Today:
- Matrix Lie groups
- Lie algebras
- Representations

Next time
- symmetries in QM
- role of representation theory

Good reference: Lie Groups, Lie Algebras, and Representations
Brian C. Hall

Def: A matrix Lie group is a closed subgroup $G \leq GL(n,\mathbb{C})$ for some $n \in \mathbb{N}$ (closed w.r.t. the topology induced from $M_n(\mathbb{C})$) $\rightarrow$ using operator norm

$$\|A\| = \sup\left\{ \frac{\|Ax\|}{\|x\|} \,\middle|\, x \in \mathbb{C}^n \setminus \{0\} \right\}$$

or use the norm from $\mathbb{C}^{n^2}$

$$GL(n,\mathbb{C}) = \{ A \in \underbrace{M_n(\mathbb{C})}_{n\times n \text{ matrices}} \mid \det A \neq 0 \}$$

examples

- $GL(n,\mathbb{C})$ general linear group over $\mathbb{C}$
- $SL(n,\mathbb{C}) = \{ A \in GL(n,\mathbb{C}) \mid \det A = 1 \}$ special linear group over $\mathbb{C}$
- $GL(n,\mathbb{R}) = \{ A \in GL(n,\mathbb{C}) \mid \bar{A} - A = 0 \}$ general linear group over $\mathbb{R}$
- $SL(n,\mathbb{R}) = \{ A \in GL(n,\mathbb{R}) \mid \det A = 1 \}$ special linear group over $\mathbb{R}$
- $O(n) = \{ A \in GL(n,\mathbb{R}) \mid A^t A = \mathbb{1} \}$ orthogonal group
- $SO(n) = \{ A \in O(n) \mid \det A = 1 \}$ special orthogonal group
- $U(n) = \{ A \in GL(n,\mathbb{C}) \mid A^* A = \mathbb{1} \}$ unitary group
- $SU(n) = \{ A \in U(n) \mid \det A = 1 \}$ special unitary group

real Lie groups despite having complex matrices!

Some familiar examples:

- $U(1) = \{ z \in \mathbb{C} \setminus \{0\} \mid |z|^2 = 1 \}$ circle group
- $SO(2) = \left\{ \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} \,\middle|\, \vartheta \in [0,2\pi) \right\}$ 2D rotation group

$\Rightarrow \{ A \in M_2(\mathbb{R}) \mid A^t A = \mathbb{1},\ \det A = 1 \}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1,\ \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Def: A real or complex Lie algebre is a vector space V over $\mathbb{R}$ or $\mathbb{C}$ with an operation $[\cdot,\cdot]: V \times V \to V$ (Lie bracket) satisfying

- $[\alpha x + \beta y, z] = \alpha [x,z] + \beta [y,z]$
- $[z, \alpha x + \beta y] = \alpha [z,x] + \beta [z,y]$

$\Big\}$ compatibility

- $[y,x] = -[x,y]$
- $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ Jacobi identity

Prototype: $M_n(\mathbb{C})$ with $[X,Y] = XY - YX$

Note: $[\cdot,\cdot]$ not associative

Matrix Lie Group

Def: Let $G \leq GL(n,\mathbb{C})$ be a MLG. The associated Lie algebre is the set $\underline{Lie(G)} = \{X \in M_n(\mathbb{C}) \mid e^{\varepsilon X} \in G \ \forall \varepsilon \in \mathbb{R}\}$ with Lie bracket $[X,Y] = XY - YX$

↳ matrix exponential $e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!}$

Note: $Lie(G)$ is usually denoted by $\mathfrak{g}$ → you can use $\underline{\mathfrak{g}}$

proof can be found in "extra" folder on google drive

$\underline{o}(n) = \underline{so}(n)$?

$\underline{o}(n) = \{X \in M_n(\mathbb{R}) \mid X^T = -X\}$

$\underline{so}(n) = \{X \in \underline{o}(n) \mid tr\,X = 0\}$

$X^T = -X \Rightarrow tr\,X = 0$

$tr\,X^T = tr\,X$
$\quad\shortparallel$
$tr(-X) = -tr\,X$

$O(n) \neq SO(n)$

but their Lie algebres are the same!

Some more notions about MLGs:

- $G \leq GL(n, \mathbb{C})$ is called compact if it is closed and bounded (w.r.t. to $M_n(\mathbb{C})$)

  $\rightarrow$ $O(n)$, $SO(n)$, $U(n)$, $SU(n)$ are compact

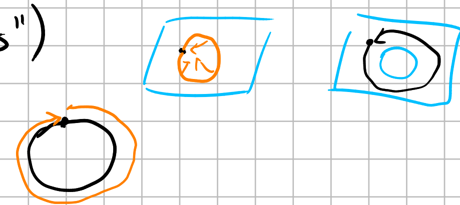
  Heine-Borel theorem

- G connected if for every $g \in G$ there is a continuous path connecting it to the identity ($\gamma: [0,1] \rightarrow G$ with $\gamma(0) = e$, $\gamma(1) = g$)

  $\rightarrow$ $SO(n)$ is connected, $O(n)$ is not

- G simply connected if connected and every loop ($\gamma: [0,1] \rightarrow G$ with $\gamma(0) = \gamma(1)$) can be shrunk to a point ("no holes")

  $\rightarrow$ $U(1)$ is not simply connected



  $(e^{\varepsilon X})^{-1} = e^{-\varepsilon X}$

- if $X \in Lie(G)$ then $\{e^{\varepsilon X} \mid \varepsilon \in \mathbb{R}\}$ is a one-parameter subgroup of G.

  In fact, every (smooth) one-parameter subgroup of G is of this form.

  $\rightarrow$ X generates the subgroup $\{e^{\varepsilon X} \mid \varepsilon \in \mathbb{R}\}$

- if G is connected, every group element $g \in G$ can be written as

  $$g = e^{X_1} e^{X_2} \cdots e^{X_k} \quad \text{with } X_i \in Lie(G)$$

  $\rightarrow$ G is generated by $Lie(G)$

  $O(n)$ is not connected but $SO(n) \leq O(n)$ is

  $\mathfrak{so}(n)$ generates $SO(n)$ || $\mathfrak{o}(n)$ generates $SO(n)$

- MLG homomorphism: continuous group homomorphism $\varphi: G \rightarrow H$

- MLG isomorphism: group isomorphism $\varphi: G \rightarrow H$ with φ and φ^{-1} continuous

ex: $\varphi: z \in U(1) \mapsto \begin{pmatrix} Re(z) & Im(z) \\ -Im(z) & Re(z) \end{pmatrix} \in SO(2)$ isomorphism