ECON 6110

Problem Set 4

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Problem 1

(a) If both players play C, then we will have G with probability p and B with probability $1-p$. The expected payoff will be

$$g_i(C,C) = p \cdot \left(1 + \frac{2(1-p)}{p-q}\right) + (1-p) \cdot \left(1 - \frac{2p}{p-q}\right) = \frac{p-q}{p-q} = 1$$

If both players play D, then we will have G with probability r and B with probability $(1-r)$. The expected payoff will be

$$g_i(D,D) = r \cdot \frac{2(1-r)}{q-r} + (1-r) \cdot \frac{-2r}{q-r} = \frac{0}{q-r} = 0$$

If one player plays C and the other plays D, we will have G with probability q and B with probability $1-q$. The expected payoff for each of the two player types will be

$$g_i(C,D) = q \cdot \left(1 + \frac{2(1-p)}{p-q}\right) + (1-q) \cdot \left(1 - \frac{2p}{p-q}\right) = \frac{q-p}{p-q} = -1$$
$$g_i(D,C) = q \cdot \frac{2(1-r)}{q-r} + (1-q) \cdot \frac{-2r}{q-r} = \frac{2q-2r}{q-r} = 2$$

The interpretation of this is that this game is, in stages, a Prisoner's Dilemma. Each player is strictly (in expectation) incentivized to defect, as that will net them a higher expected payoff.

(b) We will use $a = (D,D)$ and $w(G) = v$, $w(B) = v'$. We will first show that $v = (1-\delta)g(a) + \delta \sum_y \pi(y \mid a)w(y)$ (for each player, but the game is symmetric so this is without loss). We have that

$$(1-\delta)g(D,D) + \delta \left((1-r) \cdot \frac{\delta r}{1-\delta(p-r)} + r \cdot \frac{1-\delta+\delta r}{1-\delta(p-r)}\right) = 0 + \delta \cdot \frac{r}{1-\delta(p-r)} = v$$

It remains to show incentive compatibility, which will hold as long as

$$\begin{aligned}
v &\geq (1-\delta)\cdot(-1) + \delta v + \delta q \cdot (v'-v) \\
(1-\delta)(v+1) &\geq \delta q \cdot (v'-v) \\
(1-\delta)\frac{1-\delta p + 2\delta r}{1-\delta(p-r)} &\geq (1-\delta)\frac{\delta q}{1-\delta(p-r)} \\
1 - \delta p + 2\delta r &\geq \delta q \\
1 &\geq \delta(p+q-2r)
\end{aligned}$$

So this is enforceable as long as $\delta \leq \frac{1}{p+q-2r}$.

(c) We will use $a = (C,C)$ and the same $w(\cdot)$ as above. First, we need to show that $v' = (1-\delta)g(a) + \delta \sum_y \pi(y \mid a)w(y)$. We have that

$$(1-\delta)g(C,C) + \delta \cdot ((1-p)\cdot v + p \cdot v') = (1-\delta) + \delta \cdot v - \delta \cdot p \cdot v + \delta \cdot p \cdot v'$$

which becomes the extremely gross

$$\frac{1-\delta p+\delta r-\delta+\delta^2 p-\delta^2 r+\delta^2 r-\delta^2 pr+\delta p-\delta^2 p+\delta^2 pr}{1-\delta(p-r)}=\frac{1-\delta+\delta r}{1-\delta(p-r)}=v'$$

It remains to show incentive compatibility. We need that

$$\begin{aligned}
v' &\geq 2(1-\delta)+\delta v+\delta q(v'-v)\\
(1-\delta\cdot q)v' &\geq 2(1-\delta)+(1-q)\delta\cdot v\\
(1-\delta+\delta r)(1-\delta q) &\geq 2(1-\delta)(1-\delta(p-r))+(1-q)\delta^2 r\\
1-\delta+\delta r-\delta q+\delta^2 q-\delta^2 rq &\geq 2-2\delta-2\delta p+2\delta^2 p+2\delta r-2\delta^2 r+\delta^2 r-\delta^2 rq\\
\delta(2p-r-q) &\geq 1-\delta+\delta^2(2p-q-r)\\
\delta(2p-r-q)(1-\delta) &\geq 1-\delta\\
\delta &\geq \frac{1}{2p-r-q}
\end{aligned}$$

So this is enforceable as long as $\delta \geq \frac{1}{2p-r-q}$.

Problem 2

(a) The action spaces are technically $p_i \in \mathbb{R}_+$. Since any $p_i > a$ is trivially not rationalizable, we restrict attention to $p_i \in [0, a]$, which is without loss. The state space is $\{b_L, b_H\}^2$, the Cartesian product of the types – specifically, the state is a tuple of types for each firm. The type space is simply $\{b_L, b_H\}$, and the prior beliefs are the same for each agent, where $p_i(t_j = b_L) = \theta$ and $p_i(t_j = b_H) = 1-\theta$ for each i. Pure strategies map each type b_i to a price p_i, and are well-defined, so each pure strategy is a tuple $(p_i(b_L), p_i(b_H))$. Finally, utility functions for a type b_i agent are expected utility functions given an agents type and the strategy that the opponent plays. Formally, given a strategy p,

$$u_i(p; b_i) = \theta \cdot p_i(b_i) \cdot q_i(p_i(b_i), p_j(b_L)) + (1-\theta) \cdot q_i(p_i(b_i), p_j(b_H))$$

(b) Observe that there can be no symmetric strategies where $p_i = 0$ for all i, since $0 \cdot q_i(0,0) = 0 < \varepsilon \cdot q_i(a-\varepsilon, 0) > 0$ for any $\varepsilon < a$. For this reason we restrict attention to strictly positive prices. Firm i maximizes their expected utility under symmetric strategies, so they choose p_i to maximize each of

$$u_i(p; b_L) = \theta \cdot p_i(b_L) \cdot (a - p_i(b_L) - b_L \cdot p_j(b_L)) + (1-\theta) \cdot p_i(b_L) \cdot (a - p_i(b_L) - b_L \cdot p_j(b_H))$$

and

$$u_i(p; b_H) = \theta \cdot p_i(b_H) \cdot (a - p_i(b_H) - b_H \cdot p_j(b_L)) + (1-\theta) \cdot p_i(b_H) \cdot (a - p_i(b_H) - b_H \cdot p_j(b_H))$$

Since these functions are concave, we can find the maximum using the first order conditions:

$$\frac{\partial u_i(\sigma; b_L)}{\partial p_i(b_L)} = \theta \cdot (a - 2p_i(b_L) - b_L \cdot p_j(b_L)) + (1-\theta) \cdot (a - 2p_i(b_L) - b_L \cdot p_j(b_H)) = 0$$

so we have that

$$a - 2p_i(b_L) - b_L \cdot (\theta \cdot p_j(b_L) + (1-\theta) \cdot p_j(b_H)) = 0 \Longrightarrow p_i^\star(b_L) = \frac{a - b_L \cdot (\theta \cdot p_j(b_L) + (1-\theta) \cdot p_j(b_H))}{2}$$

Similarly, the second equation gives us that

$$p_i^\star(b_H) = \frac{a - b_H \cdot (\theta \cdot p_j(b_L) + (1-\theta) \cdot p_j(b_H))}{2}$$

Define $\mathbb{E}[p_k] = \theta \cdot p_k(b_L) + (1-\theta) \cdot p_k(b_H)$. Since we are restricting our attention to symmetric strategies, we have that

$$\mathbb{E}[p_j^\star] = \theta \cdot p_i^\star(b_L) + (1-\theta) \cdot p_i^\star(b_H)$$

So

$$\mathbb{E}[p_j^\star] = \theta \cdot \frac{a - b_L \mathbb{E}[p_j^\star]}{2} + (1-\theta) \cdot \frac{a - b_H \mathbb{E}[p_j^\star]}{2} \Longrightarrow \mathbb{E}[p_j^\star] = \frac{a}{2 + \theta \cdot b_L + (1-\theta) \cdot b_H}$$

Finally, we get that

$$\begin{aligned} p_i^\star(b_L) &= \frac{a - b_L \cdot \mathbb{E}[p_j^\star]}{2} = \frac{a}{2}\left[1 - \frac{b_L}{2 + \theta \cdot b_L + (1-\theta) \cdot b_H}\right] \\ p_i^\star(b_H) &= \frac{a - b_H \cdot \mathbb{E}[p_j^\star]}{2} = \frac{a}{2}\left[1 - \frac{b_H}{2 + \theta \cdot b_L + (1-\theta) \cdot b_H}\right] \end{aligned}$$

and we can verify that these are strictly positive by the assumption that $2 - \theta(b_H - b_L) > 0$. Thus, this is a symmetric pure-strategy Bayesian Nash equilibrium.

Problem 3

Observe first that for type b, D strictly dominates U, so $a_1(b) = D$ always. We first consider pure strategy Bayesian Nash equilibria. For player 2, if $a_1(b) = D$, the best response is to play R when $t_1 = b$. Suppose that player 2 always plays R. Then type a also prefers D, so one pure strategy Bayesian Nash equilibrium is $(a_1(a) = D, a_1(b) = D, a_2 = R)$.

Suppose that player 2 always plays L. Then type a prefers U strictly, so $a_1(a) = U$, and again $a_1(b) = D$. The expected utility of each action for player 2 is thus:

$$\begin{aligned} u(L) &= 0.9 \cdot 2 + 0.1 \cdot -2 = 1.6 \\ u(R) &= 0.9 \cdot 0 + 0.1 \cdot 0 = 0 \end{aligned}$$

so this is incentive compatible for player 2 as well. Thus, another pure strategy Bayesian Nash equilibrium is $(a_1(a) = U, a_1(b) = D, a_2 = L)$.

We now move to mixed strategy equilibria. Recall that type b will always play D, so in any mixed strategy it must be that player 2 plays L with some probability p, and type a plays U with some probability q. For type a of player 1 to be indifferent, we need that

$$\begin{aligned} u_1(U; a) &= p \cdot 2 + (1-p) \cdot (-2) = 4p - 2 \\ u_1(D; a) &= p \cdot 0 + (1-p) \cdot 0 = 0 \\ \text{indifference } &\implies p = 0.5 \end{aligned}$$

For player 2 to be indifferent, we need that

$$\begin{aligned} u_2(L) &= 0.9 \cdot (q \cdot 2 + (1-q) \cdot -2) + 0.1 \cdot (-2) = 3.6 \cdot q - 2 \\ u_2(R) &= 0.9 \cdot (q \cdot 0 + (1-q) \cdot 0) + 0.1 \cdot 0 = 0 \\ \text{indifference } &\implies q = \frac{5}{9} \end{aligned}$$

So we have exactly one mixed strategy. In summation, we have three equilibria:

$$\begin{gathered} (a_1(a) = D, a_1(b) = D, a_2 = R) \\ (a_1(a) = U, a_1(b) = D, a_2 = L) \\ \left(a_1(a) = \frac{5}{9}U + \frac{4}{9}D, a_1(b) = D, a_2 = \frac{1}{2}L + \frac{1}{2}R\right) \end{gathered}$$