Today: • symmetries of an action
• Noether's theorem

"Extra" folder: I will put additional notes in here, in case you are interested. Feel free to make suggestions!

Symmetry of an action

$L(q, \dot{q}, t)$ Lagrangian

action $S[q] = \int_a^b L(q(t), \dot{q}(t), t)\, dt$

symmetry: transform $\underbrace{\text{something}}_{\Downarrow \; t,\, q}$ → keep action invariant

$T_0 = id \quad T_{-\varepsilon} = T_\varepsilon^{-1} \quad T_{\varepsilon+\varepsilon'} = T_\varepsilon \circ T_{\varepsilon'}$

→ one-parameter subgroup

continuous (smooth) symmetries

$\tilde{t} = T_\varepsilon(t) \qquad \tilde{q}(\tilde{t}) = Q_\varepsilon(q(t)) = Q_\varepsilon(q(T_{-\varepsilon}(\tilde{t})))$

new time (at $\tilde{t}$) old time (at t)

how does S change?

$\frac{d\tilde{q}}{d\tilde{t}}$

$$\tilde{S}[\tilde{q}] = \int_{T_\varepsilon(a)}^{T_\varepsilon(b)} d\tilde{t}\, L(\tilde{q}(\tilde{t}), \dot{\tilde{q}}(\tilde{t}), \tilde{t})$$

S invariant if $\underbrace{\tilde{S}[\tilde{q}] = S[q]}$ → we have a symmetry

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there is also a notion of quasi-symmetry

Noether's theorem (1918)

Suppose $\tilde{t} = T_\varepsilon(t)$ $\quad \tilde{q}(\tilde{t}) = Q_\varepsilon(q(t))$ is a symmetry of

$$S[q] = \int_a^b L(q(t), \dot{q}(t), t)\, dt \qquad \big(\tilde{S}[\tilde{q}] = S[q] \big)$$

↳ for all a, b

for all q's
→ not just solutions to EL eqs.

Then if we denote

$$\delta t = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} T_\varepsilon \qquad \delta q = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} Q_\varepsilon$$

the quantity $\left[L(q(t), \dot{q}(t), t)\, \delta t + \frac{\partial L}{\partial \dot{q}} (\delta q - \dot{q}\, \delta t) \right]$ is

conserved if $\frac{\partial L}{\partial q} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}}$ (on physical trajectories)

↳ constant in time

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ex: $L = L(q, \dot{q})$ time independent

$\tilde{t} = T_\varepsilon(t) = t + \varepsilon$ (time translation)
$\tilde{q}(\tilde{t}) = q(t) \quad \rightarrow Q_\varepsilon = \mathrm{id}$

} → 1-parameter subgroups

$$\tilde{S}[\tilde{q}] = \int_{a+\varepsilon}^{b+\varepsilon} L(\underbrace{\tilde{q}(\tilde{t})}_{q(t)}, \underbrace{\dot{\tilde{q}}(\tilde{t})}_{\dot{q}(t)})\, d\tilde{t}$$

$\leftarrow \frac{d\tilde{t}}{dt} dt \text{ "=" } d\tilde{t}$

$\rightarrow \frac{d}{d\tilde{t}} = \frac{d}{dt}$

$$= \int_a^b L(q(t), \dot{q}(t))\, dt = S[q]$$

$\delta t = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} T_\varepsilon(t) = 1$

$\delta q = 0$

→ symmetry!

conserved: $L(q, \dot{q})\, \overset{=1}{\delta t} + \frac{\partial L}{\partial \dot{q}} (\overset{=0}{\delta q} - \dot{q}\, \overset{=1}{\delta t})$

$\rightarrow \boxed{L(q, \dot{q}) - \frac{\partial L}{\partial \dot{q}} \dot{q}}$ – energy

Proof of Noether's theorem

$\tilde t = T_\varepsilon(t)$ $\qquad$ $\tilde q(\tilde t) = Q_\varepsilon(q(t)) = Q_\varepsilon(q(T_{-\varepsilon}(\tilde t)))$

with T_ε and Q_ε smooth one-parameter subgroups ($T_0 = id$ $\quad T_{-\varepsilon} = T_\varepsilon^{-1}$ $\quad T_{\varepsilon+\varepsilon'} = T_\varepsilon \circ T_{\varepsilon'}$)

→ i.e. $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} T_\varepsilon$ makes sense

$$S[q] = \int_a^b L(q(t), \dot q(t), t)$$

$$\tilde S[\tilde q] = \int_{T_\varepsilon(a)}^{T_\varepsilon(b)} L(\tilde q(\tilde t), \overset{\frac{d\tilde q}{d\tilde t}}{\dot{\tilde q}(\tilde t)}, \tilde t)\, d\tilde t = \int_a^b \frac{dT_\varepsilon}{dt} L(\tilde q(T_\varepsilon(t)), \dot{\tilde q}(T_\varepsilon(t)), T_\varepsilon(t))\, dt$$

suppose that $\tilde S[\tilde q] = S[q]$ for all paths (even those that are not solutions of E-L eqs.) and for all choices of a, b.

Then $\underbrace{\delta S := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \tilde S[\tilde q] = 0}$ → can use this as assumptio

Some notation:

- $\delta t := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} T_\varepsilon$ $\quad$ (function of t)
- $\delta q := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} Q_\varepsilon$ $\quad$ (function of q)

generators of T_ε and Q_ε (see Lie algebras later)

Let's apply $D := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}$ everywhere we can!

double chain rule

- $D\frac{dT_\varepsilon}{dt} = \frac{d}{dt} DT_\varepsilon = \frac{d}{dt}\delta t$
- $D\tilde q(\tilde t) = DQ_\varepsilon(q(t)) = \delta q(q(t))$
- $\frac{d\tilde q}{d\tilde t}(\tilde t) = \frac{d}{d\tilde t} Q_\varepsilon(q(t)) = \frac{\partial Q_\varepsilon}{\partial q}(q(t))\, \dot q(t) \frac{dt}{d\tilde t}(\tilde t)$

→ $= \dfrac{1}{\frac{d\tilde t}{dt}(t)} = \dfrac{1}{\frac{dT_\varepsilon}{dt}(t)}$ ← derive of inverse function

$$\Rightarrow \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \dot{\tilde q}(\tilde t) = \frac{\partial}{\partial q} \underbrace{DQ_\varepsilon(q(t))}\, \dot q(t) \frac{1}{\underbrace{\frac{dT_0}{dt}}_{=1}} + \underbrace{\frac{\partial}{\partial q} Q_0(q(t))}_{=1}\, \dot q(t)\, D\frac{1}{\frac{dT_\varepsilon}{dt}}$$

$\varepsilon = 0$

→ $-\frac{d}{dt}\delta t$ → chain rule

$$\Rightarrow D\dot{\tilde q}(\tilde t) = \frac{\partial}{\partial q}\delta q\, \dot q(t) - \dot q(t)\frac{d}{dt}\delta t = \frac{d}{dt}\delta q - \dot q(t)\frac{d}{dt}\delta t$$

Now suppose that q satisfies $\boxed{\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q} = 0}$ EL eqs

$$\underbrace{0 = D\tilde S[\tilde q]}_{\delta S = 0} = \int_a^b dt\, D\left[\frac{dT_\varepsilon}{dt} L(\tilde q(\tilde t), \dot{\tilde q}(\tilde t), \tilde t)\right]$$

$$\swarrow D\frac{dT_\varepsilon}{dt}$$

$$= \int_a^b dt \left\{ \frac{d\delta t}{dt} L(\underbrace{q,\dot q,t}_{\varepsilon=0}) + \underbrace{\frac{dT_0}{dt}}_{=1} DL(\tilde q(\tilde t), \dot{\tilde q}(\tilde t), \tilde t)\right\}$$

$$DL(\tilde q(\tilde t), \dot{\tilde q}(\tilde t), \tilde t) = \frac{\partial L}{\partial t} D\tilde t + \frac{\partial L}{\partial q} D\tilde q(\tilde t) + \frac{\partial L}{\partial \dot q} D\dot{\tilde q}(\tilde t)$$ chain rule again

$$A\frac{dB}{dt} = \frac{d}{dt}(AB) - B\frac{dA}{dt}$$

$$(*) = \frac{\partial L}{\partial t}\delta t + \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\left(\frac{d}{dt}\delta q - \dot q \frac{d}{dt}\delta t\right)$$

$$= \frac{\partial L}{\partial t}\delta t + \frac{\partial L}{\partial q}\delta q + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\delta q\right) - \delta q \frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\dot q\,\delta t\right) + \delta t\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\dot q\right)$$

$$= \frac{d}{dt}\left[\frac{\partial L}{\partial \dot q}(\delta q - \dot q\,\delta t)\right] + \frac{\partial L}{\partial t}\delta t + \delta t\left(\frac{d}{dt}\frac{\partial L}{\partial \dot q}\right)\dot q + \delta t\frac{\partial L}{\partial \dot q}\ddot q$$

EL

$$= \frac{d}{dt}\left[\frac{\partial L}{\partial \dot q}(\delta q - \dot q\,\delta t)\right] + \delta t\left[\underbrace{\frac{\partial L}{\partial t} + \frac{\partial L}{\partial q}\dot q + \frac{\partial L}{\partial \dot q}\ddot q}\right]$$

$$= \frac{d}{dt}\left[\frac{\partial L}{\partial \dot q}(\delta q - \dot q\,\delta t)\right] + \delta t\frac{d}{dt}L(q(t),\dot q(t),t)$$

δS

$$\Rightarrow 0 = \int_a^b \left\{\underbrace{L(q(t),\dot q(t),t)\frac{d}{dt}\delta t + \delta t\frac{d}{dt}L(q(t),\dot q(t),t)}_{\text{product rule}} + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot q}(\delta q - \dot q\,\delta t)\right]\right\}dt$$

$$= \int_a^b \frac{d}{dt}\left[L\,\delta t + \frac{\partial L}{\partial \dot q}(\delta q - \dot q\,\delta t)\right]dt$$ zero for all a, b

$$\Rightarrow 0 = \frac{d}{dt}\left[L\,\delta t + \frac{\partial L}{\partial \dot q}(\delta q - \dot q\,\delta t)\right] \quad \text{or} \quad \boxed{L\,\delta t + \frac{\partial L}{\partial \dot q}(\delta q - \dot q\,\delta t)}$$ conserved

Note: for this to work we need to check that $\tilde S[\tilde q] = S[q]$ or $D\tilde S[\tilde q] = 0$ without using E-L eqs!