[BAUPC 2002 3번]

3. Let V be the initial speed. The horizontal speed and initial vertical speed are then $V\cos\theta$ and $V\sin\theta$, respectively. You can easily show that the distance traveled in the air is the standard

$$d_{\text{air}} = \frac{2V^2\sin\theta\cos\theta}{g}. \tag{5}$$

To find the distance traveled along the ground, we must determine the horizontal speed just after the impact has occurred. The normal force, N, from the ground is what reduces the vertical speed from $V\sin\theta$ to zero, during the impact. So we have

$$\int N\,dt = mV\sin\theta, \tag{6}$$

where the integral runs over the time of the impact. But this normal force (when multiplied by μ, to give the horizontal friction force) also produces a sudden decrease in the horizontal speed, during the time of the impact. So we have

$$m\Delta v_x = -\int(\mu N)\,dt = -\mu mV\sin\theta \qquad \Longrightarrow \qquad \Delta v_x = -\mu V\sin\theta. \tag{7}$$

(We have neglected the effect of the mg gravitational force during the short time of the impact, since it is much smaller than the N impulsive force.) Therefore, the brick begins its sliding motion with speed

$$v = V\cos\theta - \mu V\sin\theta. \tag{8}$$

Note that this is true only if $\tan\theta \le 1/\mu$. If θ is larger than this, then the horizontal speed simply becomes zero, and the brick moves no further. (Eq. (8) would give a negative value for v.)

The friction force from this point on is μmg, so the acceleration is $a = -\mu g$. The distance traveled along the ground can easily be shown to be

$$d_{\text{ground}} = \frac{(V\cos\theta - \mu V\sin\theta)^2}{2\mu g}. \tag{9}$$

We want to find the angle that maximizes the total distance, $d_{\text{total}} = d_{\text{air}} + d_{\text{ground}}$. From eqs. (5) and (9) we have

$$\begin{aligned} d_{\text{total}} &= \frac{V^2}{2\mu g}\left(4\mu\sin\theta\cos\theta + (\cos\theta - \mu\sin\theta)^2\right) \\ &= \frac{V^2}{2\mu g}(\cos\theta + \mu\sin\theta)^2. \end{aligned} \tag{10}$$

Taking the derivative with respect to θ, we see that the maximum total distance is achieved when

$$\tan\theta = \mu. \tag{11}$$

Note, however, that the above analysis is valid only if $\tan\theta \le 1/\mu$ (from the comment after eq. (8)). We therefore see that if:

- $\mu < 1$, then the optimal angle is given by $\tan\theta = \mu$. (The brick continues to slide after the impact.)

[BAUPC 2004 2번]

2. Let the total mass of the rope be m, and let a fraction f of it hang in the air. Consider the right half of this section. Its weight, $(f/2)mg$, must be balanced by the vertical component, $T\sin\theta$, of the tension at the point where it joins the part of the rope touching the right platform. The tension at that point is therefore $T = (f/2)mg/\sin\theta$.

Now consider the part of the rope touching the right platform, which has mass $(1-f)m/2$. The normal force from the platform is $N = (1-f)(mg/2)\cos\theta$, so the maximal friction force also equals $(1-f)(mg/2)\cos\theta$, because $\mu = 1$. This fiction force must balance the sum of the gravitational force component along the plane, which is $(1-f)(mg/2)\sin\theta$, plus the tension at the lower end, which we found above. Therefore,

$$\frac{1}{2}(1-f)mg\cos\theta = \frac{1}{2}(1-f)mg\sin\theta + \frac{fmg}{2\sin\theta}. \tag{1}$$

This gives

$$f = \frac{F(\theta)}{1+F(\theta)}, \qquad \text{where} \quad F(\theta) \equiv \cos\theta\sin\theta - \sin^2\theta. \tag{2}$$

This expression for f is a monotonically increasing function of $F(\theta)$, as you can check. The maximal f is therefore obtained when $F(\theta)$ is as large as possible. Using the double-angle formulas, we can rewrite $F(\theta)$ as

$$F(\theta) = \frac{1}{2}(\sin 2\theta + \cos 2\theta - 1). \tag{3}$$

The derivative of this is $\cos 2\theta - \sin 2\theta$, which equals zero when $\tan 2\theta = 1$. Therefore,

$$\theta_{\max} = 22.5^\circ. \tag{4}$$

Eq. (3) then yields $F(\theta_{\max}) = (\sqrt{2}-1)/2$, and so eq. (2) gives

$$f_{\max} = \frac{\sqrt{2}-1}{\sqrt{2}+1} = (\sqrt{2}-1)^2 = 3 - 2\sqrt{2} \approx 0.172. \tag{5}$$

5. Let the angular speeds of the tops be ω_i, starting with the top one (so $\omega_1 \equiv \omega$). Let I be the moment of inertia of each top around its symmetry axis. Let Ω be the angular speed of precession.

We will use $\vec{\tau} = d\vec{L}/dt$ on each top. We therefore must determine $d\vec{L}/dt$ and the torque $\vec{\tau}$ for each top.

- $d\vec{L}/dt$:
  If the ω_i's are large enough (as we are assuming), then the angular momentum of the ith top will have magnitude essentially equal to $L_i = I\omega_i$, and $\vec{L_i}$ will point along the symmetry axis. (In other words, we can neglect the angular momentum due to the slow angular velocity of precession. We will see below that $\Omega \propto 1/\omega$.)
  The tip of $\vec{L_i}$ will trace out a circle of radius $L_i \sin\theta$, with angular speed Ω. Therefore,
  $$\left|\frac{d\vec{L_i}}{dt}\right| = \Omega L_i \sin\theta = \Omega I \omega_i \sin\theta, \tag{20}$$
  and $d\vec{L_i}/dt$ points tangentially around the circle.
- $\vec{\tau}$:
  None of the N tops are accelerating in the vertical direction. Therefore, the forces on the bottom top are NMg upward (to balance the weight of all the tops) at its lower end, and $(N-1)Mg$ downward (to keep up the other $N-1$ tops) at its upper end. The torque on the bottom top (around its CM) therefore has magnitude $(2N-1)Mgr\sin\theta$, where r is half the length of a top. It points perpendicular to the page.
  It is easy to see that the torque on the second-to-bottom top has magnitude $(2N-3)Mgr\sin\theta$, and so on, until the torque on the top top is $Mgr\sin\theta$.
  So the torque on the ith top has magnitude $(2i-1)Mgr\sin\theta$.

Equating $\vec{\tau}$ with $d\vec{L_i}/dt$ gives

$$(2i-1)Mgr\sin\theta = \Omega I \omega_i \sin\theta. \tag{21}$$

Therefore,

$$\omega_i = (2i-1)\omega_1 \equiv (2i-1)\omega. \tag{22}$$

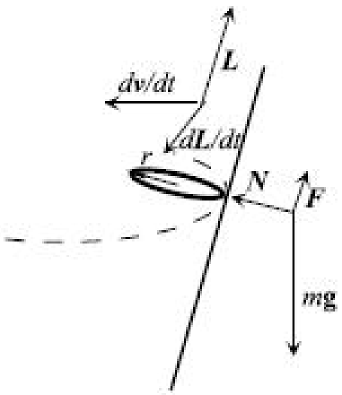
NOTE: As a double-check, the reader can verify that these ω_i's make $\vec{\tau} = d\vec{L}/dt$ true, where $\vec{\tau}$ and $\vec{L}$ are the total torque and angular momentum about the CM of the entire system.

5. (a) The forces on the particle are gravity (mg) and the normal force (N) from the cone. In our situation, there is no net force in the vertical direction, so

$$N \sin\theta = mg, \tag{17}$$

i.e., $N = mg/\sin\theta$. Therefore, the inward horizontal force, $N\cos\theta$, equals $mg/\tan\theta$. This force must account for the centripetal acceleration of the particle moving in a circle of radius $h\tan\theta$. Hence, $mg/\tan\theta = m(h\tan\theta)\omega^2$, and

$$\omega = \sqrt{\frac{g}{h}}\frac{1}{\tan\theta}. \tag{18}$$

(b) 

The forces on the ring are gravity (mg), the normal force (N) from the cone, and a friction force (F) pointing up along the cone. In our situation, there is no net force in the vertical direction, so

$$N\sin\theta + F\cos\theta = mg. \tag{19}$$

The fact that the inward horizontal force accounts for the centripetal acceleration yields

$$N\cos\theta - F\sin\theta = m(h\tan\theta)\omega^2. \tag{20}$$

We must now consider the torque, $\vec{\tau}$, on the ring. The torque is due solely to F (because gravity provides no torque, and N points though the center of the ring, by assumption (2) in the problem). So

$$\tau = rF, \tag{21}$$

and it points in the direction along the circular motion. Since $\vec{\tau} = d\vec{L}/dt$, we must now find $d\vec{L}/dt$.

$\vec{L}$ is made up of two pieces. One comes from the center of mass motion of the ring, which revolves around the axis of the cone. This part of $\vec{L}$ does not change, so we may neglect it in calculating $d\vec{L}/dt$. The other piece comes from the rotation of the ring. Let us call this part $\vec{L}'$. It points up along the cone, so the $\vec{L}'$ vector traces out a cone in which the tip of $\vec{L}'$ moves in a circle of radius $L' \sin\theta$. The frequency of this circular motion is of course the same ω as above. Therefore,

$$\frac{d\vec{L}}{dt} = \frac{d\vec{L}'}{dt} = \omega L' \sin\theta \tag{22}$$

(in the direction of the circular motion).
Thus, $\vec{\tau} = d\vec{L}/dt$ gives

$$rF = \omega L' \sin\theta. \tag{23}$$

But $L' = mr^2\omega'$, where ω' is the angular speed of the ring. And we know that ω' and ω are related by $r\omega' = (h\tan\theta)\omega$ (the rolling-without-slipping condition)[1]. Therefore $L' = mr(h\tan\theta)\omega$. Using this in eq. (23) yields

$$F = m\omega^2(h\tan\theta)\sin\theta. \tag{24}$$

Eqs. (19), (20), and (24) have the three unknowns, N, F, and ω. We can solve for ω by multiplying eq. (19) by $\cos\theta$, and eq. (20) by $\sin\theta$, and taking the difference, to obtain

$$F = mg\cos\theta - m\omega^2(h\tan\theta)\sin\theta. \tag{25}$$

Equating this expression for F with that in eq. (24) gives

$$\omega = \sqrt{\frac{g}{2h}\frac{1}{\tan\theta}}. \tag{26}$$

This frequency is $1/\sqrt{2}$ times the frequency found in part (a).
Remark: If one considers an object with moment of inertia $\rho m r^2$ (our ring has $\rho = 1$), then one can show by the above reasoning that the "2" in eq. (26) is simply replaced by $(1+\rho)$.