ECON 6110, Microeconomic Theory III	**Section 12**
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1 Continued: Signaling Game

Consider the Spence's job-market signaling model with a discrete set of effort choices. The sender is a student, the receiver an employer. There are two types of students, defined by the value of their innate talent, $\theta \in \{2, 3\}$. Nature chooses θ with probability p that $\theta = 2$. The student chooses an effort level in college, $a_1 \in \{0, 1\}$. After observing a_1, the employer chooses a wage $a_2 \in [0, \infty)$. The student maximizes wage less cost of effort, the latter inversely related to talent:

$$v_1(a_1, a_2, \theta) = a_2 - \frac{ca_1}{\theta} \tag{1}$$

for some $c > 0$. The employer minimizes the expected squared difference between the wage and the student's innate talent.[1]

$$v_2(a_1, a_2, \theta) = -\mathbb{E}(a_2 - \theta)^2 \tag{2}$$

(a) Define a Bayesian extensive game with the information above. Specify the players, set of types, prior on types, player's actions, and utility functions. What are player's strategies and beliefs? Represent it with a graph.

(b) Does the above signaling game have a **separating PBE** where the low type chooses the low action and the high type chooses the high action?

(c) Does the above signaling game have a **separating PBE** where the low type chooses the high action and the high type chooses the low action?

(d) Does the above signaling game have a **pooling PBE** where both types chooses the low action?

(e) Does the above signaling game have a **pooling PBE** where both types chooses the high action?

(f) Does the above signaling game have a **semi-seperating PBE** where one type mixes?

[1] Note that the employer doesn't want to *underpay* the student either, perhaps because the student would then choose an alternative employer.

Solution:

(a) (i) Doubleton set of players, $N = \{1, 2\}$; player 1 is the student, player 2 is the employer.

(ii) Set of types of player 1, $\Theta = \{2, 3\}$.

(iii) Prior on types $p = \Pr(\theta = 2)$.

(iv) Set of actions of player i, $A_1 = \{0, 1\}$ low or high effort. $A_2 = [0, \infty)$ wage.

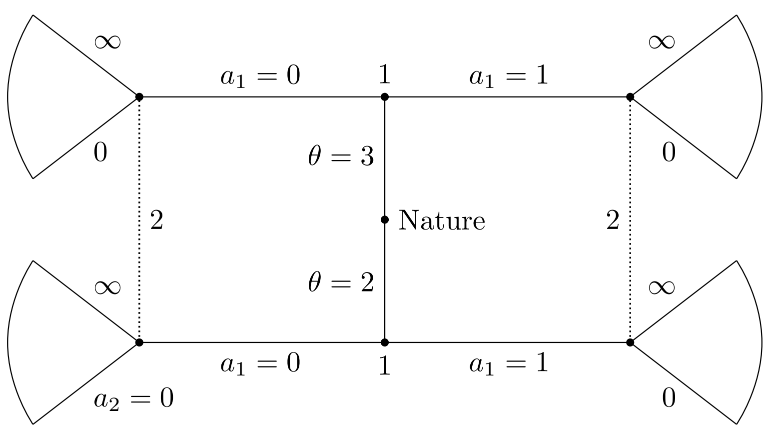
(v) Payoff function of player i, for $c > 0$:

$$v_1(a_1, a_2, \theta) = a_2 - \frac{ca_1}{\theta}$$
$$v_2(a_1, a_2, \theta) = -\mathbb{E}(a_2 - \theta)^2$$

(vi) Strategy of player i,

$$\sigma_1 : \Theta \to \Delta(A_1)$$
$$\sigma_2 : A_1 \to \Delta(A_2)$$

(vii) Player 2's belief on player 1's type $\mu : A_1 \to \Delta(\Theta)$.



(b) One candidate separating PBE is

$$s_1^*(\theta) = \begin{cases} 1 & \text{if } \theta = 3 \\ 0 & \text{if } \theta = 2 \end{cases} \tag{3}$$

and

$$s_2^*(a_1) = \mu(\theta \mid a_1) = \begin{cases} 3 & \text{if } a_1 = 1 \\ 2 & \text{if } a_1 = 0 \end{cases} \tag{4}$$

Given σ_2^*, σ_1^* is optimal if

$$3 - \frac{c}{3} \geq 2 \tag{5}$$

$$\implies c \leq 3 \tag{6}$$

and

$$2 \geq 3 - \frac{c}{2} \tag{7}$$

$$\implies c \geq 2 \tag{8}$$

Thus, this is a valid PBE if $c \in [2, 3]$.

(c) The other potential separating PBE is

$$s_1^*(\theta) = \begin{cases} 0 & \text{if } \theta = 3 \\ 1 & \text{if } \theta = 2 \end{cases} \tag{9}$$

and

$$s_2^*(a_1) = \begin{cases} 2 & \text{if } a_1 = 1 \\ 3 & \text{if } a_1 = 0 \end{cases} \tag{10}$$

The associated restrictions on c are then

$$2 - \frac{c}{3} \leq 3 \tag{11}$$

$$\implies c \geq -3 \tag{12}$$

and

$$3 \leq 2 - \frac{c}{2} \tag{13}$$

$$\implies 2 \leq -c \tag{14}$$

The latter is impossible, so this is not a PBE.

(d) The candidate pooling PBE is $s_1^*(2) = s_1^*(3) = 0$ and

$$s_2^*(a_1) = \begin{cases} 3 - p & \text{if } a_1 = 0 \\ w & \text{otherwise} \end{cases} \tag{15}$$

where $w \in [2,3]$ is such that

$$3 - p \geq w - \frac{c}{2} \tag{16}$$

$$6 + c - 2p \geq 2w \tag{17}$$

$$6 + c - 2p \geq 4 \tag{18}$$

$$2 + c \geq 2p \tag{19}$$

$$c \geq 2p - 2 \tag{20}$$

which is always true, as $2p - 2 \leq 0$. Therefore, this is a valid PBE with belief function

$$\mu(\theta \mid a_1) = \begin{cases} p & \text{if } \theta = 2 \text{ and } a_1 = 0 \\ 1 - p & \text{if } \theta = 3 \text{ and } a_1 = 0 \\ 0 & \text{if } \theta = 2 \text{ and } a_1 = 1 \\ 1 & \text{if } \theta = 3 \text{ and } a_1 = 1 \end{cases} \tag{21}$$

(e) The candidate pooling PBE is $s_1^*(2) = s_1^*(3) = 1$ and

$$s_2^*(a_1) = \begin{cases} 3 - p & \text{if } a_1 = 1 \\ w & \text{otherwise} \end{cases} \tag{22}$$

where $w = 2\mu(2 \mid 0) + 3\mu(3 \mid 0) \in [2,3]$ is such that

$$3 - p - \frac{c}{2} \geq w \tag{23}$$

There exists a PBE if $p + \frac{c}{2} \leq 1$.

(f) Finally, a semi-separating equilibrium will arise if one type mixes with $\alpha \in (0,1)$. This requires that, given the strategy of the employer, this type is indifferent between $a_1 = 0$ and $a_1 = 1$. This could only be the case if the employer pays a c/θ unit higher wage upon observing $a_1 = 1$. Given the employer optimally pays her conditional expectation of θ, we must have

$$\mathbb{E}(\theta \mid 1) = 2[1 - \mu(3 \mid 1)] + 3\mu(3 \mid 1) = \mu(3 \mid 1) + 2 \tag{24}$$

$$= \mathbb{E}(\theta \mid 0) + \frac{c}{\theta} = \mu(3 \mid 0) + 2 + \frac{c}{\theta} \tag{25}$$

implying

$$\mu(3 \mid 1) = \mu(3 \mid 0) + \frac{c}{\theta} \tag{26}$$

If the high-productivity type is mixing, then this becomes

$$1 - \frac{c}{3} = \mu(3 \mid 0) \tag{27}$$

Seeing as the right-hand side is nonnegative, we must have $c \leq 3$. Applying Bayes' rule, we then have

$$\begin{aligned}
1 - \frac{c}{3} &= \frac{\sigma_1^*(0 \mid 3)p(3)}{\sum_{\theta=2,3} \sigma_1^*(0 \mid \theta)p(\theta)} \qquad (28) \\
&= \frac{\alpha(1-p)}{\alpha(1-p)+p} \qquad (29) \\
\frac{c}{3} &= \frac{p}{\alpha(1-p)+p} \qquad (30) \\
\alpha &= \frac{3-c}{c} \cdot \frac{p}{1-p} \qquad (31)
\end{aligned}$$

It is apparent that higher values of c will lead to the high-productivity type playing $a_1 = 0$ with lower probability. A higher prior probability of θ being 3 will have a similar effect.

If the low-productivity type is mixing, we have

$$\mu(3 \mid 1) = \frac{c}{2} \tag{32}$$

and c must not exceed 2. Then

$$\begin{aligned}
\frac{c}{2} &= \frac{\sigma_1^*(1 \mid 3)p(3)}{\sum_{\theta=2,3} \sigma_1^*(1 \mid \theta)p(\theta)} \qquad (33) \\
&= \frac{1-p}{1-p+\alpha p} \qquad (34) \\
\alpha &= \frac{2-c}{c} \cdot \frac{1-p}{p} \qquad (35)
\end{aligned}$$

In this case, higher values of c lead the low-productivity type to choose high effort with lower probability. A higher prior probability of θ being 2 will have a similar effect.

2 PBE for general games and the Sequential Equilibrium (SE)

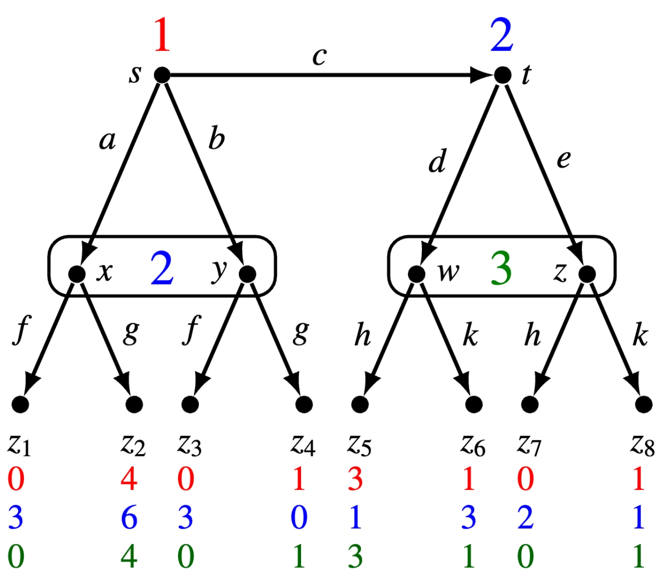
Definition 2.1 (Informal)**.** An assessment (σ, μ), where σ is a profile of behavioral strategies and μ is a list of probability distributions, one for every information set, over the nodes in that information set, is a **Perfect Bayesian Equilibrium** if

1. Sequentially rational: For each type θ_i, σ_i is the best response given μ_i and σ_{-i} at every information set I_i.

2. Bayesian updating whenever possible. (But no restriction on off-path beliefs)

Remark. This definition applies to general extensive games with imperfect information, not just Bayesian extensive games with observed actions. In the literature, this is sometimes called the "weak sequential equilibrium."

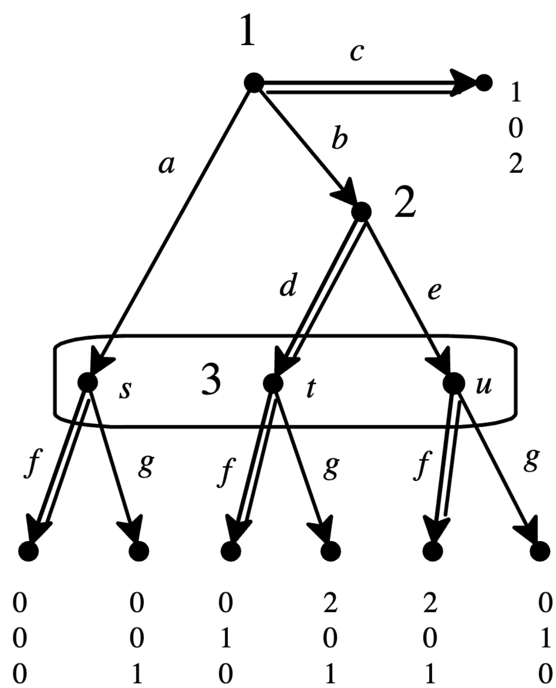
Example 2.1.



Is the following assessment sequentially rational and follows Bayesian updating whenever possible?

Example 2.2.

$$\sigma = \left(\begin{array}{ccc|cc|cc|cc} a & b & c & f & g & d & e & h & k \\ \frac{1}{8} & \frac{3}{8} & \frac{4}{8} & 1 & 0 & \frac{3}{4} & \frac{1}{4} & \frac{1}{5} & \frac{4}{5} \end{array} \right) \quad \text{and} \quad \mu = \left(\begin{array}{cc|cc} x & y & w & z \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{2} & \frac{1}{2} \end{array} \right),$$



Consider assessment consisting of the pure-strategy profile $\sigma = (c, d, f)$ (highlighted as double edges) and the system of beliefs that assigns probability 1 to node u. Is it plausible?

Thus we need to impose some restrictions on beliefs to ensure that they are consistent with the strategy profile with which they are paired (in the assessment under consideration). At reached information sets this is achieved by requiring Bayesian updating, but so far we have imposed no restriction on beliefs at unreached information sets. We want these restrictions to be "just like Bayesian updating". Kreps and Wilson (1982) proposed a restriction on beliefs that they called consistency.

Note that if σ is a completely mixed strategy profile (in the sense that $\sigma(a) > 0$, for every choice a) then the issue disappears, because every information set is reached with positive probability and Bayesian updating yields unique beliefs at every information set.