MA2031: **Linear Algebra**
Assignment-6

Submission Date: 09/10/2017

Q 1. What are the eigenvalues and eigenvectors of an $n \times n$ diagonal matrix $\mathbf{D}$?

Q 2. Prove that λ^k (here k is a nonnegative integer) is an eigenvalue of the matrix $\mathbf{M}^k$ if λ is an eigenvalue of the matrix $\mathbf{M}$

Q 3. Let $\mathbf{T} : \mathbf{R}^3 \to \mathbf{R}^3$ given by $\mathbf{T}(x_1, x_2, x_3) = (x_1, x_2, 0)$ be a linear transformation. What is the matrix $\mathbf{M} = [\mathbf{T}]_{\mathbf{B},\mathbf{B}}$ with respect to the natural basis $\mathbf{B} = \{(1,0,0),(0,1,0),(0,0,1)\}$ of $\mathbf{R}^3$? What are the eigenvalues and corresponding eigenvectors of $\mathbf{M}$?

Q 4. Let $\mathbf{P}$ be a real square matrix that satisfies $\mathbf{P}^2 = \mathbf{P}$. Such matrices are called idempotent, and also orthogonal projectors. Show that all eigenvalues of $\mathbf{P}$ are either zero or one.

Q 5. Let $\mathbf{T} : \mathbf{C}^2 \to \mathbf{C}^2$ given by $\mathbf{T}(x_1, x_2) = (-x_2, x_1)$ be a linear transformation. What is the matrix $\mathbf{M} = [\mathbf{T}]_{\mathbf{B},\mathbf{B}}$ with respect to the natural basis $\mathbf{B} = \{(1,0),(0,1)\}$ of $\mathbf{C}^2$? What are the eigenvalues and corresponding eigenvectors of $\mathbf{M}$?

Q 6. Let $\mathbf{V} \neq \{0\}$ be a finite dimensional vector space and $\mathbf{T} \in \mathbf{L}(\mathbf{V}, \mathbf{V})$. Prove that $\mathbf{T}$ has at least one eigenvalue.

Q 7. Show that the characteristic polynomials of $\mathbf{A}$ and $\mathbf{S}^{-1}\mathbf{A}\mathbf{S}$, in which $\mathbf{S}$ is a nonsingular transformation matrix, are identical. Conclude that the similarity transformation $\mathbf{A} \to \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ preserves eigenvalues and their algebraic multiplicities.

Q 8. let $\mathbf{M} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$.

(a) Obtain the characteristic polynomial $p_M(\lambda) = det(\mathbf{M} - \lambda \mathbf{I})$.

(b) Determine the eigenvalues λ_1, λ_2 & λ_3 and the corresponding eigenvectors v_1, v_2 & v_3.

(c) Let $\mathbf{S}$ be the matrix $[v_1 v_2 v_3]$ with the eigenvectors of $\mathbf{M}$ as the columns. Calculate $\mathbf{S}^{-1}$.

(d) Let row vectors w_1^t, w_2^t & w_3^t be defined as the first, second and third rows, respectively, of $\mathbf{S}^{-1}$. Define projection operators $\mathbf{P}_j = v_j w_j^t$ for $j = 1, 2, 3$. Show that

$$\mathbf{P}_j \mathbf{P}_k = \mathbf{P}_j \delta_{jk}, \text{ where } \delta_{jk} = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j \neq k. \end{cases} .$$

(e) Show that $\mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 = \mathbf{I}$.

(f) Verify that $\mathbf{M} = \lambda_1 \mathbf{P}_1 + \lambda_2 \mathbf{P}_2 + \lambda_3 \mathbf{P}_3$.

(g) Verify Cayley-Hamilton theorem $p_M(\mathbf{M}) = \mathbf{0}$.

Q 9. let $\mathbf{M} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Repeat parts (a) -(g) of the previous problem. In addition, show that $\mathbf{M}^2 - 3\mathbf{M} = \mathbf{0}$ and explain what does this imply.

Q 10. Show that the eigenvalues of a real symmetric square matrix are real, and also that all eigenvectors can be chosen to be real.

Q 11. Let the real symmetric $n \times n$ matrix $\mathbf{A}$ have r nonzero eigenvalues and let the remaining $n - r$ eigenvalues be equal to zero. Show that $\mathbf{A}$ has rank r.

Q 12. Let $\mathbf{Q}$ be a real orthogonal matrix: $\mathbf{Q}^t \mathbf{Q} = \mathbf{I}$. Show that all of its eigenvalues λ_i are unimodular (i.e., they have unit modulus).

Q 13. Let $\mathbf{A}$ be real skew-symmetric, that is, $\mathbf{A}^t = -\mathbf{A}$. Show that all eigenvalues of $\mathbf{A}$ are pure imaginary or zero.

Q 14. Two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ are said to commute if $\mathbf{AB} = \mathbf{BA}$. Show that the necessary and sufficient condition for this to happen is that $\mathbf{A}$ and $\mathbf{B}$ should have a set of n simultaneous (common) eigenvectors.

Q 15. Let $\mathbf{A}$ and $\mathbf{B}$ be two diagonalizable square matrices of equal order that commute. Show that the eigenvalues of $\mathbf{AB}$ are the product of the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$.

Q 16. Let $\mathbf{M} = \begin{bmatrix} a & c & 0 \\ 0 & a & d \\ 0 & 0 & b \end{bmatrix}$. Determine the eigen values of $\mathbf{M}$ and the corresponding eigenvectors. Can you diagonalize $\mathbf{M}$ by using a similarity transformation?

Q 17. A matrix whose elements are equal on any line parallel to the main diagonal is called a Toeplitz matrix. Show that if $\mathbf{A}$ and $\mathbf{B}$ are any two Toeplitz matrices of the same order, they commute: $\mathbf{AB} = \mathbf{BA}$.

Q 18. Consider the set of coupled ordinary differential equations

$$\begin{aligned}\frac{d}{dt}x(t) &= x(t) + y(t)\\ \frac{d}{dt}y(t) &= x(t) + 2y(t)\end{aligned}$$

together with the initial condition $x(t=0) = x_0$ and $y(t=0) = y_0$. This can be presented in the form of a matrix equation

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \mathbf{M}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \text{ where, } \mathbf{M} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \tag{1}$$

(a) Determine the eigenvalues λ_1 & λ_2 and corresponding eigenvectors v_1 & v_2 of the matrix $\mathbf{M}$.

(b) Use the above information to obtain the similarity transformation $\mathbf{M} \to \mathbf{S}^{-1}\mathbf{M}\mathbf{S} = \mathbf{D} = diag\,(\lambda_1, \lambda_2)$.

(c) Use the above to recast the initial value problem Eq.(1) as

$$\frac{d}{dt}\begin{bmatrix} \rho(t) \\ \eta(t) \end{bmatrix} = \mathbf{D}\begin{bmatrix} \rho(t) \\ \eta(t) \end{bmatrix} \tag{2}$$

where

$$\begin{bmatrix} \rho(t) \\ \eta(t) \end{bmatrix} = \mathbf{S}^{-1}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}. \tag{3}$$

(d) Solve Eq.(2) and then obtain the solution for the original initial value problem Eq. (1).

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