Auctions

Optimal Auctions

We assume independent, private values.

We assume I bidders with preferences:

$$v_i\phi_i - t_i$$

where v_i is the marginal valuation, ϕ_i is the probability that i receives the good, t_i is the expected payment.

Bidders have a reservation utility U_i.

For simplicity here we assume $U_i = 0 \; \forall i$.

$P_i(v_i)$ is the distribution of type v_i with support $[\underline{v}, \bar{v}]$

Density $p_i(v_i)$ and $p_{-i}(v_{-i}) = \prod_{j \neq i} p_i(v_i)$.

The seller has a valuation v_0 for the good.

A direct mechanism is a $\phi_i(v), t_i(v)$ that maps type reports to an allocation.

Naturally the allocation must be feasible so $\sum \phi_i(v) \leq 1$.

The goal of this section is to design an optimal auction, that is the auction that maximize revenues.

The seminal work is Myerson [1981].

Incentive compatibility

Let us denote:

$$\phi_i(v_i) = E_{v_{-i}}(\phi_i(v))$$
$$t_i(v_i) = E_{v_{-i}}(t_i(v))$$

The indirect utility function of a type v_i who reports $\widehat{v}_i$ is:

$$U_i(\widehat{v}_i; v_i) = \phi_i(\widehat{v}_i)v_i - t_i(\widehat{v}_i)$$

Let $U_i(v_i) = U_i(v_i; v_i)$.

The truth telling condition is:

$$U_i(v_i) \geq U_i(\widehat{v}_i; v_i) \text{ for any } \widehat{v}_i,\ v_i$$

The first main result should be familiar by now:

Proposition. *An auction mechanism with $\phi_i(v_i)$ continuous and absolutely continuous first derivative is incentive compatible if and only if:*

$$\frac{\partial U_i(v_i)}{\partial v_i} = \phi_i(v_i)$$

and:

$$\phi_i(v_i) \text{ is nondecreasing in } v_i$$

The principal's problem is to solve:

$$\max_{\phi\in\Delta,\text{ and } t} E_v\Big[\Big(1-\sum \phi_i(v_i)\Big)v_0 + \sum t_i(v_i)\Big]$$

$$st.\ IC,\ IR$$

Since $t_i(v_i) = \phi_i(v_i)v_i - U_i(v_i)$:

$$\max_{\phi\in\Delta,t} E_v\left[\Big(1-\sum \phi_i(v_i)\Big)v_0 + \sum \phi_i(v_i)v_i - \sum_i U_i(v_i)\right]$$

$$st.\ IC,\ IR$$

Let us now rewrite $E_v U_i(v_i)$:

$$
\begin{aligned}
E_v U_i(v_i) &= \int_0^1 U_i(x) p_i(x) dx \\
&= -|U_i(x)[1 - P_i(x)]|_0^1 + \int_0^1 U_i'(x) \frac{1 - P_i(x)}{p_i(x)} p_i(x) dx \\
&= U_i(0) + \int_0^1 \phi_i(x) \frac{1 - P_i(x)}{p_i(x)} p_i(x) dx
\end{aligned}
$$

Note that we are using the fact that $U_i'(x) = \phi_i(x)$

Substituting the IC, we have:

$$\max_{\phi\in\Delta,t} E_v\left[\begin{array}{c}\left(1-\sum\phi_i(v_i)\right)v_0+\sum\phi_i(v_i)v_i \\ -\sum\left(\phi_i(v_i)\frac{1-P_i(v_i)}{p_i(v_i)}+U_i(0)\right)\end{array}\right]$$

$st.\ \phi_i(v_i)$ is monotonic and IR

or:

$$\max_{\phi\in\Delta,U_i(0)} E_v\left[v_0+\sum\phi_i(v)\left(v_i-\frac{1-P_i(v_i)}{p_i(v_i)}-v_0\right)\right]$$

$st.\ \phi_i(v_i)$ is monotonic

What do we learn from this representation of the optimal auction?

Revenue equivalence

From:

$$\max_{\phi \in \Delta} E_v \left[v_0 + \sum \phi_i(v) \left(v_i - \frac{1 - P_i(v)}{p_i(v)} - v_0 \right) - \sum U_i(0) \right]$$

$st.\ \phi_i(v_i)$ is monotonic

The objective function is the revenue of a general auction implementing a monotonic allocation $\phi_i(v)$.

Revenues depend only on the allocation $\phi_i(v)$ and $U_i(0)$.

The implication is that:

- all auctions that induce the same allocation of the good;
- and that provide the same rents to types 0.

generate the same expected revenues for the seller.

What does this mean?

First price Auction (and Dutch Auction) is a mechanism that allocates the good to the bidder with the highest value and provides zero utility to the buyer with lowest type.

Second price auction (and English Auction) is a mechanism allocates the good to the bidder with the highest value and provides zero utility to the buyer with lowest type (at least when we look at the WUS eq.).

Implication: FPA and SPA must generate the same revenues.

These are examples of the *Revenue Equivalence Theorem*:

Theorem. *The seller's expected utility from an implementable auction is completely determined by the allocation function* ϕ *and the reservation utilities* $U_i(0)$.

Optimal design

What is the revenue maximizing auction? Again:

$$\max_{\phi \in \Delta} E_v \left[v_0 + \sum \phi_i(v) \left(v_i - \frac{1 - P_i(v_i)}{p_i(v_i)} - v_0 \right) - \sum U_i(0) \right]$$

$st.\ \phi_i(v_i)$ is monotonic

We only need to chose $\phi \in \Delta$.

Let $J_i(v_i) = v_i - \frac{1-P_i(v_i)}{p_i(v_i)}$. This is the virtual utility.

First if $\max_i J_i(v_i) < v_0$, then set $\sum \phi_i(v) = 0$, i.e. do not sell.

If $\max_i J_i(v_i) \geq v_0$, then allocate the good to the bidders with the highest virtual utility.

Is this the solution?

We need to make sure that the monotonicity constraint is satisfied.

This is the case when $J_i(v_i)$ is monotonic non decreasing in $J_i(v_i)$.

A sufficient condition for this the MHRC:

$$\frac{1 - P_i(v_i)}{p_i(v_i)} \text{ non increasing in } v_i$$

Are the standard auctions optimal?

Assume a symmetric environment in which $P_i(v) = P_j(v)$:

- In this case $J_i(v) = J_j(v)$ so the optimal allocation is to sell the good to the agents with the highest value.
- This is achieved in all of the four standard auctions.
- The four standard auctions also satisfy $U_i(0) = 0$
- The standard auctions are therefore optimal.

Revenue equivalence however fails with asymmetric distributions.

Assume for example that $P_1(v)$ is uniform in $[0, 2]$ and $P_2(v)$ is uniform in $[2, 4]$. We have:

$$J_1(v) = v - \frac{\frac{1}{2}(2 - v)}{1/2} = v - (2 - v) = 2v - 2$$

and

$$J_2(v) = v - \frac{\frac{1}{2}(4 - v)}{1/2} = 2v - 4 = J_1 - 2$$

Bidder 2 is handicapped by 2 relative to bidder 1.

Standard auctions (that are symmetric) do not implement this optimal handicap.

Why do we need an handicap? To better provide incentive for type revelation encouraging 2 to bid more aggressively.

Efficiency

In general we might have that:

$$\max_i J_i(v_i) < v_0 < \max_i v_i$$

In this case, we do not sell the object, but it would be efficient to sell it.

This is a distortion introduced to maximize revenues.

Notes

When the **MHRC does not hold**, the unconstrained solution is not necessarily optimal since it may suggest an allocation that is not monotonic.

In this case we have a constrained solution that "forces" the solution to be monotonic, this is refereed to as "ironing."

When the allocation is not efficient we have a problem of **commitment**, since after the good is allocated there is room for a reallocation of the good.

All the analysis presented above holds with risk neutral bidders.

With **risk averse** bidders, revenue equivalence does not hold true anymore.

The *first price auction* generally **outperforms** the *second price auction*:

- your bid is closer to the price you pay, so there is less uncertainty.
- this leads to more aggressive bids.

An advantage of the second price bid is that it has an equilibrium with a stronger equilibrium concept.

A problem with the second price auction is that there are other Nash equilibria (not in WUS).

There is a "**collusive**" **equilibrium** in which the good is sold to some bidder at a price of zero.