STAT547C: Topics in Probability **Fall 2019-20**

Assignment 1

Giorgio Viancha Sgarbi giorgio.sgarbi@stat.ubc.ca

1 PARTITION GENERATED SIGMA ALGEBRA

SOLUTION:

(1a)

Let K be the set of all possible countable unions of A, B and C, i.e:

$\mathcal{K} = \{A, B, C, A \cup B, A \cup C, B \cup C, A \cup B \cup C, \varnothing\}$

I will show $\mathcal{K} \subset \sigma\mathcal{D}, \sigma\mathcal{D} \subset \mathcal{K}$. Hence, $\mathcal{K} = \sigma\mathcal{D}$.

$\mathcal{K}$ is a σ-algebra on E :
$E \in \mathcal{K}$:
$E = A \cup B \cup C \in \mathcal{K}$ since D is a partition of E.

$\mathcal{K}$ is closed under union:
By construction, K closed under union (finite/countable union)

$\mathcal{K}$ is closed under complement:
Since $E = A \cup B \cup C$ and $\varnothing = A \cap B = A \cap C = B \cap C$:
$E \setminus A = B \cup C$
$E \setminus B = A \cup C$
$E \setminus C = A \cup B$
$E \setminus A \cup B \cup C = \varnothing$

$D \subset \mathcal{K}$:
$A, B, C \in \mathcal{K}$

$\sigma D \subset \mathcal{K}$:
$\mathcal{K}$ is a σ-algebra on E that contains D, so $\sigma D \subset K$

$\mathcal{K} \subset \sigma\mathcal{D}$:
$A, B, C \in \sigma D$ by definition of σD.
$A \cup B, A \cup C, B \cup C, A \cup B \cup C \in sD$ because σD is a σ-algebra on E and, thus, needs to be closed under countable unions. Therefore, all elements of $\mathcal{K}$ are elements of σD.

Hence, the list of elements in $\sigma\mathcal{D}$ are:
$\{A, B, C, A \cup B, A \cup C, B \cup C, A \cup B \cup C, \varnothing\}$, where $E = A \cup B \cup C$

SOLUTION:

(1b)

Notation: $C = \mathcal{C}$

C is a countable partition of E. Let S be the collection of all sets that are countable unions of elements in C (C_i's, $\varnothing$ included).

Will show $S = \sigma\mathcal{C}$

Proof template:
- S is a σ-algebra on E .
- $\sigma\mathcal{C} \subset \mathcal{S}$
- $S \subset \sigma\mathcal{C}$

Lemma: The union of a countable collection of countable sets is countable.

$E \in S$:
$C = \{C_1, C2, ...\}$ is a countable partition of E $\Rightarrow$ $E = \bigcup C_i \in S$

S is closed under countable unions:
The union of a countable collection of countable sets is countable and S is the collection of all sets that are countable unions of elements from C. So any countable union of elements in S will again be a countable union of elements from C, i.e, an element of S.

S is closed under complement:
Any A from S can be written as a countable union of a collection of elements in $C^* \subset C_i$'s. A^c is then the union of the C_i's not in C^* because C is a partition of E. Therefore, by our lemma, A^c is also a countable union of elements from C and hence is also in S.

$\sigma\mathcal{C} \subset \mathcal{S}$:
$\sigma\mathcal{C}$ is the intersection of all σ-algebra on E containing C. S is one of those σ-algebra on E , so $\sigma\mathcal{C} \subset \mathcal{S}$

$S \subset \sigma\mathcal{C}$:
By definition, $\sigma\mathcal{C}$ needs to be closed under countable union, so it has to contain C and own ($\ni$) all possible countable unions of elements taken from C. So, by the definition of S, $S \subset \sigma\mathcal{C}$.

$\sigma\mathcal{C} \subset \mathcal{S} \wedge \mathcal{S} \subset \sigma\mathcal{C} \iff \mathcal{S} = \sigma\mathcal{C}$ $\square$

SOLUTION:

(1c)

$\mathcal{B}(\mathbb{R})$ is generated by, for example, the collection of sets of the form (q, ∞), where $q \in \mathbb{Q}$. Since that collection has the same cardinality of Q and Q is countable, $\mathcal{B}(R)$ is countably generated.

SOLUTION:

(1d)

This proof will use the following:
Prop. 2.3 from book with B_0 as the collection of sets of the form (q, ∞). We know, from the previous question, that B_0 generates $\mathcal{B}(\mathbb{R})$.

$(E, \mathcal{E}), (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$f : E \to \mathbb{R}$

$E = \bigcup_n C_n$

$C_i \cap C_j = \varnothing \ \forall i, j \geq 1 \land i \neq j$

f is constant on each member of the partition $\implies$ f is $\mathcal{E}$-measurable :

Assumption:
$\forall i \in 1, 2, ... :$
$f(x) = k_i \in \mathbb{R} \ \forall x \in C_i$

Now, for each set B of the form (q, ∞): $f^{-1}B = \{x \in E : f(x) \in B\} = \{x \in E : f(x) > q\} = \bigcup C_j : k_j > q \ \forall j \in \mathbb{N}$

I can write $f^{-1}B$ in the way above because C is a partition of E.

Furthermore, since E is the collection of countable unions of elements taken from C (with $\varnothing$ of course), we have that: $f^{-1}B \in \mathcal{E} \ \forall B \in B_0$

f is $\mathcal{E}$-measurable $\implies$ f is constant on each member of the partition :

Will prove the counter positive:
If f is **NOT** constant on at least one member of the partition, then f is **NOT** $\mathcal{E} - measurable$.

Assume f is not constant on at least one member C_d in C. This means: $\exists x_1, x_2 \in C_k : f(x_2) < f(x_1)$.

Then take $B = (q, \infty)$ with $f(x_2) < q < f(x_1)$. [1]

Using the fact that C is a partition of E, we can write $f^{-1}B = C_1 \cup C_2 \cup ... \cup S_2 ...$, where:
$S_2 = \{x \in C_d : f(x) \in B\}$.

However, S_2 is a proper subset of C_d as its elements are taken from C_d and it can't be the whole C_d since $x_2 \in C_d$, but $x_2 \notin S_2$.

Since C is a partition of E and S_2 is a **proper** subset of C_d , $f^{-1}B$ will only be a member of $\mathcal{E}$, i.e a countable union of elements taken from C, if $S_2 = \varnothing$, but $x_1 \in S_2$, so $f^{-1}B \notin \mathcal{E}$. Therefore, f is **NOT** $\mathcal{E} - measurable$.
Since we've proved the counter positive, we have:
f is $\mathcal{E}$-measurable $\implies$ f is constant on each member of the partition

□

[1] This rational q exists because $f(x_1)$ and $f(x_2)$ are real numbers and Q is dense in $\mathbb{R}$, but I won't say q can be written as the average between those two values as said average may not be rational.

2 Towards conditioning and sufficiency

SOLUTION:

(2a)

For each $A \in \mathcal{B}(\mathbb{R})$, in order for X to be a random variable, we need $X^{-1}A \in \mathcal{H}$.

What must be true of the relationship between $\mathcal{C}$ and $\mathcal{H}$?
$\mathcal{C} \subset \mathcal{H}$

What must be true of X on each member of $\mathcal{C}$?
X needs to be constant on each element of $\mathcal{C}$.

SOLUTION:

(2b)

$\mu_n(A) = P(A), A \in \mathcal{C}_n$

Show that μ_n is a measure:

Since P is a probability measure, $\mu_n : \mathcal{C}_n \to [0,1] \subset \mathbb{R}_+$

$\mu_n(\varnothing) = P(\varnothing) = 0$ since P is a measure.

Let $\{A_n\} \subset C_n$ be a disjoint collection. Using that P is a measure:
$\mu(\bigcup_n A_n) = \mathbb{P}(\bigcup_n A_n) = \sum_n P(A_n) = \sum_n \mu(A_n)$

Hence, μ_n is a measure.

Not necessarily a probability measure:

There's no guaranteed that $\mu_n(C_n) = 1$ so far. For instance, let $C = \{C_1, C_2\}, n = 1, \mu_1(C_1) = 0.6, \mu_1(C_2) = 0.4$ Since $\mu_1(C_1) \neq 1$, μ_1 is not a probability measure on $(C_1, \mathcal{C}_1)$.

For μ_n to be a a probability measure on $(C_n, \mathcal{C}_n)$, **we need** $\mu_n(C_n) = 1$, i.e, $P(C_n) = 1$.

SOLUTION:

(2c)

Define a probability measure P_n on $(C_n, \mathcal{C}_n)$.

Let $P_n(A) = \frac{\mu_n(A)}{\mu_n(C_n)}$. Now we have all the requirements for a **probability measure** as long as this is well defined, i.e, $\mu_n(C_n) \neq 0$

(Ie., we have the requirements for a measure, but now we also have: $P_n(C_n) = \frac{\mu_n(C_n)}{\mu_n(C_n)} = 1$)

SOLUTION:

(2d)

$\hat{P}_n(A) = P_n(A \cap C)$ for $A \in \mathcal{H} = \frac{\mu_n(A \cap C_n)}{\mu_n(C_n)} = \frac{P(A \cap C_n)}{P(C_n)}$

In an undergrad course, this would be known as a **conditional probability**.

SOLUTION:

(2e)

$P(A) = P(\cup_{n\geq 1}(A \cap C_n)) =$ (countable additivity as the $A \cap C_n$ are pairwise disjoint) $\sum_{n\geq 1} P(A \cap C_n) = \sum_{n\geq 1} P(C_n)\hat{P}_n(A) = \sum_{n\geq 1} c_n \hat{P}_n(A)$, where $(c_n)_{n\geq 1} = P(C_n)$.

Show convexity:

$\sum_{n\geq 1} c_n = \sum_{n\geq 1} P(C_n) = P(\cup_{n\geq 1} C_n) = P(E) = 1$ (using that P is a probability measure)

Also, notice $0 = P(\varnothing) \leq c_n \leq P(\Omega) = 1$ $\forall n \geq 1$.

Therefore, P can be recovered as a mixture of the set $\{\hat{P}_n\}_{n\geq 1}$

SOLUTION:

(2f)

v and ν are synonyms here.

$\nu(A) = \sum_{n\geq 1} c_n \hat{P}_n(A), A \in \mathcal{H}$

Since each c_n is between 0 and 1, and the sum of the c_n's is 1, we can notice that the smallest value that $\nu(A)$ can take is 0 (for instance, when $A = \varnothing$) while the maximum value is 1, attained when $\hat{P}_n(A)$ (for instance, when $A = \Omega$). Therefore:
$v : \mathcal{H} \to [0, 1]$

$\nu(\varnothing) = \sum_{n\geq 1} c_n \hat{P}_n(\varnothing) = \sum_{n\geq 1} c_n 0 = 0$

Take a disjoint collection (A_m) from $\mathcal{H}$:
$\nu(\cup_{m\geq 1} A_m) =$
$\sum_{n\geq 1} c_n \hat{P}_n(\cup_{m\geq 1} A_m) =$ (used disjoint and P is a measure here)
$\sum_{n\geq 1} c_n (\sum_{m\geq 1} \hat{P}_n(A_m)) =$
$\sum_{n\geq 1} (\sum_{m\geq 1} c_n \hat{P}_n(A_m)) =$
$\sum_{m\geq 1} (\sum_{n\geq 1} c_n \hat{P}_n(A_m)) =$
$\sum_{m\geq 1} \nu(A_m)$

$\nu(\Omega) = \sum_{n\geq 1} c_n \hat{P}_n(\Omega) = \sum_{n\geq 1} c_n = 1$

SOLUTION:

(2g)

$\Rightarrow$

We know:
$v_n(A) = \frac{v(A)}{v(C_n)} = \frac{P(A)}{P(C_n)} = P_n(A), A \in \mathcal{C}_n$

We want:
1. $P(C_n) = 0 \implies v(C_n) = 0$

2. $\nu(A) = \sum_{n\geq 1} c_n \hat{P}_n(A),$ where $\hat{P}_n = P_n(A \cap C_n), A \in \mathcal{H}$

Proof:

1. $\frac{v(A)}{v(C_n)} = \frac{P(A)}{P(C_n)} \implies v(C_n) = P(C_n)\frac{v(A)}{P(A)}$ Therefore, $P(C_n) = 0 \implies v(C_n) = 0$

2.
Assume $P(C_n) \neq 0$. Otherwise, from (1), $v(A) = 0$, and we can take $(c_n) = (1, 0, 0, ...)$ and we're done.

$v(A) = v(C_n)\frac{P(A)}{P(C_n)}$ for any $A \in \mathcal{C}_n$

$A \in \mathcal{C}_n$ and $C_n \in \mathcal{C}_n$. Therefore, since sigma-algebras are closed under countable intersection, $A \cap C_n \in \mathcal{C}_n$ and we can write: $v(A \cap C_n) = v(C_n)\frac{P(A\cap C_n)}{P(C_n)} = v(A \cap C_n) = v(C_n)\hat{P}_n(A)$
$\implies \sum_{n\geq 1} v(A \cap C_n) = \sum_{n\geq 1} v(C_n)\hat{P}_n(A)$
$\implies v(A) = \sum_{n\geq 1} c_n\hat{P}_n(A)$ where, $c_n = v(C_n)$.

v is a probability measure on $(\Omega, \mathcal{H}) \implies 0 \leq c_n \leq 1, \sum_{n\geq 1} c_n = 1$

$\Leftarrow$

Fix n, write the mixture using index m, and assume $P(C_n) \neq 0$:

$$\frac{v(A)}{v(C_n)} = \frac{\sum_{m\geq 1} c_m\hat{P}_m(A)}{\sum_{m\geq 1} c_m\hat{P}_m(C_n)} = \frac{\sum_{m\geq 1} c_m\frac{P(A\cap C_m)}{P(C_m)}}{\sum_{m\geq 1} c_m\frac{P(C_n\cap C_m)}{P(C_m)}} = \frac{\sum_{m\geq 1} P(A\cap C_m)}{c_n\frac{P(C_n)}{P(C_n)}} = \frac{P(A)}{P(C_n)}$$

Now, if $P(C_n) = 0,$ then $v(C_n) = 0$, and we have:
$A \subset C_n \implies P(A) = 0, v(A) = 0.$ Therefore:
$\frac{v(A)}{v(C_n)} = \frac{0}{0} = \frac{P(A)}{P(C_n)}$ $\square$

3 Continuous functions

SOLUTION:

(3)

$(E, \mathcal{E}), (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$f : E \to \mathbb{R}$ continuous

E is topological

Hint: f continuous $\implies f^{-1}B$ is open for every open subset $B \subset \mathbb{R}$ (1)

Show $\forall B \in \mathcal{B}(\mathbb{R}) : f^{-1}B \in \mathcal{E}$

If $(E, \mathcal{E})$ is topological, $\mathcal{E}$ is a collection of open sets in E. Therefore, we have:

$\forall B \in \mathcal{B}(\mathbb{R})$:
$B\ open \implies {}^{(1)} f^{-1}B\ open \implies f^{-1}B \in \mathcal{E} \quad \square$

4 Functional representation

SOLUTION:

(4)

$\Leftarrow$:

We have: $\exists h : G \to F : h$ is $\mathcal{G}$-measurable and $f = h \circ g$

We want: f is $g^{-1}\mathcal{G} - measurable$

h is $\mathcal{G}$-measurable $: \forall B \in \mathcal{F}, h^{-1}B \in \mathcal{G}$
g is $\mathcal{E}$-measurable $: \forall B \in \mathcal{G}, g^{-1}B \in \mathcal{E}$
f is $\mathcal{E}$-measurable $: \forall B \in \mathcal{F}, f^{-1}B \in \mathcal{E}$

Take any $F_1 \in \mathcal{F}$:
h is $\mathcal{G}$-measurable $: h^{-1}F_1 \in \mathcal{G}$
g is $\mathcal{E}$-measurable $: g^{-1}h^{-1}F_1 \in g^{-1}\mathcal{G}$
$\implies g^{-1} \circ h^{-1}F_1 \in g^{-1}\mathcal{G}$
$\implies (h \circ g)^{-1}F_1 \in g^{-1}\mathcal{G}$
$\implies f^{-1}F_1 \in g^{-1}\mathcal{G}$
$\implies$ f is $\in g^{-1}\mathcal{G}$-measurable

$\Rightarrow$: