STAT305 2018-2019 WINTER 2 - QUIZ 2 SOLUTIONS (BY GIO) MARCH 7, 2019

Important:

I'm using a shorter notation for some items. Let me know if you have questions about the material. Questions about marking can be done by following the instructions here (TBA) **after** I hand back your quizzes. Please don't post questions related to marking on this post.

Marking scheme:

For questions worth **1 mark**: Either right or wrong as they usually require short and straightforward answers.

For questions worth **2 marks**: Solution with small mistakes, but sensitive answer and logical reasoning will be generally awarded [1] mark.

For questions worth **3 marks or more**: I detailed the marking scheme for those in the solutions below. You can find the marks in this format "[mark]".

Solutions:

a) This question basically asks to show that $Beta(1,\theta) = \frac{1}{\theta}$. Notice that $\Gamma(a) = (a-1)\Gamma(a-1)$. Therefore:

$B(1,\theta) = \frac{\Gamma(1)\Gamma(\theta)}{\Gamma(\theta+1)} = \frac{0!\Gamma(\theta)}{\theta\Gamma(\theta)} = \frac{1}{\theta}$

Remark: $\theta!$ does not make sense as θ is not necessarily an integer. However, we will not deduct marks for that this time.

b) $L(\theta) = \theta^n(\prod y_i)^{\theta-1}$

c) $log(L(\theta)) = nlog\theta + (\theta-1)log(\prod y_i) = nlog\theta + (\theta-1)\sum log(y_i)$

d) I will simplify notation a bit and use simple derivative notation as the questions uses $L(\theta)$:
$dl/d\theta = \frac{n}{\theta} + \sum log(y_i)$

Setting the derivative equal to 0:
$\hat{\theta} = \frac{n}{-\sum log(y_i)}$ **[2] [-1 if θ instead of $\hat{\theta}$]**

$d^2l/d\theta^2 = \frac{-n}{\theta^2} < 0$ for all θ. In particular, $d^2l/d\theta^2 < 0$ at $\theta = \hat{\theta}$ **[1]**

e) We don't know the distribution of $\tilde{\theta}$ and $Var(\tilde{\theta})$ is not a linear combination of Y_i. [Either is OK]

f) $-d^2l/d\theta^2 = \frac{n}{\theta^2}$

g) $\frac{\theta^2}{n}$

h) (Non-exact) Approximation. Justification won't be marked, but check question j for the exact expression for $Var(\tilde{\theta})$.

i)

i. See table for pdf of $Gamma(n,\theta)$.

ii. See my post with tittle "Gamma properties" on piazza: (The answer is on page 2. We also discussed it during office hours and labs.)

iii. $E(\tilde{\theta}) = \frac{n\theta}{n-1}$

iv. No. $E(\tilde{\theta}) \neq \theta$

j) A sufficient condition for an estimator to be consistent is:

$MSE(\tilde{\theta}) = Bias^2(\tilde{\theta}) + Var(\tilde{\theta}) \to 0$ as n increases. [**1**]

From the question, $Var(\tilde{\theta}) = o(1/n) \to 0$. [**1**]

Additionally, $Bias^2(\tilde{\theta}) = (\frac{n\theta}{n-1} - \theta)^2 \to 0$ (i.e, $\tilde{\theta}$ is not unbiased, but is asymptotically unbiased: $E(\tilde{\theta}) = \frac{n\theta}{n-1} \to \theta$) [**1**]

k) **(Bonus)** Following the hint:

$F_{X_i}(x) = P(X_i \leq x) = P(X_i < x) = P(-log(Y_i) < x) = P(log(Y_i) > -x) = P(Y_i > e^{-x}) = 1 - P(Y_i \leq e^{-x}) = 1 - F_{Y_i}(e^{-x})$ [**1**]

$F_{Y_i}(y) = \int_0^y \theta t^{\theta-1} dt = y^\theta$

Therefore: $F_{X_i}(x) = 1 - (e^{-x})^\theta = 1 - e^{-\theta x}$, which is the CDF of Exp(θ) [**1**]

Finally, since $X = \sum_{i=1}^n X_i$, $X \sim Gamma(n, \theta)$ as required. [**1**]