Repeated Games I

A general framework

Let G be a normal form game with action spaces $A_1, \ldots, A_I$, payoff functions $g_i : A \to R$, where $A = A_1 \times \ldots \times A_I$.

Let $G^\infty(\delta)$ be the infinitely repeated version of G played at $t = 0, 1, 2, \ldots$ where players discount at δ and observe all previous actions.

A history is $H^t = \{(a_1^0, \ldots, a_I^0), \ldots, (a_1^{t-1}, \ldots, a_I^{t-1})\}$.

A (pure) strategy is $s_{i,t} : H^t \to A_i$.

The average discounted payoff is:

$$u_i(s_i, s_{-i}) = (1-\delta)\sum_{t=0}^{\infty} \delta^t g_i(s_i(h^t), s_{-i}((h^t))).$$

Our goal is to study the set of average payoffs that are associated to SPE of the repeated game as a function of δ.

A few constraints immediately bound this set:

Definition. *The set of feasible payoffs is the set of* R^I *vectors* C*:*

$$(v_1,\ldots,v_I) \in Co\Big((v_1,\ldots,v_I)\Big|\exists(a_1,\ldots,a_I) \text{ with } g_i(a) = v_i \forall i\Big)$$

Naturally the set of equilibria must be included in this set.

Another constraint is individual rationality.

Definition. *A player's min-max payoff is:*

$$\underline{v}_i = \min_{s_{-i}} \max_{s_i} g_i(s_i, s_{-i})$$

Here s_i is a mixed strategy.

Definition. *A payoff vector is individually rational if $v_i \geq \underline{v}_i$ $\forall i$.*

It is easy to see that in any Nash equilibrium payoffs must be individually rational.

To see this suppose s_i^*, s_{-i}^* is a Nash equilibrium. If $v_i < \underline{v}_i$ then let s_i^{**} be the best response to s_{-i}^*:

$$\begin{aligned}
u_i(s_i^{**}, s_{-i}^*) &= \max_{\tilde{s}_i} u_i(\tilde{s}_i, s_{-i}^*) \\
&\geq (1-\delta) \max_{s_i} \sum_{t=0}^{\infty} \delta^t g_i(s_i, s_{-i}(h^t)) \\
&\geq (1-\delta) \sum_{t=0}^{\infty} \delta^t \min_{s_{-i}} \max_{s_i} g_i(s_i, s_{-i})) \\
&= \min_{s_{-i}} \max_{s_i} g_i(s_i, s_{-i}) = \underline{v}_i > v_i = g_i(s_i^*, s_{-i}^*),
\end{aligned}$$

so s_i^* is not a best response to s_{-i}^*.

Classic Folk Theorem: perfect monitoring

We start from the first basic result, *Folk theorem in Nash equilibrium*.

This will highlight some key ideas in a simple setting.

But it will also highlight what is missing from a more satisfying result, i.e. the Folk Theorem in SPE.

Theorem. (Nash Folk Theorem) *If* $(v_1, \ldots, v_I)$ *is feasible and strictly individually rational, then there exists* $\delta^* < 1$ *such that for all* $\delta > \delta^*$*, there is a Nash Equilibrium of* $G^\infty(\delta)$ *with average payoffs* $(v_1, \ldots, v_I)$*.*

Assume there exists a profile $a = (a_1, \ldots, a_I)$ such that $g_i(a) = v_i$ for all i.

We do this for simplicity, such an action profile may not exist. We will return on this later.

Let m_{-j}^{j} denote the strategy profile of players other than j that holds j to at most $\underline{v}_j$ and write m_j^j for i's best-response to m_{-j}^{j}. Let $m^j = \left(m_j^j, m_{-j}^j\right)$

Now consider the following strategies:

State I. Play $a = (a_1, \ldots, a_I)$ if there was no deviation or if there was more than one deviation.

State II. If j deviates, play m^j forever.

Let us verify this is a Nash equilibrium using the one-stage-deviation principle.

If a is played, then j receives $(1-\delta)\left(v_i + \delta \frac{v_i}{1-\delta}\right) = v_i$.

With a deviation: $(1-\delta)\left(\bar{v}_i + \delta \frac{\underline{v}_i}{1-\delta}\right)$.

So the deviation is not profitable:

$$v_i + \delta \frac{v_i}{1-\delta} \geq \bar{v}_i + \delta \frac{\underline{v}_i}{1-\delta}$$

$$\Leftrightarrow (1-\delta)(\bar{v}_i - v_i) \leq \delta(v_i - \underline{v}_i).$$

As $\delta \to 1$, we have $(1 - \delta)(\bar{v}_i - v_i) \to 0$, so this condition is always verified.

Note that here we are using the fact that v_i is *strictly* IR.

What is the problem here?

The problem is that we are asking players $-j$ to minimax j after j's deviation.

The action profile that minimaxes j, i.e. m^j, may be associated to payoffs that are below the minmax value of some $i \neq j$.

So the subgame corresponding to Stage II may not be a SPE.

Consider this example:

	A	B
A	$8, 8$	$0, -50$
B	$10, 1$	$0, -50$

Note that here $\underline{v}_1 = 0$ and $\underline{v}_2 = 1$.

So $8, 8$ is feasible and individually rational, the Nash Folk Theorem says that we can achieve it as a Nash equilibrium.

After a deviation by 1, however, the strategies seen above call for minmaxing 1. forever.

This implies that 2 chooses B forever, implying a payoff of $-50 < \underline{v}_2$.

The strategies in the subgame after a deviation cannot be a NE.

Theorem (Fudenberg and Maskin's (1986) Folk Theorem)
Let V^* be the set of feasible and strictly individually rational payoffs.

Assume that $\dim V^* = I$.

Then for any $(v_1, \ldots, v_I) \in V^*$, there exists a $\delta^{**} < 1$, such that for any $\delta > \delta^{**}$, there is a subgame perfect equilibrium of $G^\infty(\delta)$ with average payoffs $(v_1, \ldots, v_I)$.

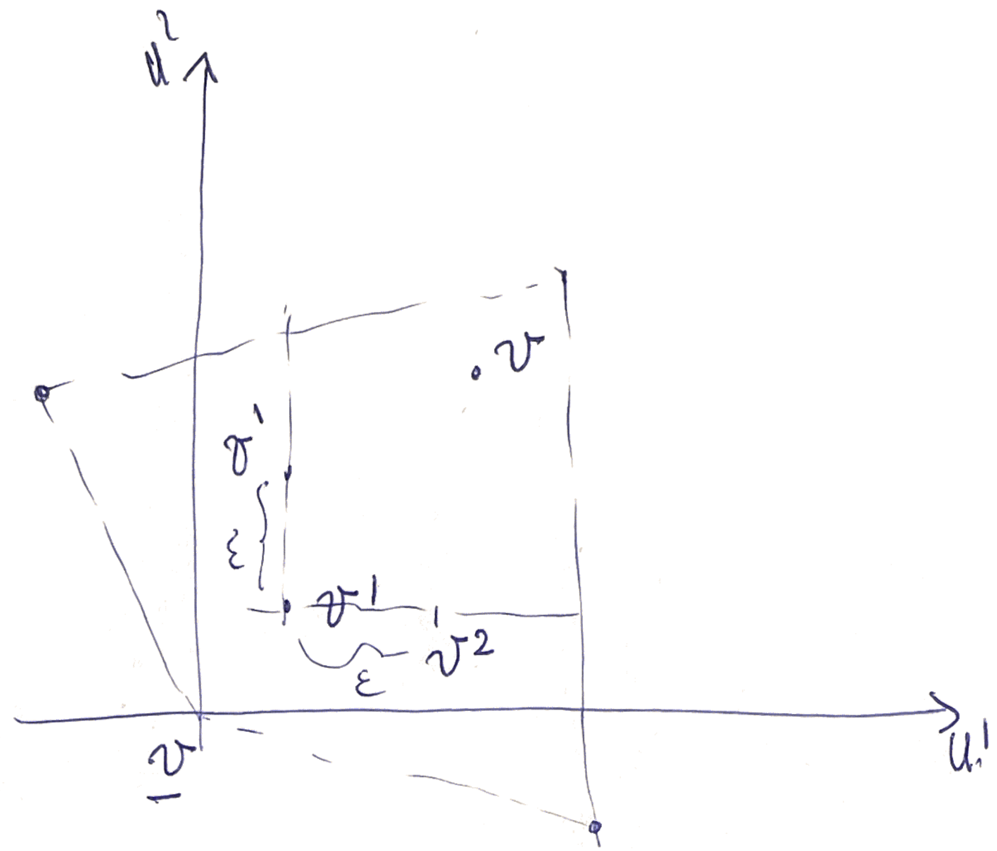
Fixing a payoff vector $(v_1, \ldots, v_I) \in V^*$, we construct a SPE that achieves it.

For convenience, let's assume that there is some profile $(a_1, \ldots, a_I)$ such that $g_i(a) = v_i$ for all i.

Choose $v' \in Int(V^*)$ such that $\underline{v}_i < v'_i < v_i$ for all i.

Choose N such that:

$$\max_a g_i(a) + N\underline{v}_i < \min_a g_i(a) + Nv'_i$$


u^2
v
v'
ε
v^1
v^2
ε
$\underline{v}$
u^1

Choose $\varepsilon > 0$ such that for each i:

$$v'(i) = \left(v'_1 + \varepsilon, \ldots, v'_{i-1} + \varepsilon, \;\; \mathbf{v}'_i, \;\; v'_{i+1} + \varepsilon, \ldots, v'_I + \varepsilon\right).$$

Assume there is an a^i such that $g(a^i) = v'(i)$

Assume there is a pure strategy profile m^i that minimaxes i, so $g_i(m^i) = \underline{v}_i$. We will return on this later.

We now construct the following "carrot and stick" strategies:

Stage I.

- Play a_i so long as no player deviates from $(a_1, \ldots, a_I)$.
- If j alone deviates, go to II_j . (If two or more players simultaneously deviate, play stays in I.)

Stage II_j.

- Play m^j for N periods, then go to III_j if no one deviates.
- If k deviates, re-start II_k.

Stage III_j.

- Play a^j so long as no one deviates.
- If k deviates, go to II_k.

To check that these are equilibrium strategies, we verified that in all subgames it is optimal to follow them.

Subgame I. Consider i's payoff to playing the strategy and deviating:

If i follows the strategy : $(1-\delta)[v_i + \delta v_i + \ldots] \;=\; v_i$

If i deviates : $(1-\delta)(\max_a g_i(a) + \delta \underline{v}_i + \ldots + \delta^N \underline{v}_i + \delta^{N+1} v_i' \ldots)$

The second of which is obviously lower for large δ since $\underline{v}_i < v_i' < v_i$.

Subgame II_i. (suppose there are $N' \leq N$ periods left)

i follows strategy:

$$(1-\delta^{N'})g_i(m^i) + \delta^{N'} v_i' = q(N')$$
$$= (1-\delta)g_i(m^i) + \delta q(N'-1)$$

where $g_i(m^i)$ is the payoff at the mimimax strategy m^i for i.

If i deviates :

$$(1-\delta)\max_a g_i(a, m_{-i}^i) + \delta(1-\delta^N)\underline{v}_i + \delta^{N+1} v_i'$$
$$= (1-\delta)g_i(m^i) + \delta q(N) < (1-\delta)g^i(m^i) + \delta q(N'-1).$$

Subgame II_j. (suppose there are $N' \le N$ periods left)

i follows strategy : $(1 - \delta^{N'})g_i(m^j) + \delta^{N'}(v_i' + \varepsilon)$

i deviates : $(1 - \delta)\max_a g_i(a, m_{-i}^j) + \delta(1 - \delta^N)\underline{v}_i + \delta^{N+1}v_i'$

Subgame III_i. Consider i's payoff to playing the strategy and deviating:

i follows strategy : v_i'

i deviates : $(1-\delta)\max_a g_i(a, a_{-i}^i) + \delta(1-\delta^N)\underline{v}_i + \delta^{N+1}v_i'$.

But

$$
\begin{aligned}
&\leq (1-\delta)\max_a g_i(a) + \delta(1-\delta^N)\underline{v}_i + \delta^{N+1}v_i' \\
&= (1-\delta)\left[\max_a g_i(a) + \delta\frac{1-\delta^N}{1-\delta}\underline{v}_i\right] + \delta^{N+1}v_i' \\
&\simeq (1-\delta)\left[\max_a g_i(a) + N\underline{v}_i\right] + \delta^{N+1}v_i' \\
&< (1-\delta)\left[\min_a g_i(a) + Nv_i'\right] + \delta^{N+1}v_i' < v_i'
\end{aligned}
$$

Where we are using:

$$\max_a g_i(a) + N\underline{v}_i < \min_a g_i(a) + Nv_i'.$$

For $j \neq i$, it also can be verified deviating is unprofitable for δ large.

Notes

At two steps we assumed the existence of action profiles a, a^i and m^i that generates utility v, v^i and $\underline{v}_i$.

If this is not the case, we can generates the payoffs if we have **public randomizations**.

In the punishment phase II, we are assuming that the **minimax punishment** can be implemented with a **pure strategy**.

If this is not the case, we need to make sure that the players are willing to use a mixed minimax strategy.

To this goal, a player i must be willing to mix over a set of actions. This is possible only if the player is indifferent among the actions.

For this to be the case, the payoff promised to i after II_j may have to depend on the realization of minimax actions in phase II_j.

This is possible since the exact value of $v'(i)$ is not important.

This of course considerably complicates the proof.

Fudenberg and Maskin's theorem generalizes the Folk theorem under a mild assumption: full rank, i.e. $\dim V^* = I$.

The assumption may be weakened (what is really necessary is that no two players have payoffs that are **affine transformations** of each others).

However a qualification on payoffs is necessary.

Consider the following example.

P1 selects rows, P2 column and P3 matrix (A on the left and B on the right).

A

	A	B
A	$\mathbf{1, 1, 1}$	$\mathbf{0, 0, 0}$
B	$\mathbf{0, 0, 0}$	$\mathbf{0, 0, 0}$

B

	A	B
A	$\mathbf{0, 0, 0}$	$\mathbf{0, 0, 0}$
B	$\mathbf{0, 0, 0}$	$\mathbf{1, 1, 1}$

In this game the minimax is 0 for all players.

The set of feasible, individually rational payoffs is
$V^* = \{(v, v, v) : v \in [0, 1]\}$.

Can we obtain all of these payoffs as SPE?

Let $\underline{v} = \inf\left\{v \text{ s.t. } (v, v, v) \text{ is a SPE payoff}\right\}$.

For v to be a SPE we need:

$$v \geq \frac{1}{4}(1-\delta) + \delta \underline{v}$$

since there must be at least two players among the 3 with $s_i(A) \geq 1/2$ or $s_i(B) \geq 1/2$ in the first period

Say $s_1(A) \geq 1/2$ or $s_2(B) \geq 1/2$.

But then $\underline{v} \geq \frac{1}{4}(1-\delta) + \delta \underline{v} \Leftrightarrow \underline{v} \geq \frac{1}{4}$, since 3 can choose A in the first period.

So there is no equilibrium with payoff, say, $(1/8, 1/8, 1/8)$.