Path Integrals

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1 Instantons

(a) We want to solve the equation

$$-\frac{d^2\bar{q}}{d\tau^2} + V'(\bar{q}) = 0 \tag{1}$$

with boundary conditions

$$\bar{q}\left(\frac{\pm T_E}{2}\right) = q_0 \,. \tag{2}$$

These equations are the equations of motion of a particle with mass $m = 1$ moving in a potential. Make the analogy with a particle moving in a potential explicit, paying attention to the signs.

(b) There is a trivial solution to equations of motion and boundary conditions. What is it? What is the action of this trivial solution?

(c) Using your knowledge of a particle moving in a potential, sketch a plot of a non-trivial zero-energy solution to the equations of motion and boundary conditions.

(d) Using conservation of energy, show that a solution to the equation of motion with zero energy has

$$\bar{q}(\tau) - q_0 \approx e^{-\omega|\tau|} \tag{3}$$

for large $|\tau|$ (that is $|\tau| \approx \frac{T_E}{2}$).

(e) Refine your sketch if necessary.

(f) We are interested in the $T_E \to \infty$ limit. Is the action of the solution you found finite and non-zero in this limit?

(g) Change integration variables and use conservation of energy to show that the action

$$S_0 = \lim_{T_E \to \infty} \int_{-T_E/2}^{T_E/2} d\tau \left(\frac{1}{2}\left(\frac{d\bar{q}}{d\tau}\right)^2 + V(\bar{q})\right) \tag{4}$$

of the solution you found can be expressed as

$$S_0 = 2 \int_{q_0}^{q_*} d\bar{q} \sqrt{2V(\bar{q})} \tag{5}$$

where q_* is the other point where the potential vanishes $V(q_*) = 0$.

(h) Are there any solutions with non-zero energy with finite action in the $T_E \to \infty$ limit?

(i) Are there any other solutions with zero energy and finite action?