Midterm

Macroeconomics I

6130

December 2023

On the productivity of capital and capital-labor ratios.

The economy is populated by a continuum of infinitely lived households and firms. Assume that time is discrete and households consume and invest in capital to maximize utility. They rent capital to the firms in the economy. Households' preferences are given by

$$U_i = \sum_{t=0}^{\infty} \beta^t u(c_{it}) \quad \text{with } \beta \in (0,1)$$

Assume that u is strictly concave, increasing in both arguments, and twice differentiable. The household takes initial capital as given $k_0 > 0$ and makes decisions for consumption and capital accumulation each period. The household derives income from renting capital to the firms at rate r_t as well as from supplying n_i units of labor at a market wage w_t. Labor in efficiency units is heterogeneous across households i. The average labor supply in the population is N, constant in time.
Firms produce output with a technology

$$Y_t = [N_t^\rho + b^k K_t^\rho]^{\frac{1}{\rho}}$$

for $\rho \in (-\infty, 1)$, $\rho \neq 0$ and $b^k > 0$.

1. (20 points) Define the problem of the households and firms as well as a competitive equilibrium.

   - The problem of the household is

   $$\max_{c_{it}, k_{it+1}} \sum_{t=0}^{\infty} \beta^t u(c_{it}) \quad \text{with } \beta \in (0,1)$$

   subject to

   $$c_{it} + k_{it+1} - (1-\delta)k_{it} = r_t k_{it} + w_t n_i. \qquad k_0 > 0.$$

(or one can write the law of motion for capital as a different constraint and let the consumer also choose investment x_{it}).

One can also let the household accumulate "assets" a so that the budget constraint looks like

$$c_{it} + a_{it+1} = R_t a_{it} + w_t n_i. \qquad a_0 > 0.$$

plus a no-ponzi scheme condition

$$\lim_{T\to\infty} u'(c_{iT}) a_{iT+1} = 0.$$

and then needs to change the market clearing condition accordingly.

- the problem of the firm is

$$\max_{N_t, K_t} [N_t^\rho + b^k K_t^\rho]^{\frac{1}{\rho}} - w_t N_t - r_t K_t$$

- A competitive equilibrium is an allocation and a price for labor and the rental rate of capital such that (a) households maximize utility; (b) firms maximize profits; and the following market clearing conditions are satisfied.

$$K_t = \int_0^1 k_{it} di \qquad N_t = N = \int_0^1 n_i di \qquad C_t = \int_0^1 c_{it} di \qquad C_t + K_{t+1} - (1-\delta) K_t = Y_t.$$

if one writes the budget constraint of the household in terms of assets, then the market clearing condition for capital needs to be modified to

$$K_t = \int_0^1 a_{it} di$$

2. (10 points) Explain whether this economy displays aggregation and why.

   This economy displays aggregation because technologies and preferences u are identical across households and markets are complete. Then, heterogeneity in labor income leads to different levels of consumption and investment across households, but the aggregate economy behaves as the average household in the economy.

3. (20 points) Write down the planner's problem for this economy and describe the phase diagram associated to it.

- The planner's problem associated to this economy is

$$\max_{C_t, K_{t+1}} \sum_{t=0}^{\infty} \beta^t u(C_t) \quad \text{with } \beta \in (0,1)$$

subject to

$$C_t + K_{t+1} - (1-\delta) K_t = [N^\rho + b^k K_t^\rho]^{\frac{1}{\rho}}. \qquad k_0 > 0.$$

- The phase diagram associated to this economy is characterized by the optimality conditions to the problem

$$K_{t+1} = Y_t + (1-\delta)K_t - C_t$$

$$\lambda_{t+1} = \frac{\lambda_t}{\beta[\partial Y_{t+1}/\partial K_{t+1} + (1-\delta)]}$$

and

$$\lambda_t = \beta^t \frac{\partial u(C_t)}{\partial C_t}$$

which describe the saddle path.
We also need two loci as follows. One where capital is constant. λ_1

$$C(\lambda_1) = Y - \delta K$$

and one where marginal utilities are constant, λ_2. To characterize it, notice that then capital one period ahead corresponds to the steady state capital. This is pin down by the Euler equation as follows

$$\beta[(\frac{Y^\star}{K^\star})^{1-\rho} b^k + (1-\delta)] = 1$$

Then capital tomorrow is the capital in steady and therefore,

$$K^\star = Y + (1-\delta)K - C(\lambda_2)$$

$$C(\lambda_2) = C(\lambda_1) + K - K^\star$$

4. (10 points) Describe an algorithm to compute the saddle path of this economy in the computer.

   - This is a standard shooting algorithm, see description in class notes.

5. (15 points) Describe what is the equilibrium interest rate of the economy along the steady state and how it depends on the efficiency of capital in production. Explain the economic intuition behind your result.

   - The equilibrium interest rate is the **marginal product of capital in steady state**, so from the Euler equation

$$\beta[(\frac{N}{K^\star}^\rho + 1)^{\frac{1}{\rho}-1}(b^k)^{\frac{1}{\rho}} + (1-\delta)] = 1$$

and therefore

$$r_t = \frac{1}{\beta} - (1-\delta).$$

In other words, it is independent of the efficiency of capital in production. the reason is that a higher efficiency of capital increases the marginal product of capital, everything else equal. But a higher marginal product induces a larger steady state level of capital which offsets this effect. The marginal product of capital along the steady state is only a function of the depreciation rate and the impatience parameter.

6. (10 points) Describe what is the equilibrium level of capital in the economy along the steady state.

   - The Euler equation can be solved for $K^\star$ as

$$K^\star = \frac{N}{[(\frac{\frac{1}{\beta}-(1-\delta)}{(b^k)^{\frac{1}{\rho}}})^{\frac{\rho}{1-\rho}} - 1]^{\frac{1}{\rho}}}$$

7. (15 points) Describe how the level of capital per worker in steady state depends on the efficiency of capital b^k. Do improvements in the efficiency of capital induce capital intensification in the economy?

   The relationship between the capital in steady state and its efficiency depends on whether capital and labor are substitutable or not.

   - If $\rho \in (0,1)$ then capital and labor are substitutable and the improvement in efficiency of capital induces higher capital (and capital per worker) in steady state.
   - If $\rho < 0$ them capital and labor are complements, and the improvement in efficiency of capital induces lowers capital (and lower capital per worker) in steady state. ($\frac{\rho}{1-\rho} < 0$).

   The model says that the statement above is only valid if capital and labor are substitutes in production.