Macroeconomics, PhD core

Lecture #9

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Intro

- *"The standard version of the natural class of models to look at have the property that in the long run there is no growth and that—under small discounting—output per capita converges to its steady state level independently of initial conditions"*

  Growth in Convex Economies: Manuelli & Jones (1990) "A Convex Model of Equilibrium Growth: Theory and Policy Implications" Journal of Political Economy

- Models of long-term growth $\leftrightarrow$ models of catch up growth

Facts: Growth Accounting

- Suppose that production technology is cobb-douglas

$$Y_t = A_t M_t K_t^{\alpha} H_t^{1-\alpha}$$

where H_t is human capital, M_t is a "measure of our uncertainty" and A_t is the level of technology

$$\frac{Y_t}{L_t} = Z_t \frac{H_t}{L_t} (\frac{K_t}{Y_t})^{\frac{\alpha}{1-\alpha}} \qquad \text{and } Z_t = (A_t M_t)^{\frac{1}{1-\alpha}}$$

- RCK: all long-term growth from A or population L, exogenous!

Model predictions and data:

- From the one sector growth model (with growth!)
  1. $\frac{y_t}{l_t}$ and $\frac{k_t}{l_t}$ grow over time at roughly constant equal rates.
     Implication: $\frac{k_t}{y_t}$ constant in time.
  2. $\frac{i_t}{y_t}$ constant
  3. $\frac{r_t k_t}{y_t}$ and $\frac{w_t l_t}{y_t}$ constant (input shares)
- These predictions are consistent with the Kaldor facts (advanced economies)
- And also with some catch-up dynamics, but transition is too fast!

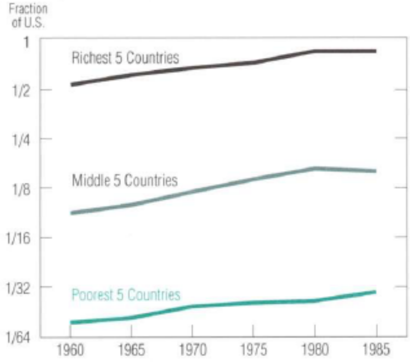
Parente & Prescott, '95

from Summers & Heston, Penn World Tables

Chart 8

A Widespread Upward Shift

Average Real GDP Relative to 1985 U.S. Level for Selected Wealth Groups in the 102-Country Data Set During 1960–85



Source of basic data: Summers and Heston 1991

Outline

- Ramsey-Cass-Koopmans growth model (we did all the work in discrete time)
  - the AK model (no transition dynamic)
  - the general case (consumption and capital transition)
- Characterization optimal paths

RCK

- Representative, infinitely lived family (dynasty)
- Population growth $n > 0$
- Labor force: $L(t) = \exp(nt)$ where $L(0) = 1$ by assumption.
- Constant returns technology, $Y(t) = F(K(t), A(t)L(t))$
- Capital Dynamics,

$$\dot{K}(t) + \delta K(t) = F(K(t), A(t)L(t)) - C(t)$$

RCK

- Define, $c(t) = \frac{C(t)}{L(t)}$ and $\zeta(t) = \frac{C(t)}{A(t)L(t)}$ (efficiency units)
- $\kappa(t) = \frac{K(t)}{A(t)L(t)}$

$$\frac{\dot{K}(t)}{A(t)L(t)} + \delta \frac{K(t)}{A(t)L(t)} = F\left(\frac{K(t)}{A(t)L(t)}, 1\right) - \frac{C(t)}{A(t)L(t)}$$

$$\dot{\kappa}(t) = F(\kappa(t)) - \zeta(t) - (n + g + \delta)\,\kappa(t)$$

$$\dot{\kappa}(t) = \left(\frac{\dot{K(t)}}{A(t)L(t)}\right) = \frac{\dot{K(t)}}{A(t)L(t)} - n\kappa(t) - g\kappa(t)$$

- Problem: F is Decreasing Returns in Capital. No endogenous long run growth
- Solution: Constant Returns to Capital, i.e. the "AK model"

AK

- Simplest version. $Y(t) = AK(t)$
- Capital per capita $k(t) = \frac{K(t)}{L(t)}$
- Feasibility constraint of the economy

$$\dot{k}(t) = Ak(t) - c(t) - \delta k(t)$$

or

$$\dot{k}(t) = Ak(t) - c(t) - (n + \delta)k(t)$$

if there is population growth.

- Family values **per capita** consumption as

$$u(c) = \int_0^\infty \exp(-\rho t) U(c(t)) dt$$

ρ is the time discount factor

- Assume that the per period utility function is CRRA,

$$\begin{aligned} U(c) &= \frac{c^{1-\sigma}}{1-\sigma} \quad \text{if } \sigma \neq 1 \\ &= \ln(c) \qquad \text{if } \sigma = 1 \end{aligned}$$

AK

- You can show that an allocation (κ^*, ζ^*) is Pareto optimal IFF it solves the planner's problem (can you prove this?), i.e.

$$\max_{\zeta,\kappa\geq 0} \int_0^\infty \exp(-\rho t) U(c(t)) dt$$

subject to

$$\dot{k}(t) = Ak(t) - c(t) - (n+\delta)k(t)$$
$$k(0) = k_0$$

- The problem can be solves using the Pontryagin's maximum principle (in Review Session, last Friday!)
  - Control variable, c
  - State Variable, k
  - Co-state Variable, λ

AK

- Present Value Hamiltonian (I am ignoring the non-negativity constraints, why can I do that?)

$$\begin{aligned}\mathcal{H}(t,k,c,\lambda) &= \exp(-\rho t)U(c(t)) \\ &\quad +\lambda(t)\left[f(k(t)) - c(t) - (n+\delta)\,k(t)\right]\end{aligned}$$

- Sufficient conditions

$$\exp(-\rho t)U'(c(t)) = \lambda(t)$$

$$\dot{\lambda}(t) = -\left[f'(k(t)) - (n+\delta)\right]\lambda(t)$$

$$\lim_{t\to\infty} \lambda(t)k(t) = 0 \qquad \text{TVC}$$

plus the constraint

$$\dot{k}(t) = Ak(t) - c(t) - (n+\delta)k(t)$$

AK

- Get rid of the co-state variable by differentiating

$$\dot{\lambda}(t) = \exp(-\rho t) U''(c(t)) \dot{c}(t) - \rho \exp(-\rho t) U'(c(t))$$

$$\frac{\dot{\lambda}(t)}{\lambda(t)} = \frac{U''(c(t))}{U'(c(t))} \dot{c}(t) - \rho$$

- Replacing

$$\frac{U''(c(t))}{U'(c(t))} c(t) \dot{c}(t) = - \left[f'(k(t)) - (n + \delta + \rho) \right] c(t)$$

- Because U is CRRA

$$\sigma \dot{c}(t) = \left[A - (n + \delta + \rho) \right] c(t)$$

Balanced Growth Path (BGP)

Definition

A balance growth path is an allocation such that consumption, capital and output grow at a constant (possibly different) growth rate

$$\frac{\dot{x}(t)}{x(t)} = \gamma_x > 0 \quad \text{for any variable } x.$$

AK model , BGP

- In this particular problem, the dynamic of consumption is independent of that of capital (not true in the standard RKC, remember our sequential economy)

$$c(t) = \exp(\frac{1}{\sigma}\left[A - (n + \delta + \rho)\right] t) c(0)$$

  Hence, consumption should grow at a constant rate from the beginning!
- Capital dynamics

$$\frac{\dot{k}(t)}{k(t)} = A - \frac{c(t)}{k(t)} - (n + \delta)$$

- Along the BGP, consumption and capital have to grow at the same rate.
- The economy is on the BGP from $t = 0$.

AK model , BGP

- The economy is on the BGP from $t = 0$.
- We need an additional condition such that the utility does not "blow up", U is well defined.

$$\int_0^\infty \exp(-\rho t) U(c(t)) dt$$

$$= \quad c(0)^{\frac{1-\sigma}{\sigma}} \int_0^\infty \exp(-\rho t) \frac{\exp(\frac{1-\sigma}{\sigma} \left[A - (n + \delta + \rho)\right] t) dt}{1 - \sigma}$$

We need

$$\frac{1-\sigma}{\sigma} \left[A - (n + \delta) - \frac{\rho}{1-\sigma} \right] < 0$$

AK model , Analysis

- Implication: No country can ever catch up.
- But we see some of those in the data (i.e. Asian tigers)
- What happens if we add labor?
- With increasing returns to scale we don't have a well defined equilibrium in the standard AK model.
- Romer (1986, JPE): IRS but the agents in the economy behave as if it were CRS (next class).

RCK with productivity growth

Consumption and capital dynamic around the steady state

- Remember?
- The optimal allocation satisfies,

$$\dot{\zeta}(t) = \frac{1}{\sigma}\left[f'(\kappa(t)) - (n + \sigma g + \delta + \rho)\right]\zeta(t)$$

$$\dot{\kappa}(t) = f(\kappa(t)) - \zeta(t) - (n + g + \delta)\,\kappa(t)$$

$$\lim_{t\to\infty} \exp(-\widehat{\rho}t)U'(\zeta(t))\kappa(t) = 0 \qquad \text{TVC}$$

  where g is an exogenous growth of technology, and $\kappa(t) = \frac{K(t)}{A(t)L(t)}, \zeta(t) = \frac{C(t)}{A(t)L(t)}$
- Steady state analysis $x^* = (\zeta^*, \kappa^*)$ / Saddle paths.

RCK, linear approximation

Dynamic around the steady state

- From the theory of linear approximation, we know that when the path is "nice" (IMPORTANT!)...
  .... the behavior **around the steady state** is well approximated by the behavior of a linearized system around the steady state.
- First order taylor approximation:

$$f(x) = f(x^*) + \nabla f(x^*) \cdot (x - x^*)$$

$\nabla f(x^*)$ is the $\mathbb{R}^n$ gradient of f at x^*.

RCK, linear approximation

Dynamic around the steady state

- In our case, $x^* = (\zeta^*, \kappa^*)$ and two functions g_1 and g_2 that solve

$$g_1(\zeta(t), \kappa(t)) = \dot{\zeta}(t) = -\frac{1}{\sigma}\left[f'(\kappa(t)) - (n + g + \delta + \rho)\right]\zeta(t)$$

$$g_2(\zeta(t), \kappa(t)) = \dot{\kappa}(t) = f(\kappa(t)) - \zeta(t) - (n + g + \delta)\,\kappa(t)$$

where, $g_1(\zeta^*, \kappa^*) = g_2(\zeta^*, \kappa^*) = 0$.... why?

- Linear approx. at (ζ^*, κ^*)

$$\begin{pmatrix} \dot{\zeta}(t) \\ \dot{\kappa}(t) \end{pmatrix} \approx \begin{pmatrix} -\frac{1}{\sigma}\left[f'(\kappa(t)) - (n+g+\delta+\rho)\right] & -\frac{1}{\sigma} f''(\kappa(t))\zeta(t) \\ -1 & f'(\kappa(t)) - (n+g+\delta) \end{pmatrix} \Big|_{(\zeta^*,\kappa^*)} \begin{pmatrix} \zeta(t) - \zeta^* \\ \kappa(t) - \kappa^* \end{pmatrix}$$

$$\begin{pmatrix} \dot{\zeta}(t) \\ \dot{\kappa}(t) \end{pmatrix} \approx \begin{pmatrix} 0 & \frac{1}{\sigma} f''(\kappa^*)\zeta^* \\ -1 & \rho \end{pmatrix} \Big|_{(\zeta^*,\kappa^*)} \begin{pmatrix} \zeta(t) - \zeta^* \\ \kappa(t) - \kappa^* \end{pmatrix}$$

- This two dimensional difference equation can be solved analytically.
- Look at the eigenvalues of A, λ, which satisfy

$$0 = \det(A - \lambda I) = -\lambda(\rho - \lambda) + \frac{1}{\sigma} f''(\kappa^*)\zeta^*$$

- There are two roots to this quadratic equation, one negative and one positive.

- Let l be the number of negative eigenvalues
- Let m be the number of state variables of the problem
- Stability
  - If $l = m$: "saddle-path stable", unique optimal trajectory. Negative eigenvalue governs the speed of convergence.
  - If $l < m$: unstable, no convergence to steady state.
  - If $l > m$: indeterminacy, multiple optimal trajectories.
- Speed of convergence:

$$\kappa(t) - \kappa^* \approx e^{-|\lambda_1|t}(\kappa(0) - \kappa^*)$$

- Half life

$$\kappa(t_{1/2}) - \kappa^* \approx \frac{1}{2}(\kappa(0) - \kappa^*) \quad \text{hence} \quad t_{1/2} = \frac{\ln(2)}{|\lambda_1|}$$

- First reduce it to a single equation,

$$\ddot{\kappa}(t) = -\dot{\zeta}(t) + \rho\dot{\kappa}(t)$$

- Let $\beta = \frac{1}{\sigma} f''(\kappa^*)\zeta^*$

$$\ddot{\kappa}(t) = \beta\left(\kappa(t) - \kappa^*\right) + \rho\dot{\kappa}(t)$$

$$\ddot{\kappa}(t) = \beta\left(\kappa(t) - \kappa^*\right) + \rho\dot{\kappa}(t)$$
$$\ddot{\kappa}(t) - \rho\dot{\kappa}(t) - \beta\kappa(t) = -\beta\kappa^*$$

- We need a solution to the homogeneous and non-homogenous part

$$\kappa(t) = \kappa_p(t) + \kappa_h(t)$$

- A solution to the non-homogeneous part, $\kappa_p(t) = \kappa^*$ Check it!
- A solution to the non-homogeneous part has the form $\kappa_h(t) = C_1 \exp(\lambda_1 t) + C_2 \exp(\lambda_2 t)$
- The characteristic equations is

$$\lambda^2 - \rho\lambda - \beta = 0$$

- Two roots, $\lambda_1 < 0$ and $\lambda_2 > 0$

- Two roots,

$$\lambda_{1,2} = \frac{\rho}{2} \pm \sqrt{\beta + \frac{\rho^2}{4}}$$

- $\lambda_2 > 1$ ¿¿ Locally around the steady state, the dynamic system is saddle-path stable.
- We need $C_2 = 0$, otherwise $\lim_{t \to \infty} \kappa(t) = \infty$ (or $-\infty$) which violates feasibility.
- C_1 is solved using the initial condition

$$\kappa(0) = \kappa^* + C_1 \exp(\lambda_1 t) = \kappa^* + C_1$$

$$C_1 = \kappa(0) - \kappa^*$$

- Solution,

$$\kappa(t) = \kappa^* + (\kappa(0) - \kappa^*)\exp(\lambda_1 t)$$

- Solution for $\zeta(t)$ can be found from

$$\dot{\kappa}(t) = -\zeta(t) + \zeta^* + \rho(\kappa(t) - \kappa^*)$$

  using the solution for $\kappa(t)$
- The unique optimal path $(\zeta(t), \kappa(t))$ converges to the steady state at rate $\lambda_1 = \left|\frac{\rho}{2} - \sqrt{\frac{1}{\sigma} f''(\kappa^*)\zeta^* + \frac{\rho^2}{4}}\right|$
- Decreasing in $\frac{1}{\sigma}$. Higher intertemporal elasticity of substitution ¿¿ consumption today is foregone for later, and capital accumulates faster.
- Decreasing in ρ. Higher discount, more impatient households, more current consumption, slower growth rate.

- We can't solve the system of non-linear equations explicitly.
- BUT.. you can do it in the computer.
- Finite difference methods (shooting algorithm)
- Approximate $k(t)$ and $c(t)$ with N discrete points in the time dimension. Denote the distance between grid points Δt.
- Let $k_n = k(t_n)$ and approximate derivatives as

$$\dot{k}(t) \approx \frac{k_{n+1} - k_n}{\Delta t}$$

- Approximate ODE

$$\begin{aligned}
\frac{c_{n+1} - c_n}{\Delta t} \frac{1}{c_n} &= \frac{1}{\sigma}(f'(k_n) - \rho - \delta) \\
\frac{k_{n+1} - k_n}{\Delta t} &= f(k_n) - \delta k_n - c_n
\end{aligned}$$

- Approximate ODE

$$\begin{aligned} c_{n+1} &= c_n(\Delta t \frac{1}{\sigma}(f'(k_n) - \rho - \delta) + 1) \\ k_{n+1} &= \Delta t(f(k_n) - \delta k_n - c_n) + k_n \end{aligned}$$

  with $k_0 = k(0)$ given.

- Algorithm:
  1. Guess c_0
  2. obtain (c_n, k_n) for $n = 1, \ldots, N$ by running the equations above forward
  3. If the sequence converges to c^*, k^* then you have the correct saddle path. If not, update c_0 and go back to 1.