

Noether's theorem for a scalar field in flat space

Note: Einstein's implicit sum notation is used here.

Def: The smooth one-parameter subgroup of transformations

$$\tilde{x}^\mu = X_\epsilon^\mu(x) \quad \tilde{\phi}(\tilde{x}) = F_\epsilon(\phi(x))$$

is an infinitesimal symmetry of the action $S[\phi] = \int_{U \subseteq \mathbb{R}^n} d^n x \mathcal{L}(\phi(x), \partial\phi(x), x)$

$$\text{if } \delta S = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \tilde{S}[\tilde{\phi}] = 0 \text{ for all } \phi(x) \quad \left(\tilde{S}[\tilde{\phi}] = \int_{X_\epsilon(U)} d^n \tilde{x} \mathcal{L}(\tilde{\phi}(\tilde{x}), \tilde{\partial} \tilde{\phi}(\tilde{x}), \tilde{x}) \right)$$

↗ derivatives w.r.t. \tilde{x}

Theorem: if $\tilde{x}^\mu = X_\epsilon^\mu(x)$, $\tilde{\phi}(\tilde{x}) = F_\epsilon(\phi(x))$ is an infinitesimal symmetry

of $S[\phi] = \int_{U \subseteq \mathbb{R}^n} d^n x \mathcal{L}(\phi(x), \partial\phi(x), x)$ for all (nice) $U \subseteq \mathbb{R}^n$

then $\partial_\mu \left[\mathcal{L} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} (\delta\phi - \phi_{,\nu} \delta x^\nu) \right] = 0$ when ϕ satisfies the

↗ integration over U should be defined

↗ notation for $\partial_\mu \phi$

Euler-Lagrange eqs. $\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} = 0$. $\left(\delta x^\mu = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} X_\epsilon^\mu(x), \delta\phi(\phi) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} F_\epsilon(\phi) \right)$

Note: in Minkowski space this is a continuity equation $\partial_\mu j^\mu = 0$ ↗ conserved current

proof Very similar to the particle case. Let's set $D_{\epsilon=0} = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0}$

By assumption $\delta S = D_{\epsilon=0} \tilde{S}[\tilde{\phi}] = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \tilde{S}[\tilde{\phi}] = 0$.

↘
Note that

$$\tilde{S}[\tilde{\phi}] = \int_{X_\epsilon(U)} d^n \tilde{x} \mathcal{L}(\tilde{\phi}(\tilde{x}), \tilde{\partial} \tilde{\phi}(\tilde{x}), \tilde{x})$$

$$D_{\epsilon=0} (A(\epsilon) B(\epsilon))$$

$$\stackrel{!!}{=} (D_{\epsilon=0} A(\epsilon)) B(0) + A(0) (D_{\epsilon=0} B(\epsilon))$$

$$= \int_U d^n x |\det J_\epsilon| \mathcal{L}(\tilde{\phi}(\tilde{x}), \tilde{\partial} \tilde{\phi}(\tilde{x}), \tilde{x})$$

where J_ϵ is the Jacobian matrix, $(J_\epsilon)^\mu_\nu = \frac{\partial X_\epsilon^\mu}{\partial x^\nu} = \frac{\partial \tilde{x}^\mu}{\partial x^\nu}$

Since everything is smooth, $(J_\epsilon^{-1})^\mu_\nu = \frac{\partial x^\mu}{\partial \tilde{x}^\nu}$

Some information that we need: Let A be an invertible matrix.

$$\cdot \frac{d}{d\varepsilon} \det(A) = \det(A) \operatorname{tr}\left(A^{-1} \frac{dA}{d\varepsilon}\right) \quad (\text{Jacobi's formula})$$

$$\cdot \frac{d}{d\varepsilon} A^{-1} = -A^{-1} \frac{dA}{d\varepsilon} A^{-1}$$

Now:

$$\cdot D_{\varepsilon=0} (J_\varepsilon)^\mu_\nu = D_{\varepsilon=0} \frac{\partial X^\mu_\varepsilon}{\partial x^\nu} = \partial_\nu \delta x^\mu$$

$$\cdot D_{\varepsilon=0} \det J_\varepsilon = \det(J_0) \operatorname{tr}(J_0^{-1} D_{\varepsilon=0} J_\varepsilon) = \operatorname{tr}(D_{\varepsilon=0} J_\varepsilon) = \partial_\mu \delta x^\mu$$

$$\cdot D_{\varepsilon=0} |\det J_\varepsilon| = \frac{|\det J_0|}{\det J_0} D_{\varepsilon=0} \det J_\varepsilon = \partial_\mu \delta x^\mu$$

$$\cdot D_{\varepsilon=0} J_\varepsilon^{-1} = -J_0^{-1} (D_{\varepsilon=0} J_\varepsilon) J_0^{-1} = -D_{\varepsilon=0} J_\varepsilon$$

$$\cdot D_{\varepsilon=0} \tilde{\phi}(\tilde{x}) = D_{\varepsilon=0} F_\varepsilon(\phi(x)) = \delta\phi(\phi(x))$$

$$\cdot \frac{\partial \tilde{\phi}}{\partial \tilde{x}^\mu} = \frac{\partial}{\partial \tilde{x}^\mu} F_\varepsilon(\phi(x)) = \frac{dF_\varepsilon}{d\phi}(\phi(x)) \partial_\nu \phi(x) \left(\frac{\partial x^\nu}{\partial \tilde{x}^\mu} \right) \rightarrow (J_\varepsilon^{-1})^\nu_\mu$$

$$\begin{aligned} \rightarrow D_{\varepsilon=0} \frac{\partial \tilde{\phi}}{\partial \tilde{x}^\mu} &= \left(\frac{d}{d\phi} \delta\phi \right) \underbrace{\partial_\nu \phi}_{\delta^\nu_\mu} (J_0^{-1})^\nu_\mu - \underbrace{\frac{dF_0}{d\phi}}_{\rightarrow 1}(\phi(x)) \partial_\nu \phi \partial_\mu \delta x^\nu \\ &= \partial_\mu \delta\phi - \phi_{,\nu} \partial_\mu \delta x^\nu \end{aligned}$$

$$\begin{aligned} 0 = \delta S &= \int_\mu d^4x \left\{ (D_{\varepsilon=0} |\det J_\varepsilon|) \mathcal{L}(\phi(x), \partial\phi(x), x) + |\det J_0| D_{\varepsilon=0} \mathcal{L}(\tilde{\phi}(\tilde{x}), \tilde{\partial}\tilde{\phi}(\tilde{x}), \tilde{x}) \right\} \\ &= \int_\mu d^4x \left\{ \mathcal{L} \partial_\mu \delta x^\mu + \frac{\partial \mathcal{L}}{\partial x^\mu} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} (\partial_\mu \delta\phi - \phi_{,\nu} \partial_\mu \delta x^\nu) \right\} \end{aligned}$$

\rightarrow use this to check if $\delta S = 0$ for the given transformation!

Here is where we use the EL eqs.

$$\cdot \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \partial_\mu \delta\phi = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \delta\phi \right) - \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \right) \delta\phi = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \delta\phi \right) - \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi$$

$$\begin{aligned}
\bullet \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \phi_{,\nu} \delta x^\nu &= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \phi_{,\nu} \delta x^\nu \right) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \phi_{,\nu} \right) \delta x^\nu \\
&= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \phi_{,\nu} \delta x^\nu \right) - \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \right) \phi_{,\nu} \delta x^\nu - \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \phi_{,\mu\nu} \delta x^\nu \\
&= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \phi_{,\nu} \delta x^\nu \right) - \frac{\partial \mathcal{L}}{\partial \phi} \phi_{,\mu} \delta x^\mu - \frac{\partial \mathcal{L}}{\partial \phi_{,\nu}} \phi_{,\nu\mu} \delta x^\mu \rightarrow \text{swapped dummy indices}
\end{aligned}$$

Putting everything together:

$$\begin{aligned}
0 = \delta S &= \int_U d^4x \left\{ \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} (\delta \phi - \phi_{,\nu} \delta x^\nu) \right] + \mathcal{L} \partial_\mu \delta x^\mu + \underbrace{\frac{\partial \mathcal{L}}{\partial x^\mu} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial \phi} \phi_{,\mu} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial \phi_{,\nu}} \phi_{,\nu\mu} \delta x^\mu}_{\delta x^\mu \partial_\mu \mathcal{L}(\phi(x), \partial \phi(x), x)} \right\}
\end{aligned}$$

$$= \int_U d^4x \partial_\mu \left[\mathcal{L} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} (\delta \phi - \phi_{,\nu} \delta x^\nu) \right]$$

Since this holds for all U , it must be $\partial_\mu \left[\mathcal{L} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} (\delta \phi - \phi_{,\nu} \delta x^\nu) \right] = 0$

Example: If \mathcal{L} does not depend on x then

$X_\epsilon^\mu(x) = x^\mu + \epsilon a^\mu$ ($a \in \mathbb{R}^n$), $F_\epsilon = \text{id}$ is a symmetry of $S[\phi]$ for every $U \subseteq \mathbb{R}^n$.

In fact $\delta x^\mu = a^\mu$ $\delta \phi = 0$

$$\delta S = \int_U d^4x \left[\underbrace{\mathcal{L}}_{=0} \partial_\mu \delta x^\mu + \frac{\partial \mathcal{L}}{\partial x^\mu} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} (\partial_\mu \delta \phi - \phi_{,\nu} \partial_\mu \delta x^\nu) \right] = 0$$

which means that

$$\begin{aligned}
0 &= \partial_\mu \left[\mathcal{L} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} (\delta \phi - \phi_{,\nu} \delta x^\nu) \right] \\
&= \partial_\mu \left[a^\mu \mathcal{L} - a^\nu \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \phi_{,\nu} \right] = a^\nu \partial_\mu \left[\delta^\mu_\nu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \phi_{,\nu} \right]
\end{aligned}$$

since a^ν is arbitrary, this tells us that the energy-momentum tensor

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \phi_{,\nu} - \delta^\mu_\nu \mathcal{L} \quad \text{satisfies} \quad \partial_\mu T^\mu_\nu = 0$$