ECON 6100

Problem Set 6

Gabe Sekeres

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1. Find cost functions for the production functions. For every production function, the cost function is determined by
$$C(w_k, w_l, q) = \min_{k,l} w_k \cdot k + w_l \cdot l \text{ s.t. } f(k,l) \geq q$$
We will use the fact that many cost functions are always homogenous of degree 1, to find $C(\cdot,\cdot,1)$.

   (a) Cobb-Douglas: The cost function is:
   $$\min_{k,l} w_k \cdot k + w_l \cdot l \text{ s.t. } k^\alpha l^{1-\alpha} \geq q$$
   so we have that
   $$C(w_k, w_l, 1) = \alpha^{-\alpha}\beta^{-\beta} w_k^\alpha w_l^\alpha \Longrightarrow C(w_k, w_l, q) = \alpha^{-\alpha}\beta^{-\beta} w_k^\alpha w_l^\alpha q$$

   (b) CES: The cost function is
   $$C(w_k, w_l, 1) = \min_{k,l} w_k \cdot k + w_l \cdot l \text{ s.t. } ak^r + bl^r \geq 1$$
   so we have that
   $$C(w_k, w_l, 1) = \left[\left(\frac{w_k^r}{a}\right)^{\frac{1}{r-1}} + \left(\frac{w_l^r}{b}\right)^{\frac{1}{r-1}}\right]^{\frac{r-1}{r}} \Longrightarrow C(w_k, w_l, q) = q \cdot \left[\left(\frac{w_k^r}{a}\right)^{\frac{1}{r-1}} + \left(\frac{w_l^r}{b}\right)^{\frac{1}{r-1}}\right]^{\frac{r-1}{r}}$$

   (c) Linear: The cost function is
   $$C(w_k, w_l, q) = \min_{k,l} w_k \cdot k + w_l \cdot l \text{ s.t. } ak + bl \geq q$$
   which simplifies to
   $$C(w_k, w_l, q) = \min\left\{\frac{w_k \cdot q}{a}, \frac{w_l \cdot q}{b}\right\}$$

   (d) Leontief: We must have, in a Leontief model, that $k = \frac{q}{a}$ and $l = \frac{q}{b}$, so the cost function is
   $$C(w_k, w_l, q) = \frac{w_k \cdot q}{a} + \frac{w_l \cdot q}{b}$$

   (e) von Thünen: There is no closed form in general for this cost function, as this problem is not convex and not homogeneous of degree 1.

2. Two-country world.

   (a) ***Proof.*** FSOC, assume that the price of a factor g is strictly less than its marginal product in equilibrium, meaning that there is some profit, which does not go to the factors. Then we will have that $c_g(\cdot) > p_g$, meaning that the factor demand is not profit-maximizing and thus this is not an equilibrium. That is a contradiction. $\square$

(b) If p_A increases, the capital share of the national product will increase as long as the price of capital decreases, which will happen (by Stolper-Samuelson) as long as good A is labor-intensive. This implies that $\nabla_\ell f_A < \nabla_\ell f_B$.

3. Assume that an endowment is in the cone in the two-sector model, and assume that the quantity of a factor g slightly increases. Output will strictly increase, since we assume that f is strictly increasing in both factors and that output prices are positive. This means that the country can produce more output while profit-maximizing, so they will do so.

4. Here is another two-sector model. Sector 1 produces investment goods (capital goods). Sector 2 produces consumption goods. Each sector is characterized by a neoclassical production function (strictly concave, $\mathbf{C}^2$, Inada conditions at 0) with constant returns to scale. Write $Y_i = F_i(K_i, L_i)$ for output in sector i as a function F_i of capital K_i and labor L_i employed in sector i.

   (a) We have that $y_i = F_i(k_i, 1) = f_i(k_i)$. The previous assumptions imply that F_i is homogeneous of degree 1, so f_i is also homogeneous of degree 1, strictly concave, $\mathbf{C}^2$, and has Inada conditions at $k_i = 0$.

   (b) The conditions imply that the factor input costs are equal to the marginal product (scaled to dollar terms) of the input, that the markets for capital and labor clear in every individual market, and that the total demand for capital goods is equal to the total product of capital goods, with the same for the consumption goods being equal to the total product of labor.

   (c) These conditions become

$$\begin{aligned}
y_i &= f(k_i) & & \\
f_i'(k_i) &= \frac{r}{P_i} & \frac{w}{P_i} &= f_i(k_i) - k_i f_i'(k_i) \\
\implies \omega &= \frac{f_i(k_i) - k_i f_i'(k_i)}{f_i'(k_i)} & & \\
k_1 + k_2 &= k & 1 &= \ell_1 + \ell_2 \\
y_1 &= \frac{k}{\omega} & y_2 &= \omega \\
\implies \frac{y_1}{y_2} &= k & &
\end{aligned}$$

   (d) Using Implicit Function Theorem, we define

$$G(k_i, \omega) = \omega \cdot f_i'(k_i) - f_i(k_i) + k_i f_i'(k_i)$$

   and since the conditions hold, we can say that

$$\frac{\partial k_i}{\partial \omega} = -\frac{\frac{\partial G}{\partial \omega}}{\frac{\partial G}{\partial k_i}} = -\frac{f_i'(k_i)}{f_i''(k_i)(\omega + k_i)}$$

   By the Implicit Function Theorem, there is a unique function $k_i(\omega)$ that determines the optimal capital level for any given wage / rent ratio.

   (e) We have that from part (c), $\omega = \frac{k}{f(k_1)}$. This implicitly defines the wage / rent ratio as a function of the capital / labor ratio, $k = \frac{K}{L}$.