Income Fluctuations Problem

Consider an economy populated by a continuum of households that live forever and have per period preferences

$$u(c_{it}) = \frac{c_{it}^{1-\gamma}}{1-\gamma}$$

with $\gamma = 1.5$ and discount factor $\beta = 0.98$. The household has a savings technology with return r, and initial asset holdings that for simplicity we set equal to zero for all households. Households are constrained in the amount of borrowing by b

$$a_{t+1} \geq -b.$$

Labor income is a two-state Markov chain with state $y_H = 2$ and $y_L = 0$ with transition probabilities $\pi_{HH} = 0.95$ and $\pi_{LL} = 0.4$.

1. Assume an interest rate of $r = 0.01$. Solve for the savings and consumption function of the household assuming no-borrowing is allowed whatsoever ($b = 0$).

The utility maximization problem is

$$\begin{aligned} v(a, y) = \max_c u(c) + \beta \sum \pi(y'|y) v(a', y') \\ \text{s.t.} \quad c + a' = y + (1+r)a, \\ a' \geq -b. \end{aligned}$$

The Euler equation is

$$\frac{\partial u(c)}{\partial c} \geq \beta \sum \pi(y'|y) \frac{\partial u(c')}{\partial c'}(1+r),$$

with equality when the borrowing constraint is not binding.

By combining this with the feasibility condition, we can identify the policy functions for consumption ($c(a, y)$) and savings ($a'(a, y)$).

One way to solve this problem is to apply the VFI algorithm. The key challenge here is carefully handling corner cases where the borrowing constraint binds.

2. Plot the policies in the asset space.

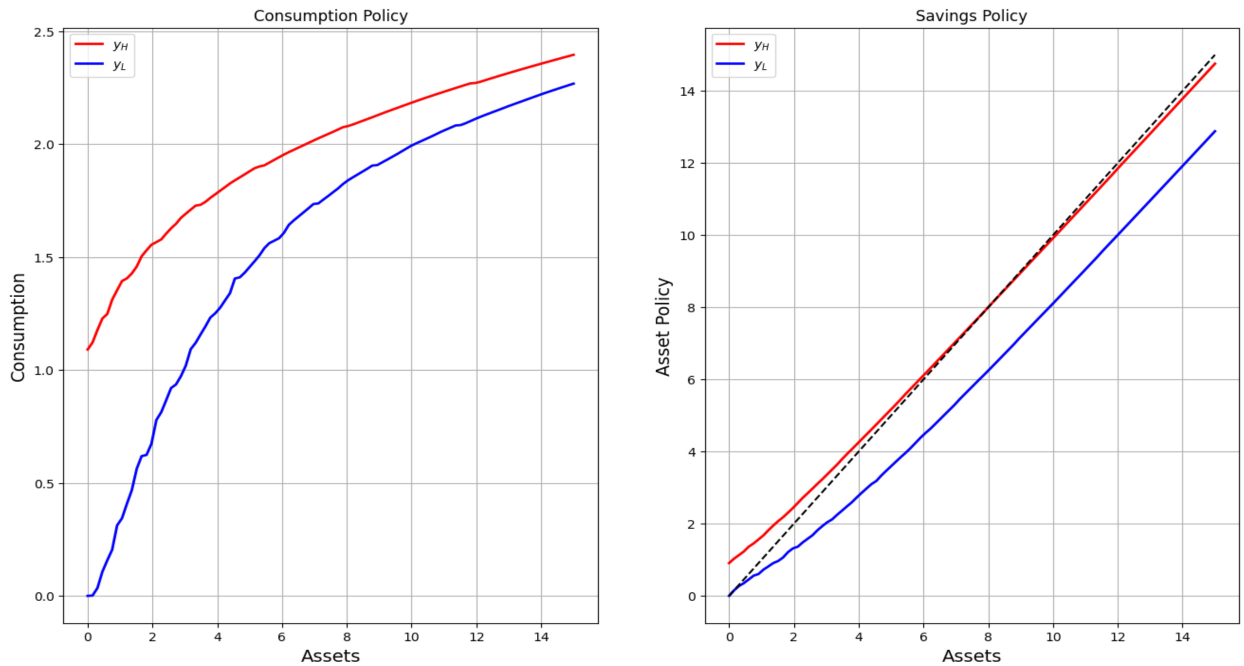

Figure 1: Consumption and savings policies (baseline).

3. Increase the curvature of the preferences to $\gamma = 3$, plot the new policies and explain why they move in such a direction.

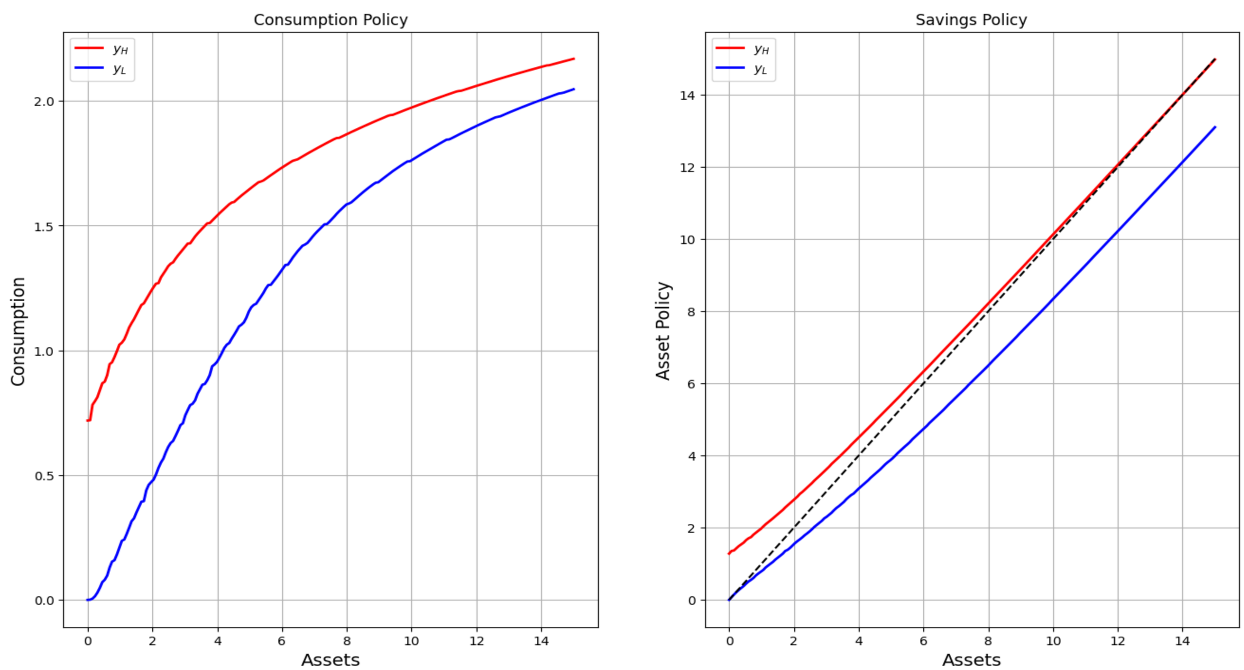

Figure 2: Consumption and savings policies ($\gamma = 3$).

When γ increases, the agent becomes more risk-averse, leading them to save more and consume less. Consequently, the asset policy function (savings) shifts upwards whereas the consumption policy function shifts downwards.

4. Now go back to the baseline framework and set the borrowing constraint $b = 4$. Plot the new policies and explain why they move in such a direction.

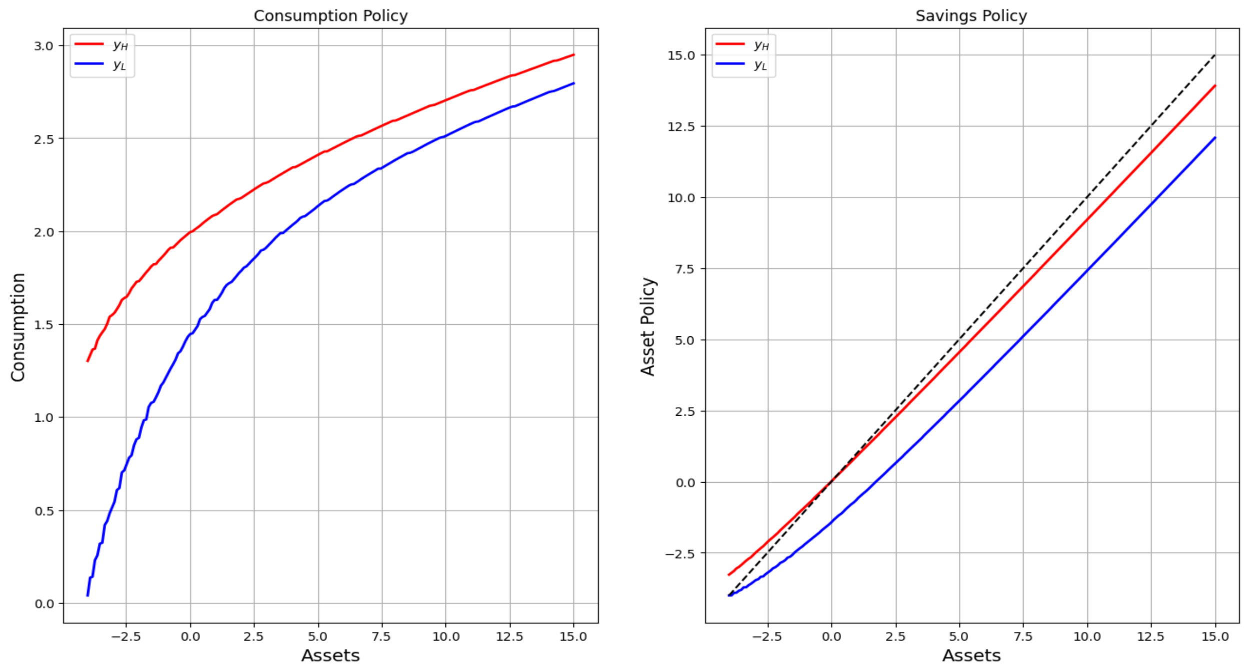

Figure 3: Consumption and savings policies ($b = 4$).

As the agents are now allowed to borrow, they can increase their consumption. Intuitively, consumption level becomes higher as households are allowed to borrow. Correspondingly, households are now willing to hold slightly fewer assets as some of those assets are now negative.

5. Now compute the aggregate assets of the economy in the stationary distribution. Plot the level of aggregate assets for different levels of the interest rate.

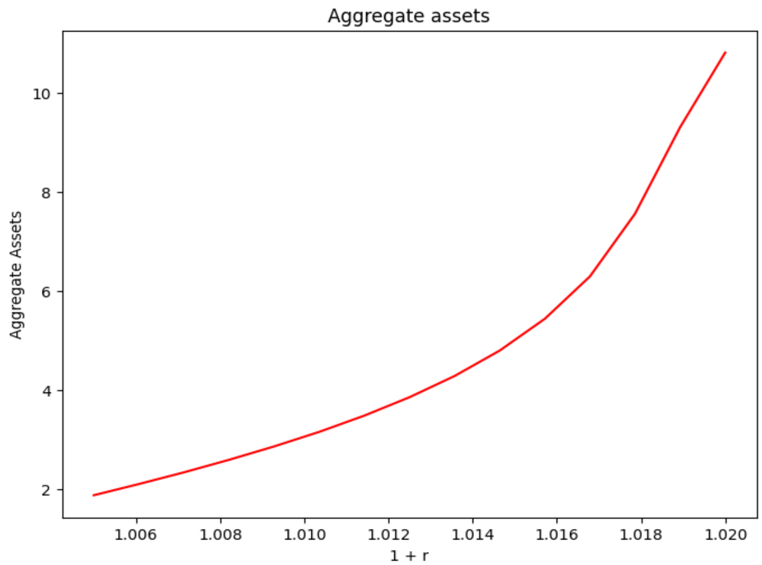

Figure 4: Consumption and savings policies ($b = 4$).

Credit Markets and Economic Growth

Consider an economy where individuals can either supply their labor to firms or operate an individual-specific technology. Individuals in this economy are heterogeneous in terms of their wealth and entrepreneurial ability. However, everyone has the same labor productivity and behaves competitively.

Output is produced using labor (l), capital (k), and entrepreneurial input (e_i):

$$f(e_i, k, l) = e_i^v l^{(1-v)(1-\alpha)} k^{\alpha(1-v)}$$

Individuals' preference is given by

$$\int_0^\infty e^{-\rho t} \frac{c_i(t)^{1-\sigma}}{1-\sigma} dt$$

To simplify, assume that the entrepreneurial ability can either be $e_L = 0$ or $e_H > 0$. Note that an individual's ability does not change over time. The measure of the high-ability individuals is normalized to one, and that of the low-ability ones is π. Assume that π is large enough that high-ability types always choose to operate their technology. Further assume that all the initial capital stock in the economy is owned by the individuals with high ability and is equally distributed among them.

1. Argue that you can compute the competitive equilibrium by solving the planner's problem.

In this setting:

1. Markets are complete.

2. No externalities.

Then, by the First Welfare Theorem, competitive equilibrium allocation is Pareto optimal. Consequently, the competitive equilibrium allocation can be characterized by the solution to the planner's problem[1]:

$$\max_{\{c_i(t), K(t)\}} \int_0^\infty e^{-\rho t} \sum_{i \in \{H,L\}} \mu_i \, \lambda_i \frac{c_i(t)^{1-\sigma}}{1-\sigma} \, dt,$$

subject to the aggregate constraints, where μ_i are Pareto weights and λ_i represent the measures of

[1] I introduced Pareto weights here, but in general, we can assume the planner is utilitarian and values both types equally. In that case, the only relevant weights for aggregate welfare are the population measures of the different household types.

households of each type (since households within each type are identical). Since the planner internalizes all market interactions and constraints, the solution to the SPP yields the same allocations as the decentralized competitive equilibrium, up to the choice of the planner's weights.

2. Show that it is possible to write the aggregate production function as

$$F(K_t) = e_H^v \pi^{(1-v)(1-\alpha)} K_t^{\alpha(1-v)}$$

where K_t is the aggregate capital stock at time t.

Thus, the resource constraint of the planner will be

$$\dot{K}(t) = F(K_t) - \delta K(t) - C(t)$$

where δ is the depreciation rate of capital and $C(t)$ stands for aggregate consumption.

We know that

- high-ability individual always choose to operate their technology,
- low-ability individual always choose to supply labor (because otherwise their output is zero),
- all the initial capital stock in the economy is owned by the individuals with high ability and is equally distributed among them.

Then, aggregate output can be written as

$$Y(t) = \int_0^1 e_i^v \left(\int_0^\pi 1 \, dj \right)^{(1-v)(1-\alpha)} k_i(t)^{\alpha(1-v)} \, di = \int_0^1 e_i^v \, \pi^{(1-v)(1-\alpha)} \, k_i(t)^{\alpha(1-v)} \, di =$$

$$= \pi^{(1-v)(1-\alpha)} \int_0^1 e_i^v \, k_i(t)^{\alpha(1-v)} \, di.$$

Since only high–ability agents operate their technology, we have $e_i = e_H$ for all i in the integral, and by aggregating capital we denote

$$\int_0^1 k_i(t)^{\alpha(1-v)} \, di = K(t)^{\alpha(1-v)}.$$

Thus,

$$Y(t) = F(K(t)) = e_H^v \, \pi^{(1-v)(1-\alpha)} \, K(t)^{\alpha(1-v)}.$$

3. Why is it acceptable to write the resource constraint in terms of aggregate consumption instead of c_H and c_L, the consumption of each ability type household?

Since within each type all individuals are identical

$$C(t) \equiv \int_0^1 c_{iH}(t)di + \int_0^\pi c_{iL}di = c_H(t) + \pi c_L(t).$$

If we consider c_H and c_L separately, the planner's problem is written as:

$$\max_{c_H(t), c_L(t), K(t)} \int_0^\infty e^{-\rho t}\Big(\mu_H \frac{c_H(t)^{1-\sigma}}{1-\sigma} + \mu_L \frac{\pi c_L(t)^{1-\sigma}}{1-\sigma}\Big)dt,$$

subject to

$$\dot{K}(t) = F(K(t)) - \delta K(t) - C(t),$$

$$C(t) = c_H(t) + \pi c_L(t),$$

$$F(K(t)) = e_H^v \, \pi^{(1-v)(1-\alpha)} \, K(t)^{\alpha(1-v)},$$

$$K(0) > 0.$$

Since the utility function is strictly increasing, if we are only interested in aggregate allocation, it is equivalent to rewrite this problem as

$$\max_{C(t), K(t)} \int_0^\infty e^{-\rho t} \frac{C(t)^{1-\sigma}}{1-\sigma} dt,$$

subject to

$$\dot{K}(t) = F(K(t)) - \delta K(t) - C(t),$$

$$F(K(t)) = e_H^v \, \pi^{(1-v)(1-\alpha)} \, K(t)^{\alpha(1-v)},$$

$$K(0) > 0.$$

The allocation of consumption across types in optimum is determined by the FOCs of the first problem and is given by

$$c_H(t) = \Big(\frac{\mu_L \pi}{\mu_H}\Big)^{-\frac{1}{\sigma}} c_L(t).$$

If the planner is utilitarian, Pareto weights just cancel out, so the allocation only depends on the measures of different types of households.

4. Write down the Hamiltonian of the problem and derive the first-order conditions. Then substitute out the co-state variable to obtain a pair of first-order differential equations in $C(t)$ and $K(t)$.

The planner's problem is to maximize

$$\max_{\{C(t),K(t)\}} \int_0^\infty e^{-\rho t} \frac{C(t)^{1-\sigma}}{1-\sigma}\, dt,$$

subject to

$$\dot{K}(t) = F(K(t)) - \delta K(t) - C(t),$$

with

$$F(K(t)) = e_H^v \, \pi^{(1-v)(1-\alpha)} \, K(t)^{\alpha(1-v)},$$

and

$$K(0) > 0.$$

The current-value Hamiltonian is given by

$$\mathcal{H} = \frac{C(t)^{1-\sigma}}{1-\sigma} + \lambda(t)\Big[F(K(t)) - \delta K(t) - C(t)\Big].$$

FOCs:

$$\frac{\partial \mathcal{H}}{\partial C(t)} = C(t)^{-\sigma} - \lambda(t) = 0 \quad \Longrightarrow \quad \lambda(t) = C(t)^{-\sigma}.$$

$$\dot{\lambda}(t) = \rho\,\lambda(t) - \frac{\partial \mathcal{H}}{\partial K(t)} = \rho\,\lambda(t) - \lambda(t)\Big[F'(K(t)) - \delta\Big].$$

TVC:

$$\lim_{t\to\infty} \lambda(t)K(t) = 0$$

Since $\lambda(t) = C(t)^{-\sigma}$, we differentiate to get

$$\dot{\lambda}(t) = -\sigma C(t)^{-\sigma-1}\dot{C}(t).$$

Equating the two expressions for $\dot{\lambda}(t)$:

$$-\sigma C(t)^{-\sigma-1}\dot{C}(t) = C(t)^{-\sigma}\Big[\rho - \big(F'(K(t)) - \delta\big)\Big].$$

Multiplying by $C(t)^{\sigma+1}$ yields

$$-\sigma\dot{C}(t) = C(t)\Big[\rho - \big(F'(K(t)) - \delta\big)\Big].$$

Thus, the equilibrium dynamics are given by the system

$$\begin{aligned}
&\frac{\dot{C}(t)}{C(t)} = \frac{1}{\sigma}\Big(\alpha(1-v)e_H^v\,\pi^{(1-v)(1-\alpha)}\,K(t)^{\alpha(1-v)-1} - \delta - \rho\Big),\\
&\dot{K}(t) = e_H^v\,\pi^{(1-v)(1-\alpha)}\,K(t)^{\alpha(1-v)} - \delta K(t) - C(t),\\
&\lim_{t\to\infty}\lambda(t)K(t) = 0.
\end{aligned}$$

5. If we assume that all the initial capital stock belongs to the low-ability agents, how should the dynamics of aggregate variables change?

Since low-ability agents have $e_L = 0$, they do not operate technology. Under complete markets, transfers or borrowing/lending allow for a reallocation of capital, so high-ability individuals borrow as much capital as they need to operate. Hence, in equilibrium, the aggregate capital stock $K(t)$ in the resource constraint remains the same:

$$\dot{K}(t) = F(K(t)) - \delta K(t) - C(t),$$

with

$$F(K(t)) = e_H^v\,\pi^{(1-v)(1-\alpha)}\,K(t)^{\alpha(1-v)}.$$

Thus, while the distributional dynamics (i.e., individual consumption and wealth) will reflect the initial imbalance, the aggregate variables remain unchanged.