Path Integrals

Dan Wohns

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1 Semi-classical Limit

(a) Let $f(x)$ be a real valued function. By expanding $f(x)$ around its minimal values x_c show that we can approximate

$$I = \int_{-\infty}^{\infty} dx e^{-f(x)/\hbar} \tag{1}$$

by

$$I \approx \sum_{x_c} \sqrt{\frac{2\pi\hbar}{f''(x_c)}} e^{-f(x_c)/\hbar}(1 + \mathcal{O}(\hbar^a)) \tag{2}$$

(b) Express the most important corrections to equation (2) in the form

$$\int_{-\infty}^{\infty} dx C_1 x^n e^{-C_2 x^2} \tag{3}$$

where C_1 and C_2 are constants.

(c) Determine the value of a in equation (18) by making a change of variables.

2 Perturbation Theory

(a) Show that

$$\exp\left[-\lambda\left(\frac{d}{dJ}\right)^4\right]\int_{-\infty}^{\infty} dx\exp[-x^2+xJ] = \int_{-\infty}^{\infty} dx\exp[-x^2-\lambda x^4+xJ]\,. \tag{4}$$

Path integrals with interactions can be computed by taking derivatives of free path integrals with a source J using generalizations of this formula.

(b) Use induction to show that

$$\int_{-\infty}^{\infty} dx x^{2n}\exp\left[-\frac{1}{2}ax^2\right] = \left(\frac{2\pi}{a}\right)^{1/2}\frac{1}{a^n}(2n-1)!! \tag{5}$$

where $(2n-1)!! = (2n-1)(2n-3)\cdots 5\cdot 3\cdot 1$. Hint: differentiate with respect to a.

(c) Show that

$$\int_{-\infty}^{\infty} dx\exp[-x^2-\lambda x^4] = \sqrt{\pi}\sum_{j=0}^{\infty}\frac{(-\lambda)^j(4j-1)!!}{2^{2j}j!}\,. \tag{6}$$

(d) Show that the series in part (c) diverges.

(e) Which step in the derivation was not justified?