Exercise

$L(q,\dot{q},t) = \frac{1}{2}(m\dot{q}^2 - \kappa q^2)\, e^{\alpha t} \quad (\alpha \in \mathbb{R}) \qquad T_\varepsilon(t) = t + \varepsilon \qquad Q_\varepsilon(q) = q\, e^{-\frac{\varepsilon\alpha}{2}}$

$\tilde{t} = T_\varepsilon(t)$

$\tilde{q}(\tilde{t}) = Q_\varepsilon(q(t))$

1. Show that T_ε and Q_ε are one-parameter subgroups
2. Show that $\delta S = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \tilde{S}[\tilde{q}] = 0$ (without using E-L eqs)
3. Find the associated conserved quantity
4. Find the Euler-Lagrange equations
5. Show that the conserved quantity is indeed conserved if the E-L eqs hold

Note: for point 2 use the fact that

$$\delta S = \int_a^b \left[L(q,\dot{q},t)\frac{d}{dt}\delta t + \frac{\partial L}{\partial t}\delta t + \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\left(\frac{d}{dt}\delta q - \dot{q}\frac{d}{dt}\delta t\right)\right] dt$$

and substitute L, δt, δq