# Macroeconomics, PhD core

## Lecture #2

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# Lecture Road Map

- Transition paths
- Recursive formulation
- Computation
- How do we confront the neoclassical growth model to the data?
- Calibration and accounting exercises.

- Add a technology level $z$ so that output is $zf(k)$
- FOC

$$
\begin{aligned}
c_t &: \quad u'(c_t) = \lambda_t \\
k_{t+1} &: \quad \beta\lambda_{t+1}\left[zf'(k_{t+1}) + (1-\delta)\right] = \lambda_t \\
TVC &: \quad \lim_{T\to\infty} \beta^T \lambda_T k_{T+1} = 0 \\
c_t + x_t + g_t &\leq zf(k_t) \\
k_{t+1} &\leq x_t + (1-\delta)k_t
\end{aligned}
$$

- One could describe the dynamics in terms of consumption or of its shadow value.
- Define

$$u'(c(\lambda)) \equiv \lambda$$

and assume $g_t = g$ "small" for now.

- Key Conditions

$$\beta\lambda_{t+1}\left[zf'(k_{t+1})+(1-\delta)\right]=\lambda_t \qquad \text{(Euler)}$$

$$c_t+k_{t+1}-(1-\delta)k_t+g_t\leq zf(k_t) \qquad \text{(Feasibility)}$$

- Phase Diagram? (review?)
- We can't solve the system of non-linear equations explicitly.
- BUT.. you can do it in the computer. `https://python.quantecon.org/cass_koopmans_1.html`

- Finite difference methods (shooting algorithm)
- Approximate $k$ and $c$ (or $\lambda$) with $N$ discrete points in the time dimension. Denote the distance between grid points $\Delta t$.
- Let $(k_n, c_n)$ be a point in the grid
- Use the equations in difference that characterize the optimum

$$c_{n+1} = \beta c_n \left[ z f'(f(k_n) - (1-\delta)k_n - c_n) + (1-\delta) \right]$$
$$k_{n+1} = f(k_n) - (1-\delta)k_n - c_n$$

and $k_0$ given

- Algorithm:
  1. Guess $c_0$
  2. obtain $(c_n, k_n)$ for $n = 1, ..., N$ by running the equations above forward
  3. If the sequence converges to $c^*, k^*$ then you have the correct saddle path. If not, update $c_0$ and go back to 1.

# Growth Model

## Recursive representation

- Sequential problem

$$\max_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$

$$c_t,\, k_{t+1} \geq 0 \text{ and } k_0 \; > 0 \text{ given}$$

- Value function

$$V(k_0) = \max_{\{(c_t,k_{t+1})\in\Gamma(k_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

where

$$\Gamma(k_t) = \{(c_t, k_{t+1}) : c_t \geq 0,\; k_{t+1} \geq 0,$$
$$c_t + k_{t+1} \leq f(k_t) + (1-\delta)k_t\}$$

- Recursive formulation

$$V(k_0) = \max_{(c_0,k_1)\in\Gamma(k_0)} u(c_0) + \beta V(k_1)$$

- In general,

$$V(k) = \max_{(c,k')\in\Gamma(k)} u(c) + \beta V(k')$$

where $k$ is the state variable (or a set of state variables).

- Using differentiability of $u$, the solution to this problem satisfies,

$$k' = f(k) + (1-\delta)k - c$$

$$\frac{\partial u(c)}{\partial c} = \beta \frac{\partial V(k')}{\partial k}$$

- and from the envelope condition

$$\frac{\partial V(k')}{\partial k} = \frac{\partial u(c')}{\partial c}[z\frac{\partial f(k')}{\partial k} + (1-\delta)]$$

you can recover the Euler equation.

- The solution to this problem is a function $g : K \to K$ such that $k' = g(k)$ and $V^* : K \to \mathbb{R}^+$ (or some bounded subset).
- We typically construct approximations of $g$ and $V$ numerically. Methods differ on how to approximate function of interest.
- General algorithms:
  - value function iteration
  - policy function can be solved from euler equations directly

- Set of non-linear equations to be solved over the state space $K$ (euler eq.)

$$\frac{\partial u(c)}{\partial c} = \beta \frac{\partial u(c')}{\partial c}[z\frac{\partial f(k')}{\partial k} + (1-\delta)]$$

$$\frac{\partial u(f(k) + (1-\delta)k - k')}{\partial c} =$$
$$\beta u'(f(k')(1-\delta)k' - k'')[\frac{\partial f(k')}{\partial k} + (1-\delta)]$$

- Discretize $K$ in $N$ nodes, solve the sytem of equations. Make sure $k = k^*$ belongs to the set $K$.

$$\frac{\partial u(f(k) + (1-\delta)k - g(k))}{\partial c} =$$

$$\beta u'(f(g(k)) + (1-\delta)g(k) - g(g(k)))[\frac{\partial f(g(k))}{\partial k} + (1-\delta)]$$

1. Discretize the state space $K$.
2. Guess for the value function $V_0(k)$
3. Use an optimization algorithm to solve for $k'(k)$
4. Compute

$$V_1(k) = u(f(k) + k(1-\delta) - k'(k)) + \beta V_0(k'(k))$$

   - various methods to approximate $V_0(k')$ for "off-grid" points.
5. Check the distance between $V_0(k)$ and $V_1(k)$. If larger than a given tolerance update $V_0(k) = V_1(k)$ and go back to 3. Otherwise, stop.

With $u(c) = \ln(c)$ and full depreciation $\delta = 1$,you can do it by hand!

- Suppose we want to use the neoclassical model to provide quantitative assessments.
  1. What is the role of the capital output ratio ...
  2. What is the role of productivity ....

  for differences in output per capita across countries?
- Choose functional forms for technology and preferences where parameters have clear economic interpretations, i.e.

$$u(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

$$f(k) = zk_t^{\alpha}$$

- Set $z = 1$ in the US for now, (level shifter only).
- We have four parameters to choose, $\sigma, \delta, \beta, \alpha$.

# Growth Model

## Bringing it to the data

- Literature suggest two approaches:
  - Estimation
  - Calibration
- Why not (always) estimate?
  - model is an abstraction in which some features have been deliberately abstracted away from.
  - (many) standard formal statistical procedures use criteria that does not necessarily make economic sense.
- Alternative: choose the aspects of the data that your model was designed to capture.
- Key idea of calibration: choosing parameters boils down to choosing moments in the data to match.
- You can think of calibration as an exactly identified GMM estimation.

- Model is designed to explain the capital accumulation process.
- Key stats:

$$\frac{k}{y}, \frac{x}{y}, \; r$$

- Think of US post WWII as fluctuations around s.s., i.e. take averages to calibrate the s.s.
- 1 period, 1 year, which gives (data)

$$\frac{k}{y} \approx 2.5, \; \frac{x}{y} \approx 0.2, \; r \approx 0.04$$

- 4 parameters:$\sigma, \delta, \beta, \alpha$, but 3 moments!

- Identification: $\sigma$ does not affect the s.s., so it cannot be identified from s.s. moments.
  Estimates in the data range $\sigma = [1, 2.5]$. Use $\sigma = 1$, log utility
- Targets
$$\frac{k}{y} \approx 2.5, \ \frac{x}{y} \approx 0.2, \ r \approx 0.04$$
- Steady state conditions
$$\frac{x}{y} = \delta \frac{k}{y} \approx 0.2$$
$$r = \alpha \frac{y}{k} \approx 0.04$$
$$\beta[r + (1 - \delta)] = 1$$

# Income accounting: framework.

## full lecture STEG

How much of the observed differences in income per capita are accounted for...

- differences in $z$?
- differences in $k/y$?

$$Y_t = A_t K_t^{\alpha} H_t^{1-\alpha}$$

$Y_t$ output, $K_t$ physical capital, $H_t$ "quality-adjusted" labor force, $\alpha$ capital share, $A_t$ TFP, Total Factor Productivity.

- Estimate? *Orthogonality between TFP and inputs,...VERY unlikely.*

$$\ln(Y_t) = \underbrace{\alpha}_{\beta_k} \ln(K_t) + \underbrace{(1-\alpha)}_{\beta_H} \ln(H_t) + \underbrace{\ln(A_t)}_{\epsilon}.$$

# Income accounting: framework.

Important take-away from one-sector growth model:

differences in TFP induce differences in K.

but $\frac{K}{Y}$ is independent from TFP in steady state (s.s.)!

$$1 = \text{discount}[\underbrace{\alpha \frac{Y}{K}}_{\text{MPK}} + (1 - \text{depreciation})] \qquad \text{(Euler)}$$

Output per worker

$$\frac{Y}{L} = Z \left( \frac{K}{Y} \right)^{\frac{\alpha}{1-\alpha}} \frac{H}{L} \qquad \text{where } Z = A^{\frac{1}{1-\alpha}}.$$

In logs

$$\ln \left( \frac{Y}{L} \right) = \ln(Z) + \frac{\alpha}{1-\alpha} \ln \left( \frac{K}{Y} \right) + \ln \left( \frac{H}{L} \right).$$

# Growth Model

## Income Accounting: Results

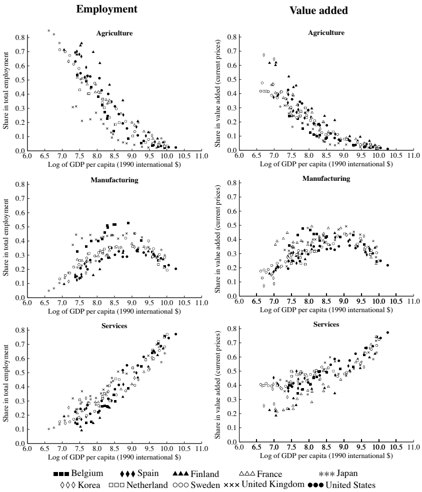
| Country | $\frac{Y}{L}$ | $\left(\frac{K}{Y}\right)^{\frac{\alpha}{1-\alpha}}$ | $\frac{H}{L}$ | $Z$ | share due to TFP |
|---|---|---|---|---|---|
| Singapore | 1.01 | 1.12 | 1.06 | 0.84 | 0.59 |
| **United States** | **1.00** | **1.00** | **1.00** | **1.00** | |
| France | 0.82 | 1.33 | 0.85 | 0.72 | 0.61 |
| Germany | 0.77 | 1.18 | 0.98 | 0.67 | 0.64 |
| China, Hong Kong SAR | 0.76 | 1.41 | 0.87 | 0.62 | 0.67 |
| United Kingdom | 0.72 | 1.21 | 1.01 | 0.59 | 0.67 |
| Republic of Korea | 0.63 | 1.21 | 0.99 | 0.52 | 0.69 |
| Japan | 0.59 | 1.22 | 0.96 | 0.50 | 0.70 |
| Argentina | 0.40 | 0.98 | 0.81 | 0.50 | 0.61 |
| Mexico | 0.35 | 1.16 | 0.73 | 0.41 | 0.67 |
| Botswana | 0.32 | 1.17 | 0.77 | 0.35 | 0.72 |
| South Africa | 0.30 | 1.08 | 0.75 | 0.37 | 0.69 |
| Brazil | 0.25 | 1.17 | 0.79 | 0.27 | 0.77 |
| Thailand | 0.24 | 1.18 | 0.73 | 0.28 | 0.76 |
| China | 0.19 | 1.04 | 0.71 | 0.26 | 0.74 |
| Indonesia | 0.18 | 1.30 | 0.62 | 0.22 | 0.79 |
| India | 0.13 | 1.06 | 0.57 | 0.22 | 0.73 |
| Kenya | 0.07 | 0.85 | 0.62 | 0.13 | 0.80 |
| Malawi | 0.02 | 0.61 | 0.52 | 0.06 | 0.84 |
| Average | 0.35 | 1.14 | 0.71 | 0.40 | |
| 1/Average | 2.88 | 0.88 | 1.40 | 2.48 | 0.67 |

Github repo to play with the data.

# Under the hood of the aggregates

## Structural change: reallocation out of agriculture, into manufacturing, into services

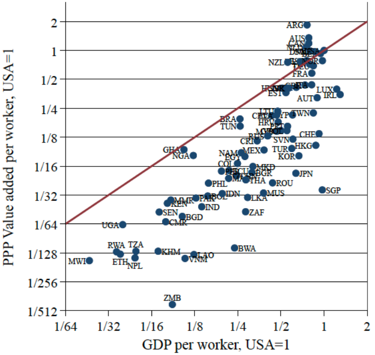
**Figure 1: Sectoral Shares of Employment and Value Added – Selected Developed Countries 1800–2000**




**Source:** Various historical statistics, see Appendix A.

Larger prod, gaps in agriculture than elsewhere in the economy.

**Productivity Gaps in Agriculture vs. Aggregate in 2017**




**Source**: Productivity Level Database, 2023, GGDC