ECON6110: Problem Set 3

Spring 2025

This problem set is due on at 23:59 on March 14, 2024. Every student must write their own solution and submit it individually. Problem set submissions are submitted electronically and may be typed or handwritten. If handwritten, please ensure your work and scan are legible. **Illegible submissions will not be graded.**

Problem 1

Consider a repeated game with finite horizon $T < \infty$ and discount factor $\delta = 1$. The stage game is as follows:

	X	Y	Z
A	$5,5$	$0,7$	$2,5$
B	$5,2$	$4,3$	$2,1$
C	$7,0$	$3,3$	$3,4$

(a) Suppose that $T = 1$, that is, the game is played two times. Is there a SPE in which (A, X) is played in the first period? Explain.

(b) Find a value of $T < \infty$ and a SPE in which (A, X) is played in the first period.

Solution:

(a) The answer is no. In the last period, the players must play one of the Nash EQ: (B, Y), (C, Z), or $(\frac{1}{3}B + \frac{2}{3}C, \frac{1}{2}Y + \frac{1}{2}Z)$. This means that in the last period player 1 gets at most 4 and at least 3. So the SPE must leverage on the different payoffs in the Nash EQ in the second stage to incentivize the Non-Nash play in the first period. However, the gain is 2 when the player deviates in the first period to her strictly dominating strategy. Therefore, playing (A, X) in the first period cannot be sustained by any strategy profile.

(b) Consider the following strategy for $T = 4$ (5 periods) for the two players:

- On-path strategy:

$$(A, X) \to (B, Y) \to (C, Z) \to (B, Y) \to (C, Z)$$

  In payoffs:

$$(5, 5) \to (4, 3) \to (3, 4) \to (4, 3) \to (3, 4)$$

- Off-path strategy:
  If player 1 deviates at any t, play (C, Z) for all following periods.
  If player 2 deviates at any t, play (B, Y) for all following periods

Verify this is a SPE. First note that for $t \geq 1$ the players play a stage Nash. So any one-shot deviation is not profitable for the current period, and is also not strictly profitable for future periods (stuck with a worse Nash). So we only need to check devitions at $t = 0$.

First, we calculate that the on-path payoff for both players are 5+4+3+4+3 $= 19$. Suppose player 1 deviates, her best shot is to deviate to (C, X) with a payoff of 7, and they would play (C, Z) afterwards. The total payoff of player 1 is $7+3+3+3+3 = 19$. Suppose instead player 2 deviates, the best she can get is also $7+3+3+3+3 = 19$. In both cases, there is not a strict incentive for any player to deviate. Indeed, this strategy is a SPE for any $T \geq 4$.

Problem 2

Consider the infinitely repeated game where the stage game is Bertrand's duopoly game. In the stage game, each firm i chooses simultaneously a price $p_i \in [0,1]$. If the market price is p_i, the demand is $1 - p_i$. The marginal cost of production is constant and equal to $c = 0.5$. The payoff in the stage game is per-period profit:

$$\pi_i(p_i, p_{-i}) = \begin{cases} (1-p_i)(p_i - 0.5) & \text{if } p_i < p_{-i} \\ \frac{1}{2}(1-p_i)(p_i - 0.5) & \text{if } p_i = p_{-i} \\ 0 & \text{otherwise.} \end{cases}$$

Denote by $p_i^* \in [0,1]$ the price that firm i would choose if she was a monopolist, that is, p_i^* is the price that maximizes $(1-p_i)(p_i - 0.5)$ over $p_i \in [0,1]$.

(a) Consider the following strategy profile: "Each firm i chooses $p_i = p_i^*$ in the first period and subsequently as long as the other firm continues to charge $p_{-i} = p_{-i}^*$ and punishes any deviation from $p_{-i} = p_{-i}^*$ by the other firm by choosing the price $p_i = c$ for a finite number $T \geq 1$ of periods, then reverting to $p_i = p_i^*$." Given any $\delta \in (1/2, 1)$, for what values of T this strategy profile is a subgame perfect equilibrium?

(b) Consider the following strategy profile: "In the first period, each firm i chooses $p_i = p_i^*$. In every subsequent period, each firm i charges the lowest of all the prices charged by the other firm in all previous periods." Is there a value of $\delta \in (0,1)$ for which this strategy profile is a SPE? Explain.

Solution:

(a) Assume that firm i deviates from $p_i^* = 0.75$ by choosing $p_i < 0.75$, thus undercutting the opponent and getting the entire market. The payoff from this one-shot deviation is

$$(1-\delta)(1-p_i)(p_i - 0.5) + \delta^{T+1}\frac{1}{2}(0.25)^2.$$

For the deviation not to be profitable, it must be that

$$(1-\delta)(1-p_i)(p_i - 0.5) + \delta^{T+1}\frac{1}{2}(0.25)^2 \leq \frac{1}{2}(0.25)^2.$$

This inequality must hold for all $p_i < 0.75$. Thus it must be true that

$$(1-\delta)(0.25)^2 + \delta^{T+1}\frac{1}{2}(0.25)^2 \leq \frac{1}{2}(0.25)^2.$$

This simplifies to

$$(1-\delta) \leq \frac{1}{2}(1-\delta^{T+1}).$$

The inequality is satisfied for T sufficiently large. More precisely,

$$T \geq \frac{\log(2\delta - 1)}{\log \delta} - 1.$$

(b) This strategy profile is never a SPE. Since a SPE must be Nash equilibrium in any subgame following any history, a simple way to see this is to consider a history of length one where both firms choose the same price below marginal cost; for example, $p_1^0 = p_2^0 = 0.25$. In all subsequent periods $t \geq 1$, the strategy profile recommends $p_1^t = p_2^t = 0.25$. But this would give the firms negative profits, and the firms can always obtain zero profits by pricing at marginal cost. Thus the suggested strategy profile cannot be a SPE, no matter what the discount factor is.

Problem 3

Consider an infinite repetition of the following stage game:

	A	B	C
A	$4,4$	$3,0$	$1,0$
B	$0,3$	$2,2$	$1,0$
C	$0,1$	$0,1$	$0,0$

Assume the discount factor is very close to one; to be concrete, assume $\delta = 0.99$.

(a) Find a subgame perfect equilibrium where both players get a payoff of 4.

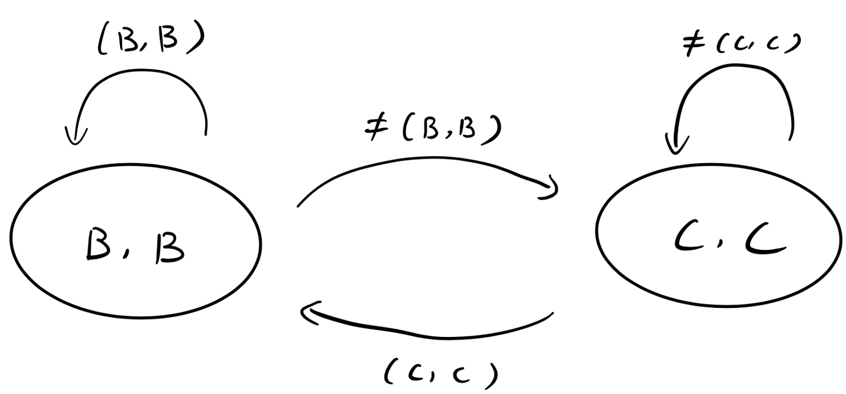
Answer. (A, A) is a Nash equilibrium of the stage game. Thus "always A" is a SPE.

(b) Is there a SPE where player 1 gets a payoff of 0.5?

Answer. The answer is no: player 1 can follow the strategy "always A" and obtain a payoff of at least 1.

(c) Find a SPE where both players get a payoff of 2.

Answer. Here is one SPE where both players get a payoff of 2. Consider the following simple limited punishment strategy. We verify it is a SPE by showing the one-shot deviations cannot be profitable.



1. Consider the one-shot deviations when the players are supposed to play (B, B). The game is symmetric so consider player 1. Her best deviation is A, which gives a payoff of 3. For this to be unprofitable,

$$(1-\delta)3 + 2\delta^2 \leq 2 \tag{1}$$

2. Consider the one-shot deviations when the players are supposed to play (C, C). For any deviation to be unprofitable,

$$(1-\delta) + 2\delta^2 \leq 2\delta \tag{2}$$

We can verify that $\delta = 0.99$ satisfies both inequalities.

Problem 4

Consider the following infinite-horizon game between a single firm and a sequence of workers, each of whom lives for one period. In each period the worker chooses either to expend effort and so produce output y at effort cost c or to expend no effort, produce no output, and incur no cost. If output is produced, the firm owns it but can share it with the worker by paying a wage w, as described next. Assume that at the beginning of the period the worker has an alternative opportunity worth zero (net of effort cost) and that the worker cannot be forced to accept a wage less than zero. Assume also that $y > c$ so that expending effort is efficient. Within each period, the timing of events is as follows: first the worker chooses an effort level, then output is observed by both the firm and the worker, and finally the firm chooses a wage to pay the worker. Assume that no wage contracts can be enforced by law: the firms choice of wage is completely unconstrained. In a one-period game, therefore, subgame-perfection implies that the firm will offer a wage zero independent of the workers output, so the worker will not expend any effort.

Now consider the infinite-horizon problem. Recall that each worker lives for only one period. Assume, however, that at the beginning of period t; the history of the game through period $t-1$ is observed by the worker who will work in period t. Suppose the firm discounts the future according to the discount factor δ per period. Describe strategies for the firm and each worker in a subgame-perfect equilibrium in the innite horizon game in which in equilibrium each worker expends effort and so produces output y; provided the discount factor is high enough. Give a necessary and sufficient condition for your equilibrium to exist.

Solution

A SPE in which each worker at every period exerts effort could only be if the worker expect to receive a wage of at least c, to cover her effort costs. Thus, effort has to be conditional on a reputation of the firm to pay wages above c, where the reputation is some function of the history of wages the company has given in response to previous workers' efforts. For instance, let $w \in [c, y)$ and consider the following strategies for the workers:

$$s_w\left(h^t\right)=\begin{cases}\text{Effort (E)} & \text{if } t=0 \text{ or } \left[t>0 \text{ and } h^t=\left(\left(w_E^0\geq w,E\right),...,\left(w_E^{t-1}\geq w,E\right)\right)\right]\\ \text{No effort (NE)} & else\end{cases}$$

and the firm:

$$s_F(h^t)=\begin{cases}w, & \text{if } t=0 \text{ and } s_w(h^0)=E\\ & \text{or } t>0 \text{ and } h^t=((w_E^0\geq w,E),\ldots,(w_E^{t-1}\geq w,E),s_w(h^t)=E)\\ 0, & \text{else}\end{cases}$$

After a good reputation of paying at least w, playing the equilibrium strategy, the worker will make an effort and will get $w-c$ and the firm will have at time t a profit of $y-w$. By following the equilibrium strategy, the firm will receive $\frac{y-w}{1-\delta}$ and the worker $w-c$: A deviation of the firm at time t to pay zero (deviation that gives the greatest possible one-period payoff) would yield a profit of y and 0 afterwards. Therefore, if:

$$y\leq\frac{y-w}{1-\delta}\iff\delta\geq\frac{w}{y}$$

the firm has no incentive to deviate.

For the worker to have incentive to sustain the above strategy in equilibrium it has to be that $w\geq c$: Thus, for any $1>\delta\geq\frac{c}{y}$ one can find a SPE in which the worker exerts effort in every period and receives a wage of $\delta y\geq w\geq c$.