ECON 6110, Microeconomic Theory III — **Section 11**

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1 Bayesian Extensive Games and the Perfect Bayesian Equilibrium (PBE)

Definition 1.1. A **Bayesian extensive game with observed actions** is a tuple $\langle N, H, P, (\Theta_i), (p_i), (u_i)\rangle$ where:

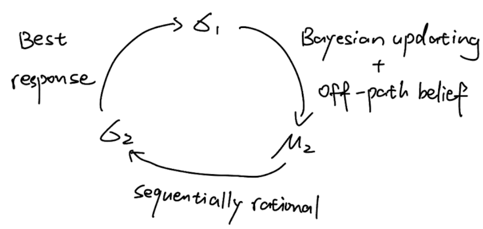
1. Set of N players, set of histories H, and player function P.
2. For each i:
   (a) A finite set of types Θ_i.
   (b) A probability measure p_i over Θ_i. (Assume independent types and common prior)
   (c) A preference relation $\succsim_i$ over $Z \times \Theta$.

Remark. In solving the game, we often recast the game as an extensive game with imperfect information, which is a tuple $\langle N, H, P, f_c, (\mathcal{I}_i), (u_i)\rangle$. We introduce Nature as another player, selecting types at time 0. (It will become clearer in the signaling game)

Definition 1.2 (Informal)**.** An assessment (σ, μ) is a **perfect Bayesian equilibrium** if

1. Sequentially rational: For each type θ_i, σ_i is the best response given μ_i and σ_{-i} at every information set I_i.
2. Bayesian updating whenever possible. (But no restriction on off-path beliefs)
3. Action determine beliefs: beliefs on i's type can only be changed by i's action. (True when independent types)

Remark. Solving for PBE often proceeds in a "loop":



2 Signaling Game

Consider the Spence's job-market signaling model with a discrete set of effort choices. The sender is a student, the receiver an employer. There are two types of students, defined by the value of their innate talent, $\theta \in \{2, 3\}$. Nature chooses θ with probability p that $\theta = 2$. The student chooses an effort level in college, $a_1 \in \{0, 1\}$. After observing a_1, the employer chooses a wage $a_2 \in [0, \infty)$. The student maximizes wage less cost of effort, the latter inversely related to talent:

$$v_1(a_1, a_2, \theta) = a_2 - \frac{ca_1}{\theta} \tag{1}$$

for some $c > 0$. The employer minimizes the expected squared difference between the wage and the student's innate talent.[1]

$$v_2(a_1, a_2, \theta) = -\mathbb{E}(a_2 - \theta)^2 \tag{2}$$

(a) Define a Bayesian extensive game with the information above. Specify the players, set of types, prior on types, player's actions, and utility functions. What are player's strategies and beliefs? Represent it with a graph.

(b) Does the above signaling game have a **separating PBE** where the low type chooses the low action and the high type chooses the high action?

(c) Does the above signaling game have a **separating PBE** where the low type chooses the high action and the high type chooses the low action?

(d) Does the above signaling game have a **pooling PBE** where both types chooses the low action?

(e) Does the above signaling game have a **pooling PBE** where both types chooses the high action?

(f) Does the above signaling game have a **semi-seperating PBE** where one type mixes?

[1]Note that the employer doesn't want to *underpay* the student either, perhaps because the student would then choose an alternative employer.

Solution:

(a) (i) Doubleton set of players, $N = \{1, 2\}$; player 1 is the student, player 2 is the employer.

(ii) Set of types of player 1, $\Theta = \{2, 3\}$.

(iii) Prior on types $p = \Pr(\theta = 2)$.

(iv) Set of actions of player i, $A_1 = \{0, 1\}$ low or high effort. $A_2 = [0, \infty)$ wage.

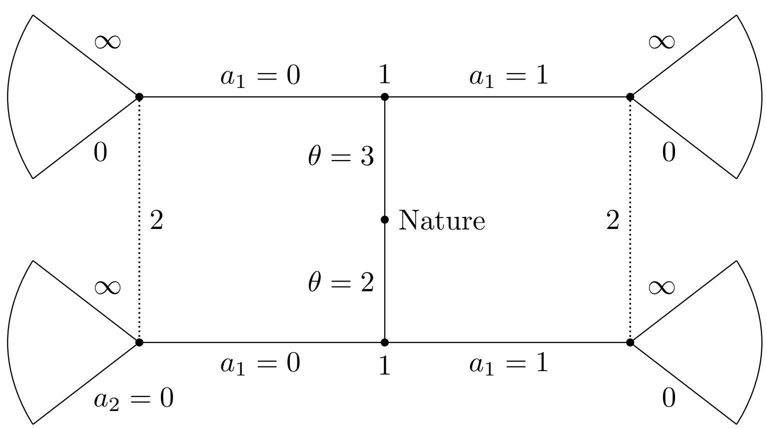
(v) Payoff function of player i, for $c > 0$:

$$v_1(a_1, a_2, \theta) = a_2 - \frac{ca_1}{\theta}$$
$$v_2(a_1, a_2, \theta) = -\mathbb{E}(a_2 - \theta)^2$$

(vi) Strategy of player i,

$$\sigma_1 : \Theta \to \Delta(A_1)$$
$$\sigma_2 : A_1 \to \Delta(A_2)$$

(vii) Player 2's belief on player 1's type $\mu : A_1 \to \Delta(\Theta)$.



(b) One candidate separating PBE is

$$s_1^*(\theta) = \begin{cases} 1 & \text{if } \theta = 3 \\ 0 & \text{if } \theta = 2 \end{cases} \tag{3}$$

and

$$s_2^*(a_1) = \mu(\theta \mid a_1) = \begin{cases} 3 & \text{if } a_1 = 1 \\ 2 & \text{if } a_1 = 0 \end{cases} \tag{4}$$

Given σ_2^*, σ_1^* is optimal if

$$3 - \frac{c}{3} \geq 2 \tag{5}$$

$$\implies c \leq 3 \tag{6}$$

and

$$2 \geq 3 - \frac{c}{2} \tag{7}$$

$$\implies c \geq 2 \tag{8}$$

Thus, this is a valid PBE if $c \in [2, 3]$.

(c) The other potential separating PBE is

$$s_1^*(\theta) = \begin{cases} 0 & \text{if } \theta = 3 \\ 1 & \text{if } \theta = 2 \end{cases} \tag{9}$$

and

$$s_2^*(a_1) = \begin{cases} 2 & \text{if } a_1 = 1 \\ 3 & \text{if } a_1 = 0 \end{cases} \tag{10}$$

The associated restrictions on c are then

$$2 - \frac{c}{3} \leq 3 \tag{11}$$

$$\implies c \geq -3 \tag{12}$$

and

$$3 \leq 2 - \frac{c}{2} \tag{13}$$

$$\implies 2 \leq -c \tag{14}$$

The latter is impossible, so this is not a PBE.

(d) The candidate pooling PBE is $s_1^*(2) = s_1^*(3) = 0$ and

$$s_2^*(a_1) = \begin{cases} 3 - p & \text{if } a_1 = 0 \\ w & \text{otherwise} \end{cases} \tag{15}$$

where $w \in [2,3]$ is such that

$$3 - p \geq w - \frac{c}{2} \tag{16}$$

$$6 + c - 2p \geq 2w \tag{17}$$

$$6 + c - 2p \geq 4 \tag{18}$$

$$2 + c \geq 2p \tag{19}$$

$$c \geq 2p - 2 \tag{20}$$

which is always true, as $2p - 2 \leq 0$. Therefore, this is a valid PBE with belief function

$$\mu(\theta \mid a_1) = \begin{cases} p & \text{if } \theta = 2 \text{ and } a_1 = 0 \\ 1 - p & \text{if } \theta = 3 \text{ and } a_1 = 0 \\ 0 & \text{if } \theta = 2 \text{ and } a_1 = 1 \\ 1 & \text{if } \theta = 3 \text{ and } a_1 = 1 \end{cases} \tag{21}$$

(e) The candidate pooling PBE is $s_1^*(2) = s_1^*(3) = 1$ and

$$s_2^*(a_1) = \begin{cases} 3 - p & \text{if } a_1 = 1 \\ w & \text{otherwise} \end{cases} \tag{22}$$

where $w = 2\mu(2 \mid 0) + 3\mu(3 \mid 0) \in [2,3]$ is such that

$$3 - p - \frac{c}{2} \geq w \tag{23}$$

There exists a PBE if $p + \frac{c}{2} \leq 1$.

(f) Finally, a semi-separating equilibrium will arise if one type mixes with $\alpha \in (0,1)$. This requires that, given the strategy of the employer, this type is indifferent between $a_1 = 0$ and $a_1 = 1$. This could only be the case if the employer pays a c/θ unit higher wage upon observing $a_1 = 1$. Given the employer optimally pays her conditional expectation of θ, we must have

$$\mathbb{E}(\theta \mid 1) = 2[1 - \mu(3 \mid 1)] + 3\mu(3 \mid 1) = \mu(3 \mid 1) + 2 \tag{24}$$

$$= \mathbb{E}(\theta \mid 0) + \frac{c}{\theta} = \mu(3 \mid 0) + 2 + \frac{c}{\theta} \tag{25}$$

implying

$$\mu(3 \mid 1) = \mu(3 \mid 0) + \frac{c}{\theta} \tag{26}$$

If the high-productivity type is mixing, then this becomes

$$1 - \frac{c}{3} = \mu(3 \mid 0) \tag{27}$$

Seeing as the right-hand side is nonnegative, we must have $c \leq 3$. Applying Bayes' rule, we then have

$$\begin{align}
1 - \frac{c}{3} &= \frac{\sigma_1^*(0 \mid 3)p(3)}{\sum_{\theta=2,3} \sigma_1^*(0 \mid \theta)p(\theta)} \tag{28} \\
&= \frac{\alpha(1-p)}{\alpha(1-p)+p} \tag{29} \\
\frac{c}{3} &= \frac{p}{\alpha(1-p)+p} \tag{30} \\
\alpha &= \frac{3-c}{c} \cdot \frac{p}{1-p} \tag{31}
\end{align}$$

It is apparent that higher values of c will lead to the high-productivity type playing $a_1 = 0$ with lower probability. A higher prior probability of θ being 3 will have a similar effect.

If the low-productivity type is mixing, we have

$$\mu(3 \mid 1) = \frac{c}{2} \tag{32}$$

and c must not exceed 2. Then

$$\begin{align}
\frac{c}{2} &= \frac{\sigma_1^*(1 \mid 3)p(3)}{\sum_{\theta=2,3} \sigma_1^*(1 \mid \theta)p(\theta)} \tag{33} \\
&= \frac{1-p}{1-p+\alpha p} \tag{34} \\
\alpha &= \frac{2-c}{c} \cdot \frac{1-p}{p} \tag{35}
\end{align}$$

In this case, higher values of c lead the low-productivity type to choose high effort with lower probability. A higher prior probability of θ being 2 will have a similar effect.