Signalling games

Spence’s model of education

A signaling game is a Bayesian extensive game with observable actions in which:

- There are two players, a "sender" and a "receiver".
- The sender is informed of the value of an uncertain parameter θ_1 and then chooses an action m (referred to as a message).
- The receiver observes the message (but not θ_1) and takes an action a.

- Each player's payoff depends upon the value of θ_1, the message m sent by the sender, and the action a taken by the receiver: $u_i(\theta_1, m, a)$.

What makes this problem interesting is that the sender has the information, but does not take any action; the receiver controls the action, but has no information.

Specifically, let us assume two players a worker ($i = 1$, sender), whose ability is θ_1.

Two types: $\theta_1 \in \{\theta_1^L, \theta_1^H\}$, with $\theta_1^H > \theta_1^L$, and p_L is the probability of a low type.

The employer's payoff is $u_2(\theta_1, e, w) = -(w - \theta_1)^2$

The worker's payoff is: $u_1(\theta_1, e, w) = w - \frac{e}{\theta}$

$e \geq 0$ is effort.

Here effort e is the message and w is the action a.

Pooling equilibria

We now show that this game we have a pooling equilibrium with $e^H = e^L = e^*$ and wage $w^* = E(\theta_1) = p_L\theta_1^L + (1 - p_L)\theta_1^H$.

This is an equilibrium in which there is no signalling, the types behave in the same way.

When such an equilibrium exists?

First of all note that the description so far is incomplete: what beliefs follow an effort $e \neq e^*$?

For this equilibrium, let us assume that $\mu(\theta_1^L) = 1$ if $e \neq e^*$.

In this case the wage after a deviation is $w(e) = \theta_1^L$, so we have that a deviation is unprofitable if and only if:

$$\theta_1^L \leq p_L \theta_1^L + (1 - p_L)\theta_1^H - \max\left\{ \frac{e^*}{\theta_1^H}, \frac{e^*}{\theta_1^L} \right\}$$
$$= p_L \theta_1^L + (1 - p_L)\theta_1^H - \frac{e^*}{\theta_1^L}$$

or:

$$e^* \leq \theta_1^L \bullet p^H(\theta_1^H - \theta_1^L).$$

So this equilibrium is a Perfect Bayesian equilibrium.

Is the assumption on out of equilibrium beliefs consistent?

Let us consider the fully mixed strategies:

- $\sigma_1(\theta_1^L)(e^*) = 1 - \varepsilon$ and $\sigma_1(\theta_1^L)(e \neq e^*) = \varepsilon$ with $\sigma_1(\theta_1^L)(e) = \sigma_1(\theta_1^L)(e')$ for $e, e' \neq e^*$.

- $\sigma_1(\theta_1^H)(e^*) = 1 - \varepsilon^2$ and $\sigma_1(\theta_1^L)(e \neq e^*) = \varepsilon^2$ with $\sigma_2(\theta_1^H)(e) = \sigma_2(\theta_1^H)(e')$ for $e, e' \neq e^*$.

Then we have $\mu(\theta_1^L; e, \sigma) = \frac{\varepsilon p_L}{\varepsilon^2(1-p_L)+\varepsilon p_L} \to 1$ as $\varepsilon \to 0$.

We conclude that the pooling equilibria are sequential equilibria.

Separating equilibrium

We now show that this game admits a very different type of equilibrium in which the types reveal themselves.

In this equilibrium $e_1^L = 0$ with associated $w(e_1^L) = \theta_1^L$; and $e_1^H > 0$, with associated $w(e_1^H) = \theta_1^H$.

Let us assume beliefs are $\mu(\theta_1^L) = 1$ if $e < e_1^H$.

For this to be an equilibrium the low type does not have a deviation if:

$$\theta_1^L \geq \theta_1^H - \frac{e^H}{\theta_1^L}$$

For the high type:

$$\theta_1^H - \frac{e^H}{\theta_1^H} \geq \theta_1^L$$

So:

$$\theta_1^L \bullet p^H(\theta_1^H - \theta_1^L) \leq e^H \leq \theta_1^H \bullet p^H(\theta_1^H - \theta_1^L)$$

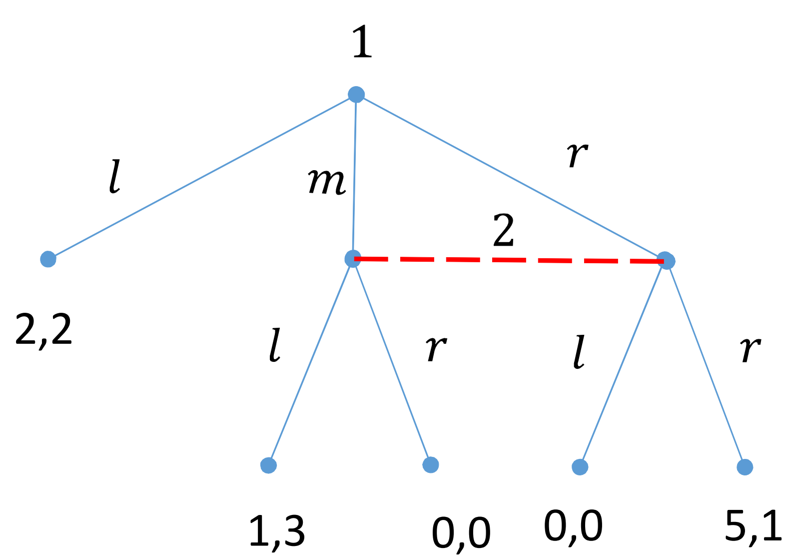
Again, we can prove this is a sequential equilibrium by assuming that the probability of a deviation from a low type converges to zero at a speed that slower than the probability of a deviation from a high type.

Refinements

The general idea

Are there ways to refine all these equilibria?

Here is an idea. Consider this game:



This game has a sequential equilibrium with outcome (r, r).

In this equilibrium player 2 believes that R is chosen, since R is chosen with probability 1.

It also has a sequential equilibrium with outcome (l, l).

This equilibrium is supported by the belief that with high probability m is selected by 1.

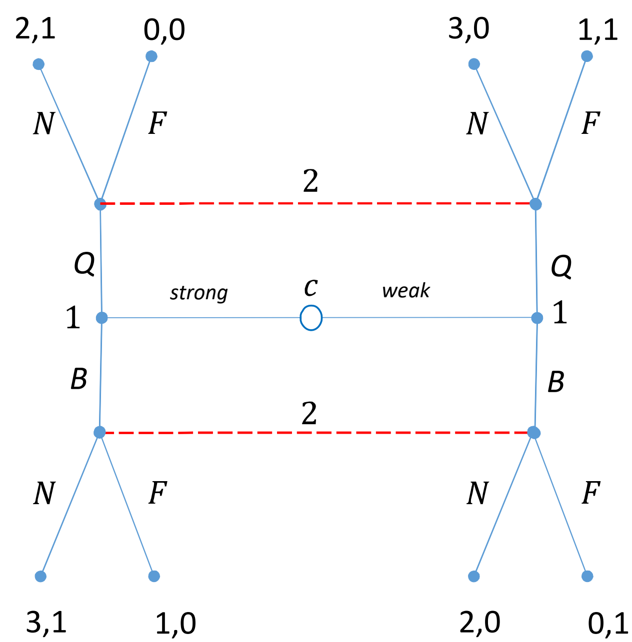
This second equilibrium, however has a problem.

m is strictly dominated by l; r is not strictly dominated.

So if I_2 is reached, 2 should believe that r was the choice rather than m.

Beer or Quiche?

Let us first extend this idea to a simple signaling game.



The strong type has probability 0.9.

There are two classes of equilibria:

- Both types of player 1 choose B, and player 2 fights if he observes Q and not if he observes B. If player 2 observes Q then he assigns probability of at least 0.5 that player 1 is weak.

- Both types of player 1 choose Q, and player 2 fights if he observes B and not if he observes Q. If player 2 observes B then he assigns probability of at least 0.5 that player 1 is weak.

To rule out the second class of equilibria, we ask: who benefits form a deviation?

- A weak type deviates, s/he obtains less than in equilibrium no mater what 2 does.
- On the contrary, if 1 is strong and the deviation works, so after B, 2 selects N, then 1 benefits.
- Thus it is (perhaps) reasonable to conclude that after B, 2 should attribute the deviation to a strong type.

The argument on who benefits from the deviation is done in reference to the prevailing equilibrium.

Idea: For weak Q in equilibrium is strictly better than B for any reaction of 2.

But the equilibrium is already specifying what happens in case of B.

Back to Spence's signaling model

We can apply the ideas above to Spence's model.

Consider a e' such that:

$$E\theta - \frac{e^*}{\theta_1^L} \geq \theta_1^H - \frac{e'}{\theta_1^L} \text{ and } E\theta - \frac{e^*}{\theta_1^H} < \theta_1^H - \frac{e'}{\theta_1^H}$$

where e^* is a pooling equilibrium ($e^* \leq \theta_1^L \bullet p^H(\theta_1^H - \theta_1^L)$).

- A low type is worse off if believed; a high type is better off;
- ⇒A receiver would believe that the deviator is of high type.

We conclude that there are no pooling equilibria.

Consider now a separating equilibrium $e^L = 0, e^H$

In such an equilibrium $e^L = 0$ and $\theta_1^H - \frac{e^H}{\theta_1^H} \geq \theta_1^L$.

Assume $\theta_1^H - \frac{e^H}{\theta_1^H} > \theta_1^L$.

If the receiver receives a deviation e' such that $e' < e^H$, but still $\theta_1^H - \frac{e'}{\theta_1^H} \geq \theta_1^L$:

- A high type benefits from the deviation if believed;
- A low type does not benefit if believed or not.
- Receiver concludes: deviation comes from the high type.

We conclude that there is only one separating equilibrium that survives in which:

$$e^L = 0 \text{ and } \theta_1^H - \frac{e^H}{\theta_1^H} = \theta_1^L$$

Trembling Hand Perfect Equilibrium

Strategic games

In a trembling hand perfect equilibrium we contemplate robustness of an equilibrium to small perturbations of the equilibrium strategies.

Recall that a strategy is completely mixed if it assigns positive probability on all actions.

Definition *A trembling hand perfect equilibrium of a finite strategic game is a mixed strategy profile* σ *with the property that there exists a sequence* $(\sigma^k)_{k=0}^{\infty}$ *of completely mixed strategy profiles that converges to* σ *such that for each player* i *the strategy* σ_i *is a best response to* σ_{-i}^k *for all values of* k.

Note that σ must be a Nash equilibrium.

Some games however show that not all Nash equilibria are Trembling Hand perfect.

	A	B	C
A	0,0	0,0	0,0
B	0,0	1,1	2,0
C	0,0	0, 2	2,2

BB is the only Trembling Hand eq.

Recall the definition of a weakly dominated action (there is another action of mixed strategy that generates weakly better payoff, strictly for some profile of action of the opponents).

We can have Nash equilibria in which weakly dominated actions are played.

But a Trembling hand equilibrium can not assign positive probability on a weakly dominated action.

For two player games there is an even tighter relationship between weak domination and trembling hand perfection.

Proposition. *A strategy profile in a finite two-player strategic game is a trembling hand perfect equilibrium if and only if it is a mixed strategy Nash equilibrium and the strategy of neither player is weakly dominated*.

This, however. is not true with more than 2 players. Consider:

	L	R
T	$1, 1, 1$	$1, 0, 1$
B	$1, 1, 1$	$0, 0, 1$

l

	L	R
T	$1, 1, 0$	$0, 0, 0$
B	$0, 1, 0$	$1, 0, 0$

r

The **Nash eq**. B, L, l is undominated but not trembling hand perfect since 1's payoff to T exceed her payoff to B when **2** and **3** assign **small probability to** R **and** r.

We have:

Proposition. *Every finite strategic game has a trembling hand perfect equilibrium.*

Proof. Define a perturbation of the game by letting the set of actions of each player i be the set of mixed strategies of player i that assign probability of at least ε_j^i to each action j of player i, for some collection (ε_j^i) with $\varepsilon_j^i > 0 \ \forall i,j$.

That is, constrain each player to use each action available to him with some minimal probability.)

Every such perturbed game has a Nash equilibrium by a standard argument.

Consider a sequence of such perturbed $\varepsilon_j^i(n)\ \forall i,j$ games in which $\varepsilon_j^i(n) \to 0\ \forall i,j$.

By the compactness of the set of strategy profiles, we can find a converging subsequence of strategies, converging to say to some $\sigma*$.

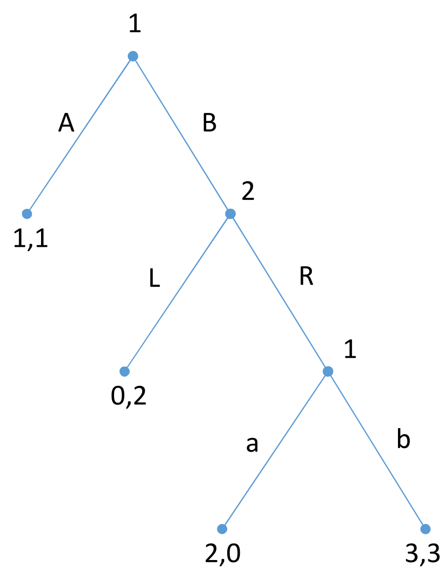
This is a Trembling Hand Equilibrium.

Extensive Games

We can extend the idea of trembling hand perfection to extensive games.

Problem however arise.

Consider this game.



	L	R
A,a	**1, 1**	**1,1**
A,b	**1, 1**	**1, 1**
B,a	**0,2**	**2,0**
B,b	**0,2**	**3,3**

$(B,b),R$ is the unique subgame perfect equilibrium.

This seems the plausible prediction.

However the profile $(A,a),L$ is a Trebling Hand Perfect equilibrium of the strategic form on the right:

- (A,a) is a best response for 1 to any strategy in which 2 put enough probability on L;
- L is a best response to a strategy with enough probability on (A,a). Note on (A,a) not on a alone...

The problem is that when evaluating the optimality of her strategy:

- 1 *does not* consider the possibility that she *herself* will make mistakes when carrying out this strategy.

- If she considered mistakes (and B receives positive probability), then (A, a) would not be optimal (specifically a would never be optimal in the last node).

This motivates to study the trembling hand perfect equilibria not of the strategic form but of the *agent strategic form* of the game.

In the *agent strategic form* there is one player for each information set in the extensive game.

Each player in the extensive game is split into a number of agents, one for each of his information sets, all agents of a given player having the same payoffs.

Any mixed strategy $\sigma(I_i)$ in the agent strategic form corresponds to the behavioral strategy $\beta(I_i)$ of the corresponding game.

Definition. *A trembling hand perfect equilibrium of a finite extensive game is a behavioral strategy profile that corresponds to a trembling hand perfect equilibrium of the agent strategic form of the game.*