Supermodular games

In a number of applications we have "strategic complementarities".

This means that each player’s marginal utility of choosing a higher action is increasing in the actions of the other players.

Games with this property have nice characteristics.

Lattices

Let's start from a few useful concepts.

Assume the strategy space of i is S_i a subspace of R^{m_i}.

For $x, y \in R^K$, we say that $x = (x_1, \ldots, x_K) \geq y$ if $x_k \geq y_k$ for all k.

For $x, y \in R^K$, we say that $x = (x_1, \ldots, x_K) > y$ if $x_k \geq y_k$ for all k, and $x_k > y_k$ for at least one k.

We say that the meet $x \wedge y$ is:

$$x \wedge y := (\min(x_1, y_1), \ldots, \min(x_K, y_K))$$

The join is:

$$x \vee y := (\max(x_1, y_1), \ldots, \max(x_K, y_K))$$

A subset S of R^m is a sublattice if $x, y \in S$, imply $x \vee y \in S$ and $x \wedge y \in S$.

If S is a non-empty, compact sublattice of R^m, then it has a greatest element, and a least element.

Examples.

Formalizing strategic complementarity

Definition $u_i(s_i, s_{-i})$ *has (strict)* ***increasing differences*** *in* (s_i, s_{-i}) *if for all* $(s_i, \tilde{s}_i)$ *and* $(s_{-i}, \tilde{s}_{-i})$ *such that* $s_i \geq \tilde{s}_i$ *and* $s_{-i} \geq \tilde{s}_{-i}$ *(resp.* $s_i > \tilde{s}_i$ *and* $s_{-i} > \tilde{s}_{-i}$*), we have:*

$$u_i(s_i, s_{-i}) - u_i(\tilde{s}_i, s_{-i}) \geq u_i(s_i, \tilde{s}_{-i}) - u_i(\tilde{s}_i, \tilde{s}_{-i})$$

(resp., $u_i(s_i, s_{-i}) - u_i(\tilde{s}_i, s_{-i}) > u_i(s_i, \tilde{s}_{-i}) - u_i(\tilde{s}_i, \tilde{s}_{-i})$*).*

With increasing differences, and increase in the strategies of the opponents raises the desirability of choosing a higher strategy.

This however does not imply that all the dimensions of s_i increase.

For example, say $s_i = (s_{i,1}, s_{i,2})$. Then if $s_{-i} \uparrow$, and we keep $s_{i,2}$ fixed, then $s_{i,1} \uparrow$.

Similarly, if $s_{-i} \uparrow$, and we keep $s_{i,1}$ fixed, then $s_{i,2} \uparrow$.

But the increase in $s_{i,1}$ may be so strong to induce $s_{i,2} \downarrow$, if we do not make additional assumptions.

Here is what we need.

Definition. $u_i(s_i, s_{-i})$ *is **supermodular** in* s_i *if for each* s_{-i}*:*

$$u_i(s_i, s_{-i}) + u_i(\widetilde{s}_i, s_{-i}) \leq u_i(s_i \wedge \widetilde{s}_i, s_{-i}) + u_i(s_i \vee \widetilde{s}_i, s_{-i})$$

Supermodularity means that there is a complementarity among the components of a player's strategy.

The components move together when the rivals strategies move together.

If $S_i = R^{m_i}$ and u_i is twice continuously differentiable in s_i, then u_i is supermodular in s_i if and only if for any two components $s_{i,j}$ $s_{i,k}$ of s_i with j,k, $\partial^2 u_i / \partial s_{i,j} \partial s_{i,k} \geq 0$.

Intuitively, supermodularity implies:

$$\begin{bmatrix} u_i(s_i + \varepsilon e_l, s_{-i}) \\ +u_i(s_i + \kappa e_k, s_{-i}) \end{bmatrix} \leq \begin{bmatrix} u_i(s_i, s_{-i}) \\ +u_i(s_i + \kappa e_k + \varepsilon e_l, s_{-i}) \end{bmatrix}$$

where e_l is a vector equal to zero except at the lth dimension, where it is 1; and κ and ε are small variables.

Multiply and divide by $\kappa\varepsilon$, and you get $\kappa\varepsilon \bullet \partial^2 u_i / \partial s_{i,l} \partial s_{i,k} \geq 0$.

Similarly as above we can define supermodularity in s:

$$u_i(s) + u_i(\widetilde{s}) \leq u_i(s \wedge \widetilde{s}) + u_i(s \vee \widetilde{s})$$

for all s, $\widetilde{s}$.

Note that supermodularity in s implies increasing differences in (s_i, s_{-i}) and supermodularity in s_i

Supermodular games

Definition. *A (resp., strictly) supermodular game is a game in which for each* i*:*

- S_i *is a sublattice of* R^{m_i},
- u_i *has (resp., strictly) increasing differences in* (s_i, s_{-i})
- *and* u_i *is (resp., strictly) supermodular in* s_i.

Example of supermodular game

Bertrand competition

Consider an oligopoly with linear demand functions:

$$D_i(p_i, p_{-i}) = a_i - b_i p_i + \sum_{j \neq i} d_{i,j} p_j$$

with $b_i > 0$ and $d_{i,j} > 0$.

Constant marginal cost c_i.

Profits are:

$$\pi_i(p_i, p_{-i}) = (p_i - c_i) D_i(p_i, p_{-i})$$

We can verify that $\partial^2 \pi / \partial p_i \partial p_j \geq 0$ so the game has increasing differences (and trivially is supermodular since the strategy is unidimensional).

Diamond's search model

A player's utility depends on his search intensity, and other player's intensities:

$$u_i(s_i, s_{-i}) = \alpha s_i \cdot \sum s_j - c(s_i)$$

where $c(s_i)$ is a convex cost function.

Here we have strategic complementarities.

One-dimensional strategies

Solving the Bertrand example

Assume $N = 2$ $A_i = [0,1]$ and $D_i(p_i, p_j) = 1 - 2p_i - p_j$.

So we have $\pi_i(p_i, p_j) = p_i[1 - 2p_i - p_j]$

and:

$$\frac{\partial}{\partial p_i}\pi_i(p_i, p_j) = 1 - 4p_i + p_j$$

We can use this expression to eliminate strictly dominated strategies:

- If $p_i < 1/4$, then $\frac{\partial}{\partial p_i}\pi_i(p_i, p_j) > 0$ for any p_j, so they are strictly dominated by $p_i = 1/4$.
- If $p_i > 1/2$, then $\frac{\partial}{\partial p_i}\pi_i(p_i, p_j) < 0$ for any p_j, so they are strictly dominated by $p_i = 1/2$

We can therefore restrict attention to $X_i^1 = [1/4, 1/2] = [\underline{p}^1, \overline{p}^1]$.

Iterate this approach to define $X_i^k = [\underline{p}^k, \overline{p}^k]$

Now note that:

$$\underline{p}^k = \frac{1}{4} + \frac{1}{4}\underline{p}^{k-1} = \frac{1}{4} + \frac{1}{16} + \frac{1}{4}\underline{p}^{k-2} + \ldots \to_{k\to\infty} \sum_{l=1}^{\infty}\left(\frac{1}{4}\right)^{l} = 1/3$$

and by a similar logic: $\overline{p}^k \to \frac{1}{3}$.

So the game is solvable by iterated deletion of strictly dominated strategies.

Generalizing these results

Suppose $a_i \in A_i \subset R$, compact for all i

$u_i(a_i, a_{-i})$ has increasing differences in (a_i, a_{-i})

$u_i(a_i, a_{-i})$ is continuous.

Best responses

Then $BR_i(a_{-i}) = \arg\max_{a_i} u_i(a_i, a_{-i})$:

- $BR_i(a_{-i})$ is non empty with a greatest and least elements, respectively $\overline{BR}_i(a_{-i})$ and $\underline{BR}_i(a_{-i})$.

- $a'_{-i} \geq a_{-i}$ implies that $\overline{BR}_i(a'_{-i}) \geq \overline{BR}_i(a_{-i})$ and $\underline{BR}_i(a'_{-i}) \geq \underline{BR}_i(a_{-i})$

The fact that $BR_i(a_{-i})$ is non empty and has a greatest and least element follows from continuity and compactness (easy here since actions are one-dimensional).

Consider monotonicity.

Let $a_i \in BR_i(a_{-i})$ and $a_i' \in BR_i(a_{-i}')$ with $a_{-i}' \geq a_{-i}$.

It is easy to see that:

$$u_i(\max(a_i, a_i'), a_{-i}) - u_i(a_i', a_{-i}) \geq 0$$

(if $a_i = \max(a_i, a_i')$, by definition, if $a_i' = \max(a_i, a_i')$, then it is 0).

Using increasing differences:

$$u_i(\max(a_i, a_i'), a_{-i}') - u_i(a_i', a_{-i}') \geq 0$$

So $\max(a_i, a_i') \in BR_i(a_{-i}')$, implying that $\overline{BR}_i(a_{-i}') \geq \overline{BR}_i(a_{-i})$.

A similar argument can be used to show that
$\underline{BR}_i(a'_{-i}) \geq \underline{BR}_i(a_{-i})$

Equilibria

Theorem. *Let* (S,u) *be a supermodular game. Then:*

- *the set of strategies surviving iterated strict dominance has greatest and least elements* $\bar{a},\underline{a}$.
- *and* $\bar{a},\underline{a}$ *are both Nash equilibria.*

Let's start from $A = A^0$ and let $\overline{a}^0 = (\overline{a}_1^0, \ldots, \overline{a}_n^0)$ be the largest element.

Define $\overline{a}_i^1 = \overline{BR}_i(\overline{a}_{-i}^0)$.

Then any $a_i > \overline{a}_i^1$ is strictly dominated by $\overline{a}_i^1$.

Iterate and obtain $\overline{a}_i^k$, note that the sequence is decreasing in k.

Define $\overline{a}_i = \lim_{k\to\infty} \overline{a}_i^k$.

This has the property that $\overline{a}_i = \overline{BR}_i(\overline{a}_{-i})$. To see this note that

$$u_i(\overline{a}_i^{k+1}, \overline{a}_{-i}^k) \geq u_i(a_i, \overline{a}_{-i}^k)$$

for any k. By continuity:

$$\begin{aligned} u_i(\lim_{k\to\infty} \overline{a}_i^{k+1}, \lim_{k\to\infty} \overline{a}_{-i}^k) &\geq u_i(a_i, \lim_{k\to\infty} \overline{a}_{-i}^k) \\ &\Leftrightarrow u_i(\overline{a}_i, \overline{a}_{-i}) \geq u_i(a_i, \overline{a}_{-i}) \end{aligned}$$

We can then do the same for the smallest element.

We could also have used a different type of fixed-point theorem.

Theorem. (Tarski). *If S is a non-empty, compact sublattice of R^m and $f : S \rightarrow S$ is increasing then f has a fixed point.*

When strategies are one-dimensional, this theorem can be applied (almost) immediately with increasing differences since $BR_i(a_{-i})$ is non a non empty, compact, monotonic sublattice and it has a monotone selection (say $\overline{BR}_i(a_{-i})$).

(When strategies are multidimensional, we need to prove $f(S)$ is a sublattice, that is $s_i \in r_i(s_{-i})$ and $\tilde{s}_i \in r_i(s_{-i})$ implies that $s_i \wedge \tilde{s}_i \in r_i(s_{-i})$ and $s_i \vee \tilde{s}_i \in r_i(s_{-i})$)

Back to multidimensional strategies

Properties of the equilibrium set

Theorem. *Consider a supermodular game such that:*

- S_i *is a complete sublattice and bounded above,*
- *and* u_i *is continuous and bounded above.*

Then iterated deletion of strictly dominated strategies yields a set of strategies in which the greatest and the least elements are Nash equilibria $\bar{s}$ *and* $\underline{s}$*.*

Proof. Since S is a complete lattice, there is a greatest element $s^0 = (s_1^0, \ldots, s_I^0)$.

Let s_i and s_i' be two strategies in $r_i^*(s_{-i}^0)$ such that there is no s_i'' s.t. $s_i'' > s_i$ or $s_i'' > s_i'$.

If $s_i \neq s_i'$, consider $s_i \wedge s_i'$. We have:

$$u_i(s_i, s_{-i}^0) - u_i(s_i \wedge s_i', s_{-i}^0) \leq u_i(s_i \vee s_i', s_{-i}^0) - u_i(s_i', s_{-i}^0) < 0$$

where the weak inequality follows from *supermodularity* and strict inequality follows from $s_i \vee s_i' > s_i'$ (since by assumption there is *no strictly larger element in the best response*).

So we have a contradiction.

We conclude that $r_i^*(s_{-i}^0)$ has a greatest element, that we call s^1.

We can repeat this logic and define s^n.

Consider an element that is not $s_i \leq s_i^n$, then it is dominated by $s_i \wedge s_i^n < s_i$ (when $s_{-i} \leq s_{-i}^n$).

To see this note that:

$$
\begin{aligned}
u_i(s_i, s_{-i}) - u_i(s_i \wedge s_i^n, s_{-i}) &\leq u_i(s_i, s_{-i}^{n-1}) - u_i(s_i \wedge s_i^n, s_{-i}^{n-1}) \\
&\leq u_i(s_i \vee s_i^n, s_{-i}^{n-1}) - u_i(s_i^n, s_{-i}^{n-1}) < 0
\end{aligned}
$$

The first inequality follow from *increasing differences*, the second from *supermodularity*, the third from the fact that s_i^n is the greatest best response to s_i^{n-1} and $s_i \vee s_i^n > s_i$.

(s_i^n) is bounded below and decreasing in n, so it converges to some $\bar{s}$.

We now show $\bar{s}$ is a Nash equilibrium. Fix a s_i, by optimality of s_i^{n+1} against s_i^n:

$$u_i(s_i^{n+1}, s_{-i}^n) \geq u_i(s_i, s_{-i}^n)$$

By continuity this implies:

$$\rightarrow u_i(\bar{s}_i, \bar{s}_{-i}) \geq u_i(s_i, \bar{s}_{-i})$$

Similarly, we obtain $\underline{s}$ as the lower bound on the set of strategies that survives iterated deletion of strictly dominated strategies.