ECON 6100

Problem Set 0

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1. ***Proof.***[1] Define $\mathcal{H} = \{H : C \subseteq H, H \text{ is a closed half-space}\}$. We want to show that $C = \bigcap_{H \in \mathcal{H}} H$. We will do so with double set containment.

   First, $C \subseteq \bigcap_{H\in\mathcal{H}} H$ simply – for any $c \in C$, since $C \subseteq H \;\forall\; H \in \mathcal{H}$, $c \in H \;\forall\; H \in \mathcal{H}$, so $c \in \bigcap_{H\in\mathcal{H}} H$. Thus, $C \subseteq \bigcap_{H\in\mathcal{H}} H$.

   Next, $\bigcap_{H\in\mathcal{H}} H \subseteq C$. We will show by contrapositive. Take some $x \notin C$. Then since singleton sets are closed and convex, since C is closed and convex by assumption, and since $\{x\} \cap C = \varnothing$, we have that the Strong Separating Hyperplane Theorem applies, meaning that there exists a hyperplane $P \neq 0$ which strongly separates $\{x\}$ and C. Since this is strong separation, $x \notin P$, meaning that the (weak) halfspace generated by P that contains C does not contain x. Since $P \in \mathcal{H}$, we have that $x \notin \bigcap_{H\in\mathcal{H}} H$.

   Since we have that $C \subseteq \bigcap_{H\in\mathcal{H}} H$ and $\bigcap_{H\in\mathcal{H}} H \subseteq C$, we have that $C = \bigcap_{H\in\mathcal{H}} H$. $\square$

2. We have the concave support function of C, $e_C(p) = \inf\{p \cdot x : x \in C\}$.

   (a) ***Proof.***Fix some $p, p' \in \mathbb{R}^n$ and some $\alpha \in [0,1]$, and define $p'' = \alpha p + (1-\alpha)p'$. If either $e_C(p)$ or $e_C(p')$ are equal to $-\infty$, then trivially $e_C(p'') \geq -\infty = \alpha e_C(p) + (1-\alpha)e_C(p')$. If either $e_C(p)$ or $e_C(p')$ are equal to ∞, then $C = \varnothing$ and so $e_C(p'') = \infty \geq \alpha e_C(p) + (1-\alpha)e_C(p')$. From here, assume that $e_C(p), e_C(p')$ are finite. Define $x, x', x'' \in C$ where $p \cdot x = e_C(p)$, $p' \cdot x' = e_C(p')$, and $p'' \cdot x'' = e_C(p'')$. Existence follows from closed and a finite infimum, meaning that extrema are attained. (Uniqueness isn't necessarily true, but not necessary here). We have that

   $$e_C(p'') = p'' \cdot x'' = \alpha p \cdot x'' + (1-\alpha)p' \cdot x'' \geq \alpha p \cdot x + (1-\alpha)p' \cdot x' = \alpha e_C(p) + (1-\alpha)e_C(p')$$

   where the inequality follows from the attained infimum. Thus, $e_C(\cdot)$ is concave. $\square$

   (b) ***Proof.***Fix some $p \in \mathbb{R}^n$ and some $\lambda \in \mathbb{R}_{++}$. If $e_C(p) = \infty$, then $C = \varnothing$ so $e_C(\lambda p) = \infty \equiv \lambda\infty$. If $e_C(p) = -\infty$, then there exists a sequence $\{x_n\} \in C$ such that $\lim_{n\to\infty} p \cdot x_n = -\infty$, so $\lim_{n\to\infty} \lambda p \cdot x_n = \lambda \lim_{n\to\infty} p \cdot x_n = \lambda \cdot (-\infty)$, From here, assume that $e_C(p)$ is finite. Then since C is closed, there exists $x \in C$ such that $p \cdot x = e_C(p)$. It follows directly that $e_C(\lambda p) \leq \lambda p \cdot x = \lambda e_C(p)$. It remains to show that $\lambda e_C(p) \leq e_C(\lambda p)$. FSOC, assume that there exists $x' \in C$ such that $\lambda p \cdot x' < \lambda e_C(p)$. Then we would have that $p \cdot x' < p \cdot x$, contradicting the definition of $e_C(p)$ as the minimum. Thus, $e_C(\lambda p) = \lambda e_C(p)$. $\square$

   (c) If $e_C(p) = -\infty$, then C is unbounded in at least one dimension i, specifically in the opposite direction of p_i, where p_i is nonzero. In this dimension, there exists a sequence $\{x_n\} \in C$ such that the ith coordinate of x diverges, so that $\lim_{n\to\infty} p \cdot x_n = -\infty$.

   (d) ***Proof.*** First, assume that the halfspace $[p \geq \alpha] \subseteq \mathbb{R}^n$ contains C. Then for all $x \in C$, $p \cdot x \geq \alpha$, meaning that $\alpha \leq \inf\{p \cdot x : x \in C\} = e_C(p)$. Next, assume that $\alpha \leq e_C(p)$. Then either $e_C(p) = \infty$, meaning that C is empty and contained in any nonempty set, including the halfspace, or $e_C(p)$ is finite, so there exists $x \in C$ such that $p \cdot x = e_C(p)$. Since $\alpha \leq p \cdot x$, from the definition of extrema $\alpha \leq p \cdot y \;\forall\; y \in C$, so $C \subseteq [p \geq \alpha]$. $\square$

[1] Partially from Patrick, who had a really nice proof in the solutions to the 6170 final. His second part is way cleaner than mine was, I ended up proving strong and strict separation, for no real reason.

3. ***Proof.*** Assume that f is concave. Take some (x, y) and (x', y') such that $y \leq f(x)$ and $y' \leq f(x')$, and fix $\alpha \in [0, 1]$. Then we have that since f is concave,

$$\alpha y + (1-\alpha)y' \leq \alpha f(x) + (1-\alpha)f(x') \leq f(\alpha x + (1-\alpha)x')$$

so $\alpha(x, y) + (1-\alpha)(x', y')$ is in the subgraph of f, and it is convex. Next, assume that the subgraph of f is convex. Take some $x, x' \in \mathbb{R}^n$, and fix $\alpha \in [0, 1]$. Set $y = f(x)$ and $y' = f(x')$. Since $y \leq f(x)$ and $y' \leq f(x')$, (x, y) and (x', y') are both in the subgraph. Then we have that since the subgraph is convex, $\alpha y + (1-\alpha)y' \leq f(\alpha x + (1-\alpha)x')$, meaning that $f(\alpha x + (1-\alpha)x') \geq \alpha f(x) + (1-\alpha)f(x')$, so f is concave. $\square$

4. Two simple examples: $X = [0, 1] = Y$, which are both closed and convex and cannot be separated as they are the same set. Or, X is nonempty closed and convex and $Y = \varnothing$, which is vacuously closed and convex. These cannot be separated since each hyperplane (again, vacuously) contains Y.

5. ***Proof.*** If $yA \ll 0$, then $yAx < 0$ for any $x > 0$, meaning that $Ax \neq 0$ for any $x > 0$. On the other side, if $Ax = 0$ for some $x > 0$, taking some element i of x where $x_i > 0$, we have that the ith column of A must necessarily be 0, for complimentary slackness. Thus, the ith element of yA must be 0 for any y, so $yA \ll 0$ has no solutions. Thus, the two results are mutually exclusive. $\square$