

Ex 1

$$\textcircled{1} \quad \text{suppose } \begin{cases} g e = e g = g \\ g \tilde{e} = \tilde{e} g = g \end{cases} \quad \forall g \in G.$$

$$\text{Then } e \tilde{e} = \tilde{e} \quad \text{but also } e \tilde{e} = e \Rightarrow e = \tilde{e}$$

$$\textcircled{2} \quad \text{Let } g \in G \quad \text{and suppose } \begin{cases} g \cdot a = a \cdot g = e \\ g \cdot b = b \cdot g = e \end{cases}$$

$$\text{Then } b \cdot g \cdot a = b \cdot e = b \quad \text{but also } b \cdot g \cdot a = e \cdot a = a \Rightarrow a = b$$

$$\textcircled{3} \quad \text{Since the inverse is unique and } g \cdot g^{-1} = g^{-1} \cdot g = 1, \text{ then } g \text{ is the inverse of } g^{-1}$$

$$\textcircled{4} \quad \left. \begin{aligned} (h^{-1} g^{-1})(gh) &= h^{-1} g^{-1} g h = h^{-1} e h = h^{-1} h = e \\ (gh)(h^{-1} g^{-1}) &= g h h^{-1} g^{-1} = g e g^{-1} = g g^{-1} = e \end{aligned} \right\} \Rightarrow (gh)^{-1} = h^{-1} g^{-1}$$

Ex 2

$$\textcircled{1} \quad \varphi(e_G) = \varphi(\overset{e_G}{e_G e_G}) = \varphi(e_G) \varphi(e_G)$$

$$\Rightarrow \varphi(e_G)^{-1} \varphi(e_G) = \varphi(e_G)^{-1} \varphi(e_G) \varphi(e_G) \Rightarrow e_H = e_H \varphi(e_G) = \varphi(e_G)$$

$$\textcircled{2} \quad \begin{aligned} e_H &= \varphi(e_G) = \varphi(g^{-1} g) = \varphi(g^{-1}) \varphi(g) \\ e_H &= \varphi(e_G) = \varphi(g g^{-1}) = \varphi(g) \varphi(g^{-1}) \end{aligned} \Rightarrow \varphi(g^{-1}) = \varphi(g)^{-1}$$

Ex 3

$$\mathbb{Z}_2 = \{ [0], [1] \} \quad \text{with} \quad \begin{aligned} [0] + [0] &= [0] \\ [0] + [1] &= [1] + [0] = [1] \\ [1] + [1] &= [0] \end{aligned}$$

$$\text{let } \varphi: \mathbb{Z}_2 \rightarrow \{ \mathbb{1}_n, -\mathbb{1}_n \}$$

it must be $\varphi([0]) = \mathbb{1}_n$ since homomorphisms send identity to identity.

↓

We want φ to be invertible $\Rightarrow \varphi([1]) \neq \varphi([0]) \Rightarrow \varphi([1]) = -1_n$

is it a homomorphism?

$$\varphi([0] + [0]) = \varphi([0]) = 1_n = 1_n 1_n = \varphi([0]) \varphi([0]) \quad \checkmark$$

$$\varphi([1] + [0]) = \varphi([1]) = -1_n = (-1_n) 1_n = \varphi([1]) \varphi([0]) \quad \checkmark$$

$$\varphi([0] + [1]) = \varphi([1]) = -1_n = 1_n (-1_n) = \varphi([0]) \varphi([1]) \quad \checkmark$$

$$\varphi([1] + [1]) = \varphi([0]) = 1_n = (-1_n)(-1_n) = \varphi([1]) \varphi([1]) \quad \checkmark$$

$$\rightarrow \text{Yes} \Rightarrow \mathbb{Z}_2 \cong \{1_n, -1_n\}$$

Ex 4 What does a group G with two elements look like?

- one of the elements is the identity, call it 1
- the other element is not the identity, call it a

$$\text{it must be } \begin{cases} 1 * a = a * 1 = a \\ 1 * 1 = 1 \end{cases}$$

what about $a * a$? it must be in G , so it's either 1 or a

- if $a * a = a$ then there is no g in G such that $a * g = g * a = 1$
- but a must have an inverse, so the only option is $a * a = 1$

This looks just the same as \mathbb{Z}_2 ! In fact, choosing

$\varphi: G \rightarrow \mathbb{Z}_2$ such that $\varphi(1) = [0]$ and $\varphi(a) = [1]$ we get

φ invertible

$$\varphi(1 \cdot a) = \varphi(a) = [1] = [0] + [1] = \varphi(1) + \varphi(a)$$

$$\varphi(a \cdot 1) = \varphi(a) = [1] = [1] + [0] = \varphi(a) + \varphi(1)$$

$$\varphi(a \cdot a) = \varphi(1) = [0] = [1] + [1] = \varphi(a) + \varphi(a)$$

$$\varphi(1 \cdot 1) = \varphi(1) = [0] = [0] + [0] = \varphi(1) + \varphi(1)$$

$$\Rightarrow G \cong \mathbb{Z}_2$$