Midterm
ECON 6140
Spring 2019

March 15

This is your midterm exam. You have 1 hour and 15 minutes (75 minutes) to answer these questions. The "relative prices" can be found next to each question for a total of 85 points.
If you get stuck at some point in the exam, keep moving and come back to those questions at the end.
Read the instructions carefully before writing anything. If you need to add any additional assumptions, please state them clearly.

(50 pt) Growth with investment-specific technical change

Time is discrete and indexed by t. There are two goods produced in the economy, final goods and investment goods.
There is a continuum of measure one of households with preferences,

$$\sum_{t=0}^{\infty} \beta^t \log(c_t)$$

Each household provides L units of labor to the market which can be allocated to either sector of production. She uses her income to purchase consumption, c_t, and investment goods, x_t. The household owns the economy's stock of capital and the law of motion is:

$$k_{t+1} - (1-\delta)k_t = x_t \text{ for } \delta \in (0,1)$$

The initial level of capital, k_0, is given.
A representative firm produces final goods using an homogeneous of degree one function F

$$y_t = A_t^y F(l_t^y, k_t^y)$$

where A_t^y is a productivity shifter that evolves exogenously according to

$$A_{t+1}^y = A_t^y(1+\gamma^y)$$

The firm maximizes profits and takes prices as given.

A representative firm produces investment goods using an homogeneous of degree one function G

$$x_t = A_t^x G(l_t^x, k_t^x)$$

where A_t^x is a productivity shifter that evolves exogenously according to

$$A_{t+1}^x = A_t^x(1+\gamma^x)$$

The firm maximizes profits and takes prices as given.
Assume that factor shares are identical across sectors.

1. [5 pt] Describe the problem of households, and that of investment and consumption goods firms. Define a recursive competitive equilibrium.

   - Firms problem is to maximize profits (std) plus take as given the law of motion for sectorial productivity.
   - Household problem

     $$\max_{c_t, x_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t \log(c_t)$$

     subject to

     $$p_t^y c_t + p_t^x x_t = w_t L + r_t k_{t+1}$$
     $$k_{t+1} - (1-\delta)k_t = x_t \text{ for } \delta \in (0,1)$$

   - Std. definition of RCE

2. [10 pt] Prove that, in equilibrium, capital labor ratios are equalized across sectors irrespective of the path of productivity. What features of the model render this result?

   - This comes straight from the optimality conditions of the firms and the HOD1 assumption on the production functions

     $$A_t^y F_k(\frac{k_t^y}{l_t^y}) = r(t) = A_t^x G_k(\frac{k_t^x}{l_t^x})$$

     $$A_t^y F_l(\frac{k_t^y}{l_t^y}) = w(t) = A_t^x G_l(\frac{k_t^x}{l_t^x})$$

     Consider sector y

     $$A_t^y F_l(\frac{k_t^y}{l_t^y}) l_t^y = w(t) l_t^y$$

     $$A_t^y F_k(\frac{k_t^y}{l_t^y}) k_t^y = r(t) k_t^y$$

     Because of the HOD1 assumption, $F_k(\frac{k_t^y}{l_t^y})k_t^y = F(\frac{k_t^y}{l_t^y}, 1)l_t^y - F_l(\frac{k_t^y}{l_t^y})l_t^y$. Replacing and computing the ratio between the last two equations

     $$\frac{F(\frac{k_t^y}{l_t^y}, 1)}{F_l(\frac{k_t^y}{l_t^y})} - 1 = \frac{r(t)k_t^y}{w(t)l_t^y}$$

We can multiply and divide the first term by l^y to obtain,

$$\frac{F(k_t^y, l_t^y)}{F_l(\frac{k_t^y}{l_t^y})l_t^y}$$

which is the inverse of the labor share. The same arguments follow from the investment sector. Because factor shares are identical, the result follows.

3. [15 pt] Characterize the BGP under Cobb-Douglas production functions in final and investment goods with equal capital share of α. In particular, characterize the equilibrium growth rate of consumption, capital, investment and output. (HINT: it might be easier to solve for the planner's problem and characterize gross growth rates, that is $\frac{h_{t+1}}{h_t} = (1+\gamma_h)$ for an arbitrary variable h).

    - The planner's problem in this economy is

    $$\max_{c_t, l_t^y, k_t^y, l_t^x, k_t^x, k_{t+1}} \sum_{t=0}^{\infty} \beta^t \log(c_t),$$

    subject to

    $$\begin{aligned} c_t &= A_t^y F(l_t^y, k_t^y) & (\lambda)\\ k_{t+1} - (1-\delta)k_t &= A_t^x G(l_t^x, k_t^x) & (\mu)\\ k_t^x + k_t^y &= k_t & (\kappa)\\ A_{t+1}^x &= A_t^x(1+\gamma^x) \\ A_{t+1}^y &= A_t^y(1+\gamma^y) \end{aligned}$$

    - The key optimality conditions are

    $$\lambda_t = \frac{1}{c_t}$$

    $$\mu_t = \beta[(1-\delta)\mu_{t+1} + \kappa_{t+1}] \tag{1}$$

    $$\kappa_t = \mu_t A_t^x G_k(l_t^x, k_t^x) \tag{2}$$

    $$\kappa_t = \lambda_t A_t^y F_k(l_t^y, k_t^y) \tag{3}$$

    Rewrite the Euler equation as follows

    $$\mu_t = \beta\mu_{t+1}[(1-\delta) + A_{t+1}^x G_k(\frac{k_{t+1}^x}{l_{t+1}^x})]$$

    - Hence, for the shadow price of investment to grow at a constant rate, we need that the marginal product of capital is constant, $A_{t+1}^x G_k(\frac{k_t^x}{l_{t+1}^x})$. Using the Cobb-Douglas structure,

    $$A_{t+1}^x(\frac{l_{t+1}^x}{k_{t+1}^x})^{1-\alpha}$$

    Since labor is constant along the BGP (the feasible set is bounded), the gross growth rates satisfy

    $$(1+\gamma^x)^{\frac{1}{1-\alpha}} = (1+\gamma_{k^x})$$

    Because capital labor ratios are equalized across sectors, then $\gamma_{k^y} = \gamma_{k^x} = \gamma_k$.

- Using the ratio of the optimality conditions with respect to capital, the euler equation can be alternatively written as

$$\mu_t = \beta\mu_{t+1}[(1-\delta) + \frac{\lambda_{t+1}}{\mu_{t+1}} A^y_{t+1} F_k(l^y_{t+1}, k^y_{t+1})]$$

See derivations in question 4. so that that

$$\frac{\mu_{t+1}}{\lambda_{t+1}} = \frac{A^y_{t+1}}{A^x_{t+1}}$$

So for $\frac{\lambda_{t+1}}{\mu_{t+1}} A^y_{t+1} F_k(l^y_{t+1}, k^y_{t+1})$ to be constant we need, $A^x_{t+1} F_k(l^y_{t+1}, k^y_{t+1})$ to be constant which is satisfied, because $\gamma_{k^y} = \gamma_{k^x}$.

- From the feasibility condition for output, we can compute the growth rate of output and consumption

$$1 + \gamma_c = 1 + \gamma_Y = (1+\gamma^y)(1+\gamma^x)^{\frac{\alpha}{1-\alpha}}$$

The growth rate of investment is characterized by the production structure

$$1 + \gamma_x = (1+\gamma^x)(1+\gamma_{k^x})^{\alpha} = (1+\gamma^x)^{\frac{1}{1-\alpha}}$$

4. [10 pt] Go as far as you can to provide conditions on the exogenous productivity trends such that the relative price of investment to consumption falls along the BGP of the economy. What is the intuition behind your result?

- The ratio of the shadow values of consumption and investment (the lagrange multipliers in the problem above), λ_t, μ_t give us the ratio of the price of consumption to investment in the decentralized economy (Second welfare theorem holds). Then, combining the optimality conditions for capital in each sector,

$$\frac{\mu_t}{\lambda_t} = \frac{A^y_t F_k(\frac{k^y_t}{l^y_t})}{A^x_t G_k(\frac{k^x_t}{l^x_t})}$$

Capital labor ratios are identical across sectors (we proved this in 2) as well as factor shares. Then, the ratio of relative prices is the inverse of the ratio of productivities,

$$\frac{\mu_t}{\lambda_t} = \frac{A^y_t}{A^x_t}$$

which implies that for the relative price to fall along the BGP, $\gamma^x > \gamma^y$.

5. [10 pt] Show how to detrend the economy using two auxiliary variables, $Q_1 = A^y_t(A^x_t)^{\frac{\alpha}{1-\alpha}}$ and $Q_2 = (A^x_t)^{\frac{1}{1-\alpha}}$. Describe the planner's problem associated to it.

- Given the results of question 3, we can detrend consumption and output dividing by $Q_1 = A^y_t(A^x_t)^{\frac{\alpha}{1-\alpha}}$ and investment and capital by $Q_2 = (A^x_t)^{\frac{1}{1-\alpha}}$.

- Call detrended variables $\widehat{c}_t = \frac{c_t}{A_t^y (A_t^x)^{\frac{\alpha}{1-\alpha}}}$, $\widehat{k}_t = \frac{k_t}{(A_t^x)^{\frac{1}{1-\alpha}}}$. Then $\widehat{Y}_t = \frac{Y_t}{A_t^y (A_t^x)^{\frac{\alpha}{1-\alpha}}} = (l^y)^{1-\alpha}(\widehat{k}_t^y)^\alpha$ and $\widehat{x}_t = \frac{x_t}{(A_t^x)^{\frac{1}{1-\alpha}}} = A_t^x(\frac{l^x}{k_t})^{1-\alpha}(\frac{k_t^x}{(A_t^x)^{\frac{1}{1-\alpha}}}) = (l^y)^{1-\alpha}(\widehat{k}_t^x)^\alpha$
- The planner's problem is

$$\max_{c_t, l_t^y, k_t^y, l_t^x, k_t^x, k_{t+1}} \sum_{t=0}^{\infty} \beta^t \log(\widehat{c}_t),$$

subject to

$$\widehat{c}_t = (l^y)^{1-\alpha}(\widehat{k}_t^y)^\alpha \qquad (\lambda)$$

$$(1+\gamma^x)^{\frac{1}{1-\alpha}}\widehat{k}_{t+1} - (1-\delta)\widehat{k}_t = (l^y)^{1-\alpha}(\widehat{k}_t^x)^\alpha \qquad (\mu)$$

$$\widehat{k}_t^x + \widehat{k}_t^y = \widehat{k}_t \qquad (\kappa)$$

which is a standard two sector economy.

(35 pt) Transition paths in the one sector growth model with distortionary taxes

Consider the one sector growth model (Ramsey-Cass-Koopmans). Suppose that the economy is initiated with 85% of the steady state level of capital, $k_0 = 0.85k^*$ and that there is government expenditure that is fully financed with labor taxes, τ_l. Assume the representative household supplies one unit o labor each period.

1. [5 pt] Describe the feasibility constraints of the economy. At the same time, describe budget constraint of the representative household

   - Feasibility

   $$c_t + x_t + g_t \leq F(k_t) +$$

   as well as the law of motion for capital

   $$k_{t+1} = (1-\delta)k_t + x_t$$

   - The BC of the household is

   $$c_t + x_t + b_{t+1} = w_t(1-\tau_{l_t}) + r_t k_t + (1+r_t^b)b_t$$

2. [10 pt] Describe the shooting algorithm that could be implemented to solve for the transition path to a steady state of this economy assuming that the path of government expenditure does not change in time

   - Identical to lecture notes

3. [10 pt] Describe how would the government finance a tax break of one period.

- The intertemporal budget constraint of the government is

$$b_1 + \tau_{l_0} w_0 = g_0 + (1 + r_0^b) b_0$$

and under a one period tax break,

$$\widehat{b}_1 = g_0 + (1 + r_0^b) b_0$$

so that government debt should go up, and future taxes go up too. However, the NPV of the debt is unchanged.

4. [10 pt] Go as far as you can describing the path of consumption under the tax break policy.

- The key here is that the tax on labor when the supply of labor is inelastic, is actually not distortionary. It looks like a lumpsum tax. Because the present discounted value of government expenditure doesn't change, the path of consumption is left unaffected. Ricardian equivalence holds.