Lecture 1 problems

Note: you will be working on these exercises in groups throughout the lecture. You are **not** expected to do them all at once!

Exercise 1. Prove the following consequences of the definition of a group:

1. the identity element is unique (if two elements satisfy the identity property, they are necessarily equal)
2. for each $g \in G$ the inverse g^{-1} is unique
3. $(g^{-1})^{-1} = g$
4. $(gh)^{-1} = h^{-1}g^{-1}$

Exercise 2. Prove that if $\varphi : G \to H$ is a group homomorphism, then

1. $\varphi(e_G) = e_H$ (Hint: look at $\varphi(e_G e_G)$)
2. $\varphi(g^{-1}) = \varphi(g)^{-1}$ for all $g \in G$.

Exercise 3. Prove that $\mathbb{Z}_2$ is isomorphic to the subgroup $\{\mathbb{I}_n, -\mathbb{I}_n\} \leq \mathrm{GL}(n, \mathbb{C})$. While you are at it, prove that the latter is indeed a subgroup!

Exercise 4. I'll do you one better: prove that *any* group with only two elements is isomorphic to $\mathbb{Z}_2$.