Extendability of Graphs with Perfect Matchings

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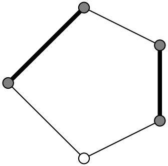
Matching and Perfect Matching

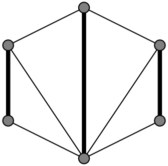
Definition

Let $G = (V, E)$ be a graph. A *matching* is a set $M \subseteq E$ of independent edges.

Definition

A *perfect matching* is a matching that matches all the vertices.





Extendability

Definition

Let k be a positive integer and $G = (V, E)$ be a graph with $|V| \geq 2k + 2$. G is *k-extendable* if G has a perfect matching and **every** matching of G of cardinality k is a subset of a perfect matching in G.

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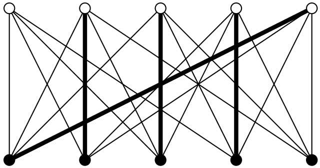

Algorithmic Version

EXTENDABILITY	
Input:	A graph G and a positive integer k.
Question:	Is the graph G k-extendable?

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Input:	A graph G and a positive integer k.
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- In the general case the problem is coNP-complete.
- In the case where the input graph is bipartite, there is a polynomial algorithm that decides it.

Alternating Paths

Definition

Let $G = (V, E)$ be a graph and let M be a perfect matching of G. An M-alternating path P of G is a path in G where edges in M and edges in $E \setminus M$ appear on P alternately.

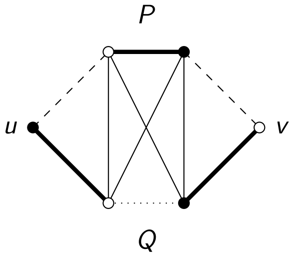

Figure: P, Q are M-alternating paths in G.

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Menger's Theorem

Theorem

Let $G = (V, E)$ be a graph. G is k-vertex-connected if and only if there are k internally disjoint paths between every pair of two distinct vertices.

Theorem I (Aldred et al., Discrete Math., 2003)

$G = (X \cup Y, E)$ bipartite with perfect matching. G is k-extendable if and only if

- for **every** perfect matching M and
- for **every** pair of vertices $x \in X$ and $y \in Y$

there are k internally disjoint free M-alternating paths connecting x and y.

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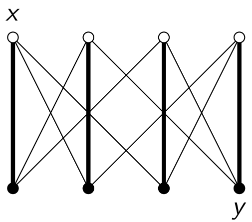


Figure: Graph G is 2-extendable.

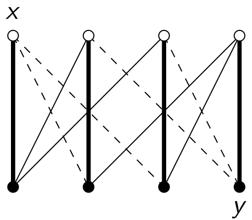


Figure: P_1, P_2 are M-alternating paths connecting x and y.

Theorem II (Disgjun Lou et al., Discrete Math., 2005)

$G = (X \cup Y, E)$ bipartite with perfect matchings M_0, M. Let $x \in X$ and $y \in Y$.

G has k internally disjoint free M_0-alternating paths connecting x and y

$$\downarrow$$

G has k internally disjoint free M-alternating paths connecting x and y

Theorem II (Disgjun Lou et al., Discrete Math., 2005)

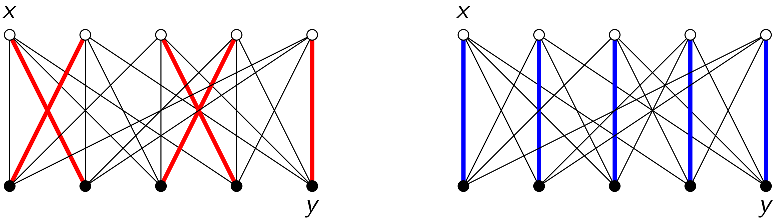
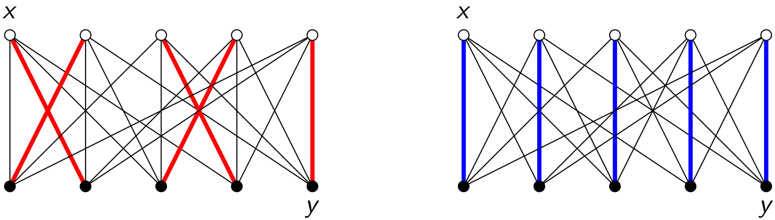


Figure: Graph G is 3-extendable with perfect matchings M_0, M.

Theorem II (Disgjun Lou et al., Discrete Math., 2005)



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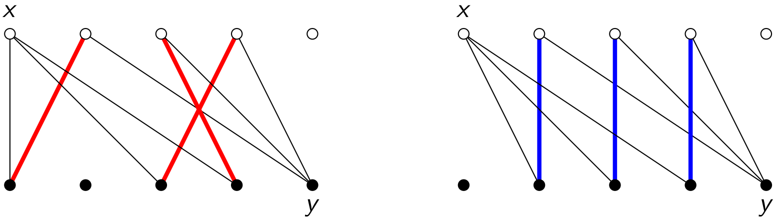


Figure: Graphs H, K as obtained by 3-extendable graph G.

Theorem III (J Lakhal et al., IPL, 1998)

$G = (X \cup Y, E)$ bipartite with perfect matching. G is k-extendable if and only if

- for **every** perfect matching M
- for **every** pair of vertices $x \in X$ and $y \in Y$

there are k internally disjoint free alternating paths and 1 saturated connecting x and y.

Theorem III - Proof of Only If

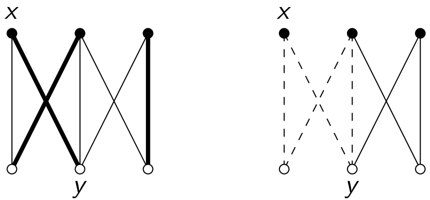
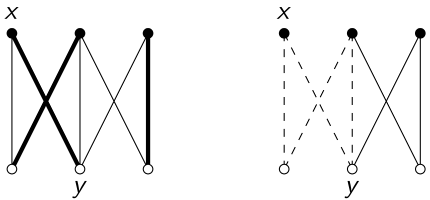


Figure: An 1-extendable graph G in case $\{x, y\} \in M$.

Theorem III - Proof of Only If



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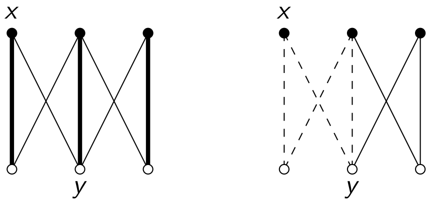

Figure: An 1-extendable graph G in case $\{x, y\} \notin M$.

Theorem III - Proof of Only If

Lemma

Let p and k be two integers such that $0 < p < k < n$. G is k-extendable if and only if for every matching $M_p = \{(u_1, v_1), \ldots, (u_p, v_p)\}$ of p edges $G \setminus u_1 \setminus v_1 \setminus \cdots \setminus u_p \setminus v_p$ is $(k-p)$-extendable.

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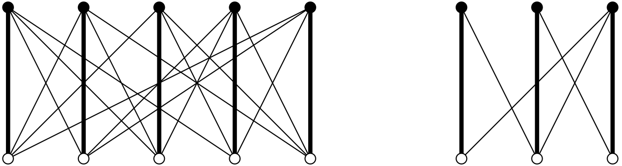

Figure: A 3-extendable graph G.

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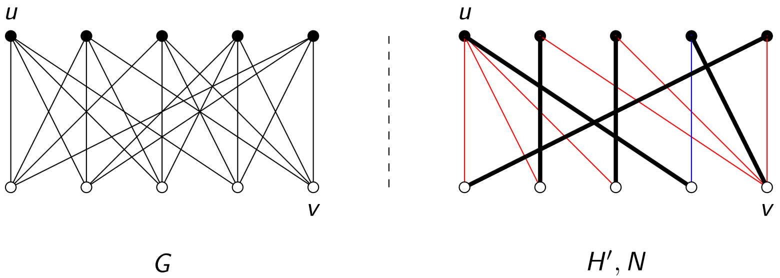

Figure: A 3-extendable graph G and paths P_1, P_2, P_3, Q in case $\{u, v\} \notin E'$.

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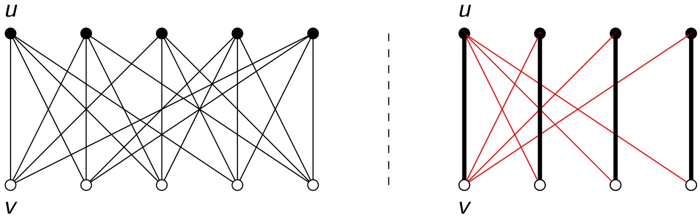

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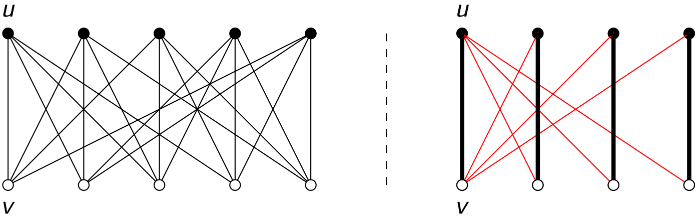

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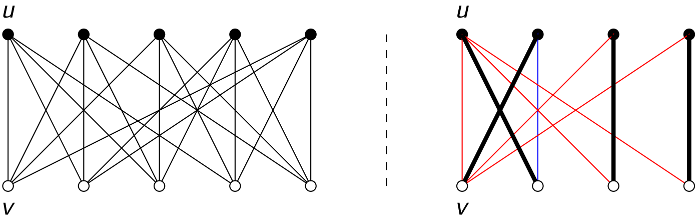

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Obtaining Directed Graphs from Undirected

Definition

We call *residual graph* of G, denoted by G_M, the graph obtained from G by directing the unmatched edges from X to Y and the matched edges from Y to X.

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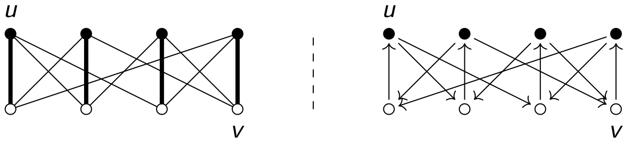

Figure: A 2-extendable graph G and its residual graph G_M.

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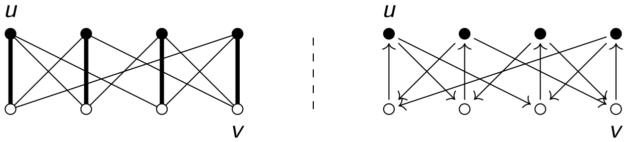

Figure: A 2-extendable graph G and its residual graph G_M.

Theorem

Let M be a perfect matching of G. G is k-extendable if and only if its residual graph G_M is strongly connected and there are k vertex internally disjoint directed paths between every vertex of X and every vertex of Y.

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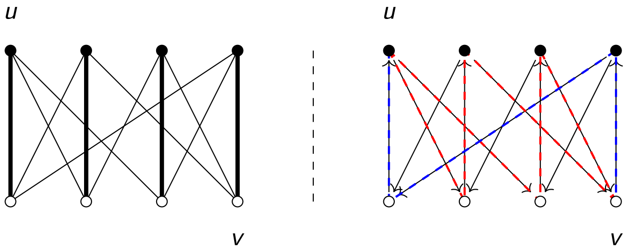
The Polynomial Algorithm on Bipartite Graphs (J Lakhal et al., IPL, 1998)



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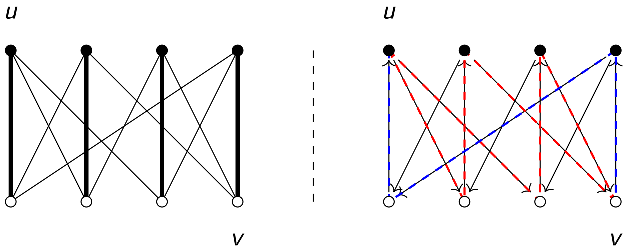

Figure: A 2-extendable graph G and its residual graph G'.

for $u \in U$ **do**
  for $v \in V$ **do**
    $paths \leftarrow \text{max-disjoint-paths}(G', u, v)$
    $k \leftarrow \min(k, paths)$
  end for
end for

coNP-completeness on General Graphs (Jan Hackfeld et al., J Comb Optim, 2018)

Theorem

$\overline{\textsc{Extendability}}$ is in NP.

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Theorem

$\overline{\textsc{Extendability}}$ is NP-hard.

Reduction from Vertex Cover.
Let (G_{VC}, s) be an instance of Vertex Cover and let r be the total number of vertices of G_{VC}.

coNP-completeness - The Construction

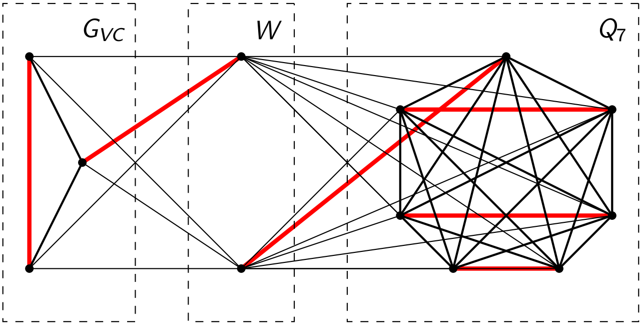

Figure: An obtained graph G from G_{VC} with a perfect matching M.

coNP-completeness - Part I

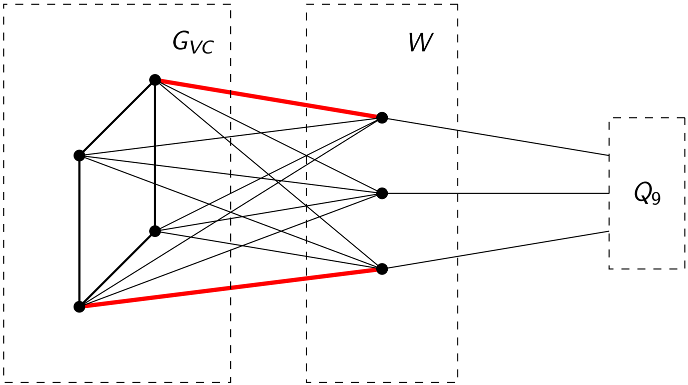

Figure: G_{VC} has a vertex cover of size at most 2 and G is not 2-extendable.

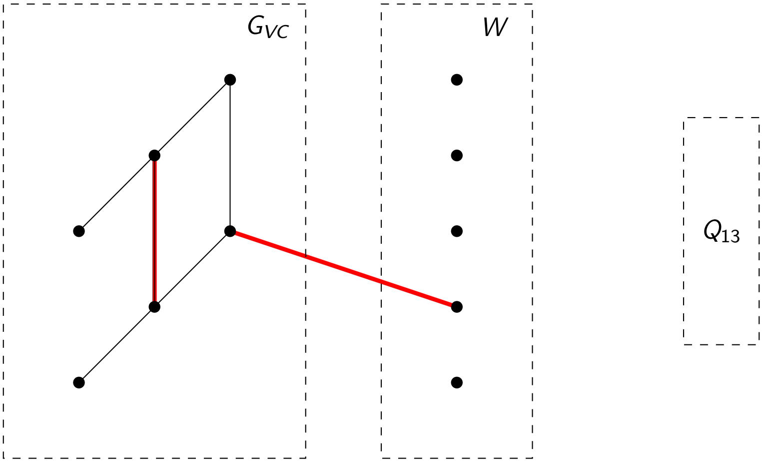

Figure: G_{VC} does not contain a vertex cover of size at most 2 and G is 2-extendable.

coNP-completeness - Part II

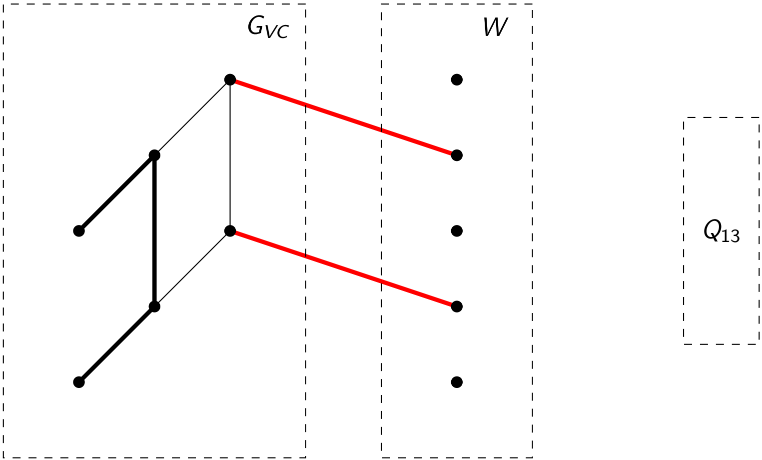

Figure: G_{VC} does not contain a vertex cover of size at most 2 and G is 2-extendable.

Thank You!!!