Macroeconomics, PhD core

Lecture #3-4

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Lecture Road Map

- Competitive equilibrium.
- Recursive equilibrium.
- Taxes.

Competitive Equilibrium

- How many markets are open?
- The simplest case: all markets are open
- This case is cumbersome and not very realistic.
- Instead we look at a (recursive) market structure.

- Firms rent inputs in spot markets (do not own capital)

$$\max_{\{k_t, n_t\}} c_t + p_{k_t} x_t - r_t k_t - w_t n_t$$

  subject to

$$c_t + x_t \leq F(k_t, n_t) \qquad t = 0, 1, \ldots$$

  and a bunch of non-negativity constraints.

- **Assumption:** F is strictly increasing in each argument, concave, twice continuously differentiable and homogeneous of degree 1 (HOD1)
- Why do we assume this?

Competitive Equilibrium

Firm Problem

- In an interior solution $p_{k_t} = 1$ (why?) and

$$r_t = F_k(k_t, n_t)$$

$$w_t = F_n(k_t, n_t)$$

  Assume that the number of firms equal the number of workers (why? is it ok?)

- Let $f(k) = F(k, 1)$
- Equilibrium factor prices satisfy

$$r_t = f'(k_t)$$

$$w_t = f(k_t) - k_t f'(k_t)$$

- Firms choose the level of employment

Competitive Equilibrium

Household Problem

- Households solve

$$\max_{\{c_t, x_t, k_{t+1}, b_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$c_t + p_{k_t} x_t + b_{t+1} \leq w_t + r_t k_t + R_t b_t, \qquad t = 0, 1, \ldots$$

$$k_{t+1} \leq x_t + k_t (1 - \delta) \qquad t = 0, 1, \ldots$$

$$\lim_{T \to \infty} \beta^T u'(c_T) b_{T+1} = 0$$

$$(c_t, x_t, k_{t+1}) \geq (0, 0, 0) \qquad t = 0, 1, \ldots.$$

- Why do we need to impose a No-ponzi scheme condition on b_t?
- Alternatively, one could impose that $b_t \geq \underline{b}$ for some value $0 > \underline{b} > -\infty$

- Lagrangean

$$
\begin{aligned}
L(\mathrm{c}, \mathrm{x}, \mathrm{k}, \breve{}, \grave{}, \mathrm{fl}) \quad = \quad & \sum_{t=0}^{\infty} \beta^t \{ u(c_t) + \\
& + \lambda_t \left[w_t + r_t k_t + R_t b_t - c_t - p_{k_t} x_t - b_{t+1} \right] \\
& + \theta_t \left[(1-\delta) k_t + x_t - k_{t+1} \right] \\
& + \gamma_{1t} c_t + \gamma_{2t} x_t + \gamma_{3t} k_{t+1} \}
\end{aligned}
$$

- Conditions for the Kuhn-Tucker theorem to be satisfied.

- FOCs

$$c_t : u'(c_t) = \lambda_t$$

$$x_t : \lambda_t = \theta_t$$

$$k_{t+1} : \theta_t = \beta\left[\theta_{t+1}(1-\delta) + \lambda_{t+1} r_{t+1}\right]$$

$$b_{t+1} : \lambda_t = \beta R_{t+1} \lambda_{t+1}$$

$$TVC_k : \lim_{T\to\infty} \beta^T \lambda_T k_{T+1} = 0$$

$$TVC_b : \lim_{T\to\infty} \beta^T \lambda_T b_{T+1} = 0$$

- Euler equation

$$u'(c_t) = \beta u'(c_{t+1}) \left[1 - \delta + r_{t+1}\right]$$

One could use the FOC of the optimal choice of capital by the representative firm to obtain

$$u'(c_t) = \beta u'(c_{t+1}) \left[1 - \delta + f'(k_{t+1})\right]$$

- The Euler equation for bonds is

$$u'(c_t) = \beta u'(c_{t+1}) R_{t+1}$$

Definition

A recursive competitive equilibrium (RCE) is a collection of price sequences $\left[\{w_t^*\}, \{r_t^*\}, \{p_{kt}^*\}, \{R_t^*\}\right]$, $t = 0, 1, \ldots$, an allocation $\left[\{c_t^*\}, \{x_t^*\}, \left\{k_{t+1}^*\right\}\right]$, $t = 0, 1, \ldots$, and a sequence of bond holdings $\left\{b_{t+1}^*\right\}$ such that,

a) Given prices, the allocation and the sequence of bonds $\left\{b_{t+1}^*\right\}$ is utility maximizing.

b) Given prices, the allocation is profit maximizing.

c) Markets clear.

d) $b_0^* = b_0 = 0$, $k_0^* = k_0 > 0$ is given. (Why is b_0 given?)

- **Claim:** Any budget feasible allocation must satisfy

$$\sum_{t=0}^{\infty} q_t c_t = \sum_{t=0}^{\infty} q_t w_t + q_0 \left[1 - \delta + r_0\right] k_0 + q_o R_0 b_0$$

  where $q_t / q_0 = \Pi_{j=1}^{t} R_j^{-1}$ for $t \geq 1$.
- q_t / q_0 is the price of consumption at time t (in terms of consumption at time 0).
- These are the prices in an Arrow-Debreu economy with time zero trading.

Proof.
(brute force) Budget constraint at $t = 0$

$$b_1 = w_0 + r_0 k_0 - c_0 - x_0 \tag{1}$$

while at $t = 1$ is

$$c_1 + x_1 + b_2 = w_1 + r_1 k_1 + R_1 b_1 \tag{2}$$

Solve (2) for b_1 and replace in (1)

$$\begin{aligned} c_0 + x_0 + R_1^{-1}(c_1 + x_1) &= w_0 + r_0 k_0 + R_1^{-1}(w_1 + r_1 k_1) \\ &\quad + R_0 b_0 - R_1^{-1} b_2 \end{aligned}$$

$\square$

...Repeated substitution implies (use the next budget constraint to eliminate b_2 and "pick up" b_3)

$$\sum_{t=0}^{T} q_t \left(c_t + x_t\right) = \sum_{t=0}^{T} q_t (w_t + r_t k_t) + q_o R_0 b_0 - q_T b_{T+1}$$

Proof.

Claim: *Feasibility requires that* $\lim_{T\to\infty} q_T b_{T+1} = 0$
No-Ponzi implies $\lim_{T\to\infty} \beta^T \lambda_T b_{T+1} = 0$. The Euler equation on bonds dictates $\beta^T \lambda_T = \lambda_0 \Pi_{j=1}^T R_j^{-1}$ or $\beta^T \lambda_T = \lambda_0 q_T$.
As $T \to \infty$

$$\sum_{t=0}^{\infty} q_t(c_t + x_t) = \sum_{t=0}^{\infty} q_t(w_t + r_t k_t) + q_o R_0 b_0$$

Are we done? Not yet .... □

Proof.

...Next we want to show

$$
\begin{aligned}
& \sum_{t=0}^{\infty} q_t \left(r_t k_t - x_t \right) \\
= \; & \sum_{t=0}^{\infty} q_t \left(r_t k_t - k_{t+1} + (1-\delta) k_t \right) = q_0 \left[1 - \delta + r_0 \right] k_0
\end{aligned}
$$

$\square$

Proof.

Rearrange the terms in the second sum to read

$$\begin{aligned}
&\lim_{T\to\infty} q_0 \left[1-\delta+r_0\right] k_0 + \\
&+k_1 \left[-q_0 + q_1 \left(1-\delta+r_1\right)\right] + \dots \\
&.. + k_T \left[-q_{T-1} + q_T \left(1-\delta+r_T\right)\right] - q_T k_{T+1}
\end{aligned}$$

If $-q_t + q_{t+1}\left(1-\delta+r_{t+1}\right) > 0 \to$ optimal choice of k_{t+1} is ∞ (look at the euler equation)
If $-q_t + q_{t+1}\left(1-\delta+r_{t+1}\right) < 0 \to$ optimal choice of k_{t+1} is 0
If $-q_t + q_{t+1}\left(1-\delta+r_{t+1}\right) = 0 \to$ we are done!
Next, $\lim_{T\to\infty} q_T k_{T+1} = 0$. Why? TVC $\square$

- **Claim:** In an equilibrium, $b_t^* = 0$ for all t.

Proof.

Household budget constraint

$$w_t^* + r_t^* k_t^* + R_t^* b_t^* = c_t^* + x_t^* + b_{t+1}^*$$

or, using the equilibrium values of (w_t^*, r_t^*)

$$f(k_t^*) - k_t^* f'(k_t^*) + k_t^* f'(k_t^*) + R_t^* b_t^* = c_t^* + x_t^* + b_{t+1}^*$$

In equilibrium, market cleariing dictates $f(k_t^*) = c_t^* + x_t^*$
Hence, the budget constraint for the "private sector" yields

$$R_t^* b_t^* = b_{t+1}^* \text{ and } b_0 = 0 \quad \text{yields the result}$$

□

Taxes and Deficits

- How do deficits affect the economy?
- Result (Ricardian Equivalence):
  - budget policy "does not matter", i.e. for a given sequence $\{g_t\}$ all **non-distortionary tax structures** that raise the appropiate level of revenue are associated to the **same real** allocation.
  - timing of the tax collection is irrelevant, in the sense that prices and allocations are independent of the timing. This result is often described as the Ricardian proposition

Proof.

Let g $= \{g_t\}$ a sequence of expenditure. Define $\o = \{\tau_t\}$ the sequence of lump sum taxes to finance g and any initial debt. Consider an alternative sequence of taxes $\widehat{\o}$ such that $\widehat{\tau}_0 < \tau_0$

$$\sum_{t=0}^{\infty} q_t c_t = \sum_{t=0}^{\infty} q_t (w_t - \tau_t) + q_0 \left[1 - \delta + r_0\right] k_0 + q_o R_0 b_0 \quad \text{(HH)}$$

(assume one-period bonds only)

$$R_t b_t + g_t = \tau_t + b_{t+1}$$

□

Proof.

$$\sum_{t=0}^{\infty} q_t(\tau_t - g_t) = q_0 R_0 b_0 - q_T b_{T+1}$$

Impose a No-Ponzi scheme condition, $\lim_{T\to\infty} q_T b_{T+1} = 0$
Then $\sum_{t=0}^{\infty} q_t \tau_t = \sum_{t=0}^{\infty} q_t \widehat{\tau}_t$, and

$$\sum_{t=0}^{\infty} q_t c_t = \sum_{t=0}^{\infty} q_t(w_t - g_t) + q_0 \left[1 - \delta + r_0\right] k_0$$

$\square$

Proof.

Guess and Verify: Assume prices don't change after the tax change
Then given that the budget constraint of the household is the same, it choose the same consumption and investment sequences.
Firms maximize profits at the same allocations than before.
Market clearing requires

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t - g_t$$

which is satisfied since the sequence of g did not change. $\square$

Taxes and Deficits

Ricardian Equivalence

- Why is that higher deficit does not generate shifts in the interest rate?
- For $\widehat{\tau}_0 < \tau_0$
$$b_1 + \tau_0 = g_0 + R_0 b_0$$
$$\widehat{b}_1 + \widehat{\tau}_0 = g_0 + R_0 b_0$$
so
$$\widehat{b}_1 - b_1 = \tau_0 - \widehat{\tau}_0$$
Debt goes up!
- But agents understand that they are not richer because lower taxes today will be compensated with higher taxes in the future
- Households save (they increase their demand for loans in the same amount that the government wishes to increase its debt)

Distortionary taxes

Representative Household

$$\max_{\{c_t, n_t, x_t, k_{t+1}, b_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_t)$$

subject to

$$(1+\tau_t^c)c_t + (1+\tau_t^x)x_t + b_{t+1} \leq$$
$$(1-\tau_t^n)w_t n_t + r_t k_t - \tau_t^k (r_t - \delta_t) k_t + (1 + (1-\tau_t^b) r_t^b) b_t,$$

$$k_{t+1} \leq x_t + k_t (1-\delta) \qquad t = 0, 1, \ldots$$

$$\lim_{T \to \infty} \beta^T u'(c_T) b_{T+1} = 0$$

$$(c_t, x_t, k_{t+1}) \geq (0, 0, 0) \qquad t = 0, 1, \ldots.$$

Representative firm

$$\max_{k_t, n_t} = F(k_t, n_t) - r_t k_t - w_t n_t$$

Household

$$u'(c_t) = \lambda_t(1+\tau_t^c) \tag{c_t}$$

$$(1+\tau_t^x)\lambda_t = \beta\lambda_{t+1}\left[(1-\delta_k)(1+\tau_{t+1}^x) + (1-\tau_{t+1}^k)r_{t+1} + \tau_{t+1}^k\delta_k\right] \tag{k_{t+1}}$$

$$\lambda_t = \beta\lambda_{t+1}\left[1 + (1-\tau_{t+1}^b)r_{t+1}^b\right] \tag{b_{t+1}}$$

$$n_t : u_{lt} = \lambda_t(1-\tau_t^w)w_t \tag{n_t}$$

$$TVC_k : \lim_{T\to\infty}\beta^T\lambda_T k_{T+1} = 0 \tag{TVC}$$

$$\lim_{T\to\infty}\beta^T\lambda_T b_{T+1} = 0 \tag{No ponzi}$$

Firms

$$F_k(k_t, n_t) = r_t \qquad F_n(k_t, n_t) = w_t$$

Distortionary taxes

Steady states: taxes are constant (why?)

$$\rho + \tau^x(\rho + \delta_k) = (1 - \tau^k)(F_k(k, n) - \delta_k) \quad \text{(k)}$$

$$1 = \beta [1 + \rho] \quad \text{(b)}$$

where $\rho \equiv (1 - \tau^b) r^b$

$$n_t : u_l = u_c \frac{1 - \tau^w}{1 + \tau^c} F_l(k, n) \quad \text{(n)}$$

Feasibility

$$F(k, n) = c + g + \delta_k k$$

Distortionary taxes

Steady states: taxes are constant (why?)

$$\rho + \tau^x(\rho + \delta_k) = (1 - \tau^k)(F_k(k, n) - \delta_k) \tag{k}$$

$$1 = \beta\,[1 + \rho] \tag{b}$$

where $\rho \equiv (1 - \tau^b) r^b$

$$n_t : u_l = u_c \frac{1 - \tau^w}{1 + \tau^c} F_l(k, n) \tag{n}$$

Feasibility

$$F(k, n) = c + g + \delta_k k$$

- Taxes on gvment bonds just change r^b.
- Consumption and income taxes are "equivalent".
- Investment and capital taxes are "equivalent".

Theorem

Let $\left[\left(\mathrm{w}^*, \mathrm{r}^*, \mathrm{p}_k^*, \mathrm{R}^*\right), \left(\mathrm{c}^*, \mathrm{x}^*, \mathrm{k}^*\right), \mathrm{b}^*\right]$ *be an interior competitive equilibrium, then* $\left(\mathrm{c}^*, \mathrm{x}^*, \mathrm{k}^*\right)$ *solves the planner''s problem (First Welfare Theorem)*

Proof.

Since $b^* = 0$ in any equilibrium, the household budget constraint is

$$f(\mathrm{k}^*) = \mathrm{c}^* + \mathrm{x}^*$$

$$\mathrm{k}^{*\prime} = (1-\delta)\,\mathrm{k}^* + \mathrm{x}^*$$

where $\mathrm{k}^{*\prime}$ is a vector of capital stocks with first element k_1^*. Also, the optimality conditions for the household and the firm together imply

$$u'(c_t) = \beta u'(c_{t+1})\left[(1-\delta) + f'(k_{t+1})\right]$$

$$\lim_{T\to\infty} \beta^T u'(c_T) k_{T+1} = 0$$

$\square$

Theorem

Let $(\mathrm{c}^*, \mathrm{x}^*, \mathrm{k}^*)$ *be an interior solution to the planner's problem. Then, there exists prices,* $(\mathrm{w}^*, \mathrm{r}^*, \mathrm{p}_k^*, \mathrm{R}^*)$, *and a sequence of bond holdings* b^* *such that* $[(\mathrm{w}^*, \mathrm{r}^*, \mathrm{p}_k^*, \mathrm{R}^*), (\mathrm{c}^*, \mathrm{x}^*, \mathrm{k}^*), \mathrm{b}^*]$ *is a competitive equilibrium of an economy with a representative agent who has initial financial wealth* $b_0 = 0$, *and initial capital holdings* $k_0 > 0$ *equal to the initial endowment of capital of the economy (Second Welfare Theorem)*

Proof.

Let $b^* = 0$. Since the solution to the planner's problem is feasible conditions c) and d) of the definition of CE are automatically satisfied.

Define

$$w^* = f(k^*) - f'(k^*)k^*$$

$$r^* = f'(k^*)$$

$$R^* = (1 - \delta) + f'(k^*)$$

At this prices the optimality conditions of the firms and households are satisfied. $\square$

Proof.

We need to check that the candidate solution satisfies the TVCs. The one on bonds is trivially satisfied as $b^* = 0$. For the second one, set $\lambda_t = u'(c_t)$ and then the TVC of the consumer problem in capital is the same as the one the planner faces.

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