CS 5003: Parameterized Algorithms

Lectures 10-11

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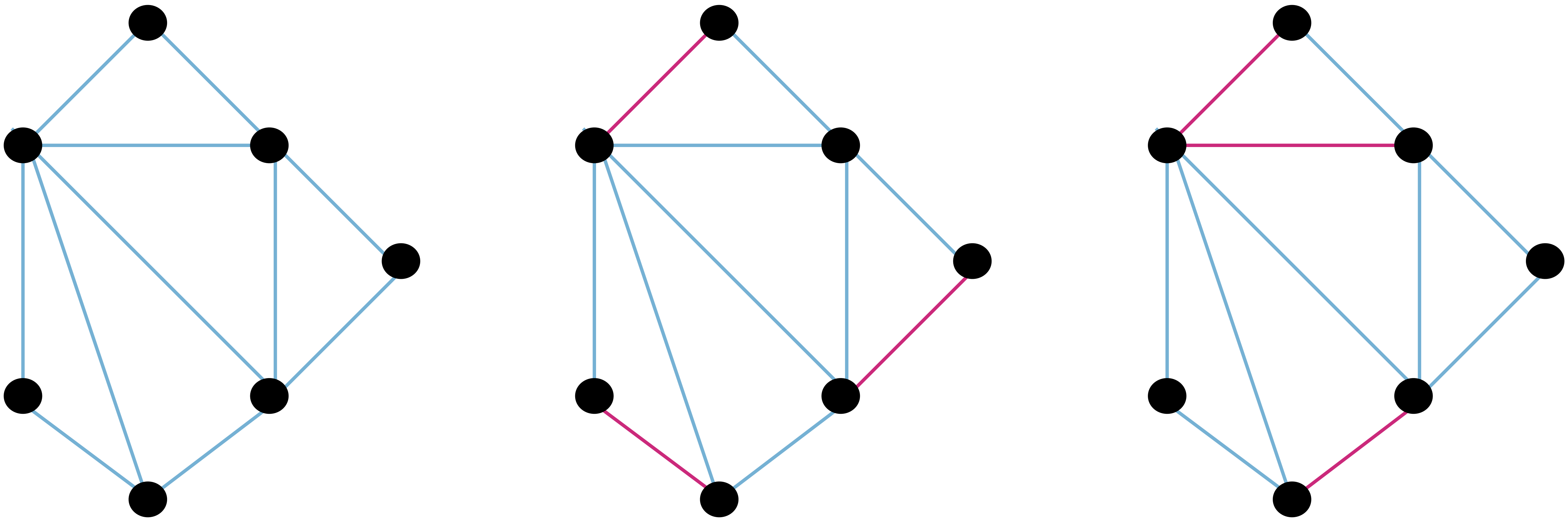
Reference Books: Parameterized Algorithms by Cygan et al. and Kernelization by Fomin et al.

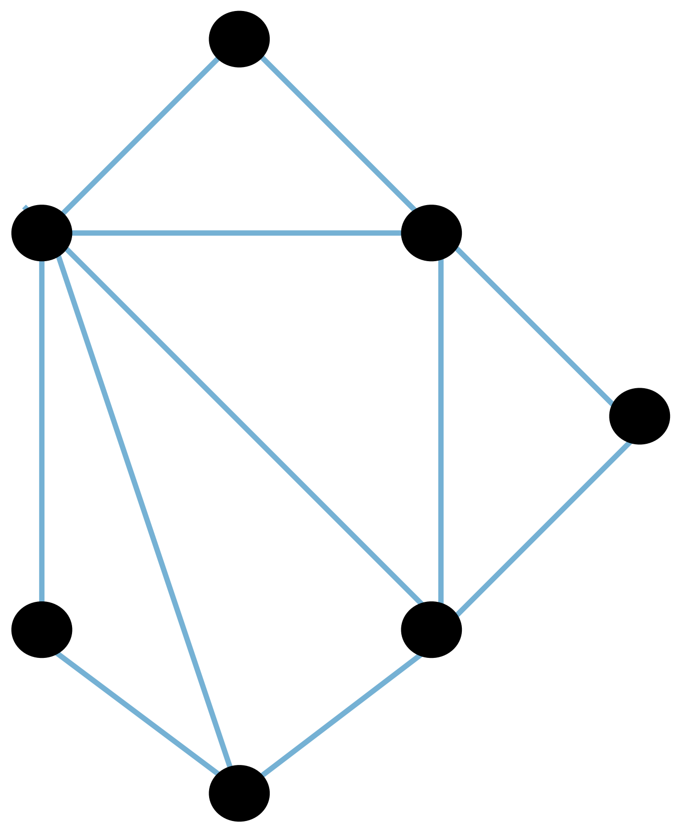
Matching

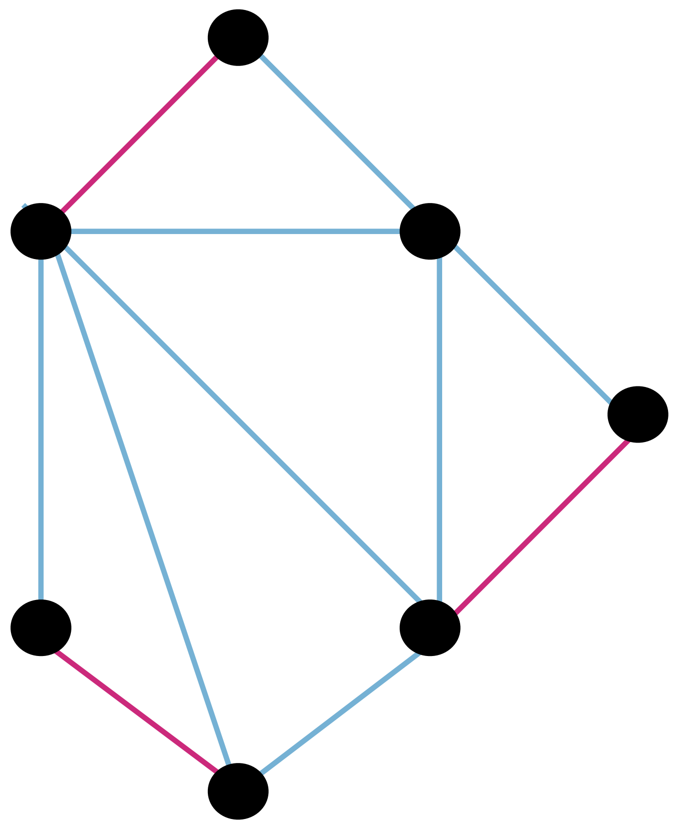
Matching - set of edges that do not share any endpoint



Matching and Vertex Cover

Matching - set of edges that do not share any endpoint

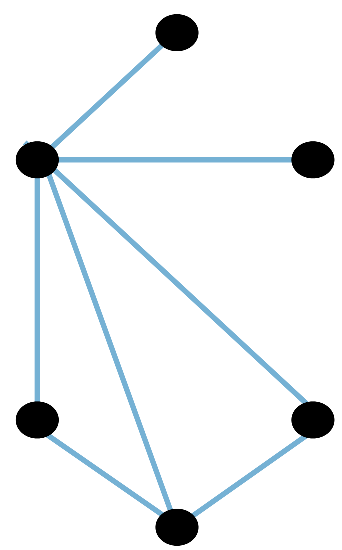


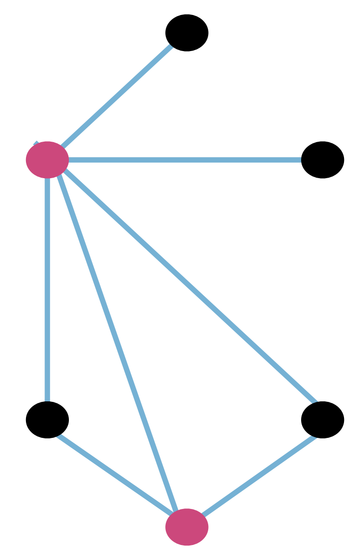


Matching of size x => Any vertex cover has size >= x

Vertex Cover

Vertex cover - set of vertices that has at least one endpoint of each edge





Instance: A graph G on n vertices m edges and integer k
Question: Does G have a vertex cover of size at most k?
Parameter: k

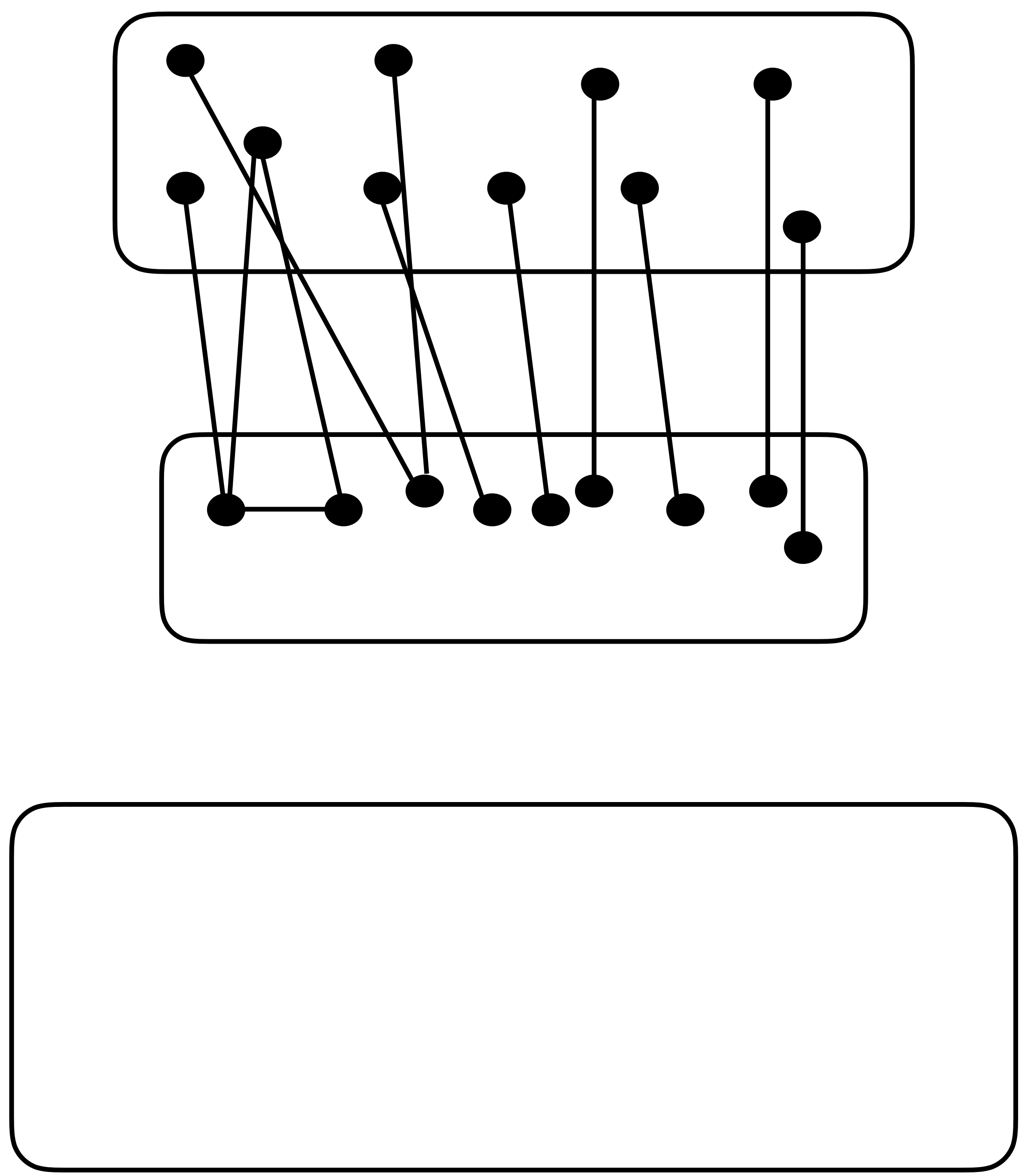
* Kernel with k^2 edges and $2k^2/3$ vertices
* $O(n^3+1.4656^k k^3)$ time algorithm

Vertex Cover: 3k vertex kernel

Crown Decomposition

Crown C

Independent set



Head H

$N(C) \subseteq H$

Rest R

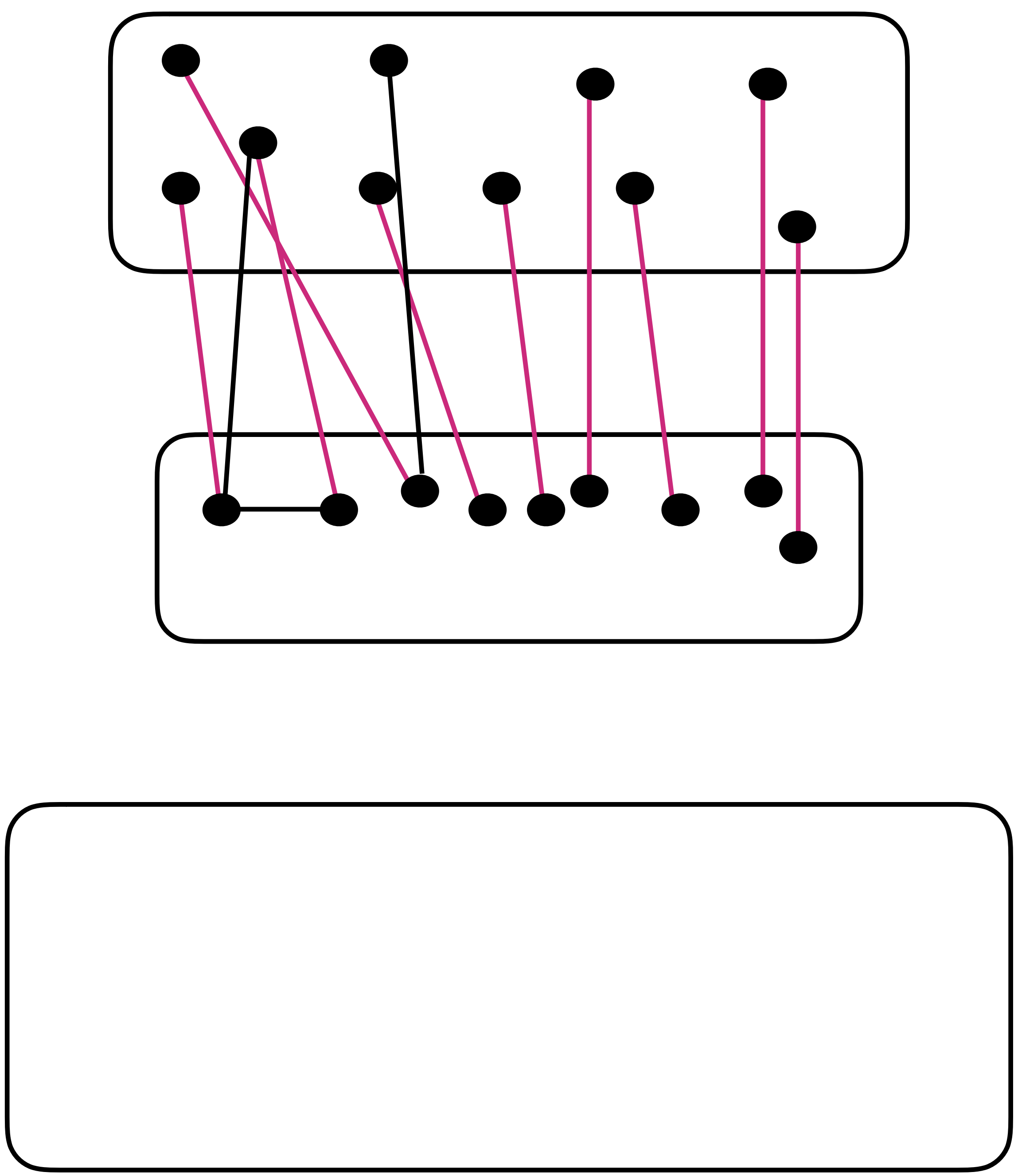
Vertex Cover: 3k vertex kernel

Crown Decomposition

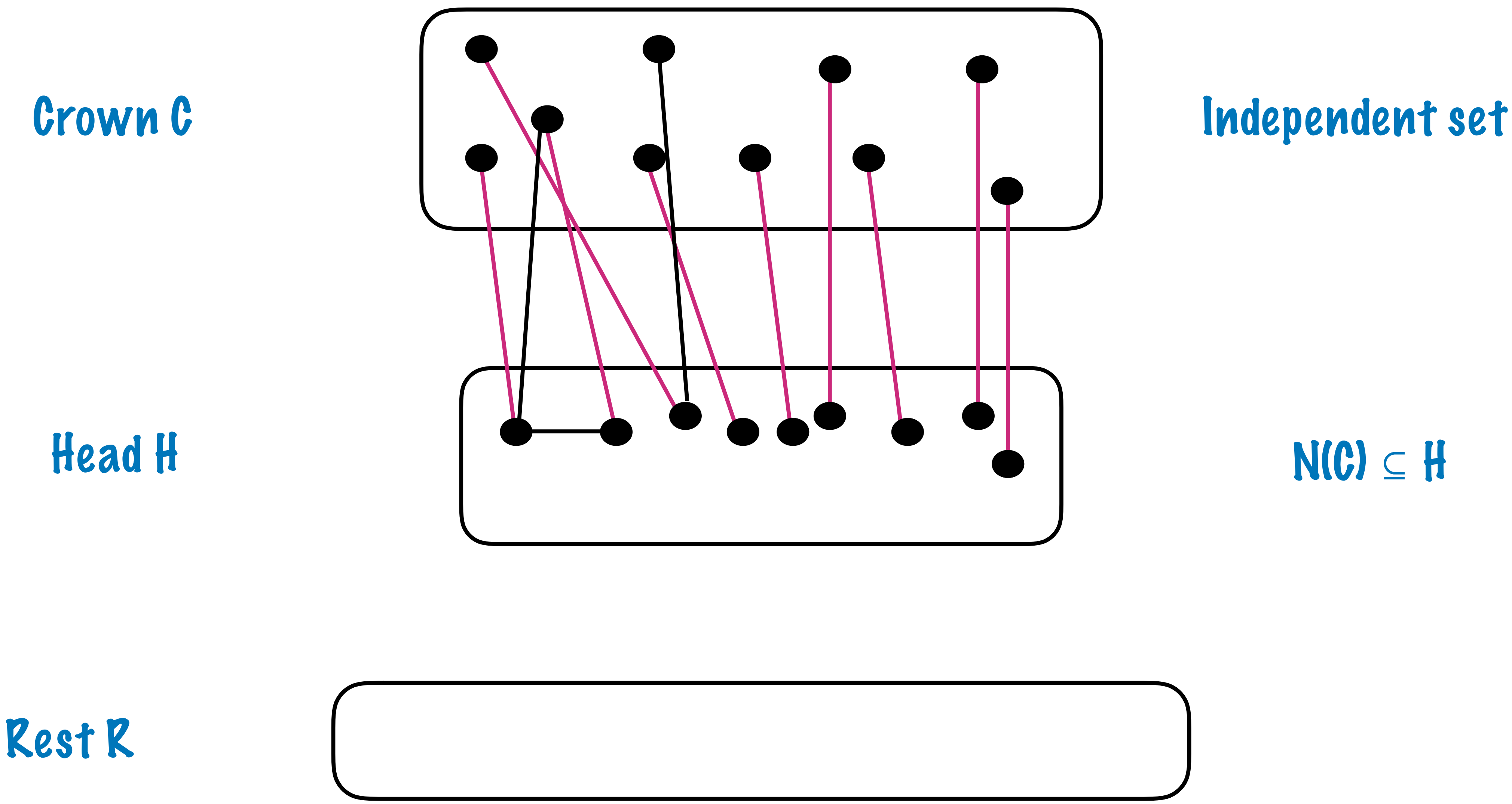
Crown C — Independent set

Head H — $N(C) \subseteq H$

Rest R — Matching saturating H



Vertex Cover: 3k vertex kernel



(G,k) is an yes-instance iff (G-(H∪C), k-|H|) is an yes-instance

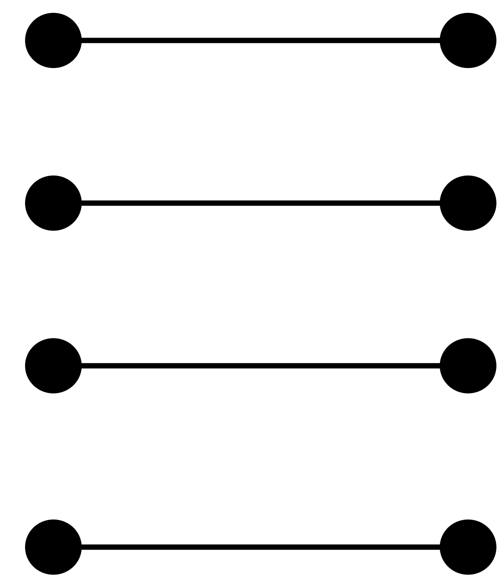
Vertex Cover: 3k vertex kernel

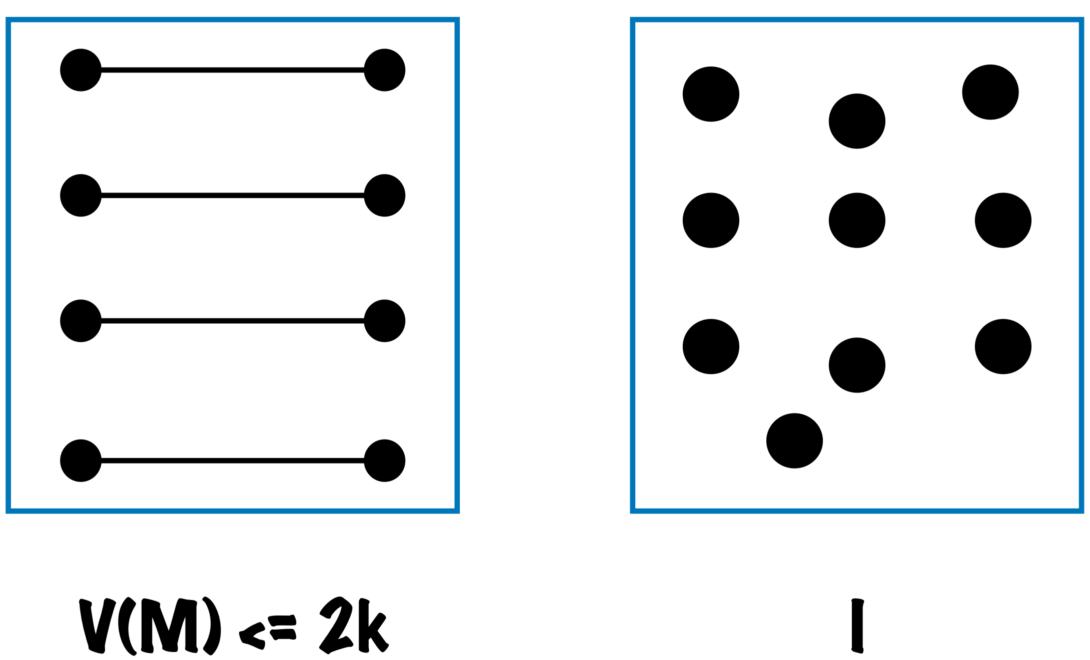
Crown Lemma: Let G be a graph without isolated vertices and with at least $3k + 1$ vertices. Then, there is a polynomial time algorithm that either finds a matching of size $k + 1$ in G, or finds a crown decomposition of G.

* **Reduction Rule 1:** Delete isolated vertices
* **Reduction Rule 2:**
  * If Crown Lemma finds a mat of size $k+1$, then (G,k) is a no-instance
  * Otherwise, (C,H,R) is a crown
    * Add H into the solution, delete $H \cup C$, reduce k by $|H|$
* If Reduction Rules 1 & 2 can't be applied, then G has at most 3k vertices

Proof of Crown Lemma

* G is a graph without isolated vertices and with >= 3k+1 vertices
* Find a maximal matching M in G
* If |M| >= k+1, then (G,k) is a no-instance
* Otherwise,

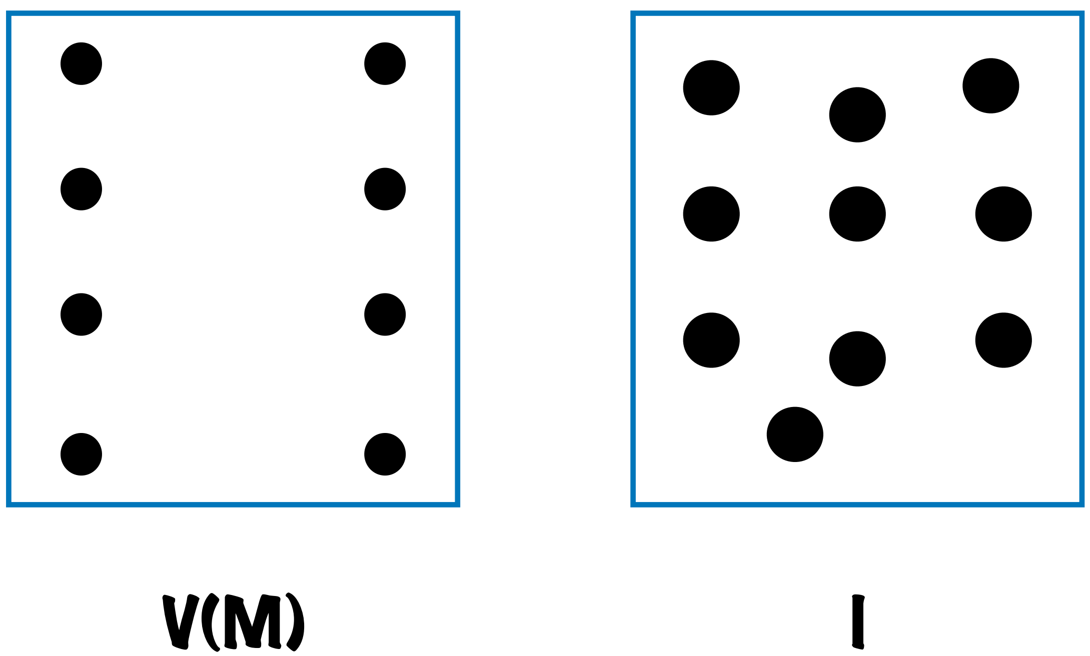




Proof of Crown Lemma

* Bipartite graph B

König's Theorem: For a bipartite graph, |Max Mat| = |Min VC|



X: min vertex cover

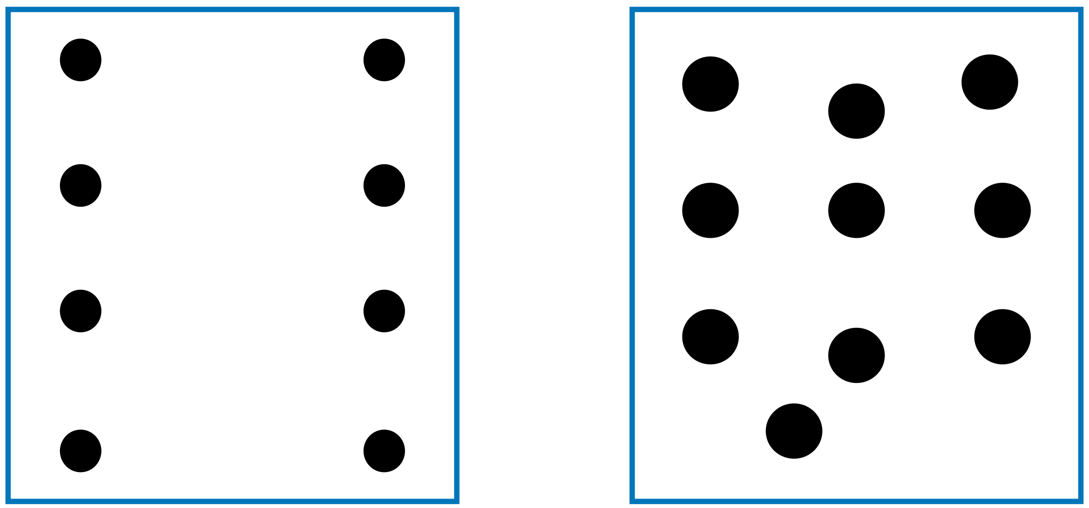
M': max matching

* Find a maximum matching M' and minimum vertex cover of B
* If |M'| >= k+1, then (G,k) is a no-instance

Proof of Crown Lemma

- Otherwise, $|M'| <= k$

König's Theorem: For a bipartite graph, |Max Mat| = |Min VC|



V(M) I

X: min vertex cover

M': max matching

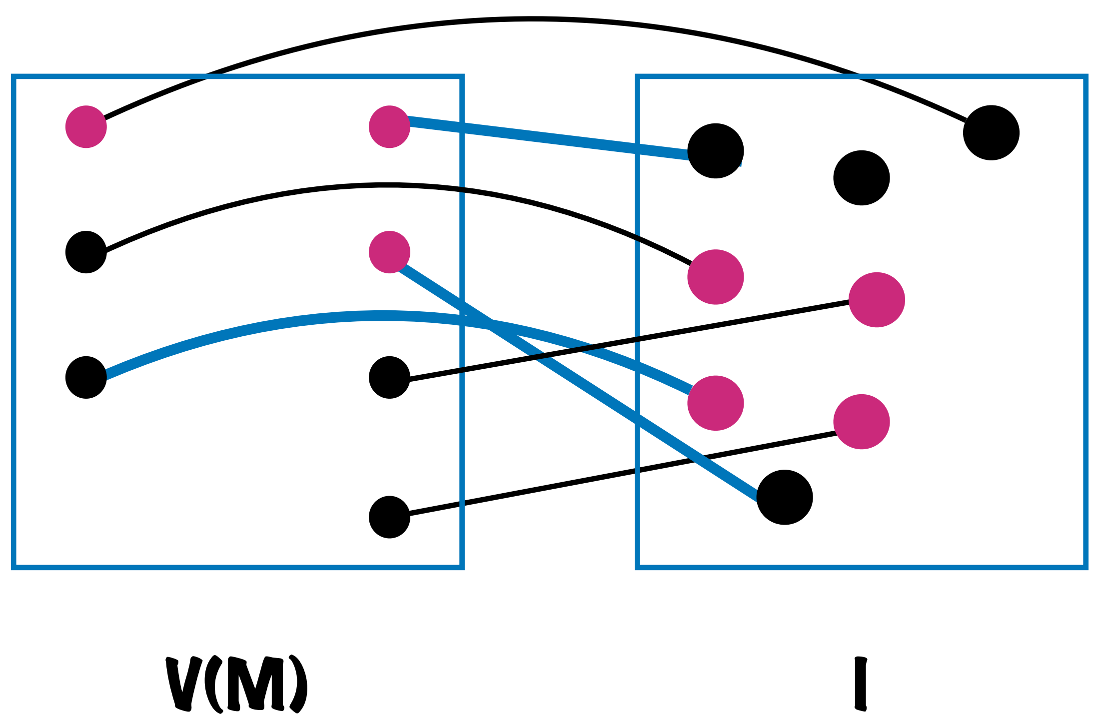
$|X| = |M'| <= k$

Claim: X has at least one vertex from V(M)

- If $X \subseteq I$, then $X = I$. Then, $|V(M)| + |I| <= 2k + k$. A contradiction!

Proof of Crown Lemma

* X has at least one vertex from V(M)



X: min vertex cover

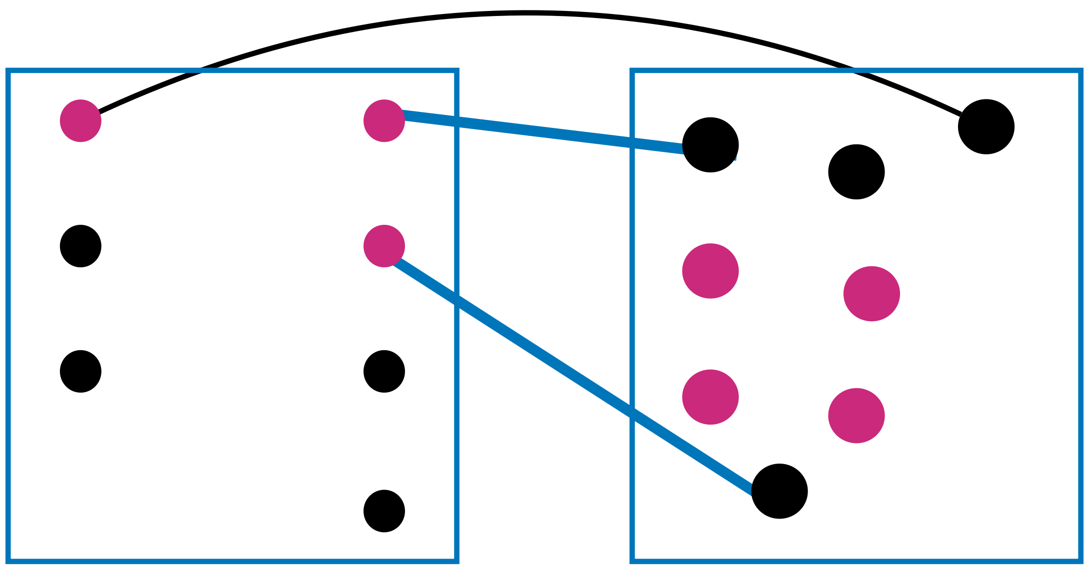
M': max matching

| X | = | M' | <= k

* Every edge of M′ has exactly one endpoint in X
* M′′ ⊆ M′ such that each edge in M′′ has an endpoint in X ∩ V(M)

Proof of Crown Lemma

* $M'' \subseteq M'$ such that each edge in M'' has an endpoint in $X \cap V(M)$



V(M)   I

X: min vertex cover

M': max matching

$|X| = |M'| \le k$

$V(M'') \cap I$ $V(M'') \cap X$ Rest

C   H