STAT547C: Topics in Probability **Fall 2019-20**

Assignment 1

Giorgio Viancha Sgarbi giorgio.sgarbi@stat.ubc.ca

1 PARTITION GENERATED SIGMA ALGEBRA

(a) Let $D = \{A, B, C\}$ be a partition of E. List the elements of σD.
D is a partition of E:
$E = A \cup B \cup C$
$\emptyset = A \cap B = A \cap C = B \cap C$

SOLUTION:

Let K be the set of all possible countable unions of A, B and C, i.e:

$\mathcal{K} = \{A, B, C, A \cup B, A \cup C, B \cup C, A \cup B \cup C, \emptyset\}$

I will show $\mathcal{K} \subset \sigma\mathcal{D}, \sigma\mathcal{D} \subset \mathcal{K}$. Hence, $\mathcal{K} = \sigma\mathcal{D}$.

$\mathcal{K}$ is a σ-algebra on E :
$E \in \mathcal{K}$:
$E = A \cup B \cup C \in \mathcal{K}$ since D is a partition of E.

$\mathcal{K}$ is closed under union:
By construction, K closed under union (finite/countable union)

$\mathcal{K}$ is closed under complement:
Since $E = A \cup B \cup C$ and $\emptyset = A \cap B = A \cap C = B \cap C$:
$E \setminus A = B \cup C$
$E \setminus B = A \cup C$
$E \setminus C = A \cup B$
$E \setminus A \cup B \cup C = \emptyset$

$D \subset \mathcal{K}$:
$A, B, C \in \mathcal{K}$

$\sigma D \subset \mathcal{K}$:
$\mathcal{K}$ is a σ-algebra on E that contains D, so $\sigma D \subset K$

$\mathcal{K} \subset \sigma\mathcal{D}$:
$A, B, C \in \sigma D$ by definition of σD.
$A \cup B, A \cup C, B \cup C, A \cup B \cup C \in sD$ because σD is a σ-algebra on E and, thus, needs to be closed under countable unions. Therefore, all elements of $\mathcal{K}$ are elements of σD.

Hence, the list of elements in $\sigma\mathcal{D}$ are:
$\{A, B, C, A \cup B, A \cup C, B \cup C, A \cup B \cup C, \emptyset\}$, where $E = A \cup B \cup C$

b) Let C be a countable partition of E. Show that every element of $\sigma\mathcal{C}$ is a countable union of elements taken from C. Hint: Let S be the collection of all sets that are countable unions of elements taken from C. Show that S is a σ-algebra on E and argue that $S = \sigma\mathcal{C}$ Hint: You might find it useful to recall the fact that the union of a countable collection of sets, each of which is countable, is again countable.

SOLUTION:

Notation: $C = \mathcal{C}$

C is a countable partition of E. Let S be the collection of all sets that are countable unions of elements in C (C_i's, $\emptyset$ included).

Will show $S = \sigma\mathcal{C}$

Proof template:
- S is a σ-algebra on E .
- $\sigma\mathcal{C} \subset \mathcal{S}$
- $S \subset \sigma\mathcal{C}$

Lemma: The union of a countable collection of countable sets is countable.

$E \in S$:
$C = \{C_1, C2, ...\}$ is a countable partition of E $\Rightarrow$ $E = \bigcup C_i \in S$

S is closed under countable unions:
The union of a countable collection of countable sets is countable and S is the collection of all sets that are countable unions of elements from C. So any countable union of elements in S will again be a countable union of elements from C, i.e, an element of S.

S is closed under complement:
Any A from S can be written as a countable union of a collection of elements in $C^* \subset C_i$'s. A^c is then the union of the C_i's not in C^* because C is a partition of E. Therefore, by our lemma, A^c is also a countable union of elements from C and hence is also in S.

$\sigma\mathcal{C} \subset \mathcal{S}$:
$\sigma\mathcal{C}$ is the intersection of all σ-algebra on E containing C. S is one of those σ-algebra on E , so $\sigma\mathcal{C} \subset \mathcal{S}$

$S \subset \sigma\mathcal{C}$:
By definition, $\sigma\mathcal{C}$ needs to be closed under countable union, so it has to contain C and own ($\ni$) all possible countable unions of elements taken from C. So, by the definition of S, $S \subset \sigma\mathcal{C}$.

$\sigma\mathcal{C} \subset \mathcal{S} \wedge \mathcal{S} \subset \sigma\mathcal{C} \iff \mathcal{S} = \sigma\mathcal{C}$ $\square$

(c) Show that the Borel σ-algebra on $\mathbb{R}$, $\mathcal{B}(\mathbb{R})$, is countably generated.

SOLUTION:

(checked) $\mathcal{B}(R)$ is generated by, for example, the collection of sets of the form (q, ∞), where $q \in Q$. Since that collection has the same cardinality of Q and Q is countable, $\mathcal{B}(R)$ is countably generated.

1(d) Suppose $\mathcal{E}$ is generated by a countable partition C of E. Show that, in this case, a R function is $\mathcal{E}$ -measurable if and only if it is constant on each member of that partition.

SOLUTION:

This proof will use the following:
Prop. 2.3 from book with B_0 as the collection of sets of the form (q, ∞). We know, from the previous question, that B_0 generates $\mathcal{B}(\mathbb{R})$.

$(E, \mathcal{E}), (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$f : E \to \mathbb{R}$

$E = \bigcup_n C_n$

$C_i \cap C_j = \emptyset \ \forall i, j \geq 1 \land i \neq j$

f is constant on each member of the partition $\implies$ f is $\mathcal{E}$-measurable :

Assumption:
$\forall i \in 1, 2, ... :$
$f(x) = k_i \in \mathbb{R} \ \forall x \in C_i$

Now, for each set B of the form (q, ∞): $f^{-1}B = \{x \in E : f(x) \in B\} = \{x \in E : f(x) > q\} =$
$\bigcup C_j : k_j > q \ \forall j \in \mathbb{N}$

I can write $f^{-1}B$ in the way above because C is a partition of E.

Furthermore, since E is the collection of countable unions of elements taken from C (with $\emptyset$ of course), we have that: $f^{-1}B \in \mathcal{E} \ \forall B \in B_0$

f is $\mathcal{E}$-measurable $\implies$ f is constant on each member of the partition :

Will prove the counter positive:
If f is **NOT** constant on at least one member of the partition, then f is **NOT** $\mathcal{E} - measurable.$

Assume f is not constant on at least one member C_d in C. This means: $\exists x_1, x_2 \in C_k : f(x_2) < f(x_1)$.

Then take $B = (q, \infty)$ with $f(x_2) < q < f(x_1)$. [1]

Using the fact that C is a partition of E, we can write $f^{-1}B = S_1 \cup S_2$, where:

$S_1 = \bigcup C_j : k_j > q \ \forall j \in \mathbb{N} \setminus \{d\}$ and $S_2 = \{x \in C_d : f(x) \in B\}$ with $S_1 \cap S_2 = \emptyset$

S_1 is defined in a similar fashion as the first case and uses the fact that f takes constant values on partition elements, *with the exception of* C_d.

However, S_2 is a **proper** subset of C_d as its elements are taken from C_d while S_2 can't be the whole C_d since $x_2 \in C_d$, but $x_2 \notin S_2$.

Since C is a partition of E and S_2 is a **proper** subset of C_d , $f^{-1}B$ will only be a member of $\mathcal{E}$, i.e a countable union of elements taken from C, if $S_2 = \emptyset$, but $x_1, x_2 \in S_2$, so $f^{-1}B \notin \mathcal{E}$. $\square$

[1] This rational q exists because $f(x_1)$ and $f(x_2)$ are reals and Q is dense in $\mathbb{R}$, but I won't say q can be written as the average between those two values as said average may not be rational.

2 Towards conditioning and sufficiency

2a) Suppose there is random variable X : $\Omega \to \mathbb{R}$ taking values in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ that induces a measurable (countable) partition, $C = (C_n)_{n\geq 1}$ of Ω.

SOLUTION:

2a) sol For each $A \in \mathcal{B}(\mathbb{R})$, in order for X to be a random variable, we need:

$X^{-1}A$ to ...

2b) Recall that for D

SOLUTION:

(b)

Proof template:
$\mu(\emptyset) = 0$
$\mu(\bigcup_n A_n) = \sum_n \mu(A_n)$ $\mu = P(A)$

2c)Define a probability measure constructed from n. What must be true of n(Cm) in order for this to be well-deaned?

SOLUTION:

2c) sol

2(d)

SOLUTION:

2(d) sol

2e question

SOLUTION:

2e) sol

2f) question

SOLUTION:

2f)sol

2g question

SOLUTION:

2g) sol

question 3 Show every continuous functions from a topological space E to R is a Borel function.

SOLUTION:

solution 3

$(E, \mathcal{E}), (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$f : E \to \mathbb{R}$ continuous

E is topological

Hint: f continuous $\implies f^{-1}B$ is open for every open subset $B \subset \mathbb{R}$ (1)

Show $\forall B \in \mathcal{B}(\mathbb{R}) : f^{-1}B \in \mathcal{E}$

If $(E, \mathcal{E})$ is topological, $\mathcal{E}$ is a collection of open sets in E. Therefore, we have:

$\forall B \in \mathcal{B}(\mathbb{R})$:
$B \ open \implies^{(1)} f^{-1}B \ open \implies f^{-1}B \in \mathcal{E} \quad \square$

question 4

SOLUTION:

solution 4