Extendability of Graphs with Perfect Matchings

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ABSTRACT

A matching of a graph is a set of pairwise disjoint edges and it is called perfect if every vertex of the graph is incident to some edge of the matching. The purpose of this thesis is the study of structural and algorithmic properties of graphs with perfect matchings. In particular, we focus on the following question: Assuming that k is a positive integer and G is a graph with perfect matching, is G k-extendable? That is, is it true that for every matching M of cardinality k in G there exists a perfect matching that entirely contains M?

There is a detailed structural characterization of bipartite graphs G with perfect matchings in terms of the existence of disjoint paths with certain properties which is a direct analogue of Menger's theorem. Let (U, V) be the bipartition of G and M be a perfect matching of G. Graph G is k-extendable if and only if there are k internally disjoint M-alternating paths between every vertex of U and every vertex of V. More strongly, it has been proven that someone can obtain the respective k paths for every other perfect matching M_0 by using the k paths for a specific perfect matching M.

From a computational perspective, the Extendability problem focuses on the question whether a graph G is k-extendable or not, where pair (G, k) is the input. The extendability of a graph G, denoted by *ext*(G), is defined as the maximum k for which G is k-extendable. In the general case, this problem is coNP-complete. In the case where graph G is bipartite, there is a polynomial algorithm that computes *ext*(G). Thus, the aforementioned problem can be decided in a polynomial amount of time on the number of vertices and edges of G.

ΣΥΝΟΨΗ

Ταίριασμα ενός γραφήματος είναι ένα σύνολο ακμών οι οποίες δεν έχουν κανένα κοινό άκρο και λέγεται τέλειο εάν κάθε κορυφή του γραφήματος προσπίτει σε κάποια ακμή του ταιριάσματος. Σκοπός της διπλωματικής είναι η μελέτη αλγοριθμικών και δομικών ιδιοτήτων γραφημάτων με τέλεια ταιριάσματα. Συγκεκριμένα, εστιάζουμε στην ακόλουθη ερώτηση: Υποθέτοντας ότι το k είναι ένας θετικός ακέραιος και G είναι ένα γράφημα, είναι το G k-επεκτάσιμο; Δηλαδή, είναι αλήθες ότι για κάθε ταίριασμα M στο G πληθυκότητας k υπάρχει κάποιο τέλειο ταίριασμα που περιέχει όλες τις ακμές του M;

Υπάρχει άμεση συσχέτιση στον δομικό χαρακτηρισμό των k-επεκτάσιμων διμερών γραφημάτων G με τέλεια ταιριάσματα και στην ύπαρξη k ξένων μονοπατιών, που είναι ανάλογο του θεωρήματος του Menger. Υποθέτοντας ότι το ζεύγος (U, V) είναι μία διαμέριση των κορυφών του G και M είναι ένα τέλειο ταίριασμα του, το G είναι $k-$επεκτάσιμο εάν και μόνον εάν υπάρχουν k εσωτερικώς διακεκριμένα M-εναλλασόμενα μονοπάτια μεταξύ κάθε κορυφής του U και κάθε κορυφής του V. Ισχυρότερα, αποδεικνύεται ότι είναι δυνατόν να βρεθούν αυτά τα k μονοπάτια για οποιοδήποτε άλλο ταίριασμα M_0 του G χρησιμοποιώντας τα γνωστά k μονοπάτια του τέλειου ταιριάσματος M.

Από υπολογιστικής απόψεως, το Extendability πρόβλημα εστιάζει στο εάν ένα γράφημα G είναι $k-$επεκτάσιμο, όπου (G, k) είναι η είσοδος. Η επεκτασιμότητα ενός γραφήματος G, η οποία συμβολίζεται *ext*(G), ορίζεται ως η μέγιστη τιμή του k για το οποίο το G είναι k-επεκτάσιμο. Στην γενική περίπτωση, αυτό το πρόβλημα είναι coNP-πλήρες. Στην περίπτωση όπου το G είναι διμερές, υπάρχει πολυωνυμικός αλγόριθμος που υπολογίζει το *ext*(G). Συνεπώς, το προαναφερθέν πρόβλημα μπορεί να αποφασιστεί σε πολυωνυμικό χρόνο ως προς τον αριθμό των κορυφών και των ακμών του G.

ACKNOWLEDGEMENTS

First and foremost i would like to express my deep gratitude to my advisor, Professor Archontia C. Giannopoulou for her valuable guidance, encouragement, patience and helpful critique throughout all stages of this work.

I would also like to give my sincere thanks to the other two members of my thesis examination committee, Professor Stavros G. Kolliopoulos and Dr. Dimitris Zoros, for their time, patience and support.

Also, i am thankful to the committee of the postgraduate program "Algorithms, Logic and Discrete Mathematics" for recognizing the course "Data Analytics - Big Data Mining Techniques". They gave me the opportunity to break into applications which are used in real world problems.

Furthermore, i would like to thank a professor from my undergraduate studies whose name is Nikolaos Tzanakis. He is the person who made mathematics even more attractive to me and suggested me to participate in this specific program for my graduate studies.

Finally, there are no words to thank my parents for being supportive and patient during the program.

CONTENTS

CHAPTER 1

PRELIMINARY DEFINITIONS

1.1 General graphs

Definition 1.1. A *graph* is a pair $G = (V, E)$, where V is a set whose elements are called *vertices* or *nodes* and E is a set whose elements are sets of two distinct vertices and they are called *edges* or *lines*. We can also write $V(G), E(G)$ instead of V, E respectively.

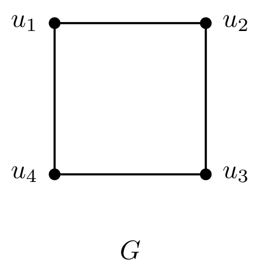

Figure 1.1: A graph $G = (V, E)$ with vertex set $V = \{u_1, u_2, u_3, u_4\}$ and edge set $E = \{\{u_1, u_2\}, \{\{u_2, u_3\}, \{\{u_3, u_4\}, \{\{u_1, u_4\}\}$.

There are different kinds of graphs G according to the properties of the set of edges E. For instance, a graph can contain loops, i.e. at least one edge that connects a vertex with itself, or parallel edges, i.e. two or more edges that connect two distinct vertices. Also, a graph can contain a set of either directed or undirected edges but not both.

Definition 1.2. A *directed graph* is a graph, where set E contains directed edges, i.e. every edge of E is an ordered pair of vertices of the graph.

For abbreviation, we will write digraphs instead of directed graphs.

Definition 1.3. An *undirected graph* is a graph, where set E contains undirected edges.

Definition 1.4. A *simple graph* is a graph that does not contain loops and parallel edges.

From now on, when we refer to a general graph without additional restrictions, we will mean a simple and undirected graph.

Now, we proceed with the terminology "neighborhood of a vertex in a graph". First of all, if $\{u, v\} \in E$ for some graph $G = (V, E)$, then u, v are called *adjacent*. Fix the vertex u. Let $\{v_1, \ldots, v_r\}$ be the maximum set of vertices of G such that $\{u, v_i\} \in E$ for every $i = 1, \ldots, r$. The elements of the set form the neighborhood of u in G.

Definition 1.5. Let $G = (V, E)$ be a graph and $u \in V$. The *neighborhood* of u in G, denoted by $N_G(u)$, is the set of vertices connected with u by an edge from E.

Observe that $N_G(u) = \{v \in V | \{u, v\} \in E\}$. In Figure 1.2, $N_G(u_1) = \{u_2, u_4\}$, $N_G(u_2) = \{u_1, u_3, u_4\}$, $N_G(u_3) = \{u_2, u_4\}$ and $N_G(u_4) = \{u_1, u_2, u_3\}$.

Definition 1.6. Let $G = (V, E)$ be a graph and $u \in V$. The *degree* of vertex u, denoted by $deg_G(u)$, is the total number of edges which are incident to it.

The *minimum degree* of graph is defined as $\delta(G) = \min\{deg_G(u) | u \in V\}$.

The following observation is a direct result from Definition 1.5 and Definition 1.6. It holds that $deg_G(u) = |N_G(u)|$. That is, the degree of a vertex in a graph equals the total number of its neighbors.

Definition 1.7. Let $G = (V, E)$ be a graph and let $S \subseteq V$. We define

$$G \setminus S = (V \setminus S, \{\{u, v\} \in E | \{u, v\} \cap S = \emptyset\}).$$

Definition 1.8. Let $G = (V, E)$ be a graph and let $S \subseteq V$. We consider the graph $G[S] = (S, E(S) = \{\{u, v\} \in E | u, v \in S\})$. Then $G[S]$ is called *induced subgraph* of graph G.

Observe that $G \setminus (V \setminus S) = G[S]$. Figure 1.2 shows an example of the process of deletion of a vertex set. Notice that a graph is an induced subgraph of itself.

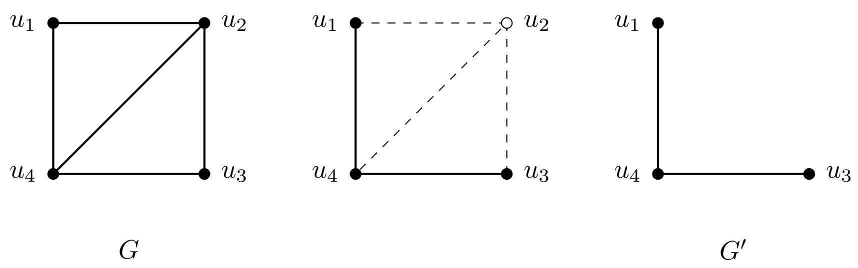

Figure 1.2: Graph G' is an induced subgraph of the initial graph G and it's obtained by deleting vertex u_2. Notice that $S = \{u_1, u_3, u_4\}$.

Definition 1.9. Let $G = (V, E)$ be a graph and $u_1, u_l \in V$. We define a $\{u_1, u_l\}$-*path* P to be a sequence of edges $\{u_1, u_2\}, \ldots, \{u_{l-1}, u_l\}$, which joins a sequence of distinct vertices $\{u_1, u_2, \ldots, u_{l-1}, u_l\}$. We will write $P = u_1 u_2 \ldots u_l$. The number of edges defines the *length* of a path.

Let $P = u_1 u_2 \dots u_l$ be a path in a graph G. Observe that the length of P is equal to $|V(P)| - 1$, where $V(P)$ denotes the set of vertices of this path. If $u_1 = u_l$, we say that the length of P is equal to zero and we call it a *trivial path*. Also, if we write $u_i \overrightarrow{P} u_j$, we mean the part of path P from u_i to u_j.

Let G be a graph and u, v be two distinct vertices of G. Furthermore, let P, Q be two $\{u, v\}$-paths. P, Q are *internally disjoint* if $V(P) \cap V(Q) = \{u, v\}$, i.e. they share only the start and end vertex.

Note that a *directed path* in a digraph is a sequence of edges which joins a sequence of distinct vertices, but with the additional restriction that the edges must be all directed in the same direction.

Definition 1.10. Let $G = (V, E)$ be a graph. Then G is *connected* if and only if there is a $\{u, v\}$-path for all pair of distinct vertices u, v of V.

If there is a pair of vertices such that there is no path between them, then the graph is called *disconnected*.

Definition 1.11. Let $G = (V, E)$ be a digraph. G is *strongly connected* if and only if for every pair of distinct vertices u, v of V there is a path from u to v and there is another path from v to u.

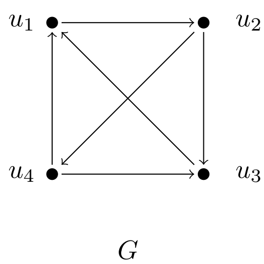

Figure 1.3: A strongly connected digraph G.

Definition 1.12. Let $G = (V, E)$ be a connected graph and let $S \subseteq V$. We call S a *separator* of G if the subgraph $G \setminus S$ of G is disconnected.

Definition 1.13. Let $G = (V, E)$ be a graph. G is *k-vertex-connected* if $|V| \geq k + 1$ and every separator of G has at least k vertices. We define the *connectivity* of a graph G to be $\kappa(G) = \max\{k | G \text{ is } k\text{-vertex-connected}\}$.

Now, we will define a class of graphs that it is going to concern us in the following chapters. Before that, we define the term ”independence of vertices” in a given graph.

Definition 1.14. Let $G = (V, E)$ be a graph and let $S \subseteq V$. We say that S is an *independent set* of G if there is no edge between any pair of two distinct vertices of S. Specifically, for every $u, v \in S$ with $u \neq v$ it holds that $\{u, v\} \notin E$.

Definition 1.15. A graph $G = (V, E)$ is called *bipartite* if there are two sets $S_1, S_2 \subseteq V$ such that (i) $S_1 \cup S_2 = V$, (ii) $S_1 \cap S_2 = \emptyset$ and (iii) S_1, S_2 are independent sets of G.

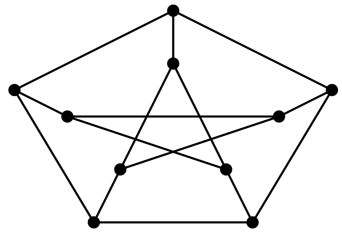

Figure 1.4: Petersen graph is 3-vertex-connected..

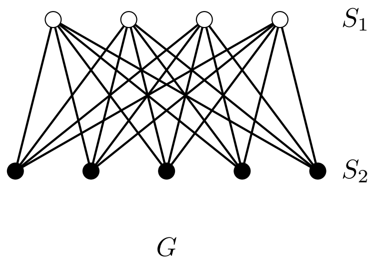

Figure 1.5: The graph G is a bipartite graph, since the sets S_1, S_2 satisfy the desired conditions.

If we refer to a bipartite graph $G = (V, E)$ with bipartition (S_1, S_2), we can write, for abbreviation, $G = (S_1, S_2, E)$. This alternative method provide us a way to easily understand that the given graph is bipartite.

Definition 1.16. Let $G = (V, E)$ be a graph. Let $\{u, v\}$ be an edge from E. Let P_1 be a path in G of odd length from u to v in such a way that it does not use the edge $\{u, v\}$. Observe that $G_1 = P_1 + \{u, v\}$ is an even cycle. Thus, G_1 is a bipartite graph. We proceed inductively to construct a sequence of bipartite graphs. Let the bipartite graph $G_r = \{u, v\} + P_1 + \cdots + P_r$, where P_r is a path of odd length joining two vertices of different partitions of G and having no other common vertex with G_{r-1}. If $G_r = G$, then G_r is called an *ear decomposition* of G.

Figure 1.6 illustrates an ear decomposition of a graph G.

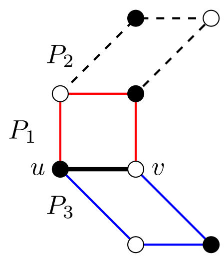

Figure 1.6: An ear decomposition of G with $G_r = \{u, v\} + P_1 + P_2 + P_3$.

1.2 Perfect matching

Definition 1.17. Let $G = (V, E)$ be a graph. A *matching* of G is a set $M \subseteq E$ of vertex-disjoint or *independent* edges.

We will call a vertex *matched* with respect to a specific matching if it is an endpoint of an edge of this matching. Otherwise, we will call it *unmatched*. Furthermore, we will call an edge *matched* with respect to a specific matching if it belongs to this matching. Otherwise, we will call it *unmatched*.

Definition 1.18. A *perfect matching* is a matching that matches all the vertices.

A direct observation is that a graph must have an even number of vertices in order to contain a perfect matching. Otherwise, it is impossible. But, we have to be careful, since this is not the only condition.

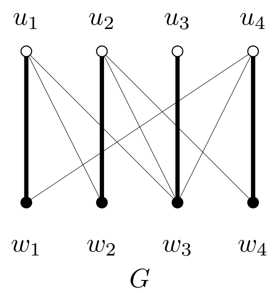

Figure 1.7: The set $M = \{\{u_1, w_1\}, \{u_2, w_2\}, \{u_3, w_3\}, \{u_4, w_4\}\} \subseteq E$ is a perfect matching of the graph G.

Definition 1.19. Let k be a positive integer and G be a graph with $|V(G)| \geq 2k + 2$. G is k*-extendable* if G has a perfect matching and any k independent edges of G can be extended to a perfect matching of G. That is, every matching of G of cardinality k is a subset of a perfect matching in G.

Definition 1.20. The *extendability* of a graph G is defined as the maximum value of k for which G is k-extendable. It is denoted by $ext(G)$.

The table in Figure 1.8 describes the main problem of this thesis.

Extendability	
Input:	A graph G and a natural k.
Question:	Is the graph G k-extendable?

Figure 1.8: Description of the problem.

We remind you that in the previous section we defined the term path of a graph. Now, we present alternative definitions about what a path is with respect to some perfect matching.

Definition 1.21. Let G be a connected and bipartite graph and M be a perfect matching of G. An *M-alternating path* P of G is a path in G where edges in M and edges in $E \setminus M$ appear on P alternately.

Let P be an M-alternating path of odd length. If the edges at the extremities of P are unmatched then P is called *free* otherwise it is called *saturated*.

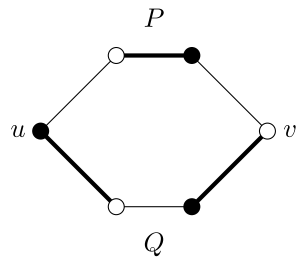

Figure 1.9: P is a free M-alternating $\{u, v\}$-path whereas Q is a saturated M-alternating $\{u, v\}$-path.

Definition 1.22. Let G be a connected and bipartite graph and M be a perfect matching of G. An *M-alternating cycle* is an M-alternating path where the first and last vertices of the path are the same.

Let $G = (S_1, S_2, E)$ be a graph with a perfect matching M. Let $P = u_0u_1 \dots u_l$ be an M-alternating path and let $C = v_0v_1 \dots v_rv_0$ be a M-alternating cycle. The *predecessor* of a vertex is defined as follows:

i. For each $1 \leq i \leq l$, we define $u_i^{(-P)} = u_{i-1}$.

ii. For each $1 \leq i \leq r-1$, the vertex v_i has exactly two neighbors v_{i-1}, v_{i+1} in C with, without loss of generality, $\{v_{i-1}, v_i\} \in M$ and $\{v_i, v_{i+1}\} \in E \setminus M$. Then, we define $v_i^{(-C)} = v_{i-1}$, if $v_i \in S_1$, and $v_i^{(-C)} = v_{i+1}$, if $v_i \in S_2$.

Definition 1.23. Let $G = (S_1, S_2, E)$ be a graph and M be a perfect matching of G. We define as the *residual graph* of G, denoted by G_M, the graph obtained from G by directing the edges in $E \setminus M$ from S_1 to S_2 and the edges in M from S_2 to S_1.

Figure 1.10 illustrates the construction of a residual graph by a graph with a perfect matching using the previous definition.

1.3 Another way to see bipartite graphs with perfect matchings

A useful observation is that we can obtain a digraph by a bipartite graph with perfect matching by following specific rules of construction and vice versa. We will explain the first method of construction, where given a graph as described on the title of the section, we obtain a digraph. Definition 1.24 defines the procedure of this construction. Figure 1.11 illustrates an example of this construction in act.

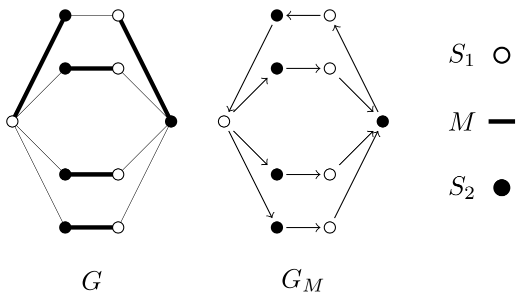

Figure 1.10: A graph $G = (S_1, S_2, E)$ with perfect matching M and its residual graph G_M.

Definition 1.24. Let $G = (S_1, S_2, E)$ be a bipartite graph and let $M \in \mathcal{M}(G)$ be a perfect matching of G, where $\mathcal{M}(G)$ is a family that consists of all perfect matchings of G. The M-*digraph* $\mathcal{D}(G, M)$ is defined as follows. Suppose that M contains the edges $\{a_1, b_1\}, \dots, \{a_{|M|}, b_{|M|}\}$ with $a_i \in S_1, b_i \in S_2$ for $i = 1, \dots, |M|$. Then,

i. $V(\mathcal{D}(G, M)) := \{u_1, \dots, u_{|M|}\}$

ii. $E(\mathcal{D}(G, M)) := \{\{u_i, u_j\} | \{a_i, b_j\} \in E, i \neq j\}$.

Observe that the edges of M transform into vertices in $\mathcal{D}(G, M)$. Intuitively, we give direction on the edges from S_1 to S_2. Two vertices u_i, u_j of $\mathcal{D}(G, M)$ connect by an edge if there is an edge between a_i and b_j in G.

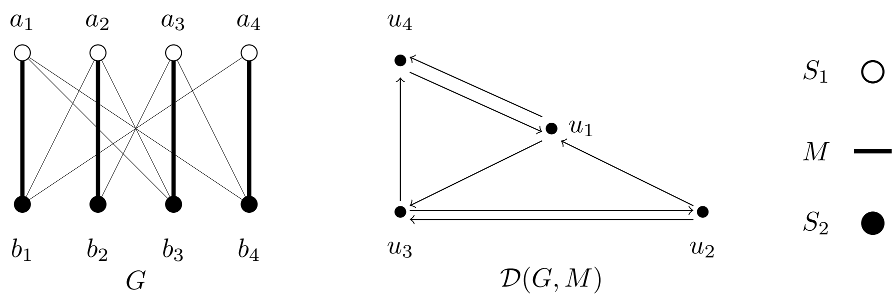

Figure 1.11: The bipartite graph $G = (S_1, S_2, E)$ and the M-digraph $\mathcal{D}(G, M)$.

When we say that a bipartite graph with perfect matching is k-extendable, it's like we speak about the connectivity of an undirected graph. Furthermore, there is a corresponding relation between the extendability of bipartite graphs and the strong connectivity of digraphs. The last correlation is described in [3] and we are going to deeply explore its usefulness in section 3.3.

As you have probably already noticed, there is an extensive reference to the specific class of graphs which are bipartite with perfect matching. We assure you that there is a reason about it. In Chapter 2, we prove an important theorem which is quite similar to Menger's theorem for general graphs, whereas in Chapter 3 we prove the hardness of Extendability problem depending on the input graph.

CHAPTER 2

STRUCTURAL CHARACTERIZATION OF K-EXTENDABLE BIPARTITE GRAPHS

In this chapter, we assume that a graph G is always undirected, simple, connected and bipartite. First, we present some basic theorems regarding graphs with perfect matchings and k-extendability.

2.1 Basic theorems

Theorem 2.1 (Plummer [1]). Let G be a graph on n vertices with bipartition (S_1, S_2). Suppose that k is a positive integer such that $k \leq \frac{n-2}{2}$. The following are equivalent:

i. G is k-extendable,

ii. $|S_1| = |S_2|$ and for each $X \subseteq S_1$ such that $|X| \leq |S_1| - k$, $|N_G(X)| \geq |X| + k$,

iii. For all $s_1^1, \ldots, s_k^1 \in S_1$ and $s_1^2, \ldots, s_k^2 \in S_2$, $G' = G \setminus s_1^1 \setminus \cdots \setminus s_k^1 \setminus s_1^2 \setminus \cdots \setminus s_k^2$ has a perfect matching.

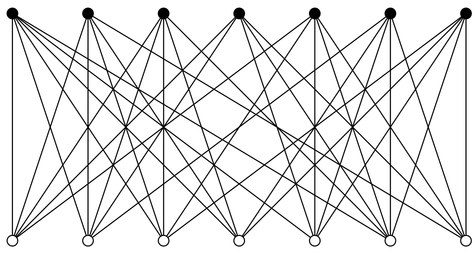

Figure 2.1: A 4-extendable graph G.

Theorem 2.2 (Dingjun Lou [7]). Let $G = (S_1, S_2, E)$ be a graph. If G is k-extendable, then for each $X \subseteq S_1$ such that $|S_1| - k < |X| \leq |S_1|$, $|N_G(X)| = |S_2|$.

Theorem 2.3 ([2])**.** Let $G = (S_1, S_2, E)$ be a k-extendable graph for a positive integer k. Then for any $X \subseteq S_1$, if $N_G(X) \neq S_2$, then $|N_G(X)| \geq |X| + k$.

Proof. Let $X \subseteq S_1$. We consider only the case where $|X| \leq |S_1| - k$, because if $|S_1| - k < |X| \leq |S_1|$, then $N_G(X) = S_2$. Since G is k-extendable, it follows directly from Theorem 2.1 that $|N_G(X)| \geq |X| + k$. □

Theorem 2.4 (Plummer [6])**.** If G is k-extendable, then $\kappa(G) \geq k + 1$.

Observe that $\delta(G) \geq \kappa(G)$. This observation together with Theorem 2.4 implies that a k-extendable graph has $\delta(G) \geq k+1$. Furthermore, the extendability of a graph is strictly smaller that $\delta(G)$. The last observation is a direct result from Theorem 2.1.

Theorem 2.5 (Plummer [6])**.** Let k be an integer such that $0 < k < n$. If G is k-extendable, then G is $(k-1)$-extendable.

Lemma 2.6. (Lovasz, Plummer [10]) G is 1-extendable if and only if G has an ear decomposition.

Lemma 2.7. (Lovasz, Plummer [10]) G is 1-extendable if and only if every edge of G belongs to an alternating cycle.

Lemma 2.8. (Plummer [1]) Let p, k be two integers such that $0 < p < k < |V(G)|$. G is k-extendable if and only if for every $s_1^1, \ldots, s_p^1 \in S_1$ and for every $s_1^2, \ldots, s_p^2 \in S_2$, $G \setminus s_1^1 \setminus s_1^2 \setminus \cdots \setminus s_p^1 \setminus s_p^2$ is $(k-p)$-extendable.

Lemma 2.9. ([3]) Let p, k be two integers such that $0 < p < k < |V(G)|$. G is k-extendable if and only if for every matching $M_p = \{\{s_1^1, s_2^1\}, \ldots, \{s_p^1, s_p^2\}\}$ of p edges, $G \setminus s_1^1 \setminus s_1^2 \setminus \cdots \setminus s_p^1 \setminus s_p^2$ is $(k-p)$-extendable.

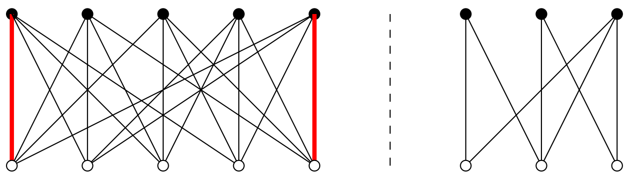

Figure 2.2: G is 3-extendable, $p = 2$ and M_p is a matching of 2 edges. The deletion of the vertices incident to the edges of M_p creates a 1-extendable graph.

Proof. Let $G = (S_1, S_2, E)$ be a graph. Fix a random matching M_p of p edges as described. Further, let $H = G \setminus s_1^1 \setminus s_1^2 \setminus \cdots \setminus s_p^1 \setminus s_p^2$.

Assume that G is k-extendable. By Lemma 2.8, for the particular subset of vertices $s_1^1, \ldots, s_p^1 \in S_1$ and $s_1^2, \ldots, s_p^2$ such that $\{s_1^1, s_1^2\}, \ldots, \{s_p^2, s_p^2\} \in M_p$, H is $(k-p)$-extendable.

Assume that H is $(k-p)$-extendable. Then H has a perfect matching and every matching of size $k - p$ can be extended to a perfect matching. Let M_{k-p} be such a matching of H and let M be the perfect matching of H that contains M_{k-p}. Observe that $M' = M \cup M_p$ is a perfect matching of G. It follows that every matching composed of M_p and any other M_{k-p} extends to a perfect matching in G. Thus, G is k-extendable. □

2.2 Alternating paths on a fixed perfect matching

Here, we focus on the following structural characterization of bipartite graphs with perfect matchings.

Theorem 2.10 ([2])**.** Let $G = (S_1, S_2, E)$ be a graph with perfect matching. G is k-extendable if and only if for any perfect matching M and for each $x \in S_1, y \in S_2$, there are k internally disjoint M-alternating paths $P_1, \ldots, P_k$ connecting x and y. These paths start and end with edges in $E \setminus M$.

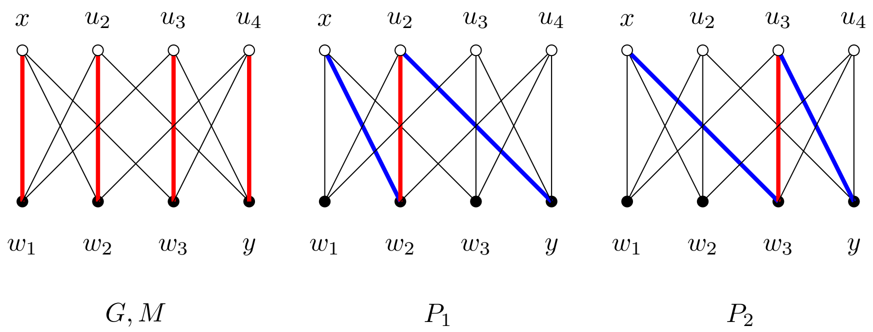

Figure 2.3: A 2-extendable graph G with perfect matching M and M-alternating paths P_1, P_2.

Proof. Let S be a matching in G of k edges such that it is not contained in a perfect matching. Suppose towards a contradiction that there are k internally disjoint M-alternating paths between every pair of two distinct vertices of different bipartition in G, where M is a perfect matching in G which contains as many edges of S as possible. Observe that there is an edge $e = \{u, v\} \in E$ such that $e \in S$ and $e \notin M$. Let $u \in S_2$ and $v \in S_1$. Since M is a perfect matching, there are vertices x, y in S_1, S_2 respectively such that $\{u, x\}, \{v, y\} \in M$. Let $P_1, \ldots, P_k$ be the paths joining x and y such that each P_i starts and ends with edges in $E \setminus M$. Since $|S \setminus e| = k - 1$, there is at least one path that does not contain any edge from S. Let P_j be this path. We consider $C = P_j + yvux$. Observe that C is an M-alternating cycle. Let $M' = M \triangle E(C) = (M \setminus E(C)) \cup (E(C) \setminus M)$. Then M' is also a perfect matching of G. The crucial observation is that every edge in $M \cap S$ and e belong to M'. Thus, M' contains strictly more edges from S than M. This result contradicts the choice of M. Thus, there are no k internally disjoint M-alternating paths between every vertex of S_1 and every vertex of S_2.

Let $G = (S_1, S_2, E)$ be a k-extendable graph, M be a perfect matching of G, $x \in S_1$ and $y \in S_2$. We proceed with the introduction of the following terminology and notation before we prove this part of the theorem.

Let $P = x_1 x_2 \ldots x_l$. Then $\{x_i, x_{i+1}\} \in E \setminus M$, if i is odd, and $\{x_i, x_{i+1}\} \in M$, if i is even. At this point, we suggest the reader to recall the definition of the predecessor of a vertex in a path. For abbreviation, we omit the phrase "with respect to M". Let y' be the unique vertex such that $\{x, y'\} \in M$. It is possible $y = y'$. The following paragraph describes the construction of a useful tool for the proof of this direction.

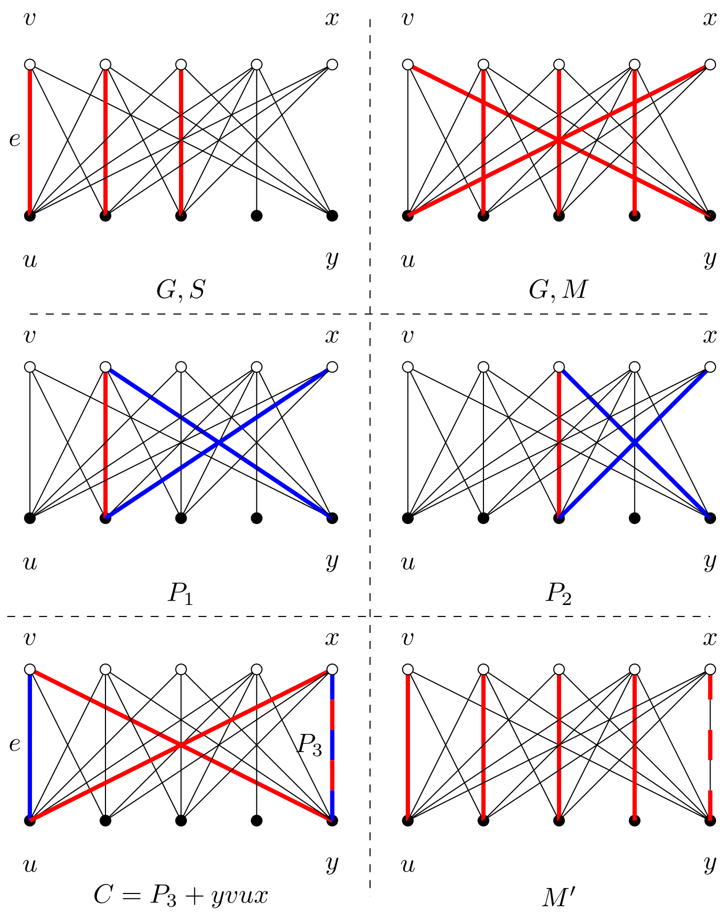

Figure 2.4: A sketch of the first part of the proof given a graph G which is not 3-extendable. Observe that $|M' \cap S| > |M \cap S|$.

Let $P_1, \ldots P_{k-1}$ be alternating paths from x to y. Let Q be an alternating path from x to some vertex $v \in S_1$. Note that if $v = x$, then Q is a trivial path. Also, let Γ be a set of alternating cycles in G. Γ may be an empty set. We say that $K = (P_1, \ldots, P_{k-1}, Q, \Gamma)$ is a k-*system* if the following conditions hold:

i. $P_1, \ldots, P_{k-1}$ are alternating internally disjoint paths from x to y.

ii. For each $1 \leq i \leq k-1$, $V(P_i) \cap V(Q) = \{x\}$.

iii. Every pair of two elements of Γ are vertex-disjoint.

iv. For each $C \in \Gamma$, $(\bigcup_{i=1}^{k-1} V(P_i) \cup V(Q)) \cap V(C) \subseteq \{x, y'\}$.

Let K be a k-system. We define

$$V(K) = \bigcup_{i=1}^{k-1} V(P_i) \cup V(Q) \cup \bigcup_{C \in \Gamma} V(C)$$

and

$$E(K) = \bigcup_{i=1}^{k-1} E(P_i) \cup E(Q) \cup \bigcup_{C \in \Gamma} E(C)$$

.

Let $v \in S_2 \setminus y$. We define the predecessor of v with respect to K as follows:

i. If $v \in V(P_i)$, then $v^{-(K)} = v^{-(P_i)}$.

ii. If $v \in V(Q)$, then $v^{-(K)} = v^{-(Q)}$.

iii. If $v \in V(C)$, then $v^{-(K)} = v^{-(C)}$.

iv. If $v \notin V(K)$, then $v^{-(K)} = u$, where u is a vertex such that $\{u, v\} \in M$.

Moreover, we define $V^{-(K)} = \{v^{-(K)} | v \in V\}$, for each $V \subseteq S_2 \setminus y$.

Now, we are ready to continue with the proof. We proceed by induction on k. If $k = 0$, then theorem is true. Suppose that $k \geq 1$ and theorem is true for $k-1$. Assume that there are no k alternating paths joining x and y in G. Since G is k-extendable, it follows by Theorem 2.5 that G is $(k-1)$-extendable. By induction hypothesis, there are $k-1$ alternating paths $P_1^0, \ldots, P_{k-1}^0$ from x to y. Let $Q^0 = x$ be a trivial path.

Let $K^0 = (P_1^0, \ldots, P_{k-1}^0, Q^0, \emptyset)$. Observe that K^0 is a k-system.

For a natural i, we recursively define:

$$A_i = \begin{cases} \{x\}, & i = 0 \\ A_{i-1} \cup B_i^{-(K^0)}, & i \geq 1 \end{cases}$$

and

$$B_i = \begin{cases} \emptyset, & i = 0 \\ N_{G \setminus y}(A_{i-1}), & i \geq 1 \end{cases}$$

Observe that this construction defines two infinite chains $\emptyset = B_0 \subseteq B_1 \subseteq \ldots$ and $\{x\} = A_0 \subseteq A_1 \subseteq \ldots$. Let $A = \bigcup_{i=0}^{\infty} A_i$ and $B = \bigcup_{i=0}^{\infty} B_i$. Observe that $A \subseteq S_1$ and $B \subseteq S_2$. Let $h : A \cup B \to \mathbb{N}$ be the function that follows. Alternatively, we can refer to this function as the height function of a vertex. For every $w \in A \cup B$,

$$h(w) = \begin{cases} \min\{i | w \in A_i\}, & w \in A \\ \min\{i | w \in B_i\}, & w \in B \end{cases}$$

Intuitively, the height of such a vertex w is equal to the length of $x\overrightarrow{P}w$, where P is the path that contains w. We proceed by proving three claims.

Claim 2.11 ([2]). *For each $u \in A$, there exist a k-system $K = (P_1, \ldots, P_{k-1}, Q, \Gamma)$ such that:*

(1) u is the terminal vertex of Q.

(2) for each $v \in S_2 \setminus y$, if $h(v) > h(u)$ then $v^{-(K)} = v^{-(K^0)}$.

Proof. We prove this claim by induction on $h(u)$. Let $h(u) = 0$. By the previous terminology, it follows that $u = x$. Observe that K^0 is the required k-system for the base case. Suppose that this claim holds in every case where $h(u) < t$ for $t > 0$. Now, let $h(u) = t$. Then $u \in A_t \setminus A_{t-1}$. Since $A_t = A_{t-1} \cup B_t^{-(K^0)}$,

$u \in B_t^{-(K^0)}$. There exists a vertex $v_0 \in B_t$ such that $u = v_0^{-(K^0)}$. Observe that $v_0 \notin B_{t-1}$, because, otherwise, we would have $u \in B_{t-1}^{-(K^0)} \subseteq A_{t-1}$ which contradicts the hypothesis. Hence, $h(v_0) = t$. Let u_0 be a vertex in A_{t-1} such that $v_0 \in N_{G\setminus y}(u_0)$. Observe that $u_0 \notin A_{t-2}$, because, otherwise, we would have that $v_0 \in N_{G\setminus y}(A_{t-2}) = B_{t-1}$. Hence, $h(u_0) = t-1$. By induction hypothesis, there exist a k-system $K' = (P_1^{'}, \dots, P_{k-1}^{'}, Q', \Gamma')$ such that (1) u_0 is the terminal vertex of Q' and (2) for each $v \in S_2 \setminus y$, if $h(v) > h(u_0)$, then $v^{-(K')} = v^{-(K^0)}$. Since $h(v_0) = t$ and $h(u_0) = t-1$, $v_0^{-(K')} = v_0^{-(K^0)} = u$. We consider two cases depending on whether $\{u_0, v_0\} \notin M$ or $\{u_0, v_0\} \in M$ and we prove that there exists a desired k-system for vertex u. The Figures of each different case can be found in Figure 2.1. The paths $P_1', \dots, P_{k-1}'$ are colored red. The path Q' is colored blue. The alternating cycles that can be found in Γ' are colored brown. And finally, we make the edge connecting the vertices u_0 and v_0 dashed. These illustrations aim to provide intuition behind to understand the construction of the desired k-system.

Firstly, let us assume that $\{u_0, v_0\} \notin M$. Then we have to consider the following four cases.

(i) Let $v_0 \in V(P_i^{'})$. Then $v_0^{-(K')} = v_0^{-(P_i^{'})} = u$. Then $(P_1, \dots P_{k-1}, Q, \Gamma)$ is a k-system, where $P_i = x\overrightarrow{Q'}u_0v_0\overrightarrow{P_i^{'}}y$, $P_j = P_j^{'}$, for $j \neq i$, $Q = x\overrightarrow{P_i^{'}}u$ and $\Gamma = \Gamma'$.

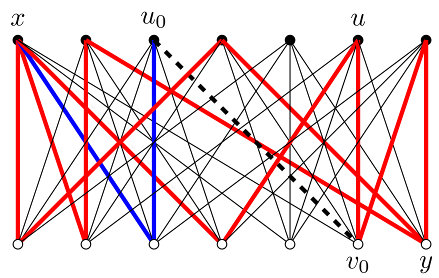

Figure 2.5: The case where $\{u_0, v_0\} \notin M$ and $v_0 \in V(P_i')$ for some index i.

(ii) Let $v_0 \in V(Q')$. Then $v_0^{-(K')} = v_0^{-(Q')} = u$. If $v_0 = u_0^{-(K')}$, then $\{u_0, v_0\} \in M$. This contradicts the hypothesis of this case. Thus, $v_0 \neq u_0^{-(K')}$. Observe that $C = v_0\overrightarrow{Q'}u_0v_0$ is an alternating cycle. Furthermore, $V(C) \cap V(C') = \emptyset$, for every $C' \in \Gamma'$. $(P_1, \dots, P_{k-1}, Q, \Gamma)$ is a k-system, where $P_i = P_i^{'}$, for every $1 \leq i \leq k-1$, $Q = x\overrightarrow{Q'}u$ and $\Gamma = \Gamma' \cup \{C\}$ (see Figure 2.6).

(iii) Let $v_0 \in V(C')$, for some $C' \in \Gamma'$. Then $v_0^{-(K')} = v_0^{-(C')} = u$. We have two cases to consider. If $x \in V(C)$, then let $P_i = P_i^{'}$, for every i, $Q = x\overrightarrow{C'}u$ and $\Gamma = (\Gamma' \setminus C') \cup C$, where $C = x\overrightarrow{Q'}u_0v_0\overrightarrow{C'}x$ is a new alternating cycle. If $x \notin V(C')$, then let $P_i = P_i^{'}$, for every i, $Q = x\overrightarrow{Q'}u_0v_0\overrightarrow{C'}u$ and $\Gamma = \Gamma' \setminus C'$.

(iv) Let $v_0 \notin V(K')$. Let u' be the unique vertex such that $\{v_0, u'\} \in M$. Then $v_0^{-(K')} = u' = u$. We have two cases to consider depending on whether v_0 is

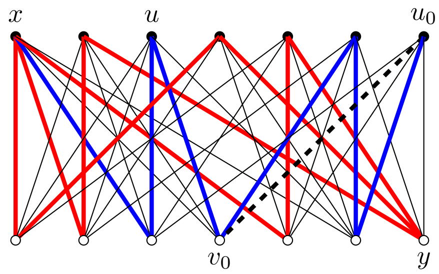

Figure 2.6: The case where $\{u_0, v_0\} \notin M$ and $v_0 \in V(Q')$.

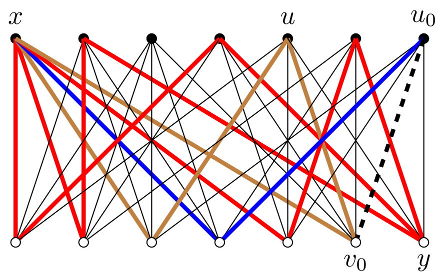

Figure 2.7: The case where $\{u_0, v_0\} \notin M$, $v_0 \in V(C')$ and $x \in V(C')$.

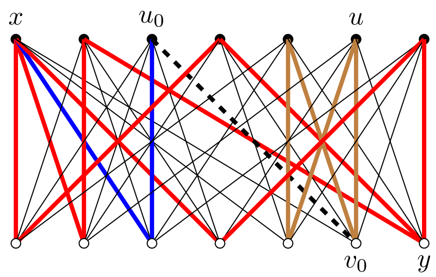

Figure 2.8: The case where $\{u_0, v_0\} \notin M$, $v_0 \in V(C')$ and $x \notin V(C')$.

equal or not to y'. If $v_0 \neq y'$, then $u \notin V(K')$. Suppose that $u \in V(K')$. We will prove that this assumption leads us to the contradiction $v_0 \in V(K')$. We have that $u \in V(P_i^{'})$, for some i, or $u \in V(Q')$ or $u \in V(C')$, for some $C' \in \Gamma'$. If $u \in V(P_i^{'})$, then $v_0 \in V(P_i^{'})$. If $u \in V(Q')$, then $v_0 \in V(Q')$. If $u \in V(C')$, then $v_0 \in V(C')$. Let $P_i = P_i^{'}$, $Q = x\overrightarrow{Q'}u_0v_0u$ and $\Gamma = \Gamma'$. Assume that $v_0 = y'$. Then $v_0^{-(K')} = y'^{-(K')}$. Since $v_0^{-(K')} = u$ and $y'^{-(K')} = x$, it follows that $x = u$. Let $P_i = P_i^{'}$, for every i, $Q = x$ and $\Gamma = \Gamma' \cup \{C\}$, where $C = x\overrightarrow{Q'}u_0y'x$ is a new alternating cycle (see Figures 2.9 and 2.10).

Now, let us assume that $\{u_0, v_0\} \in M$. The assumption $u_0 = x$ leads us to a contradiction. If the equality holds, then $v_0 = y'$. That is because $\{x, y'\} \in M$. It

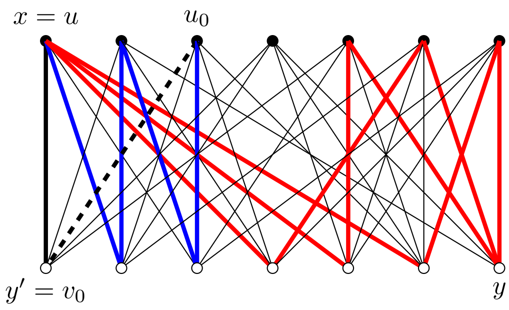

Figure 2.9: The case where $\{u_0, v_0\} \notin M$, $v_0 \notin V(K')$ and $v_0 = y'$.

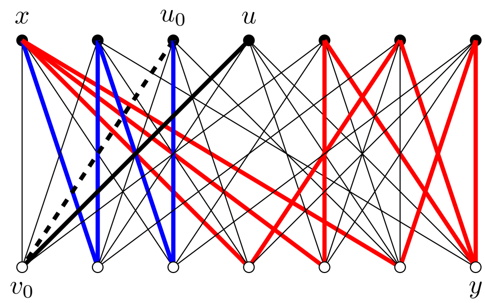

Figure 2.10: The case where $\{u_0, v_0\} \notin M$, $v_0 \notin V(K')$ and $v_0 \neq y'$.

follows that $u = y'^{-(K^0)} = x$. This implies $h(u) = h(x) = 0$. This contradicts the hypothesis that $h(u) > 0$. Hence, $u_0 \in V(Q') \setminus x$. Since $\{u_0, v_0\} \in M$, $v_0 \in V(Q')$. Furthermore, $u_0^{-(Q')} = v_0$. Since $v_0 \in V(Q')$ and $u = v_0^{-(K')}$, we have that $u = v_0^{-(Q')}$. Let $P_i = P_i'$, for every i, $Q = x\overrightarrow{Q'}u$ and $\Gamma = \Gamma'$.

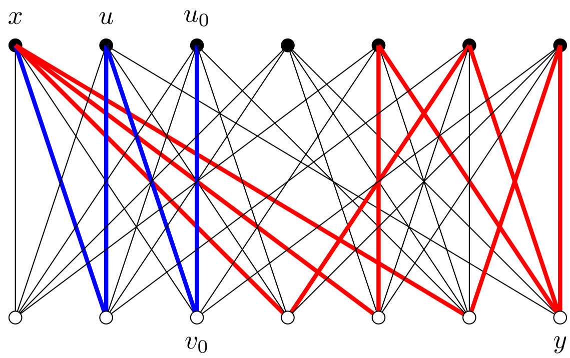

Figure 2.11: The case where $\{u_0, v_0\} \in M$.

We proved that there exist a k-system such that condition (1) of the claim holds. Let $K = (P_1, \ldots, P_{k-1}, Q, \Gamma)$ be this k-system such that u is the terminal vertex of

Q. For every $v \in S_2 \setminus y$, observe that:

$$v^{-(K)} = \begin{cases} v^{-(K')}, & v \neq v_0 \\ u_0, & v = v_0 \end{cases}$$

Let $h(v) > h(u) = t$. Since $h(v_0) = t$, $v \neq v_0$. This implies $v^{-(K)} = v^{-(K')}$. On the other hands, since $h(v) > h(u_0)$, $v^{-(K')} = v^{-(K^0)}$. Thus, $v^{-(K)} = v^{-(K^0)}$ and condition (2) holds as well. $\square$

Claim 2.12 ([2]). *(1) $y \notin N_G(A \setminus x)$.*

(2) If $\{x, y\} \notin M$, then $y \notin N_G(A)$.

Proof. (1) Suppose towards a contradiction that $y \in N_G(A \setminus x)$. Let u be a vertex of $A \setminus x$ in G. By Claim 2.7, there exist a k-system $K = (P_1, \ldots, P_{k-1}, Q, \Gamma)$ such that u is the terminal vertex of Q. Observe that $\{u, y\} \in E \setminus M$ and, thus, $x\overrightarrow{Q}uy$ is an alternating path which is internally disjoint to every one of $P_1, \ldots, P_{k-1}$. This contradicts the initial assumption that there are no k alternating paths connecting x and y in G. Thus, $y \notin N_G(A \setminus x)$.

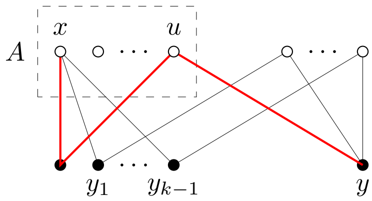

Figure 2.12: The forbidden path in case $y \in N_G(A \setminus x)$.

(2) Let $y \in N_G(A)$. Then there exist a vertex $u \in A$ such that $y \in N_G(u)$. By Claim 2.7, there exist a k-system $K = (P_1, \ldots, P_{k-1}, Q, \Gamma)$ such that u is the terminal vertex of Q. We remind you that there are no k alternating paths from x to y. Hence, $x\overrightarrow{Q}uy$ is not an alternating path. This occurs only if $x = u$ and $x\overrightarrow{Q}uy = \{x, y\} \in M$. $\square$

Claim 2.13 ([2]). *(1) $N_{G\setminus y}(A) = B$.*

(2) $N_G(A \setminus x) \subseteq B$.

(3) If $\{x, y\} \notin M$, then $B = N_G(A)$.

(4) $A = B^{-(K^0)}$.

Proof. (1) Let $v \in N_{G\setminus y}(A)$. There exist a vertex $u \in A$ such that $v \in N_G(u)$. Let $h(u) = s$, where s is a positive integer. Equivalently, $u \in A_s$. Hence, $v \in N_{G\setminus y}(A_s)) = B_{s+1} \subseteq B$. Thus, $N_{G\setminus y}(A) \subseteq B$. Let $v \in B$ and let $h(v) = t$, where t is a positive integer. Then $v \in B_t = N_{G\setminus y}(A_t) \subseteq N_{G\setminus y}(A)$. Thus, $B \subseteq N_{G\setminus y}(A)$ and the equality holds.

(2) Recall that $y \notin N_G(A \setminus x)$. This implies $N_G(A \setminus x) = N_{G\setminus y}(A \setminus x)$. Since $N_{G\setminus y}(A \setminus x) \subseteq N_{G\setminus y}(A)$ and $N_{G\setminus y}(A) = B$, $N_G(A \setminus x) \subseteq B$.

(3) Recall that if $\{x, y\} \notin M$, then $y \notin N_G(A)$. This implies $N_G(A) = N_{G\setminus y}(A)$. Since $N_{G\setminus y}(A) = B$, $B = N_G(A)$.

(4) Let $u \in A$ and let $h(u) = s$, where s is a positive integer. By the definition of the height function, $u \in A_s \setminus A_{s-1}$. Hence, $u \in B_s^{-(K^0)} \subseteq B^{-(K^0)}$. Let $u \in B^{-(K^0)}$. Equivalently, there exist a vertex $v \in B$ such that $u = v^{-(K^0)}$. Let $h(v) = t$, where t is a positive integer. Then $v \in B_t$. Hence, $u \in B_t^{-(K^0)} \subseteq A_t \subseteq A$. Thus, the equality holds. □

Suppose that y_i is the second vertex vertex of P_i^0, for every $1 \leq i \leq k-1$. Observe that for every pair of distinct vertices v_1, v_2 of $S_2 \setminus y$, the equality $v_1^{-(K^0)} = v_2^{-(K^0)}$ holds only if $v_1, v_2 \in \{y', y_1, \dots, y_{k-1}\}$. Furthermore, $\{y', y_1, \dots, y_{k-1}\}^{-(K^0)} = \{x\}$. By combining two previous notations, it holds that $|B^{-(K^0)}| \geq |B| - k + 1$. Since $A = B^{-(K^0)}$, $|A| \geq |B| - k + 1$. The equality holds if $\{y', y_1, \dots, y_{k-1}\} \subseteq B$.

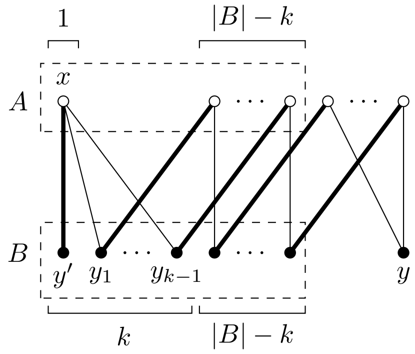

Figure 2.13: An illustration of the case where $|A| = |B| - k + 1$.

Let $\{x, y\} \notin M$. Then $y \notin N_G(A)$ and therefore $N_G(A) \neq S_2$. By Theorem 2.3, this implies that $|N_G(A)| \geq |A| + k$. On the other hands, recall that $N_G(A) = B$. Hence, $|N_G(A)| = |B| \leq |A| + k - 1$.

Let $\{x, y\} \in M$. Since $N_G(A \setminus x) \subseteq B$, $|N_G(A \setminus x)| \leq |B|$. Since $y \notin N_G(A \setminus x)$, $N_G(A \setminus x) \neq S_2$. Hence, by Theorem 2.3, $|N_G(A \setminus x)| \geq |A \setminus x| + k = |A| + k - 1$ and therefore $|B| \geq |A| + k - 1$. However, recall that $|B| \leq |A| + k - 1$. Thus, $|B| = |A| + k - 1$. This implies that $\{y', y_1, \dots, y_{k-1}\} \subseteq B$. Since y' is the unique vertex such that $\{x, y'\} \in M$ and $\{x, y\} \in M$, the equality $y = y'$ follows. At this point, observe that $y = y' \in \{y', y_1, \dots, y_{k-1}\} \subseteq B \subseteq S_2 \setminus y$.

Observe that either case leads to a contradiction. Therefore, the theorem follows. □

The following theorem tells us something stronger. Given a k-extendable bipartite graph with a perfect matching, not only are there k internally disjoint M-alternating paths between every pair of vertices of two different partitions of G, but also one alternating path that starts and ends with an edge in M.

Theorem 2.14. ([3]) Let $G = (S_1, S_2, E)$, k be a positive integer such that $0 < k < |V(G)|$ and M be a perfect matching of G. G is k-extendable if and only if for every pair of vertices u, v such that $u \in S_1, v \in S_2$ there are k-vertex-disjoint free M-alternating paths and one saturated M-alternating path between u and v.

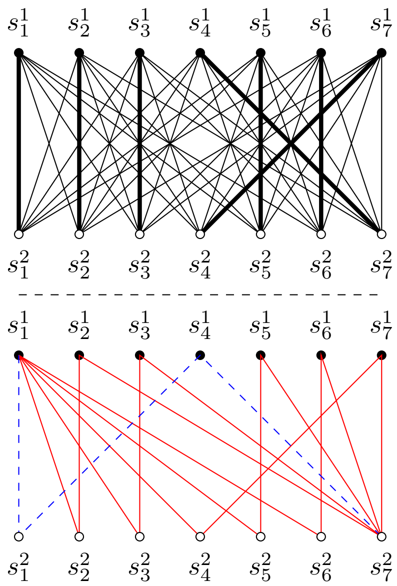

Figure 2.14: A 5-extendable graph with five free M-alternating paths and one saturated M-alternating path.

Proof. Assume first that there are k-vertex-disjoint free M-alternating paths and one saturated M-alternating path between every vertex of S_1 and every vertex of S_2. We will show by induction on k that G is k-extendable.

Let $k = 1$. Let $\{u, v\} \in M$ and let P be the free M-alternating path from u to v. Then $P \cup \{u, v\}$ is an M-alternating cycle. Let $\{u, v\} \in E \setminus M$ and let Q be the saturated M-alternating path from u to v. Then $Q \cup \{u, v\}$ is an M-alternating cycle. Thus, every edge of G belongs to an M-alternating cycle. By Lemma 2.7, this implies that G is 1-extendable.

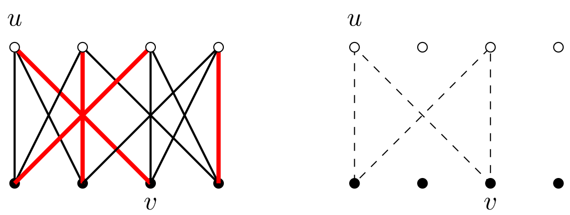

Figure 2.15: A 2-extendable graph G in case $\{u, v\} \in M$.

Suppose that the proposition is true for every $p \leq |V(G)|-2$. Recall that the maximum value of k such that G is k-extendable is at most $|V(G)| - 1$. This is the reason for considering the specific upper bound of p. We will show that the proposition holds for the value $p + 1$.

Assume that there are $p + 1$-vertex-disjoint free M-alternating paths and exactly one saturated M-alternating path between every vertex of S_1 and every vertex of S_2.

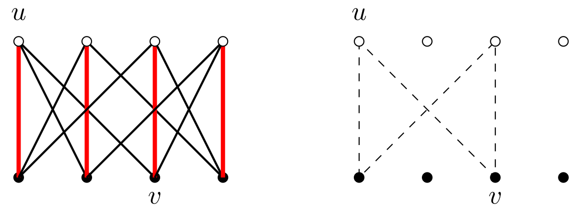

Figure 2.16: A 2-extendable graph G in case $\{u, v\} \in E \setminus M$.

Let $M_p = \{\{s_1^1, s_1^2\}, \ldots, \{s_p^1, s_p^2\}\}$ be a matching with p edges. Furthermore, let $H = G \setminus s_1^1 \setminus s_1^2 \setminus \cdots \setminus s_p^1 \setminus s_p^2$. We would like to show that G is $(p+1)$-extendable. By Lemma 2.9, it suffices to show that H is 1-extendable.

By the induction hypothesis, G is p-extendable. Thus, we can assume for simplicity that M contains every edge of M_p. Also, notice that for every edge $\{u, v\}$ of H, there is at least one free M-alternating path between u and v. Then every matched edge of H belongs to an M-alternating cycle. Now, let $\{w, z\} \notin M$. Let w', z' be vertices of G such that $\{w, w'\}, \{z, z'\} \in M$. Observe that these edges belong to H. Furthermore, there is at least one free M-alternating path P in H between w' and z'. Observe that $P \cup \{w, z\} \cup \{w, w'\} \cup \{z, z'\}$ is an M-alternating cycle in H that contains $\{w, z\}$. Hence, H is 1-extendable.

For the opposite direction, assume that G is k-extendable. By Theorem 2.4, this implies that G is $k+1$-vertex-connected. Let $u \in S_1$ and $v \in S_2$. By Menger's theorem, there are $k+1$ vertex-disjoint paths $P_1, \ldots, P_{k+1}$ joining these two vertices. Observe that the length of these paths is odd. By applying Theorem 2.5 $k-1$ times, G is 1-extendable. It follows by Lemma 2.6 that G has an ear decomposition. Let $H' = (S_1', S_2', E')$ be a subgraph of G formed by u, v and $P_1, \ldots, P_{k+1}$. Then H' is 1-extendable([10]). Let N be a perfect matching of H'.

Assume that $\{u, v\} \notin E'$. Since the vertex-disjoint paths $P_1, \ldots, P_{k+1}$ have odd length, then they are alternating paths. Let u', v' be two vertices such that $\{u, u'\}, \{v, v'\} \in N$. Without loss of generality, let $\{u, u'\} \in P_1$. Since P_1 is an alternating path of odd length, then $\{v, v'\} \in P_1$. Observe that $P_2, \ldots, P_{k+1}$ are k-vertex-disjoint free alternating paths and P_1 is saturated alternating path between u and v.

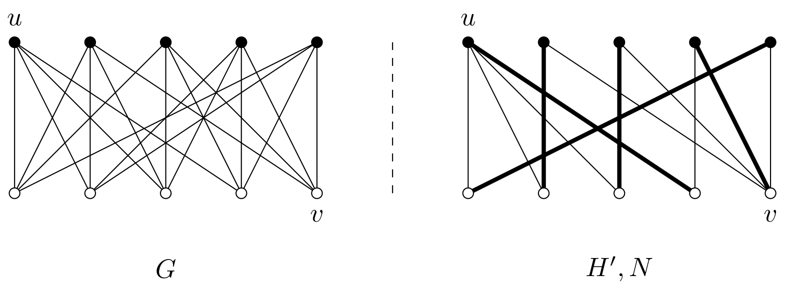

Figure 2.17: A 3-extendable graph G and paths P_1, P_2, P_3, Q in case $\{u, v\} \notin E'$.

Assume that $\{u, v\} \in N$. This edge is a saturated alternating path between u

and v. Let P_1 be this path. Then $P_2, \dots, P_{k+1}$ are the desired k-vertex-disjoint free alternating paths.

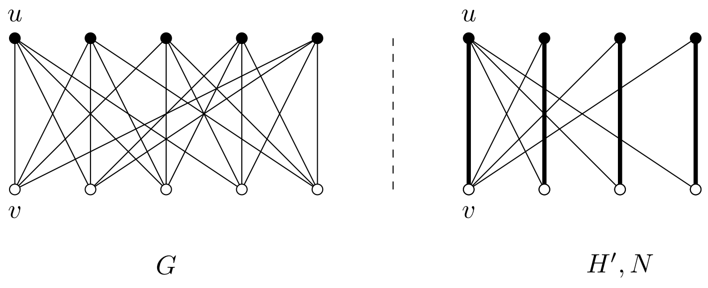

Figure 2.18: A 3-extendable graph G and paths P_1, P_2, P_3, Q in case $\{u, v\} \in N$.

Assume that $\{u, v\} \notin N$. Let u', v' be two vertices such that $\{u, u'\}, \{v, v'\} \in N$. Without loss of generality, let $\{u, u'\} \in P_1$. Then $\{v, v'\} \in P_1$ as well. Since $\{u, v\} \in E'$, then this edge is a free alternating path. Let P_2 be this path. Now, observe that $P_3, \dots, P_{k+1}$ are the other $k-1$-vertex-disjoint free alternating paths. Thus, $P_2, \dots, P_{k+1}$ are the desired k-vertex-disjoint free alternating paths and P_1 is the saturated alternating path between u and v. $\square$

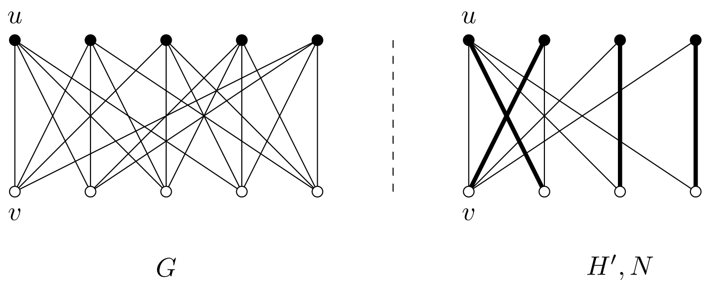

Figure 2.19: A 3-extendable graph G and paths P_1, P_2, P_3, Q in case $\{u, v\} \notin N$.

2.3 Alternating paths on any perfect matching

In the proof of sufficiency of Theorem 2.10 we proved that for a random perfect matching M of a k-extendable graph $G = (S_1, S_2, E)$ there exist k alternating paths with respect to M between every pair of vertices $x \in S_1$ and $y \in S_2$. If our target was to find these paths for every possible perfect matching of G, the first idea would be to check every perfect matching separately. Theorem 2.15 help us to avoid such a situation. It guarantees us that the existence of paths with respect to a perfect matching is sufficient in order to find the paths for every other perfect matching.

Theorem 2.15 ([4]). Let $G = (S_1, S_2, E)$ be a graph with a perfect matching and let x, y be two vertices such that $x \in S_1, y \in S_2$. Let M, M_0 be perfect matchings of G. If G has k internally disjoint alternating $\{x, y\}$-paths with respect to M_0, then G has k internally disjoint alternating $\{x, y\}$-paths with respect to M.

Proof. Suppose that G contains k internally disjoint alternating $\{x, y\}$-paths with respect to M_0. Let $P_1, \ldots, P_k$ be these paths. Let $H = (V(G), \bigcup_{i=1}^{k} E(P_i))$. If v is a random vertex of H, then its degree equals to either $0, 2$ or k. Specifically,

$$deg_H(v) = \begin{cases} 0, & v \notin \bigcup_{i=1}^{k} V(P_i) \\ 2, & v \in \bigcup_{i=1}^{k} V(P_i) \setminus \{x, y\} \\ k, & v = x \text{ or } v = y \end{cases}$$

Furthermore, let $K = (V(G), E(K))$, where $E(K) = E(H) \triangle M_0 \triangle M$. It obviously holds that $E(K) \subseteq E(G)$. Let J be the intersection of the sets $E(H), M$ and M_0. Particularly, the set $E(K)$ does not contain edges from $(E(H) \cap M) \setminus J$, $(E(H) \cap M_0) \setminus J$ and $(M \cap M_0) \setminus J$. Figure 2.20 is crucial for understanding the proofs that follow.

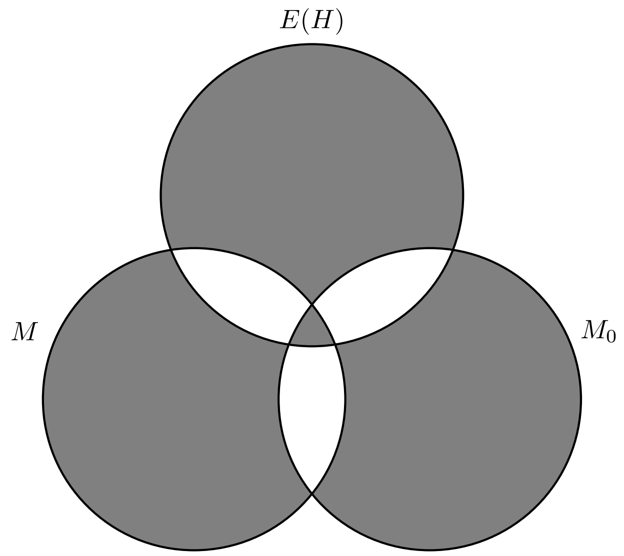

Figure 2.20: A set representation of $E(K)$. All the edges of graph K belong to the gray part.

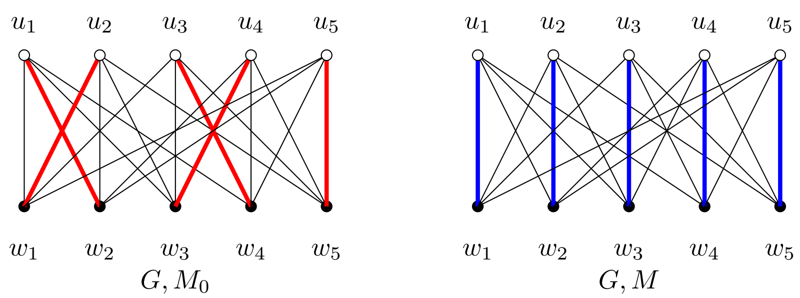

Figure 2.21: A 3-extendable graph G with perfect matchings M_0, M.

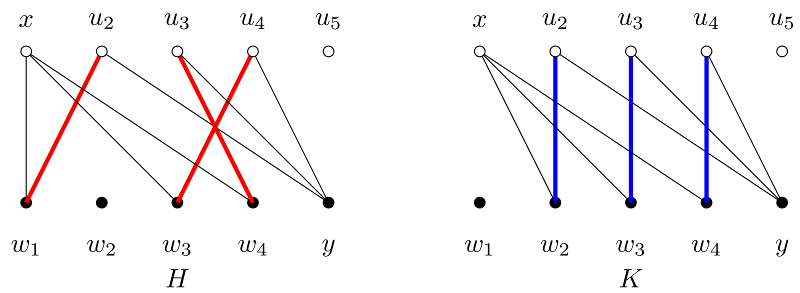

Figure 2.22: Graphs H, K as obtained by 3-extendable graph G.

Claim 2.16 ([4]). *For each $v \in V(G) \setminus \bigcup_{i=1}^{k} V(P_i)$, $deg_K(v) = 0$ or $deg_K(v) = 2$. Furthermore, if $deg_K(v) = 2$, then exactly one of the two edges of K incident with v belongs to M.*

Proof. Let $v \in V(G) \setminus \bigcup_{i=1}^{k} V(P_i)$. Observe that v is not incident to any edge of H. Since M_0, M are two different perfect matchings of G, there exist vertices v_0, v' such that $\{v, v_0\} \in M_0$ and $\{v, v'\} \in M$. Notice that these edges do not belong to $E(H)$. We have two cases to consider. If $v_0 = v'$, then $\{v, v'\} \in (M_0 \cap M) \setminus E(H)$. This implies $\{v, v'\} \notin E(K)$. Thus, $deg_K(v) = 0$. If $v_0 \neq v'$, then $\{v, v_0\} \in M_0 \setminus (M \cup E(H))$ and $\{v, v'\} \in M \setminus (M_0 \cup E(H))$. By construction of graph K, $N_K(v) = \{v_0, v'\}$. Thus, $deg_K(v) = 2$ and, furthermore, the last property holds trivially. □

Claim 2.17 ([4]). *For each $v \in \bigcup_{i=1}^{k} V(P_i) \setminus \{x, y\}$, $deg_K(v) = 0$ or $deg_K(v) = 2$. Furthermore, if $deg_K(v) = 2$, then exactly one of the two edges of K incident with v belongs to M.*

Proof. Let $v \in \bigcup_{i=1}^{k} V(P_i) \setminus \{x, y\}$. Specifically, v is a vertex of a path P_j, where $j \in \{1, \ldots, k\}$. It follows directly by construction of graph H that $deg_H(v) = 2$. Let $N_H(v) = \{v_1, v_2\}$. Since P_j is a $\{x, y\}-$alternating path with respect to M_0, either $\{v, v_1\} \in M_0$ or $\{v, v_2\} \in M_0$. We suppose that $\{v, v_1\} \in E(H) \cap M_0$ and $\{v, v_2\} \in E(H) \setminus M_0$. Furthermore, let v_3 be a vertex such that $\{v, v_3\} \in M$. The following paragraph describes three different cases.

Let $v_1 = v_3$. Then $\{v, v_1\} \in E(H) \cap M_0 \cap M$ and $\{v, v_2\} \in E(H) \setminus (M_0 \cup M)$. Thus, $N_K(v) = \{v_1, v_2\}, deg_K(v) = 2$ and $\{v, v_1\} \in M$. Let $v_3 = v_2$. Then $\{v, v_1\} \in (E(H) \cap M_0) \setminus M$ and $\{v, v_2\} \in (E(H) \setminus M_0) \cap M$. Observe that $\{v, v_1\}, \{v, v_2\} \notin E(K)$. Thus, $deg_K(v) = 0$. Let $v_3 \neq v_1$ and $v_3 \neq v_2$. Then $\{v, v_1\} \in (E(H) \cap M_0) \setminus M$ and $\{v, v_2\} \in E(H) \setminus (M_0 \cup M)$. Furthermore, $\{v, v_3\} \in M \setminus (E(H) \cup M_0)$. Thus, $N_K(v) = \{v_2, v_3\}$, $deg_K(v) = 2$ and $\{v, v_3\} \in M$. □

Claim 2.18 ([4]). *Let $u \in \{x, y\}$. Then $deg_K(u) \in \{k, k + 2\}$. If $deg_K(u) = k$, then none of k edges in K incident with u belong to M. If $deg_K(u) = k + 2$, then exactly one of the $k + 2$ edges in K incident with u belongs to M.*

Proof. We consider only the case where $u = y$. By symmetry, we proceed in the same way in case $u = x$. Let $N_H(y) = \{x_1, \ldots, x_k\}$ and $\{y, x_0\} \in M_0$. Observe that $x_0 \notin \{x_1, \ldots, x_k\}$. Hence, for $i \in \{1, \ldots, k\}$ it holds that $\{y, x_i\} \in E(H) \setminus M_0$ and $\{y, x_0\} \in M_0 \setminus E(H)$. Furthermore, let $\{y, x'\} \in M$. We have three cases to take under consideration.

Assume that $x' = x_0$. It holds that, for $i \in \{1, \ldots, k\}$, $\{y, x_i\} \in E(H) \setminus (M \cup M_0)$ and $\{y, x_0\} \in (M_0 \setminus E(H)) \cap M$. Equivalently, $\{y, x_i\} \in E(K)$ and $\{y, x_0\} \notin E(K)$. Thus, $N_K(y) = \{x_1, \ldots, x_k\}$. Notice that none of k edges $\{y, x_1\}, \ldots, \{y, x_k\}$ belong to the perfect matching M.

Suppose that $x' \in \{x_1, \ldots, x_k\}$. We may assume that $x' = x_1$. Then $\{y, x_0\} \in M_0 \setminus (E(H) \cup M)$, $\{y, x'\} \in (E(H) \setminus M_0) \cap M$ and $\{y, x_i\} \in E(H) \setminus (M_0 \cup M)$, for $i \in \{2, \ldots, k\}$. Observe that $N_K(y) = \{x_0, x_2, \ldots, x_k\}$. Notice that none of the k edges $\{y, x_0\}, \{y, x_2\}, \ldots, \{y, x_k\}$ belong to the perfect matching M.

Finally, let $x' \notin \{x_0, x_1, \ldots, x_k\}$. Then $\{y, x_0\} \in M_0 \setminus (E(H) \cup M)$, $\{y, x'\} \in M \setminus (E(H) \cup M_0)$ and $\{y, x_i\} \in E(H) \setminus (M_0 \cup M)$, for $i \in \{1, \ldots, k\}$. In this case, observe that $N_K(y) = \{x', x_0, x_1, \ldots, x_k\}$. Furthermore, $\{y, x'\} \in M$. □

We are now ready to proceed with the proof of the theorem. It suffices to consider only two cases.

Case 1: $deg_K(x) = k$ or $deg_K(y) = k$.

We assume that $deg_K(x) = k$ and $N_K(x) = \{x_1, \ldots, x_k\}$. By Claims 2.11 and 2.12, for every $i \in \{1, \ldots, k\}$, there exists $T_i = a_0^{(i)} a_1^{(i)} \ldots a_{l_i}^{(i)}$ in K with $a_0^{(i)} = x$, $a_1^{(i)} = x_i$ and $a_{l_i}^{(i)} \in \{x, y\}$. Each T_i is an alternating path with respect to M. By taking one T_i which is as small as possible, we may assume that T_i is either a cycle or a $\{x, y\}$-path. Suppose, for contradiction, that T_i is a cycle and $a_{l_i}^{(i)} = x$. Then $\{a_{l_1-1}^{(i)}, a_{l_i}^{(i)}\} \notin M$ by composing the assumption of the case and Claim 2.18. Since T_i is alternating, T_i is an odd cycle. This contradicts the fact that G is bipartite. Therefore, each T_i is an alternating $\{x, y\}$-path. Furthermore, $T_1, \ldots, T_k$ are internally disjoint. By assuming that there are two paths with a common vertex u, we conclude to the contradiction that $deg_K(u) = 4$.

Case 2: $deg_K(x) = deg_K(y) = k + 2$.

Let $N_K(x) = \{x', x_0, x_, \ldots, x_k\}$ and $\{x, x'\} \in M$. For each $i \in \{0, 1, \ldots, k\}$, we can construct an alternating path $T_i = a_0^{(i)} a_1^{(i)} \ldots a_{l_i}^{(i)}$ in K with $a_0^{(i)} = x$, $a_1^{(i)} = x_i$ and $a_{l_i}^{(i)} \in \{x, y\}$. We can assume that each T_i is either a cycle or a $\{x, y\}$-path and that $T_0, T_1, \ldots, T_k$ are internally disjoint. Suppose that $a_{l_k}^{(k)} = x$. Since G is bipartite, l_k is even and $\{a_{l_k-1}^{(k)}, a_{l_k}^{(k)}\} \in M$. It follows that $a_{l_k-1}^{(k)} = x'$. Observe that we still have k internally disjoint $\{x, y\}$-paths $T_0, \ldots, T_{k-1}$ with respect to M. □

CHAPTER 3

COMPLEXITY OF MATCHING EXTENDABILITY PROBLEM

3.1 Computational complexity theory

Definition 3.1. A decision problem is a problem that can be posed as a yes-no question on the input values.

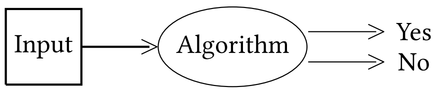

Figure 3.1: If the input belongs to the decision problem, then the algorithm return Yes. Otherwise, it returns No.

P and NP are the most famous among all the complexity classes. It still remains an open problem whether P=NP or not. Researchers believe that the inequality most probably holds.

Definition 3.2. Class P contains all decision problems that can be solved by a Turing machine deterministically using a polynomial amount of computational time.

One problem which lies in P is Maximum Matching(Edmonds [9]). This problem accepts as input a coding of a graph G and returns a maximum matching in G. In graph theory, maximum matching of a graph is a matching of maximum size, i.e. no matching in the graph has strictly more elements than it.

Definition 3.3. A problem Π is in NP if there exists a polynomial $p : \mathbb{N} \to \mathbb{N}$ and a polynomial time Turing machine V (called the verifier for the problem) such that for every input x it holds that $x \in \Pi$ if and only if $\exists u$ (called the certificate for the input) of size at most $p(x)$ such that $V(x, u) = 1$.

Polynomial time reductions is an interesting concept of Computational Complexity Theory. Intuitively, it is a method for solving one problem using another. If there exists a hypothetical algorithm that solves the second problem, then the first problem can be solved by transforming its input into a new one for the second problem and calling the algorithm one or multiple times. If the previous procedure is done in polynomial time, then the first problem is polynomial time reducible to the second.

Definition 3.4. Let A,B be two problems. Then A *reduces* to B in polynomial time if there exists a computable polynomial function f such that $x \in A$ if and only if $f(x) \in B$.

For abbreviation, we will write A $\leq$ B.

Theorem 3.5. ([14]) Let A,B,C be three problems. If A $\leq$ B and B $\leq$ C, then A $\leq$ C.

Theorem 3.6. ([15]) Let A,B be two problems. Then A $\leq$ B if and only if $\overline{\text{A}} \leq \overline{\text{B}}$.

It is known that there exist problems that are at least as hard as any other problem in NP. This property is called NP-*hardness*.

Definition 3.7. A problem L is NP-*hard* if for every L' in NP it holds that L' $\leq$ L.

Definition 3.8. A problem is NP-*complete* if it is in NP and it is NP-hard.

One famous NP-complete problem is Vertex Cover(Karp [8]). It contains pairs (G, k) for which graph G has a vertex cover of size at most k. In graph theory, a vertex cover of a graph $G = (V, E)$ is a subset $S \subseteq V$ such that for every $e = \{u, v\} \in E$ it holds $u \in S$ or $v \in S$.

Assume that A,B are two problems such that A is NP-complete and we would like to prove that B is NP-complete. By Definition 3.8, we have to prove two properties. Firstly, we prove that B is in NP. Afterwards, notice that it is sufficient to reduce A to B in order to prove the second property. That is because Theorem 3.5 holds and A is NP-complete by assumption.

We define a new complexity class which contains the complements of problems that are in NP.

Definition 3.9. A problem L is in coNP if $\overline{\text{L}}$ is in NP.

We define coNP-*hardness* and coNP-*completeness* in analogous way as NP-hardness and NP-completeness respectively.

Theorem 3.10. A problem is NP-complete if and only if its complement is coNP-complete.

Proof. Let L be a problem such that L is NP-complete. By Definition 3.8, L is in NP and every other problem L' in NP reduces to L in polynomial time. By Definition 3.9 and Theorem 3.6, it follows that $\overline{\text{L}}$ is in coNP and every other problem $\overline{\text{L'}}$ in coNP reduces to $\overline{\text{L}}$ in polynomial time. Thus, $\overline{\text{L}}$ is coNP-complete. The opposite direction is proved symmetrically. ☐

3.2 coNP-completeness on general graphs

Firstly, we mention a lemma which is going to be used afterwards.

Lemma 3.11. ([5]) Let $G = (V, E)$ be a graph and k an integer. If there is no vertex cover of size at most k in G, then there is a matching M in G which matches at least $k + 1$ vertices.

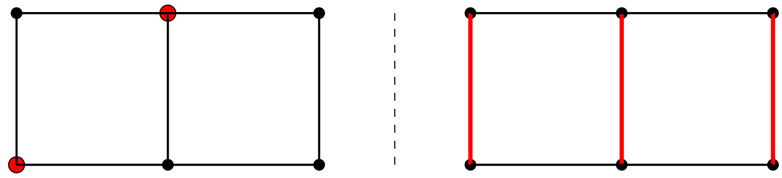

Figure 3.2: There is no vertex cover of size at most 2 in G. Therefore, there is a matching of size at least 3.

Proof. The Maximum Matching problem, on input G, returns a maximum matching M of G. Let $S \subseteq V$ be the set of vertices incident to edges in M. Observe that S is a vertex cover in G. If S was not a vertex cover in G, then there would exist an edge $e \in E$ such that no endpoint of it would belong to S and, as a result, $M \cup e$ would be a matching strictly larger than M. This outcome contradicts the fact that M is a maximum matching in G. By hypothesis, there is no vertex cover of size at most k in G. Thus, $|S| \geq k+1$ and M matches at least $k+1$ vertices. $\square$

We will show that $\overline{\text{Extendability}}$ is NP-complete. It consists of pairs (G, k) with the property that there exists a matching of size k which can not be extended to a perfect matching in G. This proof is sufficient for proving that Extendability is coNP-complete.

Firstly we prove that $\overline{\text{Extendability}}$ is in NP. We construct a verifier N that accepts as input an encoding $((G = (V, E), k), M)$, where G is a graph, k is a natural number and M is our certificate and verifies in polynomial time whether (G, k) is a yes-instance of $\overline{\text{Extendability}}$ or not. Since $|M| = k \leq |V| - 1$, M has polynomial length. The verifier N works as follows:

N= "On input $((G = (V, E), k), M)$:"

1. $G' := G \setminus V_M$, where V_M contains the vertices which are incident to an edge of M
2. Run Maximum Matching on input (G') and obtain M'
3. If M' is a perfect matching in G, then reject. Otherwise, accept."

If M' is a perfect matching in G', then $M \cup M'$ is a perfect matching in G. Thus, (G, k) is a no-instance of $\overline{\text{Extendability}}$. If M' is not a perfect matching in G', then M can not be extended to a perfect matching in G. Thus, (G, k) is a yes-instance of $\overline{\text{Extendability}}$. Observe that the verifier N runs in polynomial time.

We proceed with the proof of NP-hardness. We prove it by reducing Vertex Cover in polynomial time to it. Let (G_{VC}, s) be a general instance of Vertex Cover and $\{v_1, \dots, v_r\}$ be the set of vertices of G_{VC}. We assume that $0 < s < r - 1$ and $E(G_{VC}) \neq \emptyset$. This is because if $s = 0$ or $s = r - 1$ we can decide in polynomial time whether G_{VC} has a vertex cover of at most the given size or not. We will map (G_{VC}, s) to a suitable instance (G, k). The following function f maps (G_{VC}, s) to a new instance (G, k):

$f(G_{VC}, s)$:
1. Set $k := s$.
2. Let $W := \{w_1, \dots, w_{r-1}\}$ and $Q_{2r+1} := \{q_1, \dots, q_{2r+1}\}$.
3. Let $E_{VW} = \{\{v, w\} | v \in V(G_{VC}), w \in W\}$.
4. Let $E_{WQ} = \{\{w, q\} | w \in W, q \in Q_{2r+1}\}$.

5. Let $E_Q = \{\{q, q'\} | q, q' \in Q_{2r+1}, q \neq q'\}$.
6. Initialize graph $G := \emptyset$.
7. Set $V(G) := V(G_{VC}) \cup W \cup Q_{2r+1}$.
8. Set $E(G) := E(G_{VC}) \cup W_{VW} \cup E_{WQ} \cup E_Q$.
9. Output (G, k)."

Observe that (G, k) is constructed in polynomial time. Clearly, G is connected and has $4r$ vertices, i.e. an even number. Furthermore, notice that a perfect matching G can be found in the following way: Since $E(G_{VC}) \neq \emptyset$, we randomly choose one edge in G_{VC}. Then we add $r - 2$ independent edges from E_{VW}. Again, we take one edge from E_{WQ} for the remaining node in W. Finally, we add r independent edges from E_Q.

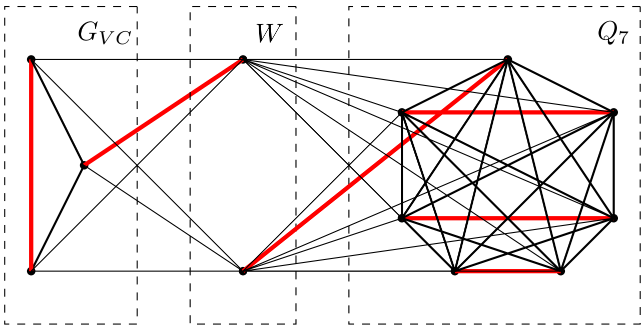

Figure 3.3: An obtained graph G from f on input G_{VC} with a perfect matching M.

We show that if G_{VC} has a vertex cover S of size at most s, then G is not k-extendable. If $|S| < k$, then extend S to a set of k vertices. The new S is obviously still a vertex cover. Assume that $S = \{v_1, \ldots, v_k\}$. Let $M = \{\{v_i, w_i\} | i = 1, \ldots, k\}$. Since $0 < k < r - 1$, M is well defined and $|M| = k$. We show that M can not be extended to a perfect matching. By assuming the opposite, we observe that every vertex of $V(G_{VC})$ must be matched with a vertex of W. Since $|V(G_{VC})| = r$ and $|W| = r - 1$, this is not possible. Suppose that two distinct vertices $v, v' \in V(G_{VC})$ can be matched. Since S is a vertex cover in G_{VC}, it holds that $v \in S$ or $v' \in S$. Thus, v or v' are already matched to some vertex $w \in W$. Therefore, no edge in $E(G_{VC})$ can belong to a perfect matching that contains M as a subset. Since we found a matching M of k vertices such that there is no perfect matching that contains it, G is not k-extendable (see Figure 3.4).

Assume now that G_{VC} has no vertex cover of size at most s. We will show that G is k-extendable. Let M be a matching in G such that $|M| = k$. We set

$$k_V = |M \cap E(G_{VC})|, k_{VW} = |M \cap E_{VW}|, k_{WQ} = |M \cap E_{WQ}|, k_Q = |M \cap E_Q|.$$

Observe that $k = k_V + k_{VW} + k_{WQ} + k_Q$. Furthermore, the number of unmatched vertices in $V(G_{VC})$ is given by $K_V^{free} = r - 2k_V - k_{VW}$. Similarly, the number of unmatched vertices in W is given by the formula $k_W^{free} = r - 1 - k_{VW} - k_{WQ}$. We have two cases to take under consideration.

Let $k_V^{free} \leq k_W^{free}$. Then at least one edge in G_{VC} belongs to M. We can extend M to a perfect matching.The idea is to match every remaining $v \in V(G_{VC})$ with an unmatched $w \in W$, then match any remaining $w \in W$ to some $q \in Q$ and finally

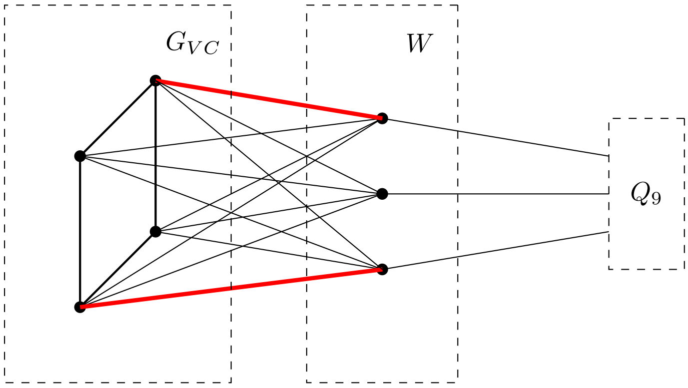

Figure 3.4: A case where G_{VC} has a vertex cover of size at most 2 and G is not 2-extendable.

match the remaining vertices in Q. The crucial observation is that the number of unmatched vertices in W after the first step is odd. In what follows we discuss the reason of this situation. For simplicity, let M' be the perfect matching that contains M and let M'_{VW} be a matching such that $M'_{VW} \subset M' \cap E_{VW}$ and $M'_{VW} \cap M = \emptyset$.

- Assume that r is even.
  - Let k_V^{free} is odd. Then observe that necessarily an odd number of edges in E_{VW} can be in M. After the first step, notice that $|M'_{VW}|$ is odd. Since k_V^{free} is odd, $r-1$ is odd and $|M'_{VW}|$ is odd, it follows that the number of the remaining unmatched vertices in W is odd (see Figure 3.5).

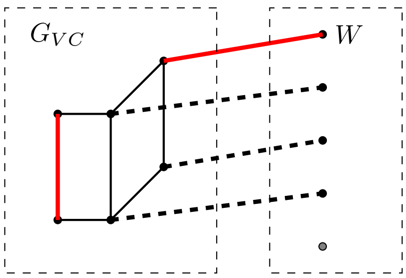

Figure 3.5: A case where G_{VC} is has no vertex cover of size 2, $k_V^{free} \leq k_W^{free}$, r is even and k_V^{free} is odd. Color red the edges of M, dashed black the edges of M'_{VW} and gray the remaining unmatched vertices in W.

  - Let k_V^{free} is even. Then observe that necessarily an even number of edges in E_{VW} can be in M. After the first step, notice that $|M'_{VW}|$ is even. Since k_V^{free} is even, $r-1$ is odd and $|M'_{VW}|$ is even, it follows that the number of the remaining unmatched vertices in W is odd (see Figure 3.6).

- Assume that r is odd.

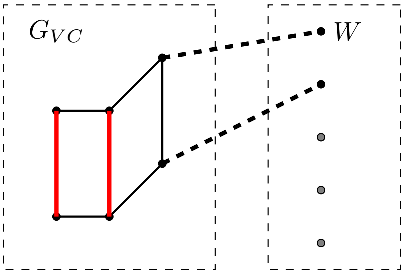

Figure 3.6: A case where G_{VC} is has no vertex cover of size 2, $k_V^{free} \leq k_W^{free}$, r is even and k_V^{free} is even. Color red the edges of M, dashed black the edges of M'_{VW} and gray the remaining unmatched vertices in W.

– Let k_V^{free} is odd. Then observe that necessarily an even number of edges in E_{VW} can be in M. After the first step, notice that $|M'_{VW}|$ is odd. Since k_V^{free} is odd, $r-1$ is even and $|M'_{VW}|$ is even, it follows that the number of the remaining unmatched vertices in W is odd (see Figure 3.7).

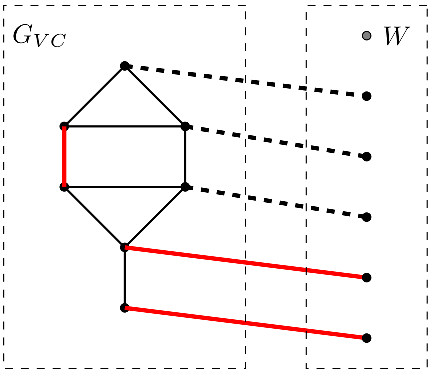

Figure 3.7: A case where G_{VC} is has no vertex cover of size 3, $k_V^{free} \leq k_W^{free}$, r is odd and k_V^{free} is odd. Color red the edges of M, dashed black the edges of M'_{VW} and gray the remaining unmatched vertices in W.

– Let k_V^{free} is even. Then observe that necessarily an odd number of edges in E_{VW} can be in M. After the first step, notice that $|M'_{VW}|$ is even. Since k_V^{free} is even, $r-1$ is even and $|M'_{VW}|$ is odd, it follows that the number of the remaining unmatched vertices in W is odd (see Figure 3.8).

Let $k_V^{free} > k_W^{free}$. Then G_{VC} has $1 - 2k_V + k_{WQ}$ more unmatched vertices than W. Figure 3.9 illustrates a case where $k_V^{free} = 7 > 4 = k_W^{free}$.

Let G'_{VC} be a subgraph of G_{VC} induced by all the vertices which are not incident to an edge in M. Observe that vertices in G'_{VC} can be matched with either a distinct vertex in G'_{VC} or an unmatched vertex in W. We want to find a matching M' in G'_{VC} which matches at least $1 - 2k_V + k_{WQ}$ vertices.

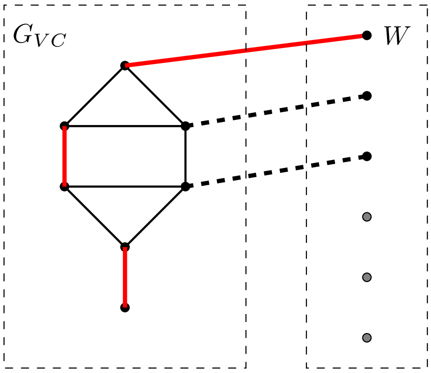

Figure 3.8: A case where G_{VC} is has no vertex cover of size 3, $k_V^{free} \leq k_W^{free}$, r is odd and k_V^{free} is even. Color red the edges of M, dashed black the edges of M'_{VW} and gray the remaining unmatched vertices in W.

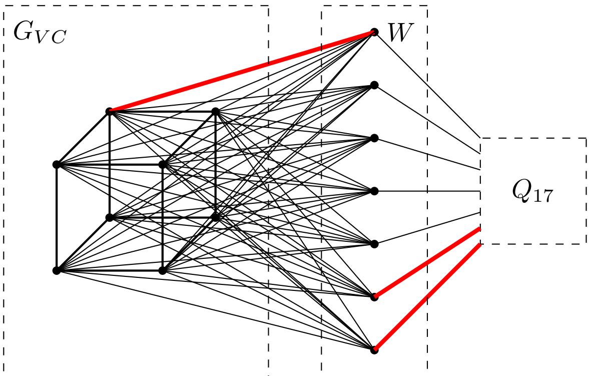

Figure 3.9: A case where G_{VC} has no vertex cover of size 3 and G is 3-extendable.

Since G_{VC} does not contain a vertex cover of size k and G'_{VC} has $2k_V + k_{VW}$ vertices less than G_{VC}, it follows directly that G'_{VC} does not have a vertex cover of size $k - 2k_V - k_{VW} = k_Q + k_{WQ} - k_V$. Notice that the right term is non-negative. This is because $k_V \geq 0$ and $k_V \leq k_{WQ} - k_V$. By using Lemma 3.11, we get that there is a matching M' in G'_{VC} which matches at least $1 + k_Q + k_{WQ} - k_V$. Observe that the following inequalities hold: $1 + k_Q + k_{WQ} - k_V \geq 1 + k_{WQ} - k_V \geq 1 - 2k_V + k_{WQ}$. Thus, M' matches the desired number of vertices in G'_{VC}.

Now M can be extended to a perfect matching. Firstly, we add the edges in M' to M. After that we match every remaining unmatched $v \in V(G_{VC})$ with some $w \in W$. Notice that the second step is now possible. Next, we match every remaining $w \in W$ with some $q \in Q$. Finally, we match the remaining even number of vertices in Q.

3.3 A polynomial algorithm for bipartite graphs

In this section the graph G is undirected, simple, connected and bipartite and has a perfect matching.

Lemma 3.12. ([3]) Let M be a perfect matching of $G = (S_1, S_2, E)$. G is k-extendable if and only if its residual graph G_M is strongly connected and there are k-vertex-disjoint directed paths between every vertex of S_1 and every vertex of S_2 in G_M.

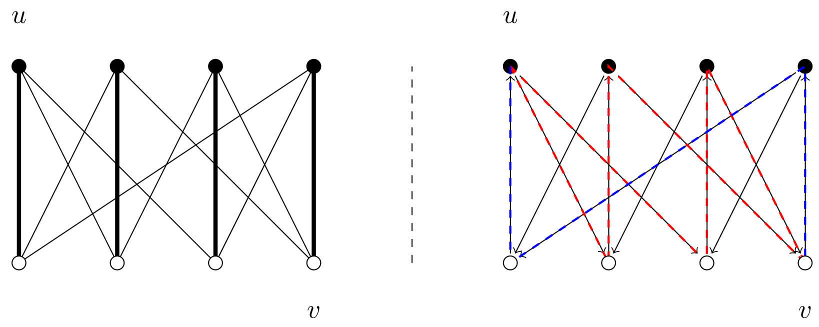

Figure 3.10: A 2-extendable graph G and its residual graph G_M.

Proof. Firstly, we discuss about a direct observation. Let $u \in S_1$ and $v \in S_2$. A free alternating path in G from u to v becomes a directed path from u to v in G_M. Furthermore, a saturated alternating path in G from u to v becomes a directed path from v to u in G_M. Consequently, G has k internally disjoint free M-alternating paths and one saturated between every vertex $u \in S_1$ and every vertex $v \in S_2$ if and only if there are k internally disjoint directed paths from u to v and one directed path from v to u in G_M(see Figure 3.10).

Assume that G is k-extendable. By Theorem 2.14, there are k internally disjoint free M-alternating paths and one saturated between u and v. Observe that there are k internally disjoint directed paths from u to v and one directed path from v to u in G_M. Thus, G_M is strongly connected.

For the other direction, we assume that there are k internally disjoint directed paths from u to v and G_M is strongly connected. These paths are free M-alternating paths in G. Since G_M is strongly connected, there is a directed path from v to u. This path is saturated M-alternating path in G and is disjoint with every aforementioned path. Thus, G is k-extendable. $\square$

The maximization version of Extendability problem focuses on finding the maximum value of k for which the input graph G is k-extendable. Initially, we describe the operation of some functions which are used in the algorithm. Then, the algorithm follows and finally we describe its time complexity.

- find-perfect-matching(G) searches for a perfect matching in G. It returns $\emptyset$ if G does not contain a perfect matching.
- is-perfect-matching(G, M) returns true if M is a perfect matching of G. If M is not a perfect matching in G, then it returns false.
- direct(G, M) returns the residual graph of $G = (S_1, S_2, E)$.
- is-strongly-connected(G) returns true if G is strongly connected. Otherwise, it returns false.

- max-disjoint-paths(G, s, t) returns the maximum number of vertex-disjoint paths in G between s and t, where s is the source node and t is the target node.

Algorithm 1 finds the extendability of the input graph $G = (S_1, S_2, E)$

```
MAIN FUNCTION: find-extendability(G)
k ← +∞
M ← find-perfect-matching(G)
perfect_matching ← is-perfect-matching(G, M)
if perfect_matching then
    G′ ← direct(G, M)
    strongly_connected ← is-strongly-connected(G′)
    if strongly_connected then
        for u ∈ S1 do
            for v ∈ S2 do
                paths ← max-disjoint-paths(G′, u, v)
                k ← min(k, paths)
            end for
        end for
    else
        k ← 0
    end if
else
    k ← 0
end if
return k
```

It is known that finding a perfect matching in a bipartite graph can be done in $O(E\sqrt{V})$(Hopkroft, Karp [11]). We can decide in $O(E)$ time whereas M is a perfect matching in G. We simply take all the vertices incident to some edge of M and check their number equals the total number of vertices. G' can be constructed in $O(E)$. Checking if G' is strongly connected can be done in $O(E)$([12]). Finding the maximum number of vertex-disjoint paths between every vertex of S_1 and every vertex of S_2 can be done in $O(E \cdot min(k^3 + V, k \cdot V))$([13]). Thus, the total running time of the above algorithm is $O(E \cdot min(k^3 + V, k \cdot V))$.

Let G be a bipartite graph and k be a positive integer. Let (G, k) be the input of the Extendability. Observe that in this particular case it is very easy to decide whether this input is yes or no instance of the problem. It suffices to compute the extendability of the graph G, denoted by *ext*(G), and check whether k is at most *ext*(G) or not.

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