DIFFERENCE EQUATIONS
LECTURE NOTES

First order scalar difference equations. Consider the linear first order difference equation

$$x_t = c + \rho x_{t-1} + \varepsilon_t \tag{0.1}$$

where ε_t is a white noise shock process with mean zero and variance σ^2. The equation (0.1) describes the process for how the variable x_t evolves over time. That is, it is valid for any period t so that

date	Equation
0	$x_0 = c + \rho x_{-1} + \varepsilon_0$
1	$x_1 = c + \rho x_0 + \varepsilon_1$
2	$x_2 = c + \rho x_1 + \varepsilon_2$
$\vdots$	$\vdots$
t	$x_t = c + \rho x_{t-1} + \varepsilon_t$

By recursive substitution, we can write x_t as a function of some initial condition x_0 and the shocks $\varepsilon_t : t > 0$. For period 1 we thus get

$$x_1 = c + \rho x_0 + \varepsilon_1 \tag{0.2}$$

for period 2

$$\begin{aligned} x_2 &= c + \rho\left(c + \rho x_0 + \varepsilon_1\right) + \varepsilon_2 \\ &= c + \rho c + \rho^2 x_0 + \rho\varepsilon_1 + \varepsilon_2, \end{aligned} \tag{0.3}$$

and for period t

$$x_t = c + \rho c + \rho^2 c + ... + \rho^{t-1} c + \rho^t x_0 + \rho^{t-1}\varepsilon_1 + ... + \rho\varepsilon_{t-1} + \varepsilon_t \tag{0.4}$$

$$= c\frac{1-\rho^t}{1-\rho} + \rho^t x_0 + \sum_{s=1}^{t} \rho^{t-s}\varepsilon_s \tag{0.5}$$

When $|\rho| < 1$ the initial condition x_0 becomes irrelevant for large enough t since $\lim_{t\to\infty} \rho^t = 0$.

Impulse response functions (dynamic multipliers). The impulse response function describes the effect on x_{t+s} of a unit change in ε_t. From

$$x_{t+s} = c\frac{1-\rho^s}{1-\rho} + \rho^{s+1} x_{t-1} + \rho^s \varepsilon_t + \rho^{s-1}\varepsilon_{t+1} + ... + \rho\varepsilon_{t+s-1} + \varepsilon_s \tag{0.6}$$

we get that

$$\frac{\partial x_{t+s}}{\partial \varepsilon_t} = \rho^s. \tag{0.7}$$

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Higher order difference equations

A p^{th} order difference equation of the form

$$x_t = c + \rho_1 x_{t-1} + \rho_1 x_{t-2} + \cdots + \rho_p x_{t-p} + \varepsilon_t \tag{0.8}$$

is difficult to manipulate. (Try recursive substitution....). It is often easier to simply rewrite it as an equivalent vector valued first order difference equation using the companion form. That is, define the vectors ξ_t, $\mathbf{c}$ and $\mathbf{v}_t$ and the matrix F in

$$\xi_t = \mathbf{c} + F\xi_{t-1} + \mathbf{v}_t \tag{0.9}$$

as follows

$$\xi_t \equiv \begin{bmatrix} x_t \\ x_{t-1} \\ x_{t-2} \\ \vdots \\ x_{t-p+1} \end{bmatrix}, \mathbf{c} = \begin{bmatrix} c \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{F} \equiv \begin{bmatrix} \rho_1 & \rho_2 & \cdots & \rho_{p-1} & \rho_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & & & 1 & 0 \end{bmatrix}, \mathbf{v}_t \equiv \begin{bmatrix} \varepsilon_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{0.10}$$

The equation then represents the same process as (0.8).

We can now do recursive substitution just like before.

$$\begin{aligned} \xi_t &= \left(I - F^{t-1}\right)(I-F)^{-1}\mathbf{c} + F^t\xi_0 + F^{t-1}\mathbf{v}_1 + ... + F\mathbf{v}_{t-1} + \mathbf{v}_t & (0.11) \\ &= \left(I - F^{t-1}\right)(I-F)^{-1}\mathbf{c} + F^t\xi_0 + \sum_{s=0}^{t} F^{t-s}\mathbf{v}_s & (0.12) \end{aligned}$$

For $\max |eig(F)| < 1$ we can again ignore ξ_0 for large for large enough t.

The impulse response function is now given by

$$\frac{\partial x_{t+s}}{\partial \varepsilon_t} = e_1' F^s e_1 \tag{0.13}$$

where e_i is a vector with a 1 in the i^{th} position and zeros elsewhere.

Variances. The variance of a process x_t is defined as $var(x_t) \equiv E(x_t - \mu)^2$ where μ is the mean (or expected value) of x_t. For a stable process that started infinitely far back in the past it can be computed by noting that

$$x_t = c + \rho c + \rho^2 c + ... + \rho^t c + \rho^t x_0 + \rho^{t-1}\varepsilon_1 + ... + \rho\varepsilon_{t-1} + \varepsilon_t \tag{0.14}$$

so that

$$\mu = \frac{c}{1-\rho} \tag{0.15}$$

and

$$\begin{aligned} E(x_t - \mu)^2 &= \sigma^2\left(\rho^2 + \rho^4 + ...\right) & (0.16) \\ &= \frac{\sigma^2}{1-\rho^2} & (0.17) \end{aligned}$$

An alternative way to find the variance is to define

$$\sigma_x^2 \equiv var(x_t) \tag{0.18}$$

and use that x_{t-1} and ε_t are uncorrelated so that taking variance of both sides of (0.8) gives

$$\sigma_x^2 = \rho^2 \sigma_x^2 + \sigma^2 \tag{0.19}$$

and solving for σ_x^2.

For vector valued processes ξ_t one cannot solve directly for the covariance, but again defining $\Sigma_\xi \equiv cov\,(\xi_t, \xi_t)$ and using that ξ_{t-1} and $\mathbf{v}_t$ are uncorrelated, we get

$$\Sigma_\xi = F \Sigma_\xi F' + E\,(\mathbf{v}_t \mathbf{v}_t') \tag{0.20}$$

which can be solved by function iteration if $\max |eig\,(F)| < 1$. (Alternatively, one can use the Matlab command `dlyap`.)