MA2031: **Linear Algebra**
Assignment-5

Submission Date: 09/09/2017

Note: In what follows, $[\mathbf{T}]$ represents the matrix corresponding to the linear operator $\mathbf{T}$.

Q 1. For each real number θ let $\mathbf{T}_\theta : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear map represented by the matrix

$$[\mathbf{T}]_\theta = \begin{pmatrix} cos\,(\theta) & -sin\,(\theta) \\ sin\,(\theta) & cos\,(\theta) \end{pmatrix}.$$

Show that if θ and θ' are real numbers, then $[\mathbf{T}]_\theta\,[\mathbf{T}]_{\theta'} = [\mathbf{T}]_{\theta+\theta'}$. Also show that $[\mathbf{T}]_\theta^{-1} = [\mathbf{T}]_{-\theta}$.

Q 2. Let $\mathbf{T}_\theta$ be the rotation through an angle θ. Let (x, y) be the coordinates of a point of the plane with respect to the standard basis. Let (x', y') be the coordinates of this point in the rotated system. Express x', y' in terms of x, y, and θ.

Q 3. Let $\mathbf{T} : \mathbf{V} \to \mathbf{V}$ be a linear map, where $\mathbf{V}$ is a vector space over the field $\mathbf{F}$. Let $\mathbf{B} = \{v_1, \cdots, v_n\}$ be a basis of $\mathbf{V}$. Suppose that there are scalars $c_1, \cdots, c_n \in \mathbf{F}$ such that $\mathbf{T}\,(v_i) = c_i v_i$ for $i = 1, \cdots, n$. What is the matrix $[\mathbf{T}]$ corresponding to the linear operator $\mathbf{T}$?

Q 4. Consider the vector space $\mathbf{V} = \mathbf{R}^3$ over the field $\mathbf{R}$. Let $v \in \mathbf{V}$ and let $\mathbf{S}_{\mathbf{B},\mathbf{B}'}$ represent the matrix that relates coordinate vector $[v]_{B'}$ of the vector v with respect to basis $\mathbf{B}'$ with the coordinate vector $[v]_B$ with respect to basis $\mathbf{B}$. Given $\mathbf{B}$ and $\mathbf{B}'$ calculate $\mathbf{S}_{\mathbf{B},\mathbf{B}'}$ in each of the following cases.

(a) $\mathbf{B} = \{(1, 1, 0), (-1, 1, 1), (0, 1, 2)\}$, $\mathbf{B}' = \{(2, 1, 1), (0, 0, 1), (-1, 1, 1)\}$.

(b) $\mathbf{B} = \{(3, 2, 1), (0, -2, 5), (1, 1, 2)\}$, $\mathbf{B}' = \{(1, 1, 0), (-1, 2, 4), (2, -1, 1)\}$.

Q 5. Let $\mathbf{P}_3$ be the set of all polynomials of degree ≤ 3 of the real variable t. Define $\mathbf{T} : \mathbf{P}_3 \to \mathbf{R}$ by $\mathbf{T}f(t) = f(3)$ for all $f \in \mathbf{P}_3$. Obtain the matrix $[\mathbf{T}]$ corresponding to the linear operator $\mathbf{T}$ using the standard bases of the spaces.

Q 6. Define the linear transformation $\mathbf{T} : \mathbf{R}^2 \to \mathbf{R}^3$ by $\mathbf{T}\,(x_1, x_2) = (x_1 - x_2, x_1 + x_2, 2x_1 + x_2)$. Let $\mathbf{B} = \{(1, 0), (0, 1)\}$, $\mathbf{C} = \{(1, 2), (2, 3)\}$ and $\mathbf{D} = \{(1, 1, 0), (0, 1, 1), (1, 0, 1)\}$. Calculate transformation matrices (a) $[\mathbf{T}]_{\mathbf{B},\mathbf{D}}$ and (b) $[\mathbf{T}]_{\mathbf{C},\mathbf{D}}$ associated with $\mathbf{T}$. Please note that in case (a), $\mathbf{B}$ and $\mathbf{D}$ are to be used as the bases of respectively $\mathbf{R}^2$ and $\mathbf{R}^3$. In case (b), $\mathbf{C}$ is to be used as the basis of $\mathbf{R}^2$.

Q 7. Let $\mathbf{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$, $\mathbf{C} = \{1, t, t^2\}$ and $\mathbf{D} = \{1\}$. Let $\mathbf{R}^{2\times 2}$ the set of all all 2×2 real matrices and $\mathbf{P}_2$ the set of all polynomials of degree ≤ 2 of the real variable t.

(a) Let $\mathbf{T} : \mathbf{P}_2 \to \mathbf{R}^{2\times 2}$ given by $\mathbf{T}f(t) = \begin{pmatrix} f(0) & f'(1) \\ f''(1) & f(1) \end{pmatrix}$ for all $f \in \mathbf{P}_2$. Here f' and f'' correspond to first and second derivatives of f. Calculate the matrix $[\mathbf{T}]_{\mathbf{C},\mathbf{B}}$ corresponding to the transformation $\mathbf{T}$.

(b) Let $\mathbf{T} : \mathbf{R}^{2\times 2} \to \mathbf{R}$ given by $\mathbf{TA} = trace\,(\mathbf{A})$ for all $\mathbf{A} \in \mathbf{R}^{2\times 2}$. Calculate the matrix $[\mathbf{T}]_{\mathbf{B},\mathbf{D}}$ corresponding to the transformation $\mathbf{T}$.

Q 8. In each of the following cases, let $\mathbf{B}$ be a set of linearly independent functions of the real variable t and $\mathbf{D} = d/dt$ be the derivative. The basis $\mathbf{B}$ generates a vector space $\mathbf{V}$, and $\mathbf{D}$ is a linear map from $\mathbf{V}$ into itself. Find the matrix associated with $\mathbf{D}$ relative to the bases $\mathbf{B}, \mathbf{B}$ (that is, the domain and codomain are described by the same basis).

(a) $\{1, t\}$

(b) $\{e^t, e^{2t}\}$

(c) $\{e^t, te^t\}$

(d) $\{1, t, t^2\}$

(e) $\{1, t, e^t, e^{2t}, te^{2t}\}$

(f) $\{sin(t), cos(t)\}$

Q 9. Let $\mathbf{B} = \{1+t, 1-t, t^2\}$ and $\mathbf{C} = \{1, 1+t, 1+t+t^2, 1+t+t^2+t^3\}$ be bases of $\mathbf{P}_2$ and $\mathbf{P}_3$. Here $\mathbf{P}_n$ is the set of all polynomials of degree $\leq n$ of the real variable t. Let $\mathbf{T} : \mathbf{P}_2 \to \mathbf{P}_3$ be given by $\mathbf{T}f(t) = tf(t)$. Calculate the corresponding matrix $[\mathbf{T}]_{\mathbf{B},\mathbf{C}}$.

Q 10. Let $\mathbf{M}$ be a square matrix. $\mathbf{M}$ is nilpotent if there exists a positive integer r such that $\mathbf{M}^r = \mathbf{O}$, where $\mathbf{0}$ is the zero matrix of the same size as $\mathbf{M}$.

(a) Prove that if $\mathbf{M}$ is nilpotent, then $\mathbf{I} - \mathbf{M}$ is invertible.

(b) State and prove the analogous statement for linear maps of a vector space into itself.

Q 11. Let $\mathbf{P}_n$ be the vector space of polynomials of degree $\leq n$ of the real variable t. Then the derivative $\mathbf{D} : \mathbf{P}_n \to \mathbf{P}_n$ given by $\mathbf{D}f(t) = df(t)/dt$ for all $f \in \mathbf{P}_n$ is a linear map of $\mathbf{P}_n$ into itself. Let $\mathbf{I}$ be the identity mapping. Prove that the following linear maps are invertible:

(a) $\mathbf{I} - \mathbf{D}^2$.

(b) $\mathbf{D}^m - \mathbf{I}$ for any positive integer m.

(c) $\mathbf{D}^m - c\mathbf{I}$ for any number $c \neq 0$.

Q 12. Let $[\mathbf{T}]$ be the $n \times n$ matrix

$$[\mathbf{T}] = \begin{pmatrix} 0 & 1 & 0\cdots & \cdots & 0 & 0 \\ 0 & 0 & 1\cdots & \cdots & 0 & 0 \\ 0 & 0 & 0\cdots & \cdots & 0 & 0 \\ 0 & 0 & 0\cdots & \cdots & 0 & 0 \\ 0 & 0 & 0\cdots & \cdots & 1 & 0 \\ 0 & 0 & 0\cdots & \cdots & 0 & 0 \end{pmatrix}$$

$[\mathbf{T}]$ is upper triangular, with zeros on the diagonal, 1 just above the diagonal, and zeros elsewhere as shown.

(a) How would you describe the effect of the corresponding linear $\mathbf{T} : \mathbf{F}^n \to \mathbf{F}^n$ on the standard basis vectors $\{e_1, \cdots, e_n\}$ of $\mathbf{F}^n$? Here F is a field.

(b) Show that $[\mathbf{T}]^n = \mathbf{0}$ and $[\mathbf{T}]^{n-1} \neq \mathbf{0}$ using the effect of powers of $[\mathbf{T}]$ on the basis vectors.

Q 13. Let $\mathbf{V}$ be an inner product space over the field $\mathbf{F}$ with basis $\mathbf{B} = \{e_1, \cdots, e_n\}$.

(a) Show that the matrix $\mathbf{A}$ given by $\mathbf{A}_{i,j} = \langle u_i, u_j \rangle$ is invertible.

(b) Let $c_1, \cdots, c_n \in \mathbf{F}$. Show that there is exactly one vector $v \in \mathbf{V}$ such that $\langle e_j, v \rangle = c_j$ for all $j = 1, \cdots, n$.