STAT547C: Topics in Probability **Fall 2019-20**

Assignment 1

Due date: Sept. 23 before class. *Instructor: Benjamin Bloem-Reddy*

Instructions

Academic integrity policy: I encourage you to discuss verbally with other students about the assignment. However, you should write your answers by yourself. For example, copying (either manually or electronically) part of a function or a LaTeXequation is not permitted; and if you use online resources, you must cite them. If you discuss the assignment with anyone (a classmate or anyone else), you must say so at the top of your solutions. Also, refrain from looking at answer keys from other schools, previous years, or the Math Stack Exchange. For more information, see:
`http://learningcommons.ubc.ca/guide-to-academic-integrity/`

1. The assignment due date is available from the course webpage (*before the lecture begins, as I may sometimes go over solutions in class*), see the Schedule section.
2. You only have to do the non-optional questions for full marks. While the other questions are optional, we encourage you to at least attempt them if you are motivated to learn the subject in depth. Effort to do so is one of the ways to get participation points (the main other ways are the exercises, the learning logs, and participation during the lecture and the office hours).
3. *Justify your answers formally.*

1 Partition-generated σ-algebras

(a) Let $\mathcal{D} = \{A, B, C\}$ be a partition of E. List the elements of $\sigma\mathcal{D}$.

(b) Let $\mathcal{C}$ be a countable partition of E. Show that every element of $\sigma\mathcal{C}$ is a countable union of elements taken from $\mathcal{C}$.

Hint: Let $\mathcal{S}$ be the collection of of all sets that are countable unions of elements taken from $\mathcal{C}$. Show that $\mathcal{S}$ is a σ-algebra, and argue that $\mathcal{S} = \sigma\mathcal{C}$.

Hint: You might find it useful to recall the fact that the union of a countable collection of sets, each of which is countable, is again countable.

(c) A σ-algebra is **countably generated** if it is generated by countably many sets. Argue that the Borel σ-algebra on $\mathbb{R}$, $\mathcal{B}(\mathbb{R})$, is countably generated.

Hint: There is not much to prove; we essentially answered this in an exercise.

(d) Suppose that $\mathcal{E}$ is generated by a countable partition $\mathcal{C}$ of E. Show that in this case, a $\mathbb{R}$-valued function is $\mathcal{E}$-measurable if and only if it is constant on each member of that partition.

2 Towards conditioning and sufficiency

Throughout this question, let $(\Omega, \mathcal{H}, P)$ be a probability space.

(a) Suppose there is random variable $X : \Omega \to \mathbb{R}$ taking values in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ that induces a measurable (countable) partition, $\mathcal{C} = (C_n)_{n \geq 1}$, of Ω. Turn this into a mathematical statement.

(b) Recall that for $D \subset \Omega$,

$$\mathcal{D} = \mathcal{H} \cap D = \{A \cap D : A \in \mathcal{H}\}$$

is a σ-algebra called the trace of $(\Omega, \mathcal{H})$ on D. Let $\mathcal{C}$ be a countable partition of Ω. Consider a set $C_n \in \mathcal{C}$ and the trace $\mathcal{C}_n$ of $(\Omega, \mathcal{H})$ on C_n. Define $\mu_n(A) = P(A)$ for any $A \in \mathcal{C}_n$, called the restriction of P to C_n. Show that μ_n is a measure, but not necessarily a probability measure, on $(C_n, \mathcal{C}_n)$. What is a necessary and sufficient condition for μ_n to be a probability measure on $(C_n, \mathcal{C}_n)$?

(c) Define a probability measure P_n on $(C_n, \mathcal{C}_n)$ constructed from μ_n. What must be true of $\mu_n(C_n)$ in order for this to be well-defined?

(d) P_n can be extended to be a probability measure $\hat{P}_n$ on $(\Omega, \mathcal{H})$ by defining $\hat{P}_n(A) = P_n(A \cap C_n)$ for any $A \in \mathcal{H}$.[1] Write down $\hat{P}_n(A)$, for any $A \in \mathcal{H}$, in terms of P. If A and C_n were random variables, what would we call $\hat{P}_n(A)$ in an undergraduate course in probability?

(e) Show that our original probability measure, P, can be recovered as a convex combination, or *mixture*, of the set $\{\hat{P}_n\}_{n \geq 1}$.

(f) Let $(c_n)_{n \geq 1}$ be *mixing coefficients*, i.e., they satisfy $0 \leq c_n < 1$ and $\sum_{n \geq 1} c_n = 1$. Define $\nu(A) = \sum_{n \geq 1} c_n \hat{P}_n(A)$ for $A \in \mathcal{H}$. Show that ν is a probability measure on $(\Omega, \mathcal{H})$.

(g) Now let ν be an arbitrary probability measure on $(\Omega, \mathcal{H})$. Define $\frac{0}{0} = 0$. Show that the restriction of ν to C_n is equal to that of P, for all $n \in \mathbb{N}$, that is,

$$\nu_n(A) = \frac{\nu(A)}{\nu(C_n)} = \frac{P(A)}{P(C_n)} = P_n(A) \ , \qquad \text{for all } A \in \mathcal{C}_n, \tag{1}$$

for all $n \in \mathbb{N}$, if and only if both of the following are true:

(a) $P(C_n) = 0 \Rightarrow \nu(C_n) = 0$.

(b) $\nu(A) = \sum_{n \geq 1} c_n \hat{P}_n(A)$ for each $A \in \mathcal{H}$, for some mixing coefficients $(c_n)_{n \geq 1}$.

Note: We will revisit these ideas in the context of conditioning.

3 Continuous functions

Suppose that E is a toplogocial space. Show that every continuous function $f : E \to \mathbb{R}$ is a Borel function (i.e., it is $\mathcal{B}(E)/\mathcal{B}(\mathbb{R})$-measurable).

Hint: If f is continuous, then $f^{-1}B$ is open for every open subset $B \subset \mathbb{R}$.

[1] It's worth proving this to yourself for general measures; see Çinlar, Exercise I.3.12.

4 Functional representation (optional)

Recall that for a set E and a measurable space $(F, \mathcal{F})$, the σ-algebra $f^{-1}\mathcal{F}$ on E generated by a function $f : E \to F$ is $f^{-1}\mathcal{F} = \{f^{-1}B : B \in \mathcal{F}\}$.

Let $(E, \mathcal{E})$ and $(G, \mathcal{G})$ be measurable spaces, and $(F, \mathcal{B}(F))$ a standard Borel space. Fix two $\mathcal{E}$-measurable functions $f : E \to F$ and $g : E \to G$. Then f is $g^{-1}\mathcal{G}$-measurable if and only if there exists some $\mathcal{G}$-measurable mapping $h : G \to F$ with $f = h \circ g$.

Hint: Use Theorem 3.7 in the lecture notes. First, prove for indicator and simple functions, then for positive and arbitrary functions.