ECON 6110, Microeconomic Theory III	Section 1
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1 A few results about mixed strategies

Definition 1.1.

- $N = \{1, \dots, n\}$: set of players.
- A_i: set of actions of player i.
- $A = \Pi_{i=1}^n A_i$: set of profiles of actions. $\times$ is the Cartesian product of sets. $A \times B$ is the set of all ordered pairs of (a, b) where $a \in A, b \in B$.
- $a = (a_1, \dots, a_n) \in A$.

Definition 1.2.

- $\alpha_i : A_i \to [0, 1]$ such that $\sum_{a_i} \alpha_i(a_i) = 1$: probability distribution over i's actions. We also call α_i a mixed action.
- $\Delta(A_i)$: set of probability distribution over i's actions. $\alpha_i \in \Delta(A_i)$.
- $\Pi_{i=1}^n \Delta(A_i)$: set of profiles of mixed actions.
- $\alpha = (\alpha_1, \dots, \alpha_n) \in \Pi_{i=1}^n \Delta(A_i)$.

Remark. $\alpha_i(U) = \alpha_i(D) = \frac{1}{2}$, can also write as a shorthand $\alpha_i = \frac{1}{2}U + \frac{1}{2}D$.

Remark. $\Pi_{i=1}^n \Delta(A_i) \neq \Delta(\Pi_{i=1}^n A_i) = \Delta(A)$.

Remark. In the definition of Nash equilibrium, the domain of the best response correspondence is $\Pi_{j\neq i}\Delta(A_j)$. In correlated equilibrium and rationalizable actions, the domain of the belief is $\Delta(A_{-i}) = \Delta(\Pi_{j\neq i}A_j)$.

Definition 1.3 (Mixed Nash equilibrium)**.** A mixed Nash equilibrium of a game $\langle N, (A_i), (u_i)\rangle$ is a profile $\alpha^* = (\alpha_1^*, \ldots, \alpha_n^*)$ of mixed actions such that for every $i \in N$:

$$U_i(\alpha_i^*, \alpha_{-i}^*) \geq U_i(\alpha_i, \alpha_{-i}^*)$$

for all $\alpha_i \in \Delta(A_i)$.

Definition 1.4 (Two ways to decompose the utility of mixed actions)**.**

$$\begin{aligned} U_i(\alpha) &= \sum_{a_1,\ldots,a_n} u_i(a_1,\ldots,a_n)\cdot\alpha_1(a_1)\cdots\alpha_n(a_n) \\ &= \sum_{a_i} U_i(a_i, \alpha_{-i})\cdot\alpha_i(a_i) \end{aligned}$$

Definition 1.5. $supp(\alpha_i) = \{a_i \in A_i : \alpha_i(a_i) > 0\}$.

Proposition 1.1 (Randomize out of indifference)**.** *If $\alpha_i^* = (\alpha_i^*, \alpha_{-i}^*)$ is a Nash equilibrium, then*

$$U_i(a_i^*, \alpha_{-i}^*) = U_i(\alpha_i^*, \alpha_{-i}^*)$$

for all $a_i^* \in supp(\alpha_i^*)$.

Proposition 1.2. *For every i and $\alpha = (\alpha_i, \alpha_{-i})$, the followings are equivalent:*

1. *For all $\alpha_i' \in \Delta(A_i)$,*

$$U_i(\alpha_i, \alpha_{-i}) \geq U_i(\alpha_i', \alpha_{-i})$$

2. *For all $a_i \in supp(\alpha_i)$ and $\alpha_i' \in \Delta(A_i)$,*

$$U_i(a_i, \alpha_{-i}) \geq U_i(\alpha_i', \alpha_{-i})$$

3. *For all $a_i \in supp(\alpha_i)$, $a_i' \in A_i$, and $\alpha_i' \in \Delta(A_i)$,*

$$U_i(a_i, \alpha_{-i}) \geq U_i(a_i', \alpha_{-i})$$

2 Exercise

Find all mixed strategy Nash equilibria of the following game.

	L	R
U	$8,3$	$3,1$
M	$7,5$	$4,4$
D	$3,3$	$7,5$

Proof. First step: check if any strategies are strictly dominated, because strictly dominated strategies cannot be part of NE. None of the strategies are dominated by a pure strategy. Check if M is dominated by a mixed strategy between U and D? Suppose the probability on U is x. For M to be dominated, we would need:

$$\begin{cases} 8x + 3(1-x) > 7 \\ 3x + 7(1-x) > 4 \end{cases}$$

Simplifying, we get:

$$\begin{cases} x > 4/5 \\ x < 3/4 \end{cases}$$

which is not possible. So M is not strictly dominated.

Since we are interested in the mixed NEs only, we assume player 2 mixes between L and R with probability q and $1-q$. But with three strategies for player 1, we need to guess and verify all possible supports for Player 1.

Case 1: Can a mixed NE exist with support in all three strategies? By Proposition 1.2, this means player 1 is indifferent between all three strategies when player 2 is mixing with probability q.

$$8q + 3(1-q) = 7q + 4(1-q) = 3q + 7(1-q)$$

Not possible.

Case 2: Can a mixed NE exist with support in $\{U, M\}$? Player 1 must be indifferent between U and M when player 2 is mixing with probability q:

$$8q + 3(1-q) = 7q + 4(1-q) \implies q = 1/2$$

Check when $q = 1/2$, U (or M) gives a higher payoff than D.

$$8q + 3(1-q) \geq 3q + 7(1-q)$$

Also, player 2 must be indifferent between L and R when player 1 is mixing with p on U and $1-p$ on M.

$$3p + 5(1-p) = p + 4(1-p)$$

No solution!

Case 3: Can a mixed NE exist with support in $\{M, D\}$?

$$7q + 4(1-q) = 3q + 7(1-q) \implies q = 3/7$$

Check with $q = 3/7$, M (or D) gives a higher payoff than U.

$$7q + 4(1-q) \geq 8q + 3(1-q)$$

Player 2 must be indifferent between L and R when player 1 is mixing with p on M and $1-p$ on D.

$$5p + 3(1-p) = 4p + 5(1-p) \implies p = 2/3$$

Case 4: Can a mixed NE exist with support in $\{U, D\}$?

$$8q + 3(1-q) = 3q + 7(1-q) \implies q = 4/9$$

Check with $q = 4/9$, U (or D) gives a higher payoff than M.

$$8q + 3(1-q) \geq 7q + 4(1-q)$$

This doesn't hold!

Taking together, there is only one mixed NE: player 1 mixing between M and D with probability $(2/3, 1/3)$ and player 2 mixing with probability $(3/7, 4/7)$.

Remark. If player 1 has m strategies and player 2 has n strategies, need to check $O(2^{m+n})$ possible combination of supports! For this reason, finding Nash equilibria takes exponential time in the worst case. The hardness of finding Nash equilibria is an important result in complexity theory and algorithmic game theory.

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