Today:

- group theory (continued)
- Discrete, continuous, infinitesimal symmetries
- ~~Symmetries of an action~~

Quotient group and isomorphism theorem

Definition (normal subgroup)

Let G be a group. A subgroup $N \leq G$ is called *normal* if

$$gng^{-1} \in N, \quad \forall g \in G, \quad \forall n \in N.$$

The notation $N \trianglelefteq G$ is commonly used to indicate that N is a normal subgroup of G.

← closed under conjugation

Definition (quotient group)

Let N be a normal subgroup of a group G. We can define an equivalence relation on G as

$$g \sim h \iff h^{-1}g \in N,$$

→ $g \sim h$ if $\exists a \in N$ s.t. $h = ga$

can go from g to h by multiplying with something in N

• we did not use the fact that N is normal (yet)

with equivalence classes

$$[g] = \{h \in G \mid h^{-1}g \in N\}.$$

The quotient group G/N (pronounced "G mod N") is the set of equivalence classes

$$G/N = \{[g] \mid g \in G\}$$

which is made into a group by defining

$$[g][h] = [gh], \quad [g]^{-1} = [g^{-1}], \quad e_{G/N} = [e_G].$$

need to make sure that $[gh] = [g][h]$ does not depend on representatives

→ is ok if N normal

Exercise

Show that the $2\mathbb{Z} = \{2n \mid n \in \mathbb{Z}\}$ is a normal subgroup of $(\mathbb{Z}, +)$ and that $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

$\tilde{h} \sim h$ $\quad$ $\tilde{g} \sim g$

$\tilde{h}^{-1} h \in N$ $\quad$ $\tilde{g}^{-1} g \in N$

$$[gh] = [\tilde{g}\tilde{h}] \iff gh \sim \tilde{g}\tilde{h} \iff \underbrace{(\tilde{g}\tilde{h})^{-1} gh}_{} \in N$$

$$\tilde{h}^{-1}\tilde{g}^{-1} g h = \underbrace{\tilde{h}^{-1} h}_{\in N} \overbrace{h^{-1} \underbrace{\tilde{g}^{-1} g}_{\in N} h}^{\in N} \quad \rightarrow \text{because } N \text{ is normal}$$

$\in N$

Theorem (first isomorphism theorem)

Let $\varphi : G \to H$ be a group homomorphism. Then:

- $\operatorname{Im}\varphi$ *is a subgroup of* H
- $\ker\varphi$ *is a normal subgroup of* G
- $\operatorname{Im}\varphi$ *is isomorphic to the quotient group* $G/\ker\varphi$

Exercise

Prove the first two points of the isomorphism theorem.

$N = \ker\varphi$

$[g] \overset{?}{=} \{h \in G \mid h^{-1}g \in \ker\varphi\}$

$h^{-1}g \in \ker\varphi \Leftrightarrow \varphi(h^{-1}g) = e_H$

$\Leftrightarrow \varphi(h)^{-1}\varphi(g) = e_H$

$\Rightarrow \varphi(g) = \varphi(h)$

$[g] = \{h \in G \mid \varphi(h) = \varphi(g)\}$

$$\boxed{\operatorname{Im}\varphi \cong G/\ker\varphi}$$

$\mathbb{Z}_2 = \{[0],[1]\}$

$[0] = \{2k \mid k \in \mathbb{Z}\}$

$[1] = \{2k+1 \mid k \in \mathbb{Z}\}$

$[0]+[1] = [1]+[0] = [1]$

$[1]+[1] = [0]$

$[0]+[0] = [0]$

$2\mathbb{Z} = \{2k \mid k \in \mathbb{Z}\} \trianglelefteq \mathbb{Z}$

gng^{-1}

$\downarrow$

let $n = 2k \in 2\mathbb{Z}$, $a \in \mathbb{Z}$

$a + n + (-a) = a - a + n = n \in 2\mathbb{Z}$

$\mathbb{Z}/2\mathbb{Z} = \{[n] \mid n \in \mathbb{Z}\}$

$[n] + [m] = [n+m]$

$[n] = \{m \in \mathbb{Z} \mid \underbrace{-m+n}_{h^{-1}g} \in 2\mathbb{Z}\} = \{m \in \mathbb{Z} \mid m-n \text{ is even}\}$

only distinct ones: $[0] = \{2k \mid k \in \mathbb{Z}\}$

$[1] = \{2k+1 \mid k \in \mathbb{Z}\}$

$\rightarrow \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$

Discrete vs continuous symmetries

- continuous if there is a continuous parameter for the group elements
- discrete otherwise

$\psi: \varepsilon \in (\mathbb{R},+) \mapsto g_\varepsilon \in G$ (group homomorphism)

if there is a topology on G such that ψ is continuous

$\Rightarrow \underbrace{\{g_\varepsilon \mid \varepsilon \in \mathbb{R}\}}_{\text{one-parameter subgroup}} = \mathrm{Im}\,\psi$ is continuous

$g_0 = e_G$

$g_{-\varepsilon} = g_\varepsilon^{-1}$

$g_{\varepsilon+\varepsilon'} = g_\varepsilon g_{\varepsilon'}$ $\Rightarrow$ one-param subgroup is abelian

$g_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$g_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$g_{\theta+\theta'} = g_\theta g_{\theta'}$

$g_{-\theta} = g_\theta^{-1}$

infinitesimal

assume that $\varepsilon \in \mathbb{R} \mapsto g_\varepsilon \in \underbrace{G}_{\text{differentiable structure}}$ is "smooth" (as nice as we need it to be)

linearise: $g_\varepsilon = \underbrace{\overset{e}{g_0} + \varepsilon \frac{d}{d\varepsilon} g_\varepsilon \Big|_{\varepsilon=0}}_{\text{infinitesimal symmetry}} + o(\varepsilon^2)$

inf symmetry: $e + \varepsilon X + o(\varepsilon^2)$

where $X = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} g_\varepsilon$ derivative entry by entry

ex: $R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ $\quad \frac{d}{d\theta}\Big|_{\theta=0} R_\theta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$R_\theta = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\text{infinitesimal rotation}} + o(\theta^2)$$