Black–Scholes via MGFs and Geometric Brownian Motion

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1 Introduction

This paper develops the Black–Scholes option pricing formula beginning from a discrete-time log–return model and using moment generating functions (MGFs). We derive the lognormal distribution of prices, obtain geometric Brownian motion (GBM) as the continuous-time limit, adjust the drift under the risk–neutral measure, and finally price a European call by evaluating the expected discounted payoff. The objective is a clear and self-contained derivation that highlights the mathematical structure of the model from first principles.

Geometric Brownian Motion via Moment Generating Functions

1. Discrete-time log-return model

Fix a time step $\Delta t > 0$ and define $t_k = k\Delta t$ for $k = 0, 1, \ldots, N$, with $T = N\Delta t$ the maturity horizon. Let S_k denote the asset price. Define the log-return

$$Y_k = \ln\left(\frac{S_k}{S_{k-1}}\right), \qquad S_k = S_{k-1}e^{Y_k}.$$

Log-returns are additive:

$$\ln S_k = \ln S_{k-1} + Y_k,$$

so

$$\ln S_T = \ln S_0 + \sum_{k=1}^{N} Y_k.$$

2. Distributional assumptions

Assume i.i.d. normal increments:

$$Y_k \sim \mathcal{N}(m\Delta t,\ \sigma^2 \Delta t).$$

Then

$$\mathbb{E}[Y_k] = m\Delta t, \qquad \mathrm{Var}(Y_k) = \sigma^2 \Delta t.$$

Hence the cumulative log-return

$$\xi_N := \sum_{k=1}^{N} Y_k$$

satisfies

$$\xi_N \sim \mathcal{N}(mT,\ \sigma^2 T).$$

3. MGFs and the cumulative log-return

The MGF of a normal variable $X \sim \mathcal{N}(a, b^2)$ is

$$M_X(u) = \exp\!\left(au + \tfrac{1}{2} b^2 u^2\right).$$

A single increment has MGF

$$M_{Y_k}(u) = \exp\!\left(m\Delta t\, u + \tfrac{1}{2}\sigma^2 \Delta t\, u^2\right).$$

By independence,

$$M_{\xi_N}(u) = \left(M_{Y_1}(u)\right)^N = \exp\!\left(mTu + \tfrac{1}{2}\sigma^2 T u^2\right),$$

which is the MGF of $\mathcal{N}(mT, \sigma^2 T)$, confirming

$$\xi_N \sim \mathcal{N}(mT, \sigma^2 T).$$

Thus the terminal log-price satisfies

$$\ln S_T = \ln S_0 + mT + \sigma\sqrt{T}\, Z, \qquad Z \sim \mathcal{N}(0,1),$$

and therefore the price

$$S_T = S_0 \exp\!\left(mT + \sigma\sqrt{T}\, Z\right)$$

is lognormal.

4. Expected price via MGFs

Write $S_T = S_0 e^X$ with

$$X = mT + \sigma\sqrt{T}Z.$$

For standard normal Z,

$$\mathbb{E}[e^{aZ}] = e^{\frac{1}{2}a^2}.$$

Thus

$$\mathbb{E}[S_T] = S_0 e^{mT}\mathbb{E}\left[e^{\sigma\sqrt{T}Z}\right] = S_0 e^{mT} e^{\frac{1}{2}\sigma^2 T} = S_0 e^{(m+\frac{1}{2}\sigma^2)T}.$$

The price drift exceeds the log-drift m due to the convexity of e^x.

5. GBM limit

Write each increment as

$$Y_k = m\Delta t + \sigma\sqrt{\Delta t}\, Z_k, \qquad Z_k \sim \mathcal{N}(0,1).$$

Summing:

$$\sum_{k=1}^{N} Y_k = m\sum_{k=1}^{N}\Delta t + \sigma\sum_{k=1}^{N}\sqrt{\Delta t}Z_k = mT + \sigma W_T,$$

where

$$W_T := \sum_{k=1}^{N}\sqrt{\Delta t}\, Z_k \sim \mathcal{N}(0,T)$$

is the discrete approximation to Brownian motion.

Passing to the continuous limit gives

$$\ln S_t = \ln S_0 + mt + \sigma W_t, \qquad S_t = S_0 \exp(mt + \sigma W_t),$$

the standard GBM representation.

Risk-Neutral Drift Adjustment

In the real-world measure,

$$S_T = S_0 \exp\left(mT + \sigma\sqrt{T}Z\right).$$

Risk-neutral pricing requires the discounted price to be a martingale:

$$\mathbb{E}[e^{-rT}S_T] = S_0.$$

Compute:

$$\mathbb{E}[e^{-rT}S_T] = S_0 e^{(m-r)T}\mathbb{E}[e^{\sigma\sqrt{T}Z}] = S_0 e^{(m-r)T}e^{\frac{1}{2}\sigma^2 T}.$$

Setting this equal to S_0 gives

$$e^{(m-r+\frac{1}{2}\sigma^2)T} = 1, \qquad \Rightarrow \qquad m = r - \tfrac{1}{2}\sigma^2.$$

Thus under the risk-neutral measure:

$$S_T = S_0 \exp\Big((r - \tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z\Big).$$

European Call Pricing via MGFs

The discounted value of a call option with strike K is

$$C_0 = e^{-rT}\mathbb{E}[(S_T - K)^+].$$

1. Rewriting the payoff

Using the indicator function,

$$(S_T - K)^+ = (S_T - K)\mathbf{1}_{\{S_T>K\}} = S_T\mathbf{1}_{\{S_T>K\}} - K\mathbf{1}_{\{S_T>K\}}.$$

Thus

$$C_0 = e^{-rT}\mathbb{E}[S_T\mathbf{1}_{\{S_T>K\}}] - Ke^{-rT}\mathbb{P}(S_T > K).$$

2. Probability of finishing in the money

We have

$$\ln S_T = \ln S_0 + (r - \tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z.$$

Solve $S_T > K$:

$$\ln S_T > \ln K \iff Z > -d_2,$$

where

$$d_2 = \frac{\ln(S_0/K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

Thus

$$\mathbb{P}(S_T > K) = \Phi(d_2).$$

3. Expectation of S_T over the ITM region

Compute

$$\mathbb{E}[S_T \mathbf{1}_{\{S_T>K\}}] = S_0 e^{(r-\frac{1}{2}\sigma^2)T} \mathbb{E}\left[e^{\sigma\sqrt{T}Z} \mathbf{1}_{\{Z>-d_2\}}\right].$$

Use the truncated-normal identity

$$\mathbb{E}\left[e^{aZ} \mathbf{1}_{\{Z>b\}}\right] = e^{\frac{1}{2}a^2} \Phi(a-b).$$

Here $a = \sigma\sqrt{T}$ and $b = -d_2$, giving

$$\mathbb{E}[S_T \mathbf{1}_{\{S_T>K\}}] = S_0 e^{rT} \Phi(d_1),$$

where

$$d_1 = d_2 + \sigma\sqrt{T}.$$

Discounting:

$$e^{-rT} \mathbb{E}[S_T \mathbf{1}_{\{S_T>K\}}] = S_0 \Phi(d_1).$$

4. Black–Scholes formula

Putting the pieces together:

$$C_0 = S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2),$$

with

$$d_1 = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}.$$

Interpretation

- d_2 represents the standardized risk-neutral distance between S_0 and the strike K.
- $d_1 = d_2 + \sigma\sqrt{T}$ appears because the payoff depends on S_T, weighting the right tail more heavily.
- Because lognormal distributions are right-skewed, $\Phi(d_1) > \Phi(d_2)$, ensuring positive option value even when the probability of finishing in the money is modest.