Notes on Assorted Project Euler Problems

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1 Problem 140

The first step is to sum the geometric series

$$A_G(x) = \sum_{k=1}^{\infty} x^k G_k \,.$$

To do so, we need a closed form expression for G_k. Such an expression exists for the Fibonacci numbers. I did some digging and found that for a recursion series of the form $G_k = G_{k-1} + G_{k-2}$, with $G_1 = a$, $G_2 = b$, the expression is

$$G_k = \frac{1}{2}\left[(3a-b)F_k + (b-a)L_k\right] \,,$$

where F_k, L_k are the k-th Fibonacci and Lucas numbers. In this particular case,

$$G_k = \left(\frac{15-\sqrt{5}}{10}\right) y_-^k + \left(\frac{15+\sqrt{5}}{10}\right) y_+^k \,,$$

where

$$y_\pm \equiv \left(\frac{1\pm\sqrt{5}}{2}\right)^k \,.$$

Summing up the geometric series gives

$$A_G = \left(\frac{15-\sqrt{5}}{10}\right)\frac{xy_-}{1-xy_-} + \left(\frac{15+\sqrt{5}}{10}\right)\frac{xy_+}{1-xy_+} \,,$$

and we can solve for x in terms of A_G:

$$x = \frac{\sqrt{1+14A_G+5A_G^2} - (A_G+1)}{6+2A_G} \,.$$

Now our task seems easy enough, in order for x to be a "Golden Nugget", we need $(1+14A_G+5A_G^2)$ to be a perfect square so that x is rational. The problem is that the perfect square values of A_G become increasingly spread out. Below is a log plot of the first few:

We'll need a more sophisticated method to find the solution since a simple scan will take an exponentially long time. It turns out that the question of when a polynomial such as

$$z^2 = 5w^2 + 14w + 1$$

has integer solutions (for both w and z) is exactly what the Diophantine equations determine. I found that *Mathematica* has a routine capable of solving these equations and implemented it, unfortunately learning very little in the process. I should revisit this.

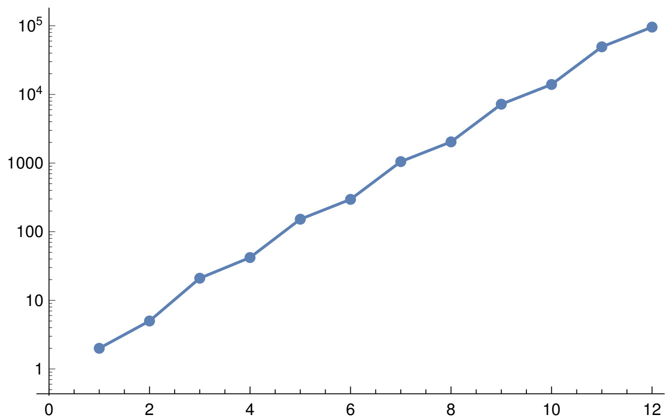

Figure 1: The first few values of $A_G(x)$ for which x is a Golden Nugget.