Exercises for Lecture 2
Foundations of Stat Mech

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1 Swap operator

One of the problems of the entanglement entropies is that to know them one has to know all the eigenvalues of the reduced density operator, which amounts to knowing the wave-function. In other words, entanglement is not the expectation value of an observable. Define the swap operator as the operator on two copies of the original Hilbert space, namely $T^{(2)} \in \mathcal{H} \otimes \mathcal{H}$. This operator is defined from the action on the basis states as

$$T^{(2)} |\phi_i \phi_j\rangle = |\phi_j \phi_i\rangle \tag{1}$$

1.1 Purity as expectation value

Given a state $\sigma \in \mathcal{B}(\mathcal{H})$, show that $\mathrm{tr}\sigma^2 = \mathrm{tr}(T^{(2)}\sigma^{\otimes 2})$.

Solution.— Let us take $\sigma = \sum_i p_i |i\rangle \langle i|$. Then $\sigma^2 = \sum_i p_i^2 |i\rangle \langle i|$ so that

$$\mathrm{tr}\sigma^2 = \sum_i p_i^2.$$

Next, let us consider $\sigma^{\otimes 2}$. We have $\sigma^{\otimes 2} = \sum_{ij} p_i p_j |ij\rangle \langle ij|$, and thus $T^{(2)}\sigma^{\otimes 2} = \sum_{ij} p_i p_j |ji\rangle \langle ij|$. Taking the trace, we find that

$$\begin{aligned}
\mathrm{tr}(T^{(2)}\sigma^{\otimes 2}) &= \sum_{k\ell} \langle k\ell| \left(\sum_{ij} p_i p_j |ji\rangle \langle ij| \right) |k\ell\rangle \\
&= \sum_{i,j,k,\ell=1} p_i p_j \delta_{kj}\delta_{\ell i}\delta_{ik}\delta_{j\ell} \\
&= \sum_i p_i^2.
\end{aligned}$$

Comparing our expressions for $\mathrm{tr}\sigma^2$ and $\mathrm{tr}(T^{(2)}\sigma^{\otimes 2})$, we now see that $\mathrm{tr}\sigma^2 = \mathrm{tr}(T^{(2)}\sigma^{\otimes 2})$.

1.2 Trace of $T^{(2)}$

Compute the trace of $T^{(2)}$.

Solution.— Direct calculation in a basis shows that

$$\mathrm{tr}T^{(2)} = \mathrm{tr} \sum_{k\ell=1}^{d} |k\ell\rangle \langle k\ell| = \sum_{k\ell}^{d} \langle k\ell \, |k\ell\rangle = d \tag{2}$$

1.3 Frobenius norm

Show that, for any linear operator X, the Frobenius norm reads $\|X\|_2^2 = \mathrm{tr}(T^{(2)} X \otimes X^\dagger)$.

Solution.— This is basically the same proof as the purity. Notice that $\mathrm{tr}\sigma^2 = \|\sigma\|_2^2$.

2 Unitary evolution

2.1 Norm invariance

Show that the Frobenius norm and the Operator norm are preserved by unitary operations.

Solution.— For the operator norm, we have

$$\begin{aligned}
\|\mathcal{U}(X)\| &= \sup_{\|v\|=1} \|UXU^\dagger |v\rangle \| = \sup_{\|v\|=1} \sqrt{\langle v| UX^\dagger U^\dagger UXU^\dagger |v\rangle} \\
&= \sup_{\|Uv\|=1} \sqrt{\langle v| UX^\dagger XU^\dagger |v\rangle} = \sup_{\|w\|=1} \sqrt{\langle w| X^\dagger X |w\rangle} = \|X\| \qquad (3)
\end{aligned}$$

For the Frobenius norm, we have

$$\|\mathcal{U}(X)\|_2^2 = \mathrm{tr}[(U^\dagger XU)^\dagger U^\dagger XU] = \mathrm{tr}[U^\dagger X^\dagger UU^\dagger XU] = \mathrm{tr}X^\dagger X\|X\|_2^2$$

Corollary. Purity is preserved by unitary evolution.

2.2 Entropy invariance

Show that all Entropies are preserved by unitary evolution.

Solution.— The state ψ and $\mathcal{U}\psi$ have the same spectrum and therefore same Rényi entropies.

3 Entanglement and partial trace

Show that, if a state $\sigma(t)$ is evolving in time, the purity of the marginal state $\sigma_A(t)$ is given by

$$\mathrm{tr}\sigma_A^2(t) = \mathrm{tr}(T^{(2)} \otimes I_B^{\otimes 2} U^{\otimes 2} \sigma^{\otimes 2} U^{\dagger \otimes 2}) \qquad (4)$$

where U is the unitary evolution operator.

Solution.— Use the identity $\mathrm{tr}(X_A\sigma_A) = \mathrm{tr}(X \otimes I_B \sigma)$ and then recall $\sigma(t) = U\sigma U^\dagger$ and take the tensor power two.

4 Time evolution by Hamiltonian

Be $\mathcal{U}(X) = e^{iHt}Xe^{-iHt} \equiv X_t$ the time evolution generated by a non degenerate Hamiltonian $H = \sum_n E_n P_n$

4.1 Time averaged operator

define the time average as $\overline{X} = \lim_{T\to\infty} \frac{1}{T}\int_0^T X_t dt$. Show that

$$\overline{X} = \sum_n P_n X P_n \tag{5}$$

Solution.—

$$\bar{X} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \sum_{nn'} e^{-i(E_n - E_{n'})t} X_{nn'} |E_n\rangle \langle E_{n'}| \, dt \tag{6}$$

If the Hamiltonian is non-degenerate, $\overline{e^{i\omega t}} = 0$ and we are left with just the diagonal part, that is,

$$\bar{X} = \sum_n X_{nn} |E_n\rangle \langle E_n| = \sum_n \Pi_n \phi \Pi_n \tag{7}$$

5 Quantum measurement

Show that, after measuring on the state σ some observable $\Omega = \sum_i o_i |i\rangle\langle i| \equiv \sum_i o_i \omega_i$ the resulting state is

$$\sigma' = \sum_i \omega_i \sigma \omega_i \tag{8}$$

Solution.— We have

$$\sigma' = \sum_i \mathrm{tr}(\sigma\omega_i)\omega_i = \sum_i \langle i|\,\sigma\,|i\rangle \, |i\rangle \langle i| = \sum_i |i\rangle \langle i|\,\sigma\,|i\rangle \langle i| = \sum_i \omega_i \sigma \omega_i$$