

$$\begin{aligned} \textcircled{1} \quad T_0(t) &= t+0 = t \rightarrow T_0 = \text{id} \\ T_{\epsilon+\epsilon'}(t) &= t+\epsilon+\epsilon' = (T_\epsilon \circ T_{\epsilon'})(t) \end{aligned} \quad \left. \vphantom{\begin{aligned} T_0(t) &= t+0 = t \\ T_{\epsilon+\epsilon'}(t) &= t+\epsilon+\epsilon' = (T_\epsilon \circ T_{\epsilon'})(t) \end{aligned}} \right\} \rightarrow T_{-\epsilon} = T_\epsilon^{-1}$$

$$\begin{aligned} Q_0(q) &= e^0 q = q \rightarrow Q_0 = \text{id} \\ Q_{\epsilon+\epsilon'}(q) &= e^{-\alpha(\epsilon+\epsilon')/2} q = e^{-\frac{\alpha\epsilon}{2}} e^{-\frac{\alpha\epsilon'}{2}} q = (Q_\epsilon \circ Q_{\epsilon'})(q) \end{aligned} \quad \left. \vphantom{\begin{aligned} Q_0(q) &= e^0 q = q \\ Q_{\epsilon+\epsilon'}(q) &= e^{-\alpha(\epsilon+\epsilon')/2} q = e^{-\frac{\alpha\epsilon}{2}} e^{-\frac{\alpha\epsilon'}{2}} q = (Q_\epsilon \circ Q_{\epsilon'})(q) \end{aligned}} \right\} \rightarrow Q_{-\epsilon} = Q_\epsilon^{-1}$$

$$\textcircled{2} \quad \delta t = \left. \frac{d}{d\epsilon} (t+\epsilon) \right|_{\epsilon=0} = 1 \quad \delta q = \left. \frac{d}{d\epsilon} e^{-\frac{\alpha\epsilon}{2}} q \right|_{\epsilon=0} = -\frac{\alpha}{2} e^{-\frac{\alpha\epsilon}{2}} q \Big|_{\epsilon=0} = -\frac{\alpha}{2} q$$

$$\begin{aligned} \delta S &= \int_a^b \left[L \underbrace{\frac{d}{dt} \delta t}_0 + \underbrace{\frac{\partial L}{\partial t} \delta t}_{\alpha L} + \underbrace{\frac{\partial L}{\partial q} \delta q}_{-kq e^{\alpha t}} + \underbrace{\frac{\partial L}{\partial \dot{q}} \left(\frac{d}{dt} \delta q - \dot{q} \frac{d}{dt} \delta t \right)}_{m\dot{q} e^{\alpha t} - \frac{\alpha}{2} \dot{q}} \right] dt \\ &= \int_a^b e^{\alpha t} \left[\cancel{\frac{\alpha m}{2} \dot{q}^2} - \cancel{\frac{\alpha k}{2} q^2} + \cancel{\frac{\alpha k}{2} q^2} - \cancel{\frac{\alpha m}{2} \dot{q}^2} \right] dt = 0 \end{aligned}$$

$\textcircled{3}$ the conserved quantity is

$$\begin{aligned} L \delta t + \frac{\partial L}{\partial \dot{q}} (\delta q - \dot{q} \delta t) &= L - \frac{\partial L}{\partial \dot{q}} \left(\frac{\alpha}{2} q + \dot{q} \right) \\ &= e^{\alpha t} \left(\frac{m}{2} \dot{q}^2 - \frac{k}{2} q^2 - \frac{m\alpha}{2} q \dot{q} - m \dot{q}^2 \right) \\ &= -\frac{e^{\alpha t}}{2} (m \dot{q}^2 + k q^2 + m\alpha q \dot{q}) \end{aligned}$$

$$\textcircled{4} \quad \frac{\partial L}{\partial q} = -kq e^{\alpha t} \quad \frac{\partial L}{\partial \dot{q}} = m\dot{q} e^{\alpha t} \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = m\ddot{q} e^{\alpha t} + \alpha m\dot{q} e^{\alpha t}$$

$$EL \text{ eq: } e^{\alpha t} (m\ddot{q} + \alpha m\dot{q} + kq) = 0 \Leftrightarrow m\ddot{q} + \alpha m\dot{q} + kq = 0$$

$\textcircled{5}$ Suppose that $m\ddot{q} + \alpha m\dot{q} + kq = 0$. Then

$$\begin{aligned} \frac{d}{dt} \left[e^{\alpha t} (m\dot{q}^2 + kq^2 + m\alpha q \dot{q}) \right] &= e^{\alpha t} \left[\alpha (m\dot{q}^2 + kq^2 + m\alpha q \dot{q}) + 2m\dot{q}\ddot{q} + 2kq\dot{q} + m\alpha \dot{q}^2 + m\alpha q \ddot{q} \right] \\ &= e^{\alpha t} \left[\underbrace{2\alpha m\dot{q}^2 + \alpha kq^2 + m\alpha^2 q \dot{q}}_{\text{orange}} + \underbrace{2m\dot{q}\ddot{q} + 2kq\dot{q} + m\alpha q \ddot{q}}_{\text{purple}} \right] \\ &= e^{\alpha t} \left[2\dot{q} (\cancel{m\ddot{q}} + \alpha \cancel{m\dot{q}} + kq) + \alpha q (\cancel{m\ddot{q}} + m\alpha \cancel{\dot{q}} + kq) \right] = 0 \end{aligned}$$