Exercises for Lecture 3,4,5
Foundations of Stat Mech

May 29, 2020

1 Time averages by Hamiltonian evolution

Be $\mathcal{U}(X) = e^{iHt} X e^{-iHt} \equiv X_t$ the time evolution generated by a non degenerate Hamiltonian $H = \sum_n E_n P_n$ and $\phi_t = \mathcal{U}\phi = e^{1iHt}\phi e^{iHt}$ the time evolved state. Let p_n be the probability of being in the state P_n, namely $p_n = \mathrm{tr}(\phi P_n)$.

1.1 Time averaged state

Let ϕ be a pure state with amplitudes $|\phi\rangle = \sum_n a_n |E_n\rangle$. Show that

$$\overline{\phi} = \sum_n P_n \phi P_n = \sum_n p_n P_n = \sum_n |a_n|^2 P_n = D_H \phi \tag{1}$$

and compute its purity.

Solution.— Using results from last problem set, just use the fact that $\phi_{nn} = |a_n|^2$. The purity then reads

$$\mathrm{Tr}\overline{\phi}^2 = \sum_n p_n^2 = \sum_n |a_n|^4 \tag{2}$$

1.2 Average Loschmidt Echo

Define the Loschmidt echo as $\mathcal{L}_t = |\langle\phi|\,\phi_t\rangle|^2$. Show that $\overline{\mathcal{L}_t} = \mathrm{Tr}\overline{\phi}^2$.

Solution.— We have

$$\overline{\mathcal{L}_t} = \overline{\mathrm{tr}(\phi\phi_t)} = \mathrm{tr}(\phi\overline{\phi_t}) = \sum_n p_n \mathrm{tr}(\phi P_n) = \sum_n p_n^2 \tag{3}$$

1.3 Typicality in time (challenge)

Define $\omega = D_H\phi$. Be H non-degenerate and having non degenerate gaps. Show that

$$\overline{D(\phi_{St}, \omega_S)} \leq \frac{1}{2}\sqrt{d_S^2 \mathrm{tr}\omega^2} \tag{4}$$

Solution.— This problem is hard and the proof is long. Let's see where we get to.

2 Averages in $\mathcal{H}$

2.1 Haar-average Loschmidt Echo

Show that

$$\langle\overline{\mathcal{L}_t}\rangle_\phi = \frac{2}{d+1} \tag{5}$$

Solution.— First of all, given $P_s = |s\rangle\langle s|$ notice that $P_k \otimes P_l = |kl\rangle\langle kl|$. Also, $T^{(2)}P_k \otimes P_l = |lk\rangle\langle kl|$. Now, use the result

$$\langle\phi^{\otimes 2}\rangle_\phi = \frac{I^{\otimes 2} + T^{(2)}}{d(d+1)} \tag{6}$$

we obtain

$$\begin{aligned}
\langle\overline{\mathcal{L}_t}\rangle_\phi &= \frac{\mathrm{tr}[T^{(2)}D_H^{\otimes 2}(I^{\otimes 2} + T^{(2)})]}{d(d+1)} = \frac{\mathrm{tr}[T^{(2)}\sum_{kl} P_k \otimes P_l(I^{\otimes 2} + T^{(2)})P_k \otimes P_l]}{d(d+1)} \\
&= \frac{1}{d(d+1)}\mathrm{tr}\sum_{kl}\left[T^{(2)}\left(|kl\rangle\langle kl| + |kl\rangle\langle kl|lk\rangle\langle kl|\right)\right] = \frac{2}{d+1}
\end{aligned} \tag{7}$$

2.2 Haar-average purity

Show that

$$\langle\mathrm{tr}\omega_A^2\rangle_\phi = \frac{d_A + d_B}{d_A d_B + 1} \tag{8}$$

Solution.— We use the result

$$\langle \phi^{\otimes 2} \rangle_\phi = \frac{I^{\otimes 2} + T^{(2)}}{d(d+1)} \tag{9}$$

We have

$$\begin{aligned}
\langle \mathrm{tr}\phi_A^2 \rangle_\phi &= \langle \mathrm{tr}(T_A^{(2)} \phi_A^{\otimes 2}) \rangle_\phi = \langle \mathrm{tr}(T_A^{(2)} \otimes I_B^{\otimes 2} \phi^{\otimes 2}) \rangle_\phi = \mathrm{tr}(T_A^{(2)} \otimes I_B^{\otimes 2} \langle \phi^{\otimes 2} \rangle_\phi) \\
&= \mathrm{tr}\left(T_A^{(2)} \otimes I_B^{\otimes 2} \frac{I^{\otimes 2} + T^{(2)}}{d(d+1)} \right) = \mathrm{tr}\left(\frac{T_A^{(2)} \otimes I_B^{\otimes 2} + I_A^{\otimes 2} \otimes T_B^{(2)}}{d(d+1)} \right) \\
&= \frac{d_A + d_B}{d_A d_B + 1}
\end{aligned} \tag{10}$$

2.3 Purity of the dephased state

Let $\omega = D_H \phi$. Show that

$$\langle \mathrm{tr}\omega^2 \rangle_\phi < \frac{2}{d} \tag{11}$$

Solution.— We use the result

$$\langle \mathrm{tr}\omega^2 \rangle_\phi = \mathrm{tr}\left[D_H^{\otimes 2} \langle \phi^{\otimes 2} \rangle_\phi \right] = \mathrm{tr}\left[\sum_{kl} P_k \otimes P_l \frac{I^{\otimes 2} + T^{(2)}}{d(d+1)} P_k \otimes P_l \right] = \sum_k \frac{2}{d(d+1)} < \frac{2}{d}$$

2.4 Typicality of canonical state

Let $\omega = D_H \phi$. Also define $\Omega_S \equiv \langle \mathrm{tr}_B \Omega \rangle_\phi = \langle \omega_S \rangle_\phi$. Be $d(\cdot, \cdot)$ the trace distance. Show that

$$\langle d\left(\omega_S, \frac{I_S}{d_S} \right) \rangle_\phi \leq \frac{1}{2} \sqrt{\frac{1}{d_B}} \tag{12}$$

Solution.— Notice that $\Omega_S = I_S/d_S$. We start using the inequality discussed in class

$$\langle d(\omega_S, \Omega_S \rangle_\phi \leq \langle \frac{1}{2} \sqrt{d_S \mathrm{tr}(\omega_S - \Omega_S)^2} \rangle_\phi \leq \frac{1}{2} \sqrt{d_S \langle \mathrm{tr}(\omega_S - \Omega_S)^2 \rangle_\phi} \tag{13}$$

Now, let's compute these '$\phi-$fluctuations':

$$\langle \mathrm{tr}(\omega_S - \Omega_S)^2\rangle_\phi = \langle \mathrm{tr}[T_S^{(2)}(\omega_S - \Omega_S)^{\otimes}2]\rangle_\phi = \mathrm{tr}[T_S^{(2)}(\langle\omega_S^{\otimes 2}\rangle_\phi - \Omega_S^{\otimes 2})]$$

where the operator $T_S^{(2)}$ is acting on $\mathcal{H}_S^{\otimes 2}$. Now, use

$$\langle\omega_S^{\otimes 2}\rangle_\phi = \mathrm{Tr}_B[D_H^{\otimes 2}\langle\phi^{\otimes 2}\rangle_\phi] = \mathrm{Tr}_B\left[D_H^{\otimes 2}\frac{I^{\otimes 2}+T^{(2)}}{d(d+1)}\right] \tag{14}$$

$$\Omega_S^{\otimes 2} = \langle\omega_S\rangle_\psi^{\otimes 2} = \mathrm{tr}_{BB'}\left[D_H^{\otimes 2}\left(\frac{I}{d}\otimes\frac{I}{d}\right)\right] \tag{15}$$

Then we have

$$\begin{aligned}
\mathrm{tr}[T_S^{(2)}(\langle\omega_S^{\otimes 2}\rangle_\phi - \Omega_S^{\otimes 2})] &= \mathrm{tr}_{SS'}\left[\mathrm{tr}_{BB'}\left[D_H^{\otimes 2}\left(\frac{I^{\otimes 2}+T^{(2)}}{d(d+1)} - \frac{I^{\otimes 2}}{d^2}\right)\right]T_S^{(2)}\right] \\
&\leq \mathrm{tr}_{SS'}\left[\mathrm{tr}_{BB'}\left[D_H^{\otimes 2}\left(\frac{T^{(2)}}{d^2}\right)\right]T_S^{(2)}\right] \\
&= \sum_{kl}\mathrm{tr}_{SS'}\left[\mathrm{tr}_{BB'}\left(|kl\rangle\langle kl|\frac{T^{(2)}}{d^2}|kl\rangle\langle kl|\right)T_S^{(2)}\right] \\
&= \frac{1}{d^2}\sum_{kl}\mathrm{tr}_{SS'}\left[\mathrm{tr}_{BB'}\left(|kl\rangle\langle kl|lk\rangle\langle kl|\right)T_S^{(2)}\right] \\
&= \frac{1}{d^2}\sum_{k}\mathrm{tr}_{S}\left[\mathrm{tr}_{B}\left(|kk\rangle\langle kk|\right)T_S^{(2)}\right] \leq \frac{1}{d}
\end{aligned} \tag{16}$$

where the last inequality follows from the fact that $\mathrm{tr}_S\left[\mathrm{tr}_B\left(|kk\rangle\langle kk|\right)T_S^{(2)}\right]$ is the purity of a state and is therefore upper bounded by 1.

We can now conclude that

$$\langle d\left(\omega_S, \frac{I}{d_S}\right)\rangle_\phi \leq \frac{1}{2}\sqrt{\frac{d_S}{d}} = \frac{1}{2}\sqrt{\frac{1}{d_B}} \tag{17}$$

2.5 General canonical state

Consider a subspace $\mathcal{H}_R \subset \mathcal{H}$ and let Π_R the projector on it. If we take the Haar average on $\mathcal{H}_R$ one has

$$\langle\phi^{\otimes 2}\rangle_\phi = \frac{\Pi_R^{\otimes 2}\left(I^{\otimes 2}+T^{(2)}\right)}{d_R(d_R+1)} \tag{18}$$

Then show that, in this subspace, the bound of the previous exercise holds as

$$\langle d\left(\omega_S, \frac{\Omega_S}{d_S}\right)\rangle_\phi \leq \frac{1}{2}\sqrt{\frac{d_S}{d_R}} \tag{19}$$

Also show that, in the same subspace, the bound on purity Eq.(11) reads

$$\langle \mathrm{tr}\omega^2\rangle_\phi < \frac{2}{d_R} \tag{20}$$

Solution.— All the proofs follow the same techniques by using Eq.(18) for the Haar average instead.