Assignment-5 Answers

Q1. Given $[T]_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ ①

$$① \Rightarrow [T]_\theta [T]_{\theta'} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta' & -\sin\theta' \\ \sin\theta' & \cos\theta' \end{bmatrix}$$

$$= \begin{bmatrix} (\cos\theta\cos\theta' - \sin\theta\sin\theta') & -(\sin\theta\cos\theta' + \cos\theta\sin\theta') \\ (\sin\theta\cos\theta' + \cos\theta\sin\theta') & (\cos\theta\cos\theta' - \sin\theta\sin\theta') \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta+\theta') & -\sin(\theta+\theta') \\ \sin(\theta+\theta') & \cos(\theta+\theta') \end{bmatrix} \equiv [T]_{\theta+\theta'}$$

$$① \Rightarrow \det [T]_\theta = \cos^2\theta + \sin^2\theta = 1 \Rightarrow [T]_\theta^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = [T]_{-\theta}$$

Q2. Rotation $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ implies $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x' \\ y' \end{bmatrix} = [T]_\theta \begin{bmatrix} x \\ y \end{bmatrix}$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Thus $x' = x\cos\theta - y\sin\theta$ & $y' = x\sin\theta + y\cos\theta$

Q3. Given: $T: V \to V$ is a linear map of vector space V over $\mathbb{F}$ into V, and $B = \{v_1, \ldots, v_n\}$ is a basis of V; $T(v_i) = c_i v_i$, $c_i \in \mathbb{F}$, $i = 1, \ldots, n$.

Let $x \in V$. There exist scalars $x_1, \ldots, x_n \in \mathbb{F}$ such that

$x = v_1 x_1 + \cdots + v_n x_n$ ① so that the coordinate vector $[x]_B$ of x w.r.t. basis B is $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

Upon the transformation considered here,

$$x \to Tx = T(v_1 x_1 + \cdots + v_n x_n) = T(v_1)x_1 + \cdots + T(v_n)x_n$$

$$= (c_1 v_1)x_1 + \cdots + (c_n v_n)x_n$$

$$= v_1(c_1 x_1) + \cdots + v_n(c_n x_n)$$

$\therefore$ coordinate vector of Tx w.r.t. basis B is

$$[Tx]_B = \begin{bmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & c_n \end{bmatrix}$$

Q4. Given $V = \mathbb{R}^3$, Let $v \in V$, and let B and B' be two bases of V. The coordinate vector $[v]_B$ and $[v]_{B'}$ of vector v corresponding to bases B and B' are related by $[v]_{B'} = S_{B,B'} [v]_B$

General treatment

$$\begin{tikzcd} V \arrow[r, "I"] \arrow[d, "{[\,]_B}"'] & V \arrow[d, "{[\,]_{B'}}"] \\ F^n \arrow[r, "S_{B,B'}"'] & F^n \end{tikzcd}$$

commutativity of this diagram $\Rightarrow$

$$[\,]_{B'} \circ I = [\,]_{B'} = S_{B,B'} [\,]_B. \quad (1)$$

(2)

Let $B = \{e_1, \dots, e_n\}$, $B' = \{e_1', \dots, e_n'\}$,

$$v = e_1 v_1 + \dots + e_n v_n = [e_1 \dots e_n] [v]_B = [e_1' \dots e_n'] [v]_{B'} \quad (3)$$

put $[e_1 \dots e_n] = [e_1' \dots e_n'] S_{B,B'}$ in (3) to get

$$[v]_{B'} = S_{B,B'} [v]_B$$

(a) $e_1 = (1,1,0)$, $e_2 = (-1,1,1)$, $e_3 = (0,1,2)$

$e_1' = (2,1,1)$, $e_2' = (0,0,1)$, $e_3' = (-1,1,1)$

Determination of S can be done in 2-ways.

Long route

$$e_1 = (1,1,0) = e_1' S_{11} + e_2' S_{21} + e_3' S_{31}$$

$$= (2,1,1) S_{11} + (0,0,1) S_{21} + (-1,1,1) S_{31}$$

$$\Rightarrow \left.\begin{array}{l} 2S_{11} - S_{31} = 1 \\ S_{11} + S_{31} = 1 \end{array}\right\} \Rightarrow S_{11} = \frac{2}{3}, \; S_{31} = \frac{1}{3}$$

$$S_{11} + S_{21} + S_{31} = 0 \longrightarrow S_{21} = -1$$

$e_2 = (-1,1,1) = e_1' S_{12} + e_2' S_{22} + e_3' S_{32}$
$= (2,1,1) S_{12} + (0,0,1) S_{22} + (-1,1,1) S_{32}$

$\Rightarrow \left.\begin{matrix} 2S_{12} - S_{32} = 1 \\ S_{12} + S_{32} = 1 \end{matrix}\right\} \Rightarrow S_{12}=0, \; S_{32}=1$

$S_{12} + S_{22} + S_{32} = 1 \Rightarrow S_{22}=0$

$e_3 = (0,1,2) = e_1' S_{13} + e_2' S_{23} + e_3' S_{33}$

$= (2,1,1) S_{13} + (0,0,1) S_{23} + (-1,1,1) S_{33}$

$\Rightarrow \left.\begin{matrix} 2S_{13} - S_{33} = 0 \\ S_{13} + S_{33} = 1 \end{matrix}\right\} \Rightarrow S_{13} = 1/3, \; S_{33} = 2/3$

$S_{13} + S_{23} + S_{33} = 2 \Rightarrow S_{23} = 1$

$$S = \begin{bmatrix} \frac{2}{3} & 0 & \frac{1}{3} \\ -1 & 0 & 1 \\ \frac{1}{3} & 1 & \frac{2}{3} \end{bmatrix}$$

Shorter method

$[e_1 \; e_2 \; e_3] = [e_1' \; e_2' \; e_3'] S$

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} S$$

or in short $E = E'S$

$\Rightarrow S = (E')^{-1} E$

$\det E' = 2(-1) - 1(1) = -3$

$(E')^{-1} = -\frac{1}{3} \begin{bmatrix} -1 & -1 & 0 \\ 0 & 3 & -3 \\ 1 & -2 & 0 \end{bmatrix}$

$$\Rightarrow S = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -1 & 1 \\ -\frac{1}{3} & \frac{2}{3} & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} & 0 & \frac{1}{3} \\ -1 & 0 & 1 \\ \frac{1}{3} & 1 & \frac{2}{3} \end{bmatrix}$$ Same as what we got earlier

Part b $\quad e_1 = (3,2,1), \; e_2 = (0,-2,5), \; e_3 = (1,1,2)$

$e_1' = (1,1,0), \; e_2' = (-1,2,4), \; e_3' = (2,-1,1)$

$S = (E')^{-1} E$

$E = \begin{bmatrix} 3 & 0 & 1 \\ 2 & -2 & 1 \\ 1 & 5 & 2 \end{bmatrix}$

$E' = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \\ 0 & 4 & 1 \end{bmatrix}$

$$S = \frac{1}{15}\begin{bmatrix} 6 & 9 & -3 \\ -1 & 1 & 3 \\ 4 & -4 & 3 \end{bmatrix}\begin{bmatrix} 3 & 0 & 1 \\ 2 & -2 & 1 \\ 1 & 5 & 2 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 33 & -33 & 9 \\ 2 & 13 & 6 \\ 7 & 23 & 6 \end{bmatrix}$$

det E' : wrt 3rd row
$= -4(-1-2)+1(2+1) = 15$

$$(E')^{-1} = \frac{1}{15}\begin{bmatrix} 6 & 9 & -3 \\ -1 & 1 & 3 \\ 4 & -4 & 3 \end{bmatrix}$$

Q5. $P_3 = \{ c_0 + c_1 t + c_2 t^2 + c_3 t^3 \mid c_0, c_1, c_2, c_3 \in \mathbb{R} \}$

$T : P_3 \to \mathbb{R}$ given by $Tf(t) = f(3) \quad \forall f \in P_3$.

Let $f(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3$

$$= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} \to f(3) = c_0 + 3c_1 + 9c_2 + 27c_3$$

$$= \begin{bmatrix} 1 & 3 & 9 & 27 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$\Longrightarrow \quad [T] = \begin{bmatrix} 1 & 3 & 9 & 27 \end{bmatrix}$

Another way Basis of P_3 : $B = \{ e_1 = 1, e_2 = t, e_3 = t^2, e_4 = t^3 \}$

$T(e_1) = 1, \quad T(e_2) = 3, \quad T(e_3) = 9, \quad T(e_4) = 27$

$\therefore [T]_B = \begin{bmatrix} 1 & 3 & 9 & 27 \end{bmatrix}$ since the basis of the codomain $\mathbb{R}$ is 1.

Q6. $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by $T(x_1, x_2) = (x_1 - x_2, x_1 + x_2, 2x_1 + x_2)$ (1)

$B = \{ e_1 = (1,0), e_2 = (0,1) \}$, $C = \{ f_1 = (1,2), f_2 = (2,3) \}$,

$D = \{ g_1 = (1,1,0), g_2 = (0,1,1), g_3 = (1,0,1) \}$.

Part a to calculate $[T]_{B,D}$

$Te_1 = (0,1,2)$ (by substituting $x_1 = 1$ & $x_2 = 0$ in (1))

$= g_1 T_{11} + g_2 T_{21} + g_3 T_{31}$

$= (1,1,0) T_{11} + (0,1,1) T_{21} + (1,0,1) T_{31}$

$$\Rightarrow \left.\begin{matrix} T_{11} + T_{31} = 1 \\ T_{11} + T_{21} = 1 \end{matrix}\right\} \left.\begin{matrix} T_{31} = T_{21} \\ \\ \end{matrix}\right\} \left.\begin{matrix} T_{21} = T_{31} = 1 \end{matrix}\right\} \quad T_{11} = 0$$

$T_{21} + T_{31} = 2$

$Te_2 = (-1,1,1) = g_1 T_{12} + g_2 T_{22} + g_3 T_{32}$

$= (1,1,0) T_{12} + (0,1,1) T_{22} + (1,0,1) T_{32}$

$$\Rightarrow \left.\begin{aligned} T_{12} + T_{32} &= -1 \\ T_{12} + T_{22} &= 1 \end{aligned}\right\} \left. \begin{aligned} T_{22} - T_{32} = 2 \\ \\ \end{aligned} \right\} \quad T_{22} = 3/2, \; T_{3/2} = -1/2$$

$$T_{22} + T_{32} = 1 \qquad\qquad T_{12} = -1/2$$

$$\therefore [T]_{B,D} = \begin{bmatrix} 0 & -1/2 \\ 1 & 3/2 \\ 1 & -1/2 \end{bmatrix}$$

Part b to calculate $T_{C,D}$ $\quad || \quad C = \{f_1, f_2\}, \; D = \{g_1, g_2, g_3\}$

$$Tf_1 = (-1, 3, 4) = g_1 T_{11} + g_2 T_{21} + g_3 T_{31}$$

$$= (1,1,0) T_{11} + (0,1,1) T_{21} + (1,0,1) T_{31}$$

$$\Rightarrow \left.\begin{aligned} T_{11} + T_{31} &= -1 \\ T_{11} + T_{21} &= 3 \end{aligned}\right\} \left. \begin{aligned} T_{21} - T_{31} = 4 \\ \\ \end{aligned} \right\} \quad T_{21} = 4, \; T_{31} = 0 \; \Big\} \; T_{11} = -1$$

$$T_{21} + T_{31} = 4$$

$$Tf_2 = (-1, 5, 7) = g_1 T_{12} + g_2 T_{22} + g_3 T_{32}$$

$$= (1,1,0) \; T_{12} + (0,1,1) \; T_{22} + (1,0,1) \; T_{32}$$

$$\Rightarrow \left.\begin{aligned} T_{12} + T_{32} &= -1 \\ T_{12} + T_{22} &= 5 \end{aligned}\right\} \left. \begin{aligned} T_{22} - T_{32} = 6 \\ \\ \end{aligned} \right\} \quad T_{22} = \frac{13}{2}, \; T_{32} = \frac{1}{2}$$

$$T_{22} + T_{32} = 7 \qquad\qquad T_{12} = -\frac{3}{2}$$

$$\therefore [T]_{C,D} = \begin{bmatrix} -1 & -\frac{3}{2} \\ 4 & \frac{13}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$$

Q7. Given $B = \left\{ e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

$C = \{f_1 = 1, \; f_2 = t, \; f_3 = t^2\}, \quad D = \{g_1 = 1\}$

$$\mathbb{R}^{2\times 2} = \{ e_1 x_1 + e_2 x_2 + e_3 x_3 + e_4 x_4 \mid x_1, x_2, x_3, x_4 \in \mathbb{R} \}$$

$P_2 = \{ f_1 x_1 + f_2 x_2 + f_3 x_3 \mid x_1, x_2, x_3 \in \mathbb{R} \}$

$\mathbb{R}' = \{ g_1 x \mid x \in \mathbb{R} \}$

<u>Part-a</u> $T: P_2 \to \mathbb{R}^{2\times 2}$ is given by

$$T\,g(t) = \begin{bmatrix} g(0) & g'(1) \\ g''(1) & g(1) \end{bmatrix} \quad \forall\, g \in P_2.$$

<u>Action of T on the basis vectors</u>

$$T f_1(t) = T \cdot 1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = e_1 \cdot 1 + e_4 \cdot 1$$

$$T f_2(t) = T\, t = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = e_2 \cdot 1 + e_4 \cdot 1$$

$$T f_3(t) = T\, t^2 = \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix} = e_2 \cdot 2 + e_3 \cdot 2 + e_4 \cdot 1$$

$$\therefore \; [T]_{C,B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

<u>Part b</u> $T: \mathbb{R}^{2\times 2} \to \mathbb{R}'$ given by $TA = tr(A) \quad \forall\, A \in \mathbb{R}^{2\times 2}$.

<u>Action of T on the basis elements</u>

$$T e_1 = tr\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 = g_1 \cdot 1$$

$$T e_2 = tr\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0 = g_1 \cdot 0$$

$$T e_3 = tr\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0 = g_1 \cdot 0$$

$$T e_4 = tr\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1 = g_1 \cdot 1$$

$$\therefore \; [T]_{B,D} = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}$$

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8. (a) $B=\{e_1=1, e_2=t\}$ $[D]_{B,B}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$

$De_1=0, \quad De_2=1=e_1\cdot 1+e_2\cdot 0$

(b) : $B=\{e_1=e^t, e_2=e^{2t}\}$

$De_1=e_1$ $[D]_{B,B}=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$

$De_2=2e_2$

(c) $B=\{e_1=e^t, e_2=te^t\}$

$De_1=e_1$ $[D]_{B,B}=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$

$De_2=e_1+e_2$

(d) $B=\{e_1=1, e_2=t, e_3=t^2\}$

$De_1=0$ $[D]_{B,B}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 2\\ 0 & 0 & 0\end{bmatrix}$

$De_2=1=e_1$

$De_3=2t=2e_2$

(e) $B=\{e_1=1, e_2=t, e_3=e^t, e_4=e^{2t}, e_5=te^{2t}\}$

$De_1=0$

$De_2=1=e_1$ $[D]_{B,B}=\begin{bmatrix}0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 2 & 1\\ 0 & 0 & 0 & 0 & 2\end{bmatrix}$

$De_3=e^t=e_3$

$De_4=2e^{2t}=2e_4$

$De_5=e^{2t}+2te^{2t}=e_4+2e_5$

(f) $B=\{e_1=\sin t, e_2=\cos t\}$

$De_1=\cos t=e_2$ $[D]_{B,B}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$

$De_2=-\sin t=-e_1$

Q9. $B = \{ e_1 = 1+t,\ e_2 = 1-t,\ e_3 = t^2 \}$

$C = \{ f_1 = 1,\ f_2 = 1+t,\ f_3 = 1+t+t^2,\ f_4 = 1+t+t^2+t^3 \}$

$T : P_2 \to P_3$ given by $Tf(t) = t f(t)$

$$Te_1 = t(1+t) = f_1 T_{11} + f_2 T_{21} + f_3 T_{31} + f_4 T_{41}$$

$$t + t^2 = T_{11} + (1+t) T_{21} + (1+t+t^2) T_{31}$$ (it is obvious that $T_{41} = 0$)

$$T_{11} + T_{21} + T_{31} = 0$$

$$\left.\begin{aligned} T_{21} + T_{31} &= 1 \\ T_{31} &= 1 \end{aligned}\right\} \quad T_{21} = 0, \quad T_{11} = -1$$

$$Te_2 = t(1-t) = f_1 T_{12} + f_2 T_{22} + f_3 T_{32} + f_4 T_{42}$$ (again, $T_{42} = 0$, obviously)

$$t - t^2 = T_{12} + (1+t) T_{22} + (1+t+t^2) T_{32}$$

$$T_{12} + T_{22} + T_{32} = 0$$

$$\left.\begin{aligned} T_{22} + T_{32} &= 1 \\ T_{32} &= -1 \end{aligned}\right\} \quad T_{32} = -1,\ T_{22} = 2,\ T_{12} = -1$$

$$Te_3 = t^3 = f_1 T_{13} + f_2 T_{23} + f_3 T_{33} + f_4 T_{43}$$

$$t^3 = T_{13} + (1+t) T_{23} + (1+t+t^2) T_{33} + (1+t+t^2+t^3) T_{43}$$

$$\left.\begin{aligned} T_{13} + T_{23} + T_{33} + T_{43} &= 0 \\ T_{23} + T_{33} + T_{43} &= 0 \end{aligned}\right\} \quad \begin{aligned} T_{13} &= 0 \\ T_{23} &= 0 \end{aligned}$$

$$\left.\begin{aligned} T_{33} + T_{43} &= 0 \\ T_{43} &= 1 \end{aligned}\right\} \quad T_{33} = -1$$

$$[T]_{B,C} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Q10: Given M is a square matrix and that it is nilpotent of order r. i.e. $M^r = 0$.

Part a: to prove that $I-M$ is invertible.

Let X be a general power series in M. Since $M^r=0$, X is a polynomial of degree $\leq r-1$. Demand X to be the inverse of $I-M$.

$$X = c_0 I + c_1 M + \dots + c_{r-1} M^{r-1}$$

$$X(I-M) = \begin{array}{l} c_0 I + c_1 M + \dots \qquad + c_{r-1} M^{r-1} \\ \quad - c_0 M - \dots \qquad - c_{r-2} M^{r-1} \end{array}$$

$$= c_0 I + (c_1 - c_0) M + \dots + (c_{r-1} - c_{r-2}) M^{r-1}$$

$$= I \quad \text{if} \quad c_0 = c_1 = \dots = c_{r-1} = 1$$

$\therefore \sum_{j=0}^{r-1} M^j$ is the inverse of $I-M$.

Part b Let $T: V \to V$ be a linear map with $\ker(T^r) = V$.

Then $\ker(Id - T) = \{0\}$ so that $Id - T$ is invertible.

Q11. $D: P_n \to P_n$ given by $Df(t) = \frac{d}{dt} f(t) \quad \forall f \in P_n$.

It is clear that $D^j = 0$ on P_n if $j > n$.

Part a. $T = I - D^2$

Let $m_0 = \left[\frac{n}{2}\right]$: (integer part of $\frac{n}{2}$) $= \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$

Let $X = I + D^2 + \dots + D^{2m_0}$

$$X(I-D^2) = \begin{array}{l} I + D^2 + \dots + D^{2m_0} \\ \quad - D^2 - \dots - D^{2m_0} - D^{2m_0+2} \end{array}$$

$= I$, since $2m_0 + 2 > n$, $D^{2m_0+2} f = 0 \ \forall f \in P_n$

$\therefore \sum_{j=0}^{\left[\frac{n}{2}\right]} D^j$ is the inverse of $I - D^2$ on P_n.

part b. $T = D^m - I$, m is a positive integer.

If $m > n$, then $D^m f = 0 \;\forall f \in P_n$

$\Rightarrow T = -I$ for $m > n$ & hence T is invertible.

Let $k = \left[\frac{n}{m}\right]$ (integer part of $\frac{n}{m}$).

(~~thereuskmer~~ Then $0 \le n - km \le m-1$)

By inspection $X = -(I + D^m + \dots + D^{km})$ is the inverse of T.

$$XT = (I + D^m + \dots + D^{km})(I - D^m)$$

$$= \begin{array}{l} I + D^m + \dots + D^{km} \\ \quad -D^m - \dots - D^{km} - D^{km+m} \end{array}$$

$$= I, \quad \text{since } D^{km+m} f = 0 \;\forall f \in P_n$$

Part C $T = D^m - cI = -c\left[I - \frac{D^m}{c}\right]$

has the inverse $\left(-\frac{1}{c}\right)\sum_{j=0}^{k}\left(\frac{D^m}{c}\right)^k = -\sum_{j=0}^{k}\frac{1}{c^{k+1}} D^{mk}$

using the solution of part b.

Q12 Please note that $[T]$ has 0 except along the diagonal above the main diagonal.

$$[T] = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Standard basis B of $\mathbb{R}^n$ is $\{e_1, e_2, \dots, e_n\}$, where e_j is an $(n\times 1)$ column vector with 1 at j^{th} row and 0 elsewhere.

$e_1 = (1, 0, \dots, 0)^t$, $e_2 = (0, 1, \dots, 0)^t$, $\dots$ $e_n = (0, 0, \dots, 1)^t$.

Part 1: $[T]e_1 = 0$ vector, $[T]e_2 = e_1$, $\dots$ $[T]e_j = e_{j-1}$ $\dots$ $[T]e_n = e_{n-1}$

$[e_1\; e_2\; e_3 \dots e_n] \longrightarrow [0\; e_1\; e_2 \dots e_{n-1}]$ (1)

Part b Let $v = v_1 e_1 + \dots + v_n e_n$ be a general vector $\in \mathbb{R}^n$.

$[T]v = T[v_1 e_1 + v_2 e_2 + v_3 e_3 + \dots + v_n e_n]$

$= v_2 e_1 + v_3 e_2 + \dots + v_n e_{n-1}$ (using ①)

$\therefore [T]^2 v = v_3 e_1 + \dots + v_n e_{n-2}$

$\vdots$

$[T]^j v = v_{j+1} e_1 + \dots + v_n e_{n-j}$

$\vdots$

$[T]^{n-1} v = v_n e_1$

& $[T]^n v = 0$

$\therefore [T]^n = 0, \quad [T]^{n-1} \neq 0.$

In terms of matrix multiplication

$[T]_{jk} = \delta_{j+1,k} \qquad 1 \le j, k \le n$

$\Rightarrow [T^2]_{jk} = \sum_{l=1}^{n} [T]_{jl} [T]_{lk} = \sum_{l=1}^{n} \delta_{j+1,l} \, \delta_{l+1,k} = \delta_{j+2,k}$

$\Rightarrow [T^3]_{jk} = \delta_{j+3,k}$

$[T^{n-1}]_{jk} = \delta_{n-1+j,k} \Rightarrow [T^{n-1}]_{1n} = 1$, all other elements are T^{n-1} are zero

$[T^n]_{jk} = \delta_{n+j,k} \Rightarrow [T^n] = 0$

Q13: V is an inner product space with basis $B = \{e_1, \dots, e_n\}$

Part a Show that the matrix A given by $A_{ij} = \langle e_i, e_j \rangle$ is invertible.

By definition of a basis, $e_1, \dots, e_n$ are linearly independent. It is possible to construct an ortho-normal basis $B' = \{e_1', \dots e_n'\}$ for V. Then by definition, there exists an invertible matrix S such that $e_i = \sum_{j=1}^{n} e_j' S_{ji}$ ① [each e_i is a unique linear combination of $e_1', \dots, e_n'$]

$\therefore A_{ij} = \langle e_i, e_j \rangle = \langle \sum_{k=1}^{n} e'_k S_{ki}, \sum_{l=1}^{n} e'_l S_{lj} \rangle$

$= \sum_{k=1}^{n} \sum_{l=1}^{n} \bar{S}_{ki} S_{lj} \langle e'_k, e'_l \rangle$. But $\langle e'_k, e'_l \rangle = \delta_{kl}$

$\Rightarrow A_{ij} = \sum_{k=1}^{n} \bar{S}_{ki} S_{kj} = \sum_{k=1}^{n} (S^+)_{ik} S_{kj} \Rightarrow$

$A = S^+ S$; since S is invertible, A is also invertible.

Part b Let $\langle e_j, v \rangle = c_j$ ② The idea is to express e_j in terms of the orthonormal basis.

using eq (1), ② $\Rightarrow \langle \sum_{k=1}^{n} e'_k S_{kj}, v \rangle = c_j$

i.e., $\sum_{k=1}^{n} \bar{S}_{kj} \langle e'_k, v \rangle = c_j$

i.e. $\sum_{k=1}^{n} S^+_{jk} b_k = c_j$, where $b_k = \langle e'_k, v \rangle$

$\Rightarrow b_k = \sum_{j=1}^{n} (S^+)^{-1}_{kj} c_j$ is the unique k^{th} component of v wrt the orthonormal basis $e'_1 \dots e'_n$

$v = e'_1 b_1 + \dots + e'_n b_n = \sum_{j=1}^{n} e'_j b_j$

$= \sum_{j=1}^{n} \left(\sum_{k=1}^{n} S^{-1}_{kj} e_k \right) \sum_{l=1}^{n} (S^+)^{-1}_{jl} c_l$

Another equivalent proof

Let $v = \sum_{k=1}^{n} e_k \alpha_k$, ($\alpha_1, \dots, \alpha_n \in \mathbb{F}$) be one vector which satisfies

$\langle e_j, v \rangle = c_j$.

This implies $\langle e_j, \sum_{k=1}^{n} e_k \alpha_k \rangle = \sum_{k=1}^{n} A_{jk} \alpha_k = c_j$

$\Rightarrow \{\alpha_j\}$ satisfies $\begin{bmatrix} A_{11} & \dots & A_{1n} \\ \vdots & & \\ A_{n1} & \dots & A_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = A^{-1} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Since A^{-1} exists, there is a unique set $\alpha_1, \dots \alpha_n$ corresponding to $c_1, \dots, c_n$.