ECON 6110, Microeconomic Theory III	**Section 14**
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1 Knowledge and Equilibrium

Definition 1.1. A model of knowledge consists of:

1. Set of states Ω. Only one of the state can be true (but unobservable).

2. An **information function** for each agent h such that $h(\omega) \subset \Omega$. When a true state $\omega \in \Omega$ occurs, any state in $h(\omega)$ is deemed possible by the agent.

3. A subset of states $E \subset \Omega$ is an **event**. If $h(\omega) \subset E$, then we say in state ω, the agent **knows** E. We define the agent's **knowledge function** K by:

$$K(E) = \{\omega \in \Omega : h(\omega) \subset E\}.$$

   $K(E)$ is the set of states where the agent knows E.

Definition 1.2. An information function is **partitional** if there is some partition of Ω such that for any $\omega \in \Omega$, $h(\omega)$ is the element of the partition that contains ω.

Proposition 1.1. *An information function is* ***partitional*** *iff it satisfies the following two properties:*

P1 $\omega \in h(\omega)$ *for every* $\omega \in \Omega$.

P2 *If* $\omega' \in h(\omega)$, *then* $h(\omega') = h(\omega)$.

Example 1.2. Suppose $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and that the agent's partition is $\{\{\omega_1, \omega_2\}, \{\omega_3\}, \{\omega_4\}\}$. Then $h(\omega_3) = \{\omega_3\}$, while $h(\omega_1) = \{\omega_1, \omega_2\}$.

Suppose $E = \{\omega_3\}$. Then $K(E) = \{\omega_3\}$. Similarly, $K(\{\omega_3, \omega_4\}) = \{\omega_3, \omega_4\}$ and $K(\{\omega_1, \omega_3\}) = \{\omega_3\}$.

1.1 Axioms of Knowledge

K1 (Axiom of Awareness) $K(\Omega) = \Omega$. In all states, agents knows he is in set Ω.

K2 $K(E) \cap K(F) = K(E \cap F)$. In a state, if the agent knows both E and F then he knows $E \cap F$.

K3 (if h satisfies **P1**) (Axiom of Knowledge) $K(E) \subset E$. Whenever the agent knows E then indeed some state in E must be the true state.

K4 (if h satisfies **P1-2**) (Axiom of Transparency) $K(E) \subset K(K(E))$. If the agent knows E then he knows that he knows E.

K5 (if h satisfies **P1-2**) (Axiom of Wisdom) $\Omega \backslash K(E) \subset K(\Omega \backslash K(E))$. If agent does not know E then he knows that he does not know E.

1.2 Common Knowledge

Suppose there are I agents with partitional information functions $h_1, \ldots, h_I$ and associated knowledge functions $K_1, \ldots, K_I$.

Definition 1.3. We say that an event $E \subset \Omega$ is **mutual knowledge** in state ω if it is known to all agent. In math, let

$$K_1(E) \cap K_2(E) \cap \cdots \cap K_I(E) \equiv K^1(E)$$

Then $\omega \in K^1(E)$.

Definition 1.4. An event F is **self-evident** if for all $\omega \in F$ and $i = 1, \ldots, I$, we have $h_i(\omega) \subset F$.

Remark. A "self-evident" event is one whose occurrence is immediately and transparently known to each agent, purely by virtue of their information partitions. Formally, an event F is self-evident for agent i if, whenever the true state ω lies in F, the entire information cell $h_i(\omega)$ is contained in F, which implies $K_i(F) = F$.

Definition 1.5. The following two definitions of **common knowledge** is equivalent:

1. An event $E \subset \Omega$ is common knowledge in state ω if $\omega \in K^1(E) \cap K^1 K^1(E) \cap \cdots$. (An event is "common knowledge" if is it mutual knowledge and each individual knows that all other individuals know it, each individual knows that all other individuals know that all the individuals know it, and so on.)

2. An event $E \subset \Omega$ is common knowledge in state $\omega \in \Omega$ if there is a self-evident event F for which $\omega \in F \subset E$.

Example 1.3. In a two-player example, an event $E \subseteq \Omega$ is common knowledge between 1 and 2 in the state $\omega \in \Omega$ if ω is a member of every set in the infinite sequence $K_1(E),\ K_2(E),\ K_1(K_2(E)),$ $K_2(K_1(E)), \ldots.$

1.3 Epistemic Foundations for Equilibrium

Fix a game $G = (I, \{S_i\}, \{u_i\})$. Let Ω be a set of states. Each state is a complete description of each player's knowledge, action and belief. Formally, each state $\omega \in \Omega$ specifies for each i,

- $h_i(\omega) \subset \Omega$, i's knowledge in state ω.
- $s_i(\omega) \in S_i$, i's pure strategy in state ω.
- $\mu_i(\omega) \in \Delta(S_{-i})$, i's belief about the actions of others (note that i may believe other players actions are correlated).

We assume that among the players, it is common knowledge that the game being played is G.

Proposition 1.4. *Suppose that in state $\omega \in \Omega$, each player i:*

1. *knows the others' actions: $h_i(\omega) \subset \{\omega' \in \Omega : s_{-i}(\omega') = s_{-i}(\omega)\}$.*
2. *has a belief consistent with this knowledge: supp$(\mu_i(\omega)) \subset \{s_{-i}(\omega') \in S_{-i} : \omega' \in h_i(\omega)\}$*
3. *is rational: $s_i(\omega)$ is a best response to $\mu_i(\omega)$,*

Then $s(\omega)$ is a pure strategy Nash equilibrium of G.

Proof. By *(iii)*, $s_i(\omega)$ is a best response for i to his belief, which by *(ii)* and *(i)* assigns probability one to the profile $s_{-i}(\omega)$. $\square$

Remark. Previously, we defined Nash equilibrium as "mutual best responses." Here, we are able to arrive at the same place by using the common knowledge assumption of rationality.

With a bit more work, we can also provide an epistemic characterization of mixed Nash equilibrium, correlated equilibrium, and rationalizability.

Example 1.5. It seems natural to extend this reasoning to dynamic games.

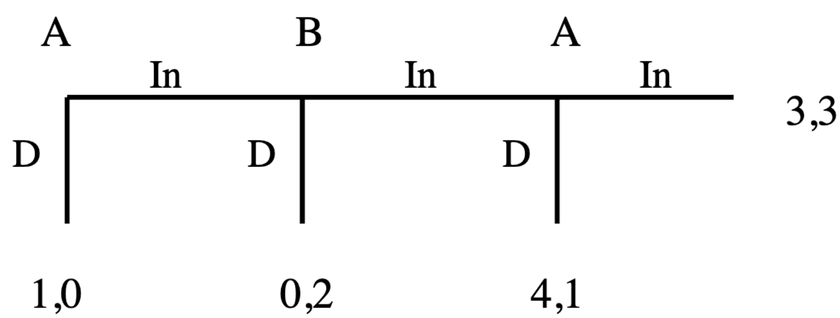

Figure 1: The Centepede Game

Consider the following argument for why common knowledge of rationality should imply the backward induction solution that A play D immediately:

If A is rational, A will play D at the last node; if B knows A is rational, then B knows this; if B herself is rational, she must then play D at the second to last node; if A knows B is rational, and that B knows that A is rational, then A knows that B will play D at the second to last node; thus, if A is rational, A must play D immediately.

The subtlety, however, is *what will happen if* A plays In? At that point, how will B assess A's rationality?