ECON6110: Problem Set 4

Spring 2025

This problem set is due on at 23:59 on April 11, 2024. Every student must write their own solution and submit it individually. Problem set submissions are submitted electronically and may be typed or handwritten. If handwritten, please ensure your work and scan are legible. **Illegible submissions will not be graded.**

Problem 1: Repeated Games with Imperfect Public Monitoring

Consider an infinitely repeated game with imperfect public monitoring. There are two players $i = 1, 2$ and $A_i = \{C, D\}$. The public signals are $Y = \{G, B\}$ (good and bad) where:

$$\Pr(G \mid a) = \begin{cases} p & a = (C,C) \\ q & a = (C,D),(D,C) \\ r & a = (D,D) \end{cases}$$

with $p > q > r$ and $p + q + r = 1$. Assume $p - q > q - r$. The payoffs are:

$$r_i(a_i, y) = \begin{cases} 1 + \frac{2(1-p)}{p-q} & (a_i, y) = (C, G) \\ 1 - \frac{2p}{p-q} & (a_i, y) = (C, B) \\ \frac{2(1-r)}{q-r} & (a_i, y) = (D, G) \\ \frac{-2r}{q-r} & (a_i, y) = (D, B) \end{cases}$$

(a) What is the expected stage-game payoff of an action profile (C, C), i.e., $g_i(C, C)$, for each player? What is the expected payoff of the other action profiles? Discuss and provide an interpretation.

(b) Let

$$v = \frac{\delta r}{1 - \delta(p - r)} \quad \text{and} \quad v' = \frac{1 - \delta + \delta r}{1 - \delta(p - r)}$$

Find a condition on δ under which the payoff v is generated by δ and the set of continuation payoffs $W = \{(v, v), (v', v')\}$.

Hint: to generate v use the stage game action profile $a = (D, D)$ and the map $w(y) = \frac{\delta r}{1-\delta(p-r)}$ if $y = B$ and $w(y) = \frac{1-\delta+\delta r}{1-\delta(p-r)}$ if $y = G$.

(c) Let v, v' be the same as in (b). Find a condition on δ under which the payoff v' is generated by δ and the set of continuation payoffs $W = \{(v, v), (v', v')\}$.

Hint: use the same $w(y)$ as above and $a = (C, C)$.

Solution:

(a) Consider $a_i = C$ when $a_j = C$. The expected payoff is:

$$g_i(C, C) = \mathbb{E}_y \left[r_i(a_i, y); a = (D, D)\right] = p(1 + \frac{2(1-p)}{p-q}) + (1 - p) \left(1 - \frac{2p}{p-q}\right) = 1$$

Similarly we have $g_1(C,D) = -1$, $g_1(D,C) = 2$ and $g_1(D,D) = 0$. So the expected payoffs give us a prisoners' dilemma game.

(b) Since the payoffs and the stage game actions are symmetric, we only need to verify the two conditions for player 1. First:

$$\begin{aligned} v &= (1-\delta)\, g_1(D,D) + \delta\, [(1-r)v + rv'] \\ &= \delta\, [v + r\,(v'-v)] \end{aligned}$$

which can be checked to be true.

Second, we need:

$$\begin{aligned} v &\geq (1-\delta) g_1(C,D) + \delta \sum_y \pi(y \mid C,D) w_1(y) \\ &= -(1-\delta) + \delta q v' + \delta(1-q)v \\ &= -(1-\delta) + \delta v + \delta q(v'-v) \end{aligned}$$

This condition is verified if:

$$\delta \leq \frac{1}{p+q-2r}$$

(c) We need to verify that:

$$\begin{aligned} v' &= (1-\delta) + \delta\, [(1-p)\, v + pv')] \\ v' &\geq 2(1-\delta) + \delta\, [(1-q)\, v + qv')] \end{aligned}$$

The first condition is always verified, the second requires:

$$\delta \geq \frac{1}{2p-q-r}$$

Problem 2: Bayesian Games

Consider the following model of Bertrand duopoly with differentiated products and asymmetric information. Demand for firm $i = 1, 2$ is equal to:

$$q_i(p_i, p_j) = \begin{cases} a - p_i - b_i p_j & p_i \leq a - b_i p_j \\ 0 & \text{else} \end{cases}$$

Costs are zero for both firms. The sensitivity of firm i's demand to firm j's price is either high or low. That is, b_i is either b_H or b_L; where $1 > b_H > b_L > 0$. We also assume

$$2 - \theta (b_H - b_L) > 0.$$

For each firm i: $b_i = b_L$ with probability θ and $b_i = b_H$ with probability $1 - \theta$; independent of the realization of b_j. Each firm knows its own b_i but not its competitor's. All of this is common knowledge.

(a) What are the action spaces, state spaces, type spaces, prior beliefs, and utility functions in this game? What do pure strategies in this game look like?

(b) Find the symmetric pure-strategy Bayesian Nash equilibrium of this game, i.e. when both firms have the same strategy given their type.

Hint. Assume an interior solution (i.e. that equilibrium prices and quantities are positive), and verify this is indeed the case.

Solution:

(a) We have:

$$\begin{aligned} N &= 1, 2 \\ A_i &= [0, \infty) \\ \Omega &= \{b_L, b_H\} \times \{b_L, b_H\} \\ T_i &= \{b_L, b_H\} \\ p(b_L) &= \theta \\ u_i(b_i, p_i, p_j) &= \begin{cases} (a - p_i - b_i p_j)\, p_i & p_i \leq a - b_i p_j \\ 0 & \text{else} \end{cases} \end{aligned}$$

Pure strategies are functions $s_t : T_i \to A_i$. Since T_i is discrete, a pure strategy can be represented by two numbers p_L, p_H.

(b) Let us assume (and verify later) an interior equilibrium with $p_L^* > 0$, $p_H^* > 0$.

The best response of a type b_i is:

$$p_i^* \in \arg\max_{p_i} \left\{ \begin{array}{c} \theta\left(\left(a - p_i - b_i p_L^*\right) p_i\right) \\ +\left(1-\theta\right) \cdot \left(\left(a - p_i - b_i p_H^*\right) p_i\right) \end{array} \right\}$$

Using FOCs, we have:

$$p_i^* = \frac{a - \theta b_i p_L^* - \left(1-\theta\right) b_i p_H^*}{2}$$

For $b_i = b_L$ and $p_i = p_L$:

$$p_L^* \left(2 + \theta b_L\right) = a - \left(1-\theta\right) b_L p_H^*$$

Similarly, for $b_i = b_H$ and $p_i = p_H$:

$$p_H^* \left(2 + (1-\theta) b_H\right) = a - \theta b_H p_L^*$$

Doing the algebra:

$$\begin{aligned} p_L^* &= \frac{a - \left(1-\theta\right) b_L p_H^*}{2 + \theta b_L} \\ p_H^* &= \frac{a - \theta b_H p_L^*}{2 + \left(1-\theta\right) b_H} \end{aligned}$$

Solving for the system, we get:

$$\begin{aligned} p_L^* &= \frac{a}{2} \cdot \frac{\left[2 + \left(1-\theta\right)\left(b_H - b_L\right)\right]}{2 + \left(1-\theta\right) b_H + \theta b_L} \\ p_H^* &= \frac{a}{2} \cdot \frac{\left[2 - \theta\left(b_H - b_L\right)\right]}{2 + \left(1-\theta\right) b_H + \theta b_L} \end{aligned}$$

We now need to verify that we indeed have an interior equilibrium, meaning that $p_H^* > 0$, $p_L^* > 0$ and $q(p_H^*, p_H^*) > 0$, $q(p_H^*, p_L^*) > 0$, $q(p_L^*, p_H^*) > 0$, $q(p_L^*, p_L^*) > 0$.

Note that $p_L^* > 0$ is always true and $p_H^* > 0$, if

$$2 - \theta \left(b_H - b_L\right) > 0$$

as assumed.

The quantities are positive if:

$$\begin{aligned}
q(p_H^*, p_H^*) &= a - p_H^* - b_H p_H^* > 0 \\
q(p_H^*, p_L^*) &= a - p_H^* - b_H p_L^* > 0 \\
q(p_L^*, p_H^*) &= a - p_L^* - b_L p_H^* > 0 \\
q(p_L^*, p_L^*) &= a - p_L^* - b_L p_L^* > 0
\end{aligned}$$

All these inequalities hold if

$$q(p_H^*, p_L^*) = a - p_H^* - b_H p_L^* > 0$$

or:

$$p_H^* < a - b_H p_L^*$$

Which can be verified to be true if $b_H < 1$, as assumed.

Problem 3: Bayesian Games

Consider a Bayesian game in which player 1 may be either type a or type b; where type a has probability 0.9 and type b has probability 0.1. Player 2 has no private information. Depending on player 1's types, the payoffs to the two players depend on their actions in $A_1 = \{U, D\}$ and $A_1 = \{R, L\}$ as shown in the following tables:

$t_1 = a$

	L	R
U	$2, 2$	$-2, 0$
D	$0, -2$	$0, 0$

$t_1 = b$

	L	R
U	$0, 2$	$1, 0$
D	$1, -2$	$2, 0$

Find **all** (pure and mixed) Bayesian Nash equilibria of this game.

Solution:

First of all notice that in any BNE, player 1 chooses D when her type is b: for any action of player 2, the payoff of type b (of player 1) from action D strictly dominates the payoff from action U.

Player 2 has no private information so she has to choose a single strategy regardless of the states. Suppose that player 2 chooses L. Then type a (of player 1) chooses U and type b chooses D. The expected payoff of player 2 is

$$0.9 \cdot (2) + 0.1 \cdot (-2) = 1.6$$

Note that if player 2 plays R, holding the other player's strategy constant, her (expected) payoff is 0. So the strategy profile $((U, D), L)$ is a BNE.

Suppose that player 2 chooses R. Then both type a and type b (of player 1) choose D. The (expected) payoff of player 2 is 0. If player 2 plays L, her (expected) payoff is -2. Thus we have another BNE: $((D, D), R)$.

Finally, suppose that player 2 randomizes between L and R. Let β denote the probability that player 2 chooses L. At least one type of player 1 must also randomize to make player 2 indifferent between L and R. Remember that type b chooses D in any BNE. Let α denote the probability that type a chooses U. Player 2 is willing to randomize if and only if she is indifferent between L and R:

$$0.9(2\alpha - 2\,(1-\alpha)) + 0.1 \cdot -2 = 0$$

that is if: $\alpha = 5/9$. Type a of player 1 is willing to randomize if and only if she is indifferent between U and D:

$$2\beta - 2\,(1-\beta) = 0$$

so if $\beta = 1/2$.

In summary we have 3 BNE: $((U, D), L)$, $((D, D)\,, R)$ and $\left(\left(\frac{5}{9}U + \frac{4}{9}D, D\right), \frac{1}{2}L + \frac{1}{2}R\right)$.