Macroeconomics, PhD core

Lecture #6-7

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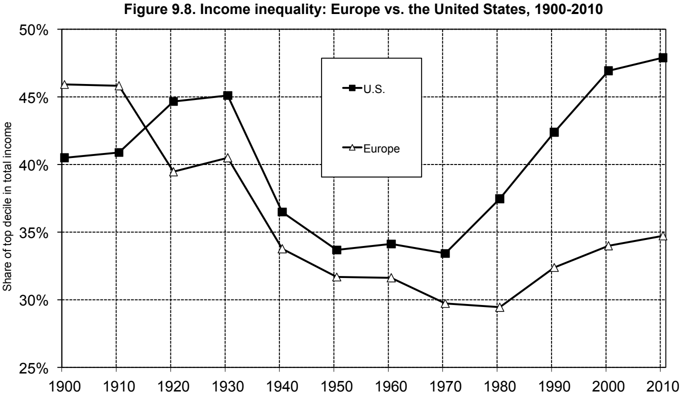
Lecture Road Map

- Heterogeneity and consumption distribution.
- Gorman aggregation.
- Variance of consumption.

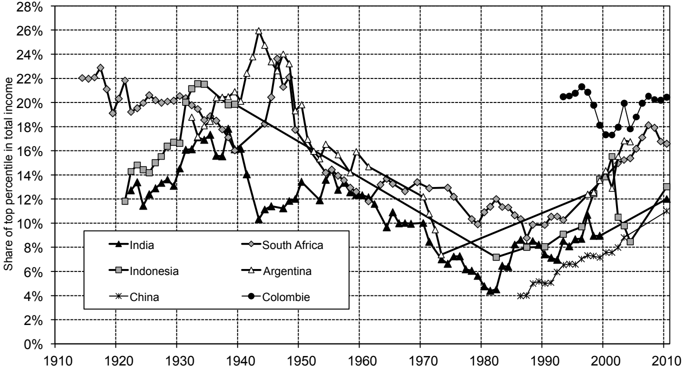
Facts:



The top decile income share was higher in Europe than in the U.S. in 1900-1910; it is a lot higher in the U.S. in 2000-2010. Sources and series: see piketty.pse.ens.fr/capital21c.

Facts:

Figure 9.9. Income inequality in emerging countries, 1910-2010



Measured by the top percentile income share, income inequality rose in emerging countries since the 1980s, but ranks below U.S. level in 2000-2010. Sources and series: see piketty.pse.ens.fr/capital21c.

Heterogeneity

- Is the one sector growth model consistent with some degree of heterogeneity across households?
- Under some conditions, one can show that heterogeneity in initial wealth, effective labor (i.e. human capital) and (limited) differences in utility do not affect equilibrium.
- Why? because "averages" of the economy with heterogeneity are identical to those of the representative agent economy.
- The basic ideas go back to the work of Gorman on aggregation.

Heterogeneity

Household

- Individual utility: $u_i(c) = \frac{(c+\theta_i)^{1-\eta}}{1-\eta}$, $\eta > 0$ and θ_i can be either positive or negative.
- N households, each characterized by a vector (θ_i, a_i, e_i) where a_i are assets and e_i endowment of labor.
- Household Problem

$$\max_{\{c_{ti}, a_{it+1}\}} \sum_{t=0}^{\infty} u_i(c_{it})$$

subject to

$$\begin{aligned} c_{it} + a_{it+1} &\leq w_t e_i + R_t a_{it} \qquad \text{t=0,1,....} \\ \lim_{T\to\infty} \beta^T u_i'(c_{iT}) a_{iT+1} &= 0 \end{aligned}$$

- Present Value Budget constraint

$$\sum_{t=0}^{\infty} q_t c_{it} \leq \sum_{t=0}^{\infty} q_t w_t e_i + q_0 a_{i0} \qquad \text{where } q_t \equiv \Pi_{j=0}^{t} R_j^{-1}$$

- The representative firm solves,

$$\max_{c_t, x_t} c_t + p_{kt} x_t - w_t e_t - r_t k_t$$

subject to

$$c_t + x_t = F(k_t, e_t)$$

- Aggregate law of motion for capital

$$k_{t+1} = (1 - \delta) k_t + x_t$$

- Let N the number of households in the economy
- For any variable z_{it}, let $z_t = N^{-1} \sum z_{it}$; i.e. the population average.
- Population moments: for any variables z_{it} and b_{it} let

$$var(z_t) \equiv \sigma(z_t) \equiv N^{-1} \sum_{i=1}^{N} (z_{it} - z_t)^2$$

$$cov(z_t, b_t) \equiv \sigma(z_t, b_t) \equiv N^{-1} \sum_{i=1}^{N} (z_{it} - z_t)(b_{it} - b_t)$$

- Also

$$\theta \equiv N^{-1} \sum_{i=1}^{N} \theta_i$$

and

$$a_t \equiv N^{-1} \sum_{i=1}^{N} a_{it} \qquad e \equiv N^{-1} \sum_{i=1}^{N} e_i$$

Definition

A competitive equilibrium is a collection of price sequences $\left[\{q_t, w_t, r_t, R_t\}_{t=0}^{\infty}\right]$, an allocation $\left[\{x_t, k_t\}_{t=0}^{\infty}, \{c_{it}\}_{t=0}^{\infty},\ \text{i=1,...,N}\right]$ and a sequence of asset holdings $\{a_{it}\}_{t=0}^{\infty}$, i=1,...,N such that,

a) Given the equilibrium prices, the allocation and the sequence of asset maximizes utility

b) Given the equilibrium prices, the allocation maximizes profits

c)The allocation is feasible: market clearing + aggregate law of motion for capital

d) $a_0 = k_0 > 0$ is given

Claim average quantities corresponding to a competitive equilibrium also solve the following planner's problem

$$\max_{\{c_t, x_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \frac{(c_t + \theta)^{1-\eta}}{1-\eta}$$

subject to

$$c_t + x_t \leq F(k_t, e_t)$$

$$k_{t+1} = (1-\delta) k_t + x_t$$

Proof.

Assume an interior solution (not necessary). The first order condition for the planner's problem is just

$$(c_t + \theta)^{-\eta} = \beta (c_{t+1} + \theta)^{-\eta} [1 - \delta + F_k(k_{t+1}, e_{t+1})]$$

Euler Equation for family i

$$(c_{it} + \theta_i)^{-\eta} = \beta (c_{it+1} + \theta_i)^{-\eta} R_{t+1}$$

But $R_{t+1} = [1 - \delta + F_k(k_{t+1}, e_{t+1})]$ which implies

$$(c_{it} + \theta_i) = \beta^{-\frac{1}{\eta}} (c_{it+1} + \theta_i) [1 - \delta + F_k(k_{t+1}, e_{t+1})]^{-\frac{1}{\eta}}$$

...

□

Proof.
Averaging

$$N^{-1}\sum_{i=1}^{N}(c_{it}+\theta_i)=N^{-1}\sum_{i=1}^{N}(c_{it+1}+\theta_i)\,\beta^{-\frac{1}{\eta}}\left[1-\delta+F_k(k_{t+1},e_{t+1})\right]^{-\frac{1}{\eta}}$$

$$(c_t+\theta)=\beta^{-\frac{1}{\eta}}(c_{t+1}+\theta)\left[1-\delta+F_k(k_{t+1},e_{t+1})\right]^{-\frac{1}{\eta}}$$

□

- Thus, all economies that share the same aggregate θ, k_0, e will display the same aggregate behavior, independently of the distribution a, e, θ.
- Cross sectional pattern of consumption and wealth is quite different across economies, but aggregate behavior is the same.

- Implications for "consumption mobility" and cross sectional dispersion of consumption?
- FOC of the household (PV version of the budget constraint)

$$(c_{it} + \theta_i)^{-\eta} = \lambda_i \frac{q_t}{\beta^t}$$

or

$$c_{it} + \theta_i = \left(\lambda_i \frac{q_t}{\beta^t}\right)^{\frac{-1}{\eta}} \tag{1}$$

- Use this expression in the budget constraint to solve for λ_i

$$\lambda_i^{\frac{-1}{\eta}} \sum_{t=0}^{\infty} q_t \left(\frac{q_t}{\beta^t}\right)^{\frac{-1}{\eta}} = e_i \sum_{t=0}^{\infty} q_t w_t + \theta_i \sum_{t=0}^{\infty} q_t + q_0 a_{i0}$$

- For any sequence $\mathbf{z} \equiv \{z_t\}_{t=0}^{\infty}$, let $v(\mathbf{z}, \mathbf{q}) \equiv \sum_{t=0}^{\infty} q_t z_t$ be the value of the sequence $\mathbf{z}$ at prices $\mathbf{q}$.

- Solve for λ_i

$$\lambda_i^{-\frac{1}{\eta}} v(\left(\frac{\mathbf{q}}{\beta}\right)^{\frac{-1}{\eta}}, \mathbf{q}) = e_i v(\mathbf{w}, \mathbf{q}) + \theta_i v(\mathbf{1}, \mathbf{q}) + q_0 a_{i0}$$

- Let $\phi_t \equiv \frac{q_t}{\beta^t}$, and $\phi_0 \equiv 1$ (i.e. without loss of generality we assume $q_0 = 1$). It follows,

$$\lambda_i^{\frac{-1}{\eta}} = \frac{e_i v(\mathbf{w}, \mathbf{q}) + \theta_i v(\mathbf{1}, \mathbf{q}) + a_{i0}}{v(\phi^{\frac{-1}{\eta}}, \mathbf{q})}$$

- Use this multiplier in the FOC of the household (**??**)

$$c_{it} = \frac{e_i v(\mathbf{w}, \mathbf{q}) + \theta_i v(\mathbf{1}, \mathbf{q}) + a_{i0}}{v(\phi^{\frac{-1}{\eta}}, \mathbf{q})} \phi^{\frac{-1}{\eta}} - \theta_i$$

- Note that consumption for each household is a **linear function of** e_i, θ_i, a_{i0}

- Let M_i

$$M_i \equiv m_e e_i + m_\theta \theta_i + m_a a_{i0}$$

where

$$m_e = \frac{v(\mathbf{w}, \mathbf{q})}{v(\phi^{\frac{-1}{\eta}}, \mathbf{q})}, \quad m_\theta = \frac{v(\mathbf{1}, \mathbf{q})}{v(\phi^{\frac{-1}{\eta}}, \mathbf{q})}, \quad m_a = \frac{1}{v(\phi^{\frac{-1}{\eta}}, \mathbf{q})}$$

- Also, let M

$$M \equiv m_e e + m_\theta \theta + m_a a_0$$

[M is M_i evaluated at the average vector (e, θ, a_0)]

- M is the economy wide population average of M_i too (why? linearity!)

- Write the optimal consumption schedule in terms of M_i.

$$c_{it} = M_i \phi_t^{\frac{-1}{\eta}} - \theta_i$$

while the aggregate per capita consumption is

$$c_t = M \phi_t^{\frac{-1}{\eta}} - \theta$$

- Key features:
  - All individuals face the same prices
  - Aggregate demand functions are independent of the distribution of (e_i, θ_i, a_{i0})
  - A sufficient (and it turns out necessary condition as well) is that the Engel curves be affine.

(If this is still not clear to you I would recommend you check out Gorman and Engel curves in Mas-Collel Winston and Greene)

- Household's i relative consumption

$$c_{it}^R = \frac{c_{it}}{c_t}$$

- **Claim:** The long run distribution of consumption is non-degenerate. [It's not true that as $t \to \infty$, $c_{it}^R = 1$]
- What do we know?
  - From the analysis of the growth model, we know that c_t converges monotonically to c^*
  - If $k_0 < k^*$, the sequence c_t is monotonically increasing.
  - $\left(\frac{q_t}{\beta^t}\right)^{\frac{-1}{\eta}} = \phi_t^{\frac{-1}{\eta}} \to \phi^{*\frac{-1}{\eta}} > 0$ as $t \to \infty$, for some ϕ^*
  - $c_i^{R*} = \frac{M_i \phi^{*\frac{-1}{\eta}} - \theta_i}{M \phi^{*\frac{-1}{\eta}} - \theta}$ which is in general different from 1.

- When do we observe $c_i^{R*} < 1$

$$\begin{aligned} c_i^{R*} &< 1 \leftrightarrow M_i \phi^{*\frac{-1}{\eta}} - \theta_i < M\phi^{*\frac{-1}{\eta}} - \theta \\ &\leftrightarrow \phi^{*\frac{-1}{\eta}}\left[m_e(e_i - e) + m_a(a_{i0} - a_0)\right] < (\theta_i - \theta)\left(1 - \phi^{*\frac{-1}{\eta}} m_\theta\right. \end{aligned}$$

- Comparative Statics
  - Impact of initial wealth: (assume $\theta_i = \theta$) higher e_i or a_{i0} higher c_i^{R*}
  - Impact of minimum consumption (assume $e_i = e$ and $a_{i0} = a_0$): since $1 - \phi^{*\frac{-1}{\eta}} m_\theta < 0$ the household with lower minimum consumption: lower θ_i (since $c_{it} = M_i \phi_t^{\frac{-1}{\eta}} - \theta_i$) has higher c_i^{R*}.

- Second moments (painful algebra)

$$
\begin{aligned}
\sigma^2(c_i^{R*}) &= \frac{1}{\left[M\phi^{*\frac{-1}{\eta}} - \theta\right]^2}\{\phi^{*\frac{-2}{\eta}}\left[m_e^2\sigma^2(e) + m_a^2\sigma^2(a_0)\right] \\
&\quad + \left(\phi^{*\frac{-1}{\eta}} m_\theta - 1\right)^2 \sigma^2(\theta) + \\
&\quad + 2\phi^{*\frac{-1}{\eta}}\left[m_e m_a \sigma(e, a_0) + \right. \\
&\quad \left. + (m_\theta - 1)\left(m_e \sigma(e, \theta) + m_a \sigma(\theta, a)\right)\right]\}
\end{aligned}
$$

- Notice that $\sigma^2(c_i^*) = \sigma^2(c_i^{R*}) \left[M {\phi^*}^{\frac{-1}{\eta}} - \theta \right]^2$
- Note that even if we have two economies A and B such that $\sigma_A^2(e) = \sigma_B^2(e)$ and $\sigma_A^2(a_0) = \sigma_B^2(a_0)$; differences in covariances matter for variance in consumption.
- A positive covariance between any two of the elements that determine household's type (e_i, θ_i, a_{i0}) increases long run variance in consumption.
- What do you think about the statement: "controlling for initial wealth inequality, all countries display the same amount of consumption inequality"?

Extra Slides: Heterogeneity

Time path of consumption inequality

- One can rewrite consumption as

$$c_{it} = \frac{M_i}{M} c_t + \frac{M_i}{M} \theta - \theta_i$$

- Special case in which $e_i = e$ and $a_{i0} = a_0$. Individual consumption is

$$c_{it} = \widetilde{m} \phi_t^{-\frac{1}{\eta}} + \left(m_\theta \phi_t^{-\frac{1}{\eta}} - 1 \right) \theta_i$$

  where $\widetilde{m} \equiv m_e e + m_a a_0$.

- Thus $c_{it} = c_t \Leftrightarrow m_\theta \phi_t^{-\frac{1}{\eta}} - 1 = 0$

- Suppose that at $t = \widehat{t}$ $c_{it} = c_t$ for all i.
- Since

$$c_{it} - c_t = \left(m_\theta \phi_t^{-\frac{1}{\eta}} - 1 \right) (\theta_i - \theta)$$

and $\phi_0^{-\frac{1}{\eta}} = 1$ and $m_\theta - 1 > 0$. It follows that if $\theta_i < \theta$

$$c_{it} \gtreqless c_t \Leftrightarrow t \lesseqgtr \widehat{t}$$

- Go back to the expression for the variance of consumption

$$
\begin{aligned}
\sigma^2(c_i^{R*}) \quad = \quad & \frac{1}{\left[M\phi^{*\frac{-1}{\eta}} - \theta\right]^2}\{\phi^{*\frac{-2}{\eta}}\left[m_e^2\sigma^2(e) + m_a^2\sigma^2(a_0)\right] \\
& + \left(\phi^{*\frac{-1}{\eta}} m_\theta - 1\right)^2 \sigma^2(\theta) + \\
& + 2\phi^{*\frac{-1}{\eta}}\left[m_e m_a \sigma(e, a_0) + \right. \\
& \left. + \left(m_\theta - 1\right)\left(m_e\sigma(e,\theta) + m_a\sigma(\theta, a)\right)\right]\}
\end{aligned}
$$

- In the previous case we assumed,
$\sigma^2(e) = \sigma^2(a_0) = \sigma(e, a_0) = \sigma(e, \theta) = \sigma(\theta, a_0) \rightarrow$
$\sigma^2(c_i^{R*}) = 0 \Leftrightarrow m_\theta \phi_t^{-\frac{1}{\eta}} - 1 = 0$

- Suppose there is only initial wealth inequality, i.e. $e_i = e$ and $\theta_i = \theta$.

$$\sigma^2(c_{it}^R) = \frac{\phi_t^{\frac{-2}{\eta}} m_a^2 \sigma^2(a_0)}{\left[M\phi_t^{\frac{-1}{\eta}} - \theta\right]^2}$$

Thus

$$\frac{\partial \sigma(c_{it}^R)}{\partial t} = -\theta \frac{m_a \sigma(a_0)}{\left[M\phi_t^{\frac{-1}{\eta}} - \theta\right]^2} \frac{\partial \phi_t^{\frac{-1}{\eta}}}{\partial t} > 0$$

where $\frac{\partial \phi_t^{\frac{-1}{\eta}}}{\partial t} < 0$