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| **ECON 6110, Microeconomic Theory III** | **Section 10** |
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## 1 Bayesian games and Bayesian Nash equilibrium (BNE)

**Definition 1.1.**

1. Set of $N$ players.
2. Finite set of state of nature $\Omega$.
3. For each $i$:
   (a) Action set $A_i$.
   (b) Set of types $T_i$ and a signal function $\tau_i : \Omega \to T_i$.
   (c) A probability measure $p_i$ over $\Omega$
   (d) A preference relation $\succsim_i$ over $A \times \Omega$

**Definition 1.2.** A **Bayesian Nash Equilibrium** is the Nash Equilibrium of the strategic game defined as:

- The set of players is the set of pairs $(i, t_i)$ for each $i \in N$ and $t_i \in T_i$.
- The set of actions of player $i, t_i$ is $A_i$.
- The preference of $(i, t_i)$ is defined as:

$$a^* \succsim b^* \iff L_i(a^*, t_i) \succsim L_i(b^*, t_i)$$

where $L_i(a, t_i)$ is a lottery over $A \times \Omega$ that assigns probability $\frac{p_i(\omega)}{p_i(\tau_i^{-1}(t_i))}$ to $(a^*_{\tau_j(\omega)}, \omega)$ if $\omega \in \tau_i^{-1}(t_i)$ and 0 otherwise.

*Example* 1.1. There are two players 1 and 2, each can be of types $\{l, h\}$. A strategy profile $a$ is a length 4 vector $(a_{l_1}, a_{h_1}, a_{l_2}, a_{h_2})$. Suppose there are three states $\{\omega_1, \omega_2, \omega_3\}$ with equal prior probability. The signaling function $\tau_1$ maps state 1 and 2 to $l_1$ and 3 to $h_1$. The signaling function $\tau_2$ maps state 1 to $l_2$ and 2 and 3 to $h_2$. Suppose now player 1 receives that her type is $h_1$. What should she make of the posterior probability of the states, the other player's type, and the strategy of the other player?

*Remark.* Here, the action profile $a^*$ is of length $\sum_i |T_i|$, i.e., it collects actions for each $(i, t_i)$ pair. For example, for a given state $\omega$, $a^*_{\tau_j(\omega)}$ denotes the action of $(j, \tau_j(\omega))$. Therefore, when a player receives a signal $t_i$, she updates her posterior belief over states, which gives the probability distribution over the opponents' types according to the (deterministic) function $\tau$ and the opponents' actions given by the action profile.

**Definition 1.3** (Equivalent formulation)**.** In a Bayesian game, each (player, type) pair has a strategy. Equivalently, we can say that each player's strategy is a function that maps from his type to his action.

$$\sigma_i : T_i \to \Delta(A_i)$$

Let $\sigma(t) = (\sigma_1(t_1), \ldots, \sigma_n(t_n))$ denote the strategy profiles. Player $i$'s expected payoff from playing $a_i$ when receiving a signal $t_i$ is:

$$u_i(a_i, \sigma_{-i} \mid t_i) = \sum_{\omega, t_{-i}} u_i(\omega, a_i, \sigma_{-i}(t_{-i}))p(\omega, t_{-i} \mid t_i)$$

So a strategy profile $\sigma$ is a BNE if and only if for all $i, t_i$, and $a_i$ such that $\sigma_i(a_i \mid t_i) > 0$:

$$u_i(a_i, \sigma_{-i} \mid t_i) = \max_{a'_i} u_i(a'_i, \sigma_{-i} \mid t_i)$$

*Remark.* In a Nash equilibrium of a Bayesian game each player chooses the best action available to him given the signal that he receives and his belief about the state and the other players' actions that he deduces from this signal.

## 2 Exercise: Existence of Pure Strategy BNE in a Simple Game of Incomplete Information

Consider the following static game of incomplete information. There are two states, two players, and two pure actions for each player. In state $\theta_1$, the payoffs for the players are given by

| | $L$ | $R$ |
|---|---|---|
| $U$ | 3,1 | 0,0 |
| $D$ | 4,0 | 1,3 |

whereas in state $\theta_2$, the payoffs for the players are given by

| | $L$ | $R$ |
|---|---|---|
| $U$ | 5,0 | 2,2 |
| $D$ | 2,2 | 0,0 |

Player 2 receives a signal informing him of the state, whereas Player 1 receives no such signal. Finally, both players agree that the prior probability of state $\theta_1$ is $p \in (0, 1)$ and state $\theta_2$ is $1 - p$.

**(a)** For what values of $p$ are there no pure strategy BNE?

**(b)** For what values of $p$ are there multiple pure strategy BNE?

**(c)** For what values of $p$ is there a unique pure strategy BNE?

## Solution:

**(a)** Given the setup of this problem, Player 1's pure strategies are just her pure actions whereas Player 2's pure strategies are functions that map each signal to either $L$ or $R$. From the payoff matrix, it is clear that if Player 1 plays $U$, then 2 should play $L$ in state $\theta_1$ and $R$ in state $\theta_2$. Conversely, if Player 1 plays $D$, then 2 should play $R$ in state $\theta_1$ and $L$ in state $\theta_2$. This knowledge is sufficient to derive the Player 1's condition for deviation at any pure strategy:

Suppose that Player 1 plays $U$. Then her expected payoff for not deviating is $3p + 2(1-p)$ whereas her expected payoff for deviating is $4p$. Thus, 1 has a strict incentive to deviate from $U$ provided that

$$3p + 2(1-p) < 4p$$

or

$$\frac{2}{3} < p.$$

Conversely, suppose that Player 1 plays $D$. Then her expected payoff for not deviating is $p + 2(1-p)$ whereas her expected payoff from deviating is $5(1-p)$. Thus, 1 has a strict incentive to deviate from $D$ provided that

$$p + 2(1-p) < 5(1-p)$$

or

$$p < \frac{3}{4}.$$

If both of these conditions hold simultaneously, then there cannot be a pure strategy BNE as player 1 would necessarily have a strict incentive to deviate from any pure strategy. Thus, we conclude that for

$$p \in \left(\frac{2}{3}, \frac{3}{4}\right),$$

there are no pure strategy BNE.

**(b)** Using the work done in part (a), we have that 1 does not have an incentive to deviate from $U$ if

$$3p + 2(1-p) \geq 4p$$

or

$$\frac{2}{3} \geq p.$$

Similarly, 1 does not have an incentive to deviate from $D$ if

$$p + 2(1-p) \geq 5(1-p)$$

or

$$p \geq \frac{3}{4}.$$

If both of these conditions hold simultaneously, then there are pure strategy BNE for Player 1 playing $U$ and $D$. However, both of these conditions cannot hold simultaneously. Thus, we conclude that there are no values of $p$ for which the provided game has multiple pure strategy BNE.

**(c)** From the previous questions, it is clear that if

$$p \in \left(0, \frac{2}{3}\right]$$

then the unique pure strategy BNE sees Player 1 play $U$, and for

$$p \in \left[\frac{3}{4}, 1\right)$$

the unique pure strategy BNE sees Player 1 play $D$.

## 3 A Simple Model of Adverse Selection

Firm 1 (the "acquirer") is considering taking over firm 2 (the "target"). Firm 1 does not know firm 2's value; it believes that this value, when firm 2 is controlled by its own management, is uniformly distributed between 0\$ and 100\$. Firm 2 will be worth 50% more under firm 1's management than it is under its own management. Suppose that firm 1 bids $y \geq 0$ to take over firm 2, and firm 2 is worth $x \in [0, 100]$ (under its own management). Then if 2 accepts 1's offer, 1's payoff is $\frac{3}{2}x - y$ and 2's payoff is $y$; if 2 rejects 1's offer, 1's payoff is 0 and 2's payoff is $x$.

(a) Model this situation as a *static* game of incomplete information in which *simultaneously* firm 1 chooses how much to offer and firm 2 decides the lowest offer to accept. Specify players, actions, information structure, etc.

(b) Find the Bayes-Nash equilibrium of this game.

(c) Explain why the logic behind the equilibrium is called "adverse selection."

## Solution:

(a)
- Set of players: $\mathrm{N} = \{\mathrm{Firm1}, \mathrm{Firm2}\}$.
- Set of state of nature $\Omega = [0, 100]$. Let $x \in \Omega$ be the realized state.
- Prior $p$ is the uniform distribution.
- Actions: $A_1 = A_2 = [0, \infty)$ where $y \in A_1$ represents the offer made by firm 1 and $c \in A_2$ represents the lowest offer that firm 2 accepts.
- Set of types $T_2 = \Omega = [0, 100]$, signal function $\tau_2(x) = x$.
- Payoffs: $v_1(y, c, x) = \begin{cases} \frac{3}{2}x - y & \text{if } y \geq c \\ 0 & \text{if } y < c \end{cases}$; $v_2(y, c, x) = \begin{cases} y & \text{if } y \geq c \\ x & \text{if } y < c \end{cases}$

(b) Firm 1 doesn't receive any signal, so its strategy is a single number $y$. Firm 2's strategy would be a function $s_2 : T_2 \to A_2$. Inspecting Firm 2's payoff function gives that setting the acceptance threshold to $x$ is the optimal strategy. With this at hand, we evaluate

Firm 1's problem:

$$\begin{aligned}
&\max_{y\geq 0} \mathbb{E}_x(\frac{3}{2}x - y)\cdot \mathbb{1}\{y \geq x\} \\
&= \max_{y\geq 0} \int_0^y (\frac{3}{2}x - y)\cdot \frac{1}{100}dx \\
&= \max_{y\geq 0} \frac{1}{100}(\frac{3}{4}x^2 - yx)\Big|_0^y \\
&= \max_{y\geq 0} \frac{1}{100}(-\frac{1}{4}y^2)
\end{aligned}$$

So the optimal $y^* = 0$, i.e., offering the lowest possible acquisition price and the market essentially collapses.

*Alternatively, one can calculate the expected payoff using law of iterated expectations:

$$\begin{aligned}
\mathbb{E}[v_2] &= \mathbb{E}_x\left[\frac{3}{2}x - y|y > x\right] P(y > x) + \mathbb{E}_x\left[0|y \leq x\right] P(y \leq x) \\
&= \left(\frac{3}{2}\mathbb{E}_x\left[x|y > x\right] - y\right)\frac{y}{100} \\
&= \left(\frac{3}{2}\cdot\frac{y}{2} - y\right)\frac{y}{100} \\
\Rightarrow y^* &= 0
\end{aligned}$$

The second to last line uses the property of the uniform distribution on $[a, b]$ – conditional on being below a certain point $y$, the expected value of the uniform variable is $\frac{y-a}{2}$.

(c) Suppose Firm 1 makes an optimal offer $y$ with the entire distribution of $x$ in mind. This would be $\frac{3}{2}\cdot\mathbb{E}[X] = 75$. But only when Firm 2 has a realization of value lower than 75 it would accept the offer. Anticipating this, Firm 1 adjusts its offer based on the lower three fourth of the value distribution and offers a lower $y'$. But this offer will again only attract Firm 2 with lower valuation than what Firm 1 has used to make its offer, so Firm 1 will again adjust the offer. This 'mental iteration' process continues until Firm 1 offers 0 and the market collapses.