① $T_0(t) = t + 0 = t \;\rightarrow T_0 = id$

$T_{\varepsilon+\varepsilon'}(t) = t+\varepsilon+\varepsilon' = (T_\varepsilon \circ T_{\varepsilon'})(t)$ $\Big\} \rightarrow T_{-\varepsilon} = T_\varepsilon^{-1}$

$Q_0(q) = e^0 q = q \;\rightarrow Q_0 = id$

$Q_{\varepsilon+\varepsilon'}(q) = e^{-\alpha(\varepsilon+\varepsilon')/2} q = e^{-\frac{\alpha\varepsilon}{2}} e^{-\frac{\alpha\varepsilon'}{2}} q = (Q_\varepsilon \circ Q_{\varepsilon'})(q)$ $\Big\} \rightarrow Q_{-\varepsilon} = Q_\varepsilon^{-1}$

② $\delta t = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (t+\varepsilon) = 1$ $\qquad \delta q = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} e^{-\frac{\alpha\varepsilon}{2}} q = \left. -\frac{\alpha}{2} e^{-\frac{\alpha\varepsilon}{2}} q \right|_{\varepsilon=0} = -\frac{\alpha}{2} q$

$$\delta S = \int_a^b \Big[L \underbrace{\frac{d}{dt}\delta t}_{=0} + \underbrace{\frac{\partial L}{\partial t}}_{\alpha L} \overbrace{\delta t}^{=1} + \underbrace{\frac{\partial L}{\partial q}}_{-\kappa q e^{\alpha t}} \overbrace{\delta q}^{-\frac{\alpha}{2} q} + \overbrace{\frac{\partial L}{\partial \dot q}}^{m\dot q e^{\alpha t}} \Big(\underbrace{\frac{d}{dt}\delta q}_{-\frac{\alpha}{2}\dot q} - \dot q \underbrace{\frac{d}{dt}\delta t}_{=0} \Big) \Big] dt$$

$$= \int_a^b e^{\alpha t} \Big[\cancel{\frac{\alpha m}{2}} \dot q^2 - \cancel{\frac{\alpha \kappa}{2}} q^2 + \cancel{\frac{\alpha\kappa}{2}} q^2 - \cancel{\frac{\alpha m}{2}} \dot q^2 \Big] dt = 0$$

③ the conserved quantity is

$$L\,\delta t + \frac{\partial L}{\partial \dot q}(\delta q - \dot q\, \delta t) = L - \frac{\partial L}{\partial \dot q}\Big(\frac{\alpha}{2} q + \dot q \Big)$$

$$= e^{\alpha t}\Big(\frac{m}{2}\dot q^2 - \frac{\kappa}{2} q^2 - \frac{m\alpha}{2} q \dot q - m \dot q^2 \Big)$$

$$= -\frac{e^{\alpha t}}{2}\big(m\dot q^2 + \kappa q^2 + m\alpha q \dot q \big)$$

④ $\frac{\partial L}{\partial q} = -\kappa q e^{\alpha t}$ $\qquad \frac{\partial L}{\partial \dot q} = m \dot q e^{\alpha t}$ $\qquad \frac{d}{dt}\frac{\partial L}{\partial \dot q} = m \ddot q e^{\alpha t} + \alpha m \dot q e^{\alpha t}$

EL eq: $\quad e^{\alpha t}(m\ddot q + \alpha m \dot q + \kappa q) = 0 \iff m\ddot q + \alpha m \dot q + \kappa q = 0$

⑤ Suppose that $\; m\ddot q + \alpha m \dot q + \kappa q = 0$. Then

$$\frac{d}{dt}\Big[e^{\alpha t}\big(m\dot q^2 + \kappa q^2 + m\alpha q\dot q \big) \Big] = e^{\alpha t}\Big[\alpha\big(m\dot q^2 + \kappa q^2 + m\alpha q \dot q \big) + 2m\dot q \ddot q + 2\kappa q \dot q + m\alpha \dot q^2 + m\alpha q \ddot q \Big]$$

$$= e^{\alpha t}\Big[2\alpha m \dot q^2 + \alpha\kappa q^2 + m\alpha^2 q \dot q + 2m\dot q\ddot q + 2\kappa q \dot q + m\alpha q \ddot q \Big]$$

$$= e^{\alpha t}\Big[2\dot q\big(m\ddot q + \cancel{\alpha m \dot q} + \kappa q \big) + \alpha q \big(m\ddot q + \cancel{m\alpha \dot q} + \kappa q \big) \Big] = 0$$