Exercises for Lecture 1
Fundamentals of Quantum Theory

May 25, 2020

1 Density operators

Show that given the distribution of states $(q_j, |\psi_j\rangle)$ the probability of getting an outcome a_i when measuring $A = \sum_i a_i |a_i\rangle\langle a_i|, \equiv \sum_i a_i P_i$ is given by

$$\Pr(a_i) = \mathrm{Tr}(\psi P_i) \tag{1}$$

where we have defined the density matrix of the system as

$$\psi = \sum_j q_j \psi_j \tag{2}$$

Solution.— Since the trace is linear, the probability of getting the outcome a_i is

$$\Pr(a_i) = \mathrm{Tr}\left(\sum_j q_j \psi_j P_i\right) = \mathrm{Tr}(\psi P_i) \tag{3}$$

2 Partial Trace

Mixed states appear naturally also when considering subsystems of a composite quantum system. Consider

$$A = \tilde{A}_S \otimes I_{\overline{S}} \tag{4}$$

in a Hilbert space with tensor product structure $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_{\overline{S}}$. It is a natural question to ask what is the state $\rho_S \in \mathcal{B}(\mathcal{H}_S)$ onto the subsystem $\mathcal{H}_S$ that can return all the right expectation values when performing any measurement of operators with that support, namely the $\tilde{A}_S$. We ask that

$$tr(A\rho) = tr_S(\tilde{A}_S \rho_S) \tag{5}$$

Show that the solution of this equation is

$$\rho_S = tr_{\overline{S}}\rho \tag{6}$$

where the operation of partial trace is defined by

$$tr_{\overline{S}} X = tr_{\overline{S}} \sum_{i_S i_{\overline{S}} j_S j_{\overline{S}}} X_{i_S i_{\overline{S}} j_S j_{\overline{S}}} \left| i_S i_{\overline{S}} \right\rangle \left\langle j_S j_{\overline{S}} \right| = \sum_{i_S i_{\overline{S}} j_S} X_{i_S i_{\overline{S}} j_S i_{\overline{S}}} \left| i_S \right\rangle \left\langle j_S \right| \tag{7}$$

Solution.— Just write the left hand side in a basis and read out solution by inspection. Notice that ρ_S is still a state, that is, it is positive and has trace one.

3 Entropies

The Rényi entropy for Rényi index q is defined as

$$S_q(\rho) = \frac{1}{1-q} \log \mathrm{Tr}\rho^q = \frac{1}{1-q} \log \left(\sum_i \lambda_i^q \right), \tag{8}$$

where the logarithm is again taken base two and λ_i are the eigenvalues of ρ.

3.1 Properties of Rényi entanglement entropies

a) Show that for the completely mixed density operator $\rho = I/d$ in d-dimensional Hilbert space, the von Neumann entropy is given by

$$S(I/d) = \log d. \tag{9}$$

Solution.— The eigenvalues of $\rho = I/d$ are $\lambda_i = 1/d$ for $i = 1, \ldots, d$. We then have

$$S(I/d) = -\sum_i \frac{1}{d} \log \frac{1}{d} = -d \times \left(\frac{1}{d} \log \frac{1}{d} \right) = -\log \frac{1}{d} = \log d$$

b) Show that

$$S_0(\rho) = \lim_{q \to 0^+} S_q(\rho) = \log(\operatorname{rank} \rho), \tag{10}$$

where $\operatorname{rank} \rho$ is the rank of the density matrix ρ.

Solution.— The rank of ρ is equal to the number of non-zero eigenvalues λ_i. We then find that

$$\begin{aligned}
\lim_{q \to 0^+} S_q(\rho) &= \lim_{q \to 0^+} \frac{1}{1-q} \log\left(\sum_i \lambda_i^q\right) \\
&= \log\left(\sum_{i \text{ with } \lambda_i \neq 0} \lambda_i^0 + \sum_{i \text{ with } \lambda_i = 0} 0\right) \\
&= \log\left(\sum_{i \text{ with } \lambda_i \neq 0} 1\right) \\
&= \log(\operatorname{rank} \rho).
\end{aligned}$$

c) Show that

$$\lim_{q \to 1} S_q(\rho) = S_1(\rho), \tag{11}$$

where $S_1(\rho)$ is the von Neumann entropy.

Solution.— Regarding the second question, using L'Hospital's rule,

$$\begin{aligned}
\lim_{q\to 1} S_q(\rho) &= \frac{1}{\ln 2} \lim_{q\to 1} \left[\frac{\log\left(\sum_i \lambda_i^q\right)}{1-q} \right] \\
&= \frac{1}{\ln 2} \lim_{q\to 1} \left[\frac{\frac{1}{\left(\sum_i \lambda_i^q\right)} \sum_i \frac{\mathrm{d}}{\mathrm{d}q} \lambda_i^q}{-1} \right] \\
&= \frac{1}{\ln 2} \lim_{q\to 1} \left[\frac{\frac{1}{\left(\sum_i \lambda_i^q\right)} \sum_i \lambda_i^q \ln \lambda_i}{-1} \right] \\
&= -\frac{1}{\left(\sum_i \lambda_i\right)} \sum_i \lambda_i \frac{\ln \lambda_i}{\ln 2} \\
&= -\sum_i \lambda_i \log \lambda_i \\
&= S(\rho),
\end{aligned}$$

d) Consider the Rényi entropy for the case where the Rényi index is a integer n with $n \geq 2$. If ρ corresponds to a pure state, calculate $S_n(\rho)$.

   Solution.— Regarding the higher Rényi entriopy of a pure state, if ρ is pure, then $\rho^n = \rho$ for integer n with $n \geq 2$. It follows that

$$S_n(\rho) = \frac{1}{1-n} \log \mathrm{Tr}\rho^n = \frac{1}{1-n} \log \mathrm{Tr}\rho = \frac{1}{1-n} \log 1 = 0$$

3.2 Entropy of the Gibbs state

Compute the von Neumann entropy of the Gibbs state.

$S_{Gibbs} = S(Z^{-1}e^{-\beta H}) = -\sum_i p_i \log p_i = \log Z + \beta\langle E\rangle$