Representations

Let G be a MLG. A finite dimensional complex vector space V is a $\boxed{G\text{-module}}$ if there is a continuous action of G on V such that

- $g \triangleright (\alpha|\psi\rangle + \beta|\varphi\rangle) = \alpha(g\triangleright|\psi\rangle) + \beta(g\triangleright|\varphi\rangle)$
- $g \triangleright h \triangleright |\psi\rangle = (gh)\triangleright|\psi\rangle$

the map $G\times V \to V$ is continuous
$(g,|\psi\rangle) \mapsto g\triangleright|\psi\rangle$

($g\triangleright|\psi\rangle$ denotes the action of g on $|\psi\rangle$)

Note: if V is infinite-dimensional we have to be careful about its topology!

Note 2: The action of G on V is also called a representation of G on V (same thing, different point of view)

example: Let $V_n = \text{span}\{|n\rangle\}$ be a one-dimensional vector space with the action of $e^{i\theta}\in U(1)$ on $|n\rangle$ defined by

$$e^{i\theta}\triangleright|n\rangle = (e^{i\theta})^n|n\rangle = e^{in\theta}|n\rangle \qquad \text{for some fixed } n\in\mathbb{Z}$$

The action is extended to the other vectors by linearity. → $e^{i\theta}\triangleright\alpha|n\rangle$ is defined to be $\alpha(e^{i\theta}\triangleright|n\rangle)$

When $z,w\in U(1)$ we get

$$z\triangleright w\triangleright|n\rangle = z\triangleright w^n|n\rangle = z^n w^n|n\rangle = (zw)^n|n\rangle$$

→ only works because $n\in\mathbb{Z}$!

example: Let $V_n = \text{span}\{|n\rangle\}$ as before, with $n\in\mathbb{Z}$

We can extend the action defined for $U(1)$ to $\mathbb{C}\setminus\{0\} = GL(1,\mathbb{C})$ by defining

$$z\triangleright|n\rangle = z^n|n\rangle$$

- A G-module V is unitary if it has an inner product and G acts unitarily, that is $\langle\varphi|g\triangleright|\psi\rangle = \langle\psi|g^{-1}\triangleright|\varphi\rangle \quad \forall\, |\psi\rangle,|\varphi\rangle\in V$

→ with abuse of notation, we can say "$g^\dagger = g^{-1}$"

example: Define an inner product on V_n as $\langle n|n\rangle = 1$.

- For the action of $U(1)$ we get $\overline{\langle n| e^{i\theta} \triangleright |n\rangle} = \overline{\langle n| e^{in\theta} |n\rangle} = \overline{e^{in\theta}} \langle n|n\rangle = e^{-in\theta}$

  and $\langle n| (e^{i\theta})^{-1} \triangleright |n\rangle = \langle n| e^{-i\theta} \triangleright |n\rangle = \langle n| e^{-in\theta} |n\rangle = e^{-in\theta}$

  $\Rightarrow$ V_n is unitary

- For the action of $GL(1,\mathbb{C})$ we get $\langle n| z^{-1} \triangleright |n\rangle = z^{-n}$ (same steps as above)

  but $\overline{\langle n| z \triangleright |n\rangle} = \overline{\langle n| z^n |n\rangle} = \bar{z}^n \neq z^{-n}$ in general (unless $n=0$)

  $\Rightarrow$ V_n is not unitary

Lie algebras

Let $\mathfrak{g}$ be a Lie algebra. A finite-dimensional complex vector space V is a $\mathfrak{g}$-module if there is an action of $\mathfrak{g}$ on V such that

- $X \triangleright (\alpha|\psi\rangle + \beta|\varphi\rangle) = \alpha(X \triangleright |\psi\rangle) + \beta(X \triangleright |\varphi\rangle)$
- $(\alpha X + \beta Y) \triangleright |\psi\rangle = \alpha(X \triangleright |\psi\rangle) + \beta(Y \triangleright |\psi\rangle)$
- $[X,Y] \triangleright |\psi\rangle = X \triangleright Y \triangleright |\psi\rangle - Y \triangleright X \triangleright |\psi\rangle$

- V is unitary if it has an inner product and

  $\overline{\langle \varphi | X \triangleright |\psi\rangle} = - \langle \psi | X \triangleright |\varphi\rangle$ " $X^\dagger = -X$ " (action is anti-hermitian)

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example: $su(2) = \{ X \in M_n(\mathbb{C}) \mid X^* = -X, tr(X) = 0 \} = \text{span}\{X_1, X_2, X_3\}$

where $X_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ $X_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $X_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

are a basis. Since $[\cdot,\cdot]$ is bi-linear, we only need to know

Symmetries in Quantum Mechanics

Wigner's theorem. If $T: H \to H$ is an invertible transformation of an Hilbert space into itself that preserves transition amplitudes

$$\frac{|\langle T(\psi), T(\varphi)\rangle|^2}{\|T(\psi)\|^2\,\|T(\varphi)\|^2} = \frac{|\langle \psi, \varphi\rangle|^2}{\|\psi\|^2\,\|\varphi\|^2} \quad \text{for all } \psi, \varphi \in H$$

(bra-ket notation doesn't work here!)

then one of the following happens:

- T is linear and unitary (up to a multiplicative constant)
- T is anti-linear and anti-unitary (up to a multiplicative constant)

Since symmetries should (at the very least!) preserve transition amplitudes, then they should act a unitary or anti-unitary operators.

Since the identity is unitary, if G is a connected group of symmetries, by continuity G must act <u>unitarily</u>

(anti-unitary ones are used for time reversal)

This sure sounds like unitary representations!

Back to representations

if V is a G-module, we can make it into a $\mathrm{Lie}(G)$-module by defining

$$X \triangleright |\psi\rangle = \frac{d}{d\varepsilon}\left(e^{\varepsilon X} \triangleright |\psi\rangle\right)\Big|_{\varepsilon=0} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\Big(e^{\varepsilon X} \triangleright |\psi\rangle - \underbrace{e^{0X}}_{\mathbb{1}} \triangleright |\psi\rangle\Big)$$

$X \in \mathrm{Lie}(G)$

$X = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} e^{\varepsilon X}$ $\qquad e^{\varepsilon X} \in G$ $\qquad \boxed{e^{\varepsilon X}\,|\psi\rangle}$

example: if $U(1)$ acts on $V_n = \text{span}\{|n\rangle\}$ as $e^{i\theta} \triangleright |n\rangle = e^{in\theta}|n\rangle$

$n \in \mathbb{Z}$ fixed

and $i = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} e^{i\varepsilon} \in \underline{u}(1)$, then

$\underline{u}(1) = \text{span}\{i\} = \{i\theta \mid \theta \in \mathbb{R}\}$

$\underline{u}(1)$

$$i \triangleright |n\rangle = \frac{d}{d\varepsilon} e^{i\varepsilon} \triangleright |n\rangle \Big|_{\varepsilon=0} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\Big(\underbrace{e^{i\varepsilon}\triangleright|n\rangle}_{e^{in\varepsilon}|n\rangle} - \underbrace{e^{i0}\triangleright|n\rangle}_{|n\rangle}\Big) = \boxed{\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(e^{in\varepsilon}-1\right)}|n\rangle$$

$\frac{d}{d\varepsilon} e^{in\varepsilon}\Big|_{\varepsilon=0}$

$$= in\,|n\rangle$$

$\in \mathbb{C}$

which is indeed an action of $\boxed{\underline{u}(1)}$ (check it!)

Note that the action of $\underline{u}(1)$ is $\boxed{\text{anti-hermitian}} \to \boxed{\text{unitary } \underline{u}(1)\text{-module}}$

Does the converse work? In other words, do all the representations of $\text{Lie}(G)$ come from representations of G?

Let's find out!

1-dim.

- $V_\alpha = \text{span}\{|\alpha\rangle\}$ $\alpha \in \mathbb{R}$ with $\underline{u}(1)$ acting as $i \triangleright |\alpha\rangle = i\alpha\,|\alpha\rangle$

$\in \underline{u}(1)$, $\in \mathbb{C}$

This is a unitary $\underline{u}(1)$ module, since

$$\overline{\langle\alpha| i \triangleright |\alpha\rangle} = \overline{i\alpha\langle\alpha|\alpha\rangle} = -i\alpha\langle\alpha|\alpha\rangle = -\langle\alpha| i \triangleright|\alpha\rangle$$

anti-hermitian

Can we say that $e^{i\theta}\triangleright|\alpha\rangle =$ "$\exp(i\theta \triangleright |\alpha\rangle)$" ?

reverse of $X \triangleright |n\rangle = \frac{d}{d\varepsilon} e^{\varepsilon X}\triangleright|n\rangle\Big|_{\varepsilon=0}$

Suppose we say that $e^{i\theta}\triangleright|\alpha\rangle = \sum_{n=1}^{\infty} \frac{(i\theta)^n}{n!} \triangleright |\alpha\rangle$

$(i\theta)^2 \triangleright |\alpha\rangle \equiv i\theta \triangleright i\theta \triangleright |\alpha\rangle$

$\boxed{i\theta \triangleright |\alpha\rangle = i\alpha\theta|\alpha\rangle}$

$(i\theta)^2 \triangleright |\alpha\rangle = i\theta \triangleright \boxed{i\alpha\theta|\alpha\rangle} = (i\alpha\theta)^2|\alpha\rangle$ etc.

$$\to \sum_{n=1}^{\infty}\frac{(i\theta)^n}{n!}\triangleright|\alpha\rangle = \underbrace{\left(\sum_{n=1}^{\infty}\frac{(i\alpha\theta)^n}{n!}\right)}_{\in\mathbb{C}}|\alpha\rangle = \boxed{e^{i\theta\alpha}|\alpha\rangle}$$

$$e^{i\theta}\triangleright|\alpha\rangle \overset{?}{=} e^{i\theta\alpha}|\alpha\rangle$$

everything seems fine, but there's a problem: the choice of θ is ambiguous! $\quad e^{i\theta} = e^{i\theta + i2\pi k} \quad \forall k \in \mathbb{Z}$

standard choice in complex analysis

We need to make a choice, say $\boxed{\theta \in (-\pi, \pi]}$

$\rightarrow$ if $\boxed{z \in U(1)}$, $\boxed{z \triangleright |\alpha\rangle = e^{i\alpha \operatorname{Arg}(z)} |\alpha\rangle}$ $\rightarrow$ principal argument, returns something in $(-\pi, \pi]$ with $z = |z| e^{i \operatorname{Arg}(z)}$

$\operatorname{Arg}(e^{i3\pi}) = \pi$

All seems well. Is this a $U(1)$-module?

- linear ✓
- $z \triangleright w \triangleright |\alpha\rangle = z \triangleright e^{i\alpha \operatorname{Arg}(w)} |\alpha\rangle = e^{i\alpha \operatorname{Arg}(z)} e^{i\alpha \operatorname{Arg}(w)} |\alpha\rangle = e^{i\alpha(\operatorname{Arg}(z) + \operatorname{Arg}(w))} |\alpha\rangle$

but in general $\operatorname{Arg}(z) + \operatorname{Arg}(w) = \operatorname{Arg}(z w) + 2\pi k \quad , \; k \in \{-1, 0, 1\}$ depending on what z and w are.

for example, $e^{i\pi} \triangleright e^{i\pi} \triangleright |\alpha\rangle = e^{i\alpha 2\pi} |\alpha\rangle \overset{?}{\underset{\shortparallel}{=}} (e^{i\pi} e^{i\pi}) \triangleright |\alpha\rangle \overset{?}{=} |\alpha\rangle$

$$e^{i\alpha(\operatorname{Arg}(z) + \operatorname{Arg}(w))} = e^{i\alpha \operatorname{Arg}(z w)} \underbrace{e^{i2\pi\alpha k}}_{\neq 1 \text{ if } k \neq 0,\ \alpha \notin \mathbb{Z}}$$

for example if $\alpha = 1/2$, $e^{i\pi} \triangleright e^{i\pi} \triangleright |1/2\rangle = e^{i\pi} |1/2\rangle = -|1/2\rangle \neq (e^{i\pi} e^{i\pi}) \triangleright |1/2\rangle = |1/2\rangle$

Does this mean that things are broken?

No!

Remember that physical states are only defined up to a non-zero scalar $\left(|\psi\rangle \sim \lambda |\psi\rangle \quad \text{if } \lambda \neq 0 \right)$

$\rightarrow \quad [e^{i\pi} \triangleright e^{i\pi} \triangleright |1/2\rangle] = [-|1/2\rangle] = [|1/2\rangle] = [e^{i2\pi} \triangleright |1/2\rangle]$

and in general $\quad [z \triangleright w \triangleright |\alpha\rangle] = [(zw) \triangleright |\alpha\rangle]$

$\leftarrow$ projective representation

in general $\quad g \triangleright h \triangleright |\psi\rangle = e^{i\omega(g,h)} (gh) \triangleright |\psi\rangle$ projective rep.

$\rightarrow$ still describes a symmetry

Nice things that happen: (assume V is finite-dimensional)

- projective reps of G are all obtained by "exponentiating" (pure) reps of $\mathrm{Lie}(G)$ (G connected)
    - pure reps of $\mathrm{Lie}(G)$ instead of proj. reps of G
    - reps of $\mathrm{Lie}(G)$ are easier to find than reps of G

- projective reps of G are in "one-to-one" correspondence with pure reps of $\tilde{G}$ (simply connected cover of G)

universal cover

$\hookrightarrow$ topological concept

$$\widetilde{SO(3)} = SU(2)$$

$\rightarrow$ instead of proj reps of $SO(3)$, we can look at reps of $SU(2)$

pure reps of $SO(3)$ labelled by $j = 0, 1, 2, 3, \ldots$

proj reps of $SO(3)$ $\quad j = 0, 1/2, 1, 3/2, 2, \ldots$

$$\left.\begin{aligned} J_z |jm\rangle &= m |jm\rangle \\ J_\pm |jm\rangle &= C_\pm(j,m) |jm\pm 1\rangle \end{aligned}\right\} \text{rep. of } \underline{so}(3)$$

$$\boxed{\text{pure reps of } SU(2) \text{ labelled by } j = 0, 1/2, 1, 3/2, \ldots}$$

$\longleftrightarrow$

pure reps of $\underline{so}(3) \cong \underline{su}(2)$

$j = 0, 1/2, 1, 3/2, \ldots$