Bayesian Games and Mechanism Design

A model of nonlinear pricing

Model

Buyer's preferences: $u(q, T, \theta) = \theta v(q) - T$.

The buyer's type is θ: observed by the buyer but not by the seller.

There are two types: θ_L, with probability β and $\theta_H > \theta_L$.

We assume $v(0) = 0$, $v'(q) > 0$, $v''(q) < 0$.

Seller's profits: $\pi = T - cq$, so constant marginal costs.

We assume that the buyer can always avoid buying the good, so $q = 0, T = 0$ is always an option.

What contract should the seller offer in order to maximize profits?

Benchmark: complete information

In this case the seller can offer a price an quantity to each type of buyer.

Seller's problem is:

$$\max_{T_i, q_i} \{T_i - cq_i\}$$

$$s.t.\ \theta_i v(q_i) - T_i \geq \theta_i v(0) = 0$$

To solve this problem note we can assume wlg that the constraint is binding:

$$\theta_i v(q_i) - T_i = 0.$$

So we have:

$$\max_{T_i, q_i} \{\theta_i v(q_i) - c q_i\}$$

Implying:

$$\theta_i v'(q_i) = c \rightarrow q_i^*$$
$$T_i^* = \theta_i v(q_i^*) - c q_i^*$$

Selling Mechanisms

Attempt 1: **linear pricing**

The simplest way to sell the good is to set a price P and forget about the types.

With a price P demand is given by:

$$\theta_i v'(q_i) = P \to q_i = D_i(P).$$

Seller's problem is:

$$\max_P \{(P - c)D(P)\}$$

where $D(P)$ is expected demand.

Note

$$D(P) = \beta D_L(P) + (1 - \beta) D_H(P),$$

where $D_i(P)$ is the demand by type i.

We obtain:

$$P_m = c - \frac{D(P_m)}{D'(P_m)}.$$

Notes:

- This is the price that a monopolist would charge with expected demand $D(P)$;
- It is distorted above the efficient price;
- It generates positive profits.
- An implicit assumption in what we have done above is that $D_L(P_m) > 0$. But this may not be the case. Depending on the parameters we might have $D_L(P_m) = 0$. For simplicity we assume this is not the case.

Attempt 2: Two-part Tariff

In a two part tariff we charge a fixed fee (Z), plus a marginal fee (P).

Let $S_L(P) = \theta_i v(D_i(P)) - PD_i(P)$ be the surplus of agent i.

It would be irrational to set $Z < S_L(P)$.

It may be optimal to set $Z = S_H(P) > S_L(P)$, but in this case we would serve only high types.

Lets assume this is not optimal.

So we have $Z = S_L(P)$ and the seller's problem is:

$$\max_P \{S_L(P) + (P - c)D(P)\}$$

From the foc we have:

$$S_L'(P) + D(P) + (P - c)D'(P) = 0$$

Implying:

$$P = c - \frac{D(P) + S_L'(P)}{D'(P)}$$

Notes:

- Profits must of course be larger than with linear prices (Since we could have set Z=0, but we did not).
- P is distorted above again. Note $S_L'(P) = -D_L(P)$ (why?). So

$$\begin{aligned} D(P) + S_L'(P) &= D(P) - D_L(P) \\ &= (1-\beta)(D_H(P) - D_L(P)) > 0 \\ &\to P > c \end{aligned}$$

- $P \in (c, P_m)$

Attempt 3: A direct, truthful mechanism

Definition. A *direct revelation mechanism* is a mapping $g(\cdot)$ from the space of types to the space of outcomes which writes as $g(\theta_i) = (q(\theta_i), T(\theta_i))$ for all θ_i.

A direct revelation mechanism can also be called an *allocation rule* since it maps types to outcomes.

The principal commits to offer $q(\theta_i)$ at a price $T(\theta_i)$ if the agent reports to be of type θ_i.

Definition. An agent finds it *incentive compatible* to announce his/her type in correspondence to g if and only if:

$$\theta_L v(q(\theta_L)) - T(\theta_L) \geq \theta_L v(q(\theta_H)) - T(\theta_H)$$

$$\theta_H v(q(\theta_H)) - T(\theta_H) \geq \theta_H v(q(\theta_L)) - T(\theta_L)$$

Definition. A direct revelation mechanism $g(\bullet)$ is *truthful* if it is incentive compatible for the agent to announce his true type for any type.

Definition. A direct revelation mechanism $g(\bullet)$ is *individually rational* if $\theta_i v(q(\theta_i)) - T(\theta_i) \geq 0$ for any type.

Lets now try selling the good with a direct, incentive compatible and individually rational mechanism.

The seller's problem can be written as:

$$\max_{T_i, q_i} \beta(T_L - cq_L) + (1-\beta)(T_H - cq_H)$$

$$s.t. \quad \begin{aligned} &\theta_L v(q(\theta_L)) - T(\theta_L) \geq \theta_L v(q(\theta_H)) - T(\theta_H) \; IC_L \\ &\theta_H v(q(\theta_H)) - T(\theta_H) \geq \theta_H v(q(\theta_L)) - T(\theta_L) \; IC_H \\ &\theta_H v(q(\theta_H)) - T(\theta_H) \geq 0 \; IR_H \\ &\theta_L v(q(\theta_L)) - T(\theta_L) \geq 0 \; IR_L \end{aligned}$$

To solve this problem we proceed in steps.

Step 1

Note that IR_L and IC_H implies IR_H:

$$
\begin{aligned}
\theta_H v(q(\theta_H)) - T(\theta_H) &\geq \theta_H v(q(\theta_L)) - T(\theta_L) \\
&\geq \theta_L v(q(\theta_L)) - T(\theta_L) \geq 0
\end{aligned}
$$

Step 2

Consider a relaxed version of the problem in which we ignore IC_L:

$$\max_{T_i, q_i} \beta(T_L - cq_L) + (1-\beta)\beta(T_H - cq_H)$$

$$s.t. \quad \begin{array}{c} \theta_H v(q(\theta_H)) - T(\theta_H) \geq \theta_H v(q(\theta_L)) - T(\theta_L) \; IC_H \\ \theta_L v(q(\theta_L)) - T(\theta_L) \geq 0 \; IR_L \end{array}$$

Note that the value of this program is not lower than the value of the original program.

If, once we have solved it, we can prove that indeed IC_L is satisfied at the solution, then the two values coincide.

Step 3

Note that if $\theta_L v(q(\theta_L)) - T(\theta_L) > 0$, then we can increase $T(\theta_L)$ without violating any other constraint (indeed relaxing IC_H).

This change increases the payoff, a contradiction

The case for IC_H is similar.

Step 4

Note that from IC_H we can write:

$$\begin{aligned}
\theta_H v(q(\theta_H)) - T(\theta_H) &= \theta_H v(q(\theta_L)) - T(\theta_L) \\
&= \theta_L v(q(\theta_L)) - T(\theta_L) + (\theta_H - \theta_L) v(q(\theta_L)) \\
&= (\theta_H - \theta_L) v(q(\theta_L))
\end{aligned}$$

Substituting this and IR_L, seller's program becomes:

$$\max_{T_i, q_i} \beta(\theta_L v(q(\theta_L)) - cq(\theta_L)) + (1-\beta)\begin{pmatrix} \theta_H v(q(\theta_H)) - cq_H \\ -(\theta_H - \theta_L) v(q_L) \end{pmatrix}$$

For the contract we solve this problem.

Note that the objective function above, W, is not necessarily concave.

Observe that:

$$W_{q_H,q_H} = (1-\beta)(\theta_H v''(q(\theta_H))) < 0$$

$$W_{q_H,q_L} = W_{q_L,q_H} = 0$$

So this problem is concave if the hessian is negative (semi-)definite:

$$W_{q_L,q_L} = \beta(\theta_L v''(q(\theta_L)) - (1-\beta)(\theta_H - \theta_L)v''(q_L) < 0$$

This is however no always the case.

It is the case if β is high enough, or $\theta_H - \theta_L$ is small enough.

We will see more examples below.

We assume here that concavity is satisfied.

Our focs are:

$$\begin{aligned}
\theta_H v'(q(\theta_H)) &= c \\
\theta_L v'(q(\theta_L)) &= \frac{c}{1 - \left(\frac{1-\beta}{\beta}\frac{\theta_H - \theta_L}{\theta_L}\right)}
\end{aligned}$$

Note that: $q(\theta_H) > q(\theta_L)$.

For this to be a solution we need to verity that IC_L is satisfied.

From the binding IC_H we have:

$$\theta_H v(q(\theta_H)) - T(\theta_H) = \theta_H v(q(\theta_L)) - T(\theta_L)$$
$$\rightarrow \theta_H[v(q(\theta_H)) - v(q(\theta_L))] = T(\theta_H) - T(\theta_L)$$
$$\rightarrow \theta_L[v(q(\theta_H)) - v(q(\theta_L))] \leq T(\theta_H) - T(\theta_L)$$

Implying:

$$\theta_L v(q(\theta_L)) - T(\theta_L) \geq \theta_L v(q(\theta_H)) - T(\theta_H) \quad IC_L$$

We conclude that the solution of the relaxed problem is a solution of the original problem.

Why did I need to wait until now to establish IC_L?

Because I needed to show $q(\theta_H) \geq q(\theta_H)$ for the argument.

Solution then is:

$$
\begin{aligned}
\theta_H v'(q(\theta_H)) &= c \\
\theta_L v'(q(\theta_L)) &= \frac{c}{1 - \left(\frac{1-\beta}{\beta}\,\frac{\theta_H - \theta_L}{\theta_L}\right)}
\end{aligned}
$$

Note:

- High types buy more than low types.
- High types buy the efficient quantity; low types less than efficient.
- The low type receives a surplus of zero; the high type receives a positive surplus.

The optimal contract

The mechanisms described in the previous section generates increasing revenues for the seller.

Can we do better than the direct truthful incentive compatible, individually rational mechanisms?

We can indeed define more general mechanisms.

Definition. A mechanism is a message space M and a mapping $h(\bullet)$ from M to the space of outcomes which writes as $h(m) = (Q(m), t(m))$ for all m belonging to M.

Any mechanism induces an allocation rule.

Let:

$$m^*(\theta) \in \arg\max_{m \in M} \theta v(Q(m)) - t(m)$$

Then a mechanisms M induces the allocation rule:

$$a(\theta) = Q(m^*(\theta)), t(m^*(\theta)).$$

The seller’s problem is to find the mechanisms that induces the profit maximizing allocation rule.

Is there loss of generality in restricting attention to direct mechanisms?

Theorem (**Revelation Principle**). Any possible allocation rule $a(\theta)$ obtained with a mechanism $\{M, h(\bullet)\}$ can also be implemented with a truthful direct revelation mechanism.

Proof. We will show that if an outcome function is implemented by a mechanism, then it can be implemented by a direct mechanism as well.

This implies that there is no loss of generality in studying direct mechanisms.

Mechanism $\{M, h(\bullet)\}$ induces an outcome function

$$g(\theta) = Q(m^*(\theta)), T(m^*(\theta)).$$

Construct the functions $\widehat{Q} = Q \circ m^*$, $\widehat{T} = T \circ m^*$, so that:

$$\widehat{Q}(\theta), \widehat{T}(\theta) = Q(m^*(\theta)), T(m^*(\theta))$$

This is a direct mechanism implementing outcome $g(\theta)$.

Is it truthful?

To verify that $g(\theta)$ is a direct, truthful mechanisms we need to verify truthfulness. Since:

$$m^*(\theta_i) \in \arg\max_{m \in M} \theta_i v(Q(m)) - t(m)$$

We must have:

$$\theta_i v(Q(m^*(\theta_i))) - TQ(m^*(\theta_i)) \geq \theta_i v(Q(m^*(\theta_j))) - TQ(m^*(\theta_j))$$

$$\Rightarrow \theta_i v(\widehat{Q}(\theta_i)) - \widehat{T}(\theta_i) \geq \theta_i v(\widehat{Q}(\theta_j)) - \widehat{T}(\theta_j)$$

for any θ_i, θ_j.

Other Applications

Quality and non-linear pricing

A seller with utility: $\Pi(p, q) = p - \frac{q^2}{2}$.

A buyer: $u(p, q) = \theta q - p$

Two types: θ_H with probability μ; θ_L with probability $1 - \mu$.

Symmetric information

Monopolist's problem:

$$\max_{p,q}\{p_i - q_i^2/2\}$$

$$st\ \theta q_i - p_i \geq \underline{u}$$

That is:

$$\max_{q}\{\theta_i q_i - q_i^2/2\}$$

so:

$$q_i = \theta_i$$

Asymmetric information

We apply the revelation principle and look for the best direct mechanism: $\{p_i, q_i\}_{i=H,L}$

$$\max_{p,q}\{\mu(p_H - q_H^2/2) + (1-\mu)(p_L - q_L^2/2)\}$$

$$\theta_H q_H - p_H \geq \theta_H q_L - p_L$$

$$\theta_L q_L - p_L \geq \theta_L q_H - p_H$$

$$\theta_H q_H - p_H \geq \underline{u}$$

$$\theta_L q_L - p_L \geq \underline{u}$$

We proceed as standard.

$PC_L + IC_H \rightarrow PC_H$:

$$\theta_H q_H - p_H \geq \theta_H q_L - p_L \geq \theta_L q_L - p_L \geq 0$$

We now focus on the relaxed problem in which we ignore IC_L.

Now we have:

$$\max_{p,q}\{\mu(p_H - q_H^2/2) + (1-\mu)(p_L - q_L^2/2)\}$$

$$\theta_H q_H - p_H \geq \theta_H q_L - p_L$$

$$\theta_L q_L - p_L \geq \underline{u}$$

It is easy to see that both constraints hold as equalities:

$$\theta_H q_H - p_H = \theta_L q_L - p_L + (\theta_H - \theta_L) q_L$$

$$= \underline{u} + (\theta_H - \theta_L) q_L$$

and $\theta_L q_L - p_L = \underline{u}$

Note that the high type receives an informational rent.

We can write:

$$\max_{p,q}\left\{\mu\begin{bmatrix}\theta_H q_H - q_H^2/2 \\ -(\theta_H q_H - p_H)\end{bmatrix} + (1-\mu)\begin{bmatrix}\theta_L q_L - q_L^2/2 \\ -(\theta_L q_L - p_L)\end{bmatrix}\right\}$$

or:

$$\max_{p,q}\left\{\mu\begin{bmatrix}\theta_H q_H - q_H^2/2 \\ -(\underline{u} + (\theta_H - \theta_L)q_L)\end{bmatrix} + (1-\mu)\begin{bmatrix}\theta_L q_L - q_L^2/2 \\ -\underline{u}\end{bmatrix}\right\}$$

Note that this is a concave program.

The first order conditions give us:

$$q_H = \theta_H$$
$$q_L = \theta_L - \frac{\mu}{1-\mu}\Delta\theta$$

Here again the high type is un-distrorted, the low type is distorted below.

Is this the optimal contract? Yes, we can check that the IC_L constraint is satisfied.

Optimal income taxation

Model

Income q is produced with effort e

We assume $q = f(e,\theta) = \theta e$ where θ is the agent's type

Two types: θ_H and $\theta_L < \theta_H$

A proportion β is of type L.

Agents' utility is $u(q - t - \psi(e))$ with $\psi'(e) > 0$, $\psi''(e) > 0$.

The government maximizes a utilitarian social welfare function (i.e. the sum of utilities).

The government has a budget constraint: tax revenues can not be negative.

Benchmark: complete information

If the government could observe types and choose effort, the welfare maximizing problem would be:

$$\max_{t_i, e_i} \{\beta u(\theta_L e_L - t_L - \psi(e_L)) \; + (1-\beta) u(\theta_H e_H - t_H - \psi(e_H))\}$$

$$s.t. \quad \beta t_L + (1-\beta) t_H \geq 0$$

This is a concave maximization problem characterized by the first order necessary and sufficient conditions.

The focs are:

$$
\begin{aligned}
u'(\theta_L e_L - t_L - \psi(e_L)) &= u'(\theta_H e_H - t_H - \psi(e_H)) \\
\psi'(e_L) &= \theta_L \\
\psi'(e_H) &= \theta_H
\end{aligned}
$$

Note:

- In the utilitarian optimum marginal utilities are equalized
- The marginal cost of effort is equalized to the correspondent marginal benefit.

Optimal Taxation

Let us apply the revelation principle here.

We can restrict attention to a direct mechanism $h(\theta) = (q(\theta), t(\theta))$.

To each type we have an associated allocation in terms of observable output and taxes.

What about effort? We have that if you are a type θ and report to be a type θ' your effort must be: $e(\theta, \theta') = q(\theta')/\theta$

The government's problem can be written as:

$$\max_{t_i, q_i} \left\{ \begin{array}{c} \beta u(q_L - t_L - \psi(f^{-1}(q_L, \theta_L))) \\ +(1-\beta) u(q_H - t_H - \psi(f^{-1}(q_H, \theta_H)) \end{array}) \right\}$$

$$s.t. \quad \begin{array}{rl} q_L - t_L - \psi(f^{-1}(q_L, \theta_L)) & \geq q_H - t_H - \psi(f^{-1}(q_H, \theta_L)) \\ q_H - t_H - \psi(f^{-1}(q_H, \theta_H)) & \geq q_L - t_L - \psi(f^{-1}(q_L, \theta_H)) \\ \beta t_L + (1-\beta) t_H \geq 0 & \end{array}$$

Note that we have no participatioon constraints, but we have a balanced budget condition

Using the technology, we can express this in terms of effort:

$$\max_{t_i, e_i}\{\beta u(\theta_L e_L - t_L - \psi(e_L)) \; + (1-\beta)u(\theta_H e_H - t_H - \psi(e_H))\}$$

$$q_L - t_L - \psi(e_L) \geq q_H - t_H - \psi(\frac{e_H \theta_H}{\theta_L})$$

$$s.t. \quad q_H - t_H - \psi(e_H)) \quad \geq q_L - t_L - \psi(\frac{e_L \theta_L)}{\theta_H})$$

$$\beta t_L + (1-\beta) t_H \geq 0$$

Can we achieve the first best?

From IC_H we have:

$$q_H - t_H - \psi(e_H) \geq q_L - t_L - \psi(e_L) + \psi(e_L) - \psi(\frac{e_L \theta_L}{\theta_H})$$

So:

$$\begin{aligned} u'(\theta_H e_H - t_H - \psi(e_H)) &\leq u'(q_L - t_L - \psi(e_L) + \psi(e_L) - \psi(\frac{e_L \theta_L)}{\theta_H})) \\ &< u'(q_L - t_L - \psi(e_L)) \end{aligned}$$

So the the marginal utility of the high type must be lower than the marginal utility of the low type: this is incompatible with the utilitarian solution.

Characterization

Lets now make the guess that IC_L is not binding and consider the relaxed problem without IC_L.

We can show that in the relaxed problem IC_H is binding.

It follows that:

$$\theta_H e_H - t_H - \psi(e_H) = \theta_L e_L - t_L - \psi(\frac{e_L\theta_L)}{\theta_H})$$

Implying:

$$t_H = t_L + \theta_H e_H - \theta_L e_L - \psi(e_H) + \psi(\frac{e_L\theta_L}{\theta_H})$$

$$t_L = -\frac{(1-\beta)}{\beta} t_H$$

Solving the system we obtain:

$$t_L(e_H, e_L) = -(1-\beta)\left(\theta_H e_H - \theta_L e_L - \psi(e_H) + \psi(\frac{e_L\theta_L)}{\theta_H})\right)$$

We arrive to the following problem:

$$\max_{e_i} \left\{ \begin{array}{c} \beta u(\theta_L e_L - t_L(e_H, e_L) - \psi(e_L)) \\ +(1-\beta)u(\theta_L e_L - t_L(e_H, e_L) - \psi(\frac{e_L \theta_L}{\theta_H})) \end{array} \right\}$$

with:

$$t_L(e_H, e_L) = -(1-\beta)\left(\theta_H e_H - \theta_L e_L - \psi(e_H) + \psi(\frac{e_L \theta_L)}{\theta_H})\right)$$

The focs are:

$$\psi'(e_H) = \theta_H$$

$$\psi'(e_L) = \theta_L - (1-\beta)\gamma \bullet \left[\psi'(e_L)) - \frac{\theta_L}{\theta_H}\psi'(\frac{e_L\theta_L)}{\theta_H}) \right]$$

where

$$\gamma = \frac{(u'_L - u'_H)}{\beta u'_L + (1-\beta)u'_H}$$

Observation 1:

- The high type chooses the efficient level of effort.
- The low type is distorted below, meaning that is induced to choose less than efficient effort.

Why?

Remember that in the first best we would like to equalize the marginal utilities but by the binding IC_H we have:

$$u'(q_H - t_H - \psi(e_H)) = u'(q_L - t_L - \psi(e_L) + \boldsymbol{\psi}(\mathbf{e}_L) - \boldsymbol{\psi}(\frac{\mathbf{e}_L\boldsymbol{\theta}_L}{\boldsymbol{\theta}_H}))$$

We would improve welfare if we could reduce:

$$\boldsymbol{\psi}(\mathbf{e}_L) - \boldsymbol{\psi}(\frac{\mathbf{e}_L\boldsymbol{\theta}_L}{\boldsymbol{\theta}_H}).$$

Suppose we choose the efficient level of effort $\mathbf{e}_L$:
$\psi'(e_L) = \theta_L$

Then if we marginally reduced $\mathbf{e}_L$ we would reduce efficiency by $\theta_L - \psi'(e_L) \approx 0$.

But we have a first order reduction in $\boldsymbol{\psi}(\mathbf{e}_L) - \boldsymbol{\psi}(\frac{\mathbf{e}_L \boldsymbol{\theta}_L}{\boldsymbol{\theta}_H})$:
$\boldsymbol{\psi}'(\mathbf{e}_L) - \frac{\boldsymbol{\theta}_\mathbf{L}}{\boldsymbol{\theta}_\mathbf{H}} \boldsymbol{\psi}'(\frac{\mathbf{e}_L \boldsymbol{\theta}_L}{\boldsymbol{\theta}_H}) > \mathbf{0}$.

This implies that it must be optimal to effe less then the efficient $\mathbf{e}_L$

Observation 2:

It implies: top (bottom) income agents pay a zero (positive) marginal tax rate.

What is L's marginal tax rate? It must be such that:

$$(1-t)\theta_L = \psi'(e_L)$$

But:

$$\psi'(e_L) = \theta_L - (1-\beta)\gamma \bullet \left[\psi'(e_L)) - \frac{\theta_L}{\theta_H}\psi'(\frac{e_L\theta_L)}{\theta_H}) \right]$$

So:

$$(1-t)\theta_L = \theta_L - (1-\beta)\gamma \bullet \left[\psi'(e_L)) - \frac{\theta_L}{\theta_H}\psi'(\frac{e_L\theta_L)}{\theta_H}) \right]$$

We conclude that:

$$(1-t) = 1 - \frac{(1-\beta)\gamma \bullet \left[\psi'(e_L)) - \frac{\theta_L}{\theta_H}\psi'(\frac{e_L\theta_L)}{\theta_H})\right]}{\theta_L}$$

$$\rightarrow t = \frac{(1-\beta)\gamma \bullet \left[\psi'(e_L)) - \frac{\theta_L}{\theta_H}\psi'(\frac{e_L\theta_L)}{\theta_H})\right]}{\theta_L} > 0$$

As we will see, with many types only the highest type has zero marginal taxation.

With a continuum of types the measure of types with zero tax rate is zero.

Optimal regulation

Consider the problem of a government who buys a good from a monopolist contractor (e.g., the construction of a bridge).

The government is interested in choosing a contract that maximizes welfare.

For simplicity we assume that this coincides with the minimization of the cost of the realization.

In general the cost of a dollar in public expenditure is higher than a dollar for society because of the deadweight loss of taxation.

The government can see the books of the company and so can see the cost.

The government pays $s + c$, where s is a bonus.

The cost is $c = \theta - e$:

- θ is the type of the contractor, it can be θ_L for an efficient contractor or $\theta_H > \theta_L$ for an inefficient contractor.
- The contractor however can exert effort to reduce costs. This effort is e.
- The cost of effort is $\psi(e) = e^2/2$.

First best

In the first best the government observes both θ and e.

The problem is:

$$\min_{e,s}\{s + \theta - e\}$$

$$s.t.\ s - e^2/2 \geq \underline{u}$$

where $\underline{u}$ is the reservation value for the contractor.

So we have:

$$\min_{e}\{e^2/2 + \underline{u} + \theta - e\}$$

The foc implies $e = 1$, $s = 1/2$ (plus the costs reinbursement).

Optimal contract

Of course the previous contract would not be incentive compatible.

To find best contract, let's apply the revelation principle again. The problem becomes:

$$\min_{s,e} \beta(s_L + c_L) + (1-\beta)(s_H + c_H)$$

$$s.t. \begin{cases} s_H - e_H^2/2 \geq s_L - e(\theta_L;\theta_H)^2/2 \\ s_L - e_L^2/2 \geq s_H - e(\theta_H;\theta_L)^2/2 \\ s_H - e_H^2/2 \geq \underline{u},\ s_L - e_L^2/2 \geq \underline{u} \end{cases}$$

where $e(\theta_j;\theta_i)$ is the effort that a type θ_i needs to exert to achieve a cost c_j.

Note that:

$$\begin{aligned}\theta_L - e(\theta_H;\theta_L) &= c_H = \theta_H - e_H \\ &\rightarrow e(\theta_H;\theta_L) = e_H - \Delta\theta\end{aligned}$$

and:

$$\begin{aligned}\theta_H - e(\theta_L;\theta_H) &= c_L = \theta_L - e_L \\ &\rightarrow e(\theta_L;\theta_H) = e_L + \Delta\theta\end{aligned}$$

So we can write:

$$\min_{s,e} \beta(s_L - e_L) + (1-\beta)(s_H - e_H)$$

$$s.t. \begin{cases} s_H - e_H^2/2 \geq s_L - \frac{(e_L+\Delta\theta)^2}{2} \\ s_L - e_L^2/2 \geq s_H - \frac{(e_H-\Delta\theta)^2}{2} \\ s_H - e_H^2/2 \geq \underline{u} \\ s_L - e_L^2/2 \geq \underline{u} \end{cases}$$

Proceeding as usual, we first get rid of IR_L:

$$s_L - e_L^2/2 \geq s_H - \frac{(e_H - \Delta\theta)^2}{2}$$

$$\geq s_H - \frac{(e_H)^2}{2} \geq 0$$

Then we relax the program ignoring IC_H and note that constraint are now binding:

$$\min_{s,e} \beta(s_L - e_L) + (1-\beta)(s_H - e_H)$$

$$s.t. \begin{cases} s_L - e_L^2/2 = s_H - \frac{(e_H - \Delta\theta)^2}{2} \\ s_H - e_H^2/2 \geq \underline{u} \end{cases}$$

Substitute and obtain:

$$\min_{e_i} \left\{ \begin{array}{c} \beta\left(e_L^2/2 - e_L + e_H^2/2 - \frac{(e_H - \Delta\theta)^2}{2} \right) \\ +(1-\beta)(e_H^2/2 - e_H) \end{array} \right\}$$

This is concave problem.

The focs are:

$$\begin{aligned} e_L &= 1 \\ e_H &= 1 - \frac{\beta}{1-\beta}\Delta\theta \end{aligned}$$

Notes:

- Once more, no distortion at the "top," downward distortion at the bottom.
- How do we decentralize this contract?

Consider a price scheme in which we offer a menu of two regimes:

- One regime has a constant price $P_L = s_L + \theta_L - e_L = s_L + \theta_L - 1$, no cost reimbursement.
- The other regime has a cost sharing arrangement in which a share $1 - \alpha$ of the cost is reimbursed and a lower price P' is paid. How do we choose α?

The H type must choose e_H, so:

$$u_H = P' - c + (1 - \alpha)c - \frac{(e)^2}{2} = P' - \alpha(\theta_H - e) - \frac{(e)^2}{2}$$

We must have:

$$\alpha - e_H = 0 \rightarrow \alpha = e_H$$

And:

$$P' + \alpha(\theta_H - e_H) = e_H^2/2 + \underline{u}$$

$$\Leftrightarrow P' = e_H^2/2 + \underline{u} - e_H(\theta_H - e_H)$$

The lesson from all this is that it can be optimal to reimburse less efficient companies, because they allow to screen the companies that are more efficient.

Would the efficient type want to deviate?

If L does not deviate, s/he selects $e_L = 1$ and $P_L = s_L + \theta_L - e_L = s_L + \theta_L - 1$, as in the optimal contract.

If he devviates, s/he selects e_L and receives P', so a utility corresponding to S_H and e_H