About TA sections:

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Section time and location: 8:40am - 9:55am Rockefeller Hall 132

Office hours: Tuesday 4:30-5:30 pm in Uris Hall 451; other times available by appointment (just send me an email).

Remark: This set of notes provides supplementary material that is optional and not required for the exam.

In these notes:

1 Additional notes on the Hamilton-Jacobi-Bellman (HJB)

1.1 Derivation

Recall the problem we discussed in Section 4. We have:

$$
\begin{aligned}
\max_{x(\cdot),\,y(\cdot)} \quad W\big(x(\cdot), y(\cdot)\big) &\equiv \int_0^\infty e^{-\rho t}\, f\big(t, x(t), y(t)\big)\, dt, \\
\text{subject to} \quad \dot{x}_i(t) &= g_i\big(t, x(t), y(t)\big), \quad \text{for } i = 1, 2, \ldots, n, \\
x_i(0) &= x_{i0} \quad \text{(given initial conditions)}, \\
x(t) \in \mathcal{X}, \quad y(t) &\in \mathcal{Y}, \quad \text{for all } t \geq 0.
\end{aligned}
$$

Here:

- $x(t) \in \mathbb{R}^n$ is the state vector (e.g., capital stock, population, etc.) at time t.
- $y(t) \in \mathbb{R}^m$ (or a more general set) is the control or decision variable (e.g., consumption, investment, policy instrument).
- $f(\cdot)$ represents the instantaneous payoff (or utility) function.
- $g_i(\cdot)$ are the components of the dynamic constraints describing how the state evolves over time.

To study this problem via dynamic programming, define the value function:

$$
V(x, t) = \max_{x(\cdot),\,y(\cdot)} \left\{ \int_t^\infty e^{-\rho(\tau - t)}\, f\big(\tau, x(\tau), y(\tau)\big)\, d\tau \;\middle|\; x(t) = x \right\}.
$$

If the problem is stationary (i.e., f and g do not depend explicitly on time), then we often drop the time argument and write

$$
V(x) = \max_{y(\cdot)} \int_0^\infty e^{-\rho t}\, f\big(x(t), y(t)\big)\, dt,
$$

with $\dot{x}(t) = g\big(x(t), y(t)\big)$ and $x(0) = x$.

The **dynamic programming principle** states that, if we look at an optimal control policy over $[0, \infty)$, any initial segment of that policy (over $[0, \Delta t]$) combined with an optimal policy starting from the state at Δt will also be optimal. Formally, for small Δt:

$$
V\big(x(0)\big) = \max_{y(\cdot):\ 0 \leq \tau \leq \Delta t} \left\{ \int_0^{\Delta t} e^{-\rho\tau}\, f\big(x(\tau), y(\tau)\big)\, d\tau + e^{-\rho \Delta t}\, V\big(x(\Delta t)\big) \right\}.
$$

Because the same problem structure applies after Δt, the maximum over $[\Delta t, \infty)$ is exactly $V\big(x(\Delta t)\big)$, multiplied by the discount factor $e^{-\rho\Delta t}$.

Let x denote the current state. Over the short interval $[0, \Delta t]$, assume the control $y(\tau) \approx y$ (constant) and the state evolves approximately as

$$x(\tau) \approx x + g(x, y)\,\tau \quad \text{for small } \tau.$$

At $\tau = \Delta t$,

$$x(\Delta t) \;\approx\; x + g(x, y)\,\Delta t.$$

Hence, the payoff from 0 to Δt can be approximated by:

$$\int_0^{\Delta t} e^{-\rho\tau} f\big(x(\tau), y(\tau)\big)\, d\tau \;\approx\; f(x, y) \int_0^{\Delta t} e^{-\rho\tau}\, d\tau \;=\; f(x, y)\, \frac{1 - e^{-\rho\,\Delta t}}{\rho}.$$

For small Δt, we may use $1 - e^{-\rho\,\Delta t} \approx \rho\,\Delta t$, so this becomes approximately $f(x, y)\,\Delta t$.

Next, expand $V\big(x(\Delta t)\big)$ around x:

$$V\big(x(\Delta t)\big) \;\approx\; V(x) \;+\; \nabla V(x)^\top \big[g(x, y)\big]\,\Delta t.$$

Thus,

$$e^{-\rho\Delta t}\, V\big(x(\Delta t)\big) \;\approx\; e^{-\rho\,\Delta t}\Big[V(x) + \nabla V(x)^\top g(x, y)\,\Delta t\Big].$$

For small Δt, $e^{-\rho\Delta t} \approx 1 - \rho\,\Delta t$.

Then, the expression for small Δt becomes:

$$V(x) \approx \max_{y \in \mathcal{Y}} \left\{ f(x, y)\,\Delta t \;+\; \big(1 - \rho\,\Delta t\big)\Big[V(x) + \nabla V(x)^\top g(x, y)\,\Delta t\Big] \right\}.$$

Rearrange and divide both sides by Δt. We obtain the **Hamilton-Jacobi-Bellman equation**:

$$\rho\, V(x) \;=\; \max_{y \in \mathcal{Y}} \Big\{ f(x, y) \;+\; \nabla V(x)^\top g(x, y) \Big\}.$$

Intuition:

- $\rho\, V(x)$ is the "opportunity cost" or the discount-weighted value of being in state x,
- $f(x, y)$ is the instantaneous reward,
- $\nabla V(x)^\top g(x, y)$ represents how much the future value changes if you shift the state in the direction of $g(x, y)$,

- the maximum operator reflects that we choose the policy y that yields the best immediate payoff plus the best improvement in future value.

1.2 Application to the neoclassical growth model and connection to the Hamiltonian

$$\max_{c(\cdot)} \int_0^\infty e^{-\rho t} \frac{c(t)^{1-\theta}}{1-\theta}\, dt$$

subject to

$$\dot{k}(t) = F\big(k(t)\big) - \delta\, k(t) - c(t),$$

and $k(0) = k_0$ given.

The HJB equation:

$$\rho\, V(k) = \max_{c\geq 0}\Big\{U\big(c\big) + V'(k)\left[F(k) - \delta\, k - c\right]\Big\}.$$

Denote an optimal choice by c^*. Then:

$$\frac{\partial}{\partial c}\Big[U(c) + V'(k)(F(k) - \delta k - c)\Big] = U'(c) - V'(k) = 0 \implies U'\big(c^*(k)\big) = V'(k).$$

Hence, at optimum, the marginal utility of consumption equals the shadow value of capital, i.e., the co-state λ in the Hamiltonian.

Substituting $c^*(k)$ back into the HJB equation gives

$$\rho\, V(k) = U\big(c^*(k)\big) + V'(k)\Big[F(k) - \delta\, k - c^*(k)\Big].$$

Envelope theorem:

$$\rho\, V'(k) = V''(k)\Big[F(k) - \delta\, k - c^*(k)\Big] + V'(k)\Big[F'(k) - \delta\Big].$$

Note that $F(k) - \delta\, k - c^*(k) = \dot{k}(t)$ and $\lambda(t) = V'(k) \implies \dot{\lambda} = V'(k)\dot{k}$. Thus,

$$\rho\lambda - \dot{\lambda} = \lambda\Big[F'(k) - \delta\Big],$$

which is the same optimality condition as we would obtain by taking FOC of the **current-value** Hamiltonian with respect to the state variable k.

2 Exam Check List*

* Prepared by the previous TA, Zhuoheng Xu, and slightly updated by me.

Consider this the minimum you should be familiar with before taking the exam. It provides a foundation to build upon.

Classes 1-4: Discrete Time One Sector Growth Model

- Apply Kuhn-Tucker Theorem to find optimality conditions
- Know additional assumptions needed for the interior solution
- Understand First and Second Welfare Theorem
- "Derive" the Transversality Condition
- Know assumptions required for the existence and the uniqueness of the steady state
- Find the steady state of the model
- Know how to use the shooting method to find the saddle path numerically
- Find the recursive representation of the social planner's problem + Envelope Theorem
- Know how to use value function iteration to numerically solve the model
- Define the competitive equilibrium
- Ricardian Equivalence: If taxes are non-distortionary, households consumption decisions do not depend on the timing of tax collection

Classes 4-5: OLG model

- Know how to define competitive equilibrium and characterize its allocation
- Understand the special case of this model with CRRA utility and a Cobb-Douglas production function
- Understand the source of dynamic inefficiency
- Know how to distinguish between fully funded and unfunded social security systems and their implications for generational welfare

Class 6-8: Heterogeneity, Aggregation, Incomplete Markets

- Understand the result that if the model exhibits aggregation, the average quantities corresponding to a competitive equilibrium also solve the planner's problem
- Know conditions required for aggregation
- Understand how incomplete markets arise by restricting the borrowing behavior of the household
- Know the definition of stationary equilibrium
- Know how to numerically find the stationary equilibrium
- Understand the main mechanism of the model (e.g., how policy functions respond to a tighter/slacker borrowing constraint)

Class 9-11: Continuous Time Long Run Growth Model

- Write down the Hamiltonian and apply the Maximum Principle of Pontryagin to find optimality conditions
- Understand the difference between present-value and current-value Hamiltonian
- Know the definition of Balanced Growth Path (BGP)
- Know how to derive the optimal consumption growth rate by solving the optimization problem
- Understand how to identify the relationship between the growth rates of different variables by using feasibility constraint/capital dynamics/...
- Know how to do log-linearization around the steady state and describe the dynamics
- Understand the source of long run growth in AK model
- Understand the connection between Hamiltonian and HJB
- Understand the differences and similarities between Romer's model and Lucas's model
- Know how to detrend the economy and analyze transition dynamics

- Know how to derive equilibrium conditions and determine the growth rates of key variables in the social planner's problem (SPP) and the decentralized economy in
  - Romer (1986)
  - Lucas (1988)
  - Romer (1990)