ECON 6200 : Section 1

1/24/25

TA Info

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Email me questions in advance!

Homework

- graded by the TA
- will be graded generously
- work in groups!

Note: HW1 due Jan 31

Sections

- Upload notes in advance
- Reupload annotated notes after class

Data Matrix & Vector Notation

$$
\begin{aligned}
\hat{\beta} &= (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y} \\
&= (\mathbb{E}_n XX')^{-1}\mathbb{E}_n XY \\
&= \left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^{n} X_i Y_i \\
&= \left(\sum_{i=1}^{n} X_i X_i'\right)^{-1} \sum_{i=1}^{n} X_i Y_i.
\end{aligned}
$$

x_1

$$
\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{k1} \\ x_{21} & \ddots & & \\ \vdots & & \ddots & \\ x_{n1} & \cdots & \cdots & x_{nk} \end{bmatrix}
$$

$n \times k$

$$
= \begin{bmatrix} \text{——} & x_1^T & \text{——} \\ \text{——} & x_2^T & \text{——} \\ & \vdots & \\ \text{——} & x_k^T & \text{——} \end{bmatrix}
$$

WTS : $\sum_{i=1}^{n} x_i x_i' = \mathbf{X}'\mathbf{X}$

$$\sum_{i=1}^{n} x_i x_i' = \underbrace{\begin{bmatrix} | \\ x_1 \\ | \end{bmatrix}}_{k\times 1} \underbrace{\begin{bmatrix} — & x_1 & — \end{bmatrix}}_{1\times k} + \underbrace{\begin{bmatrix} | \\ x_2 \\ | \end{bmatrix}}_{k\times 1} \underbrace{\begin{bmatrix} — & x_2 & — \end{bmatrix}}_{1\times k} + \cdots$$

$$\text{where } \begin{bmatrix} | \\ x_i \\ | \end{bmatrix} = \begin{bmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{ik} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} x_{11}^2 & x_{11}x_{12} & \cdots & x_{11}x_{1k} \\ x_{12}x_{11} & & & \vdots \\ \vdots & & & \vdots \\ x_{1k}x_{11} & \cdots & & x_{k1}^2 \end{bmatrix}}_{k\times k} + \underbrace{\begin{bmatrix} & & \\ & & \end{bmatrix}}_{k\times k} + \cdots$$

$$= \begin{bmatrix} \sum_{i=1}^{n} x_{i1}^2 & \sum x_{i1}x_{i2} & \cdots & \sum x_{i1}x_{ik} \\ \sum x_{i2}x_{i1} & \cdots & & \vdots \\ \vdots & & & \\ \sum x_{ik}x_{i1} & \sum x_{ik}x_{i2} & \cdots & \sum x_{ki}^2 \end{bmatrix}$$

$$X'X = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_k \\ | & | & & | \end{bmatrix} \begin{bmatrix} \text{---} & x_1 & \text{---} \\ \text{---} & x_2 & \text{---} \\ & \vdots & \\ \text{---} & x_k & \text{---} \end{bmatrix}$$

$$\underset{k \times n}{} \qquad \underset{n \times k}{}$$

$$= \begin{bmatrix} x_{11} & x_{21} & \cdots & x_{n1} \\ x_{12} & & & \\ \vdots & & & \\ x_{1k} & \cdots & & x_{nk} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1k} \\ x_{21} & & & \vdots \\ \vdots & & & \\ x_{n1} & \cdots & & x_{nk} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^{n} x_{i1}^2 & \sum_{i=1}^{n} x_{i1}x_{i2} & \cdots & \sum x_{i1}x_{ik} \\ \sum x_{i2}x_{i1} & & \ddots & \\ \vdots & & & \\ \sum x_{ik}x_{i1} & \cdots & & \sum x_{ik}^2 \end{bmatrix}$$

Linear Model

n : # of observations / sample size

k : # of regressors / covariates

X, Y, Z : vectors (capitalized letters)

$\mathbf{X}, \mathbf{Y}, \mathbf{Z}$: Matrix (bold)

$\mathbb{E}_n$, $\hat{Q}$: sample analogs

In data matrix notation:

$$\underset{n\times 1}{\mathbf{Y}} = \underset{n\times k}{\mathbf{X}} \underset{k\times 1}{\beta} + \underset{n\times 1}{\mathbf{e}}$$

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} X_{11} & X_{21} & \cdots & X_{1k} \\ X_{21} & \cdots & \cdots & X_{2k} \\ \vdots & \ddots & & \vdots \\ X_{n1} & \cdots & \cdots & X_{nk} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

* show them matrix diff rules (Prop 5, 9)

⊛ Derive OLS Estimator in Matrix Form

$$\min_{\beta} (Y - X\beta)'(Y - X\beta)$$

Recall $(AB)^T = B^T A^T$

$$= Y'Y - Y'X\beta - \beta'X'Y + \beta'X'X\beta$$

$$= Y'Y - Y'X\beta - (Y'X\beta) + \beta'X'X\beta$$

$$= Y'Y - 2Y'X\beta + \beta'\underbrace{X'X}_{\text{symmetric}}\beta$$

FOC

$[\beta]$ $$-2Y'X + 2X'X\beta = 0$$

If $(X'X)^{-1}$ exists, then $\hat{\beta} = (X'X)^{-1}(X'Y)$

Simple Linear Regression

$$y = \beta_0 + \beta_1 X + \mu$$

The OLS estimator is defined as

$$(\hat{\beta}_0, \hat{\beta}_1) \equiv \underset{b_0, b_1}{\arg\min} \sum_{i=1}^{n} (y_i - b_0 - b_1 x_i)^2$$

FOC

$$[b_0] \quad -2 \sum_{i=1}^{n} (y_i - b_0 - b_1 x_i) = 0 \quad ①$$

$$[b_1] \quad -2 \sum_{i=1}^{n} (y_i - b_0 - b_1 x_i) x_i = 0 \quad ②$$

If we simplify ①, we get

$$\sum_{i=1}^{n} (y_i - b_0 - b_1 x_i) = 0$$

$$\frac{1}{n} \sum_{i=1}^{n} (y_i - b_0 - b_1 x_i) = 0$$

multiply both sides by $\frac{1}{n}$

$$\bar{y} - b_0 - b_1 \bar{x} = 0$$

$$\Rightarrow \hat{\beta}_0 = \bar{y} - b_1 \bar{x}$$

Substituting $\hat{\beta}_0$ into ②

$$\sum_{i=1}^{n} (y_i - b_0 - b_1 x_i) x_i = 0$$

$$\sum_{i=1}^{n} ((y_i - \bar{y}) + b_1 \bar{x} - b_1 x_i) x_i = 0$$

$$\sum_{i}^{n} x_i (y_i - \bar{y}) + b_1 \sum x_i (\bar{x} - x_i) = 0$$

$$\Rightarrow \hat{\beta}_1 = \frac{\sum x_i (y_i - \bar{y})}{\sum x_i (x_i - \bar{x})}$$

OR you could do some more work and get

$$\hat{\beta}_1 = \frac{\sum_{i}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i} (x_i - \bar{x})^2}$$

Frisch - Waugh - Lowell Thm

Intuition: FWL Thm simplifies how we interpret multiple regression models

- To get the effect of X_1 on Y (holding X_2 fixed), we must remove the correlation between X_1 and X_2

(Hansen, pg 82)

Theorem 3.5 Frisch-Waugh-Lovell (FWL)
In the model (3.31), the OLS estimator of β_2 and the OLS residuals $\widehat{\boldsymbol{e}}$ may be computed by either the OLS regression (3.32) or via the following algorithm:

1. Regress $\boldsymbol{Y}$ on $\boldsymbol{X}_1$, obtain residuals $\widetilde{\boldsymbol{e}}_1$;
2. Regress $\boldsymbol{X}_2$ on $\boldsymbol{X}_1$, obtain residuals $\widetilde{\boldsymbol{X}}_2$;
3. Regress $\widetilde{\boldsymbol{e}}_1$ on $\widetilde{\boldsymbol{X}}_2$, obtain OLS estimates $\widehat{\beta}_2$ and residuals $\widehat{\boldsymbol{e}}$.

Suppose we want to find $\hat{\beta}_2$ where the model

$$Y = \beta_1 X_1 + \beta_2 X_2$$

1) $Y = \tilde{\beta}_1 X_1 + \tilde{e}_1$; $\tilde{e}_1$ = effect of Y not explained by X_1

2) $X_2 = \tilde{\gamma} X_1 + \tilde{X}_2$; $\tilde{X}_2$ = effect of X_2 not explained by X_1

3) $\tilde{e}_1 = \hat{\beta}_2 \tilde{X}_2 + \hat{e}$

Since we have eliminated all the variation of Y explained by X_1 and any correlation between X_1 and X_2, our $\hat{\beta}_2$ coefficient is equivalent to that of our original model

$$\hat{Y} = \hat{\beta}_1 X_1 + \hat{\beta}_2 X_2 + \bar{e}$$

2. (from an old exam) A researcher separately regresses an outcome Y on two different sets of covariates. Specifically, consider separate estimation by OLS of the models

$$\begin{aligned} Y &= \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \varepsilon, \\ Y &= \gamma_0 + \gamma_1 Z_1 + \gamma_2 Z_2 + \eta, \end{aligned}$$

where the outcome variable Y is the same in both regressions and where the covariates are deterministically linked by the equalities

$$\begin{aligned} Z_1 &= X_1 - 2X_2, \\ Z_2 &= X_1 + 4X_2. \end{aligned}$$

2.1 Show that (in data matrix notation) one can write

$$\boldsymbol{Z} = \boldsymbol{X}\boldsymbol{A}.$$

What is $\boldsymbol{A}$? (IE, please give the exact matrix.) Is $\boldsymbol{A}$ invertible? If so, give $\boldsymbol{A}^{-1}$. (Solution of this question does not require mechanical inversion of a 3×3 matrix.)

2.2 How are $\hat{\beta}$ and $\hat{\gamma}$ related? Let $\hat{\boldsymbol{Y}}_Z$ denote the vector of fitted values from the regression of $\boldsymbol{Y}$ on $\boldsymbol{Z}$ and $\hat{\boldsymbol{Y}}_X$ the analogous vector but for the regression of $\boldsymbol{Y}$ on $\boldsymbol{X}$. Prove: $\hat{\boldsymbol{Y}}_Z = \hat{\boldsymbol{Y}}_X$.

2.3 What can you say about SST, SSR, SSE, and R^2 from the regression on $\boldsymbol{X}$, compared to the same quantities from the regression on $\boldsymbol{Z}$?

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \varepsilon$$

$$\underset{n\times 1}{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}} = \underset{n\times 3}{\begin{array}{ccc} & X_1 & X_2 \\ \left[\begin{matrix} 1 \\ 1 \\ \vdots \end{matrix}\right. & \begin{matrix} \vdots \\ \\ \vdots \end{matrix} & \left.\begin{matrix} \vdots \\ \\ \vdots \end{matrix}\right] \end{array}} \underset{3\times 1}{\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

$$Z_1 = X_1 - 2X_2$$

$$Z_2 = X_1 + 4X_2$$

$$\mathbf{Z} = \mathbf{XA}$$

$$\underset{1\times 3}{\begin{bmatrix} 1 & Z_1 & Z_2 \end{bmatrix}} = \underset{1\times 3}{\begin{bmatrix} 1 & X_1 & X_2 \end{bmatrix}} \underset{3\times 3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}}$$

2.2)

$$\hat{Y}_Z = \hat{\gamma}_0 + \hat{\gamma}_1 Z_1 + \hat{\gamma}_2 Z_2 = \mathbf{Z}\hat{\gamma}$$

$$\hat{Y}_X = \hat{\beta}_0 + \hat{\beta}_1 X_1 + \hat{\beta}_2 X_2 = \mathbf{X}\hat{\beta}$$

$$\mathbf{Z} = \mathbf{X}\mathbf{A}$$

(pretend bold)

$$\begin{aligned}
\hat{\gamma} &= (Z'Z)^{-1} Z'Y \\
&= ((XA)'XA)^{-1}(XA)'Y \\
&= (A'X'XA)^{-1}(XA)'Y \\
&= ((A'X'XA)')^{-1}(XA)'Y \\
&= ((A'X'XA)^{-1})'(XA)'Y \\
&= A^{-1}(X'X)^{-1}\cancel{(A^{-1})'A'}X'Y \\
&= A^{-1}\underbrace{(X'X)^{-1}X'Y}_{\hat{\beta}} \\
&= A^{-1}\hat{\beta}
\end{aligned}$$

Prove $\hat{Y}_Z = \hat{Y}_X$

$$\hat{Y}_Z = Z\hat{\gamma}$$

$$= Z(A^{-1}\hat{\beta})$$

$$= X\hat{\beta} = \hat{Y}_X$$

$Z = XA$

$ZA^{-1} = X$

2.3)

$$\sum_i (y_i - \bar{y})^2$$

$$= \sum ((y_i - \hat{y}_i) + \hat{y}_i - \bar{y})^2$$

$$= \sum (\hat{\varepsilon}_i + (\hat{y}_i - \bar{y}))^2$$

$$= \sum \hat{\varepsilon}_i^2 + \sum (\hat{y}_i - \bar{y})^2 + \cancelto{0}{2\sum \hat{\varepsilon}_i(\hat{y}_i - \bar{y})}$$

since $\hat{\varepsilon} \perp X$ and $\hat{Y} = X\hat{\beta}$

$\Rightarrow \hat{\varepsilon} \perp \hat{Y}$

$\Rightarrow$ SSE, SSR, R^2 same