Macroeconomics, PhD core

Lecture #1

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Lecture Road Map

- Administrative details and approach
- Main questions
- The one-sector growth model (review)

Administrative Details

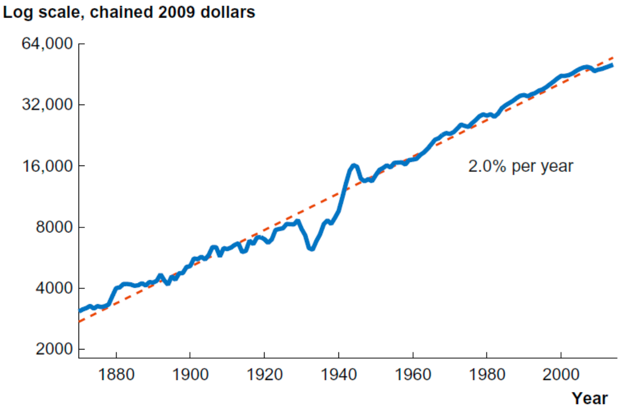
- HW are due at the beginning of the lecture.
- No late homework (only N-1 are counted).
- Work in groups!... **but** turn your own version
- Midterm Exam: Tuesday March 11, 2025

Motivation

- Why are people in the US today much richer that they were in 1800?
- Why are Germany and France much richer than Argentina and Kenya?
- Does growth generate inequality?
- What is the role of frictions in hindering growth?
- Macroeconomists aim at answering these questions building **quantitative** models - models that can be contested with empirical facts.

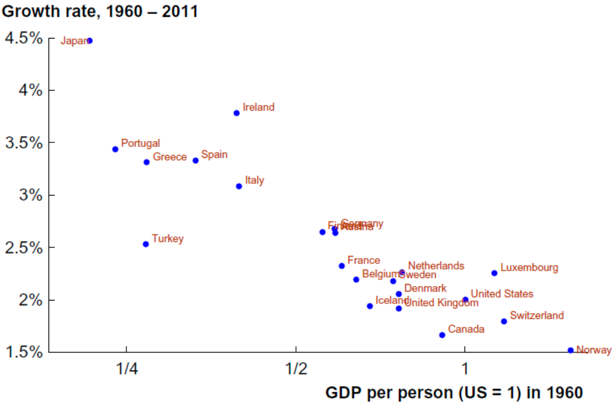
" *The weight of evidence for an extraordinary claim must be proportioned to its strangeness*" Laplace, 1812

Motivation
GDP in the US



Growth rates and levels of income

advanced economies



Growth rates and levels of income

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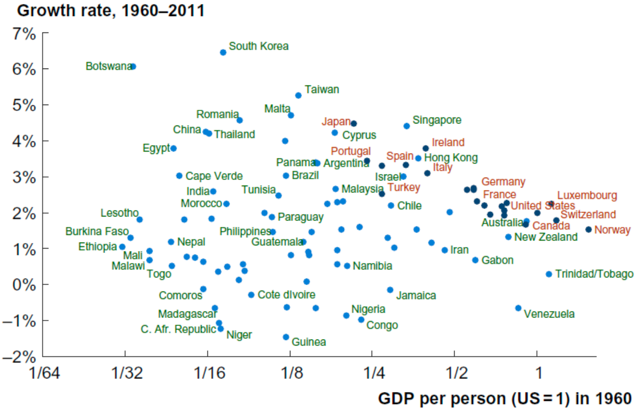

Questions in this course

1. Why are some countries richer than others?
2. Why do some countries grow faster than others?
3. What is the effect of wealth and consumption inequality for cross-country differences in income per capita?
4. What is the role of financial frictions?
5. What is the role of education for long run growth?
6. How do firms innovate, and what's their impact on economic growth?
7. Why do some firms operate older technologies while better ones are available?

Topics for this half of the course

- One sector growth model (Ramsey-Cass-Koopmans) revision?
  - Computation
  - Extensions to multiple sectors (ISTC and structural change)
- Competitive equilibrium
  - Heterogeneity and aggregation.
  - The income fluctuations problem, incomplete markets.
- Overlapping generations
  - Recursive representation
  - Dynamic inefficiencies
- Long run growth
  - Human capital
  - AK model
  - Endogenous growth: externalities and innovation

Growth Model

Environment: Who does what, when?

- Time?
- Preferences
- Technology

- Time: Discrete, Infinite Horizon
- Preferences: representative dynasty preferences

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

  - $\beta \in (0,1)$
  - u is strictly concave, increasing, and differentiable.
- Technology

$$c_t + x_t + g_t \leq f(k_t)$$

$$k_{t+1} \leq x_t + (1-\delta) k_t$$

  - x_t is investment, g_t is government spending, $f(k_t)$ output.

- Technology

$$c_t + x_t + g_t \leq f(k_t)$$

$$k_{t+1} \leq x_t + (1-\delta) k_t$$

  - $f(k_t)$ concave, strictly increasing, s.t. $f(0) = 0$
  - $\lim_{k \to 0} f'(k_t) > \frac{1}{\beta} - (1-\delta)$
  - $\lim_{k \to \infty} f'(k_t) < \frac{1}{\beta} - (1-\delta)$
- Assumption: All quantities must be non-negative

- Given a sequence $\{g_t\}_{t=0}^{\infty}$ and k_0

$$\max_{\{c_t, x_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$c_t + x_t + g_t \leq f(k_t)$$

$$k_{t+1} \leq x_t + (1-\delta) k_t$$

$$(c_t, x_t, k_{t+1}) \geq (0, 0, 0)$$

Definition

An **allocation** is a set of sequences $\{c_t, x_t, k_{t+1}\}_{t=0}^{\infty}$

Definition

An allocation is **feasible** if it satisfies

$$c_t + x_t + g_t \leq f(k_t)$$

$$k_{t+1} \leq x_t + (1-\delta) k_t$$

$$(c_t, x_t, k_{t+1}) \geq (0, 0, 0)$$

1. Solve a finite horizon problem
2. Wave hands
3. "Derive" and use the TVC

$$\max_{\{c_t, x_t, k_{t+1}\}} U^T = \sum_{t=0}^{T} \beta^t u(c_t)$$

subject to

$$c_t + x_t + g_t \leq f(k_t) \quad (1)$$

$$k_{t+1} \leq x_t + (1-\delta)k_t \quad (2)$$

$$(c_t, x_t, k_{t+1}) \geq (0,0,0) \text{ with } k_0 \text{ given.} \quad (3)$$

- How do we solve it?
- Application of K-T, the lagrangean is

$$
\begin{aligned}
\mathcal{L}(\mathrm{c},\mathrm{x},\mathrm{k},\breve{\ },\grave{\ },\mathrm{fl}) \quad = \quad & \sum_{t=0}^{T} \beta^{t} \{ u(c_t) + \lambda_t \left[f(k_t) - (c_t + x_t + g_t) \right] + \\
& \theta_t \left[x_t + (1-\delta) k_t - k_{t+1} \right] + \\
& + \gamma_{1t} c_t + \gamma_{2t} x_t + \gamma_{3t} k_{t+1} \}
\end{aligned}
$$

where c, x, k are vectors of size $T+1$ (k includes k_0).

- The Lagrange multipliers are defined as time t multipliers, $\lambda_t = \beta^{-t} \widehat{\lambda}_t$.
- Why? No reason. But in this way they are stationary

- FONC

$$
\begin{aligned}
c_t &: \quad u'(c_t) - \lambda_t + \gamma_{1t} = 0 \qquad\qquad t = 0, 1, ..., T \\
x_t &: \quad -\lambda_t + \theta_t + \gamma_{2t} = 0 \qquad\qquad t = 0, 1, ..., T \\
k_{t+1} &: \quad -\theta_t + \gamma_{3t} + \beta\lambda_{t+1} f'(k_{t+1}) + \beta(1-\delta)\theta_{t+1} = 0 \qquad t = 0, \\
k_{T+1} &: \quad -\theta_T + \gamma_{3T} = 0
\end{aligned}
$$

- $+$ complementary slackness conditions
- Why is the last period different?
- What's the payoff of accumulating capital at that point?

Infinite Horizon

An "In-house" Approach

1. Truncate the problem at $t = T$
2. Stare at the complementary slackness (CS) condition for capital in the last period, k_{T+1}, i.e. $\beta^T \gamma_{3T} k_{T+1}$
3. Find the lagrange multiplier (LM) γ_{3T} in another optimality condition, so that you can write it in terms of an alternative LM, i.e $-\theta_T + \gamma_{3T} = 0$.
4. Write the CS condition in terms of this new multiplier, and ask that the solution to the problem satisfies

$$\lim_{T \to \infty} \beta^T \theta_T k_{T+1} = 0$$

5. This is what is commonly called, a transversality condition in calculus of variations (a final condition)
6. Last, and most important! Do not ever repeat this "approach" outside this class!

Infinite Horizon

Optimality conditions, interior solution

- FONC

$$
\begin{aligned}
c_t &: \quad u'(c_t) = \lambda_t \\
x_t &: \quad \theta_t = \lambda_t \\
k_{t+1} &: \quad \beta\lambda_{t+1}\left[f'(k_{t+1}) + (1-\delta)\right] = \lambda_t \\
TVC &: \quad \lim_{T\to\infty} \beta^T \lambda_T k_{T+1} = 0 \\
c_t + x_t + g_t &\leq f(k_t) \\
k_{t+1} &\leq x_t + (1-\delta)k_t
\end{aligned}
$$

Infinite Horizon

Characterization, interior solution

- Rearranging (getting rid of the lagrange multiplier)

$$
\begin{aligned}
k_{t+1} &: \quad \beta u'(c_{t+1})\left[f'(k_{t+1})+(1-\delta)\right]=u'(c_t) \\
TVC &: \quad \lim_{T\to\infty}\beta^T u'(c_T)k_{T+1}=0
\end{aligned}
$$

- Micro 101

$$
\begin{aligned}
\frac{u'(c_t)}{\beta u'(c_{t+1})} &= \left[f'(k_{t+1})+(1-\delta)\right] \\
MRS &= MRT
\end{aligned}
$$

- TVC?

Suppose that $g_t = g \; \forall t$.

Definition

A **steady state** is an allocation such that, for all t, $c_t = c$, $x_t = x$, $k_{t+1} = k$.

- At the steady state,

$$\begin{aligned} c_t &: \quad u'(c^*) = \lambda^* \\ k_{t+1} &: \quad \beta\left[f'(k^*) + (1-\delta)\right] = 1 \\ c^* + x^* + g &= f(k^*) \\ \delta k^* &= x^* \end{aligned}$$

- To simplify

$$\beta\left[f'(k^*) + (1-\delta)\right] = 1 \qquad \text{(Euler)}$$

$$c^* + \delta k^* + g = f(k^*) \qquad \text{(Feasibility)}$$

- To simplify

$$\beta\left[f'(k^*)+(1-\delta)\right]=1 \qquad \text{(Euler)}$$

$$c^*+\delta k^*+g=f(k^*) \qquad \text{(Feasibility)}$$

- Interpretation?
- Existence, Uniqueness?
- Define $L(k)=\beta\left[f'(k)+(1-\delta)\right]$, Properties?

$$\lim_{k\to 0} L(k)>1,\ \lim_{k\to\infty} L(k)<1$$

  - What else do we need? (Continuity)
- Uniqueness? Monotonicity of $f'(k)$?

- Are we done yet?
- Need to check feasibility! is $c^* > 0$?
  - Suppose $g = 0$
  - We need to check $f'(k) < \delta$ at the point $\widehat{k}$ such that $f(\widehat{k}) = \delta\widehat{k}$ $\widehat{c} = 0$. Why? Plot
  - Hence, for $k < \widehat{k}$, $f(k) > \delta k$.
  - So... is $k^* < \widehat{k}$? Yes!

$$f'(k^*) = \frac{1}{\beta} - (1 - \delta) > \delta$$

for $\beta \in (0, 1)$.