① $\underline{gl}(n,\mathbb{R}) = \{ X \in M_n(\mathbb{C}) \mid e^{\varepsilon X} \in GL(n,\mathbb{R}) \ \forall \varepsilon \in \mathbb{R} \}$

$e^{\varepsilon X} \in GL(n,\mathbb{R}) \ \forall \varepsilon \iff \overline{(e^{\varepsilon X})} = e^{\varepsilon X} \ \forall \varepsilon \iff e^{\varepsilon \bar{X}} = e^{\varepsilon X} \ \forall \varepsilon \iff \bar{X} = X$ (X is real)

$\Rightarrow \underline{gl}(n,\mathbb{R}) = M_n(\mathbb{R}) \Rightarrow \dim(\underline{gl}(n,\mathbb{R})) = n^2$

② $\underline{sl}(n,\mathbb{R}) = \{ X \in M_n(\mathbb{C}) \mid e^{\varepsilon X} \in SL(n,\mathbb{R}) \ \forall \varepsilon \in \mathbb{R} \}$

$= \{ X \in M_n(\mathbb{R}) \mid \det(e^{\varepsilon X}) = 1 \ \forall \varepsilon \in \mathbb{R} \}$

$\det(e^{\varepsilon X}) = 1 \ \forall \varepsilon \iff e^{\varepsilon \, tr(X)} = 1 = e^{\varepsilon 0} \ \forall \varepsilon \iff tr(X) = 0$

$\Rightarrow \underline{sl}(n,\mathbb{R}) = \{ X \in M_n(\mathbb{R}) \mid tr X = 0 \}$

if X has entries X_{ij}, $tr X = 0 \iff \sum_i X_{ii} = 0 \rightarrow$ we can write $X_{nn} = -\sum_{i=1}^{n-1} X_{ii}$

So X_{nn} depends on the other entries $\rightarrow n^2 - 1$ independent entries

$\dim(\underline{sl}(n,\mathbb{R})) = n^2 - 1$

③ $\underline{sl}(n,\mathbb{C}) = \{ X \in M_n(\mathbb{C}) \mid \det(e^{\varepsilon X}) = 1 \ \forall \varepsilon \in \mathbb{R} \}$

$= \{ X \in M_n(\mathbb{C}) \mid tr X = 0 \}$

We want its dimension as a $\underline{\text{real}}$ vector space. if $X = A + iB$, $A, B \in M_n(\mathbb{R})$

then $tr(X) = tr(A) + i\, tr(B) = 0 \iff \underbrace{tr(A) = 0}_{n^2-1 \text{ d.o.f.}}, \ \underbrace{tr(B) = 0}_{n^2-1 \text{ d.o.f.}}$

$\Rightarrow \dim_{\mathbb{R}}(\underline{sl}(n,\mathbb{C})) = 2(n^2 - 1)$

since $O(n) \leq GL(n,\mathbb{R})$

④ $\underline{o}(n) = \{ X \in M_n(\mathbb{C}) \mid e^{\varepsilon X} \in O(n) \ \forall \varepsilon \in \mathbb{R} \} = \{ X \in M_n(\mathbb{R}) \mid (e^{\varepsilon X})^t e^{\varepsilon X} = \mathbb{1} \ \forall \varepsilon \in \mathbb{R} \}$

$(e^{\varepsilon X})^t e^{\varepsilon X} = \mathbb{1} \ \forall \varepsilon \iff e^{\varepsilon X^t} e^{\varepsilon X} = \mathbb{1} \ \forall \varepsilon \iff e^{\varepsilon X^t} = e^{-\varepsilon X} \ \forall \varepsilon \iff \underbrace{X^t = -X}_{\frac{n(n-1)}{2} \text{ d.o.f.}}$

$\Rightarrow \underline{o}(n) = \{ X \in M_n(\mathbb{R}) \mid X^T = -X \}$

$\dim(\underline{o}(n)) = \frac{n(n-1)}{2}$

⑤ $\underline{so}(n) = \{ X \in M_n(\mathbb{C}) \mid e^{\varepsilon X} \in SO(n) \ \forall \varepsilon \in \mathbb{R} \} = \{ X \in \underline{o}(n) \mid \det(e^{\varepsilon X}) = 1 \ \forall \varepsilon \in \mathbb{R} \}$

$= \{ X \in \underline{o}(n) \mid \operatorname{tr} X = 0 \}$

However, if $X^T = -X$ then $\operatorname{tr} X = \operatorname{tr} X^T = -\operatorname{tr} X \Rightarrow \operatorname{tr} X = 0$

so $\underline{so}(n) = \underline{o}(n)$

⑥ $\underline{su}(n) = \{ X \in M_n(\mathbb{C}) \mid e^{\varepsilon X} \in SU(n) \ \forall \varepsilon \in \mathbb{R} \} = \{ X \in \underline{u}(n) \mid \det(e^{\varepsilon X}) = 1 \ \forall \varepsilon \in \mathbb{R} \}$

$= \{ X \in \underline{u}(n) \mid \operatorname{tr} X = 0 \}$

This time $\operatorname{tr} X = 0$ is not trivial, since $\operatorname{tr}(X) = \overline{\operatorname{tr}(X^*)} = -\overline{\operatorname{tr}(X)}$

so $\operatorname{tr}(X)$ can be any imaginary number.

if $X = A + iB$, A, B real with $A^T = -A$, $B^T = B$ like before,

$\operatorname{tr} X = 0 \iff \underbrace{\operatorname{tr} A = 0}_{\text{always true}}, \ \underbrace{\operatorname{tr} B = 0}$ → B loses one d.o.f.

$\dim(\underline{su}(n)) = n^2 - 1$

<u>Extra exercise:</u>

① $\underline{so}(n, \mathbb{C}) = \{ X \in M_n(\mathbb{C}) \mid \det(e^{\varepsilon X}) = 1 \text{ and } (e^{\varepsilon X})^T e^{\varepsilon X} = \mathbb{1} \ \ \forall \varepsilon \in \mathbb{R} \} =$

$= \{ X \in M_n(\mathbb{C}) \mid X^T = -X, \operatorname{tr} X = 0 \} = \{ X \in M_n(\mathbb{C}) \mid X^T = -X \}$

if $X = A + iB$, A, B real we have $\underbrace{A^T = -A}_{\frac{n(n-1)}{2}}, \ \underbrace{B^T = -B}_{\frac{n(n-1)}{2}}$

$\Rightarrow \dim_{\mathbb{R}}(\underline{so}(n, \mathbb{C})) = n(n-1)$

② $\underline{so}(p,q) = \{ M \in \underbrace{M_{p+q}(\mathbb{R})}_{\text{since } SO(p,q) \le GL(p+q, \mathbb{R})} \mid \underbrace{\det(e^{\varepsilon M}) = 1}_{\Downarrow \ \operatorname{tr} M = 0} \text{ and } (e^{\varepsilon M})^T \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} e^{\varepsilon M} = \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} \ \forall \varepsilon \in \mathbb{R} \}$

$(e^{\varepsilon M})^T \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} e^{\varepsilon M} = \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} \ \forall \varepsilon \iff e^{\varepsilon M^T} = \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} e^{-\varepsilon M} \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} \ \forall \varepsilon$

$\iff M^T = -\begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} M \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix}$

Note that $\quad tr(M) = tr(M^T) = -tr\left[\begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} M \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix}\right] \underset{\text{cyclic property}}{=} -tr\left[M \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix}^2\right] = -tr(M)$

$\Rightarrow tr(M) = 0$

in block form we have $M = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$

$$\begin{pmatrix} X^T & Z^T \\ Y^T & W^T \end{pmatrix} = -\begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix}\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}\begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} = -\begin{pmatrix} X & -Y \\ -Z & W \end{pmatrix}$$

$$\Rightarrow \quad \underbrace{X^T = -X}_{\frac{p(p-1)}{2}} \quad \underbrace{W^T = -W}_{\frac{q(q-1)}{2}}, \quad \underbrace{Z^T = Y}_{pq}$$

$$\underline{so}(p,q) = \left\{\begin{pmatrix} X & Y \\ Y^T & W \end{pmatrix} \middle| \; X \in \underline{o}(p), W \in \underline{o}(q), Y \in M_{p\times q}(\mathbb{R})\right\}$$

$$\dim(\underline{so}(p,q)) = \frac{p(p-1)}{2} + \frac{q(q-1)}{2} + pq = \frac{1}{2}(p+q)(p+q-1)$$

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③ $\underline{sp}(2n,\mathbb{R}) = \left\{ M \in M_{2n}(\mathbb{R}) \middle| \; trM = 0 \text{ and } e^{\varepsilon M^T}\begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} e^{\varepsilon M} = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} \quad \forall \varepsilon \in \mathbb{R}\right\}$

Let $\Omega = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}$ and note that $\Omega^2 = -\mathbb{1}_{2n} \Rightarrow \Omega^{-1} = -\Omega$

$$e^{\varepsilon M^T}\Omega e^{\varepsilon M} = \Omega \quad \forall \varepsilon \iff e^{\varepsilon M^T} = \Omega e^{-\varepsilon M}(-\Omega) \quad \forall \varepsilon \iff e^{\varepsilon M^T} = e^{-\Omega \varepsilon M(-\Omega)} \quad \forall \varepsilon$$

$\iff M^T = \Omega M \Omega$

Note that $\quad tr(M) = tr(M^T) = tr(\Omega M \Omega) = tr(M\Omega^2) = -tr(M) \Rightarrow tr(M) = 0$

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if $M = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$ ($n\times n$ blocks) then

$$\begin{pmatrix} X^T & Z^T \\ Y^T & W^T \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}\begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} = \begin{pmatrix} -W & Z \\ Y & -X \end{pmatrix}$$

$$\Rightarrow X^T = -W \quad Z^T = Z \quad \underbrace{Y^T = Y} \rightarrow \frac{n(n+1)}{2} \text{ d.o.f.}$$

$$\underline{sp}(2n,\mathbb{R}) = \left\{\begin{pmatrix} X & Y \\ Z & -X^T \end{pmatrix} \middle| \; X \in M_n(\mathbb{R}), Y^T = Y, Z^T = Z\right\}$$

$$\Rightarrow \dim(\underline{sp}(2n,\mathbb{R})) = n^2 + \frac{n(n+1)}{2} + \frac{n(n+1)}{2} = n(2n+1)$$

④ Similarly to $\underline{sp}(2n,\mathbb{R})$, we get $\underline{sp}(2n,\mathbb{C}) = \left\{ \begin{pmatrix} X & Y \\ Z & -X^T \end{pmatrix} \middle| \; X \in M_n(\mathbb{C}),\ Y^T = Y,\ Z^T = Z \right\}$

$\rightarrow \dim_{\mathbb{R}}(\underline{sp}(2n,\mathbb{C})) = 2n(2n+1)$

⑤ $\underline{sp}(n) = \{ M \in M_{2n}(\mathbb{C}) \mid e^{\varepsilon M} \in Sp(2n,\mathbb{C}) \cap U(2n) \ \ \forall \varepsilon \in \mathbb{R} \}$

$= \underline{sp}(2n,\mathbb{C}) \cap \underline{u}(2n)$

Now, if $M = \begin{pmatrix} X & Y \\ Z & -X^T \end{pmatrix} \in \underline{sp}(2n,\mathbb{C})$ then $M^* = -M \Leftrightarrow \begin{pmatrix} X^* & Z^* \\ Y^* & -\bar{X} \end{pmatrix} = \begin{pmatrix} -X & -Y \\ -Z & X^T \end{pmatrix}$

$\Leftrightarrow \begin{cases} X^* = -X \\ Z^* = -Y \end{cases}$

$\underline{sp}(n) = \left\{ \begin{pmatrix} X & Y \\ -Y^* & -X^T \end{pmatrix} \middle| \; X \in \underline{u}(n),\ \overbrace{Y^T = Y}^{n(n+1) \text{ d.o.f since complex}} \right\}$

$\rightarrow \dim(\underline{sp}(n)) = n^2 + n(n+1) = n(2n+1)$