

How do we find the Lie algebra of a MLG?

Some useful properties of the matrix exponential:

$$\begin{aligned} \cdot (e^X)^{-1} &= e^{-X} & \cdot \overline{(e^X)} &= e^{\bar{X}} \\ \cdot (e^X)^T &= e^{X^T} & \cdot \det(e^X) &= e^{\text{tr}(X)} \\ \cdot (e^X)^* &= e^{X^*} & \cdot e^{\epsilon X} &= e^{\epsilon Y} \quad \forall \epsilon \in \mathbb{R} \Leftrightarrow X=Y \end{aligned}$$

→ proof: (1) trivial
(2) take derivative on both sides and evaluate at $\epsilon=0$

Example: $U(n) = \{A \in GL(n, \mathbb{C}) \mid A^* A = \mathbb{1}\}$

$$\Rightarrow U(n) = \{X \in M_n(\mathbb{C}) \mid e^{\epsilon X} \in U(n) \quad \forall \epsilon \in \mathbb{R}\}$$

$$e^{\epsilon X} \in U(n) \quad \forall \epsilon \in \mathbb{R} \Leftrightarrow (e^{\epsilon X})^* e^{\epsilon X} = \mathbb{1} \quad \forall \epsilon \in \mathbb{R} \Leftrightarrow e^{\epsilon X^*} e^{\epsilon X} = \mathbb{1} \quad \forall \epsilon \in \mathbb{R}$$

$$\Leftrightarrow e^{\epsilon X^*} = e^{-\epsilon X} \quad \forall \epsilon \in \mathbb{R} \Leftrightarrow X^* = -X$$

$$\Rightarrow U(n) = \{X \in M_n(\mathbb{C}) \mid X^* = -X\} \quad *$$

Dimension as a real vector space? Let $X = A + iB$ with A and B real.

$$X^* = -X \Leftrightarrow A^T - iB^T = -A - iB \Leftrightarrow A^T = -A, \quad B^T = B \rightarrow \frac{n(n+1)}{2} \text{ independent entries}$$

$$\downarrow$$
$$\frac{n(n-1)}{2} \text{ independent entries}$$

$$\Rightarrow \dim(U(n)) = \frac{n(n-1)}{2} + \frac{n(n+1)}{2} = n^2$$

Exercise: Find out the explicit form (the equivalent of $*$) and dimension of the Lie algebras of the following groups:

$$GL(n, \mathbb{R}), SL(n, \mathbb{R}), SL(n, \mathbb{C}), O(n), SO(n), SU(n)$$

More exercises (you can do this on your own later)

Do the same for the following groups

$$SO(n, \mathbb{C}) = \{ A \in SL(n, \mathbb{C}) \mid A^T A = \mathbb{1} \} \quad \text{complex special orthogonal group}$$

$$SO(p, q) = \left\{ A \in SL(p+q, \mathbb{R}) \mid A^T \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} A = \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} \right\} \quad \text{indefinite special orthogonal group}$$

$$Sp(2n, \mathbb{R}) = \left\{ A \in SL(2n, \mathbb{R}) \mid A^T \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} A = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} \right\}$$

$$Sp(2n, \mathbb{C}) = \left\{ A \in SL(2n, \mathbb{C}) \mid A^T \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} A = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} \right\} \quad \left. \begin{array}{l} \text{real and complex} \\ \text{symplectic group} \end{array} \right\}$$

$$Sp(n) = Sp(2n, \mathbb{C}) \cap U(2n) \quad \text{compact symplectic group}$$

example: $U(p, q) = \left\{ A \in GL(p+q, \mathbb{C}) \mid A^* \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} A = \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} \right\}$

$$e^{\varepsilon A^*} \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} e^{\varepsilon A} = \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} \quad \forall \varepsilon \in \mathbb{R}$$

$$\downarrow C e^A C^{-1} = e^{C A C^{-1}}$$

$$\Leftrightarrow e^{\varepsilon A^*} = \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} e^{-\varepsilon A} \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} = \exp \left[-\varepsilon \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} A \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} \right] \quad \forall \varepsilon \in \mathbb{R}$$

$$\Leftrightarrow A^* = - \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} A \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} \quad \text{or} \quad A^* \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} = - \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} A$$

if we write A in block form $A = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$ we get

$$\begin{pmatrix} X^* & Z^* \\ Y^* & W^* \end{pmatrix} \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} = - \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} X^* & -Z^* \\ Y^* & -W^* \end{pmatrix} = \begin{pmatrix} -X & -Y \\ Z & W \end{pmatrix} \Rightarrow \begin{array}{l} X^* = -X \rightarrow p^2 \text{ independent real entries} \\ W^* = -W \rightarrow q^2 \text{ independent real entries} \\ Z^* = Y \rightarrow 2pq \text{ independent real entries} \end{array}$$

$$\Rightarrow \mathcal{U}(p, q) = \left\{ \begin{pmatrix} X & Y \\ Y^* & W \end{pmatrix} \mid X \in \mathcal{U}(p), W \in \mathcal{U}(q), Y \in M_{p \times q}(\mathbb{C}) \right\}$$

$$\dim(\mathcal{U}(p, q)) = (p+q)^2$$