Rationalizability and dominance

Rationalizability

Motivation

In a Nash equilibrium we assume that:

- Each player optimally respond given his/her beliefs about the other players' behavior;
- Moreover, we assume that beliefs are correct.
- Each players "knows" the other players' equilibrium behavior.

This is an heroic assumption in one-shot environments.

Nash equilibrium conditions are best understood as necessary conditions for stationary outcomes.

A different approach is to give up on "correct" beliefs and rely only on rationality:

- players’s actions are optimal given their beliefs;
- each player believes that the actions of the other players is a best response to some belief;
- ...which in turn is backed by optimal behavior supported by "backed" beliefs...

We call outcomes supported by this type of reasoning "rationalizable".

Naturally a Nash equilibrium must satisfy these conditions, so rationalizability is a weaker equilibrium concept.

Examples

For most games rationalizability provide no bite.

- In the prisoner's dilemma C, C is the unique rationalizable outcome, but it is also the unique Nash equilibrium.
- In, say, Hawk-Dove, (H, D) and (D, H) are pure Nash equilibria, so for both players $Z_j = \{D, H\}$ are rationalizable sets.

There are however examples in which it has a bite.

	b_1	b_2	b_3	b_4
a_1	0,7	2, 5	7,0	0,1
a_2	5,2	3,3	5,2	0,1
a_3	7, 0	2, 5	0,7	0,1
a_4	0,0	0,-2	0,0	10,-1

Lets first prove that b_4 is not rationalizable.

- If $\mu_2(b_4)(a_4) > \frac{1}{2}$ (i.e. the belief of 2 on a_4) then $u_2(b_3, \mu_2(b_4)) > u_2(b_4, \mu_2(b_4))$
- If $\mu_2(b_4)(a_4) \leq \frac{1}{2}$ then $u_2(b_2, \mu_2(b_4)) > u_2(b_4, \mu_2(b_4))$.

Given that b_4 is not rationalizable, then a_4 is also not rationalizable since strictly dominated by a_2.

a_2, b_2 is a Nash equilibrium so rationalizable.

$a_1 \in B_1(b_3)$, $b_3 \in B_2(a_3)$, $a_3 \in B_1(b_1)$, $b_1 \in B_2(a_1)$: so a_1, a_3, b_1, b_3 are all rationalizable.

Two equivalent definitions

A belief of a player i is a probability measure on A_{-i}.

Note that this probability measure does not need to treat players' actions as independent, it is a general probability measure on A_{-i}.

Or at least we have a choice: we could define beliefs as non necessarily independent or assume they are independent. We go with the first for now (see more later).

Definition 1. *An action* $a_i \in A_i$ *is rationalizable in the strategic game* $\langle N, (A_i), (u_i) \rangle$ *if there exists:*

- *A collection* $\left((X_j^t)_{j \in N} \right)_{t=1}^{\infty}$ *of sets with* $X_j^1 \subseteq A_j$ *for all j and t;*
- *A belief* μ_i^1 *of player i whose support is a subset of* X_{-i}^1
- *for each player* $j \in N$ *and* $t \geq 1$ *and each* $a_j \in X_j^t$ *a belief* $\mu_j^{t+1}(a_j)$ *of player j with support* X_{-j}^{t+1};

such that:

- $a_i = a_i^0$ *is a best response to the belief* μ_i^1 *of player i, ...*

- *for every j and* $t \geq 1$, *every action* $a_j \in X_j^t$ *is a best response to the belief* $\mu_j^{t+1}(a_j)$ *of player j.*

- The sets X_j^1 for $j \in N \backslash \{i\}$ are defined as the set of a_j' such that there is a $\mathbf{a}_{-i}$ in the support of $\mu_i^1(a_1)$ for which $a_j' = (\mathbf{a}_{-i})_j$, i.e. a_j' is the jth element of $\mathbf{a}_{-i}$ (and $X_i^1 = \varnothing$ by convention).

- The sets X_j^t for $t \geq 2$ are defined as the set of a_j' such that there is some player $k \in N \backslash \{j\}$ some action $a_k \in X_k^{t-1}$ and some $\mathbf{a}_{-k}$ in the support of $\mu_k^t(a_k)$ for which $a_j' = (\mathbf{a}_{-k})_j$.

Assume there are 3 players $\{a, b, c\}$, and 2 actions, $\{A, B\}$.

Assume A is rationalizable for a, so:

- we have a belief μ_a^1 on the actions of b and c that makes A optimal for a, say $\mu_a^1(A, A) = \alpha$, $\mu_a^1(B, B) = 1 - \alpha$.
- In this case $X_b^1 = X_c^1 = \{A, B\}$ and X_a^1 is undefined (and irrelevant).
- We need beliefs that make the beliefs rational.
- So μ_b^2 which is a belief on X^2_{-b} and μ_c^2 which is a belief on X^2_{-c} ...

This definition is a bit cumbersome.

The next definition is equivalent, but easier to remember and to check.

Definition 2. *An action* $a_i \in A_i$ *is rationalizable in the strategic game* $\langle N, (A_i), (u_i) \rangle$ *if for each* $j \in N$ *there is a set* $Z_j \subseteq A_j$ *such that:*

- $a_i \in Z_i$;
- *Every action* $a_j \in Z_j$ *is a best response to a belief* $\mu_j(a_j)$ *of player j whose support is a subset of* Z_{-j}.

We now prove that the two definitions are equivalent.

If $a_i \in A_i$ is rationalizable according to Definition 1, then define:

- $Z_i = \{a_i\} \cup (\bigcup_{t=1}^{\infty} X_i^t)$
- and $Z_j = (\bigcup_{t=1}^{\infty} X_j^t)$ for each $j \in N \setminus \{i\}$, for this define $X_i^1 = \varnothing$.

And we are done proving that Definition 1⇒Definition 2!

If $a_i \in A_i$ is rationalizable according to Definition 2, then define $\mu_i^1 = \mu_i(a_i)$ and $\mu_j^t = \mu_j(a_j)$ for $t \geq 2$ and $j \in N$. (note that Definition 2 gives us $\mu_j(a_i)$).

We now need to define the X_j^ts:

- The sets X_j^t for $t \geq 2$ are defined as the set of a_j' such that there is some player $k \in N \setminus \{j\}$ some action $a_k \in X_k^{t-1}$ and some $\mathbf{a}_{-k}$ in the support of $\mu_k(a_k)$ for which $a_j' = (\mathbf{a}_{-k})_j$.

- The sets X_j^1 for $j \in N \setminus \{i\}$ are defined as the set of a_j' such that $\mathbf{a}_{-i}$ in the support of $\mu_i(a_1)$ for which $a_j' = (\mathbf{a}_{-i})_j$ (and $X_i^1 = \varnothing$ by convention).

Note that every action used with positive probability by some player in a correlated equilibrium of a finite strategic game is rationalizable.

Hint: the support of actions Z_i is the set of action in the support of the strategies, and the beliefs are the distributions over A_{-i} generated by the equilibrium strategies.

However the set of rationalizable actions can be larger than the support of correlated equilibria.

Cournot revisited

In a previous lecture we have studied the Nash equilibrium in the Cournot model.

This model can also be studied using rationalizability.

Consider the game with $N = \{1, 2\}$, $A_i = [0, 1]$ and:

$$u_i(a_1, a_2) = a_i(1 - \sum_{j=1,2} a_j).$$

Player i best response is $B_i(a_j) = (1 - a_j)/2$, so the Nash equilibrium is $a_i = a_j = \frac{1}{3}$.

Lets consider the set of rationalizable strategies $Z_i = Z_j = Z$ (by symmetry).

Define $m = \inf Z$, and $M = \sup Z$.

A best response by i is a maximum of:

$$a_i(1 - a_i - E(a_j))$$

Thus:

$$B_i(E(a_j)) \in \left[\frac{1-M}{2}, \frac{1-m}{2} \right]$$

We need to have:

$$m \geq \frac{1-M}{2},\ M \leq \frac{1-m}{2}$$

So:

$$2m \geq 1-M \geq 1-\frac{1-m}{2} = \frac{1+m}{2}$$

$$\Leftrightarrow m \geq \frac{1}{3}$$

And proceeding similarly, $M \leq \frac{1}{3}$, so $M = m = \frac{1}{3}$.

Correlated vs. independent beliefs

In the definitions presented above beliefs of a player i is a general probability distribution on A_{-i} (so possible with correlated actions).

Alternatively we may assume that agents randomize in an independent way.

The definitions are not equivalent.

Consider this game (the number in a cell is the common payoff of all players). Player 3 selects the matrix.

	L	R
U	8	0
D	0	0

M_1

	L	R
U	4	0
D	0	4

M_2

	L	R
U	0	0
D	0	8

M_3

	L	R
U	3	3
D	3	3

M_4

In this game M_2 is a rationalizable choice by 3.

$$U \in B_1(L, M_2), D \in B_1(R, M_2),\ L \in B_2(U, M_2), R \in B_2(D, M_2)$$

$$M_2 \in B_3(\frac{1}{2}(UL) + \frac{1}{2}(DR))$$

So M_2 is a rationalizable with $Z_1 = \{U, D\}, Z_2 = \{L, R\}$ and $Z_3 = \{M_2\}$.

However M_2 is not rationalizable if we require beliefs in which the actions of 1 and 2 are independent.

Let p be the probability with which 1 chooses U and q the probability with which 2 selects L.

For M_2 to be optimal we need:

$$4pq + 4(1-p)(1-q) \geq \max\{8pq, 8(1-p)(1-q), 3\}$$

But this inequality is impossible.

Dominance

We start with two definitions of dominance.

Definition. *An action of a player i is a never-best response if it is not a best response to any belief of player i.*

An action that is never a best response, cannot be rationalized.

Definition. *An action* a_i *of* i *is strictly dominated if there is a mixed strategy* α_i *of a player* i *such that*

$$U_i(a_{-i}, \alpha_i) > U_i(a_{-i}, a_i)$$

for all a_{-i}.

We have:

Lemma. *An action of a player in a finite game is a never best response if and only if it is strictly dominated.*

See Osborne and Rubinstein Lemma 60.1.

Iterated Elimination of Strictly Dominated Actions

An example

	L	R
T	3,0	0, 1
M	0,0	3,1
B	1,1	1, 0

In this game B is not dominated by any other action, but it is dominated by $\frac{1}{2}T + \frac{1}{2}M$.

Once B is eliminated then L is strictly dominated, and eliminated L, then T is dominated, so the outcome is M, R.

Let us formalize this process.

Definition. *The set $X \subseteq A$ of outcomes of a strategic game survives iterated elimination of strictly dominated actions if $X = \times_{j\in N} X_j$ and there is a collection $\left((X_j^t)_{j\in N}\right)_{t=0}^{T}$ of sets that satisfies the following conditions for each $j \in N$:*

- $X_j^0 = A_j$ *and* $X_j^T = X_j$; *and* $X_j^{t+1} \subseteq X_j^t$ *for each* $t = 0, T-1$;
- *For each* $t = 0, .., T-1$ *every action of player* j *in* $X_j^t \backslash X_j^{t+1}$ *is strictly dominated in the game* $\langle N, (X_i^t), (u_i^t) \rangle$, *where* u_j^t *is the function* u_i *restricted to* $X^t = \times_{i\in N} X_i^t$.

Proposition. *If* $X = \times_{j \in N} X_j$ *survives iterated deletion of strictly dominated actions, in a finite strategic game, then* X_j *is the set of player's* j *rationalizable actions for each* j*.*

Proof. ($Z_j \subseteq X_j$). Assume a_i is rationalizable with supporting sets $(Z_j)_{j \in N}$.

Then for any t we must have $Z_j \subseteq X_j^t$ since each action in Z_j is a best response in A_j to some belief over Z_{-j}, hence not strictly dominated in game $\langle N, (X_j^t), (u_j^t) \rangle$.

Note we are using the fact that if an action is sometimes a best response, then it is not strictly dominated.

We now show $X_j \subseteq Z_j$, i.e. X_j is rationalizable.

Every action in X_j is a best response in X_j to some belief in X_{-j}. However, we need to show it is a best response in A_j.

Assume not. Then there an $a_j \in X_j$ that is a best response in X_j^t to a belief μ_j in X_{-j} but not in X_j^{t-1}.

So there is a $b_j \in X_j^{t-1} \backslash X_j^t$ that is a best response in X_j^{t-1} to μ_j in X_{-j}.

But then $b_j \in X_j^t$, a contradiction.

This finding relies on the result that an action is strictly dominated if and only if it is a never best response.

This finding, however, relies on the fact that for rationalizability we allow beliefs on the action of other players to be correlated.

If we require beliefs to respect independence of the action of the players, then X_j may not be rationalizable.

Note that this may be a problem only with more than 2 players.

An example

	L	R
U	8	0
D	0	0

M_1

	L	R
U	4	0
D	0	4

M_2

	L	R
U	0	0
D	0	8

M_3

	L	R
U	3	3
D	3	3

M_4

When we allow for the beliefs that opponents actions are correlated, M2 is rationalizable, however it is not when we impose independence.

Still M2 survives Iterated deletion of strictly dominated strategies.

Iterated Deletion of Weakly Dominated Actions

Definition. *The action a_i is weakly dominated if there is a mixed strategy α_i of player i such that:*

$$U_i(a_{-i}, \alpha_i) \geq U_i(a_{-i}, a_i)$$

for all a_{-i} and

$$U_i(a_{-i}, \alpha_i) > U_i(a_{-i}, a_i)$$

for some a_{-i}.

An action that is weakly dominated is a (weak) best response to some belief.

Eliminating such actions is therefore not always a rational decision.

There is however no strict advantage to using such actions.

Definition. *The set* $X \subseteq A$ *of outcomes of a strategic game survives iterated elimination of weakly dominated actions if* $X = \times_{j\in N} X_j$ *and there is a collection* $\left((X_j^t)_{j\in N}\right)_{t=0}^{T}$ *of sets that satisfies the following conditions for each* $j \in N$*:*

- $X_j^0 = A_j$ *and* $X_j^T = X_j$*;*
- $X_j^{t+1} \subseteq X_j^t$ *for each* $t = 0, T-1$*;*
- *For each* $t = 0,..,T-1$ *every action of player* j *in* $X_j^t \backslash X_j^{t+1}$ *is weakly dominated in the game* $\langle N, (X_i^t), (u_i^t)\rangle$, *where* u_j^t *is the function* u_i *restricted to* $X^t = \times_{i\in N} X_i^t$.

The order of deletion now matters:

	L	R
T	1,1	0, 0
M	1,1	2,1
B	0,0	2, 0

Eliminate T (by M), then L (by R): the outcome is (M,R)

But also: eliminate B (by M), then R (by L): resulting set does not include (M,R).

Dominance solvability

Definition. *A strategic game is said to be dominance solvable if all players are indifferent between all outcomes that survive the iterative procedure in which **all** the weakly dominated actions of **each** player are eliminated **at each stage**.*

Consider this game:

	L	R
U	1,0	0, 0
D	0,1	0, 0

The game is dominance solvable, with only one surviving outcome (U,L).

But if we delete D, then neither L, nor R is dominated So (U,L) and (U,R) survive iterated elimination of weekly dominated actions.