Review

Equivalence relations and equivalence classes

(Equivalence) relations

An equivalence relation is a special kind of binary relation, so let's review that first. A (homogeneous[1]) binary relation on a set X is a way of encoding a relationship between two elements of X. Given a relation R we use the notation xRy to mean "x is related to y".

Examples of relations are $=$, $>$, $<$, $\geq$, and $\leq$ on $\mathbb{R}$. Note that I've been extremely vague on what a relation *actually is*. If you are interested in the rigorous definition, I'm adding it below, but you can skip it if you're happy with the previous paragraph.

Definition. A *homogeneous binary relation* on a set X is a set $R \subseteq X \times X$ of pairs of elements of X. When $(x, y) \in R$ we say that x is related to y, which we also indicate with the less cumbersome notation xRy.

We can now define equivalence relations.

Definition. An *equivalence relation* on a set X is homogeneous binary relation R that satisfies the following properties:

- xRx for all $x \in X$ (reflexive)
- $xRy \Rightarrow yRx$ (symmetric)
- xRy and $yRz \Rightarrow xRz$ (transitive)

These properties are chosen as they encode the fundamental features of the notion of things being "equivalent".

Equivalence relations are often denoted by the symbol $\sim$. In this case we read $x \sim y$ as "x is equivalent to y".

Equivalence classes

Now that we know what an equivalence relation is, we can define the extremely useful concept of equivalence class.

[1] Non-homogeneous binary relations allow to consider relations between objects belonging to different sets, but are not needed for our purposes.

Definition. Let $\sim$ be an equivalence relation on X. The *equivalence class* of $x \in X$ is the set

$$[x]_\sim = \{y \in X \,|\, y \sim x\} \tag{1}$$

of all the elements in X equivalent to x. We often drop the subscript and just write $[x]$ if there is no risk of confusion. Note that

$$[x]_\sim = [y]_\sim, \quad \forall y \in [x]_\sim, \tag{2}$$

that is we can label the equivalence class using any of its elements. The element that we choose as the label is called the *representative* of the equivalence class.

Equivalence classes are essentially buckets in which we put elements of X that are equivalent to each other. Inequivalent elements end up in different equivalence classes. One way of thinking about this is that an equivalence class is a way to encode as a single object the properties common to its elements, which are otherwise indistinguishable.

Example: integers modulo 2

An example of equivalence relation appears in modular arithmetic. Let's consider the simplest case of the *integers modulo* 2. We define an equivalence relation $\sim$ on $\mathbb{Z}$ by

$$x \sim y \quad \Rightarrow \quad x - y = 2k, \quad k \in \mathbb{Z}, \tag{3}$$

that is x and y are equivalent if they differ by a multiple of two[2]. If you want, you can check that this is indeed an equivalence relation as an exercise. There are only two distinct equivalence classes,

$$[0] = \{2k \,|\, k \in \mathbb{Z}\} \tag{4}$$

$$[1] = \{2k + 1 \,|\, k \in \mathbb{Z}\}, \tag{5}$$

which are respectively the sets of even and odd integers. The sets of integers modulo 2 is defined as

$$\mathbb{Z}_2 = \{[0], [1]\}. \tag{6}$$

Although this is a set of sets, we really want to pretend that it's a set of numbers. To do that, we define a way to add elements in $\mathbb{Z}_2$. The obvious choice is

$$[a] + [b] = [a + b], \tag{7}$$

but at this stage this expression is purely formal: since we are specifying this operation through the sum of the representatives of each class, we need to make sure that the definition is actually independent of the choice of representatives (since they should be equivalent). This happens *all the time* when dealing with equivalence classes! In this case we are fine, since

$$\begin{cases} \text{even} + \text{even} = \text{even} \\ \text{even} + \text{odd} = \text{odd} \\ \text{odd} + \text{odd} = \text{even}. \end{cases} \tag{8}$$

[2] This is also called *congruence modulo* 2, denoted by $x \equiv y \pmod 2$.

Example: projective Hilbert space

Here's an example from quantum physics. This example is optional—we'll see it again during the lectures—but I added it anyway in case the previous content gave you a pure math overdose.

Let $\mathcal{H}$ be the Hilbert space of a quantum theory. While extremely useful, the states $|\psi\rangle \in \mathcal{H}$ are not *physical*, where physical is meant as something that can be *measured*. The physical quantities in a quantum theory are the expectation values of the observables in a given state, that is

$$\langle A \rangle_\psi = \frac{\langle \psi | A | \psi \rangle}{\langle \psi | \psi \rangle}. \tag{9}$$

Now consider another state $|\phi\rangle = \lambda |\psi\rangle$, with $\lambda \in \mathbb{C} \setminus \{0\}$. For any observable A we have

$$\begin{aligned} \langle A \rangle_\phi &= \frac{|\lambda|^2}{|\lambda|^2} \frac{\langle \psi | A | \psi \rangle}{\langle \psi | \psi \rangle} \\ &= \frac{\langle \psi | A | \psi \rangle}{\langle \psi | \psi \rangle} \\ &= \langle A \rangle_\psi. \end{aligned} \tag{10}$$

This means that the states $|\psi\rangle$ and $|\phi\rangle$ are *physically indistinguishable*! It is then natural to consider them equivalent, and define the equivalence relation

$$|\psi\rangle \sim |\phi\rangle \quad \Leftrightarrow \quad |\phi\rangle = \lambda |\psi\rangle, \quad \lambda \in \mathbb{C} \setminus \{0\}. \tag{11}$$

The set

$$P(\mathcal{H}) = \{ [|\psi\rangle] \,|\, |\psi\rangle \in \mathcal{H} \setminus \{0\} \} \tag{12}$$

is called the *projective Hilbert space* associated to $\mathcal{H}$, and can be interpreted as the set of all distinguishable quantum states. The zero vector was removed as it does not describe any physical state (expectation values don't make sense for the zero vector!).

In a sense $P(\mathcal{H})$ is the true set of states of a quantum theory. The reason why we work with $\mathcal{H}$ instead is that the latter is a vector space, which comes with a lot of nice properties and allows for an easy description of the superposition principle.