ECON 6100

Problem Set 3

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1. We know that a matrix A is productive if and only if the $(I-A)^{-1}$ has non-negative columns and is non-singular. We have that

$$I - A_1 = \begin{bmatrix} 0.4 & -0.2 & -0.1 \\ -0.3 & 0.8 & -0.4 \\ -0.2 & -0.4 & 0.7 \end{bmatrix} \text{ and so } (I-A_1)^{-1} \approx \begin{bmatrix} 5.405 & 2.432 & 2.162 \\ 3.919 & 3.514 & 2.568 \\ 3.784 & 2.703 & 3.514 \end{bmatrix}$$

which has strictly positive columns and has full rank, so the A_1 is productive. We also have that

$$I - A_2 = \begin{bmatrix} 0.4 & -0.5 \\ -0.1 & 0.5 \end{bmatrix} \text{ and so } (I-A_2)^{-1} = \begin{bmatrix} 10/3 & 10/3 \\ 2/3 & 8/3 \end{bmatrix}$$

which has strictly positive columns and is full rank, so A_2 is productive.

2. ***Proof.*** ($\Rightarrow$) We have that A is productive, meaning that there exists $x^\star \gg 0$ such that $x^\star \gg Ax^\star$. Note that $Ix^\star \gg 0$, so $(I-A)^{-1}(I-A)x^\star \gg 0$, and since $(I-A)x^\star \gg 0$ since $x^\star \gg Ax^\star$, we have that $(I-A)^{-1} \gg 0$.

   ($\Leftarrow$) We have that $(I-A)^{-1}$ has non-negative columns, $(I-A)^{-1}x \geq 0$ for any $x \geq 0$. Define $x^\star = (I-A)^{-1}e_j$ for some index j, and since $e_j \geq 0$, $x^\star \geq 0$. Further, $(I-A)x^\star = e_j \implies x^\star \gg Ax^\star$, so A is productive. $\square$

3. ***Proof.*** FSOC, assume that every column sum of $A \in \mathbb{R}^{n\times n}$ is greater than 1. Then taking the system of equations required for A to be productive (for the assumed $x^\star \in \mathbb{R}^n_+$), we have that

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &< x_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &< x_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &< x_n \end{aligned}$$

   Summing each equation, this becomes

$$\sum_{i=1}^n a_{i1}x_1 + \sum_{i=1}^n a_{i2}x_2 + \cdots + \sum_{i=1}^n a_{in}x_n < x_1 + x_2 + \cdots + x_n$$

   and since $\sum_{i=1}^n a_{ij} > 1$ for every j, this implies that $Ax \not\ll x$, contradicting the assumption that A is productive. $\square$

4. We need a price vector $(p_0, p) \in \mathbb{R}_+ \times \mathbb{R}^3_+$ such that the profit matrix π is the zero matrix. Formally, we want

$$p \cdot (I-A) - a_0 = 0 \implies a_0 = p \cdot (I-A) \implies \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \begin{bmatrix} 0.9 & -0.4 & -0.3 \\ -0.2 & 0.3 & 0 \\ -0.1 & -0.1 & 0.5 \end{bmatrix}$$

so the equilibrium prices are the solution to the following system of equations:

$$\begin{aligned} 0.9p_1 - 0.4p_2 - 0.3p_3 &= 1 \\ -0.2p_1 + 0.3p_2 &= 1 \\ -0.1p_1 - 0.1p_2 + 0.5p_3 &= 1 \\ \Longrightarrow (p_1, p_2, p_3) &= (5.875, 7.25, 4.625) \end{aligned}$$

where we are assuming $p_0 = 1$ to guarantee a unique solution.

5. Irreducible matrices.

   (a) ***Proof.*** ($\Rightarrow$) Assume we have an irreducible square matrix A. This means that the graph is strongly connected. Take some i, j. Since the graph is strongly connected, there is a path of length m from i to j. If the path is of length 1, then $a_{ij} > 0$. If the path is of length 2, then there exists k such that $a_{ik} > 0$ and $a_{kj} > 0$, so $A^2_{ij} = a_{i1} \cdot a_{1j} + \cdots + a_{ik} \cdot a_{kj} + \cdots > 0$. If the path is of length m, then there exist $m-1$ intermediate points (indexed $n_1, \ldots, n_{m-1}$ such that $a_{in_1} > 0$, $a_{n_x n_{x+1}} > 0$, and $a_{n_{m-1}j} > 0$ for all $x = 1, \ldots, m-2$. Thus, $A^m_{ij} = \cdots + a_{in_1} \cdot a_{n_1 n_2} \cdots \cdot a_{n_{m-1}j} + \cdots > 0$.

   ($\Leftarrow$) Assume that for each i, j there exists m such that $A^m_{ij} > 0$. This means that there is at least one product in the sum A^m_{ij} that is strictly positive. As with all products, it will begin with a_{ix} for some x and end with a_{yj} for some j. All of the intermediate terms connecting x to y will be strictly positive, as will a_{ix} and a_{yj}. Thus, they constitute a connected path from i to j, so the graph is strongly connected and the matrix is irreducible. □

   (b) ***Proof.*** Assume that $a_0 > 0$ but $a_0 \not\gg 0$, meaning that for at least one i, $a_{0i} = 0$, and further assume that A is irreducible. Recall that per-unit profits are $\pi = p(I - A) - a_0$. Take $p = (I - A)^{-1} a_0$, so $\pi = 0$. It remains to show that p is strictly positive, for equilibrium to exist. Recall that a_0 has at least one non-zero element. Since $(I - A)^{-1} = I + A + A^2 + \cdots + A^n + \cdots$, and since A is irreducible, there exists n sufficiently large that all elements of A are strictly positive. Thus, $(I - A)^{-1}$ has all elements strictly positive, so p is strictly positive by $a_0 > 0$. □