

Variáveis Complexas

List 1:

1-

$$\begin{aligned} \text{a)} (2+3i) \cdot (7+i) \\ &= 14 + 2i + 3i^2 + 21i \\ &= (14 - 3) + i(2 + 22) \\ &= 11 + 23i \end{aligned}$$

$$\begin{aligned} \text{b)} (8+6i)^2 \\ &= (8+6i)(8+6i) \\ &= 64 + 2 \cdot 48i + 36i^2 \\ &= 64 + 96i - 36 \\ &= 28 + 96i \end{aligned}$$

$$\text{c)} \frac{5}{-3+4i}$$

$$\begin{aligned} &\frac{5(-3+4i)}{(-3+4i)(-3-4i)} \\ &= \frac{-15+20i}{9+12i-12i+16} \end{aligned}$$

$$\frac{-15}{25} + \frac{20}{25}i$$

$$-\frac{3}{5} + \frac{4}{5}i$$

$$d) \frac{4+i}{2-3i} = \frac{4+i}{2-3i} \cdot \frac{(2+3i)}{(2+3i)}$$

$$\frac{8+12i+2i-3}{4+9} = \frac{5+14i}{13}$$

$$e) \frac{-1+3i}{2-i} = \frac{-1+3i}{2-i} \cdot \frac{(2+i)}{(2+i)}$$

$$\frac{-2-i+6i-3}{4+1} = \frac{-5+5i}{5}$$

$$-2+5i$$

$$f) \left(\frac{2+i}{i-2} + \frac{3}{(1+i)^2} \right)^2$$

$$\left(\frac{2+i+3i}{i-2} \right)^2$$

$$\frac{(2+4i)^2}{(i-2)^2}$$

$$\frac{2+8i-16}{1-2i-1}$$

$$\frac{-15+8i}{-2i}$$

$$-2i$$

$$\frac{-4-\frac{15}{2}i}{2}$$

$$c) \frac{5}{(1-i)(2-i)(3-i)} = \frac{5}{(2-i-2i+1)(3-i)}$$

$$\frac{5}{(1-3i)(3-i)} = \frac{5}{3-i-9i-3} = \frac{5}{10i} = \frac{-1}{2}i$$

2-a) $(x+yi)^2$
 $x^2 + 2xyi - y^2$
 $x^2 - y^2 + 2xyi$

b) $(x+yi)^3$
 $(x+yi)^2(x+yi)$
 $(x^2 + 2xyi - y^2)(x+yi)$
 $x^3 + 2x^2yi - 2xy^2 - y^3i$
 $(x^3 - 2xy^2) + i(2x^2y - y^3)$

c) $\frac{1}{z} = \frac{1}{x+yi} = \frac{x-yi}{x^2-y^2}$

D) $1/z^2$
 $(x+yi)^2$
 $(x^2 + 2xyi - y^2)^{-1}$
 $\left[(x^2 - y^2) + i(2xy) \right]^{-1}$

$$\frac{\left[(x^2 - y^2) - i(2xy) \right]}{(x^2 - y^2)^2 + (2xy)^2}$$

$$\frac{\left[(x^2 - y^2) - i(2xy) \right]}{x^4 + 2x^2y^2 + y^4}$$

$$e) \frac{z-2}{z+2}$$

$$\frac{x+yi-2}{x+yi+2}$$

$$\frac{(x-1)+yi}{(x+1)+yi} \cdot \frac{(x+1)-yi}{(x+1)-yi}$$

$$\frac{x^2 - 1 - xyi + yi + xyi + yi + y^2}{(x+1)^2 + y^2}$$

$$\frac{x^2 - 1 + 2yi + y^2}{x^2 + 2x + 1 + y^2}$$

4.

$$a) -2 + i\sqrt{3}$$

$$|z| = \sqrt{(-2)^2 + \sqrt{3}^2}$$

$$|z| = \sqrt{4+3} = 2$$

$$z = a + bi$$

$$a = -1 = 2 \cos(\theta)$$

$$\theta = 2 \cos(-1/2)$$

$$b = \sqrt{3} = 2 \sin(\theta)$$

$$\theta = 2 \sin\left(\frac{\sqrt{3}}{2}\right)$$

$$\theta = \pi - \frac{\pi}{3} + 2k\pi$$

$$\theta = \frac{2\pi}{3} + 2k\pi$$

$$b) |z| = \sqrt{(-1/2)^2 + (\sqrt{3}/2)^2}$$

$$|z| = \sqrt{1/4 + 3/4} = 1$$

$$\begin{cases} a = -1/2 = \cos \theta \\ b = \sqrt{3}/2 = \sin \theta \end{cases} \quad \theta = \pi - \frac{\pi}{3} + 2k\pi$$

$$\theta = \frac{2\pi}{3} + 2k\pi$$

$$c) -\cos\left(\frac{\pi}{7}\right) + i \sin\left(\frac{\pi}{7}\right)$$

$$1 < \pi - \frac{\pi}{7} + 2k\pi = 1 < 6\pi/7$$

$$d) |z| = \sqrt{(3/2)^2 + (\sqrt{3}/2)^2}$$

$$|z| = \sqrt{(9/4) + (3/4)}$$

$$|z| = \sqrt{12/4} = \sqrt{3}$$

$$a = 3/2 = \sqrt{3} \cos(\theta)$$

$$b = -\sqrt{3}/2 = \sqrt{3} \sin(\theta)$$

$$\begin{cases} \cos \theta = \sqrt{3}/2 \\ \sin \theta = -1/2 \end{cases} \quad \theta = \frac{\pi}{6} + 2k\pi$$

S-

$$a) z^3 = (1+i)^3$$

$$|z| = \sqrt{2}$$

$$\theta = \pi/4$$

$$z^3 = \sqrt[2]{2^3} \left[\cos(3\pi/4) + i \sin(3\pi/4) \right]$$

$$2\sqrt{2} (-\sqrt{2}/2 + i\sqrt{2}/2)$$

$$z^3 = -2 + 2i$$

$$b) z = (1+i)^{25}$$

$$|z| = \sqrt{2^{25}} = \sqrt{2^{24} \cdot 2} = 2^{12} \sqrt{2} = 4096\sqrt{2}$$

$$z = 4096 [\cos(25\pi/4) + i \sin(25\pi/4)] \sqrt{2}$$

$$z = 4096\sqrt{2} \left[\cos\left(\frac{24\pi}{4} + \frac{\pi}{4}\right) + i \sin\left(\frac{24\pi}{4} + \frac{\pi}{4}\right) \right]$$

$$z = 4096\sqrt{2} [\cos(6\pi + \pi/4) + i \sin(6\pi + \pi/4)]$$

$$z = 4096\sqrt{2} (\sqrt{2}/2 + i\sqrt{2}/2)$$

$$z = 4096 + 4096i$$

$$c) z = (-1+i)^7$$

$$|z| = \sqrt{2^7} = \sqrt{2^6 \cdot 2} = 8\sqrt{2}$$

$$\Theta = 7\pi/4 = 8\pi/4 - \pi/4$$

$$z = 8\sqrt{2} [\cos(-\pi/4) + i \sin(-\pi/4)]$$

$$z = 8\sqrt{2} (\sqrt{2}/2 - i\sqrt{2}/2)$$

$$z = 8 - 8i$$

$$d) z^{99} = (1+i\sqrt{3})^{99}$$

$$|z| = \sqrt{1+3} = 2$$

$$\left. \begin{array}{l} 2 \cos(\theta_2) = 1 \\ 2 \sin(\theta_2) = \sqrt{3} \end{array} \right\} \theta_2 = \pi/3$$

$$z^{99} = 2^{99} [\cos(99\pi/3) + i \sin(99\pi/3)]$$

$$z^{99} = 2^{99} [\cos(32\pi + \pi) + i \sin(32\pi + \pi)]$$

$$z^{99} = 2^{99} (-1 + i \cdot 0)$$

$$z^{99} = -2^{99}$$

$$e) \frac{(1+i)^9}{(1-i)^7} = \frac{z_1^9}{z_2^7} = \frac{\sqrt{2}^9}{\sqrt{2}^7} \left[\frac{\cos(9\pi/4) + i \sin(9\pi/4)}{\cos(7\pi/4) + i \sin(7\pi/4)} \right]$$

$$|z_1| = |z_2| = \sqrt{2}$$

$$2 \frac{\cos(\pi/4) + i \sin(\pi/4)}{\cos(3\pi/4) + i \sin(3\pi/4)} = 2\sqrt{2}/2 + i\sqrt{2}/2$$

$$\theta_1 = \pi/4$$

$$\theta_2 = -\pi/4$$

$$\frac{2\sqrt{2}/2 + i\sqrt{2}/2}{\sqrt{2}/2 - i\sqrt{2}/2} = \frac{2(\sqrt{2}/2 + i\sqrt{2}/2)^2}{1/2 + 1/2} = 2(1/2 + 2i \cdot 1/2 - 1/2) = 2i$$

f) $z^{30} = \left(\frac{1+i\sqrt{3}}{1-i}\right)^{30}$

$$z = \frac{1+i\sqrt{3}}{1-i} \Rightarrow \begin{cases} z_1 = 1+i\sqrt{3} \\ z_2 = 1-i \end{cases} \Rightarrow \begin{cases} |z_1| = 2, \theta_1 = \pi/3 \\ |z_2| = \sqrt{2}, \theta_2 = -\pi/4 \end{cases}$$

$$z = \frac{2}{\sqrt{2}} \left[\cos(\pi/3) + i \sin(\pi/3) \right]$$

$$z^{30} = \left(\frac{2^0}{2^5}\right)^3 \left[\begin{array}{c} \cos(10\pi) + i \sin(10\pi) \\ \cos(\pi/2) + i \sin(\pi/2) \end{array} \right] = 2^{\frac{15}{5}} = -i2^{\frac{15}{5}} = -i32 + 0i$$

6- a) $|z| - z = 1+2i$

$$\sqrt{a^2+b^2} - a - bi = 1+2i$$

$$b = 2$$

$$\sqrt{a^2+4} - a = 1$$

$$\sqrt{a^2+4} = 1+a$$

$$a^2+4 = (1+a)^2$$

$$a^2+4 = (1+a)^2$$

$$a^2+4 = a^2+2a+1$$

$$2a = 3$$

$$a = 3/2$$

b) $z^2 - 2z + 2 = 0$

$$z = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

$$c) z^2 = \bar{z}$$

$$(a+bi)^2 = a - bi$$

$$a^2 + 2ab i - b^2$$

$$a^2 - b^2 = a$$

$$2ab = -b$$

$$2a = -1$$

$$a = -1/2$$

$$\left(\frac{-1}{2}\right)^2 - b^2 = -\frac{1}{2}$$

$$\frac{1}{4} + \frac{1}{2} = b^2$$

$$b = \frac{\sqrt{3}}{2}$$

$$d) z^3 = z$$

$$r^3 e^{i3\theta} = r e^{-i\theta}$$

$$\frac{r^2 e^{i3\theta}}{e^{-i\theta}} = 1$$

$$r^2 e^{i4\theta} = 1$$

$$r^2 (\cos 4\theta + i \sin 4\theta) = 1$$

$$r^2 \cos(4\theta) = 1$$

$$r^2 \sin(4\theta) = 0$$

$$\sin(4\theta) = 0 \Rightarrow \theta = k\pi/4$$

$$r^2 \cos(4\theta) = 1$$

$$r^2 \cos(k\pi) = 1$$

$$r^2 > 0$$

$$k = 2m, m \in \mathbb{Z} \setminus \{m=0\}$$

$$e) z^2 + 1 = 0$$

$$z^2 = -1$$

$$z = \pm \sqrt{-1}$$

$$z = \pm i$$

$$f) z^2 - 2i = 0$$

$$z^2 = 2i$$

$$z^2 = r^2 [\cos(2\theta) + i \sin(2\theta)] = 2i$$

$$r^2 = 2$$

$$2 \cos(2\theta) = 0$$

$$2 \sin(2\theta) = 2$$

$$\sin(2\theta) = 1$$

$$2\theta = \frac{\pi}{2} + 2k\pi$$

$$\theta = \frac{\pi}{4} + k\pi$$

$$g) z^3 + 1 = 0$$

$$z^3 = -1$$

$$r^3 [\cos(3\theta) + i \sin(3\theta)] = -1$$

$$r = 1$$

$$r^3 = 1$$

$$\cos(3\theta) = -1$$

$$3\theta = \pi + 2k\pi$$

$$\theta = \frac{\pi}{3} + \frac{2}{3}k\pi$$

$$\theta_0 = \frac{\pi}{3}$$

$$\theta_1 = \frac{\pi}{3} + \frac{2}{3}\pi = \frac{7}{3}\pi$$

$$\theta_2 = \frac{\pi}{3} + \frac{4}{3}\pi = \frac{5}{3}\pi$$

$$h) z^3 + i = 0$$

$$z^3 = -i$$

$$z^3 = -i$$

$$r^3 \cos(3\theta) = 0$$

$$\cos(3\theta) = 0$$

$$3\theta = \frac{\pi}{2} + k\pi$$

$$\theta = \frac{\pi}{6} + k\pi/3$$

$$r^3 \sin 3\theta = -1$$

$$r^3 \sin \left[3(\pi/2 + k\pi) / 3 \right] = -1$$

$$r^3 \sin (\pi/2 + k\pi) = -1$$

Since $r^3 > 0$

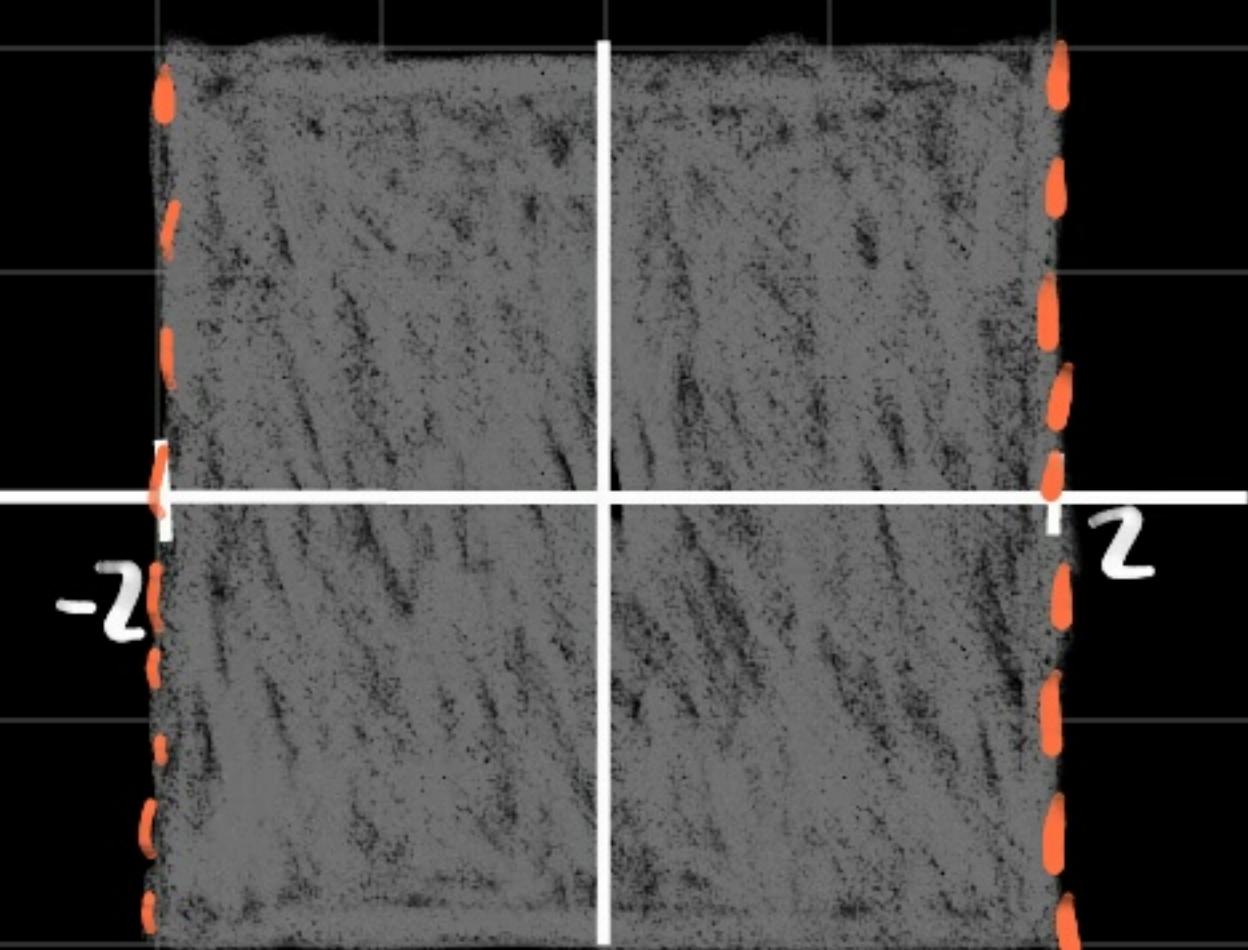
$\sin(\pi/2 + k\pi) = -1$, for this

to be true

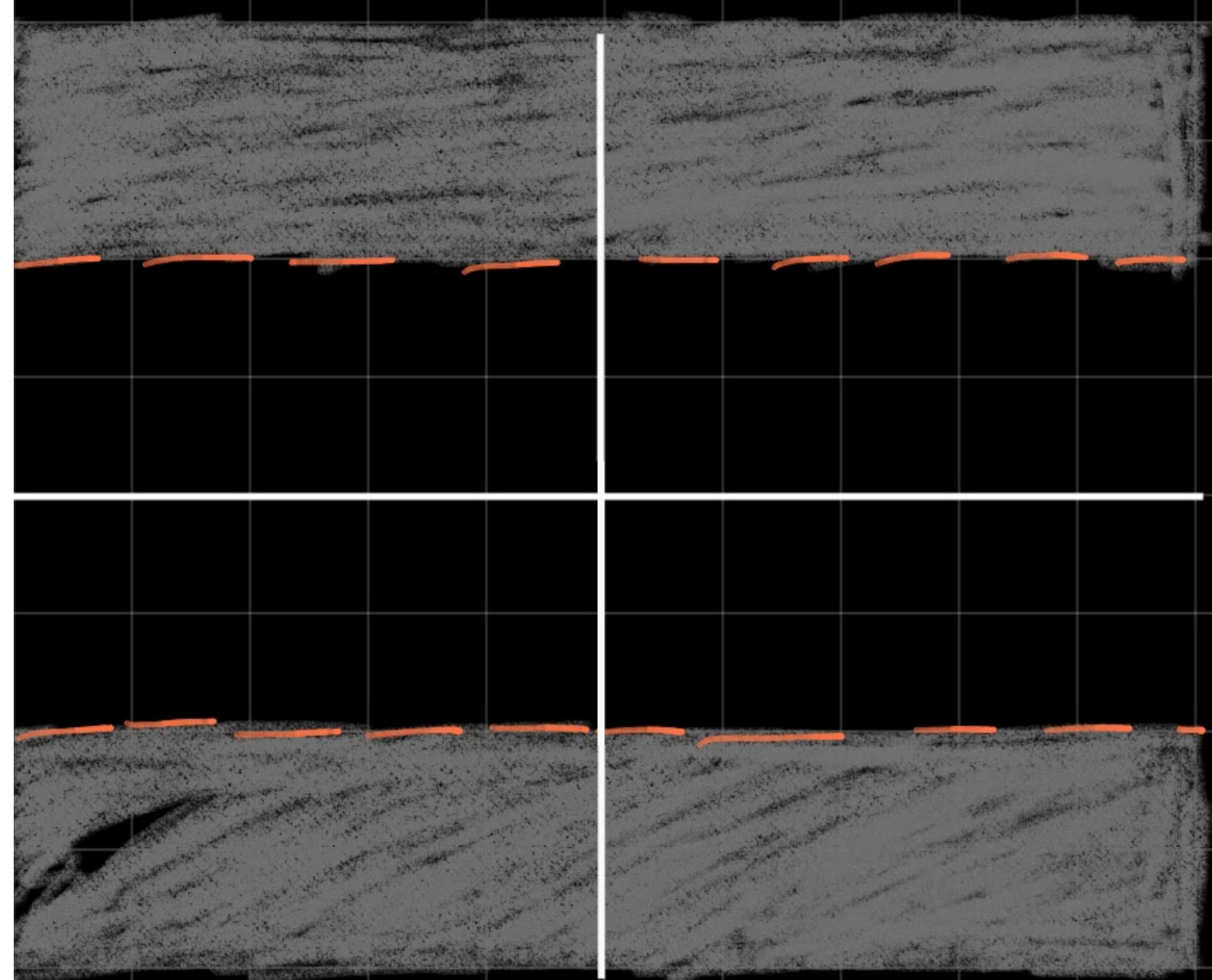
$$k \geq (2n+1)$$

8-

a) $\{z \in \mathbb{C} \mid |Re(z)| <$



b) $\{z \in \mathbb{C} \mid |\operatorname{Im}(z)| > 3\}$



c) $\{z \in \mathbb{C} \mid |z-1| = |z+i|\}$

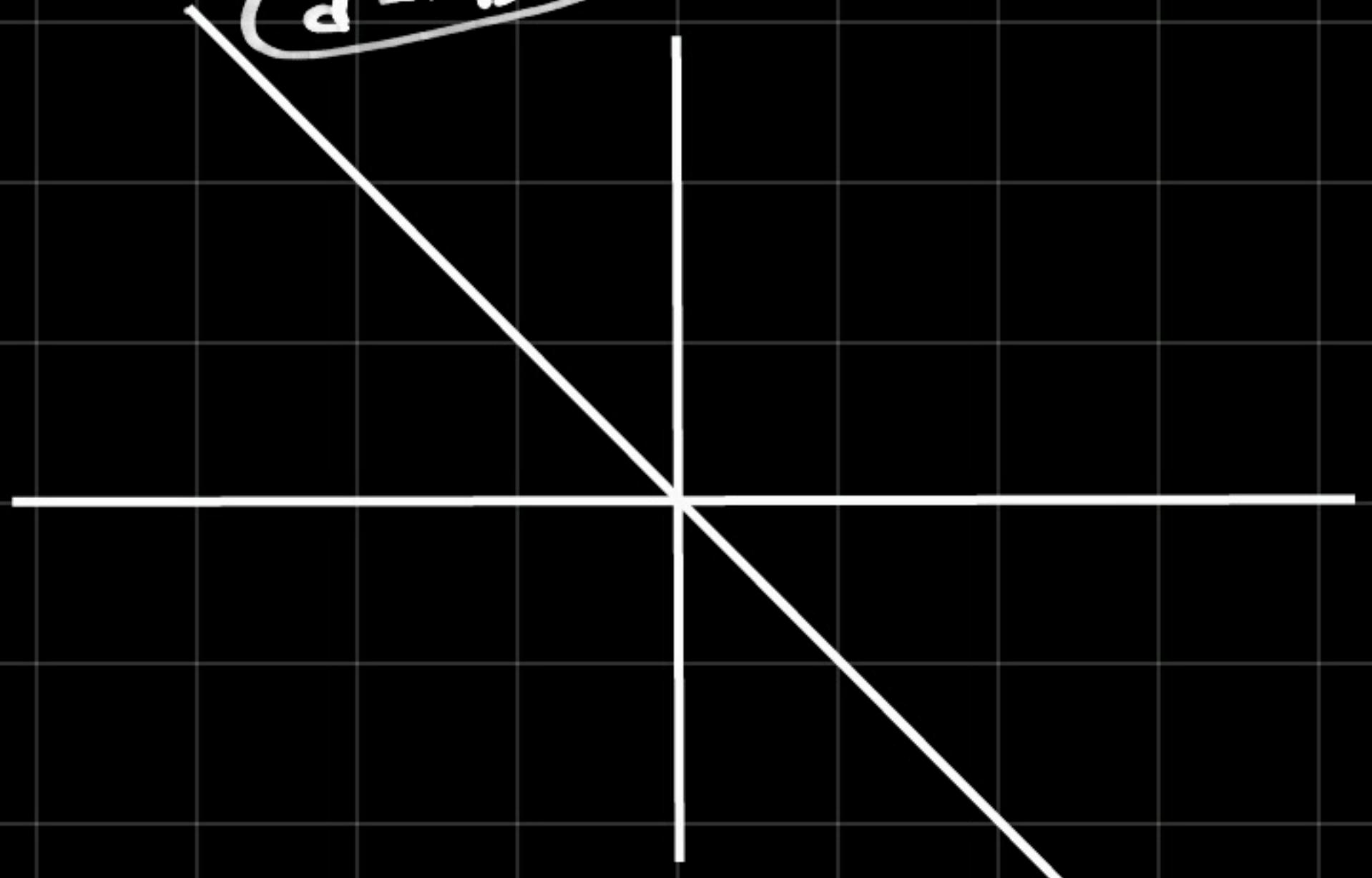
$$\sqrt{(a-1)^2 + b^2} = \sqrt{a^2 + (b+1)^2}$$

$$a^2 - 2a + 1 + b^2 = a^2 + b^2 + 2b + 1$$

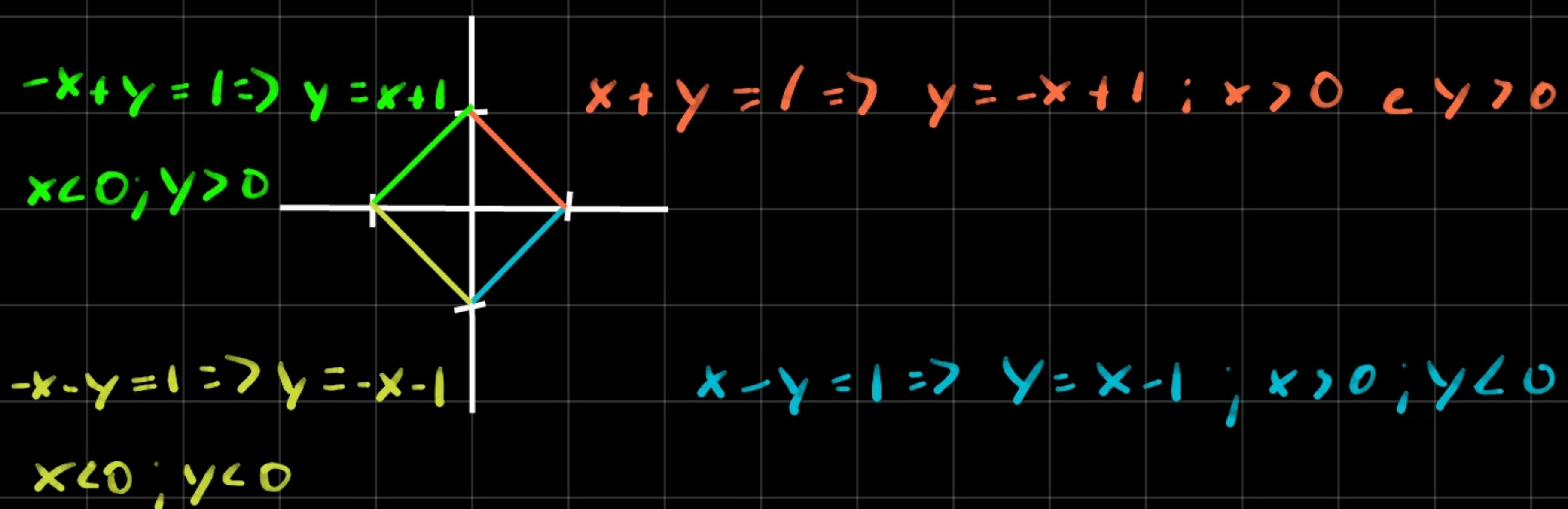
$$-2a = 2b$$

$$a = -b$$

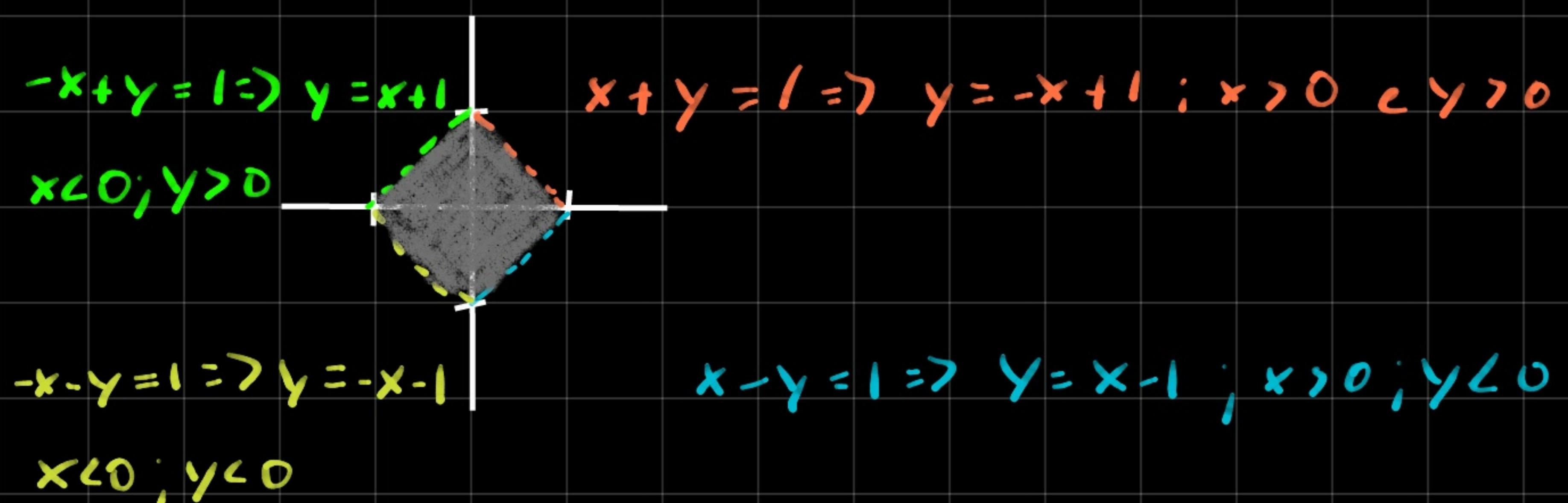
∴ $a = -b$



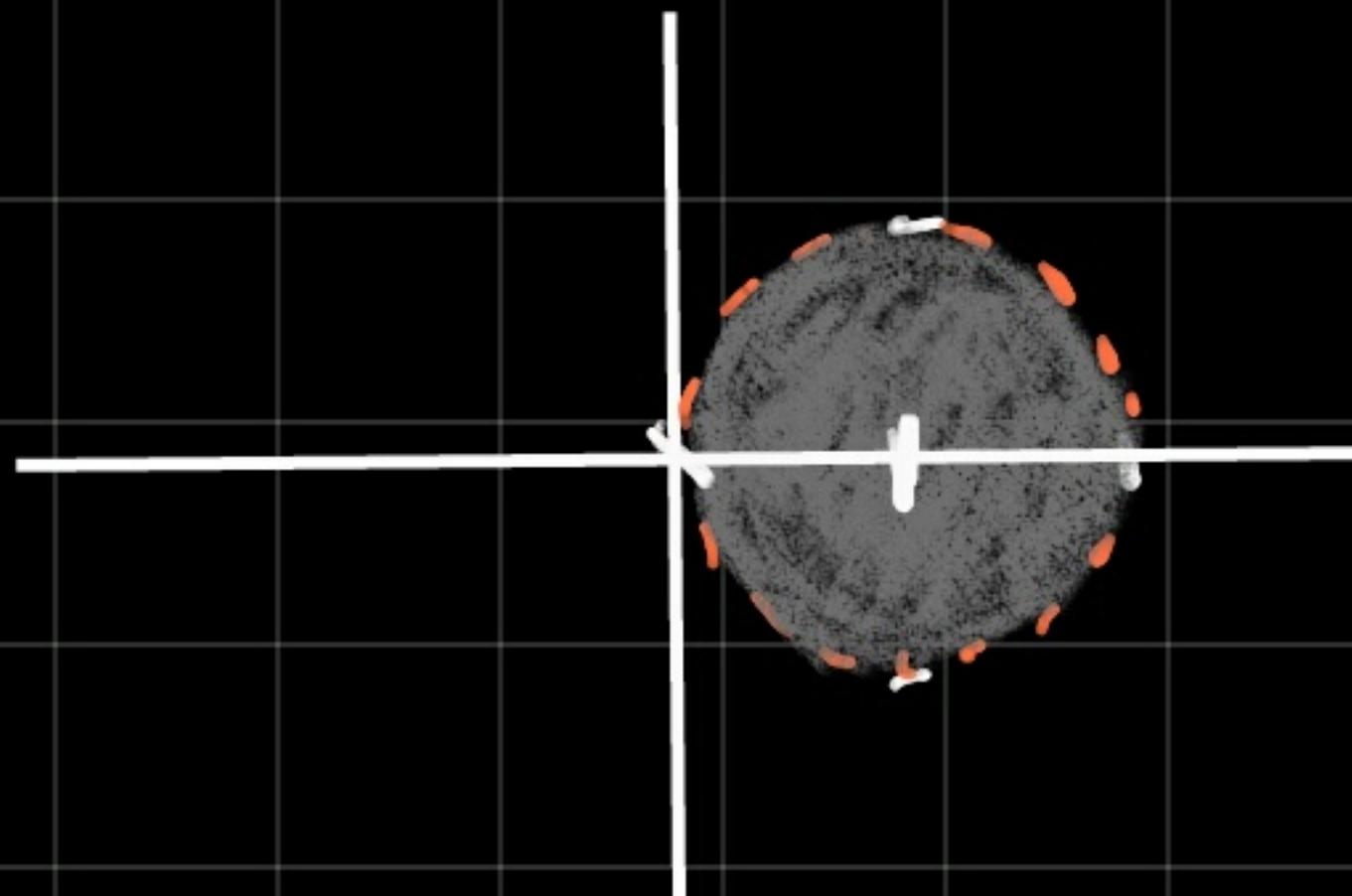
$$d) \{z \in \mathbb{C} ; |\operatorname{Re}(z)| + |\operatorname{Im}(z)| = 1\}$$



$$e) \{z \in \mathbb{C} ; |\operatorname{Re}(z)| + |\operatorname{Im}(z)| < 1\}$$



$$f) z \in \mathbb{C} ; |z-1| < 1$$



$$g) \{z \in \mathbb{C} ; |z+i| < 2\}$$

