

STAT305 2018-2019 WINTER 2 - QUIZ 2 SOLUTIONS (BY GIO) MARCH 7, 2019

**Important:**

I'm using a shorter notation for some items. Let me know if you have questions about the material. Questions about marking can be done by following the instructions here (TBA) **after** I hand back your quizzes. Please don't post questions related to marking on this post.

**Marking scheme:**

For questions worth **1 mark**: Either right or wrong as they usually require short and straightforward answers.

For questions worth **2 marks**: Solution with small mistakes, but sensitive answer and logical reasoning will be generally awarded [1] mark.

For questions worth **3 marks or more**: I detailed the marking scheme for those in the solutions below. You can find the marks in this format "[ mark ]".

**Solutions:**

a) This question basically asks to show that  $Beta(1, \theta) = \frac{1}{\theta}$ . Notice that  $\Gamma(a) = (a-1)\Gamma(a-1)$ . Therefore:

$$B(1, \theta) = \frac{\Gamma(1)\Gamma(\theta)}{\Gamma(\theta+1)} = \frac{0!\Gamma(\theta)}{\theta\Gamma(\theta)} = \frac{1}{\theta}$$

**Remark:**  $\theta!$  does not make sense as  $\theta$  is not necessarily an integer. However, we will not deduct marks for that this time.

b)  $L(\theta) = \theta^n (\prod y_i)^{\theta-1}$

c)  $\log(L(\theta)) = n\log\theta + (\theta - 1)\log(\prod y_i) = n\log\theta + (\theta - 1)\sum \log(y_i)$

d) I will simplify notation a bit and use simple derivative notation as the questions uses  $L(\theta)$ :  
 $dl/d\theta = \frac{n}{\theta} + \sum \log(y_i)$

Setting the derivative equal to 0:  
 $\hat{\theta} = -\frac{n}{\sum \log(y_i)}$  [2] [-1 if  $\theta$  instead of  $\hat{\theta}$ ]

$d^2l/d\theta^2 = -\frac{n}{\theta^2} < 0$  for all  $\theta$ . In particular,  $d^2l/d\theta^2 < 0$  at  $\theta = \hat{\theta}$  [1]

e) We don't know the distribution of  $\tilde{\theta}$  and  $Var(\tilde{\theta})$  is not a linear combination of  $Y_i$ . [Either is OK]

f)  $-d^2l/d\theta^2 = \frac{n}{\theta^2}$

g)  $\frac{\theta^2}{n}$

h) (Non-exact) Approximation. Justification won't be marked, but check question j for the exact expression for  $Var(\tilde{\theta})$ .

i)

i. See table for pdf of  $Gamma(n, \theta)$ .

ii. See my post with tittle "Gamma properties" on piazza: (The answer is on page 2. We also discussed it during office hours and labs.)

iii.  $E(\tilde{\theta}) = \frac{n\theta}{n-1}$

iv. No.  $E(\tilde{\theta}) \neq \theta$

j) A sufficient condition for an estimator to be consistent is:

$$MSE(\tilde{\theta}) = Bias^2(\tilde{\theta}) + Var(\tilde{\theta}) \rightarrow 0 \text{ as } n \text{ increases. [1]}$$

From the question,  $Var(\tilde{\theta}) = o(1/n) \rightarrow 0$ . [1]

Additionally,  $Bias^2(\tilde{\theta}) = (\frac{n\theta}{n-1} - \theta)^2 \rightarrow 0$  (i.e,  $\tilde{\theta}$  is not unbiased, but is asymptotically unbiased:  
 $E(\tilde{\theta}) = \frac{n\theta}{n-1} \rightarrow \theta$ ) [1]

k) **(Bonus)** Following the hint:

$$F_{X_i}(x) = P(X_i \leq x) = P(X_i < x) = P(-\log(Y_i) < x) = P(\log(Y_i) > -x) = P(Y_i > e^{-x}) = 1 - P(Y_i \leq e^{-x}) = 1 - F_{Y_i}(e^{-x}) \text{ [1]}$$

$$F_{Y_i}(y) = \int_0^y \theta t^{\theta-1} dt = y^\theta$$

Therefore:  $F_{X_i}(x) = 1 - (e^{-x})^\theta = 1 - e^{-\theta x}$ , which is the CDF of  $\text{Exp}(\theta)$  [1]

Finally, since  $X = \sum_{i=1}^n X_i$ ,  $X \sim \text{Gamma}(n, \theta)$  as required. [1]