Prop: if $G \leq GL(n, \mathbb{C})$ is a matrix Lie group then

$Lie(G) = \{X \in M_n(\mathbb{C}) \mid e^{\varepsilon X} \in G,\ \forall \varepsilon \in \mathbb{R}\}$ with $[X,Y] = XY - YX$ is a Lie algebra

proof:

- $[\cdot,\cdot]$ is anti-commutative, bilinear, and satisfies Jacobi identity
- 0 matrix belongs to $Lie(G)$ since $e^{\varepsilon 0} = e^0 = \mathbb{1} \in G \quad \forall \varepsilon \in \mathbb{R}$
- if $X \in Lie(G)$ and $\alpha \in \mathbb{R}$ then $e^{\varepsilon \alpha X} \in G \quad \forall \varepsilon \in \mathbb{R}$ since $\varepsilon \alpha \in \mathbb{R}$
  $\Rightarrow \alpha X \in \mathbb{R}$

Lie product formula

- if $X, Y \in Lie(G)$ then $e^{\varepsilon(X+Y)} = \lim_{k \to \infty} \Big(\underbrace{e^{\varepsilon X/k}}_{\in G} \underbrace{e^{\varepsilon Y/k}}_{\in G} \Big)^k$

  which is a limit of products of elements of G. Since G is closed, the limit converges to something in G $\Rightarrow$ $X + Y \in Lie(G)$
- This means that $Lie(G)$ is a subspace of $M_n(\mathbb{C})$ (as a real vector space)

- Last we need to show that $X, Y \in Lie(G) \Rightarrow [X,Y] \in Lie(G)$
  First note that if $A \in G$ then $A e^{\varepsilon Y} A^{-1} \in G$ for all ε. However

  $A e^{\varepsilon Y} A^{-1} = e^{\varepsilon A Y A^{-1}}$ (property of matrix exponential) so $AYA^{-1} \in Lie(G)$.

  In particular $e^{\varepsilon X} Y e^{-\varepsilon X} \in Lie(G)$ and since $Lie(G)$ is a vector space $\frac{d}{d\varepsilon}\left(e^{\varepsilon X} Y e^{-\varepsilon X}\right)\Big|_{\varepsilon=0} \in Lie(G)$ (this is $\lim_{\varepsilon \to 0} \frac{e^{\varepsilon X} Y e^{-\varepsilon X} - Y}{\varepsilon}$, which is in $Lie(G)$)

  However, $\frac{d}{d\varepsilon}\left(e^{\varepsilon X} Y e^{-\varepsilon X}\right)\Big|_{\varepsilon=0} = \left(X e^{\varepsilon X} Y e^{-\varepsilon X} - e^{\varepsilon X} Y X e^{-\varepsilon X}\right)\Big|_{\varepsilon=0} = XY - YX$

$\rightarrow$ $Lie(G)$ is a real Lie algebra.