ECON6110: Problem Set 1 Answers

Spring 2025

This problem set is due at 23:59 on Friday February 14, 2025. You are encouraged to work in groups, but every student has to write their own solution and submit it individually. Problem set submissions are submitted electronically and may be typed or handwritten. If handwritten, please ensure your work and scan are legible. **Illegible submissions will not be graded.**

Problem 1

The answer of this problem may be useful to solve some of the following problems (but the proofs of these findings are not essential for the next problems).

Question 1: Prove that a profile is a Nash equilibrium of a game only if it is the Nash equilibrium of the game in which strategies have been removed by iterated **strict** dominance.
Question 2: Prove that a profile is a Nash equilibrium of a game if it is the Nash equilibrium of the game in which strategies have been removed by iterated **strict** dominance.
Question 3: Prove that a profile is a Nash equilibrium of a game if it is the Nash equilibrium of a game in which strategies have been removed by iterated **weak** dominance.
Question 4: Give an example of a Nash equilibrium of a game that is not a Nash equilibrium of the game where strategies have been removed by iterated **weak** dominance.

Solution:

Question 1: Denote the original game using Γ and the game after N iteration of deletions of strictly dominated strategies using Γ^N. Let the set of mixed strategies in the original game Γ be Σ and the set of mixed strategies in game Γ^N be Σ^N.

We want to show the following: if σ^* is a Nash equilibrium in Γ, then it is a Nash equilibrium in Γ^N.

1. The result is obvious if $\sigma^* \in \Sigma^N$, since $\Sigma_i^N \subseteq \Sigma_i$ for all i

2. We need to prove that $\sigma^* \in \Sigma^N$. If not then there is a game Γ^n with $0 < n < N$ in which σ_i^* is eliminated by some σ_i', i.e.:

$$u_i(\sigma_i', \sigma_{-i}^*) > u_i(\sigma_i^*, \sigma_{-i}^*)$$

since otherwise, σ_i^* would be a best response to σ_{-i}^*. But then we have contradiction to σ^* being a Nash equilibrium in the original game.

Question 2: We now prove that if σ^* is a Nash equilibrium in Γ^N, then it is a Nash equilibrium in Γ. Assume not. Then there is a $\sigma_i^{(1)} \in \Sigma$ such that:

$$u_i(\sigma_i^{(1)}, \sigma_{-i}^*) > u_i(\sigma_i^*, \sigma_{-i}^*)$$

If $\sigma_i^{(1)} \in \Sigma^N$, we are done since it is a contradiction to σ^* being a Nash equilibrium in Γ^N. If $\sigma_i^{(1)} \notin \Sigma^N$, then there is a $n < N$ and a $\sigma_i^{(2)}$ such that:

$$u_i(\sigma_i^{(2)}, \sigma_{-i}^*) > u_i(\sigma_i^{(1)}, \sigma_{-i}^*) > u_i(\sigma_i^*, \sigma_{-i}^*)$$

If $\sigma_i^{(2)} \in \Gamma^N$, we are done. If not we can find a $n' \in (n, N)$ and a $\sigma_i^{(3)}$ such that:

$$u_i(\sigma_i^{(3)}, \sigma_{-i}^*) > u_i(\sigma_i^{(2)}, \sigma_{-i}^*) > u_i(\sigma_i^{(1)}, \sigma_{-i}^*) > u_i(\sigma_i^*, \sigma_{-i}^*)$$

we can repeat this process and we must find a strategy profile $\sigma_i^{(k)} \in \Sigma^N$ that strictly dominates σ^*. Thus the contradiction.

Question 3: This question can be proved in the same way as Question 2, replacing strict with weak inequalities as needed. Here we provide an alternative proof:

Consider the game in which strategies have been removed by iterated weak dominance, call it Γ^{NW} with strategies $\Sigma^{NW} \subseteq \Sigma$. If a strategy profile σ^* is a Nash equilibrium of Γ^{NW}, then it remains a Nash equilibrium if we add back the strategy eliminated at stage $N-1$ for some player i, since at best this strategy generates the same payoff as σ_i^*. To see this, suppose that a strategy $\sigma_i^{(N-1)}$ was eliminated at stage $N-1$. Then there must exists a strategy $\hat{\sigma}_i \in \Gamma^{NW}$ that weakly dominates $\sigma_i^{(N-1)}$ for all opponent's strategies, which includes σ_{-i}^*. Therefore,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\hat{\sigma}_i, \sigma_{-i}^*) \geq u_i(\sigma_i^{(N-1)}, \sigma_{-i}^*)$$

The first inequality stems from that σ^* is a Nash equilibrium of Γ^{NW}. We have shown that σ^* remains a Nash equilibrium if we add back the strategy $\sigma_i^{(N-1)}$. We can do this iteratively for N steps and recover the original game. So σ^* is a NE of Γ.

Question 4: Consider the game:

	L	R
U	$5, 1$	$4, 0$
M	$6, 0$	$3, 1$
D	$6, 4$	$4, 4$

Here we have 2 Nash equilibria: (D, L) and (D, R). Note that U and M are weakly dominated by D. If we eliminate U, then L is eliminated and we get (D, R). If we eliminate M, then R is eliminated and we get (D, L).

Problem 2

Find all the equilibria of these games:

Question 1:

	L	R
U	$4, 5$	$3, 1$
D	$4, 0$	$0, 6$

Question 2:

	L	R
U	$3, 4$	$-2, 6$
D	$0, 3$	$-5, 1$

Question 3:

	L	C	R
U	$6, 6$	$1, 2$	$3, 3$
M	$2, 1$	$4, 7$	$4, 3$
D	$3, 4$	$2, 5$	$3, 9$

Solution:

Question 1: There is a pure strategy (U, L). Consider a totally mixed strategy equilibrium in which 2 selects L with probability β.

This is possible only if $\beta = 1$, since U weakly dominates D and strictly of any positive probability is assigned on L.

So the only other possibility is a partially mixed equilibrium in which in which 2 selects L and 1 selects U with probability α.

For this to be possible we need

$$5\alpha \geq \alpha + 6(1 - \alpha)$$

So $\alpha \geq \frac{3}{5}$.

Question 2: D is strictly dominated by U, we can eliminate D. At this point, we can also eliminate L: (U, R) is the unique equilibrium.

Question 3: Strategy D is strictly dominated by a strategy $\alpha, 1-\alpha, 0$ for $\alpha \in (1/4, 2/3)$. So we can eliminate D.

At this point, R is strictly dominated by a mixes strategy $\beta, 1-\beta, 0$ with $\beta \in (1/4, 2/3)$.

We are left with the game:

	L	C
U	$6, 6$	$1, 2$
M	$2, 1$	$4, 7$

This game has 2 pure strategy equilibria: (U, L), (C, M); and a mixed equilibrium: $\alpha = 3/5$, $\beta = 3/7$.

Problem 3

Consider this game:

	L	R
U	0, 0	2, 1
D	1, 2	0, 0

Consider a correlated equilibrium with this distribution over actions (the table represents the joint distribution):

	L	R
U	1/3	1/3
D	1/3	0

We want to precisely define the correlated equilibrium that generates this outcome.

Question 1: Define the probability space (Ω, π). Argue that the state space Ω can be restricted to coincide the action space A.

Question 2: Define the partitions for each player associated to the state space described above.

Question 3: Define the strategies associated to the probability space described above and the distribution described in the table.

Question 4: Are the strategies a correlated equilibrium? Prove your answer.

Solution:

Question 1: Without loss of generality we can define $\Omega \equiv A$, so $\Omega = \{UL, UR, DL, DR\}$. The associated probabilities are $\pi(UL) = \pi(UR) = \pi(DL) = 1/3$ and $\pi(DR) = 0$.

Question 2: $P_1 = \{\{UL, UR\}, \{DL, DR\}\}$, $P_2 = \{\{UL, DL\}, \{UR, DR\}\}$.

Question 3: The strategies are:

$$\sigma_1(UL) = \sigma_1(UR) = U$$
$$\sigma_1(DL) = \sigma_1(DR) = D$$

and:

$$\sigma_2(UL) = \sigma_2(DL) = L$$
$$\sigma_2(UR) = \sigma_2(DR) = R$$

Question 4: We now check it is an equilibrium. When the state is $\{UL\}$, P1 assigns probability $1/2$ to UL and $1/2$ to UR. So he believes that 2 is choosing L and R with probabilities $1/2$ and $1/2$. If 1 plays U, the payoff is:

$$0 \times 1/2 + 2 \times 1/2 = 1$$

If he plays D the payoff is:

$$1 \times 1/2 + 0 \times 1/2 = 1/2$$

It follows that $BR_1(UL) = U$.

Similarly for 2, when the state is $\{UL\}$, 2 assigns probability $1/2$ to UL and $1/2$ to DL. The payoff of playing L is:

$$0 \times 1/2 + 2 \times 1/2 = 1$$

The payoff of playing U is $1/2$. Therefore $L \in BR_2(UL)$

If the state is UR, then $BR_1(UR) = U$, same as the best response when the state is UL. For 2, now 2 knows that the state is UR (since they agree that $\pi(DR) = 0$) and 1 plays U, so $BR_2(UR) = R$.

In state DL, 1 knows the state and finds it optimal to play D (since he knows that 2 plays L) so $BR_1(DL) = D$. For 2, he thinks the state is either UL or DL and thus believes that 1 randomizes between U and D with probability 1/2 and 1/2 so $BR_2(DL) = L$.

We have verified that the correlated strategies described above are an equilibrium. The equilibrium payoff (as it can be verified) is 1,1.

Problem 4

Consider an election with two candidates, Ann and Bob. There is a continuum of citizens, whose most preferred policies are distributed on $[0,1]$ according to a cumulative distributive function $F : [0,1] \to [0,1]$. We assume that F is continuous and strictly increasing.[1] Candidates choose their policy simultaneously from $[0,1]$. Each citizen votes for the candidate whose policy is closer to his or her most preferred policy. The candidate with majority votes wins.

This game can be formalized as follows. The player set is $N = \{Ann, Bob\}$. The set of actions is $[0,1]$ for both players. Let's call $s_i(a_i, a_{-i}) \in [0,1]$ the share of votes for candidate i:

$$s_i(a_i, a_{-i}) = \begin{cases} 1 - F\left(\frac{1}{2}a_i + \frac{1}{2}a_{-i}\right) & \text{if } a_i > a_{-i}, \\ 1/2 & \text{if } a_i = a_{-i}, \\ F\left(\frac{1}{2}a_i + \frac{1}{2}a_{-i}\right) & \text{if } a_i < a_{-i}. \end{cases}$$

The payoff function is

$$v_i(a_i, a_{-i}) = \begin{cases} 1 & \text{if } s_i(a_i, a_{-i}) > 1/2, \\ 1/2 & \text{if } s_i(a_i, a_{-i}) = 1/2, \\ 0 & \text{else.} \end{cases}$$

Hint. It could be useful to keep in mind a concrete example for F. For example, F could be the uniform distribution. In such case,

$$s_i(a_i, a_{-i}) = \begin{cases} 1 - \frac{1}{2}a_i + \frac{1}{2}a_{-i} & \text{if } a_i > a_{-i}, \\ 1/2 & \text{if } a_i = a_{-i}, \\ \frac{1}{2}a_i + \frac{1}{2}a_{-i} & \text{if } a_i < a_{-i}. \end{cases}$$

Question 1: What policies are rationalizable?

Question 2: Find a Nash equilibrium in "weakly dominant actions," that is, find a

[1] Recall that, in addition, $F(0) = 0$ and $F(1) = 1$.

pure Nash equilibrium a^* such that for all $i \in N$ and $a_i, a_{-i} \in [0,1]$

$$v_i(a_i^*, a_{-i}) \geq v_i(a_i, a_{-i}).$$

Question 3: Show that the equilibrium above is the unique (pure or mixed) Nash equilibrium.

Solution:

First, we prove a lemma that will be useful throughout this problem:

Lemma 1: Let $m \in (0,1)$ such that $F(m) = \frac{1}{2}$, i.e. m is the median of F. The player whose action is strictly closer to m wins the election.

To see this, without loss of generality suppose a_i is closer to m. If $a_i > a_{-i}$, $\frac{1}{2}a_i + \frac{1}{2}a_{-i} < m$ and $1 - F\left(\frac{1}{2}a_i + \frac{1}{2}a_{-i}\right) > 1 - F(m) = \frac{1}{2}$. If $a_i < a_{-i}$, $\frac{1}{2}a_i + \frac{1}{2}a_{-i} > m$ and $F\left(\frac{1}{2}a_i + \frac{1}{2}a_{-i}\right) > F(m) = \frac{1}{2}$. In both cases, player i wins the election.

Question 1: With this lemma, we consider three cases.

1. Case 1: $m = \frac{1}{2}$. In this case, $(0,1)$ is rationalizable. $\forall a_i \in (0,1)$ is not strictly dominated because $\exists a_{-i} \in (0,1)$ such that a_i is closer to m and is thus a best response to a_{-i}. However, 0 or 1 is strictly dominated by $a_i = m$. For any $a_{-i} \in [0,1]$ that the opponent plays, player i is always strictly better payoff playing $a_i = m$ than playing $a_i \in \{0,1\}$ (verify this!).

2. Case 2: $m > \frac{1}{2}$. In this case, $(0,1]$ is rationalizable. The argument for the interior stays the same, and now 1 is rationalizable because it is a best response to any point strictly within 2ε distance to 0, where $\varepsilon = m - \frac{1}{2}$. In other words, 1 is a best response to any point $\in [0, 2m-1)$.

3. Case 3: $m < \frac{1}{2}$. In this case, $[0,1)$ is rationalizable. 0 is a best response to any point $\in (2m, 1]$. The argument is symmetric.

Question 2: We claim that $a^* = (m,m)$ is a pure Nash equilibrium that also weakly dominates all other pure actions. First, from the lemma, it is easy to see that for any given a_{-i}, m gives a weakly better payoff than any $a_i \in [0,1]$. Then, (m,m) is indeed a Nash equilibrium because while this strategy profile gives both players a payoff of $\frac{1}{2}$, any deviation would give a payoff of 0 to the deviating player.

Question 3: For the sake of contradiction, suppose that $\alpha = (\alpha_i, \alpha_{-i})$ is an equilibrium where $\alpha_i(m) < 1$. Player $-i$ can obtain a payoff of $\frac{1}{2}\alpha_i(m) + 1 - \alpha_i(m)$ by playing $a_{-i} = m$. This means that

$$v_{-i}(\alpha_i, \alpha_{-i}) \geq \frac{1}{2}\alpha_i(m) + 1 - \alpha_i(m).$$

Moreover, notice that $v_i + v_{-i} = 1$. Thus,

$$v_i(\alpha_i, \alpha_{-i}) = 1 - v_{-i}(\alpha_i, \alpha_{-i}) \leq 1 - \left(\frac{1}{2}\alpha_i(m) + 1 - \alpha_i(m)\right) = \frac{1}{2}\alpha_i(m) < \frac{1}{2}.$$

But player i can obtain a payoff of at least $\frac{1}{2}$ by playing $a_i = m$: contradiction.

Problem 5

Consider a voting game with $M+N$ voters. The first M voters $M = \{1, ..., M\}$ are in team T_1, the remaining N voters in team T_2. Each voter can either abstain or vote for candidate 1 or 2. The candidate with the highest number of votes is elected. In case of a tie candidate 2 wins. A member of team i receives a payoff of 1 if candidate i is elected and a payoff of 0 otherwise. If a voter votes, then s/he pays a participation cost $c \in (0, 1)$.

Question 1. Is it possible that a member of team 1 (resp. team 2) votes for candidate 2 (resp. candidate 1)?

Question 2. Lets try to construct an Nash equilibrium in which each member of team 1 votes with probability q for candidate 1, and the members of team 2 abstain with certainty. Under what conditions does such an equilibrium exists? What is the strategy used by the member of team 1? What is the probability that a voter of team 1 votes if $c = 1/2$ and $M = 25$?

Question 3. Lets study the existence of a Nash equilibrium in which the members of team 1 (i.e., T_1) vote with probability q for 1 and abstain otherwise; and exactly $k \in (0, \min(M-1, N)]$ members of T_2 vote with probability 1 for candidate 2, and all the others abstain. Characterize q and establish a necessary and sufficient condition for such an equilibrium to exist.

Question 4. Use the answer of Q2-3 to show that an equilibrium in which the members of T1 adopt a totally mixed strategy q and k members of T_2 always vote exists for all $M, N \geq 2$ and $c \in (0, 1)$ for some $k \geq 0$.

Solution:

Question 1. No. Voting for the other candidate is strictly dominated by abstaining. It is always $-c$ worse than abstaining no matter which candidate wins.

Question 2. If the members of team 2 do not vote, the equilibrium condition for the member of team 1 is:

$$c = \Pr(m_i = 0) = (1-q)^{M-1}$$

where m_i is the number of voters in team 1 when voter i does not vote. This gives us:

$$q(c) = 1 - c^{\frac{1}{M-1}}$$

For this to be an equilibrium, it must be that the members of team 2 do not want to vote, so:

$$c \geq \Pr(m = 1) = Mq\,(1-q)^{M-1}$$

where m is the number of voters in Team 1. Substituting $q(c)$and using the fact that $c = (1-q)^{M-1}$, we get:

$$\begin{aligned} M\left(1 - c^{\frac{1}{M-1}}\right) &\leq 1 \\ &\Leftrightarrow \quad c \geq (1 - \frac{1}{M})^{M-1} = c^*_{0,M} \end{aligned}$$

Moreover $q_{25}(1/2) = 1 - (1/2)^{\frac{1}{24}} = 2.846\,8 \times 10^{-2}$.

Question 3. For the mixed strategy of the members of T_1, q needs to satisfy:

$$c = \Pr(m^i = k) = \binom{M-1}{k} q^k (1-q)^{M-k-1}$$

The incentive compatibility (IC) condition for non-voters in Team 2 is (IC1):

$$c \geq \Pr(m = k+1) = \binom{M}{k+1} q^{k+1} (1-q)^{M-k-1}$$

The IC for the k voters in Team 2 is (IC2):

$$c \leq \Pr(m = k) = \binom{M}{k} q^k (1-q)^{M-k}.$$

Note that:

$$\begin{aligned}\binom{M}{k+1} &= \frac{M!}{(k+1)!(M-k-1)!} = \frac{M(M-1)!}{(k+1)k!(M-k-1)!} \\ &= \frac{M}{(k+1)}\binom{M-1}{k}\end{aligned}$$

So we can write IC1:

$$\begin{aligned}c &\geq \frac{M}{(k+1)}q \cdot \binom{M-1}{k} q^k(1-q)^{M-k-1} \\ &\Leftrightarrow\ 1 \geq \frac{M}{(k+1)}q \\ &\Leftrightarrow\ q \leq \frac{k+1}{M}\end{aligned}$$

Similarly, IC2 can be rewritten as:

$$q \leq \frac{k}{M}$$

which implies IC1. The equilibrium is characterized by:

$$\begin{aligned}c &= \binom{M-1}{k} q^k(1-q)^{M-k-1} = B_{M-1,k}(q) \\ \text{for } q &\leq \frac{k}{M}\end{aligned}$$

For a sufficient condition of existence, note that differentiating the first equation we can prove that $B_{M-1,k}(q)$ is increasing in q for $q \leq \frac{k}{M}$. So:

$$\begin{aligned}q &\leq \frac{k}{M} \\ &\Leftrightarrow\ c(q) \leq B_{M-1,k}(\frac{k}{M})\end{aligned}$$

It follows that:

$$c \leq B_{M-1,k}(\frac{k}{M}) = \binom{M-1}{k}\left(\frac{k}{M}\right)^k(\frac{M-k}{M})^{M-k-1} = c^*_{k,M}$$

is a sufficient condition for the existence of such an equilibrium.

Question 4. Note that $c^*_{1,M} = c^*_{0,M}$. So for $c \geq c^*_{0,M}$, we have an equilibrium with $k = 0$; and for $c \leq c^*_{0,M}$we have equilibria with $k \in (0, \min(M-1, N)]$.