CS 5003: Parameterized Algorithms

Lectures 26-27

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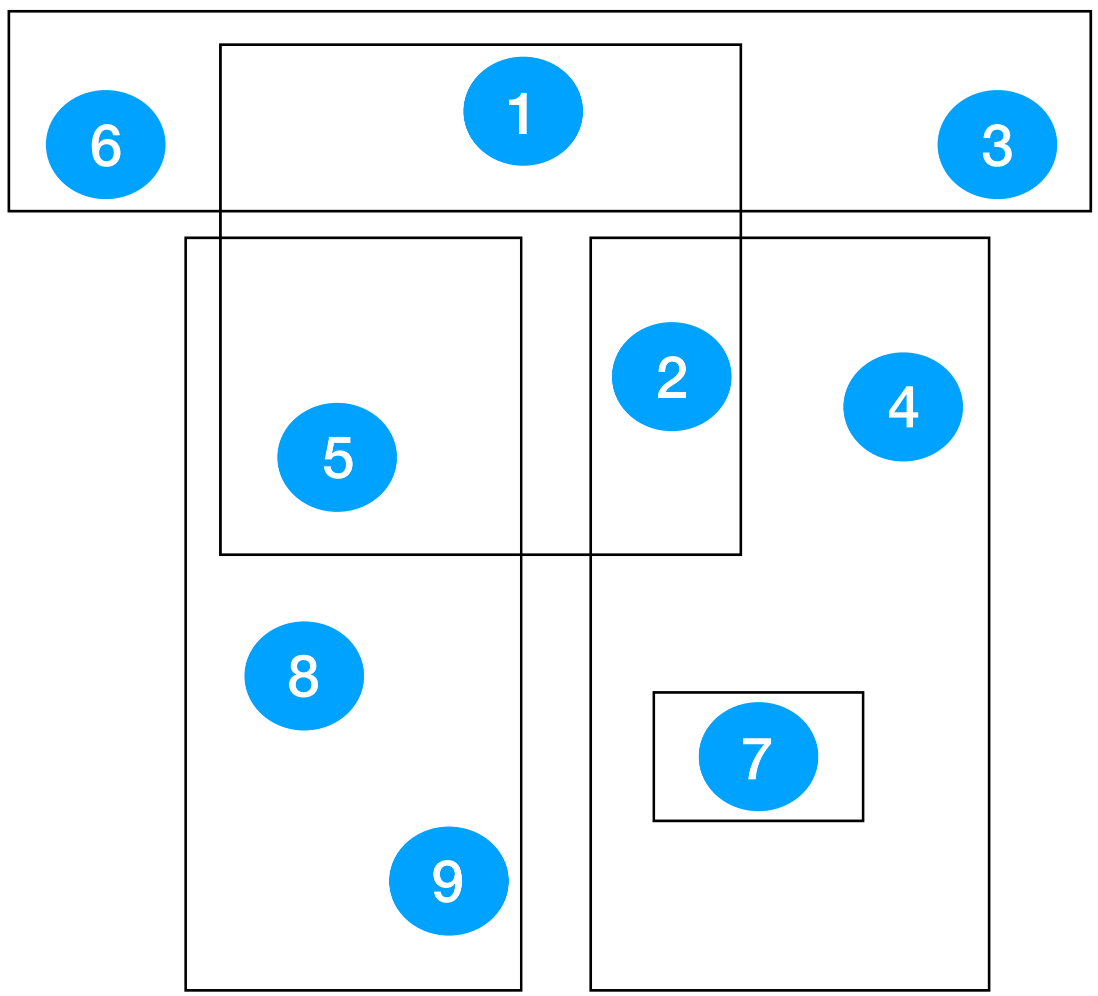
Reference Books: Parameterized Algorithms by Cygan et al. and Kernelization by Fomin et al.

Sunflower Lemma

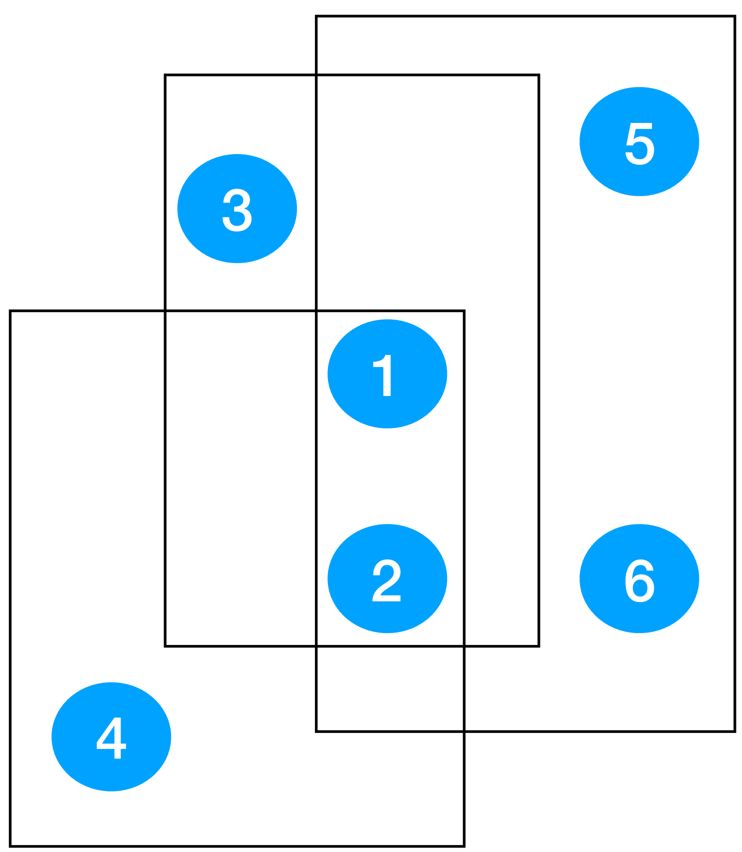
* Universe $U = \{1, 2, 3, \ldots, n\}$
* Family of subsets of U: $F = \{S_1, S_2, \ldots, S_m\}$



Sunflower Lemma

* Universe $U = \{1, 2, 3, \ldots, n\}$, Family of subsets of U: $F = \{S_1, S_2, \ldots, S_m\}$
* A sunflower with k petals and core Y is a collection $C \subseteq F$ of sets s.t
  * $S \cap S' = Y$ for any two distinct S, S' in C
  * $S \setminus Y$ is non-empty for every S in C

Sunflower



* 3 Petals {3}, {4}, {5, 6}
  * Each petal is non-empty
* Core = {1, 2}

* A set of pairwise disjoint sets is a sunflower with empty core

Sunflower Lemma

* Universe $U = \{1, 2, 3, \ldots, n\}$
* Family of subsets of U: $F = \{S_1, S_2, \ldots, S_m\}$ (no duplicates)
* Each set in F has size d
* $|F| > d!\,(k-1)^d$

$$\downarrow$$

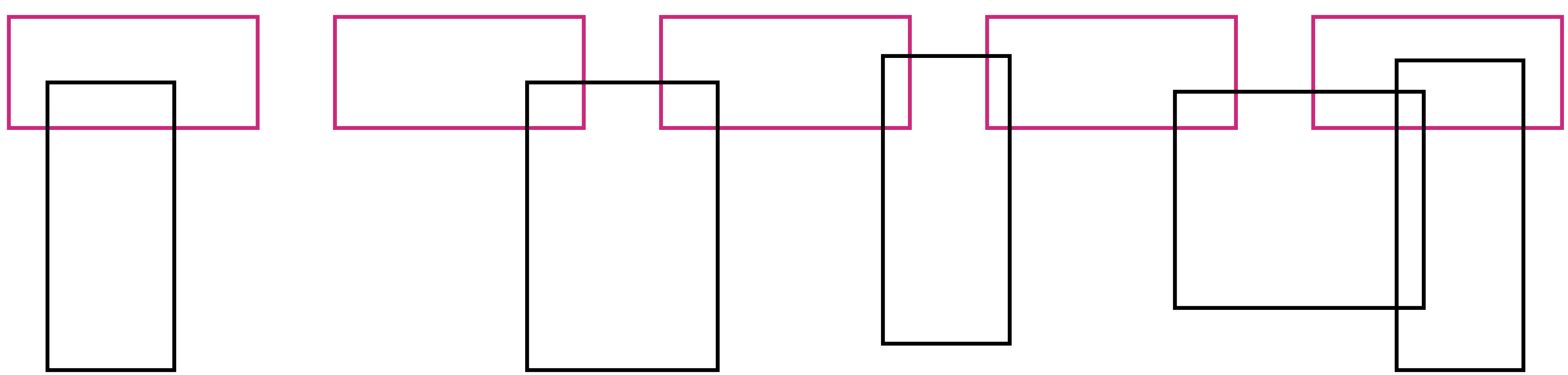
* F has a sunflower with k petals that can be obtained in polynomial time

Proof of Sunflower Lemma

- Universe $U = \{1, 2, 3, \ldots, n\}$
- Family of subsets of U: $F = \{S_1, S_2, \ldots, S_m\}$
  - No duplicates
- Each set in F has size d, $|F| > d!\,(k-1)^d$

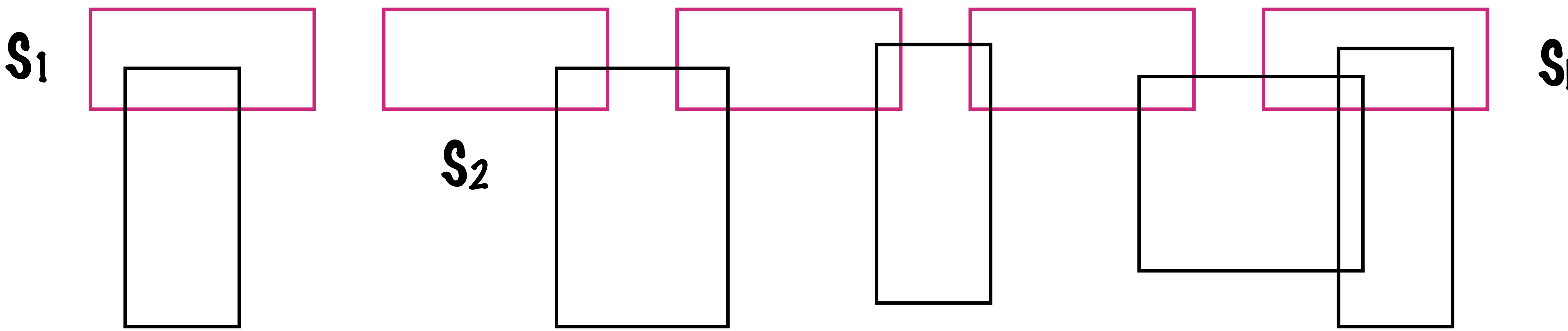
Induction on d

- $d = 1$ (singletons)
- $d >= 2$
  - $\{S_1, S_2, \ldots, S_l\}$ maximal set of disjoint sets in F
  - If $l >= k$, $\{S_1, S_2, \ldots, S_l\}$ is the sunflower with k petals
  - Otherwise, $l <= k$



Proof of Sunflower Lemma

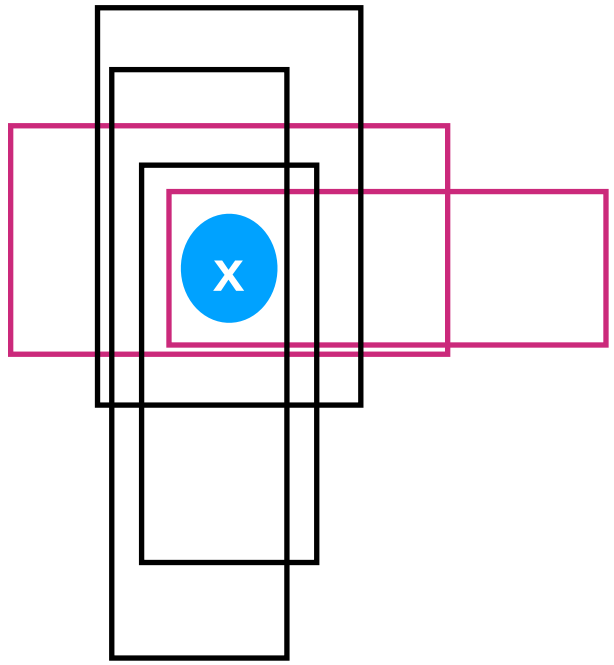
- Universe $U = \{1, 2, 3, \ldots, n\}$
- Family of subsets of U: $F = \{S_1, S_2, \ldots, S_m\}$
  - No duplicates
- Each set in F has size d, $|F| > d!\,(k\text{-}1)^d$



- $S = S_1 \cup S_2 \cup \ldots \cup S_l$ and $|S| <= d(k\text{-}1)$
- Every set in F has an element in S
- There is an element x that is in $>= |F|/|S| > d!\,(k\text{-}1)^d / d(k\text{-}1)$ sets of F

Proof of Sunflower Lemma

- Universe $U = \{1, 2, 3, \ldots, n\}$
- Family of subsets of U: $F = \{S_1, S_2, \ldots, S_m\}$
  - No duplicates
- Each set in F has size d, $|F| > d!\,(k-1)^d$



- There is an element x that is in $>= |F|/|S| > d!\,(k-1)^d / d(k-1)$ sets of F
- F' = sets in F containing x
- F'' : obtained from F' by deleting x (no duplicates)
  - $|F''| > (d-1)!\,(k-1)^{d-1}$
- By induction hypothesis, F'' has a sunflower with k petals
  - $\{S'_1, S'_2, \ldots, S'_k\}$
- $\{S'_1 \cup \{x\}, S'_2 \cup \{x\}, \ldots, S'_k \cup \{x\}\}$ is a sunflower with k petals in F

Computing a Sunflower

- Universe $U = \{1, 2, 3, \ldots, n\}$
- Family of subsets of U: $F = \{S_1, S_2, \ldots, S_m\}$
  - No duplicates
- Each set in F has size d, $|F| > d!\,(k-1)^d$

- Find a maximal set $\{S_1, S_2, \ldots, S_l\}$ of disjoint sets in F
- Let x be an element that is in a maximum number of sets in F
  - x is in >= |F|/|S| sets of F
- F´ : sets in F containing x
- F´´ : obtained from F´ by deleting x (no duplicates)
  - $|F''| > (d-1)!\,(k-1)^{d-1}$ as $|F|/|S| > d!\,(k-1)^d / d(k-1)$
- Recurse on F´´ to find a sunflower $\{S'_1, S'_2, \ldots, S'_k\}$ with k petals
- $\{S'_1 \cup \{x\}, S'_2 \cup \{x\}, \ldots, S'_k \cup \{x\}\}$ is a sunflower with k petals in F

Sunflower Lemma (Variant)

- Universe $U = \{1, 2, 3, \ldots, n\}$
- Family of subsets of U: $F = \{S_1, S_2, \ldots, S_m\}$ (no duplicates)
- Each set in F has size at most d
- $|F| > d * d!\,(k-1)^d$

$$\downarrow$$

- F has a sunflower with k petals that can be obtained in polynomial time

Hint: There exists $r \leq d$ s.t $|F_r| > d!\,(k-1)^d$ where F_r is the subset of F containing sets of size $=r$

d-Hitting Set

Input:

- Universe $U = \{1, 2, 3, \ldots, n\}$
- Family $F = \{S_1, S_2, \ldots, S_m\}$ of <= d sized subsets of U
- A non-negative integer k

Question: Does there exist $V \subseteq U$ with $|V| \leq k$ s.t for each S in F, $S \cap V \neq \emptyset$?

Some Common Hitting Sets

- Vertex Cover
- Feedback Vertex Set
- Odd Cycle Transversal
- Cluster Vertex Deletion

d-Hitting Set

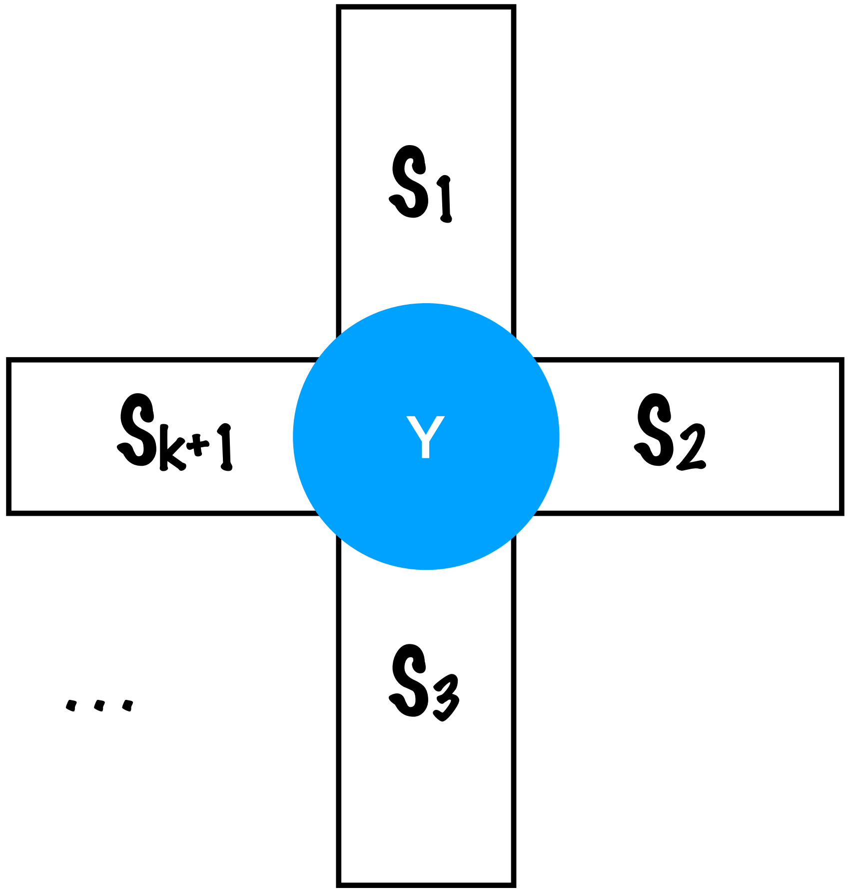
Input:

* Universe $U = \{1, 2, 3, \ldots, n\}$
* Family $F = \{S_1, S_2, \ldots, S_m\}$ of <= d sized subsets of U
* A non-negative integer k

Question: Does there exist $V \subseteq U$ with $|V| <= k$ s.t for each S in F, $S \cap V \neq \emptyset$?

* If $|F| <= d * d!\, k^d$
    * Kernel with $d * d!\, k^d$ sets and $d^2 * d!\, k^d$ elements
* Otherwise
    * find a sunflower with (k+1) petals

d-Hitting Set



* If Y is empty then there is no (U,F) has no hitting set of size <=k
* Otherwise,
  * Any hitting set of size <= k has a non-empty intersection with Y
  * Delete $S_1, S_2, \ldots, S_{k+1}$ from F and add Y to F to get resultant instance (U',F',k)

(U,F,k) is a yes-instance iff (U',F',k) is a yes-instance

d-Set Packing

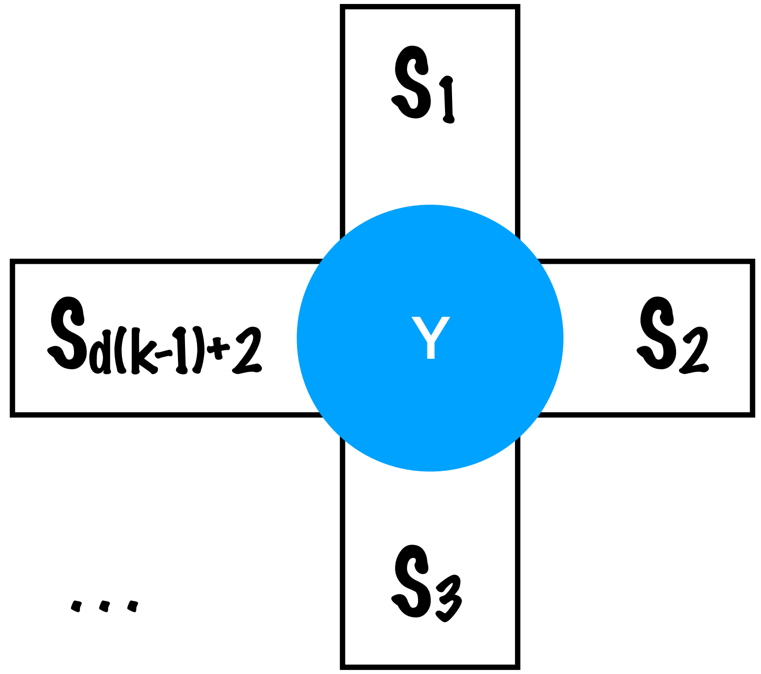
Input:

* Universe $U = \{1, 2, 3, \ldots, n\}$
* Family $F = \{S_1, S_2, \ldots, S_m\}$ of <= d sized subsets of U
* A non-negative integer k

Question: Does there exist $F' \subseteq F$ of pairwise disjoint elements s.t $|F'| >= k$?

* If $|F| <= d * d! (d(k-1)+1)^d$
  * Kernel with $d * d! (d(k-1)+1)^d$ sets and $d^2 * d! (d(k-1)+1)^d$ elements
* Otherwise
  * find a sunflower with d(k-1)+2 petals

d-Set Packing



* Delete S_1 from F to get resultant instance (U',F',k)
* (U,F,k) is a yes-instance iff (U',F',k) is a yes-instance
  * Reverse direction is easy
  * Forward direction
    * Suppose P is a k-set packing containing S_1
    * $P \setminus \{S_1\}$ has <= d(k-1) elements (set X)
    * There is a set S_i that has no element from X for some $2 \le i \le d(k-1)+2$
    * $P \setminus \{S_1\} \cup \{S_i\}$ is a k-set packing not containing S_1