Extensive Games with Imperfect Information

We have previously studied extensive games with perfect information.

We now extend the definition of extensive games to include situations in which players are imperfectly informed about past event when taking actions.

We also allow for exogenous uncertainty.

We will first restrict to situation in which one player selects an action at a time, but we will see this is w.l.g.

Basic Definitions

Definition. *An extensive game with imperfect information* $\langle N, H, P, f_c, (\Im_i), (u_i)\rangle$ *is the following:*

- *A set N of players;*
- *A set H of histories with the following 3 properties:*
  - $\varnothing \in H$
  - $(a^k)_{k=1,\ldots,K} \in H$, *where K may be infinite, then* $(a^k)_{k=1,\ldots,L} \in H$ *for* $L \le K$
  - *if an infinite sequence* $(a^k)_{k=1}^{\infty}$ *satisfies* $(a^k)_{k=1,\ldots,K} \in H$

for every L, then $(a^k)_{k=1}^{\infty} \in H$

- *A history is terminal of it is infinite or there is a K such that* $(a^k)_{k=1,\ldots,K} \in H$, *but* $(a^k)_{k=1,\ldots,K}, a^{K+1} \notin H$ *for any* a^{K+1}.

- Note: $A(h) = \{a\ s.t.(h,a) \in H\}$

- *A function* $P : h \to N \bigcup \{c\}$ *that assigns to each non-terminal history a member of* $N \bigcup \{c\}$. (Player function).

- A function f_c that associates with every history h in which chance moves ($P(h) = c$), a probability measure $f_c(\bullet; h)$ on $A(h)$.

- For each player i, a partition $\Im_i$ of $\{h \in H : P(h) = i\}$, with the property that $A(h) = A(h')$ if the two histories are in the same member of the partition.
  - $\Im_i$ is i's information partition;
  - $I_i \in \Im_i$ is an information set.
- *Preferences over lotteries over terminal histories*
  $u_i : \Delta Z \to R$

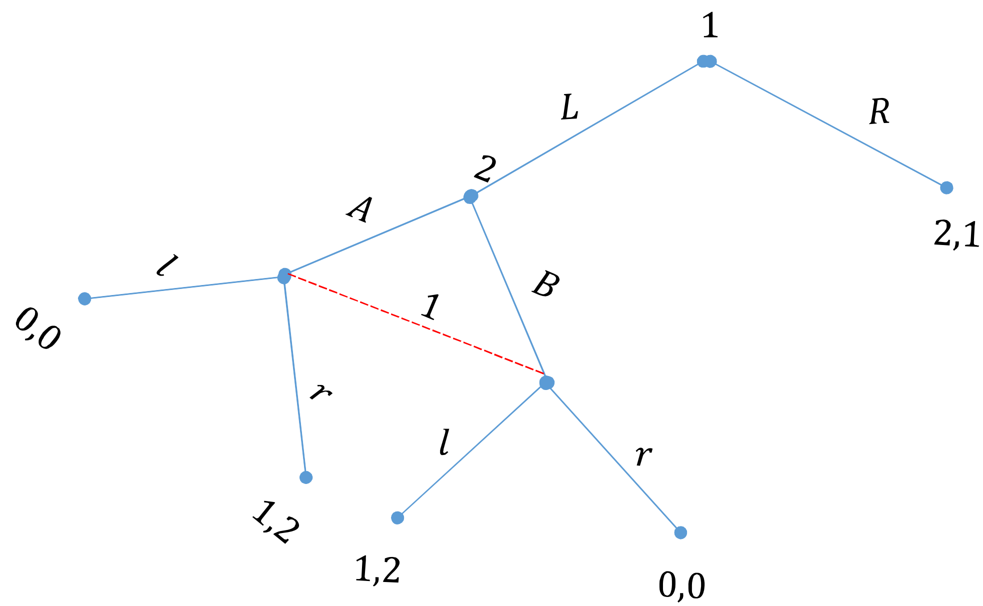
The key innovation with respect to extensive games with perfect information are the information partitions.

The interpretation is that a player cannot distinguish two histories in the same member of the partition.

This may include histories h and $h' = (h, h'')$.

In this case we have imperfect recall (although often these situations are ruled out by assumption).

An example (Example 1):



Here: $P(\emptyset) = P(L, A) = P(L, B) = 1$, $P(L) = 2$

$\mathfrak{I}_1 = \{\emptyset, \{(L, A), (L, B)\}\}$, $\mathfrak{I}_2 = \{\{L\}\}$.

With this model we can describe situations in which players **play simultaneously**.

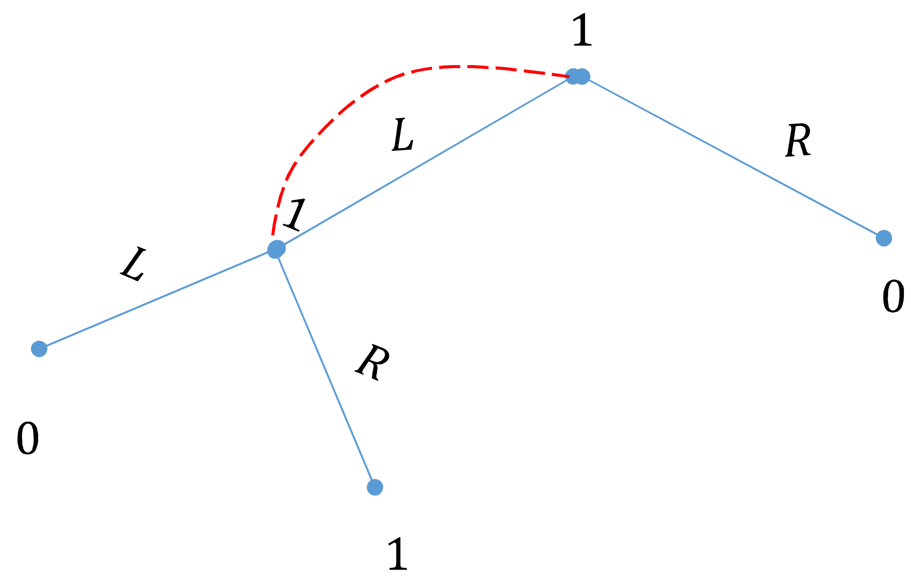
These are equivalent to situations in which players play sequentially, but do not observe the actions of the other players.

For example, consider a simultaneous game in pure strategies.

Information partitions are a primitive of the game, but they can be **refined by using reasoning**.

Again, consider a simultaneous game in pure strategies.

Games with **imperfect recall**. An example (Example 2):



Let $X_i(h)$ be the sequence of of information sets and corresponding actions encountered by i in history h.

$X_i(h)$ is a record of i's experience along history h.

In Example 1: $X_1((L,A)) = (\emptyset, L, \{(L,A),(L,B)\})$.

Definition. *An extensive game form has perfect recall if for each player i we have $X_i(h) = X_i(h')$ if h and h' are in the same information set of i.*

In Example 2 we have:

$$X_1(\emptyset) = \{\emptyset, L\} \neq X_1(L) = (\{\emptyset, L\}, L)$$

Definition. *A pure strategy of player i in an extensive game*

$$\{N, H, P, f_c, (\Im_i), u_i\}$$

is a function mapping each information set I_i to an action $A(I_i)$.

Here we define $A(I_i)$, $P(I_i)$ intuitively.

Strategic equivalence

We say that two extensive games are *strategically equivalent* if they have the **same** strategic form.

Thomson [1952]: 4 transformations on the extensive form that preserve the strategic form, and thus strategic equivalence:

- Inflation-deflation,
- Coalescing of moves,
- Interchange of moves,

- Addition of superfluous moves.

Inflation-deflation principle

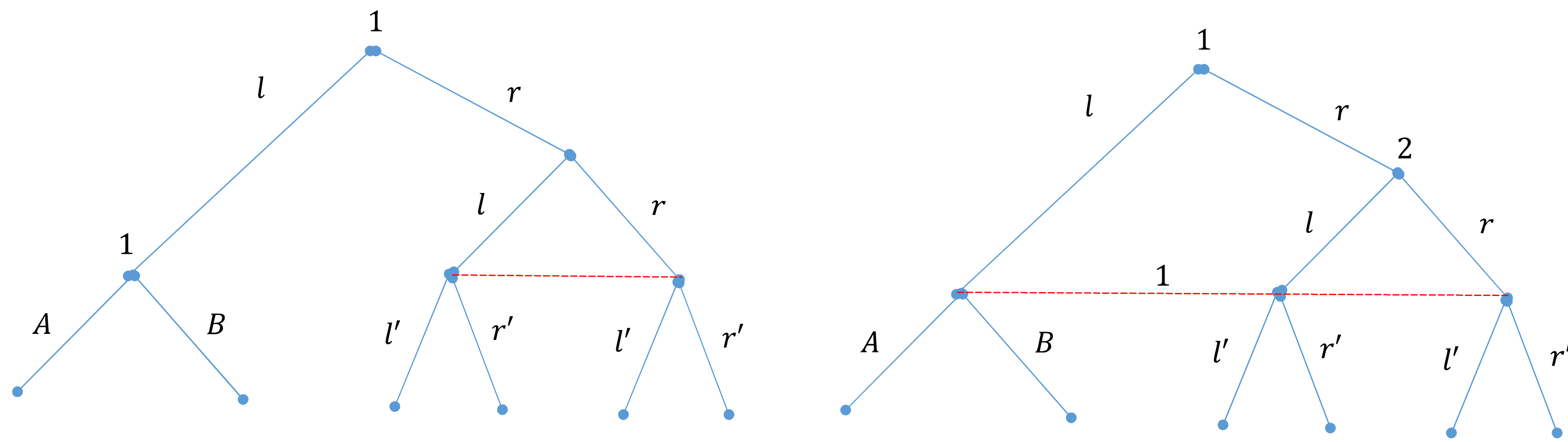
A game Γ is equivalent to the extensive game Γ' if:

- Γ' differs from Γ only in that there is an information set of some player i in Γ that is a union of information sets of player i in Γ'.

- and **any** two histories h and h' in different members of the union:

  - have sub-histories that are in the same information set of player i

  - and player i's action at this info. set is $\neq$ in h and h'.



Take $h = l$ and $h' = (r, l)$

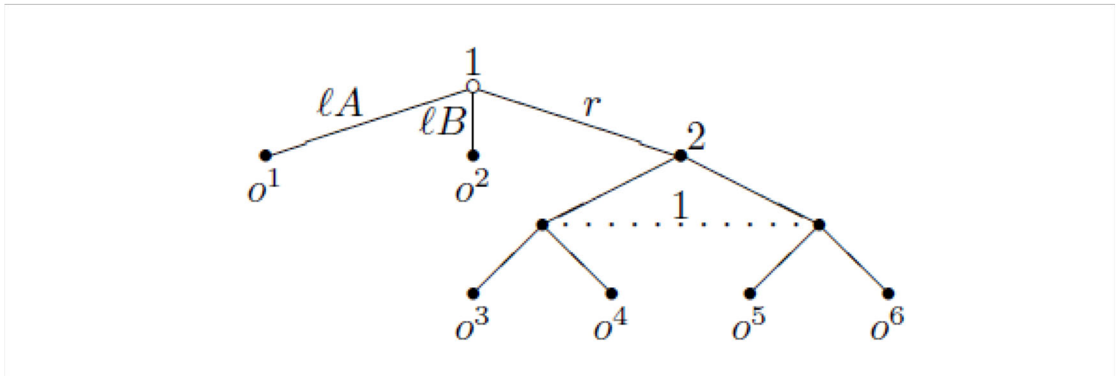
They have a common subhistory $h'' = \emptyset$

At this history 1 takes different actions.

According to the inflation deflation principle:

- Γ is strategically equivalent to the inflated Γ'.
- And similarly Γ' is equivalent to the deflated Γ.

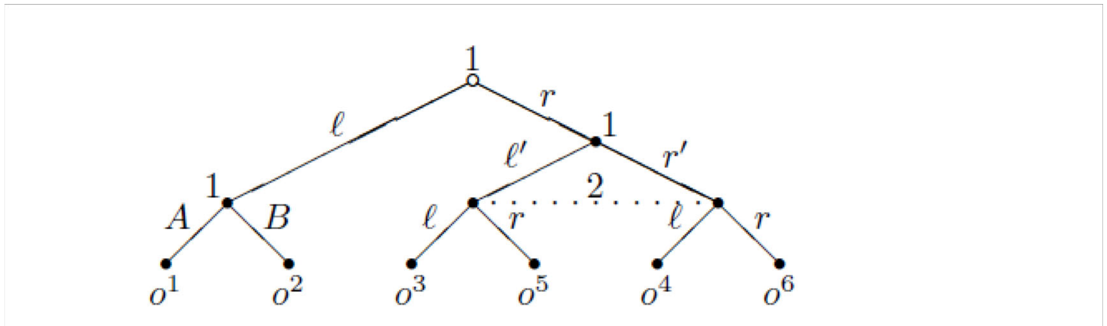
Coalescing moves



Game Γ is equivalent to the above.

Idea: it is the outcomes that matter, not the sequence of actions per se.

Interchange of moves

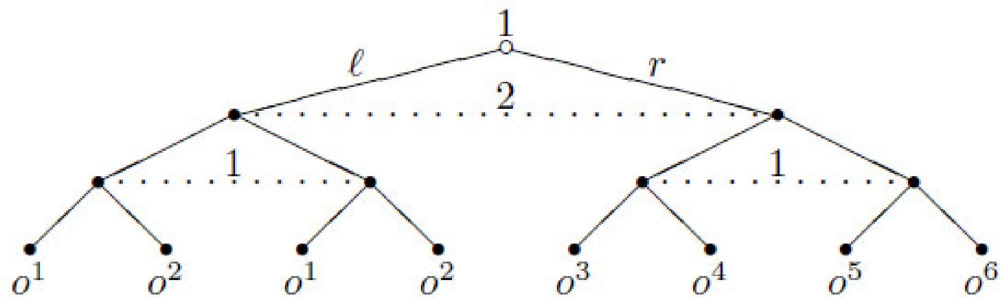


Game Γ is equivalent to the above.

Idea: order of moves is immaterial when players do not have information about the other player's action.

Addition of superfluous moves

Game Γ is equivalent to this:



Idea: After l is chosen, 2's action is irrelevant (i.e. it has no effect on the outcome), thus whether 2 is informed or not about the action of 1 at Ø is irrelevant.

Theorem (Thompson 1952):

Consider finite extensive games in which no information set contains both a history h and some sub-history of h.

If*...any two such games have the same reduced strategic form...*

Then*...one can be obtained from the other by a sequence of the four transformations.*

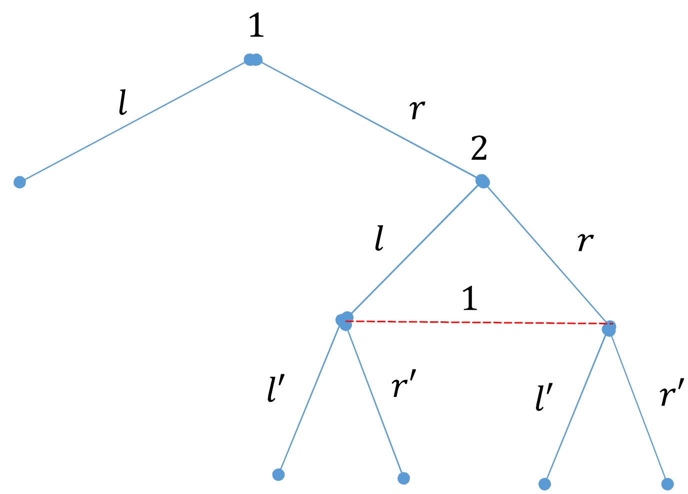
Mixed and Behavioral strategies

We have two ways to describe a mixed strategy in an extensive game.

Definition. *A mixed strategy of player* i *in an extensive game* $\langle N, H, P, f_c, (\Im_i), (u_c)\rangle$ *is a probability measure over the set of player* i*'s pure strategies.*

Definition. *A behavioral strategy of player* i *is a collection* $(\beta_i(I_i))_{I_i \in \Im_i}$ *of independent probability measures, where* $\beta_i(I_i)$ *is a probability measure over* $A(I_i)$*.*

Consider this game:



Pure strategies assign an action to each information set. For 1: $\emptyset, \{(r,l),(r,r)\}$. For 2: $\{\{l\},\{r\}\}$.

So for 1 a strategy is and ordered pair: $(r,l'),(r,r'),(l,l')$, (l,r').

A mixed strategy is a prob. dist. on these strategies.

A behavioral strategy instead is a collection of $\beta_i(I_i)(a_i)$.

For any strategy profile σ, we define the outcome $O(\sigma)$ to be the probability distribution over the terminal histories that results when each player i follows the precepts of i.

Let us formalize this idea.

For any history $h = (a^1, \ldots, a^k)$ define a **pure strategy** s_i of player i to be **consistent** with h if for every sub-history $(a^1, \ldots, a^t)$ of h for which $P(a^1, \ldots, a^t) = i$ we have $s_i(a^1, \ldots, a^t) = a^{t+1}$.

For any history h let $\boldsymbol{\pi}_i(\mathbf{h})$ be the sum of the probabilities according to σ_i of all the pure strategies of player i that are consistent with h.

Then for any profile of **mixed strategies** the probability that $O(\sigma)$ assigns to any terminal history h is $\prod_{i \in N \cup \{c\}} \pi_i(h)$.

For any profile of **behavioral strategies** the probability that $O(\sigma)$ assigns to the terminal history $h = (a^1, \ldots, a^K)$ is

$$\prod_{k=0}^{K-1} \beta_{P(a^0,\ldots,a^k)}(a^{k+1})$$

where $a^0 = \{\emptyset\}$ by definition.

Two strategies are **outcome equivalent** if for any profile of pure strategies of the other players, they induce the same outcome distribution.

When is it the case that for any mixed strategy, there is an outcome equivalent behavioral strategy and the reverse?

Let see when for any behavioral strategy we can define a mixed strategy.

A “natural” definition of a mixed strategy is:

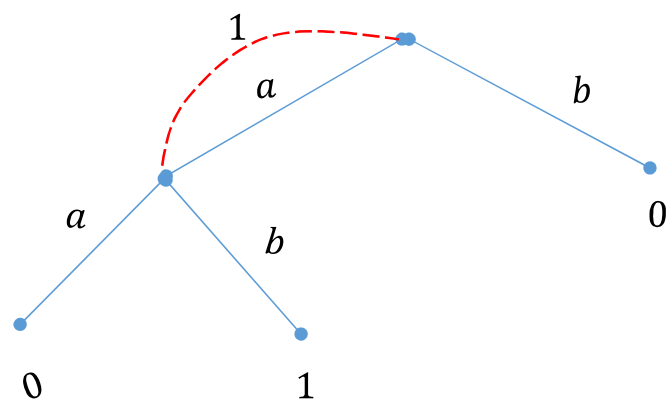
$$\prod_{I_i \in \Im_i} \beta_i(I_i)(s_i(I_i))$$

This however works only if the randomizations at the information sets are independent.

This does **not work** if some **history and an associated sub-history are in the same information set**.

This property is implied by **perfect recall**.

For example:

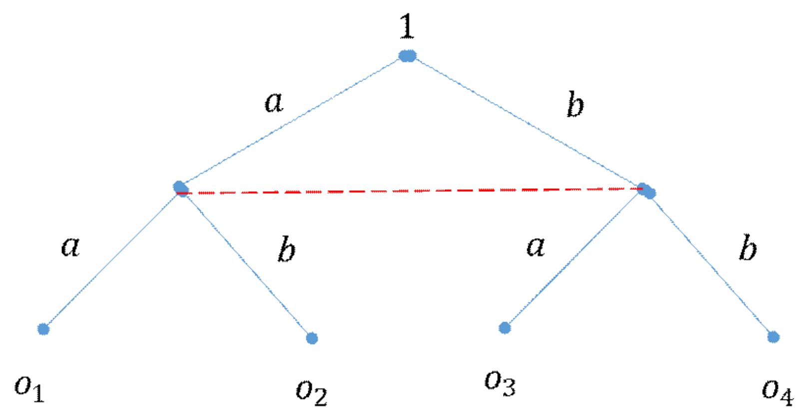


In this case a behavioral strategy $\beta_i(I_i)(a) = p$ induces a distribution $p^2, p(1-p), (1-p)$.

A mixed strategy assigns positive probability only on (a, a) or (b, b).

This because a mixed str. is a distrib. over pure strategies.

Perfect recall is also necessary to guarantee that a mixed strategy defines an outcome equivalent behavioral strategy.



Consider a mixed strategy $\sigma((a, a)) = 1/2, \sigma((b, b)) = 1/2$, this generates the outcome $(1/2, 0, 0, 1/2)$.

There is no behavioral strategy that can induce this outcome distribution.

Theorem (Kuhn). *For any mixed strategy of a player in a finite extensive game with perfect recall there is an outcome-equivalent behavioral strategy.*

Proof. Recall that $\pi_i(h)$ is the sum of probabilities according to σ_i of all the pure strategies consistent with h.

Because we have perfect recall, if h and h' are in the same information set I_i, then $X_i(h) = X_i(h')$.

Recall $X_i(h)$ is the record of i's experiences: the information sets and respective actions taken.

It follows that $\pi_i(h) = \pi_i(h')$.

Specifically, two h, h' in the same information set I_i for i, are such that:

- $\pi_i(h) = \pi_i(h')$,
- $A_i(h) = A_i(h')$;
- and $\pi_i(h, a) = \pi_i(h', a)$.

We can thus define the behavioral strategy

$$\beta_i(I_i)(a) = \frac{\pi_i(h, a)}{\pi_i(h)}$$

for $h \in I_i$ such that $\pi_i(h) > 0$.

Note that for outcome equivalence it is irrelevant how we define $\beta_i(I_i)(a)$ when $\pi_i(h) = 0$.

Nash Equilibrium

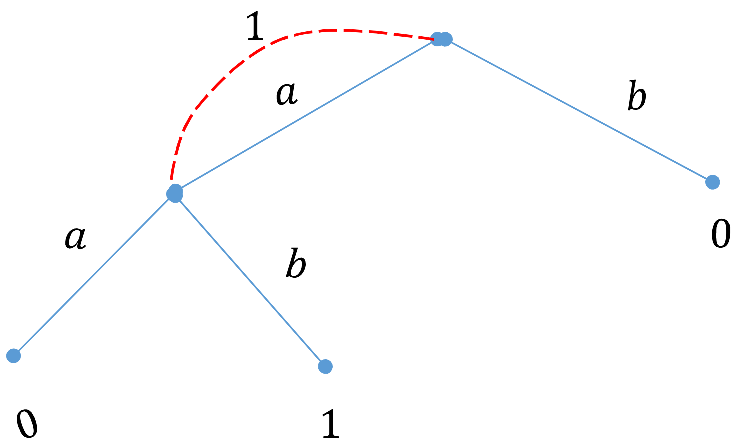
Definition. *A Nash equilibrium in mixed strategies of an extensive game is a profile of mixed strategies with the property that for every player* $i \in N$ *we have:*

$$O(\sigma_i^*, \sigma_{-i}^*) \succeq_i O(\sigma_i, \sigma_{-i}^*)$$

for any σ_i.

From the analysis above, the definition with behavioral is analogous if the game has perfect recall, but not if it does not have perfect recall.

Consider this game again:



Here, the player is indifferent between mixed strategies: the outcome puts only probability on (a, a) or (b, b), so they all induce a payoff of 0.

A behavioral strategy $\beta_i(a) = p$ however achieves a payoff of 1 with probability $p(1-p)$.