV. Clearly, since it is moving *northeast* while in III, this can only happen if the capital stock is greater than k_a; that is, in the upward sloping branch of $\lambda_1(k)$. Thus, we assume that $k > k_a$. Our strategy is to show that, if we start right on $\lambda_1(k)$, the system "kicks" us back into III, and **never** into IV. Let (k, λ) be on $\lambda_1(k)$. Then, by construction, k' (next period's capital stock) is $k' = k$. What about λ'? It is given by,

$$\lambda' = \lambda\{\beta[1 - \delta + f'(k)]\}^{-1} > \lambda\{\beta[1 - \delta + f'(k_a)]\}^{-1} > \lambda,$$

where the last inequality follows because $\beta[1-\delta+f'(k_a)] < 1$. Thus, starting on any point (k, λ) on $\lambda_1(k)$, the next point (k', λ') **must** be in region III. Thus, an "arrow" (or a path) does not cross from III to IV. ■

Since candidate solutions cannot enter regions I or III, and since regions II and IV are "transient" except for paths that converge to (k^*, λ^*), we have shown that all solution paths will converge to the steady state given by (k^*, λ^*). We now argue that there is only one such path. The reason is simple: The programming problem (P.1) consists of maximizing a strictly concave function over a convex set and this program has a unique solution. Formally we have

Claim 37 *The solution to (P.1) is unique.*

Proof. To see this suppose to the contrary that there are two solutions $\mathbf{c}^a = \{c_t^a\}$ and $\mathbf{c}^b = \{c_t^b\}$. Since they are both solutions, it must be that $U(\mathbf{c}^a) = U(\mathbf{c}^b) \equiv U^0$. Since they are both feasible and the feasible set is convex then, for any $0 < \varphi < 1$, the sequence $\mathbf{c}^\varphi = \varphi \mathbf{c}^a + (1 - \varphi)\mathbf{c}^b$ is feasible as well. Given strict concavity of the utility function it follows that

$$U(\mathbf{c}^\varphi) > \varphi U(\mathbf{c}^a) + (1 - \varphi)U(\mathbf{c}^b) = U^0,$$

which contradicts the assumption that $\mathbf{c}^a$ and $\mathbf{c}^b$ are maximal. ■

In addition to the uniqueness result, we can show that convergence is "monotone," in the sense that if the system starts in region II on the stable manifold it will not "jump" to some other point on the same stable manifold but in region IV. Formally, we can show

Claim 38 *The sequence $\{k_t\}$ that solves (P.1) is monotone.*

Proof. To see this assume that the pair (k, λ) is in region II. However, since $\lambda_2(k)$ satisfies,

$$k^* = (1 - \delta)k + f(k) - c(\lambda_2(k)),$$

and, by assumption $\lambda < \lambda_2(k)$ (this is an implication of being in region II), it follows that $c(\lambda) > c(\lambda_2(k))$ and,

$$k' = (1 - \delta)k + f(k) - c(\lambda) < k^*.$$

Thus, the system cannot jump from region II to region IV (actually to *any* point to the right of k^*) along *any* path. Thus, this must also be true for the stable manifold. ■

In summary, we have managed to characterize the solution of a system of two first order nonlinear difference equations, even though we only had one initial condition. In fact, we used another condition about feasible growth rates of the variables as a sort of "end point" condition. Thus, in mathematical jargon, two "boundary" conditions are necessary to solve the system. In many applications, those boundary conditions are initial conditions. This need not be the case. In this example a condition about the limits of the product of the two variables plays the role of the second boundary condition.

3.1.1 The Effects of Permanent Shocks

In this section we study the effects of once and for all totally unanticipated "shocks" to the economy. These shocks can be of different types. For example, there could be an unexpected and permanent increase in government spending, g, which we could interpret as a permanent war, or a permanent productivity increase (i.e. an increase in z) when the economy's production function is of the form $zf(k)$.

It turns out to be relatively easy to study the effect of a permanent war. At the risk of creating some confusion let's indicate that the λ_i loci depend on g. Specifically,

$$\begin{aligned} c(\lambda_1(k;g)) &\equiv f(k) - \delta k - g \\ c(\lambda_2(k;g)) &\equiv f(k) - \delta k - g + k - k^* \end{aligned}$$

It follows that an increase in g (from g to g')shifts both loci without changing the value of k^* where they intercept. Figure 3.4 shows the two sets of $(\lambda_1(k;g), \lambda_2(k;g))$ corresponding to the two values of g. It also displays the two stable manifolds, labeled $\varphi(g)$ and $\varphi(g')$. It is clear that $\varphi(g')$ lies above $\varphi(g)$.

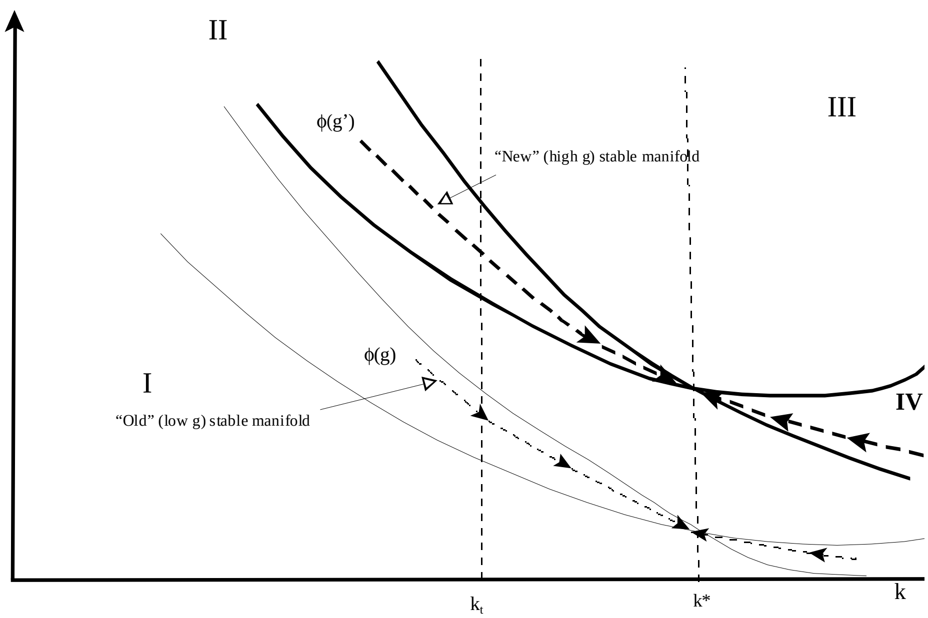


Figure 3.4: A Permanent Increase in g

Consider what happens if at time t the economy —which expects the level of government spending to remain forever constant at g— finds itself with a capital stock equal to k_t. In the morning of date t the planner learns that the new permanent level of g is $g' > g$. The problem that the planner faces is how to choose consumption and saving but, from a formal point of view, this "new" situation is identical to the problem faced by a planner that starts with initial capital equal to k_t and expects the (constant) level of government consumption to be g'. Optimal behavior requires that the planner choose λ on the stable manifold $\phi(g')$. Since $\phi(g') > \phi(g)$ the planner cuts down consumption (recall that $c'(\lambda) < 0$) instantaneously to finance the new higher level of spending.

Even though it is not possible to determine what happens to investment in the short run, it follows that the capital stock continues to increase and it converges to k^*.

3.1.2 The Effects of Temporary Shocks

The previous section considered the impact of a once and for all unanticipated change in some exogenous variable. In that analysis it was critical that the change was both completely unanticipated and permanent. This section considers changes that are transitory and —in some cases— anticipated as well. The key result is that anticipated changes in some exogenous variable will not result —at the time at which the *actual change occurs*— in a "jump" in the Lagrange multiplier λ_t (the marginal utility of income). However, the level of consumption (or, equivalently, the Lagrange multiplier) can —and usually will— jump at the time that the future *change is announced.* Thus, in this class of models, announcements have instantaneous effects, in anticipation of actual changes.

First, consider an extension of the planner's problem (P.1). The simple extension is that the production function is now allowed to depend on time. We do this to accommodate arbitrary changes in the resource constraint. (It is easy to check that the same result holds for time varying preferences.) The problem faced by the planner is,

$$\max_{\{c_t\},\{x_t\},\{k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\begin{aligned} c_t + x_t + g_t &\leq f_t(k_t) \\ k_{t+1} &\leq (1-\delta)k_t + x_t \\ k_0 &> 0 \qquad \textit{given.} \end{aligned}$$

Since all our assumptions are satisfied (provided that each f_t is C^2, increasing and concave), the Kuhn-Tucker theorem implies that the first order conditions are,

$$\begin{aligned} k_{t+1} &= (1-\delta)k_t + f_t(k_t) - c(\lambda_t) \\ \lambda_{t+1} &= \lambda_t\{\beta[(1-\delta) + f_t'((1-\delta)k_t + f_t(k_t) - c(\lambda_t))]\}^{-1}, \end{aligned}$$

where, as before, $u'(c(\lambda)) \equiv \lambda$. In addition to these two conditions, the TVC must be satisfied by any candidate solution.

We consider a special case of time varying production functions. For concreteness, let

$$\begin{aligned} f_t(k) &= f_a(k) \qquad if\ t < T, \\ f_t(k) &= f_b(k) \qquad if\ t \geq T. \end{aligned}$$

Then, three relevant first order conditions are,

$$\begin{aligned} \lambda_{T-1} &= \lambda_{T-2}\{\beta[(1-\delta) + f_a'((1-\delta)k_{T-2} + f_a(k_{T-2}) - c(\lambda_{T-2}))]\}^{-1} \\ \lambda_T &= \lambda_{T-1}\{\beta[(1-\delta) + f_b'((1-\delta)k_{T-1} + f_a(k_{T-1}) - c(\lambda_{T-1}))]\}^{-1} \\ \lambda_{T+1} &= \lambda_T\{\beta[(1-\delta) + f_b'((1-\delta)k_T + f_b(k_T) - c(\lambda_T))]\}^{-1} \end{aligned}$$

Note that the first and last equations are similar to the Euler equations corresponding to either an "always f_a" or "always f_b" economy. The "problem" is the transition period, which is captured by the middle condition.

What do we know about the nature of the solution? The intuition (and this can be formalized easily in continuous time, but it is harder in discrete time) is that when switching from the "a" to the "b" regime, the Lagrange multiplier *does not* jump. If we could ignore the transition period (from $T-1$ to T) this would be very clear. To simplify matters we will pretend that the intermediate, or transition period, does not exist and require the Lagrange multiplier from the "a" regime in the last period to coincide with the Lagrange multiplier of the "b" regime in its first period of operation.

In practical terms, this means that if we consider the solution of the problem starting in period T, the unique solution is described by a stable manifold. The reason is simple: after T the problem is a standard time-invariant planner's problem. The difficult part is to determine the solution from 0 to T. Our argument implies that we have to pick a path for (λ_t, k_t) such that the last λ_t picked under the "a" regime —this is λ_{T-1}— "lands" on the stable manifold that would be picked after period T. Thus, we seek a solution in which the Lagrange multipliers do not "jump" at the time of the regime switch.

To illustrate this point consider the effect of a temporary increase in government spending. To simplify ideas suppose that the economy starts at $t = 0$ with a high level of government spending which we denote g_a. At time T it is known that this level will switch to $g_b < g_a$. We want to describe the behavior of consumption and investment over time.

To do this we use the following procedure.

a) First, we determine how a change in g affects the $(\lambda_1(k; g), \lambda_2(k; g))$ loci.

b) Second, we pick a solution path for periods 0 to $T-1$ which is determined by the dynamics corresponding to $(\lambda_1(k; g_a), \lambda_2(k; g_a))$.

c) Third, what criteria do we use to choose such a path? We choose it so that the Lagrange multiplier of the "last period" of the high government spending regime (this is $t = T-1$) is on the stable manifold corresponding to the loci $(\lambda_1(k; g_b), \lambda_2(k; g_b))$.

d) Fourth. For $t \geq T$ the loci $(\lambda_1(k; g_b), \lambda_2(k; g_b))$ determines the dynamic behavior of the system.

From our discussion in the previous section we have already determined that since $g_b < g_a$ the stable manifold corresponding to the initial situation (i.e. the manifold associated with $(\lambda_1(k; g_a), \lambda_2(k; g_a))$, which we denote $\varphi(g_a)$ lies above $\varphi(g_b)$.

The relevant versions of the two loci are presented in Figure 3.5.

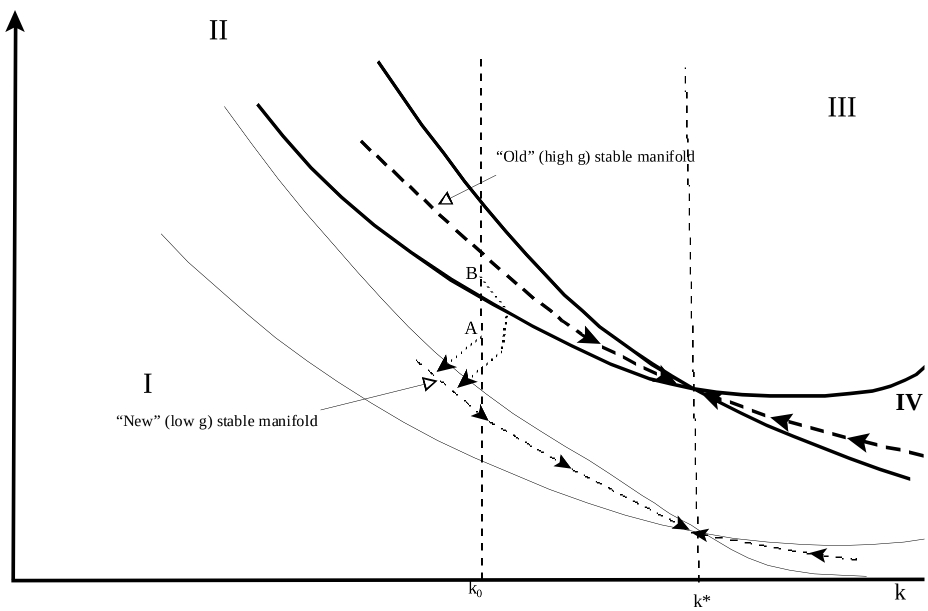


Figure 3.5: Temporary Increases in g

The heavy lines represent the current, high g, level of government spending, while the thin lines correspond to the $(\lambda_1(k; g_b), \lambda_2(k; g_b))$ loci corresponding to the (future) low level of government spending. At time 0 the stock of capital is k_0. If no change had been announced (this corresponds to the case $T = \infty$) the planner would have chosen the marginal utility of income (λ_0) on the stable manifold indicated by the heavy arrows. On the other hand, we know that if the unexpected change occurred today (this corresponds to $T = 0$), the planner would pick λ_0 on the "new" stable manifold indicated by the "thin" arrows.

What happens when T is finite (but greater than 0)? Our rule says that the planner should choose a value of λ_0 such that after exactly T periods the corresponding path would "hit" the new stable manifold, that is, the manifold indicated by the thin arrow. Without knowing T we cannot precisely describe the solution (after all, each different T gives rise to a different planner's problem), however it is possible to argue that λ_0 would be *below* what the planner would have chosen in the case of no change $(T = \infty)$. Why? If the planner chooses λ on or above the heavy dashed manifold, the solution would never intersect the "thin" manifold. The only paths consistent with intersection the new stable manifold have to start below the old one.

In Figure 3.5 there two such paths shown. The one labeled A starts "close" to the new stable manifold, while the one labeled B starts "close" to the old stable manifold. They both can be candidate solutions. What determines whether the unique solution looks more like A or B? The intuition is as follows (this happens to be true but we have not shown this, it is just intuition): If T is large, we choose something that is close to the $T = \infty$ solution. Thus, when T is large the solution is close to B. On the other hand, when the horizon is short (T close to 0), the optimal path starts close to the new stable manifold, and the solution looks more like A. Thus, the general rule is that the shorter the horizon, the closer to the new stable manifold the solution will start.

The time path for consumption and capital implied by our model is shown in Figure 3.6.

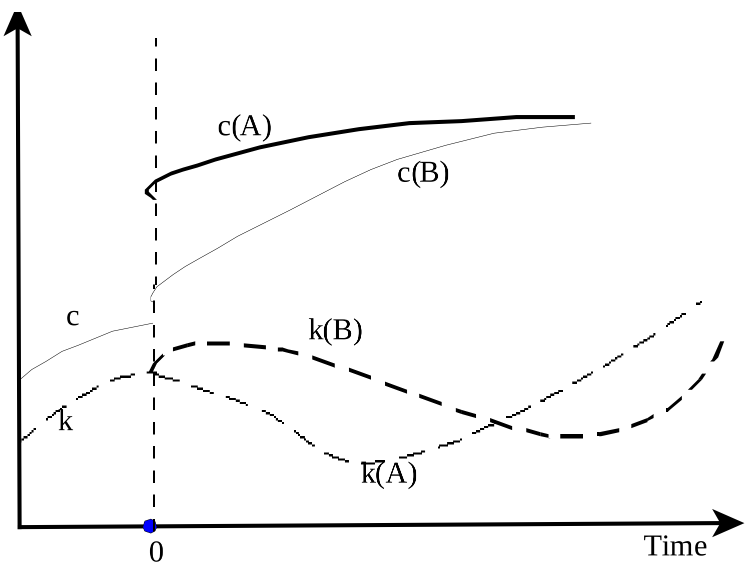


Figure 3.6: Time Paths of Aggregate Variables

Note that for both A and B, there is an instantaneous —at time 0— impact upon consumption and investment of the *announcement* that government spending will decrease at T. Consumption increases and investment decreases. Note that the longer the horizon, the smaller the impact on consumption and investment relative to the "old" levels (the levels that would have been chosen had no change been announced).

After the initial impact consumption keeps increasing for all periods (note that even when the solution hits the new stable manifold at time T, the Lagrange multiplier keeps decreasing, and hence, consumption increases). Capital, on the other hand, behaves differently. If T is small (case A) it decreases until time T and then increases. The intuition is that the announcement of a lower level of g makes the planner wealthier and he/she celebrates this by consuming more. If this comes at a cost of investment so be it. When g is lower investment can increase, as the lower g allows for more c and more x.

If T is large (case B) the forces are the same, but since this "gain" in the form of lower g will not occur for a long time, the planner has to keep investing since otherwise it would not be possible to sustain an increasing path of consumption. Thus, capital can increase for some periods. However, note that, eventually, the solution enters in region III (of the "old" curves, which, of course, determine the dynamics between 0 and T) and the capital stock decreases. Thus, when the planner gets close to the time at which g will decrease, he/she starts to consume more even at the cost of lower levels of capital.

In both cases, the time path of capital over time is not monotone. Thus, announcement effects can substantially change the dynamic behavior that we described in the case of no changes. If one thinks of actual economies being subject to these types of shocks fairly often, it is not hard to see that even a simple model like this can accommodate a rich time path of the endogenous variables.

Exercise 39 *Consider the basic economy of Exercise 27 (part ii). Assume that given $z = z_1$, the economy is at its steady state. At time zero (which we take to be the present) there is an unexpected increase in z to $z' = z_1 + a$, for some $a > 0$. We assume that $g = 0$.*

i) Describe the behavior of output and investment in the transition to the new steady state. Go as far as you can giving an economic interpretation of your results.

ii) Is it possible to say anything about the behavior of the consumption/investment ratio?

Exercise 40 *Consider the economy described in Exercise 28 (part iii). Assume that the economy is initially at its steady state when $z = z_1$. At time zero (which we take to be the present) there is an unexpected increase in z to $z' = z_1 + a$, for some $a > 0$.*

i) Describe the behavior of output, government spending and investment in the transition to the new steady state. Go as far as you can giving an economic interpretation of your results.

ii) Go as far as you can analyzing the behavior –along the dynamic path to the new steady state– of the ratio c_t/x_t.

Exercise 41 (Taxation) *Consider a simple* two planner *economy. The first planner picks taxes (we denote the tax rate by τ) and makes transfers to the representative agent (which we denote v_t). The second planner takes the tax rates and the transfers as given. That is, even though* we *know the connection between tax rates and transfers, the second planner does not; he/she takes the sequence of tax rates and transfers as given and beyond his/her control when* *solving for the optimal allocation. Thus the problem faced by the second planner (the only one we will analyze for now) is,*

$$\max_{\{c_t\},\{x_t\},\{k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\begin{aligned} c_t + x_t + g_t &\leq (1-\tau_t) f(k_t) + \nu_t, \\ k_{t+1} &\leq (1-\delta)k_t + x_t, \\ k_0 &> 0 \qquad \text{given}, \\ (c_t, x_t, k_{t+1}) &\geq (0,0,0) \qquad t \geq 0. \end{aligned}$$

It is assumed that the functions u and f have the same properties as in the general problem (problem (P.1)).

i) Assume that $0 < \tau_t = \tau < 1$ (constant tax rates), and that $v_t = \tau f(k_t)$ (remember that we know this, but the second planner takes v_t as given at the time he/she maximizes). Show that there exists a steady state, and that for any initial condition $k_0 > 0$ the economy converges to the steady state.

ii) Assume now that the economy has reached the steady state you analyzed in i). The first planner decides to change the tax rate to $0 < \tau' < \tau$. (Of course, the first planner and us know that this will result in a change in v_t; however, the second planner —the one that solves the maximization problem— acts as if v_t is a given sequence which is independent *of his/her decisions.) Describe the new steady state as well as the dynamic path followed by the economy to reach this new steady state. Be as precise as you can about, consumption, investment and output.*

Exercise 42 (Temporary Tax Increases) *Describe the impact of a temporary increase (from 0 to T) of government spending (a temporary war).*

Exercise 43 (Anouncement Effects of Tax Cuts) *Consider the economy Exercise 41. Suppose that at time zero there is an announcement that taxes (τ) will be lowered in period T (to simplify let's make them zero). What is the impact upon consumption and investment? Are announcements of future tax cuts always expansionary for the capital goods industry?*

Exercise 44 (Temporary Wars Financed by Distortionary Taxes) *(Harder) Suppose that at $t = 0$ there is an announcement that for T periods there will be a war. It turns out that the necessary expenditures are given by g_t and "financed" by a tax τ. More precisely, let $g_t = \tau_t f(k_t)$, where k_t is the equilibrium level of k. As in Exercise 41, the planner is "given" the sequences $\{\tau_t\}, \{g_t\}$ but he/she is unaware of the connection between the two. Go as far as you can describing how consumption and investment respond to this news.*

Exercise 45 (Expansionary Effects of Temporary Wars) *Some economists argue that temporary wars –if they are not too long– are expansionary. Do you agree or disagree? What does the evidence suggest for the US?*

Exercise 46 (Endogenous Government Spending) *Consider the following optimal growth problem. A planner solves,*

$$\max_{\{c_t\},\{x_t\},\{k_{t+1}\}\{g_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\begin{aligned} c_t + x_t + g_t &\leq z f(k_t), \\ k_{t+1} &\leq (1-\delta)k_t + x_t, \\ k_0 &> 0 \qquad \textit{given}, \\ (c_t, x_t, k_{t+1}, g_t) &\geq (0,0,0,0) \qquad t \geq 0. \end{aligned}$$

It is assumed that the function u is twice differentiable, strictly increasing and strictly concave, and such that $\lim_{c\to 0} u'(c) = \infty$. The function f is twice differentiable, strictly concave and strictly increasing in each argument. In addition, assume that there exist values of (k,g) denoted (k_m, g_m), such that for all $(k,g) \geq (k_m, g_m)$, $zf(k,g) - g - \delta k \leq 0$. We also assume that,

$$\begin{aligned} \forall g &> 0, \lim_{k\to 0} z f_k(k,g) = \infty \\ \forall k &> 0, \lim_{g\to 0} z f_g(k,g) = \infty \end{aligned}$$

i) Go as far as you can describing the optimal solution to the planner's problem.

*ii) Consider the special case $f(k,g) = zk^{\alpha} g^{\eta}$, with $0 < (\alpha, \eta) < 1$, and $\alpha + \eta < 1$. Can you tell whether the optimal ratio of government is increasing or decreasing along a development path? (*Note*: Assume k_0 is small and consider what happens along a path in which k_t is increasing.)*

iii) For the technology described in ii), explore the existence of a steady state. Go as far as you can developing the implications of this model for the differences in capital/output ratios —$k/zf(k,g)$— and government spending/output ratios —$g/zf(k,g)$— for high and low technology economies (high or low z's).

*iv) (*Optional*. It could be hard and I have not solved it. On the other hand, it is probably fun) Consider an economy with initial technology level zA. Assume that this economy is at the steady state. Assume that due to exogenous political changes this economy has now access to a new level of technology z' given by $z' = zA + a$, for some $a > 0$. Describe —for the technology described in section ii— the dynamic adjustment path to the new steady state.*

Exercise 47 (Choice of Technology and the Industrial Revolution) *Consider the following two technologies: The "a" technology uses land (L) and labor (n_t) to produce output. The "b" technology uses capital and labor (but no land) to produce output. In both cases the production function $f_i, i = a, b$, exhibits decreasing returns to scale (it is increasing and concave). In addition, suppose* *that even though there is no technological change in the a technology, output using the b technology is given by,*

$$y_t \leq \gamma^t f_b(k_t, n_t)$$

where $\gamma > 1$. Assume that land is in fixed supply but that capital can be accumulated. In particular assume that given output y (no matter what technology produced it) the uses of output satisfy,

$$\begin{aligned} c_t + x_t &\leq y_t, \\ k_{t+1} &\leq (1-\delta)k_t + x_t, \end{aligned}$$

where either

$$y_t = f_a(L, n_t),$$

or

$$y_t = \gamma^t f_b(k_t, n_t),$$

i) Show conditions under which a planner would choose the primitive technology "a."

ii) Assume that $k_0 = 0$. Is it true that all economies will always use the "a" technology.

iii) According to this model, what are the "immediate causes" of the industrial revolution (defined here as a persistent increase in per capita output)?

iv) Go as far as you can analyzing the role of the per capita supply of land (L) on the speed of transition to the "b" technology.

Exercise 48 (Open Economies and Terms of Trade Shocks) *Consider a standard optimal growth model with only one difference: A country only produces a good that it does not consume; that is, it sells all of its output in international markets, and it buys both consumption and investment in the "world" market. In addition, we assume that this country's planner can borrow and lend at the international interest rate given by r^*. Let the price, at time t, of the good produced (but not consumed) by this country be p_t. We normalize the price of the international good –the one consumed by this country– at one. Let b_{t+1} be the stock of international bonds purchased (issued if negative) by the planner at time t denominated in terms of the international good. These bonds will pay $(1+r^*)b_{t+1}$ units of international goods at time $t+1$. Assume that $\beta(1+r^*) = 1$.*

The planner's problem for this economy is,

$$\max_{\{c_t\},\{x_t\},\{k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t, g_t)$$

subject to

$$\begin{aligned} c_t + x_t + g_t + b_{t+1} &\leq p_t f(k_t) + (1+r^*)b_t, \\ k_{t+1} &\leq (1-\delta)k_t + x_t, \\ k_0 &> 0 \text{ and } b_0 \qquad \text{given}, \\ (c_t, x_t, k_{t+1}) &\geq (0,0,0) \qquad t \geq 0. \end{aligned}$$

i) Formulate the optimal growth problem. Make sure that you describe both the Euler equations and the transversality condition (TVC). *[*Note*: The planner takes the sequence* $\{g_t\}$ *as given.]*

ii) Assume that $g_t = g$, *and that* $p_t = p_L$ *for* $t = 0, 1, ..., N-1$, *and* $p_t = p_H$ *for* $t = N, N+1, ...,$ *with* $p_H > p_L$. *Describe, as carefully as you can, the time path of consumption, investment and the trade balance. If we assume that* $p_t = p_H, t \leq 0$, *then the exercise captures the impact of a temporary deterioration of the terms of trade. [*Note*: In this exercise a country has a* trade surplus *if GDP is greater than the sum of consumption, investment and government spending. Thus, if* $p_t f(k_t) - c_t - x_t - g_t$ *is positive, this country has a trade surplus, and if it is negative it has a trade deficit.]*

iii) Assume that $g_t = g$, *and that* $p_t = p_L$ *for* $t = 0, 1, ..., N-1$, ***and*** $t = N+T, N+T+1, ...$ *For* $t = N, N+1, ..., N+T-1$, *it is assumed that* $p_t = p_H$ *with* $p_H > p_L$. *Describe, as carefully as you can, the time path of consumption, investment and the trade balance.*

iv) Consider the model in iii). You are asked to evaluate these two arguments:

a) When countries enjoy a temporary increase in their terms of trade, the optimal policy is to increase investment during the good times in order to have more capital (and, hence higher output) when the price of exports decreases.

b) An exogenous government spending policy given by "high g whenever p is low, and low g whenever p is high" (this is a countercyclical policy) results in higher level of output relative to a constant g policy.

Exercise 49 (Interest Rates and Output) *Consider an economy populated by a large number of identical households that maximize the utility function,*

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\theta}}{1-\theta} \qquad 0 < \beta < 1, \theta > 0.$$

Each household is endowed with one unit of labor which is supplied inelastically. Output in this economy is produced using capital, k, and labor, n, with a concave, differentiable, and homogeneous of degree one function $F(k, n)$. *Assume that firms and households behave competitively.*

This economy has access to an international bond market with gross interest rates given by $1 + r_t$ *where*

$$\begin{aligned} 1 + r_t &= \beta^{-1}(1+\gamma) \qquad t = 0, 1, ..., T-1, \text{ and } \gamma > 0 \\ 1 + r_t &= \beta^{-1} \qquad\qquad\quad\;\; t = T, T+1, ... \end{aligned}$$

i) Describe the steady state of this economy.

ii) Describe the dynamic path of consumption, domestic interest rates and wage rates. Go as far as you can determining the effect of T *upon the initial* ($t = 0$) *level of consumption.*

iii) What is the relationship, if any, between:

a) world interest rates and domestic output?

b) world interest rates and wages?

c) world interest rates and consumption growth?

d) consumption growth and domestic output?

iv) Consider an economy that, at $t = 0$, has a stock of capital per worker that is "small." In this economy there are two views about the optimal timing of globalization, given that the objective is to maximize the utility of the representative agent. One view –let's call it L– argues that the economy should wait until time T to allow its residents to borrow and lend in international capital markets, so that they can profit from the lower interest rates. The other view –let's call it R– is that the representative agent will be better off by opening up the economy immediately (i.e. at $t = 0$). Which of these two views would you favor? Justify your answer.

Exercise 50 (Habit Persistence) *Consider an economy populated by a large number of households with utility functions given by*

$$\sum_{t=0}^{\infty} \beta^t u(c_t - \phi z_t), \qquad 0 < \beta < 1,$$

where u is twice differentiable, increasing and strictly concave (if necessary you may assume that it satisfies the Inada condition). The variable c_t is individual consumption, and z_t is a measure of lagged consumption. To be precise,

$$z_t \geq \sum_{j=0}^{\infty} (1-\delta_c)^j c_{t-1-j}.$$

It follows that, alternatively, it is possible to describe the law of motion for z_t as

$$z_{t+1} \geq (1-\delta_c) z_t + c_t.$$

In this setting, z_t is a measure of 'habit persistence,' as it implies that the marginal utility of any given level of consumption decreases the higher the level of past consumption. The technology in this economy is standard and given by

$$\begin{aligned} c_t + x_t &\leq f(k_t), \\ k_{t+1} &\leq (1-\delta_k) k_t + x_t, \end{aligned}$$

where the functions f is strictly concave, increasing and satisfies Inada conditions.

i) Let the planner maximize the utility of the representative agent subject to all the feasibility constraints. Argue that, under some condition on (ϕ, δ_c) an ***interior*** *steady state exists and is unique. Describe the condition that (ϕ, δ_c) has to satisfy.*

ii) What does the model say about the impact of cross-country differences in how much people care about past consumption —as measured by ϕ— on the steady state output per worker.

Exercise 51 (Terms of Trade Shocks) *Let preferences over two goods, c and y, be given by*

$$\sum_{t=0}^{\infty} \beta^t \frac{[c_t^{-\eta} + y_t^{-\eta}]^{-(1-\theta)/\eta}}{1-\theta}, \qquad \eta > -1,\ \theta > 0.$$

Assume that both goods can be traded internationally. Let the price of good y_t be p_t. The planner's feasibility constraints are

$$\begin{aligned} c_t + p_t y_t + x_t &\leq f(k_t) + p_t e, \\ k_{t+1} &\leq (1-\delta)k_t + x_t, \\ (e, k_0) &> 0 \qquad \textit{given} \\ (c_t, y_t, x_t, k_{t+1}) &\geq (0,0,0,0) \qquad t \geq 0. \end{aligned}$$

The interpretation is as follows: domestic production of good c (which is also the investment good) is given by $f(k)$. Domestic production of good y is constant and equal to e. Since the economy can trade with the rest of the world the appropriate feasibility constraint requires that the ***value*** *of the consumption bundle (i.e. $c_t + p_t y_t + x_t$) be less than or equal to the value of production (i.e.$f(k_t) + p_t e$)*

1. *Assume that $p_t = p$. Go as far as you can describing the steady state.*

2. *Assume that $p_t = p$.Are there conditions under which this country is an exporter of good y (i.e. $e > y$) in the steady state.*

3. *Assume that the economy is at the steady state. Go as far as you can describing the effects of a permanent increase in p.*

4. *Assume that at $t = 0$ the economy starts with k_0 units of capital (k_0 less than the steady state). There is a temporary decrease in p, which is expected to last for T periods. Go as far as you can describing the impact of this shock. In particular, discuss if it is possible that during the period in which the price of y is low, the economy disinvests (i.e. the capital stock decreases).*

Exercise 52 (Productivity and Durable Goods) *Consider an economy populated by a large number of identical dynasties with utility functions given by*

$$U \equiv \sum_{t=0}^{\infty} \beta^t [u(c_t) + v(z_t)], \qquad 0 < \beta < 1,$$

where $u(c)$ and $v(z)$ are strictly increasing and strictly concave functions. In this context, c_t denotes consumption of non-durable (e.g. food) goods, while z_t is the stock of durables (e.g. houses) at time t. The economy's aggregate feasibility *constraint is given by*

$$\begin{aligned} c_t + qx_{zt} + x_{kt} &\leq Af(k_t), \\ k_{t+1} &\leq (1-\delta_k)k_t + x_{kt}, \\ z_{t+1} &\leq z_t h(\frac{x_{zt}}{z_t}), \qquad h(0)=1,\ h(\delta_z)=1,\ 0<\delta_z<1 \\ 0 &< (k_0, N_0), \qquad \textit{given} \end{aligned}$$

where f is a standard (i.e. twice differentiable, increasing, strictly concave, $\lim_{k\to 0} f'(k) = \infty$ and $\lim_{k\to\infty} f'(k) = 0$) production function, and $0 < \delta_k < 1$ is the depreciation rate corresponding to capital. The quantity x_{zt} is interpreted as investment (purchases) of new durables. The function h is assumed to be strictly increasing and strictly concave. In this specification, δ_z corresponds to the depreciation rate of the durable good given that, when $x_{zt} = \delta_z z_t$, the stock of the durable good is unchanged (i.e. $z_{t+1} = z_t$). Let the non-durable good be the numeraire.

1. *Define a competitive equilibrium in which households own the stocks of capital and durables and trade (at least) one period bonds*

2. *Show that a steady state exists and is unique. What does the model imply about the effect of changing TFP (the constant A) upon the steady state quantities consumed of durables and non-durables? Explain your argument.*

3. *Does theory pin down the impact of an increase in TFP (A) on the **ratio** of durables to nondurables (i.e. c^*/z^*, where a * denotes a steady state value). Explain your argument. To answer this section you may assume that the relevant utility functions are*

$$u(c) = \frac{c^{1-\theta}}{1-\theta}, \qquad v(z) = \frac{z^{1-\eta}}{1-\eta}, \qquad (\theta, \eta) > 0.$$

4. *Assume now that the economy is open and that individuals can borrow and lend (subject to the standard transversality condition) in the international bond market. Assume that the world interest rate, R^*, satisfies $R^*\beta = 1$. It is claimed that, since the interest rate is constant, the country's level of output and stock of durables will converge in **one period** to their steady state values (not necessarily the same steady state as the closed economy). Discuss this claim.*

5. *For the isoelastic utility functions described above, go as far as you can describing the dynamic behavior of the stock of durables for an economy that starts with the steady state level of capital (but not durables) and is open to international borrowing and lending with $R^*\beta = 1$.*

6. *Consider next the effect of a permanent increase in productivity on the dynamic path of output and the stock of durables in an open economy similar to the one described in the previous section.*

Exercise 53 (Allocating Talent) *Consider an economy in which all individuals rank consumption streams according to the function*

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t), \qquad 0 < \beta < 1,$$

where u *is a bounded and strictly concave* C^2 *function. The production function uses intermediate inputs produced with capital and labor using the production function* $F(k,n)$*, and entrepreneurial skills. Assume that* $F(k,n)$ *is a* C^2 *function, which is concave, increasing in each argument, and homogeneous of degree one. If an individual is a "worker," he/she supplies one unit of labor. Let the cross-sectional distribution of skills be given by* G *which satisfies*

$$G(0) = 0,\ G(\bar{e}) = 1,\ G'(e) = g(e) > 0,\ for\ e \in (0, \bar{e}).$$

Thus, $G(\hat{e})$ *is interpreted as the fraction of the population with skill level less than or equal to* $\hat{e}$*. Note that if all individuals with skill level greater than or equal to* e^* *are employed as managers, the available number of workers is* $N = G(e^*)$*. Thus, individuals not employed as managers are employed as workers.*

Consider the problem faced by a planner that in every period has to decide how to allocate individuals (they can be managers or workers), capital (the total stock of capital must be allocated to the different firms) and workers so as to maximize the utility of the representative agent. If Y_t *is aggregate output at time* t*, the aggregate resource constraint is*

$$C_t + K_{t+1} - (1-\delta)K_t \leq Y_t.$$

Let M *be a subset of* $[0, \bar{e}]$ *such that if* $e \in M$ *then* e *is a manager. (Of course, if* $e \in M^c$ *—the complement of* M *in* $[0, \bar{e}]$*—,* e *is a worker.) If* $k(e)$ *and* $n(e)$ *are the amounts of capital and labor allocated to a firm managed by a manager with skill level* e*, output of that firm is*

$$y(e) = e^{\theta} F(k(e), n(e))^{1-\theta},$$

and aggregate output is

$$Y = \int_M y(e)dG(e) = \int_M y(e)g(e)de.$$

Given the set M *and an initial stock of capital* K *a feasible allocation is a pair of functions* $k(e)$ *and* $n(e)$ *satisfying*

$$\begin{aligned}
K &= \int_M k(e)dG(e) = \int_M k(e)g(e)de, \\
N &= \int_M n(e)dG(e) = \int_M n(e)g(e)de, \\
N &= \int_{M^c} dG(e) = \int_{M^c} g(e)de.
\end{aligned}$$

1. *State the planner's problem.*

2. *Argue that any optimal allocation is such that, at time* t*, there exist a cutoff skill level,* e_t^**, such that* $M = [e_t^*, \bar{e}]$*, and* $M^c = [0, e_t^*]$*.*

3. *Assume that the optimal rule is to find a single cutoff point* e_t^* *(even if you could not show this to be the case). Go as far as you can characterizing the functions* $k(e)$ *and* $n(e)$*.*

4. *Assume that the optimal rule is to find a single cutoff point* e_t^* *(even if you could not show this to be the case). Since the total number of firms is* $1 - G(e_t^*)$*, and the total number of workers is* $G(e_t^*)$*, the average size of a firm (when size is measured using employment) is*

$$S_t = \frac{1 - G(e_t^*)}{G(e_t^*)}.$$

   It follows that the implications of the model for the size distribution are summarized by the time path of e_t^**. Show that, if* $F(k, n) = Ak^\alpha n^{1-\alpha}$*,* $e_t^* = e^*$ *independently of* K_t*.*

5. *Assume that the solution to the planner's problem is such that the capital stock,* K_t*, is increasing. Let the technology be given by* $F(k, n) = A[\alpha k^{-\rho} + (1 - \alpha)n^{-\rho}]^{-1/\rho}$*. Go as far as you can describing the predictions of the model for the time path of* S_t *as a function of* ρ*.*

Exercise 54 (Vintage Capital) *Consider an economy in which productivity depends on the vintage of the capital used. To be precise, the amount produced by a stock of capital of vintage* τ *at time* t*,* K_τ*, is*

$$Y_{\tau,t} = \gamma^\tau \min\{K_\tau, L_{\tau,t}\}, \ \gamma > 1$$

where $L_{\tau,t}$ *is the amount of labor allocated to the* τ *vintage. Assume that labor is supplied inelastically and the total number of people is normalized to one. Preferences are given by*

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t),$$

and the resource constraint is

$$c_t + \gamma^t K_{t+1} \le Y_t = \sum_{\tau=t}^{0} Y_{\tau,t},$$

and

$$\sum_{\tau=t}^{0} L_{\tau,t} \le 1.$$

In your answers consider only the solution to the planner's problem. Consider an economy in which productivity depends on the vintage of the capital used. To *be precise, the amount produced by a stock of capital of vintage τ at time t, K_τ, is*

$$Y_{\tau,t} = \gamma^\tau \min\{K_\tau, L_{\tau,t}\},\ \gamma > 1$$

where $L_{\tau,t}$ is the amount of labor allocated to the τ vintage. Assume that labor is supplied inelastically and the total number of people is normalized to one. Preferences are given by

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t),$$

and the resource constraint is

$$c_t + \gamma^t K_{t+1} \leq Y_t = \sum_{\tau=t}^{0} Y_{\tau,t},$$

and

$$\sum_{\tau=t}^{0} L_{\tau,t} \leq 1.$$

In your answers consider only the solution to the planner's problem.

1. *Go as far as you can showing that there is a balanced growth path defined as an allocation in which objects that grow do so at the same rate. What is the growth rate along the balanced growth path?*

2. *Go as far as you can showing that, along the balanced growth path, only a finite number of vintages are used.*

3. *Go as far as you can characterizing how labor is allocated across vintages of capital, and the impact of increases in the (exogenous) rate of technological change, γ, on the number of vintages in use and the fraction of the labor force allocated to each vintage.*

3.2 Applications

In this section we develop the implications of the model in some special cases of interest. We consider constant saving rates, as well as the role of technical progress and population growth.

3.2.1 Constant Saving Rates

In his seminal 1956 article, Robert Solow first studied the dynamic behavior of aggregate economic variables and –effectively– created the field of growth theory which forms an important fraction of modern macroeconomics. Solow did not work with the optimizing framework we use, although in other respects the models are quite similar. Instead Solow assumed that the saving rate is constant. A natural question to ask is whether it is possible in a fully optimizing framework to rationalize Solow's assumption of a constant saving rate. In other words, is it possible to find preferences and technologies such that when used in our model they render the Solow model of constant saving rates as the equilibrium outcome?

It turns out that the answer is affirmative. To "discover" the right preferences and technologies it is sufficient to show that a constant saving rate, s, satisfies the Euler equation (3.1b) which can also be written as

$$u'(c_t) = \beta u'(c_{t+1})[1 - \delta + zf'(k_{t+1})].$$

In order to find the right preferences and technology, we impose on the Euler equation Solow's optimal consumption-saving plan; that is, we guess that

$$\begin{aligned} c_t &= (1-s)zf(k_t) \\ k_{t+1} &= (1-\delta)k_t + szf(k_t). \end{aligned}$$

Using these two guesses in the Euler equation we obtain

$$u'((1-s)zf(k_t)) = \beta u'((1-s)zf((1-\delta)k_t + szf(k_t)))[1-\delta+zf'((1-\delta)k_t+szf(k_t))],$$

where the same value of s must satisfy this equation for all values of k_t. (It is easy to check that the TVC is satisfied.) To show that there is at least one economy that is a "Solow" economy, it suffices to present an example. Consider the case $u(c) = \ln(c)$, $\delta = 1$ and $zf(k) = zk^{\alpha}$, for $0 < \alpha < 1$. Substituting in these functions in the above expression it is possible to see that the Euler equation is satisfied if $s = \alpha\beta$ which, as required, is between zero and one. It turns out that within the class of economies that have steady states, this is the only one displaying the property of constant saving rates. Thus, although not theoretically impossible, the assumption of constant saving is quite restrictive.

3.2.2 Population Growth and Technical Change

So far in our basic model we have assumed a constant level of population and no technical progress. It turns out that —at least in some cases— it is possible to show that both population growth and technical progress can be accommodated quite easily by *reinterpreting* the different parameters. Let us consider the case in which both population and technology grow at a given rate. Specifically, we assume that

$$\begin{aligned} L_t &= (1+n)^t L_0, \\ A_t &= (1+\gamma)^t A_0, \end{aligned}$$

represent the population level and the state of technology at time t, respectively. Assume that preferences are given by,

$$\sum_{t=0}^{\infty} \beta'^t L_t u(\frac{C_t}{L_t}).$$

This formulation captures the idea that total welfare depends on consumption per worker times the number of workers. The technology is given by,

$$\begin{aligned} C_t + X_t &\leq F(K_t, A_t L_t), \\ K_{t+1} &\leq (1-\delta')K_t + X_t. \end{aligned}$$

There are two things to note about this technology. First, we assume that technological change is "labor augmenting" in the sense that it is not an overall increase in productivity, but only an increase in the productivity of labor. Second, we will assume that F is concave, twice differentiable, homogeneous of degree one and increasing in each argument.

For any variable Z_t, define $z_t = Z_t/(A_t L_t)$. Thus, z_t is measured in per unit of "effective" labor. With this notation the technology is given by,

$$\begin{aligned} c_t + x_t &\leq F(k_t, 1) = f(k_t), \\ k_{t+1} &\leq \frac{(1-\delta')k_t}{(1+n)(1+\gamma)} + \frac{x_t}{(1+n)(1+\gamma)}. \end{aligned}$$

Note that by defining δ so that $1-\delta = \frac{1-\delta'}{(1+n)(1+\gamma)}$ the technology is similar to that used in problem (P.1) except for the constant dividing x_t. It is easy to check that such constant does not affect the analysis. (It does affect the definition of the steady state k^*.)

Using the same transformation, preferences can be written as,

$$\sum_{t=0}^{\infty} [\beta'(1+n)]^t L_0 u(c_t(1+\gamma)^t A_0).$$

If $u(c_t(1+\gamma)^t A_0) = \varphi^t v(c_t)$ for *some* function v we can directly use the original formulation of the planner's problem. It turns out that it is easy to prove that for a utility function to satisfy that property it must be of the form $u(c) = \frac{c^{1-\theta}}{1-\theta}$ for some $\theta > 0$. In this case,

$$u(c_t(1+\gamma)^t A_0) = u(c_t)(1+\gamma)^{(1-\theta)t} A_0^{1-\theta}.$$

If we define $\beta = \beta'(1+n)(1+\gamma)^{(1-\theta)}$, the planner's objective function is well defined provided that $\beta < 1$ (which we will assume), and given by,

$$\sum_{t=0}^{\infty} \beta^t u(c_t).$$

Simple inspection shows that the transformed economy has the same structure as the standard one sector growth model described in (P.1).

Exercise 55 *Evaluate the following claims in the context of a model with positive population growth and exogenous technological change.*

i) Changes in the population growth rate do not change the steady state capital per effective worker.

ii) An increase in the rate of technical progress may increase or decrease the steady state capital per effective worker.

iii) The growth rate of capital per worker (not per effective worker) in the steady state is equal to the rate of exogenous technical change.

Exercise 56 *Consider a model in which the planner cares only about the utility of the* average *individual. Specifically, the planner's utility function is of the form*

$$\sum_{t=0}^{\infty} \beta'^{t} u(\frac{C_t}{L_t}).$$

i)Discuss claims i)-iii) from Exercise 55 in the context of this model.

ii) Are there economic arguments for preferring one of the two specifications of preferences?

Exercise 57 *In the text we considered the case in which exogenous technological change is "labor augmenting." In this setting we showed that –after appropriate reinterpretation– the planner's problem is stationary in the transformed variables. In this exercise you are asked to consider two alternative specifications of technical change.*

i) Modify the basic model of this section by letting technological progress to be "neutral." By neutral we mean technical change that does not affect the relative marginal products of the different inputs. Specifically, let the technology be $A_t F(K_t, L_t)$. Show that –in general– it is not possible to redefine variables so as to turn this problem into a stationary problem.

ii) Modify the basic model to allow for "capital augmenting" technical change. More precisely. let the technology be given by $F(A_t K_t, L_t)$. Show that –in general– it is not possible to redefine variables so as to turn this problem into a stationary problem.

iii) Show that in the special case of a Cobb-Douglas production function $F(K, L) = AK^{\alpha}L^{1-\alpha}$, for $0 < \alpha < 1$, all three forms of technical change are equivalent.

Exercise 58 (Dynamics of a Baby Boom) *Consider an economy populated by a large number of identical households. The utility function of each household is,*

$$\sum_{t=0}^{\infty} \beta^t N_t u(c_t), \qquad 0 < \beta < 1,$$

and u is differentiable, increasing and strictly concave, and N_t is the number of members of the household . In this notation, c_t is consumption per family member. All households, and all individuals within a household, are endowed with one unit of labor that is supplied inelastically.

Assume that there is a large number of firms that produce output using capital and labor. Each firm has a production function given by $F(K, N)$ which is increasing, differentiable, concave and homogeneous of degree one. Here K is total (not per capita) capital, and N total population (or workforce).

i) Assume that $N_{t+1} = (1+n)N_t$*. Describe the steady state of this model. Is the capital per worker different in "high* n*" and "low* n*" economies.*

ii) Assume that $N_0 = 1$*, and that* $N_j = 1$*, for* $j < 0$*. For the purposes of this question you may assume that the planner thought that population was going to be constant forever. At* $t = 0$*, the planner "discovers" (this is a surprise announcement) that there is a baby boom described by,* $N_t = (1+n)^t$ *for* $t = 1, 2, ..., T$*, and* $N_t = (1+n)^T$ *for* $t > T$*, for some* $n > 0$*. Thus, population grows between time* 0 *and* T*, and it stabilizes afterwards. Describe as carefully as you can:*

a) The impact (relative to what had been planned) on consumption and investment at time 0*.*

b) The time path of capital per worker.

c) The time path of interest and wage rates.

Chapter 4

Competitive Equilibrium

So far we have studied the allocation chosen by a planner. However, this is not an intrinsically interesting exercise in positive economics. More precisely, we are interested in how markets work. In this section we define an equilibrium for the economy given by the preferences and technology corresponding to the simple growth model, and we show that the planner's allocation can be decentralized as a competitive equilibrium.

4.1 Basic Results

The first issue we need to decide is how many markets will be open. In principle the simplest case is one in which all markets are open. This case turns out to be rather cumbersome and not very realistic. Instead, we will look at the following market structure:

- At each date t there are "spot" markets for all commodities (e.g. different types of consumption goods and different types of labor).
- At each date t there is a market for bonds or deposits, and capital.
- At each date t, firms rent labor and capital from households.

We now describe —for this particular market structure— the maximization problems solved by households and firms. The representative household solves the following problem,

$$\max_{\{c_t\},\{x_t\},\{k_{t+1}\}\{b_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{HH.1}$$

subject to

$$
\begin{array}{rcll}
c_t + p_{kt}x_t + b_{t+1} & \leq & w_t + r_t k_t + R_t b_t & t = 0, 1, ... \\
k_{t+1} & \leq & (1-\delta)k_t + x_t & t = 0, 1, ... \\
\lim_{T\to\infty} \beta^T u'(c_T) b_{T+1} & = & 0, & \\
k_0 & > & 0 \textit{ and } b_0 \quad \textit{given}, & \\
(c_t, x_t, k_{t+1}) & \geq & (0,0,0) & t = 0, 1, ...
\end{array}
$$

In this setting, p_{kt} is the relative (to consumption) price of new capital, w_t is the wage rate, r_t is the rental price of existing capital, R_t is the gross (one period) interest rate, and b_t is the stock of bonds or deposits owned by the household at the beginning of period t. Although the "real" variables are required to be nonnegative, the financial variable, b_t, can take any sign. In particular, a negative value of b_t is interpreted as borrowing by the household.

Note that although the imposition of a TVC on b_t is somewhat artificial it is clear why we need it from an economic point of view. If the household could choose the sequence $\{b_t\}$ unconstrained it would always choose $b_t = -\infty$. In other words, it would borrow an infinite amount today and would "repay" tomorrow by borrowing (again) an infinite amount. This scheme would allow the household to have infinite consumption at all times. Since this cannot be an equilibrium we need to put "bounds" on the sequence of borrowing. One would be tempted to impose a maximum amount that can be borrowed (i.e. a credit limit), and this strategy would work. For example, we could specify that $b_t \geq \underline{b}$ for some value $|\underline{b}| < \infty$; however, our condition —satisfaction of the TVC— is even weaker than imposing a higher bound on borrowing and also "works," in the sense of discouraging infinite borrowing or Ponzi games.

Since firms rent inputs in spot markets (they do not own the capital) their problem is quite simple. It is given by,

$$\max c_t + p_{kt}x_t - w_t n_t - r_t k_t \tag{FF.1}$$

subject to

$$c_t + x_t \leq F(k_t, n_t)$$

as well as the appropriate nonnegativity constraints. We assume that the function F is strictly increasing in each argument, concave, twice continuously differentiable and homogeneous of degree one. These assumptions imply (see Exercise 64) that profits are zero in any equilibrium and that the number of firms does not matter. It is because of this that we neglected to include profits in the consumer's budget constraint and we did not consider the market for shares in the representative firm. We assume without loss of generality (again, see Exercise 64) that the number of consumers and firms is the same.

Let's consider the firm's problem first. In an interior solution (which we assume in this section) it follows that $c_t > 0$ and $x_t > 0$ if and only if $p_{kt} = 1$. Note that, in general (i.e. if $x = 0$) the price of new capital p_{kt} is less than or equal to one. In addition, we have,

$$\begin{aligned} r_t &= F_k(k_t, n_t) \\ w_t &= F_n(k_t, n_t). \end{aligned}$$

In equilibrium we can assume that there are as many firms as there are workers (Why?) and, hence, the per firm level of employment is 1. Thus, in equilibrium, factor prices obey

$$\begin{aligned} r_t &= f'(k_t) \\ w_t &= f(k_t) - k_t f'(k_t), \end{aligned}$$

where we have used the definition $f(k) \equiv F(k, 1)$.[1] Note that we allowed the firm to "chose" its level of employment. It was only after we discovered what is required by the firm in order to hire a certain amount of labor that we imposed the equilibrium condition $n_t = 1$.

To solve the household problem (HH.1) we form the Lagrangean, as we did in Chapter 1. In this case, the Lagrangean function is given by,

$$\begin{aligned} L(\mathbf{x}, \mathbf{c}, \mathbf{k}, \boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\gamma}) &= \sum_{t=0}^{\infty} \beta^t \{u(c_t) + \lambda_t [w_t + r_t k_t + R_t b_t - c_t - x_t - b_{t+1}] \\ &\quad + \theta_t [(1-\delta) k_t + x_t - k_{t+1}] + \gamma_{1t} c_t + \gamma_{2t} x_t + \gamma_{3t} k_{t+1}\}. \end{aligned}$$

Since all the concavity assumptions required to use the Kuhn-Tucker theorem are satisfied, we have that the first order conditions that characterize a saddle point (for an interior solution) are the constraints at equality and

$$\begin{aligned} c_t &: u'(c_t) = \lambda_t, \\ x_t &: \lambda_t = \theta_t, \\ k_{t+1} &: \theta_t = \beta[\theta_{t+1}(1-\delta) + \lambda_{t+1} r_{t+1}], \\ b_{t+1} &: \lambda_t = \beta \lambda_{t+1} R_{t+1}, \\ TVC_k &: \lim_{T \to \infty} \beta^T \lambda_T k_{T+1} = 0, \\ TVC_b &: \lim_{T \to \infty} \beta^T \lambda_T b_{T+1} = 0. \end{aligned}$$

Before we proceed it is useful to rewrite the Euler equation for capital in the household problem as,

$$u'(c_t) = \beta u'(c_{t+1})[1 - \delta + r_{t+1}],$$

or, using the first order condition corresponding to the optimal choice of capital by the representative firm we obtain

$$u'(c_t) = \beta u'(c_{t+1})[1 - \delta + f'(k_{t+1})], \tag{4.1}$$

[1] It is a straightforward consequence of Euler's Theorem that, with the definition $f(k) = F(k, 1)$, $F_n(k, 1) = f(k) - k f'(k)$.

while the Euler equation for bonds is just,

$$u'(c_t) = \beta u'(c_{t+1})R_{t+1}. \tag{4.2}$$

We are now ready to define an equilibrium.

Definition 59 (Recursive Competitive Equilibrium) *A recursive competitive equilibrium is a collection of price sequences* $[\{w_t^*\}, \{r_t^*\}, \{p_{kt}^*\}, \{R_t^*\}]$, $t = 0, 1, ...,$ *an allocation* $[\{c_t^*\}, \{x_t^*\}, \{k_{t+1}^*\}]$, $t = 0, 1, ...,$ *and a sequence of bond holdings* $\{b_{t+1}^*\}$ *such that,*

a) Given prices, the allocation and the sequence $\{b_{t+1}^*\}$ *solve (HH.1) [utility maximization].*

b) Given prices, the allocation solves (FF.1) [profit maximization].

c) The allocation is feasible [market clearing].

d) $b_0^* = b_0 = 0, k_0^* = k_0 > 0$ *is given.*

Most of the conditions are quite standard and require no further elaboration. One that may seem peculiar is $b_0 = 0$. Note that in this world of representative agents b_0 is both the individual initial value of bonds (or outstanding loans if negative) and the economy wide average. In this world without outside bonds (because there is no government and no international borrowing and lending), it must be the case that the economy's net asset position is zero. Thus, $b_0 = 0$ simply says that at time 0 financial markets are in equilibrium. In an economy with heterogeneous agents it is possible to let the b_{i0} be different from zero. In this case it is required that the sum of b_{i0} over all individuals i be equal to zero. From now on we will use bold letters without a subscript to denote an infinite sequence; that is, for any $\{z_t\}$, let $\mathbf{z} = \{z_t\}$, $t = 0, 1, ...$

Before we explore the connection between equilibrium and optimal allocations, it is useful to derive some implications of the sequence of budget constraints faced by the household. The key property is that —under our assumptions— the sequence of budget constraints is equivalent to one budget constraint. We formalize this in,

Claim 60 (Budget Constraints: Recursive and Present Value) *Any allocation that solves the household problem must satisfy the following present value budget constraint:*

$$\sum_{t=0}^{\infty} q_t c_t = \sum_{t=0}^{\infty} q_t w_t + q_0[1 - \delta + r_0]k_0 + q_0 R_0 b_0 \tag{4.3}$$

where $q_t/q_0 = \Pi_{j=1}^{t} R_j^{-1}$, *for* $t \geq 1$ *is the price –in terms of time zero consumption– of consumption at time* t. *In addition the initial price* q_0 *can be chosen equal to one without loss of generality.*

Proof. First note that the first budget constraint is just,

$$b_1 = w_0 + r_0 k_0 + q_0 R_0 b_0 - c_0 - x_0,$$

while the second is

$$c_1 + x_1 + b_2 = w_1 + r_1 k_1 + R_1 b_1.$$

Combining these two, we get,

$$(c_0 + x_0) + R_1^{-1}(c_1 + x_1) = (w_0 + r_0 k_0) + R_1^{-1}(w_1 + r_1 k_1) + R_0 b_0 - R_1^{-1} b_2.$$

Repeated substitution (use the next budget constraint to eliminate b_2 and "pick up" a b_3) implies that,

$$\sum_{t=0}^{T} q_t(c_t + x_t) = \sum_{t=0}^{T} q_t(w_t + r_t k_t) + q_0 R_0 b_0 - q_T b_{T+1}. \tag{4.4}$$

Next we want to argue that $\lim_{T\to\infty} q_T b_{T+1} = 0$ for any feasible allocation. Recall that a feasible allocation must satisfy $\lim_{T\to\infty} \beta^T \lambda_T b_{T+1} = 0$. However, from Euler equation corresponding to the optimal choice of bonds it follows that $\beta^T \lambda_T = \lambda_0 \Pi_{j=1}^{t} R_j^{-1}$ or, $\beta^T \lambda_T = \lambda_0 q_T$. Thus, the transversality condition —which we imposed as an additional condition to be satisfied by any candidate allocation— implies $\lim_{T\to\infty} q_T b_{T+1} = 0$. Next taking limits on both sides of (4.4) as $T \to \infty$ we get,

$$\sum_{t=0}^{\infty} q_t(c_t + x_t) = \sum_{t=0}^{\infty} q_t(w_t + r_t k_t) + q_0 R_0 b_0. \tag{4.5}$$

Next we want to show that under reasonable conditions (4.5) is equivalent to (4.3). To do this it suffices to show that,

$$\sum_{t=0}^{\infty} q_t(r_t k_t - x_t) = \sum_{t=0}^{\infty} q_t(r_t k_t - k_{t+1} + (1-\delta)k_t) = q_0[1 - \delta + r_0]k_0.$$

To prove this, rearrange the expression in the middle to read,

$$\lim_{T\to\infty} \left\{ q_0[1-\delta+r_0]k_0 + k_1[-q_0 + q_1(1-\delta+r_1)] + ...k_T[-q_{T-1} + q_T(1-\delta+r_T)] - q_T k_{T+1} \right\}.$$

Consider first any term of the form $-q_t + q_{t+1}(1 - \delta + r_{t+1})$. If this term is strictly positive then the optimal choice of k_{t+1} is infinite. The reason is simple: it allows infinite consumption without an infinite cost. Put differently, any constant returns to scale activity (in this case a linear activity) cannot generate positive profits because, if it does its optimal scale is infinite. However, an infinite demand for capital is inconsistent with an equilibrium. Thus, any candidate equilibrium price system must have $-q_t + q_{t+1}(1 - \delta + r_{t+1}) \le 0$. If this quantity is strictly negative the optimal level of k_{t+1} is zero. This cannot be an equilibrium under the Inada condition which we have assumed. Thus, we conclude that for all $t \ge 0$, $-q_t + q_{t+1}(1 - \delta + r_{t+1}) = 0$. Next, consider the term $\lim_{T\to\infty} q_T k_{T+1}$. The same argument that we used for b_{T+1} (i.e. the transversality condition) implies that this limit is zero. Thus, our desired result follows. ■

This result simply says that under our assumptions we can either work with an infinite sequence of budget constraints or, simply, one present value budget constraint. Even though the present value version of the budget constraint is often interpreted (and correctly so) to imply the existence of infinitely many markets for future consumption —markets that we do not seem to observe in the real world— our argument shows that the existence of a market for bonds is sufficient to replicate the infinitely many markets for consumption and labor at different points in time that are assumed in the standard formulation of competitive equilibria.

We next show that in any equilibrium it must be the case that, for all t, $b_t^* = 0$. Formally,

Claim 61 *In any competitive equilibrium* $\mathbf{b}^* = 0$*.*

Proof. Consider the budget constraint of the household. It is given by,

$$w_t^* + r_t^* k_t^* + R_t^* b_t^* = c_t^* + x_t^* + b_{t+1}^*,$$

or, using the equilibrium values of (w_t^*, r_t^*),

$$f(k_t^*) - k_t^* f'(k_t^*) + f'(k_t^*) k_t^* + R_t^* b_t^* = c_t^* + x_t^* + b_{t+1}^*.$$

However, in any equilibrium,

$$f(k_t^*) = c_t^* + x_t^*,$$

it follows that the budget constraint for the "private sector" implies

$$R_t^* b_t^* = b_{t+1}^*.$$

Since in any equilibrium $b_0 = 0$, the result follows trivially

Note: Even though this result uses the assumption $b_0 = 0$, it is easy to extend it to an economy with heterogeneous agents. In such a case what can be proven —by adding all the budget constraints— is that the sum of b_{it}^* over all households i is zero. ■

We next show that any competitive equilibrium is a solution to the planner's problem. Formally, we prove,

Claim 62 *Let* $[(\mathbf{w}^*, \mathbf{r}^*, \mathbf{p}_k^*, \mathbf{R}^*), (\mathbf{c}^*, \mathbf{x}^*, \mathbf{k}^*), \mathbf{b}^*]$ *be an interior competitive equilibrium, then* $(\mathbf{c}^*, \mathbf{x}^*, \mathbf{k}^*)$ *solves the planners problem (P.1).*

Proof. In Claim 61 we showed that $b^* = 0$ in any equilibrium. From the consumer budget constraint it follows that,

$$\begin{aligned} f(\mathbf{k}^*) &= \mathbf{c}^* + \mathbf{x}^* \\ (1-\delta)\mathbf{k}^* + \mathbf{x}^* &= \mathbf{k}'^*, \end{aligned}$$

where $\mathbf{k}'^*$ is the vector of capital stocks that has as its first element k_1^*. It follows that the feasibility constraints of the planner's problem are satisfied. To complete the proof suffices to show that the Euler equation and the transversality conditions are satisfied. It is sufficient to show that,

$$\begin{aligned} u'(c_t) &= \beta u'(c_{t+1})[1-\delta+f'(k_{t+1})] \\ \lim_{T\to\infty} \beta^T u'(c_T)k_{T+1} &= 0. \end{aligned}$$

The first of these conditions is the Euler equation satisfied by the solution to the consumer problem (see (4.1)), while the second condition is the transversality condition corresponding the household utility maximization problem. This completes the proof. ■

Even though we stated the result for the interior case —the most relevant in macro— it can be extended to the boundary case. However, in this situation one needs to verify some infinite dimensional version of Slater's condition and this turns out to be quite involved. Next we prove the converse: Given a solution to the planner's problem, it is possible to find prices such that such an allocation is a competitive equilibrium allocation.

Claim 63 *Let $(\mathbf{c}^*, \mathbf{x}^*, \mathbf{k}^*)$ be an interior solution to the planner's problem (P.1). Then there exist prices, $(\mathbf{w}^*, \mathbf{r}^*, \mathbf{p}_k^*, \mathbf{R}^*)$, and a sequence of bond holdings, $\mathbf{b}^*$, such that $[(\mathbf{w}^*, \mathbf{r}^*, \mathbf{p}_k^*, \mathbf{R}^*), (\mathbf{c}^*, \mathbf{x}^*, \mathbf{k}^*), \mathbf{b}^*]$ is a competitive equilibrium of an economy with a representative agent who has initial financial wealth $b_0 = 0$, and initial capital holdings $k_0 > 0$ equal to the initial endowment of capital of the economy.*

Proof. Let $\mathbf{b}^* = 0$. Since the solution to the planner's problem is feasible, conditions c) and d) of the definition of competitive equilibrium are automatically satisfied. We need to check that the candidate allocation is both utility and profit maximizing (i.e. it solves Problems HH.1 and FF.1). Let prices be given by,

$$\begin{aligned} \mathbf{w}^* &= f(\mathbf{k}^*) - \mathbf{k}^* f'(\mathbf{k}^*) \\ \mathbf{r}^* &= f'(\mathbf{k}^*) \\ \mathbf{R}^* &= 1-\delta+f'(\mathbf{k}^*). \end{aligned}$$

It is clear that at these prices, the firm's first order conditions for profit maximization are satisfied (i.e. the candidate allocation solves Problem FF.1). To check that the consumer's first order conditions are satisfied it suffices to verify that (4.1) holds because, by definition, R_t^* is chosen to satisfy (4.2) if (4.1) is satisfied. However, because the allocation solves the planner's problem it satisfies (2.2b). However, (2.2b) and (4.1) are identical since $\lambda_t = u'(c_t)$. It follows that (4.1) holds. Finally, we need to check that our candidate solution satisfies the TVC of the consumer's problem. One of them —the one involving b_t— is trivially satisfied by our choice of $\mathbf{b}^* = 0$. The second is equivalent to the planner's transversality condition given by (2.2c) since $\lambda_t = u'(c_t)$. This completes the argument. ■

Can this result be extended to the case of many heterogeneous consumers? Even though we do not deal with this in macro very often, the proper statement of the result is that, given the appropriate convexity in feasible sets and preferences, *any* allocation that solves the planner's problem —which in this case involves weighted averages of individual utility functions— can be supported as a competitive equilibrium outcome provided that the planner can redistribute initial income. For a general statement of this result see Debreu (1954).

To summarize, in this section we have shown that the solution to the planner's problem is also a competitive equilibrium allocation even in the case in which there are no "future" markets and households are restricted to a set of markets that allows only one period ahead trades.

Exercise 64 *Consider a firm that has a production function given by $F(x)$, where $x \in \Re^n_+$. Assume that the function F is strictly increasing in each argument, concave, twice continuously differentiable and homogeneous of degree one. Let the price of the good produced be given and equal to p, and let the vector of input prices be $w \in \Re^n_+$.*

i) Show that if the firm acts as a price taker profits are zero.

ii) Show that if the firm has some market power in the sense that the market price depends on the quantity produced by this firm (i.e. $p = g(q)$, where $q = F(x)$ is the quantity produced) then profits are not necessarily zero.

iii) Show that if there are N firms with the same production function, that are at an interior point in the input space and facing the same input prices, then given the total amount of all inputs used, the total output of the industry is independent of N, regardless of whether they behave as price takers.

iv) In what sense does the previous result justify the usual assumption that there is only one firm, i.e. N=1

Exercise 65 *Consider an economy in which there are two capital stocks, physical and human capital, k and h , respectively. The feasibility constraints are given by,*

$$\begin{aligned}
c_t + x_{kt} + x_{ht} + g_t &\leq F(k_t, h_t), && t = 0, 1, ... \\
k_{t+1} &\leq (1 - \delta_k)k_t + x_{kt} && t = 0, 1, ... \\
h_{t+1} &\leq (1 - \delta_h)h_t + x_{ht} && t = 0, 1, ... \\
(c_t, x_{kt}, x_{ht}, k_{t+1}, h_{t+1}) &\geq (0, 0, 0, 0, 0) \\
k_0 &> 0 \text{ and } h_0 > 0 \text{ given.}
\end{aligned}$$

The function F is strictly increasing, strictly concave, twice differentiable, and such that the marginal product of both forms of capital converge to zero as the stocks go to infinity. Assume that $g_t = 0$, for all t. The function F is some sort of reduced form, with the "real" production function is given by $\hat{F}(k, h, a)$, where a is interpreted as land which is in fixed supply. We assume without loss of generality that the supply of land per capita is normalized to one. The function $\hat{F}$ is assumed to be strictly increasing in each argument, twice differentiable and *concave. It is also assumed to display constant returns to scale in all three inputs.*

i) Define a competitive equilibrium for this economy. Go as far as you can describing the behavior of prices.

ii) Assume that $\delta_h = \delta_k$ *and that the technology is given by* $F(k, h) = zk^{\alpha}h^{\epsilon}$, *with* $0 < \alpha, \epsilon < 1$, *and* $\alpha + \epsilon < 1$. *Go as far as you can describing the equilibrium path of* w_t/r_t *—the wage rate relative to the rental rate of capital.*

iii) Consider two economies that are identical except for their "scale." The large economy —economy A— has initial capital stocks (k_A, h_A), *while the small economy has initial capital stocks given by* $(k_B, h_B) = b(k_A, h_A)$, *for some number* $0 < b < 1$. *In addition, assume that* (k_A, h_A) *are below their steady state values. Go as far as you can comparing the predictions of the model for the wage rates in the two countries. In particular, consider the differences in wage rates between two individuals who have the same level of human capital –say* h_r*– but one of them lives in A, and the other in B. (I know this does not seem to agree with the representative story, but representative is not essential.) Let the wage rate –the rental price of human capital– be* w_A *and* w_B. *Thus, the wage per hour for these individuals is* $w_A h_r$, *and* $w_B h_r$, *respectively. Which one of these two individuals earns a higher hourly wage? Do you think that in this model there are incentives for individuals with the same level of human capital to migrate?*

iv) Consider two individuals each living in a different country which we denote by $i, i = A, B$. *Let one of them have* m *times more human capital than the other. What does this theory say about their relative wage rates? Are these different between rich and poor countries?*

Exercise 66 *Consider a standard one sector growth model. Preferences of the representative household are given by,*

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \qquad 0 < \beta < 1.$$

The function u *is assumed concave, increasing and twice continuously differentiable. The representative household is endowed with one unit of time per period and it does not derive utility from leisure. The household owns the capital stock and rents it to firms. There is tax levied on net capital income at the rate* τ. *Thus, the household's budget constraint —in sequence form— is,*

$$\begin{aligned} w_t + r_t k_t - \tau(r_t - \delta)k_t + R_t b_t &\geq c_t + x_t + b_{t+1} \\ (1-\delta)k_t + x_{kt} &\geq k_{t+1}, \end{aligned}$$

where r_t *is the rental price of capital,* R_t *is the gross interest rate on bonds, and* w_t *is the wage rate. Note that the base of capital income tax is rental income net of economic depreciation. Firms maximize profits. They hire capital and labor and face the following technological constraint:*

$$c_t + x_t + g_t \leq zF(k_t, n_t),$$

and it is assumed that F is homogeneous of degree one, concave, twice differentiable and increasing in each argument. The function $zf(k) \equiv zF(k,1)$ satisfies $\lim_{k\to 0} zf'(k) = \infty$ and $\lim_{k\to\infty} zf'(k) = 0$. In addition, consumption, investment, government spending and the capital stock must be nonnegative. Assume that given the sequence of tax rates, the sequence $\{g_t\}$ satisfies $g_t = \tau(r_t - \delta)k_t$.

i) Define a competitive equilibrium for this economy.

ii) Argue that there is steady state. If necessary provide additional conditions for its existence. Is it unique?

iii) Suppose that the economy is at the steady state and g is constant. Assume that the production function is Cobb-Douglas and equal to $zF(k,n) = zk^{\alpha}n^{1-\alpha}$. At $t = 0$, a new administration that announces a tax cut of ϕ% is voted into office. (Note that g is unchanged.) Basically a new tax rate τ' is announced, with $\tau' = \phi\tau (0 < \phi < 1)$. Any revenue shortfall will be made up using lump-sum taxes. Go as far as you can describing the impact of the tax cut on tax revenue at the "new" steady state. Is your estimate dependent on the size of capital's share of income (α in this example). If so, does a bigger α increase or decrease the impact of a tax cut on tax revenue.

Exercise 67 (Heterogeneity and Growth) *Consider an economy populated by a large number of households indexed by i. The utility function of household i is,*

$$\sum_{t=0}^{\infty} \beta^t u_i(c_t) \qquad 0 < \beta < 1.$$

where u_i is differentiable, increasing and strictly concave. Note that although we allow the utility function to be "household specific," all households share the same discount factor. All households are endowed with one unit of labor that is supplied inelastically.

Assume that in this economy capital markets are perfect and that households start with initial capital given by $k_{i0} > 0$. Let total capital in the economy at time t be denoted k_t and assume that total labor is normalized to 1.

Assume that there is a large number of firms that produce output using capital and labor. Each firm has a production function given by $F(k,n)$ which is increasing, differentiable, concave and homogeneous of degree one. Firms maximize the present discounted value of profits. Assume that initial ownership of firms is uniformly distributed across households.

i) Define a competitive equilibrium.

ii) Discuss the following two statements and justify your answer. In doing so, be as formal as you can:

Statement a) *"Economist A argues that the steady state of this economy is unique and independent of the u_i functions, while B says that without knowledge of the u_i functions it is impossible to calculate the steady state interest rate."*

Statement b) *"Economist A says that if k_0 coincides with the steady state aggregate stock of capital, then the pattern of "consumption inequality" will mirror exactly the pattern of "initial capital inequality" (given by the distribution of*

k_{i0}), even though capital markets are perfect. Economist B argues that for all k_0, in the long run, per capita consumption will be the same for all households."

iii) Assume that the economy is at the steady state. Describe the effects of these three policies.

a) At time zero, capital is redistributed across households (i.e. some people get taxed and others get their capital).

b) Half the households are required to pay a lump sum tax. The proceeds of the tax are used to finance a transfer program to the other half the population.

c) Two thirds of the households are required to pay a lump sum tax. The proceeds of the tax are used to finance the purchase of a public good –say g– which does not enter in either preferences or technology.

Exercise 68 (Natural Resources Monopolies, Land Prices and Output) *In this problem you will explore the impact of economic liberalization in an economy in which a monopolist controls the supply of a natural resource.*

Consider an economy populated by N identical individuals (farmers) and a monopolist who owns the supply of a natural resource. (Pick your friendly dictator here.) All $N+1$ individuals have the same preferences given by,

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \qquad 0 < \beta < 1.$$

where u is a strictly concave, increasing, and twice differentiable utility function. (If necessary make assumptions to guarantee interiority.), and c_t is per-capita consumption. Each farmer is endowed with a units of land.

Output is produced using land and natural resources. There is no capital in this economy. There is only one consumption good which is produced according to the following production function,

$$y \leq zF(a, \gamma n),$$

where z is a productivity factor, a is land per farmer, and n is the amount of natural resource per farmer. Assume that F is strictly concave, twice differentiable, with positive partial derivatives. Consider a closed economy in which farmers can rent —in a spot market— natural resources. In every period, farmers can buy and sell land and bonds, rent natural resources, and consume.

We assume that there is a monopolist that owns $N\bar{n}$ units of a (perfectly durable) natural resource. This monopolist decides in each period how much of the natural resource to make available. Thus, in every period, the monopolist announces the amount available per farmer for that period, say n^*(this means that total supply is Nn^*). The monopolist maximizes utility (which coincides with the present discounted value of the rents). Given n^*, all farmers participate in competitive markets.

i) Describe the equilibrium in this economy. Make sure that you make enough assumptions about F to guarantee that an equilibrium exists. Assume that, in equilibrium, $n^* < \bar{n}$.

ii) How would the equilibrium that you described in i) change if the monopolist sold —once and for all— part or all of his stock of the natural resource at time zero? [Assume that the monopolist cannot make sales at a latter date.]

iii) An economist at the World Bank indicates that the resource is underutilized, and that opening up the economy to international capital flows (but not to trade in the resource) would solve the problem. Do you agree? [To answer this question assume that the world interest rate, R^*, satisfies $\beta R^* = 1$]

iv) Consider the economy in i). Assume that at time $J > 0$ there is an unexpected permanent increase in γ, say $\gamma' > \gamma$. What is the effect of this form of technological progress on land prices, the equilibrium supply of the natural resource, its rental price, and total output? Would your answer be the same if the economy was open and the world interest rate, R^*, satisfied $\beta R^* = 1$? [Make additional assumptions if necessary]

v) Consider the economy in i). Suppose that at $t = 0$ it is announced that at time $T > 0$, the natural resource will be nationalized and the full supply, $N\bar{n}$, will be made available for production. Describe, as precisely as you can, the monopolist's optimal decision about how much to supply (per farmer) to the market at time t, n_t^*, for $t = 0, 1, ..., T-1$. What are the implications for output, land prices and interest rates from $t = 0$ on?

vi) What are your predictions about the effects on output and land prices of technological change of the type discussed in iv) after time T. [Basically using the notation we have used so far, $J > T$]

Extra Credit*: Consider the economy in i) and assume that the equilibrium supply is $n^* < \bar{n}$. An economist at the IMF claims that allowing for a moderate amount of imports of the natural resource, say $\tilde{n} < n^*$, will provide a "competitive fringe" and will bring about an increase in output and a decrease in the price of the natural resource. Discuss.*

Exercise 69 (Taxes and Growth) *Consider a simple two planner economy. The first planner picks "tax rates" (we denote the tax rate by τ) and makes transfers to the representative agent (which we denote v_t). The second planner takes the tax rates and the transfers as given. That is, even though we know the connection between tax rates and transfers, the second planner does not, he/she takes the sequence of tax rates and transfers as given and beyond his/her control when solving for the optimal allocation. Thus the problem faced by the second planner (the only one we will analyze for now) is,*

$$\max_{\{c_t\},\{x_t\},\{k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\begin{aligned}
c_t + x_t + g_t &\leq (1-\tau_t) f(k_t) + \nu_t, \\
k_{t+1} &\leq (1-\delta) k_t + x_t, \\
k_0 &> 0 \qquad \text{given}, \\
(c_t, x_t, k_{t+1}) &\geq (0,0,0) \qquad t \geq 0.
\end{aligned}$$

It is assumed that the functions u and f have the same properties as in the general problem (problem (P.1)).

i) Assume that $0 < \tau_t = \tau < 1$ (constant tax rates), and that $v_t = \tau f(k_t)$ (remember that we know this, but the second planner takes v_t as given at the time he/she maximizes). Show that there exists a steady state, and that for any initial condition $k_0 > 0$ the economy converges to the steady state.

ii) Assume now that the economy has reached the steady state you analyzed in i). The first planner decides to change the tax rate to $0 < \tau' < \tau$. (Of course, the first planner and us know that this will result in a change in v_t; however, the second planner —the one that solves the maximization problem— acts as if v_t is a given sequence which is independent *of his/her decisions.) Describe the new steady state as well as the dynamic path followed by the economy to reach this new steady state. Be as precise as you can about, consumption, investment and output.*

iii) Consider now a competitive economy in which households —but not firms— pay income tax at rate τ_t on both labor and capital income. In addition, each household receives a transfer, v_t, that it takes to be given and independent *of its own actions. Let the aggregate per capita capital stock be k_t. Then, balanced budget on the part of the government implies $v_t = \tau_t(r_t k_t + w_t n_t)$, where r_t and w_t are the rental prices of capital and labor, respectively. Assume that the production function is $F(k, n)$, with F homogeneous of degree one, concave and "nice." In this setting $F(k, 1) \equiv f(k)$. Go as far as you can describing the impact of the change described in part ii) upon the equilibrium interest rate.*

Exercise 70 (Land Prices, Slave Prices, Emancipation and Growth) *Consider an economy in which output is produced using capital, k_t, land, a_t, and labor, n_t according to the production function,*

$$y_t \leq B k_t^{\alpha} a_t^{\theta} n_t^{1-\alpha-\theta}, \qquad 0 < \alpha, \theta < 1 \qquad \alpha + \theta < 1,$$

The workforce in this economy, n_t, is composed of free workers, z_t, and slaves, s_t. Thus, at any time t, $n_t = s_t + z_t$. Assume that landowners rank consumption streams according to the function,

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \qquad 0 < \beta < 1,$$

where u is differentiable, increasing and strictly concave, and c_t is consumption per landowner family. Assume that each landowner can buy and sell both *land and slaves. Thus, landowners are also slave-owners. If s_t slaves are used in production at time t, net output is*

$$B k_t^{\alpha} a_t^{\theta} n_t^{1-\alpha-\theta} - m s_t,$$

where m is the minimum consumption needed to keep each slave alive. Assume that both new slaves and new land purchased at time t are not productive until time $t+1$. Let the supply of land per landowner be $\bar{a}$ (you may assume that, in *equilibrium, all landowners end up with the same amount of land). Let the total number of free workers per landowner be* $\bar{z}$*, and let the wages of free workers be denoted by* w_t*.*

To simplify, assume that only landowners face a truly dynamic problem: at any time t*, they decide how many free workers to hire, how many slaves to buy or sell, how many acres of land to buy or sell, and how much of the output is invested in capital in order to maximize their utility. Assume that slaves and free workers consume in every period their "income,"* m *and* w_t*, respectively.*

i) Consider an economy that is open to trade in slaves, but closed to international capital movements (no international bond market). Let the international (given to the economy) price of slaves be p_s^**. Let the price of land at time* t *be* p_{at}*. Describe the problem faced by landowners.*

ii) Go as far as you can describing the steady state (assume that it exists) of the economy.

iii) Compare this economy with another economy that has an endowment of $\bar{a}'$ *units of land, with* $\bar{a}' > \bar{a}$*. Go as far as you can describing the impact of the higher endowment of land upon:*

a) the steady state stock of slaves per landowner, s^**.*

b) the steady state capital stock per landowner, k^**.*

c) the steady state wage rate, w^**.*

d) the steady state level of output per landowner, $Bk^{*\alpha}a^{*\theta}n^{1-\alpha-\theta}$*.*

iv) Compare this economy with another economy with $\bar{z}'$ *free workers, with* $\bar{z}' > \bar{z}$*. Go as far as you can describing the impact of the higher number of free workers upon the steady state stock of slaves,* s^**, the steady state capital stock* k^**, and the steady state wage rate,* w^**. Show your work.*

v) Go as far as you can describing the impact on the steady state of an increase in the price of slaves, p_s^**.*

vi) Let the economy be at the steady state that you described in ii). Assume that at $t = 0$ *there is an* unexpected *change in the legal status of slaves: all former slaves are declared free workers. Go as far as you can describing the dynamic path of output after emancipation (i.e. after* $t = 0$*). Determine, if possible, the impact of emancipation upon land prices and wage rates. Do you have enough information to determine if free workers and or landowners would have favored emancipation?*

vii) ***Extra Credit****: Suppose that, at time* 0*, the economy starts at* $k_0 < k^*$*, where* k^* *is the level of capital corresponding to the steady state analyzed in ii). At that time, it is announced (this is an* unexpected *shock) that the maximum number of slaves per landowner will be capped at* $\hat{s}$ *where* $\hat{s} < s^*$*. Go as far as you can determining the effect of this trade restriction on the steady state value of landowner's wealth (land plus capital plus slaves), and on the steady state wages of free workers.*

Exercise 71 (Habit Persistence II) *Consider an economy populated by a* *large number of households with utility functions given by*

$$\sum_{t=0}^{\infty} \beta^t u(c_t - \phi z_t), \qquad 0 < \beta < 1,$$

where u is twice differentiable, increasing and strictly concave (if necessary you may assume that it satisfies the Inada condition). The variable c_t is individual consumption, and z_t is a measure of lagged consumption. To be precise,

$$z_t \geq \sum_{j=0}^{\infty} (1-\delta_c)^j c_{t-1-j}.$$

It follows that, alternatively, it is possible to describe the law of motion for z_t as

$$z_{t+1} \geq (1-\delta_c) z_t + c_t.$$

In this setting, z_t is a measure of 'habit persistence,' as it implies that the marginal utility of any given level of consumption decreases the higher the level of past consumption. The technology in this economy is standard and given by

$$\begin{aligned} c_t + x_t &\leq f(k_t), \\ k_{t+1} &\leq (1-\delta_k) k_t + x_t, \end{aligned}$$

where the functions f is strictly concave, increasing and satisfies Inada conditions.

i) Let the planner maximize the utility of the representative agent subject to all the feasibility constraints. Argue that, under some condition on (ϕ, δ_c) an ***interior*** *steady state exists and is unique. Describe the condition that (ϕ, δ_c) has to satisfy.*

ii) What does the model say about the impact of cross-country differences in how much people care about past consumption —as measured by ϕ— on the steady state output per worker.

iii) Define a competitive equilibrium in which, in each period, households trade (at least) one period bonds, capital, consumption and investment goods. Assume that consumption is taxed at the rate τ, that is, the cost of purchasing c units of consumption is $(1+\tau)c$. The revenue produced by this tax is rebated in a lump-sum fashion to the households.

iv) Economist A argues that current consumption produces an 'externality,' in the sense that it lowers the marginal utility of future consumption. Given this, he/she suggests that a tax on consumption (with proceeds rebated to the consumer in lump-sum fashion) will guarantee that the ***steady state*** *of the competitive equilibrium of this economy will coincide with the* ***steady state*** *of the planner's problem, and that the tax rate that attains this equality of the two steady states minimizes the value of z_t (at the steady state). Go as far as you can analyzing this claim.*

Problem 72 ***Note:*** *Section iii) requires you to describe the steady state version of the competitive equilibrium with constant consumption taxes in this economy. You need not need to show that every competitive equilibrium converges to a steady state. It suffices to assume that a steady state exists, and derive its properties.*

4.2 Taxes and Deficits

The impact of budget deficits on the economy (both on the allocation and on prices) is one of the traditional questions that macroeconomists are interested in. In this section we derive the implications of our model for changes in the time structure of taxation (deficit policy) holding the level of expenditures constant. The result that we will prove —which is sometimes attributed to David Ricardo and, in its modern reincarnation was described by Barro (1974)— is that budget policy does not matter in the sense that for a given sequence of government expenditures $\{g_t\}$ all *nondistortionary* tax structures that raise the appropriate level of revenue are associated with the same real equilibrium. Thus the timing of tax collection is irrelevant. The irrelevance result implies that the allocations and prices (e.g. the wage rate and the real interest rate) are independent of the specific timing of taxes. This result is often described as the Ricardian proposition.

Consider our basic economy, and let a feasible sequence $\mathbf{g} = \{g_t\}$ be given. Let us assume that the government uses lump sum taxes to finance this sequence of expenditures as well as any initial outstanding debt. (In this section we allow the government to issue debt.) Let this sequence of taxes be $\boldsymbol{\tau} = \{\tau_t\}$. The government now considers another sequence of taxes $\bar{\boldsymbol{\tau}} = \{\bar{\tau}_t\}$, such that $\bar{\tau}_0 < \tau_0$ —that is, taxes are lowered, and the time zero deficit increases. Will this stimulate the economy? Will consumption increase? Will interest rates increase? Will consumers buy fewer durable goods?

In order to explore these issues it is convenient to work with the present value version of both the household and the government budget constraint. The relevant version of equation (4.3) is,

$$\sum_{t=0}^{\infty} q_t c_t = \sum_{t=0}^{\infty} q_t (w_t - \tau_t) + q_0[1 - \delta + r_0]k_0 + q_0 R_0 b_0 \tag{4.6}$$

where now we do not require $b_0 = 0$. If $b_0 > 0$, the private sector has an aggregate claim on the government (government debt is positive as in the U.S.), while if $b_0 < 0$ the government is a net lender (e.g. Norway in the early 90s). Now let's consider the government's budget constraint. In its recursive or "flow" form it simply says that expenditures —given by purchases and repayment of principal and interest on existing debt— must equal resources —given by tax revenue and bond issues. To simplify the algebra we assume that the government issues only one period bonds. In Exercise (73) the reader is asked to show that the maturity structure of the government debt does not matter. Thus the government's flow budget constraint is,

$$b_{t+1} + \tau_t = g_t + R_t b_t \qquad t = 0, 1, ... \tag{4.7}$$

Using the same manipulations we used to prove Claim 60, it is easy to show that putting together T of these budget constraints and eliminating redundant terms we get,

$$\sum_{t=0}^{T} q_t(\tau_t - g_t) = q_0 R_0 b_0 - q_T b_{T+1}.$$

Since we showed (see Claim 60) that $\lim_{T\to\infty} q_T b_{T+1} = 0$, it follows that taking the limit as $T \to \infty$ in the previous equation one obtains,

$$\sum_{t=0}^{\infty} q_t(\tau_t - g_t) = q_0 R_0 b_0. \tag{4.8}$$

It follows that, unless the government changes its consumption sequence or it defaults on its initial debt —a possibility that we ignore for now— all feasible tax sequences $\boldsymbol{\tau}$ and $\bar{\boldsymbol{\tau}}$, given the equilibrium prices, must satisfy

$$\sum_{t=0}^{\infty} q_t \bar{\tau}_t = \sum_{t=0}^{\infty} q_t \tau_t = \sum_{t=0}^{\infty} q_t g_t + q_0 R_0 b_0. \tag{4.9}$$

Next, use (4.8) in (4.6) to get

$$\sum_{t=0}^{\infty} q_t c_t = \sum_{t=0}^{\infty} q_t(w_t - g_t) + q_0[1 - \delta + r_0]k_0. \tag{4.10}$$

Equation (4.10) is a first step in proving our claim. It shows that given a sequence of government spending, any feasible sequence of taxes (and feasible means satisfying (4.8)) leaves the household's budget constraint unchanged. Thus, if prices remain unchanged (a big if for now), the household faces *exactly* the same budget set now as it did before, even though the tax sequence may have changed.

We now show that the equilibrium prices are indeed the same. The argument is very simple and consists of making a guess and then verifying it. (Incidentally, this guess and verify approach has a long history in science and economics; it is a great —and totally correct from a formal point of view— approach; the only problem is that it is not available all the time.) So, we guess that, after a change in the tax sequence (say, moving from $\boldsymbol{\tau}$ to $\bar{\boldsymbol{\tau}}$) prices are unchanged. Given this guess, and since the budget constraint (4.10) is unchanged, consumers demand the same quantities as before. Similarly, given the assumption that prices have not changed, firms supply the same quantities as before. If we can show that markets clear, we are done. (In fact, we have just gone trough the definition of competitive equilibrium.) Market clearing is summarized by the two feasibility constraints,

$$\begin{aligned} c_t + x_t + g_t &\leq f(k_t) \\ k_{t+1} &\leq (1-\delta)k_t + x_t. \end{aligned}$$

However, since $\{g_t\}$ is unchanged, if the "old" quantities satisfied these constraints so do the "new" quantities which, we guessed, are just the same. This completes the argument.

What is going on? Very simply, if two alternative tax regimes are feasible in the sense of satisfying (4.8) it must be the case that they also satisfy (4.9). In other words, both tax sequences must have exactly the same present value. From the consumer's budget constraint, it is clear that only the present value of taxes matter. Thus, alternative tax sequences with the same present value do not change the set of feasible choices.

Let us assume that there is a tax cut at $t = 0$. We know from (4.9) that this will be followed by a tax increase in the future to keep the present value of tax revenue constant. At $t = 0$ the government has to issue more bonds. Its budget constraint is,

$$\bar{b}_1 + \bar{\tau}_0 = g_0 + R_0 b_0,$$

which implies that

$$\bar{b}_1 - b_1 = \tau_0 - \bar{\tau}_0,$$

which is just the statement that a tax cut is financed by higher borrowing. Thus, the demand for loans on the part of the government has increased. Why is it that the interest rate —intuitively the price of loans— does not increase as well? Recall that the household knows that it is not any richer and hence it does not change either its consumption level or its investment in capital, which is driven by the equality between the marginal product of capital and the market interest rate (see (4.1) and (4.2)). It must then decide to save more. For example, this takes the form of an increase in the household's savings account balance. Thus, the supply of loans increases (given the increase in loanable funds). Our arguments show that the household increases its savings in the form of higher demand for deposits exactly by the same amount that the government increases its demand for loans. This is the reason why the interest rate does not change.

It is easy to show that changes in the sequence $\{g_t\}$ will have real effects. This follows from the feasibility constraint. Thus, this model does not imply that what governments do is irrelevant. It just says that, under some conditions, the timing of taxes is irrelevant. Even though we were very explicit about the conditions under which the result holds, the exercises at the end of this section explore the role of some of these assumptions.

Exercise 73 (The Maturity Structure of Government Debt) *Let the government issue two types of bonds —one period and two period bonds. The relevant version of the government budget constraint is,*

$$p_{1t}b_{1t+1} + p_{2t}b_{2t+2} + \tau_t = g_t + b_{1t} + b_{2t},$$

where b_{is} is the amount of i period bonds that will mature at time s. Note that in this formulation we are assuming that government bonds are zero coupon bonds. In other words, when they are first issued (say, at time t), bonds of maturity i with face value equal to 1 sell for p_{it}. At time 0 there is an initial stock of both types of bonds.

i) Ignoring two period bonds for now, prove that $p_{1t} = R_{t+1}^{-1}$, *where* R_{t+1} *is the interest rate defined in (4.2).*

*ii) Derive an expression for the price of two period bonds. [*Hint*: Add two period bonds to the household's problem and derive the appropriate Euler equation. Remember that today's two period bond will be a one period bond tomorrow.]*

iii) Derive the present value forms of both the household and the government's budget constraint.

iv) Does the Ricardian proposition hold?

Exercise 74 *Consider the model described in Exercise* **??**. *Does the Ricardian proposition hold in this model? Explain.*

Exercise 75 *Consider the economy described in Exercise* **??**. *Does the Ricardian proposition hold in this economy? Explain.*

Exercise 76 *Consider the economy described in Exercise* **??**. *Does the Ricardian proposition hold in this economy? Explain.*

Exercise 77 *Consider a version of the economy described in Exercise 41 with one variation: The sequence* $\{\tau_t\}$ *is allowed to change in such a way that the present value of the sequence* $\{\upsilon_t\}$ *at the original prices remains unchanged. Will these changes in tax rates be neutral?*

Exercise 78 *[Hard?] Consider an economy just like the one described in Exercise* **??**. *Suppose that the government does not have access to lump-sum taxes and, instead, relies on bonds and labor income taxes.*

i) Go as far as you can formulating this problem as a competitive equilibrium.

*ii) Try to come up with some "pseudo planner's" problem whose solution is the competitive equilibrium with labor income taxes. [*Hint*: See the pseudo planner's problem in Exercise 41]*

iii) Do you expect any version of the Ricardian proposition to hold in this economy? Explain your argument.

Exercise 79 *Consider the basic model described in this section. Assume that consumers can save/lend but cannot borrow (thus* $b_t \geq 0$ *in the household's problem).*

i) Suppose that the government considers a policy that lowers period 0 *taxes, and increases period* 1 *taxes to retire the additional debt (i.e. the present value of taxes is constant at the original prices). Will this "deficit" policy have any effects upon the economy?*

ii) Consider a situation that is the opposite to the one described in i). The government increases taxes at $t = 0$, *and simultaneously announces tax cuts in period* 1. *Assume that, at the original prices —the prices prevailing before this change— the present value of taxes has not changed. Do you expect the Ricardian proposition to hold?*

iii) If you find differences between cases i) and ii), go as far as you can interpreting the role of "credit market imperfections" in your analysis.

Exercise 80 (Slavery and Land Prices) *In this problem you will explore the impact of emancipation of slaves upon land prices. This is a purely fictional exercise. Any resemblance to an actual economy is purely coincidental.*

Consider an economy populated by a large number of individuals. They all have the same preferences which are given by,

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

where $0 < \beta < 1$ *is a discount factor,* c_t *is consumption at time* t*, and* u *is a strictly concave, increasing, and twice differentiable utility function. (If necessary make additional assumptions to guarantee interiority. See however, the little twist about minimum consumption described later.) Some individuals are endowed with land (*a *units each) and —initially— with the rights to force other individuals work, i.e. they own slaves. Landowners do not work (they are worthless in production). Let each landowner own* n *slaves. Thus, the slave to land ratio is* $x = n/a$*. Slaves do not get paid salaries. Instead, in order to survive they have to consume at least* ϵ *units of consumption per period (we could "load" this requirement into the definition of* u*, but this is somewhat cumbersome). Assume that landowners allocate the minimum feasible consumption to their slaves. To make things simple assume that there are no free workers.*

Output is produced using land and workers. There is no capital in this economy. There is only one consumption good which is produced according to the following production function,

$$y \leq zf(a, n),$$

where z *is a productivity factor, and* f *is a concave, twice differentiable, with increasing marginal products, and homogeneous of degree one production function. Assume that* $zf(a, n) > \epsilon n$ *(in general, you may assume that* ϵ *is as small as desired). Consider a closed economy in which landowners have access to three markets in addition to the market for consumption: a land market, a credit market, and a slave market.*

i) Describe the competitive problem faced by a landowner.

ii) Go as far as you can describing the equilibrium. In particular, describe the interest rate and the factors that influence the prices of land and slaves.

iii) What is the effect of an increase in the productivity parameter z *(which in the real world could be a relative price) on land and slave prices?*

iv) Consider now an unanticipated *emancipation. At* $t = 0$ *all slaves are freed. Instead of a slave market, there is a labor market in which former slaves can sell their labor. Describe the new equilibrium. In particular, discuss what happens to the price of land and describe conditions under which former slaves are better off. Does emancipation affect interest rates or aggregate output? Discuss your results.*

v) Suppose that at $t = 0$ *the economy is in its "slave mode" and is in the steady state. At that point it is learned that at* $t = T$ *slaves will be emancipated.*

Go as far as you can describing the "time series" for output and land prices in three periods: before $t = 0$ *(the steady state of the slave economy),* $0 < t < T$ *(the slave economy which is known to disappear at* T*), and* $t > T$ *(the market economy).*

vi) Consider a slave economy which is in steady state. Suppose that new land becomes available. What is the effect of this land discovery upon land prices and slave prices?

vii) Suppose that to import a new slave costs k *units of consumption. Describe —from the point of view of slave owners— the optimal importation policy. Would slave owners like to restrict imports? If so, why?*

viii) Suppose that —in addition to slaves— there are m *free workers in the slave economy (*m *small relative to* n*). Do these free workers support emancipation of slaves? Do they support a free importation policy for slaves? Assume that these free workers have the same preferences as landowners and slaves.*

ix) [Hard?] Suppose that at time $t = 0$ *is learned that with probability* p *slaves will be emancipated the next period, and with probability* $1 - p$ *the slave regime continues. In subsequent years —conditional on not having emancipated the slaves— the situation is the same. In other words, there is a constant probability of emancipation in every period, until emancipation occurs, in which case it is irreversible. Describe how land and slave prices are affected by* p*.*

x) Suppose that there is capital in addition to land. The production function is homogeneous of degree one in all three factors. Go as far as you can describing the steady state of the slave and market economies. Go as far as you can describing the transition from the steady state of the slave economy to the steady state of the market economy.

xi) [Hard?] Consider the scenario described in ix) with the technology given in x). Consider a slave economy at its steady state. Suppose that at time $t = 0$ *it is learned that with probability* p *slaves will be emancipated in the next period. Go as far as you can describing the dynamics of capital accumulation in this case.*

xii) Consider a slave economy with capital but with no markets for slaves. Compare the steady state of the slave economy with the steady state of the market economy. Is it true that the slave economy displays capital under accumulation?

xiii) Consider the slave economy with capital. What is the optimal importation policy from the point of view of capital owners? What is the optimal policy if capital owners are landowners as well?

xiv) Consider an economy with two types of technology: a slave intensive technology (that requires a minimum plantation size equal to a^**), and a linear technology that can be operated at all plantation sizes. Go as far as you can describing technologies that "capture" this description. Is it possible to find an equilibrium in which both "large" and "small" plantations coexist?*

Exercise 81 (Terms of Trade Shocks) *Consider a standard optimal growth model with only one difference: A country only produces a good that it does not consume; that is, it sells all of its output in international markets, and it buys both consumption and investment in the "world" market. In addition, we assume* *that this country's planner can borrow and lend at the international interest rate given by r^*. Let the price, at time t, of the good produced (but not consumed) by this country be p_t. We normalize the price of the international good –the one consumed by this country– at one. Let b_{t+1} be the stock of international bonds purchased (issued if negative) by the planner at time t denominated in terms of the international good. These bonds will pay $(1+r^*)b_{t+1}$ units of international goods at time $t+1$. Assume that $\beta(1+r^*)=1$. The planner's problem for this economy is,*

$$\max_{\{c_t\},\{x_t\},\{k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t, g_t)$$

subject to

$$\begin{aligned} c_t + x_t + g_t + b_{t+1} &\leq p_t f(k_t) + (1+r^*)b_t, \\ k_{t+1} &\leq (1-\delta)k_t + x_t, \\ k_0 &> 0 \text{ and } b_0 \qquad \text{given}, \\ (c_t, x_t, k_{t+1}) &\geq (0,0,0) \qquad t \geq 0. \end{aligned}$$

*i) Formulate the optimal growth problem. Make sure that you describe both the Euler equations and the transversality condition (TVC). [*Note*: The planner takes the sequence $\{g_t\}$ as given.]*

*ii) Assume that $g_t = g$, and that $p_t = p_L$ for $t = 0, 1, ..., N-1$, and $p_t = p_H$ for $t = N, N+1, ...$, with $p_H > p_L$. Describe, as carefully as you can, the time path of consumption, investment and the trade balance. If we assume that $p_t = p_H, t \leq 0$, then the exercise captures the impact of a temporary deterioration of the terms of trade. [*Note*: In this exercise a country has a* trade surplus *if GDP is greater than the sum of consumption, investment and government spending. Thus, if $p_t f(k_t) - c_t - x_t - g_t$ is positive, this country has a trade surplus, and if it is negative it has a trade deficit.]*

*iii) Assume that $g_t = g$, and that $p_t = p_L$ for $t = 0, 1, ..., N-1$, **and** $t = N+T, N+T+1, ...$ For $t = N, N+1, ..., N+T-1$, it is assumed that $p_t = p_H$ with $p_H > p_L$. Describe, as carefully as you can, the time path of consumption, investment and the trade balance.*

iv) Consider the model in iii). You are asked to evaluate these two arguments:

a) When countries enjoy a temporary increase in their terms of trade, the optimal policy is to increase investment during the good times in order to have more capital (and, hence higher output) when the price of exports decreases.

b) An exogenous government spending policy given by "high g whenever p is low, and low g whenever p is high" (this is a countercyclical policy) results in higher level of output relative to a constant g policy.

Exercise 82 (Interest Rate Shocks) *Consider an economy populated by a large number of identical households that maximize the utility function,*

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\theta}}{1-\theta} \qquad 0 < \beta < 1, \theta > 0.$$

Each household is endowed with one unit of labor which is supplied inelastically. Output in this economy is produced using capital, k, and labor, n, with a concave, differentiable, and homogeneous of degree one function $F(k,n)$. Assume that firms and households behave competitively. This economy has access to an international bond market with gross interest rates given by $1+r_t$ where

$$\begin{aligned} 1+r_t &= \beta^{-1}(1+\gamma) \qquad t=0,1,...,T-1, \text{ and } \gamma>0 \\ 1+r_t &= \beta^{-1} \qquad t=T,T+1,... \end{aligned}$$

i) Describe the steady state of this economy.

ii) Describe the dynamic path of consumption, domestic interest rates and wage rates. Go as far as you can determining the effect of T upon the initial $(t=0)$ level of consumption.

iii) What is the relationship, if any, between:

1. *world interest rates and domestic output?*
2. *world interest rates and wages?*
3. *world interest rates and consumption growth?*
4. *consumption growth and domestic output?*

iv) Consider an economy that, at $t=0$, has a stock of capital per worker that is "small." In this economy there are two views about the optimal timing of globalization, given that the objective is to maximize the utility of the representative agent. One view –let's call it L– argues that the economy should wait until time T to allow its residents to borrow and lend in international capital markets, so that they can profit from the lower interest rates. The other view –let's call it R– is that the representative agent will be better off by opening up the economy immediately (i.e. at $t=0$). Which of these two views would you favor? Justify your answer.

4.3 The Structure of Taxation: Steady State Implications

In this section we introduce a variety of distortionary taxes, and we study how they affect the equilibrium allocation. For the most part attention is restricted to steady states. We assume that the representative household solves the following maximization problem

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_t) \tag{4.11}$$

subject to,

$$\begin{aligned} (1+\tau_t^c)c_t + (1+\tau_t^x)x_t + b_{t+1} \quad \leq \quad & r_t k_t - \tau_t^k (r_t - \delta_k) k_t - T_t \\ & +(1-\tau_t^n) w_t n_t + (1+(1-\tau_t^b) r_t^b) b_t \end{aligned} \tag{4.12}$$

where τ_t^j denotes the tax rate on item $j = c, x, b, k, n$ and T corresponds to lump-sum taxes. The stock of capital —which we assume is directly owned by the household— evolves according to the usual capital accumulation equation,

$$k_{t+1} = (1-\delta_k)k_t + x_t \tag{4.13}$$

where x_t denotes investment. Thus, in this economy there are consumption taxes, investment taxes, capital income taxes, bond interest income taxes and labor income taxes.

There is a representative firm that rents all inputs in spot markets. The firm maximizes profits. Thus, in every period, it solves

$$\Pi = \max_{k_t, n_t} F(k_t, n_t) - r_t k_t - w_t n_t,$$

where —as before— we assume that F is concave and homogeneous of degree one.

The Lagrangean for the household problem is,

$$\begin{aligned} L \;=\; & \sum_{t=0}^{\infty} \beta^t \{u(c_t, 1-n_t) + \lambda_t[r_t k_t - \tau_t^k (r_t - \delta_k)k_t + (1 + (1-\tau_t^b) r_t^b) b_t \\ & +(1-\tau_t^n) w_t n_t - (1+\tau_t^c)c_t - (1+\tau_t^x)(k_{t+1} - (1-\delta_k)k_t) - b_{t+1}]\} \end{aligned} \tag{4.14}$$

where the capital accumulation equation, (4.13), has been substituted in the budget constraint. The first order conditions for the household problem are the constraints and

$$c_t \; : \; u_c(t) = \lambda_t(1+\tau_t^c) \tag{4.15a}$$

$$k_{t+1} \; : \; (1+\tau_t^x)\lambda_t = \beta\lambda_{t+1}[(1-\delta_k)(1+\tau_{t+1}^x) + (1-\tau_{t+1}^k)r_{t+1} + \tau_{t+1}^k \delta_k] \tag{4.15b}$$

$$b_{t+1} \; : \; -\lambda_t + \beta[1 + (1-\tau_{t+1}^b) r_{t+1}^b]\lambda_{t+1} = 0 \tag{4.15c}$$

$$n_t \; : \; u_\ell(t) = \lambda_t(1-\tau_t^n) w_t. \tag{4.15d}$$

Profit maximization by firms imply that,

$$F_k(k_t, n_t) = r_t \tag{4.16a}$$

$$F_n(k_t, n_t) = w_t. \tag{4.16b}$$

Finally, aggregate feasibility implies that,

$$c_t + x_t + g_t \leq F(k_t, n_t).$$

Note that by Walras law it is not necessary to explicitly describe the government budget constraint as the consumer's budget constraint and aggregate feasibility imply that the government's accounts must be balanced as well.

4.3.1 Basic Steady State Analysis

In this section we assume that the economy has reached a steady state. Of course, this requires that all tax rates be constant as well. After substituting (4.16a) and (4.16b) in (**??**), the relevant marginal conditions are:

$$\rho + \tau^x(\rho + \delta_k) = (1 - \tau^k)(F_k(k, n) - \delta_k) \tag{4.17a}$$

$$u_\ell(c, 1-n) = u_c(c, 1-n)F_n(k, n)\frac{1 - \tau^n}{1 + \tau^c} \tag{4.17b}$$

$$F(k, n) = c + g + \delta_k k \tag{4.17c}$$

$$\rho = (1 - \tau^b)r^b. \tag{4.17d}$$

From a formal point of view the system of equations (4.20) contains 4 equations in four unknowns. We denote the *allocation* that solves this system as $(k(\tau), n(\tau), c(\tau))$, where τ is the vector of tax rates. It is clear that given quantities (c, k, n) compatible with the first three equations, (4.17d) just *defines* the interest rate on government bonds. This condition says that changes in the tax rate on interest income levied on government bonds has a direct impact on the interest rate that the government must pay but that it does not the after tax real rate of return or the steady state allocation.

Consider next (4.17a); it shows that —given quantities of k an n— there are a number of taxes on investment (τ^x) and capital (τ^k) that give rise to *exactly* the same outcome. For example, consider an economy that taxes capital at the rate τ^k_A and that it does not tax investment; i.e. $\tau^x_A = 0$. Consider now another economy that has a higher tax rate on capital income —say a higher tax rate on dividends— given by $\tau^k_B > \tau^k_A$. However, if economy B also imposes an investment subsidy given by,

$$\tau^x_B = \frac{(1 - \tau^k_B)(F_k(k, n) - \delta_k) - \rho}{\rho + \delta_k} \tag{4.18}$$

then *both economies have the same effective tax rate on capital income.* Note that since

$$0 = (1 - \tau^k_A)(F_k(k, n) - \delta_k) - \rho \tag{4.19}$$

and $\tau^k_B > \tau^k_A$, it follows that $\tau^x_B < 0$. Thus, the economy with the higher formal tax on capital income could have an effective lower tax rate on capital, provided that it subsidizes the purchases of investment goods. Even though in applications (that is, when dealing with real world economies) one must use the actual tax code, and equation (4.18), to calculate the impact of a tax system on the steady state, in these notes (i.e. for this model economy) we will take a shortcut: we will assume that whenever we have an economy with taxes given by $(\hat{\tau}^x, \hat{\tau}^k)$ —and, hence, that is such that its taxes and quantities of capital and labor satisfy (4.18)— we can transform it into an economy with taxes $(0, \tau^k)$ such that τ^k satisfies (4.19). In other words, we will transform a regime with both investment and capital income taxes into a regime with just capital income taxes.

Exercise 83 *Show that if the "undistorted" capital-labor ratio defined by $\kappa^* = k^*/n^*$ (where k^* and n^* are steady state capital and labor corresponding to the solution to the planner's problem) can be supported with a tax rate on capital income given by τ^k then, the tax (a subsidy really) rate on investment equals $-\rho\tau^k/(\rho+\delta_k)$*

From (4.17b) it follows that in order to determine the impact on steady state quantities all that matters is the ratio $\frac{1-\tau^n}{1+\tau^c}$. This shows that, in this simple setting, *a consumption tax is equivalent to tax on labor income.* As before, it pays to simplify: For any economy with consumption and labor income taxes given by $(\hat{\tau}^c, \hat{\tau}^n)$ one can define another economy with tax rates given by $(0, (\hat{\tau}^c + \hat{\tau}^n)/(1+\hat{\tau}^c)$ that has the property that the equilibrium quantities in the steady state coincide. Thus, it is possible to "transform" an actual tax code —that includes both consumption, or value added taxes— into a simplified tax code with only labor income taxes. In what follows I will work with the version of the economy in which τ^c is set equal to zero. However, it is useful to remember that a "real world" increase in consumption taxes is equivalent to a "model" increase in labor income taxes.

Finally, let us consider what happens when the government uses just lump-sum taxes and, consequently, it sets all other tax rates to zero. In this case, (4.20) is simply

$$\begin{aligned}
\rho &= F_k(k,n) - \delta_k \\
u_\ell(c, 1-n) &= u_c(c, 1-n)F_n(k,n) \\
F(k,n) &= c + g + \delta_k k \\
\rho &= r^b.
\end{aligned}$$

The solution to these equations —except for the interest rate on government bonds— coincides with the solution to the planner's problem, (k^*, n^*, c^*). Thus, in terms of steady state allocations, the triplet (k^*, n^*, c^*) yields the highest possible level of utility. This, of course, proves that lump-sum taxes support the optimal allocation; in other words, they are the "best" tax system.

Let the marginal rate of substitution be defined by $\Phi(c,\ell)$ where $\Phi(c,\ell) = u_\ell(c,1-n)/u_c(c,1-n)$. Then we assume:

Condition 84 (MRS) *The function $\Phi(c,\ell)$ is increasing in c and decreasing in ℓ.*

This assumption corresponds to the following inequalities

$$\begin{aligned}
\frac{u_{\ell\ell}u_c - u_\ell u_{c\ell}}{u_c^2} &< 0 \\
\frac{u_c u_{c\ell} - u_\ell u_{cc}}{u_c^2} &> 0
\end{aligned}$$

which are satisfied by standard utility functions. For example, a sufficient condition is that the cross partial, $u_{c\ell}$, be positive.

We are now ready to study the effects of changes in tax rates holding the level of government spending, g, constant. Thus, we interpret the change in tax rates as being matched by a change in transfers to the private sector (i.e. lump-sum taxes).

Substituting (4.17c) into (4.17b) and recalling that, by convention, we set $\tau^c = \tau^x = 0$, we have that the equilibrium is completely described by

$$\Phi(F(k,n) - \delta_k k - g, 1-n) = F_n(k,n)(1-\tau^n) \quad (4.21a)$$
$$(1-\tau^k)(F_k(k,n) - \delta_k) = \rho. \quad (4.21b)$$

The system (4.21) is a system of two non-linear equations and two unknowns. Totally differentiating this system we get,

$$\begin{bmatrix} \Phi_c(F_k - \delta_k) - (1-\tau^n)F_{kn} & \Phi_c F_n - \Phi_\ell - (1-\tau^n)F_{nn} \\ (1-\tau^k)F_{kk} & (1-\tau^k)F_{kn} \end{bmatrix} \begin{bmatrix} dk \\ dn \end{bmatrix} = \begin{bmatrix} -F_n d\tau^n \\ (F_k - \delta_k)d\tau^k \end{bmatrix}.$$

A little algebra shows that the determinant is given by

$$\Delta = \Phi_c(F_k - \delta_k)(1-\tau^k)F_{kn} - (1-\tau^k)F_{kk}(\Phi_c F_n - \Phi_\ell) > 0. \quad (4.22)$$

Exercise 85 *Prove that (4.22) is as stated. Hint: It is necessary to use the assumption that F is homogeneous of degree one.*

We can now explore the effects of changing the tax rates on capital and labor.

4.3.2 Changing the Tax Rate on Capital Income

Consider first the case in which $d\tau^n = 0$ and $d\tau^k > 0$. It follows that,

$$\frac{\partial k}{\partial \tau^k} = -\frac{(F_k - \delta_k)(\Phi_c F_n - \Phi_\ell - (1-\tau^n)F_{nn}}{\Delta} \quad (4.23)$$

and

$$\frac{\partial n}{\partial \tau^k} = \frac{(F_k - \delta_k)(\Phi_c(F_k - \delta_k) - (1-\tau^n)F_{kn})}{\Delta}. \quad (4.24)$$

It is clear that given condition (MRS) and equation (4.22) that $\partial k/\partial \tau^k < 0$. Thus, an increase at the rate at which capital is taxed decreases the stock of capital. Moreover, from (4.21b), and since homogeneity of degree one implies that $F_k(k,n)$ is a function of just the capital-labor ratio, $\kappa \equiv k/n$, it follows that increases in the tax rate on capital decreases the capital-labor ratio.

What happens to employment? Even though we have shown that the denominator in (4.24) is positive, the numerator can be either positive or negative. To see this possibility consider two extreme cases. First, assume that the production function is "almost" separable. Basically, if $F_{kn} \approx 0$, then $\partial n/\partial \tau^k > 0$. To see why this happens suppose to the contrary that as τ^k increases, n does not change. Since an increase in τ^k results in a decrease in k this must imply a decrease in consumption, given that government spending is constant (see (4.17c)). However, this decrease in c implies that $\Phi(c, 1-n)$ decreases (see

(MRS)). However, this results in (4.21a) being violated: the left hand side has decreased but that right hand side has not changed (here is where the separability assumption is used). Thus, we get a contradiction! Intuitively, and increase in capital taxes that leads to a decrease in output (and consumption) results in a lower demand for leisure. This, of course, corresponds to an increase in the number of hours worked.

We next show that the opposite, i.e. that $\partial n/\partial \tau^k < 0$, is possible as well. From (4.24) it suffices to find conditions under which $\Phi_c \approx 0$. In this case $\partial n/\partial \tau^k < 0$ since $F_{kn} > 0$. It turns out (see the exercises at the end of this section), that if leisure and consumption are "almost" perfect substitutes in utility, then $\Phi_c \approx 0$. The intuition is as follows: An increase in τ^k increases the cost of producing market output (and consumption), since consumption and leisure are "almost" perfect substitutes (i.e. the indifference curves are close to straight lines) the consumer finds it optimal to substitute away from c and into ℓ. The increase in the demand for leisure corresponds to a decrease in the supply of labor. Consequently, employment decreases. The formal result is derived in the following exercise

Exercise 86 *Assume that the utility function is of the form* $u(c, 1-n) = \left[\alpha c^{-\theta} + (1-\alpha)(1-n)^{-\theta}\right]^{-1/\theta}$*, where* $1/(1+\theta)$ *is the elasticity of substitution between consumption and leisure, and it is restricted to be between 0 (perfect complements) and* ∞ *(perfect substitutes). Show that* $\lim_{\theta\to\infty} \Phi_c(c, 1-n) = 0$.

It follows that even in as simple model as the standard growth model it is not possible to determine the impact of an increase in the tax rate on capital income (e.g. taxes on dividends) on employment. The key features are the degree of substitutability between capital and labor in production, and between consumption and leisure in preferences. If the two inputs are very good substitutes in production, an increase in τ^k increases employment. On the other hand, if consumption and leisure are very good substitutes in preferences, an increase in τ^k decreases employment.

In order to predict the impact of the type of tax changes that we are discussing here, it is necessary to identify the relevant parameters of the production function and the utility function, and then to solve the system of equations (4.21).

Exercise 87 *Consider an economy similar to the one described in the text with one exception: the depreciation allowance for tax purposes,* $\hat{\delta}_k$*, need not coincide with the "true" rate of depreciation. It follows that the total return, at time t+1, from investing one unit of consumption at time t in capital is just* $(1-\tau^k)r_{t+1} + 1 - \delta_k + \tau^k\hat{\delta}_k$*. Go as far as you can analyzing the impact of changes in* $\hat{\delta}_k$ *on the steady state quantities holding all tax rates constant. Is there a sense in which an increase in depreciation allowances,* $\hat{\delta}_k$*, is akin to an increase in the tax rate of capital income?*

Exercise 88 *Suppose that you are advising the Finance Minister of a country that wants to reduce its tax rate on investment goods from* τ^x *to 0, but it is* *willing to decrease depreciation allowance so as to kept the steady state quantities unchanged. Go as far as you can providing this government official with a formula that can be used to calculate the required level of depreciation allowances.*

Exercise 89 *Let the production function be* $F(k,n) = \left[\alpha k^{-\theta} + (1-\alpha)n^{-\theta}\right]^{-1/\theta}$, *where* $1/(1+\theta)$ *is the elasticity of substitution between capital and labor which is restricted to be between 0 (perfect complements) and* ∞ *(perfect substitutes). Argue that if the two inputs are perfect complements, then* $dn/d\tau^k < 0$.

4.3.3 A Digression: The User Cost of Capital

In this section we present the notion of the user *cost of capital*, or *cost of funds*, and discuss how it is affected by taxes and inflation. A simple way of thinking about the cost of funds is to imagine a firm that buys a capital good —say a machine— and then rents it to another firm. The amount that it charges for the lease is what we define as the cost of capital. To simplify the presentation we assume that the good depreciates geometrically at the rate δ_k. This implies that the good lasts forever, although in every period it only retains $1-\delta_k$ of its previous productive capacity.

Assume that the price of the machine is p_k, that the tax rate on capital income (paid by the owner of the machine) is τ^k, and that the tax code allows a fraction $\hat{\delta}_k$ of the purchase price of the machine to be deducted from the income generated by the machine for tax purposes (i.e. depreciation allowance). Let the nominal interest rate be constant and equal to $1+i = (1+r)(1+\pi)$, where r is the real interest rate, and π is the inflation rate. We will assume that the owner of the machine charges q per period of time (in real terms) for the lease. That is, j periods into the future the nominal value of the lease will be $(1+\pi)^j q$. In this setting q is the *cost of capital*.

How much has the seller have to charge in order to break even? The value q must be sufficient so that the present discounted value of after tax returns equal the price of capital. In symbols this requires,

$$\begin{aligned} p_k \quad = \quad & \frac{(1-\tau^k)(1+\pi)q + \tau^k p_k \hat{\delta}_k}{1+i} + \frac{(1-\tau^k)(1-\delta_k)(1+\pi)^2 q + \tau^k p_k \hat{\delta}_k (1-\hat{\delta}_k)}{(1+i)^2} \qquad (4.25) \\ & \frac{(1-\tau^k)(1-\delta_k)^2(1+\pi)^3 q + \tau^k p_k \hat{\delta}_k (1-\hat{\delta}_k)^2}{(1+i)^3} + ... \end{aligned}$$

The interpretation is relatively simple: After j periods, the owner does not have a "new" machine any more. It has just $(1-\delta_k)^{j-1}$ machines. This "fraction" of a machine brings $(1-\tau^k)(1+\pi)^j q$ after tax dollars in direct revenue. In addition, the depreciation allowance is worth $\tau^k p_k \hat{\delta}_k (1-\hat{\delta}_k)^{j-1}$ additional dollars. Thus, the total after tax return is the numerator of the fractions in (4.25). This value must be discounted —using the nominal interest rate $1+i$— to determine the present value of the lease. Note that the depreciation allowance is not indexed by inflation. Thus, even though the owner of the machine is free to adjust the value of the lease —as we assume— to account for inflation, the tax code provides a tax break that is fixed in nominal terms.

Computing the infinite sum in (4.25), one gets that,

$$p_k = \frac{(1-\tau^k)q}{r+\delta_k} + \frac{\tau^k p_k \hat{\delta}_k}{i+\hat{\delta}_k},$$

or,

$$q = p_k \left[\frac{r+\delta_k}{1-\tau^k} \frac{i+(1-\tau^k)\hat{\delta}_k}{i+\hat{\delta}_k} \right]. \tag{4.26}$$

The expression (4.26) summarizes the factors that affect the cost of capital. First, it is clear that increases in the tax rate on capital income, τ^k, increase the cost of capital, q. Increases in inflation (π) also increase the real cost of capital through their impact on the nominal interest rate (recall that $1+i = (1+r)(1+\pi)$). The reason is simple: An increase in the nominal interest rate (associated with higher inflation) lowers the present value of depreciation allowances. This reduces, from the point of view of the owner of the machine, the profitability of the project. In order to compensate for this loss, an increase in q is necessary. An increase in the rate at which a machine can be depreciated for tax purposes, $\hat{\delta}_k$, decreases the cost of capital, and has an effect that is similar to that of a decrease in the tax rate.

Our discussion shows that when trying to determine the impact of policies on the cost of capital, the explicit tax rate on capital income is just *one* factor that must be taken into account. Other dimensions of the tax code (e.g. whether it allows for accelerated depreciation or not), and of the macroeconomy (e.g. the rate of inflation) also play an important role.

Exercise 90 *A trade industry group argues that increases in inflation can have no impact in the cost of capital if depreciation allowances ($\hat{\delta}_k$ in (4.26)) are properly adjusted. Suppose you are a government economist. Go as far as you can deriving a formula that adjusts depreciation allowances, $\hat{\delta}_k$, as a function of the inflation rate to keep the cost of capital, q, constant.*

Exercise 91 *Derive a version of (4.26) for an economy in which there is tax on purchases of capital goods. The tax rate is τ^x.*

Exercise 92 *Assume that capital goods last only n periods. Depreciation is exponential, but after n periods the machine's productive capacity is reduced to zero. Derive the appropriate version of (4.26)*

4.3.4 Changing the Tax Rate on Labor Income

In this section we study the effect of changing τ^n. Formally, we consider the case in which $d\tau^k = 0$ and $d\tau^n > 0$. It follows that the effects on capital and labor are given by,

$$\frac{\partial k}{\partial \tau^n} = -\frac{F_n F_{kn}(1-\tau^k)}{\Delta}$$

and

$$\frac{\partial n}{\partial \tau^n} = \frac{F_n F_{kk}(1-\tau^k)}{\Delta}$$

which are both negative. Thus, an increase in tax rate on labor income has a negative impact on the amounts of capital and labor used in the economy.

Exercise 93 *Consider an economy with a small tax rate on labor income. The government has decided to increase the value added tax. One economist —we will call him A— argues that a tax on consumption will increase investment. Another economist —we will call her B— insists that this will ultimately decrease investment. Who is correct? Justify your answer.*

4.3.5 Digression: Partial Equilibrium Analysis

In this section we discuss the effect of changes in the tax rate on labor income on labor supply and consumption under the assumption that factor prices remain unchanged. There are two reasons for this. First, it will show that some results in the static literature are essentially analysis of permanent changes in tax rates. Second, it will highlight the differences between temporary and permanent changes in tax rates.

Consider a representative household that maximizes

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_t)$$

subject to,

$$(1+\tau_t^c)c_t + a_{t+1} \leq (1-\tau_t^n)w_t n_t + (1+(1-\tau_t^a)r_t^a)a_t$$

where a_t denotes the stock of assets at the beginning of period t, and $(1+(1-\tau_t^a)r_t^a)$ is the after tax rate of return on all assets[2]. In order to concentrate on price systems that are consistent with the existence of steady states, we assume that $1 = \beta[1+(1-\tau^a)r^a]$. The first order conditions are

$$\Phi(c_t, 1-n_t) = \frac{1-\tau^n}{1+\tau^c} w \tag{4.27a}$$

$$u_c(c_t, 1-n_t) = u_c(c_{t+1}, 1-n_{t+1}) \tag{4.27b}$$

$$(1+\tau_t^c)c_t + a_{t+1} = (1-\tau_t^n)w_t n_t + (1+(1-\tau_t^a)r_t^a)a_t. \tag{4.27c}$$

Any solution to the system (**??**) summarizes consumer choice given wages and interest rates.

[2] In the absence of uncertainty all assets must have the same rate of return. Thus, to simplify, we consider just one asset (saving).

Lon Run Analysis: Permanent Changes in Tax Rates

Consider the impact of a permanent increase in τ^n upon consumption and labor supply, assuming that saving stays constant. Since the economy starts at the steady state, the relevant conditions are

$$\Phi(c, 1-n) = \frac{1-\tau^n}{1+\tau^c} w \tag{4.28a}$$
$$(1+\tau^c)c = (1-\tau^n)wn + (1-\tau^k)r^k a. \tag{4.28b}$$

where we have imposed that the stocks of assets, a, be constant. Totally differentiating this system with respect to τ^n one gets that

$$\begin{bmatrix} \Phi_c & -\Phi_\ell \\ 1 & \frac{(1-\tau^n)w}{1+\tau^c} \end{bmatrix} \begin{bmatrix} dc \\ dn \end{bmatrix} = \begin{bmatrix} -\frac{w}{1+\tau^c} d\tau^n \\ -\frac{w}{1+\tau^c} n d\tau^n \end{bmatrix}$$

or that,

$$\frac{\partial c}{\partial \tau^n} = \frac{w}{(1+\tau^c)\Delta^*}[\frac{(1-\tau^n)w}{1+\tau^c} - n\Phi_\ell] < 0$$
$$\frac{\partial n}{\partial \tau^n} = \frac{w}{(1+\tau^c)\Delta^*}[1 - n\Phi_c] \lesseqgtr 0,$$

where,

$$\Delta^* = (1+\tau^c)^{-1}[-\Phi_c w(1-\tau^n) + (1+\tau^c)\Phi_\ell] < 0.$$

Thus, even though the sign of the change in consumption is not ambiguous (consumption decreases when the tax rate on labor income increases), the impact on labor supply depends on parameters.

It is useful to consider the case in which the utility function is separable. This implies that $u_{c\ell} = 0$. In this case, simple algebra shows that

$$\frac{\partial n}{\partial \tau^n} < 0 \Leftrightarrow 1 - n\Phi_c > 0 \Leftrightarrow 1 > -\frac{u_{cc}c}{u_c} \frac{(1-\tau^n)wn}{(1-\tau^n)wn + (1-\tau^k)r^k a}.$$

If we denote the elasticity of the marginal utility of consumption with respect to consumption (or income in the steady state) by $\epsilon_1$then

$$\frac{dn}{d\tau^n} < 0 \Leftrightarrow 1 > \epsilon_1 \frac{(1-\tau^n)wn}{(1-\tau^n)wn + (1-\tau^k)r^k a}. \tag{4.29}$$

Thus, (4.29) shows that if the elasticity of the marginal utility of income multiplied by the share of labor income in total income is less than one, and increase in the tax rate on labor income decreases labor supply. This condition coincides with the "standard" results in Public Economics (e.g. See Atkinson and Stiglitz, *Lectures on Public Economics*, equation (2.10)). It follows then that the "static" analysis in the standard textbook treatment of the impact of income taxes on labor supply is similar to the effect of a permanent increase in income tax rates when saving does not respond.

Exercise 94 *Analyze the impact of an increase in τ^a assuming that a is unchanged. How does this differ from an increase in lump-sum taxes?*

Exercise 95 *Assume that* $u(c, 1-n) = \log(c) + v(1-n)$*, for some function* v*. Argue that, in this case, permanent increases in the tax rate on labor income result in decreases in labor supply holding saving constant.*

Exercise 96 *Argue that an increase in* τ^c *is similar to an increase in* τ^n *combined with a lump-sum tax.*

Consider next what happens when individuals are allowed to change their saving levels. In this case, it is clear that c and n will be constant over time, and that they will have to satisfy the present value version of the budget constraint,

$$(1+\tau^c)c = (1-\tau^n)wn + (1-\beta)[1+(1-\tau^a)r^a]a = (1-\tau^n)wn + \rho a,$$

where the last equality uses the assumption that $1 = \beta[1+(1-\tau^a)r^a]$. In addition, consumption and leisure still need to satisfy (4.28a). It follows that letting saving adjust does not change the results. Thus, we conclude that *permanent increases in labor income taxes in a dynamic setting are similar to tax changes in a static framework.*

Short Run Analysis: Temporary Changes in Tax Rates

In this section we consider the following experiment: the tax rate on labor income, τ^n, is increased for just one period, and it returns to its old value starting tomorrow. This is a stylized version of a tax rate change that is temporary (e.g. a special surcharge). The question one is interested in is whether the change in labor supply associated with a temporary increase in taxes is bigger or smaller than the change associated with a permanent change. To keep notation simple we set the tax rate on consumption, τ^c, equal to zero.

Let the first period tax rate be τ^n, and the tax rate in periods 2, 3, 4, ... be $\hat{\tau}^n > \tau^n$. It is easy to show that in this setting, the consumer converges to the new steady state after one period. Let the first period values of consumption and labor be (c, n), and the second (and third, and all future values) be $(\hat{c}, \hat{n})$. The first order conditions for utility maximization include

$$\begin{aligned}
\Phi(c, 1-n) &= (1-\tau^n)w \\
\Phi(\hat{c}, 1-\hat{n}) &= (1-\hat{\tau}^n)w \\
u_c(c, 1-n) &= u_c(\hat{c}, 1-\hat{n}),
\end{aligned}$$

and the present discounted value of the budget constraint,

$$c + \frac{\beta}{1-\beta}\hat{c} = (1-\tau^n)wn + \frac{\beta}{1-\beta}(1-\hat{\tau}^n)w\hat{n} + [1+(1-\tau^a)r^a]a.$$

If the utility function is separable in consumption and leisure, then $u_c(c, 1-n) = u_c(\hat{c}, 1-\hat{n}) \Leftrightarrow c = \hat{c}$. This implies that the relevant set of equations is,

$$\begin{aligned}
\Phi(c, 1-n) &= (1-\tau^n)w \\
\Phi(c, 1-\hat{n}) &= (1-\hat{\tau}^n)w,
\end{aligned}$$

and

$$c = (1-\beta)(1-\tau^n)wn + \beta(1-\hat{\tau}^n)w\hat{n} + \rho a,$$

where we have repeatedly used the property that the after tax rate of return $[1+(1-\tau^a)r^a] = 1+\rho = \beta^{-1}$. In general, once functional forms are specified these equations can be solved numerically. However to illustrate the difference between transitory and permanent changes in tax rates, we now study a simple example. Let $u(c, 1-n) = \ln(c) + \ln(1-n)$. In this case one can show that the previous system can be simplified to,

$$\begin{aligned}
(1-n)(1-\tau^n) &= (1-\hat{n})(1-\hat{\tau}^n) \\
(1-\hat{n})(1-\hat{\tau}^n)w &= (1-\beta)(1-\tau^n)wn + \beta(1-\hat{\tau}^n)w\hat{n} + \rho a.
\end{aligned}$$

Simple algebra shows that the equilibrium supply of labor in the first period is,

$$n = \frac{(1-\hat{\tau}^n)w - \rho a - \beta w[(1-\tau^n) - (1-\hat{\tau}^n)]}{2w(1-\tau^n)}, \tag{4.30}$$

where we assume that $(1-\hat{\tau}^n)w - \rho a - \beta w[(1-\tau^n)-(1-\hat{\tau}^n)] > 0$, to guarantee that the supply of labor is positive (a negative supply of labor is interpreted as "hiring" people to do work around the house).

It is useful to derive, for this example, the impact of a permanent change in the tax rate, as we did in the previous section. In order to do this we simply set $\hat{\tau}^n = \tau^n$ in (4.30). We label the resulting labor supply function n^P to indicate that it captures the effect of a permanent change in tax rates. The resulting function is

$$n^P = \frac{(1-\tau^n)w - \rho a}{2w(1-\tau^n)}. \tag{4.31}$$

We are now in position to compute the responsiveness of labor supply to a change in taxes. It is simpler to determine the elasticity of n with respect to $(1-\tau^n)w$. In the case of a permanent change (4.31) implies that the elasticity is

$$\varepsilon^P_{(1-\tau^n)w} \equiv \frac{(1-\tau^n)w}{n^P}\frac{\partial n^P}{\partial(1-\tau^n)w} = \frac{\rho a}{(1-\tau^n)w - \rho a}, \tag{4.32}$$

while in the case of a temporary change the elasticity —evaluated at the point $\hat{\tau}^n = \tau^n$— is

$$\varepsilon^S_{(1-\tau^n)w} \mid_{\tau^n=\hat{\tau}^n} \equiv \frac{(1-\tau^n)w}{n^S}\frac{\partial n^S}{\partial(1-\tau^n)w} = \frac{\rho a + \beta w(1-\tau^n)}{(1-\tau^n)w - \rho a}. \tag{4.33}$$

It is straightforward to check that the elasticities with respect to the tax rate —instead of the after tax wage— are given by

$$\varepsilon^P_{\tau^n} = \varepsilon^P_{(1-\tau^n)w}\frac{\tau^n}{1-\tau^n},$$

and

$$\varepsilon^S_{\tau^n} \mid_{\tau^n=\hat{\tau}^n} = \varepsilon^S_{(1-\tau^n)w} \mid_{\tau^n=\hat{\tau}^n} \frac{\tau^n}{1-\tau^n},$$

are just proportional to the elasticities with respect to after tax wages. It is clear from (4.32) and (4.33) that labor supply responds more to transitory changes in tax rates. Thus, a temporary tax increase induces a larger decrease in the supply of labor than a permanent one.

The intuition for this result is simple: In the case of a temporary tax change income effects are relatively unimportant, since the tax affects one out of many periods. On the other hand, there is the possibility of intertemporal substitution; as the tax rate is high today it is optimal to postpone working until the future, when tax rates will be lowered. From the point of view of the government, a temporary tax increase will raise less revenue than a permanent tax increase due to the larger reduction in hours worked.

This intuition suggests that changes in tax rates will have different effects on different age groups. In the case of people who are near retirement, there are very few possibilities of intertemporal substitution. Thus, a temporary increase in income taxes resembles, for that group, a permanent increase. On the other hand, young people —who have many periods of labor market participation ahead of them— react more

Exercise 97 *Consider an individual who has preferences described by* $\sum_{t=0}^{N}\beta^t[\ln(c_t)+\ln(1-n_t)]+\beta^{N+1}\ln(c_{N+1})$. *Go as far as you can computing the effect on the first period labor supply,* $t=0$, *of a temporary (lasts one period) increase in* τ^n *as a function of* N, *the planning horizon. If one interprets* N *as the number of periods until retirement (minus one), what do your results say about the impact on hours worked of a temporary increase in the tax rate on labor income for young and mature workers?*

4.3.6 Individual and Corporate Taxation

Until this point we have ignored the effects of corporate taxes. That is, we assumed that consumers own the capital stock and they rent it out to firms. This, of course, implies that taxation at the firm level is irrelevant given constant returns to scale. In this section we explicitly model firms' investment decisions and explore the impact of the corporate income tax.

The representative consumer solves a standard dynamic problem. Since households do not own capital directly, they can hold a portfolio of government bonds, corporate bonds and equity. Thus, the representative consumer faces the following maximization problem

$$\max\sum_{t=0}^{\infty}\beta^t u(c_t,1-n_t)$$

subject to,

$$\begin{aligned}c_t+s_{t+1}p_{et}+b_{t+1}^g+b_{t+1}^f \quad &\leq \quad (1-\tau_t^n)w_t n_t+(1+(1-\tau_t^g)r_t^g)b_t^g \\ &\quad +(1+(1-\tau_t^f)r_t^f)b_t^f+s_t p_{et}+s_{t+1}(1-\tau_t^s)d_{et},\end{aligned}$$

where b^g_{t+1} is the stock of one period government bonds held by the household at the end of period t, $b^f_{t+1}$is the stock of one period corporate bonds held by the household at the end of period t, s_{t+1} is the number of shares purchased by the household in period t, p_{et} is the price per share, and d_{et} is dividends per share. Note that this formulation implicitly assumes that shares are purchased before they pay dividends; that is, a consumer who buys s_{t+1} shares at time t immediately collects $s_{t+1}d_{et}$ in dividends paid by this firm and, next period, it can sell its shares for $s_{t+1}p_{et+1}$ before they pay dividends.

Each asset, and labor income, is taxed at its own rate. Note that the notation is sufficiently flexible to allow different tax rates for different assets. However, this does not imply that the rates have to be different. For example, if the economy has a single tax rate on income (i.e. an income tax) levied at rate τ^i_t, then it follows that $\tau^i_t = \tau^n_t = \tau^g_t = \tau^f_t = \tau^s_t$.

The first order conditions corresponding to the solution of the consumer problem at the steady state include (among others)

$$\rho = (1-\tau^f)r^f \tag{4.34a}$$
$$\rho = (1-\tau^g)r^g \tag{4.34b}$$
$$\rho = (1+\rho)(1-\tau^s)\frac{d_e}{p_e}. \tag{4.34c}$$

These conditions simply state that in a world without uncertainty all assets have to have the same after tax rate of return.

Let's consider the corporate sector next. We have to distinguish profits (or dividends) from taxable income. We consider a firm that has produced (and sold) output given by $F(k,n)$, has to pay a wage bill given by wn, owns capital given by k, has a stock of one period debt given by b^f, and has decided to buy new capital (investment) equal to x. Let's assume that, for tax purposes the existing capital stock can be depreciated at rate $\hat{\delta}_k$, that a certain fraction, λ, of interest paid can be deducted, and that a fraction μ of purchases of new capital can be expensed (deducted) in the period in which they occur. It follows that business taxable income is

$$F(k,n) - wn - \hat{\delta}_k k - \lambda r^f b^f - \mu x$$

while profits —which we equate to distributed dividends— are

$$F(k,n) - wn - (1+r^f)b^f + b^{f'} - x - \tau^\pi[F(k,n) - wn - \hat{\delta}_k k - \lambda r^f b^f - \mu x],$$

where τ^π is the tax rate on corporate income, and $b^{f'}$ is new one period debt issued by the firm. Thus, total dividends paid by this firm are given by

$$D_e = (1-\tau^\pi)(F(k,n)-wn)+\tau^\pi\hat{\delta}_k k+b^{f'}-b^f-r^f(1-\lambda\tau^\pi)b^f-(1-\mu\tau^\pi)x. \tag{4.35}$$

Let the value of the firm be denoted P_e. We conjecture that a firm that invests optimally and that it keeps the ratio of debt to capital constant, say

$b^f/k = \varphi$, has a value given by $P_e = p_e k$. Under this guess (that will have to be verified) the representative firm solves

$$p_e k = \max_x D_e + q^f p_e k' \tag{4.36}$$

subject to,

$$k' \leq (1-\delta_k)k + x,$$

where q^f is the relevant discount factor —to be determined— that is used by firms in discounting profits.

Unfortunately our model is not rich enough to endogenously determine the debt-equity ratio. Thus, for now, we ignore the optimal choice of $b^{f'}$. The first order condition for the optimal choices of labor and new capital are

$$n \quad : \quad F_n(k,n) = w \tag{4.37a}$$
$$k \quad : \quad (1-\mu\tau^\pi) = q^f p_e. \tag{4.37b}$$

The first condition simply says that firms choose employment so that the marginal product of labor equals the wage rate. The second condition says the after tax price of new capital goods, $(1-\mu\tau^\pi)$, must be equated to the marginal contribution of capital to the value of the firm properly discounted.

Under the assumption that $F(k,n)$ is homogeneous of degree one, it follows that at the steady state (which corresponds to the case $k = k'$ and $b^f = b^{f'}$)

$$D_e = (1-\tau^\pi)F_k k + \tau^\pi \hat{\delta}_k k - r^f(1-\lambda\tau^\pi)b^f - (1-\mu\tau^\pi)\delta_k k.$$

Given the assumption that corporate debt is a fraction φ of total assets (capital), i.e. $b^f = \varphi k$, it follows that dividends per unit of own capital are

$$d_e \equiv \frac{D_e}{k} = (1-\tau^\pi)F_k + \tau^\pi \hat{\delta}_k - r^f(1-\lambda\tau^\pi)\varphi - (1-\mu\tau^\pi)\delta_k. \tag{4.38}$$

We next use (4.38) in (4.36) to get

$$p_e = d_e + q^f p_e. \tag{4.39}$$

We now want to derive an expression for $q^f p_e$ so that we can evaluate the right hand side of (4.37b) and determine the impact of the tax code on investment. From (4.39) it follows that

$$q^f = 1 - \frac{d_e}{p_e},$$

and using (4.34c) in the above expression we get that

$$q^f = \frac{1-\tau^s(1+\rho)}{(1+\rho)(1-\tau^s)}. \tag{4.40}$$

This expression shows that the appropriate discount factor that firms must use to discount profits is *independent* of the details of corporate taxation. It differs from the consumers' discount factor, $1/(1+\rho)$, only because dividends are taxed at the *individual* level. Given that $p_e = (1+\rho)(1-\tau^s)d_e/\rho$, then

$$q^f p_e = (1 - \tau^s(1+\rho))\frac{d_e}{\rho},$$

and using (4.38) and (4.34a) in the above expression one gets,

$$q^f p_e = (1 - \tau^s(1+\rho))\frac{(1-\tau^\pi)F_k + \tau^\pi\hat{\delta}_k - \frac{\rho}{1-\tau^f}(1-\lambda\tau^\pi)\varphi - (1-\mu\tau^\pi)\delta_k}{\rho}. \tag{4.41}$$

Finally, substituting (4.41) in the first order condition for the optimal investment decision (4.37b) and manipulating the resulting expression the equilibrium condition is

$$\rho = \frac{1-\tau^s(1+\rho)}{1-\mu\tau^\pi}\left[(1-\tau^\pi)F_k + \tau^\pi\hat{\delta}_k - \frac{\rho}{1-\tau^f}(1-\lambda\tau^\pi)\varphi - (1-\mu\tau^\pi)\delta_k\right]. \tag{4.42}$$

The previous condition completely summarizes the implication of the model for the optimal choice of the capital labor ratio. Note that in (4.42), the term F_k depends on $\kappa \equiv k/n$, while all other terms are just constants defined by the tax code or consumers' preferences. Equation (4.42) is the appropriate version of (4.21b) in this more complicated economy with a corporate sector. However, with some manipulation, it is possible to describe the sense in which (4.42) and (4.21b) are equivalent. To see this we consider a number of special cases:

Case 98 : $\tau^s = 0, \hat{\delta}_k = \delta_k, \mu = 0, \varphi = 0$

This case corresponds to an economy in which all firms are 100% equity financed (there is no corporate debt and $\varphi = 0$), investment expenses are not tax deductible ($\mu = 0$), real economic depreciation and depreciation allowances in the tax code coincide ($\hat{\delta}_k = \delta_k$), and dividends are not taxed at the individual level ($\tau^s = 0$). It follows from (4.42) that the optimal capital-labor ratio must satisfy

$$\rho = (1-\tau^\pi)(F_k - \delta_k),$$

which is identical to (4.21b) with the identification $\tau^\pi = \tau^k$. Thus, in this case, the corporate tax rate is what we call the tax rate on capital income.

Case 99 $\hat{\delta}_k = \delta_k, \mu = 0, \varphi = 0$

This case is similar to the one before except that dividends are taxed at the household level. It follows that (4.42) is given by

$$\rho = (1-\tau^s(1+\rho))(1-\tau^\pi)(F_k - \delta_k).$$

Since (4.21b) is just,

$$\rho = (1-\tau^k)(F_k - \delta_k),$$

it follows the what we have defined as the tax rate on capital income is satisfies,

$$(1-\tau^k) = (1-\tau^s(1+\rho))(1-\tau^\pi),$$

or,

$$\tau^k = \tau^s + \tau^\pi + \tau^s(\rho - \tau^\pi(1+\rho)).$$

Note that if we discard terms that involve the product of two small quantities (i.e. $\tau^s(\rho - \tau^\pi(1+\rho))$ in this example), the previous expression shows that the effective tax rate on capital is the sum of the tax rate on dividends at the household level (the income tax rate) and the corporate tax rate. For example, in an economy with a 20% tax rate on ordinary income (this includes dividends) and a 30% tax rate on corporate profits the effective tax rate on capital is 50%.

Case 100 $\hat{\delta}_k = 0, \mu = 1, \lambda = 1$

In this case purchases of capital goods can be expensed in the period in which they are made ($\mu = 1$). Thus, is a firm purchases a new piece of equipment it can deduct the full purchase price from taxable income in the year in which the purchase was made. Since purchases of capital goods are expensed, we assume that there are no capital consumption allowances ($\hat{\delta}_k = 0$). In this case we do not need to assume that the firm is 100% equity financed; that is, we allow for an arbitrary debt-equity ratio ($\varphi \neq 0$) provided that interest payments by the firms are fully tax deductible ($\lambda = 1$).

In this case, (4.42) is

$$\rho = (1-\tau^s(1+\rho))[F_k - \delta_k - \frac{\rho\varphi}{1-\tau^f}].$$

In this case, the equivalent effective tax rate on capital income, $\hat{\tau}^k$, is

$$\hat{\tau}^k = \tau^s(1+\rho) + (1-\tau^s(1+\rho))\frac{\rho\varphi}{(1-\tau^f)(F_k-\delta_k)}.$$

Thus, the effective tax rate on capital income always exceeds the tax rate at which dividends are taxed, but the magnitude depends on a number of variables. First, the higher is the tax rate on interest paid by corporations the higher the equivalent effective tax rate on capital. The basic force at work here is that increases in the tax rate on interest income increase the real rate that firms have to offer on their debt. Second,the higher is the debt-equity ratio (φ) the higher is the equivalent effective tax rate. The reason for this is simple: while dividends are taxed twice (first as corporate profits, and then as household income), interest payments are taxed only once (they are deductible at the corporate level and taxed at the household level). Thus, the higher the fraction of the capital that is debt financed the higher the equivalent (corresponding to the case in which firms are 100% equity financed) effective tax rate on capital income.

This result shows why our theory is incomplete: In this economy all firms would choose to be 100% debt financed. In this way they could altogether avoid the corporate income tax. There are arguments related to the details of bankruptcy codes as well as moral hazard considerations that might explain less than full debt-equity ratios, but ours is too simple to capture those effects. Thus, we will assume that φ is exogenous. This "parameter" is however quite important in quantitative analysis of the impact of the tax code. For example consider an economy with an annual pure discount rate of 5% ($\rho = 0.05$), a before tax return on capital equal to 20% ($F_k - \delta_k = 0.20$), and a tax rate on personal income equal to 30% ($\tau^s = \tau^f = \tau^i = 0.30$). In this case $\tau^k = .315 + 0.245\varphi$. If firms are 100% equity financed ($\varphi = 0$), then the equivalent effective tax rate on capital income is very close to the rate at which dividends are taxed at the personal level (30%). However, at the other end, when firms are heavily leveraged ($\varphi \approx 1$), the equivalent effective tax rate on capital (τ^k) is over 46%!

The last remarkable feature of this case is that the tax rate on corporate income (τ^π) is completely neutral in terms of the rate at which capital is taxed. The somewhat surprising result is driven by our assumption that purchases of capital goods can be expensed fully. In this case investment is akin to a "static" input and since the corporate tax rate does not distort the choice of static inputs, it has no effect on the decision to invest.

The basic message from this section is that in actual economies, determining the effective tax rate on capital income is not a simple matter. It is necessary to take into account the full complexities of the tax code in order to transform statutory tax rates in what, in the model, we call *the* tax rate on capital.

Exercise 101 *Consider an economy in which financial liberalization results is firms being able to increase the fraction of their capital stock that can be financed through borrowing. Go as far as you can describing the impact of this development on the capital labor ratio, employment and output.*

Exercise 102 *Go as far as you can analyzing the impact on the equilibrium capital-labor ratio of an increase in depreciation allowances ($\hat{\delta}_k$).*

Exercise 103 *Go as far as you can analyzing the impact on the equilibrium capital-labor ratio of an increase in the fraction of investment purchases that can be expensed (μ).*

Exercise 104 *Go as far as you can analyzing the impact on the equilibrium capital-labor ratio of an increase in the fraction of interest costs that firms can deduct (λ).*

Exercise 105 *Discuss the following claim: An increase in the fraction of interest costs that firms can deduct (λ) will result in a decrease in the interest rate on firm issued bonds.*

Exercise 106 *Discuss the following claim: An increase in the personal income tax rate (common to all forms of income) will have a larger impact in an economy in which firms are 100% equity financed relative to another economy in which firms are only $(1-\varphi)$% equity financed.*

4.3.7 A Balanced Budget (Revenue Neutral) Change in Tax Rates

In this section, we specialize the model in order to study the effects changes in the tax rates of capital and labor that are *revenue neutral*; that is, changes in the two tax rates consistent with financing the same level of investment.

We consider the case in which the utility function is Cobb-Douglas in consumption and leisure and the production function is also a Cobb-Douglas in capital and labor. In particular, we assume

$$u(c, 1-n) = c^{\theta}(1-n)^{1-\theta}$$

and,

$$F(k, n) = Ak^{\alpha}n^{1-\alpha}.$$

As before, we let the base of the capital income tax to be the gross return minus a depreciation allowance, $r - \hat{\delta}_k$. To simplify, we set the *permissible* depreciation allowance, $\hat{\delta}_k$, equal to the *true* depreciation rate, δ_k. Given the chosen functional forms, the first order conditions corresponding to the system (4.21) are,

$$\frac{1-\theta}{\theta}\frac{Ak^{\alpha}n^{1-\alpha} - \delta_k k - g}{1-n} = (1-\tau^n)(1-\alpha)\frac{Ak^{\alpha}n^{1-\alpha}}{n} \quad (4.43a)$$

$$\rho = (1-\tau^k)(\alpha\frac{Ak^{\alpha}n^{1-\alpha}}{k} - \delta_k). \quad (4.43b)$$

In addition, the two tax rates cannot be set independently. If we assume that government debt is zero,[3] the government budget constraint is

$$g = \tau^k(\alpha Ak^{\alpha}n^{1-\alpha} - \delta_k k) + \tau^n(1-\alpha))Ak^{\alpha}n^{1-\alpha}$$

which implies that, given τ^n, the tax rate on capital income, τ^k, is given by

$$\tau^k = \frac{g - \tau^n(1-\alpha)Ak^{\alpha}n^{1-\alpha}}{\alpha Ak^{\alpha}n^{1-\alpha} - \delta_k k}. \quad (4.44)$$

For future reference note that (4.44) implies that, given τ^n, the higher the level of output, the lower the required tax rate on capital income, τ^k. Substituting (4.44) in (4.43b), solving for $n^{1-\alpha}$, and then substituting the resulting expression in (4.43a) it is possible to reduce the characterization of the steady state —after some manipulation— to the solution of the following two equations

$$n = M^b(k, \upsilon) \equiv \left[\frac{(\rho+\delta_k)k + g}{Ak^{\alpha}(1-\upsilon)}\right]^{\frac{1}{1-\alpha}} \quad (4.45a)$$

$$n = M^a(k, \upsilon) \equiv \frac{\upsilon\theta[(\rho+\delta_k)k + g]}{(1-\theta)(1-\upsilon)\rho k + \upsilon[(\rho+\delta)k + g]}, \quad (4.45b)$$

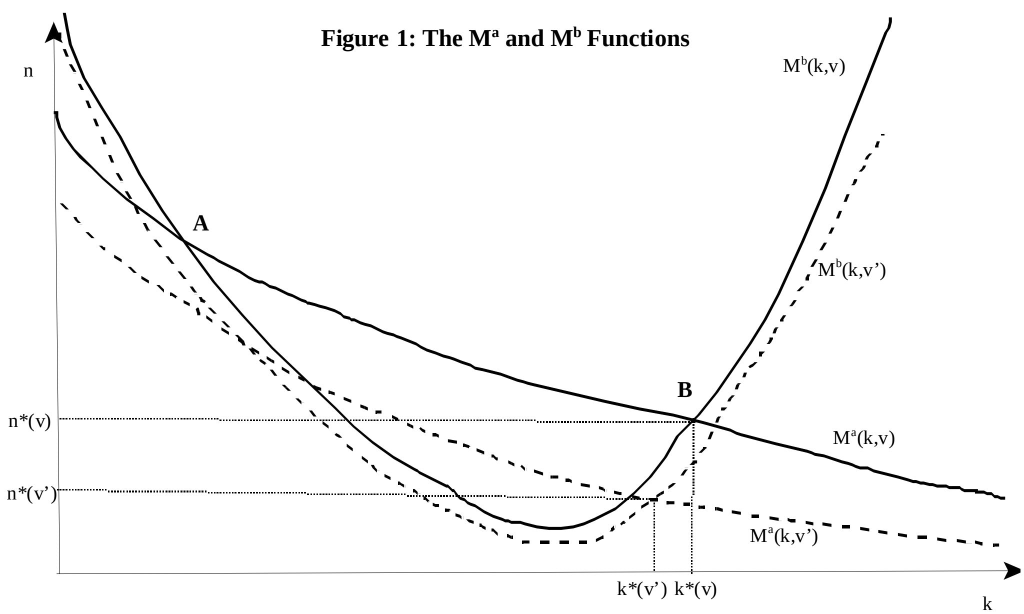
[3] This is done to simplify the algebra. In the the exercises at the end of this section, the reader is asked to consider the impact of a non-zero amount of debt.

where $\upsilon \equiv (1-\alpha)(1-\tau^n)$. The system (4.45) consists of a pair of equations in two unknowns, n and k. It is useful to summarize the properties of the $M^a(k,\upsilon)$ and $M^b(k,\upsilon)$ functions. Simple calculations show that these functions have the following properties,

$$\begin{aligned}
M^a(0,\upsilon) &= \theta \\
\lim_{k\to\infty} M^a(k,\upsilon) &= \frac{\upsilon\theta(\rho+\delta)}{(1-\theta)\rho+\upsilon(\rho+\delta)} < \theta \\
\frac{\partial M^a(k,\upsilon)}{\partial \upsilon} &= M^a \frac{(1-\theta)\rho k}{\upsilon[(1-\upsilon)(1-\theta)\rho k+\upsilon[(\rho\theta+\delta_k)k+g]]} > 0 \\
\frac{\partial M^a(k,\upsilon)}{\partial k} &= -\frac{M^a}{(\rho+\delta_k)k+g}\frac{g\rho(1-\theta)(1-\upsilon)}{(1-\upsilon)(1-\theta)\rho k+\upsilon[(\rho\theta+\delta_k)k+g]} < 0 \\
M^b(0,\upsilon) &= \infty, \qquad \lim_{k\to\infty} M^b(k,\upsilon) = \infty \\
\frac{\partial M^b(k,\upsilon)}{\partial \upsilon} &= \frac{M^b}{(1-\alpha)(1-\upsilon)} > 0 \\
\frac{\partial M^b(k,\upsilon)}{\partial k} &= \frac{M^b}{(1-\alpha)k}\frac{(\rho+\delta_k)(1-\alpha)k-\alpha g}{(\rho+\delta)k+g} \lesseqqgtr 0 \quad if\ k \lesseqqgtr \tilde{k} \equiv \frac{\alpha g}{(\rho+\delta_k)(1-\alpha)}.
\end{aligned}$$

Figure 1 displays the M^a and M^b functions (the reader should ignore the dashed lines for now).

Figure 4.1: The M^a and M^b Functions



Given the properties described above, the two functions intersect twice (that they intersect at all requires that g —the level of government spending— not be too high). One of the intersections —which we label A— corresponds to a low level of capital and output. The other one —labeled as B in Figure 1— corresponds to a higher level of capital and output. Both can be steady states.

Why are there two possible steady states? It is a standard result in public economics that, in general, there is more than one tax rate that can generate a given level of revenue. Specifically, consider a tax rate τ on some input, and let the market price of that input be p. It follows that government revenue is $R(\tau) = \tau p Q((1-\tau)p)$, where $Q((1-\tau)p)$ is the quantity supplied of the input. It is clear that if $\tau = 0$, then $R(\tau) = 0$, while if $\tau = 1$ (a tax rate of 100%) then $Q(0) = 0$, and $R(\tau) = 0$ as well. For intermediate values of τ the revenue function is positive. Then, it follows that in order to raise a certain amount of revenue, there are at least two possible taxes —one relatively high and one relatively low— that can be chosen. In our economy, point A corresponds to a "high" tax rate equilibrium while point B is associated with a "low" tax rate equilibrium. To see this, recall (4.44) where a high level of output (i.e. point B) yields a low tax rate on capital income, while a low level of output (i.e. point A) requires a high tax rate on capital income. Since there are inefficiencies associated with points like A —and much empirical evidence suggests that it is not plausible to assume that most economies are at points like A— we will assume that the economy is at B. The steady state capital stock is denoted by $k^*(\upsilon)$.

What happens when the tax rate on labor income is increased? This corresponds to a decrease in υ from υ to υ'. This shifts the $M^a(k,\upsilon)$ and the $M^b(k,\upsilon)$ functions down. To determine the new equilibrium levels of capital and labor it is necessary to evaluate the magnitude by which each of the two functions shifts. In Figure 1 we consider the case in which as a result of a change in υ the $M^a(k,\upsilon)$ changes more —at the original equilibrium level $k^*(\upsilon)$— than the function $M^b(k,\upsilon)$.

Formally, this corresponds to the case in which $\frac{\partial M^a(k,\upsilon)}{\partial \upsilon} > \frac{\partial M^b(k,\upsilon)}{\partial \upsilon}$. Some straightforward calculations show that this is the case if

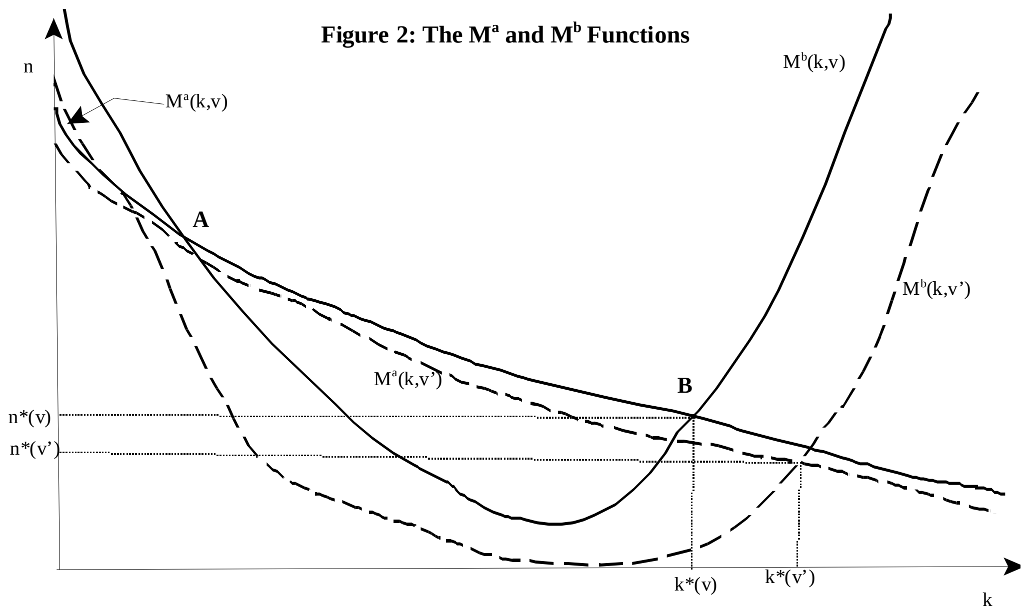
$$\rho k(1-\theta)(1-\alpha)(1-\upsilon) > \upsilon[\rho k(1-\theta)(1-\upsilon) + \upsilon((\rho+\delta_k)k + g)]. \qquad (4.46)$$

It follows that (4.46) holds if υ is sufficiently small (i.e. close to zero). This corresponds to the case in which the tax rate on labor income, τ^n, is close to 1.

Assume that (4.46) holds and let $\tau^{n\prime} > \tau^n$, and, correspondingly, let $\upsilon' < \upsilon$. The capital stock associated with the new (higher) tax, $k^*(\upsilon')$, is lower than the original capital stock, $k^*(\upsilon)$. Moreover, the new level of employment, $n^*(\upsilon')$, is also lower than the initial level, $n^*(\upsilon)$. Thus, in this case —which corresponds to a *high* initial tax rate on labor income— further increases in the tax rate on labor income reduce output. The intuition is that when a tax rate is already at a very high level, the distortionary costs associated with increasing it can result in losses that offset any revenue gains.

Is it possible to get the opposite result? In principle the two functions could move as shown in Figure 2, and this will be the case if the inequality (4.46) is reversed. The reader can check that if τ^n is close to 0 (this implies that $\upsilon \approx 1-\alpha$), the inequality (4.46) is reversed. In this case the function $M^b(k,\upsilon)$ shifts more than the function $M^a(k,\upsilon)$ as a result of an increase in τ^n.

Figure 4.2: The M^a and M^b Functions: The Case of "Small" τ^n



The new capital stock, $k^*(\upsilon')$, is higher the the initial capital stock, but the level of employment, $n^*(\upsilon')$, falls short of the initial level. Thus, in this case, which corresponds to a "small" starting tax rate on labor income, a change in the composition of taxes that increases the reliance on labor income taxes can have a positive effect on investment —and the capital stock— but a negative effect on employment.

What happens to output? From (4.45a) it is possible to derive that,

$$Ak^*(\upsilon)^\alpha n^*(\upsilon)^{1-\alpha} \equiv y^*(\upsilon) = \frac{(\rho+\delta_k)k^*(\upsilon)+g}{1-\upsilon}.$$

Some simple algebra shows that $\partial y^*/\partial\upsilon < 0$ if

$$\frac{\partial k^*(\upsilon)}{\partial\upsilon} < -\frac{(\rho+\delta_k)k+g}{(\rho+\delta_k)(1-\upsilon)}. \tag{4.47}$$

Since from our analysis it follows that

$$\frac{\partial k^*(\upsilon)}{\partial\upsilon} = \frac{\frac{\partial M^b(k,\upsilon)}{\partial\upsilon} - \frac{\partial M^a(k,\upsilon)}{\partial\upsilon}}{\frac{\partial M^a(k,\upsilon)}{\partial k} - \frac{\partial M^b(k,\upsilon)}{\partial k}},$$

the reader is asked to check what is the effect on output of an increase in τ^n when $\tau^n \approx 0$.

To summarize, even in simple settings as the ones in the example discussed in this section, changes in the composition of the tax code —what we have termed "balanced budget" changes in taxes— can either increase or decrease output. In the model of this section increases in τ^n (and recall that these are associated with decreases in τ^k) initially (i.e. when τ^n is small) could increase output. However, further increases in the tax rate of labor income eventually lead to decreases in output.

One of the messages from this section is that analytical results are hard to come by. However, useful tax analysis can be conducted by specifying production and utility functions and deriving the analog of (4.43); then, numerically, it is possible to evaluate proposals for tax reform that are revenue neutral.

Exercise 107 *Derive the system of equations (4.45).*

Exercise 108 *Consider an economy similar to the one studied in this section — same utility and productions functions— with two exceptions. First, assume that capital income taxes are levied on capital income gross of depreciation (the base of the tax is rk and not $(r - \delta_k)k$). Second, the level of government spending is not fixed. Rather, it is the ratio of government spending to output, $g/(Ak^\alpha n^{1-\alpha})$ that is held constant. Go as far as you can describing the effect of an increase in the tax rate on labor income on the stock of capital, the level of employment and the level of output.*

Exercise 109 *Consider an economy with arbitrary utility and production functions. Go as far as you can analyzing this statement: Holding the level of government debt constant, changes in the tax rate on interest income on government debt, τ^b, do not have any effects in the long-run (steady state)*

Exercise 110 *Consider an economy with parameterized utility and production functions (not necessarily Cobb-Douglas). Select a country and estimate its capital and labor income tax rates. Numerically compute the steady state and go as far as you can analyzing (also numerically) the impact of a change in the structure of the tax system.*

4.3.8 Taxation and Human Capital

In this section we expand the model to include individual decisions about how much human capital to accumulate. It turns out that a critical detail of the model is the form of the human capital accumulation equation. Unfortunately, very little is known about it. This is not surprising since the notion of human capital exceeds that of schooling and encompasses increases in experience, training (in firms and in educational institutions), and could also include health status. In the text, we will discuss a fairly popular specification, and the reader will be asked to explore alternative assumptions in the exercises.

We will assume that to produce new human capital both existing human capital and hours are used. The human capital accumulation equation is given by,

$$h_{t+1} = (1-\delta_h)h_t + Bn_{ht}^{\psi}h_t^{1-\psi} \tag{4.48}$$

where h_t is the stock of human capital and n_{ht} is the number of hours allocated to producing more human capital. The representative household uses its stock of human capital and "raw" hours to produce effective labor, z_t, according to the production function

$$z_t = Zn_{mt}^{\eta}h_t^{1-\eta}. \tag{4.49}$$

Thus the household's utility maximization problem is given by,

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_{mt}-n_{ht}) \tag{4.50}$$

subject to,

$$\begin{aligned}
(1+\tau_t^c)c_t + (1+\tau_t^x)x_t + b_{t+1} &\leq r_t k_t - \tau_t^k(r_t-\delta_k)k_t - T_t && (4.51a)\\
&\quad +(1-\tau_t^n)w_t z_t + (1+(1-\tau_t^b)r_t^b)b_t &&\\
k_{t+1} &\leq (1-\delta_k)k_t + x_t && (4.51b)\\
h_{t+1} &\leq (1-\delta_h)h_t + Bn_{ht}^{\psi}h_t^{1-\psi} && (4.51c)\\
z_t &\leq Zn_{mt}^{\eta}h_t^{1-\eta}. && (4.51d)
\end{aligned}$$

Embedded in this formulation is the assumption that firms do not care about effort or human capital separately. All they care about is how many units of effective labor the household delivers. Thus, in this section w_t is interpreted as wage rate per unit of effective labor. Hourly wages for an individual with human capital h_t and who supplies n_{mt} hours to the market is just $w_t z_t/n_{mt} = w_t Z n_{mt}^{\eta} h_t^{1-\eta}/n_{mt}$.

The relevant Lagrangean for the household is,

$$\begin{aligned}
L \;=\; & \sum_{t=0}^{\infty} \beta^t \{u(c_t, 1-n_t) + \lambda_t[r_t k_t - \tau_t^k(r_t-\delta_k)k_t + (1+(1-\tau_t^b)r_t^b)b_t\\
& +(1-\tau_t^n)w_t Z n_{mt}^{\eta} h_t^{1-\eta} - (1+\tau_t^c)c_t - T_t\\
& -(1+\tau_t^x)(k_{t+1}-(1-\delta_k)k_t) - b_{t+1}] + \mu_t[(1-\delta_h)h_t + Bn_{ht}^{\psi}h_t^{1-\psi} - h_{t+1}]\}.
\end{aligned}$$

As in section 2, we assume that taxes on investment goods, τ_t^x, and consumption taxes, τ_t^c, are both zero. The assumption that $\tau_t^x = 0$ is without loss of generality. However, assuming that $\tau_t^c = 0$ is not without consequences depending on the specification of the human capital accumulation technology.

The first order conditions for the household problem are:

$$
\begin{aligned}
c_t &: \quad u_c(t) = \lambda_t && \text{(4.52a)}\\
k_{t+1} &: \quad \lambda_t = \beta\lambda_{t+1}[(1-\delta_k) + (1-\tau^k_{t+1})r_{t+1} + \tau^k_{t+1}\delta_k] && \text{(4.52b)}\\
b_{t+1} &: \quad -\lambda_t + \beta[1 + (1-\tau^b_{t+1})r^b_{t+1}]\lambda_{t+1} = 0 && \text{(4.52c)}\\
n_{mt} &: \quad u_\ell(t) = \lambda_t(1-\tau^n_t)w_t\eta Z n^{\eta-1}_{mt} h^{1-\eta}_t && \text{(4.52d)}\\
n_{ht} &: \quad u_\ell(t) = \mu_t\psi B n^{\psi-1}_{ht} h^{1-\psi}_t && \text{(4.52e)}\\
h_{t+1} &: \quad \mu_t = \beta\{\mu_{t+1}[1-\delta_h + (1-\psi)Bn^{\psi}_{ht}h^{-\psi}_t] && \text{(4.52f)}\\
& \qquad +\lambda_{t+1}(1-\tau^n_{t+1})w_{t+1}(1-\eta)Zn^{\eta}_{mt+1}h^{-\eta}_{t+1}\},
\end{aligned}
$$

and the feasibility and budget constraints. Firms maximize profits given by $F(k_t, z_t) - r_t k_t - w_t z_t$. The first order conditions are,

$$
\begin{aligned}
F_k(k_t, z_t) &= r_t && \text{(4.53a)}\\
F_n(k_t, z_t) &= w_t. && \text{(4.53b)}
\end{aligned}
$$

Finally, aggregate feasibility requires that,

$$c_t + x_t + g_t \leq F(k_t, z_t).$$

As before, we concentrate on the steady state. This, it turns out, simplifies the presentation substantially. First, from equation (4.51c) and the assumption that $h_t = h_{t+1} = h$ we get,

$$1 = (1-\delta_h) + Bn^{\psi}_h h^{-\psi},$$

or,

$$n_h = h\left(\frac{\delta_h}{B}\right)^{1/\psi} \equiv h\xi_h. \tag{4.54}$$

Next, consider the steady state version of (4.52f) where we eliminate u_c/μ using (4.52d) and (4.52e) and we impose (4.54). It is given by,

$$\rho + \delta_h = B(1-\psi)\left(\frac{n_h}{h}\right)^{\psi} + \psi B\left(\frac{n_h}{h}\right)^{\psi-1}\frac{1-\eta}{\eta}\frac{n_m}{h},$$

or,

$$n_m = h\frac{\rho + \psi\delta_h}{\psi\delta_h(B/\delta_h)^{1/\psi}\frac{1-\eta}{\eta}} \equiv h\xi_m. \tag{4.55}$$

An interesting result —which depends on the specification of technology— is that the details of the tax code do not affect the hours-human capital ratios in either the market or the human capital accumulation sectors. It follows that leisure is given by $\ell = 1 - (\xi_h + \xi_m)h$, and that the supply of effective labor is $z = hZ(n_m/h)^{\eta} = hZ\xi^{\eta}_m$. Substituting these conditions in the remaining first order conditions (4.52), and imposing feasibility as well as the firms' first order conditions (4.53) one obtains,

$$\Phi(F(k, hZ\xi_m^\eta) - \delta_k k - g, 1 - (\xi_h + \xi_m)h) = F_n(k, hZ\xi_m^\eta)(1 - \tau^n) \quad (4.56a)$$

$$(1 - \tau^k)(F_k(k, hZ\xi_m^\eta) - \delta_k) = \rho. \quad (4.56b)$$

The system of equations (4.56) completely summarizes the implications of the model about the impact of changes in tax rates. From a formal point of view it is *identical* (up to some constants) to the system (4.21), with h taking the place of n. Thus, our analysis of the effects of changing the tax rates on capital and labor income to finance changes in transfers goes through with no modifications. The major findings are:

- An increase in τ^k decreases the stock of capital
- An increase in τ^k has ambiguous effects on the steady state level of h (and given (4.54) and (4.55) on the number of hours worked). If effective labor and capital are almost perfect substitutes in production, increases in τ^k increase h (and n). If, on the other hand, consumption and leisure are almost perfect substitutes in preferences, increases in τ^k decrease h (and n).
- Increases in τ^n decrease the levels of both k and h (and, of course, n).

In summary, adding human capital with an investment (in human capital) technology that is "labor intensive" does not change the *qualitative* response of the model to changes in tax rates compared to the case in which human capital is ignored. However, this does not mean that the *quantitative* implications are the same. To assess this it is necessary to numerically analyze the model.

The analysis of section 4.3 can easily be extended to this case as well. The results are basically the same.

Exercise 111 *Consider a model similar to the one described in this section except that the human capital accumulation technology is given by $h_{t+1} \leq (1 - \delta_h)h_t + x_{ht}$, where x_{ht} is market goods. In this case, aggregate feasibility is : $c_t + x_{kt} + x_{ht} + g_t \leq F(k_t, z_t)$. Go as far as you can analyzing the impact of changes in τ^k and τ^n (compensated by changes in transfers) on the steady state equilibrium quantities.*

Exercise 112 *Consider a model similar to the one described in this section except that the human capital accumulation technology is given by $h_{t+1} \leq (1 - \delta_h)h_t + x_{ht}$, where x_{ht} is market goods. In this case, aggregate feasibility is : $c_t + x_{kt} + x_{ht} + g_t \leq F(k_t, z_t)$. Allow the government to levy a consumption tax at the rate τ^c. Discuss the role of the consumption tax. In particular, determine if —as in section 3— all that matters for the steady state quantities is the ratio $(1 - \tau^n)/(1 + \tau^c)$. If not, analyze the impact of a change in the consumption tax, holding τ^n constant.*

Exercise 113 *Consider a model similar to the one described in this section except that the tax code allows for "educational rebates." Formally, assume that the base of the labor income tax is no longer wz but $wz - \epsilon(\frac{wz}{n_m})n_h$. Since the term $\frac{wz}{n_m}$ corresponds to hourly wages, $(\frac{wz}{n_m})n_h$ gives the market value of the hours allocated to human capital accumulation. We assume that only a fraction ϵ of this expense is deductible, where $0 < \epsilon < 1$. Go as far as you can analyzing the impact of changes in τ^k and τ^n (compensated by changes in transfers) on the steady state equilibrium quantities. Discuss the impact of changes in ϵ.*

Exercise 114 *Consider a model similar to the one described in this section except that the human capital accumulation technology is given by $h_{t+1} \leq (1 - \delta_h)h_t + Bn_{ht}^{\psi}h_t$. Go as far as you can analyzing the impact of changes in τ^k and τ^n (compensated by changes in transfers) on the steady state equilibrium quantities.*

4.4 Heterogeneity

So far our model has relied on the existence of a representative consumer. Even though this is quite a convenient assumption it is not very realistic. A natural question that arises is whether this model is consistent with some degree of heterogeneity across families. Fortunately, the answer is affirmative. In this section we will describe a class of economies populated by individuals who differ in terms of their initial asset holdings, their stock of "usable" labor and the form of the function $u(c)$. We will show that this heterogeneity has little impact on the aggregate behavior of the economy. More precisely, we will show that *average* variables in this economy are *identical* to those of another economy populated by a large number of representative agents. Thus, the representative agent model can be used to describe the behavior of economies with heterogeneous agents.

In addition to formally proving the equivalence between a representative agent economy and a heterogeneous agent economy, we show that —depending on the source of heterogeneity— these economies will display "consumption mobility. The basic ideas go back to the work of Gorman on aggregation.

Let the individual utility function be of the form $u_i(c) = (c+\theta_i)^{1-\eta}/(1-\eta)$, where $\eta > 0$ and θ_i can be either positive or negative. In this setting a negative θ_i can be interpreted as indicating that the minimum level of consumption is $-\theta_i$. A positive θ_i simply indicates that this household can survive with zero (market) consumption. For the equivalence result we will have to restrict all families to this class of one period utility functions, with the parameter θ_i allowed to vary across families.

Let's assume that there are N families each characterized by a vector (θ_i, a_i, e_i), where θ_i is a parameter of the utility function, a_i is the initial level of assets (which can be negative) and e_i is the endowment of labor. Thus, family i solves the following utility maximization problem,

$$\max_{[\{c_{it}\},\{a_{it+1}\}]} \sum_{t=0}^{\infty} \beta^t u_i(c_{it}) \qquad \text{(P.4)}$$

subject to

$$c_{it} + a_{it+1} \leq w_t e_i + R_t a_{it} \qquad t = 0, 1, ...$$

As before, we need to guarantee that household i will not run Ponzi schemes. To prevent this we impose the additional restriction that any feasible consumption-saving plan must satisfy $\lim_{T\to\infty} \beta^T u'(c_{iT}) a_{iT+1} = 0$. Recall that —as shown in the section on competitive equilibria— there is an alternative way of describing the consumer's problem. The key difference relative to (P.4) is that instead of facing a *sequence* of budget constraints, the family faces a *single* budget constraint. Since, this alternative representation will prove useful, we restate the utility maximization problem here. Thus, instead of (P.4) we could have described the household's problem as follows

$$\max_{[\{c_{it}\},\{a_{it+1}\}]} \sum_{t=0}^{\infty} \beta^t u_i(c_{it}) \tag{P.4'}$$

subject to

$$\sum_{t=0}^{\infty} q_t c_{it} \leq \sum_{t=0}^{\infty} q_t w_t e_i + q_0 a_{i0}. \tag{4.57}$$

In this representation, q_t is simply given by $q_t \equiv \Pi_{j=0}^{t} R_j^{-1}$, for $t \geq 1$, and, without loss of generality, we can set $q_0 = 1$. In addition to consumers, we assume that there is a large number of firms that have access to the same technology. These firms rent both capital and labor in spot markets and, hence, they do not face dynamic problems. We will assume (and this is just for convenience) that the technology exhibits constant returns to scale. In this setting there is no loss of generality in considering a representative firm. The representative firm solves,

$$\max_{[c_t,x_t,e_t,k_t]} c_t + p_k x_t - w_t e_t - r_t k_t \tag{P.5}$$

subject to,

$$c_t + x_t \leq F(k_t, e_t).$$

The (economy wide) law of motion for the capital stock is,

$$k_{t+1} \leq (1-\delta) k_t + x_t.$$

Since we have a potentially large number of different individuals, it is convenient to use a special notation for population averages. For any variable z_{it}, let $z_t = \sum_{i=1}^{N} z_{it}/N$. Thus, z_t is just the *population average* of the z_{it}. With this notation, the solution to the firm's optimization problem can be interpreted as delivering economy wide averages of each of the relevant variables. It is also convenient to define some other population moments. For any variables z_{it} and b_{it} let,

$$\begin{aligned} var(z_t) &\equiv \bar{\sigma}(z_t) \equiv \frac{\sum_{i=1}^{N} (z_{it} - z_t)^2}{N}, \\ cov(z_t, b_t) &\equiv \bar{\sigma}(z_t, b_t) \equiv \frac{\sum_{i=1}^{N} (z_{it} - z_t)(b_{it} - b_t)}{N}. \end{aligned}$$

In particular, note that

$$\theta \equiv \frac{\sum_{i=1}^{N} \theta_i}{N},$$

and

$$\begin{aligned} a_t &\equiv \frac{\sum_{i=1}^{N} a_{it}}{N}, \\ e &\equiv \frac{\sum_{i=1}^{N} e_i}{N}. \end{aligned}$$

We are now ready to define an equilibrium in this heterogeneous agent economy.

Definition 115 *A competitive equilibrium is a collection of price sequences* $[\{q_t\}_{t=0}^{\infty}, \{w_t\}_{t=0}^{\infty}, \{r_t\}_{t=0}^{\infty}, \{R_{t+1}\}_{t=0}^{\infty}]$, *an allocation* $[\{c_{it}\}_{t=0}^{\infty}, \{x_t\}_{t=0}^{\infty}, \{k_t\}_{t=0}^{\infty}, i = 1, ...N]$, *and a sequence of asset holdings* $\{a_{it}\}_{t=0}^{\infty}, i = 1, ...N$ *such that,*

a) Given the equilibrium prices, the allocation and the sequence $\{a_{it+1}\}_{t=0}^{\infty}$ *solve (P.4) for each i [utility maximization].(Note: An equivalent condition is that the allocation solves (P.4').)*

b) Given the equilibrium prices, the allocation solves (P.5) [profit maximization].

c) The allocation is feasible [market clearing, and it satisfies the law of motion for capital].

d) $a_0 = k_0 > 0$ *is given.*

We want to show that the *average quantities* corresponding to a competitive equilibrium also solve the following planner's problem,

$$\max_{[\{c_t\},\{x_t\},\{k_{t+1}\}]} \sum_{t=0}^{\infty} \beta^t \frac{(c_t + \theta)^{1-\eta}}{1-\eta},$$

subject to

$$\begin{aligned} c_t + x_t &\leq F(k_t, e_t) \\ k_{t+1} &\leq (1-\delta)k_t + x_t. \end{aligned}$$

To simplify the argument (this is not necessary but makes the algebra considerably less cumbersome) we will consider interior equilibria. The first order condition for the planner's problem is just,

$$(c_t + \theta)^{-\eta} = \beta(c_{t+1} + \theta)^{-\eta}[1 - \delta + f'(k_{t+1})],$$

or

$$c_t + \theta = (c_{t+1} + \theta)(\beta[1 - \delta + f'(k_{t+1})])^{-1/\eta}. \tag{4.58}$$

Consider next the first order condition for the utility maximization problem corresponding to family i. For this class of preferences, it is just,

$$(c_{it} + \theta_i)^{-\eta} = \beta(c_{it+1} + \theta_i)^{-\eta} R_{t+1}. \tag{4.59}$$

In any interior equilibrium, the gross real return on assets must equal the return on capital. (Note that this requires that just one family invests in capital.) It follows that $R_{t+1} = [1 - \delta + f'(k_{t+1})]$. Thus, (4.59) can be written as,

$$c_{it} + \theta_i = (c_{it+1} + \theta_i)(\beta[1 - \delta + f'(k_{t+1})])^{-1/\eta} \tag{4.60}$$

Averaging (4.60) over all individuals i, we get,

$$\frac{1}{N}\sum_{i=1}^{N}(c_{it} + \theta_i) = \frac{1}{N}\sum_{i=1}^{N}(c_{it+1} + \theta_i)(\beta[1 - \delta + f'(k_{t+1})])^{-1/\eta},$$

which, given that $\theta = N^{-1}\sum_{i=1}^{N}\theta_i$ and $c_t = N^{-1}\sum_{i=1}^{N}c_{it}$, coincides with (4.58). This proves that the average quantities of the heterogeneous agent economy are identical to the competitive allocation of a representative agent economy with homogeneous agent with "type" given by the average type of the heterogeneous agent economy. This implies that all economies that share the same values of (θ, k_0, e) have *exactly* the same aggregate behavior, *independently* of the distribution of assets, skills and preferences. Of course, the cross sectional pattern of consumption and wealth is quite different across all these economies, but aggregate behavior remains constant. This result also says that an initial asset redistribution will not affect aggregate behavior.

4.4.1 The Distribution of Individual Consumption

In order to understand the implications of the model for issues like "consumption mobility" and the cross sectional dispersion of consumption it is useful to derive a version of the consumption function. To do this, it turns out to be more convenient to work with the present value version of the budget constraint (i.e. Problem (P.4')). The first order condition corresponding to the optimal choice of consumption by family i at time t is,

$$(c_{it} + \theta_i)^{-\eta} = \lambda_i \frac{q_t}{\beta^t},$$

or,

$$c_{it} + \theta_i = \left(\lambda_i \frac{q_t}{\beta^t}\right)^{-1/\eta} \tag{4.61}$$

Using this expression in the budget constraint (4.57) we can solve for the Lagrange multiplier λ_i. A simple calculation shows that it is given by,

$$\lambda_i^{-\eta}\sum_{t=0}^{\infty} q_t(\frac{q_t}{\beta^t})^{-1/\eta} = \sum_{t=0}^{\infty} q_t w_t e_i + \sum_{t=0}^{\infty} q_t\theta_i + q_0 a_{i0}.$$

For any sequence $\mathbf{z} = \{z_t\}$, let $v(\mathbf{z}, \mathbf{q}) \equiv \sum_{t=0}^{\infty} q_t z_t$, be the *value* of the sequence $\{z_t\}$ at prices $\{q_t\}$. It follows then that λ_i can be expressed as

$$\lambda_i^{-\eta} v((\mathbf{q}/\boldsymbol{\beta})^{-1/\eta}, \mathbf{q}) = v(\mathbf{w}, \mathbf{q})e_i + v(\mathbf{1}, \mathbf{q})\theta_i + q_0 a_{i0}.$$

Let $\phi_t \equiv q_t/\beta^t$, and $\phi_0 = 1$ (i.e. we assume, without loss of generality, $q_0 = 1$ from now on). It follows that,

$$\lambda_i^{-\eta} = \frac{v(\mathbf{w},\mathbf{q})e_i + v(\mathbf{1},\mathbf{q})\theta_i + a_{i0}}{v(\boldsymbol{\phi}^{-1/\eta},\mathbf{q})}.$$

Using this expression in (4.61) we get,

$$c_{it} = \frac{v(\mathbf{w},\mathbf{q})e_i + v(\mathbf{1},\mathbf{q})\theta_i + q_0 a_{i0}}{v(\boldsymbol{\phi}^{-1/\eta},\mathbf{q})}\phi_t^{-1/\eta} - \theta_i.$$

In order to simplify the presentation, let

$$M_i \equiv \frac{v(\mathbf{w},\mathbf{q})e_i + v(\mathbf{1},\mathbf{q})\theta_i + q_0 a_{i0}}{v(\boldsymbol{\phi}^{-1/\eta},\mathbf{q})} = m_e e_i + m_\theta \theta_i + m_a a_{i0}$$

where

$$\begin{aligned} m_e &\equiv \frac{v(\mathbf{w},\mathbf{q})}{v(\boldsymbol{\phi}^{-1/\eta},\mathbf{q})} \\ m_\theta &\equiv \frac{v(\mathbf{1},\mathbf{q})}{v(\boldsymbol{\phi}^{-1/\eta},\mathbf{q})} \\ m_a &\equiv \frac{1}{v(\boldsymbol{\phi}^{-1/\eta},\mathbf{q})}. \end{aligned}$$

Also, let

$$M \equiv \frac{v(\mathbf{w},\mathbf{q})e + v(\mathbf{1},\mathbf{q})\theta + q_0 a_0}{v(\boldsymbol{\phi}^{-1/\eta},\mathbf{q})} = m_e e + m_\theta \theta + m_a a_0$$

be the economy wide population average of the M_i. (Note that since the M_i are linear functions of (e_i, θ_i, a_{i0}) it follows that the average of the M_i coincides with the function M evaluated at the average.)

With this notation, consumption of household i at time t is

$$c_{it} = M_i \phi_t^{-1/\eta} - \theta_i. \tag{4.62}$$

while aggregate per capita consumption (or average consumption) is

$$c_t = M \phi_t^{-1/\eta} - \theta. \tag{4.63}$$

Comparing (4.62) and (4.63) reveals the role played by several assumptions in order to get the aggregation result.

a) First, note that it is essential that *all individuals face the same prices.* If some families faced different sequences $\{q_t\}$ or $\{w_t\}$, then it would not be possible to aggregate in such a way that the outcome, say their consumption, is independent of the proportion of the population which faces each price.

b) The second essential feature is that aggregation of demand functions does not require knowledge of the distribution of (e_i, θ_i, a_{i0}) across households. A sufficient (and, it turns out necessary as well) condition for this is that the *Engel curves be affine*. It is this property that is the critical property delivered by our assumption of isoelastic preferences.

Define family $i's$ relative consumption as $c_{it}^R = c_{it}/c_t$. We now argue that —in the long run— the cross sectional distribution of relative consumption *is not* degenerate. That is, it is not the case that all c_{it}^R converge to one as $t \to \infty$. From our analysis of the growth model we know that c_t —interpreted as the level of consumption chosen by the representative agent— converges monotonically to c^* —the steady state level of consumption. If we assume that $k_0 < k^*$, the sequence c_t is monotone increasing. From (4.63) this implies that $\phi_t^{-1/\eta} \to \phi^{*-1/\eta} > 0$ as $t \to \infty$, for some ϕ^*. From (4.62) and (4.63) it also follows that $c_i^{R*} = [\phi^{*-1/\eta} M_i - \theta_i]/[\phi^{*-1/\eta} M - \theta]$ which, in general, is different from one.

It is also easy to determine the factors that result in a household's relative long-run consumption, c_i^{R*}, to be below one (i.e. below the economy's average). Simple calculations show that,

$$c_i^{R*} < 1 \Leftrightarrow \phi^{*-1/\eta}[m_e(e_i - e) + m_a(a_{i0} - a_0) < (\theta_i - \theta)(1 - \phi^{*-1/\eta} m_\theta). \quad (4.64)$$

From (4.64) we can infer how the different elements of a household "type" (e_i, θ_i, a_{i0}) affect c_i^{R*}.

1. *The impact of initial wealth.* To see the effect of both human (e_i) non-human wealth (a_{i0}), consider first the case in which all families have the same preferences $(\theta_i = \theta)$. In this case (4.64) implies that if the value of the human and nonhuman wealth of family i is less than the average, then its consumption will be lower as well. This, not surprisingly, just captures standard income effects.

2. *The impact of the level of minimum consumption.* Recall that, for the class of preferences that we are considering, $-\theta_i$ is interpreted as the minimum feasible consumption for family i. Even though differences in θ_i can capture a variety of effects, there are two fairly standard interpretations. First, a low θ_i household is a "large" household relative to the number of workers (e.g. many children). Second, a low θ_i household is a household that has high demand for consumption permanently (e.g. a member of the household needs some continuous medical treatment). (See Exercise 117 for other interpretations.) To isolate the effect of differences in θ_i we consider two households, i and j, such that both have levels of initial wealth and human capital equal to the economy wide average (i.e. $a_{i0} = a_{j0} = a_0$ and $e_i = e_j = e$). Since it is possible to show (see Exercise 116) that the term $1 - \phi^{*-1/\eta} m_\theta$ is negative, it follows from (4.64) that the household with higher minimum consumption (lower θ_i) has lower relative long-run consumption.

In addition to the *mean* level of long-run relative consumption, it is possible to calculate the long run cross sectional dispersion of c_i^* (of course, here we mean population dispersion since this is not a random variable). In this case the population variance of $c_i^{R*}$is

$$\bar{\sigma}^2(c_i^{R*}) = N^{-1}\sum_{i=1}^{N}\left(\frac{[\phi^{*-1/\eta}M_i - \theta_i] - [\phi^{*-1/\eta}M - \theta]}{[\phi^{*-1/\eta}M - \theta]}\right)^2,$$

or (after some really painful calculations),

$$\begin{aligned}\bar{\sigma}^2(c_i^{R*}) \quad = \quad & \frac{1}{[\phi^{*-1/\eta}M - \theta]^2}\{\phi^{*-2/\eta}m_e^2\bar{\sigma}^2(e) + \phi^{*-2/\eta}m_a^2\bar{\sigma}^2(a_0) \\ & +(\phi^{*-1/\eta}m_\theta - 1)^2\bar{\sigma}^2(\theta) + 2\phi^{*-1/\eta}m_e m_a\bar{\sigma}(e, a_0) \\ & +(\phi^{*-1/\eta}m_\theta - 1)2\phi^{*-1/\eta}[m_e\bar{\sigma}(e,\theta) + m_a\bar{\sigma}(a_0,\theta)]\}.\end{aligned} \tag{4.65}$$

Equation (4.65) summarizes the implications of the model for the dispersion of household consumption. (Note that $\bar{\sigma}(c_i^*) = \bar{\sigma}(c_i^{R*})[\phi^{*-1/\eta}M - \theta]^2$.) and highlights the role played by different factors in contributing to overall inequality. Some of the interesting implications are:

1. Two economies, A and B, that have the *same* cross sectional variability of human and non-human wealth, i.e. $\bar{\sigma}^2(e^A) = \bar{\sigma}^2(e^B)$ and $\bar{\sigma}^2(a_0^A) = \bar{\sigma}^2(a_0^B)$, can have different degrees of relative consumption variability depending on the covariances of these two variables, even if there are no differences in preferences.

2. A *positive* covariance between any two of the elements that determine a household's type (e_i, θ_i, a_{i0}) increases the long run cross sectional variance of consumption.

3. The model does not imply that statements like "controlling for initial wealth inequality, all countries display the same amount of consumption inequality," are to be taken seriously. The reason for this is that —as shown above— the model implies that, in addition to measures of dispersion, the degree of long run inequality also depends on the population covariances of the inequality-causing factors.

4.4.2 The Time Path of Consumption Inequality

In this section we consider the implications of the model for the time path of c_{it}. From (4.62) and (4.63) individual and aggregate consumption are related according to,

$$c_{it} = \frac{M_i}{M}c_t + \frac{M_i}{M}\theta - \theta_i. \tag{4.66}$$

It follows that there is no presumption that all individual household consumption will track aggregate consumption in any particular way. More precisely, (4.66) is consistent with $c_{it} > c_t$ for some t and $c_{it} < c_t$ for some other

t. To see this consider the special case in which $e_i = e$, and $a_{i0} = a_0$, i.e. households differ only in terms of their minimum consumption, $-\theta_i$. Individual consumption is given by

$$c_{it} = \tilde{m}\phi_t^{-1/\eta} + (m_\theta \phi_t^{-1/\eta} - 1)\theta_i,$$

where

$$\tilde{m} \equiv m_e e + m_a a_0,$$

which, by assumption, is common to all households. Thus, from (4.62) and (4.63) it follows that

$$c_{it} = c_t \quad \Leftrightarrow \quad (m_\theta \phi_t^{-1/\eta} - 1) = 1.$$

Let's ignore the "integer" problem associated with the assumption that in a discrete time model, t can only be integer valued, and let's assume that $\exists \hat{t}$ such that $m_\theta \phi_{\hat{t}}^{-1/\eta} = 1$. Then, at $t = \hat{t}$ $c_{it} = c_t$ for all i, and the economy displays no consumption inequality. Since

$$c_{it} - c_t = (m_\theta \phi_t^{-1/\eta} - 1)(\theta - \theta_i),$$

$m_\theta - 1 > 0$, and $\phi_0^{-1/\eta} = 1$, it follows that if $\theta_i < \theta$

$$c_{it} - c_t \gtreqless 0 \Leftrightarrow t \lesseqgtr \hat{t}.$$

Thus, households with high minimum consumption (low θ_i) initially —at time 0— consume more than the average household. At $t = \hat{t}$, every household, independently of θ_i, consumes exactly the same amount, and for $t > \hat{t}$, low θ_i households consume less than the average. In this sense, low θ_i households have a relatively flat time profile of consumption. Of course, high θ_i households behave in exactly the opposite way, displaying time profiles of consumption that are steeper than the average.

This model displays a significant amount of "consumption mobility," in the sense that the families that at time zero have consumption higher than the average will, in the long run, have consumption levels that fall short of the aggregate level of consumption. Moreover, cross-sectional measures of consumption dispersion will display a U-shape since the initial disparity across households is completely eliminated at $t = \hat{t}$, only to observe further increases for $t > \hat{t}$.

In the more general case in which all three elements of a household type are allowed to vary, the cross sectional (population) variance of relative consumption at time t is just the obvious analog of (4.65) and it is given by,

$$\begin{aligned}\bar{\sigma}^2(c_{it}^R) \quad = \quad & \frac{1}{[\phi_t^{-1/\eta} M - \theta]^2}\{\phi_t^{-2/\eta} m_e^2 \bar{\sigma}^2(e) + \phi_t^{-2/\eta} m_a^2 \bar{\sigma}^2(a_0) + \\ & (\phi_t^{-1/\eta} m_\theta - 1)^2 \bar{\sigma}^2(\theta) + 2\phi_t^{-1/\eta} m_e m_a \bar{\sigma}(e, a_0) + \\ & (\phi_t^{-1/\eta} m_\theta - 1) 2\phi_t^{-1/\eta}[m_e \bar{\sigma}(e,\theta) + m_a \bar{\sigma}(a_0,\theta)]\}.\end{aligned} \tag{4.67}$$

The case analyzed above corresponds to $\bar{\sigma}^2(e) = \bar{\sigma}^2(a_0) = \bar{\sigma}(e, a_0) = \bar{\sigma}(e, \theta) = \bar{\sigma}(a_0, \theta)$, and in this case it is clear that $\bar{\sigma}^2(c_i^{R*}) = 0 \Leftrightarrow \phi_t^{-1/\eta} m_\theta - 1 = 0$. Equation (4.67) also illustrates how other factors affect inequality.

For example, in an economy with no differences in preferences ($\theta_i = \theta$) or in human capital ($e_i = e$) the (population) standard deviation of relative consumption is $\bar{\sigma}(c_{it}^R) = \phi_t^{-1/\eta} m_a \bar{\sigma}(a_0)/[\phi_t^{-1/\eta} M - \theta]$, which is always positive. Ignoring that t is discrete, and taking the derivative with respect to time it follows that

$$\frac{\partial \bar{\sigma}(c_{it}^R)}{\partial t} = -\theta \frac{m_a \bar{\sigma}(a_0)}{[\phi_t^{-1/\eta} M - \theta]^2} \frac{\partial \phi_t^{-1/\eta}}{\partial t}.$$

Since $\partial \phi_t^{-1/\eta}/\partial t < 0$ (see Exercise 116) cross sectional consumption inequality may increase or decrease over time depending on the value of the economy-wide θ. If $\theta < 0$ preferences are characterized by a relatively low degree of intertemporal substitution or, equivalently, savings are not very responsive to changes in interest rates. In this case consumption inequality grows. Conversely, when $\theta > 0$, the interest elasticity of savings is high and consumption inequality decreases over time.

To sum up, the representative agent model is consistent with a variety of patterns of consumption and income inequality. Even though fairly special preferences are needed in order to obtain the appropriate aggregation result, those preferences are not "special enough" to have sharp implications about the time path of inequality.

Exercise 116 *Prove that* $1 - m_\theta > 0$, *and that* $1 - \phi^{*-1/\eta} m_\theta < 0$. *Show that there is a unique* t *(consider* t *to be a continuous variable) such that* $1 - \phi_t^{-1/\eta} m_\theta = 0$. *This value of* t *is what the text labeled* $\hat{t}$.

Exercise 117 *Consider a two period model. Preferences of the representative agent are given by the following separable utility function*

$$U = \frac{(c_1 - \theta)^{1-\eta}}{1-\eta} + \frac{\beta (c_2 - \theta)^{1-\eta}}{1-\eta}.$$

Assume that individuals are endowed with $w > 0$ *units of consumption in the first period and zero in the second. Individuals can borrow and lend at the gross rate* R. *Assume that* $R\beta > 1$.

i) Show that if s *is saving, then an increase in* θ *decreases saving.*

ii) What does i) say about the saving behavior of families that have some members who require high levels of minimum consumption (young and elderly).

iii) What is the prediction of this model for the relative consumption in both the first and second period of two families; one with θ *less than the average and the other has* θ *greater than the average.*

iv) What can you say about the connection between minimum consumption levels, θ, *and the interest elasticity of saving?*

v) Can you give some economic intuition for your findings in i)-iv)?

Exercise 118 *Consider a special case of the model in this section in which there are no differences in human and non-human wealth (i.e. $e_i = e$ and $a_{i0} = a_0$).*

i) An economist claims that the model must imply cross sectional convergence in the sense that $\bar{\sigma}(c_i^*) < \bar{\sigma}(c_{i0})$. Do you agree?

Exercise 119 *Consider the economy described in Exercise 67. and suppose that*

$$u_i(c) = \frac{(c-\theta)^{1-\eta}}{1-\eta},$$

where $\eta > 0$, and $\theta_i \geq 0$. Do not assume that the economy is at the steady state. However, retain the assumption of complete markets. Evaluate the following claims:

i) Economist A argues that the aggregate behavior of this economy is indistinguishable from that of a single agent economy. Economist B disagrees.

ii) Economist B argues that a valid test of the perfect market hypothesis is that every household's consumption should grow at the same rate. Economist A disagrees.